Assessing Regression Methods via Monte Carlo Simulations

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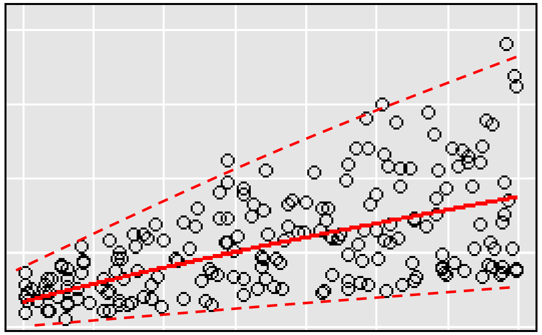
Motivation

Multiplicative error regressions are common



Many techniques exist

LOLS

MUPE

ZMPE

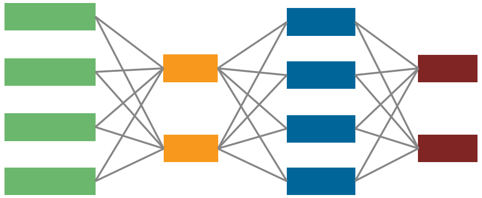
PING

MRLN

This begs the question: *Which is "the best"?*

Prior comparisons have relied on theoretical arguments and analyses of limited datasets

This investigation utilizes Millions of samples with controlled parameters to assess the techniques



The recommendations provided will be relevant to any and all efforts that utilize estimating relationships with multiplicative errors.

Questions We Are Trying To Answer

- On average, which regression method(s) estimate the **most accurate model parameters**?
- On average, which regression method(s) estimate the **most precise model parameters**?
- What are typical **convergence rates** for each regression method?
- **What are the factors** that determine whether a regression method tends to work better or worse than other methods?

Outline

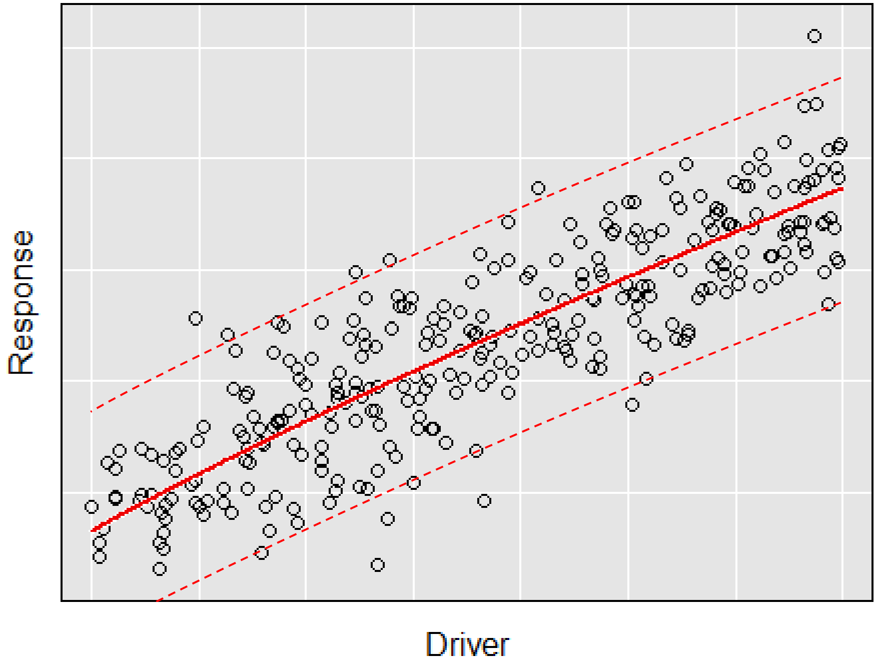
- Background
- Experimental Design
- Code Snippets
- Results
- Conclusions

Background

Additive vs. Multiplicative Error Models

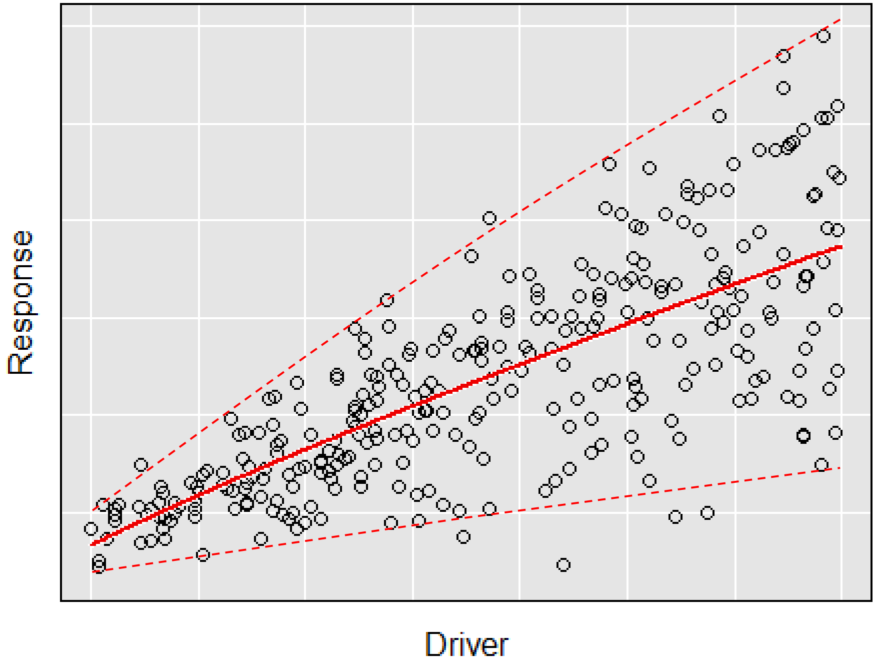
<u>Additive</u>: magnitude of residuals is **independent** of true model value.

$$y = f(\boldsymbol{X}) + \varepsilon$$
$$\varepsilon = y - f(\boldsymbol{X})$$



<u>Multiplicative</u>: magnitude of residuals is **proportional** to true model value.

$$y = f(\boldsymbol{X}) * \varepsilon$$
$$\varepsilon = \frac{y}{f(\boldsymbol{X})}$$



This investigation is concerned solely with **multiplicative error models**.

Alternative representation of multiplicative errors:

$$y = f(\boldsymbol{X}) * (1 + e) \Rightarrow e = \frac{y - f(\boldsymbol{X})}{f(\boldsymbol{X})} = \%\, error$$

where:
$e = \varepsilon - 1$

Log Error Model

If the multiplicative error term follows a **Lognormal distribution**, then the error can be measured as:[1]

$$e_i = \ln(\varepsilon_i) = \ln(y_i) - \ln\big(f(\boldsymbol{X}_i, \boldsymbol{\beta})\big)$$

Then we can solve for the model parameters, β, using **least-squares optimization** via this minimization:

$$\text{min.} \sum_{i=1}^{n} \Big(ln(y_i) - ln\big(f(\boldsymbol{X}_i, \boldsymbol{\beta})\big)\Big)^2$$

For models that can be linearized in log space, we can apply **Log-transformed Ordinary Least Squares** (LOLS). Otherwise, an optimization algorithm such as Levenberg-Marquardt must be used.

Common Criticism: although this solution is unbiased in log space, it is biased low when transformed back to unit space

- This is not a problem for generating prediction intervals, or while integrating a CER into a cost model – *so long as you correctly specify the uncertainty distribution.*[2]
- If you must have accurate point estimates within an individual regression, a multiplicative correction factor can be applied.[2]

PING correction factor:

p = # parameters
n = sample size
s = standard error of fit in log space

$$PING = \exp\left(\left(1 - \frac{p}{n}\right)\frac{s^2}{2}\right)$$

⇨ see reference #1 for derivation

[1] Hu, S. (2005), "The Impact of Using Log-Error CERs Outside the Data Range", SCEA Conference

[2] Jonov, B., Hu, S., and Smith, A. (2016), "How Regression Methods Impact Uncertainty Results", ICEAA Conference

Zero-bias Minimum Percent Error (ZMPE, or "zimpy")

Rather than assume a particular error distribution, one can directly **minimize the squared percent errors**:

$$\min. \sum_{i=1}^{n} \left(\frac{y_i - f(\boldsymbol{X}_i, \boldsymbol{\beta})}{f(\boldsymbol{X}_i, \boldsymbol{\beta})} \right)^2$$

Subject to the **zero-bias constraint**:

$$\sum_{i=1}^{n} \frac{y_i - f(\boldsymbol{X}_i, \boldsymbol{\beta})}{f(\boldsymbol{X}_i, \boldsymbol{\beta})} = 0$$

Without the forced constraint, the solution would be biased high, because the parameter estimates exist in both the numerator and denominator.

Additional recommended resources on ZMPE:

- Jonov, B., Hu, S., and Smith, A. (2016), "How Regression Methods Impact Uncertainty Results", ICEAA Conference
- Hu, S. (2016), "Generalized Degrees of Freedom", Journal of Cost Analysis and Parametrics, 9:93-111
- Smart, C. and Culver, G. (2009), "An Analytical Framework for CER Development", ISPA/SCEA Conference
- Hu, S. and Smith, A. (2007), "Why ZMPE When You Can MUPE?", ISPA/SCEA Conference

A Word on Constrained Optimization

The authors are aware of four **optimization algorithms** that can enable ZMPE regression:

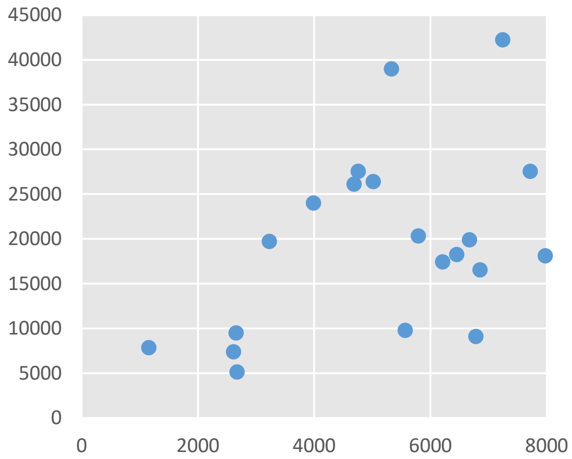
- Generalized Reduced Gradient (GRG) → from Excel Solver
- Constrained Optimization By Linear Approximation (COBYLA)
- Sequential Least-Squares Quadratic Programming (SLSQP)
- Augmented Lagrangian (AugLag)*

from the open-source NLopt library [1]

All methods were tested on a simulated dataset of the form $y = ax^b$ with a variety of starting guesses to test convergence behavior.

X	Y
3987	24002
2610	7411
7246	42239
6852	16528
7976	18126
5330	38972
6209	17432
2673	5133
4755	27547
5790	20332
6780	9114
6672	19905
1155	7875
7718	27548
4683	26109
6450	18242
5563	9772
2654	9510
5014	26408
3227	19721

the data



the results

		Starting Guess	Final Solution			
			GRG	COBYLA	SLSQP	AugLag
	a =	50	50.01	36.60	36.60	38.37
	b =	0.7	0.7009	0.7378	0.7378	0.7322
	a =	40	36.60	36.60	36.60	error
	b =	0.7	0.7378	0.7378	0.7378	error
	a =	40	36.62	36.61	36.60	36.60
	b =	1	0.7377	0.7378	0.7378	0.7378
	a =	1	36.65	36.60	error	error
	b =	1	0.7377	0.7378	error	error
	a =	10	36.68	36.60	36.60	36.60
	b =	1	0.7375	0.7378	0.7378	0.7378
	a =	100	error	36.60	36.60	36.60
	b =	1	error	0.7378	0.7378	0.7378
(LOLS sol'n)	a =	55.018	36.64	36.60	36.60	36.60
	b =	0.6777	0.7377	0.7378	0.7378	0.7378
(PING sol'n)	a =	61.178	36.60	36.60	36.60	error
	b =	0.6777	0.7378	0.7378	0.7378	error

COBYLA is recommended for the ZMPE method.

[1] https://nlopt.readthedocs.io/en/latest/

* AugLag utilizing LBFGS local optimizer

Minimum Unbiased Percent Error (MUPE, or “moopy”)

MUPE is a specific type of **Iteratively Re-weighted Least Squares** (IRLS), with weights equal to the squared inverse predictions from the prior iteration:

$$\text{min.} \sum_{i=1}^{n} \left(\frac{y_i - f(\boldsymbol{X}_i, \boldsymbol{\beta}_k)}{f(\boldsymbol{X}_i, \underbrace{\widehat{\boldsymbol{\beta}}_{k-1}}_{\text{parameter estimates from prior iteration}})} \right)^2$$

MUPE : IRLS :: square : rectangle

Iterative re-weighting decouples the numerator from the denominator, thereby eliminating bias without the need to explicitly impose a constraint.

IRLS is also commonly used to solve Generalized Linear Models (GLM).

Additional recommended resources on MUPE:

- Jonov, B., Hu, S., and Smith, A. (2016), “How Regression Methods Impact Uncertainty Results”, ICEAA Conference
- Smart, C. and Culver, G. (2009), “An Analytical Framework for CER Development”, ISPA/SCEA Conference
- Hu, S. and Smith, A. (2007), “Why ZMPE When You Can MUPE?”, ISPA/SCEA Conference

MLE Regression for Log Normal Error (MRLN, or “Merlin”)

For **power equations** of the form $Y = \beta_0 X_1^{\beta_1} \dots X_p^{\beta_p}$, with a lognormal-distributed multiplicative error, one can solve for the model parameters by minimizing the following quantity:[1]

$$\text{min.}\left[\frac{n}{2} ln(\theta) + \frac{1}{2\theta}\sum_{i=1}^{n}\left(ln(y_i) - ln(\beta_0) - \sum_{j=1}^{p}\beta_j ln\left(X_{ij}\right) + \frac{\theta}{2}\right)^2\right]$$

This is a **Maximum Likelihood Estimation** (MLE) solution.

See reference below for full explanation and derivation.

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[1] Smart, C. (2017), “Cutting the Gordian Knot: Maximum Likelihood Estimation for Regression of Log Normal Error”, ICEAA Conference

Introducing: Generalized Regression with MLE for Log Normal Errors (GRMLN, or “gremlin”)

We propose that MRLN can be generalized to any equation form, $Y = f(\boldsymbol{X}_i, \boldsymbol{\beta})$, by minimizing the following quantity:

$$\min.\left[\frac{n}{2}ln(\theta) + \frac{1}{2\theta}\underbrace{\sum_{i=1}^{n}\left(ln(y_i) - ln\big(f(\boldsymbol{X}_i, \boldsymbol{\beta})\big) + \frac{\theta}{2}\right)^2}\right]$$

estimates **mean** of lognormal distribution[1]

estimates **median** of lognormal distribution[1,2]

Note the relationship between GRMLN and the standard Log Error model:

$$\min.\overbrace{\sum_{i=1}^{n}\left(ln(y_i) - ln\big(f(\boldsymbol{X}_i, \boldsymbol{\beta})\big)\right)^2}$$

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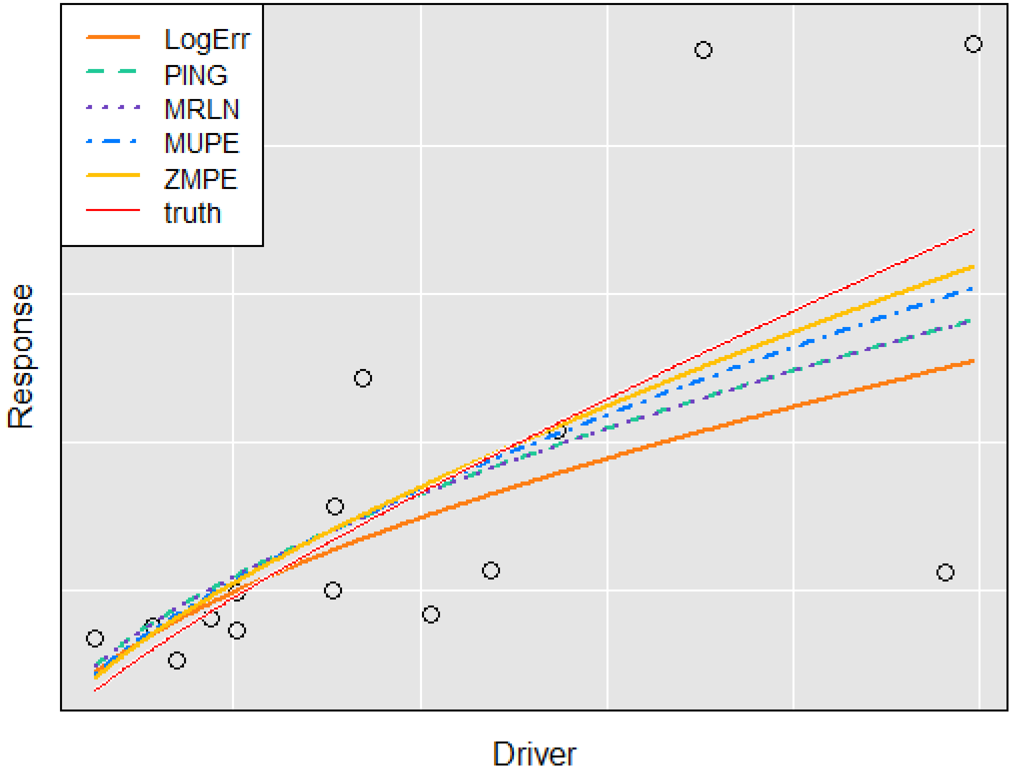
[1] Smart, C. (2017), “Cutting the Gordian Knot: MLE for Regression of Log Normal Error”, ICEAA Conference

[2] Jonov, B., Hu, S., and Smith, A. (2016), “How Regression Methods Impact Uncertainty Results”, ICEAA Conference

Experimental Design

Why Monte Carlo Simulations?

- **Sample statistics are merely proxies** for true population behavior
- A regression model with better goodness-of-fit measures than other models is **not necessarily optimal**
- Rather, the regression model that **most closely fits the true population** behavior is optimal
- With real data, true population behavior is typically **unknowable**



Why Monte Carlo Simulations?

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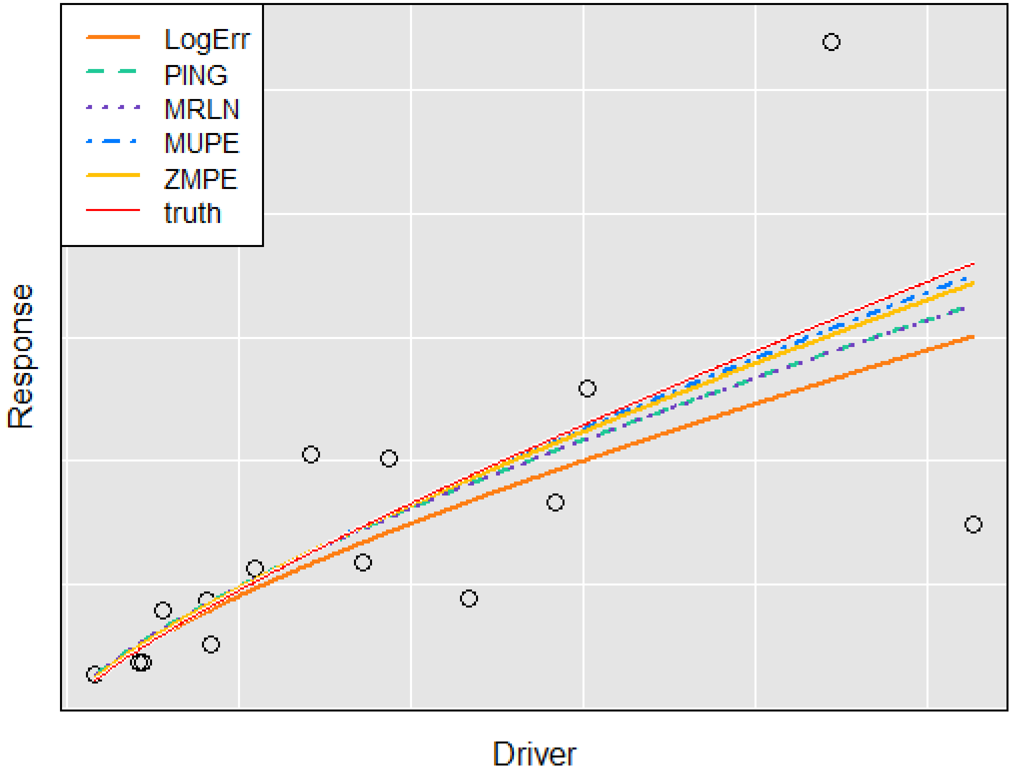
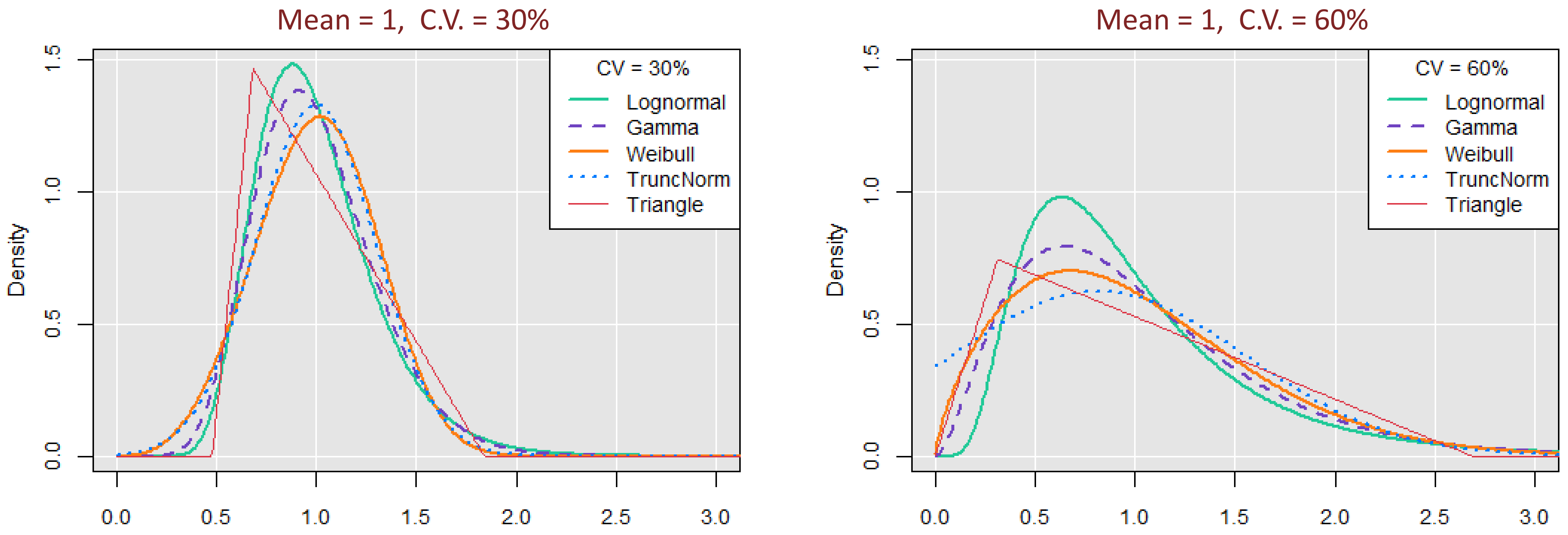
[view in slideshow mode for animation]



By **repeatedly** simulating datasets with known equations, parameters, and error terms, one can directly estimate how well regression methods model the true population behavior.

Multiplicative Error Distributions



Distributions were fit to **58** parametric CERs from the Unmanned Space Vehicle Cost Model (USCM11) database. The table shows how many passed the **Cramer-von Mises** test at 90% significance for each error distribution.

Gamma	Lognormal	Weibull	Trunc. Normal	Triangle
41	39	36	30	26

11 CERs yielded no acceptable fit!

It's **not** safe to assume Lognormal or any other specific distribution. One should always *fit rather than assume*.[1]

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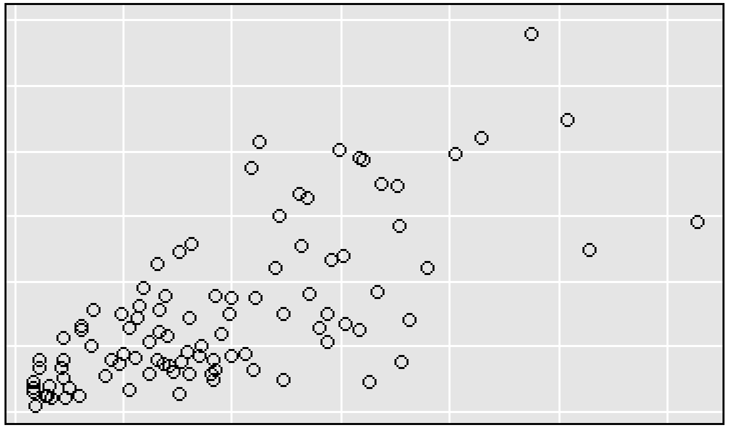
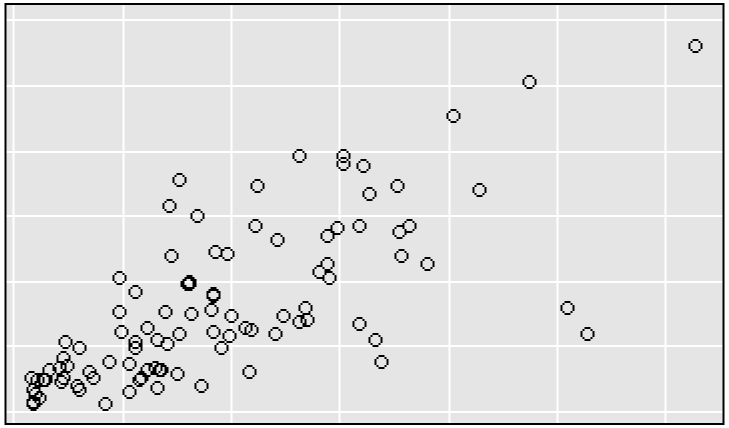
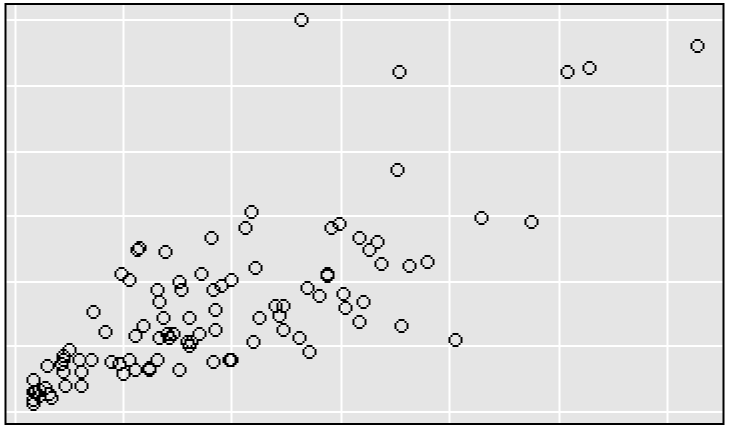
[1] Hu, S. (2013), "Fit, Rather Than Assume, a CER Error Distribution", ICEAA Conference

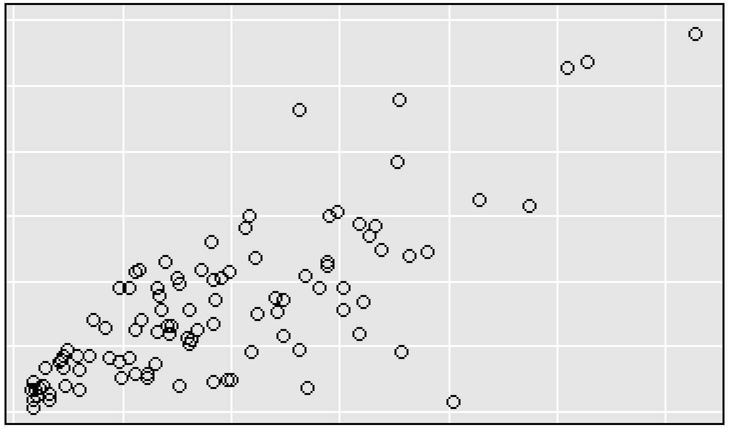
Quiz: Can you identify the error distribution?





