LAST WEEK

COMPLEX FUNCTIONS: $f(z) = u(x,y) + i\,v(x,y)$

- ANALYTIC : "nice" $\leftrightarrow$ DIFFERENTIABLE
  $\Rightarrow$ CAUCHY-RIEMANN EQ
  $\partial_x u = \partial_y v$
  $\partial_y u = -\partial_x v$

how to use it in a sentence:
"$f(z)$ is analytic in a region R of the complex plane."

eg. $f(z) = \frac{1}{z}$ is analytic on $\mathbb{C}/\{0\}$

the $\mathbb{C}$ plane w/o the origin

BUT $f(z)$ HAS A SINGULARITY @ $z=0$

Meromorphic : analytic up to discrete points.

singularities : line integrals of closed paths around them pick up a non-zero contribution.

$$\oint_c f(z)\,dz = 2\pi i \sum_j \mathrm{Res}_f(z_j)$$

POLES ENCLOSED BY C.

RESIDUE of $f(z)$ AT z_j

$\rightarrow$ a_{-1} IN LAURENT EXP: $f(z) = \sum_n a_n (z-z_j)^n$

$\rightarrow$ may have to do some work if pole is higher order.

FROM BOAS CH. 14.6 (3RD ED)

finding the RESIDUE:

- $\mathrm{Res}_f(z_j) = \underbrace{\lim_{z\to z_j} (z-z_j) f(z)}$ IF z_j IS A SIMPLE POLE

is the POLE SIMPLE? if this is not ∞, then you had a simple pole.

eg $f(z) = \cot z$ find residue @ $z=0$

$$\mathrm{Res}_f(0) \overset{?}{=} \lim_{z\to 0} \frac{z\cos z}{\sin z} = \underbrace{\cos 0}_{1} \underbrace{\lim \frac{z}{\sin z}}_{1} = 1$$

MAY NOT BE SIMPLE POLE?

eg. $f(z) = \cot^2 z$ find res @ $z=0$

$$\frac{(1 - z^2/2 + \cdots)^2}{(z - z^3/3! + \cdots)^2} \sim \frac{1}{z^2} + O(1) \to \mathrm{Res}_f(0) = 0$$

eg $f(z) = \underbrace{z}\cot^2 z$

$$\frac{z(1 - z^2/2 + \cdots)^2}{(z - z^3/3! + \cdots)} \sim \frac{z}{z^2} + O(z)$$

$\sim \frac{1}{z} \to \mathrm{Res}_f(0) = 1$

eg $f(z) = \left(\sum_{n=0}^{\infty} c_n z^n\right) \cot^2 z$

$c_0 + c_1 z + \underbrace{c_2 z^2 + c_3 z^3 + \cdots}_{\text{no singularity @ } z=0}$

$\boxed{\mathrm{Res}_f(0) = c_1}$

zero from $\cot^2 z$ eg

<u>on homework</u>: what if the pole is higher order?

$$(z-z_j)f(z_j) = \infty \ldots$$

$$(z-z_j)^2 f(z_j) = \infty \ldots$$

$$\vdots$$

$$\boxed{(z-z_j)^m f(z_j) \neq 0}$$

eg: $f(z) = \dfrac{z \sin z}{(z-\pi)^3}$, $\operatorname{Res}_f(\pi)$?

Method 1

using formula from HW:

$$\operatorname{Res}_f(\pi) = \frac{1}{2!}\frac{d^2}{dz^2}\left(z \sin z\right)_{z=\pi}$$

$$= \frac{1}{2!}\frac{d}{dz}\left(\sin z + z\cos z\right)_{z=\pi}$$

$$= \frac{1}{2!}\Big[\underbrace{\cos z + \cos z}_{2\cos\pi = -2} - z \underbrace{\sin z}_{0}\Big]_{z=\pi}$$

$$= -1$$

Method 2

COMPARISON TO LAURENT EXP.

