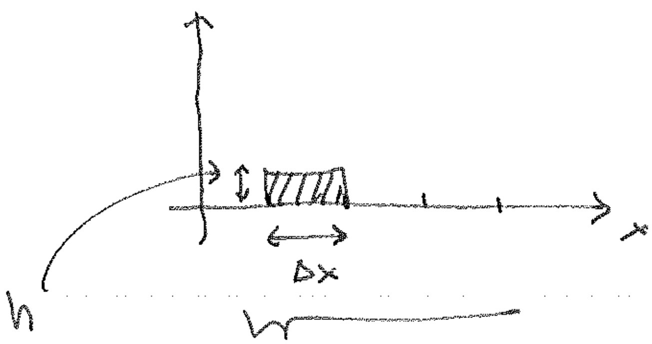
LAST TIME: FUNCTION SPACE

∞ DIM VECTOR SPACE WITH:

- inner product, like L_2 $\langle g|f\rangle = \int dx\, g^* f$
- DOMAIN — limits of integral
- BOUNDARY CONDITIONS

PUZZLES from LAST TIME:

1. WHAT'S WRONG W/ THE "LEGO" BASIS?



DISCRETIZE THE X-VARIABLE

POSITION SPACE; IN CONTRAST TO MOMENTUM SPACE (FOURIER) BASIS

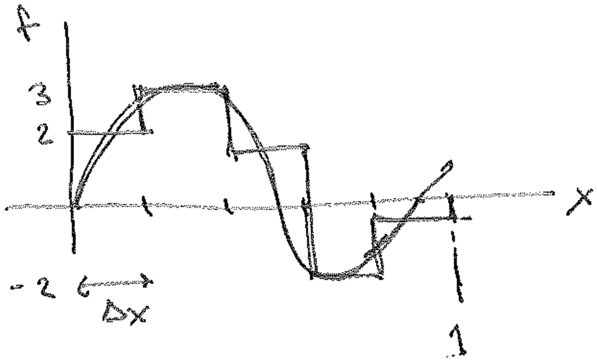
2. WHAT DO LINEAR OPERATORS LOOK LIKE?

3. WHAT DOES THE IDENTITY OPERATOR LOOK LIKE?

THE POSITION SPACE / LEGO BASIS IS ACTUALLY QUITE NICE

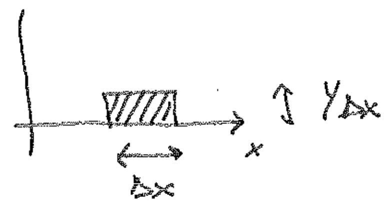
↳ IT IS HOW THE SIMPLEST NUMERICAL ALGORITHMS CALCULATE DERIVATIVES

WHAT'S WRONG? THIS IS BASICALLY A HISTOGRAM.



$$f^n = \begin{pmatrix} 2 \\ 3 \\ 1.5 \\ -2 \\ -.5 \end{pmatrix}$$

$|e_i\rangle$ is a unit blip



$$e_i(x) = \begin{cases} \frac{1}{\Delta x} & \text{if in window} \\ 0 & \text{otherwise} \end{cases}$$

can write w/ Θ functions

COEFFICIENTS:

$$\langle e^i | f \rangle = \int_0^1 e^i(x)\, f(x)\, dx$$

i^{th} coeff.

$$= \underbrace{\frac{1}{\Delta x} \int_{\text{WINDOW}} f(x)\, dx}_{\text{AVG } f(x) \text{ OVER THE WINDOW}}$$

can see: $e_i(x)$ has unit area "under the curve"
gets taller as $\Delta x \to 0$

→ this is a discretization of Dirac δ-function

NOTE: DISCRETIZATION OF $\delta(x)$ IS NOT $\delta^i{}_j$!

