- HW 2 UP.
- NOW IS A GOOD TIME TO MAKE SURE YOU HAVE A TEXTBOOK

BOAS → GOAL is §15, integral transforms
w/ §14 ($\mathbb{C}$ analysis) as necessary

BYRON & FULLER → CH. 5, 7 ; ch. 6 as nec.
$\uparrow$ HILBERT $\uparrow$ GREEN $\uparrow$ $\mathbb{C}$

- LAST WEEK : VECTOR SPACES, review
why? TO GET TO THIS WEEK

- this week : $\boxed{\text{FUNCTIONS as } \infty\text{-DIM VECTOR SPACES}}$

some aspects of vector-ness are obvious.

eg can pick a basis of monomials

$$M_i = x^i$$

eg
$$f(x) = 3x^2 + 7x + 2$$
$$= 3|M_2\rangle + 7|M_1\rangle + 2|M_0\rangle$$
$$= f^i|M_i\rangle$$
$$\uparrow \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$$

THIS is LINEAR: if
$$f(x) = f^i|M_i\rangle$$
$$g(x) = g^i|M_i\rangle$$

$$(f+g)(x) = (f^i + g^i)|M_i\rangle$$

$\uparrow$ eg

$$\begin{array}{rll} & 3x^2 + 7x + 2 & \\ + & 2x & \leftarrow g(x) \\ \hline & 3x^2 + 9x + 2 & \leftarrow (f+g)(x) \end{array}$$

OTHER PROPERTIES ARE NOT IMMEDIATELY OBVIOUS

- IS THIS BASIS ORTHONORMAL?

$\hookrightarrow$ WHAT ARE THE DUAL VECTORS?
WHAT IS THE INNER PRODUCT?
$\uparrow$
HAVE TO DEFINE

convenient definition : L^2 norm

$$\| f \|^2 \text{ or } |f|^2 = \int |f|^2 \, dx$$

$$\boxed{\langle g | f \rangle = \int g^*(x) f(x) \, dx}$$

$f(x)$

$g(x)^\dagger$

[so $\langle g| = g(x)^*$
acts on $|f\rangle$ as integral.

$$\langle g | f \rangle = \langle f | g \rangle^*$$

[$\longrightarrow$ DO SINES & COSINES]

GENERALIZATION :

$$\langle g | f \rangle_W = \int dx \, W(x) \, g^*(x) \, f(x)$$

WEIGHT

$W(x) \neq$ const

when do you get nontrivial weights?

$\llcorner$ 3D spaces : $d^3x = \underline{\underline{r^2}} dr \, d\Omega$

or curvy spaces

$\hookrightarrow$ near black holes & related things (including dual systems!)

- <u>BASIS VECTORS</u> (w/ $\langle\cdot|\cdot\rangle$, also gives 1-forms)

the monomial basis is neither <u>orthogonal</u> nor <u>normal</u>.

$$\langle m^i | m_j \rangle \neq \delta^i{}_j \qquad \underbrace{\langle m^i | m_i \rangle}_{\text{no sum}} \neq 1$$

$m_i(x)^*$

of course I mean "astro" as well

as physics students, we have an intuitive example of an orthogonal basis

<u>FOURIER SERIES</u>

$$|e_i\rangle = \sqrt{2}\,\sin(n\pi x)$$

hint for something important.

clearly $f = f^i |e_i\rangle$ is linear just like monomial case

ALSO LIKE MONOMIAL CASE: A DISCRETE BASIS OF ∞ DIMENSIONAL FUNCTION SPACE.

countably ∞ for something which is not necessarily so.

