ANNOUNCEMENTS

1. OFFICE HOUR / DISCUSSION: 3:10pm in my office / conf. room
2. NO CLASS ON WED
3. → LONG HW REVIEWING GREEN'S FUNCTIONS
4. NEXT WEEK: PROBABILITY + STATISTICS

GAUSSIAN INTEGRALS

something different. We will relate to DIFF EQ. + PROBABILITIES @ END.

REF: Zee, QFT in a Nutshell APPENDIX 1

$$G = \int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2}x^2} \quad \leftarrow \text{how to solve?}$$

TRICK:

$$G^2 = \int \underbrace{dx\,dy}_{r\,dr\,d\theta}\, e^{-\frac{1}{2}\underbrace{(x^2+y^2)}_{r^2}}$$

let $u = \frac{1}{2}r^2$
$du = r\,dr$

$$= \int_0^{\infty} du\, e^{-u} \,(2\pi)$$

$$= (-e^{-\infty} + 1)(2\pi) = 2\pi \Rightarrow \boxed{G = \sqrt{2\pi}}$$

SIMILARLY: SUPPOSE $x \equiv \sqrt{a}y$

$$G = \int_{-\infty}^{\infty} \sqrt{a}\,dy\, e^{-\frac{1}{2}ay^2} = \sqrt{2\pi}$$

$$\rightarrow \int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}ay^2} = \boxed{\sqrt{\frac{2\pi}{a}}}$$

could guess $a^{-1/2}$ from dimensional analysis

MORE VARIANTS

$$\int_{-\infty}^{\infty} dx \; e^{\underbrace{-\frac{1}{2}ax^2 + Jx}}$$

COMPLETE THE SQUARE

$$= -\left(\tfrac{a}{2}\right)\left(x^2 - 2Jx/a\right) = -\left(\tfrac{a}{2}\right)\underbrace{\left(x - \tfrac{J}{a}\right)}_{y = x - J/a}{}^2 + \frac{J^2}{2a}$$

$$= \underbrace{\int_{-\infty}^{\infty} dy \; e^{-\frac{1}{2}ay^2}} \; \underbrace{e^{J^2/2a}}_{\text{const}}$$

$$= \boxed{\sqrt{\frac{2\pi}{a}} \; e^{J^2/2a}}$$

IN N DIMENSIONS : $N \times N$ SYMMETRIC MATRIX A_{ij}

invertible

$$\int_{-\infty}^{\infty} dx_1\, dx_2 \cdots dx_N \; e^{\underbrace{-\frac{1}{2}\underline{x}\cdot\underline{A}\cdot\underline{x}}_{x_i A^i{}_j x^j} + \underbrace{\underline{J}\cdot\underline{x}}_{J_i x^i}}$$

LET $\underline{A}$ BE DIAGONALIZED BY AN ORTHOGONAL TRANSFORMATION , $\underline{R}$.

$$\underline{A} = \underline{R}^{-1} \cdot \underbrace{\hat{\underline{A}}}_{\text{DIAGONAL}} \cdot \underline{R} \qquad \hat{A} = \begin{pmatrix} \hat{a}_1 & & \\ & \hat{a}_2 & \\ & & \hat{a}_3 \; \ddots \end{pmatrix}$$

THEN CHANGE VARIABLES : $\underline{y} \equiv \underline{R}\cdot\underline{x}$

MEASURE UNCHANGED

$$\prod_i dx_i = \prod_i dy_i$$

$$= \int_{-\infty}^{\infty} dy_1 \cdots dy_N \; e^{\underbrace{-\frac{1}{2}\underline{y}\cdot\hat{\underline{A}}\cdot\underline{y}}_{\hat{a}_{11} y_1^2 + \hat{a}_{22} y_2^2 + \cdots} + \underbrace{\underline{J}\cdot(\underline{R}^{-1}\underline{y})}_{\underline{J}' \,\to\, J'_1 y_1 + J'_2 y_2 + \cdots}}$$

$$= \left(\int dy_1\, e^{-\frac{1}{2}\hat{a}_{11} y_1^2 + J'_1 y_1}\right)\left(\int dy_2 \cdots\right)\cdots$$

$$= \prod_i^N \sqrt{\frac{2\pi}{\hat{a}_i}}\; e^{(J'_i)^2/2\hat{a}_i}$$

$$\sum_i (J'_i)^2 = \underline{J}' \cdot \underline{J}'$$

I'LL BE EXPLICIT W/ T FOR THIS LINE

$$= \underline{J}^T \cdot \underline{R}^T \cdot \underline{R}\, \underline{J}$$

$$= \underline{J}^T \cdot \underline{J}$$

note: $J^T \to J$

$$\prod_i^N \hat{a}_i = \det \hat{A} = \det A$$

FURTHER :

