16

Mass Matrix Construction Overview

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§16.1. Introduction

This Part focuses on the construction of the mass matrix for FEM dynamic analysis of structures. Chapters 16 through 20 present and illustrate the two standard methods used to accomplish that goal: direct mass lumping and variational mass lumping. Those methods gave been used for decades, are well understood by now, and are implemented in production FEM programs. Chapters 21ff deal with a more general and advanced technique: templates. This approach produces *parametrized* mass matrices that include the standard ones as special instances. This newer scheme is still undergoing development.

For the convenience of the reader, Table 16.1 collects acronyms often used in this Part.

§16.2. Mass Matrix Construction Steps

In Part II the master (system) mass matrix of a structural FEM model emerged as the discrete operator that converts nodal accelerations to inertial nodal forces:

$$\mathbf{f}_I = \mathbf{M}\,\ddot{\mathbf{u}}. \tag{16.1}$$

This relation expresses Newton's second law for a discrete dynamic system with masses constant in time.[1] In the framework of the Direct Stiffness Method (DSM), the construction of **M** is done through three steps:

Step 1: **Localization**: Form the mass matrix of each element, using a local frame if convenient.

Step 2: **Globalization**: Transform element mass matrices to global coordinates if necessary.

Step 3: **Assembly**: Merge the globalized element mass matrices to form **M**.

In practice these operations are carried out concurrently within an element-by-element loop. On loop exit, the master mass matrix is complete.

Readers familiar with the Direct Stiffness Method (DSM) for FEM static analysis [255] may notice that the formation of **M** through the preceding steps largely parallels that of the master stiffness matrix **K**. In particular, merging of element mass matrices into the master mass matrix follows exactly the same techniques. Consequence: assemblers for **K** and **M**, before application of boundary conditions, can be made identical — except for obvious indexing shortcuts in the case of diagonal mass matrices. This procedural uniformity is one of the strengths of DSM.

A notable difference with the stiffness matrix is the possibility of using a *diagonally lumped mass matrix* (DLMM) based on the *direct mass lumping* scheme described below. A master DLMM can be stored as a vector. If all entries are nonzero, it is easily inverted in place, since the inverse of a diagonal matrix is diagonal. Plainly using a DLMM entails significant advantages in computations that involve $\mathbf{M}^{-1}$; for example explicit time integration [74,766] as well as symmetric eigenproblem solution [542]. Those benefits are counteracted by some negative features discussed later.

[1] For the relatively rare cases in which the mass varies with time, the law must be used in the original form stated by Newton: the time derivative of momentum equals the inertial force.

Table §16.1. Acronyms Used in Part III

Acronym	*Stands for*
AB	Acoustic branch in DDD: has physical meaning in continuum models
ABTS	AB Taylor series in DWN κ, centered at $\kappa = 0$
BLCD	Best linear combination (LFF sense) of the CMM and a selected DLMM
BLFD	Best possible DLMM (LFF sense); acronym also applies to MS pair with this mass
BLFM	Best possible FPMM (LFF sense); acronym also applies to MS pair with this mass
CMM	Consistent mass matrix: a special VDMM in which VSM and DSF coalesce
CMS	Component Mode Synthesis: model reduction framework for structural dynamics
CMT	Congruential (also spelled congruent) mass transformation
COB	Constant optical branch: OB frequency is independent of wavenumber
COF	Cutoff frequency: OB frequency at zero wavenumber (lowest one if multiple OB)
DDD	Dimensionless dispersion diagram: DCF Ω vs. DWN κ
DGVD	Dimensionless group velocity diagram: $\gamma_c = c/c_0$ vs. DWN κ
DIMM	Directionally invariant mass matrix: repeats with respect to any RCC frame
DLMM	Diagonally lumped mass matrix; qualifier "diagonally" is often omitted
DML	Direct mass lumping
DSF	Displacement shape functions to interpolate displacements over element
DTI	Direct time integration of EOM
DCF	Dimensionless circular frequency, always denoted by Ω
DWN	Dimensionless wavenumber, always denoted by κ
FFB	Flexural frequency branch in Bernoulli-Euler or Timoshenko beam models
FPMM	Fully populated mass matrix (at element level); includes CMM as special case
HF	High frequency: short wavelength, small DWN, typically $\kappa > 1$
LCD	Mass matrix obtained as linear combination of the CMM and a selected DLMM
LF	Low frequency: long wavelength, small DWN, typically $\kappa < 1$
LFF	Low frequency fitting of AB to that of continuum
LLMM	Lobatto lumped mass matrix: a DLMM based on a Lobatto quadrature rule
MOF	Maximum overall frequency: largest frequency in DDD over Brillouin zone
NCT	Non-continuum term: a term in the ABTS that is not present in the continuum
OB	Optical branch (or branches) in DDD: no physical meaning in continuum models
OBTS	OB Taylor series in DWN κ, centered at $\kappa = 0$
SDAV	Structural dynamics and vibration applications: low frequency range important
SFB	Shear frequency branch in the Timoshenko beam model
SLMM	Simpson lumped mass matrix: a LLMM based on Simpson's 3-pt quadrature rule
SMS	Selective mass scaling: modifying a mass matrix by adding a scaled stiffness
TML	Template mass lumping
VDMM	Variational derived mass matrix: Hessian of discretized kinetic energy
VML	Variational mass lumping
VSF	Velocity shape functions to interpolate velocities and produce a VDMM

