LEC 18

nb: solutions to hw on ilearn (by lon)

11/13/2017

GREEN'S FUNCTIONS

THE STORY SO FAR : some time: PATH INTEGRALS → EOM

PLAN
W: GREEN'S IN 1D.
F: OTHER SYS
M: ACTION PRINC
M: STATS.

WANT TO SOLVE : $\underbrace{\mathcal{O}}_{\text{DIFF. OPERATOR}} \underbrace{f(x)}_{\text{STATE}} = \underbrace{s(x)}_{\text{SOURCE}}$

want to know how source (KNOWN) affects state.

STRATEGY: $G(x,x')$ s.t. $\mathcal{O}G(x,x') = \delta(x-x')$

FUNCTION OF x & DIFFERENTIAL OPERATORS OF x

I CAN CONSTRUCT $s(x)$ USING BUILDING BLOCKS OF $\delta(x-x')$

$$s(x) = \underbrace{\int dx' \, s(x')}_{\text{APPLY THIS TO BOTH SIDES OF GREEN'S FUNCTION EQUATION}} \delta(x-x')$$

$$\int dx' \, s(x') \, \mathcal{O}_x G(x,x') = \int dx' \, s(x') \, \delta(x-x')$$

$$\mathcal{O}_x \boxed{\int dx' \, s(x') \, G(x,x')} = s(x)$$

$\equiv f(x)$, what we wanted to find!

in other words, given "PHYSICS" ($\mathcal{O}$) ← DYNAMICS
& some physical scenario $s(x)$ ← SOURCE

then $\mathcal{O} \to G(x,x')$ ← trick: fourier transform
TURNS $\mathcal{O}$ INTO ALGEBRAIC OPERATOR. THEN TRANSFORM BACK
↳ let CAUSALITY DEAL W/ POLES.

& so:

$$\boxed{f(x) = \int dx' \, s(x') \, G(x,x')}$$

EXAMPLE SO FAR: HARMONIC OSCILLATOR, $\mathcal{O} = \left(\frac{d}{dt}\right)^2 + \omega^2$

WE SET THE SOURCE AT $t' = 0$

$$\mathcal{O}_t\, G(t, t'=0) = \delta(t-t') = \delta(t) \Rightarrow \frac{1}{\omega}\sin(\omega t)$$

DIRICHLET BC, RET. POT

WHAT if we RESTORE t'? $\longrightarrow$ $\boxed{\frac{1}{\omega}\sin[\omega(t-t')]}$

WHY? TRANSLATION INVARIANCE if we shifted the origin of our coordinate system ("ZERO" of our clock), the SOURCE-AND-RESPONSE SHOULDN'T CHANGE

!! following is from: courses.physics.ucsd.edu/2009/Fall/physics130b/Green-functions.pdf !!

in general: $G(x, x')$

these should match if translation inv.

shift in coordinate system: $G(x+a, x'+a)$

thus: $$\frac{\partial G(x+a, x'+a)}{\partial a} = 0 = \frac{\partial G(x,x')}{\partial x}\underbrace{\cancel{\frac{\partial(x+a)}{\partial a}}}_{1} + \frac{\partial G}{\partial x'}\underbrace{\cancel{\frac{\partial(x'+a)}{\partial a}}}_{1}$$

(*) $\rightarrow \boxed{\frac{\partial G}{\partial x} + \frac{\partial G}{\partial x'} = 0}$

CHANGE VARS AGAIN: $y_+ = x + x'$ ← DEP ON THIS
$y_- = x - x'$

expect: G is INDEP OF THIS

$$\frac{\partial}{\partial x} = \overset{+}{\left(\frac{\partial y_+}{\partial x}\right)}\frac{\partial}{\partial y_+} + \overset{+}{\left(\frac{\partial y_-}{\partial x}\right)}\frac{\partial}{\partial y_-}$$

$$\frac{\partial}{\partial x'} = \underset{+}{\left(\frac{\partial y_+}{\partial x'}\right)}\frac{\partial}{\partial y_+} + \underset{-}{\left(\frac{\partial y_-}{\partial x'}\right)}\frac{\partial}{\partial y_-}$$

$$(*)\quad \underbrace{\frac{\partial G}{\partial y_+}\cancel{\frac{\partial y_+}{\partial x}}^{1} + \frac{\partial G}{\partial y_-}\cancel{\frac{\partial y_-}{\partial x}}^{1}}_{\frac{\partial G}{\partial x}} + \underbrace{\frac{\partial G}{\partial y_+}\cancel{\frac{\partial y_+}{\partial x'}}^{1} + \frac{\partial G}{\partial y_-}\boxed{\frac{\partial y_-}{\partial x'}}^{-}}_{\frac{\partial G}{\partial x'}} = 0$$

CANCEL

$\boxed{2\frac{\partial G}{\partial y_+} = 0}$ $\Rightarrow G(x,x')$ IS INDEP. OF $(x+x')$

<u>So</u>: if $G(x,x')$ is indep of $(x+x')$, then it is only a function of $(x-x')$

$\rightarrow G(x,x') = G(x-x')$
AS WE OBSERVED.