$$\sum_i (J'_i) \frac{1}{\hat{a}_i} = \underline{J}^T \cdot \underline{R}^T (\underline{\hat{A}})^{-1} \underline{R}\, J$$

$$\underline{A}^{-1} = \underline{R}^{-1} \cdot \underline{\hat{A}}^{-1} \cdot \underline{R}$$

$$\sum_i (J'_i) \frac{1}{\hat{a}_i} = \boxed{\underline{J} \cdot \underline{A}^{-1} \cdot \underline{J}}$$

$$= \boxed{\sqrt{\frac{(2\pi)^N}{\det \underline{A}}}\; e^{\frac{1}{2} \underline{J} \cdot \underline{A}^{-1} \underline{J}}}$$

cute result. so what?

Gaussians show up all the time as distributions

WE OFTEN TAKE MOMENTS OR CORRELATION FUNCTIONS OF DISTRIBUTIONS.

eg. $\int_{-\infty}^{\infty} dx \; x \; e^{-\frac{1}{2}ax^2} = 0$ by symmetry

NEGATIVE PART CANCELS POSITIVE

eg $\int_{-\infty}^{\infty} dx \; x^2 \; e^{-\frac{1}{2}ax^2} = ?$

LIKE AN EXPECTATION OF x^2

OBSERVE: $= -2 \frac{d}{da} \boxed{\int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2}ax^2}}$

$= -2 \frac{d}{da} \sqrt{\frac{2\pi}{a}}$

$= \sqrt{2\pi} \; \boxed{a^{-3/2}}$

COULD GUESS FROM DIM. ANALYSIS

DEFINE EXPECTATION VALUE OF x^2 WRT DIST

$\langle x^2 \rangle = \frac{1}{Z} \left(-2 \frac{d}{da}\right) Z$ $\quad Z = \int dx \; e^{-\frac{1}{2}ax^2}$

$= \boxed{\frac{1}{a}}$

SIMILARLY

$\langle f(x^2) \rangle = \frac{1}{Z} \underbrace{f\left(-2 \frac{d}{da}\right)}_{\text{as taylor expansion}} Z$

when we take the INTEGRAL w/ A SOURCE:

$Z = \int dx \; e^{-\frac{1}{2}ax^2 + Jx}$ $\quad \sqrt{\frac{2\pi}{a}} \, e^{J^2/2a}$

$\langle x \rangle = \frac{1}{Z} \int dx \; x \; e^{-\frac{1}{2}ax^2 + Jx} = \frac{1}{Z} \frac{d}{dJ} Z = \frac{J}{a}$

J IS A SOURCE FOR x

LET ME NOW CHANGE VARIABLES FROM x TO q
CONSIDER OUR MOST COMPLICATED GAUSSIAN

$$Z = \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q \cdot A \cdot q + J \cdot q} = \sqrt{\frac{(2\pi)^N}{\det A}} \, e^{\frac{1}{2} J \cdot A^{-1} \cdot J}$$

some kind of distribution

bunch of objects like sequence of coupled springs that have heights distributed according to Z

Z encodes physics

THEN CAN ASK: $\langle q_i q_j \rangle = \frac{1}{Z} \int dq_1 \cdots dq_N \, (q_i q_j) \, e^{\cdots}$

What is the correlation btwn spring i & j?
IF SPRING i IS SQUISHED, IS SPRING j LIKELY TO BE

SQUISHED ALSO $(\langle q_i q_j \rangle > 0)$
PULLED INSTEAD $(\langle q_i q_j \rangle < 0)$
COMPLETELY INDEP $(\langle q_i q_j \rangle = 0)$

$$\langle q_i q_j \rangle = \frac{1}{Z} \frac{d}{dJ_i} \frac{d}{dJ_j} Z$$

$$= \frac{1}{Z} \sqrt{\frac{(2\pi)^N}{\det A}} \frac{d}{dJ_i} \frac{d}{dJ_j} \exp\left[\frac{1}{2} J_a A^{-1}_{ab} J_b\right]$$

$$\cdots + \frac{1}{2} J_i A^{-1}_{ij} J_j + \frac{1}{2} J_j A^{-1}_{ji} J_i + \cdots$$

(no sum over repeated)

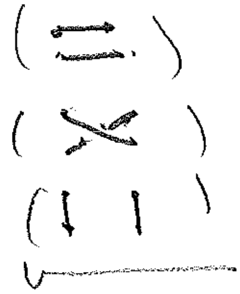
nb: $A^{-1}_{ij} = A^{-1}_{ji}$
BY SIM. OF A

$$= \underbrace{\left(\frac{1}{Z} \left(\frac{(2\pi)^N}{\det A} \right) e^{\frac{1}{2} J A^{-1} J} \right)}_{=1} \cdot \boxed{A^{-1}_{ij}}$$

SO (A^{-1}_{ij}) TELLS YOU HOW INFORMATION AT q_i PROPAGATES TO q_j.

