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Distributed consensus of linear multi-agent systems with adaptive dynamic protocols☆

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ABSTRACT

This paper considers the distributed consensus problem of multi-agent systems with general continuous-time linear dynamics for both the cases without and with a leader whose control input might be nonzero and time varying. For the case without a leader, based on the relative output information of neighboring agents, two types of distributed adaptive dynamic consensus protocols are proposed, namely, the edge-based adaptive protocol which assigns a time-varying coupling weight to each edge in the communication graph and the node-based adaptive protocol which uses a time-varying coupling weight for each node. These two adaptive protocols are designed to ensure that consensus is reached in a fully distributed fashion for all undirected connected communication graphs. It is shown that the edge-based adaptive consensus protocol is applicable to arbitrary switching connected graphs. For the case where there exists a leader whose control input is possibly nonzero and bounded, a distributed continuous adaptive protocol is designed to guarantee the ultimate boundedness of the consensus error with respect to any communication graph which contains a directed spanning tree with the leader as the root and whose subgraph associated with the followers is undirected, requiring neither global information of the communication graph nor the upper bound of the leader's control input. A distributed discontinuous protocol is also discussed as a special case. Simulation examples are finally given to illustrate the theoretical results.

1. Introduction

Consensus is an important problem in the area of cooperative control of multi-agent systems. The main idea of consensus is to develop distributed control policies that enable a group of agents to reach an agreement on certain quantities of interest. Due to its potential applications in broad areas such as spacecraft formation flying and sensor networks, the consensus problem has been extensively studied by numerous researchers from various perspectives; see Jadbabaie, Lin, and Morse (2003), Li, Fu, Xie, and Zhang (2011), Olfati-Saber, Fax, and Murray (2007), Olfati-Saber and Murray (2004), Ren and Beard (2005), Ren, Beard, and Atkins (2007) and references therein. Existing consensus algorithms can be roughly categorized into two classes, namely, consensus without a leader (i.e., leaderless consensus) and consensus with a leader. The case of consensus with a leader is also called leader–follower consensus or distributed tracking.

A pioneering work on consensus is (Jadbabaie et al., 2003) which provides a theoretical explanation for the linearized Vicsek model (Vicsek, Czirók, Ben-Jacob, Cohen, & Shochet, 1995) by using tools from algebraic graph theory. In Olfati-Saber and Murray (2004), a general framework of the consensus problem for networks of integrators with fixed or switching topologies is proposed. The controllability of leader–follower multi-agent systems is considered in Rahmani, Ji, Mesbahi, and Egerstedt (2009) from a graph-theoretic perspective. Distributed tracking control for multi-agent consensus with an active leader is addressed in Hong, Chen, and Bushnell (2008) and Hu and Feng (2010) by using neighbor-based state estimators. Consensus of networks of double- and high-order integrators is studied in Jiang

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and Wang (2010) and Ren and Atkins (2007). Consensus algorithms are designed in Carli, Bullo, and Zampieri (2009) and Li et al. (2011) for a group of agents with quantized communication links and limited data rate. In Cao and Ren (2012), discontinuous controllers are studied in the absence of velocity or acceleration measurements. The authors in Mei, Ren, and Ma (2011) address the distributed coordinated tracking problem for multiple Euler–Lagrange systems with a dynamic leader. The readers can refer to the surveys (Olfati-Saber et al., 2007; Ren et al., 2007) for more recent works on consensus. In most existing studies on consensus, the agent dynamics are assumed to be first-, second-, or high-order integrators, which might be restrictive in many cases.

This paper considers the distributed consensus problem of multi-agent systems with general continuous-time linear dynamics. Previous works along this line include Li, Duan, Chen, and Huang (2010), Li, Liu, Lin, and Ren (2011), Ma and Zhang (2010), Scardovi and Sepulchre (2009), Seo, Shim, and Back (2009), Tuna (2009) and Zhang, Lewis, and Das (2011). One common feature in Li et al. (2010, 2011), Ma and Zhang (2010), Seo et al. (2009) and Zhang et al. (2011) is that at least the smallest nonzero eigenvalue of the Laplacian matrix associated with the communication graph is required to be known for the consensus protocol design. However, the smallest nonzero eigenvalue of the Laplacian matrix is global information in the sense that each agent has to know the entire communication graph to compute it. Therefore, the consensus protocols given in Li et al. (2010, 2011), Ma and Zhang (2010), Seo et al. (2009) and Zhang et al. (2011) cannot be implemented by the agents in a fully distributed fashion, i.e., using only the local information of its own and neighbors. To overcome this limitation, distributed adaptive static consensus protocols are proposed in Li, Liu, Ren, and Xie (2013) and Li, Ren, Liu, and Fu (in press). Similar adaptive schemes are presented to achieve second-order consensus with nonlinear dynamics in Su, Chen, Wang, and Lin (2011) and Yu, Zheng, Lü, and Chen (2011). Note that the protocols in Li et al. (2013, in press), Su et al. (2011) and Yu et al. (2011) rely on the relative states of neighboring agents, which however might not be available in many circumstances.

In this paper, we intend to present a unified framework to address the consensus problem for both the cases without and with a leader whose control input might be nonzero and time varying by proposing fully distributed adaptive protocols, when the relative state information of neighboring agents is not accessible. For the case without a leader, based on the relative outputs of neighboring agents, two types of distributed adaptive dynamic consensus protocols are proposed, namely, the edge-based adaptive protocol which assigns a time-varying coupling weight to each edge in the communication graph and the node-based adaptive protocol which uses a time-varying coupling weight for each node. These two adaptive protocols are designed to ensure that consensus is reached in a fully distributed fashion for any undirected connected communication graph without using any global information about the communication graph. The case with switching communication graphs is also studied. It is shown that the edge-based adaptive consensus protocol is applicable to arbitrary switching connected graphs. For the case where there exists a leader whose control input is possibly nonzero and bounded, a distributed continuous adaptive protocol is designed to guarantee the ultimate boundedness of the consensus error and the adaptive weights with respect to any communication graph which contains a directed spanning tree with the leader as the root and whose subgraph associated with the followers is undirected, requiring neither global information of the communication graph nor the upper bound of the leader's control input. As a special case, a distributed discontinuous adaptive protocol is also discussed, in which case the consensus error converges to zero. A sufficient condition for the existence of these adaptive protocols is that each agent is stabilizable and detectable.

It is worth mentioning that the dynamic consensus protocol in Scardovi and Sepulchre (2009) does not need any global information only when the eigenvalues of the state matrix A of each agent lie in the closed left half-plane. As stated in Remark 1 of Scardovi and Sepulchre (2009), global information of the communication graph is required for the case where A has unstable eigenvalues. Compared to the dynamic protocol in Scardovi and Sepulchre (2009), the adaptive consensus protocols proposed in this paper have advantages in three respects. First, by proposing adaptive consensus protocols, we do not impose the assumption that A has no unstable eigenvalues. Second, the dimension of the node-based adaptive protocol here is lower than that of the consensus protocol in Scardovi and Sepulchre (2009). Third, contrary to the results in Scardovi and Sepulchre (2009) which are applicable to only the case of a leader with zero control input, the proposed framework in this paper can address the consensus problem for the case with a leader whose control input is nonzero and time varying. It should be mentioned that an advantage of the results in Scardovi and Sepulchre (2009) is that they are applicable to the case with jointly connected switching graphs.

The rest of this paper is organized as follows. Some useful preliminary results are reviewed in Section 2. The leaderless consensus problems under the proposed distributed adaptive protocols are investigated in Section 3. The leader–follower consensus problem with a leader of possibly nonzero control input is studied in Section 4. Simulation examples are presented for illustration in Section 5. Section 6 concludes the paper.

2. Mathematical preliminaries

2.1. Notation and graph theory

Let $\mathbf{R}^{n\times n}$ be the set of $n \times n$ real matrices. The superscript T means the transpose for real matrices. I_p represents the identity matrix of dimension p. Denote by $\mathbf{1}$ a column vector with all entries equal to one. The matrix inequality $A > (\geq) B$ means that $A - B$ is positive (semi-)definite. $A \otimes B$ denotes the Kronecker product of matrices A and B. For a vector x, let $\|x\|$ denote its 2-norm. A matrix is Hurwitz if all of its eigenvalues have negative real parts.

A directed graph $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \ldots, v_N\}$ is a nonempty finite set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of edges, in which an edge is represented by an ordered pair of distinct nodes. For an edge (v_i, v_j), node v_i is called the parent node, node v_j the child node, and v_i is a neighbor of v_j. A graph with the property that $(v_i, v_j) \in \mathcal{E}$ implies $(v_j, v_i) \in \mathcal{E}$ for any $v_i, v_j \in \mathcal{V}$ is said to be undirected. A path from node v_{i_1} to node v_{i_l} is a sequence of ordered edges of the form $(v_{i_k}, v_{i_{k+1}})$, $k = 1, \ldots, l-1$. An undirected graph is connected if there exists a path between every pair of distinct nodes, otherwise it is disconnected. A directed graph contains a directed spanning tree if there exists a node called the root, which has no parent node, such that the node has directed paths to all other nodes in the graph.

The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbf{R}^{N\times N}$ associated with the directed graph $\mathcal{G}$ is defined by $a_{ii} = 0$, $a_{ij} = 1$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L} = [\mathcal{L}_{ij}] \in \mathbf{R}^{N\times N}$ is defined as $\mathcal{L}_{ii} = \sum_{j\neq i} a_{ij}$ and $\mathcal{L}_{ij} = -a_{ij}$, $i \neq j$.

Lemma 1 (*Agaev & Chebotarev, 2005, Olfati-Saber & Murray, 2004, Ren & Beard, 2005*). (1) *Zero is an eigenvalue of the Laplacian matrix $\mathcal{L}$ with $\mathbf{1}$ as a corresponding right eigenvector and all nonzero eigenvalues have positive real parts. Furthermore, zero is a simple eigenvalue of $\mathcal{L}$ if and only if $\mathcal{G}$ has a directed spanning tree.* (2) *For an undirected graph $\mathcal{G}$, the smallest nonzero eigenvalue λ_2 of $\mathcal{L}$ satisfies $\lambda_2 = \min_{x\neq 0, \mathbf{1}^T x=0} \frac{x^T \mathcal{L} x}{x^T x}$.*

2.2. Ultimate boundedness

Lemma 2 (Corless & Leitmann, 1981). *For a system $\dot{x} = f(x,t)$, where $f(\cdot)$ is locally Lipschitz in x and piecewise continuous in t, assume that there exists a continuously differentiable function $V(x,t)$ such that along any trajectory of the system,*

$$\alpha_1(\|x\|) \le V(x,t) \le \alpha_2(\|x\|),$$
$$\dot{V}(x,t) \le -\alpha_3(\|x\|) + \epsilon,$$

where $\epsilon > 0$, α_1 and α_2 are class $\mathcal{K}_\infty$ functions, and α_3 is a class $\mathcal{K}$ function. Then, the solution $x(t)$ of $\dot{x} = f(x,t)$ is uniformly ultimately bounded.

