LAST TIME: CAUCHY-RIEMANN EQ; $z = x + iy$

$$f(x,y) = u(x,y) + iv(x,y)$$

ANALYTIC $\leftrightarrow$ $\partial_x u = \partial_y v$, $\partial_y u = -\partial_x v$ $\leftrightarrow$ $f(z)$

differentiable $\updownarrow$ TAYLOR EXPANDABLE

"one variable" vs. $f(z, \bar{z})$

CAUCHY INTEGRAL THEOREM

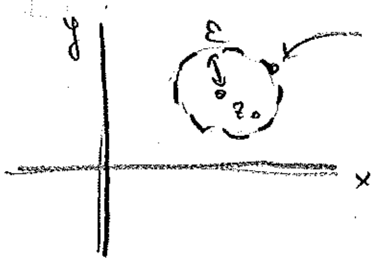
CAHILL P.160 SKETCH PF.

CLAIM: IF f IS ANALYTIC IN A REGION $R \subset \mathbb{C}$ WITH A BOUNDARY $C = \partial R$, THEN

$$\boxed{\oint_C f(z)\,dz = 0}$$

ANALYTIC FUNCTIONS ARE "NICE"... MAY BE TOO NICE... BORING!

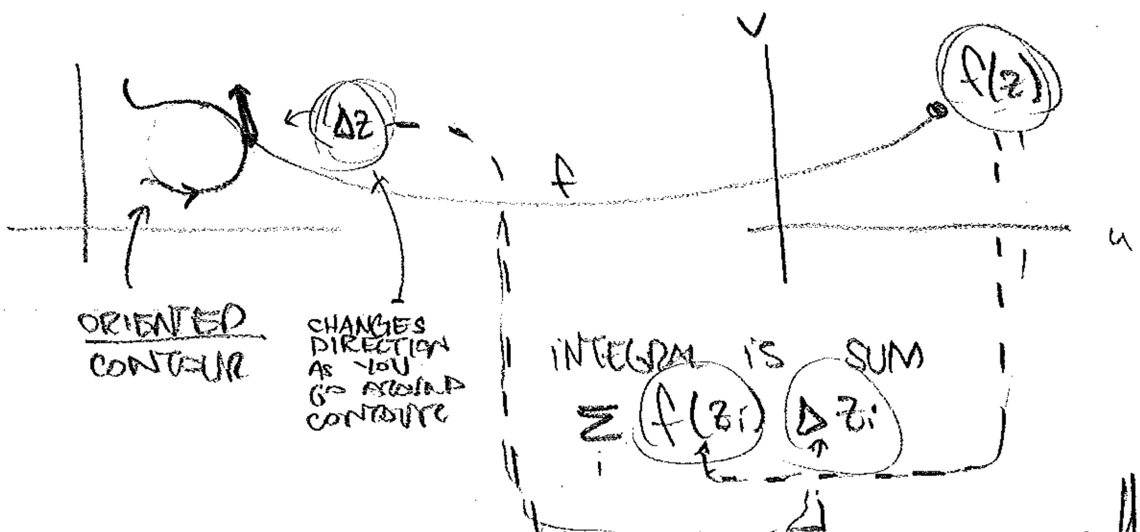
SKETCH PROOF: FIRST CONSIDER A SMALL CIRCLE AROUND z_0



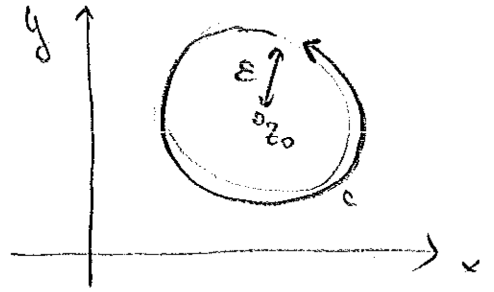
POINTS ON CIRCLE:

$$z = z_0 + \varepsilon e^{i\theta}$$

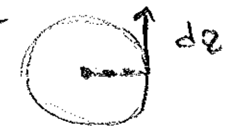
WHAT DOES INTEGRAL MEAN?