WHERE DOES IT COME FROM?

$$\delta(x-x') = \int dk\, e^{ik(x-x')}$$

$$\mathcal{O}_x \int dk\, e^{-ikx}\,\tilde{G}(k) = \int dk\, e^{-ik(x-x')}$$

$$\int dk\, e^{-ikx}\, P(k)\tilde{G}(k) = \int dk\, e^{-ikx}\, e^{ikx'}$$

POLYNOMIAL

$$\tilde{G}(k) = \frac{e^{ikx'}}{P(k)}$$

<u>then</u>: $G(x,x') = \int dk\, e^{-ikx}\, \dfrac{e^{ikx'}}{P(k)}$

$$= \boxed{\int dk\, e^{-ik(x-x')}\, \frac{1}{P(k)}}$$

END UP W/ FUNCTION OF $(x-x')$

just have to <u>do</u> this integral
w/ some care for pole prescription
(important for causality.)

EXAMPLE: THE DAMPED SPRING

SAME AS USUAL HARMONIC OSCILLATOR, BUT W/ DAMPING COEFFICIENT γ

$$(\ddot\smile) \quad \ddot{x}(t) + 2\gamma \dot{x}(t) + \omega^2 x(t) = \underset{\text{DRIVING FORCE}}{F(t)}$$

// EXPECTED BEHAVIOR:
γ DAMPS RESPONSE; GUITAR STRING VIBRATIONS GET SMALLER IN TIME.

TOTALLY not rigorous diagnosis:
ASSUME PLANE WAVE $\rightarrow x(t) = e^{ikt}$

& SEE WHAT IT WOULD LOOK LIKE FAR FROM SUPPORT OF SOURCE

$x(t) = e^{ikt}$ (ansatz)

$$(\ddot\smile): \underbrace{(-k^2 + 2i\gamma k + \omega^2)}_{=0} e^{ikt} \overset{?}{=} 0 \Rightarrow k = k_R + i\bar{k}$$

$$\text{so } e^{ikt} = \underbrace{e^{ik_R t}}_{\text{PLANE WAVE}} \underset{\text{DAMPING}}{e^{-\bar{k}t}}. \quad //$$

okay. so we can guess that something happens to our poles.

$$\left[\left(\frac{d}{dt}\right)^2 + 2\gamma\frac{d}{dt} + \omega^2\right] G(t,t') = \delta(t-t')$$

$$\int dk\, e^{ikt}\, \tilde{G}(k) \qquad \int dk\, e^{-ik(t-t')}$$

$$(-k^2 - 2i\gamma k + \omega^2)\,\tilde{G}(k) = e^{ikt'}$$

$$\tilde{G}(k) = \frac{-e^{ikt'}}{k^2 + 2i\gamma k - \omega^2}$$

$\mathbb{R}$ for $\boxed{\omega^2 > \gamma^2}$

POLES @ $k = \cancel{2}\,-i\gamma \pm \sqrt{\cancel{4}\,\gamma^2 + \cancel{4}\,\omega^2}\;/\cancel{2} : k_\pm$

IMAGINARY PART

$$\boxed{\pm\sqrt{\omega^2-\gamma^2} - i\gamma}$$

RECALL: $G(t-t') = \int dk\, \frac{-e^{-ik(t-t')}}{k^2 + \dots}$

for $(t-t') > 0$, want $\mathrm{Re}(-ik) < 0$

$+k\sin\theta$

LOWER HALF PLANE

FORTUNATELY: POLES ARE ALREADY AUTOMATICALLY IN LOWER HALF PLANE FOR $\gamma > 0$! $\rightarrow$ $\boxed{\text{no } \varepsilon \text{ PRESCRIPTION}}$

