LAST TIME : HW3 #1

THIS WEEK : $\mathbb{C}$ ANALYSIS CRASH COURSE

→ MON: DISSECTION?

WHERE ARE WE GOING W/ THIS?

HW: will keep having Green's questions

MAIN GOAL : $\mathbb{C}$ CONTOUR INTEGRATION

→ why? powerful tool for doing difficult integrals... IN PARTICULAR, THOSE THAT SHOW UP IN GREEN'S FUNCTION EQUATIONS

this lec; APPEL Ch. 4 & MY 2016 NOTES

[I SHOULD REVIEW WHY $G(x,y) = G(x-y)$.]

$$L_x G(x,y) = \delta(x-y)$$

$$\sum_n p_n(x)\left(\frac{d}{dx}\right)^n$$

FOURIER TRANSFORM : $G(x,y) = \int e^{ikx} \tilde{G}(k)\, dk$

then $L_x G(x,y) = \int dk\; e^{ikx} \tilde{G}(k) \left[\sum_n \tilde{p}_n (ik)^n \right]$

$= \int e^{ik(x-y)} dk$

→ BECOMES ALGEBRAIC PROBLEM TO SOLVE FOR $\tilde{G}(k)$ ... BUT TRICKY INTEGRAL TO GET $G(x,y)$

contour int!

OTHER REASONS WHY THIS IS WORTHWHILE

NATURE "KNOWS" ABOUT $\mathbb{C}$ #'s!

→ ANALYTICITY ("good behavior") IS IMPORTANT IN PHYSICS

eg. GIVES A HANDLE FOR DIVERGENCES & WHAT THEY MEAN (missing important dynamics)

→ eg. UNITARITY of WW SCATTERING @ TeV SCALE

really neat observation, so we will examine

eg CAUSALITY IN DISPERSION RELATIONS

eg conformal mapping : see E&M

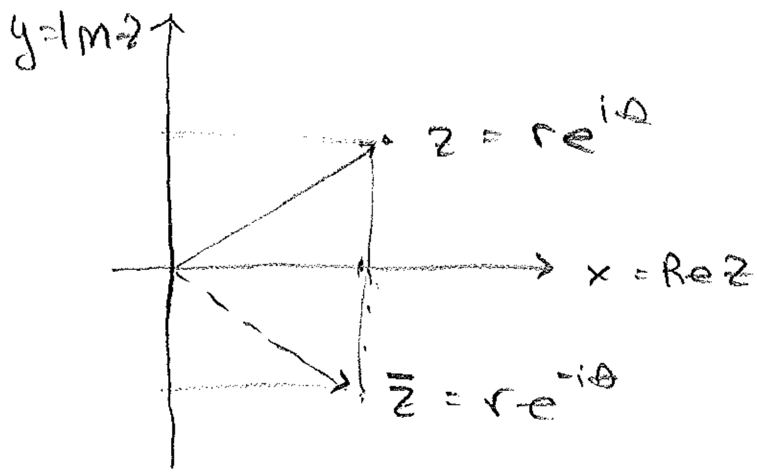
COMPLEX VARIABLES

$z = x + iy$

$\bar{z} = x - iy$ ← or z^*

$\mathbb{C}$ is a 2D vector space (like $\mathbb{R}^2$) WITH AN ADDITIONAL RULE FOR MULTIPLICATION: $V \times V \to V$.

↑ "complex structure"



COMPLEX FUNCTIONS : $F(x,y) \leftrightarrow$ "$f(z, \bar{z})$"

THERE IS AN IMPORTANT SENSE OF NICENESS :

