Consensus of Linear Multi-Agent Systems by Distributed Event-Triggered Strategy

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***Abstract*—This paper studies the consensus problem of multi-agent systems with general linear dynamics. We propose a novel event-triggered control scheme with some desirable features, namely, distributed, asynchronous, and independent. It is shown that consensus of the controlled multi-agent system can be reached asymptotically. The feasibility of the event-triggered strategy is further verified by the exclusion of both singular triggering and Zeno behavior. Moreover, a self-triggered algorithm is developed, where the next triggering time instant for each agent is determined based on its local information at the previous triggering time instant. Continuous monitoring of measurement errors is thus avoided. The effectiveness of the proposed control schemes is demonstrated by two examples.**

***Index Terms*—Consensus, event-triggered control, general linear dynamics, multi-agent systems.**

I. Introduction

OVER the past decade, various control problems of networked multi-agent systems have been extensively studied. These problems include formation, synchronization, flocking, swarming, and rendezvous, to name just a few. One reason is the multi-agent systems can be potentially applied in such broad areas as spacecraft formation flying [1], distributed sensor networks [2], and cooperative surveillance [3]. In many circumstances, the states of all agents need to reach a common quantity of interest while each agent only has access to information of its neighboring agents. This is the so-called consensus problem of multi-agent systems under distributed framework. Typical results on this topic can be found in [4]–[15] and references therein. Many of them studied multi-agent systems with single- or double-integrator dynamics [5]–[7], [9]–[13].

It should be noted that each individual agent is usually equipped with simple embedded microprocessors, onboard communication modules, and actuation modules, which have limited energy resources to perform such functions as gathering information, communicating with neighboring agents, and driving the agent. For most existing control schemes in consensus problems, an agent needs to measure its state, send the state to its neighbors, and update its control signal continuously or in a prespecified sampling rate. Such control laws might become infeasible or impractical in many applications due to their excessive consumption of on-board energy resources. It is thus desirable to design novel control schemes, such that the load of communication and controller update for each agent can be reduced significantly. In this way, limited energy resources of agents can be greatly saved and operational lifespan of multi-agent systems can be thus prolonged. To address this issue, Tabuada [16] introduced an event-triggered strategy for a stabilization problem, where the control actuation is triggered whenever a defined error exceeds a threshold with respect to the norm of the state. Dimarogonas *et al.* [17] proposed a distributed event-driven scheme to solve a first order agreement problem in multi-agent systems and extended the results to a self-triggered setup, where continuous tracking of the state error can be avoided. Most recently, a new combined measurement approach to event-based design was developed in [18]. As a result, control of agents is only triggered at their own event times, which is a significant improvement. Seyboth *et al.* [19] presented an event-based broadcasting strategy for multi-agent systems with double-integrator dynamics, where a time-dependent threshold is used to bound each agent's measurement error. Comparison between event-triggered sampling and traditional periodical sampling for certain isolated systems was made in [20]–[22]. It follows from [20] and [22] that event-triggered sampling has some advantages over periodic sampling in a stochastic setting. Some other relevant studies on the topic can be found in [23]–[27].

From the review of existing works on event-triggered consensus of multi-agent systems, it can be observed that there are still some challenging issues to be addressed. One of the issues is to consider more general agent dynamics as most existing works focus on single- or double-integrator dynamics [17]–[19]. To our best knowledge, there are only a few results on the consensus of linear multi-agent systems without considering the event-triggered strategy, such as [28]–[30]. Another issue is to develop more efficient triggering mechanisms that have the following features. First, each agent only needs the information of its neighbors and itself, termed as distributed. Second, all the agents are not required to be triggered in a synchronous way, termed as asynchronous. Third, triggering of an agent should not affect or be affected by triggering of other agents, termed as independent. However, the triggering mechanisms in most existing works only have

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some but not all of these features. For example, all the agents are triggered at the same time, namely, in a synchronous fashion in [23], [24], and [26]. The controllers in [17] are required to be updated at the neighbors' event times. Besides, when the event-triggered mechanism is considered, it is desirable that continuous monitoring of measurement errors can be avoided. Unfortunately, this is usually neglected in most existing works.

Motivated by the above-mentioned considerations, in this paper, we propose a novel distributed event-triggered scheme with those desirable features for the consensus problem of multi-agent systems with general linear dynamics. Moreover, a self-triggered algorithm is also proposed such that continuous monitoring of measurement errors can be avoided. The main advantages of the proposed approaches in comparison to existing ones can be summarized as follows. In the proposed event-triggered scheme and self-triggered control scheme, the update of controller and the triggering times of any agent are independent of triggering time instants of all other agents. In addition, the proposed schemes can be applied to the multi-agent systems with general linear dynamics, which cover the cases in [17]–[19].

The rest of this paper is organized as follows. Section II introduces some preliminaries and problem formulation. In Section III, we propose novel distributed event-triggered control schemes for multi-agent systems with general linear dynamics and systems with a special form, respectively, with their feasibility analyzed. The self-triggered algorithm is developed in Section IV. We provide two examples to demonstrate the effectiveness of the proposed control schemes in Section V and the conclusion is drawn in Section VI.