so integrating f over the small circle:



$$\left[\begin{array}{l} z = z_0 + \varepsilon e^{i\theta} \\ dz = i\varepsilon e^{i\theta}\, d\theta \end{array}\right.$$



$$\oint_c f(z)\,dz = \underbrace{\int_0^{2\pi} d\theta}_{\text{const radius}} \underbrace{f(z)\big|_c}\, i\varepsilon e^{i\theta}$$

analytic ⟺ TAYLOR EXP.

$$f(z_0) + f'(z_0)\underbrace{(z-z_0)}_{\mathcal{O}(\varepsilon)} + \mathcal{O}(\varepsilon^2)$$

$$= \int_0^{2\pi} d\theta\; i\varepsilon e^{i\theta}\left[f(z_0) + f'(z_0)\underbrace{(z-z_0)}_{\varepsilon e^{i\theta}}\right]$$

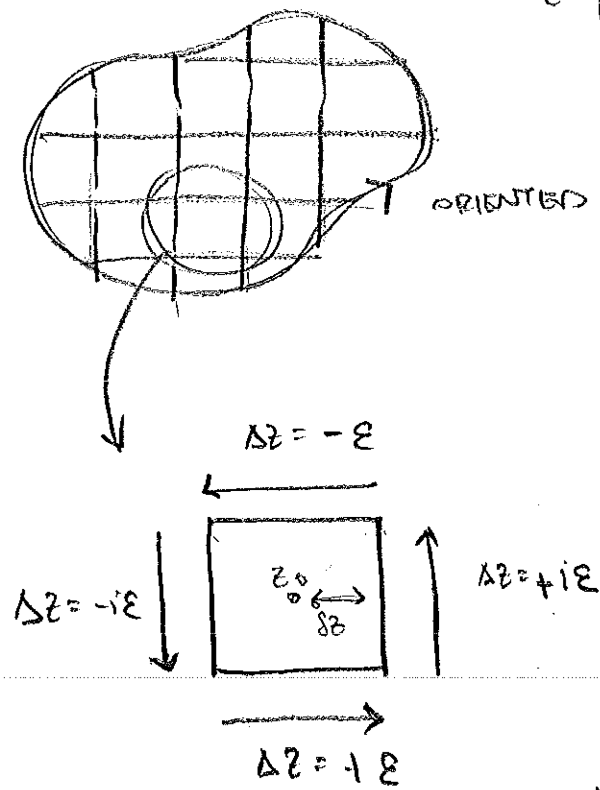
$$= \int_0^{2\pi} d\theta\; \underbrace{\underbrace{\left[f(z_0)\, i\varepsilon\right]}_{\text{const}} e^{i\theta}}_{=0} + \underbrace{\underbrace{\left[f'(z_0)\, i\varepsilon^2\right]}_{\text{const.}} e^{2i\theta}}_{=0}$$

$$\int d\theta\; e^{a\theta} = \frac{1}{a} e^{a\theta}\Big|_0^{2\pi} = 0$$

$= 0$, in a small circle around any point z_0

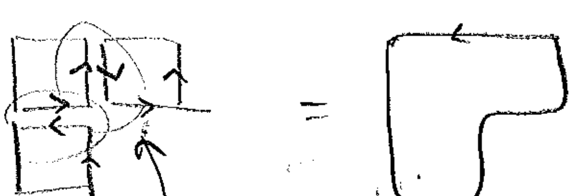
convince yourself: didn't matter the shape!

