first part of LECTURE: finish LEC 18.

SALIENT POINTS

- INVARIANCE + COVARIANCE

4D FOURIER TRANSFORM : $K \cdot x$ is invariant

$$\underbrace{e^{-iK\cdot x}}_{\text{4-vectors}} = e^{-iEt} e^{i\underline{k}\cdot\underline{x}}$$

4-vectors (actually dual vec & vector ...)

this is a basis of plane waves

WE FOUND POLES @ $E = \pm K$ — from $\partial^2 = \partial_t^2 - \partial^2$, $\left(\frac{1}{c}\right)^2$

RESTORING c: $E = \pm cK$

SO WE EXPAND AS PLANE WAVES TRAVELLING @ SPEED OF LIGHT.

WE COULD HAVE EXPANDED IN SOMETHING WEIRD, eg EUCLIDEAN $\exp(-i\underbrace{k\cdot x}_{Et + \underline{k}\cdot\underline{x}})$

but would miss manifest Lorentz inv.

→ ALL THIS TO DERIVE $\boxed{G(x,x')}$ — SCALAR FUNCTION ASSOCIATED w/ ∂^2

$$A_\mu(x) = \int d^4x' \, G(x,x') \, j_\mu(x')$$

COVARIANT QUANTITY — you can tell by the index

INVARIANT

$$G(x,x') = G(x-x')$$
$$= G(\underbrace{|x-x'|^2}_{\text{invariant}})$$

- ACTUALLY, TO BE COMPLETE,

$$A_\mu(x) = \int d^4x'\, G(x,x')\, j_\mu(x') + \underbrace{b B_\mu(x) + c C_\mu(x)}_{\text{SOLUTION TO } \partial^2 B_\mu = \partial^2 C_\mu = 0 \text{ (HOMOGENEOUS)}}$$

BUT: FINDING FREE SOLUTIONS CAN BE DIFFICULT GIVEN BC @ ∞.

<u>FLATLAND</u>: what if we had EM in 2+1 DIM?

WHAT CHANGES? $\partial^2 = \left(\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial x}\right)^2 - \left(\frac{\partial}{\partial y}\right)^2$

IN (3+1)D: $\Delta \frac{1}{r} = -4\pi\delta(r)$

Δ = $(\partial)^2$ (SPATIAL)

IN (2+1)D $\Delta \underbrace{\log|r|}_{} = 2\pi\delta(r)$

EVIDENTLY $\varphi \sim \log r$ (COULOMB POT)

<u>ALSO</u>: $\underline{B} \sim \underbrace{\nabla\times\underline{A}}_{}$

no $\nabla\times$ for 2D

TURNS OUT B IS A SCALAR.

why: $F_{\mu\nu} \sim \begin{pmatrix} 0 & E & E & E \\ & 0 & B & B \\ & & 0 & B \\ & & & 0 \end{pmatrix}$ vs. $F_{ij} = \begin{pmatrix} 0 & E_1 & E_2 \\ & 0 & B \\ & & 0 \end{pmatrix}$

(DIFFERENTIAL 2-FORM)

FOLLOWING SAME STEPS TO FIND GREEN'S FUNCTION IN (3+1)D,

$$G(\underbrace{u, \underline{s}}_{\text{? vec.}}) = \frac{1}{(2\pi)^3} \int \frac{e^{-iEu} e^{i\underline{k}\cdot\underline{s}}}{\underline{k}^2 - E^2} \, d^2\underline{k} \, dE$$

3 fourier transforms rather than 4

$x - x' = (u, \underline{s})$

$t-t'$

$\begin{pmatrix} x-x' \\ y-y' \end{pmatrix}$

$k \, dk \, d\theta$

$iks \cos\theta$

many ways to do $dk \, d\theta \, dE$... order doesn't change result, but can simplify work.