3. Leaderless consensus with undirected communication graphs

Consider a group of N identical agents with general linear dynamics. The dynamics of the i-th agent are described by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i &= Cx_i, \quad i = 1, \ldots, N, \end{aligned} \tag{1}$$

where $x_i \in \mathbf{R}^n$ is the state, $u_i \in \mathbf{R}^p$ is the control input, $y_i \in \mathbf{R}^q$ is the measured output, and A, B, C are constant matrices with compatible dimensions.

In this section, the communication graph among the agents is represented by an undirected graph $\mathcal{G}$. It is assumed that each agent knows the relative outputs, rather than the relative states, of its neighbors with respect to itself. Based on the relative output information of neighboring agents, two novel distributed adaptive dynamic consensus protocols are proposed, namely, the edge-based and node-based adaptive consensus protocols.

The edge-based adaptive consensus protocol dynamically updates the coupling weight for each edge (i.e., each communication link) and is given by

$$\begin{aligned} \dot{v}_i &= (A+BF)v_i + L\sum_{j=1}^N c_{ij}a_{ij}[C(v_i - v_j) - (y_i - y_j)], \\ \dot{c}_{ij} &= \varepsilon_{ij}a_{ij}\begin{bmatrix} y_i - y_j \\ C(v_i - v_j)\end{bmatrix}^T \Gamma \begin{bmatrix} y_i - y_j \\ C(v_i - v_j)\end{bmatrix}, \\ u_i &= Fv_i, \quad i = 1, \ldots, N, \end{aligned} \tag{2}$$

where $v_i \in \mathbf{R}^n$ is the protocol state, $i = 1, \ldots, N$, a_{ij} is the (i,j)-th entry of the adjacency matrix $\mathcal{A}$ associated with $\mathcal{G}$, $c_{ij}(t)$ denotes the time-varying coupling weight for the edge (i,j) with $c_{ij}(0) = c_{ji}(0)$, $\varepsilon_{ij} = \varepsilon_{ji}$ are positive constants, and $F \in \mathbf{R}^{p\times n}$, $L \in \mathbf{R}^{n\times q}$, and $\Gamma \in \mathbf{R}^{2q\times 2q}$ are the feedback gain matrices.

The node-based adaptive consensus protocol assigns a time-varying coupling weight to each node (i.e., each agent) and is described by

$$\begin{aligned} \dot{\tilde{v}}_i &= (A+BF)\tilde{v}_i + d_iL\sum_{j=1}^N a_{ij}[C(\tilde{v}_i - \tilde{v}_j) - (y_i - y_j)], \\ \dot{d}_i &= \tau_i\left[\sum_{j=1}^N a_{ij}\begin{bmatrix} y_i - y_j \\ C(\tilde{v}_i - \tilde{v}_j)\end{bmatrix}\right]^T \Gamma \left[\sum_{j=1}^N a_{ij}\begin{bmatrix} y_i - y_j \\ C(\tilde{v}_i - \tilde{v}_j)\end{bmatrix}\right], \\ u_i &= F\tilde{v}_i, \quad i = 1, \ldots, N, \end{aligned} \tag{3}$$

where $d_i(t)$ denotes the coupling weight for agent i, τ_i is a positive constant, and the rest of the variables are defined as in (3).

Note that the terms $\sum_{j=1}^N c_{ij}a_{ij}C(v_i - v_j)$ in (2) and $\sum_{j=1}^N a_{ij}C(\tilde{v}_i - \tilde{v}_j)$ in (3) imply that the agents need to use the virtual outputs of the consensus protocols from their neighbors via the communication topology $\mathcal{G}$.

The objective in this section is to find proper feedback gain matrices in (2) and (3) such that the N agents in (1) achieve consensus in the sense of $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$, $\forall\, i,j = 1, \ldots, N$.

3.1. Consensus under the edge-based adaptive protocol

In this subsection, we study the consensus problem of the agents in (1) under the edge-based adaptive protocol (2). Let $z_i = [x_i^T, v_i^T]^T$, $e_i = z_i - \frac{1}{N}\sum_{j=1}^N z_j$, $z = [z_1^T, \ldots, z_N^T]^T$, and $e = [e_1^T, \ldots, e_N^T]^T$. Then, we get $e = [(I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T)\otimes I_{2n}]z$. It is easy to see that 0 is a simple eigenvalue of $I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T$ with $\mathbf{1}$ as a corresponding eigenvector and 1 is the other eigenvalue with multiplicity $N-1$. Then, it follows that $e = 0$ if and only if $z_1 = \cdots = z_N$. Therefore, the consensus problem of agents (1) under the protocol (2) is solved if e converges to zero. Because $\varepsilon_{ij} = \varepsilon_{ji}$ and $c_{ij}(0) = c_{ji}(0)$, it follows from (2) that $c_{ij}(t) = c_{ji}(t)$, $\forall\, t \ge 0$. It is not difficult to obtain that e_i and c_{ij} satisfy

$$\begin{aligned} \dot{e}_i &= \mathcal{M}e_i + \sum_{j=1}^N c_{ij}a_{ij}\mathcal{H}(e_i - e_j), \\ \dot{c}_{ij} &= \varepsilon_{ij}a_{ij}(e_i - e_j)^T\mathcal{R}(e_i - e_j), \quad i = 1, \ldots, N, \end{aligned} \tag{4}$$

where $\mathcal{M} = \begin{bmatrix} A & BF \\ 0 & A+BF\end{bmatrix}$, $\mathcal{H} = \begin{bmatrix} 0 & 0 \\ -LC & LC\end{bmatrix}$, and $\mathcal{R} = (I_2\otimes C^T)\Gamma(I_2\otimes C)$.

The following theorem presents a sufficient condition for designing (2) to solve the consensus problem.

Theorem 3. *Suppose that the communication graph $\mathcal{G}$ is undirected and connected. Then, the N agents in (1) reach consensus under the edge-based adaptive protocol (2) with F satisfying that $A+BF$ is Hurwitz, $\Gamma = \begin{bmatrix} I_q & -I_q \\ -I_q & I_q\end{bmatrix}$, and $L = -Q^{-1}C^T$, where $Q > 0$ is a solution to the following linear matrix inequality (LMI):*

$$A^TQ + QA - 2C^TC < 0. \tag{5}$$

Moreover, the protocol states v_i, $i = 1, \ldots, N$, converge to zero and each coupling weight c_{ij} converges to some finite steady-state value.

Proof. Consider the Lyapunov function candidate

$$V_1 = \frac{1}{2}\sum_{i=1}^N e_i^T\mathcal{Q}e_i + \sum_{i=1}^N\sum_{j=1,j\ne i}^N \frac{(c_{ij}-\alpha)^2}{4\varepsilon_{ij}}, \tag{6}$$

where $\mathcal{Q} \triangleq \begin{bmatrix} \varsigma\tilde{Q} + Q & -Q \\ -Q & Q\end{bmatrix}$, $\tilde{Q} > 0$ satisfies that $\tilde{Q}(A+BF) + (A+BF)^T\tilde{Q} < 0$, and α and ς are positive constants to be determined later. In light of the Schur Complement Lemma (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994), it is easy to know that $\mathcal{Q} > 0$.

The time derivative of V_1 along the trajectory of (4) can be obtained as

$$\begin{aligned} \dot{V}_1 &= \sum_{i=1}^N e_i^T\mathcal{Q}\left[\mathcal{M}e_i + \sum_{j=1}^N c_{ij}a_{ij}\mathcal{H}(e_i - e_j)\right] \\ &\quad + \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N (c_{ij}-\alpha)a_{ij}(e_i - e_j)^T\mathcal{R}(e_i - e_j). \end{aligned} \tag{7}$$

Let $\tilde{e}_i = Te_i$, $i = 1, \ldots, N$, with $T = \begin{bmatrix} I & 0 \\ -I & I\end{bmatrix}$. Then, (7) can be rewritten as

$$\begin{aligned} \dot{V}_1 &= \sum_{i=1}^N \tilde{e}_i^T\widetilde{\mathcal{Q}}\left[\widetilde{\mathcal{M}}\tilde{e}_i + \sum_{i=1}^N\sum_{j=1}^N c_{ij}a_{ij}\widetilde{\mathcal{H}}(\tilde{e}_i - \tilde{e}_j)\right] \\ &\quad + \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N (c_{ij}-\alpha)a_{ij}(\tilde{e}_i - \tilde{e}_j)^T\widetilde{\mathcal{R}}(\tilde{e}_i - \tilde{e}_j), \end{aligned} \tag{8}$$

where $\widetilde{\mathcal{Q}} = T^{-T}\mathcal{Q}T^{-1} = \begin{bmatrix} \varsigma\tilde{Q} & 0 \\ 0 & Q \end{bmatrix}$, $\widetilde{\mathcal{M}} = T\mathcal{M}T^{-1} = \begin{bmatrix} A+BF & BF \\ 0 & A \end{bmatrix}$, $\widetilde{\mathcal{H}} = T\mathcal{H}T^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & LC \end{bmatrix}$, and $\widetilde{\mathcal{R}} = T^{-T}\mathcal{R}T^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & C^TC \end{bmatrix}$. Because $c_{ij}(t) = c_{ji}(t)$, $\forall\, t \geq 0$, we have

$$\sum_{i=1}^{N}\sum_{j=1}^{N}(c_{ij}-\alpha)a_{ij}(\tilde{e}_i-\tilde{e}_j)^T\widetilde{\mathcal{R}}(\tilde{e}_i-\tilde{e}_j) = 2\sum_{i=1}^{N}\sum_{j=1}^{N}(c_{ij}-\alpha)a_{ij}\tilde{e}_i^T\widetilde{\mathcal{R}}(\tilde{e}_i-\tilde{e}_j). \tag{9}$$

It is easy to see that $\widetilde{\mathcal{Q}}\widetilde{\mathcal{H}} = -\widetilde{\mathcal{R}}$. Then, by letting $\tilde{e} = [\tilde{e}_1^T, \ldots, \tilde{e}_N^T]^T$, it follows from (8) and (9) that

$$\begin{aligned}\dot{V}_1 &= \sum_{i=1}^{N}\tilde{e}_i^T\widetilde{\mathcal{Q}}\widetilde{\mathcal{M}}\tilde{e}_i - \alpha\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\tilde{e}_i^T\widetilde{\mathcal{R}}(\tilde{e}_i-\tilde{e}_j)\\ &= \frac{1}{2}\tilde{e}^T[I_N\otimes(\widetilde{\mathcal{Q}}\widetilde{\mathcal{M}}+\widetilde{\mathcal{M}}^T\widetilde{\mathcal{Q}}) - 2\alpha\mathcal{L}\otimes\widetilde{\mathcal{R}}]\tilde{e},\end{aligned} \tag{10}$$

where $\mathcal{L}$ is the Laplacian matrix associated with $\mathcal{G}$.