IN FACT, HERE'S THE PROBLEM:

$$\langle e^i | e_j \rangle = \left(\frac{1}{\Delta x}\right)^2 \Delta x \; \delta^i{}_j$$

$$= \underbrace{\frac{1}{\Delta x}}_{\text{NOT NORMAL!}} \delta^i{}_j$$

δ-functions are not functions, they're DISTRIBUTIONS.

they really only make sense when you integrate them over something — another function.

→ then they spit out the value of the function @ a point ... a #

INDEED: δ-functions are really more like funny elements of the DUAL SPACE.

see Stone & Goldbart "RIGGED HILBERT SPACE"

YOU CAN'T "FIX" THIS NORMALIZATION.

COULD TRY $|\tilde{e}_i\rangle$ → $\tilde{e}_i(x) = \frac{1}{\sqrt{x}}$ in window

then $\langle \tilde{e}^i | \tilde{e}_j \rangle = \delta^i{}_j$

BUT: $\tilde{e}(x)$ DOESN'T INTEGRATE TO 1 (norm?)

$$|f\rangle = \sum_i \underbrace{f(x_i)}_{\text{COMPONENT}} \sqrt{\Delta x} \, |\tilde{e}_i\rangle$$

WEIRD PREFACTOR

THE REAL PROBLEM IS

$$\lim_{x_i \to x_j} \int dx\, \underbrace{\delta(x-x_i)\,\delta(x-x_j)} = \delta(x_i - x_j)$$

disaster when $x_i \to x_j$!

DISTRIBUTIONS ARE WEIRD.

~~MATRICES~~
EVEN THOUGH THE LEGO BASIS IS STRANGE, IT IS VERY INSTRUCTIVE ? Riemann Integral

So let's push on!

<u>IDENTITY</u> : $\mathbb{1} = |e_i\rangle\langle e^i|$

function

functional

$$\mathbb{1}|f\rangle = |e_i\rangle\langle e^i|\underbrace{f^j|e_j\rangle}_{|f\rangle}$$

$$= f^j|e_i\rangle \underbrace{\langle e^i|e_j\rangle}_{\delta^i{}_j}$$

$$f(x) = \int_D \delta(x-y)\, \underbrace{f(y)\, dy}$$

functional acting on f

spits out function of x

Derivatives - IN DISCRETIZED SPACE

act on functions, returns function

$$\underbrace{\left.\frac{df(x)}{dx}\right|_{x_i}}_{f'(x_i)} \to \frac{f(x_{i+1}) - f(x_i)}{\underset{x_{i+1}-x_i}{\Delta x}} = \frac{1}{\Delta x} f^{i+1} - \frac{1}{\Delta x} f^{i}$$

nb: we chose x_{i+1} ... could have use x_{i-1}, ...

IN MATRIX FORM:

$$\underbrace{\begin{array}{c c} & \begin{array}{c c c c} \quad & (i) & (i+1) & \quad \end{array} \\ (i) & \begin{pmatrix} \ddots & & & \\ & -\frac{1}{\Delta x} & \frac{1}{\Delta x} & \\ & & & \ddots \end{pmatrix} \end{array}}_{D^i{}_j} \begin{pmatrix} \vdots \\ f^i \\ f^{i+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \frac{1}{\Delta x} f^{i+1} - \frac{1}{\Delta x} f^i \\ \vdots \end{pmatrix}$$

$$D^i{}_j = \begin{cases} -1/\Delta x & \text{if } j = i \\ +1/\Delta x & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

SPARSE MATRIX ("JACOBI" mostly along diagonal & next-to-diagonal

~~WHAT ABOUT~~

is this linear? YES. (obvious once we wrote as a matrix)

$D(f) + D(g) = D(f+g)$

etc.

SECOND DERIVATIVE

→ is it linear?