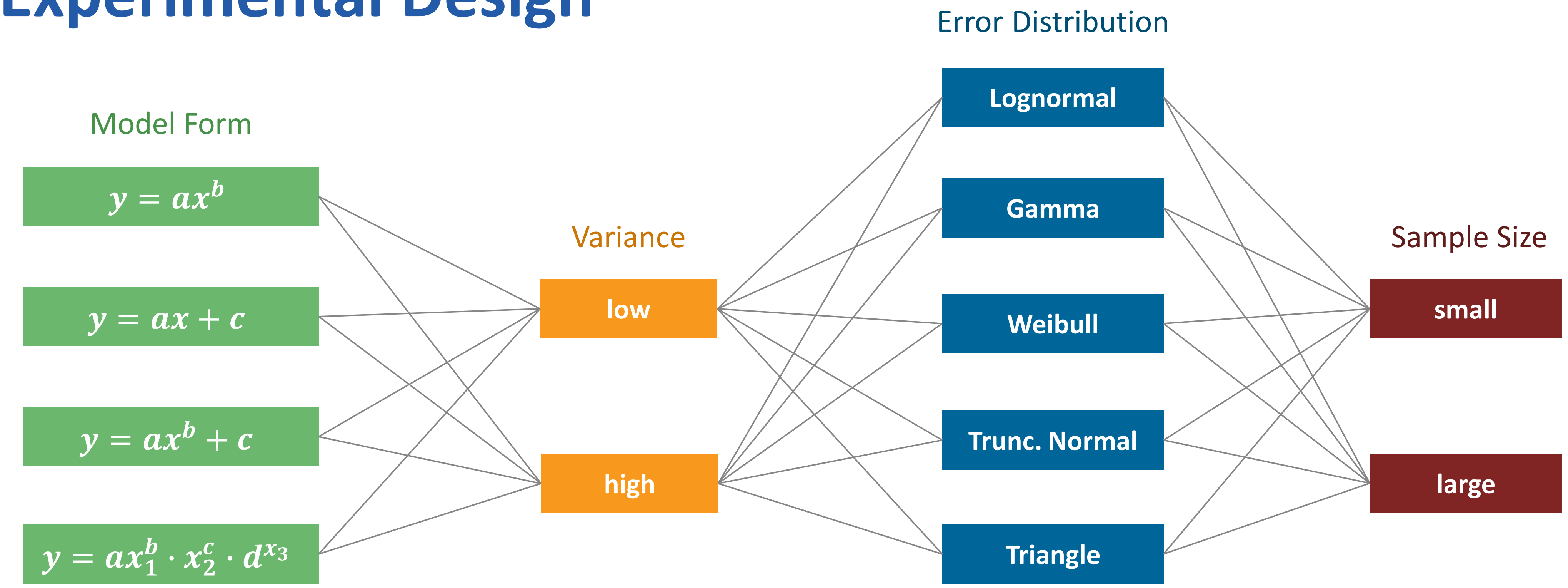
See backup for answer key.





All scatterplots generated with the same model form, parameters, random seed, sample size, sample variance, and axes limits – only difference is choice of multiplicative error distribution.

Experimental Design



Error Distribution

Model Form

$y = ax^b$

$y = ax + c$

$y = ax^b + c$

$y = ax_1^b \cdot x_2^c \cdot d^{x_3}$

Variance

low

high

Lognormal

Gamma

Weibull

Trunc. Normal

Triangle

Sample Size

small

large

$$(4\ model\ forms) \times (2\ variances) \times (5\ error\ distributions) \times (2\ sample\ sizes)$$
$$= 80\ distinct\ Monte\ Carlo\ simulations$$

Variance

	low	high
Std. % Error	30%	60%

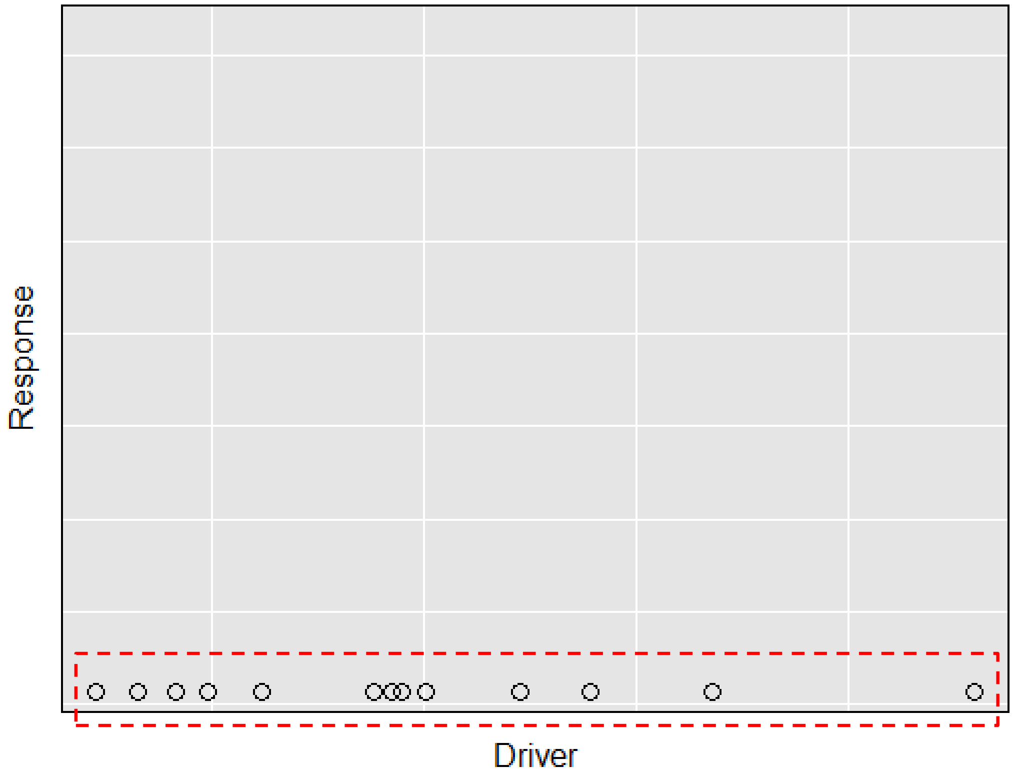
Sample Size

	small	large
n	15	50

$$(80\ sims) \times (16{,}000\ iterations\ each) \times (5\ regression\ methods)$$
$$= 6.4\ Million\ regressions\ calculated!$$

Walking Through a Simulation Iteration (1)

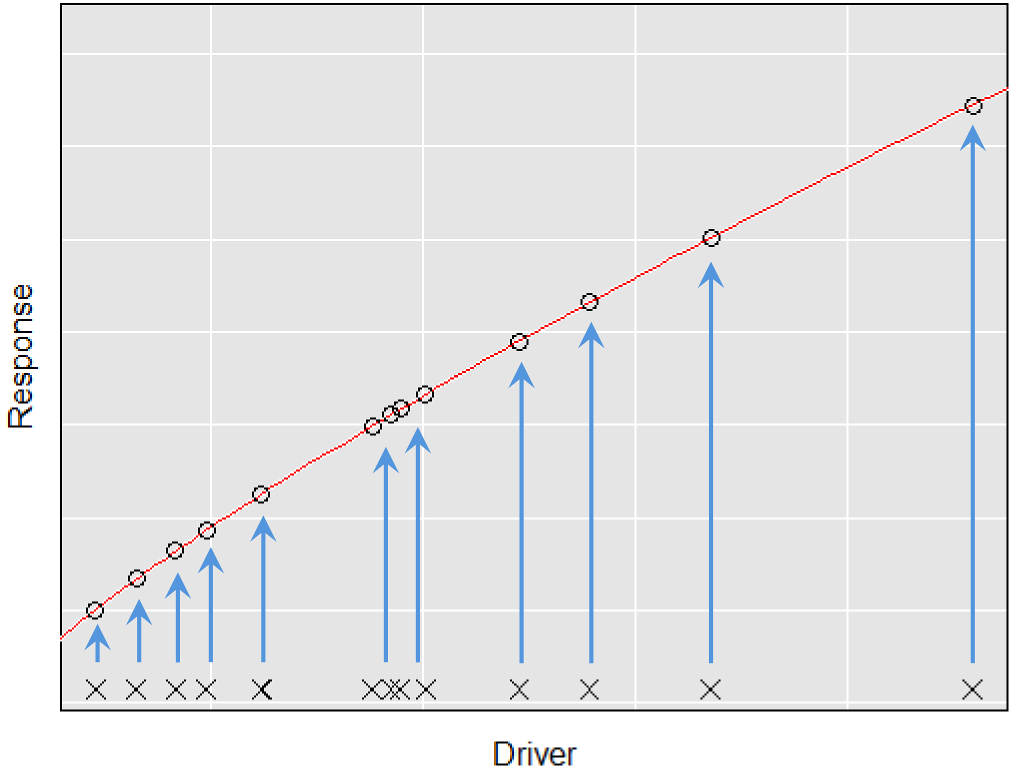
1) Generate random sample for driver variable, x
   - **Weibull** distribution used, which fits USCM cost drivers more often than Lognormal, Gamma, Truncated Normal, Triangle, or Uniform.



Walking Through a Simulation Iteration (2)

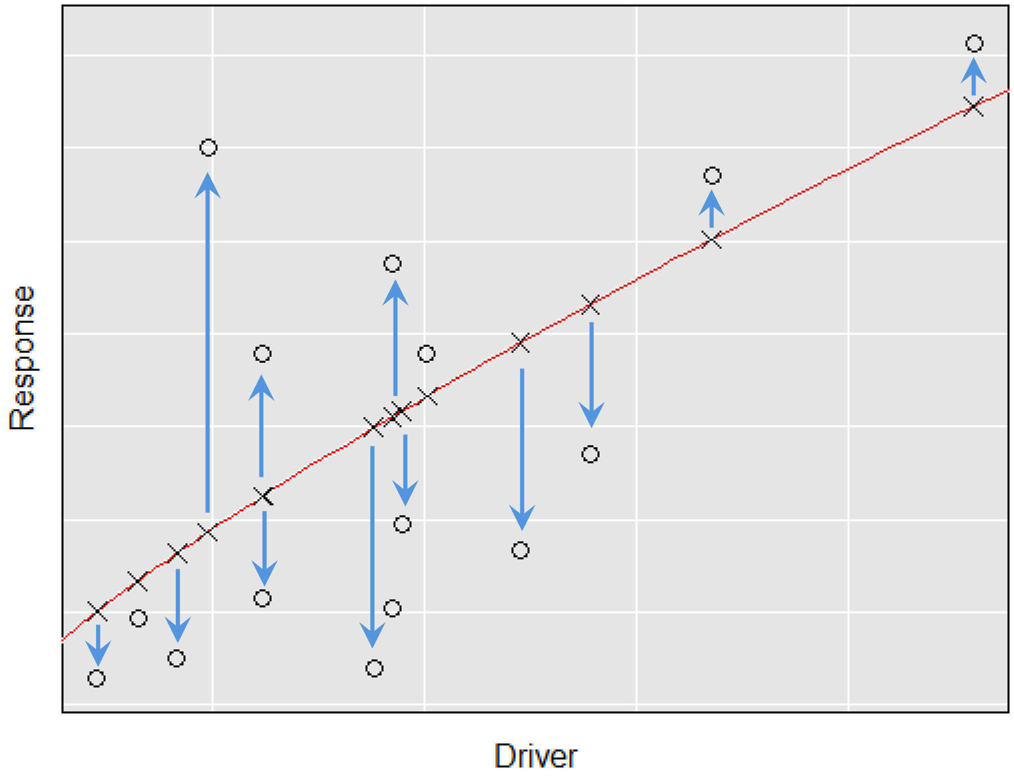
1) Generate random sample for driver variable, x

2) Define true population behavior (e.g. $y = 13.6\,x^{0.84}$)



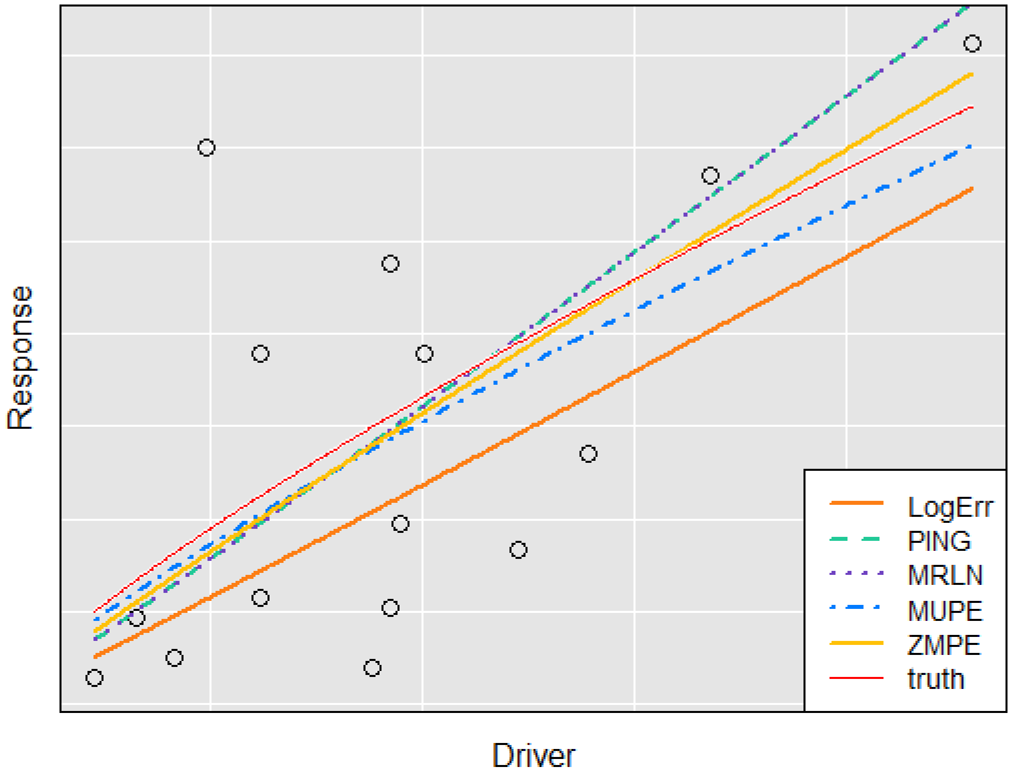
Walking Through a Simulation Iteration (3)

1) Generate random sample for driver variable, x
2) Define true population behavior (e.g. $y = 13.6\,x^{0.84}$)
3) Generate random sample from error distribution, and apply to y



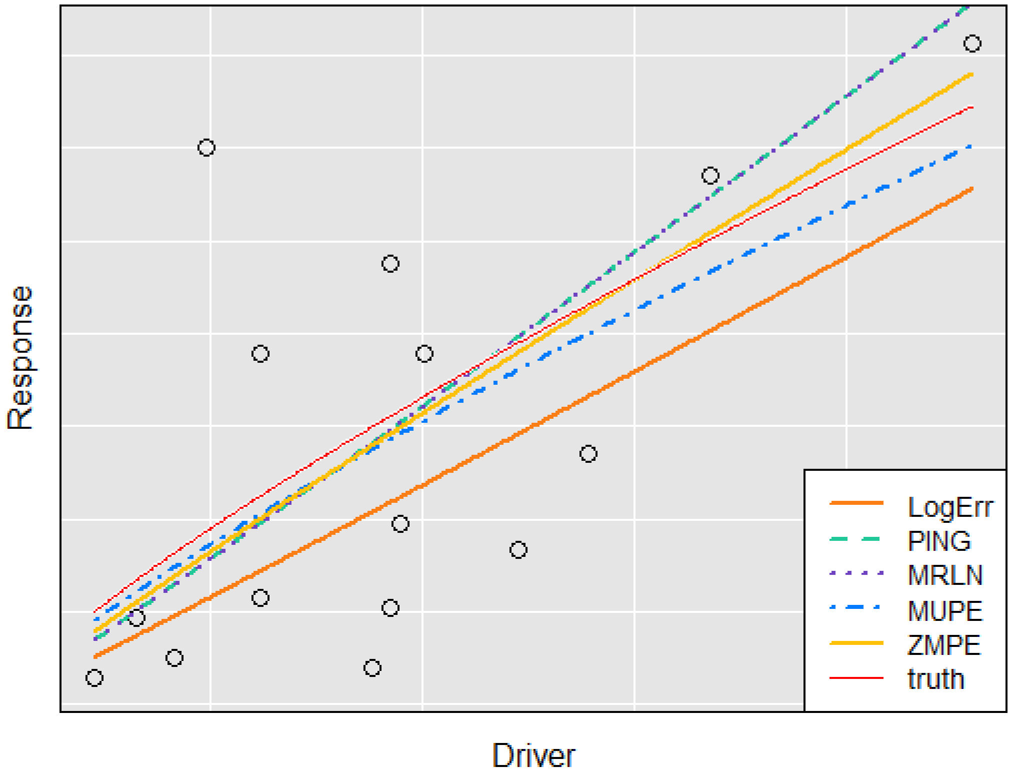
Walking Through a Simulation Iteration (4)

1) Generate random sample for driver variable, x
2) Define true population behavior (e.g. $y = 13.6\, x^{0.84}$)
3) Generate random sample from error distribution, and apply to y
4) Fit various regression models
    - Note PING and MRLN yield the same fit in this plot, and those on slides 14-15. More on that later…



Walking Through a Simulation Iteration (5)

1) Generate random sample for driver variable, x
2) Define true population behavior (e.g. $y = 13.6\ x^{0.84}$)
3) Generate random sample from error distribution, and apply to y
4) Fit various regression models
5) Compare parameter estimates to **true population parameters**

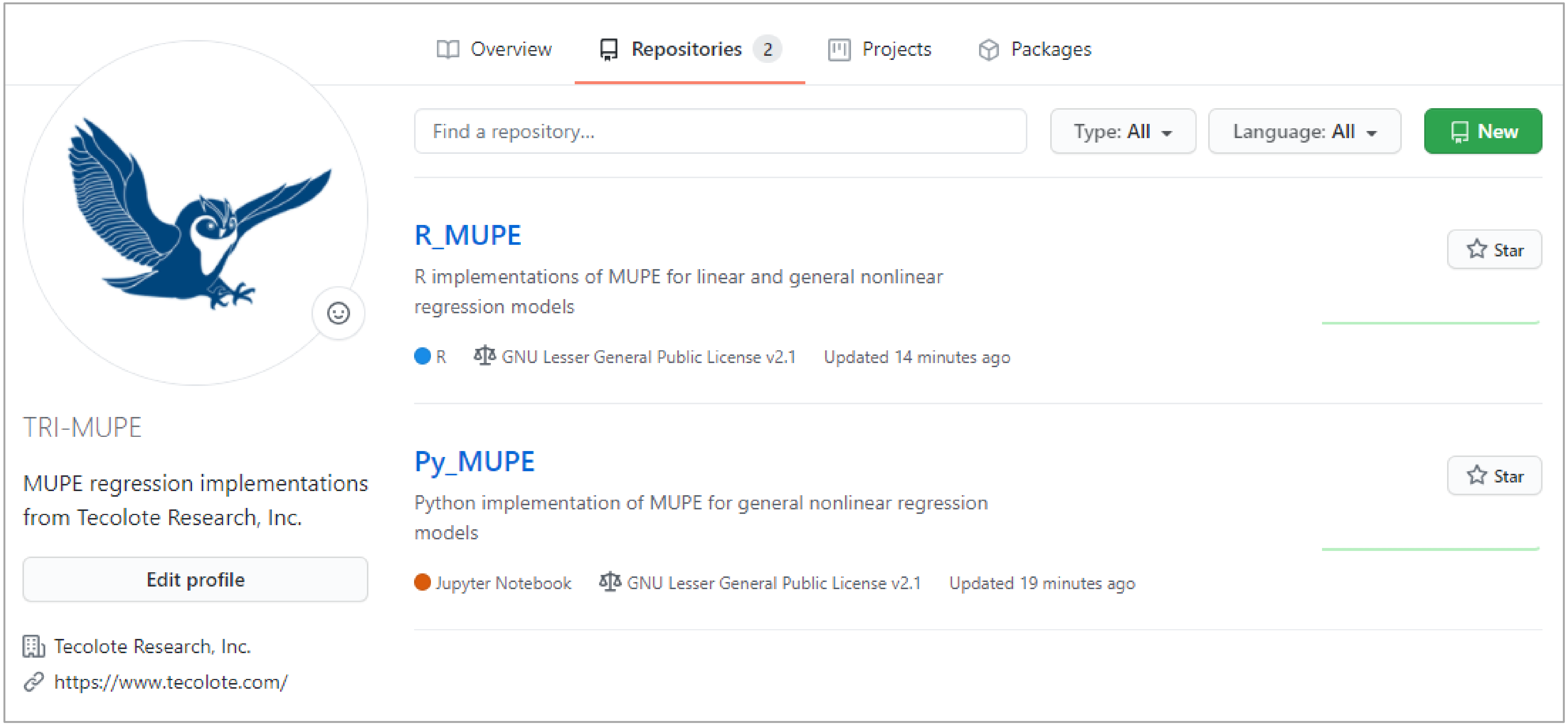


An individual iteration yields limited insights. But by aggregating results across *many* iterations, we can judge the **average fit quality** of each method.

Code Snippets

github.com/TRI-MUPE

visit me!



TRI-MUPE

MUPE regression implementations from Tecolote Research, Inc.

Tecolote Research, Inc.
https://www.tecolote.com/

R_MUPE

R implementations of MUPE for linear and general nonlinear regression models

● R GNU Lesser General Public License v2.1 Updated 14 minutes ago

Py_MUPE

Python implementation of MUPE for general nonlinear regression models

● Jupyter Notebook GNU Lesser General Public License v2.1 Updated 19 minutes ago

MUPE in R (for general nonlinear models)

```
# General nonlinear instantiation of the Minimum Unbiased Percent Error technique (MUPE)
# for multiplicative error models, which utilizes Iteratively Re-weighted Least Squares
# (IRLS) with weights equal to the squared inverse predictions from the prior iteration.
# Utilizes the MINPACK library implementation of the Levenberg-Marquardt algorithm. To
# install this, run 'install.packages("minpack.lm")' from the R console.
# Usage Example:
#   mlist = mupe_nonlinear(formula_str = "y ~ a * x1^b * c^x2", data = df,
#                          start = c(a=10, b=1, c=1))
#     - 'formula_str' must be a character string that resembles an R 'nls' formula object
#     - 'data' must be a dataframe containing the variables listed in formula_str
#     - 'start' is a named numeric vector of initial guesses (parameter names must
#       match those in formula_str). Whenever possible, provide values of the correct
#       sign and order of magnitude. For log-linear model forms, use the LOLS or PING
#       solution as the initial guess.
# Returns a list containing a standard R 'nls' object and accompanying details.
#
mupe_nonlinear = function(formula_str, data, start) {
  library(minpack.lm)                 # load Levenberg-Marquardt algorithm
  f = as.formula(formula_str)         # convert string to R formula object
  wt = rep(1, nrow(data))             # initialize weights
  pbeta = start;  conv = 1.0;  i = 0  # initialize other variables
  while (conv > 1e-5) {
    model = suppressWarnings(nlsLM(f, data, pbeta, weights=wt, control=list(maxiter=10)))
    wt = 1 / model$m$pred()^2           # calculate weights
    beta = model$m$getAllPars()         # solution of current iteration
    conv = max(abs((beta-pbeta)/beta))  # maximum fractional change in any parameter
    pbeta = beta                        # reset prior beta
    i = i + 1;  if (i == 200) break     # force stop, if necessary
  }
  return(list(model=model, start=start, mupe_iters=i))
}
```

Levenberg-Marquardt algorithm is recommended for least-squares curve fitting

Source code subject to the ***GNU LGPL v2.1*** *license.*

https://www.statisticshowto.com/levenberg-marquardt-algorithm/
https://en.wikipedia.org/wiki/Levenberg%E2%80%93Marquardt_algorithm
https://en.wikipedia.org/wiki/MINPACK

Demonstrating Nonlinear MUPE in R

Generate data to demonstrate equation of the form $y = b_0 + b_1 x_1^{b_2}$

```
# Simulate data
set.seed(0);  n = 40;  x1 = runif(n, 30, 160);  y = 42 + 3*x1^0.8
# Apply multiplicative gamma random error term with mean=1, cv=0.2
cv = 0.2;  y = y*rgamma(n, 1/cv^2, 1/cv^2);  my_df = data.frame('y'=y, 'x1'=x1)
```

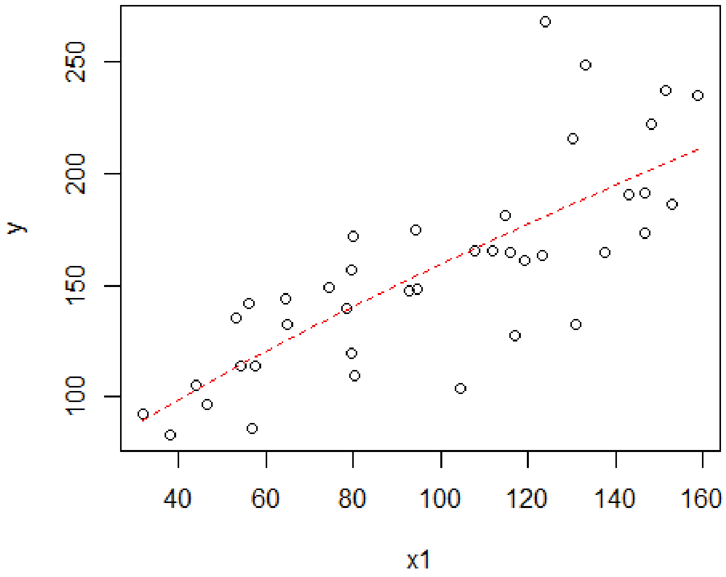
Apply method

```
my_mupe = mupe_nonlinear('y ~ b0 + b1*x1^b2', my_df, c(b0=10, b1=1, b2=1))
summary(my_mupe$model)
```

```
## Formula: y ~ b0 + b1 * x1^b2
##
## Parameters:
##    Estimate Std. Error t value Pr(>|t|)
## b0  40.8420    71.2435   0.573     0.57
## b1   3.2740    10.6432   0.308     0.76
## b2   0.7792     0.5704   1.366     0.18
##
## Residual standard error: 0.1763 on 37 degrees of freedom
```