§16.3. Mass Matrix Construction Methods

Structural elements based on continuum models have distributed mass characterized by the material density. Associated inertia forces are *body forces*: forces per unit volume triggered by motion-induced accelerations. The FEM discretization transforms those distributed forces to nodal (point) forces $\mathbf{f}_I$, which are linked to nodal accelerations by $\mathbf{M}$ as per (16.1).

The discretization process is generically called *mass lumping* or simply *lumping*.[2] A mass matrix that satisfied certain behavioral constraints, such as nonnegativity and mass conservation, will be called *admissible*.

As noted in §16.2 the construction of $\mathbf{M}$ starts at the *element* level. Three methods to construct a element mass matrix can be distinguished:

Direct Mass Lumping (DML). The total mass of the element is distributed to the nodes so that a *diagonally lumped mass matrix* (DLMM) is produced. The physical interpretation of this particular matrix configuration is that no inertial interaction occurs between the lumped masses.

Variational Mass Lumping (VML). The kinetic energy of the element is expressed in terms of the degrees of freedom (DOF) using an interpolation scheme for the nodal velocities. The Hessian of the kinetic energy taken with respect to the DOF produces a *variational mass matrix*, which is generally non-diagonal. If the element stiffness is constructed using displacement shape functions, and these are used to interpolate velocities, the so-called *consistent mass matrix* or CMM is obtained.

Template Mass Lumping (TML). This approach aims to produce an element mass matrix $\mathbf{M}^e$ that has *free parameters*. Their range is restricted so that $\mathbf{M}^e$ satisfies admissibility conditions. This is called a *mass matrix template*. Setting parameters to specific numeric values produces a *template instance* or simply *instance*. If the template happens to include *all* admissible mass matrices, it is called a *general template*.

The union of the first two methods: direct and variational mass lumping, will be collectively called the *standard approach* to the construction of mass matrices. Since both CMM and DLMM configurations are by construction admissible, they are simply instances of a general template. Why then bother to make a distinction? Practical reasons:

- Both direct and variational lumping have been used in structural FEM for a fairly long time.[3] They are straightforward to explain and implement, and are well understood after decades of experience. Furthermore, they produce specific matrices, avoiding parameters.
- The template approach is more recent and there is less experience with it. The presence of free parameters makes derivations far more involved and usually require the help of a computer algebra system (CAS) to arrive at useful results in reasonable time. Most of the derivations to date concern one-dimensional elements. Extension to multidimensional elements poses challenges outlined later.

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[2] In English the verb "to lump" conveys the following meanings: combine, put together, group, bunch, aggregate, aglomerate, unite, pool, merge, collect, throw together, consider together. Its use for mass discretization comes from the *direct lumping process* through which an object of finite extent is idealized as a point mass. The idea can be traced back to centuries of orbital computations: celestial objects such as planets were idealized as point masses.

