# PHYSICS 231 : LECTURE 1 9/29

THIS IS A "MATH METHODS" COURSE.
THE MOST IMPORTANT THING TO KNOW:

$$\boxed{\text{Physics} \neq \text{Math}}$$

HOW ARE WE DIFFERENT?
many answers, some are more insightful

1. EMPIRICAL FOUNDATION of SCIENCE
   even though PHYSICS separates

   THEORY from EXPERIMENT

   still not math — still need this course

2. RELIANCE ON TAYLOR SERIES
   or more general: PERTURBATION THEORY

   ... not just an approx,
   making the right approx

   ... sometimes even if domain of validity = 0!
   ... very few exact results

my personal favorite answer:

3. $\boxed{\text{UNITS}}$

physics relates/predicts/measures DIMENSIONFUL QUANTITIES

cm, sec, GeV, ...

TO BE VERY CLEAR: what is a unit?

 ← this is $\boxed{\text{3 apples}}$

number — standard (UNIT)

WHAT ARE WE MEASURING?

eg: COST ≈ \$3

CALORIC INTAKE ≈ 100 kcal

MASS ≈ 100 g

"3 apples" means something different depending on what we're measuring

UNIT of WHAT DIMENSION?

**CONVERTING UNITS** : multiply by 1.

$$3 \text{ apples} \times \underbrace{\left( \$1 / \text{apple} \right)}_{\substack{\text{exchange rate} \\ \text{is really multiplying by } 1}} = \$3$$

$$\underbrace{\frac{1 \text{ apple}}{\text{apple}}}_{=1} = \frac{\$1}{\text{apple}}$$

~~[illegible]~~ = 1

eg **NATURAL UNITS** : $\hbar = c = 1$

"treat these constants of nature as conversions"

SPEED OF LIGHT $= 3 \times 10^{10}$ cm/s

$\hookrightarrow$ $1 \text{ s} = 3 \times 10^{10} \text{ cm}$

$\uparrow$ DIST. TRAVELED BY LIGHT IN 1s

$1 \text{ s} = 3 \times 10^{-8} \text{ yr}$ (nb 1 sig fig)

$$\Rightarrow \boxed{1 \text{ yr} = 10^{18} \text{ cm}}$$

$\uparrow$ we often say "LIGHT YEAR"

need to know your conversion

HOW LONG is A [LIGHT] YEAR?

$$1\ \text{yr} \times \underset{\substack{\uparrow \\ \equiv 1 \text{ in natural units}}}{c} = [1\ \text{yr} \times (3\times10^{7}\ \text{s}/\text{yr})] \times (3\times10^{10}\ \text{cm}/\text{s})$$

SIMILARLY: $\hbar = 10^{-27}\ \underbrace{\text{g cm}^2/\text{s}}$

by the way: what is this a unit for?
(ANGULAR MOMENTUM ; ACTION)

$S = \int L\, dt$

<u>YOUR HW</u>: GETTING USED TO NATURAL UNITS.

# DIMENSIONAL ANALYSIS

PHYSICAL QUANTITY $q$ HAS DIMENSION $[q]$ WHICH WE TYPICALLY WRITE AS:

$$[q] = L^a M^b T^c$$

LENGTH MASS TIME

- COULD HAVE USED OTHER COMBINATIONS (eg PRESSURE), BUT TYPICALLY REDUCE TO THESE

- DIMENSIONS ARE INDEP. OF UNITS, THE LATTER "MEASURE" THE FORMER

eg. FORCE: WE KNOW $\vec{F} = m\vec{a} = m\ddot{\vec{x}}$

$$\Rightarrow [F] = L^1 M^1 T^{-2}$$

observe: $[F] = [m][a]$

eg. ENERGY: $E = \frac{1}{2}mv^2$ (OR $mc^2$)

$$[E] = M^1 L^2 T^{-2}$$

(nb. NATURAL UNITS: $c = \hbar = 1$, SO ONLY ONE "DIMENSION", MASS. $[E] = +1$)

# HOW TO USE DIM. ANALYSIS

## 1. SANITY CHECK OF EXPRESSIONS

eg $(1+x)$ vs. $(1+L)$

okay

WRONG IF $[L] \neq 1$ !

