LECTURE 16

transl. inv. $\int$

CORRECTION FROM LAST WEEK:

HARMONIC OSCILLATOR: $G''(t) + \omega^2 G(t) = \delta(t)$

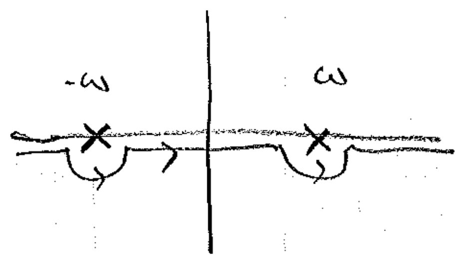
$$G(t) \equiv \int dk\, e^{-ikt}\, \tilde{G}(k)$$

$$\tilde{G}(k) = \frac{-1}{k^2-\omega^2}$$

HOW TO CLOSE:
if $t>0$: ⌣
if $t<0$: ⌢

POLES ON THE CONTOUR

OBSERVED: 2 MAIN CHOICES



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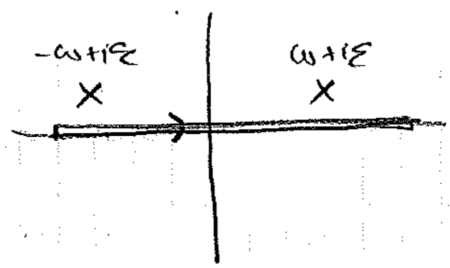
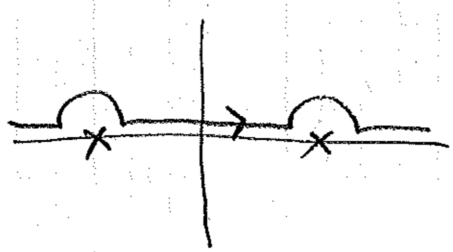


AS PRINCIPAL VALUES, THESE SHOULD BE =!



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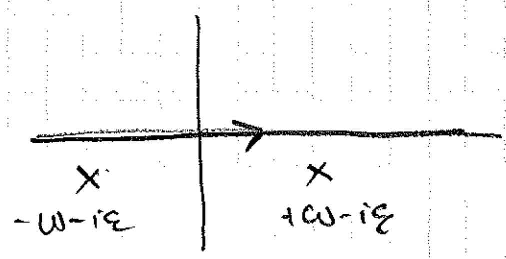


$\boxed{\text{CORRECTION}}$ ← —— my bad! sorry. :(

WAIT! HOW ARE THESE EQUAL?!

I THOUGHT LHS HAS LITTLE SEMI-CIRCLES

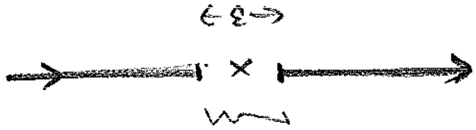
↳ CONTRIBUTE ½ RESIDUE TO CONTOUR INTEGRAL to PRINCIPAL VALUE

RHS HAS NO SUCH THING!

RESOLUTIONS:

1. equal signs are wrong
2. LHS is not A PRINCIPAL VALUE

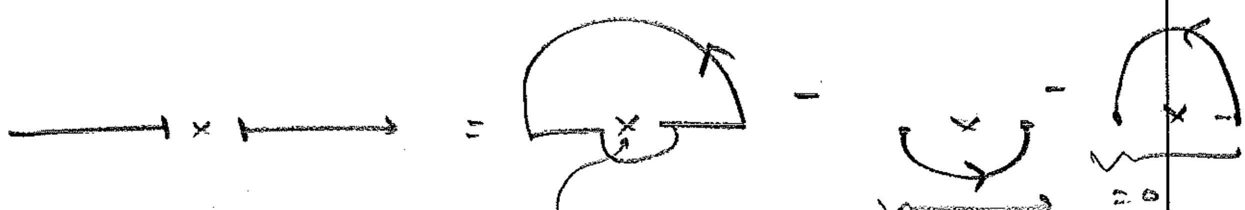
PRINCIPAL VALUE



do not integrate here

not what we're doing — we're doing the full integral, no "holes"

why the $\frac{1}{2}$ residue?



