(NB) → NO DISCUSSION SEC.

LECTURE 2: DIM. ANALYSIS, CONT'D   OCT 2

LAST TIME:

- UNITS: SEPARATE PHYSICS from MATH.
  - → DIMENSIONAL ANALYSIS
    - eg. SANITY CHECK
    - eg. ESTIMATING / GUESSING / SCALING (pendulum)
    - eg ———— " ———— (BOHR RADIUS)
      - ↳ useful for, eg: MUONIC ATOM

from lec 1 notes (OVERFLOW)

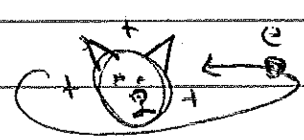
TODAY: MORE w/ DIMENSIONAL ANALYSIS
↳ SCALING & SIMILARITY

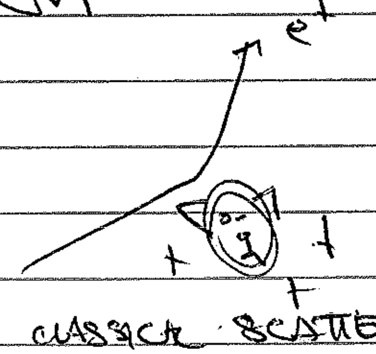
NEWTONIAN MOTION FROM A POTENTIAL:

$$m\ddot{\underline{r}} = -\frac{\partial U}{\partial \underline{r}} \qquad [U] = [F]L = ML^2T^{-2}$$

$$\dot{} = d/dt \leftrightarrow [\dot{}] = T^{-1}$$

U could be any system





CHARGED PARTICLE ORBITING CAT   CLASSICAL SCATTERING OFF CAT

LET $\underline{r}_0(t)$ BE A KNOWN TRAJECTORY
ie IT IS A SOLUTION TO THE NEWTON ODE.

DIM. ANALYSIS : (SCALING) CAN GENERATE OTHER SOLUTIONS w/o REDERIVING

CONSIDER A SCALE TRANSFORMATION OF THE TIME COORDINATE

$$t = \alpha t'$$

new TIME VARIABLE
(equivalent to new units)

$$m\ddot{\underline{r}} = -\frac{\partial U}{\partial \underline{r}}$$

in real world: "STATIC - ENOUGH"

if STATIC POTENTIAL, no t-DEPENDENCE.
[BUT RHS HAS DIM $\sim T^{-2}$? SATURATED BY, SAY, G_N]

only this side changes w/ rescaling

$$m\ddot{r}_0(t) = \underbrace{m\alpha^{-2}}_{\equiv m'}\left(\frac{d}{dt}\right)^2 r_0(\alpha t)$$

$$= \boxed{m' \, d^2/(dt)^2 \, r_0(\alpha t) = -\partial U/\partial r}$$

can drop primes

$r_0(\alpha t)$ IS ALSO A SOLUTION

but: traverse trajectory w/ different speed & for different mass

eg: $\alpha = 2$: trajectory covered twice as fast, but EOM valid for $m' = \underbrace{m/4}$

four times lighter

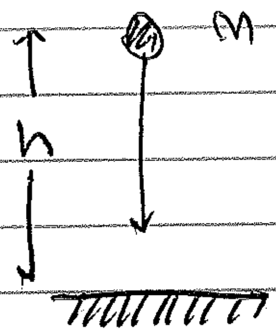
SANITY CHECK:
this does not hold for Grav. pot! (eg. PLANETARY ORBITS) why?

$F = -G_N \frac{Mm}{r^2}$ RHS HAS m-DEPENDENCE.

DIM. ANALYSIS & ERROR ANALYSIS

BOHREN, AM J. PHYS. 72 534 (vs 1102.1120)

A HIGH SCHOOL PROBLEM



FIND TIME t FOR BALL TO HIT GROUND.

(assume dropped @ rest near surface of Earth)

sign: COORD ORIENT.

HS ANSWER: SOLVE $\ddot{x} = g$

integrate: $x = \frac{1}{2} g t^2 + \cancel{ct} + \cancel{D}$

$v_0 = 0$ $\quad x_0 = 0$ (COORD CHOICE)

$$\boxed{t_0 = \sqrt{\frac{2h}{g}}}$$ ← LEADING ORDER (LO)

easy.

usually good enough.

how good?