WE CAN USE INNER PRODUCT TO IDENTIFY COEFFICIENTS

$$f^n = \langle e^n | f \rangle = \int dx\, \sqrt{2}\,\sin(n\pi x)\, f(x)$$

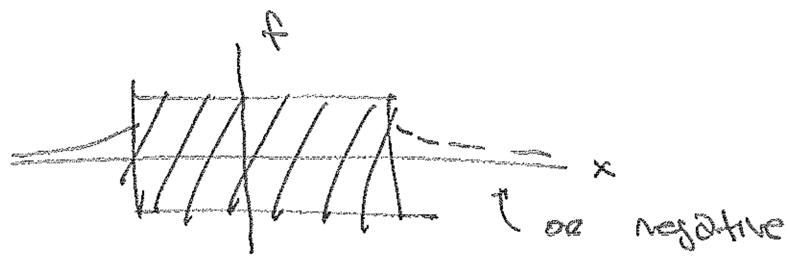
WHAT'S MISSING??

<u>AH</u>: I HAVE TO SPECIFY A <u>DOMAIN</u> FOR THE FUNCTION!

not at all trivial point!
IF THE DOMAIN IS ∞, THEN THE FUNCTIONS HAD BETTER BE SUFFICIENTLY WELL BEHAVED SO THAT

$\| f \|$ is FINITE!

BASIS FUNCTIONS MUST LOOK SOMETHING LIKE



IN OUR EXAMPLE, $|e_i\rangle = \sqrt{2} \sin(n\pi x)$

CHOSE A NORMALIZ. OF $|e_i\rangle$; must be s.t. $\|e_i\| = 1$

$$\|e_n\|^2 = 2 \int dx \sin^2(n\pi x)$$

IF DOMAIN IS $[0, 1]$, then $\|e_n\|^2 = 1$ ✓

SO: FUNCTION SPACE → define norm
orthonormal basis
define domain

there's another point to make here:

in our example, we now have

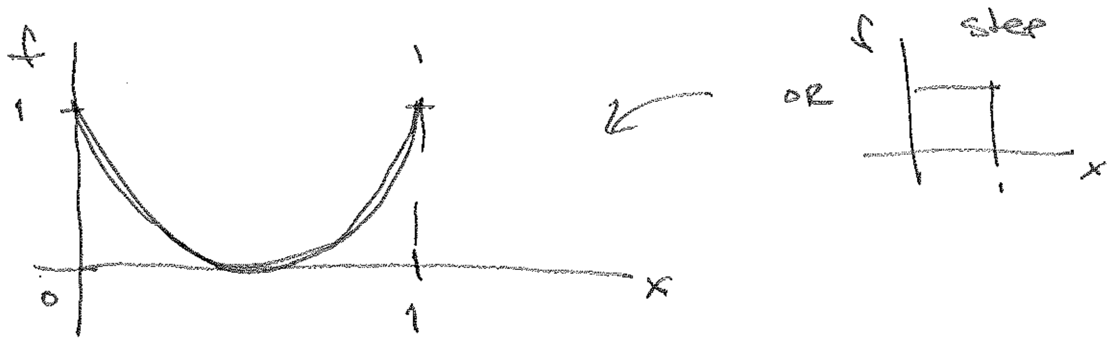
$$\langle g|f\rangle = \int_0^1 dx \; g^*(x)\, f(x)$$

DOMAIN

$w(x)=1$

$*$ is ornamental if we restrict to $\mathbb{R}$-valued f.

HOW DO I DECOMPOSE A FUNCTION THAT LOOKS LIKE:



WE CAN CERTAINLY PROJECT ONTO $|e_n\rangle = \sqrt{2}\sin(n\pi x)$

↳ get a set of #'s f^n

WHAT'S WRONG WITH IT?

↳ in some technical sense, nothing.
CAN JUST DO THE PROJECTION

BUT: for $n \in \mathbb{Z}$, $\sin(n\pi x) = 0$ @ $x=0$
@ $x=1$

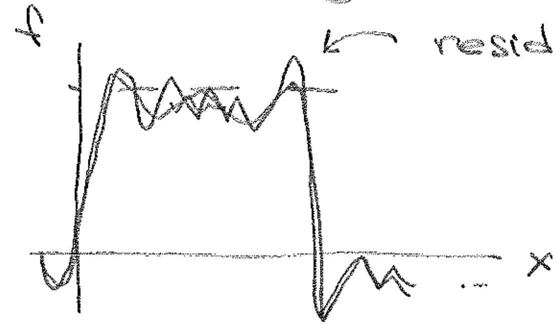
whereas $f(x)$ can be nonzero there!!
this is problematic.