By the definitions of e and $\tilde{e}$, it is easy to see that $(\mathbf{1}^T\otimes I)\tilde{e} = (\mathbf{1}^T\otimes T)e = 0$. Because $\mathcal{G}$ is connected, it then follows from Lemma 1 that $\tilde{e}^T(\mathcal{L}\otimes I)\tilde{e} \geq \lambda_2\tilde{e}^T\tilde{e}$, where λ_2 is the smallest nonzero eigenvalue of $\mathcal{L}$. Therefore, we can get from (10) that

$$\dot{V}_1 \leq \frac{1}{2}\tilde{e}^T[I_N\otimes(\widetilde{\mathcal{Q}}\widetilde{\mathcal{M}}+\widetilde{\mathcal{M}}^T\widetilde{\mathcal{Q}}-2\alpha\lambda_2\widetilde{\mathcal{R}})]\tilde{e}. \tag{11}$$

Note that

$$\widetilde{\mathcal{Q}}\widetilde{\mathcal{M}}+\widetilde{\mathcal{M}}^T\widetilde{\mathcal{Q}}-2\alpha\lambda_2\widetilde{\mathcal{R}} = \begin{bmatrix} \varsigma[\tilde{Q}(A+BF)+(A+BF)^T\tilde{Q}] & \varsigma\tilde{Q}BF \\ \varsigma F^TB^T\tilde{Q} & \Pi \end{bmatrix}, \tag{12}$$

where $\Pi \triangleq QA + A^TQ - 2\alpha\lambda_2C^TC$. By choosing α sufficiently large such that $\alpha\lambda_2 \geq 1$, it follows from (5) that $\Pi < 0$. Then, choosing $\varsigma > 0$ sufficiently small and by virtue of the Schur Complement Lemma (Boyd et al., 1994), we can obtain from (12) and (11) that $\widetilde{\mathcal{Q}}\widetilde{\mathcal{M}}+\widetilde{\mathcal{M}}^T\widetilde{\mathcal{Q}}-2\alpha\lambda_2\widetilde{\mathcal{R}} < 0$ and $\dot{V}_1 \leq 0$.

Since $\dot{V}_1 \leq 0$, $V_1(t)$ is bounded, implying that each c_{ij} is also bounded. By noting that $\mathcal{R} \geq 0$, we can see from (4) that c_{ij} is monotonically increasing. Then, it follows that each coupling weight c_{ij} converges to some finite value. Note that $\dot{V}_1 \equiv 0$ implies that $\tilde{e} = 0$ and $e = 0$. Hence, by LaSalle's Invariance principle (Krstić, Kanellakopoulos, & Kokotovic, 1995), it follows that $e(t) \to 0$, as $t \to \infty$. That is, the consensus problem is solved. By (2) and noting the fact that $A + BF$ is Hurwitz, it is easy to see that the protocol states v_i, $i = 1, \ldots, N$, converge to zero. □

Remark 1. As shown in Li et al. (2010), a necessary and sufficient condition for the existence of a $Q > 0$ to the LMI (5) is that (A, C) is detectable. Therefore, a sufficient condition for the existence of a protocol (2) satisfying Theorem 3 is that (A, B, C) is stabilizable and detectable. Because the distributed adaptive protocol (2) is essentially nonlinear, the separation principle which holds for linear systems generally does not hold anymore. A favorable feature of the adaptive protocol (2) is that its feedback gain matrices F, L, and Γ can be independently designed.

3.2. Consensus under the node-based adaptive protocol

This subsection considers the consensus problem of the agents in (1) under the node-based adaptive protocol (3). Let $\tilde{z}_i = [x_i^T, \tilde{v}_i^T]^T$, $\zeta_i = \tilde{z}_i - \frac{1}{N}\sum_{j=1}^{N}\tilde{z}_j$, and $\zeta = [\zeta_1^T, \ldots, \zeta_N^T]^T$. As shown in the last subsection, the consensus problem of agents (1) under the protocol (3) is solved if ζ converges to zero. We can obtain that ζ_i and d_i satisfy the following dynamics:

$$\begin{aligned}\dot{\zeta}_i &= \mathcal{M}\zeta_i + d_i\sum_{j=1}^{N}\mathcal{L}_{ij}\mathcal{H}\zeta_j - \frac{1}{N}\sum_{k=1}^{N}d_k\sum_{j=1}^{N}\mathcal{L}_{kj}\mathcal{H}\zeta_j,\\ \dot{d}_i &= \tau_i\left[\sum_{j=1}^{N}\mathcal{L}_{ij}\zeta_j^T\right]\mathcal{R}\left[\sum_{j=1}^{N}\mathcal{L}_{ij}\zeta_j\right],\quad i = 1,\ldots,N,\end{aligned} \tag{13}$$

where $\mathcal{M}$, $\mathcal{H}$, and $\mathcal{R}$ are defined in (4). Let $D(t) = \mathrm{diag}(d_1(t), \ldots, d_N(t))$. Then, the first equation in (13) can be rewritten into a compact form as

$$\dot{\zeta} = \left[I_N\otimes\mathcal{M} + \left(\left(I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T\right)D\mathcal{L}\right)\otimes\mathcal{H}\right]\zeta. \tag{14}$$

The following result presents a sufficient condition for designing (3).

Theorem 4. *Assume that the communication graph $\mathcal{G}$ is undirected and connected. Then, the N agents in (1) reach consensus under the node-based adaptive protocol (3) with L, F, and Γ given as in Theorem 3. Moreover, the protocol states $\tilde{v}_i$, $i = 1, \ldots, N$, converge to zero and each coupling weight d_i converges to some finite steady-state value.*

Proof. Consider the Lyapunov function candidate

$$V_2 = \frac{1}{2}\zeta^T(\mathcal{L}\otimes\mathcal{Q})\zeta + \sum_{i=1}^{N}\frac{(d_i-\beta)^2}{2\tau_i}, \tag{15}$$

where $\mathcal{Q}$ is defined in (6) and β is a positive constant to be determined later. For a connected graph $\mathcal{G}$, it follows from Lemma 1 and the definition of ζ that $\zeta^T(\mathcal{L}\otimes\mathcal{Q})\zeta \geq \lambda_2\zeta^T(I_N\otimes\mathcal{Q})\zeta$. Therefore, it is easy to see that $\Omega_c = \{\zeta, d_i | V_2 \leq c\}$ is compact for any positive c.

Following similar steps to those in the proof of Theorem 3, we can obtain the time derivative of V_2 along the trajectory of (14) as

$$\begin{aligned}\dot{V}_2 &= \zeta^T\left[\mathcal{L}\otimes\mathcal{Q}\mathcal{M} + \left(\mathcal{L}D\mathcal{L} - \frac{1}{N}\mathcal{L}\mathbf{1}\mathbf{1}^TD\mathcal{L}\right)\otimes\mathcal{Q}\mathcal{H}\right]\zeta\\ &\quad + \sum_{i=1}^{N}(d_i-\beta)\left(\sum_{j=1}^{N}\mathcal{L}_{ij}\zeta_j^T\right)\mathcal{R}\left(\sum_{j=1}^{N}\mathcal{L}_{ij}\zeta_j\right)\\ &= \tilde{\zeta}^T[\mathcal{L}\otimes\widetilde{\mathcal{Q}}\widetilde{\mathcal{M}} - \mathcal{L}D\mathcal{L}\otimes\widetilde{\mathcal{R}}]\tilde{\zeta}\\ &\quad + \sum_{i=1}^{N}(d_i-\beta)\left(\sum_{j=1}^{N}\mathcal{L}_{ij}\tilde{\zeta}_j^T\right)\widetilde{\mathcal{R}}\left(\sum_{j=1}^{N}\mathcal{L}_{ij}\tilde{\zeta}_j\right),\end{aligned} \tag{16}$$

where $\tilde{\zeta} \triangleq [\tilde{\zeta}_1^T, \ldots, \tilde{\zeta}_N^T]^T = (I_N\otimes T)\zeta$, T, $\widetilde{\mathcal{Q}}$, $\widetilde{\mathcal{M}}$, and $\widetilde{\mathcal{R}}$ are the same as in (8). Observe that

$$\tilde{\zeta}^T(\mathcal{L}D\mathcal{L}\otimes\widetilde{\mathcal{R}})\tilde{\zeta} = \sum_{i=1}^{N}d_i\left(\sum_{j=1}^{N}\mathcal{L}_{ij}\tilde{\zeta}_j^T\right)\widetilde{\mathcal{R}}\left(\sum_{j=1}^{N}\mathcal{L}_{ij}\tilde{\zeta}_j\right). \tag{17}$$

Substituting (17) into (16) yields

$$\dot{V}_2 = \frac{1}{2}\tilde{\zeta}^T[\mathcal{L}\otimes(\widetilde{\mathcal{Q}}\widetilde{\mathcal{M}}+\widetilde{\mathcal{M}}^T\widetilde{\mathcal{Q}}^T) - 2\beta\mathcal{L}^2\otimes\widetilde{\mathcal{R}}]\tilde{\zeta}. \tag{18}$$