II. Preliminaries and Problem Formulation

A. Notations

We will use $\|\cdot\|$ to denote the Euclidean norm for vectors or the induced two-norm for matrices. Given a matrix M, M^T denotes the transpose of M, and $M > 0$ (or $M \geq 0$) means that M is a positive definite (or semi positive definite) matrix. We use $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ to represent the minimum eigenvalue and the maximum eigenvalue, respectively. The notation $A \otimes B$ represents the Kronecker product of matrices A and B. For a series of column vectors $x_1, \ldots, x_n$, $\mathrm{col}\,(x_1, \ldots, x_n)$ stands for a column vector by stacking them together. The notation "$\Leftrightarrow$" means if and only if.

For easy reference of readers, we present a nomenclature including some commonly-used algebraic terms as follows.

L	Laplacian matrix
$\mathcal{A}$	adjacency matrix
N	number of agents
$\mathcal{N}$	$\{1, 2, \cdots, N\}$
$\mathcal{N}_i$	neighbor set of agent i
$\lvert\mathcal{N}_i\rvert$	cardinality of the set $\mathcal{N}_i$
I_n	identity matrix
x_i	state of agent i
q_i	combined measurement of agent i
e_i	measurement error of agent i

B. Algebraic Graph Theory

We will state some useful facts from algebraic graph theory. One can refer to [31] for more details.

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of a set of vertices or nodes $\mathcal{V} = \{1, \ldots, N\}$ and edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, if there is an edge $(i, j) \in \mathcal{E}$ between nodes i and j, then nodes i and j are called adjacent. Graph $\mathcal{G}$ is called undirected if $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. The adjacency matrix $\mathcal{A} = \mathcal{A}(\mathcal{G}) = (a_{ij})_{N\times N}$ is an $N \times N$ matrix defined by $a_{ij} = 1$ if and only if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. A path from i to j is a sequence of distinct nodes, which starts with i and ends with j while each pair of consecutive nodes is adjacent. If there is a path between any two nodes of the graph $\mathcal{G}$, then $\mathcal{G}$ is called connected. The degree matrix D of $\mathcal{G}$ is a diagonal matrix with element d_i equaling the cardinality of node i's neighbor set $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. The Laplacian matrix L of $\mathcal{G}$ is defined as $L = D - \mathcal{A}$. For undirected graphs, L satisfies $L = L^T \geq 0$ with the vector of ones $\mathbf{1}$ as an eigenvector corresponding to the eigenvalue zero. If an undirected graph is connected, the Laplacian has a single zero eigenvalue, and the other eigenvalues can be listed in an increasing order, $0 = \lambda_1(\mathcal{G}) < \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_N(\mathcal{G})$. The second smallest eigenvalue $\lambda_2(\mathcal{G})$ is called the algebraic connectivity or Fiedler eigenvalue [32].

C. Problem Formulation

Consider a multi-agent system with N agents. The dynamics of the ith agent are described by

$$\dot{x}_i = Ax_i + Bu_i, \quad i \in \mathcal{N} \tag{1}$$

where $x_i \in R^n$ is the state, $u_i \in R^m$ is the control input, and A and B are constant matrices with compatible dimensions. If agents i and j are adjacent, then the pair of agents can communicate with each other, and the communication topology among all agents is represented by an undirected graph $\mathcal{G}$.

In order to develop our event-triggered strategy, we consider the combined measurement $q_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t))$ as in [18]. In this case, the measurement error can be defined as

$$e_i(t) = q_i\left(t_k^i\right) - q_i(t). \tag{2}$$

We propose the following control law for agent i:

$$u_i(t) = Kq_i(t_k^i), \; t \in \left[t_k^i, t_{k+1}^i\right) \tag{3}$$

where K is the feedback gain matrix to be designed. The event triggering time sequence $\{t_0^i, t_1^i, \ldots\}$ for agent i will be determined by the following triggering condition which is also to be developed:

$$h\left(e_i(t), q_i(t)\right) = 0. \tag{4}$$

It is noted that (4) is also distributed, which only needs the information from neighboring agents. In the control scheme, $h(e_i(t), q_i(t)) \leq 0$ will be always enforced by (4) and at that time instant the event is triggered.

Remark 1: For agent i, $t_0^i, t_1^i, \ldots,$ are the times at which agent i will sample the information of its neighbors and update its controller accordingly. In this paper, we choose a novel control law in the form of $u_i(t) = K\sum_{j\in\mathcal{N}_i}(x_j(t_k^i) - x_i(t_k^i))$

rather than $u_i(t) = K\sum_{j\in\mathcal{N}_i}(x_j(t^j_{k'(t)}) - x_i(t^i_k))$, where $k'(t) = \mathrm{argmin}_{l\in Z^+:t\geq t^j_l}\{t - t^j_l\}$, as in [17] and [33]. This is because in these existing works, $u_i(t)$ is updated both at its own event times $t^i_0, t^i_1, \ldots$, as well as at the event times of its neighbors $t^j_0, t^j_1, \ldots, j \in \mathcal{N}_i$. In our scheme, the triggering time instants of agent i are independent of triggering time instants of all other agents, and it is expected that the number of triggering events of the whole system would be reduced.