WHAT ABOUT A FINITE REGION, R, where f is analytic?
CHOP IT UP INTO LITTLE REGIONS.
["plaquette"]



AROUND THIS PLAQUETTE: $\oint f(z)dz = f(z_0)\cdot\overbrace{[\Delta z_1 + \Delta z_2 + \cdots]}^{=0}$
$$+ f'(z_0)\left[\delta z_1\, \Delta z_1 + \cdots\right]$$

$$\left[\left(\frac{\varepsilon}{2}\right)(i\varepsilon) + \left(\frac{i\varepsilon}{2}\right)(-\varepsilon) + \left(-\frac{\varepsilon}{2}\right)(-i\varepsilon) + \left(\frac{-i\varepsilon}{2}\right)\varepsilon\right] = 0$$

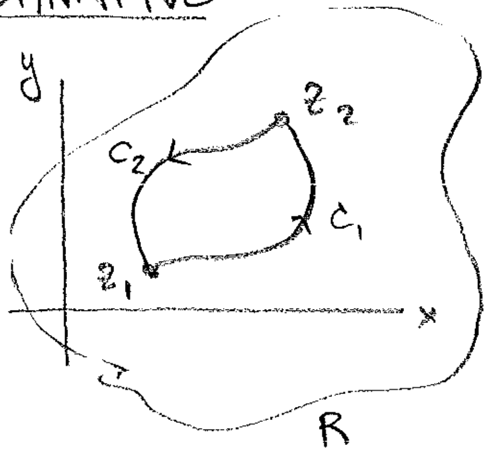


overlapping edges cancel

SO SUM OF INTEGRALS OVER EACH PLAQUETTE = INTEGRAL OVER BOUNDARY OF PLAQUETTE

$\Rightarrow$ PROVES CAUCHY INTEGRAL THM.

ALTERNATIVE



REGION OF ANALYTICITY

DIFFERENTIABLE IN R, INDEP OF PATH

↳ only dep. on endpoints

BUT ORIENTED

$$\int_{C_1} f(z)\,dz = F(z_2) - F(z_1)$$

$$\int_{C_2} f(z)\,dz = F(z_1) - F(z_2)$$

this is a general idea in diff. geometry

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

differential (n-1) form

n-dimensional space

differential n-form

(n-1) dimensional boundary of Ω

THIS IS WHY LINE INTEGRALS DEPEND ON THE ENDPOINTS

& WHY GREEN'S THM RELATES AN AREA INTEGRAL INTO A PATH INTEGRAL AROUND ITS CIRCUMFERENCE

↳ INTEGRATED VORTICITY ⟺ CIRCULATION

& WHY STOKES' THM RELATES NET CHARGE INSIDE A VOLUME TO THE POTENTIAL ON THE SURFACE.

IN CAUCHY'S INTEGRAL FORMULA

$$f(z_0) = \frac{1}{2\pi i} \oint_C \phi(z)\, dz \qquad \phi = \frac{f(z)}{z - z_0}$$

$f(z_0)$ IS THE <u>RESIDUE</u> OF $\phi(z)$ AT z_0

$= \text{Res}(\phi @ z_0)$

meromorphic w/ simple pole @ z_0

So for a function F w/ a simple pole @ z_0

$$\boxed{\oint_C F(z)\, dz = 2\pi i \, \text{Res}_F(z_0)}$$

C contains z_0

coefficient of $\left(\frac{1}{z - z_0}\right)$ in Laurent expansion

THIS IS OUR MAIN TOOL!

↳ easy to calculate Residues

→ in general, hard to do contour integrals

... but are there any contour integrals that we actually care about?

