LAST TIME: Self-Adjointness / Hermiticity of differential operators

THAT WAS A STAND-ALONE LECTURE

- HERMITICITY
  - $\to L \to L^{+}$ formally BY INTEG. BY PARTS
  - $\to$ PROPER HERMITICITY: BOUNDARY CONDITIONS TOO
- what this buys
  WE WILL PRIMARILY CARE ABOUT HERMITIAN DIFFERENTIAL OPS
  - $\to \mathbb{R}$ eigenvalues
  - $\to$ complete set of eigenfunctions

CAN INSERT $\mathbb{1} = |e^i\rangle\langle e_i|$

$$\boxed{\delta(x-y) = \sum_n e(y)^* e(x)}$$

aha! A δ-function.

THIS IS USEFUL FOR CONSTRUCTING GREEN'S FUNCTIONS.

RECALL: for some operator L, THE GREEN'S FUNCTION, G, is THE INVERSE:

sometimes I write $\mathcal{O}$

$$\boxed{LG = \delta}$$

$G(x,y)$

why? INHOMOGENOUS DIFF. EQ.

$$\underset{\substack{\uparrow \\ \text{DYNAMICS}}}{L}\ \underset{\substack{\uparrow \\ \text{PHYSICAL STATE}}}{\psi} = \underset{\substack{\uparrow \\ \text{SOURCE}}}{S}$$

$$\psi = \int \underset{\substack{\uparrow \\ \text{SOURCE POSITION}}}{dy}\ G(x,y)\, s(y)$$

$$L = \sum_n P_n(x) \left(\frac{d}{dx}\right)^n$$

STATE POSITION

So: $L\psi = \int dy\ L\,G(x,y)\, S(y)$

$$= \int dy\ \delta(x-y)\, S(y)$$

$$= S(x) \quad \checkmark$$

We haven't yet done a systematic study of solving for $G(x,y)$.

↳ OUR PLAN: $\mathbb{C}$ analysis, then use RESIDUE THEOREM. } next wk.

BIRD'S EYE VIEW

A FEW STRATEGIES

1. CONTOUR INTEGRAL

↳ FOURIER TRANSFORM CONVERTS DIFFERENTIAL EQ. INTO AN ALGEBRAIC ONE W/ AN INTEGRAL

$$L_x \int e^{ikx}\, \tilde{G}(k)\, dk = \int dk\, e^{ikx}$$

acts on e^{ikx}

eg $\left(\frac{d}{dx}\right)^2 \to -k^2$

then solve for $\tilde{G}(k)$.

BUT: need to Fourier transform BACK to position space

... requires doing a tricky integral.

eg if $L_x = (d/dx)^2 + \omega^2$

$$\int (+k^2+\omega^2)\, e^{ikx}\, \tilde{G}(k)\, dk = \int dk\, e^{ikx}$$

$$\tilde{G}(k) = \frac{1}{k^2+\omega^2}$$

$$\boxed{G(x) = \int dk\, \frac{1}{k^2+\omega^2}}$$

2. PROJECT ONTO EIGENBASIS

$$\sum_n e_n^*(y)\, e_n(x) = \delta(x-y)$$

$$L_x e_n(x) = \lambda_n$$

then:
$$L_x \underbrace{\sum_n \left(\frac{e_n^*(y)\, e_n(x)}{\lambda_n} \right)}_{G(x,y)} = \delta(x-y)$$

3. PIECEWISE SOLUTION

if you can solve the homogeneous equation $L_x \psi = 0$,
then
$$L_x G = \delta(x-y)$$
FIX y

CAN BE BROKEN INTO

$$G = \begin{cases} G^>(x) & \text{for } x>y \\ G^<(x) & \text{for } x<y \end{cases}$$

w/ matching conditions @ $x=y$ from integrating over $(y-\varepsilon, y+\varepsilon)$.

HOMEWORK 3 : PROBLEM 1.

CLAIM : THE PAULI MATRICES SPAN A VECTOR SPACE.

simpler case :

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\underbrace{|e_1\rangle \qquad |e_2\rangle \qquad |e_3\rangle}$$

A SILLY BASIS ... COULD HAVE USED

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

in fact, at this level: there is no difference.

$$\boxed{\vec{V} = V^1 |e_1\rangle + V^2 |e_2\rangle + V^3 |e_3\rangle}$$

eg if $V^i = (2, 3, 1)$

$$\vec{V} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$$

in general:

$$\vec{V} = \begin{pmatrix} V^1 & V^2 \\ V^3 & 0 \end{pmatrix}$$

IS THIS A VECTOR SPACE?

$$\vec{V} + \vec{W} = \begin{pmatrix} V^1 + W^1 & V^2 + W^2 \\ V^3 + W^3 & 0 \end{pmatrix}$$

$$= (V^1 + W^1)|e_1\rangle + (V^2 + W^2)|e_2\rangle + (V^3 + W^3)|e_3\rangle$$

$\Rightarrow$ vector space

THIS SILLY BASIS SPANS THE SPACE OF 2×2 MATRICES WITH BOTTOM RIGHT COMPONENT = 0.