Definition 1: The consensus problem for multi-agent systems described by (1) is said to be solved if and only if for any finite $x_i(0)$, $\forall i \in \mathcal{N}$, the states of agents satisfy

$$\lim_{t\to\infty}\left\|x_i(t) - x_j(t)\right\| = 0, \quad \forall i, j \in \mathcal{N}. \tag{5}$$

In this paper, the objective is to develop, for each agent, a control law and an event-triggered mechanism of form (3) and (4), respectively, such that the consensus problem is solved.

To achieve this, we introduce the following assumptions and lemma.

Assumption 1: (A, B) is stabilizable.

Assumption 2: The undirected communication graph $\mathcal{G}$ is connected.

Lemma 1 [34]: Consider a linear system (A, B, C), if (A, B) is stabilizable and (C, A) is observable, then there is a unique solution $P > 0$ to the following algebraic Riccati equation:

$$PA + A^T P - PBB^T P + C^T C = 0. \tag{6}$$

III. Distributed Event-Triggered Control Design

A. Distributed Event-Triggered Strategy

In this subsection, a distributed event-triggered control scheme for multi-agent systems with linear dynamics (1) will be developed.

Theorem 1: Under Assumptions 1 and 2, there always exists at least one solution $P > 0$ for the following inequality:

$$PA + A^T P - \alpha\mu PBB^T P + \beta\mu I_n \leq 0 \tag{7}$$

where $0 < \alpha \leq 2\lambda_2$, $\beta \geq 2\lambda_N$, and $\mu > 0$, with λ_2 and λ_N the Fiedler eigenvalue and the largest eigenvalue of the Laplacian matrix associated with graph $\mathcal{G}$, respectively. Then, letting $K = \mu B^T P$, the consensus problem of the multi-agent system (1) can be solved by control law (3) with the following triggering condition:

$$h\left(e_i(t), q_i(t)\right) = \|e_i(t)\| - \eta_i\|q_i(t)\| = 0 \tag{8}$$

where $\eta_i = \sqrt{(\sigma_i \cdot \kappa(2 - \kappa\rho)/\rho)} < 1$ with $\rho = \|PBB^T P\|$, $\sigma_i \in (0, \min(1, \rho^2))$, and κ being a positive number satisfying $\kappa < 2/\rho$.

Proof: With $K = \mu B^T P$ and (2), the closed-loop system consisting of (1) and (3) can be expressed as

$$\dot{x}_i = Ax_i + \mu BB^T P\left(e_i(t) + q_i(t)\right), \ \forall i \in \mathcal{N}. \tag{9}$$

Let $x(t) = \mathrm{col}(x_1(t), \ldots, x_N(t))$, $e(t) = \mathrm{col}(e_1(t), \ldots, e_N(t))$, and $q(t) = \mathrm{col}(q_1(t), \ldots, q_N(t))$. Then $q(t) = -(L \otimes I_n)x(t)$, and (9) can be rewritten in a compact form as follows:

$$\begin{aligned}\dot{x}(t) &= \left(I_N \otimes A - L \otimes \mu BB^T P\right)x(t)\\ &\quad + \left(I_N \otimes \mu BB^T P\right)e(t).\end{aligned} \tag{10}$$

Consider the following Lyapunov function candidate:

$$V(t) = \frac{1}{2}x^T(t)(L \otimes P)x(t). \tag{11}$$

Letting $e_{ij}(t) = x_i(t) - x_j(t)$, one has

$$V(x) = \frac{1}{2}\left[\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}e_{ij}^T Pe_{ij}\right] \geq 0. \tag{12}$$

It follows from (12) that (11) is a valid Lyapunov function candidate for the variables $e_{ij}(t)$. Denote $\hat{A} = (PA + A^T P)/2$, $\hat{B} = PBB^T P$, and the time derivative of $V(t)$ along the trajectory of (10) is

$$\begin{aligned}\dot{V}(t) &= x^T(t)\left(L \otimes \hat{A} - L^2 \otimes \mu\hat{B}\right)x(t)\\ &\quad + x^T(t)\left(L \otimes \mu\hat{B}\right)e(t).\end{aligned} \tag{13}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the eigenvalues of matrix L, satisfying $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$. Since the graph is undirected, the corresponding Laplacian matrix L is symmetric. Then, there exists an orthogonal matrix U such that

$$U^{-1}LU = U^T LU = J = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N). \tag{14}$$

One can observe that $U^T U = I_N$ and $L = UJU^T$. Define $y(t) = (U^T \otimes I_n)x(t) = \mathrm{col}(y_1(t), \ldots, y_N(t))$ and $\hat{e}(t) = (U^T \otimes I_n)e(t) = \mathrm{col}(\hat{e}_1(t), \ldots, \hat{e}_N(t))$. Equation (13) can be rewritten as

$$\begin{aligned}\dot{V}(t) &= y^T(t)\left(J \otimes \hat{A} - J^2 \otimes \mu\hat{B}\right)y(t)\\ &\quad + y^T(t)\left(J \otimes \mu\hat{B}\right)\hat{e}(t)\\ &= \sum_{i=2}^{N}y_i^T(t)\left(\lambda_i\hat{A} - \lambda_i^2\mu\hat{B}\right)y_i(t)\\ &\quad + \sum_{i=2}^{N}y_i^T(t)\left(\lambda_i\mu\hat{B}\right)\hat{e}_i(t).\end{aligned} \tag{15}$$

