LAST TIME: CONTOUR INTEGRALS

STRATEGY: GIVEN $\boxed{\int_{\mathbb{R}} f(x)\,dx}$ FIND

$$\underbrace{\oint_C f(z)\,dz}_{=2\pi i \sum_j \mathrm{Res}_f(z_j)} = \underbrace{\int_{\mathbb{R}} f(x)\,dx}_{\text{WHAT YOU WANT}} + \underbrace{\int_{C'} f(z)\,dz}$$

EASY TO CALCULATE

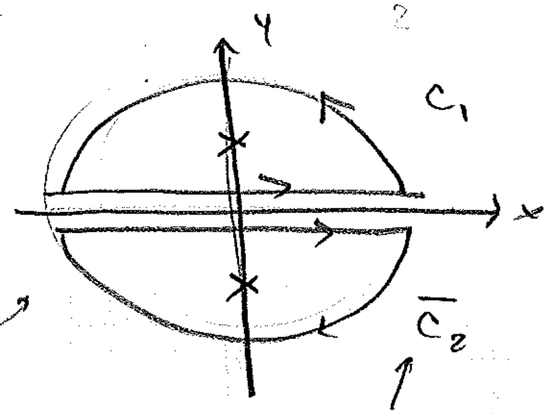
SOMETHING YOU KNOW, IDEALLY ZERO

eg: $\int_{-\infty}^{\infty} dx \boxed{\frac{2\cos x}{x^2+1}}$

$$\frac{e^{iz}+e^{-iz}}{(z+i)(z-i)}$$

$\mathbb{R}$

WHICH CONTOUR? BOTH CONTAIN $\mathbb{R}$



BAR: MY NOTATION FOR NEGATIVE ORIENTATION

☆ SEPARATE THE TERMS

$$dz\left(\frac{e^{iz}}{z^2+1}\right) + \frac{e^{-iz}}{z^2+1}\,dz$$

DEF SGN ON EA. PATH.

(-1,1) on both

WANT ARC TO GIVE ZERO CONTRIBUTION: $z = R\cos\theta + iR\sin\theta$

$$= \frac{iRe^{i\theta}\, e^{iz}}{z^2+1}\,d\theta = \frac{\boxed{iRe^{-R\sin\theta}}\boxed{e^{iR\cos\theta+i\theta}}}{R^2e^{2i\theta}+1}$$

PHASE

exponential suppression $\Rightarrow$ if $\sin\theta > 0$

MORE IMPORTANT THAN R^{-2}

$e^{-R\sin\theta} \Rightarrow C_1$ contour.

$$\oint_{C_1} \frac{e^{iz}}{z^2+1}\,dz = 2\pi i \underbrace{\left(\frac{e^{-1}}{2i}\right)}_{\mathrm{Res}_f(i)}$$

WHAT ABOUT e^{-iz} TERM?

$$\boxed{e^{+R\sin\theta}}\, e^{-iR\cos\theta}$$

controls BEHAVIOR for $R\to\infty$ ARC
for this term, the arc integral vanishes on the LOWER contour: $\boxed{\bar{C}_2}$

ORIENT.

$$\oint_{\bar{C}_2} \frac{e^{-iz}}{z^2+1}\,dz = -2\pi i \left(\frac{e^{-1}}{-2i}\right)$$

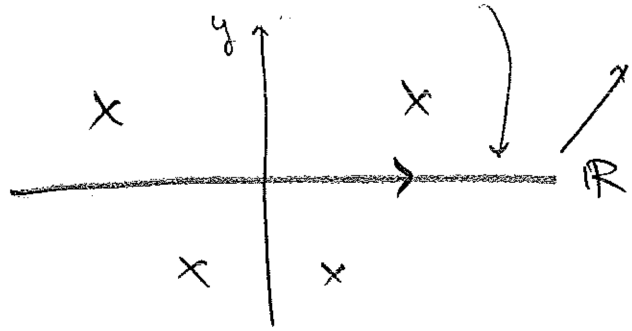
INTEGRALS ALONG ARCS

$$\underbrace{\oint_{C_1} \frac{e^{iz}}{z^2+1}\,dz + \oint_{\bar{C}_2} \frac{e^{-iz}}{z^2+1}\,dz}_{} = \int_{-\infty}^{\infty} \frac{e^{iz}+e^{-iz}}{z^2+1}\,dz + 0$$