Because $\mathcal{G}$ is connected, it follows from Lemma 1 that zero is a simple eigenvalue of $\mathcal{L}$ and all the other eigenvalues are positive. Let $U = \begin{bmatrix}\frac{\mathbf{1}}{\sqrt{N}} & Y_1\end{bmatrix}$ and $U^T = \begin{bmatrix}\frac{\mathbf{1}^T}{\sqrt{N}}\\ Y_2\end{bmatrix}$, with $Y_1 \in \mathbf{R}^{N\times(N-1)}$, $Y_2 \in \mathbf{R}^{(N-1)\times N}$, be such unitary matrices that $U^T\mathcal{L}U = \Lambda \triangleq \mathrm{diag}(0, \lambda_2, \ldots, \lambda_N)$, where $\lambda_2 \leq \cdots \leq \lambda_N$ are the nonzero eigenvalues of $\mathcal{L}$. Let $\bar{\zeta} \triangleq [\bar{\zeta}_1^T, \ldots, \bar{\zeta}_N^T]^T = (U^T\otimes I)\tilde{\zeta}$. By the definitions of ζ and $\tilde{\zeta}$, it is easy to see that $\bar{\zeta}_1 = (\mathbf{1}^T \otimes T)\zeta = 0$. Then, it follows from (18) that

$$\begin{aligned}\dot{V}_2 &= \frac{1}{2}\bar{\zeta}^T[\Lambda \otimes (\widetilde{\mathcal{Q}}\widetilde{\mathcal{M}} + \widetilde{\mathcal{M}}^T\widetilde{\mathcal{Q}}) - 2\beta\Lambda^2 \otimes \widetilde{\mathcal{R}}]\bar{\zeta} \\ &= \frac{1}{2}\sum_{i=2}^{N}\lambda_i\bar{\zeta}_i^T(\widetilde{\mathcal{Q}}\widetilde{\mathcal{M}} + \widetilde{\mathcal{M}}^T\widetilde{\mathcal{Q}} - 2\beta\lambda_i\widetilde{\mathcal{R}})\bar{\zeta}_i. \end{aligned} \tag{19}$$

As shown in the proof of Theorem 3, by choosing $\varsigma > 0$ sufficiently small and β sufficiently large such that $\beta\lambda_2 \geq 1$, we can obtain from (19) that $\dot{V}_2 \leq 0$. Note that $\dot{V}_2 \equiv 0$ implies that $\bar{\zeta}_i = 0,\ i = 2, \ldots, N$, which, together with $\bar{\zeta}_1 = 0$, further implies that $\zeta = 0$. Therefore, it follows from LaSalle's Invariance principle that $\zeta \to 0$, as $t \to \infty$. The convergence of d_i and $\tilde{v}_i,\ i = 1, \ldots, N$, can be shown by following similar steps in the proof of Theorem 3, which is omitted here for brevity. □

Remark 2. Different from the previous adaptive schemes in Li et al. (in press), Su et al. (2011) and Yu et al. (2011) which are based on the relative state information, the proposed adaptive protocols (2) and (3) rely on the relative outputs of neighboring agents. Contrary to the protocols in Li et al. (2010, 2011), Ma and Zhang (2010), Seo et al. (2009) and Zhang et al. (2011), the adaptive protocols (2) and (3) can be computed and implemented by each agent in a fully distributed fashion without using any global information of the communication graph.

Remark 3. It is worth mentioning that the dynamic consensus protocol in Scardovi and Sepulchre (2009) does not need any global information only when the eigenvalues of the state matrix A of each agent lie in the closed left half-plane. In contrast, we do not impose the assumption that A has no unstable eigenvalues. Further, the dimension of the node-based adaptive protocol (3) is lower than that of the protocol in Scardovi and Sepulchre (2009). Moreover, the adaptive protocol framework proposed in this paper can be extended to solve the consensus problem for the case where there exists a leader with nonzero control input, as detailed in Section 4. It should also be mentioned that an advantage of the results in Scardovi and Sepulchre (2009) is that they are applicable to the case with jointly connected switching graphs.

Remark 4. Some comparisons between the adaptive consensus protocols (2) and (3) are now briefly discussed. The dimension of the edge-based adaptive protocol (2) is proportional to the number of edges in the communication graph. Since the number of edges is usually larger than the number of nodes in a connected graph, the dimension of the edge-based adaptive protocol (2) is generally higher than that of the node-based protocol (3). On the other hand, the edge-based adaptive protocol (2) is applicable to the case with switching communication graphs, which will be shown in the following subsection.

3.3. *Extensions to switching communication graphs*

In the last subsections, the communication graph is assumed to be fixed throughout the whole process. However, the communication graph may change with time in many practical situations due to various reasons, such as communication constraints and link variations. In this subsection, the consensus problem under the edge-based adaptive protocol (2) with switching communication graphs will be considered.

Denote by $\mathcal{G}_N$ the set of all possible undirected connected graphs with N nodes. Let $\sigma(t) : [0, \infty) \to \mathcal{P}$ be a piecewise constant switching signal with switching times $t_0, t_1, \ldots,$ and $\mathcal{P}$ be the index set associated with the elements of $\mathcal{G}_N$, which is clearly finite. The communication graph at time t is denoted by $\mathcal{G}_{\sigma(t)}$. Accordingly, (2) becomes

$$\begin{aligned}\dot{v}_i &= (A + BF)v_i + L\sum_{j=1}^{N} c_{ij}a_{ij}(t)\left[C(v_i - v_j) - (y_i - y_j)\right], \\ \dot{c}_{ij} &= \varepsilon_{ij}a_{ij}(t)\begin{bmatrix} y_i - y_j \\ C(v_i - v_j)\end{bmatrix}^T \Gamma \begin{bmatrix} y_i - y_j \\ C(v_i - v_j)\end{bmatrix}, \\ u_i &= Fv_i, \quad i = 1, \ldots, N,\end{aligned} \tag{20}$$

where $a_{ij}(t)$ is the (i, j)-th entry of the adjacency matrix associated with $\mathcal{G}_{\sigma(t)}$ and the rest of the variables are the same as in (2).

Theorem 5. *For arbitrary switching communication graphs $\mathcal{G}_{\sigma(t)}$ belonging to $\mathcal{G}_N$, the N agents in (1) reach consensus under the edge-based protocol (20) with F, L, and Γ given as in Theorem 3. Besides, the protocol states $v_i,\ i = 1, \ldots, N$, converge to zero and the coupling weights c_{ij} converge to some finite values.*

Proof. Let e_i and e be defined as in (4). By following similar steps to those in the proof of Theorem 3, the consensus problem of the agents (1) under the protocol (20) is solved if e converges to zero. Clearly, e_i and c_{ij} satisfy

$$\begin{aligned}\dot{e}_i &= \mathcal{M}e_i + \sum_{j=1}^{N}(\tilde{c}_{ij} + \delta)a_{ij}\mathcal{H}(e_i - e_j), \\ \dot{\tilde{c}}_{ij} &= \varepsilon_{ij}a_{ij}(e_i - e_j)^T\mathcal{R}(e_i - e_j), \quad i = 1, \ldots, N,\end{aligned} \tag{21}$$

where $\tilde{c}_{ij} = c_{ij} - \delta$, δ is a positive scalar, and $\mathcal{M}$, $\mathcal{H}$, and $\mathcal{R}$ are defined as in (4).

Take a common Lyapunov function candidate

$$V_3 = \frac{1}{2}\sum_{i=1}^{N} e_i^T\mathcal{Q}e_i + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\frac{\tilde{c}_{ij}^2}{4\varepsilon_{ij}},$$

where $\mathcal{Q}$ is defined in (6). Following similar steps as in the proof of Theorem 3, we can obtain the time derivative of V_3 along (21) as

$$\dot{V}_3 = \frac{1}{2}\tilde{e}^T[I_N \otimes (\widetilde{\mathcal{Q}}\widetilde{\mathcal{M}} + \widetilde{\mathcal{M}}^T\widetilde{\mathcal{Q}}) - 2\delta\mathcal{L}_{\sigma(t)} \otimes \widetilde{\mathcal{R}}]\tilde{e}, \tag{22}$$

where $\mathcal{L}_{\sigma(t)}$ is the Laplacian matrix associated with $\mathcal{G}_{\sigma(t)}$ and the rest of the variables are the same as in (8). Since $\mathcal{G}_{\sigma(t)}$ is connected and $(\mathbf{1}^T \otimes I)\tilde{e} = 0$, it is easy to see that $\tilde{e}^T(\mathcal{L}_{\sigma(t)} \otimes I)\tilde{e} \geq \lambda_2^{\min}\tilde{e}^T\tilde{e}$, where $\lambda_2^{\min}$ denotes the minimum of the smallest nonzero eigenvalues of $\mathcal{L}_{\sigma(t)}$ for all $\mathcal{G}_{\sigma(t)} \in \mathcal{G}_N$. Therefore, we can get from (22) that

$$\dot{V}_3 \leq W(\hat{e}) \triangleq \tilde{e}^T[I_N \otimes (\widetilde{\mathcal{Q}}\widetilde{\mathcal{M}} + \widetilde{\mathcal{M}}^T\widetilde{\mathcal{Q}} - 2\delta\lambda_2^{\min}\widetilde{\mathcal{R}})]\tilde{e}.$$

As shown in the proof of Theorem 3, by choosing $\varsigma > 0$ sufficiently small and $\delta > 0$ sufficiently large such that $\delta\lambda_2^{\min} \geq 1$, we have $\widetilde{\mathcal{Q}}\widetilde{\mathcal{M}} + \widetilde{\mathcal{M}}^T\widetilde{\mathcal{Q}} - 2\delta\lambda_2^{\min}\widetilde{\mathcal{R}} < 0$. Therefore, $\dot{V}_3 \leq 0$. Note that V_3 is positive definite and radically unbounded. By the LaSalle–Yoshizawa theorem (Krstić et al., 1995), it follows that $\lim_{t\to\infty} W(\tilde{e}) = 0$, implying that $\tilde{e}(t) \to 0$, as $t \to \infty$, which further implies that $e(t) \to 0$, as $t \to \infty$. The convergence of $v_i,\ i = 1, \ldots, N$, and c_{ij} can be similarly shown as in the proof of Theorem 3. □

Remark 5. Theorem 5 shows that the edge-based adaptive consensus protocol (2) given by Theorem 3 is applicable to arbitrary switching communication graphs which are connected at any time instant. Because the Lyapunov function in (15) for the node-based adaptive protocol (3) is explicitly related with the communication graph, it cannot be taken as a feasible common Lyapunov function in the case of switching topologies.