Since $\hat{B} \geq 0$, it follows from (7) that for any $i \in \{2, \ldots, N\}$,

$$\hat{A} - \lambda_i\mu\hat{B} \leq \hat{A} - \frac{\alpha}{2}\mu\hat{B} \leq -\frac{\beta}{2}\mu I_n \leq -\lambda_i\mu I_n. \tag{16}$$

Then, noting inequality $\|\xi\| \cdot \|\zeta\| \leq \kappa/2\,\|\xi\|^2 + 1/(2\kappa)\,\|\zeta\|^2$ for any $\kappa > 0$ and any $\xi, \zeta \in R^n$, one has

$$\begin{aligned}\dot{V}(t) &\leq \sum_{i=2}^{N}-\mu\lambda_i^2 y_i^T(t)y_i(t) + \sum_{i=2}^{N}\mu\lambda_i y_i^T(t)\hat{B}\hat{e}_i(t)\\ &\leq -\mu\left(1 - \frac{\kappa\rho}{2}\right)\sum_{i=2}^{N}\lambda_i^2 \cdot \|y_i(t)\|^2\\ &\quad + \frac{\mu\rho}{2\kappa}\sum_{i=2}^{N}\left\|\hat{e}_i(t)\right\|^2\end{aligned} \tag{17}$$

where $\rho = \|\hat{B}\|$ and κ is a positive number.

Furthermore, noting $q(t) = -(L \otimes I_n)x(t)$

$$\sum_{i=1}^{N} \|q_i(t)\|^2 = x^T(t)\left(L^2 \otimes I_n\right)x(t) = \sum_{i=2}^{N} \lambda_i^2 \cdot \|y_i(t)\|^2 \tag{18}$$

$$\sum_{i=1}^{N} \left\|\hat{e}_i(t)\right\|^2 = e^T(t)\,e(t) = \sum_{i=1}^{N} \|e_i(t)\|^2 \tag{19}$$

inequality (17) can be written as follows:

$$\dot{V}(t) \leq -\mu\left(1 - \frac{\kappa\rho}{2}\right)\sum_{i=1}^{N} \|q_i(t)\|^2 + \frac{\mu\rho}{2\kappa}\sum_{i=1}^{N} \|e_i(t)\|^2. \tag{20}$$

Then, by choosing $\kappa < 2/\rho$ and enforcing the condition

$$\|e_i(t)\| \leq \sqrt{\frac{\sigma_i \cdot \kappa(2 - \kappa\rho)}{\rho}}\,\|q_i(t)\| = \eta_i\,\|q_i(t)\| \tag{21}$$

where $\sigma_i \in (0, \min(1, \rho^2))$ for all $i \in \mathcal{N}$, (20) further becomes

$$\dot{V}(t) \leq -\mu\left(1 - \frac{\kappa\rho}{2}\right)\sum_{i=1}^{N} (1 - \sigma_i)\,\|q_i(t)\|^2 \leq 0. \tag{22}$$

It is noted that $\sup \eta_i = \sqrt{\sigma_i}/\rho$ (when $\kappa = 1/\rho$), so $\eta_i < 1$ can be guaranteed. It can be further verified that

$$\|q_i(t)\| = 0,\ i = 1, \ldots, N \Leftrightarrow e_{ij} = 0,\ i, j = 1, \ldots, N. \tag{23}$$

Then, it can be concluded from (22) that $\dot{V}(x) = 0$ only when $e_{ij} = 0$, $i, j = 1, \ldots, N$.

Hence, according to the corollary of LaSalle's theorem [35], it follows that $e_{ij}(t) \to 0$, $i, j = 1, \ldots, N$ as $t \to \infty$. Thus, the consensus problem is solved.

As for the existence of the solution P of (7), letting $B' = \sqrt{\alpha\mu}B$, $C' = \sqrt{\beta\mu}I_n$, inequality (7) can be rewritten as

$$PA + A^T P - PB'B'^T P + C'^T C' \leq 0 \tag{24}$$

which has the same form as the algebraic Riccati equation (6) if the equality sign is taken. It is noted that (C', A) is observable. Thus, by Lemma 1, one knows that if (A, B) is stabilizable, so is (A, B'), then at least one solution $P > 0$ is guaranteed for inequality (7). Thus, the proof is completed. ■

Remark 2: Our result contains several existing results as its special cases. On one hand, if we let $A = 0$ and $B = 1$, for any topology with N agents, inequality (7) always holds with $P \geq \sqrt{\beta/\alpha}I_n$, thus our result can be used to tackle the problem in [18] where multi-agent systems with single-integrator dynamics are considered. On the other hand, it can be seen that when $A = [0\ 1; 0\ 0]$, $B = [0\ 1]^T$, inequality (7) can be always solved, thus our result can also be used to handle the consensus problem for double-integrator agents in [19].