EXPECT $(1+L/L_0)$

THIS IS IMPORTANT. IF $x = 0.1$ CHANGES TO $x = 0.2$, I KNOW THAT $(1+x)$ DOESN'T CHANGE MUCH. BUT IF $L = 1\text{cm}$ & CHANGES TO $L = 2\text{cm}$, I HAVE NO IDEA WHAT $(1+L/L_0)$ DOES UNLESS I KNOW $L_0$.

## egregious examples

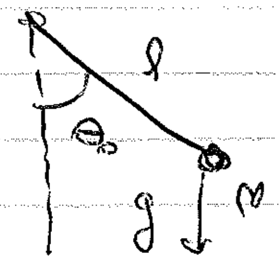
$\sin(3\text{ cm})$ ?!

$$e^L \sim 1 + L + \frac{1}{2!}L^2 + \dots$$

EACH TERM HAD BETTER HAVE THE SAME DIMENSION

## 2. ROLE OF DIMENSIONFUL QUANTITIES IN A PROBLEM

eg. What is period of a pendulum?




**RELEVANT QUANTITIES**

$[\ell] = L$

$[m] = M$

$[g] = LT^{-2}$

$[\theta_0] = 1$

$LM/T^2$

$\downarrow$

$F = -G\frac{Mm}{r^2}$

$L^3M^{-1}T^{-2}$

**IRRELEVANT QUANTITIES**

$[G] = L^3M^{-1}T^{-2}$ ← packaged into $g$

$[R_\oplus]$ RADIUS OF EARTH

$[M_\oplus]$ MASS OF EARTH

$\vdots$

PERIOD IS A TIME: START W/ $\boxed{g^{-1/2}}$

$[g^{-1/2}] = T L^{-1/2}$ can cancel w/ $\ell$

so: $\boxed{\tau \sim \sqrt{\ell/g} \times f(\theta_0)}$

don't know from D.A

key point: indep. of $m$

most important symbol in phys.

HOW TO GET THE PHYSICS OUT:

"when $l$ goes up, $\tau$ goes up
$g$ ——, $\tau$ goes down

AS SQUARE ROOT

" when $m$ goes up, $\tau$ unchanged"

THIS is PHYSICS, OVERALL PREFACTORS ARE JUST MEASUREMENTS!

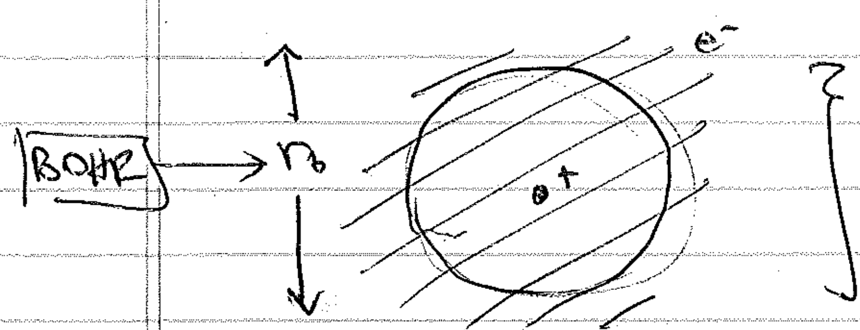
pro tip: this is an effective way to TA undergrads.

estimating { DIMENSIONLESS PREFACTORS ~ $\mathcal{O}(1)$

## A MORE ADVANCED EXAMPLE

- HOW LARGE IS HYDROGEN?
  (math.ucr.edu/home/baez/lengths.html)

→ what does this even mean?




QUANTUM FUZZ of 1S $e^-$

## WHAT COULD IT POSSIBLY DEPEND ON?

$m_e \quad e = q_e = -q_p \quad m_p \quad \hbar \quad c \quad G_N \quad \ldots$

## WHICH OF THESE DON'T MAKE SENSE?

- $m_p \gg m_e$ ... ? $m_e = 0$ limit seems fine
  PROBABLY ONLY ONE MATTERS w/ $\mathcal{O}(m_e/m_p)$ CORRECTIONS

IN FACT, WE KNOW THAT WHAT REALLY MATTERS IN A 2-BODY PROBLEM IS THE REDUCED MASS :

$$\mu \equiv \frac{m_p m_e}{(m_p + m_e)} \approx \boxed{m_e}$$

SO KEEP THIS.