Overlay fitted curve on scatterplot

```
par(mar=c(4.5,4.5,1,1))
plot(x1, y, xlab='x1', ylab='y')
xvec = seq(min(x1), max(x1), length.out=100)
lines(xvec, my_mupe$model$m$predict(data.frame('x1'=xvec)), col='red2', lty=2)
```



MUPE in R (for linear models)

```r
# Linear instantiation of the Minimum Unbiased Percent Error technique (MUPE) for
# multiplicative error models, which utilizes Iteratively Re-weighted Least Squares (IRLS)
# with weights equal to the squared inverse predictions from the prior iteration.
# Usage Example:
#   mupe = mupe_linear(formula_str = "y ~ 0 + x", data = df)
#     - 'formula_str' must be a character string that resembles an R 'lm' formula object
#       (default is no intercept, i.e. a simple factor model)
#     - 'data' must be a dataframe containing the variables listed in formula_str
# Returns a list containing a standard R 'lm' object and the number of iterations.
#
mupe_linear = function(formula_str = "y ~ 0 + x", data) {
  f = as.formula(formula_str)           # convert string to R formula object
  model = lm(f, data)                   # 1st iteration (Ordinary Least Squares)
  conv = 1.0;  i = 1                    # initialize convergence and counter
  while (conv > 1e-5) {
    wt = 1 / model$fitted^2             # calculate weights
    pbeta = model$coef                  # solution of prior iteration
    model = lm(f, data, weights=wt)     # weighted least squares
    beta = model$coef                   # solution of current iteration
    conv = max(abs((beta-pbeta)/beta))  # maximum fractional change in any parameter
    i = i + 1;  if (i == 200) break     # force stop, if necessary
  }
  return(list(model=model, mupe_iters=i))
}
```

Demonstrating Linear MUPE in R

Generate data to demonstrate equation of the form $y = b_0 + b_1 x_1$

```
# Simulate data
set.seed(0);  n = 20;  x1 = runif(n, 20, 150);  y = 180 + 6*x1
# Apply multiplicative lognormal random error term with mean=1, cv=0.3
cv = 0.3;  loc = log(1 / sqrt(cv^2 + 1));  shape = sqrt(log(1 + cv^2))
y = y*rlnorm(n, loc, shape)
my_df = data.frame('y'=y, 'x1'=x1)
```

Apply method

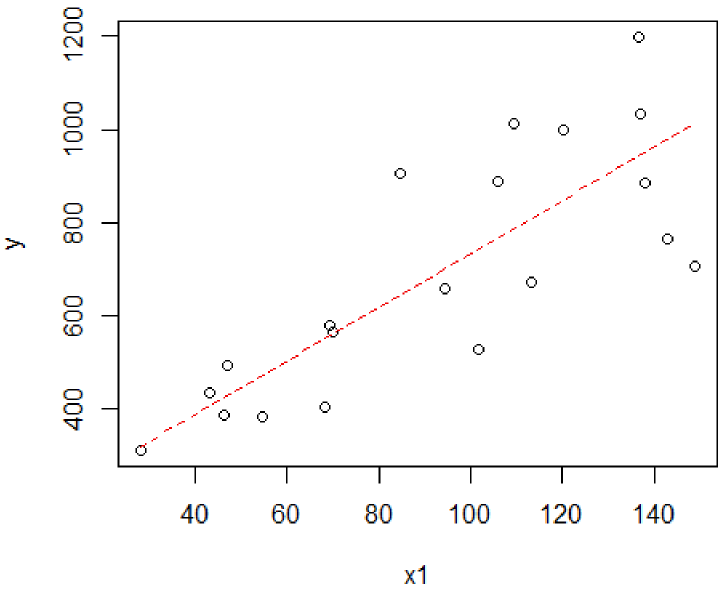
```
my_mupe = mupe_linear('y ~ x1', my_df)
```

```
summary(my_mupe$model)  # mupe$model is a standard R 'lm' object
```

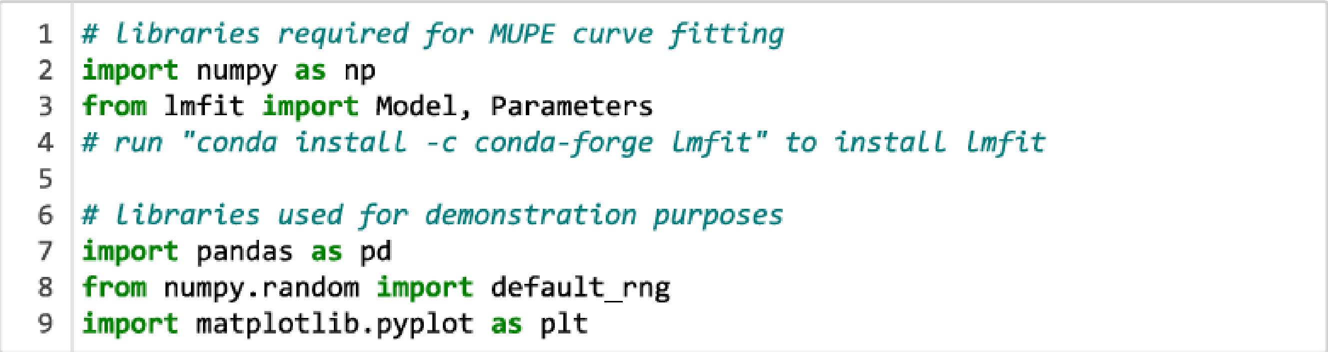
Overlay fitted curve on scatterplot

```
par(mar=c(4.5,4.5,1,1))
plot(x1, y, xlab='x1', ylab='y')
xvec = seq(min(x1), max(x1), length.out=100)
lines(xvec, predict(my_mupe$model, data.frame('x1'=xvec)), col='red2', lty=2)
```

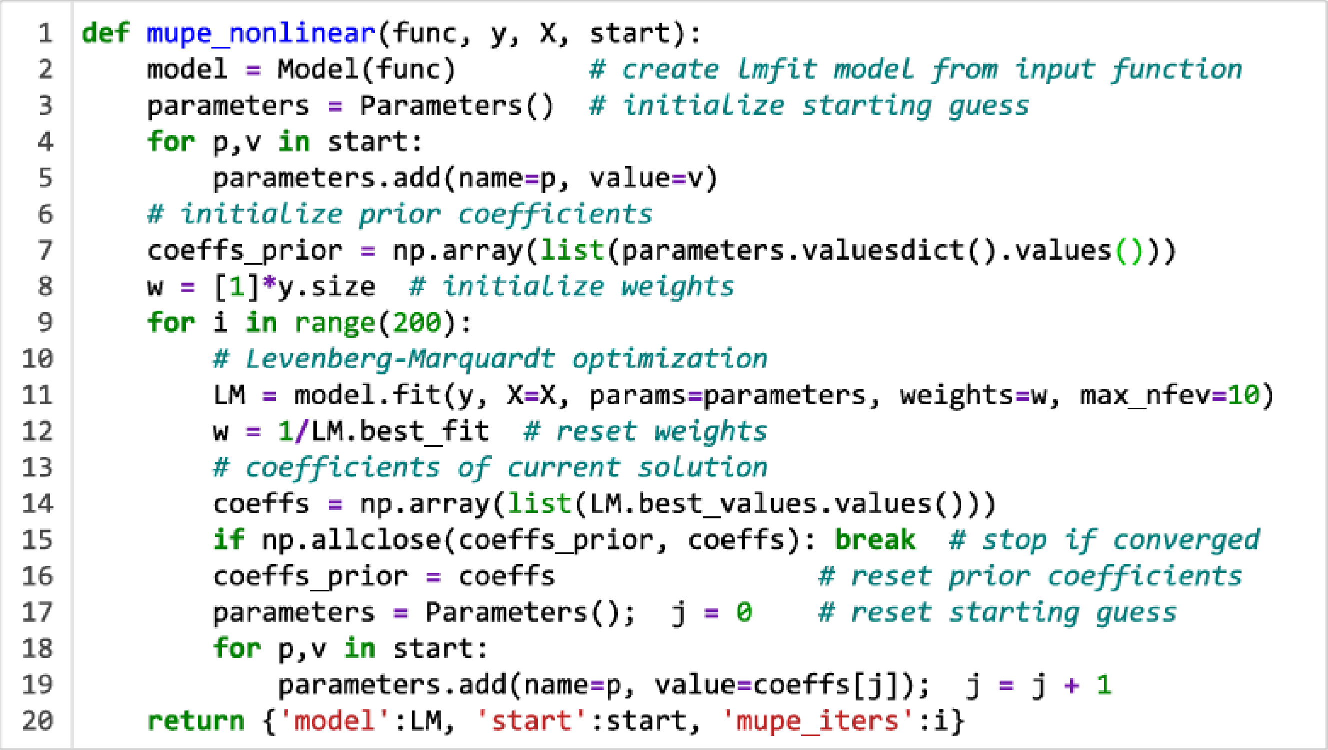
```
## lm(formula = f, data = data, weights = wt)
##
## Weighted Residuals:
##      Min       1Q   Median       3Q      Max
## -0.30411 -0.17301 -0.00728  0.15494  0.40601
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 158.7081    60.4194   2.627   0.0171 *
## x1            5.7392     0.7947   7.222 1.02e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.2102 on 18 degrees of freedom
## Multiple R-squared:  0.7434, Adjusted R-squared:  0.7292
## F-statistic: 52.15 on 1 and 18 DF,  p-value: 1.02e-06
```



MUPE in Python (for general nonlinear models)



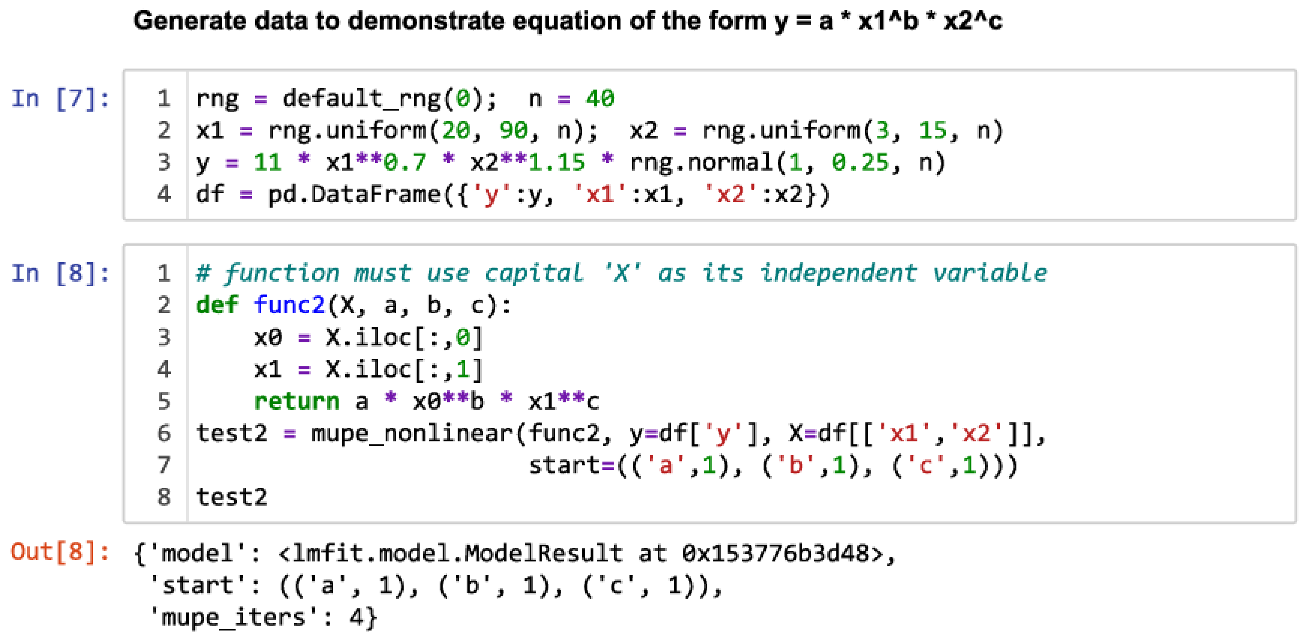
```python
# libraries required for MUPE curve fitting
import numpy as np
from lmfit import Model, Parameters
# run "conda install -c conda-forge lmfit" to install lmfit

# libraries used for demonstration purposes
import pandas as pd
from numpy.random import default_rng
import matplotlib.pyplot as plt
```



```python
def mupe_nonlinear(func, y, X, start):
    model = Model(func)        # create lmfit model from input function
    parameters = Parameters()  # initialize starting guess
    for p,v in start:
        parameters.add(name=p, value=v)
    # initialize prior coefficients
    coeffs_prior = np.array(list(parameters.valuesdict().values()))
    w = [1]*y.size  # initialize weights
    for i in range(200):
        # Levenberg-Marquardt optimization
        LM = model.fit(y, X=X, params=parameters, weights=w, max_nfev=10)
        w = 1/LM.best_fit  # reset weights
        # coefficients of current solution
        coeffs = np.array(list(LM.best_values.values()))
        if np.allclose(coeffs_prior, coeffs): break  # stop if converged
        coeffs_prior = coeffs                # reset prior coefficients
        parameters = Parameters();  j = 0    # reset starting guess
        for p,v in start:
            parameters.add(name=p, value=coeffs[j]);  j = j + 1
    return {'model':LM, 'start':start, 'mupe_iters':i}
```

*Source code subject to the **GNU LGPL v2.1** license.*

Demonstrating MUPE in Python

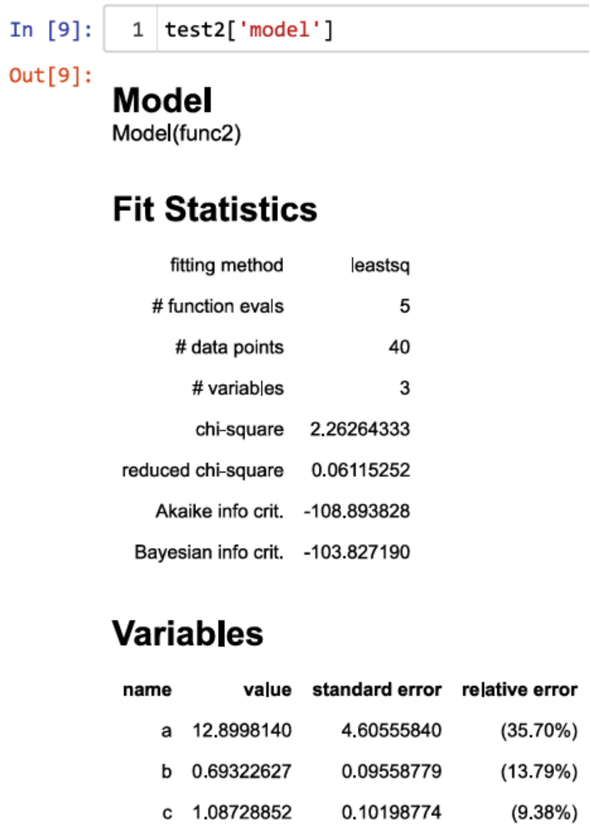


Generate data to demonstrate equation of the form y = a * x1^b * x2^c

In [7]:
```
rng = default_rng(0);  n = 40
x1 = rng.uniform(20, 90, n);  x2 = rng.uniform(3, 15, n)
y = 11 * x1**0.7 * x2**1.15 * rng.normal(1, 0.25, n)
df = pd.DataFrame({'y':y, 'x1':x1, 'x2':x2})
```

In [8]:
```
# function must use capital 'X' as its independent variable
def func2(X, a, b, c):
    x0 = X.iloc[:,0]
    x1 = X.iloc[:,1]
    return a * x0**b * x1**c
test2 = mupe_nonlinear(func2, y=df['y'], X=df[['x1','x2']],
                       start=(('a',1), ('b',1), ('c',1)))
test2
```

Out[8]:
```
{'model': <lmfit.model.ModelResult at 0x153776b3d48>,
 'start': (('a', 1), ('b', 1), ('c', 1)),
 'mupe_iters': 4}
```



In [9]:
```
test2['model']
```

Out[9]:

Model

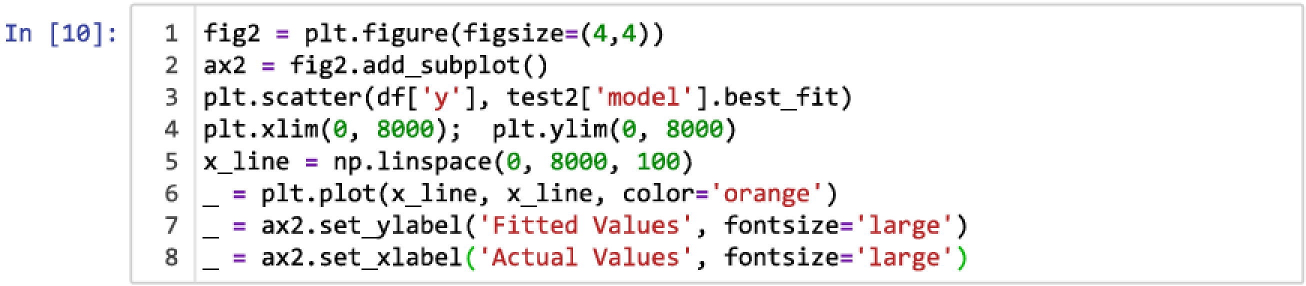
Model(func2)

Fit Statistics

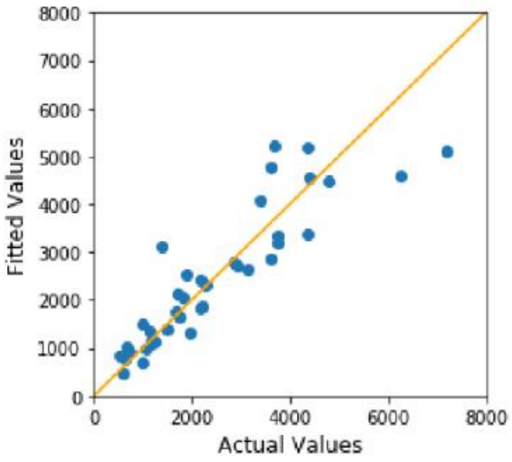
fitting method	leastsq
# function evals	5
# data points	40
# variables	3
chi-square	2.26264333
reduced chi-square	0.06115252
Akaike info crit.	-108.893828
Bayesian info crit.	-103.827190

Variables

name	value	standard error	relative error
a	12.8998140	4.60555840	(35.70%)
b	0.69322627	0.09558779	(13.79%)
c	1.08728852	0.10198774	(9.38%)



In [10]:
```
fig2 = plt.figure(figsize=(4,4))
ax2 = fig2.add_subplot()
plt.scatter(df['y'], test2['model'].best_fit)
plt.xlim(0, 8000);  plt.ylim(0, 8000)
x_line = np.linspace(0, 8000, 100)
_ = plt.plot(x_line, x_line, color='orange')
_ = ax2.set_ylabel('Fitted Values', fontsize='large')
_ = ax2.set_xlabel('Actual Values', fontsize='large')
```



See backup slides for examples of Log Error, PING, MRLN/GRMLN, and ZMPE in R

Results: Power Function

$$y = ax^b$$

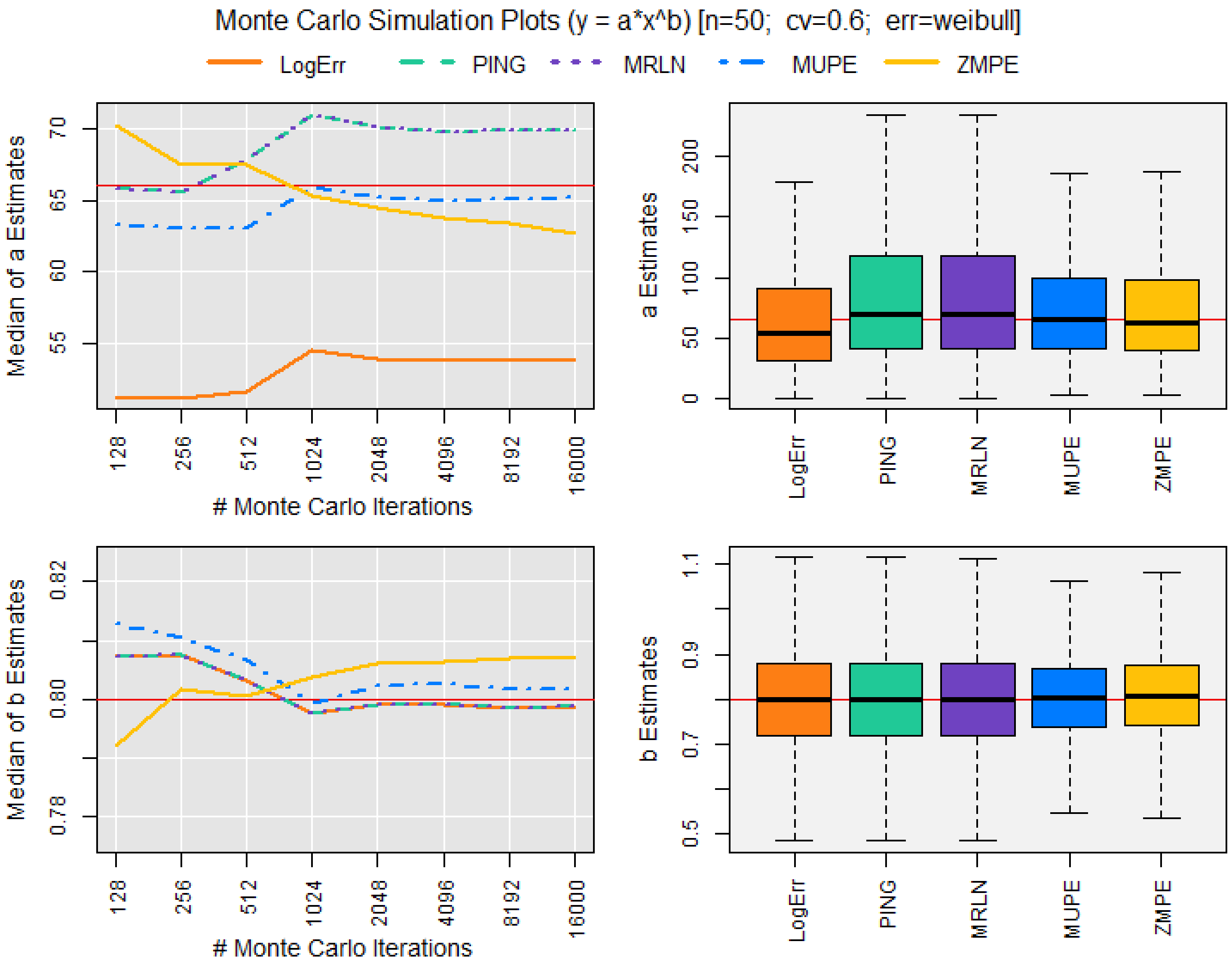
All simulations were run with 16,000 iterations.

Example Simulation Outputs: Weibull Error / Large Sample / High Variance

a = 66; *b* = 0.8

- MUPE provides most accurate estimates for *a*, followed by ZMPE
- PING and MRLN are biased high, and identical to each other
- Log-error provides lowest estimates for *a* (as expected)

- Log-error/PING/MRLN provide most accurate estimates for *b*, followed closely by MUPE
- MUPE and ZMPE estimates for both *a* and *b* have the best precision (i.e. least spread)



Results for all 20 Simulations ($y = ax^b$; parameter a)

Median of Parameter Estimates (closer to $a = 66$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	63.65	63.30	63.08	63.57	63.16	56.71	55.68	53.84	52.62	53.53	63.79	63.80	64.88	64.89	63.29	57.36	56.71	56.40	57.14	55.36
	PING	66.32	66.27	66.73	67.31	65.95	65.76	68.14	69.86	76.85	70.07	66.19	66.45	68.13	68.12	65.91	65.92	68.04	71.02	77.99	70.27
	MRLN	66.32	66.27	66.73	67.31	65.95	65.76	68.14	69.86	76.84	70.07	66.19	66.45	68.13	68.12	65.91	65.92	68.04	71.02	78.05	70.27
	MUPE	65.76	65.88	65.76	66.07	65.80	64.23	65.03	65.14	65.46	65.91	65.27	65.42	66.71	66.51	65.22	62.27	62.72	63.69	64.76	64.06
	ZMPE	64.96	64.70	65.34	65.31	65.15	61.00	62.34	62.73	63.57	64.57	63.21	64.16	64.92	64.91	63.85	56.26	58.08	60.02	59.89	59.85

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	42%	44%	50%	50%	44%	83%	96%	110%	130%	110%	84%	87%	95%	93%	88%	169%	201%	226%	257%	237%
	PING	42%	44%	50%	50%	44%	83%	96%	111%	129%	110%	84%	87%	95%	93%	88%	168%	201%	226%	265%	236%
	MRLN	42%	44%	50%	50%	44%	83%	96%	111%	129%	110%	84%	87%	95%	93%	88%	168%	201%	226%	265%	236%
	MUPE	43%	43%	45%	45%	45%	88%	89%	89%	89%	90%	85%	86%	87%	84%	88%	177%	190%	196%	191%	195%
	ZMPE	49%	47%	40%	41%	47%	126%	109%	94%	82%	79%	94%	90%	80%	80%	94%	249%	234%	200%	179%	184%

White cells are the best; Blue and Red cells are not as good.

Median and IQR used because all methods yielded highly skewed estimate distributions with many outliers.

PING solution used as starting guess for methods utilizing optimization algorithms.

