these notes overlap w LEC 14.

NOTE ON HW

HOW DO YOU INTEGRATE A GREEN'S FUNCTION TWICE?

$$\int_{y-\varepsilon}^{y+\varepsilon} \boxed{\frac{d^2G}{dx} = \delta(x)} \, dx$$

$$\frac{dG^>(y+\varepsilon)}{d(\text{arg})} - \frac{dG^<(y-\varepsilon)}{d(\text{arg})} = 1$$

$x = y+\varepsilon$, $dx = d\varepsilon$ $\qquad$ $x = y-\varepsilon$, $dx = -d\varepsilon$

treat this as: $G^>(x<y) = G^>(x>y) = 0$

NOW INTEGRATE $\boxed{\text{WRT } \varepsilon}$ ← y IS FIXED & IS A PARAMETER

$$\int_{-\delta}^{\delta} \frac{dG^>(y+\varepsilon)}{d(y+\varepsilon)} d\varepsilon - \int_{-\delta}^{\delta} \frac{dG^<(\overbrace{y-\varepsilon}^{u})}{d(y-\varepsilon)} d\varepsilon = \delta$$

$\downarrow \delta \to 0$ $\qquad$ $\delta \to 0$

$$G^>(y) - \int_{y+\delta}^{y-\delta} \frac{dG^<(u)}{-du}(-du) = 0$$

$$-\left[G^<(y-\delta) - \cancel{G^>(y+\delta)}\right]$$

$$0$$

$$\boxed{G^>(y) - G^<(y) = 0}$$

so G IS CONTINUOUS

G' IS DISCONTINUOUS

G'' HAS A δ

HARMONIC OSCILLATOR

$$\left[\left(\frac{d}{dt}\right)^2 + \underset{\substack{\uparrow \\ \text{SPRING CONST.} \\ \text{(res. freq)}}}{\omega^2}\right] \underbrace{q(t)}_{\substack{\text{POSITION OF THING} \\ \text{THAT IS OSCILLATING}}} = \underbrace{S(t)}_{\substack{\text{SOURCE} \\ \text{(DRIVING FORCE)}}}$$

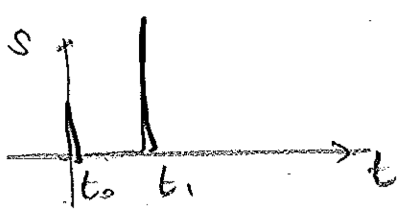
GREEN'S FUNCTION: $G(t, t_0)$ GIVES RESPONSE ($q(t)$) TO A UNIT DRIVING FORCE AT TIME t_0 (WE'LL TAKE $= 0$)

LIKE PLUCKING A GUITAR STRING.

so the Green's function eq. is

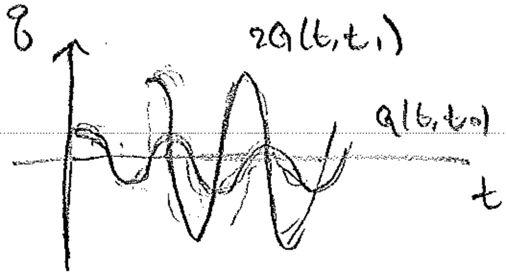
$$\boxed{G''(t) + \omega^2 G(t) = \delta(t)}$$

Why? SUPPOSE WE PLUCK GUITAR STRING TWICE, SECOND TIME IS TWICE AS HARD.



then response is a superposition

$$q(t) = G(t, t_0) + 2G(t, t_1)$$



FOURIER: GIVEN "POSITION SPACE" $f(t)$, DEFINE MOMENTUM SPACE BY

$$\boxed{f(t) = \int \frac{dk}{2\pi}\, e^{-ikt}\, \tilde{f}(k)}$$

HW6 { CHOICE: NEED TO SPLIT A $1/2\pi$; NEED TO CHOOSE SIGN (& "units")

just be consistent.

w/ these choices

$$\tilde{f}(k) = \int dx\, e^{+ikt} f(t)$$

PLUGGING IN FOURIER EXP.

$$\left(\frac{d}{dt}\right)^2 G(t) = \left(\frac{d}{dt}\right)^2 \int dk\, e^{-ikt}\, \tilde{G}(k)$$

$$= \int dk\, (-k^2)\, e^{-ikt}\, \tilde{G}(k)$$

$$\left[\left(\frac{d}{dt}\right)^2 + \omega^2\right] G(t) = \int dk\, e^{-ikt}\, \underbrace{(\omega^2 - k^2)\, \tilde{G}(k)}_{=1}$$

$$\delta(t) = \int dk\, e^{-ikt}$$

$$\Rightarrow \boxed{\tilde{G}(k) = \frac{1}{\omega^2 - k^2}}$$

done. up to integral back to t. var.

