LAST TIME: $(d/dx)^n$ AS A MATRIX

THIS TIME: Green's Functions

recall: finite-dimensional analogy

$$\underbrace{\underbrace{A}_{\text{DYNAMICS}}\;\underbrace{|\psi\rangle}_{\text{state}}}_{\text{EFFECT}} = \underbrace{|s\rangle}_{\text{SOURCE} \,\sim\, \text{CAUSE}}$$

EFFECT ⋮ derivatives on state

WANT: $|\psi\rangle = A^{-1}|s\rangle$

Dynamics is usually invertible

Green's function $\longleftrightarrow$ A^{-1} for FUNCTION SPACE

in this sense, not really a function...

OBSERVE: A^{-1} is also a linear transformation

so if $|s\rangle = |s_1\rangle + |s_2\rangle$

then $|\psi\rangle = A^{-1}|s_1\rangle + A^{-1}|s_2\rangle$

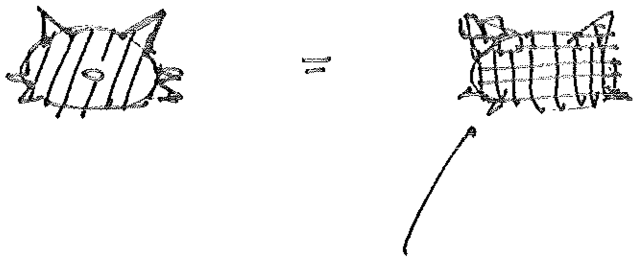
$\big\downarrow$ generalize

$= \sum_i A^{-1}|s_{(i)}\rangle$

BREAK UP SOURCE INTO LITTLE LEGO BUILDING BLOCKS.

Physics

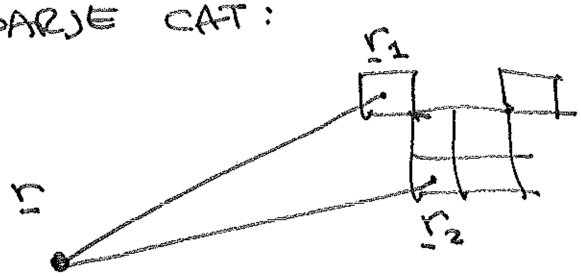
CHARGED CAT IS AN ELECTROSTATIC SOURCE



EACH UNIT BLOCK OF CHARGE
$q = \rho_0 \, d^3x$ or $\rho_0 \Delta x^3$
SOURCES AN ELECTROSTATIC POTENTIAL

$$\Delta \Phi = \frac{-\alpha q}{|\underline{r}-\underline{r}_i|} \qquad \text{(in some units)}$$

eg COARSE CAT:



PRINCIPLE OF SUPERPOSITION
$\updownarrow$
LINEARITY

$$\Phi = -\alpha q \left(\frac{1}{|\underline{r}-\underline{r}_1|} + \frac{1}{|\underline{r}-\underline{r}_2|} + \cdots \right)$$

OR IF THE CHARGE DENSITY VARIES OVER THE CAT,

$$= -\alpha \left(\frac{\rho(\underline{r}_1)}{|\underline{r}-\underline{r}_1|} + \cdots \right) \Delta x^3$$

clearly this is promoted to the usual integral when $\Delta x \to dx$

IMPORTANT: EACH $\underbrace{\frac{\rho(\underline{r}_i)}{|\underline{r}-\underline{r}_i|}(dx)^3}$

this is "A^{-1}" $|\square\rangle$

Green's function

unit source

In the continuum, the unit source lego block becomes a Dirac δ-function distribution

$$"A^{-1}"\,|\square\rangle \longrightarrow G\,\delta^{(n)}(\underline{x}-\underline{x}_i)$$

n-dimensional space (e.g. $\mathbb{R}^n$)

IF Θ IS THE DIFFERENTIAL OPERATOR $(A \to \Theta)$ THEN G IS DEFINED BY

$$\boxed{\Theta G = \delta^{(n)}}$$

$$A\,A^{-1} \sim \mathbb{1}$$

LET'S BE MORE PRECISE: THERE ARE ALWAYS 2 POSITIONS IN THIS PROBLEM:

① OBSERVATION POINT, $\underline{x}$ (what is the state @ $\underline{x}$?)

