VECTOR SPACES

IAN OH: 2pm W

LAST TIME: DIMENSIONAL ANALYSIS

TODAY: MORE SYSTEMATIC TREATMENT OF TRADITIONAL MATERIAL → LINEAR ALGEBRA (quantum mechanics)

"When the only tool you have is a hammer, then everything looks like a nail"

Better version:

"To a carpenter, even the moon looks like it is made of wood."

to a physicist, everything is QM

VECTORS ARE MATHEMATICAL ABSTRACTIONS. THEY POP UP ALL THE TIME IN PHYSICS.

obvious one: LINEAR ALGEBRA ⟷ QM

but also: VECTOR ⟷ DIRECTIONAL DERIVATIVE

notion of tangent space, fiber bundle → EVEN GROUP THEORY (Lie groups) fall under this umbrella

geometry (GENERALIZE CALCULUS) → diff. topo', Index thm, ...

DIFFERENTIAL EQUATIONS

WORTH TAKING TIME REVIEWING THIS CAREFULLY.

VECTORS LIVE IN A VECTOR SPACE, V ← where vectors (kets) live
with some dimensionality

→ eg. $V = \mathbb{R}^3$, (EUCLIDEAN) 3-SPACE

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V$$

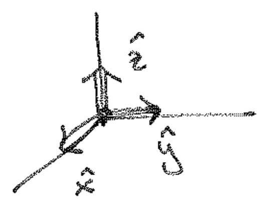
→ can also be ∞-DIM, LIKE HILBERT SPACE. (STATE SPACE)

SOME NOTATION: in highschool, we wrote

$\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ but this is problematic ...
it assumes a BASIS

↑

$\hat{x}, \hat{y}, \hat{z}$



then spherical coordinates are weird & E&M gets annoying.

WE WILL WRITE:

$$\underline{v} = \sum_i x^i \underline{e}_i = \sum_i x^i |e_i\rangle$$

or

$|v\rangle$

BASIS VECTORS

x^i IS JUST A NUMBER FOR EACH i

this is what we write in the entries of a column vector

<u>EINSTEIN CONVENTION</u>: REPEATED UPPER & LOWER INDICES ARE SUMMED OVER

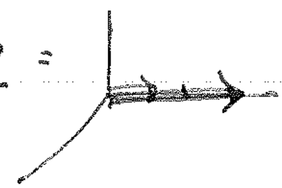
$$\underline{x} = x^i \underline{e}_i = \sum_i x^i \underline{e}_i$$

WE WRITE THE <u>BASIS</u> VECTORS WITH LOWER INDICES.

THE "COORDINATES" HAVE UPPER INDICES.

eg. IN $\mathbb{R}^3$, you can imagine

$\underline{e}_x =$ 

$v^i = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ ↳ $\underline{v} =$ 

OBSERVE: $\underline{v}$ IS BASIS INDEPENDENT ← vector

$\{v^i\}$ IS BASIS DEPENDENT ← collection of components in a specific basis

<u>Life protip</u>: DO NOT CONFUSE A VECTOR WITH ITS COMPONENTS.

↑ often clear what you mean in context, but it can lead to bad habits in more formal settings, like GR.

"LINEAR"

PROPERTIES OF VECTORS : two operations

$+ : V \times V \to V$ (2 vectors → vector)

commutative & associative

$\underline{v} + \underline{w} = \underline{w} + \underline{v}$ $(\underline{v} + \underline{w}) + \underline{u} = \underline{v} + (\underline{w} + \underline{u})$

RESCALE : vector & # → vector

IDENTITY & INVERSE ELEM. FOR BOTH.

THE OTHER THING YOU KNOW ABOUT VECTORS IS THAT YOU CAN PRODUCE SCALARS WITH THEM.

dot product, inner product

$V \times V \to V$? No

~~NO SUCH OPERATION. THAT WOULD REQUIRE ADDITIONAL STRUCTURE~~

naive picture

$$\begin{pmatrix} 3 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \\ 5 \end{pmatrix} = 3 + 14 + 30$$

$$= 47$$

"ROW VECTOR"

this is a whole different object (at least a priori)

DEFINE : V^* : DUAL SPACE

where "row vectors" live (or bras) (or 1-FORMS)

ELEMENT OF DUAL SPACE : LINEAR MAPS THAT TAKE VECTORS TO NUMBERS.