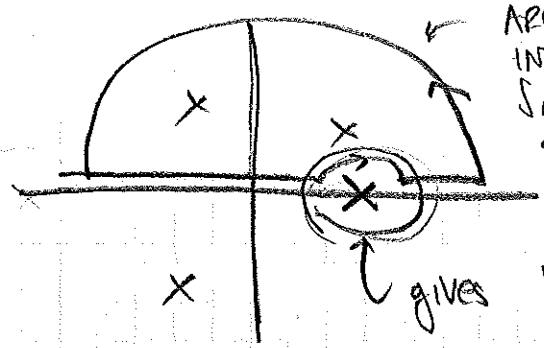
$$g(t) = \int_{-\infty}^{\infty} dk \; \frac{-e^{-ikt}}{k^2 - \omega^2}$$ ← k is the var. that's promoted to a $\mathbb{C}$ #.

$(k+\omega)(k-\omega)$: POLES @ $\pm\omega$

STOP: PHYSICAL SIGNIFICANCE IS CLEAR! RESONANCE { these are on the integration contour!!

LAST TIME : PRINCIPAL VALUE

$\dfrac{g(x)}{(x-x_0)}$

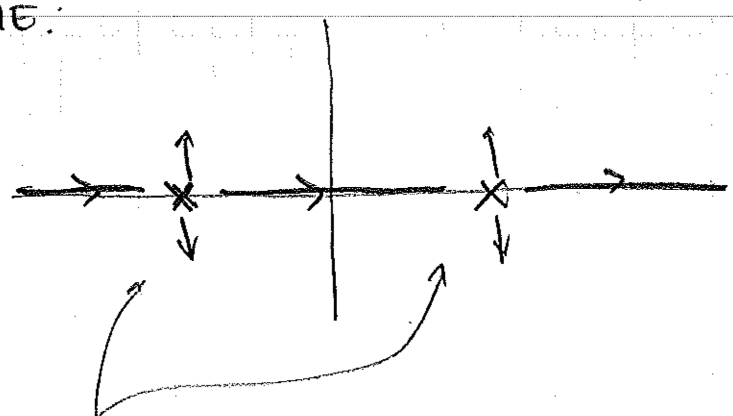


ARC CHOSEN BY INTEGRAND : WANT $\int_{arc} f(z)\,dz \to 0$ eg from e^{iz} factor

gives "half residue"

CAN GO OVER OR UNDER, GET SAME RESULT. – choice doesn't matter!

THIS TIME:



two choices : now it matters : relative to one pole, the other pole may or may not be in the contour!?

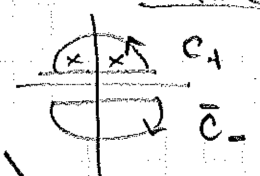
CHOICE MATTERS !

TRY: GO UNDER FOR BOTH POLES

nb: this is equivalent to moving pole an ε amount in the imaginary direction

- take $\varepsilon \to 0$ limit @ end
- secretly: same as saying "P" asterisk defined as principal val.

HOW TO CLOSE CONTOUR?



integrand: (PULLING OUT 2π)

$$f(z)\,dk = \frac{-e^{-ikt}}{\underbrace{k^2}_{R^2e^{2i\theta}} - \omega^2}\,\underbrace{dk}_{iRe^{i\theta}d\theta} \;\boxed{e^{Rt\sin\theta}}\;\overbrace{e^{-iRt\cos\theta}}^{\text{PHASE}}$$

WANT CONTOUR s.t. THIS GOES LIKE $e^{-\text{BIG}}$ ... DEPENDS ON THE SIGN OF t!

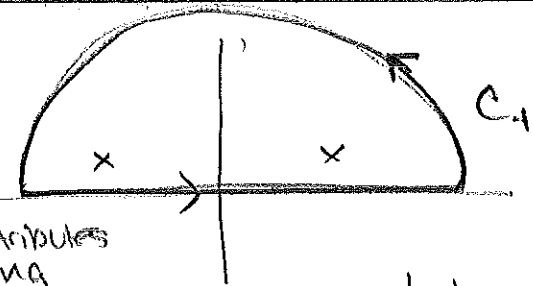
if $t>0$, need $\sin\theta<0 \to \bar{C}_-$

↳ BUT $\bar{C}_-$ ENCIRCLES NO POLES ?!

$$G^a(t>0) = \int_{\bar{C}_-} f(z)\,dk = 0$$

no 'forward propagation of information!!

WHAT ABOUT $t<0$?