SHOULD JUST BE $D^i{}_j D^j{}_k = (D^2)^i{}_k$

for fixed i

we know only $j = i$, $j = i+1$ are nonzero

$\Rightarrow \boxed{k = i,\ i+1,\ i+2}$ nonzero

SO LET'S WRITE OUT THE ELEMENTS OF THE i^{th} ROW OF (D^2). NO SUM:

$$\cancel{= D^i{}_i D^i{}_i + D^i{}_{(i+1)} D^{(i+1)}{}_{(i+1)}}$$
$$\cancel{+ D^i{}_i D^i{}_{(i+1)} + D^i{}_{(i+1)} D^{(i+1)}{}_{(i+2)}}$$

$$\cancel{= \frac{-1}{\Delta x}\left(\frac{-1}{\Delta x} +\right.}$$

$$(D^2)^i{}_i = D^i{}_i D^i{}_i + D^i{}_{(i+1)} \cancel{D^{(i+1)}{}_i} \to 0$$
$$= \left(-\frac{1}{\Delta x}\right)^2 = 1/\Delta x^2$$

$$(D^2)^i{}_{(i+1)} = D^i{}_i D^i{}_{(i+1)} + D^i{}_{(i+1)} D^{(i+1)}{}_{(i+1)}$$
$$= \left(-\frac{1}{\Delta x}\right)\left(\frac{1}{\Delta x}\right) + \left(\frac{1}{\Delta x}\right)\left(-\frac{1}{\Delta x}\right)$$
$$= -2/\Delta x^2$$

$$(D^2)^i{}_{(i+2)} = D^i{}_{(i+1)} D^{(i+1)}{}_{(i+2)} = \left(\frac{-1}{\Delta x}\right)^2 = 1/\Delta x^2$$

$$(D^2)^i{}_j = \begin{cases} \frac{1}{\Delta x^2} & \text{if } j = i,\ (i+2) \\ -\frac{2}{\Delta x^2} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

note this is shifting right...
A BETTER (more symmetric ~~def~~)

$$(D^2)^i{}_j = \begin{cases} \frac{1}{\Delta x^2} & \text{if } j = i \pm 1 \\ \frac{-2}{\Delta x} & \text{if } j = i \\ 0 & \text{other} \end{cases}$$

$$\begin{array}{cc} & \begin{array}{ccc} (i-1) & i & (i+1) \end{array} \\ i & \begin{pmatrix} \ddots & & & \\ & \frac{1}{\Delta x^2} & -2/\Delta x^2 & \frac{1}{\Delta x^2} \\ & & & \ddots \end{pmatrix} \end{array} \begin{pmatrix} \vdots \\ f^{i-1} \\ f^{i} \\ f^{i+1} \\ \vdots \end{pmatrix}$$

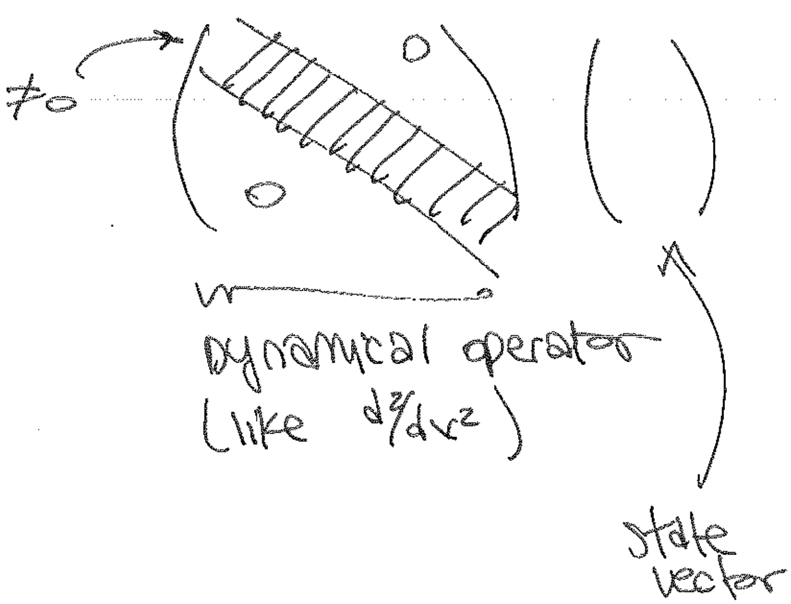
$$D^2 \qquad\qquad f$$

THIS IS, OF COURSE, CONSISTENT WITH HOW WE DISCRETIZE THE SECOND DERIVATIVE.