[3] DLMM since the 1930s and CMM since the 1960s, as narrated in Appendix H.

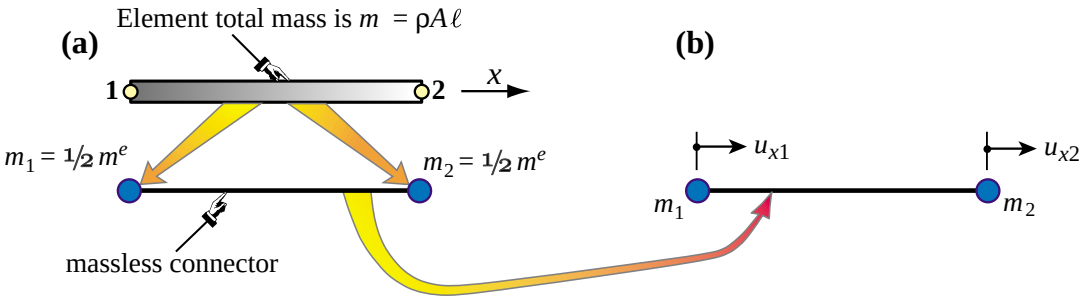

FIGURE 16.1. Direct mass lumping for two-node prismatic bar element: (a) lumping element mass to end nodes; (b) endowing the element with 2 translational degrees of freedom.

Because of the newness, dimensionality restrictions and computational demands imposed by templates, the derivation of mass matrices is restricted to the time-tested standard approach in this and following chapters.

§16.4. Element Mass Matrix Construction

The master mass matrix is built up from element contributions, and we start at that level. The construction of the mass matrix of individual elements with *distributed mass density* can be carried out through any of the three methods outlined above.[4] By now both direct and variational mass lumping enjoy extensive coverage in the structural dynamics literature at the textbook level; see e.g., [142,158,305,596,685], and references therein. They are implemented in all general purpose FEM codes.

§16.4.1. Direct Mass Lumping

This is the simplest procedure. The total mass of element e is directly apportioned to nodal freedoms, ignoring any cross coupling. The goal is to build a *diagonally lumped mass matrix* or DLMM, denoted here by $\mathbf{M}_L^e$.

As the simplest example, consider a 2-node prismatic bar element with length ℓ, cross section area A, and mass density ρ, which can only move in the axial direction x, as shown in Figure 16.1(a). We often denote this element as Bar2 in the sequel. The total element mass is $m^e = \rho A \ell$. This is divided into two equal parts and assigned to each end node. The element is endowed with the two freedoms shown in Figure 16.1(b). Thus

$$\mathbf{M}_L^e = \tfrac{1}{2}\rho A \ell \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \tfrac{1}{2} m^e \mathbf{I}_2, \tag{16.2}$$

in which $m^e = \rho A \ell$ is the element mass and $\mathbf{I}_2$ denotes the 2×2 identity matrix. As sketched in Figure 16.1, we have effectively replaced the continuum bar with a dumbbell: two masses separated by a massless connector.

This process conserves the translational kinetic energy or, equivalently, the linear momentum. To check this property for the bar example, take the constant x-velocity vector $\dot{\mathbf{u}}^e = v\,[\,1\;\;1\,]^T$. The kinetic energy of the element is $T^e = \frac{1}{2}(\dot{\mathbf{u}}^e)^T \mathbf{M}_L^e \dot{\mathbf{u}}^e = \frac{1}{2}\rho A \ell\, v^2 = \frac{1}{2} m^e v^2$. Thus the linear

[4] Beyond the element level, methods to produce the master mass matrix coalesce.

momentum $p^e = \partial T^e/\partial v = m^e v$ is preserved. When applied to simple elements that can rotate, however, the direct lumping process generally doe not preserve *angular* momentum.

Historical motivations for direct lumping are noted in §H.1. Most crucial, it covers naturally the case where concentrated (point) masses are natural part of model building. For example, in aircraft engineering it is common to idealize nonstructural masses (fuel, cargo, engines, etc.) as concentrated at given locations. (Such point masses in general have rotational freedoms; rotational inertia lumping is then part of the process.)

