GREEN'S FUNCTIONS IN (3+1)-D → Ref?

WAVE EQ ON EM POTENTIALS:

(1) $$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \underline{r}^2}\right]\varphi(\underline{r},t) = \rho(\underline{r},t)$$

(2) $$\left[\frac{1}{c^2}\frac{\partial}{\partial t^2} - \frac{\partial^2}{\partial \underline{r}^2}\right]\underline{A}(\underline{r},t) = \underline{j}(\underline{r},t)$$

} Lorentz gauge; 4 EQNS

from now on: $c = \varepsilon_0 = \mu_0 = 1$
& we work in vacuum

WHAT'S NEW: this is a PARTIAL DIFFERENTIAL EQ.

still linear, so we can still use Green's functions to solve.

SPECIAL RELATIVITY IS BUILT IN:

$$A_\mu = (\varphi(x), A_1(x), A_2(x), A_3(x))$$

$$x^\mu = (t, x, y, z)$$

$j^\mu = (\rho, \underline{j})$

$$\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$$

LORENTZ METRIC: $$x_\mu = \eta_{\mu\nu}x^\nu = (t, -x, -y, -z)$$

so: $$B^\mu B_\mu \equiv B^2 = (B^0)^2 - (\underline{B})^2 = (B^0)^2 - (B^1)^2 - (B^2)^2 - (B^3)^2$$

DERIVATIVES:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad \partial^2 = \partial^\mu\partial_\mu = \left(\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial \underline{r}}\right)^2$$

so (1) & (2) → $$\boxed{\partial^2 A_\mu(x) = j_\mu(x)}$$ ← 4 EQNS

one diff. op (G); four "states"; four sources

GIVEN THE DIFF. OPERATOR, $\boxed{\partial^2}$, CAN FIND THE GREEN'S FUNCTION

$G(x, x') = G(x - x')$ ← want to derive.

spacetime coord of source (for x')

spacetime coord of observer (for x)

s.t. $\boxed{\partial^2 G(x, x') = \delta^{(4)}(x - x')}$ ← one eqn.

then: $\boxed{A_\mu(x) = \int d^4x' \, G(x, x') \, j_\mu(x')}$

four eqns w/ the same Green's function but different sources for each component.

$\delta^{(4)}(x - x') = \delta(t - t') \, \delta^{(3)}(\underline{x} - \underline{x}')$

(x: 4-vector)

CHOOSE SIGN!

$$\text{so: } \delta^{(4)}(x - x') = \int dE \, e^{-iE(t-t')} \int dk_x \, e^{+ik_x(x-x')} \cdot \int dk_y \, e^{+ik_y(y-y')} \cdot \int dk_z \, e^{+ik_z(z-z')}$$

ENERGY IS CONJUGATE VARIABLE TO TIME

$k = (\omega, \underline{k})$

$$= \int d^4k \, e^{-ik_\mu(x-x')^\mu}$$

$$k_\mu(x-x')^\mu = k \cdot (x - x') = \omega(t - t') - k_x(x - x') - k_y(y - y') \cdots$$

nb: $e^{ik \cdot x} = e^{-i(Et - \underline{k} \cdot \underline{x})}$ ← plane wave

FOURIER TRANSFORM L.H.S of Green's func. eq.

$$\partial^2 \int d^4k\, e^{-ik\cdot x} \tilde{G}(k,x') = \int d^4k \left(-E^2 + \underline{k}^2\right) e^{-ik\cdot x} \tilde{G}(k,x')$$

$$\left[\left(\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial \underline{r}}\right)^2\right] e^{-ik\cdot x}$$

$$= (-iE)^2 - (+i\underline{k})^2$$

$$\int d^4k\, (\underline{k}^2 - E^2)\, e^{-ik\cdot x}\, \tilde{G}(k,x') = \int d^4k\, e^{-ik\cdot x}\, e^{ik\cdot x'}$$

$$\boxed{\tilde{G}(k,x') = \frac{e^{ik\cdot x'}}{\underline{k}^2 - E^2}}$$

for convenience: $y^\mu = x^\mu - x'^\mu$
$= (u, \underline{s})$

FOURIER TRANSFORM BACK:

$$\boxed{G(x,x') = \int d^4k\, \frac{e^{-ik\cdot(x-x')}}{\underline{k}^2 - E^2}}$$

now we just have to do this 4D integral

HYPER CYLINDRICAL COORDINATES:

$$d^4k = dE \boxed{d^3\underline{k}} = dE\, |\underline{k}|^2\, d|\underline{k}|\, d\cos\theta\, d\varphi$$

