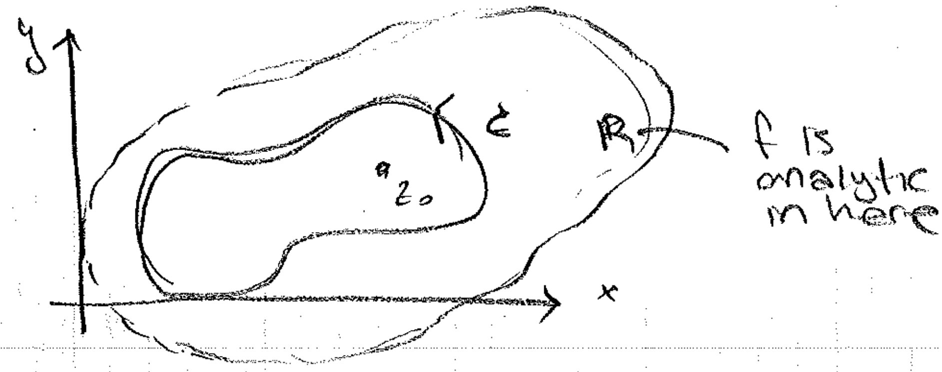
LAST TIME: INTEGRAL FORMULA – A TRICK TO DESCRIBE THE FUNCTION @ A POINT W/ AN INTEGRAL OF THE FUNCTION AROUND THE POINT.

$$f(z_0) = \frac{1}{2\pi i} \oint_C \underbrace{\frac{f(z)}{z-z_0}}_{} dz$$

ANALYTIC IN R

not nec. a small loop!

MEROMORPHIC (POLE @ $z = z_0$)



WEIRD FORMULA. KIND OF NEAT THAT IT HOLDS FOR ANY C IN THE REGION R

"information lives on the boundary"

BUT WHAT IS IT GOOD FOR?!

IF I CAN DO THE INTEGRAL ON THE RHS, THEN I ALREADY KNOW WHAT $f(z_0)$ IS!

what we want: a way to do integrals

↓

WE WANT THE RHS

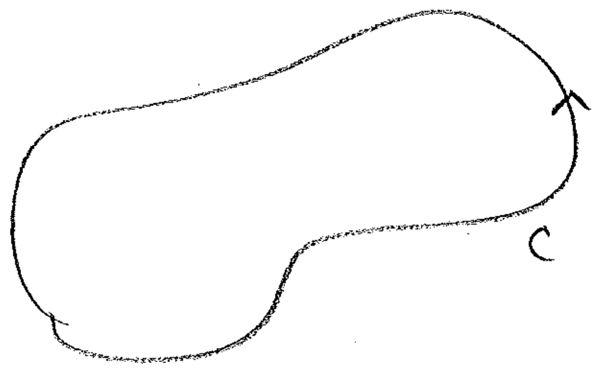
$$f(z_0) = \frac{1}{2\pi i} \boxed{\oint_C g(z)\, dz}$$

$$g(z) \equiv \frac{f(z)}{z-z_0}$$

given $g(z)$, just need $f(z)$ s.t. $g(z) = \left(f(z)/z-z_0\right)$!

ANALYTIC

IF SUCH AN f EXISTS, THEN SUPPOSE WE WANT TO PERFORM A CLOSED CONTOUR INTEGRAL OF $g(z)$



$\oint_C g(z) = ?$

WE KNOW THAT IF $g(z)$ IS ANALYTIC IN THIS REGION, THEN THE INTEGRAL IS ZERO

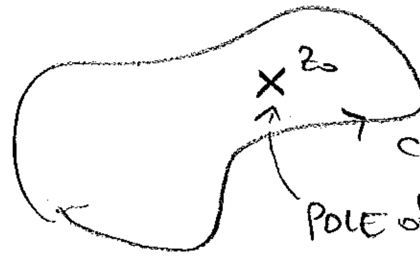
BUT WE HAVE ASSUMED THAT IT IS NOT ANALYTIC IN THE WHOLE REGION!