§16.4.2. Variational Mass Lumping

The second standard procedure is based on a variational formulation. This is done by taking the *kinetic energy* as part of the governing functional. The kinetic energy of an element of mass density ρ that occupies the domain Ω^e and moves with velocity field $\vec{\mathbf{v}}^e$ is

$$T^e = \tfrac{1}{2}\int_{\Omega^e} \rho\,(\vec{\mathbf{v}}^e)^T\vec{\mathbf{v}}^e\,d\Omega^e. \tag{16.3}$$

Following the conventional FEM philosophy, the element velocity field is interpolated using *shape functions*: $\vec{\mathbf{v}}^e = \mathbf{N}_v\,\dot{\mathbf{u}}^e$, in which $\dot{\mathbf{u}}^e$ are node DOF velocities and $\mathbf{N}_v$ a shape function matrix. (For 1D elements, $\mathbf{N}_v$ is a row vector.). Inserting into (16.3) and taking the node velocities out of the integral yields

$$T^e = \tfrac{1}{2}(\dot{\mathbf{u}}^e)^T\int_{\Omega^e} \rho\,(\mathbf{N}_v)^T\,\mathbf{N}_v\,d\Omega\,\dot{\mathbf{u}}^e \stackrel{\text{def}}{=} \tfrac{1}{2}(\dot{\mathbf{u}}^e)^T\mathbf{M}^e\,\dot{\mathbf{u}}^e, \tag{16.4}$$

whence the element mass matrix follows as the Hessian of T^e:

$$\mathbf{M}^e = \frac{\partial^2 T^e}{\partial\dot{\mathbf{u}}^e\partial\dot{\mathbf{u}}^e} = \int_{\Omega^e} \rho\,(\mathbf{N}_v^e)^T\,\mathbf{N}_v\,d\Omega. \tag{16.5}$$

If the same shape functions used in the derivation of the stiffness matrix are chosen, that is, $\mathbf{N}_v^e = \mathbf{N}^e$, (16.5) is called the *consistent mass matrix* or CMM. It is denoted here by $\mathbf{M}_C^e$. A better name for (16.5) would be stiffness-consistent mass matrix. The shorter sobricket has the unfortunate implication that other choices are "inconsistent," which is far from the truth. In fact, the consistent mass is not necessarily the best performer, a topic elaborated in Chapters that deal with templates. The shorter name is, however, by now ingrained in the FEM literature.

For the Bar2 element moving along x, pictured in Figure 16.1(a), the well known stiffness shape functions are $N_1 = 1-(x-x_1)/\ell = (1-\xi)/2$ and $N_2 = (x-x_2)/\ell = (1+\xi)/2$, in which $\xi = 2(x-x_1)/\ell - 1$ is the isoparametric natural coordinate that varies from -1 at node 1 to $+1$ at node 2. With $dx = \frac{1}{2}\ell\,d\xi$, the consistent mass is easily obtained as

$$\mathbf{M}_C^e = \int_0^\ell \rho\,A\,(\mathbf{N}^e)^T\mathbf{N}^e\,dx = \tfrac{1}{4}\rho\,\ell A\int_{-1}^{+1}\begin{bmatrix}1-\xi\\1+\xi\end{bmatrix}\,[\,1-\xi\;\;1+\xi\,]\,d\xi = \tfrac{1}{6}\,m^e\begin{bmatrix}2&1\\1&2\end{bmatrix}. \tag{16.6}$$

It can be verified that this mass matrix preserves linear momentum along x. If allowed to move in the xy plane, as considered in §16.7, it also preserves angular momentum about z.

§16.5. Mass Matrix Properties

Mass matrices must comply with conditions that can be used for verification and debugging at the element level. They are: matrix symmetry, physical symmetries, conservation and positivity.

Matrix Symmetry. This means $(\mathbf{M}^e)^T = \mathbf{M}^e$, which is easy to check. For a variationally derived mass matrix this follows directly from the definition (16.5), whereas for a DLMM is automatic.

