LAST TIME : LINEAR ALGEBRA

THE POINT : VECTORS, DUAL VECTORS, all that ARE INHERENTLY INDEX-FREE

WE, AS PHYSICISTS, PICK A BASIS (usually orthonormal) AND WRITE COMPONENTS w/rt THAT BASIS

$|V\rangle = v^i |e_i\rangle$ UPPER INDEX ON #

$\langle W| = w_i \langle e^i|$ LOWER INDEX ON #

the "cute" way of looking @ this is:

$v^i \rightsquigarrow \begin{pmatrix} v^1 \\ v^2 \\ \vdots \end{pmatrix}$ ← BTW: WORDS "DO NOT INCLUDE SPACETIME ORIGIN"

$w_i \rightsquigarrow (w_1 \ w_2 \ \cdots)$

the DUAL VECTORS are linear functions that act on vectors (& vice versa)

WE ALSO SAW MATRICES / TRANSFORMATION / OPERATOR

$A = \underset{\uparrow}{A^i{}_j} \underbrace{|e_i\rangle\langle e^j|}_{\text{"matrixy" PART}}$

JUST #'s

⤳ in fact, the $\begin{pmatrix} \vdots & \vdots & \vdots \end{pmatrix}$ representation is best understood as a look up table

YOU COULD ALSO IMAGINE OTHER TYPES OF "MATRICES"

$$B = B_{ij} \langle e^i | \langle e^j |$$

$V \to V^*$

$$C = C^{ij} | e_i \rangle | e_j \rangle \quad \leftarrow V^* \to V$$

$$D = D_i{}^j \langle e^i | \otimes | e_j \rangle$$

$V \to V^*$

REMINDER THAT THIS IS NOT $\langle e^i | e_j \rangle = \delta^i_j$

EVEN THOUGH THE INDICES ARE JUST LABELS, OUR CHOICE OF UPPER VS. LOWER CONVENTION CLARIFIES THE TYPE OF MAP.

OTHER TYPES OF OBJECTS: generalize to <u>tensors</u>

$$T^{ij}{}_k \underbrace{| e_i \rangle \otimes | e_j \rangle \otimes | e^k \rangle}$$

Shorthand: DROP THIS ... BUT UNDERSTAND THAT IT IS IMPLICITLY THERE!

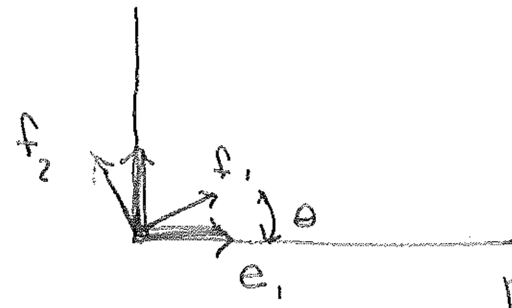
$$T^{ij}{}_k : V \to V^* \times V^*$$

$$V \times V^* \to V$$

etc.

TAKEAWAY: CAN (ought to) THINK OF

"ROTATIONS": WHY WE LIKE INDICES



$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |f_1\rangle = \begin{pmatrix} c \\ s \end{pmatrix}$

$|e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad |f_2\rangle = \begin{pmatrix} -s \\ c \end{pmatrix}$

change of orthonorm. basis by lin. trans.

$$\boxed{|f_i\rangle = R^j{}_i |e_j\rangle}$$

ODD INDEX ORDER

$$|V\rangle = v^i_{(e)} |e_i\rangle \longleftrightarrow$$

$$= v^j_{(f)} |f_j\rangle = \underbrace{v^j_{(f)} R^i{}_j}|e_i\rangle$$

$$\boxed{v^i_{(e)} = R^i{}_j v^j_{(f)}}$$

column of #'s $\longrightarrow$ $\underline{V}_{(e)} = R\, \underline{V}_{(f)}$

$$\Rightarrow \boxed{\underline{V}_{(f)} = R^{-1} \underline{V}_{(e)}}$$

COMPONENTS TRANSFORM DIFFERENTLY FROM BASIS

thing to generalize: WHY WE USE THESE SILLY INDEX CONVENTIONS:

things w/ UPPER INDICES TRANSFORM "OPPOSITELY" AS THINGS W/ LOWER INDICES.