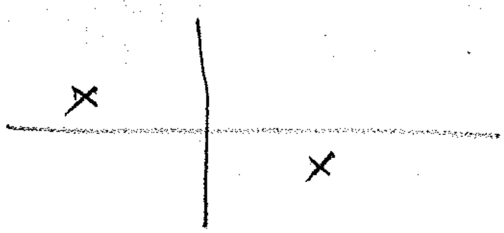
$2\pi i$ Res

$= \pi i$ Res

BUT FOR OUR PHYSICAL SYSTEM, WE ACTUALLY MAKE A PHYSICAL CHOICE OF POLE STRUCTURE!

ADVANCED & RETARDED GREEN'S FUNCTIONS ARE DIFFERENT, WE CHOOSE $G^{(R)}$.

SIDE NOTE: FEYNMAN PROPAGATOR — thanks, Jose



$$\tilde{G} = \frac{-1}{[k-(\omega-i\varepsilon)][k-(-\omega+i\varepsilon)]}$$

Denominator: $k^2 - \omega^2 \underbrace{+ 2i\varepsilon\omega}_{+\frac{1}{2}i\delta\text{, small \& positive}}$

AS IF: $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}(\omega^2 + i\delta)x^2$

$$= -\frac{1}{2}x\left[\underbrace{\left(\frac{d}{dt}\right)^2 + \omega^2}_{\Theta_{HO}} - i\delta\right]x$$

$$S = \int dt\, L = \int dt\, -\frac{1}{2}x\left[\Theta_{HO} - i\delta\right]x$$

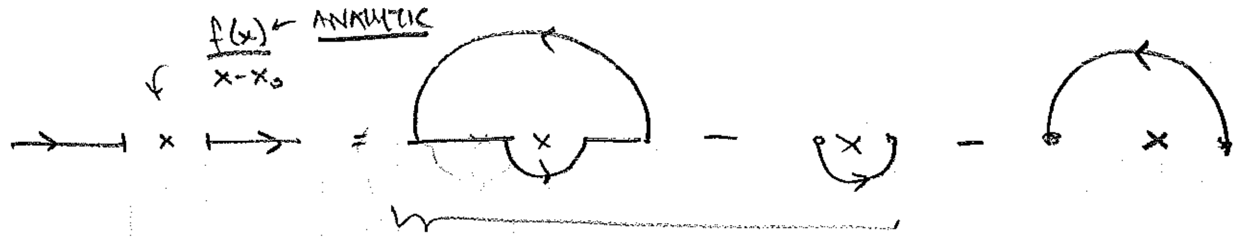
EOM $\Longleftrightarrow$ VARIATIONAL PRINCIPLE
$\Longleftrightarrow$ PATH INTEGRAL

$$Z = \int \mathcal{D}x \, e^{iS} = \int \mathcal{D}x \, e^{iS_{HO} - \frac{1}{2}\delta x^2}$$

INTEGRAL OVER ALL PATHS $x(t)$

gives convergence of path integral

LET'S GET BACK TO PRINCIPAL VALUES THOUGH, THERE IS SOMETHING TO IT:



$$\boxed{\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx = i\pi f(x_0)}$$

← interesting relation b/c of factor of i

WRITE $f(x) = u(x) + iv(x)$

COMPARE Re & Im PARTS:

$$\underbrace{\mathcal{P}\int \frac{u(x)}{x-x_0}\,dx}_{} + \underbrace{i\mathcal{P}\int \frac{v(x)}{x-x_0}\,dx}_{} = \underbrace{i\pi u(x)}_{} - \underbrace{\pi v(x)}_{}$$