Physical Symmetries. Also called *geometric* or *fabrication symmetries*. They are dictated by the physical configuration. For example, the CMM or DLMM of the prismatic Bar2 element must be symmetric about the antidiagonal: $M_{11} = M_{22}$. To see this, flip the end nodes: the element remains the same and so does the mass matrix.[5]

Conservation. At a minimum, total element mass must be preserved (we are talking about classical mechanics here; in relativistic mechanics mass and energy can be exchanged). This is easily verified by applying a uniform translational velocity and checking that linear momentum is conserved. Higher order conditions, such as conservation of angular momentum, are optional and not necessarily desirable.

Positivity. For any nonzero velocity field defined by the node values $\dot{\mathbf{u}}^e \neq \mathbf{0}$, $(\dot{\mathbf{u}}^e)^T \mathbf{M}^e \dot{\mathbf{u}}^e \geq 0$. That is, $\mathbf{M}^e$ must be nonnegative. Unlike the previous three conditions, this constraint is nonlinear in the mass matrix entries. It can be checked in two ways: through the eigenvalues of $\mathbf{M}^e$, or the sequence of principal minors. The second technique is more practical if the entries of $\mathbf{M}^e$ are symbolic.

A stricter form of the last condition requires that $\mathbf{M}^e$ be *positive definite*: $(\dot{\mathbf{u}}^e)^T \mathbf{M}^e \dot{\mathbf{u}}^e > 0$ for any $\dot{\mathbf{u}}^e \neq \mathbf{0}$. This is physically reassuring because one half of that form is the kinetic energy associated with the velocity field defined by $\dot{\mathbf{u}}^e$. In a continuum T can vanish only for zero velocities (a rest state). But allowing $T^e = 0$ for some nonzero $\dot{\mathbf{u}}^e$ makes life easier in some situations; e.g., elements with rotational or multiplier freedoms, or in the rapid-transient applications noted in §H.4.

The $\dot{\mathbf{u}}^e$ for which $T^e = 0$ collectively form the *null space* of $\mathbf{M}^e$. Because of the conservation requirement, a rigid velocity field (that is, the time derivative $\dot{\mathbf{u}}^e_R$ of a rigid body mode $\mathbf{u}^e_R$) cannot be in the mass matrix null space, as it would imply zero total mass. This scenario is dual to that of the element stiffness matrix. For the latter, $\mathbf{K}^e \mathbf{u}^e_R = \mathbf{0}$ because a rigid body motion produces no strain energy. Thus $\mathbf{u}^e_R$ must be in the null space of the stiffness matrix.

§16.6. Rank and Numerical Integration

Suppose the element has a total of n^e_F freedoms. A mass matrix $\mathbf{M}^e$ is called *rank sufficient* or of *full rank* if its rank is $r^e_M = n^e_F$. Because of the positivity requirement, a rank-sufficient mass matrix must be positive definite. Such matrices are preferred from a numerical stability standpoint.

If $\mathbf{M}^e$ has rank $r^e_M < n^e_F$ the mass is called rank deficient by $d^e_M = n^e_F - r^e_M$. Equivalently $\mathbf{M}^e$ is d^e_M times singular. For a numerical matrix the rank is easily computed by taking its eigenvalues and looking at how many of them are zero. The null space can be extracted by functions such as `NullSpace` in *Mathematica* without the need of computing eigenvalues.

The computation of $\mathbf{M}^e$ by the variational formulation (16.5) is often done using Gauss numerical quadrature. Each Gauss points adds n_D to the rank, where n_D is the row dimension of the shape

[5] The antisymmetry property would not generally hold if the element is not prismatic.

function matrix $\mathbf{N}^e$, up to a maximum of n_F^e. For most elements n_D is the same as element spatial dimensionality; that is, $n_D = 1$, 2 and 3 for 1, 2 and 3 dimensions, respectively. This property can be used to pick the minimum Gauss integration rule that makes $\mathbf{M}^e$ positive definite.