↑ nb this is how kids should learn calculus — maps on to numerical methods w/ a computer.

MANIFEST: MATRICES ARE SPARSE

⟷ PHYSICS is LOCAL.

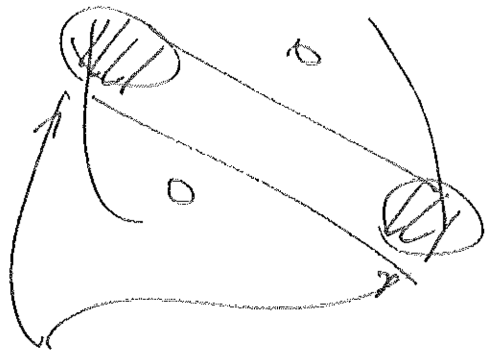


BTW: WHY SECOND ORDER?

EOM come from variation of ACTION.

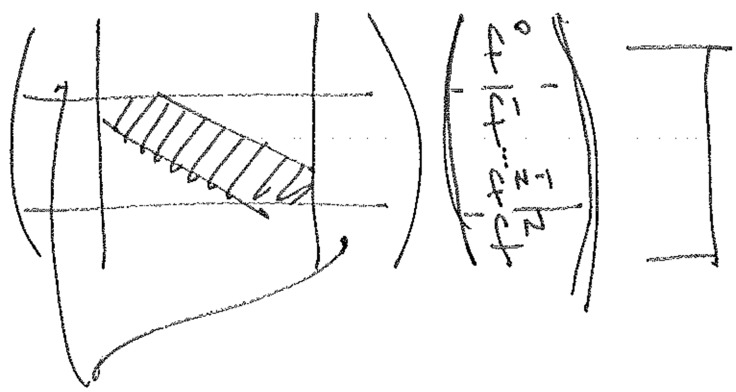
DIFF ⊛ has DIMENSION.

<u>BOUNDARY</u> → OUR LEGO BASIS DOESN'T ENFORCE BC



what happens here?

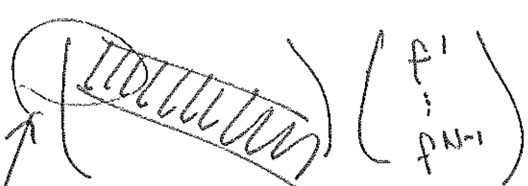
SIMPLEST EXAMPLE: DIRICHLET



set to 0

set to 0

ACTUALLY, THIS IS OF COURSE REDUNDANT.
SO CAN JUST WRITE THIS AS



= 0 ?
not in reduced space

$$(D^2)^1{}_j |f^j\rangle = \frac{1}{\Delta x^2}\left(\cancel{f^0} - 2f^1 + f^2\right)$$

$$D^2 = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & & \ddots \end{pmatrix}$$

HOW THIS LOOKS IN THE CONTINUUM:

$$A|f\rangle = |g\rangle \Longrightarrow \int dy\, A(x,y) f(y) = g(x)$$

in general, can be non-local

y can be diff. from x.

eg "PROPAGATORS" THAT REPRESENT LONG RANGE CORRELATIONS OR FORCES

(this is, of course, just a model of microphysics that is local!)

USUALLY WE CARE ABOUT

$$A = \sum_n a_n \left(d/dx\right)^n$$

may be a function.

nb: (dynamics) (state) = (source)

OUR REAL GOAL:

$$|f\rangle = (A^{-1})\, |g\rangle$$

state

inverse

as a function of source