Remark 3: It is observed that the feedback gain K contains a design parameter μ, which has an impact on the convergence rate for consensus. Generally speaking, according to (22), a smaller μ will lead to a smaller convergence rate for consensus, and vice versa.

B. Special Case

In this subsection, we will consider a special case. In this case, the dynamics of the ith subsystem are described as follows:

$$\dot{x}_i = Sx_i + u_i, \quad i \in \mathcal{N}. \tag{25}$$

The communication topology is still represented by graph $\mathcal{G}$. The following assumption is made.

Assumption 3: S is neutrally stable, namely, there exists a positive definite matrix P satisfying the linear matrix inequality

$$PS + S^T P \leq 0. \tag{26}$$

Then, we derive the following result.

Corollary 1: Under Assumptions 2 and 3, letting $K = P$, the consensus problem of the multi-agent system (25) can be solved by control law (3) with the following triggering condition:

$$\bar{h}(e_i(t), q_i(t)) = \|e_i(t)\| - \bar{\eta}_i\,\|q_i(t)\| = 0 \tag{27}$$

where $\bar{\eta}_i = \sqrt{(\sigma_i \cdot \kappa(2\lambda_m - \kappa\lambda_M)/\lambda_M)} < 1$ with $\sigma_i \in (0, 1)$, $\lambda_m = \lambda_{\min}(P^2)$, $\lambda_M = \lambda_{\max}(P^2)$, and κ being a positive number satisfying $\kappa < (2\lambda_m/\lambda_M)$.

Proof: Similar to the proof of Theorem 1, the closed-loop system consisting of (25) and (3) can be expressed in a compact form as

$$\dot{x}(t) = (I_N \otimes S - L \otimes P)\,x(t) + (I_N \otimes P)\,e(t). \tag{28}$$

We still consider the same Lyapunov function candidate (11). Calculating the derivative along the trajectory of (28) yields

$$\begin{aligned}
\dot{V}(t) &= x^T(t)\Big(L \otimes \frac{S^T P + PS}{2} - L^2 \otimes P^2\Big)x(t) \\
&\quad + x^T(t)\left(L \otimes P^2\right)e(t) \\
&\leq y^T(t)\left(-J^2 \otimes P^2\right)y(t) + y^T(t)\left(J \otimes P^2\right)\hat{e}(t) \\
&= \sum_{i=1}^{N}\left[y_i^T(t)\left(-\lambda_i^2 P^2\right)y_i(t) + y_i^T(t)\left(\lambda_i P^2\right)\hat{e}_i(t)\right] \\
&\leq \sum_{i=1}^{N}\Bigg[-\lambda_m\lambda_i^2 \cdot \|y_i(t)\|^2 \\
&\qquad + \lambda_M\left(\frac{\kappa}{2} \cdot \lambda_i^2\,\|y_i(t)\|^2 + \frac{1}{2\kappa}\left\|\hat{e}_i(t)\right\|^2\right)\Bigg]
\end{aligned} \tag{29}$$

where $y_i(t)$ and $\hat{e}_i(t)$ are the same as defined in the proof of Theorem 1. It follows from (18) and (19) that:

$$\begin{aligned}
\dot{V}(t) &\leq -\sum_{i=1}^{N}\left[\lambda_m - \frac{\kappa}{2} \cdot \lambda_M\right]\|q_i(t)\|^2 \\
&\quad + \sum_{i=1}^{N}\frac{1}{2\kappa} \cdot \lambda_M \cdot \|e_i(t)\|^2.
\end{aligned} \tag{30}$$

Similarly, enforcing the condition

$$\|e_i(t)\| \leq \sqrt{\frac{\sigma_i \cdot \kappa(2\lambda_m - \kappa\lambda_M)}{\lambda_M}}\,\|q_i(t)\| = \bar{\eta}_i\,\|q_i(t)\| \tag{31}$$

leads to

$$\dot{V}(t) \le -\sum_{i=1}^{N} (1-\sigma_i)\left[\lambda_m - \frac{\kappa}{2}\cdot\lambda_M\right]\|q_i(t)\|^2 \le 0. \quad (32)$$

It is noted that $\sup \bar{\eta}_i = (\sqrt{\sigma_i}\lambda_m/\lambda_M)$ (when $\kappa = (\lambda_m/\lambda_M)$), so $\bar{\eta}_i < 1$ can be guaranteed. Then, it can be concluded from the proof of Theorem 1 that the consensus problem can be solved. Thus, the proof is completed. ■

Remark 4: It is noted that in this special case, the feedback gain matrix K is obtained from the solution to the linear matrix inequality (26), which does not depend on the information of the communication topology. When $S = 0$, the multi-agent system (25) reduces to a single-integrator system.

