Green's functions

Green's functions:

- connect integral and differential operators via boundary conditions

- connection to QM: propagator

In general

$\mathcal{O}(x)y(x) = f(x)$

- $\psi(x,t) = \int_{-\infty}^{\infty} \langle x'|\psi\rangle \underbrace{\langle x|\mathcal{U}(t,t')|x'\rangle}_{\text{propagator},\ K(x,t;x',t')} dx'$

Simple equation

$-i\frac{dy}{dx} = f(x)$, if $y(a)=y_0$ Can write $\mathcal{O}(x)y = f(x)$, $\mathcal{O}(x) = -i\frac{d}{dx}$

$y(x) = y_0 + i\int_a^x dx' f(x')$ notice $\left(-i\frac{d}{dx} \sim p_x\right)$

- trivial, any integral can be considered the solution to a 1^{st} order differential eq.

Consider $x \in [a,b]$

$\Rightarrow y(x) = y_0 + i\int_a^b \theta(x-x')f(x')dx'$

↳ cuts of integral at $x = x'$

$\Rightarrow y(x) = y_0 + i\int_a^x dx' f(x') = y_0 + i\int_a^b \theta(x-x')f(x')dx'$

We can write, $y(x) = y_0 + Kf(x)$,

$$Kf(x) \equiv i\int_a^b \theta(x-x')f(x')dx'$$

We say $i\theta(x-x')$ is the <u>Kernal</u> of of the integral operator K, and when the Kernal comes from the solution of a defferential eq it is often refered to as the Green's function for that differential operator for the relevant boundry conditions

i.e. For $i\frac{d}{dx}$ and $y(a) = y_0$,

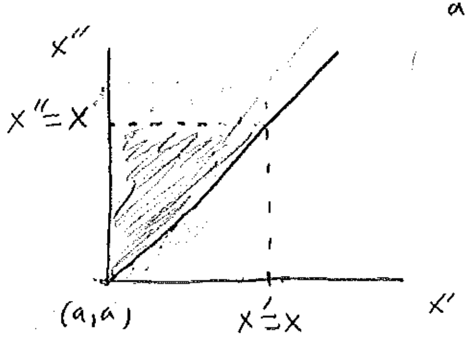
$$G_1(x,x') = i\,\theta(x-x')$$

$$y(x) = y_0 + \int_a^b G(x,x')f(x')dx'$$

$$\frac{d^2y}{dx^2} = f(x) \quad , y(a) = y_0 \quad , y'(a) = \bar{y}_0$$

$$\Rightarrow \frac{dy}{dx} = \bar{y}_0 + \int_a^x f(x')dx'$$

$$\Rightarrow y(x) = y_0 + \bar{y}_0(x-a) + \int_a^x dx'' \int_a^{x''} dx' f(x')$$



We are integrating x' over $[a, x'']$ then x'' over $[a, x]$

Equivalent to integrating x'' over $[x', x]$ then x' over $[a, x]$

$$\Rightarrow y(x) = y_0 + \bar{y}_0(x-a) + \int_a^x dx' f(x') \int_{x'}^x dx''$$

$$= y_0 + \bar{y}_0(x-a) + \int_a^x (x-x')f(x')dx' = y_0 + \bar{y}_0(x-a) + \int_a^b (x-x')\theta(x-x')f(x')dx'$$

$$\Rightarrow G_2(x, x') = (x - x')\theta(x - x')$$

G_2 is continuous while G_1 is discontinuous at $x = x'$

$y_0, \tilde{y}_0$ correspond to the homogeneous solution,

In general

$$y(x) = \alpha + \beta x + \int_a^b (x - x')\theta(x - x') f(x') dx'$$

α, β determined by boundary conditions

Now consider $\mathcal{O}(x) = -i \frac{d}{dx}$ on $[0, 1]$, $y(1) = C y(0)$

In general

$$y(x) = A + i \int_0^1 \theta(x - x') f(x') dx'$$

$$y(1) = A + i \int_0^1 f(x') dx = C y(0) = CA$$

$$\Rightarrow A = -\frac{i}{1 - C} \int_0^1 f(x') dx$$

$$\Rightarrow y(x) = -\frac{i}{1 - C} \int_0^1 f(x') dx + i \int_0^1 \theta(x - x') f(x') dx$$

$$= \int_0^1 G_4(x, x') f(x') dx'$$

$$G_q(x,x') = -i\left[\frac{1}{1-C} - \theta(x-x')\right]$$

$$= \begin{cases} \dfrac{-iC}{1-C} & x > x' \\ \dfrac{-i}{1-C} & x \le x' \end{cases}$$

If $|C| = 1 \Rightarrow C = 1/C^*$

$$\Rightarrow G_q(x,x') = \begin{cases} \dfrac{-i}{C^*-1} & x > x' \\ +\dfrac{i}{C-1} & x \le x' \end{cases}$$

$\Rightarrow G_q(x,x') = G_q^*(x',x)$ "symmetric"

$\sim a_{ij} = a_{ji}^*$

- Hermitian operator
- Boundary conditions can affect symmetry ($i\theta(x-x')$ not symmetric,