- "JUST DO IT" APPROACH:

  CONVERGENCE is not something we'll worry about.

  this is the Gibbs phenomenon

  ↳ you can technically get arbitrarily close using sines



maybe this is good enough? (no)

- "WRONG BASIS" APPROACH

  ↳ oh, in this case, use cosines...

  but we never said, in $\mathbb{R}^3$ that "in this case" the cartesian basis fails.

  ... this is closer to identifying the problem

- "OH, USE SINES AND COSINES" APPROACH

  ↳ DOUBLES the size (dimensionality) of vec space

  MAYBE WE DON'T CARE... $2 \times \infty$ is STILL ∞. (in fact, still countably infinite)

  BUT IT IS DEFINITELY AN OVERFULL BASIS

  ↳ you have twice as many coefficients to solve for @ a given truncation!

  → also, not ORTHOGONAL

KEY POINT: function space not only requires a DOMAIN, but also BOUNDARY CONDITIONS

in fact, this is what we meant when we were worried about ∞ DIM. DOMAIN.

So $|e_i\rangle$ is A GOOD BASIS FOR FUNCTIONS ON THE DOMAIN $[0,1]$ W/ BOUNDARY CONDITION

$$f(0) = f(1) = 0$$

continuous, well behaved, blah blah ...

we live in a Panglossian universe

2 KINDS OF BC THAT WE ALWAYS SEE

- DIRICHLET $\quad f|_\partial = 0$
  BOUNDARY
- NEUMANN $\quad df|_\partial = 0$

now you can see why: these are satisfied when you take linear combinations of functions.

on the other hand: $f(1) = 3$ is problematic

if f & g satisfy this, then

αf DOES NOT.
$(f+g)$ DOES NOT.

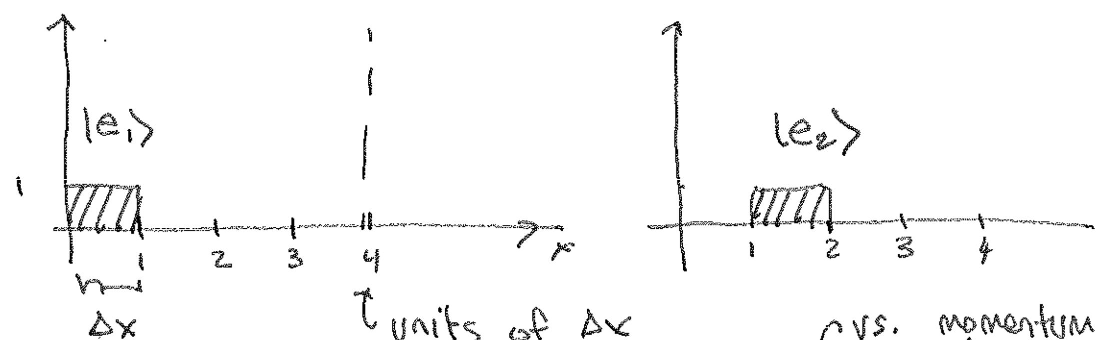
$\}$ not a VECTOR SPACE!