→ HAVE TO CONSISTENTLY EXPAND ABOUT $z=\pi$

$$\underbrace{z \sin z}_{\text{numerator}} = \underbrace{[\pi + (z-\pi)]}_{\text{expand } z \text{ about } z=\pi}\Big[\cancel{\sin\pi}^{\,0} + \cos\pi\cdot(z-\pi) - \frac{1}{2!}\cancel{\sin\pi}^{\,0}(z-\pi)^2 - \frac{1}{3!}\cos\pi\,(z-\pi)^3 + \ldots\Big]$$

overwhelms pole

$$= [\pi + (z-\pi)]\left[(-1)(z-\pi) - \frac{1}{3!}(-1)(z-\pi)^3\right]$$

DENOMINATOR: $(z-\pi)^3$.

SO RESIDUE IS PART OF NUMERATOR THAT MULTIPLIES $(z-\pi)^2$ ↳ $\boxed{(-1)}(z-\pi)^2$ ✓

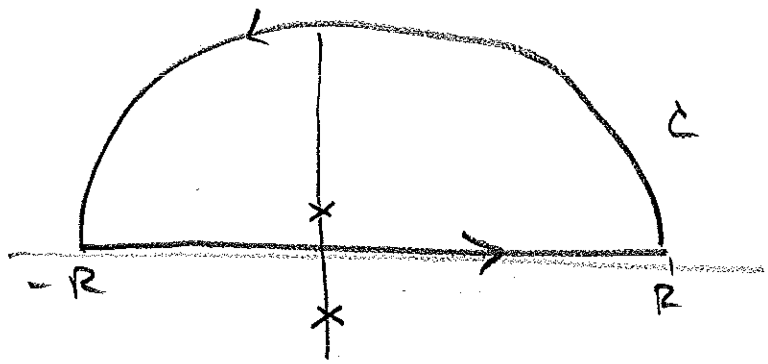
ALL OF THIS IS GREAT FOR CLOSED CONTOURS, BUT DOES NOT HOLD FOR non-closed PATHS!

↑ HW PROBLEM IS A REMINDER OF THIS.

WHY IS THIS USEFUL FOR US?

e.g. $f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$

simple POLES @ $z = \pm i$



$$\oint_C f(z)\,dz = 2\pi i \underbrace{\mathrm{Res}_f(i)}_{\frac{1}{2i}} = \pi$$

$$\oint_C dz = \underbrace{\int_{-R}^{R} dx}_{\text{along } x \text{ AXIS}} + \underbrace{\int_0^{\pi} d\theta}_{\text{along } r=R \text{ semicircle}}$$

WHAT WE'RE DOING: PARAMETERIZING A LINE INTEGRAL w/rt one $\mathbb{R}$ PARAMETER

$\Downarrow$

$$\oint_C f(z)\,dz = \underbrace{\int_{-R}^{R} f(x)\,dx}_{\text{"ORDINARY" } \mathbb{R} \text{ integral}} + \underbrace{\int_0^{\pi} f(Re^{i\theta})\,d(Re^{i\theta})}_{\int_0^{\pi} \underbrace{\frac{Re^{i\theta}}{R^2 e^{2i\theta}+1}}_{\sim \frac{1}{R} \text{ AS } R \to \infty} d\theta}$$

so: $\oint_C f(z)\, dz = \boxed{\pi = \int_{-R}^{R} \frac{dx}{x^2+1}} + \mathcal{O}(\frac{1}{R})$

$$\downarrow$$

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi}$$

PURELY $\mathbb{R}$ INTEGRAL

<u>the trick</u>: the "extra" part of the contour GOES TO ZERO

LARGE SEMICIRCLES ARE NICE B/C EASY TO PARAMETERIZE AS ANGULAR INTEGRAL.