USUAL THING YOU MAY WANT TO DO: ACTUALLY CALCULATE THE CORRECTION (NEXT-TO-LEADING ORDER, NLO) & COMPARE.

$$m\ddot{\underline{r}} = f(\ldots, \varepsilon)$$

$$\approx \underbrace{f(\ldots, 0)} + \varepsilon f'(\ldots, 0) + \cdots$$

LO: SOLVE $\underline{r}^{(0)}$ USING THIS

NLO: SOLVE CORRECTION WHEN INCLUDING THIS

eg $f(\ldots, 0)$ MAY BE SPHERICALLY SYMM. POTENTIAL, ε MAY ENCODE ANISOTROPY.

BUT ON SECOND THOUGHT: THIS IS REALLY SILLY!

→ why do the hard calc when we just want to know how good the easy calc is? (WE'RE REALLY ASKING: DO I NEED TO DO NLO?)

GOAL: ESTIMATE ERROR

$$Er = \frac{t_1 - t_0}{t_0}$$

LEADING ORDER

NLO, MORE REALISTIC

THIS IS A DIMENSIONLESS COMBINATION THAT GIVES FRACTIONAL ERROR

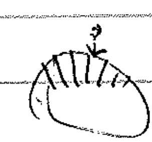
FROM NEGLECTING "MICROPHYSICS"

btw: this is the whole point of RENORMALIZATION GROUP MENTIONED IN HW!

IN OUR HS PROBLEM: HIGHER ORDER

- QM
- SPECIAL/GEN. RELATIVITY
- E+M (DIPOLES ...)

$\Rightarrow$ BIGGEST MAY BE



EARTH IS ROUND

A FANCY WAY TO DERIVE OBVIOUS RESULT

$$\epsilon_r = f(\xi)$$

↑ DIMENSIONLESS PARAMETER CHARACTERIZING HIGHER ORDER PHYSICS

PICK ξ st $\xi \to 0$ CORRESPONDS TO TURNING OFF H.O. EFFECTS

(if not, then use $\xi' = 1/\xi$

THEN $f(0) \equiv 0$ ... WE HAVE ONE TRICK

Taylor EXPAND:

$$f(\xi) = f(0) + \left.\frac{df}{d\xi}\right|_0 \xi + \mathcal{O}(\xi^2)$$

↑ DIMLESS, PRESUMABLY $\mathcal{O}(1)$

so: $\epsilon_r = \boxed{\frac{t_1 - t_0}{t_0} \sim \xi}$

APPLY TO OUR PROBLEM:

NLO: g IS NOT CONSTANT, DEPENDS ON HEIGHT.

RELEVANT DIMENSIONFUL PARAMS?
RADIUS OF THE EARTH, R.

Why not G_N?
ALREADY ENCODED IN R & g

WHAT IS DIMENSIONLESS PARAMETER?

$$\zeta = \frac{h}{R}$$

why not R/h?
BIG, BLOWS UP IN $R \to \infty$ LIMIT WHERE WE KNOW L.O. IS CORRECT.

SO:

$$t \approx t_0 \left(1 + \frac{t_1 - t_0}{t_0} + \cdots\right)$$

$$\approx \boxed{t_0 \left(1 + \mathcal{O}(h/R)\right)}$$

FYI: CHECKING THIS (just this once, to make a point)

$$\cancel{m}\frac{d^2x}{dt^2} = -\frac{G_N M \cancel{m}}{R^2 (1+x/R)^2}$$

$= g$

$$\ddot{x} = -g\left(1+x/R\right)^{-2}$$

CAN ALREADY SEE THAT THE ERROR IS $O(h/R)$

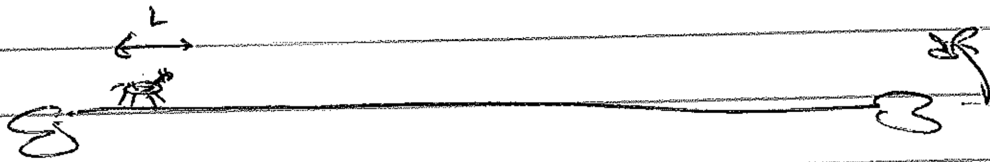
SOLVE TO NLO!

$$\ddot{x} = -g\left[1 - 2\frac{x}{R} + O\left(x^2/R^2\right)\right]$$

(SEE LEC 1 NOTES FOR DETAILS)

FROM ARNOLD (HW EXTRA CREDIT)

A DESERT ANIMAL HAS TO COVER A LARGE DISTANCE BETWEEN SOURCES OF WATER.