YOU CAN ALSO CALCULATE $\begin{pmatrix} i & j \\ k & l \end{pmatrix}$

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{il} A^{-1}_{jk} + A^{-1}_{ik} A^{-1}_{jl}$$



PROTO FEYNMAN DIAGRAMS

4-point correlation

breaks into pairs of 2-point correl.

YOU MAY RECOGNIZE Z AS A PARTITION FUNCTION IN STATISTICAL MECHANICS.

$$Z = \sum_i e^{-\beta E_i}$$

PROMOTED TO AN INTEGRAL

EXACT SAME STRUCTURE CARRIES OVER TO QUANTUM MECHANICS

quantum randomness ~ thermal randomness

<u>CLAIM</u>: $Z = \int dq_1 dq_2 \cdots e^{iS(q)}$

eg $q_i = q(t_i)$

$$S = \int_0^T dt \sum m\dot{q} - V(q)$$

HOW TO DO THESE INTEGRALS:

$$I = \int dq\, e^{-\frac{1}{\hbar} f(q)}$$

$\hbar$ SMALL PARAM

$$f(q) = \underset{\substack{\uparrow \\ \text{MINIMUM}}}{f(a)} + \frac{1}{2} f''(a)(q-a)^2 + \dots$$

$$= \int dq\, e^{-\frac{1}{\hbar} f(a)}\, e^{-\frac{1}{\hbar}\frac{1}{2} f''(a)(q-a)^2}\, e^{\cdots}$$

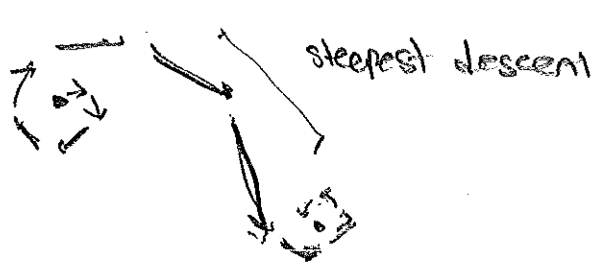
$$= e^{-\frac{1}{\hbar} f(a)} \sqrt{\frac{2\pi\hbar}{f''(a)}}\, e^{-O(\hbar^{1/2})}$$

for $\vec{q}$

$$= \underbrace{e^{-\frac{1}{\hbar} f(\vec{a})} \sqrt{\frac{(2\pi\hbar)^N}{\det f''(\vec{a})}}}_{\text{can do quadratic part.}}\, e^{-O(\hbar^{1/2})}$$

WAIT: WHAT ABOUT i?! $Z = \int \prod dq_i\, e^{iS}$

PHASOR: STILL DOMINATED BY STEEPEST DESCENT.



RELATION OF $\mathbb{R}$ & $\mathbb{IM}$ IS USED OFTEN

↓

Analytic continuation

EXTREMUM OF S → EULER-LAGRANGE EQ.

↳ WHERE ALL OF OUR GREEN'S FUNCTIONS COME FROM.

SERIES OF SPRINGS (like a bed mattress)

$$L = \frac{1}{2}\sum_i m\dot{q}_i^2 - \sum_{ij} K_{ij}\, q_i q_j$$

$[q_i(t) - q_i(t-\varepsilon)]^2$

usually only nearest neighbors to good approx (COUPLED H.O.)

$$\frac{1}{2} K_{ij} (q_i - q_j)^2$$

$$\downarrow$$

$$\frac{1}{2} K \left(q(t,x) - q(t,x-\varepsilon)\right)^2$$

$$L = \int dx \; \frac{1}{2}\dot{q}(t,x)^2 - \frac{1}{2} q'(t,x)^2$$

UP TO NORMALIZ

$$S = \int dx\,dt \; \frac{1}{2}\left[\underbrace{(\partial_t q)^2 - (\partial_x q)^2}_{(\partial q)^2}\right]$$

WHERE THE WAVE EQ COMES FROM IT'S ALL SHO.

VARIATIONAL PRINCIPLE GIVES US KG EQ: $\boxed{\partial^2 q = 0}$

WHEN THERE IS A SOURCE

$$Z = e^{iS + Jq}$$

← up to factors of i

$$\downarrow$$

$$\boxed{\partial^2 q = J}$$

We've sketched a functional approach to this problem.