Results for all 20 Simulations ($y = ax^b$; parameter a)

Median of Parameter Estimates (closer to $a = 66$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	63.65	63.30	63.08	63.57	63.16	56.71	55.68	53.84	52.62	53.53	63.79	63.80	64.88	64.89	63.29	57.36	56.71	56.40	57.14	55.36
	PING	66.32	66.27	66.73	67.31	65.95	65.76	68.14	69.86	76.85	70.07	66.19	66.45	68.13	68.12	65.91	65.92	68.04	71.02	77.99	70.27
	MRLN	66.32	66.27	66.73	67.31	65.95	65.76	68.14	69.86	76.84	70.07	66.19	66.45	68.13	68.12	65.91	65.92	68.04	71.02	78.05	70.27
	MUPE	65.76	65.88	65.76	66.07	65.80	64.23	65.03	65.14	65.46	65.91	65.27	65.42	66.71	66.51	65.22	62.27	62.72	63.69	64.76	64.06
	ZMPE	64.96	64.70	65.34	65.31	65.15	61.00	62.34	62.73	63.57	64.57	63.21	64.16	64.92	64.91	63.85	56.26	58.08	60.02	59.89	59.85

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	42%	44%	50%	50%	44%	83%	96%	110%	130%	110%	84%	87%	95%	93%	88%	169%	201%	226%	257%	237%
	PING	42%	44%	50%	50%	44%	83%	96%	111%	129%	110%	84%	87%	95%	93%	88%	168%	201%	226%	265%	236%
	MRLN	42%	44%	50%	50%	44%	83%	96%	111%	129%	110%	84%	87%	95%	93%	88%	168%	201%	226%	265%	236%
	MUPE	43%	43%	45%	45%	45%	88%	89%	89%	89%	90%	85%	86%	87%	84%	88%	177%	190%	196%	191%	195%
	ZMPE	49%	47%	40%	41%	47%	126%	109%	94%	82%	79%	94%	90%	80%	80%	94%	249%	234%	200%	179%	184%

Log Error method is consistently biased low (we already knew this, but it serves as a sanity check).
Furthermore, bias magnitude increases with increasing variance.

Results for all 20 Simulations ($y = ax^b$; parameter a)

Median of Parameter Estimates (closer to $a = 66$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	63.65	63.30	63.08	63.57	63.16	56.71	55.68	53.84	52.62	53.53	63.79	63.80	64.88	64.89	63.29	57.36	56.71	56.40	57.14	55.36
	PING	66.32	66.27	66.73	67.31	65.95	65.76	68.14	69.86	76.85	70.07	66.19	66.45	68.13	68.12	65.91	65.92	68.04	71.02	77.99	70.27
	MRLN	66.32	66.27	66.73	67.31	65.95	65.76	68.14	69.86	76.84	70.07	66.19	66.45	68.13	68.12	65.91	65.92	68.04	71.02	78.05	70.27
	MUPE	65.76	65.88	65.76	66.07	65.80	64.23	65.03	65.14	65.46	65.91	65.27	65.42	66.71	66.51	65.22	62.27	62.72	63.69	64.76	64.06
	ZMPE	64.96	64.70	65.34	65.31	65.15	61.00	62.34	62.73	63.57	64.57	63.21	64.16	64.92	64.91	63.85	56.26	58.08	60.02	59.89	59.85

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	42%	44%	50%	50%	44%	83%	96%	110%	130%	110%	84%	87%	95%	93%	88%	169%	201%	226%	257%	237%
	PING	42%	44%	50%	50%	44%	83%	96%	111%	129%	110%	84%	87%	95%	93%	88%	168%	201%	226%	265%	236%
	MRLN	42%	44%	50%	50%	44%	83%	96%	111%	129%	110%	84%	87%	95%	93%	88%	168%	201%	226%	265%	236%
	MUPE	43%	43%	45%	45%	45%	88%	89%	89%	89%	90%	85%	86%	87%	84%	88%	177%	190%	196%	191%	195%
	ZMPE	49%	47%	40%	41%	47%	126%	109%	94%	82%	79%	94%	90%	80%	80%	94%	249%	234%	200%	179%	184%

PING and MRLN results are **nearly identical**.

Results for all 20 Simulations ($y = ax^b$; parameter a)

Median of Parameter Estimates (closer to $a = 66$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	63.65	63.30	63.08	63.57	63.16	56.71	55.68	53.84	52.62	53.53	63.79	63.80	64.88	64.89	63.29	57.36	56.71	56.40	57.14	55.36
	PING	66.32	66.27	66.73	67.31	65.95	65.76	68.14	69.86	76.85	70.07	66.19	66.45	68.13	68.12	65.91	65.92	68.04	71.02	77.99	70.27
	MRLN	66.32	66.27	66.73	67.31	65.95	65.76	68.14	69.86	76.84	70.07	66.19	66.45	68.13	68.12	65.91	65.92	68.04	71.02	78.05	70.27
	MUPE	65.76	65.88	65.76	66.07	65.80	64.23	65.03	65.14	65.46	65.91	65.27	65.42	66.71	66.51	65.22	62.27	62.72	63.69	64.76	64.06
	ZMPE	64.96	64.70	65.34	65.31	65.15	61.00	62.34	62.73	63.57	64.57	63.21	64.16	64.92	64.91	63.85	56.26	58.08	60.02	59.89	59.85

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	42%	44%	50%	50%	44%	83%	96%	110%	130%	110%	84%	87%	95%	93%	88%	169%	201%	226%	257%	237%
	PING	42%	44%	50%	50%	44%	83%	96%	111%	129%	110%	84%	87%	95%	93%	88%	168%	201%	226%	265%	236%
	MRLN	42%	44%	50%	50%	44%	83%	96%	111%	129%	110%	84%	87%	95%	93%	88%	168%	201%	226%	265%	236%
	MUPE	43%	43%	45%	45%	45%	88%	89%	89%	89%	90%	85%	86%	87%	84%	88%	177%	190%	196%	191%	195%
	ZMPE	49%	47%	40%	41%	47%	126%	109%	94%	82%	79%	94%	90%	80%	80%	94%	249%	234%	200%	179%	184%

They perform best when the error distribution is indeed Lognormal
(this makes sense, since they both assume Log-normality).

Results for all 20 Simulations ($y = ax^b$; parameter a)

Median of Parameter Estimates (closer to $a = 66$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	63.65	63.30	63.08	63.57	63.16	56.71	55.68	53.84	52.62	53.53	63.79	63.80	64.88	64.89	63.29	57.36	56.71	56.40	57.14	55.36
	PING	66.32	66.27	66.73	67.31	65.95	65.76	68.14	69.86	76.85	70.07	66.19	66.45	68.13	68.12	65.91	65.92	68.04	71.02	77.99	70.27
	MRLN	66.32	66.27	66.73	67.31	65.95	65.76	68.14	69.86	76.84	70.07	66.19	66.45	68.13	68.12	65.91	65.92	68.04	71.02	78.05	70.27
	MUPE	65.76	65.88	65.76	66.07	65.80	64.23	65.03	65.14	65.46	65.91	65.27	65.42	66.71	66.51	65.22	62.27	62.72	63.69	64.76	64.06
	ZMPE	64.96	64.70	65.34	65.31	65.15	61.00	62.34	62.73	63.57	64.57	63.21	64.16	64.92	64.91	63.85	56.26	58.08	60.02	59.89	59.85

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	42%	44%	50%	50%	44%	83%	96%	110%	130%	110%	84%	87%	95%	93%	88%	169%	201%	226%	257%	237%
	PING	42%	44%	50%	50%	44%	83%	96%	111%	129%	110%	84%	87%	95%	93%	88%	168%	201%	226%	265%	236%
	MRLN	42%	44%	50%	50%	44%	83%	96%	111%	129%	110%	84%	87%	95%	93%	88%	168%	201%	226%	265%	236%
	MUPE	43%	43%	45%	45%	45%	88%	89%	89%	89%	90%	85%	86%	87%	84%	88%	177%	190%	196%	191%	195%
	ZMPE	49%	47%	40%	41%	47%	126%	109%	94%	82%	79%	94%	90%	80%	80%	94%	249%	234%	200%	179%	184%

MUPE is a strong all-around performer, regardless of conditions. It is usually among the best methods in accuracy and precision, and never the worst.

Results for all 20 Simulations ($y = ax^b$; parameter a)

Median of Parameter Estimates (closer to $a = 66$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	63.65	63.30	63.08	63.57	63.16	56.71	55.68	53.84	52.62	53.53	63.79	63.80	64.88	64.89	63.29	57.36	56.71	56.40	57.14	55.36
	PING	66.32	66.27	66.73	67.31	65.95	65.76	68.14	69.86	76.85	70.07	66.19	66.45	68.13	68.12	65.91	65.92	68.04	71.02	77.99	70.27
	MRLN	66.32	66.27	66.73	67.31	65.95	65.76	68.14	69.86	76.84	70.07	66.19	66.45	68.13	68.12	65.91	65.92	68.04	71.02	78.05	70.27
	MUPE	65.76	65.88	65.76	66.07	65.80	64.23	65.03	65.14	65.46	65.91	65.27	65.42	66.71	66.51	65.22	62.27	62.72	63.69	64.76	64.06
	ZMPE	64.96	64.70	65.34	65.31	65.15	61.00	62.34	62.73	63.57	64.57	63.21	64.16	64.92	64.91	63.85	56.26	58.08	60.02	59.89	59.85

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	42%	44%	50%	50%	44%	83%	96%	110%	130%	110%	84%	87%	95%	93%	88%	169%	201%	226%	257%	237%
	PING	42%	44%	50%	50%	44%	83%	96%	111%	129%	110%	84%	87%	95%	93%	88%	168%	201%	226%	265%	236%
	MRLN	42%	44%	50%	50%	44%	83%	96%	111%	129%	110%	84%	87%	95%	93%	88%	168%	201%	226%	265%	236%
	MUPE	43%	43%	45%	45%	45%	88%	89%	89%	89%	90%	85%	86%	87%	84%	88%	177%	190%	196%	191%	195%
	ZMPE	49%	47%	40%	41%	47%	126%	109%	94%	82%	79%	94%	90%	80%	80%	94%	249%	234%	200%	179%	184%

ZMPE accuracy is always worse than MUPE, and its precision is highly dependent on the specific conditions.

Results for all 20 Simulations ($y = ax^b$; parameter b)

Median of Parameter Estimates (closer to $b = 0.8$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	0.799	0.800	0.799	0.798	0.800	0.800	0.797	0.799	0.795	0.798	0.799	0.798	0.797	0.797	0.800	0.797	0.797	0.795	0.787	0.795
	PING	0.799	0.800	0.799	0.798	0.800	0.800	0.797	0.799	0.795	0.798	0.799	0.798	0.797	0.797	0.800	0.797	0.797	0.795	0.787	0.795
	MRLN	0.799	0.800	0.799	0.798	0.800	0.800	0.797	0.799	0.795	0.798	0.799	0.798	0.797	0.797	0.800	0.797	0.797	0.795	0.787	0.795
	MUPE	0.800	0.801	0.801	0.800	0.801	0.803	0.802	0.802	0.802	0.801	0.801	0.801	0.799	0.799	0.801	0.806	0.807	0.804	0.803	0.805
	ZMPE	0.802	0.803	0.802	0.802	0.802	0.810	0.808	0.807	0.806	0.804	0.806	0.804	0.803	0.803	0.805	0.820	0.817	0.814	0.815	0.816

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	7.8%	8.2%	9.2%	9.4%	8.1%	15%	17%	20%	23%	20%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	PING	7.8%	8.2%	9.2%	9.4%	8.1%	15%	17%	20%	23%	20%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	MRLN	7.8%	8.2%	9.2%	9.4%	8.1%	15%	17%	20%	23%	20%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	MUPE	7.9%	8.0%	8.1%	8.2%	8.2%	16%	16%	16%	16%	16%	15%	16%	16%	15%	16%	29%	32%	32%	33%	33%
	ZMPE	8.9%	8.6%	7.4%	7.6%	8.5%	22%	19%	17%	15%	14%	17%	16%	15%	15%	17%	38%	36%	33%	30%	31%

Results for all 20 Simulations ($y = ax^b$; parameter b)

Median of Parameter Estimates (closer to $b = 0.8$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	0.799	0.800	0.799	0.798	0.800	0.800	0.797	0.799	0.795	0.798	0.799	0.798	0.797	0.797	0.800	0.797	0.797	0.795	0.787	0.795
	PING	0.799	0.800	0.799	0.798	0.800	0.800	0.797	0.799	0.795	0.798	0.799	0.798	0.797	0.797	0.800	0.797	0.797	0.795	0.787	0.795
	MRLN	0.799	0.800	0.799	0.798	0.800	0.800	0.797	0.799	0.795	0.798	0.799	0.798	0.797	0.797	0.800	0.797	0.797	0.795	0.787	0.795
	MUPE	0.800	0.801	0.801	0.800	0.801	0.803	0.802	0.802	0.802	0.801	0.801	0.801	0.799	0.799	0.801	0.806	0.807	0.804	0.803	0.805
	ZMPE	0.802	0.803	0.802	0.802	0.802	0.810	0.808	0.807	0.806	0.804	0.806	0.804	0.803	0.803	0.805	0.820	0.817	0.814	0.815	0.816

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	7.8%	8.2%	9.2%	9.4%	8.1%	15%	17%	20%	23%	20%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	PING	7.8%	8.2%	9.2%	9.4%	8.1%	15%	17%	20%	23%	20%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	MRLN	7.8%	8.2%	9.2%	9.4%	8.1%	15%	17%	20%	23%	20%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	MUPE	7.9%	8.0%	8.1%	8.2%	8.2%	16%	16%	16%	16%	16%	15%	16%	16%	15%	16%	29%	32%	32%	33%	33%
	ZMPE	8.9%	8.6%	7.4%	7.6%	8.5%	22%	19%	17%	15%	14%	17%	16%	15%	15%	17%	38%	36%	33%	30%	31%

PING and MRLN results still nearly identical (and also Log Error, which is mathematically equivalent to PING for this parameter).

Results for all 20 Simulations ($y = ax^b$; parameter b)

Median of Parameter Estimates (closer to $b = 0.8$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	0.799	0.800	0.799	0.798	0.800	0.800	0.797	0.799	0.795	0.798	0.799	0.798	0.797	0.797	0.800	0.797	0.797	0.795	0.787	0.795
	PING	0.799	0.800	0.799	0.798	0.800	0.800	0.797	0.799	0.795	0.798	0.799	0.798	0.797	0.797	0.800	0.797	0.797	0.795	0.787	0.795
	MRLN	0.799	0.800	0.799	0.798	0.800	0.800	0.797	0.799	0.795	0.798	0.799	0.798	0.797	0.797	0.800	0.797	0.797	0.795	0.787	0.795
	MUPE	0.800	0.801	0.801	0.800	0.801	0.803	0.802	0.802	0.802	0.801	0.801	0.801	0.799	0.799	0.801	0.806	0.807	0.804	0.803	0.805
	ZMPE	0.802	0.803	0.802	0.802	0.802	0.810	0.808	0.807	0.806	0.804	0.806	0.804	0.803	0.803	0.805	0.820	0.817	0.814	0.815	0.816

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	7.8%	8.2%	9.2%	9.4%	8.1%	15%	17%	20%	23%	20%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	PING	7.8%	8.2%	9.2%	9.4%	8.1%	15%	17%	20%	23%	20%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	MRLN	7.8%	8.2%	9.2%	9.4%	8.1%	15%	17%	20%	23%	20%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	MUPE	7.9%	8.0%	8.1%	8.2%	8.2%	16%	16%	16%	16%	16%	15%	16%	16%	15%	16%	29%	32%	32%	33%	33%
	ZMPE	8.9%	8.6%	7.4%	7.6%	8.5%	22%	19%	17%	15%	14%	17%	16%	15%	15%	17%	38%	36%	33%	30%	31%

MUPE still the best all-around performer.

Results for all 20 Simulations ($y = ax^b$; parameter b)

Median of Parameter Estimates (closer to $b = 0.8$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	0.799	0.800	0.799	0.798	0.800	0.800	0.797	0.799	0.795	0.798	0.799	0.798	0.797	0.797	0.800	0.797	0.797	0.795	0.787	0.795
	PING	0.799	0.800	0.799	0.798	0.800	0.800	0.797	0.799	0.795	0.798	0.799	0.798	0.797	0.797	0.800	0.797	0.797	0.795	0.787	0.795
	MRLN	0.799	0.800	0.799	0.798	0.800	0.800	0.797	0.799	0.795	0.798	0.799	0.798	0.797	0.797	0.800	0.797	0.797	0.795	0.787	0.795
	MUPE	0.800	0.801	0.801	0.800	0.801	0.803	0.802	0.802	0.802	0.801	0.801	0.801	0.799	0.799	0.801	0.806	0.807	0.804	0.803	0.805
	ZMPE	0.802	0.803	0.802	0.802	0.802	0.810	0.808	0.807	0.806	0.804	0.806	0.804	0.803	0.803	0.805	0.820	0.817	0.814	0.815	0.816

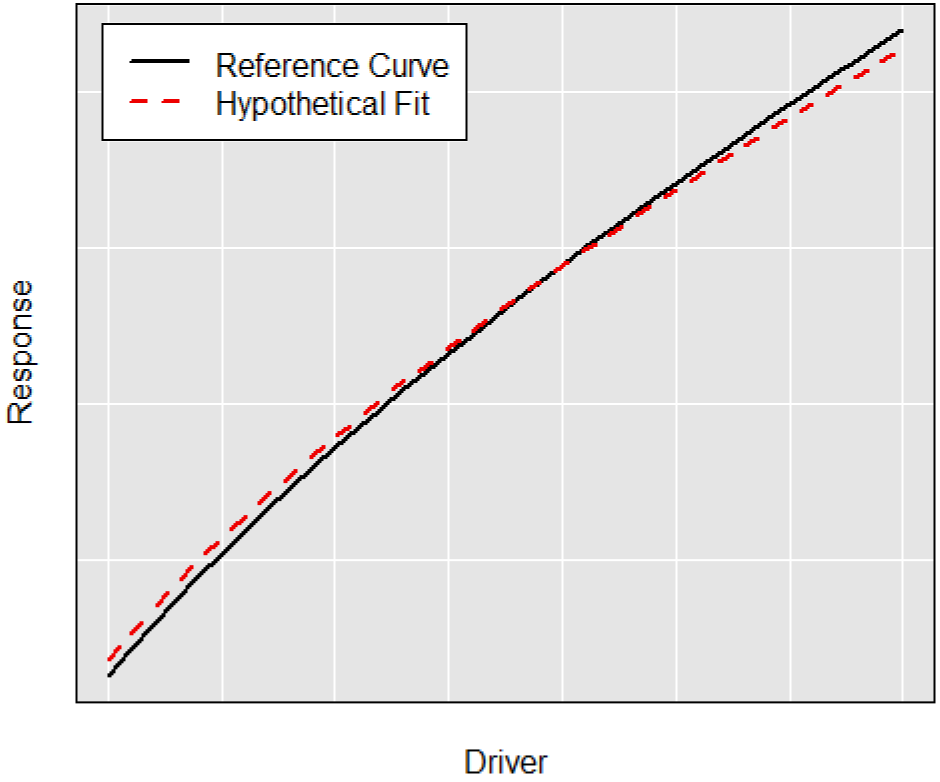
(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	7.8%	8.2%	9.2%	9.4%	8.1%	15%	17%	20%	23%	20%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	PING	7.8%	8.2%	9.2%	9.4%	8.1%	15%	17%	20%	23%	20%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	MRLN	7.8%	8.2%	9.2%	9.4%	8.1%	15%	17%	20%	23%	20%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	MUPE	7.9%	8.0%	8.1%	8.2%	8.2%	16%	16%	16%	16%	16%	15%	16%	16%	15%	16%	29%	32%	32%	33%	33%
	ZMPE	8.9%	8.6%	7.4%	7.6%	8.5%	22%	19%	17%	15%	14%	17%	16%	15%	15%	17%	38%	36%	33%	30%	31%

ZMPE tends to over-estimate parameter *b*, and its precision is condition-dependent.

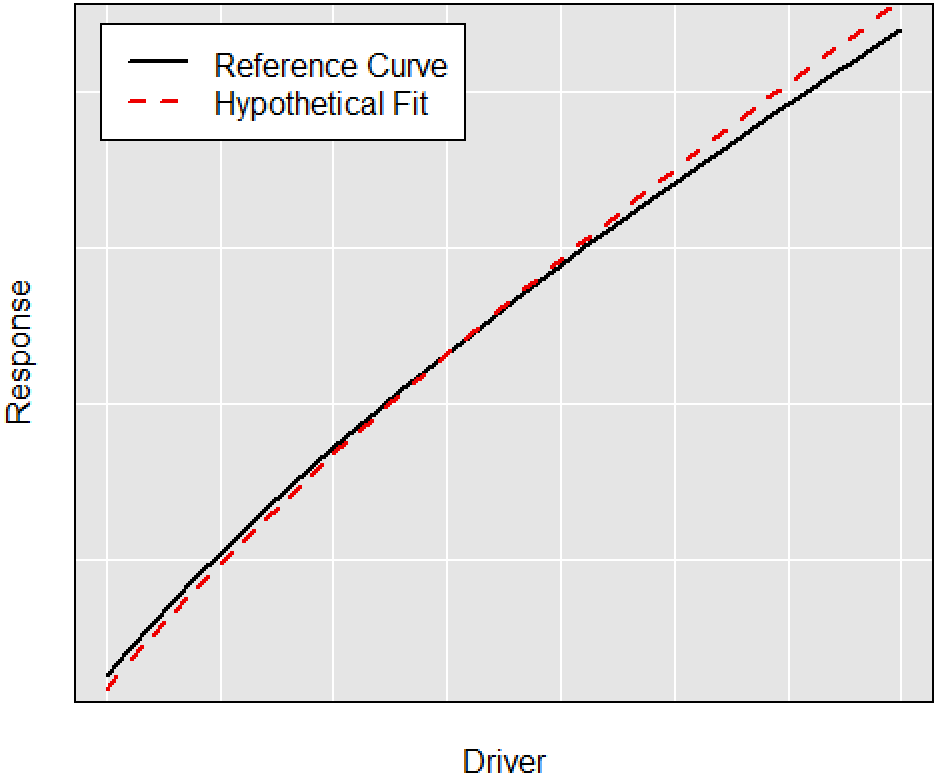
Effect of parameter bias ($y = ax^b$)

a biased high; *b* biased low



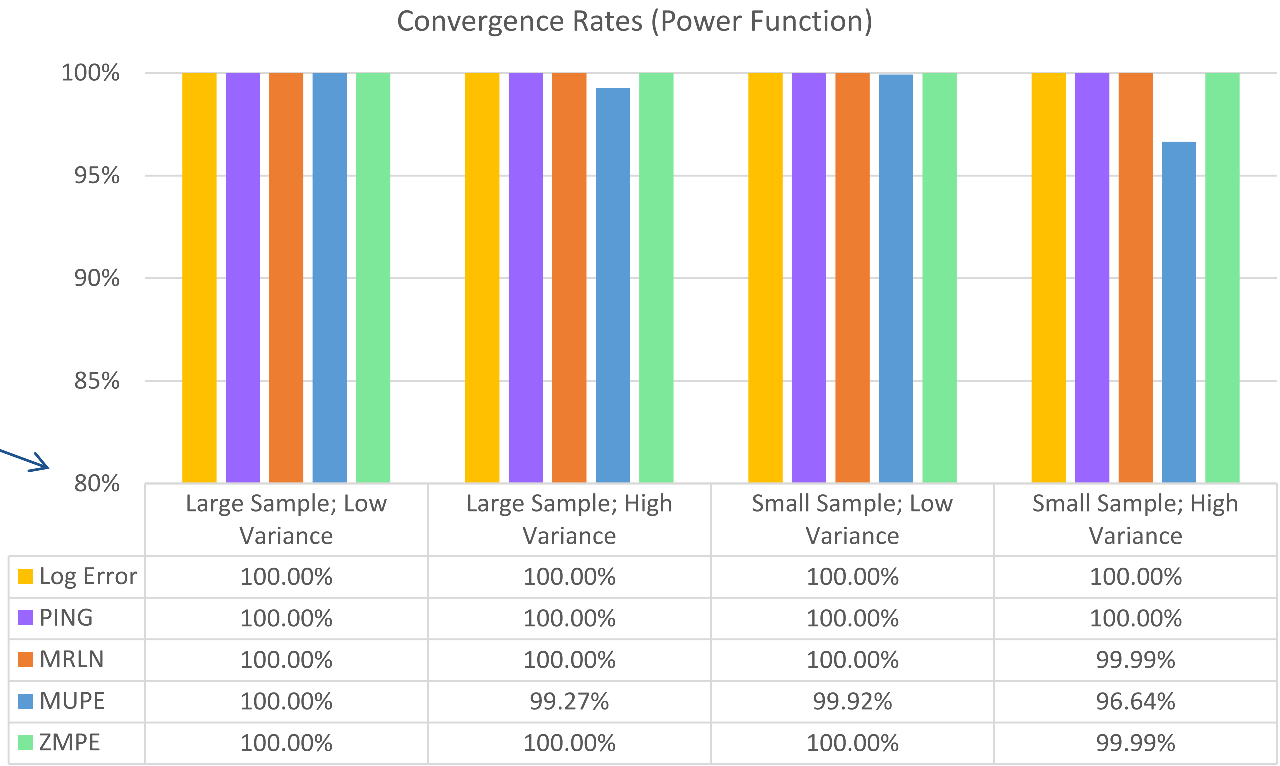
Lesser values are over-estimated, and greater values are under-estimated.

a biased low; *b* biased high



Lesser values are under-estimated, and greater values are over-estimated.