§16.7. Globalization

Like their stiffness counterparts, mass matrices are often developed in a local or element frame. Should globalization be necessary before merge, a congruent transformation is applied:

$$\mathbf{M}^e = (\mathbf{T}^e)^T \bar{\mathbf{M}}^e \mathbf{T}^e. \tag{16.7}$$

Here $\bar{\mathbf{M}}^e$ is the element mass referred to a local frame $\bar{x}_i$ (a.k.a. element frame), whereas $\mathbf{T}^e$ is the local-to-global displacement transformation matrix. The recipe (16.7) follows readily from the Principle of Virtual Work, or equivalently the invariance of the first variation of the element kinetic energy:

$$\delta \bar{T}^e = (\dot{\bar{\mathbf{u}}}^e)^T \bar{\mathbf{M}}^e \delta\dot{\bar{\mathbf{u}}}^e = (\dot{\mathbf{u}}^e)^T (\mathbf{T}^e)^T \bar{\mathbf{M}}^e \mathbf{T}^e \delta\dot{\mathbf{u}}^e = (\dot{\bar{\mathbf{u}}}^e)^T \mathbf{M}^e \delta\dot{\mathbf{u}}^e = \delta T^e. \tag{16.8}$$

Matrix $\mathbf{T}^e$ is in principle the same used for the stiffness globalization. Some procedural differences, however, must be noted. For stiffness matrices $\mathbf{T}^e$ is often *rectangular* if the local stiffness has lower dimensionality. For example, two-node bar, shaft and spar elements have 2×2 local stiffnesses. Globalization to 2D and 3D involves application of 2×4 and 2×6 transformation matrices, respectively. This works fine because the local element has zero stiffness in some directions, and those zero rows and columns may be omitted at the local level.

In contrast to stiffnesses, *translational masses never vanish*. One way to realize this is to think of an element moving in a translational rigid motion u_R with acceleration $\ddot{u}_R$. According to Newton's second law, $f_R = m^e \ddot{u}_R$, where m^e is the element translational mass. Regardless of direction, this inertia force cannot vanish.

Conclusion: *all translational masses must be retained in the local mass matrix*. A two-node prismatic bar, moving in the $\{x, y\}$ plane as in Figure 16.2, furnishes a simple illustration. With the element freedoms arranged as $\mathbf{u}^e = [\, u_{x1} \;\; u_{x2} \;\; u_{y1} \;\; u_{y2} \,]^T$, the local mass matrix constructed by variationally consistent and diagonalized lumping are, respectively,

$$\bar{\mathbf{M}}^e_C = \tfrac{1}{6} m^e \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \bar{\mathbf{M}}^e_L = \tfrac{1}{2} m^e \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \tfrac{1}{2} m^e \, \mathbf{I}_4, \tag{16.9}$$

in which $m^e = \rho \, A \, \ell$ is the total element mass. For 3D, repeat the diagonal block once more.

§16.7.1. Directional Invariance

For the case illustratedin in Figure 16.2 the local-to-global freedom transformation $\bar{\mathbf{u}}^e = \mathbf{T}^e \, \mathbf{u}^e$ is

$$\begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{x2} \\ \bar{u}_{y1} \\ \bar{u}_{y2} \end{bmatrix} = \begin{bmatrix} c & 0 & s & 0 \\ 0 & c & 0 & s \\ -s & 0 & c & 0 \\ 0 & -s & 0 & c \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{y1} \\ u_{y2} \end{bmatrix}, \quad \text{in which} \quad c = \cos\varphi, \quad s = \sin\varphi. \tag{16.10}$$

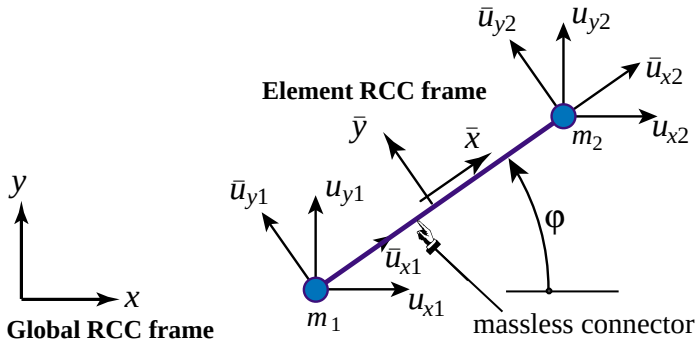

FIGURE 16.2. Bar2 element with diagonally lumped mass moving in 2D.