A FUNCTION SPACE (as VECTOR SPACE) is (physicist's definition)

1. DOMAIN
2. INNER PRODUCT
3. BOUNDARY CONDITIONS (that make sense)

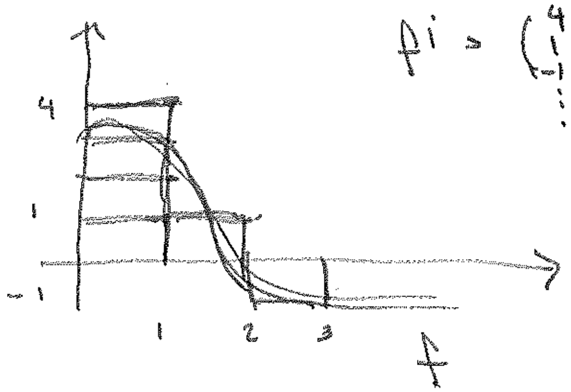
<u>Wed</u>: WHAT ARE LINEAR OPERATORS ON THIS SPACE?

ANOTHER APPROACH: LEGO SYSTEM.
↑
DISCRETIZE, then take $\Delta x \to 0$ LIMIT LATER



vs. momentum space of sines!

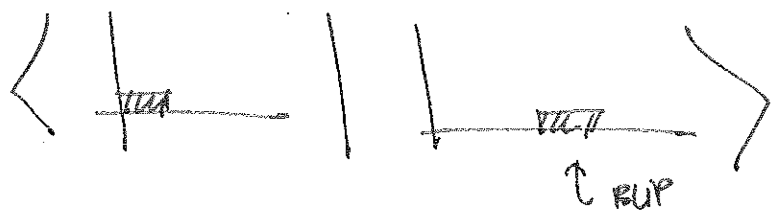
this is like going to <u>position space</u>
gives the value of the function @ each $(x, x+\Delta x)$ interval.



$f^i = \begin{pmatrix}4\\1\\-1\\\vdots\end{pmatrix}$ eg

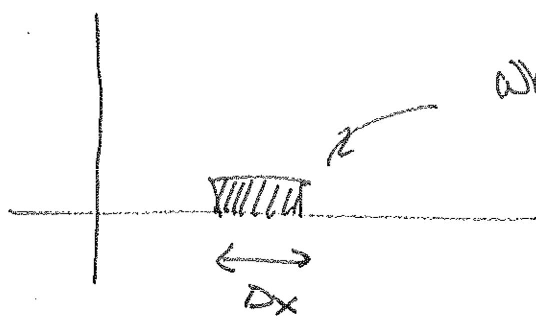
THIS IS CLEARLY LINEAR.
↳ also, seems to not require BC!
if you want $f(1) = 0$, then $f^N = 0$
↳ where $N = 1/\Delta x$.

WHAT GOES WRONG?
- CLEARLY AN ORTHOGONAL BASIS



$= 0$

- is it A NORMAL BASIS?



what is the height?

$$|e_i\rangle = \begin{cases} \# & \text{if } x \in (x_i, x_i + \Delta x) \\ 0 & \text{otherwise} \end{cases}$$

$$\langle e^i | f \rangle = f^i$$ ← value of func @ x_i (or $x_i + \Delta x/2$)

$$f(x_i) = \int_0^1 dx \, (\text{BUP}) \, f(x) \approx \Delta x \, f(x_i) \times \underset{\text{HEIGHT}}{\#}$$

EVIDENTLY: $|e_i\rangle = \begin{cases} 1/\Delta x & \text{if } x \in (x_i, x_i + \Delta x) \\ 0 & \text{otherwise} \end{cases}$

FURTHER:

$$\langle e^j | e_i \rangle = \frac{1}{\Delta x}\, \delta^j{}_i$$

this should be 1 !!

so this is not a great basis.

it IS, however the discretized version of the DIRAC δ - FUNCTION (DISTRIBUTION) that will be very important for us!

BY THE WAY: IDENTITY OPERATOR

$$\mathbb{1} \longleftrightarrow \delta^i{}_j \, |e^i\rangle\langle e_j| = \sum_i |e^i\rangle\langle e_i|$$

IN FUNCTION SPACE:

$$\sum_{n=1}^{\infty} \left(\sqrt{2}\,\sin(n\pi x)\right)\left(\sqrt{2}\,\sin(n\pi y)\right) = \delta(x-y)$$