THE PAULI MATRICES SPAN THE SPACE OF TRACELESS 2×2 HERMITIAN MATRICES.

$$\vec{V} = V^a |\sigma^a\rangle$$

I'LL USE σ INSTEAD OF $T = \frac{1}{2}\sigma$ FOR SIMPLICITY

$$= V^1 \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} + V^2 \begin{pmatrix} & -i \\ i & \end{pmatrix} + V^3 \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$= \begin{pmatrix} V^3 & V^1 - iV^2 \\ V^1 + iV^2 & -V^3 \end{pmatrix}$$

$$\equiv \begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix}$$

COLUMN VECTOR CONTAINS <u>SAME</u> INFO

AT THIS POINT, NO SIGNIFICANCE TO BASIS! COULD HAVE USED

$|e_1\rangle$ = APPLE
$|e_2\rangle$ = BANANA
$|e_3\rangle$ = CHERRY

<u>NEXT</u>: WE DEFINE A CLASS OF <u>LINEAR TRANSFORMATIONS</u>

HOW DO WE DEFINE?

↳ HOW THE TRANSFORMATION ACTS ON VECTORS

⇒ SUFFICIENT TO KNOW HOW IT ACTS ON <u>BASIS VECTORS</u>

"mind blowing" part: the PAULI MATRICES are also the transformations !

RECALL: A LINEAR TRANSFORMATION ("MATRIX"), L

- takes in a <u>vector</u>
- spits out a <u>vector</u>

for us, vectors are: $\vec{V} = V^1|\sigma^1\rangle + V^2|\sigma^2\rangle + V^3|\sigma^3\rangle$

<u>PROPOSAL</u>: $L_w = [\, w^a \sigma^a, \; \cdot \;]$

$\uparrow$ feed me a vector

eg $L_w \vec{V} = [\, W^a \sigma^a, \, V^b \sigma^b \,]$

$= W^a V^b \underbrace{[\sigma^a, \sigma^b]}_{2i\varepsilon^{abc}\sigma^c}$

$= W^a V^b \; 2i\varepsilon^{abc} \sigma^c$

$= \underbrace{2iW^a V^b \varepsilon^{abc}}_{\text{some list of 3 numbers}} \sigma^c$

$\hookrightarrow \equiv X^c$

$\equiv X^c |\sigma^c\rangle$ ← spits out a vector

D

in the problem:

$$\underbrace{\sigma^a}_{\text{OPERATOR}} \underbrace{|\sigma^b\rangle}_{\text{VECTOR}} = |\underbrace{[\sigma^a, \sigma^b]}\rangle$$

IS this a sum of PAULI MATRICES?

yes $\quad i\varepsilon^{abc}\sigma^c$

WOULD BE BAD IF THIS WERE NOT EXPRESSABLE IN OUR $|\sigma^a\rangle$ BASIS

Pauli matrices can act on Pauli matrices to yield Pauli matrices.

LIKE MAD LIBS

eg. $\sigma^1|\sigma^2\rangle = |2i\sigma^3\rangle$

$= 2i\,|\sigma^3\rangle$

what about $L = (2\sigma^1 + 3\sigma^2)$, $L|\sigma^2\rangle$?

$\underbrace{2\sigma^1|\sigma^2\rangle} + 3\sigma^2|\sigma^2\rangle$

$2\times 2i|\sigma^3\rangle + 3|\underbrace{[\sigma^2,\sigma^2]}_{=0}\rangle = \boxed{4i|\sigma^3\rangle}$

So: $\underset{\sim}{A} = A^a \sigma^a$, A LINEAR TRANSFORMATION ACTING ON THE SPACE OF 2×2, TRACELESS HERMITIAN MATRICES

$\underset{\sim}{A}$ is also a member of the vector space ... but this is irrelevant!

<u>QUESTION</u>: WRITE $\underset{\sim}{A}$ AS 3×3 MATRIX ACTING ON $|\sigma^a\rangle$ BASIS.

<u>SIMPLER EXAMPLE</u>

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

BASIS VECTOR

$$\Rightarrow \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} a & ? \\ c & ? \end{pmatrix}$$

these ELEMENTS <u>MEAN</u> "WHAT MULTIPLIES $\hat{e}_1$ TO GIVE A CONTRIBUTION TO THE OUTPUT $\hat{e}_1$."