Remark 5: As for the first triggering time for all agents in (1) or (25), each agent can randomly choose a time as the first triggering time and initiate the control scheme. Define t_0^i to be the first triggering time for agent i, $i = 1, \ldots, N$. If there exists an agent i with $q_i(t_0^i) = 0$, then there always exists $t_0^{i'} > t_0^i$, such that $q_i(t_0^{i'}) \ne 0$, since the consensus is not reached. In this case, it can be verified that $\|e_i(t_0^{i'})\| = \|q_i(t_0^i) - q_i(t_0^{i'})\| = \|q_i(t_0^{i'})\| > \eta_i\|q_i(t_0^{i'})\|$ since $\eta_i < 1$, which means (21) or (31) has already been violated at $t = t_0^{i'}$ (also note that $\bar{\eta}_i < 1$). Thus, we can reset $t_0^{i'}$ as the next triggering time for agent i with $q_i(t_0^{i'}) \ne 0$, and $t_0^{i'}$ can be regarded as the new first triggering time for agent i.

C. Feasibility

Now we investigate the feasibility of this proposed event-triggered control scheme by excluding both scenarios of singular triggering and Zeno behavior.

Singular triggering means that there will be no more triggering after a single triggering. We will prove that such scenario will not happen for our proposed control scheme in the following theorem.

Theorem 2: Consider the multi-agent systems with linear dynamics (1), controller (3), and triggering condition (8). No agent will exhibit singular triggering behavior.

Proof: For any agent i, $i \in \mathcal{N}$, assume its current triggering time is t_k^i. We need to prove that the next triggering time after t_k^i, i.e., t_{k+1}^i exists.

Since $\|e_i(t)\| \le \eta_i\|q_i(t)\|$, by utilizing $\left|\left\|q_i\left(t_k^i\right)\right\| - \|q_i(t)\|\right| \le \|e_i(t)\|$, one has

$$\frac{\left\|q_i(t_k^i)\right\|}{1+\eta_i} \le \|q_i(t)\| \le \frac{\left\|q_i(t_k^i)\right\|}{1-\eta_i}. \quad (33)$$

We first assume $q_i(t_k^i) \ne 0$. By defining $\gamma_1 = 1/(1+\eta_i)\|q_i(t_k^i)\| > 0$, $\gamma_2 = 1/(1-\eta_i)\|q_i(t_k^i)\| > 0$, one can conclude that $\|q_i(t)\|$ will always stay between γ_1 and γ_2, and events occur once $\|q_i(t)\|$ reaches the boundary values. Next, we will prove the existence of t_{k+1}^i such that $\|q_i(t_{k+1}^i)\| = \gamma_1$ or $\|q_i(t_{k+1}^i)\| = \gamma_2$ with the condition $q_i(t_{k+1}^i) \ne 0$. It follows from (12) that $V(t) \ge 1/4\lambda_{\min}(P)\|x_i - x_j\|^2$. Then, one has

$$\|q_i(t)\| \le \sum_{j=1}^{N} a_{ij}\left\|x_j(t) - x_i(t)\right\| \le d_i\sqrt{\frac{4V(t)}{\lambda_{\min}(P)}} \quad (34)$$

where $d_i = |\mathcal{N}_i|$. It follows from (22) that $V(t)$ strictly decreases to zero. Thus, $\|q_i(t)\|$ will eventually decrease to γ_1 because of (34), then at least one event will be triggered at that instant, which can be assigned to be t_{k+1}^i.

If $q_i(t_k^i) = 0$, as $k \to \infty$, then $q_i(t) = 0$ by (33). According to the triggering condition (8), the next triggering time still exists.

The proof is thus completed. ■

Remark 6: The same conclusion can be drawn for multi-agent system (25) together with controller (3) and triggering condition (27).

As for Zeno behavior, which means that there is an infinite number of triggering instants in a finite time [16], we have the following result.

Theorem 3: Consider the multi-agent systems with linear dynamics (1), controller (3), and triggering condition (8). No agent will exhibit Zeno behavior.

Proof: For any agent i, $i \in \mathcal{N}$, assume its current triggering time instant is t_k^i. We need to prove that the length of its next inter-event interval is strictly positive. We first propose the following sufficient condition to guarantee that $h(e_i(t), q_i(t)) \le 0$:

$$\|e_i(t)\| \le \frac{\eta_i}{\sqrt{2+2\eta_i^2}}\left\|q_i(t_k^i)\right\| \quad (35)$$

which follows directly from:

$$\begin{aligned}\|e_i(t)\|^2 &\le \frac{\eta_i^2}{2+2\eta_i^2}\left\|e_i(t) + q_i(t)\right\|^2\\ &\le \frac{\eta_i^2}{1+\eta_i^2}\left(\|e_i(t)\|^2 + \|q_i(t)\|^2\right). \quad (36)\end{aligned}$$