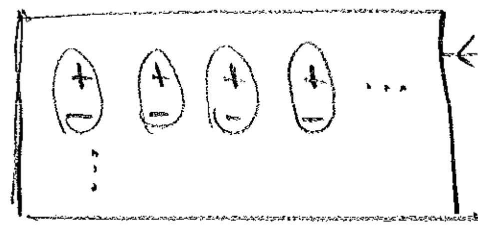
LIKE CAUCHY RIEMANN EQ: RELATES u & v TO OTHER

$$\boxed{\begin{aligned} u(x_0) &= \frac{1}{\pi}\mathcal{P}\int \frac{v(x)}{x-x_0}\,dx \\ v(x_0) &= \frac{-1}{\pi}\mathcal{P}\int \frac{u(x)}{x-x_0}\,dx \end{aligned}}$$

LIKE AN INTEGRATED VERSION OF CR EQNS.

↑ ∀ ANALYTIC FUNCTION

NOW SOMETHING CONCRETE: EM WAVES IN DIALECTRIC



$$\underline{\nabla}\cdot\underline{E} = 4\pi\left(\rho_{FREE} + \underbrace{\rho_{BOUND}}_{\text{"ENVIRONMENT"}}\right)$$

$$\equiv -\underline{\nabla}\cdot\underline{P} \quad \leftarrow \text{POLARIZATION (DEFINITION)}$$

$$\underline{\nabla}\cdot\left(\underbrace{\underline{E}+4\pi\underline{P}}_{\equiv\underline{D}}\right) = 4\pi\rho_{FREE}$$

$\equiv \underline{D}$, DIELECTRIC DISPLACEMENT

FOR not-too-big $\underline{E}$, many materials obey

$\underline{P} = \chi\underline{E}$ ← POLARIZATION FOLLOWS ELEC. FIELD

$\uparrow$ ELECTRIC SUSCEPTIBILITY $\Rightarrow \underline{D} = \underbrace{(1+4\pi\chi)}_{\equiv\varepsilon}\underline{E}$

FOR AN EM WAVE PASSING THROUGH MEDIUM, THE VALUE OF χ (OR ε) IS FREQUENCY DEPENDENT

↳ MICROPHYSICS OF BOUND CHARGES REACT TO CHANGING E-FIELD W/ SOME CHARACTERISTIC TIMESCALE

So: $\underline{P}(\omega) = \chi(\omega)\,\underline{E}(\omega)$

APPLY INVERSE FOURIER TRANSFORM TO BOTH SIDES

$\hookrightarrow \int d\omega\, e^{i\omega t}$ ////

HU
AAPT 57
P. 821
(1989)

$$\tilde{P}(t) = \int d\omega\, e^{i\omega t} \underbrace{\chi(\omega)\,\underline{E}(\omega)}$$

warning: Gymnastics ahead!

FT^{-1} OF PRODUCT
WE WILL PROVE THE CONVOLUTION THM

INSERT $\delta(\omega'-\omega')\,d\omega' = \int d\omega'\,dt'\, e^{i(\omega'-\omega')t'}$

$\S$ USE $\underline{E}(\omega) = \int d\omega'\, \delta(\omega'-\omega')\,\underline{E}(\omega')$

moved $2\pi^{-1}$

$$= \int d\omega\, d\omega'\, dt'\, \underbrace{e^{i\omega t} e^{i(\omega-\omega')t'}}_{e^{i\omega(t-t')}\, e^{i\omega t'}} \chi(\omega)\,\underline{E}(\omega')$$

$$= \int dt' \int d\omega\, e^{i\omega(t-t')}\chi(\omega) \int d\omega'\, e^{i\omega t'}\,\underline{E}(\omega')$$

$$= \int dt'\, \tilde{\chi}(t-t')\,\tilde{E}(t')$$

DROPPING $\sim$ (let argument determine it)

TIME DEP E field

$$\boxed{P(t) = \int dt'\, \chi(t-t')\, E(t')}$$

SUSCEPTIBILITY IS A GREEN'S FUNCTION !