THE EASY ONE :

$(E+k)(E-k)$, $-i\varepsilon$, $-i\varepsilon$; $k - i\varepsilon$; $k = k + i\varepsilon$

$$\int_{-\infty}^{\infty} dE \frac{-e^{-iEu}}{E^2 - k^2} = \frac{-i\pi}{2|k|}\left(e^{iku} - e^{-iku}\right)$$

SHO INTEGRAL $= 2\pi \frac{+1}{|k|} \sin ku$

FACT: $\int_0^\infty e^{iAk} \sin(ku) \, dk = \frac{u}{-A^2 + u^2} \quad (t > r|\cos\theta|)$

so:

$$G(u,s) = \frac{1}{(2\pi)^2} \int d\theta \int_0^\infty dk \, \underbrace{e^{iks\cos\theta} \sin ku}$$

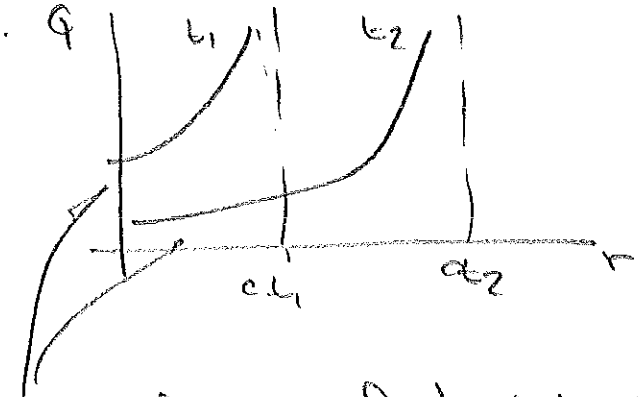
$$\frac{u}{u^2 - s^2\cos^2\theta} = \frac{1}{u} \cdot \frac{1}{1 - \underbrace{(s/u)^2}_{B}\cos^2\theta}$$

FACT $\int_0^{2\pi} d\theta \frac{1}{1 - B\cos^2\theta} = 2\pi \sqrt{\frac{1}{1-B}}$ if $B < 1$

$$\rightarrow G(u, \underline{s}) = \frac{1}{2\pi} \sqrt{\frac{1}{1 - (s/u)^2}} \cdot \frac{1}{u} = \boxed{\frac{1}{2\pi} \sqrt{\frac{1}{u^2 - s^2}}} \, \Theta(u - s)$$

$t^2 - r^2$

RESULT.



given flash of light @ (0,0), persistence of flash continues for all $r<t$!

A CLEVER / INSIGHTFUL TRICK

DIMENSIONAL REDUCTION

CLAIM: $G_{(2+1)}(\underbrace{x,y,t}) = \int_{-\infty}^{\infty} G_{(3+1)}(x,y,z,t)\, dz$

REALLY $\Delta x, \Delta y, \Delta t$, fix $x' = 0$ — integrate!

Pf [technical version]

$$\left(\int_{-\infty}^{\infty} dz\, G_{3+1}\right) = \int d^4k\, \underbrace{dz\, e^{ik_z z}}\, \tilde{G}_{3+1}(E,k)\, e^{-iEt} e^{i\vec{k}_{2D}\cdot\vec{x}}$$

DO THIS INTEGRAL GIVES $2\pi\,\delta(k_z)$ ie $k_z = 0$

$$= \underbrace{\int d^3k \left(\tilde{G}_{3+1}(E, k_x, k_y, 0)\right) e^{-iEt} e^{i\vec{k}\cdot\vec{x}}}$$

SO F.T. OF THIS IS $\tilde{G}_{3+1}(E,k_x,k_y,0)$

PRECISELY A (2+1) DIM. FOURIER

FURTHER: $\partial^2_{3+1}\tilde{G}_{3+1} = \int d^4k\, e^{ik\cdot x}(E^2-k_x^2-k_y^2-k_z^2)\,\tilde{G}_{3+1}(E,\underline{k})$

$$\Rightarrow \tilde{G}_{3+1} = \frac{1}{E^2-k_x^2-k_y^2-k_z^2}$$

SIMILARLY: $\tilde{G}_{2+1} = \frac{1}{E^2 - k_x^2 - k_y^2}$

BUT: $\tilde{G}_{3+1}(E, k_x, k_y, 0)$ IS PRECISELY THIS!