$dz = iRe^{i\theta}\, d\theta$

SO INTEGRAND NEEDS TO GO LIKE R^2 OR MORE.

so: WE START FROM A $\mathbb{R}$ INTEGRAL ... $0 \to \infty$ or $-\infty \to \infty$ THEN EXTEND TO A $\mathbb{C}$ INTEGRAL S.T.

$$\int_{\mathbb{R}} f(x)\, dx + \underbrace{\int_{C'} f(z)\, dz}_{\text{CLOSES CONTOUR}} = \oint_C f(z)\, dz = \underbrace{2\pi i \sum \text{Res}}_{\text{EASY TO CALC.}}$$

$C = \mathbb{R} + C'$ CLOSED

+ YOU CAN CALCULATE (e.g. = 0)

$\neq$

WHAT DOES IT MEAN TO GO TO $\mathbb{C}$ INTEGRAL?

WE PROMOTED A $\mathbb{R}$ FUNCTION $f(x)$ defined on the $\mathbb{R}$ line only, to a $\mathbb{C}$ function $f(z)$ on $\mathbb{C}$.

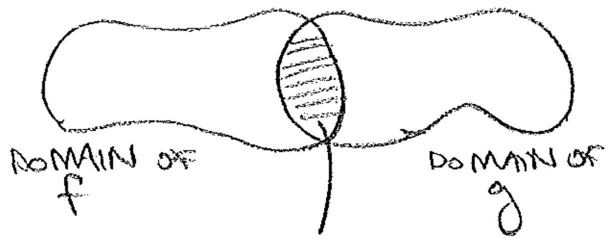
IS THAT PROMOTION $x \to z$ UNIQUE?
MAYBE $\exists$ A DIFFERENT $\mathbb{C}$ FUNCTION $g(z) \neq f(z)$ SUCH THAT g, f AGREE ON $\mathbb{R}$?

$\hookrightarrow$ BUT g HAS DIFFERENT POLES ... DIFF ANSWER?!

APPEL 5.3 RADIUS OF CONVERG. eg $\sum a_n z^n$ on $|z|<1$, BUT CAN RE-EXPAND $\sum a_n (z-z_0)^n$

<u>ANSWER</u>: ANALYTICITY PROTECTS US FROM THIS SCENARIO

<u>READ</u> (not examinable): ANALYTIC CONTINUATION

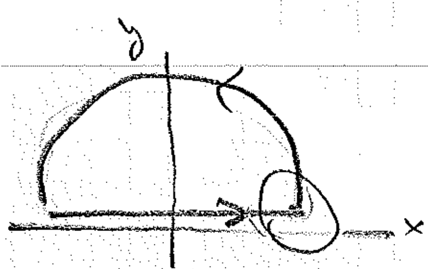


if 2 analytic functions agree on their overlap, they agree on combined domain

f & g agree here $\longrightarrow$ $f = g$

SKETCH: $(f-g)=0$ IS ANALYTIC
CAN EXTEND TAYLOR EXPANSION.

<u>ANOTHER CONCERN</u>: EDGE EFFECTS?



$$\int_0^{\pi} \frac{Rie^{i\theta}}{R^2 e^{2i\theta}+1}\, d\theta$$

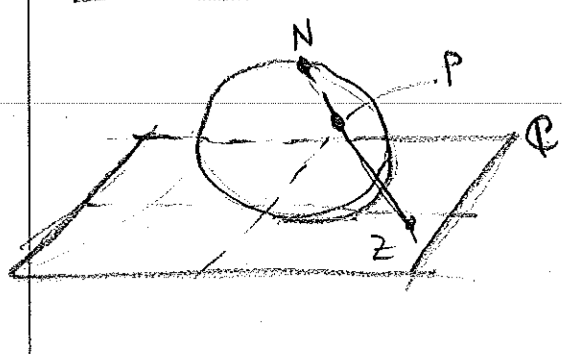
$$\sim \frac{i}{Re^{i\theta}}$$

BUT FOR <u>VERY</u> SMALL θ, DOESN'T THIS CONTRIBUTE?