no choice to make

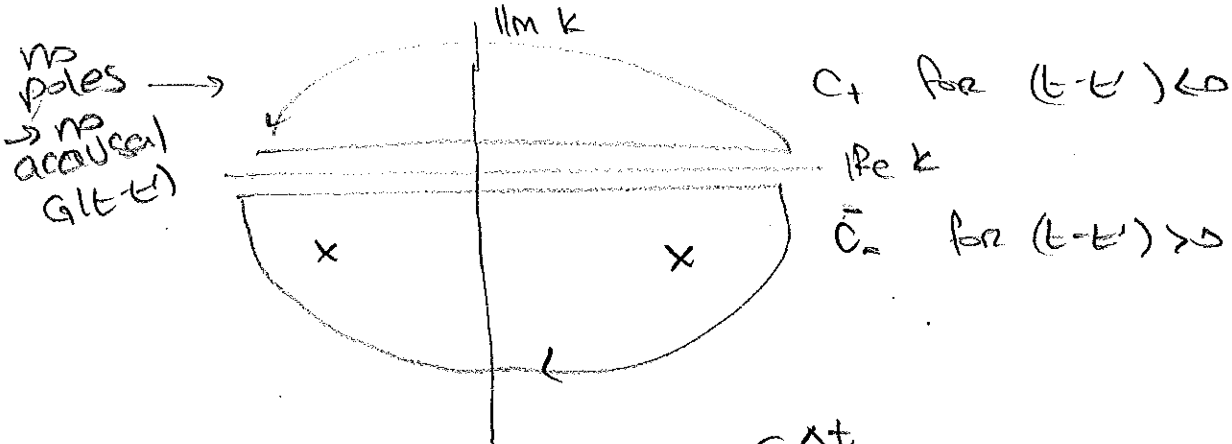
WHAT IF $\gamma < 0$? not physical.

? is every physical H.O. like this? then the ε choice is just ACADEMIC?

* HIGGS VS PHOTON.

RESIDUE THEOREM

$$G(t-t') = \int dk \, \frac{-e^{-ik(t-t')}}{(k-k_+)(k-k_-)}$$



$$G(t-t') = \cancel{2\pi} i \cdot \frac{1}{\cancel{2\pi}} \left[\frac{-e^{-ik_+\overbrace{(t-t')}^{\Delta t}}}{k_+ - k_-} + \frac{-e^{-ik_-(t-t')}}{k_- - k_+} \right]$$

$$\boxed{k_+ - k_- = 2\sqrt{\omega^2-\gamma^2}} \qquad e^{-ik_\pm \Delta t} = e^{-\gamma\Delta t} e^{\mp i\sqrt{\omega^2-\gamma^2}\Delta t}$$

$$G(t-t') = e^{-\gamma\Delta t} \frac{1}{\sqrt{\omega^2-\gamma^2}} \underbrace{\frac{-i}{2}\left[e^{-i\sqrt{\mp}\Delta t} - e^{i\sqrt{\mp}\Delta t}\right]}_{= \sin\left(\sqrt{\omega^2-\gamma^2}\, t\right)}$$

$\times \Theta(t-t')$

compare to

$$G_{SHO}(t-t') = \Theta(t-t') \frac{\sin \omega t}{\omega} \quad \checkmark$$

"I took 6 weeks of P231 & all I have is this stupid Green's function."

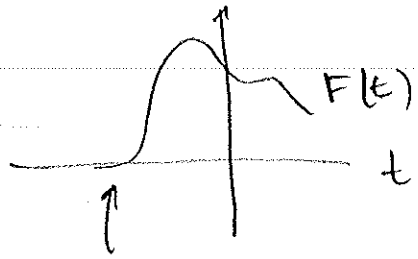
HOW TO SOLVE for A GIVEN $F(t)$

$$\boxed{x(t) = A x_1(t) + B x_2(t) + \int_{t_1}^{t_2} G(t,t')\, F(t')\, dt'}$$

SOLUTIONS TO HOMOG. EQ. : $\mathcal{O} x_{1,2} = 0$

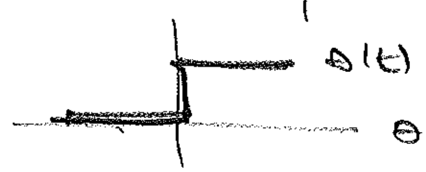
$G(t,t') = G(t-t')$

$(t_{1,2} = \mp \infty$ BUT EFFECTIVELY :



t_1 : SOME TIME WHEN SOURCE STARTED

then $t_2 = t$, BECAUSE $G(t-t') \propto \Theta(t-t')$



<u>HW</u>: ACTUALLY SOLVE FOR SOME DRIVING FORCE.

<u>ANALOGY</u>: TIME EVOLUTION IN QM e^{iHt}
HIGGS BOSON HAS $H \to H + i\Gamma$
WHAT DOES THIS MEAN? $\to$ <u>DECAYS!</u>

gets back to P.5 : MAYBE <u>EVERY</u> PHYSICAL HARMONIC OSCILLATOR IS DAMPED? eg KRAMERS - KRONIG.
BUT: PHOTON? ELECTRON?