$\exists$ A SIMPLE POLE @ z_0

$\sim 1/z$ singularity

BECAUSE WE ASSUMED WE COULD WRITE $g(z) = f(z)/z-z_0$

nb: counterexample:
$f(z) \propto (z-z_0)$, then no pole.
BUT THEN $f(z_0)=0$.
& $\oint g(z)\,dz = 0$
b/c $g(z)$ is analytic.



POLE of $g(z)$

SO:

$$\oint_C g(z) = 2\pi i \, f(z_0) = \boxed{2\pi i \, \mathrm{Res}_g(z_0)}$$

WHAT IS $f(z_0)$? IT'S THE VALUE OF $(z-z_0)g(z_0) = f(z_0)$

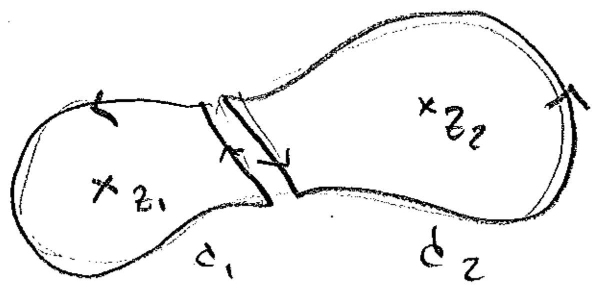
the "not diverging" factor.

"undoes" singularity

RESIDUE

IF YOU HAVE $g(z) = \dfrac{h(z)}{(z-z_1)(z-z_2)}$ $\quad z_1 \neq z_2$

THEN:



THE CONTOUR INTEGRAL OVER $C = C_1 + C_2$ IS THE SUM OF THE CONTOUR INTEGRALS OF EACH

$$\oint_C g(z)\,dz = \underbrace{\oint_{C_1} g(z)\,dz}_{2\pi i \, Res_g(z_1)} + \oint_{C_2} g(z)\,dz$$

$$2\pi i \, Res_g(z_1) + 2\pi i \, Res_g(z_2)$$

$$= 2\pi i \sum_j Res_g(z_j)$$

SUM over poles enclosed

(FOR SIMPLE POLES)

nb: $Res_g(z_1) = \dfrac{h(z_1)}{(z_1 - z_2)}$

RESIDUE THEOREM:

$$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{\substack{\text{POLES} \\ \text{enc.}}} Res_f(z_j)}$$

RESIDUES

LAST TIME: MEROMORPHIC FUNCTIONS CAN BE LAURENT EXPANDED ABOUT A POINT.

$$\sum_n a_n (z-z_0)^n$$

where n may be negative

GENERALIZES TAYLOR EXPANSION TO INCLUDE POINT SINGULARITIES.

SIMPLE POLES: $(z-z_0)^{-1}$ ← singularity goes like $1/\varepsilon$

the RESIDUE of a function f at a point z_0 is

$$\mathrm{Res}_f(z_0) = \underbrace{a_{-1}}_{\text{in LAURENT EXPANSION}}$$

WHEN THE FUNCTION ONLY HAS A SIMPLE POLE, IT'S THAT EASY.

$$f(z) = \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

$$(z-z_0)f(z) = a_{-1} + a_0(z-z_0) + \cdots$$

$$\boxed{(z-z_0)f(z)\Big|_{z=z_0}} = a_{-1} = \mathrm{Res}_f(z_0)$$

JUST MULTIPLY BY ε, only $\frac{1}{\varepsilon}$ PIECE OF RHS SURVIVES.

examples

$$f(z) = \frac{1}{z(z-1)(z-i)}$$

- WHAT ARE THE POLES?

$z_i = 0, 1, i$ ← ALL SIMPLE

- WHAT ARE THE RESIDUES?