- $G_N$ HAS TO DO W/ GRAVITY ... MUCH WEAKER THAN E&M IN THIS SYSTEM. IGNORE!

- $c$ NONRELATIVISTIC; CAN TAKE $c \to \infty$ LIMIT WHERE IT DOESN'T SHOW UP. CORRECTIONS $\sim \mathcal{O}(v/c)$.

LEFT WITH: $m_e$, $e$, $\hbar$

MASS CHARGE "QUANTUM-NESS"

DIMENSIONS: $[m_e] = M$

$[\hbar] = \underbrace{M L^2 T^{-1}}_{\text{ENERGY} \times \text{TIME}}$

$[e] = \underbrace{M^{1/2} L^{3/2} T^{-1}}$

on homework
hint: FORCE LAW
$[e] = [F]^{1/2} L$

BOHR RADIUS IS A LENGTH

$\left[\frac{\hbar}{e}\right] = M^{1/2} L^{1/2}$ (get rid of T)

$\left[\frac{\hbar}{\sqrt{m_e}\, e}\right] = L^{1/2}$ (get rid of M)

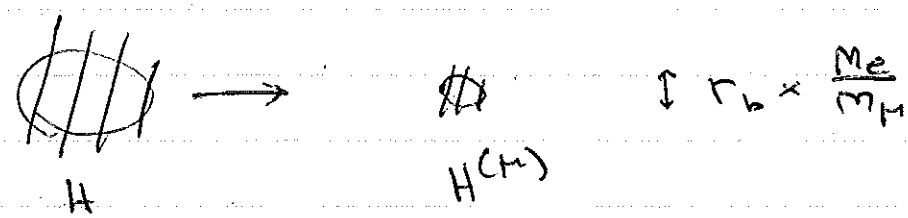
$$\Rightarrow \boxed{r_b = (\#) \frac{\hbar^2}{m_e e^2}}$$

coincidentally, the 1st principles derivation in QM gives $(\#) = 1$ ; could have had $(2\pi\text{'s})$ ...

SO WHAT?

→ CHARACTERISTIC ATOMIC SCALE FROM DIMENSIONAL ANALYSIS OF FUNDAMENTAL PARAMETERS

→ MUONIC ATOMS – how do they scale?

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DOING EVEN MORE WITH DIMENSIONAL ANALYSIS

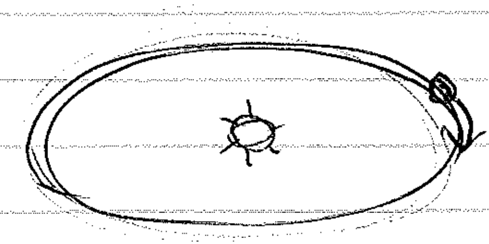
SCALING & SIMILARITY

see: V.I. ARNOLD <u>Math. Methods of Classical Mech.</u>

$$m\ddot{\underline{r}} = -\frac{\partial U}{\partial \underline{r}} \quad \leftarrow \quad [U] = [F] = ML^2T^{-2}$$

$\dot{} = d/dt$ so $[\dot{}] = T^{-1}$

SUPPOSE WE HAVE A GRAVITATIONAL SYSTEM ↳ planet w/ elliptical orbits around a star.




THIS MEANS: $\underline{r}_0(t)$ IS A SOLUTION TO PDE.

SO WHAT? CAN USE DIMENSIONAL ANALYSIS TO UNDERSTAND (generate) <u>OTHER</u> SOLUTIONS. HOW?