<u>analog:</u>

$$(\text{ROW}) \begin{pmatrix} \text{MATR.} \end{pmatrix}$$

$$\begin{pmatrix} \text{MATR} \end{pmatrix} \begin{pmatrix} \text{V} \\ \text{E} \\ \text{C} \end{pmatrix}$$

eg. ~~F~~ $\langle W | V \rangle = W_i V^i$

SUPPOSE A TRANSFORMATION TAKES

$$V^i \to \tilde{V}^i = S^i{}_j V^j$$

$\tilde{}$: new coords

THEN $W_i \to \tilde{W}_i = W_k (S^{-1})^k{}_i$

s.t. ~~WHAT~~ $W_i V^i = (\tilde{W}_i S^i{}_k)[(S^{-1})^k{}_j \tilde{V}^j]$

$$= \tilde{W}_i \underbrace{(S S^{-1})^i{}_j}_{= \mathbb{1}^i{}_j = \delta^i_j} \tilde{V}^j$$

$$= \tilde{W}_i \tilde{V}^i$$

$$\boxed{W_i V^i = \tilde{W}_i \tilde{V}^i}$$ UNDER "ROTATION"

seemingly innocuous result, but demonstrates important points

- UPPER VS LOWER INDICES TRANSFORM OPPOSITELY (contravariant vs. covariant)
- $W_i V^i$ DOESN'T TRANSFORM UNDER ROT.

  SCALAR QUANTITY

"DOES NOT TRANSFORM" $\longleftrightarrow$ CONSERVATION SYMMETRY

recall Noether's THM

this is why I ♡ indices.
IF ALL THE INDICES CONTRACT, THEN THIS IS A SCALAR W.R.T. SOME SYMMETRY OF SOME VECTOR SPACE

almost all phys. really

eg. in PARTICLE physics, WE WRITE THEORIES AS LAGRANGIANS
the "PHYSICS" is

$$Z = e^{iS} = e^{i\int dt\, L} = e^{i\int d^4x\, \mathcal{L}}$$

SANITY CHECK: UNITS?!

↳ actually, $Z = e^{i\hbar S}$

BUT I SET $\hbar = 1$

IN NATURAL UNITS, $[S] = 0$

$[L] = +1$

$[\mathcal{L}] = +4$

WHAT KIND OF TERMS IN $\mathcal{L}$?

↳ in HEP, PARTICLES ARE FIELDS

$$\mathcal{L} \supset Y_t^{ij} H^a Q_{a\alpha i A} U_j^{\alpha A}$$

SU(2) SPIN flavor color

EACH INDEX MEANS SOMETHING DIFFERENT

I'M HE'S

↳ symmetries, semi-cons. charges, struc. of thy

BUT ALL INDICES MUST CONTRACT B/C THIS TERM MUST BE INVARIANT.

SIMILARLY IN GR/SR: $G_{\mu\nu}$ skip

Einstein tensor.
EVEN IF YOU DON'T KNOW WHAT IT IS, YOU KNOW HOW IT TRANSFORMS ...

IN FACT: TRANSFORMATION IS KING

can generalize to multi-index objects

$$T^{ij}{}_{k} \quad \underbrace{\left(|e_i\rangle \otimes |e_j\rangle \otimes \langle e^k | \cdot \right)}$$

tensor product

BASIS ... WE WILL START OMITTING THIS

HOW DOES IT TRANSFORM UNDER ROTATIONS?

$$\rightarrow \underline{R^i{}_l} \; \underline{R^j{}_m (R^{-1})^n{}_k} \; T^{lm}{}_n$$

contracted indices are identified by repetition, not by "NEXT TO EACH OTHER"

Review so far

VECTORS $\longrightarrow$ in vec SPACE, COORDS have upper index

DUAL VEC $\longrightarrow$ in DIFFERENT SPACE, V^* HAVE LOWER INDEX

V & V^* are linear maps of each other.