Then, the time derivative of $\|e_i(t)\|$ over the interval $[t_k^i, t_{k+1}^i)$ is

$$\begin{aligned}\frac{d}{dt}\|e_i(t)\| &\le \frac{\left\|e_i^T\right\|}{\|e_i\|}\|\dot{e}_i\| = \left\|-\sum_{j=1}^{N} a_{ij}\left(\dot{x}_j(t) - \dot{x}_i(t)\right)\right\|\\ &= \left\|-Aq_i(t) + \mu BB^TP\sum_{j=1}^{N} a_{ij}\left(q_i\left(t_k^i\right) - q_j\left(t_{k'(t)}^j\right)\right)\right\|\\ &\le \|A\|\,\|e_i(t)\|\\ &\quad + \left\|Aq_i\left(t_k^i\right) - \mu BB^TP\sum_{j=1}^{N} a_{ij}\left(q_i\left(t_k^i\right) - q_j\left(t_{k'(t)}^j\right)\right)\right\|\\ &\le \|A\|\,\|e_i(t)\| + \alpha_k^i, \quad (37)\end{aligned}$$

where $k'(t) = \arg\max_{k\in\mathbf{N}}\{t_k^j \mid t_k^j \le t, j \in \mathcal{N}_i\}$, $d/dt\|e_i(t)\|$ denotes the right-hand derivative of $\|e_i(t)\|$ when $t = t_k^i$, and $\alpha_k^i = \max_{t\in[t_k^i, t_{k+1}^i]}\|Aq_i(t_k^i) - \mu BB^TP\sum_{j=1}^{N} a_{ij}(q_i(t_k^i) - q_j(t_{k'(t)}^j))\|$. Then, it follows that:

$$\|e_i(t)\| \le \frac{\alpha_k^i}{\|A\|}\left(e^{\|A\|(t-t_k^i)} - 1\right). \quad (38)$$

Let $s_k^i = (\eta_i/\sqrt{2+2\eta_i^2})\|q_i(t_k^i)\|$. Using (35) and (38) gives that

$$\left\|e_i(t_{k+1}^i)\right\| = s_k^i \leq \frac{\alpha_k^i}{\|A\|}\left(e^{\|A\|\left(t_{k+1}^i - t_k^i\right)} - 1\right) \tag{39}$$

which yields $t_{k+1}^i - t_k^i \geq 1/(\|A\|)\ln((\|A\|s_k^i/\alpha_k^i)+1)$.

To prove that the inter-event interval is strictly positive, we first consider the case when $q_i\left(t_k^i\right) \neq 0$. Since $q_i\left(t_k^i\right) \neq 0$, then one has $s_k^i > 0$. Thus

$$t_{k+1}^i - t_k^i \geq \frac{1}{\|A\|}\ln\left(\frac{\|A\|\, s_k^i}{\alpha_k^i} + 1\right) > 0.$$

Next, we consider the case when $q_i(t_k^i) = 0$ as $k \to \infty$. Then, it follows from (33) that $q_i(t) = 0$, and thus:

$$\begin{aligned}\dot{q}_i(t) &= Aq_i(t) - \mu BB^T P\sum_{j=1}^N a_{ij}\left(q_i\left(t_k^i\right) - q_j\left(t_{k'(t)}^j\right)\right)\\ &= 0.\end{aligned} \tag{40}$$

Simple transposition of (33) leads to

$$\lim_{k\to\infty}\frac{\left\|q_i\left(t_k^i\right)\right\|}{q_i(t)} \geq 1 - \eta_i,\ t \in \left[t_k^i, t_{k+1}^i\right]. \tag{41}$$

It also follows from (40) and the definition of α_k^i that:

$$\begin{aligned}\alpha_k^i &\leq \max_{t\in\left[t_k^i, t_{k+1}^i\right]}\left\|\mu BB^T P\sum_{j=1}^N a_{ij}\left(q_i\left(t_k^i\right) - q_j\left(t_{k'(t)}^j\right)\right)\right\|\\ &\quad + \|A\|\left\|q_i\left(t_k^i\right)\right\|\\ &= \left\|Aq_i\left(t'\right)\right\| + \|A\|\left\|q_i\left(t_k^i\right)\right\|\end{aligned} \tag{42}$$

where $t' \in [t_k^i, t_{k+1}^i]$. Together with (41), one has

$$\begin{aligned}\lim_{k\to\infty}\frac{s_k^i}{\alpha_k^i} &\geq \lim_{k\to\infty}\frac{\frac{\eta_i}{\sqrt{2+2\eta_i^2}}\left\|q_i\left(t_k^i\right)\right\|}{\left\|Aq_i\left(t'\right)\right\| + \|A\|\left\|q_i\left(t_k^i\right)\right\|}\\ &\geq \frac{\eta_i(1-\eta_i)}{\|A\|(2-\eta_i)\sqrt{2+2\eta_i^2}}.\end{aligned} \tag{43}$$

As a result

$$\begin{aligned}\tau_k^i &\triangleq \lim_{k\to\infty}(t_{k+1}^i - t_k^i)\\ &\geq \frac{1}{\|A\|}\ln\left(\frac{\|A\|\, s_k^i}{\alpha_k^i} + 1\right)\\ &\geq \frac{1}{\|A\|}\ln\left(\frac{\eta_i(1-\eta_i)}{(2-\eta_i)\sqrt{2+2\eta_i^2}} + 1\right)\end{aligned} \tag{44}$$

which is strictly positive.

Together with the case when $q_i(t_k^i) \neq 0$, the proof is thus completed. ■

Remark 7: The same conclusion can be obtained for multi-agent system (25) together with controller (3) and triggering condition (27), by letting $A = S$, $B = I$, and $\eta_i = \bar{\eta}_i$.