Power Function Regression Convergence Rates



Convergence Rates (Power Function)

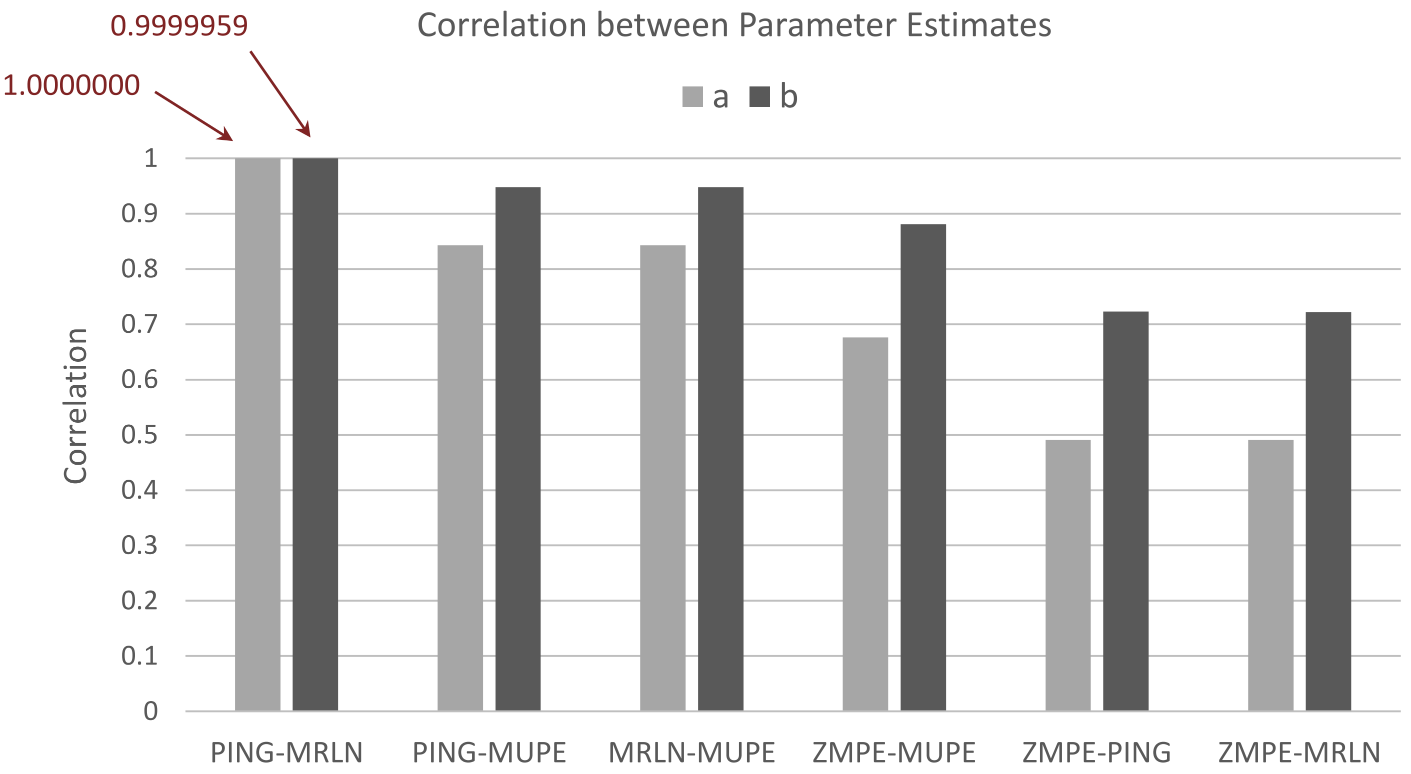
	Large Sample; Low Variance	Large Sample; High Variance	Small Sample; Low Variance	Small Sample; High Variance
🟨 Log Error	100.00%	100.00%	100.00%	100.00%
🟪 PING	100.00%	100.00%	100.00%	100.00%
🟧 MRLN	100.00%	100.00%	100.00%	99.99%
🟦 MUPE	100.00%	99.27%	99.92%	96.64%
🟩 ZMPE	100.00%	100.00%	100.00%	99.99%

Axis limits shrunk to accentuate differences.

Convergence is high across the board for univariate power functions. The worst case scenario is ~97% for high-variance small samples with MUPE.

PING solution used as starting guess for MRLN, MUPE, and ZMPE. Using a different starting guess might yield different results.

Correlation between Parameter Estimates ($y = ax^b$)



Correlation between Parameter Estimates

0.9999959

1.0000000

PING and MRLN provide approximately equivalent results.

MUPE's estimates tend to be similar to PING/MRLN.

ZMPE's estimates tend to be the most unique.

Results:
Linear Model

$$y = ax + c$$

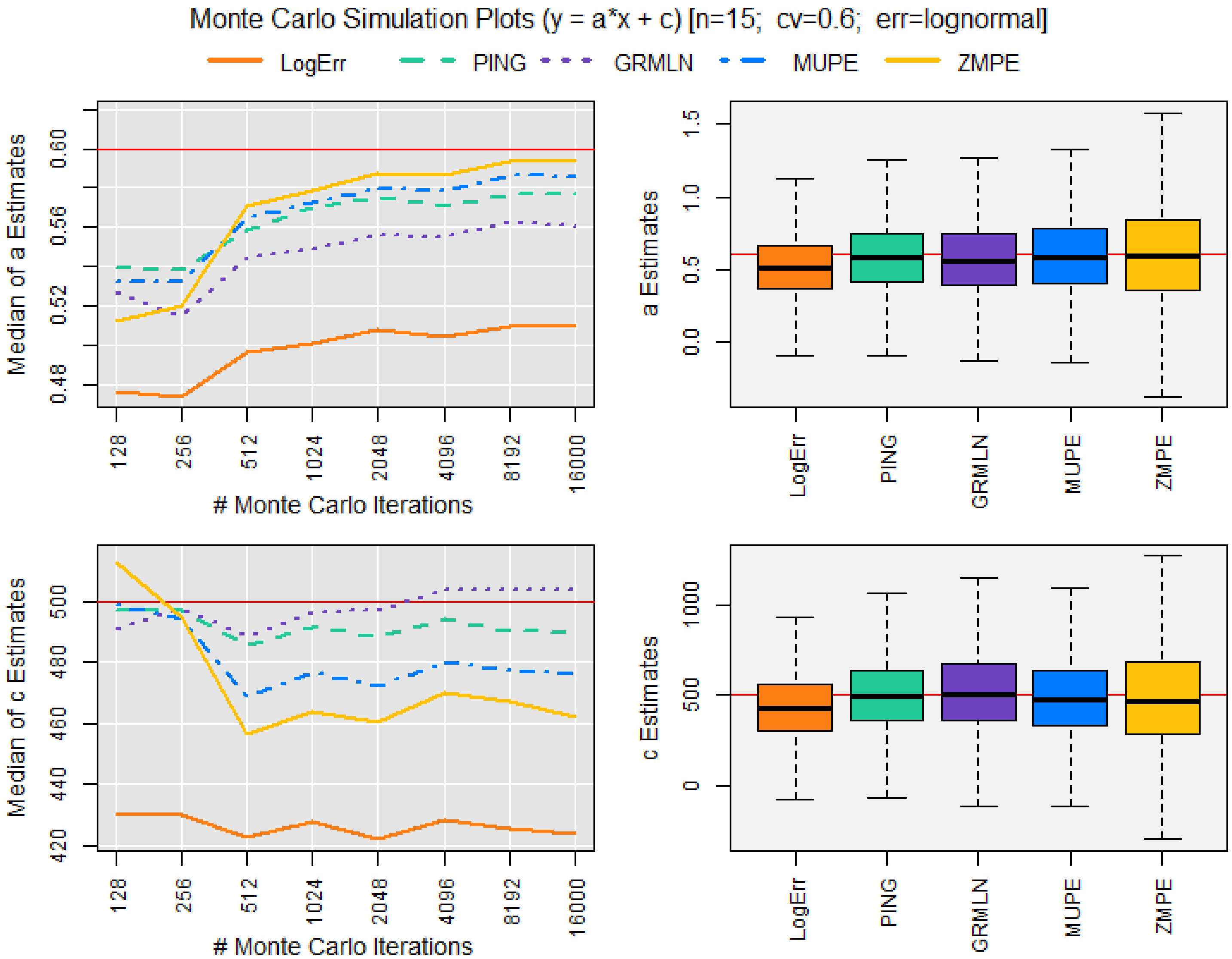
All simulations were run with 16,000 iterations.

Example Simulation Outputs: Lognormal Error / Small Sample / High Variance

a = 0.6; *c* = 500

- ZMPE and MUPE generate more accurate estimates for *a* than PING and GRMLN
- Log-error provides lowest estimates for both *a* and *c* (as expected)

- GRMLN and PING generate more accurate estimates for *c* than MUPE and ZMPE
- ZMPE has the worst precision for both *a* and *c*



OLS solution used as starting guess for all methods

See backup slides for complete linear results

Results:
Triad Model

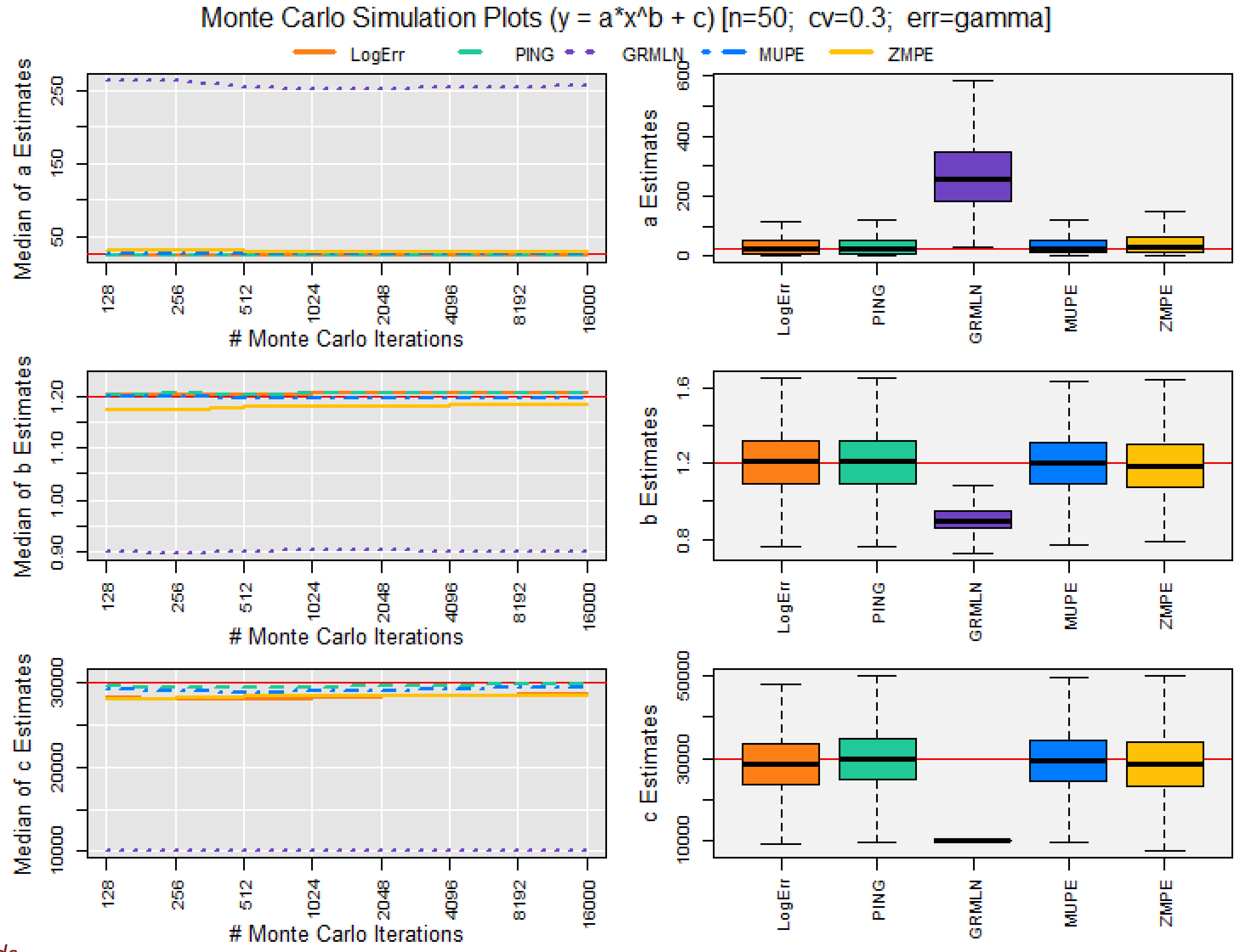
$$y = ax^b + c$$

All simulations were run with 16,000 iterations.

Example Simulation Outputs: Gamma Error / Large Sample / Low Variance

a = 25; *b* = 1.2; *c* = 30,000

- GRMLN consistently fails
  - Its estimate for *c* doesn't migrate from the initial guess.
  - As a result, *c* is biased very low, and that throws off the estimates for the remaining parameters, *a* and *b*.
  - This behavior was observed for <u>all</u> triad simulations, regardless of error distribution, sample size, or variance.

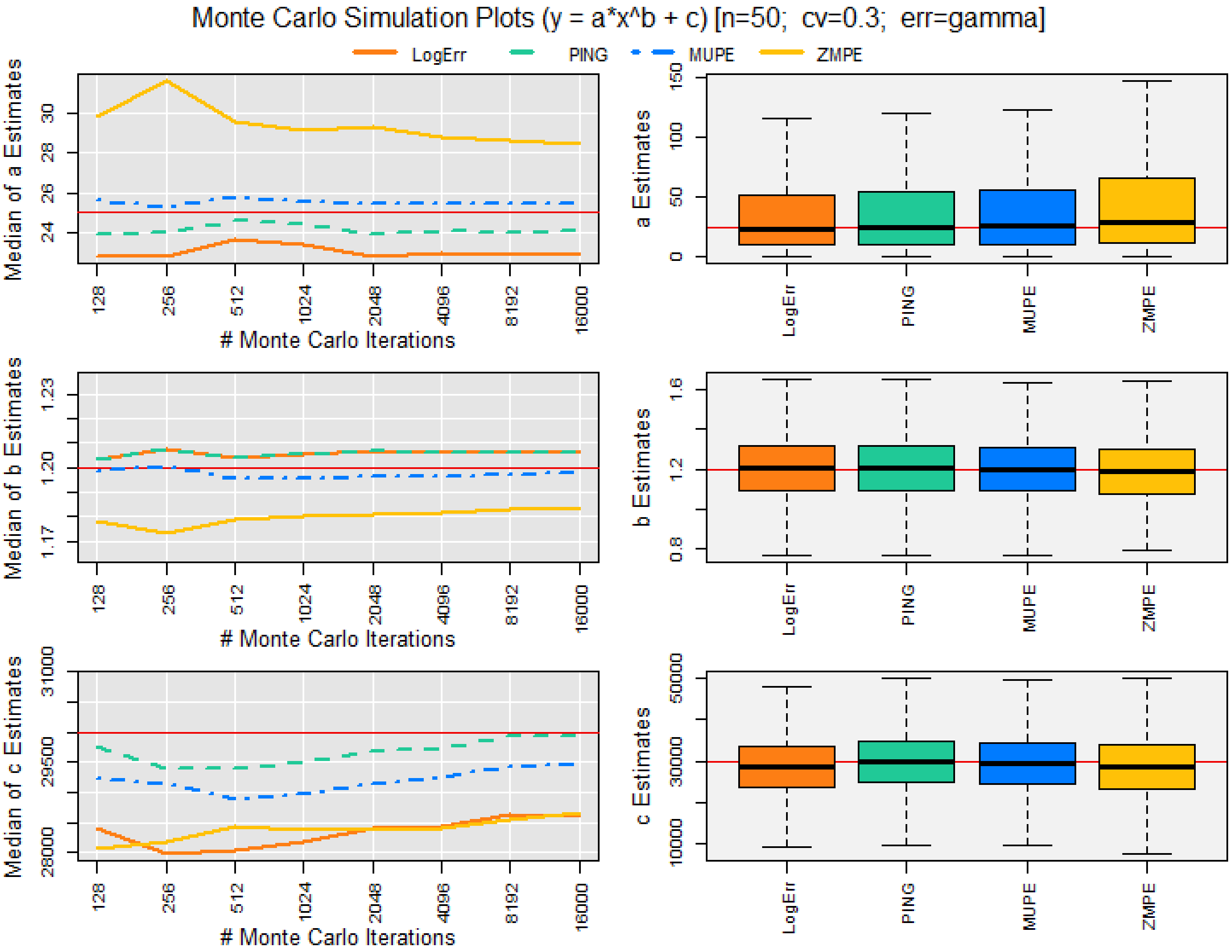


[10; 1; 10,000] used as starting guess for all methods

Example Simulation Outputs: Gamma Error / Large Sample / Low Variance (No GRMLN)

a = 25; *b* = 1.2; *c* = 30,000

- MUPE and PING exhibit the best combination of accuracy and precision
- ZMPE's parameter estimates are consistently biased one way or the other



[10; 1; 10,000] used as starting guess for all methods

See backup slides for complete triad results

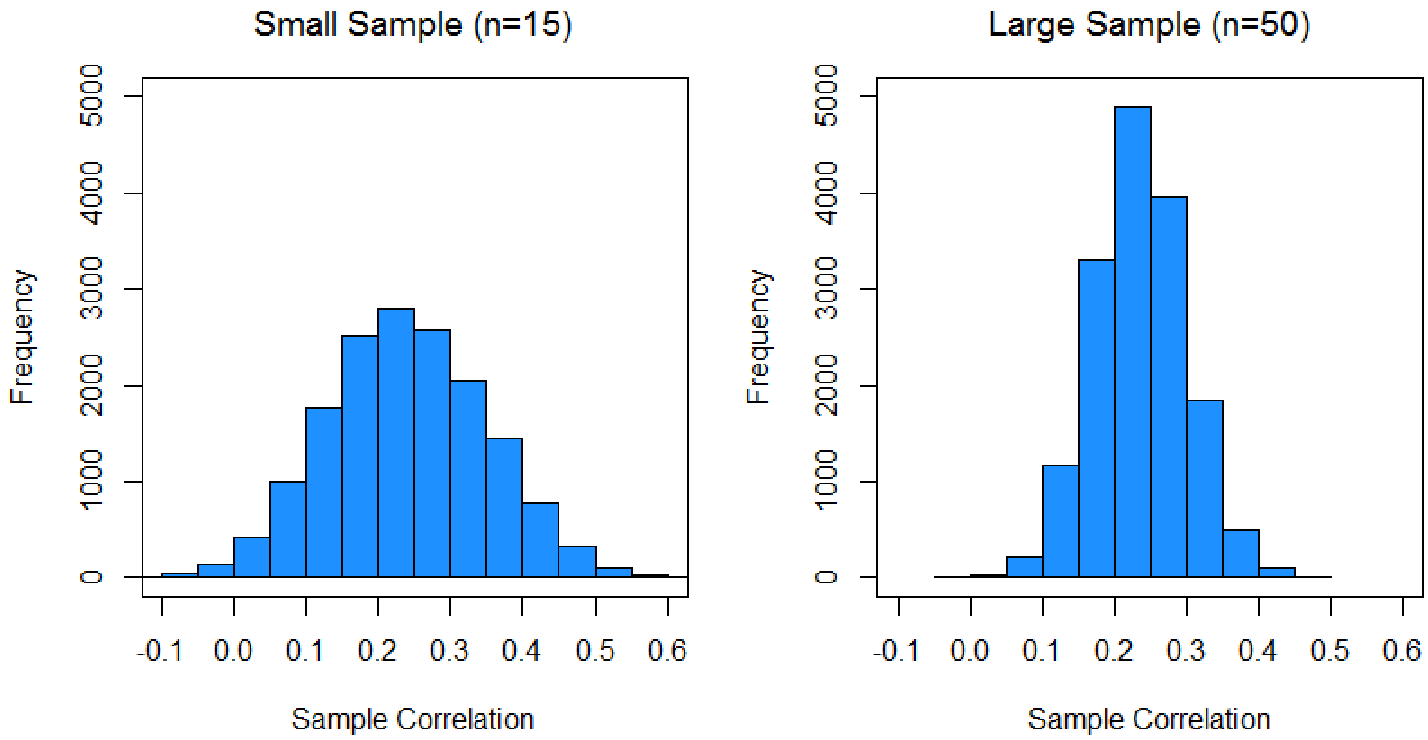
Results:
Multivariate Log-Linear

$$y = ax_1^b\, x_2^c\, d^{x_3}$$

All simulations were run with 16,000 iterations.

Details: Multivariate Model ($y = ax_1^b\, x_2^c\, d^{x_3}$)

- x_3 represents a binary dummy variable
- x_1 and x_2 represent continuous variables with a target correlation of 0.25
  - Actual correlation varies from sample to sample; see histograms for representative distributions from a simulation with 16,000 iterations



See backup slides for complete multivariate results

Recommendations & Conclusions

Recommendations & Conclusions

- Log Error method
  - Assumes lognormal error – do ***not*** use otherwise
  - Convenient for model forms that can be linearized, allowing LOLS to be used
    - Guaranteed to yield an answer in this scenario
  - Be sure to properly specify the uncertainty distribution, so as to account for the known bias (this method estimates the median rather than the mean)
- PING
  - Assumes lognormal error – do ***not*** use otherwise
  - Simple factor that corrects the bias of the Log Error method
- MRLN
  - Assumes lognormal error – do ***not*** use otherwise
  - Only applicable to power functions with continuous variables
  - **More complicated than PING, yet consistently yields nearly identical results**

Recommendations & Conclusions

- GRMLN
  - Generalization of MRLN; adds support for categorical variables and linear model forms
  - **Does not work with triad model form**
  - Assumes lognormal error – do ***not*** use otherwise
    - No better than PING under this condition
- ZMPE
  - **Not recommended as primary method**
    - Empirically proven to be inferior to MUPE in most scenarios
    - COBYLA implementation has good convergence rate – can be secondary if others fail
  - Excel implementations utilizing *GRG Nonlinear* might not be reliable
- MUPE
  - Makes no unnecessary assumptions
  - Provides good balance of accuracy and precision across scenarios
  - Easily applicable to any model form (refer to R and Python code snippets)
  - Does not always converge

In Summary

Log Error

Oldie but goodie

✓

PING

Simple yet effective

✓

MRLN

Much ado about nothing

⊖

GRMLN

Newer ≠ better

⊖

MUPE

Still top of class

✓

ZMPE

Still lagging behind

⊘

Open Questions for Future Investigations

- Which regression method(s) yield the most accurate **confidence intervals**?
- Does the choice of regression method **affect the distribution** of the resulting multiplicative errors?
- What effect do different correlation values between drivers have on regression accuracy/precision?
- Why does GRMLN fail on the triad model form?
- How do different starting guesses affect triad model performance?

Backup

About the Authors

Michael Schiavoni has supported cost/schedule research, data science, and business intelligence efforts at Tecolote for the last 4 years, and his prior professional experience includes 6 years in radar signal processing for a defense contractor and 3 years performing audits at an accounting firm. His educational background includes an M.S. in Applied Statistics from Pennsylvania State University, M.S. in Physics from the University of California at Riverside, and B.S. in Physics from the University of Delaware.

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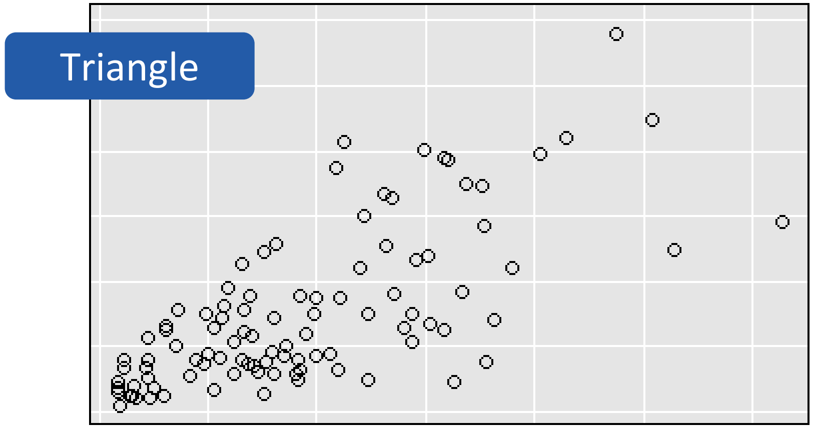
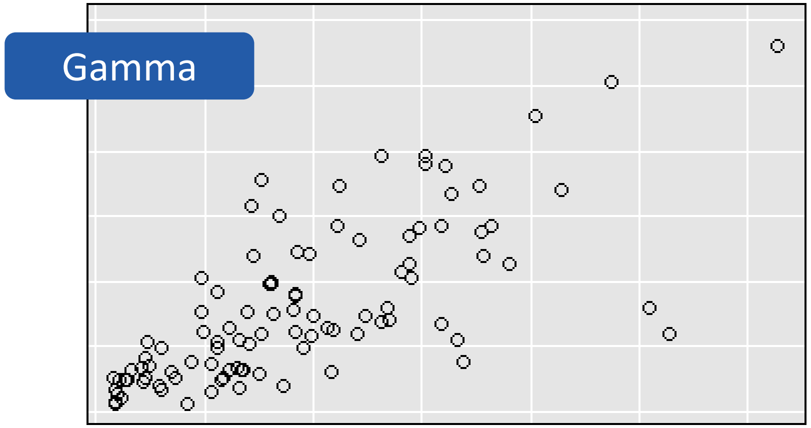
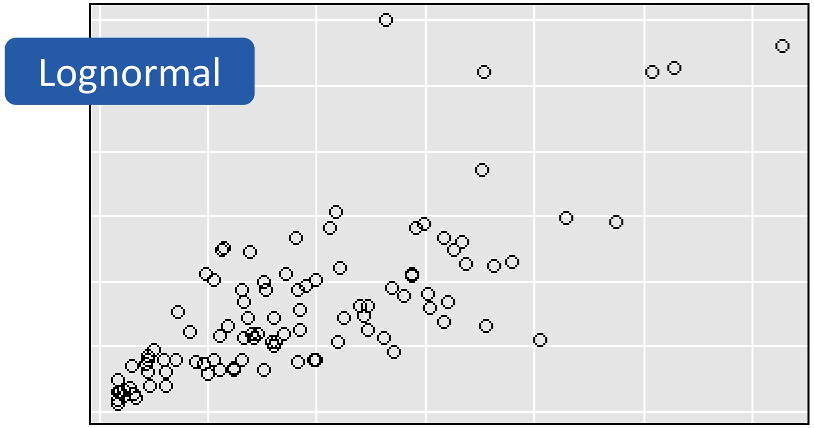
Richard Bearce is a cost analyst at Tecolote with 3 years of experience supporting the Air Force Space Command’s (now Space Force’s) Development Corps efforts, and 1.5 years prior experience working as a Research Assistant for the Economics Science Institute. His educational background includes an M.A. in Economics from the University of Arizona, an M.S. in Economics Systems Design from Chapman University, and a B.A. in economics from Chapman University.

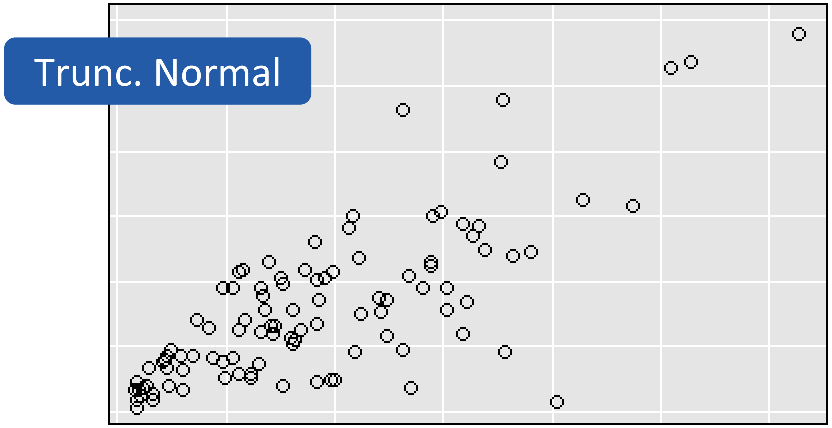
Acknowledgement: Thank you Dr. Shu-Ping Hu for helping us to identify that our PING correction factor calculation originally contained an error.