HOW DOES THE MAX TIME THE ANIMAL CAN RUN DEPEND ON THE SIZE OF THE ANIMAL?

m

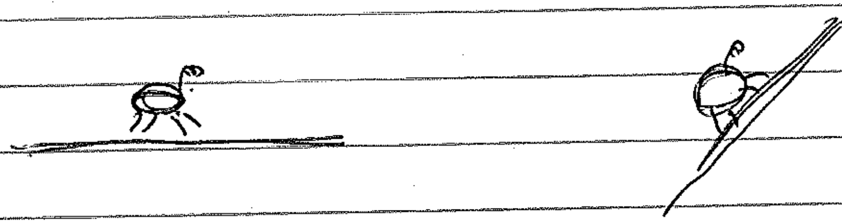
CHARACTERISTIC SCALE L

SOL: PROBLEM IMPLIES THAT THE LIMITING FACTOR IS DEHYDRATION.

AMT OF WATER $\sim$ VOL $\sim L^3$

PERSPIRATION $\sim$ AREA $\sim L^2$

FOLLOW UP: HOW DOES VELOCITY OF AN ANIMAL ON FLAT GROUND VS. UPHILL DEPEND ON L?



SOL: POWER $\propto L^2$

CHEM ENERGY → HEAT

HEAT OUTPUT $\propto$ SURFACE AREA

$$\left.\begin{aligned} \text{AIR RESISTANCE} &\sim v^2 \times \sigma \\ &\sim v^2 L^2 \end{aligned}\right\} \text{FLAT.}$$

POWER SPENT OVERCOMING IT:

$\sim (v^2 L^2)\, v$

$\sim v^3 L^2$

$$v^3 \cancel{L^2} \sim \cancel{L^2}$$

$$\Rightarrow \boxed{v \sim L^0} \quad \text{FLAT}$$

RABBITS & HORSES ARE ABOUT THE SAME SPEED ON FLAT

TO GO UPHILL, YOU'RE FIGHTING GRAVITY

POWER REQ: $mgv \sim L^3 v$

$\uparrow \sim \mathrm{Vol}$

SO $L^3 V \sim L^2$

$\uparrow$ generated power

$$\Rightarrow \boxed{V \sim L^{-1}}$$

DOGS RUN UPHILL EASILY, HORSES SLOW DOWN

you learn this in Zelda.

THE HIERARCHY PROBLEM

ELECTRON REST ENERGY : $E_0 = m_e c^2$

$$\underbrace{\Delta E_{\text{coulomb}}}_{\text{from electric self energy}} = \frac{\alpha}{r_e}$$

fictional electron radius
(smear it out)

$r_e \doteq 10^{-17}$ cm $\rightarrow \Delta E \gtrsim 10$ GeV
13 limit

$$\text{OBSERVED REST ENERGY} = \underbrace{m_e^{(0)} c^2}_{\text{unobserved "bare" mass}} + \Delta E$$

$$0.5 \text{ MeV} = (\underline{\qquad} + 10) \text{ GeV}$$

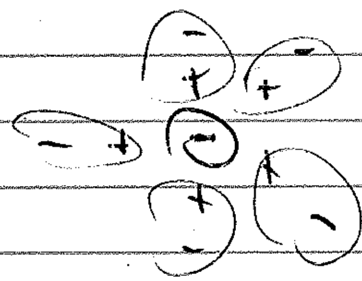
must be $\underbrace{-9.995}$ FINE TUNING @ 0.1% LEVEL

weird, BUT NO BIG DEAL

TO AVOID THIS, WE NEED NATURE TO "FIX" OUR MODEL (NEW EFFECTIVE MODEL) @ A LENGTH SCALE

$$r = \frac{e^2}{\alpha m_e c^2} \sim 3\times 10^{-13}\ \text{cm}$$

WHAT PHYSICS REGULATES $1/r$ DIVERGENCE?



VIRTUAL PARTICLES SHIELD THE POINT CHARGE

$\Delta t \Delta E \sim \hbar$ ← for VIRTUAL PARTICLE

$\Delta t \sim \hbar/(2m_e c^2)$

$d \sim c\Delta t \sim \hbar c/2m_e c^2 \sim 200 \times 10^{-13}$ cm

CHARACTERISTIC SCALE

NATURE (QM) SAVED THE DAY, W/ 2 ORDERS OF MAG TO SPARE.

SIMILAR STORY FOR HIGGS.