SOMETHING INTERESTING: <u>CAUCHY'S INTEGRAL FORMULA</u>

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, dz$$

famous factor of $2\pi i$

LOOP AROUND z_0

CURIOUS INTEGRAND: IT'S DIVIDED BY $(z - z_0)$

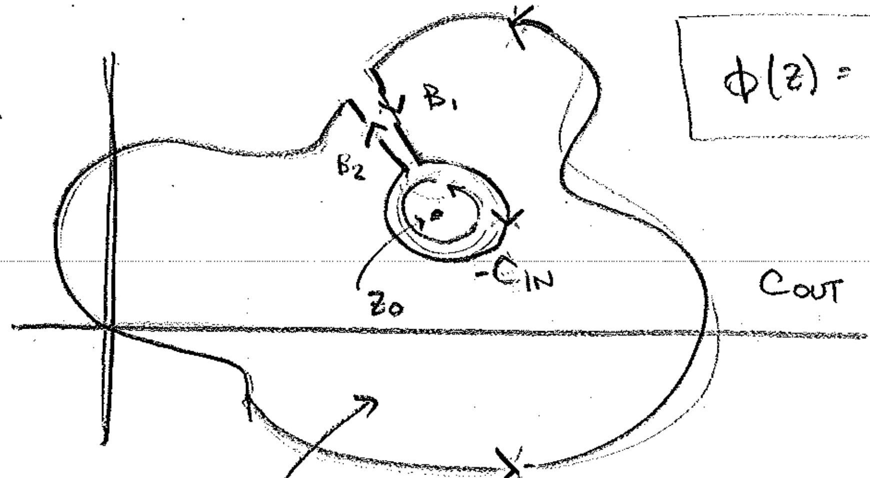
$\boxed{f(z)/(z-z_0)}$ IS NOT ANALYTIC

∴ integral need not be trivial.

ANALYTIC AROUND z_0

$$\boxed{\phi(z) = \frac{f(z)}{z - z_0}}$$

Pf.



$C_{OUT} = C$

$\phi(z)$ IS ANALYTIC IN HERE;

(because "here" doesn't include the region inside C_{in} that surrounds the <u>POLE</u> @ z_0)

THE B_1 & B_2 INTEGRALS CANCEL

$$\underbrace{\oint_C \phi(z)\, dz}_{} + \oint_{C_{in}} \phi(z)\, dz = 0$$

BY CAUCHY THM

integral on RHS of CAUCHY INT FORMULA

ORIENTATION

$C_{in}: \; z = z_0 + \varepsilon e^{i\theta}$

$$\oint_{C_m} \frac{f(z)}{\underbrace{z-z_0}_{\varepsilon e^{i\theta}}} \, \underbrace{dz}_{i\varepsilon e^{i\theta} d\theta}$$

$$= \oint_{C_m} i f(z) \, d\theta$$

} AVERAGE OF $f(z)$ AROUND z_0

↑ RECALL MEAN VALUE THM FOR HARMONIC FUNCTIONS

$$= 2\pi i \, f(z_0)$$

So: $$f(z_0) = \frac{1}{2\pi i} \oint_{C_m} \frac{f(z)}{z-z_0} \, dz$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} \, dz$$

↑ any CLOSED CURVE AROUND z_0

So there is a sense of topology ? "doesn't matter the precise path"

WE'LL USE THIS, BUT THERE'S A LESSON –
ANALYTIC ⟶ TOO BORING

INTERESTING STUFF HAPPENS WHEN THERE ARE SINGULARITIES. WE LIVE DANGEROUSLY & DANCE AROUND THE SINGULARITIES.

↑ PARTICULAR KIND OF SINGULARITY:

<u>POLES</u>: $\sim \frac{1}{(z-z_0)^n}$

POLE OF ORDER n @ $z = z_0$

$n=1$: "<u>SIMPLE POLE</u>"

A FUNCTION THAT IS ANALYTIC UP TO POLES IS CALLED MEROMORPHIC.

MEROMORPHIC FUNCTIONS ARE DESCRIBED BY LAURENT SERIES, which generalize TAYLOR SERIES

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z_0)\,(z-z_0)^n$$

USUALLY ONLY DOWN TO SOME FINITE $-N$

YOU CAN USE CAUCHY THM TO DERIVE a_n

RESIDUE THEOREM (the main tool!)

SUPPOSE $f(z)$ IS MEROMORPHIC W/ A SIMPLE POLE AT z_0. } like $\phi(z)$ that we constructed

$$\oint_C f(z)\,dz = \oint_C dz \left[\sum_{n\geq 0} a_n(z-z_0)^n + \sum_{n<0} a_n (z-z_0)^n\right]$$

$C = \partial R$ WITH $z_0 \in R$

these terms are analytic in R... INTEGRATE TO 0

SUPPOSE ONLY $n=-1$ TERM then RHS IS

$$\oint_C dz \frac{a_{-1}}{z-z_0}$$

the a_{-1} coefficient is called the RESIDUE AT z_0