SPHERICAL COORDS

angle w/r/t k_z AXIS

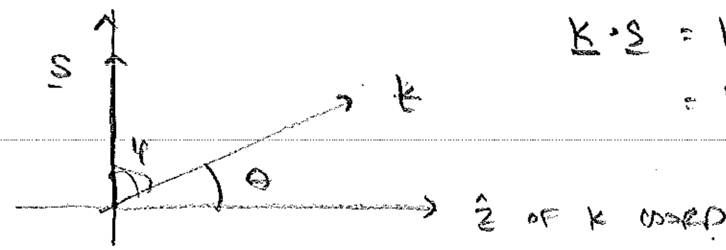
WE ARE FREE TO ALIGN THIS AXIS HOWEVER WE WANT.

$$G(x,x') = \frac{1}{(2\pi)^4}\int dE\, d|\underline{k}|\, d\cos\theta\, d\varphi\, |\underline{k}|^2 \times \frac{e^{-ik\cdot y}}{|\underline{k}|^2 - E^2}$$

$$e^{-ik\cdot y} = e^{-iEy}\, e^{i\underline{k}\cdot\underline{s}}$$

<u>TRICK:</u> there's a convenient way to align the $\underline{k}$ coordinate system

WRONG! ALIGN $\underline{K}$ COORDS W/ $\underline{S}$



$\underline{K}\cdot\underline{S} = |\underline{k}||\underline{s}|\sin\varphi$

$= KS\cos\theta$

s.t. $\underline{K}\cdot\underline{S} = KS\cos\theta$

Same as $d(\cos\theta)$!

LET'S DO SOME INTEGRALS

We have one magic bullet: Residue thm
... but many integrals! SOME WE'LL HAVE TO DO THE HARD WAY.

① $d\varphi$ integral is <u>trivial</u>, no φ-DEP. IN INTEGRAND.
↳ yields $\boxed{(2\pi)}$

② $d\cos\theta$ is now simple: let $w = \cos\theta$

$$\int_{-1}^{1} dw\, e^{iKSw} = \frac{1}{iKS}\left(e^{iKS} - e^{-iKS}\right)$$ ← ACTUALLY LEAVE IT LIKE THIS!

$$= \boxed{\frac{2}{KS}\sin(KS)}$$

$$G(x,x') = \frac{\overset{d\varphi}{\boxed{2\pi}}}{(2\pi)^4}\int dK\, dE\, K^2 \times \underbrace{\boxed{\frac{2}{KS}\sin(KS)}}_{d\cos\theta} \times \frac{e^{-iEu}}{K^2 - E^2}$$

$|E|$

$$\left[\frac{-e^{-iEu}}{E^2 - K^2}\right]$$

$$\boxed{\text{PP}: \frac{1}{8\pi^3 S}\int dk\, \frac{k}{i}\left(e^{iks} - e^{-iks}\right)}$$

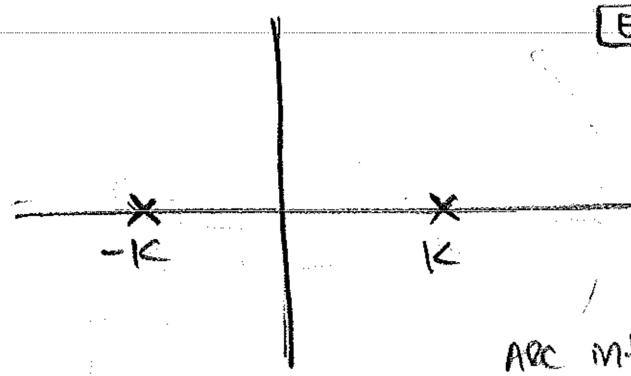
$$= \frac{1}{4\pi^3 S}\int_0^\infty dK\, K\sin(KS)\boxed{\int_{-\infty}^{\infty} dE\, \frac{e^{-iEu}}{K^2 - E^2}}$$

something familiar!

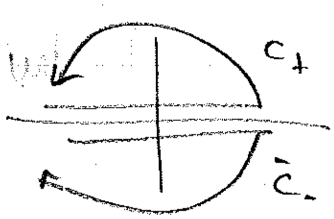
11/15

USE RESIDUE THEOREM: have to choose pole prescription

$\boxed{E}$ ← $\mathbb{C}$ E PLANE (E is integ. var.)



Arc integral: $E = R\cos\theta + iR\sin\theta$



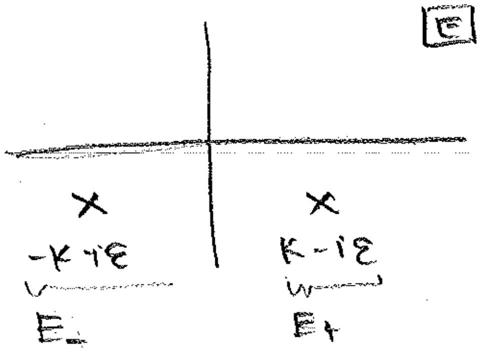
$$e^{-iEu} = e^{-iR\cos\theta \cdot u}\, e^{+R\sin\theta\, u}$$

When $u > 0$, then this is e^{∞} if $\sin\theta < 0$.

so pick $\bar{C}_-$.