4. Leader–follower consensus with a leader of possibly nonzero control input

For the undirected communication graph in the previous section, the final consensus values reached by the agents under the adaptive protocols (2) and (3), which depend on the agent dynamics, the initial conditions of the agents' states and the adaptive coupling weights c_{ij} and d_i, the communication graph, and the feedback gain matrices in (2) and (3), are generally difficult to be explicitly obtained. The main difficulty lies in that the adaptive protocols (2) and (3) are essentially nonlinear. In some cases, it might be desirable for the agents' states to converge onto a reference trajectory. This is actually the leader–follower consensus problem.

In this section, we extend to consider the case where the N agents in (1) maintain a leader–follower communication graph $\mathcal{G}$. Without loss of generality, assume that the agent indexed by 1 is the leader and the agents indexed by $2, \ldots, N$, are followers. The leader does not receive any information from the followers, i.e., it has no neighbor, while each follower can obtain the local output information from its neighbors.

In the following, the following assumption is needed.

Assumption 1. The subgraph associated with the followers is undirected and the graph $\mathcal{G}$ contains a directed spanning tree with the leader as the root.

Denote by $\mathcal{L}$ the Laplacian matrix associated with $\mathcal{G}$. Because the leader has no neighbors, $\mathcal{L}$ can be partitioned as $\mathcal{L} = \begin{bmatrix} 0 & 0_{1\times(N-1)} \\ \mathcal{L}_2 & \mathcal{L}_1 \end{bmatrix}$, where $\mathcal{L}_2 \in \mathbf{R}^{(N-1)\times 1}$ and $\mathcal{L}_1 \in \mathbf{R}^{(N-1)\times(N-1)}$ is symmetric.

It is said that the leader–follower consensus problem is solved if the states of the followers converge to the state of the leader, i.e., $\lim_{t\to\infty}\|x_i(t) - x_1(t)\| = 0,\ \forall i = 2, \ldots, N$.

In most previous works on leader–follower consensus, e.g., Hong et al. (2008), Li et al. (2010) and Zhang et al. (2011), it is usually assumed that the leader's control input u_1 is zero. This assumption might be restrictive for many cases. In practice, nonzero control actions might be implemented on the leader in order to achieve certain objectives, e.g., to reach a desirable consensus value or to avoid hazardous obstacles. Besides, it is not practical to assume that every follower knows the leader's control input. The objective of this section is to address the leader–follower consensus problem for the general case where the leader's control input is possibly nonzero and time varying and accessible to only a subset of followers, under the following mild assumption:

Assumption 2. The leader's control input u_1 is bounded, i.e., $\|u_1\| \leq \gamma$, where γ is a positive constant.

Note that the bound γ is not required to be known in the distributed consensus protocols to be developed in the following. Based on the relative estimates of the states of neighboring agents, the following distributed continuous adaptive controller is proposed for each follower:

$$
\begin{aligned}
\dot{\hat{v}}_i &= A\hat{v}_i + Bu_i + \hat{L}(C\hat{v}_i - y_i),\\
u_i &= d_i\sum_{j=1}^N a_{ij}\hat{F}(\hat{v}_i - \hat{v}_j) + d_i g\left(\sum_{j=1}^N a_{ij}\hat{F}(\hat{v}_i - \hat{v}_j)\right),\\
\dot{d}_i &= \tau_i\left[-\varphi_i d_i + \left(\sum_{j=1}^N a_{ij}(\hat{v}_i - \hat{v}_j)^T\right)\hat{\Gamma}\left(\sum_{j=1}^N a_{ij}(\hat{v}_i - \hat{v}_j)\right)\right.\\
&\quad \left. + \left\|\sum_{j=1}^N a_{ij}\hat{F}(\hat{v}_i - \hat{v}_j)\right\|\right], \quad i = 2, \ldots, N,
\end{aligned}
\tag{23}
$$

where $\hat{v}_i \in \mathbf{R}^n$ is the estimate of the state of the i-th follower, $\hat{v}_1 \in \mathbf{R}^n$ denotes the estimate of the state of the leader, given by $\dot{\hat{v}}_1 = A\hat{v}_1 + Bu_1 + L(C\hat{v}_1 - y_1)$, φ_i are small positive constants, τ_i are positive scalars, d_i denotes the time-varying coupling weight associated with the i-th follower, $\hat{L} \in \mathbf{R}^{n\times q}$, $\hat{F} \in \mathbf{R}^{p\times n}$, and $\hat{\Gamma} \in \mathbf{R}^{n\times n}$ are the feedback gain matrices, and the nonlinear function $g(\cdot)$ is defined such that for $w \in \mathbf{R}^n$,

$$
g(w) = \begin{cases} \dfrac{w}{\|w\|} & \text{if } d_i\|w\| > \kappa \\ \dfrac{w}{\kappa}d_i & \text{if } d_i\|w\| \leq \kappa, \end{cases}
\tag{24}
$$

where κ is a small positive value.

Note that the term $\sum_{j=1}^N a_{ij}(\hat{v}_i - \hat{v}_j)$ in (23) implies that the agents need to get the estimates of the states of their neighbors, which can be transmitted via the communication graph $\mathcal{G}$.

Let $\xi_i = \begin{bmatrix} x_i - x_1 \\ \hat{v}_i - \hat{v}_1 \end{bmatrix}$, $i = 2, \ldots, N$, $\xi = [\xi_2^T, \ldots, \xi_N^T]^T$, and $E(t) = \mathrm{diag}(d_2(t), \ldots, d_N(t))$. Using (23) for (1), we obtain the closed-loop network dynamics as

$$
\begin{aligned}
\dot{\xi} &= (I \otimes \mathcal{S} + E\mathcal{L}_1 \otimes \mathcal{F})\xi + (E \otimes \mathcal{B})G(\xi) - (\mathbf{1} \otimes \mathcal{B})u_1,\\
\dot{d}_i &= \tau_i\left[-\varphi_i d_i + \left(\sum_{j=2}^N \mathcal{L}_{ij}\xi_j^T\right)\mathcal{I}\left(\sum_{j=2}^N \mathcal{L}_{ij}\xi_j\right)\right.\\
&\quad \left. + \left\|\mathcal{O}\sum_{j=2}^N \mathcal{L}_{ij}\xi_j\right\|\right], \quad i = 2, \ldots, N,
\end{aligned}
\tag{25}
$$

where $G(\xi) \triangleq \begin{bmatrix} g(\mathcal{O}\sum_{j=2}^N \mathcal{L}_{2j}\xi_j) \\ \vdots \\ g(\mathcal{O}\sum_{j=2}^N \mathcal{L}_{Nj}\xi_j) \end{bmatrix}$, $\mathcal{S} = \begin{bmatrix} A & 0 \\ -\hat{L}C & A + \hat{L}C \end{bmatrix}$, $\mathcal{F} = \begin{bmatrix} 0 & B\hat{F} \\ 0 & B\hat{F} \end{bmatrix}$, $\mathcal{I} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{\Gamma} \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} B \\ B \end{bmatrix}$, and $\mathcal{O} = \begin{bmatrix} 0 & \hat{F} \end{bmatrix}$. Clearly, the leader–follower consensus problem is solved if ξ of (25) converges to zero. Hereafter, we refer to ξ as the consensus error.

The following theorem presents the ultimate boundedness of ξ and d_i.

Theorem 6. *Suppose that Assumptions 1 and 2 hold. Then, both the consensus error ξ and the coupling weights $d_i, i = 2, \ldots, N$, in (25) are uniformly ultimately bounded under the distributed continuous adaptive protocol (23) with $\hat{L}$ satisfying that $A+\hat{L}C$ is Hurwitz, $\hat{F} = -B^TP^{-1}$, and $\hat{\Gamma} = P^{-1}BB^TP^{-1}$, where $P > 0$ is a solution to the following LMI:*

$$
AP + PA^T - 2BB^T < 0. \tag{26}
$$

Proof. Let $\tilde{d}_i = d_i - \beta,\ i = 2, \ldots, N$, where β is a positive constant. Then, (25) can be rewritten as

$$
\begin{aligned}
\dot{\xi} &= (I \otimes \mathcal{S} + \widetilde{E}\mathcal{L}_1 \otimes \mathcal{F})\xi + (\widetilde{E} \otimes \mathcal{B})G(\xi) - (\mathbf{1} \otimes \mathcal{B})u_1,\\
\dot{\tilde{d}}_i &= \tau_i\left[-\varphi_i(\tilde{d}_i + \beta) + \left(\sum_{j=2}^N \mathcal{L}_{ij}\xi_j\right)^T\mathcal{I}\left(\sum_{j=2}^N \mathcal{L}_{ij}\xi_j\right)\right.\\
&\quad \left. + \left\|\mathcal{O}\sum_{j=2}^N \mathcal{L}_{ij}\xi_j\right\|\right], \quad i = 2, \ldots, N,
\end{aligned}
\tag{27}
$$

where $\widetilde{E}(t) = \mathrm{diag}(\tilde{d}_2 + \beta, \ldots, \tilde{d}_N + \beta)$.

Consider the following Lyapunov function candidate

$$
V_4 = \frac{1}{2}\xi^T(\mathcal{L}_1 \otimes \mathcal{P})\xi + \sum_{i=2}^N \frac{\tilde{d}_i^2}{2\tau_i},
$$

where $\mathcal{P} \triangleq \begin{bmatrix} \vartheta\hat{P} & -\vartheta\hat{P} \\ -\vartheta\hat{P} & \vartheta\hat{P} + P^{-1} \end{bmatrix}$, $\hat{P} > 0$ satisfies that $(A+\hat{L}C)\hat{P} + (A+\hat{L}C)^T\hat{P} < 0$, and ϑ is a positive constant to be determined later. In light of the Schur Complement Lemma (Boyd et al., 1994), it is easy to know that $\mathscr{P} > 0$. It follows directly from Assumption 1 and Lemma 1 that $\mathcal{L}_1 > 0$. Since $\mathscr{P} > 0$, it is easy to see that V_4 is positive definite.