<u>HAND WAVY ANSWER</u>: "take the $R \to \infty$ limit first"

this is <u>totally</u> unsatisfying - physics RARELY depends on the order of limits.

<u>Better answer</u>: Riemann sphere (EXTRA CREDIT on NEXT HW)



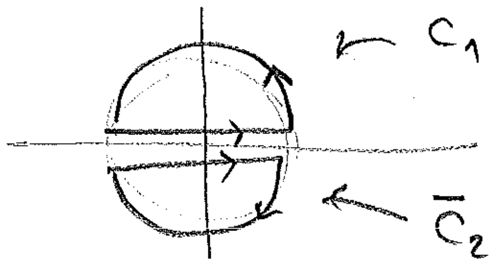
MAP EACH POINT ON $\mathbb{C}$ TO A SPHERE (STEREOGRAPHIC PROJECTION).

THEN $\int_R \leftrightarrow$ CLOSED CONTOUR ON SPHERE (no "ARC") $\to$ no edge.

CNHILL P.188

eg. $\int_{-\infty}^{\infty} \underbrace{\frac{2\cos x}{x^2+1}}\, dx$

$$f(x) \mapsto f(z) = \frac{e^{iz} + e^{-iz}}{(z+i)(z-i)}$$



note: negative orientation

$\boxed{\text{WHICH CONTOUR?}}$

CRITERIA: 1. contour includes $\mathbb{R}$ (WHAT WE WANT!)

2. LARGE ARC INTEGRATES TO ZERO

then: DO CONTOUR INTEGRAL W/ Residue thm.

$$\int_{-\infty}^{\infty} f(z)\, dz + \underbrace{\int_{ARC} \cdots dz}_{=0} = \Sigma\, 2\pi i \sum_j \mathrm{Res}_f(z_j)$$

HOW TO DECIDE: $f \sim \frac{e^{iz} + e^{-iz}}{R^2}$

$$e^{iz} = e^{i(R\cos\theta + iR\sin\theta)}$$

write w/r+ θ + R for angular integral

so $\underset{\substack{\uparrow \\ iRe^{i\theta}}}{e^{\pm iz}\, dz} = i R \underbrace{e^{\mp R\sin\theta}} \underbrace{e^{\pm iR\cos\theta + i\theta}}_{\text{PHASE}}\, d\theta$

convergence depends on sign of $\sin\theta$ (exponential!)

$$f(z) = \underbrace{\frac{e^{iz}}{(z+i)(z-i)}}_{f_1(z)} + \underbrace{\frac{e^{-iz}}{(z+i)(z-i)}}_{f_2(z)}$$

CONVERGES FOR $\sin\theta > 0$, C_1

converges for $\sin\theta < 0$, $\bar{C}_2$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f_1(x)\,dx + \int_{-\infty}^{\infty} f_2(x)\,dx$$

$$+ \int_{\substack{\text{UPPER}\\ \text{ARC}}} f_1(Re^{i\theta})\,iRe^{i\theta}\,d\theta + \int_{\substack{\text{LOWER}\\ \text{ARC}}} f_2(Re^{i\theta})\,iRe^{i\theta}\,d\theta$$

ADD ZERO TO CLOSE CONTOUR

$$\oint_{C_1} f_1(z)\,dz \qquad \oint_{C_2} f_2(z)\,dz$$

$$= 2\pi i \,\mathrm{Res}_{f_1}(i) \qquad = -2\pi i \,\mathrm{Res}_{f_2}(-i)$$

ORIENTATION

$$= 2\pi i \left[\frac{e^{-1}}{2i} - \frac{e^{-1}}{-2i} \right]$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \frac{2\pi}{e}}$$

be very clear about this example
↳ it is our main tool.