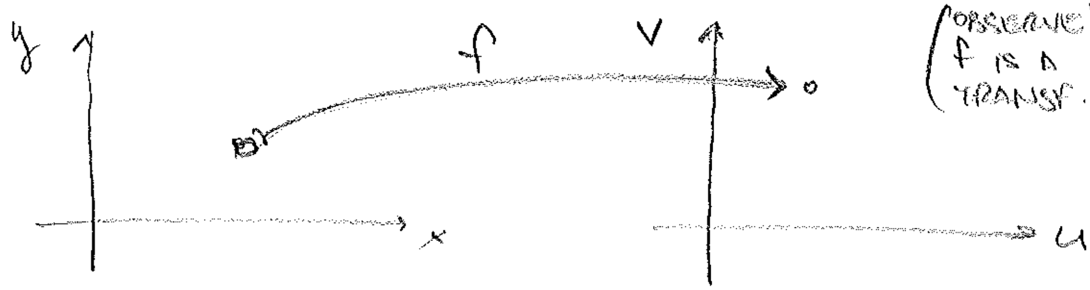
ANALYTIC : f HAS WELL DEFINED DERIVATIVE

↳ many of you already know the punchline that this boils down to "$f = f(z)$" ↑ only.

"nice" : the kind of quantity that describes actual physical quantities

↑ eg SUFFICIENTLY SMOOTH, DIFFERENTIABLE, ...

"WELL DEF. DERIVATIVE" ← what could go wrong?



(OBSERVE: f IS A TRANSF.)

A $\mathbb{C}$ function $f(x,y) = \underbrace{u(x,y)}_{\mathbb{R}} + i\underbrace{v(x,y)}_{\mathbb{R}}$

is a map from $\mathbb{C} \to \mathbb{C}$. CAN "DO CALCULUS"

BECAUSE $\mathbb{C} \sim \mathbb{R}^2$ IS A 2D SPACE, WE CAN EXAMINE INFINITESIMAL CHANGES IN DIFFERENT DIRECTIONS

DIRECTIONAL DERIVATIVE

BUT IN SOME SENSE, $\mathbb{C}$ IS A ONE DIMENSIONAL SPACE (albeit complex), namely:

$\frac{df}{dz}$ is the derivative

$$\lim_{z \to z_0} \frac{f|_z - f|_{z_0}}{z - z_0} = \frac{\Delta f}{\Delta z}$$

so plug in whatever Δz you want, and f' should give the unambiguous slope there.

WHAT COULD GO WRONG?

$$\left.\frac{df}{dz}\right|_{dz=dx} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\left.\frac{df}{dz}\right|_{dz=idy} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$z = x + iy$

LHS's HAVE TO BE CONSISTENT

CAUCHY-RIEMANN

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

ANOTHER POINT OF VIEW (GEOMETRIC) ← for culture

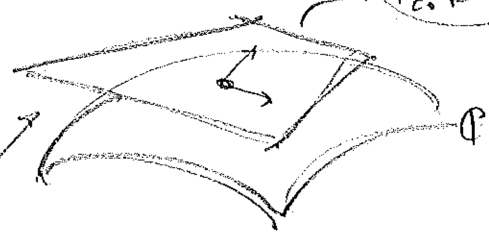
$\boxed{df(z_0)}$ IS A DIFFERENTIAL 1-FORM

ROW VECTOR.

OF WHAT VECTOR SPACE?

of the tangent space @ z_0

$T_{z_0}\mathbb{C}$.



$\mathbb{C}$ "BASE MANIFOLD"
no actual curvature,
just illustrative
[BUT GENERALIZES]

nb: the collection of all such tangent spaces for all points in $\mathbb{C}$ is called a tangent bundle

vector space "on top of" each point of the base space

MORE GENERAL TYPES OF VECTOR SPACES CAN BE PLACED ON EACH POINT
THESE CONSTRUCTIONS ARE CALLED FIBER BUNDLES

underlying structure of geometric approach to physics

inc. ANALYTICAL MECHANICS
(why are HAMILTONIANS SPECIAL?)

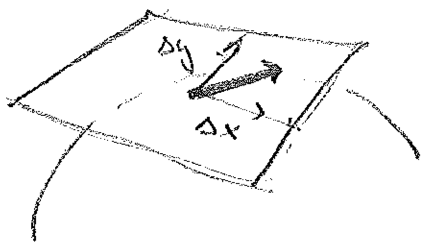
GAUGE THEORY (what is force?)

unifies diff. eq w/ geometry.

anyway: $\boxed{df(z_0)}$ is a machine that takes vectors in $T_{z_0}\mathbb{C}$ ("velocities" from z_0) & returns a #.