The time derivative of V_4 along (27) can be obtained as

$$\begin{aligned}\dot{V}_4 &= \hat{\xi}^T[(\mathcal{L}_1 \otimes \widehat{\mathscr{P}}\widehat{\mathscr{S}} + \mathcal{L}_1\widetilde{E}\mathcal{L}_1 \otimes \widehat{\mathscr{P}}\widehat{\mathscr{F}})\hat{\xi} - (\mathcal{L}_1\mathbf{1} \otimes \widehat{\mathscr{P}}\widehat{\mathscr{B}})u_1] \\ &\quad + \hat{\xi}^T(\mathcal{L}_1\widetilde{E} \otimes \widehat{\mathscr{P}}\widehat{\mathscr{B}})G(\hat{\xi}) + \sum_{i=2}^N \tilde{d}_i\Bigg[-\varphi_i(\tilde{d}_i+\beta) \\ &\quad + \left(\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j^T\right)\mathscr{I}\left(\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right) + \left\|\mathcal{O}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right\|\Bigg], \qquad (28)\end{aligned}$$

where $\hat{\xi} \triangleq [\hat{\xi}_2^T, \ldots, \hat{\xi}_N^T]^T = (I \otimes \hat{T})\xi$ with $\hat{T} = \begin{bmatrix} I & -I \\ 0 & I\end{bmatrix}$, $\widehat{\mathscr{P}} = \begin{bmatrix} \vartheta\hat{P} & 0 \\ 0 & P^{-1}\end{bmatrix}$, $\widehat{\mathscr{S}} = \begin{bmatrix} A+\hat{L}C & 0 \\ -\hat{L}C & A\end{bmatrix}$, $\widehat{\mathscr{F}} = \begin{bmatrix} 0 & 0 \\ 0 & B\hat{F}\end{bmatrix}$, and $\widehat{\mathscr{B}} = \begin{bmatrix} 0 \\ B\end{bmatrix}$. Note that

$$\begin{aligned}\hat{\xi}^T(\mathcal{L}_1\widetilde{E}\mathcal{L}_1 \otimes \widehat{\mathscr{P}}\widehat{\mathscr{F}})\hat{\xi} &= \sum_{i=2}^N(\tilde{d}_i+\beta)\left[\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j^T\right] \\ &\quad \times \widehat{\mathscr{P}}\widehat{\mathscr{F}}\left[\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right]. \qquad (29)\end{aligned}$$

By virtue of Assumption 2, we have

$$\begin{aligned}-\hat{\xi}^T(\mathcal{L}_1\mathbf{1} \otimes \widehat{\mathscr{P}}\widehat{\mathscr{B}})u_1 &= -\sum_{i=2}^N\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j^T\widehat{\mathscr{P}}\widehat{\mathscr{B}}u_1 \\ &\le \sum_{i=2}^N \left\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right\| \|u_1\| \\ &\le \gamma\sum_{i=2}^N \left\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right\|. \qquad (30)\end{aligned}$$

Next, consider the following three cases.

(i) $d_i\|\mathcal{O}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\| > \kappa,\ i = 2, \ldots, N$.

By observing that $\widehat{\mathscr{P}}\widehat{\mathscr{B}} = -\mathcal{O}^T$, it is not difficult to get from the definitions of $g(\cdot)$ and $G(\cdot)$ that

$$\hat{\xi}^T(\mathcal{L}_1\widetilde{E} \otimes \widehat{\mathscr{P}}\widehat{\mathscr{B}})G(\hat{\xi}) = -\sum_{i=2}^N(\tilde{d}_i+\beta)\left\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right\|. \qquad (31)$$

By noting that $\widehat{\mathscr{P}}\widehat{\mathscr{F}} = -\mathscr{I}$ and substituting (29)–(31) into (28), we can get that

$$\begin{aligned}\dot{V}_4 &\le Z(\hat{\xi}) - \sum_{i=2}^N \varphi_i(\tilde{d}_i^2 + \beta\tilde{d}_i) \\ &\quad - (\beta-\gamma)\sum_{i=2}^N \left\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right\|,\end{aligned}$$

where

$$Z(\hat{\xi}) \triangleq \frac{1}{2}\hat{\xi}^T[\mathcal{L}_1 \otimes (\widehat{\mathscr{P}}\widehat{\mathscr{S}} + \widehat{\mathscr{S}}^T\widehat{\mathscr{P}}) - 2\beta\mathcal{L}_1^2 \otimes \mathscr{I}]\hat{\xi}. \qquad (32)$$

Because $-\tilde{d}_i^2 - \tilde{d}_i\beta \le -\frac{1}{2}\tilde{d}_i^2 + \frac{1}{2}\beta^2$, it then follows that $\dot{V}_4 \le \Psi$, where

$$\begin{aligned}\Psi &\triangleq Z(\hat{\xi}) + \frac{1}{2}\sum_{i=2}^N \varphi_i(\beta^2 - \tilde{d}_i^2) \\ &\quad - (\beta-\gamma)\sum_{i=2}^N \left\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right\|. \qquad (33)\end{aligned}$$

(ii) $d_i\|\mathcal{O}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\| \le \kappa,\ i = 2, \ldots, N$.

From the definitions of $g(\cdot)$ and $G(\cdot)$, in this case we get that

$$\hat{\xi}^T(\mathcal{L}_1\widetilde{E} \otimes \widehat{\mathscr{P}}\widehat{\mathscr{B}})G(\hat{\xi}) = -\sum_{i=2}^N \frac{(\tilde{d}_i+\beta)^2}{\kappa}\left\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right\|^2. \qquad (34)$$

Then, it follows from (29), (30), (34), and (28) that

$$\begin{aligned}\dot{V}_4 &\le \Psi - \sum_{i=2}^N \frac{(\tilde{d}_i+\beta)^2}{\kappa}\left\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right\|^2 \\ &\quad + \sum_{i=2}^N(\tilde{d}_i+\beta)\left\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right\| \\ &\le \Psi + \frac{1}{4}(N-1)\kappa. \qquad (35)\end{aligned}$$

Note that to get the last inequality in (35), we have used the fact that for $d_i\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\xi_j\| \le \kappa$, $-\frac{d_i^2}{\kappa}\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\xi_j\|^2 + d_i\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\| \le \frac{1}{4}\kappa$.

(iii) ξ satisfies neither case (i) nor case (ii).

In this case, without loss of generality, we can assume that $d_i\|\mathcal{O}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\| > \kappa,\ i = 2, \ldots, l$, and $d_i\|\mathcal{O}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\| \le \kappa,\ i = l+1, \ldots, N$, where $3 \le l \le N-1$. Combining (31) and (34), we have

$$\begin{aligned}\hat{\xi}^T(\mathcal{L}_1\widetilde{E} \otimes \widehat{\mathscr{P}}\widehat{\mathscr{B}})G(\hat{\xi}) &= -\sum_{i=2}^l(\tilde{d}_i+\beta)\left\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right\| \\ &\quad - \sum_{i=l+1}^N \frac{(\tilde{d}_i+\beta)^2}{\kappa}\left\|\widehat{\mathscr{B}}^T\widehat{\mathscr{P}}\sum_{j=2}^N \mathcal{L}_{ij}\hat{\xi}_j\right\|^2.\end{aligned}$$

Then, by following the steps in the two cases above, it is not difficult to get that in this case $\dot{V}_4 \le \Psi + \frac{1}{4}(N-l)\kappa$.

Analyzing the above three cases, we get that $\dot{V}_4$ satisfies (35) for all $\xi \in \mathbf{R}^{2Nn}$. By choosing β sufficiently large such that $\beta\lambda_2 \ge 1$ and $\beta \ge \gamma$, we can obtain from (33) and (35) that

$$\dot{V}_4 \le Z(\hat{\xi}) + \frac{1}{2}\sum_{i=2}^N \varphi_i(\beta^2 - \tilde{d}_i^2) + \frac{1}{4}(N-1)\kappa. \qquad (36)$$

Note that

$$\begin{aligned}&\mathrm{diag}(I, P)[\widehat{\mathscr{P}}\widehat{\mathscr{S}} + \widehat{\mathscr{S}}^T\widehat{\mathscr{P}} - 2\alpha\lambda_2\mathscr{I}]\mathrm{diag}(I, P) \\ &\quad = \begin{bmatrix} \Xi & -C^TL^T \\ -LC & AP + PA^T - 2\alpha\lambda_2BB^T\end{bmatrix}, \qquad (37)\end{aligned}$$

where $\Xi = \vartheta[\hat{P}(A+LC) + (A+LC)^T\hat{P}]$. Because $\beta\lambda_2 \ge 1$, it follows from (26) that $AP + PA^T - 2\alpha\lambda_2BB^T < 0$. Then, choosing $\vartheta > 0$ sufficiently large and by virtue of the Schur Complement Lemma (Boyd et al., 1994), we can obtain from (37) that $\widehat{\mathscr{P}}\widehat{\mathscr{S}} + \widehat{\mathscr{S}}^T\widehat{\mathscr{P}} - 2\alpha\lambda_2\mathscr{I} < 0$. Observe that

$$\begin{aligned}&(\mathcal{L}_1^{-\frac{1}{2}} \otimes I)[\mathcal{L}_1 \otimes (\widehat{\mathscr{P}}\widehat{\mathscr{S}} + \widehat{\mathscr{S}}^T\widehat{\mathscr{P}}) - 2\beta\mathcal{L}_1^2 \otimes \mathscr{I}](\mathcal{L}_1^{-\frac{1}{2}} \otimes I) \\ &\quad \le I \otimes [\widehat{\mathscr{P}}\widehat{\mathscr{S}} + \widehat{\mathscr{S}}^T\widehat{\mathscr{P}} - 2\beta\lambda_2\mathscr{I}].\end{aligned}$$

Thus, we know that $\mathcal{L}_1 \otimes (\widehat{\mathscr{P}}\widehat{\mathscr{S}} + \widehat{\mathscr{S}}^T\widehat{\mathscr{P}}) - 2\beta\mathcal{L}_1^2 \otimes \mathscr{I} < 0$, implying that $Z(\hat{\xi}) - \frac{1}{2}\sum_{i=2}^N \varphi_i\tilde{d}_i^2 < 0$, where $Z(\hat{\xi})$ is defined in (32). By virtue of Lemma 2, we get from (36) that the states ξ and d_i of (25) are uniformly ultimately bounded. □

Remark 6. Note that the design of $\tilde{F}$ and $\tilde{L}$ of (23) in Theorem 6 is dual to the design of L and F of (2) in Theorem 3. Thus, as pointed out in Remark 1, a sufficient condition for the existence of a protocol (23) satisfying Theorem 6 is that (A, B, C) is stabilizable and detectable. The adaptive protocol (23) can be implemented by each agent in a fully distributed fashion requiring neither global information of the communication topology nor the upper bound for the leader's control input.