$p = -i\frac{d}{dx}$, symmetric in the sense $\langle f, pg\rangle = \langle pf, g\rangle$

"inner product"

$\langle f, g\rangle = \int dx\, \omega(x) f(x) g(x)$ ($\omega(x) = 1$ often)

Consider

$$-i\frac{dy}{dx} = f(x,y) \qquad \text{on } [0,1]$$

$$\Rightarrow y(x) = \alpha + \int_0^1 G(x,x')f[x',y(x')]\,dx'$$

If $y(0)=y_0 \Rightarrow G(x,x') = -i\,\Theta(x-x')$, $\alpha = y_0$

If $y(1) = c\,y(0)$, $|c|=1$ $\Rightarrow \alpha = 0$, $G(x,x') = -i\left[\frac{1}{1-c} - \Theta(x-x')\right]$

Consider now,

$$\frac{d^2y}{dx^2} = f(x,y) \qquad y(0)=y_0\ ,\ y(1)=y_1$$

$$\Rightarrow y(x) = y_0 + (y_1-y_0)x + \int_0^1 G(x,x')f[x',y(x')]\,dx'$$

$$G(x,x') = xx' - x'\Theta(x-x') - x\Theta(x'-x)$$

Ex

$$\frac{d^2y}{dx^2} + \lambda y = 0 \qquad , y(0) = 0 = y(1)$$

$$\Rightarrow y(x) = -\lambda\int_0^1 G(x,x')y(x')\,dx'$$

— transformed into an integral equation for $y(x)$

Green's functions

- Connect differential and integral operators via boundary conditions
- propagator in QM, QFT

$$\psi(\vec{x}'',t) = \int d^3x' \underbrace{K(\vec{x}'',t;\vec{x}',t_0)}_{\text{propagator}} \psi(\vec{x}',t_0)$$

7.2

$$K(\vec{x}'',t;\vec{x}',t_0) = \langle \vec{x}'' | \mathcal{U}(t,t_0) | \vec{x}' \rangle$$

Consider,

$Ly = f$, L : linear ordinary differential operator

Suppose L has complete set of eigenfunctions $\{\phi_n(x)\}$ such that,

$\langle \phi_n, \phi_m \rangle = \int dx\, \omega(x) \phi_n^*(x)\phi_m(x) = \delta_{nm}$

$$L\phi_n(x) = \lambda_n \phi_n(x)$$

"Basis of a hilbert space"

$$\Rightarrow y(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x) \quad , \quad f(x) = \sum_{n=1}^{\infty} \beta_n \phi_n(x)$$

$$Ly = L\sum_{n=1}^{\infty} \alpha_n \phi_n(x) = \sum_{n=1}^{\infty} \alpha_n \lambda_n \phi_n(x)$$

$$= f = \sum_{n=1}^{\infty} \beta_n \phi_n(x)$$

$$\Rightarrow \alpha_n = \frac{\beta_n}{\lambda_n}$$

$$\Rightarrow y(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\beta_n \phi_n(x) \quad , \quad \beta_n = \overbrace{\langle \phi_n , f \rangle}^{\text{inner product of space}}$$

If $\lambda_n = 0 \Rightarrow \exists\, \beta_n = 0 \Rightarrow$ non unique sol, assume $\lambda_n \neq 0$

$$y(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n(x) \langle \phi_n, f \rangle$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n(x) \int dx' \phi_n^*(x') f(x')$$

$$= \int \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n} f(x')\, dx'$$

$$= \int G(x,x') f(x') dx'$$

$$\Rightarrow \quad G(x,x') = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n}$$

$$= G(x',x)^*$$

<u>Ex</u>:

$L = \frac{d^2}{dx^2}$ for interval $[0,1]$

$$\Rightarrow \phi_n(x) = \sqrt{2}\, \sin(n\pi x)$$ "particle in a box"

$$\Rightarrow \quad G(x,x') = -\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)\sin(n\pi x')}{n^2}$$

Consider,

$$L\, G(x,x') = \sum_{n=1}^{\infty} \frac{L\,\phi_n(x)\phi_n^*(x')}{\lambda_n} = \sum_{n=1}^{\infty} \phi_n(x)\phi_n^*(x')$$

$$\Rightarrow \int dx' f(x') \sum_{n=1}^{\infty} \phi_n(x)\phi_n^*(x') = \sum_{n=1}^{\infty} \phi_n(x) \int dx' \phi_n^*(x') f(x')$$

$$= \sum_{n=1}^{\infty} \beta_n \phi_n(x)$$

$$= f(x)$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \phi_n(x)\phi_n^*(x') = \delta(x-x') \quad , \omega(x) = 1$$

$$y = L^{-1} f = K f$$

$$\Rightarrow \quad L G(x, x') = \delta(x - x') \quad , \quad L K = \delta(x - x')$$