Error Distribution Quiz: Answer Key


Triangle

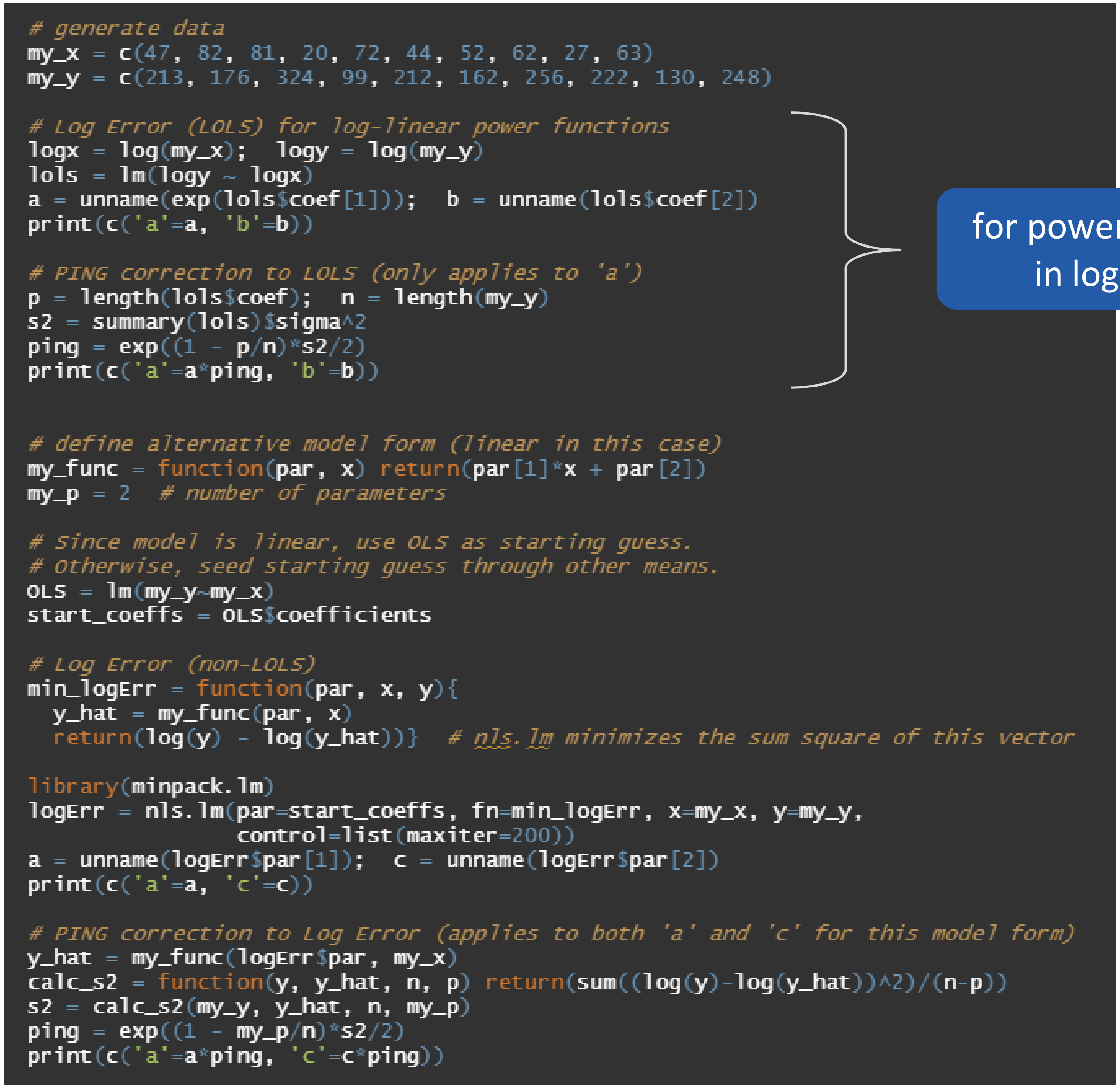

Gamma


Lognormal


Trunc. Normal

Log Error and PING in R

Source code presented without any restrictions on use.



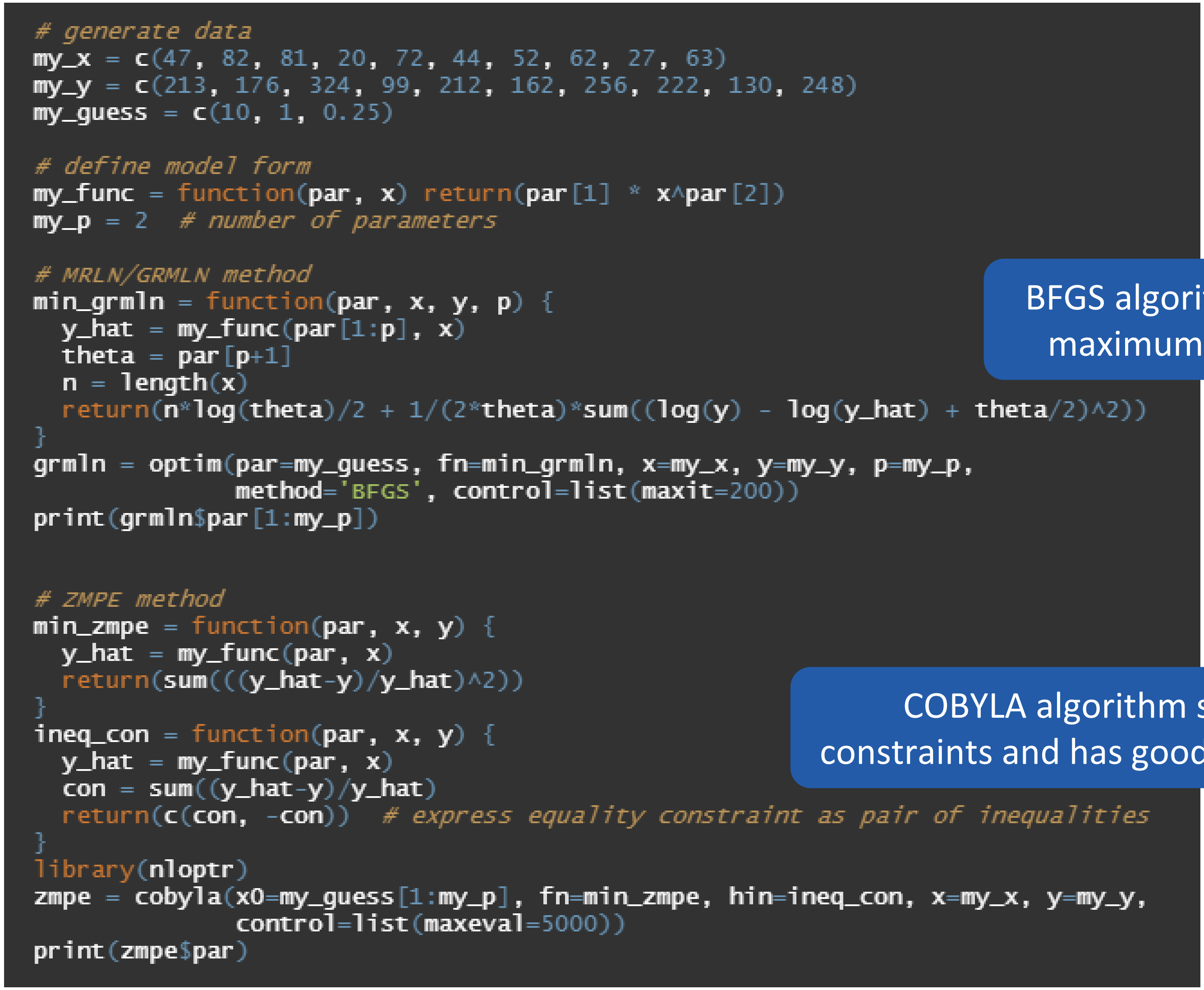
```r
# generate data
my_x = c(47, 82, 81, 20, 72, 44, 52, 62, 27, 63)
my_y = c(213, 176, 324, 99, 212, 162, 256, 222, 130, 248)

# Log Error (LOLS) for log-linear power functions
logx = log(my_x);  logy = log(my_y)
lols = lm(logy ~ logx)
a = unname(exp(lols$coef[1]));  b = unname(lols$coef[2])
print(c('a'=a, 'b'=b))

# PING correction to LOLS (only applies to 'a')
p = length(lols$coef);  n = length(my_y)
s2 = summary(lols)$sigma^2
ping = exp((1 - p/n)*s2/2)
print(c('a'=a*ping, 'b'=b))


# define alternative model form (linear in this case)
my_func = function(par, x) return(par[1]*x + par[2])
my_p = 2  # number of parameters

# Since model is linear, use OLS as starting guess.
# Otherwise, seed starting guess through other means.
OLS = lm(my_y~my_x)
start_coeffs = OLS$coefficients

# Log Error (non-LOLS)
min_logErr = function(par, x, y){
  y_hat = my_func(par, x)
  return(log(y) - log(y_hat))}  # nls.lm minimizes the sum square of this vector

library(minpack.lm)
logErr = nls.lm(par=start_coeffs, fn=min_logErr, x=my_x, y=my_y,
                control=list(maxiter=200))
a = unname(logErr$par[1]);  c = unname(logErr$par[2])
print(c('a'=a, 'c'=c))

# PING correction to Log Error (applies to both 'a' and 'c' for this model form)
y_hat = my_func(logErr$par, my_x)
calc_s2 = function(y, y_hat, n, p) return(sum((log(y)-log(y_hat))^2)/(n-p))
s2 = calc_s2(my_y, y_hat, n, my_p)
ping = exp((1 - my_p/n)*s2/2)
print(c('a'=a*ping, 'c'=c*ping))
```

for power functions that can be linearized in log space (in this case $y = ax^b$)

for other arbitrary model forms (in this case $y = ax + c$)

MRLN/GRMLN and ZMPE in R

Source code presented without any restrictions on use.



```
# generate data
my_x = c(47, 82, 81, 20, 72, 44, 52, 62, 27, 63)
my_y = c(213, 176, 324, 99, 212, 162, 256, 222, 130, 248)
my_guess = c(10, 1, 0.25)

# define model form
my_func = function(par, x) return(par[1] * x^par[2])
my_p = 2  # number of parameters

# MRLN/GRMLN method
min_grmln = function(par, x, y, p) {
  y_hat = my_func(par[1:p], x)
  theta = par[p+1]
  n = length(x)
  return(n*log(theta)/2 + 1/(2*theta)*sum((log(y) - log(y_hat) + theta/2)^2))
}
grmln = optim(par=my_guess, fn=min_grmln, x=my_x, y=my_y, p=my_p,
              method='BFGS', control=list(maxit=200))
print(grmln$par[1:my_p])


# ZMPE method
min_zmpe = function(par, x, y) {
  y_hat = my_func(par, x)
  return(sum(((y_hat-y)/y_hat)^2))
}
ineq_con = function(par, x, y) {
  y_hat = my_func(par, x)
  con = sum((y_hat-y)/y_hat)
  return(c(con, -con))  # express equality constraint as pair of inequalities
}
library(nloptr)
zmpe = cobyla(x0=my_guess[1:my_p], fn=min_zmpe, hin=ineq_con, x=my_x, y=my_y,
              control=list(maxeval=5000))
print(zmpe$par)
```

BFGS algorithm is the R default for maximum likelihood estimation

COBYLA algorithm supports nonlinear constraints and has good convergence behavior.

Results:
Linear Model

$$y = ax + c$$

All simulations were run with 16,000 iterations.

Results for all 20 Simulations ($y = ax + c$; parameter a)

Median of Parameter Estimates (closer to $a = 0.6$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	0.574	0.574	0.569	0.569	0.574	0.514	0.496	0.483	0.458	0.477	0.574	0.573	0.572	0.570	0.573	0.510	0.498	0.484	0.464	0.480
	PING	0.598	0.600	0.603	0.603	0.600	0.595	0.609	0.627	0.668	0.625	0.595	0.597	0.602	0.600	0.597	0.577	0.591	0.602	0.630	0.605
	GRMLN	0.597	0.600	0.603	0.603	0.599	0.589	0.612	0.636	0.709	0.634	0.590	0.594	0.604	0.602	0.592	0.560	0.588	0.613	0.676	0.619
	MUPE	0.599	0.600	0.600	0.600	0.601	0.596	0.598	0.600	0.601	0.599	0.598	0.598	0.602	0.599	0.600	0.586	0.595	0.596	0.600	0.600
	ZMPE	0.601	0.602	0.601	0.600	0.603	0.604	0.606	0.606	0.604	0.606	0.601	0.601	0.606	0.601	0.604	0.594	0.606	0.609	0.611	0.614

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	16%	17%	19%	19%	17%	31%	36%	41%	49%	41%	31%	33%	35%	35%	32%	59%	69%	79%	91%	78%
	PING	16%	17%	19%	19%	17%	31%	36%	40%	48%	40%	31%	33%	35%	35%	32%	58%	67%	76%	87%	75%
	GRMLN	17%	17%	19%	18%	17%	32%	36%	39%	44%	39%	33%	33%	35%	35%	34%	63%	69%	73%	78%	76%
	MUPE	17%	16%	17%	17%	17%	33%	34%	33%	34%	33%	31%	32%	32%	32%	32%	63%	65%	67%	68%	66%
	ZMPE	19%	17%	16%	16%	18%	47%	40%	35%	32%	30%	35%	34%	30%	31%	34%	82%	76%	71%	65%	65%

White cells are the best; Blue and Red cells are not as good.

Median and IQR used because all methods yielded highly skewed estimate distributions with many outliers.

OLS solution used as starting guess for methods utilizing optimization algorithms.

Results for all 20 Simulations ($y = ax + c$; parameter a)

Median of Parameter Estimates (closer to $a = 0.6$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	0.574	0.574	0.569	0.569	0.574	0.514	0.496	0.483	0.458	0.477	0.574	0.573	0.572	0.570	0.573	0.510	0.498	0.484	0.464	0.480
	PING	0.598	0.600	0.603	0.603	0.600	0.595	0.609	0.627	0.668	0.625	0.595	0.597	0.602	0.600	0.597	0.577	0.591	0.602	0.630	0.605
	GRMLN	0.597	0.600	0.603	0.603	0.599	0.589	0.612	0.636	0.709	0.634	0.590	0.594	0.604	0.602	0.592	0.560	0.588	0.613	0.676	0.619
	MUPE	0.599	0.600	0.600	0.600	0.601	0.596	0.598	0.600	0.601	0.599	0.598	0.598	0.602	0.599	0.600	0.586	0.595	0.596	0.600	0.600
	ZMPE	0.601	0.602	0.601	0.600	0.603	0.604	0.606	0.606	0.604	0.606	0.601	0.601	0.606	0.601	0.604	0.594	0.606	0.609	0.611	0.614

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	16%	17%	19%	19%	17%	31%	36%	41%	49%	41%	31%	33%	35%	35%	32%	59%	69%	79%	91%	78%
	PING	16%	17%	19%	19%	17%	31%	36%	40%	48%	40%	31%	33%	35%	35%	32%	58%	67%	76%	87%	75%
	GRMLN	17%	17%	19%	18%	17%	32%	36%	39%	44%	39%	33%	33%	35%	35%	34%	63%	69%	73%	78%	76%
	MUPE	17%	16%	17%	17%	17%	33%	34%	33%	34%	33%	31%	32%	32%	32%	32%	63%	65%	67%	68%	66%
	ZMPE	19%	17%	16%	16%	18%	47%	40%	35%	32%	30%	35%	34%	30%	31%	34%	82%	76%	71%	65%	65%

Log Error is consistently biased low, especially when the variance is high.

Results for all 20 Simulations ($y = ax + c$; parameter a)

Median of Parameter Estimates (closer to $a = 0.6$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	0.574	0.574	0.569	0.569	0.574	0.514	0.496	0.483	0.458	0.477	0.574	0.573	0.572	0.570	0.573	0.510	0.498	0.484	0.464	0.480
	PING	0.598	0.600	0.603	0.603	0.600	0.595	0.609	0.627	0.668	0.625	0.595	0.597	0.602	0.600	0.597	0.577	0.591	0.602	0.630	0.605
	GRMLN	0.597	0.600	0.603	0.603	0.599	0.589	0.612	0.636	0.709	0.634	0.590	0.594	0.604	0.602	0.592	0.560	0.588	0.613	0.676	0.619
	MUPE	0.599	0.600	0.600	0.600	0.601	0.596	0.598	0.600	0.601	0.599	0.598	0.598	0.602	0.599	0.600	0.586	0.595	0.596	0.600	0.600
	ZMPE	0.601	0.602	0.601	0.600	0.603	0.604	0.606	0.606	0.604	0.606	0.601	0.601	0.606	0.601	0.604	0.594	0.606	0.609	0.611	0.614

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	16%	17%	19%	19%	17%	31%	36%	41%	49%	41%	31%	33%	35%	35%	32%	59%	69%	79%	91%	78%
	PING	16%	17%	19%	19%	17%	31%	36%	40%	48%	40%	31%	33%	35%	35%	32%	58%	67%	76%	87%	75%
	GRMLN	17%	17%	19%	18%	17%	32%	36%	39%	44%	39%	33%	33%	35%	35%	34%	63%	69%	73%	78%	76%
	MUPE	17%	16%	17%	17%	17%	33%	34%	33%	34%	33%	31%	32%	32%	32%	32%	63%	65%	67%	68%	66%
	ZMPE	19%	17%	16%	16%	18%	47%	40%	35%	32%	30%	35%	34%	30%	31%	34%	82%	76%	71%	65%	65%

PING accuracy is equal or better than GRMLN in every case, and their precisions are similar.

Results for all 20 Simulations ($y = ax + c$; parameter a)

Median of Parameter Estimates (closer to $a = 0.6$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	0.574	0.574	0.569	0.569	0.574	0.514	0.496	0.483	0.458	0.477	0.574	0.573	0.572	0.570	0.573	0.510	0.498	0.484	0.464	0.480
	PING	0.598	0.600	0.603	0.603	0.600	0.595	0.609	0.627	0.668	0.625	0.595	0.597	0.602	0.600	0.597	0.577	0.591	0.602	0.630	0.605
	GRMLN	0.597	0.600	0.603	0.603	0.599	0.589	0.612	0.636	0.709	0.634	0.590	0.594	0.604	0.602	0.592	0.560	0.588	0.613	0.676	0.619
	MUPE	0.599	0.600	0.600	0.600	0.601	0.596	0.598	0.600	0.601	0.599	0.598	0.598	0.602	0.599	0.600	0.586	0.595	0.596	0.600	0.600
	ZMPE	0.601	0.602	0.601	0.600	0.603	0.604	0.606	0.606	0.604	0.606	0.601	0.601	0.606	0.601	0.604	0.594	0.606	0.609	0.611	0.614

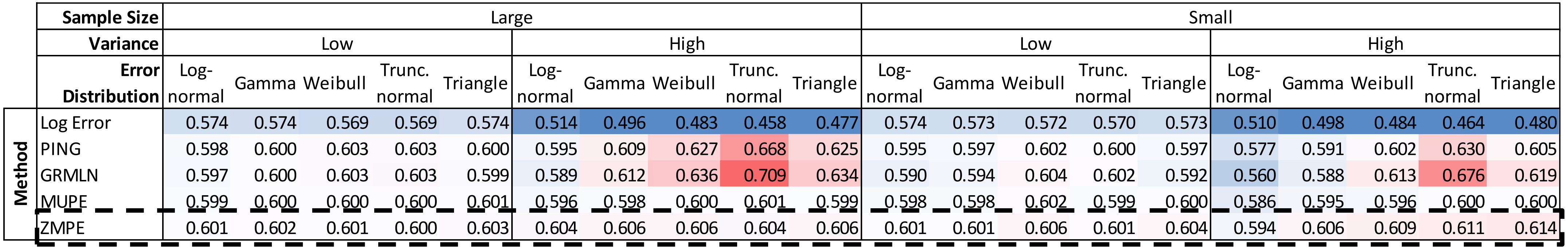
(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	16%	17%	19%	19%	17%	31%	36%	41%	49%	41%	31%	33%	35%	35%	32%	59%	69%	79%	91%	78%
	PING	16%	17%	19%	19%	17%	31%	36%	40%	48%	40%	31%	33%	35%	35%	32%	58%	67%	76%	87%	75%
	GRMLN	17%	17%	19%	18%	17%	32%	36%	39%	44%	39%	33%	33%	35%	35%	34%	63%	69%	73%	78%	76%
	MUPE	17%	16%	17%	17%	17%	33%	34%	33%	34%	33%	31%	32%	32%	32%	32%	63%	65%	67%	68%	66%
	ZMPE	19%	17%	16%	16%	18%	47%	40%	35%	32%	30%	35%	34%	30%	31%	34%	82%	76%	71%	65%	65%

MUPE tends to provide accurate and precise estimates for a.

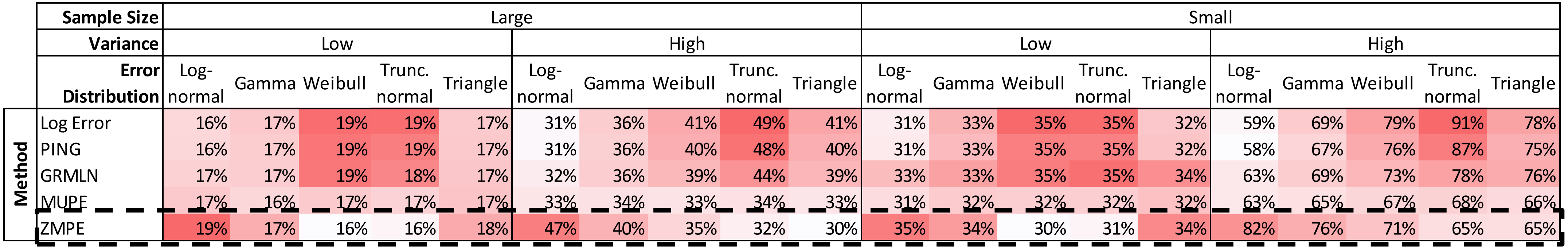
Results for all 20 Simulations ($y = ax + c$; parameter a)

Median of Parameter Estimates (closer to $a = 0.6$ is more accurate)



	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	0.574	0.574	0.569	0.569	0.574	0.514	0.496	0.483	0.458	0.477	0.574	0.573	0.572	0.570	0.573	0.510	0.498	0.484	0.464	0.480
	PING	0.598	0.600	0.603	0.603	0.600	0.595	0.609	0.627	0.668	0.625	0.595	0.597	0.602	0.600	0.597	0.577	0.591	0.602	0.630	0.605
	GRMLN	0.597	0.600	0.603	0.603	0.599	0.589	0.612	0.636	0.709	0.634	0.590	0.594	0.604	0.602	0.592	0.560	0.588	0.613	0.676	0.619
	MUPE	0.599	0.600	0.600	0.600	0.601	0.596	0.598	0.600	0.601	0.599	0.598	0.598	0.602	0.599	0.600	0.586	0.595	0.596	0.600	0.600
	ZMPE	0.601	0.602	0.601	0.600	0.603	0.604	0.606	0.606	0.604	0.606	0.601	0.601	0.606	0.601	0.604	0.594	0.606	0.609	0.611	0.614

(IQR/Median) of Parameter Estimates (lower is more precise)



	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	16%	17%	19%	19%	17%	31%	36%	41%	49%	41%	31%	33%	35%	35%	32%	59%	69%	79%	91%	78%
	PING	16%	17%	19%	19%	17%	31%	36%	40%	48%	40%	31%	33%	35%	35%	32%	58%	67%	76%	87%	75%
	GRMLN	17%	17%	19%	18%	17%	32%	36%	39%	44%	39%	33%	33%	35%	35%	34%	63%	69%	73%	78%	76%
	MUPE	17%	16%	17%	17%	17%	33%	34%	33%	34%	33%	31%	32%	32%	32%	32%	63%	65%	67%	68%	66%
	ZMPE	19%	17%	16%	16%	18%	47%	40%	35%	32%	30%	35%	34%	30%	31%	34%	82%	76%	71%	65%	65%

ZMPE's accuracy is on par with MUPE's in most cases, but its precision fluctuates, as was observed with the power function, too.

Results for all 20 Simulations ($y = ax + c$; parameter c)

Median of Parameter Estimates (closer to $c = 500$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	478.4	476.9	476.6	477.1	477.3	426.1	415.0	401.0	386.0	399.6	476.9	478.4	479.5	481.8	476.9	423.9	416.4	405.6	396.0	402.4
	PING	498.6	499.1	504.1	505.7	498.2	495.1	510.6	522.8	564.3	523.2	495.2	498.4	503.6	506.6	496.6	489.6	509.0	521.4	555.8	519.6
	GRMLN	499.6	499.7	503.9	505.6	499.0	499.2	509.5	514.0	527.0	515.1	499.1	501.0	500.2	504.9	499.9	503.7	513.4	518.0	520.9	514.5
	MUPE	498.3	497.9	500.2	501.2	498.2	490.8	495.5	493.5	498.0	498.1	492.5	496.4	498.8	501.6	495.9	476.3	485.8	486.1	491.5	494.0
	ZMPE	496.3	496.5	499.8	500.0	497.9	483.5	490.8	490.2	495.5	497.7	489.9	494.1	496.2	497.3	496.1	461.9	476.0	482.0	484.9	498.6

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	16%	17%	19%	19%	17%	31%	36%	40%	48%	41%	30%	32%	35%	34%	32%	59%	69%	76%	86%	77%
	PING	16%	17%	18%	18%	17%	31%	36%	39%	47%	40%	30%	32%	35%	34%	32%	58%	66%	72%	80%	73%
	GRMLN	17%	17%	19%	18%	17%	33%	37%	39%	43%	40%	32%	34%	35%	35%	34%	63%	70%	73%	74%	75%
	MUPE	17%	16%	17%	17%	17%	33%	34%	34%	33%	34%	31%	32%	32%	31%	32%	63%	66%	67%	67%	68%
	ZMPE	19%	17%	15%	16%	17%	47%	41%	36%	32%	31%	35%	34%	30%	30%	35%	85%	79%	73%	67%	67%

Precision is almost identical across the board for parameter *c* as with parameter *a*.