AS YOU KNOW, SINCE $df(z_0) \in T^*_{z_0}\mathbb{C}$, ← DUAL SPACE
IT IS A LINEAR MAP

$$df(z_0)[\Delta x + i\Delta y] = f'(z_0)\Delta x + f'(z_0)\, i\Delta y \qquad (\ddot\smile)$$



BUT WE ALSO KNOW

$$df\Big|_{z_0} = \frac{\partial f}{\partial x}\Big|_{z_0}\Delta x + \frac{\partial f}{\partial y}\Big|_{z_0}\Delta y \qquad (\ddot\frown)$$

COMPARING $(\ddot\smile)$ WITH $(\ddot\frown)$:

$$\boxed{\frac{\partial f}{\partial x} = f'(z_0) = -i\frac{\partial f}{\partial y}}$$

recalling $f = u + iv$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x}} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \boxed{\frac{\partial v}{\partial y}}$$

<u>CAUCHY-RIEMANN</u>

CAUCHY-RIEMANN (CR) $\Longleftrightarrow$ $\mathbb{C}$ DIFFERENTIABLE

WE USED $(\Delta x, \Delta y)$ AS A BASIS FOR $T_{z_0}\mathbb{C}$

$i\Delta y$

the infinitesimal "velocity" AT z_0

COULD USE A DIFFERENT BASIS:

$$\boxed{\begin{aligned} \Delta z &= \Delta x + i\Delta y \\ \Delta \bar{z} &= \Delta x - i\Delta y \end{aligned}}$$

$$\partial/\partial z = \partial/\partial x + \partial/\partial(iy)$$
$$\partial/\partial \bar{z} = \partial/\partial x - \partial/\partial(iy)$$

SAME   SAME

$$\frac{df}{dz} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{=0} + i\underbrace{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}_{=0} = 0$$

BY CAUCHY-RIEMANN!

$\mathbb{C}$ DIFFERENTIABLE

SO: A FUNCTION IS ANALYTIC @ z_0 if $\boxed{\text{CR EQ HOLDS}} \leftrightarrow \boxed{\partial f/\partial\bar{z} = 0}$

neat application: SUPERSYMMETRIC THEORIES ARE IMMUNE TO SOME QUANTUM CORRECTIONS; PROTECTED BY ANALYTICITY!

one last term: u + v are 2D HARMONIC IF f ANALYTIC

$$\Delta u = \partial_x^2 u + \partial_y^2 u = \partial_x(\partial_y v) + \partial_y(-\partial_x u)$$
$$= 0$$

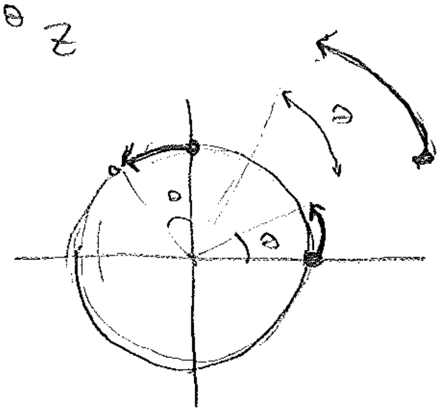
↳ SO: ANALYTIC FUNCTIONS ARE A SHORTCUT FOR 2D ELECTROSTATICS, FLUID FLOW, ...

$\mathbb{C}$ functions as maps from $\mathbb{C} \to \mathbb{C}$; see Byron & Fuller

A PRELUDE TO CONFORMAL MAPPING

$f(z)$ takes a $\underbrace{\mathbb{C} \text{ number}}_{\text{each element of } \mathbb{C}}$, gives a $\underbrace{\mathbb{C} \text{ number}}_{\text{an element of } \mathbb{C}}$