Ex

$$L = -i \frac{d}{dx}$$

$$\Rightarrow \quad -i \frac{d}{dx} G(x, x') = \delta(x - x')$$

$$\Rightarrow \quad G(x, x') = i \int_{-\infty}^{x} dx'' \, \delta(x' - x'')$$

$$= i \Theta(x - x')$$

$$-i \frac{dy}{dx} = f(x) \quad , \quad y(a) = y_0$$

$$\Rightarrow y(x) = y_0 + i \int_a^x dx' f(x')$$

$$= y_0 + i \int_a^b dx' \, \Theta(x - x') f(x')$$

$$\Rightarrow G(x, x') = i \Theta(x - x')$$

Example

$$\ddot{x}(t) + 2\gamma \dot{x}(t) + \omega_0^2 x(t) = F(t)$$

$$\Big(\overbrace{\frac{d^2}{dt^2} + 2\gamma \frac{d}{dt}}^{L} + \omega_0^2 \Big) x(t) = F(t)$$

$$f(t) = \int_{-\infty}^{\infty} đk \, \tilde{f}(k) e^{ikt} \qquad đk = \frac{dk}{2\pi}$$

$$\Rightarrow \int đk \left(\frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \omega_0^2 \right) e^{ikt} \hat{x}(k) = \int đk \, \tilde{F}(k)$$

$$= \int đk \left(-k^2 + 2i\gamma k + \omega_0^2 \right) e^{ikt} \hat{x}(k)$$

$$\Rightarrow \quad \tilde{x}(k) = \frac{\tilde{F}(k)}{-k^2 + 2i\gamma k + \omega_0^2}$$

$$\hat{F}(k) = \int_{-\infty}^{\infty} dt \, F(t) \, e^{-ikt}$$

$$\Rightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dK \, \frac{\tilde{F}(K)}{-K^2 + 2i\gamma K + \omega_0^2} \, e^{iKt}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dK \, \frac{F(t') \, e^{iK(t-t')}}{-K^2 + 2i\gamma K + \omega_0^2}$$

For completeness,

$$x(t) = A\, x_1(t) + B\, x_2(t) + \int_{-\infty}^{\infty} G(t,t') F(t') \, dt'$$

$$G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dK \, \frac{e^{iK(t-t')}}{-K^2 + 2i\gamma K + \omega_0^2} \quad \leftarrow \text{contour integral}$$

Can solve homogeneous eq to find

$$x_1(t) = e^{-\gamma t} \sin \Omega t$$

$$x_2(t) = e^{-\gamma t} \cos \Omega t \qquad \Omega \equiv \sqrt{\omega_0^2 - \gamma^2}$$

* Flip will go over how to solve contour integrals

$$G(t,t') = \theta(t-t') \, \frac{e^{-\gamma(t-t')} \sin[\Omega(t-t')]}{\Omega}$$

$$x(t) = A\, x_1(t) + B\, x_2(t) + \int_{t_0}^{t} dt' \, \frac{e^{-\gamma(t-t')}}{\Omega} \sin[\Omega(t-t')] \, F(t')$$

Let $F(t) = F_0 e^{-\alpha t}$, $t_0 = 0$

Equilibrium at $t=0 \Rightarrow A = B = 0$

$$x(t) = \frac{F_0}{\Omega} e^{-\gamma t} \int_0^t dt' \sin[\Omega(t-t')]\, e^{-(\alpha-\gamma)t'}$$

$$= \frac{F_0}{\sqrt{\omega_0^2-\gamma^2}} \frac{\sin\left[\sqrt{\omega_0^2-\gamma^2}\, t - \delta\right]}{\sqrt{\omega_0^2+\alpha^2-2\alpha\gamma}} e^{-\gamma t} + \frac{F_0}{\omega_0^2+\alpha^2-2\alpha\gamma} e^{-\alpha t}$$

$$\tan\delta = \sqrt{\omega_0^2-\gamma^2}/(\alpha-\gamma)$$

For $\gamma = 0$, $t \to \infty$

$$x(t) = \frac{F_0}{\omega_0} \frac{\sin(\omega_0 t - \delta)}{\sqrt{\omega_0^2+\alpha^2}}$$

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_0^2 + x^2 = \frac{F_0^2}{2(\omega_0^2+\alpha^2)}$$

$G(t,t')$ closely resembles homogeneous solution, It is a linear combination of x_1 and x_2

$$G(t',t') = A(t') e^{-\gamma t} \sin\Omega t + B(t') e^{-\gamma t} \cos\Omega t$$

$$A(t') = \frac{e^{\gamma t'}}{\Omega} \cos\Omega t' \qquad B(t') = -\frac{e^{\gamma t'} \sin\Omega t'}{\Omega}$$

Obvious because

$$\left(\frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \omega_0^2\right) G(t,t') = \delta(t-t')$$

$\Rightarrow$ for $t<t'$ and $t>t'$

$G(t,t')$ satisfies the same as equation as $X(t)$

$\Rightarrow$ $G(t,t')$ is a linear combination of $x_1(t)$, $x_2(t)$

w/ coefficients dependent on t'

In general

$$L \equiv f_0(t)\frac{d^2}{dt^2} + f_1(t)\frac{d}{dt} + f_2(t)$$

$x_1(t)$ and $x_2(t)$ linearly independent solutions to

$L\,x(t)=0 \quad \Rightarrow L\,G(t,t') = \delta(t-t')$