EXPAND ABOUT $z=0$

$$f(z_0) = \frac{1}{z_0}\frac{1}{(z_0-1)(z_0-i)} \quad a_{-1}$$

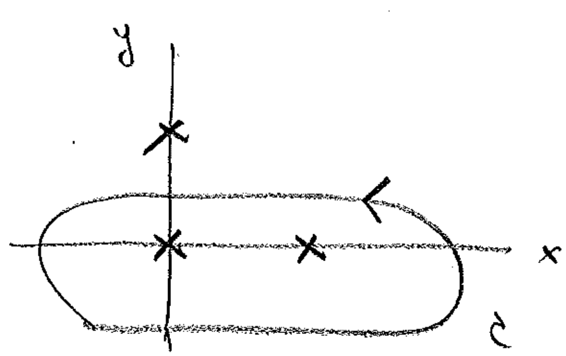
$$z_0 f(z_0) = \frac{1}{(z_0-1)(z_0-i)} = \boxed{\frac{1}{i} = Res_f(0)}$$

↑ for $z=0$

SIMILARLY

$$Res_f(1) = \frac{-1}{i} \qquad Res_f(i) = \frac{1}{i(i-1)}$$

- LET'S DO AN INTEGRAL!



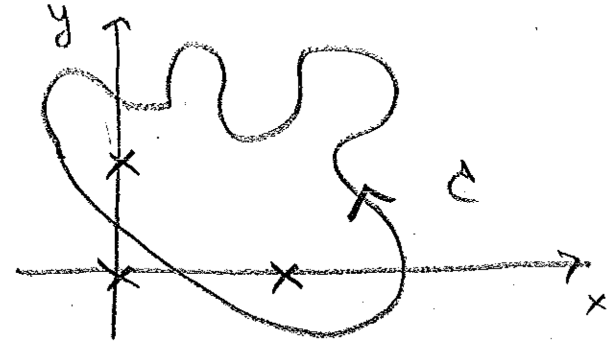
$$\oint_c \frac{dz}{z(z-1)(z-i)}$$

$$= 2\pi i\, Res_f(0) + 2\pi i\, Res_f(1)$$

$$= 2\pi i\left(\frac{1}{i} - \frac{1}{i}\right)$$

$$= 0$$

• LET'S DO ANOTHER!



$$\oint_C \frac{dz}{z(z-1)(z-i)} = 2\pi i \left(\text{Res}_f(1) + \text{Res}_f(i) \right)$$

$$= \boxed{2\pi i \left[\frac{-1}{i} + \frac{1}{i(i-1)} \right]}$$

nb: ORIENTATION MATTERS!

ANOTHER EXAMPLE

$$f(z) = \frac{z}{(2z+1)(5-z)}$$

• WHAT ARE THE POLES? $\boxed{z = -\frac{1}{2}}$, $\boxed{z = 5}$

• WHAT ARE THE RESIDUES?

$\boxed{z = -1/2}$

WRONG: $\frac{z}{\cancel{(2z+1)}(5-z)}\Big|_{z=-1/2} = \frac{z}{5-z}\Big|_{z=1/2}$ ✗

CORRECT: $\frac{(z+1/2)\,z}{(2z+1)(5-z)} = \frac{\cancel{(z+1/2)}}{\cancel{(2)(z+1/2)}} \frac{z}{2(5-z)}\Big|_{z=-1/2}$ ✓

$\boxed{z = 5}$

WRONG: $\frac{\cancel{5-z}}{\cancel{5-z}} \frac{z}{(2z+1)}\Big|_{z=5}$ ✗

CORRECT: $\frac{\cancel{(z-5)}}{\cancel{(z-5)}} \frac{z}{(-)(2z+1)}$ ✓

WHAT ABOUT NOT-SIMPLE POLES?

$$\sim \frac{1}{(z-z_0)}$$

eg: $f(z) = \dfrac{1}{(z-2i)^2}$

SECOND ORDER singularity @ $z = 2i$

RESIDUE = 0 ← no a_{-1} COEFFICIENT.