It is worth noting that the adaptive protocol (23) is actually a continuous approximation of the following distributed discontinuous adaptive protocol:

$$
\begin{aligned}
\dot{\hat{v}}_i &= A\hat{v}_i + Bu_i + \hat{L}(C\hat{v}_i - y_i),\\
u_i &= d_i \sum_{j=1}^{N} a_{ij}\hat{F}(\hat{v}_i - \hat{v}_j) + d_i \hat{g}\left(\sum_{j=1}^{N} a_{ij}\hat{F}(\hat{v}_i - \hat{v}_j)\right),\\
\dot{d}_i &= \tau_i \left[\sum_{j=1}^{N} a_{ij}(\hat{v}_i - \hat{v}_j)^T\right] \hat{\Gamma} \left[\sum_{j=1}^{N} a_{ij}(\hat{v}_i - \hat{v}_j)\right]\\
&\quad + \tau_i \left\| \sum_{j=1}^{N} a_{ij}\hat{F}(\hat{v}_i - \hat{v}_j)\right\|, \quad i = 2, \ldots, N,
\end{aligned}
\tag{38}
$$

where the nonlinear function $\hat{g}(\cdot)$ is defined such that for $w \in \mathbf{R}^n$,

$$
\hat{g}(w) = \begin{cases} \dfrac{w}{\|w\|} & \text{if } \|w\| \neq 0 \\ 0 & \text{if } \|w\| = 0, \end{cases} \tag{39}
$$

and the rest of the variables are defined as in (23).

The following result can be proved by following similar steps as in the proof of Theorem 6.

Corollary 7. *Suppose that Assumptions 1 and 2 hold. Then, the consensus error ξ converges to zero under the distributed adaptive protocol (38) with $\hat{L}$, $\hat{F}$, and $\hat{\Gamma}$ as given in Theorem 6. Moreover, each coupling weight d_i converges to some finite steady-state value.*

Note that the function $\hat{g}(\cdot)$ in (39) is discontinuous, implying that the distributed adaptive protocol (38) is discontinuous. The term $d_i\hat{g}(\sum_{j=1}^{N} a_{ij}F(\hat{v}_i - \hat{v}_j))$ in (38) is used to tackle the effect of the leader's nonzero control input u_1 on consensus. For the special case where $u_1 = 0$, we can accordingly remove from (38) the terms $d_i\hat{g}(\sum_{j=1}^{N} a_{ij}F(\hat{v}_i - \hat{v}_j))$ and $\tau_i\|\sum_{j=1}^{N} a_{ij}\hat{F}(\hat{v}_i - \hat{v}_j)\|$. For the case of $u_1 = 0$, the distributed adaptive protocols (2) and (3) in the previous section can also be extended to solve the leader–follower consensus problem. The details are omitted here for brevity.

Remark 7. One inherent drawback of the discontinuous protocol (38) is that in a real implementation, it may result in chattering due to imperfections in switching devices (Edwards & Spurgeon, 1998; Young, Utkin, & Ozguner, 1999). The function $g(\cdot)$ in (24) uses a state-dependent boundary layer to give a continuous approximation of $\hat{g}(\cdot)$ in (39), which is partly inspired by Wheeler, Su, and Stepanenko (1998). The value κ in (24) denotes the width of the boundary layer. It is worth mentioning that adding $-\varphi_i d_i$ into (23) is essentially motivated by the so-called σ-modification technique in Ioannou and Kokotovic (1984) and Wheeler et al. (1998), which plays a vital role to guarantee the ultimate boundedness of the consensus error ξ and the adaptive weights d_i. Contrary to the discontinuous protocol (38), the chattering effect can be avoided by using the continuous protocol (23). The tradeoff is that the continuous protocol (23) does not guarantee asymptotic convergence. As $\kappa \to 0$ and $\varphi_i \to 0$, the continuous protocol (23) approaches the discontinuous protocol (38). Generally speaking, φ_i and κ should be chosen to be relatively small in order to guarantee a small consensus error ξ.

Remark 8. It is worth mentioning that leader–follower consensus problems for the case with a leader of nonzero control input have been studied in previous works (Cao & Ren, 2012; Li et al., 2013; Mei et al., 2011). Compared to Cao and Ren (2012), Li et al. (2013) and Mei et al. (2011) where the controllers depend on the relative states of neighboring agents and/or the agent dynamics are restricted to integrators, Theorem 6 and Corollary 7 in this paper can solve the leader–follower consensus problem for general linear multi-agent systems without requiring the relative state information. Besides, the adaptive protocol (23) is continuous, which can avoid the undesirable chattering phenomenon caused by the discontinuous controllers in Cao and Ren (2012), Li et al. (2013) and Mei et al. (2011). To the best of knowledge of the authors, it is the first time to present a distributed continuous consensus protocol and show the ultimate boundedness of the consensus error and the adaptive weights.

Similarly as in (38) and (23), based on the estimates of the states of neighboring agents, we can propose another type of adaptive protocol to solve the leaderless consensus problem in the previous section, which can be regarded as alternatives to the protocols (2) and (3). For instance, the node-based adaptive consensus protocol based on the relative estimate information is proposed as follows:

$$
\begin{aligned}
\dot{\hat{v}}_i &= A\hat{v}_i + Bu_i + \hat{L}(C\hat{v}_i - y_i),\\
\dot{d}_i &= \tau_i \left[\sum_{j=1}^{N} a_{ij}(\hat{v}_i - \hat{v}_j)^T\right] \hat{\Gamma} \left[\sum_{j=1}^{N} a_{ij}(\hat{v}_i - \hat{v}_j)\right]\\
u_i &= d_i \sum_{j=1}^{N} a_{ij}\hat{F}(\hat{v}_i - \hat{v}_j), \quad i = 1, \ldots, N.
\end{aligned}
\tag{40}
$$

Comparing the protocols (40) and (3), to solve the leaderless consensus problem in the last section, the latter one is more preferable due to two reasons. First, the protocol (3) generally requires a lighter communication load (i.e., a lower dimension of information to be transmitted) than (40). Note that the protocol (3) exchanges y_i and $C\tilde{v}_i$ between neighboring agents while (40) exchanges $\tilde{v}_i$. The sum of the dimensions of y_i and $C\tilde{v}_i$ is generally lower than the dimension of $\tilde{v}_i$, e.g., for the single output case with $n > 2$. Second, the protocol (40) requires the absolute measures of the agents' outputs, which might not be available in some circumstances, e.g., the case where the agents are equipped with only ultrasonic range sensors. On the other hand, an advantage of (40) is that it can be modified (specifically, protocols (38) and (23)) to solve the leader–follower consensus problem for the case where the leader's control input is nonzero and time varying. In contrast, it is not an easy job to extend (3) to achieve consensus for the case of a leader with nonzero input.

5. Simulation examples

In this section, simulation examples are provided to validate the effectiveness of the theoretical results.

Example 1 (*Leaderless Consensus*). Consider a network of third-order integrators, described by (1), with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. Choose $F = -\begin{bmatrix} 3 & 6.5 & 4.5 \end{bmatrix}$ such that $A + BF$ is Hurwitz. Solving the LMI (5) by using the Sedumi toolbox (Sturm, 1999) gives the feedback gain matrix L in (20) and (3) as $L = -\begin{bmatrix} 2.1115 & 1.3528 & 0.6286 \end{bmatrix}^T$.

To illustrate Theorem 4, let the communication graph be $\mathcal{G}_1$ in Fig. 1(a) and $\tau_i = 1$, $i = 1, \ldots, 6$, in (3). The consensus errors

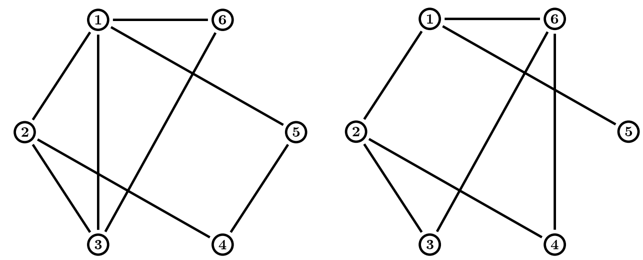
(a) $\mathcal{G}_1$. (b) $\mathcal{G}_2$.

Fig. 1. Undirected communication graphs $\mathcal{G}_1$ and $\mathcal{G}_2$.

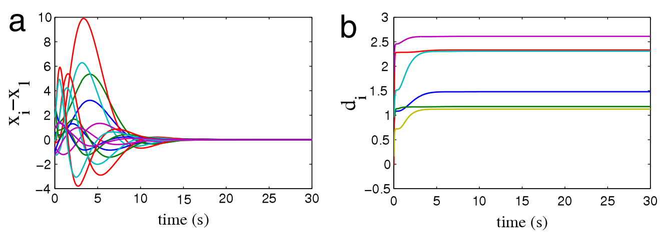

Fig. 2. (a) The consensus errors $x_i - x_1$ of third-order integrators under (3); (b) the coupling weights d_i in (3).

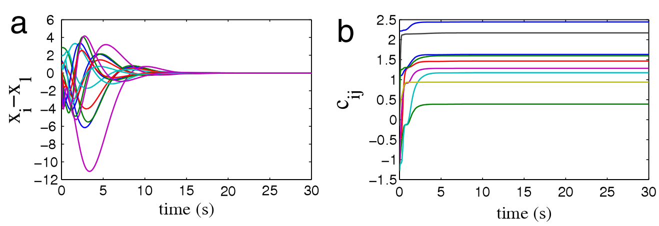

Fig. 3. (a) The consensus errors $x_i - x_1$ of third-order integrators under (20); (b) the coupling weights c_{ij} in (20).