Now apply (16.7) to either $\mathbf{M}_C^e$ or $\mathbf{M}_L^e$ of (16.9) using (16.10). The result is $\mathbf{M}_C^e = \bar{\mathbf{M}}_C^e$ and $\mathbf{M}_L^e = \bar{\mathbf{M}}_L^e$: no change. We say that these mass matrices *repeat*. Verification for the DLMM is easy because $\mathbf{T}^e$ is orthogonal: $(\mathbf{T}^e)^T\,\bar{\mathbf{M}}_L^e\,\mathbf{T}^e = \frac{1}{2}\,m^e\,(\mathbf{T}^e)^T\,\mathbf{I}_4\mathbf{T}^e = \frac{1}{2}\,m^e\,(\mathbf{T}^e)^T\,\mathbf{T}^e = \frac{1}{2}\,m^e\,\mathbf{I}_4$. For the CMM, however, repetition is not obvious. It can be shown to hold by expressing $\mathbf{M}_C^e$ and $\mathbf{T}^e$ in 2×2 partitioned form

$$\bar{\mathbf{M}}_C^e = \begin{bmatrix} \tilde{\mathbf{M}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{M}} \end{bmatrix},\ \mathbf{T}^e = \begin{bmatrix} c\mathbf{I}_2 & s\mathbf{I}_2 \\ -s\mathbf{I}_2 & c\mathbf{I}_2 \end{bmatrix},\quad \text{with } \tilde{\mathbf{M}} = \tfrac{1}{6}\,m^e \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{16.11}$$

Carrying out the congruent transformation in block form gives

$$\mathbf{M}_C^e = (\mathbf{T}^e)^T\,\mathbf{M}_C\,\mathbf{T}^e = \begin{bmatrix} (c^2+s^2)\tilde{\mathbf{M}} & (cs-cs)\tilde{\mathbf{M}} \\ (cs-cs)\tilde{\mathbf{M}} & (c^2+s^2)\tilde{\mathbf{M}} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{M}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{M}} \end{bmatrix} = \bar{\mathbf{M}}_C^e. \tag{16.12}$$

A mass matrix that repeats upon transformation to any global frame is called a *directionally invariant mass matrix*, or DIMM. Note that the contents and order of $\tilde{\mathbf{M}}$ are irrelevant to the result (16.12). Hence the following generalization follows. If upon rearranging the element DOF so that they are grouped node by node:

(i) $\bar{\mathbf{M}}^e$ has a repeating block diagonal form, and

(ii) $\mathbf{T}^e$ is configured as the block form shown above,

then local and global matrices will coalesce. For (ii) to hold, it is sufficient that all nodal DOF be translational and be referred to the same coordinate system. The same conclusion is easily extended to 3D, and to any arrangement of the element freedoms. This *repetition rule* can be summarized as follows:

> A local mass matrix is DIMM if all element DOFs are translational and all of them are referred to the same global RCC system. (16.13)

This property should be taken advantage of to skip superfluous local-to-global transformations. That operation may cost more than forming the local mass matrix. If the rule fails on actual computation, something (mass matrix or transformation) is wrong and must be fixed.

§16.7.2. Failure of Repetition Rule

The repetition rule can be expected to fail if $\bar{\mathbf{M}}^e$ is not a DIMM. This occurs under the following scenarios:

1. The element has non-translational freedoms; for example node rotations, or displacement derivatives. (Occasionally the rule may work, but that should not be taken for granted.)
2. The mass blocks are different in content and/or size. This occurs if different continuum models are used in different local directions. Examples are furnished by beam-column elements, shell elements, and elements with curved sides or faces.
3. Nodes are referred to different coordinate frames in the global system. This can happen if certain nodes are referred to special frames to facilitate the application of boundary conditions.