CONSIDER A <u>SCALE TRANSFORM</u> OF $t$:

$t = \alpha t'$ new variable
(THIS IS ESSENTIALLY USING NEW UNITS)

$$m\ddot{\underline{r}} = -\underbrace{\frac{\partial U}{\partial \underline{r}}}$$

no time dependence
(ASSUME STATIC POTENTIAL)

only this side changes

[ BUT RHS HAS DIMENSION $\sim T^{-2}$ ?
→ COMES FROM $G_N$, CONSTANT ]

$$m\,\ddot{r}_0(t) = \boxed{m\alpha^{-2}}\left(d/dt'\right)^2 \underline{r}_0(\alpha t')$$

$\equiv m'$ SOME OTHER MASS SCALE

$$= \boxed{\underbrace{m'\left(\frac{d}{dt'}\right)^2 \underline{r}_0(\alpha t')} = -\frac{\partial U}{\partial \underline{r}}}$$

ALSO A SOLUTION!

so: eg $\alpha = 2$ then: trajectory is covered twice as fast.

not true for GRAV. POT!

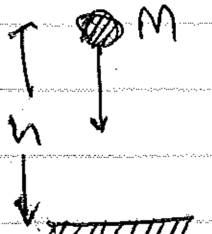
$m' = m/4$ ⇐ if MASS IS DECREASED BY A FACTOR OF 4

(ASSUMING SAME FORCE FIELD!!)

## DIMENSIONAL ANALYSIS & ERROR ESTIMATES

see: Bohren, Am. J. Phys. 72 534
[& counter point 1102. 1120]

## A HIGH SCHOOL PROBLEM




WHAT IS THE TIME, $t$, FOR OBJECT TO HIT THE GROUND AFTER BEING DROPPED (from rest) FROM HEIGHT $h$?

HIGH SCHOOL ANSWER: $|\ddot{\underline{x}}| = g$

integrate: $x = \frac{1}{2} g t^2 + \cancel{c}t + \cancel{D}$

$x(t_h) = h$     $c = 0 = v_0$     $D = 0$ (PICK COORD)

$$\boxed{t_h = \sqrt{\frac{2h}{g}}}$$

easy. often good enough

how good?

next-to-leading-order

USUAL APPROACH: DO THE HARDER ("NLO") CALCULATION & COMPARE.

BUT THAT'S CRAZY, WHY DO THE WORK OF A HARD NLO CALC WHEN YOU JUST WANT TO JUSTIFY THE EASY CALC?!

GOAL: ERROR ESTIMATE ON L.O.

LO

$$\boxed{\frac{t_{REALISTIC} - t_0}{t_0}}$$

DIMENSIONLESS COMBINATION THAT GIVES FRACTIONAL ERROR FROM NEGLECTING HIGHER ORDER CORRECTIONS

"microphysics"

By the way: this is a BIG IDEA in physics
WHY A CHEF DOESN'T NEED TO KNOW PARTICLE PHYS.

underlying idea of RENORMALIZATION GROUP

MAIN IDEA: ERROR IS SMALL.

otherwise our leading order calc. was not L.O.

$$\text{ERROR} = \frac{t_r - t_0}{t_0} \equiv f(\xi)$$

$\xi$ A DIMENSIONLESS PARAM. CHARACTERIZING THE HIGHER ORDER PHYSICS.

PICK $\xi$ s.t. $\xi \to 0$ corresponds to turning off the higher order effects.

IF NOT, THEN MAYBE USE $\xi' \equiv 1/\xi$

then $f(0) = 0$

this means we may Taylor expand:

$$f(\xi) = f(0) + \underbrace{\left.\frac{df}{d\xi}\right|_0}_{\text{DIMENSIONLESS, PRESUMABLY } \mathcal{O}(1)} \xi + \mathcal{O}(\xi^2)$$

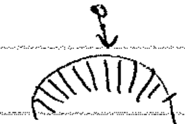
$\Rightarrow$ to leading order,

$$\boxed{\frac{t_r - t_0}{t_0} \sim \xi}$$

PRETTY SIMPLE.

eg. g is <u>not</u> constant. it varies with height (RADIAL DISTANCE FROM THE CENTER OF THE EARTH!)

Relevant dimensionful parameters? R (RADIUS OF EARTH)

$\hookrightarrow$ OUR NLO CORRECTION IS 
(not relativity, quantum, ...)