Results for all 20 Simulations ($y = ax + c$; parameter c)

Median of Parameter Estimates (closer to $c = 500$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	478.4	476.9	476.6	477.1	477.3	426.1	415.0	401.0	386.0	399.6	476.9	478.4	479.5	481.8	476.9	423.9	416.4	405.6	396.0	402.4
	PING	498.6	499.1	504.1	505.7	498.2	495.1	510.6	522.8	564.3	523.2	495.2	498.4	503.6	506.6	496.6	489.6	509.0	521.4	555.8	519.6
	GRMLN	499.6	499.7	503.9	505.6	499.0	499.2	509.5	514.0	527.0	515.1	499.1	501.0	500.2	504.9	499.9	503.7	513.4	518.0	520.9	514.5
	MUPE	498.3	497.9	500.2	501.2	498.2	490.8	495.5	493.5	498.0	498.1	492.5	496.4	498.8	501.6	495.9	476.3	485.8	486.1	491.5	494.0
	ZMPE	496.3	496.5	499.8	500.0	497.9	483.5	490.8	490.2	495.5	497.7	489.9	494.1	496.2	497.3	496.1	461.9	476.0	482.0	484.9	498.6

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	16%	17%	19%	19%	17%	31%	36%	40%	48%	41%	30%	32%	35%	34%	32%	59%	69%	76%	86%	77%
	PING	16%	17%	18%	18%	17%	31%	36%	39%	47%	40%	30%	32%	35%	34%	32%	58%	66%	72%	80%	73%
	GRMLN	17%	17%	19%	18%	17%	33%	37%	39%	43%	40%	32%	34%	35%	35%	34%	63%	70%	73%	74%	75%
	MUPE	17%	16%	17%	17%	17%	33%	34%	34%	33%	34%	31%	32%	32%	31%	32%	63%	66%	67%	67%	68%
	ZMPE	19%	17%	15%	16%	17%	47%	41%	36%	32%	31%	35%	34%	30%	30%	35%	85%	79%	73%	67%	67%

PING and GRMLN perform best when error distribution is Lognormal, but otherwise are biased high when the variance is high.

Results for all 20 Simulations ($y = ax + c$; parameter c)

Median of Parameter Estimates (closer to $c = 500$ is more accurate)

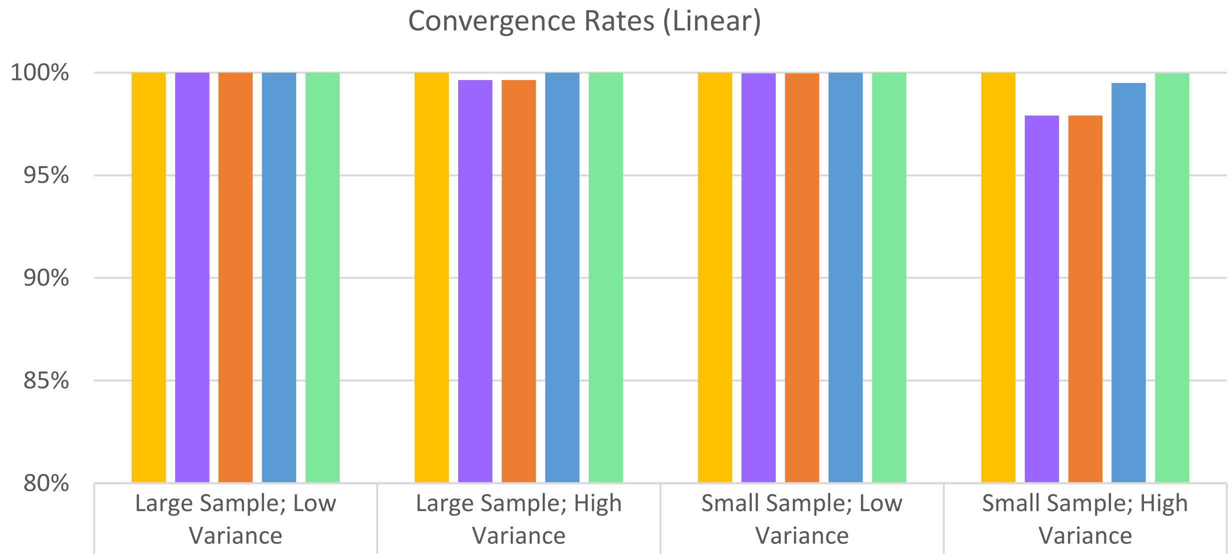
	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	478.4	476.9	476.6	477.1	477.3	426.1	415.0	401.0	386.0	399.6	476.9	478.4	479.5	481.8	476.9	423.9	416.4	405.6	396.0	402.4
	PING	498.6	499.1	504.1	505.7	498.2	495.1	510.6	522.8	564.3	523.2	495.2	498.4	503.6	506.6	496.6	489.6	509.0	521.4	555.8	519.6
	GRMLN	499.6	499.7	503.9	505.6	499.0	499.2	509.5	514.0	527.0	515.1	499.1	501.0	500.2	504.9	499.9	503.7	513.4	518.0	520.9	514.5
	MUPE	498.3	497.9	500.2	501.2	498.2	490.8	495.5	493.5	498.0	498.1	492.5	496.4	498.8	501.6	495.9	476.3	485.8	486.1	491.5	494.0
	ZMPE	496.3	496.5	499.8	500.0	497.9	483.5	490.8	490.2	495.5	497.7	489.9	494.1	496.2	497.3	496.1	461.9	476.0	482.0	484.9	498.6

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	16%	17%	19%	19%	17%	31%	36%	40%	48%	41%	30%	32%	35%	34%	32%	59%	69%	76%	86%	77%
	PING	16%	17%	18%	18%	17%	31%	36%	39%	47%	40%	30%	32%	35%	34%	32%	58%	66%	72%	80%	73%
	GRMLN	17%	17%	19%	18%	17%	33%	37%	39%	43%	40%	32%	34%	35%	35%	34%	63%	70%	73%	74%	75%
	MUPE	17%	16%	17%	17%	17%	33%	34%	34%	33%	34%	31%	32%	32%	31%	32%	63%	66%	67%	67%	68%
	ZMPE	19%	17%	15%	16%	17%	47%	41%	36%	32%	31%	35%	34%	30%	30%	35%	85%	79%	73%	67%	67%

MUPE and ZMPE are biased low with small samples when the variance is high.

Linear Function Regression Convergence Rates



Axis limits shrunk to accentuate differences.

	Large Sample; Low Variance	Large Sample; High Variance	Small Sample; Low Variance	Small Sample; High Variance
🟨 Log Error	100.00%	100.00%	100.00%	100.00%
🟪 PING	100.00%	99.64%	99.97%	97.91%
🟧 GRMLN	100.00%	99.64%	99.97%	97.91%
🟦 MUPE	100.00%	100.00%	100.00%	99.50%
🟩 ZMPE	100.00%	99.99%	100.00%	99.97%

Convergence is high across the board for univariate linear functions. The worst case scenario is 98% for high-variance small samples with PING and GRMLN.

OLS solution used as starting guess for all methods. Using a different starting guess might yield different results.

Results:
Triad Model

$$y = ax^b + c$$

All simulations were run with 16,000 iterations.

Results for all 20 Simulations ($y = ax^b + c$; parameter a)

Median of Parameter Estimates (closer to $a = 25$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	23.71	23.03	21.87	21.38	23.53	19.37	17.78	16.30	11.93	15.64	21.64	21.03	18.40	16.93	22.42	13.60	9.79	7.76	4.88	7.01
	PING	24.68	24.07	23.08	22.61	24.57	22.30	21.70	21.12	17.30	20.25	22.47	21.83	19.25	17.78	23.27	15.57	11.69	9.53	6.39	8.80
	GRMLN	255.71	256.73	247.69	248.07	257.26	147.39	139.88	142.82	120.57	141.29	124.30	118.08	114.33	114.14	120.58	59.72	44.56	37.69	30.85	35.99
	MUPE	25.84	25.47	24.84	24.20	25.47	23.02	21.74	22.08	19.86	19.89	22.80	21.89	19.52	18.95	22.99	10.67	9.70	8.95	8.32	7.86
	ZMPE	29.45	28.40	27.72	27.84	28.35	35.30	37.30	38.08	34.97	35.70	35.17	35.42	34.25	33.20	37.50	44.02	53.56	56.15	56.47	62.67

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	171%	183%	210%	218%	176%	403%	505%	651%	937%	690%	487%	516%	621%	639%	501%	1508%	2065%	2512%	3493%	3037%
	PING	171%	183%	210%	218%	175%	403%	511%	650%	918%	691%	486%	515%	620%	632%	501%	1490%	2033%	2478%	3614%	2973%
	GRMLN	62%	62%	67%	67%	63%	81%	97%	113%	126%	112%	90%	91%	99%	96%	91%	146%	166%	185%	220%	190%
	MUPE	173%	175%	180%	185%	178%	399%	414%	443%	467%	463%	467%	484%	486%	503%	469%	1571%	1836%	1897%	2044%	2252%
	ZMPE	196%	191%	167%	168%	193%	474%	413%	381%	346%	356%	450%	438%	400%	420%	428%	605%	491%	461%	427%	406%

White cells are the best; Blue and Red cells are not as good.

Median and IQR used because all methods yielded highly skewed estimate distributions with many outliers.

(10; 1; 10,000) used as starting guess for methods utilizing optimization algorithms.

Results for all 20 Simulations ($y = ax^b + c$; parameter a)

Median of Parameter Estimates (closer to $a = 25$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	23.71	23.03	21.87	21.38	23.53	19.37	17.78	16.30	11.93	15.64	21.64	21.03	18.40	16.93	22.42	13.60	9.79	7.76	4.88	7.01
	PING	24.68	24.07	23.08	22.61	24.57	22.30	21.70	21.12	17.30	20.25	22.47	21.83	19.25	17.78	23.27	15.57	11.69	9.53	6.39	8.80
	GRMLN	255.71	256.73	247.69	248.07	257.26	147.39	139.88	142.82	120.57	141.29	124.30	118.08	114.33	114.14	120.58	59.72	44.56	37.69	30.85	35.99
	MUPE	25.84	25.47	24.84	24.20	25.47	23.02	21.74	22.08	19.86	19.89	22.80	21.89	19.52	18.95	22.99	10.67	9.70	8.95	8.32	7.86
	ZMPE	29.45	28.40	27.72	27.84	28.35	35.30	37.30	38.08	34.97	35.70	35.17	35.42	34.25	33.20	37.50	44.02	53.56	56.15	56.47	62.67

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	171%	183%	210%	218%	176%	403%	505%	651%	937%	690%	487%	516%	621%	639%	501%	1508%	2065%	2512%	3493%	3037%
	PING	171%	183%	210%	218%	175%	403%	511%	650%	918%	691%	486%	515%	620%	632%	501%	1490%	2033%	2478%	3614%	2973%
	GRMLN	62%	62%	67%	67%	63%	81%	97%	113%	126%	112%	90%	91%	99%	96%	91%	146%	166%	185%	220%	190%
	MUPE	173%	175%	180%	185%	178%	399%	414%	443%	467%	463%	467%	484%	486%	503%	469%	1571%	1836%	1897%	2044%	2252%
	ZMPE	196%	191%	167%	168%	193%	474%	413%	381%	346%	356%	450%	438%	400%	420%	428%	605%	491%	461%	427%	406%

Note the poor precision across the board. GRMLN is comparatively low merely as a result of its estimates being biased so high.

Results for all 20 Simulations ($y = ax^b + c$; parameter a)

Median of Parameter Estimates (closer to $a = 25$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	23.71	23.03	21.87	21.38	23.53	19.37	17.78	16.30	11.93	15.64	21.64	21.03	18.40	16.93	22.42	13.60	9.79	7.76	4.88	7.01
	PING	24.68	24.07	23.08	22.61	24.57	22.30	21.70	21.12	17.30	20.25	22.47	21.83	19.25	17.78	23.27	15.57	11.69	9.53	6.39	8.80
	GRMLN	255.71	256.73	247.69	248.07	257.26	147.39	139.88	142.82	120.57	141.29	124.30	118.08	114.33	114.14	120.58	59.72	44.56	37.69	30.85	35.99
	MUPE	25.84	25.47	24.84	24.20	25.47	23.02	21.74	22.08	19.86	19.89	22.80	21.89	19.52	18.95	22.99	10.67	9.70	8.95	8.32	7.86
	ZMPE	29.45	28.40	27.72	27.84	28.35	35.30	37.30	38.08	34.97	35.70	35.17	35.42	34.25	33.20	37.50	44.02	53.56	56.15	56.47	62.67

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	171%	183%	210%	218%	176%	403%	505%	651%	937%	690%	487%	516%	621%	639%	501%	1508%	2065%	2512%	3493%	3037%
	PING	171%	183%	210%	218%	175%	403%	511%	650%	918%	691%	486%	515%	620%	632%	501%	1490%	2033%	2478%	3614%	2973%
	GRMLN	62%	62%	67%	67%	63%	81%	97%	113%	126%	112%	90%	91%	99%	96%	91%	146%	166%	185%	220%	190%
	MUPE	173%	175%	180%	185%	178%	399%	414%	443%	467%	463%	467%	484%	486%	503%	469%	1571%	1836%	1897%	2044%	2252%
	ZMPE	196%	191%	167%	168%	193%	474%	413%	381%	346%	356%	450%	438%	400%	420%	428%	605%	491%	461%	427%	406%

The small sample, high variance scenario is particularly challenging for this model form.

Results for all 20 Simulations ($y = ax^b + c$; parameter a)

Median of Parameter Estimates (closer to $a = 25$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	23.71	23.03	21.87	21.38	23.53	19.37	17.78	16.30	11.93	15.64	21.64	21.03	18.40	16.93	22.42	13.60	9.79	7.76	4.88	7.01
	PING	24.68	24.07	23.08	22.61	24.57	22.30	21.70	21.12	17.30	20.25	22.47	21.83	19.25	17.78	23.27	15.57	11.69	9.53	6.39	8.80
	GRMLN	255.71	256.73	247.69	248.07	257.26	147.39	139.88	142.82	120.57	141.29	124.30	118.08	114.33	114.14	120.58	59.72	44.56	37.69	30.85	35.99
	MUPE	25.84	25.47	24.84	24.20	25.47	23.02	21.74	22.08	19.86	19.89	22.80	21.89	19.52	18.95	22.99	10.67	9.70	8.95	8.32	7.86
	ZMPE	29.45	28.40	27.72	27.84	28.35	35.30	37.30	38.08	34.97	35.70	35.17	35.42	34.25	33.20	37.50	44.02	53.56	56.15	56.47	62.67

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	171%	183%	210%	218%	176%	403%	505%	651%	937%	690%	487%	516%	621%	639%	501%	1508%	2065%	2512%	3493%	3037%
	PING	171%	183%	210%	218%	175%	403%	511%	650%	918%	691%	486%	515%	620%	632%	501%	1490%	2033%	2478%	3614%	2973%
	GRMLN	62%	62%	67%	67%	63%	81%	97%	113%	126%	112%	90%	91%	99%	96%	91%	146%	166%	185%	220%	190%
	MUPE	173%	175%	180%	185%	178%	399%	414%	443%	467%	463%	467%	484%	486%	503%	469%	1571%	1836%	1897%	2044%	2252%
	ZMPE	196%	191%	167%	168%	193%	474%	413%	381%	346%	356%	450%	438%	400%	420%	428%	605%	491%	461%	427%	406%

MUPE and PING provide the best accuracy, and MUPE’s precision is more consistent.

Results for all 20 Simulations ($y = ax^b + c$; parameter b)

Median of Parameter Estimates (closer to $b = 1.2$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	1.202	1.206	1.211	1.215	1.203	1.215	1.223	1.231	1.266	1.234	1.214	1.219	1.236	1.247	1.209	1.263	1.307	1.336	1.398	1.344
	PING	1.202	1.206	1.211	1.215	1.203	1.215	1.223	1.231	1.266	1.234	1.214	1.219	1.236	1.247	1.209	1.263	1.307	1.336	1.398	1.344
	GRMLN	0.900	0.899	0.905	0.905	0.898	0.976	0.990	0.992	1.028	0.993	1.003	1.009	1.015	1.017	1.008	1.105	1.153	1.181	1.221	1.185
	MUPE	1.196	1.198	1.201	1.205	1.198	1.212	1.220	1.218	1.231	1.231	1.213	1.218	1.233	1.239	1.211	1.313	1.331	1.343	1.358	1.359
	ZMPE	1.180	1.183	1.188	1.187	1.185	1.155	1.150	1.148	1.158	1.156	1.157	1.155	1.160	1.164	1.148	1.130	1.102	1.098	1.100	1.086

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	18%	18%	20%	21%	18%	33%	39%	44%	51%	45%	37%	39%	41%	42%	38%	64%	70%	74%	79%	76%
	PING	18%	18%	20%	21%	18%	33%	39%	44%	51%	45%	37%	39%	41%	42%	38%	64%	70%	74%	79%	76%
	GRMLN	10%	10%	11%	11%	10%	12%	14%	17%	18%	16%	13%	13%	14%	14%	13%	20%	22%	23%	28%	24%
	MUPE	18%	18%	18%	18%	18%	33%	34%	35%	35%	35%	37%	38%	36%	37%	37%	63%	66%	67%	68%	69%
	ZMPE	21%	19%	17%	17%	19%	45%	40%	37%	33%	33%	40%	39%	35%	36%	39%	67%	63%	60%	55%	58%

Results for all 20 Simulations ($y = ax^b + c$; parameter b)

Median of Parameter Estimates (closer to $b = 1.2$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	1.202	1.206	1.211	1.215	1.203	1.215	1.223	1.231	1.266	1.234	1.214	1.219	1.236	1.247	1.209	1.263	1.307	1.336	1.398	1.344
	PING	1.202	1.206	1.211	1.215	1.203	1.215	1.223	1.231	1.266	1.234	1.214	1.219	1.236	1.247	1.209	1.263	1.307	1.336	1.398	1.344
	GRMLN	0.900	0.899	0.905	0.905	0.898	0.976	0.990	0.992	1.028	0.993	1.003	1.009	1.015	1.017	1.008	1.105	1.153	1.181	1.221	1.185
	MUPE	1.196	1.198	1.201	1.205	1.198	1.212	1.220	1.218	1.231	1.231	1.213	1.218	1.233	1.239	1.211	1.313	1.331	1.343	1.358	1.359
	ZMPE	1.180	1.183	1.188	1.187	1.185	1.155	1.150	1.148	1.158	1.156	1.157	1.155	1.160	1.164	1.148	1.130	1.102	1.098	1.100	1.086

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	18%	18%	20%	21%	18%	33%	39%	44%	51%	45%	37%	39%	41%	42%	38%	64%	70%	74%	79%	76%
	PING	18%	18%	20%	21%	18%	33%	39%	44%	51%	45%	37%	39%	41%	42%	38%	64%	70%	74%	79%	76%
	GRMLN	10%	10%	11%	11%	10%	12%	14%	17%	18%	16%	13%	13%	14%	14%	13%	20%	22%	23%	28%	24%
	MUPE	18%	18%	18%	18%	18%	33%	34%	35%	35%	35%	37%	38%	36%	37%	37%	63%	66%	67%	68%	69%
	ZMPE	21%	19%	17%	17%	19%	45%	40%	37%	33%	33%	40%	39%	35%	36%	39%	67%	63%	60%	55%	58%

MUPE and PING/Log-Error tend to provide the best estimates for parameter *b*.

Results for all 20 Simulations ($y = ax^b + c$; parameter c)

Median of Parameter Estimates (closer to $c = 30{,}000$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	28557	28616	28772	28839	28440	25125	24440	23758	23299	23867	28106	28413	29233	29402	28046	25054	24661	24548	23934	24410
	PING	29738	29930	30424	30540	29669	29078	29961	30792	33944	31005	29115	29520	30629	30848	29135	28270	29312	30604	32860	30688
	GRMLN	10010	10010	10009	10009	10010	10003	10003	10003	10002	10003	10001	10001	10001	10001	10001	10000	10000	10000	10000	10000
	MUPE	29377	29482	29829	29809	29444	27934	28598	28706	29396	29252	28522	28835	30040	30149	28639	26594	27256	27933	28846	28555
	ZMPE	28378	28657	29160	29104	28814	25370	25758	26229	27126	27275	26393	26657	27575	27616	26574	22982	22928	23468	23669	23708

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	33%	34%	37%	37%	34%	62%	72%	80%	90%	83%	70%	71%	72%	73%	74%	117%	128%	133%	137%	138%
	PING	33%	34%	37%	37%	34%	62%	72%	80%	90%	83%	70%	71%	72%	73%	74%	117%	128%	132%	134%	136%
	GRMLN	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
	MUPE	34%	34%	34%	34%	35%	69%	70%	72%	70%	73%	73%	73%	69%	71%	76%	131%	143%	146%	144%	150%
	ZMPE	39%	37%	33%	33%	38%	88%	80%	76%	67%	69%	78%	76%	69%	71%	79%	140%	144%	141%	135%	146%

Results for all 20 Simulations ($y = ax^b + c$; parameter c)

Median of Parameter Estimates (closer to $c = 30{,}000$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	28557	28616	28772	28839	28440	25125	24440	23758	23299	23867	28106	28413	29233	29402	28046	25054	24661	24548	23934	24410
	PING	29738	29930	30424	30540	29669	29078	29961	30792	33944	31005	29115	29520	30629	30848	29135	28270	29312	30604	32860	30688
	GRMLN	10010	10010	10009	10009	10010	10003	10003	10003	10002	10003	10001	10001	10001	10001	10001	10000	10000	10000	10000	10000
	MUPE	29377	29482	29829	29809	29444	27934	28598	28706	29396	29252	28522	28835	30040	30149	28639	26594	27256	27933	28846	28555
	ZMPE	28378	28657	29160	29104	28814	25370	25758	26229	27126	27275	26393	26657	27575	27616	26574	22982	22928	23468	23669	23708

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	33%	34%	37%	37%	34%	62%	72%	80%	90%	83%	70%	71%	72%	73%	74%	117%	128%	133%	137%	138%
	PING	33%	34%	37%	37%	34%	62%	72%	80%	90%	83%	70%	71%	72%	73%	74%	117%	128%	132%	134%	136%
	GRMLN	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
	MUPE	34%	34%	34%	34%	35%	69%	70%	72%	70%	73%	73%	73%	69%	71%	76%	131%	143%	146%	144%	150%
	ZMPE	39%	37%	33%	33%	38%	88%	80%	76%	67%	69%	78%	76%	69%	71%	79%	140%	144%	141%	135%	146%

Here it can be seen that GRMLN fails to migrate from the initial guess of 10,000 for parameter *c*.

Results for all 20 Simulations ($y = ax^b + c$; parameter c)

Median of Parameter Estimates (closer to $c = 30{,}000$ is more accurate)

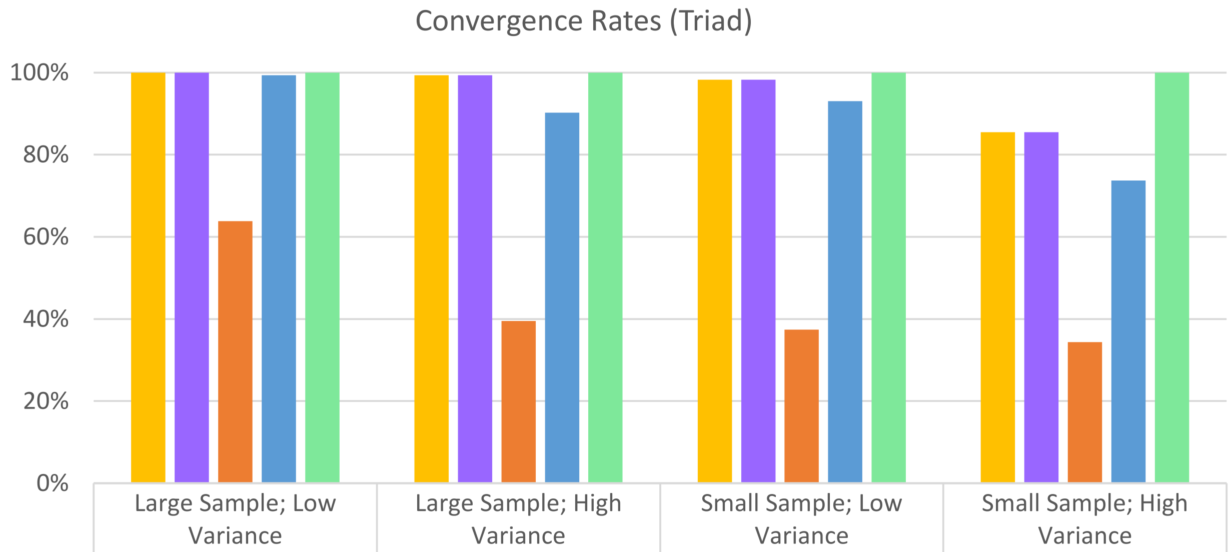
	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	28557	28616	28772	28839	28440	25125	24440	23758	23299	23867	28106	28413	29233	29402	28046	25054	24661	24548	23934	24410
	PING	29738	29930	30424	30540	29669	29078	29961	30792	33944	31005	29115	29520	30629	30848	29135	28270	29312	30604	32860	30688
	GRMLN	10010	10010	10009	10009	10010	10003	10003	10003	10002	10003	10001	10001	10001	10001	10001	10000	10000	10000	10000	10000
	MUPE	29377	29482	29829	29809	29444	27934	28598	28706	29396	29252	28522	28835	30040	30149	28639	26594	27256	27933	28846	28555
	ZMPE	28378	28657	29160	29104	28814	25370	25758	26229	27126	27275	26393	26657	27575	27616	26574	22982	22928	23468	23669	23708

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	33%	34%	37%	37%	34%	62%	72%	80%	90%	83%	70%	71%	72%	73%	74%	117%	128%	133%	137%	138%
	PING	33%	34%	37%	37%	34%	62%	72%	80%	90%	83%	70%	71%	72%	73%	74%	117%	128%	132%	134%	136%
	GRMLN	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
	MUPE	34%	34%	34%	34%	35%	69%	70%	72%	70%	73%	73%	73%	69%	71%	76%	131%	143%	146%	144%	150%
	ZMPE	39%	37%	33%	33%	38%	88%	80%	76%	67%	69%	78%	76%	69%	71%	79%	140%	144%	141%	135%	146%

MUPE and PING tend to provide the best estimates for parameter *c*.

Triad Function Regression Convergence Rates



	Large Sample; Low Variance	Large Sample; High Variance	Small Sample; Low Variance	Small Sample; High Variance
🟨 Log Error	100.00%	99.35%	98.26%	85.47%
🟪 PING	100.00%	99.35%	98.26%	85.47%
🟧 GRMLN	63.82%	39.52%	37.45%	34.34%
🟦 MUPE	99.36%	90.26%	93.05%	73.70%
🟩 ZMPE	100.00%	100.00%	100.00%	100.00%

GRMLN fails at the triad model form. Of the remaining methods, MUPE has the lowest convergence rate, although it still exceeds 90% in all cases except high variance small samples, where it is 74%.

(10; 1; 10,000) used as starting guess for all methods. These values represent the correct sign and order of magnitude of the true population parameters. Using a different starting guess might yield different results.

Results:
Multivariate Log-Linear

$$y = ax_1^b \, x_2^c \, d^{x_3}$$

All simulations were run with 16,000 iterations.