SECOND OBSERVATION

Physics is causal : $\boxed{\chi(t<0)=0}$

CAUSALITY : $G(t) = \int dk\, e^{-ikt}\, \tilde{G}(k)$

$-i\epsilon$ PRESCRIPTION { for $t>0$, CLOSE ON LOWER 1/2 PLANE
should contain poles

for $t<0$, CLOSE ON UPPER 1/2 PLANE
no poles for $G(t<0)=0$

So $\chi(t<0)=0$ $\Rightarrow$ POLES ONLY IN LOWER 1/2 PLANE
$\Rightarrow$ χ IS ANALYTIC ON UPPER 1/2 PLANE

then OUR "INTEGRAL FORM of CAUCHY-RIEMANN"
KRAMERS-KRONIG

$$\boxed{\begin{aligned} \mathrm{Re}[\chi(\omega)] &= \frac{1}{\pi} P \int \frac{\mathrm{Im}[\chi(\omega')]}{\omega'-\omega}\, d\omega' \\ \mathrm{Im}[\chi(\omega)] &= \frac{-1}{\pi} P \int \frac{\mathrm{Re}[\chi(\omega')]}{\omega'-\omega}\, d\omega' \end{aligned}} \quad (\ast)$$

in UPPER HALF PLANE INCLUDING $\mathbb{R}$ AXIS.

HEALD & MARION p. 28

SO WHAT

EM WAVE ~ $e^{ikx - i\omega t}$ $\ast$ why we chose our Fourier conventions

$k = \omega/v$ ← velocity (units of c)

IN A MEDIUM: $n = v^{-1} = \sqrt{\epsilon\mu} \xrightarrow{\mu=1} \sqrt{\epsilon}$ (PURE DIELECTRIC)
$= \sqrt{1+4\pi\chi}$

BECAUSE χ HAS $\mathbb{R}$ PART, IT MUST ALSO HAVE IMAGINARY PART (BY $\ast$)

$\mathrm{Re}[\chi(\omega)]$

frequency dependence of $n \sim v^{-1}$ is called <u>DISPERSION</u>

(how prisms & rainbows work, Snell's law, etc.)

$\mathrm{Im}[\chi(\omega)]$ $K = \frac{\omega}{v} = \omega\sqrt{1+4\pi\chi} = K_R + i K_I$

then plane wave $e^{ikx - i\omega t}$

$$\boxed{e^{-K_I x}}\, e^{iK_R x - i\omega t}$$

<u>DISSIPATION</u> : ENERGY LOST TO MEDIUM

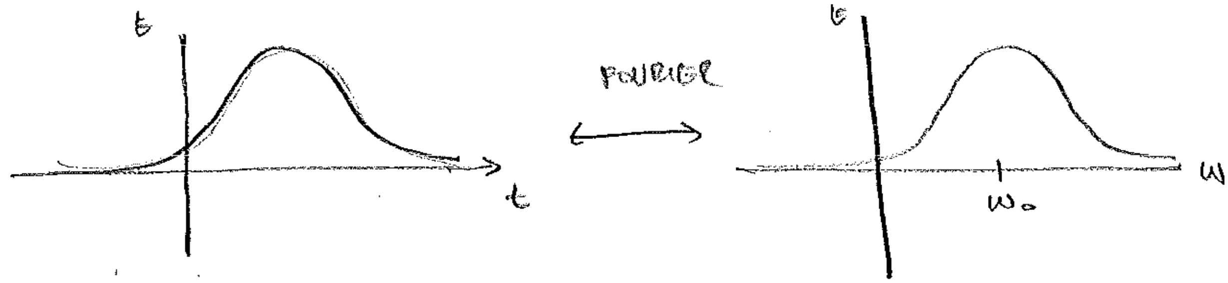
<u>KRAMERS - KRÖNIG</u> : RELATES <u>DISPERSION</u> TO <u>DISSIPATION</u>.

practical application: CAN MEASURE INDEX OF REFRACTION BY MEASURING <u>ABSORPTION</u>.

but <u>WHY</u> should dispersion & dissipation be related to each other?