BUT $u > 0 \rightarrow t - t' > 0$ so POLES SHOULD BE IN HERE.

$\boxed{E}$



$$\int dE \, \frac{-e^{iEu}}{(E-E_+)(E-E_-)} = 2\pi i \sum \mathrm{Res}$$

$$= 2\pi i \left(\frac{-e^{iE_+ u}}{E_+ - E_-} + \frac{-e^{iE_- u}}{E_- - E_+} \right)$$

$\boxed{E_+ - E_- = 2k}$

$$= \frac{-2\pi i}{E_+ - E_-} \left(e^{iE_+ u} - e^{iE_- u} \right)$$

$$= \frac{-\pi i}{k} \left(e^{iku} - e^{-iku} \right)$$

$\times \Theta(u)$

PLUG INTO $G(x,x')$:

$$G(x,x') = \frac{1}{8\pi^3 s} \int_0^\infty dk \, \frac{k}{i} \left(e^{iks} - e^{-iks} \right) \times \frac{-\pi i}{k} \left(e^{iku} - e^{-iku} \right)$$

$$= \frac{-1}{8\pi^2 s} \int_0^\infty dk \left[e^{ik(s+u)} + e^{ik(s-u)} - e^{-ik(s-u)} + e^{-ik(s+u)} \right] \Theta(u)$$

nb $\int_0^\infty dk \left(e^{ikX} + e^{-ikX} \right)$

$$= \int_{-\infty}^{\infty} dk \, e^{ikX}$$

$$= \frac{1}{8\pi^2 s} \int_{-\infty}^{\infty} dk \left(e^{ik(s-u)} - e^{ik(s+u)} \right) \Theta(u)$$

you know this integral.

$$\int \bar{d}k \, e^{ika} = \delta(a)$$

note 2π!

$$= \frac{1}{4\pi s} \left(\delta(s-u) - \cancel{\delta(s+u)} \right) \Theta(u)$$

FAMOUS 4π

because s is a radial coord $\uparrow$ $u > 0$.

REWRITING IN TERMS OF t & $\underline{x}$:

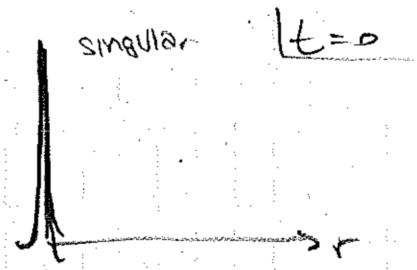
$$G(x,x') = \frac{1}{4\pi|\underline{x}-\underline{x}'|}\,\delta\big(|\underline{x}-\underline{x}'| - (t-t')\big)\,\underbrace{\Theta(t-t')}_{\text{CAUSALITY}}$$

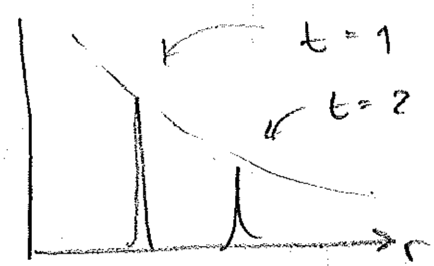
if we set $\left.\begin{matrix}\underline{x}'=0\\ t'=0\end{matrix}\right\}$ unit source @ origin of spacetime

→ restore factors of c by dimensional analysis

$$G(r,t) = \frac{1}{4\pi r}\,\delta(r-ct)\,\Theta(t)$$

SNAPSHOTS:





each of these are singular, but w/ smaller coefficient

this is a wave front.

(compare to wake of a ship)

↑ very interesting problem

SO WHAT DOES THIS MEAN? $\delta \rightarrow$ MUST BE INTEG.
THE THING WE WANT IS THE POTENTIAL:

$$A_\mu(x) = \int d^4x' \, G(x,x') \, j_\mu(x')$$

$$= \int d^3\underline{x}' \frac{1}{4\pi|\underline{x}-\underline{x}'|} \int dt' \, \delta(|\underline{x}-\underline{x}'|-(t-t')) \times j(\underline{x}', t')$$

$$= \int d^3\underline{x}' \frac{j(\underline{x}', \overbrace{|\underline{x}-\underline{x}'|}^{\text{TIME}})}{4\pi|\underline{x}-\underline{x}'|}$$

$|\underline{x}-\underline{x}'|$ is a <u>time</u> $\rightarrow \frac{1}{c}|x-x'|$

"TIME FOR LIGHT TO TRAVEL THIS DISTANCE"

SO: $j(\underline{x}', |\underline{x}-\underline{x}'|)$ is;
the SOURCE @ $\underline{x}'$ (POSITION)
@ the TIME FOR WHICH WIGGLES @ $\underline{x}'$
WOULD JUST BE REACHING ME <u>NOW</u> (@ t)
GIVEN THAT LIGHT TRAVELS @ SPEED $c=1$.

your homework: redo this
for <u>FLATLAND</u>

(2+1) DIM.