$f(a\underline{v}+b\underline{w}) = af(\underline{v}) + bf(\underline{w})$

LINEAR : DUAL VECTOR ADDITION & RESCALING ARE LINEAR

MAP : functions

$\mathbb{R}$ or $\mathbb{C}$ depending on the space

$$\langle v| \in V^* \Rightarrow \langle v| : V \rightarrow \#$$

[to make V & V^* clear, it's useful to revert to bra-ket notation]

BASIS KET HAS ↑ INDEX

$\langle x|$ is a bra $\in V^* \rightarrow = \langle e^i| x_i$

$|y\rangle$ is a ket $\in V \rightarrow = \cancel{\ } y^j |e_j\rangle$

$\langle x|y\rangle$ is a number.

$$\left(\langle e^i|x_i\right)\left(y^j|e_j\rangle\right) = x_i y^j \langle e^i|e_j\rangle$$

if $\langle e|$ & $|e\rangle$ are an orthonormal basis, then

$$\langle e^i|e_j\rangle = \delta^i_j$$

if orthonormal

$$\rightarrow \langle x|y\rangle = x_i y^j \delta^i_j = x_i y^i = \sum_i x_i y^i$$

AS YOU WOULD EXPECT IF YOU STICK x INTO A ROW VECTOR & y INTO A COLUMN VECTOR.

THE DUAL SPACE IS TOTALLY DIFFERENT FROM THE VECTOR SPACE.

WITHOUT ADDITIONAL MATHEMATICAL STRUCTURE (metric), THERE IS NO RELATION BETWEEN

$\langle e^i|$ & $|e_j\rangle$

& you certainly cannot have things like $\langle x| + |y\rangle$.

SEEMS CONFUSING – WHAT ABOUT TRANSPOSE / HERMITIAN CONJUGATE? DON'T THESE TURN VECTORS INTO DUAL VECTORS?

nb ARTICLE: 1602.03006 § 4.1.2

USUALLY IN PHYSICS, WE HAVE A METRIC

literally something that measures "DISTANCE" & "ANGLES"

Something I'm going to gloss over: DUAL FORM VS. INNER PRODUCT DISTINCTION rel to Riesz Rep. THM.

METRIC, $(\underline{v}, \underline{w}) = \underline{v}\cdot\underline{w} \quad : V\times V \to \#$

takes two vectors & spits out a number.

- bilinear : linear in each argument

$$(a\underline{v} + b\underline{u}, \underline{w}) = a(\underline{v},\underline{w}) + b(\underline{u},\underline{w})$$

& similar for $(\underline{v}, a\underline{w} + b\underline{u})$

sometimes we call this a bilinear form.

- SPECIAL TO METRIC: symmetric

$$(\underline{v}, \underline{w}) = (\underline{w}, \underline{v})$$

BUT: IF WE FEED THE METRIC ONE ARGUMENT,

$$(\cdot, \underline{w}) : V \longrightarrow \#$$

this is a linear map from $V \rightarrow \#$.

THIS IS EXACTLY A DUAL VECTOR

$$f_{\underline{w}}(\underline{x}) = (\underline{x}, \underline{w})$$

LINEAR IN x & ALL THAT

detail

Riesz thm paraphrase

$\langle \cdot , \cdot \rangle$ OR $\langle \cdot | \cdot \rangle : V^* \times V \rightarrow \#$

ENCODES HOW 1-FORM ACTS ON VECTORS

RELATION TO METRIC

$$\langle \phi_R(x) \,|\, y \rangle = (x, y)$$

ANTILINEAR: $\phi_R(a\underline{x} + b\underline{y}) = a^* \phi_R(x) + b^* \phi_R(y)$

FOR THE MOST PART, THE DISTINCTION WILL NOT SHOW UP IN THIS COURSE & WE WON'T BELABOR IT ANY FURTHER

why? s.t. $(x, x) = \|x\|^2$

norm of x
(POSITIVE DEF.)

from now on I'll use $(\cdot, \cdot)$ & $\langle \cdot , \cdot \rangle$ interchangeably

BUT THIS GIVES US A SENSE OF CONJUGATION:

$$\langle x| \sim |x\rangle^{+}$$

in the sense that

$$\dagger : |x\rangle \longrightarrow \langle x, \cdot\rangle$$

"turns it into a ROW vector".

in some sense this is purely academic... until it's not.