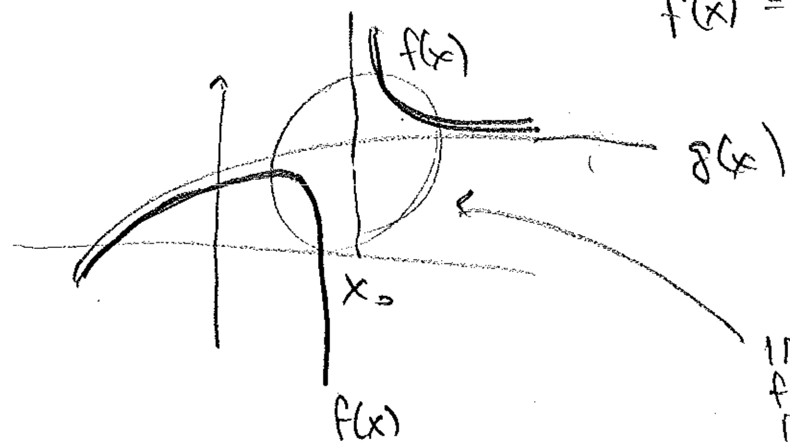
PRINCIPAL VALUE

WHAT IF YOUR CONTOUR HITS A POLE? eg $1/z$

eg. this happens when virtual particles become real

USEFUL IDEA: PRINCIPAL VALUE

$$f(x) = \frac{g(x)}{x - x_0}$$



IDEA: HERE f(x) CANCELS IN AN INTEGRAL

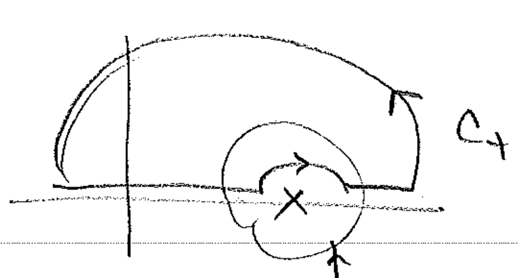
SO EVEN THOUGH f(x) IS SINGULAR, INTEGRAL NEED NOT BE. FINITE PART IS CALLED PRINCIPAL VALUE

$$\mathcal{P}\int_a^b \frac{f(x)}{x-x_0} = \int_a^{x_0-\varepsilon} \frac{f(x)}{x-x_0}\,dx + \int_{x_0+\varepsilon}^b \frac{f(x)}{x-x_0}\,dx$$

$a < x_0 < b$

ASSUME: $\displaystyle\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{f(x)}{x-x_0}\,dx = 0$

IN A CONTOUR INTEGRAL, contributes $1/2$ RESIDUE



call this c'_+

APPEL 3.16

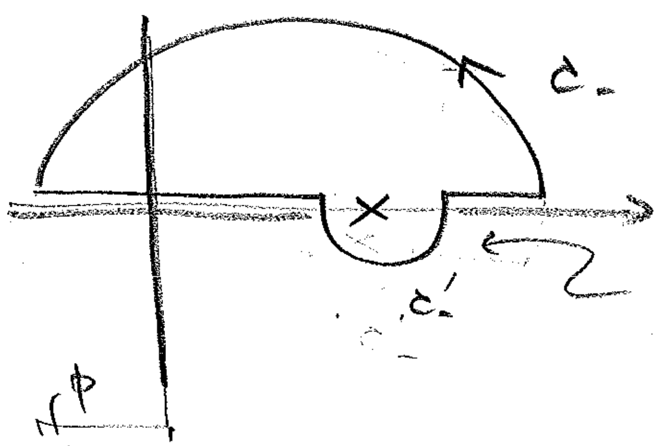
<u>WHY</u>:

$$\underbrace{\oint_{C_+} \frac{f(z)}{z-x_0}}_{= \sum_j 2\pi i \, \mathrm{Res}_\phi(z_j)} = \Big(\underbrace{\int_{-\infty}^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^{\infty}}_{\mathcal{P}\int \phi(x)\,dx} + \int_{C_+'} \Big) dz \, \overbrace{\frac{f(z)}{z-x_0}}^{\phi}$$

$$\int_{C_+'} = \int_{\pi}^{0} \frac{f(x_0)}{\cancel{\varepsilon e^{i\theta}}} \, i \cancel{\varepsilon e^{i\theta}} \, d\theta = \underbrace{-i\pi f(x_0)}_{\text{"half of a residue"}}$$

$$\mathcal{P}\int \phi(x)\,dx = \Big[\sum_j 2\pi i \, \mathrm{Res}_\phi(z_j)\Big] + i\pi f(x_0)$$