SO BLINDLY APPLYING RESIDUE THEOREM IMPLIES

$$\oint_c f(z) = 0$$

C CIRCLES

BUT DOES THIS MAKE SENSE? WE ONLY MOTIVATED THAT:

1. ANALYTIC PART HAS NO CONTRIBUTION
2. SIMPLE POLE CONTRIBUTES RESIDUE

SO MAYBE $(z-z_0)^{-2}$ POLE ALSO CONTRIBUTES?

SUBTLETY: OUR "PROOF" OF RESIDUE CONTRIBUTION:

$$\oint_c \frac{h(z)}{(z-z_0)}\, dz = \oint_{c'} \frac{h(z)}{z-z_0}\, dz$$

LITTLE CIRCLE

PARAMETERIZE THIS PATH:

$z = z_0 + \varepsilon e^{i\theta}$

$dz = i\varepsilon e^{i\theta}\, d\theta$

$$\oint_{C'} \frac{h(z)}{z-z_0}\,dz = \oint_{C'} \frac{h(z)}{\cancel{\varepsilon} e^{i\theta}} \left(i\cancel{\varepsilon} e^{i\theta}\, d\theta\right)$$

$$= \int_0^{2\pi} \underset{\approx h(z_0)}{h(z)} \underbrace{i\,d\theta}_{\text{gives } 2\pi i}$$

COMPARE TO:

$$\oint_{C'} \underbrace{h(z)}_{\text{ANALYTIC}} dz = \int_0^{2\pi} \underbrace{h(z_0)\ i\varepsilon}_{\text{CONST}} e^{i\theta}\, d\theta$$

$$= \text{const} \underbrace{\int_0^{2\pi} e^{i\theta}\, d\theta}_{=0\,!}$$

COMPARE TO:

I'M CHEATING HERE!

$$\oint_{C'} \frac{h(z)}{(z-z_0)^2} = \oint_{C'} \frac{h(z_0)}{(\varepsilon e^{i\theta})^2} \left(i\varepsilon e^{i\theta}\, d\theta\right)$$

FISHY STEP $\Longrightarrow$

$$= \int_0^{2\pi} h(z_0)\, \frac{i}{\varepsilon}\, e^{-i\theta}$$

$$= \text{const} \underbrace{\int_0^{2\pi} e^{-i\theta}\, d\theta}_{=0\,!}$$

<u>in general</u>: (<u>WRONG!</u>)

$$\oint_{C'} h(z)\,(z-z_0)^n = 0 \quad \text{for } n \neq -1$$

WHY WAS THIS WRONG??

ACTUALLY, THE PREVIOUS ARGUMENT IS WRONG! (THOUGH ILLUSTRATIVE)

what went wrong? the $\frac{1}{\epsilon}$ piece.

WHAT WE SHOULD DO:

$$\oint \frac{h(z)}{(z-z_0)^2}\, dz$$

well behaved ↳ TAYLOR EXP.

$$\frac{1}{(z-z_0)^2}\left(h(z_0) + \underbrace{h'(z_0)(z-z_0)} + \underbrace{\tfrac{1}{2}h''(z_0)(z-z_0)^2 + \dots}_{\text{integrand is analytic}}\right)$$

constant gives $1/\epsilon$ piece integrates to zero

$$\boxed{\oint \frac{h'(z)}{(z-z_0)}\, dz} = 2\pi i\, h'(z_0)$$

hey! this is <u>not</u> zero!
BY CAUCHY INTEGRAL FORMULA

SO FOR A SECOND ORDER POLE, THE RESIDUE IS DEFINED TO BE $h'(z_0)$

↳ the "multiply $f(z_0)$ BY $(z-z_0)$ & evaluate @ z_0" ALGORITHM FAILS.

YOU WILL GENERALIZE IN HW.