② SOURCE POINT, $\underline{x}'$, INTEGRATED OVER

(what is source @ x'?)

$$\Theta_x\,G(\underline{x},\underline{x}') = \delta^{(n)}(x-x')$$

derivatives on obs point

A HEURISTIC IDEA:

eg $\mathcal{O}_x = \left(\frac{d}{dx}\right)^2 + m^2$

$$\mathcal{O}_x \, \psi(x) = s(x)$$

$$\psi(x) = \mathcal{O}_x^{-1} s(x)$$

$$= \int \underbrace{\mathcal{O}_x^{-1} \delta(x-y)} s(y)\, dy$$

$$= \int \underbrace{G(x,y)} s(y)\, dy$$

"PROPAGATOR"

often
$G(x,y) = G(x-y)$
(transl. inv.)

the green's function propagates information from each infinitesimal part of the source & PROPAGATES that information to the obs. point of the state.

it integrates over the local microphysics, like dominoes.

EXPLICIT EXAMPLE: Poisson Eq. → charged cat

in COULOMB GAUGE: $\underline{\nabla} \cdot \underline{A} = 0$,

$$\underbrace{-\Delta \phi} = \sum_{i=1}^{3} \partial_i^2 \phi = \rho$$

$\partial/\partial x^i$

so want $G = (-\Delta)^{-1}$

BECAUSE THEN

$$\phi(x) = \int G(x-y) \rho(y)\, dy$$

A COMMON METHOD TO FIND G (we'll review next wk)

$$G(r) = \int e^{i\underline{k}\cdot\underline{r}}\, \tilde{G}(\underline{k})\, d^3k$$

FOURIER TRANSFORM : $\langle \tilde{e}^k | G \rangle$.

$|\tilde{e}_k\rangle$ this is just change of basis

nb we are fourier transforming $\underbrace{(\underline{x}-\underline{y})}_{\underline{r}}$, not $\underline{x}$.

FACT : $\delta^{(3)}(\underline{r}) = \int d^3k\, e^{i\underline{k}\cdot\underline{r}}$

$\frac{d^3k}{(2\pi)^3}$

THEN : GO BACK TO $(-\Delta)G(\underline{r}) = \delta^{(3)}(\underline{r})$

$$\underbrace{-\Delta \int d^3k\, e^{i\underline{k}\cdot\underline{r}}\, \tilde{G}(k)} = \int d^3k\, e^{i\underline{k}\cdot\underline{r}}$$

$$= \int d^3k\ \underline{k}^2\, e^{i\underline{k}\cdot\underline{r}}\, \tilde{G}(k)$$

BY PROJECTION (OR INSPECTION) :

$$\tilde{G}(k) = 1/|\underline{k}|^2 \quad \Big\}$$

DIFFERENTIAL OP TURNED INTO ALGEBRAIC PROBLEM

WHY? $e^{i\underline{k}\cdot\underline{r}}$ IS AN EIGENFUNCTION OF $(-\Delta)$

$\downarrow$

EIGENFUNCTION PROBLEM.

finally: $G(\underline{r}) = \int d^3k \, \frac{1}{k^2} \, e^{i\underline{k}\cdot\underline{r}}$

CAN DO CONTOUR INTEGRAL

DIMENSIONAL ANALYSIS

$e^{i\underline{k}\cdot\underline{r}} \Rightarrow [k] = [r]^{-1}$

so $\left[\frac{d^3k}{k^2}\right] \sim \boxed{\frac{1}{r}}$

& indeed, $G(\underline{r}) = \frac{1}{4\pi r}$

or: $\boxed{G(\underline{x}, \underline{y}) = \frac{1}{4\pi |\underline{x} - \underline{y}|}}$

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so the game is:

given $\mathcal{O}$, want G

s.t. $\mathcal{O} G = \delta$

we saw that an EIGENFUNCTION decomposition of $\mathcal{O}$ is helpful.

