typo: should be hw 6

Homework 5: **Fourier Review**

Course: Physics 231, *Methods of Theoretical Physics* (2017)
Instructor: Flip Tanedo (flip.tanedo@ucr.edu)

Due by: Friday, November 10

Reading: For this week, make sure you understand the example we did in Lecture 15:

$$\int_{-\infty}^{\infty} dx\, \frac{2\cos x}{x^2+1} \ .$$

This week we're pivoting back towards Green's functions now that we are armed with the power of the residue theorem. I will be assuming that you are familiar with **Fourier transforms**, if Problem 1 of this homework is not straightforward, please review them. Here are some references you may consider following:

- Boas (3rd ed): Chapters 7.7 (complex form of Fourier series), 7.12 (Fourier transforms), 8.12 (Green's functions)
- Appel: Chapters 4.6.c (Fourier integrals), 10 (Fourier transforms), 13 (Physical applications of the Fourier transform), 15 (Green's functions, especially 15.2.b)
- Matthews & Walker (2nd ed): Chapters 4.2 (Fourier transforms), 5.2 (Dispersion relations), 9 (Green's functions)
- Byron & Fuller: Chapters 5.7 (Fourier integrals; note that they use different conventions than we do—this is why Problem 1 is important for us), 6.6 (Dispersion relations), 7 (Green's functions)

We will have discussion section on Monday at 3:10pm in Chung 142. This is completely optional[1], but I welcome you to come to ask questions about the homework or the course at large.

1 A Fourier transform refresher

Fourier transforms are annoying because there are a few choices that one has to make to establish conventions. The convention that we will use is:

$$f(x) = \int \frac{dk}{2\pi} e^{-ikx}\, \widetilde{f}(k) \qquad\qquad \widetilde{f}(k) = \int dx\, e^{ikx}\, f(x) \ . \tag{1.1}$$

In this convention, the (2π) comes with the dk, so I will often write $\text{đ}k = dk/(2\pi)$.

1.1 Fourier decomposition of $\delta(x)$

What is the Fourier transform of $\delta(x)$? In other words, find $\widetilde{\delta}(k)$ in

$$\delta(x) = \int \text{đ}k\, e^{ikx}\, \widetilde{\delta}(k) \ . \tag{1.2}$$

What about $\delta(x-x_0)$?

[1] For the record, lectures are also optional.

1.2 Other conventions

A general convention for the Fourier transform is:

$$f(x) = |B|^{1/2} \int \frac{dk}{\sqrt{(2\pi)^{1+A}}} \, e^{-iBkx} \widetilde{f}(k) \qquad \widetilde{f}(k) = |B|^{1/2} \int \frac{dx}{\sqrt{(2\pi)^{1-A}}} \, e^{iBkx} f(x) \; . \tag{1.3}$$

Our conventions correspond to $B = 1$ and $A = 1$. Show that in this general form, the inverse Fourier transform of a Fourier transform is simply the original function.

1.3 Higher dimensions

In two Euclidean dimensions, the Fourier transform is

$$f(x, y) = \int \frac{dx}{2\pi} \int \frac{dy}{2\pi} \, e^{-ik_x x} e^{-ik_y y} \widetilde{f}(k_x, k_y) \; . \tag{1.4}$$

This can be rewritten in terms of a position 2-vector[2] $\mathbf{x}$ and corresponding momentum 2-vector $\mathbf{k}$,

$$f(\mathbf{x}) = \int \frac{d^2\mathbf{x}}{(2\pi)^2} \, e^{-i\mathbf{k}\cdot\mathbf{y}} \widetilde{f}(\mathbf{k}) \; . \tag{1.5}$$

Observe that the exponential is built out of the rotationally-invariant scalar quantity, $\mathbf{k} \cdot \mathbf{y}$. When we deal with *partial* differential equations, we'll need to Fourier transform in multiple dimensions. Sometimes we'll have to Fourier transform in both time and space. It is conventional to choose signs so that we may write this as

$$f(x, t) = \int \frac{dx}{2\pi} \int \frac{dt}{2\pi} \, e^{+ikx} e^{-i\omega t} \widetilde{f}(k, \omega) \; . \tag{1.6}$$

Briefly comment why this is a good idea from two points of view:

1. The idea that we are expanding about a basis of traveling plane waves.
2. Lorentz invariance, in case we want our expressions to respect special relativity.