<u>HW</u>: WHAT IF WE USED:



$$\int_{\pi}^{2\pi} \frac{f(x_0)}{\varepsilon e^{i\theta}} \, i\varepsilon e^{i\theta} \, d\theta = i\pi f(x_0)$$

$$\underbrace{\int_{C_-} \overbrace{\frac{f(z)}{z-x_0}}^{\phi}}_{\sum_j 2\pi i \, \mathrm{Res}_\phi(z_j)} = \mathcal{P}\int \phi(x)\,dx + i\pi f(x_0)$$

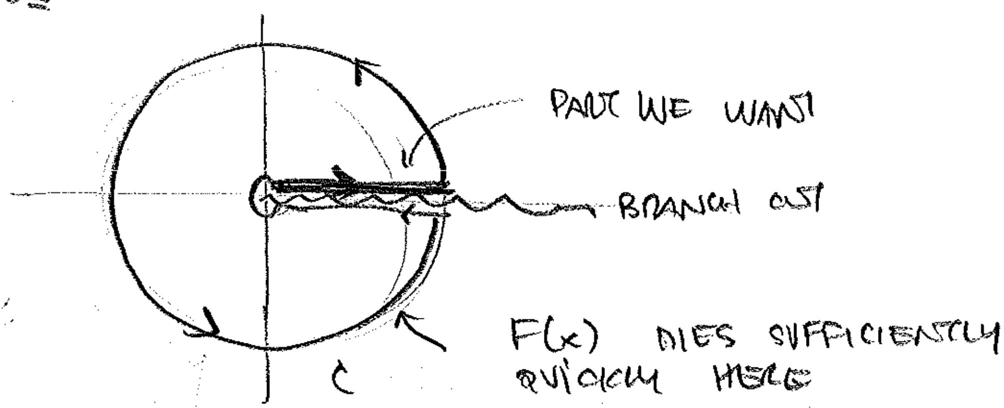
↑ includes $2\pi i f(x_0)$ — cancels half!

BRANCH CUTS

eg: $\int_0^\infty x^{1/3} \; F(x) \; dx$

∃ BRANCH CUT
SPECIFY: $0 \le \theta < 2\pi$

MEROMORPHIC
w/ POLES AWAY FROM x^+-AXIS
& DIES LIKE $1/R^2 \, e^{-R}$



$$\oint_C z^{1/3} F(z)\,dz = \int_0^\infty x^{1/3} F(x)\,dx + \underbrace{\int_{ARC} F(z)\,dz}_{=0}$$

$$\| \qquad + \int_\infty^0 \left(x e^{2\pi i}\right)^{1/3} F(x)\,dx$$

$$2\pi i \sum_j z_j^{1/3} \, \mathrm{Res}_F \, z_j$$

$$= \int_0^\infty \underbrace{\left(1 - e^{\frac{2\pi}{3} i}\right)} x^{1/3} F(x)\,dx$$

$$= 2 e^{i\pi/3} \cdot \tfrac{1}{2}\left(e^{-i\pi/3} - e^{i\pi/3}\right)$$

$$= -2 e^{i\pi/3} \sin \pi/3$$

$$= \left(-2 e^{i\pi/3}\right) \sin \pi/3 \int_0^\infty x^{1/3} F(x)\,dx$$

$$\boxed{\int_0^\infty x^{1/3} F(x)\,dx = \frac{2\pi i \sum_j z_j^{1/3} \, \mathrm{Res}_F \, z_j}{-2 e^{i\pi/3} \sin(\pi/3)}}$$