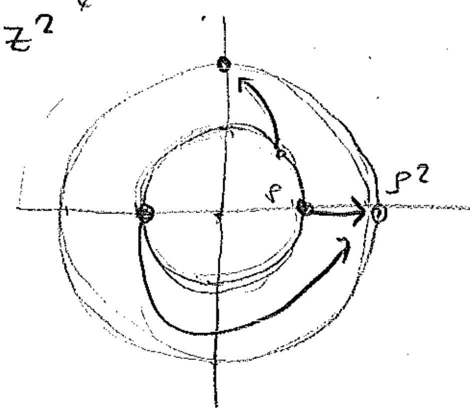
eg. $f(z) = e^{i\theta} z$



Rotates by θ in counter clockwise dir,

$\rho e^{i\theta} \to \rho^2 e^{2i\theta}$

eg. $f(z) = z^2$

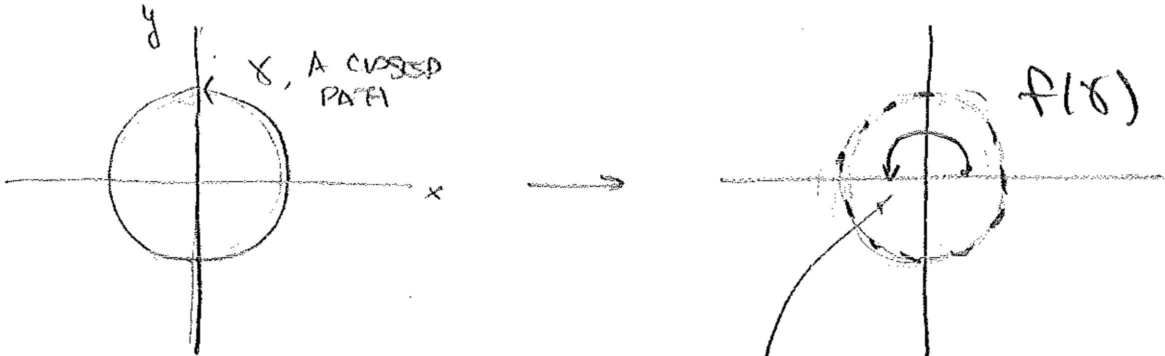


SQUARE MODULUS, DOUBLE ANGLE

eg $f(z) = z + z_0$ $\quad a + ib$

just a shift

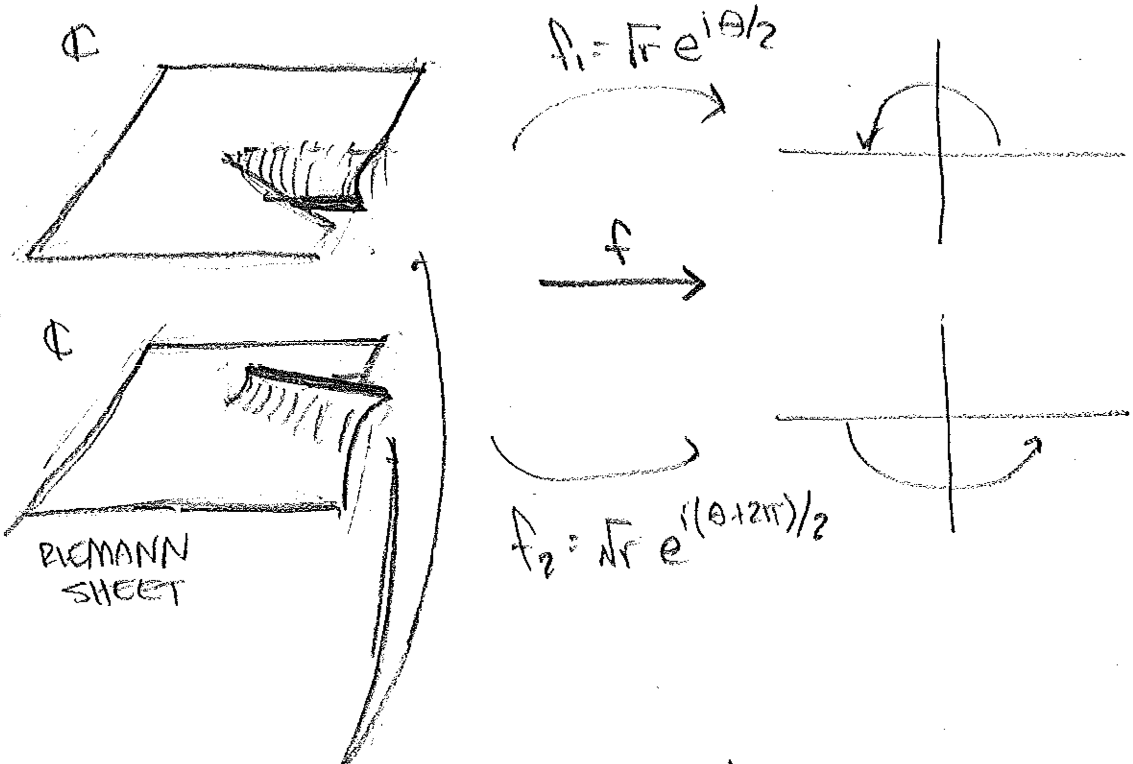
WHAT ABOUT $f(z) = \sqrt{z}$?