2 Integral representation of the step function

[Cahill, Problem 5.32] The step function is defined by $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$. Show that this is equivalent to

$$\Theta(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ixz}}{z - i\varepsilon} dz \; . \tag{2.1}$$

Hint: What does the $-i\epsilon$ mean? Compare this to the advanced and retarded Green's functions that we explored in Lecture 15.

[2]Mathematicians laugh at us when we say 'position vector.'

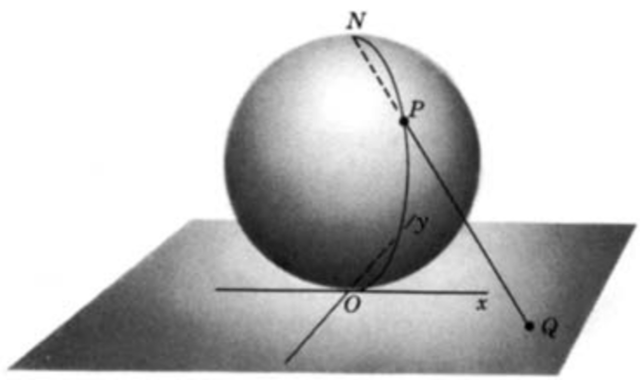
Extra Credit

These problems are not graded and are for your edification. You are strongly encouraged to explore and discuss these topics, especially if they are in a field of interest to you.

A The Riemann Sphere

[Appel Section 5.4] In class we introduced the idea of the **Riemann sphere** in the context of whether there could be 'edge effects' when converting an integral along $\mathbb{R}$ to a closed contour in the complex plane. The Riemann sphere is a completion of the complex plane that include the point at infinity. That is to say, $\infty = Re^{i\theta}$ as $R \to$ 'very big' and for any θ. Note that this means $i\infty \equiv \infty$.

The mapping is demonstrated below (image from Boas, *Mathematical Methods in the Physical Sciences*):



Imagine a sphere situated at the origin, O of the complex plane. The north pole of the sphere, N, is our reference point. Any point on the complex plane Q is mapped to a point P on the Riemann sphere given by the intersection of the line NQ with the sphere. Infinity is identified with N. Observe that an integral along the real line has an integral around the Riemann sphere. Also note that the notion of "inside" versus "outside" becomes somewhat slippery[3]. A useful convention is that 'inside' is to the left of the direction of the contour[4].

(a) Show that the residue of a function f at ∞ is

$$\text{Res}(f, \infty) = \text{Res}\left(-\frac{f(1/z)}{z^2}, 0\right), \tag{A.1}$$

Hint: we don't understand how to deal with a pole at ∞, so map that pole to a pole that's somewhere on the complex plane. Be sure to explain the sign.

[3]A physicist and a mathematician are asked to optimize the amount of land inside a fence of fixed length, L. The physicist deduces that enclosing a circular region with radius $R = L/2\pi$ optimizes the area enclosed. The mathematician sees a flaw in this logic and subsequently throws away most of the fence and encloses a circle of size $r \ll R$. The mathematician steps inside tiny region and says, "I declare myself to be on the outside."

[4]Note how *orientation* avoids the issue raised in the above footnote.

(b) Imagine a small curve C that loops once around a point P somewhere 'in the middle' in the Riemann sphere, as shown above. We know that $f(z) = 1/z$ is analytic in this region, so $\oint_C f(z)dz = 0$. However, if we reverse the orientation of the curve C and follow the 'inside is to the left' rule, this curve now encloses the pole at $z = 0$ which has $\text{Res}(f, 0) = 1$. If we're just taking $C \to -C$, we expect the integral to flip signs. How does this result make sense? HINT: What is the residue of $f(z) = 1/z$ at infinity?