$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, f(k)$$

$$= \frac{1}{2\pi}\oint dz\, f(z)$$ arc contributes nothing

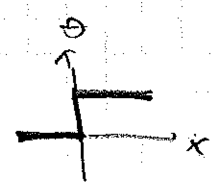
$$= \frac{1}{2\pi} \times 2\pi i \times \left(\mathrm{Res}_f\, \omega + \mathrm{Res}_f\, -\omega\right)$$ took $\varepsilon \to 0$ lim.

$$\frac{-e^{-i\omega t}}{2\omega} \qquad \frac{-e^{i\omega t}}{-2\omega}$$

$$= \frac{-1}{\omega}\underbrace{\boxed{\frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)}}_{\sin \omega t}$$

$$= \boxed{\frac{-1}{\omega}\sin \omega t} \quad \text{for } t<0$$

$\Theta(x)$



OR:

$$G^a(t) = \frac{-1}{\omega}\sin(\omega t)\,\Theta(-t)$$

wrong sign!

<u>ADVANCED</u> PROPAGATOR (Green's function)

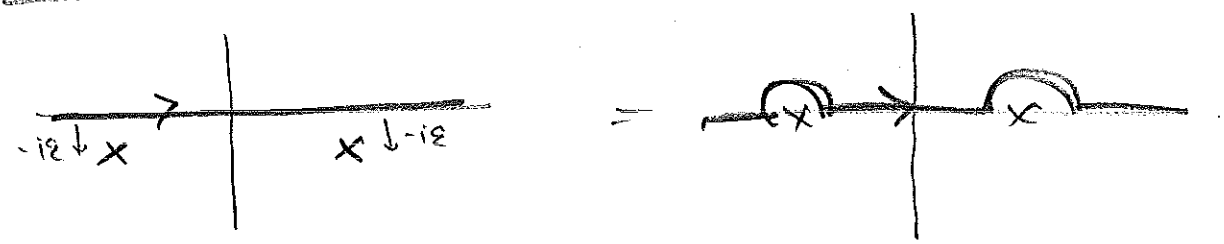
ACAUSAL. no good.

BUT THIS IS A SOLUTION MATHEMATICALLY

2> the dynamics are time reversal symmetric!

<u>INTERP</u>: $\exists$ SOME ∿∿ IN ASYMPTOTIC PAST THAT GETS CANCELED EXACTLY WHEN WE PLUCK STRING @ $t=0$.

LETS DO THE "RIGHT" THING:



PUSH BOTH POLES DOWN — equivalent to going over

CONVERGENCE STILL SET BY e^{-ikt}

$t>0 \longrightarrow \bar{C}_- \quad \leftarrow$ has poles

$t<0 \longrightarrow C_+ \quad \leftarrow$ no poles

$\Rightarrow \boxed{G^r(t<0) \equiv 0}$ ✓

first: $\oint_{\bar{C}_-} = -\oint_{C_-} \quad \leftarrow$ ORIENTATION OF CONTOUR

$$G^r(t>0) = -\frac{1}{2\pi}\oint_{C_-} f(z)\,dz$$

$$= 2\pi i \times \frac{-1}{2\pi}\left(\text{Res}_f\,\omega + \text{Res}_f\,-\omega\right)$$

sign difference w/r/t adv.

$$= \frac{1}{\omega}\sin\omega t$$

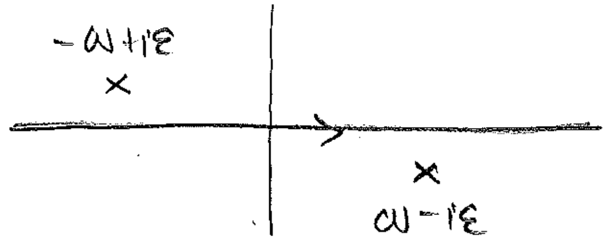
$$\boxed{G^r(t>0) = \frac{1}{\omega}\sin(\omega t)\,\Theta(t)}$$

retarded propagator

EFFECT COMES <u>AFTER</u> CAUSE.

<u>other choices?</u> eg. FEYNMAN PROPAGATOR

↳ you know it's important



one pole in C_+

one pole in C_-

$$G^F(t>0) = \frac{i}{2\omega} e^{-i\omega t}$$

$$G^F(t<0) = \underbrace{\frac{-i}{2\omega} e^{i\omega t}}_{\text{imaginary?}} \Leftarrow \text{related to } t>0 \text{ by } \boxed{\omega \to -\omega}$$

turns out: USEFUL IN RELATIVISTIC THEORIES

$t<0$ looks like a $\boxed{\text{NEGATIVE freq solution moving forward in time}}$

<u>DIRAC</u>: found negative energy states in theory of RELATIVISTIC ELECTRONS

ENERGY $\leftrightarrow$ FREQ.

so this was identified as

<u>POSITIVE ENERGY</u> <u>positron</u>

moving fwd in time.

two minus signs