Results for all 20 Simulations ($y = a x_1^b\, x_2^c\, d^{x_3}$; parameter a)

Median of Parameter Estimates (closer to $a = 35$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	33.41	33.42	33.46	33.44	33.53	30.00	29.33	28.60	28.18	28.18	33.15	33.31	34.03	34.22	33.39	29.65	28.68	30.13	30.07	28.84
	PING	34.76	34.92	35.34	35.38	34.94	34.68	35.61	36.80	39.96	36.38	34.26	34.50	35.39	35.69	34.55	33.31	33.47	36.80	38.67	35.29
	GRMLN	34.76	34.92	35.34	35.38	34.94	34.68	35.61	36.80	39.96	36.38	34.26	34.49	35.39	35.69	34.55	33.31	33.47	36.80	38.79	35.29
	MUPE	34.42	34.57	34.79	34.70	34.66	33.10	33.58	33.01	33.71	33.84	33.45	33.93	34.52	34.48	33.94	30.25	30.21	31.65	32.25	30.99
	ZMPE	33.60	33.96	34.17	34.18	33.99	31.07	31.34	31.38	31.18	32.01	32.53	32.75	33.04	32.81	32.75	27.80	26.36	26.12	25.40	25.68

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	55%	58%	64%	65%	57%	108%	126%	144%	173%	151%	113%	118%	127%	128%	120%	246%	288%	346%	404%	353%
	PING	55%	58%	64%	65%	57%	108%	126%	145%	176%	152%	113%	118%	127%	129%	120%	247%	288%	349%	417%	355%
	GRMLN	55%	58%	64%	65%	57%	108%	126%	145%	176%	152%	113%	118%	127%	129%	120%	247%	288%	349%	416%	355%
	MUPE	56%	57%	57%	57%	57%	115%	115%	124%	120%	124%	116%	117%	116%	120%	120%	258%	282%	306%	305%	309%
	ZMPE	63%	60%	52%	53%	60%	160%	139%	128%	113%	113%	126%	123%	111%	115%	129%	333%	318%	309%	298%	319%

White cells are the best; Blue and Red cells are not as good.

Median and IQR used because all methods yielded highly skewed estimate distributions with many outliers.

PING solution used as starting guess for methods utilizing optimization algorithms.

Results for all 20 Simulations ($y = ax_1^b\, x_2^c\, d^{x_3}$; parameter a)

Median of Parameter Estimates (closer to $a = 35$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	33.41	33.42	33.46	33.44	33.53	30.00	29.33	28.60	28.18	28.18	33.15	33.31	34.03	34.22	33.39	29.65	28.68	30.13	30.07	28.84
	PING	34.76	34.92	35.34	35.38	34.94	34.68	35.61	36.80	39.96	36.38	34.26	34.50	35.39	35.69	34.55	33.31	33.47	36.80	38.67	35.29
	GRMLN	34.76	34.92	35.34	35.38	34.94	34.68	35.61	36.80	39.96	36.38	34.26	34.49	35.39	35.69	34.55	33.31	33.47	36.80	38.79	35.29
	MUPE	34.42	34.57	34.79	34.70	34.66	33.10	33.58	33.01	33.71	33.84	33.45	33.93	34.52	34.48	33.94	30.25	30.21	31.65	32.25	30.99
	ZMPE	33.60	33.96	34.17	34.18	33.99	31.07	31.34	31.38	31.18	32.01	32.53	32.75	33.04	32.81	32.75	27.80	26.36	26.12	25.40	25.68

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	55%	58%	64%	65%	57%	108%	126%	144%	173%	151%	113%	118%	127%	128%	120%	246%	288%	346%	404%	353%
	PING	55%	58%	64%	65%	57%	108%	126%	145%	176%	152%	113%	118%	127%	129%	120%	247%	288%	349%	417%	355%
	GRMLN	55%	58%	64%	65%	57%	108%	126%	145%	176%	152%	113%	118%	127%	129%	120%	247%	288%	349%	416%	355%
	MUPE	56%	57%	57%	57%	57%	115%	115%	124%	120%	124%	116%	117%	116%	120%	120%	258%	282%	306%	305%	309%
	ZMPE	63%	60%	52%	53%	60%	160%	139%	128%	113%	113%	126%	123%	111%	115%	129%	333%	318%	309%	298%	319%

The small sample/high variance case is particularly challenging*.

*It should be noted that most experienced analysts would *not* apply a 3-variable, 4-parameter model with a sample size of 15.

Results for all 20 Simulations ($y = ax_1^b\, x_2^c\, d^{x_3}$; parameter a)

Median of Parameter Estimates (closer to $a = 35$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	33.41	33.42	33.46	33.44	33.53	30.00	29.33	28.60	28.18	28.18	33.15	33.31	34.03	34.22	33.39	29.65	28.68	30.13	30.07	28.84
	PING	34.76	34.92	35.34	35.38	34.94	34.68	35.61	36.80	39.96	36.38	34.26	34.50	35.39	35.69	34.55	33.31	33.47	36.80	38.67	35.29
	GRMLN	34.76	34.92	35.34	35.38	34.94	34.68	35.61	36.80	39.96	36.38	34.26	34.49	35.39	35.69	34.55	33.31	33.47	36.80	38.79	35.29
	MUPE	34.42	34.57	34.79	34.70	34.66	33.10	33.58	33.01	33.71	33.84	33.45	33.93	34.52	34.48	33.94	30.25	30.21	31.65	32.25	30.99
	ZMPE	33.60	33.96	34.17	34.18	33.99	31.07	31.34	31.38	31.18	32.01	32.53	32.75	33.04	32.81	32.75	27.80	26.36	26.12	25.40	25.68

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	55%	58%	64%	65%	57%	108%	126%	144%	173%	151%	113%	118%	127%	128%	120%	246%	288%	346%	404%	353%
	PING	55%	58%	64%	65%	57%	108%	126%	145%	176%	152%	113%	118%	127%	129%	120%	247%	288%	349%	417%	355%
	GRMLN	55%	58%	64%	65%	57%	108%	126%	145%	176%	152%	113%	118%	127%	129%	120%	247%	288%	349%	416%	355%
	MUPE	56%	57%	57%	57%	57%	115%	115%	124%	120%	124%	116%	117%	116%	120%	120%	258%	282%	306%	305%	309%
	ZMPE	63%	60%	52%	53%	60%	160%	139%	128%	113%	113%	126%	123%	111%	115%	129%	333%	318%	309%	298%	319%

In low variance scenarios, PING/GRMLN and MUPE provide the best estimates.

Results for all 20 Simulations ($y = ax_1^b\, x_2^c\, d^{x_3}$; parameter b)

Median of Parameter Estimates (closer to $b = 0.75$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	0.750	0.750	0.749	0.749	0.749	0.750	0.748	0.747	0.746	0.748	0.752	0.749	0.747	0.748	0.750	0.750	0.751	0.742	0.741	0.749
	PING	0.750	0.750	0.749	0.749	0.749	0.750	0.748	0.747	0.746	0.748	0.752	0.749	0.747	0.748	0.750	0.750	0.751	0.742	0.741	0.749
	GRMLN	0.750	0.750	0.749	0.749	0.749	0.750	0.748	0.747	0.746	0.748	0.752	0.749	0.747	0.748	0.750	0.750	0.751	0.742	0.741	0.749
	MUPE	0.751	0.751	0.749	0.750	0.749	0.752	0.751	0.751	0.752	0.750	0.754	0.751	0.748	0.750	0.751	0.757	0.755	0.752	0.748	0.759
	ZMPE	0.752	0.752	0.751	0.751	0.751	0.760	0.758	0.757	0.758	0.755	0.757	0.754	0.752	0.754	0.753	0.765	0.770	0.769	0.771	0.775

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	8.8%	9.1%	10.0%	10.2%	8.8%	16%	19%	22%	25%	22%	17%	18%	20%	19%	18%	32%	38%	43%	49%	43%
	PING	8.8%	9.1%	10.0%	10.2%	8.8%	16%	19%	22%	25%	22%	17%	18%	20%	19%	18%	32%	38%	43%	49%	43%
	GRMLN	8.8%	9.1%	10.0%	10.2%	8.8%	16%	19%	22%	25%	22%	17%	18%	20%	19%	18%	32%	38%	43%	49%	43%
	MUPE	8.9%	8.9%	8.9%	9.0%	8.9%	17%	18%	18%	18%	18%	17%	18%	18%	18%	18%	33%	37%	38%	39%	39%
	ZMPE	10.0%	9.4%	8.2%	8.5%	9.4%	23%	21%	19%	17%	17%	18%	18%	18%	17%	19%	40%	40%	39%	37%	40%

Results for all 20 Simulations ($y = ax_1^b\, x_2^c\, d^{x_3}$; parameter b)

Median of Parameter Estimates (closer to $b = 0.75$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	0.750	0.750	0.749	0.749	0.749	0.750	0.748	0.747	0.746	0.748	0.752	0.749	0.747	0.748	0.750	0.750	0.751	0.742	0.741	0.749
	PING	0.750	0.750	0.749	0.749	0.749	0.750	0.748	0.747	0.746	0.748	0.752	0.749	0.747	0.748	0.750	0.750	0.751	0.742	0.741	0.749
	GRMLN	0.750	0.750	0.749	0.749	0.749	0.750	0.748	0.747	0.746	0.748	0.752	0.749	0.747	0.748	0.750	0.750	0.751	0.742	0.741	0.749
	MUPE	0.751	0.751	0.749	0.750	0.749	0.752	0.751	0.751	0.752	0.750	0.754	0.751	0.748	0.750	0.751	0.757	0.755	0.752	0.748	0.759
	ZMPE	0.752	0.752	0.751	0.751	0.751	0.760	0.758	0.757	0.758	0.755	0.757	0.754	0.752	0.754	0.753	0.765	0.770	0.769	0.771	0.775

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	8.8%	9.1%	10.0%	10.2%	8.8%	16%	19%	22%	25%	22%	17%	18%	20%	19%	18%	32%	38%	43%	49%	43%
	PING	8.8%	9.1%	10.0%	10.2%	8.8%	16%	19%	22%	25%	22%	17%	18%	20%	19%	18%	32%	38%	43%	49%	43%
	GRMLN	8.8%	9.1%	10.0%	10.2%	8.8%	16%	19%	22%	25%	22%	17%	18%	20%	19%	18%	32%	38%	43%	49%	43%
	MUPE	8.9%	8.9%	8.9%	9.0%	8.9%	17%	18%	18%	18%	18%	17%	18%	18%	18%	18%	33%	37%	38%	39%	39%
	ZMPE	10.0%	9.4%	8.2%	8.5%	9.4%	23%	21%	19%	17%	17%	18%	18%	18%	17%	19%	40%	40%	39%	37%	40%

PING/GRMLN and MUPE provide similar performance in most scenarios, with MUPE being more accommodating of non-lognormal error distributions.

Results for all 20 Simulations ($y = a x_1^b \, x_2^c \, d^{x_3}$; parameter c)

Median of Parameter Estimates (closer to $c = 0.85$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	0.851	0.850	0.848	0.849	0.851	0.850	0.850	0.847	0.844	0.849	0.850	0.850	0.850	0.846	0.852	0.850	0.850	0.843	0.839	0.845
	PING	0.851	0.850	0.848	0.849	0.851	0.850	0.850	0.847	0.844	0.849	0.850	0.850	0.850	0.846	0.852	0.850	0.850	0.843	0.839	0.845
	GRMLN	0.851	0.850	0.848	0.849	0.851	0.850	0.850	0.847	0.844	0.849	0.850	0.850	0.850	0.846	0.852	0.850	0.850	0.843	0.838	0.845
	MUPE	0.851	0.851	0.849	0.850	0.851	0.853	0.853	0.853	0.850	0.852	0.851	0.852	0.851	0.849	0.853	0.855	0.857	0.852	0.850	0.854
	ZMPE	0.853	0.852	0.851	0.852	0.853	0.860	0.858	0.857	0.856	0.856	0.854	0.855	0.855	0.853	0.856	0.869	0.869	0.870	0.872	0.871

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	7.6%	7.9%	8.8%	8.9%	7.8%	15%	17%	19%	23%	19%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	PING	7.6%	7.9%	8.8%	8.9%	7.8%	15%	17%	19%	23%	19%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	GRMLN	7.6%	7.9%	8.8%	8.9%	7.8%	15%	17%	19%	23%	19%	15%	16%	17%	17%	16%	29%	33%	37%	43%	38%
	MUPE	7.7%	7.8%	7.9%	8.0%	7.9%	15%	16%	16%	16%	16%	15%	16%	16%	16%	16%	29%	32%	33%	34%	35%
	ZMPE	8.6%	8.2%	7.2%	7.5%	8.4%	20%	18%	17%	15%	15%	16%	16%	15%	15%	17%	34%	35%	34%	33%	35%

Results for parameter c are very similar to those for parameter b.

Results for all 20 Simulations ($y = ax_1^b\, x_2^c\, d^{x_3}$; parameter d)

Median of Parameter Estimates (closer to $d = 1.2$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	1.201	1.200	1.200	1.199	1.200	1.201	1.199	1.201	1.199	1.201	1.200	1.201	1.201	1.198	1.201	1.196	1.207	1.204	1.194	1.202
	PING	1.201	1.200	1.200	1.199	1.200	1.201	1.199	1.201	1.199	1.201	1.200	1.201	1.201	1.198	1.201	1.196	1.207	1.204	1.194	1.202
	GRMLN	1.201	1.200	1.200	1.199	1.200	1.201	1.199	1.201	1.199	1.201	1.200	1.201	1.201	1.198	1.201	1.196	1.207	1.204	1.194	1.202
	MUPE	1.201	1.200	1.200	1.200	1.200	1.201	1.199	1.201	1.199	1.200	1.201	1.201	1.201	1.197	1.201	1.193	1.204	1.200	1.197	1.198
	ZMPE	1.201	1.199	1.201	1.199	1.200	1.201	1.201	1.199	1.200	1.199	1.201	1.202	1.198	1.199	1.203	1.203	1.204	1.208	1.203	1.200

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	12%	12%	13%	14%	12%	21%	26%	29%	35%	30%	23%	24%	26%	26%	24%	44%	52%	59%	69%	59%
	PING	12%	12%	13%	14%	12%	21%	26%	29%	35%	30%	23%	24%	26%	26%	24%	44%	52%	59%	69%	59%
	GRMLN	12%	12%	13%	14%	12%	21%	26%	29%	35%	30%	23%	24%	26%	26%	24%	44%	52%	59%	69%	59%
	MUPE	12%	12%	12%	12%	12%	23%	24%	24%	25%	24%	23%	24%	24%	24%	24%	46%	49%	51%	51%	51%
	ZMPE	14%	13%	11%	11%	13%	32%	28%	25%	23%	22%	26%	25%	23%	23%	26%	55%	56%	53%	51%	53%

Results for all 20 Simulations ($y = a x_1^b \, x_2^c \, d^{x_3}$; parameter d)

Median of Parameter Estimates (closer to $d = 1.2$ is more accurate)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	1.201	1.200	1.200	1.199	1.200	1.201	1.199	1.201	1.199	1.201	1.200	1.201	1.201	1.198	1.201	1.196	1.207	1.204	1.194	1.202
	PING	1.201	1.200	1.200	1.199	1.200	1.201	1.199	1.201	1.199	1.201	1.200	1.201	1.201	1.198	1.201	1.196	1.207	1.204	1.194	1.202
	GRMLN	1.201	1.200	1.200	1.199	1.200	1.201	1.199	1.201	1.199	1.201	1.200	1.201	1.201	1.198	1.201	1.196	1.207	1.204	1.194	1.202
	MUPE	1.201	1.200	1.200	1.200	1.200	1.201	1.199	1.201	1.199	1.200	1.201	1.201	1.201	1.197	1.201	1.193	1.204	1.200	1.197	1.198
	ZMPE	1.201	1.199	1.201	1.199	1.200	1.201	1.201	1.199	1.200	1.199	1.201	1.202	1.198	1.199	1.203	1.203	1.204	1.208	1.203	1.200

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	12%	12%	13%	14%	12%	21%	26%	29%	35%	30%	23%	24%	26%	26%	24%	44%	52%	59%	69%	59%
	PING	12%	12%	13%	14%	12%	21%	26%	29%	35%	30%	23%	24%	26%	26%	24%	44%	52%	59%	69%	59%
	GRMLN	12%	12%	13%	14%	12%	21%	26%	29%	35%	30%	23%	24%	26%	26%	24%	44%	52%	59%	69%	59%
	MUPE	12%	12%	12%	12%	12%	23%	24%	24%	25%	24%	23%	24%	24%	24%	24%	46%	49%	51%	51%	51%
	ZMPE	14%	13%	11%	11%	13%	32%	28%	25%	23%	22%	26%	25%	23%	23%	26%	55%	56%	53%	51%	53%

Accuracy is good across the board, although worst in the small sample / high variance scenario.

Results for all 20 Simulations ($y = ax_1^b\, x_2^c\, d^{x_3}$; parameter d)

Median of Parameter Estimates (closer to $d = 1.2$ is more accurate)

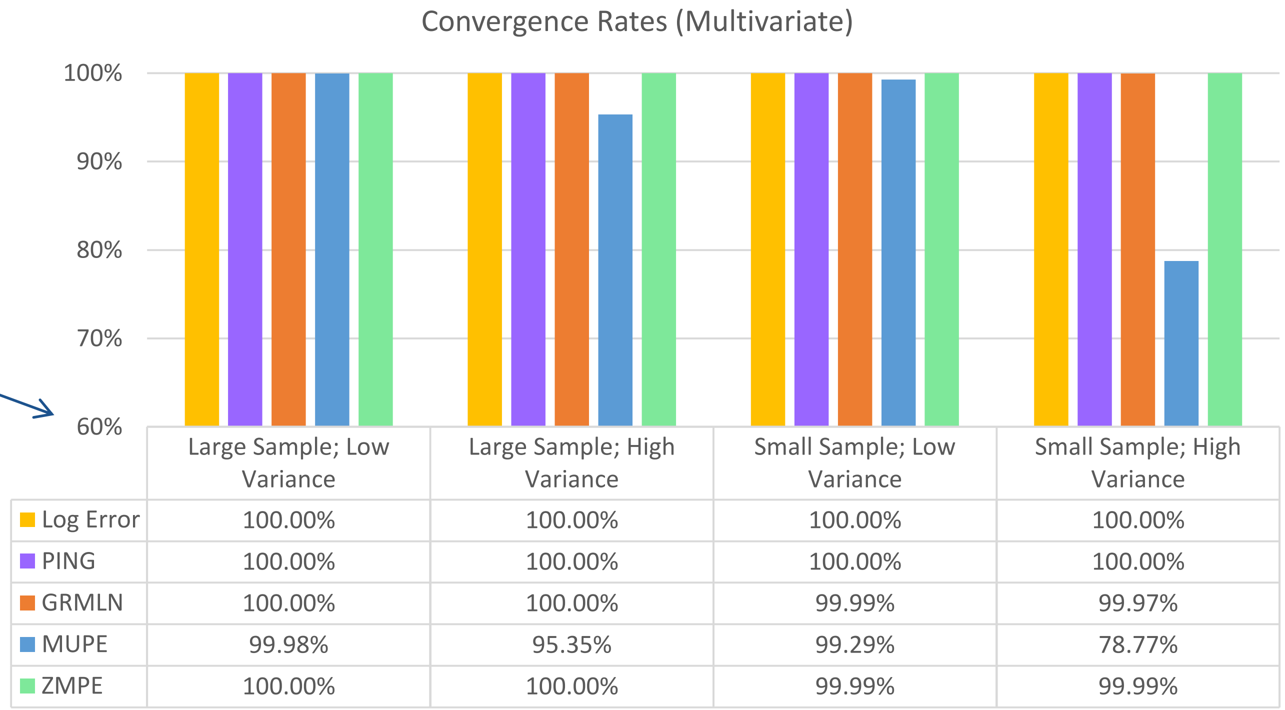
	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	1.201	1.200	1.200	1.199	1.200	1.201	1.199	1.201	1.199	1.201	1.200	1.201	1.201	1.198	1.201	1.196	1.207	1.204	1.194	1.202
	PING	1.201	1.200	1.200	1.199	1.200	1.201	1.199	1.201	1.199	1.201	1.200	1.201	1.201	1.198	1.201	1.196	1.207	1.204	1.194	1.202
	GRMLN	1.201	1.200	1.200	1.199	1.200	1.201	1.199	1.201	1.199	1.201	1.200	1.201	1.201	1.198	1.201	1.196	1.207	1.204	1.194	1.202
	MUPE	1.201	1.200	1.200	1.200	1.200	1.201	1.199	1.201	1.199	1.200	1.201	1.201	1.201	1.197	1.201	1.193	1.204	1.200	1.197	1.198
	ZMPE	1.201	1.199	1.201	1.199	1.200	1.201	1.201	1.199	1.200	1.199	1.201	1.202	1.198	1.199	1.203	1.203	1.204	1.208	1.203	1.200

(IQR/Median) of Parameter Estimates (lower is more precise)

	Sample Size	Large										Small									
	Variance	Low					High					Low					High				
	Error Distribution	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle	Log-normal	Gamma	Weibull	Trunc. normal	Triangle
Method	Log Error	12%	12%	13%	14%	12%	21%	26%	29%	35%	30%	23%	24%	26%	26%	24%	44%	52%	59%	69%	59%
	PING	12%	12%	13%	14%	12%	21%	26%	29%	35%	30%	23%	24%	26%	26%	24%	44%	52%	59%	69%	59%
	GRMLN	12%	12%	13%	14%	12%	21%	26%	29%	35%	30%	23%	24%	26%	26%	24%	44%	52%	59%	69%	59%
	MUPE	12%	12%	12%	12%	12%	23%	24%	24%	25%	24%	23%	24%	24%	24%	24%	46%	49%	51%	51%	51%
	ZMPE	14%	13%	11%	11%	13%	32%	28%	25%	23%	22%	26%	25%	23%	23%	26%	55%	56%	53%	51%	53%

MUPE exhibits the most consistent precision.

Multivariate Function Regression Convergence Rates



Axis limits shrunk to accentuate differences.

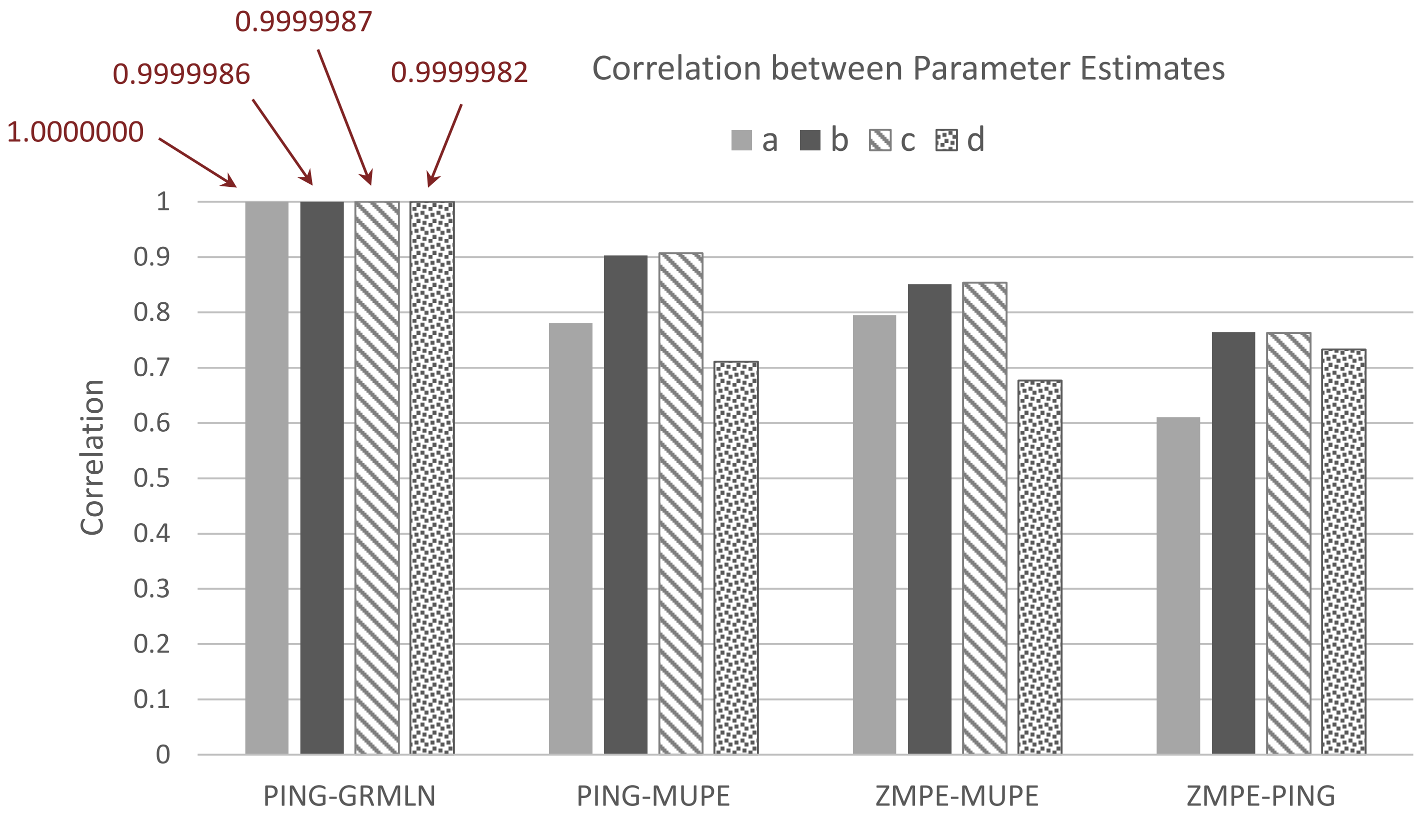
	Large Sample; Low Variance	Large Sample; High Variance	Small Sample; Low Variance	Small Sample; High Variance
🟨 Log Error	100.00%	100.00%	100.00%	100.00%
🟪 PING	100.00%	100.00%	100.00%	100.00%
🟧 GRMLN	100.00%	100.00%	99.99%	99.97%
🟦 MUPE	99.98%	95.35%	99.29%	78.77%
🟩 ZMPE	100.00%	100.00%	99.99%	99.99%

Convergence is lowest with MUPE, at 95% for high-variance large samples, and 79% for high-variance small samples*.

PING solution used as starting guess for GRMLN, MUPE, and ZMPE.
Using a different starting guess might yield different results.

*It should be noted that most experienced analysts would *not* apply a 3-variable, 4-parameter model with a sample size of 15.

Correlation between Parameter Estimates ($y = ax_1^b\, x_2^c\, d^{x_3}$)



1.0000000

0.9999986

0.9999987

0.9999982

PING and GRMLN provide approximately equivalent results for multivariate log-linear models.