$w \in V^*$ s.t. $w: V \rightarrow \#$

INNER PRODUCT: $\langle \cdot , \cdot \rangle : V \times V \rightarrow \#$ CAN BE USED TO GO $V \longleftrightarrow V^*$ (assume invertible)

$\langle a | b \rangle = \langle b | a \rangle^*$ for $\mathbb{C}$ SPACES (to give pos. norm)

SO FAR : $\langle e^i | e_j \rangle = \delta^i_j$ EUCLIDEAN

BUT YOU'RE PROB. FAMILIAR W/ <u>MINKOWSKI SPACE</u> FROM SPECIAL RELATIVITY

$$\langle e^\mu | e_\mu \rangle = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = g_{\mu\nu} \quad \text{IN 2D}$$

eg: IN SR; A VECTOR IS SOMETHING LIKE

$$p^\mu = \begin{pmatrix} E \\ p \end{pmatrix} \begin{matrix} \leftarrow \text{ENERGY} \\ \leftarrow \text{3-MOMENTUM} \end{matrix}$$

components are basis-dependent

$\longleftrightarrow$ WHAT <u>FRAME</u> ARE YOU IN.

"<u>ROTATION</u>":
$$\begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}$$

$$\begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$$

<u>INVARIANT</u> : $\|p\| = p_\mu p^\mu$

$$p_\mu = g_{\mu\nu} p^\nu$$
$$= (E, -p)$$

$$p_\mu p^\mu = E^2 - p^2 \equiv m^2$$

is invariant under boosts.

EIGENVALUE PROBLEMS

not exhaustive

things that tend to act on vectors

- "ROTATIONS" ⟵ CHANGE OF BASIS or SYMMETRY TRANSF ON STATE (PRESERVES info)
- PROJECTIONS ⟵ COLLAPSES INFO eg. OBSERVATION IN QM
- OBSERVABLE ⟵ asks a question about a state

↓

EIGENVALUE QUESTION

$$\underset{\text{OPERATOR}}{\mathcal{O}}\ \overset{\text{VECTOR}}{\underline{x}} = \underset{\#}{\lambda}\ \overset{\text{SAME VECTOR}}{\underline{x}}$$

#; RESCALING ⟷ OBSERVABLE

eg. ROTATION ABOUT z AXIS LEAVES VECTORS // TO $\hat{z}$ INVARIANT

$$R_z \underline{V} = 1 \cdot \underline{V}$$

$$\underline{V} = V^z |e_z\rangle$$

MATRIX NOTATION:

$$\mathcal{O}^i{}_j x^j = \lambda x^i$$

IN QM: HERMITIAN MATRICES ARE PHYSICAL OBSERVABLES

$A^\dagger = A$

WHY? eigenvalues are real
eigenvectors are orthogonal

eg: SPIN 1/2 IN QM: $|\uparrow\rangle \quad |\downarrow\rangle$

$$S_z \sim |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$$

this is just $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$
(so PROBABLY PREFACTOR OF 1/2)

WHAT IS SPIN OF $|\uparrow\rangle$?

$$\langle\uparrow|S_z|\uparrow\rangle = \frac{1}{2}$$

ASSUMING $|\uparrow\rangle$ IS S_z EIGENSTATE

$\langle\uparrow|S_x|\uparrow\rangle$ is not an eigenval.
not an S_z EIGENSTATE.

WHAT'S SO GREAT ABOUT EIGENSTATES?

$$i\frac{\partial}{\partial t}\underline{\psi} = H\underline{\psi}$$

$$\psi(t) = e^{iHt}\underline{\psi}_0 \quad \underline{\psi}_0 = c^i|E_i\rangle$$

$$= \sum_i c_i e^{iE_i t}|E_i\rangle$$

Preview of next week

function space

eg nice functions on $[0,1]$
s.t. $f(0) = f(1) = 0$

then: $|e_n\rangle = \sqrt{2}\,\sin(n\pi x)$

DISCRETE BASIS OF ∞ DIM SPACE

$$f(x) = \sum_n c_n \sqrt{2}\,\sin(n\pi x)$$

in this space

WHAT IS c_n?, eg. if $\underline{V} = V^i e_i$

then $V^3 = \langle e^3 | V \rangle$

$$\langle e^n | f \rangle = \int_0^1 \sqrt{2}\,\sin(n\pi x)\, f(x)\, dx$$