$$= 2\pi i\left(\frac{1}{2ie} + \frac{1}{2ie}\right) = \boxed{\frac{2\pi}{e}}$$

PRINCIPAL VALUE

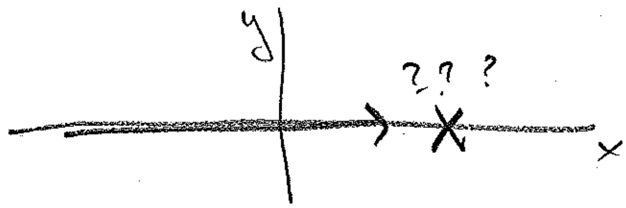
SO FAR SO GOOD: $f(x)$



then close contour as necessary to get a nice result

↑ USUALLY ALONG PATH THAT INTEGRATES TO ZERO.

WHAT IF THERE'S A POLE IN THE WAY?

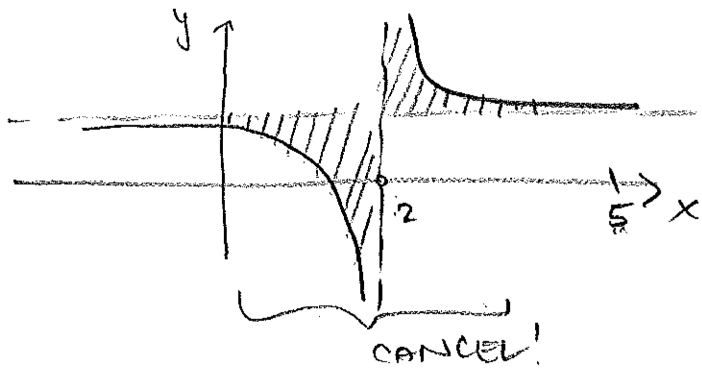


$$f(x) = \frac{g(x)}{x - x_0}$$

well behaved (for $g(x)$)

IS IT OKAY TO STILL ASK FOR $\int_{-\infty}^{\infty} f(x)\,dx$?

eg: what if $g(x) = \text{const}$? eg $x_0 = 2$



basically still true for nice $g(x)$

$$\int_0^5 \frac{c}{x-2}\,dx = c\int_4^5 \frac{1}{x-2}\,dx$$

NOTE: THE POLE ALWAYS HAS PHYSICAL MEANING! (in any actual process)

non Analyticity is Nature's way of saying something.

STILL: YOU SHOULD FEEL QUEASY DOING THIS
SO MATHEMATICALLY WE PUT AN ASTERISK ON IT:

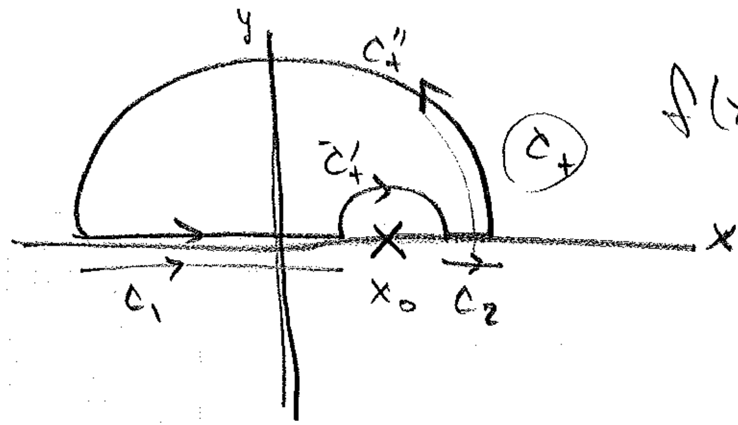
$$\mathcal{P}\int_a^b \frac{g(x)}{x-x_0}\,dx \equiv \int_a^{x_0-\varepsilon} \frac{g(x)}{x-x_0}\,dx + \int_{x_0+\varepsilon}^{b} \frac{g(x)}{x-x_0}\,dx$$

"PRINCIPAL VALUE"

MEANS: ASSUMING $\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{g(x)}{x-x_0}\,dx = 0$

(DOES THE POLE MATTER?

WHAT HAPPENS IN A CONTOUR INTEGRAL?



$$f(x) = \frac{g(x)}{x-x_0}$$

$$\oint_{C_+} f(z) = \Big(\underbrace{\int_{C_1} + \int_{C_2}}_{\mathcal{P}\int_{-\infty}^{\infty}} + \int_{\bar{C}_+'} + \int_{C_+''}\Big)dz\, f(z)$$

ASSUME $g(z)$ DIES @ $R\to\infty$ FASTER THAN R.