$\rightarrow$ WHY NOT $G_N$? ALREADY ENCODED IN R & g
WHEN YOU KNOW FORCE LAW / POTENTIAL

$\xi$ : $h/R$ or $R/h$
$\uparrow$ from $R \to \infty$ LIMIT (GIVING LO)

$$\Rightarrow \frac{t_r - t_0}{t_0} \sim \frac{h}{R}$$

just this once

CHECK: $F = m\frac{d^2x}{dt^2} = -\frac{G_N M m}{R^2(1+x/R)^2}$

$\equiv g$

$$\Rightarrow \ddot{x} = -g(1+x/R)^{-2}$$

DIFFICULT DIFF EQ. TO SOLVE. BUT WE ONLY WANT NLO CORRECTION IN $\xi = h/R$, SO TAYLOR EXPAND!

$$\ddot{x} = -g\left[1 - 2\frac{x}{R} + \mathcal{O}(x^2/R^2)\right]$$

NOW WHAT? $2^{ND}$ O INHOMOG DIFF EQ SOLVABLE, BUT THIS IS NOT OUR GOAL ☺

WE CAN USE MATHEMATICA. → FREE STUDENT LIC.
BUT FIRST: PICK CONVENIENT UNITS.

$\boxed{\xi = x/R}$ DIMENSIONLESS

$$\ddot{\xi} = -g/R\,(1-2\xi)$$

<u>then</u>: $t = (g/R)^{-1/2} s = \sqrt{\frac{R}{g}}\, s$

$$\Rightarrow \boxed{\frac{d^2q}{ds^2} = -1 + 2q}$$

<u>Mathematica</u>:

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DSolve[q''[s] == -1 + 2q[s], q[s], s]
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$\hookrightarrow q(s) = c_1 e^{\sqrt{2}s} + c_2 e^{-\sqrt{2}s} + \frac{1}{2}$

<u>INITIAL CONDITIONS</u>

$\dot{x}\big|_{t=0} = 0 \Rightarrow dq/ds = 0 \quad @ \quad s=0$

$= \sqrt{2}\left(c_1 e^{\sqrt{2}s} - c_2 e^{\sqrt{2}s}\right)\Big|_{s=0} = 0$

$\Rightarrow c_1 = c_2$

$q(s) = 2c_1 \cosh(\sqrt{2}s) + \frac{1}{2}$

$q(0) = x(0)/R = h/R \equiv q_0$

$q_0 = 2C_1 + \frac{1}{2}$

$\Rightarrow \boxed{C_1 = \frac{1}{2}\left(q_0 - \frac{1}{2}\right)}$

$q(s) = \left(q_0 - \frac{1}{2}\right)\cosh(\sqrt{2}\,s) + \frac{1}{2}$

or: $\boxed{\dfrac{2q(s)-1}{2q_0-1} = \cosh(\sqrt{2}\,s)}$

DESCENT TIME: when $q(s_D) = q_0$

$\Rightarrow \dfrac{1}{1-2q_0} = \cosh(\sqrt{2}\,s_D)$

$q_0 = h/R \ll 1$ so $\cosh(\sqrt{2}\,s_D) \approx 1$

DO POWER SERIES

$\cosh(\sqrt{2}\,s_D) \approx 1 + s_D^2 + O(s_D^4)$

EXPECT $O(h^2/R^2)$ ORDER

FROM WHICH

$$S_r^2 = \frac{1}{1-2q_0} - \frac{1-2q_0}{1-2q_0}$$

$$= \frac{2q_0}{1-2q_0}$$

COMPARE TO $t_0 = \sqrt{\frac{2h}{g}} = \sqrt{2}\,\sqrt{\frac{h}{R}\cdot\frac{R}{g}}$ $\leftarrow \sqrt{q_0}$

$$S_0 = \sqrt{\frac{g}{R}}\,t_0 = \sqrt{2q_0}$$

$$\frac{S_r - S_0}{S_0} = \frac{\sqrt{\frac{2q_0}{1-2q_0}} - \sqrt{2q_0}}{\sqrt{2q_0}}$$

$$= \sqrt{2q_0}\left(1 + q_0 + \mathcal{O}(q_0^2)\right)$$

$$\frac{t_r - t_0}{t_0} = \frac{S_r - S_0}{S_0} = q_0 = \frac{h}{R} \leftarrow \text{ERROR}$$

EVEN PREFACTOR MATCHES!