LAST WEEK: GREEN'S FUNCTIONS ARE "inverses" OF DIFFERENTIAL OPERATORS

LINEAR TRANSF. on FUNC SPACE

§4 & 5 of Stone & Goldbart

MON (W/ IAN): EXAMPLE IN PRACTICE

used result from ℂ analysis, WE WILL REVIEW NEXT WEEK

TODAY: SAME STORY, SLIGHTLY MORE FORMAL.

IN QM: OBSERVABLES ARE HERMITIAN

SELF ADJOINT: $\mathcal{O}^\dagger = \mathcal{O}$

$\mathcal{O}^\dagger = \mathcal{O}^{T*}$

THIS MAKES SENSE; HERMITIAN OPERATORS

1. HAVE $\mathbb{R}$ EIGENVALUES
2. $\leftrightarrow$ DIAGONALIZABLE
3. $\leftrightarrow$ COMPLETE BASIS OF EIGENVECTORS

WHAT IS THE ANALOG IN FUNCTION SPACE?

general linear differential operator in 1D

$$L = P_n(x)\left(\frac{d}{dx}\right)^n + P_{(n-1)}(x)\left(\frac{d}{dx}\right)^{n-1} + \cdots + P_0(x)$$

& given inner product w/ WEIGHT $w(x)$

$$\langle f|g\rangle_w = \int_D w\, f^* g\, dx$$

DOMAIN D

"formal diff. operator"

not yet thinking about function space on which it acts (eg BOUNDARIES)

<u>CLAIM</u>: $\exists$ ANOTHER FORMAL OPERATOR L^+ s.t.
FOR $\underbrace{\text{SUFFICIENTLY DIFFERENTIABLE}}_{\text{"nice"}}$ f, g

$$w\,[\,f^* L g - g(L^+ f)^*\,] = \frac{d}{dx} Q[f,g]$$

$\boxed{\text{FORMAL ADJOINT of } L}$

some bilinear functional

WHY IS THIS USEFUL OR IMPORTANT?
IF WE INTEGRATE BOTH SIDES:

$$\langle f | Lg \rangle - \langle L^+ f | g \rangle = Q[f,g]\Big|_{\partial D}$$

EVALUATED @ BOUNDARIES OF D

SO IF $D = [a,b]$ AND $Q|_b - Q|_a = 0$, THEN THE FORMAL ADJOINT IS REALLY THE ADJOINT / HERMITIAN CONJUGATE IN THE FINITE DIM / QM SENSE:

$$\langle f | Lg \rangle = \langle L^+ f | g \rangle$$

THIS IS A FANCY WAY OF SAYING:

ADJOINT : <u>integrate by parts</u>

eg. consider $L = c \frac{d}{dx}$

$\uparrow$ constant

on a function space with $w = 1$
& boundaries $[a, b]$

$$\langle f | Lg \rangle = \int_a^b dx\, f^* \left(c \frac{d}{dx} g \right)$$

$$= -\int_a^b dx\, \frac{d}{dx}(c f^*)\, g + c f^* g \Big|_a^b$$

$$= \underbrace{\int_a^b dx \left(-c^* \frac{d}{dx} f \right)^* g}_{\langle (\underbrace{-c^* \frac{d}{dx}}_{= L^\dagger}) f | g \rangle} + \underbrace{c f^* g \Big|_a^b}_{\underbrace{\int_a^b \frac{d}{dx}(c f^* g)\, dx}_{\text{BOUNDARY TERM, } Q.}}$$

is L SELF ADJOINT?

$$c \frac{d}{dx} \overset{?}{=} \underbrace{-c^*}_{\text{only if } c \propto i} \frac{d}{dx}$$

why? $c^* = -c$
WHERE MINUS
SIGN CAME FROM
INTEGRATION BY
PARTS

OF COURSE, WE RECOGNIZE $-i\, d/dx$ AS MOMENTUM op!