$\bar{C}_+'$ PARAMETERIZED BY $z = x_0 + \varepsilon e^{i\theta}$

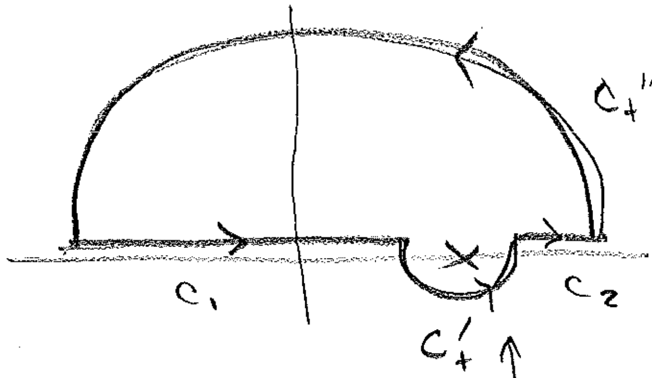
$$= \int_\pi^0 \frac{g(z)}{z-x_0}\,dz = -\int_0^\pi \frac{g(x_0)}{\cancel{\varepsilon e^{i\theta}}}\, i\cancel{\varepsilon e^{i\theta}}\,d\theta$$

$$= -ig(x_0)\int_0^\pi d\theta = -\pi i\, g(x_0)$$

$$= -i\pi\,\mathrm{Res}_f(x_0)$$ half a residue!

$$\boxed{\mathcal{P}\int_{-\infty}^{\infty} dz\, f(x) = \oint_{C_+} f(z) + \tfrac{1}{2}\,\mathrm{Res}_f(x_0)}$$

HOMEWORK: THERE'S AN OBVIOUS CONSISTENCY CHECK:



does this give the same "HALF RESIDUE"?

ANS: YOU PICK UP THE WHOLE RESIDUE FROM RESIDUE THM.

BUT: $\int_{\pi}^{2\pi} \frac{g(z)}{z - x_0}\, dz = \oplus i\pi\, g(x_0)$

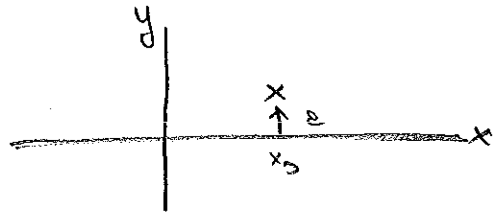
$$\oint f(z)\,dz = P\int_{-\infty}^{\infty} f(x)\,dx + i\pi\, g(x_0)$$

SUBTRACT

includes
$2\pi i\, \mathrm{Res}(x_0)$

another eg: $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$ in DIS notes

<u>Note</u>: RATHER THAN GOING AROUND POLE, CAN IMAGINE PUSHING POLE A LITTLE



then take $\varepsilon \to 0$ limit later

$$f(z) \equiv \frac{g(z)}{z - x_0 - i\varepsilon}$$

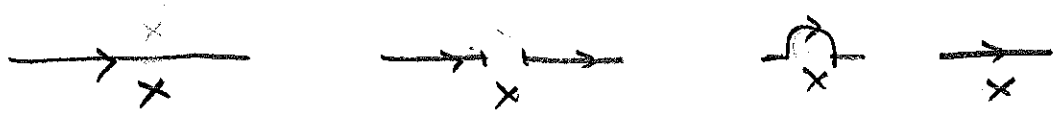
BECAUSE THE [CLOSED] CONTOUR INT. IS "TOPOLOGICAL" (DOESN'T CARE ABOUT SPECIFIC PATH), THIS DOESN'T CHANGE ANYTHING.

LIGHTNING REVIEW / SEGUE

$$\langle h | g \rangle = \int h^*(x)\, g(x)\, dx$$

SO ASSUMING g IS ANALYTIC IN CONTOUR,

$$\left\langle \frac{1}{x - x_0 + i\varepsilon} \middle| g \right\rangle = \mathcal{P}\left\langle \frac{1}{x - x_0} \middle| g \right\rangle - i\pi \left\langle \delta(x - x_0) \middle| g \right\rangle$$



ASSUMING  $= 0$

SO, AS A <u>DISTRIBUTION</u>,

$$\frac{1}{x - x_0 \pm i\varepsilon} = \mathcal{P}\left(\frac{1}{x - x_0}\right) \mp i\pi\, \delta(x - x_0)$$