$x_i - x_1,\ i = 2, \ldots, 6$, of the third-order integrators under the protocol (3) with F and L as above and $\Gamma = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ are depicted in Fig. 2(a). The coupling weights d_i associated with the nodes are drawn in Fig. 2(b). To illustrate Theorem 5, let $\mathcal{G}_{\sigma(t)}$ switch randomly every 0.1 s between $\mathcal{G}_1$ and $\mathcal{G}_2$ as shown in Fig. 1. Note that both $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected. Let $\varepsilon_{ij} = 1,\ i, j = 1, \ldots, 6$, in (20), and $c_{ij}(0) = c_{ji}(0)$ be randomly chosen. The consensus errors $x_i - x_1,\ i = 2, \ldots, 6$, under the protocol (20) designed as above are depicted in Fig. 3(a). The coupling weights c_{ij} associated with the edges in this case are shown in Fig. 3(b). Figs. 2(a) and 3(a) state that consensus is indeed achieved in both cases. From Figs. 2(b) and 3(b), it can be observed that the coupling weights converge to finite steady-state values.

Example 2 (*Consensus with a Leader of Nonzero Input*). The dynamics of the agents are given by (1), with $x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. The communication graph is given as in Fig. 4, where the node indexed by 1 is the leader which is only accessible to the second node. Clearly all the agents are unstable without control. For the leader, design a first-order controller in the form of $\dot{\upsilon} = -3\upsilon - 2.5x_{11}$, $u_1 = -8\upsilon - 6x_{11}$. Then, the closed-loop dynamics of the leader have eigenvalues as -1 and $\pm i$. In this case, u_1 is clearly bounded. However, the bound γ, for which $\|u_1\| \le \gamma$, depends on the initial state, which thereby might not be known to the followers. Here we use the adaptive protocol (23) to solve the leader–follower consensus problem.

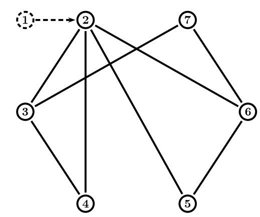

Fig. 4. The leader–follower communication graph.

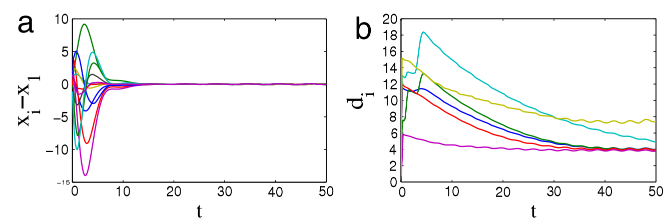

Fig. 5. (a) The consensus errors $x_i - x_1,\ i = 2, \ldots, 7$; (b) the coupling weights d_i in (23).

In (23), choose $\hat{L} = -\begin{bmatrix} 3 & 6 \end{bmatrix}^T$ such that $A + \hat{L}C$ is Hurwitz. Solving the LMI (26) gives the gain matrices $\hat{F}$ and $\hat{\Gamma}$ in (23) as $\hat{F} = -\begin{bmatrix} 1.2983 & 3.3878 \end{bmatrix}$ and $\hat{\Gamma} = \begin{bmatrix} 1.6857 & 4.3984 \\ 4.3984 & 11.4769 \end{bmatrix}$. To illustrate Theorem 6, select $\kappa = 0.1$, $\varphi_i = 0.005$, and $\tau_i = 1,\ i = 2, \ldots, 7$, in (23). The consensus errors $x_i - x_1,\ i = 2, \ldots, 7$, of the agents under (23) designed as above are shown in Fig. 5(a), implying that leader–follower consensus is indeed achieved. The coupling weights d_i associated with the followers are drawn in Fig. 5(b), which are clearly bounded.

6. Conclusion

In this paper, we have addressed the consensus problem of multi-agent systems with general linear dynamics for the cases without and with a leader whose control input is possibly nonzero, without knowing the relative states of neighboring agents. Based on the relative outputs and relative state estimates of neighboring agents, several distributed adaptive protocols have been designed to achieve leaderless and leader–follower consensus in a fully distributed fashion requiring neither global information about the communication graph nor the upper bound of the leader's control input. A sufficient condition for the existence of the proposed adaptive protocols is that each agent is stabilizable and detectable. An interesting future topic is to extend the results to the case with general directed communication graphs.

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References

Agaev, R., & Chebotarev, P. (2005). On the spectra of nonsymmetric Laplacian matrices. *Linear Algebra and its Applications*, *399*(1), 157–178.

Boyd, S., El Ghaoui, L., Feron, E., & Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory*. Philadelphia, PA: SIAM.

Cao, Y. C., & Ren, W. (2012). Distributed coordinated tracking with reduced interaction via a variable structure approach. *IEEE Transactions on Automatic Control*, *57*(1), 33–48.

Carli, R., Bullo, F., & Zampieri, S. (2009). Quantized average consensus via dynamic coding/decoding schemes. *International Journal of Robust and Nonlinear Control*, *20*(2), 156–175.

Corless, M., & Leitmann, G. (1981). Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems. *IEEE Transactions on Automatic Control*, *26*(5), 1139–1144.

Edwards, C., & Spurgeon, S. K. (1998). *Sliding mode control: theory and applications*. London: Taylor & Francis.

Hong, Y. G., Chen, G. R., & Bushnell, L. (2008). Distributed observers design for leader–following control of multi-agent networks. *Automatica*, *44*(3), 846–850.

Hu, J. P., & Feng, G. (2010). Distributed tracking control of leader–follower multi-agent systems under noisy measurement. *Automatica*, *46*(8), 1382–1387.

Ioannou, P. A., & Kokotovic, P. V. (1984). Instability analysis and improvement of robustness of adaptive control. *Automatica*, *20*(5), 583–594.

Jadbabaie, A., Lin, J., & Morse, A. S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, *48*(6), 988–1001.

Jiang, F. C., & Wang, L. (2010). Consensus seeking of high-order dynamic multi-agent systems with fixed and switching topologies. *International Journal of Control*, *85*(2), 404–420.

Krstić, M., Kanellakopoulos, I., & Kokotovic, P. V. (1995). *Nonlinear and adaptive control design*. New York: John Wiley & Sons.

Li, Z. K., Duan, Z. S., Chen, G. R., & Huang, L. (2010). Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint. *IEEE Transactions on Circuits and Systems. I. Regular Papers*, *57*(1), 213–224.

Li, T., Fu, M. Y., Xie, L. H., & Zhang, J. F. (2011). Distributed consensus with limited communication data rate. *IEEE Transactions on Automatic Control*, *56*(2), 279–292.

Li, Z. K., Liu, X. D., Lin, P., & Ren, W. (2011). Consensus of linear multi-agent systems with reduced-order observer-based protocols. *Systems and Control Letters*, *60*(7), 510–516.

Li, Z. K., Liu, X. D., Ren, W., & Xie, L. H. (2013). Distributed tracking control for linear multiagent systems with a leader of bounded unknown input. *IEEE Transactions on Automatic Control*, *58*(2), 518–523.

Li, Z. K., Ren, W., Liu, X. D., & Fu, M. Y. (2013). Consensus of multiagent systems with general linear and Lipschitz nonlinear dynamics using distributed adaptive protocols. *IEEE Transactions on Automatic Control*, in press (http://dx.doi.org/10.1109/TAC.2012.2235715).

Ma, C. Q., & Zhang, J. F. (2010). Necessary and sufficient conditions for consensusability of linear multi-agent systems. *IEEE Transactions on Automatic Control*, *55*(5), 1263–1268.

Mei, J., Ren, W., & Ma, G. F. (2011). Distributed coordinated tracking with a dynamic leader for multiple Euler–Lagrange systems. *IEEE Transactions on Automatic Control*, *56*(6), 1415–1421.

Olfati-Saber, R., Fax, J. A., & Murray, R. M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, *95*(1), 215–233.

Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*(9), 1520–1533.

Rahmani, A., Ji, M., Mesbahi, M., & Egerstedt, M. (2009). Controllability of multi-agent systems from a graph–theoretic perspective. *SIAM Journal on Control and Optimization*, *48*(1), 162–186.

Ren, W., & Atkins, E. M. (2007). Distributed multi-vehicle coordinated control via local information exchange. *International Journal of Robust and Nonlinear Control*, *17*(10–11), 1002–1033.

Ren, W., & Beard, R. W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, *50*(5), 655–661.

Ren, W., Beard, R. W., & Atkins, E. M. (2007). Information consensus in multivehicle cooperative control. *IEEE Control Systems Magazine*, *27*(2), 71–82.

Scardovi, L., & Sepulchre, R. (2009). Synchronization in networks of identical linear systems. *Automatica*, *45*(11), 2557–2562.

Seo, J. H., Shim, H., & Back, J. (2009). Consensus of high-order linear systems using dynamic output feedback compensator: low gain approach. *Automatica*, *45*(11), 2659–2664.

Sturm, J. F. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, *11*(1), 625–653.

Su, H. S., Chen, G. R., Wang, X. F., & Lin, Z. L. (2011). Adaptive second-order consensus of networked mobile agents with nonlinear dynamics. *Automatica*, *47*(2), 368–375.

Tuna, S. E. (2009). Conditions for synchronizability in arrays of coupled linear systems. *IEEE Transactions on Automatic Control*, *54*(10), 2416–2420.

Vicsek, T., Czirók, A., Ben-Jacob, E., Cohen, I., & Shochet, O. (1995). Novel type of phase transition in a system of self-driven particles. *Physical Review Letters*, *75*(6), 1226–1229.

Wheeler, G., Su, C. Y., & Stepanenko, Y. (1998). A sliding mode controller with improved adaptation laws for the upper bounds on the norm of uncertainties. *Automatica*, *34*(12), 1657–1661.

Young, K. D., Utkin, V. I., & Ozguner, U. (1999). A control engineer's guide to sliding mode control. *IEEE Transactions on Control Systems Technology*, *7*(3), 328–342.

Yu, W. W., Zheng, W. X., Lü, J. H., & Chen, G. R. (2011). Designing distributed control gains for consensus in multi-agent systems with second-order nonlinear dynamics. In *Proceedings of the 18th IFAC world congress* (pp. 1231–1236).

Zhang, H. W., Lewis, F., & Das, A. (2011). Optimal design for synchronization of cooperative systems: state feedback, observer, and output feedback. *IEEE Transactions on Automatic Control*, *56*(8), 1948–1952.

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