} SIMILAR FOR $\hat{e}_2$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \Rightarrow \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} ? & b \\ ? & d \end{pmatrix}$$

acts on $\rightarrow$	$\vert\sigma^1\rangle$	$\vert\sigma^2\rangle$	$\vert\sigma^3\rangle$
σ^1	0	$2i\sigma^3$	$-2i\sigma^2$
σ^2	$-2i\sigma^3$	0	$2i\sigma^1$
σ^3	$2i\sigma^2$	$-2i\sigma^1$	0

$$\text{"}A^1\sigma^1\text{"}\underset{\underset{|\sigma^1\rangle}{\uparrow}}{\begin{pmatrix}1\\0\\0\end{pmatrix}} = A^1\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\text{"}A^2\sigma^2\text{"}\begin{pmatrix}1\\0\\0\end{pmatrix} = \underbrace{A^2\begin{pmatrix}0\\0\\-2i\end{pmatrix}}_{-A^2\cdot 2i\,|\sigma^3\rangle}$$

$$\text{"}A^3\sigma^3\text{"}\begin{pmatrix}1\\0\\0\end{pmatrix} = \underset{\uparrow\ A^3\cdot 2i\,|\sigma^2\rangle}{A^3\begin{pmatrix}0\\2i\\0\end{pmatrix}}$$

$$\underbrace{A^a \sigma^a}_{\mathcal{A}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2iA^3 \\ -2iA^2 \end{pmatrix}$$

$$\Downarrow$$

$$\mathcal{A} = \begin{pmatrix} 0 & \cdot & \cdot \\ 2iA^3 & \cdot & \cdot \\ -2iA^2 & \cdot & \cdot \end{pmatrix}$$

make sure you understand this!

this component: WHICH PAULI MATRIX, WHEN ACTING ON σ^1 (1st col) GIVES SOMETHING $\propto \sigma^2$ (2nd row)?

WE CAN FILL IN THE REST FROM THE TABLE

$$\mathcal{A} = \begin{pmatrix} 0 & -2iA^3 & 2iA^2 \\ 2iA^3 & 0 & -2iA^1 \\ -2iA^2 & 2iA^1 & 0 \end{pmatrix}$$

COMPARE THIS TO, eg. eq. (3.5.54) of Sakurai

SO: GIVEN $\underbrace{A^1\sigma^1 + \cdots + A^3\sigma^3}_{\text{TRANSFORMATION}}$

ACTS ON $\underbrace{V^a|\sigma^a\rangle}_{\text{VECTOR}}$

AS $V^a|[A^b\sigma^b, \sigma^a]\rangle$

$$= V^a A^b\,|\,[\sigma^b, \sigma^a]\rangle$$

$$= \underbrace{V^a A^b \times 2i\varepsilon^{abc}}_{\text{sum over } a,\, b}\,|\sigma^c\rangle$$

note antisymmetry in a, b

I CAN REPRESENT THIS AS A MATRIX

$A \leftarrow 3\times 3$

ACTING ON $\begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix}$

∴ SO REDUCE THIS TO THE USUAL FORM OF A LINEAR TRANSFORMATION AS A MATRIX.

incorrect in question: σ^3 is <u>not</u> diagonal

↳ BUT WE <u>CAN</u> DIAGONALIZE IT.

$$\sigma^3 = \left(\begin{array}{cc|c} 0 & -2i & \\ 2i & 0 & \\ \hline & & 0 \end{array}\right)$$

$$= 2i \left(\begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right)$$

45° ROTATION

TRICK: USE 2x2 MATRIX "BASIS"

$$\sigma^{\pm} = \frac{1}{2}\left(\sigma^1 \pm i\sigma^2\right)$$

$$\boxed{[\sigma^3, \sigma^{\pm}] = \pm\, \sigma^{\pm}}$$

CHARGES

FOR NEXT WEEK

Please review the following topics

- COMPLEX #'s
- CAUCHY - RIEMANN EQ
  ⇒ "ANALYTIC"

- $\mathbb{C}$ functions as maps

- $\mathbb{C}$ logarithm

- SINGULARITIES
  LAURENT SERIES

- CAUCHY'S THEOREM
  $$\oint_{\partial D} f(z)\, dz = 0$$
  for f ANALYTIC IN D

- CAUCHY'S INTEGRAL FORMULA
  & RESIDUE THEOREM
  $$\oint_{\partial D} f(z)\, dz = 2\pi i \,\mathrm{Res}\,(f @ z_0)$$
  simple pole

⇒ non Green's func. Applic:
DISPERSION, CAUSALITY

BYRON & FULLER: CH. 6 [6.1, 6.3-6.4, 6.7, 6.8]
BOAS: CH. 14, ALL OF IT

MATHEWS & WALKER: 3-3; APP. A
STONE & GOLDBART ; CH. 17
APPEL: CH. 4.1-4.5, 4.6 ← GOOD TEXT
CAHILL: CH. 5.1 - 5.14