HARMONIC OSCILLATOR : $\mathcal{O} = \left(\frac{d}{dt}\right)^2 + \omega^2$

A "UNIT DISPLACEMENT" @ $t_0 = 0$ GIVES
A RESPONSE AT TIME t:

$$\boxed{G''(t) + \omega^2 G(t) = \delta(t)} \longleftarrow \begin{array}{l} G(t) = G(t,t') \\ \delta(t) = \delta(t-t') \end{array}$$

choice of sign

SOLVE : $G(t) = \int_{-\infty}^{\infty} e^{-ikt}\, \tilde{G}(k)\, dk$

| TO DO: CHECK SIGNS & 2π's |

$$\mathcal{O}G(t) = \int_{-\infty}^{\infty} (\omega^2 - k^2)\, e^{-ikt}\, \tilde{G}(k)\, dk$$

$$= \int_{-\infty}^{\infty} e^{ikt}\, dk$$

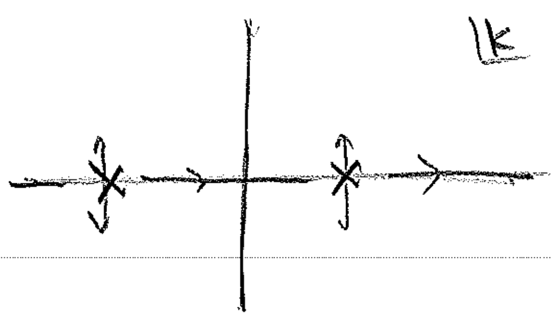
$\Downarrow$ compare coeff of $|k\rangle \sim e^{ikt}$

$$\boxed{\tilde{G} = \frac{1}{\omega^2 - k^2}}$$ feels like we're done

$$G(t) = \int_{-\infty}^{\infty} \frac{e^{ikt}}{\omega^2 - k^2}\, dk$$

POLES @ $k = \pm\omega$
ALONG CONTOUR ! :(

HAVE TO PICK A POLE PRESCRIPTION / CONTOUR

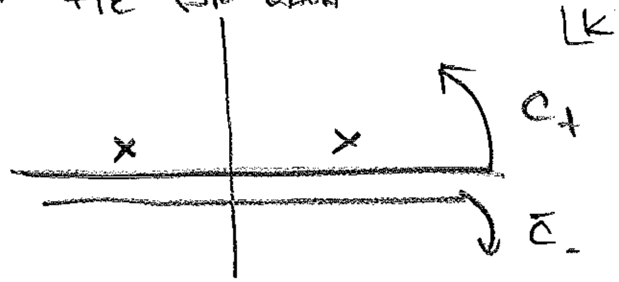


$$\frac{-e^{ikt}}{(k-\omega)(k+\omega)}$$

$+i\varepsilon$ $\quad$ $+i\varepsilon$

$(k-\omega)$ vs. $(\omega-k)$

TRY: CHOOSE $+i\varepsilon$ for EACH



WHICH CONTOUR? WANT $\tilde{G} e^{ikt} \to 0$ on arc.

BYRON & FULLER vs. BOAS

Re PART SHOULD GO LIKE e^{-R}
nb: DEPENDS ON OUR CONVENTION FOR THE FOURIER TRANSFORM
$G = \tilde{G} e^{ikt}$ or $\tilde{G} e^{-ikt}$?

SO WHICH CONTOUR DO WE WANT? coord: (k, θ)

$+ k \sin\theta \; t$

DEPENDS ON SIGN OF t!

IF $t > 0$ (after source), $\sin\theta < 0$ for $\int_{ARC} \to 0$

↳ so TAKE C_- : no poles!?

$\boxed{G(t) = 0}$ $+i\varepsilon$ w $t>0$

IF $t<0$, $\sin\theta > 0 \to$ take C_+

$$G(t) = \int_{C_+} dk\, e^{-ikt} (-1) \frac{1}{k-(\omega+i\varepsilon)} \frac{1}{k-(-\omega+i\varepsilon)}$$

$$= \cancel{2\pi} i \times \frac{-1}{\cancel{2\pi}} \left(\text{Res}(\omega+i\varepsilon) + \text{Res}(-\omega+i\varepsilon) \right)$$

$$= \frac{1}{i} \left(\frac{e^{-i\omega t}}{-2\omega} + \frac{e^{i\omega t}}{2\omega} \right)$$

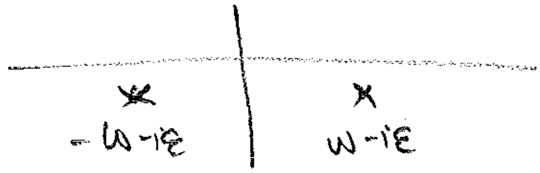
$$= \boxed{\frac{-1}{\omega} \sin(\omega t)} \quad \boxed{t<0}$$

not physical

$$\boxed{G(t) = -\frac{1}{\omega} \sin(\omega t)\, \Theta(-t)}$$

ADVANCED PROPAGATOR
ACAUSAL!!