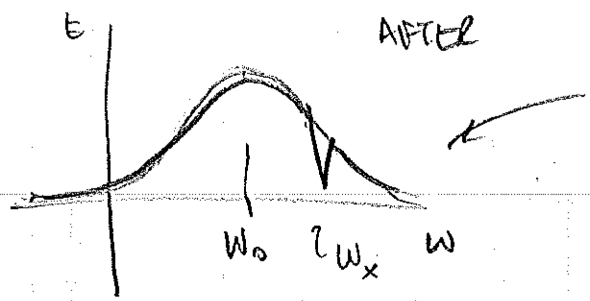
we can do a thought experiment that justifies this

IMAGINE A GAUSSIAN WAVE PACKET



fact: the FOURIER TRANSFORM of A GAUSSIAN IS ALSO GAUSSIAN.

NOW SEND THIS WAVE PACKET THROUGH AN IDEALIZED ABSORBING MATERIAL (FILTER) THAT ABSORBS ONLY <u>ONE</u> FREQUENCY.

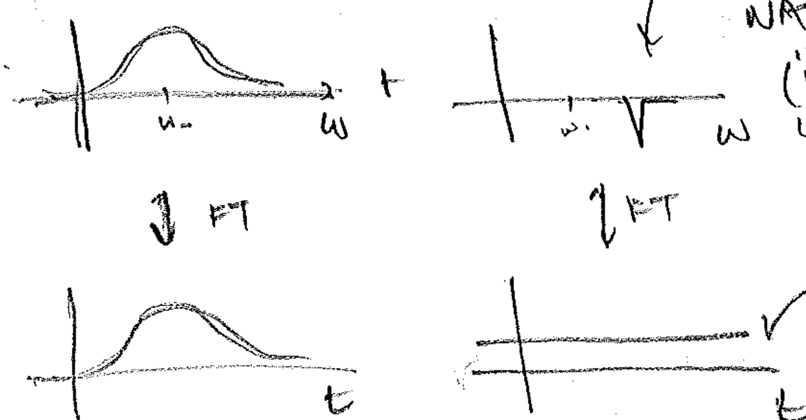


no longer Gaussian!

So F.T. to time domain is <u>not</u> GAUSSIAN.

In fact, this is a sum of PLANE WAVES!

("HEISENBERG")
LOCALIZE IN SPACE, DELOCALIZE IN MOMENTUM & V.V.



$\boxed{\text{ACAUSAL}}$

does not make sense!?

need some miracle to make it causal!

TO BETTER UNDERSTAND: DIVIDE χ INTO EVEN & ODD PARTS W/RT TIME REVERSAL SYM:

$$\chi_E = \tfrac{1}{2}\left(\chi(t) + \chi(-t)\right)$$
$$\chi_o = \tfrac{1}{2}\left(\chi(t) - \underbrace{\chi(-t)}\right)$$

for $t>0$, these PARTS CANCEL BY CAUSALITY

NOW GO TO MOMENTUM SPACE

$$\chi(\omega) = \int \underbrace{e^{-i\omega t}}\ \underbrace{\chi(t)}\ dt$$

$$\underbrace{\cos \omega t}_{\substack{\text{EVEN IN } t \\ \text{\& } \mathbb{R}}} - \underbrace{i \sin \omega t}_{\substack{\text{ODD IN } t \\ \text{\& } \boxed{\text{IMAG}}}}$$

$$\chi(t) = \tfrac{1}{2}\underbrace{(\chi_E + \chi_o)}_{\mathbb{R}}$$

BUT: $\int \underbrace{(\text{even in } t)(\text{odd in } t)}_{\text{ODD IN } t}\, dt = 0$

SO: $$\chi(\omega) = \underbrace{\int \tfrac{1}{2}\cos \omega t \cdot \chi_E\, dt}_{\mathrm{Re}\,\chi} \underbrace{- i \int \tfrac{1}{2} \sin \omega t\ \chi_o}_{\mathrm{Im}\,\chi}$$

this is exactly what KRAMERS-KRÖNIG does! Re χ & Im χ related to each other.

FROM WIKIPEDIA