~~THERE~~ ↳ this is why there are so many SPECIAL FUNCTIONS in physics (b/c many $\mathcal{O}$'s ...)

WE WILL BE MORE SYSTEMATIC NEXT WEEK, BUT I WANT TO FOCUS ON PLAYING W/ THE IDEAS.

RECALL (LECTURE 5, on MONDAY)

$$\mathbb{1} = \sum_i |e^i\rangle\langle e_i| = \sum_i |e^i\rangle\langle e_i|$$

in function space ← for $D = [0,1]$ w/ DIRICHLET B/C

$$\sum_{n=1}^{\infty} \left(\sqrt{2}\sin(n\pi x)\right)\left(\sqrt{2}\sin(n\pi y)\right) = \delta(x-y)$$

or, in other words:

$$\boxed{\underbrace{\sum_i}\ \underbrace{w(y)}\ \underline{e_i^*(y)\, e_i(x) = \delta(x-y)}}$$

WEIGHT:
$$\langle e^i | e_j\rangle = \int_D dx\, w(x)\, e_i^*(x)\, e_j(x) = \delta^i_j$$

hey, this thing has a δ on the right! I wonder if I can hack this relation to finagle a Green's function?

$$\underset{\uparrow\ \psi^i|e_i\rangle}{\mathcal{O}\psi} = \underset{\uparrow\ s^j|e_j\rangle}{s} \qquad ; \ \mathcal{O}|e_i\rangle = \lambda_i|e_i\rangle \quad \text{(no sum)}$$

$$\Rightarrow \sum_i \lambda_i \psi^i |e_i\rangle = s^j|e_j\rangle \Rightarrow \psi^i = \frac{s^i}{\lambda_i} = \frac{\langle e_i|s\rangle}{\lambda_i}$$

$$\psi = \sum_i \frac{\langle e_i|s\rangle}{\lambda_i}|e_i\rangle = \sum_i \frac{e_i(x)}{\lambda_i}\int w(y)\, e_i^*(y)\, s(y)\, dy$$

$$= \int \underbrace{\sum_i \frac{e_i(x)\, e_i^*(y)}{\lambda_i}}_{\boxed{G(x,y)}} w(y)\, s(y)\, dy$$

ANOTHER REP OF THE GREEN'S FUNCTION: (for $-\Delta$)

$$G(\underline{r},\underline{r}') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \underline{Y_{\ell,m}(\theta,\phi) Y^*_{\ell,m}(\theta',\phi')} \frac{r_<^{\ell}}{r_>^{\ell+1}}$$

PARTIAL WAVES (orbital angular momentum)

momentum m z-DIR

$r_> = \max(r, r')$
$r_< = \min(r, r')$

$= \frac{1}{4\pi|\underline{r}-\underline{r}'|}$

you can see the $e_i^*(\underline{r}') \; e_j(\underline{r})$ structure.

"ket"

~~ke~~ "BRA"
ACTS ON <u>SOURCE</u>

So that for some source $\rho(x)$, THE POTENTIAL IS

$$\Phi = \underline{\sum_\ell \sum_m} \int d^3x' \frac{r_<^\ell}{r_>^{\ell+1}} \underbrace{\left(\tfrac{1}{2\ell+1}\right) Y_{\ell m}(\theta,\phi) Y^*_{\ell m}(\theta',\phi')} \rho(\underline{x}')$$

SPECIAL NAME:
LEGENDRE POLYNOMIALS
$P_\ell(\underbrace{\hat{r}\cdot\hat{r}'}_{\text{unitvecs.}})$

WE ARE OFTEN INTERESTED IN $r > r'$ (obs far from source)
$\hookrightarrow r_> = r \quad r_< = r'$

$$\phi = \sum_\ell \frac{1}{2\ell+1} \sum_m \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} \underbrace{\int d^3r' \, r'^\ell \, Y^*_{\ell,m}(\theta',\phi') \rho(\underline{x}')}_{}$$

PROPERTY OF SOURCE DISTRIBUTION:
MULTIPOLE MOMENTS

$$\boxed{Q_\ell^m}$$