$\uparrow$
why this sign?
convention of which way
the momentum is defined

$e^{i\omega t - ikx}$

ANOTHER EXAMPLE

$$L = P_2 \left(\frac{d}{dx}\right)^2 + P_1 \frac{d}{dx} + P_0 \qquad W = 1$$

POLYNOMIAL COEFFICIENTS, ASSUME REAL $P_i^* = P_i$

$$\langle f | L g \rangle = \int_a^b dx\, f^* \left(P_2 g'' + P_1 g' + P_0 g \right)$$

$$- \int_a^b dx \left[(P_2 f)' \right]^* g' + (\text{BNDY})$$

$$+ \int_a^b dx \left[(P_2 f)'' \right]^* g + (\text{BNDY})$$

$$- \int_a^b dx \left[(P_1 f)' \right]^* g + (\text{BNDY})$$

$$= \underbrace{\int_a^b dx \left[(P_2 f)'' - (P_1 f)' + P_2 f \right]^* g}_{\langle L^\dagger f | g \rangle} + (\text{BNDY})$$

HW?

SG eq (4.25)

CLAIM: $L^\dagger = \left(\frac{d}{dx}\right)^2 P_2 - \left(\frac{d}{dx}\right) P_1 + P_0$

means:

$$\left[-\frac{d}{dx} P_1 \right] f = -\frac{d}{dx} (P_1 f)$$

$$L^{\dagger} = P_2 \left(\frac{d}{dx}\right)^2 + \left(2P_2' - P_1\right)\frac{d}{dx} + \left(P_2'' - P_1' + P_0\right)$$

WHAT REQUIREMENTS on P_i s.t. L IS SELF-ADJOINT?

$L = L^{\dagger} \longrightarrow \quad P_2 = P_2$

$$2P_2' - P_1 = P_1 \Rightarrow \boxed{P_2' = P_1}$$

$$\underbrace{P_2'' - P_1'}_{=0} + P_0 = P_0 \Rightarrow \boxed{P_2'' = P_1'}$$

1st & second deriv. match.

$$\Rightarrow \boxed{P_1 = P_2'}$$

$$\boxed{L = \frac{d}{dx}\left(P_2 \frac{d}{dx}\right) + P_0} \longleftarrow \boxed{\text{STURM-LIOUVILLE OP}}$$

$$= P_2\left(\frac{d}{dx}\right)^2 + \underbrace{\left(\frac{d}{dx}P_2\right)}_{P_1}\frac{d}{dx} + P_0$$

Why does this deserve a name?

↳ general 2ND O DIFF OP ON FLAT 1D SPACE THAT IS SELF-ADJOINT.

↑ observable

EIGENVALUES: want: nice, $\mathbb{R}$ eigenvalues
complete set of EIGENVEC.

OUR FAVORITE CONCRETE EXAMPLE

$D = [0,1]$ $\quad w = 1 \quad L = -(d/dx)^2$

W/ DIRICHLET B/C ON BOTH

MUCH OF YOUR PHYSICS LIFE: VARIANTS OF THIS

$$\langle f|g\rangle = \int_0^1 f^* g \, dx$$

$$\langle f|Lg\rangle - \langle Lf|g\rangle = [f'^* g - f^* g']_0^1$$

$= 0$ by DIRICHLET.

LET V_n be EIGENFUNCTION OF L w/ EIG. VAL λ_n

$\sqrt{2}\sin n\pi x$ $\quad$ $n^2\pi^2$

ORTHOG! $\quad$ $\mathbb{R}$!

in general.

$$\langle V_i|LV_j\rangle = \int dx\, V_i^* L V_j = \lambda_j \int dx\, V_i^* V_j$$

$$\langle V_j|LV_i\rangle = \lambda_i \int dx\, V_j^* V_i$$

IF L HERMITIAN:

$$\langle V_i|LV_j\rangle = \langle V_j|LV_i\rangle^*$$

$$\langle V_j|LV_i\rangle - \langle V_i|LV_j\rangle^* = 0 = (\lambda_i - \lambda_j^*) \underbrace{\int dx\, V_j^* V_i}_{\text{"}\delta_{ij}\text{"}}$$

$\Rightarrow \lambda \in \mathbb{R}$

eg: THIS DOESN'T WORK AUTOMATICALLY

SAME FUNCTION SPACE

$[0,1]$, DIRICHLET, $w=1$

$\boxed{L = -i\,d/dx}$ our familiar momentum op.