MAYBE WE SHOULD HAVE PUSHED POLES <u>DOWN</u> ?



convergence <u>still</u> set by e^{-ikt}, BUT POLES ARE NOW IN C_- CONTOUR:

<u>$t>0$</u> : converges (ARC) to 0 for $\bar{C}_-$

<u>$t<0$</u> : converges to 0 for C_+

BUT NOW NO POLES

$$G(t<0) = 0$$

no acausal propagation

take care of overall sign: $\oint_{\bar{C}_-} = -\oint_{C_-}$ (ORIENT CONTOUR)

FROM $(k^2-\omega^2)$ VS (ω^2-k^2)

$$G(t>0) = -\oint_{C_-} \text{đ}k\, e^{-ikt} \frac{-1}{k^2-\omega^2+i\varepsilon}$$

$$= 2\pi i \times \frac{1}{2\pi}\left(\text{Res}(\omega+i\varepsilon) + \text{Res}(-\omega+i\varepsilon)\right)$$

rel sign

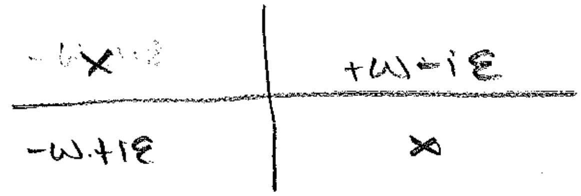
$$= i\left(\frac{e^{-i\omega t}}{-2\omega} + \frac{e^{i\omega t}}{2\omega}\right)$$

$$= \frac{1}{\omega}\sin(\omega t)$$

$$\boxed{G(t) = \frac{1}{\omega}\sin(\omega t)\,\Theta(t)}$$

RETARDED PROPAGATOR (PHYSICAL)

Feynman Propagator



one pole in C_+, one in C_-

$$G_F(t>0) = \frac{i}{2\omega} e^{-i\omega t}$$

$$G_F(t<0) = \underbrace{\frac{-i}{2\omega} e^{i\omega t}}_{\text{IMAGINARY?}}$$

TURNS OUT TO BE USEFUL IN RELATIVISTIC THEORIES. NOTE MINUS SIGN.

$t<0$ is a "NEGATIVE ENERGY" SOLUTION MOVING FWD IN TIME
FREQ.

DIRAC NOTICED THAT IN QM, $\exists$ "negative energy" states $\rightarrow$ REALIZED THEY HAD TO DO W/ ANTIPARTICLES MOVING FORWARD IN TIME.

FOURIER TRANSFORM NOTE

see physics stackexchange : 308234

two freedoms: (a) & (b)

$$\underbrace{\mathcal{F}_{a,b}[f](\omega)}_{\tilde{f}} = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} e^{ib\omega t} f(t)\, dt$$

then: $$\mathcal{F}^{-1}_{a,b}[\tilde{f}](t) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{\infty} e^{-ib\omega t} \tilde{f}(\omega)\, d\omega$$

check:

$$\mathcal{F}^{-1}_{a,b}\left[\mathcal{F}_{a,b}[f]\right] = \frac{|b|}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-ib\omega t} \int_{-\infty}^{\infty} ds\, e^{ib\omega s} f(s)$$

$$= \frac{|b|}{2\pi} \iint d\omega\, ds\, e^{ib\omega(s-t)} f(s)$$

$$= \frac{|b|}{2\pi} \int ds\, f(s) \int d\omega\, e^{ib\omega(s-t)}$$

nb: $\int d\xi\, e^{2\pi i x \xi} = \delta(x)$

then: $\xi = \frac{b}{2\pi}\omega \qquad d\omega = \left|\frac{2\pi}{b}\right| \xi$

$$= \int ds\, \delta(s-t)\, f(s)$$

FOR SPACETIME, MUST MAINTAIN LORENTZ INVARIANCE.