$\downarrow FT^{-1}$

$$\boxed{\int dz\, G_{3+1}} = \tilde{G}_{2+1}$$

PLUGGING IN G_{3+1}

THEN: $G_{2+1}(x, y\, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\underbrace{\vec{r}^{\,2}}_{x^2+y^2} + z^2}}\, \delta\left(\sqrt{\vec{r}^{\,2} + z^2} - t\right) dz$

ZEROES OF f

reminder: $\delta(f(z)) = \sum_{z_0} \frac{1}{|f'(z_0)|} \delta(z - z_0)$

$f'(z) = \frac{z}{\sqrt{\vec{r}^{\,2} + z^2}}$ for $z = \pm \sqrt{t^2 - \vec{r}^{\,2}}$

$$G_{2+1}(x, y, t) = \frac{1}{4\pi} \frac{1}{\sqrt{\vec{r}^{\,2} + z^2}} \left(\frac{\sqrt{\vec{r}^{\,2} + z^2}}{|z_+|} + \frac{\sqrt{\vec{r}^{\,2} + z^2}}{|z_-|} \right)$$

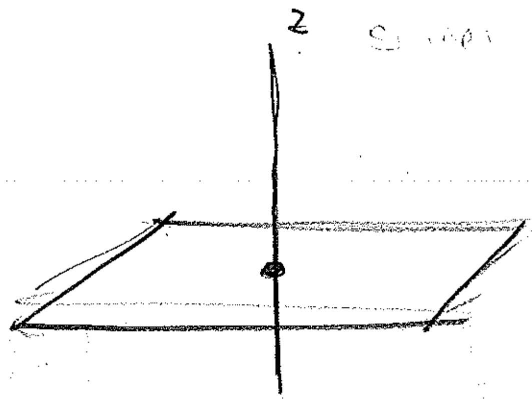
$$= \boxed{\frac{1}{2\pi} \frac{1}{\sqrt{t^2 - \vec{r}^{\,2}}}} \quad \text{for } t > |\vec{r}|$$

INTUITION:

$$G_{2+1}(x,y,t) = \int_{-\infty}^{\infty} G_{3+1}(x,y,z,t)\, dz$$

RECALL: $\varphi = \int d^4x'\, G(x-x')\, \rho(x')$

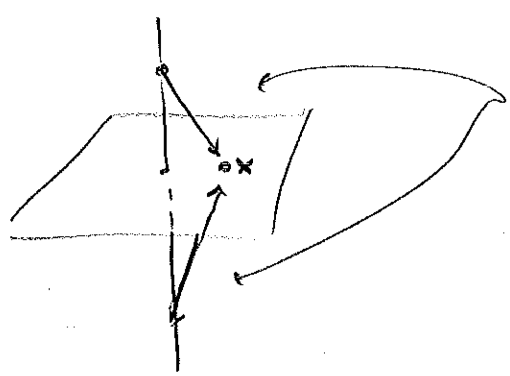
SO RHS IS LIKE AN INFINITE UNIT LINE CHARGE DENSITY ALONG Z AXIS ← WRT UNIT CHARGE



IF WE LOOK ONLY @ PLANE, WE GET 2D PHYSICS.

WHY ISN'T ELECTRIC FIELD LEAKING INTO THE THIRD DIMENSION?

No: ∞ LINE CHARGE ON EITHER END



contributions in 3rd dimension cancel each other.

NB: SOMETHING LIKE THE "LEAKING INTO AN EXTRA DIMENSION" MAY BE WHY GRAVITY IS SO WEAK COMPARED TO OTHER FORCES!