HERMITIAN, WE SHOWED THIS.

EIGENFUNCTIONS: $Lu = \lambda u$

$\Rightarrow \quad u \sim e^{i\lambda x}$

BUT $\boxed{e^{i\lambda x} \neq 0}$ — incompatible w/ B/C

so in order to define ADJOINT / HERMITIAN, need to include BOUNDARIES.

BOILS DOWN TO THE BOUNDARY TERMS

↳ see Stone & Goldbart 4.2.3 ADJOINT B.C.

$\boxed{\text{OPERATOR ON FUNC SPACE COMES w/ BC}}$

as we saw in the matrix representations

IN FINITE DIM LIN ALG

$T : V \to V$ w/ inner product $\langle \cdot | \cdot \rangle$

$\exists\, w$ s.t. $\langle w | v \rangle = \langle u | Tv \rangle$?

IF SO: u is in DOMAIN of $T^\dagger$
AND $T^\dagger u = w$.

in finite dim V, there is always such a w.

in infinite dim, cannot always do this, for FINITE LENGTH ELEMENTS.

no δ-functions, which would help w/ b/c.

then need to make sure BOUNDARY TERM VANISHES
$\Rightarrow$ PART OF DEF OF L

eg: $L = -i\,d/dx$ DOMAIN: $f,\ Tf \in L^2[0,1]$

$\underbrace{f(1) = 0}_{\text{one b/c.}}$

$$\langle u | Lv \rangle = -i\left[u^*(1)\,v(1) - u^*(0)\,v(0)\right] + \int_0^1 dx \left(-i\tfrac{d}{dx}u\right)^* v$$

$= 0$ b/c v in DOMAIN OF L.

can be anything

$\Rightarrow$ this B/C ON THIS MUST BE 0

$L^\dagger = -i\,d/dx$ DOMAIN: $f,\ Lf \in L^2[0,1]$

$\boxed{f(0) = 0}$

ADJOINT B/C

BY COMPARISON

$L = -i\,d/dx$ DOMAIN: $f, Lf \in L^2[0,1]$

$\boxed{f(0) = f(1) = 0}$

OUR OLD DIRICHLET BC

then $\int_0^1 dx\, u^* \left(-i \frac{d}{dx} v\right) = \underbrace{-i\left[u^* v\right]_0^1}_{\equiv 0} + \int dx \left(-i \frac{d}{dx} u\right)^* v$

thus NO B/C on $L^\dagger$

$L^\dagger = -i \frac{d}{dx}$ DOMAIN: $f, Lf \in L^2[0,1]$

w/ $\boxed{\text{NO B/C}}$

DOMAIN $(L^\dagger) \neq$ DOMAIN (L)

$\hookrightarrow$ not "truly" self-adjoint.

HOW TO MAKE MOMENTUM SELF ADJOINT ?

$\int_0^1 dx\, u^* \left(-i \frac{d}{dx} v\right) - \int_0^1 \left(-i \frac{d}{dx} u\right)^* v = -i \underbrace{\left(u^*(1) v(1) - u^*(0) v(0)\right)}$

want: $= 0$
FOR SAME B/C on u, v

$\Leftrightarrow \boxed{\dfrac{u^*(1)}{u^*(0)} = \dfrac{v(0)}{v(1)}}$ w/ u, v unrelated.

$f(0)/f(1) = K$

BOTH SIDES: constant, K
EQUALITY: $K^* = K^{-1}$

most general RESULT:

$$\boxed{f(0)/f(1) = e^{i\theta}}$$ ← same for u & v

$$f(0) = f(1)\underbrace{e^{i\theta}}_{\text{twist}}$$

(twisted BC)

for $\theta = 0 \longleftrightarrow$ PERIODIC B/C.
(OF COURSE! THAT'S WHERE OUR PLANE WAVES MAKE SENSE)

$$\boxed{L = -i\frac{d}{dx}}, \quad D[L] = \{f, Lf \in L^2[0,1], \; f(0) = f(1)\, e^{i\theta}\}$$

DOMAIN

W/ these: EIGENFUNC: $e^{i(2\pi n + \theta)}$ ORTHOG.

λ : $2\pi n + \theta$ $\mathbb{R}$