IN COMPONENTS: (redux) ← why mathematicians make fun of us

$$\langle v| = v_i \langle e^i|$$

$$|w\rangle = w^i |e_i\rangle$$

$$\langle v|w\rangle = v_i w^j \langle e^i|e_j\rangle$$

AS BEFORE

$$(|e_i\rangle)^{\dagger} |e_j\rangle = (e_i, e_j)$$

inner product

$$= g_{ij}$$

I'M BEING SLOPPY HERE FOR CONVENIENCE!

metric components

in $\mathbb{R}^3$ w/ CARTESIAN COORDS:

$$g_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

NORM : $\| \, |x\rangle \, \|^2 = \langle x | x \rangle$

ANGLE : $\langle x | y \rangle \equiv$ "$\cos\theta$" $\| |x\rangle \| \cdot \| |y\rangle$

LINEAR TRANSFORMATIONS

operators, matrices

THE TRANSFORMATION A ACTS ON A VECTOR :

$$A|x\rangle : V \longrightarrow V$$

SPITS OUT ANOTHER VECTOR

$$A = A^i{}_j \, |e_i\rangle\langle e^j|$$

eats vector (gives #)

is a vector

gives prefactor (#) for $|e_i\rangle$

$$A|x\rangle = A^i{}_j \, |e_i\rangle\langle e^j| \, x^k \, |e_k\rangle$$

$$= \underbrace{A^i{}_j \, x^k}_{\text{JUST \#'s, ORDER DOESN'T MATTER}} \, |e_i\rangle \, \underbrace{\langle e^j | e_k \rangle}_{\text{ALSO JUST A \#!}}$$

FOR ORTHONORMAL BASIS, THIS IS $\delta^j{}_k$

ASSUMING $\underbrace{\langle e^j | e_k \rangle = \delta^j{}_k}_{}$,

WE WILL WORK IN THIS CASE ALMOST EXCLUSIVELY.

$$A|x\rangle = \underbrace{A^i{}_j x^k \delta^j{}_k}_{} |e_i\rangle$$

$$(\underbrace{A^i{}_j x^j}_{\text{call this } y^i}) |e_i\rangle$$

$$= y^i |e_i\rangle$$

see how SUMMATION CONVENTION WORKS?

nb $\delta^i{}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

"ROTATIONS" & generalizations
for invertible linear transformation

$$A^{\dagger} A = \mathbb{1} \leftarrow \delta^{n}{}_{m} |e_n\rangle\langle e^m|$$

$$(A^{-1})^{k}{}_{\ell} |e_k\rangle\langle e^{\ell}| \, A^{i}{}_{j} |e_i\rangle\langle e^j| =$$

$$= (A^{-1})^{k}{}_{\ell} A^{i}{}_{j} |e_k\rangle \underbrace{\langle e^{\ell}||e_i\rangle}_{\delta^{\ell}{}_{i}} \langle e^j|$$

$$= (A^{-1})^{k}{}_{i} A^{i}{}_{j} \underbrace{|e_k\rangle\langle e^j|}_{\text{BASIS FOR MATRICES}}$$

observe contraction of indices

SOMETIMES, AS SHORTHAND WE SAY THAT

$$B^{i}{}_{j} C^{j}{}_{k} = D^{i}{}_{k}$$

IS MATRIX MULTIPLICATION.

IT IS — BUT WE HAVE TO REMEMBER THAT "SECRETLY" THERE ARE BASIS VECTORS.

otherwise, this is BASIS-DEPENDENT

ACTIVE VS. PASSIVE TRANSFORMATIONS

↳ eg CHANGE OF BASIS.

SUPPOSE $|f_i\rangle$ IS A DIFFERENT BASIS.
(ORTHONORM, LET'S SAY)

$$|f_i\rangle = \underbrace{|e_k\rangle\langle e^k|}\, f_i\rangle$$

$$\sum_k |e_k\rangle\langle e^k| = \mathbb{1}$$

(SUM OVER COMPLETE SET OF STATES)

$$= \underbrace{\langle e^k | f_i\rangle}_{R^k{}_i} |e_k\rangle$$

↑ "ROTATION" MATRIX

$$|V\rangle = v^i |e_i\rangle$$

$$= v^i |f_k\rangle\langle f^k|e_i\rangle$$

$$= \underbrace{v^i \langle f^k|e_i\rangle}_{v^k_{(f)}} \langle f_k|$$

$v^k_{(f)}$ ← components in $|f\rangle$ BASIS

CHANGING BASIS. (ACTING ON BASIS, not index)