not a closed path?

HOW DO WE GET TO LOWER HALF PLANE IN IMAGE?
NEED θ from $2\pi \to 4\pi$

multivalued
need 2 copies of domain



WHERE WE GLUE THE 2 SHEETS TOGETHER IS CALLED A

BRANCH CUT

IS $f(z) = \sqrt{z}$ ANALYTIC ?
... WHERE? ↑ need to specify where,

... everywhere. $\mathbb{R}^+$ axis is not special

COULD HAVE PUT BRANCH CUT ANYWHERE!
↑ OUR CHOICE!

BRANCH CUT: GLOBAL PROPERTY

WHY IT'S IMPORTANT (PRACTICAL):
DON'T LOOP AROUND A BRANCH CUT
... you kind of end up in a different place than you intended

↑ Meaning: CAREFUL w/ contour integrals!

LOGARITHM { log = ln

BYRON & FULLER:
log on $\mathbb{C}$
ln on $\mathbb{R}$

$$\log z = \ln r + i\theta$$

$$\log z_1 z_2 = \ln(r_1 r_2) + \boxed{i(\theta_1 + \theta_2)} \leftarrow v(x,y)$$

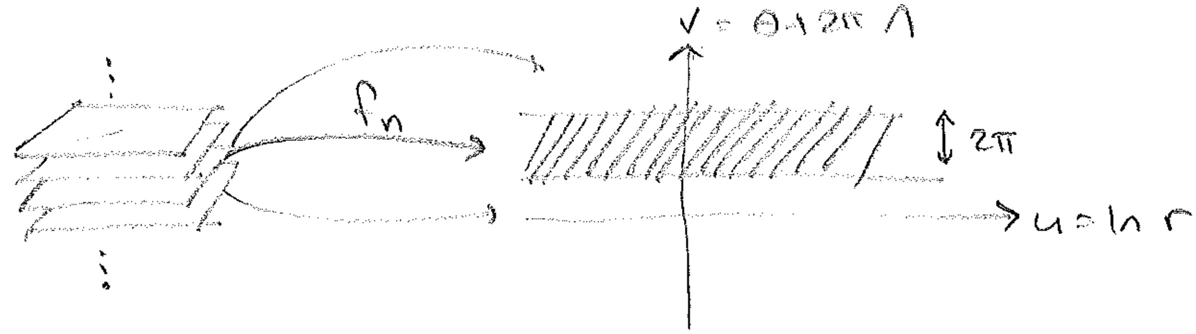
$$= \log z_1 + \log z_2$$

every time you go around, GO TO A NEW SHEET because θ just keeps on increasing

is log analytic? everywhere but $z=0$.

A "PIECEWISE" DEFINITION WRT SINGLE VALUED FUNCTIONS:

$$f_n(z) = \ln r + i\theta + 2\pi n i$$



BUT REALLY: the take-home message is

DO NOT CROSS A BRANCH CUT WHEN YOU INTEGRATE!

PREVIEW: $\mathbb{C}$ CONTOUR INTEGRALS WILL BE ALL ABOUT TAKING CLOSED LOOP PATHS IN $\mathbb{C}$ PLANE.

BUT IF INTEGRAND HAS A BRANCH CUT

you have some freedom to move it

... make sure you do not cross it!