IV. Self-Triggered Algorithm

In the event-triggered control scheme described in Theorem 1 or Corollary 1, it is apparent that continuous monitoring of measurement errors is required to check condition (8) or (27). Such continuous monitoring process imposes a heavy burden in implementation of the proposed control scheme.

Therefore, inspired by [17] and [18], we propose an improved self-triggered algorithm in this section. Unlike the event-triggered scheme, the next controller update time t_{k+1}^i of agent i in this self-triggered control scheme can be determined by the information at its own previous event time t_k^i, and thus no state or error measurements are required between two event instants of agent i.

Since (35) is a sufficient condition to guarantee (21), we can set a more conservative triggering condition as

$$h'\left(e_i(t), q_i\left(t_k^i\right)\right) = \|e_i(t)\| - s_k^i = 0. \tag{45}$$

According to (37), the increasing rate of $\|e_i(t)\|$ is

$$\frac{d}{dt}\|e_i(t)\| \leq \omega_i(t) \leq \|A\|\, s_k^i + \alpha_k^i \tag{46}$$

where $\omega_i(t) = \|A\|s_k^i + \left\|Aq_i(t_k^i) - \mu BB^T P\sum_{j=1}^N a_{ij}(q_i(t_k^i) - q_j(t_{k'(t)}^j))\right\|$. Denote $\omega_k^i = \omega_i(t_k^i)$ and $\bar{\omega}_k^i = \|A\|s_k^i + \alpha_k^i$. It is noted that $\omega_i(t)$ remains to be the same as ω_k^i until any neighboring agents' combined state $q_j(t_{k'(t)}^j)$ is updated.

Once an event is triggered at the time instant t_k^i for any agent i, agent i samples the states of its neighbors via communication. Thus the combined state $q_i(t_k^i)$ can be obtained, and $q_i(t_k^i)$ will be transmitted to its neighbors. The next triggering time of agent i, t_{k+1}^i, can be determined as the least time it takes for $\|e_i(t)\|$ to increase from 0 to s_k^i. If no state from its neighbors is received before $\|e_i(t)\|$ reaches s_k^i, one has that $t_{k+1}^i - t_k^i = s_k^i/\omega_k^i$. Otherwise, one needs to update the increasing rate $\omega_i(t)$ based on the newly received states, and calculate the remaining time it will take for $\|e_i(t)\|$ to cover the rest of s_k^i. In this way, the event triggering time sequence $T_i = \{t_0^i, t_1^i, \ldots t_k^i, \ldots\}$ can be determined for agent i.

The proposed self-triggered algorithm can be summarized in Algorithm 1, for which the consensus result is summarized in Theorem 4.

Theorem 4: Under Assumptions 1 and 2, consensus of multi-agent system (1) is reached under control law (3) with the triggering time sequence $\{t_0^i, t_1^i, \ldots t_k^i, \ldots\}$ determined by Algorithm 1.

Proof: Since $\|e_i(t_k^i)\| = 0$, for any $t \in (t_k^i, t_{k+1}^i]$, one has

$$\begin{aligned}\|e_i(t)\| &\leq \int_{t_k^i}^{t_{k+1}^i}\frac{d}{dt}\|e_i(t)\|\, dt\\ &\leq \omega_i\left(t_k^i\right)\cdot\left(tu_1 - t_k^i\right) + \omega_i(tu_1)\cdot(tu_2 - tu_1)\\ &\quad + \cdots + \omega_i(tu_m)\cdot\left(t_{k+1}^i - tu_m\right) = s_k^i\end{aligned} \tag{47}$$

where tu_i, $i = 1, \ldots, m$, is the updating time instant for $\omega_i(t)$ between t_k^i and t_{k+1}^i according to Algorithm 1.

Noting that (35) or (47) is a sufficient condition to guarantee (21), the claimed result follows directly from the proof of Theorem 1. ■

Algorithm 1 Determination of Triggering Instants for All Agents

Initialization: $time = 0$; $\mathcal{S} = \{s_1, s_2, \ldots, s_N\}$; $\mathcal{W} = \{\omega_1, \omega_2, \ldots, \omega_N\}$; $k_i = 0$; $s_i = \frac{\eta_i}{\sqrt{2+2\eta_i^2}} \left\| q_i\left(t^i_{k_i}\right)\right\|$; $\omega_i = \omega_i\left(t^i_{k_i}\right)$; $T_i = \left\{t^i_0\right\}$; for any $i = 1, \ldots, N$.
while $t < T$, T is defined as the lifespan of the whole system.
do $h = \arg\min_i (t^i_{k_i} + \frac{s_i}{\omega_i})$; $time = t^h_{k_h} + \frac{s_h}{\omega_h}$;
for $j = 1, \ldots, N$
if $j = h$ //which means agent j is triggered.
update $k_j = k_j + 1$; $T_j = \{t^j_0, \ldots, t^j_{k_j} = time\}$; $s_j = \frac{\eta_j}{\sqrt{2+2\eta_j^2}} \|q_j(t^j_{k_j})\|$; $\omega_j = \omega_j(t^j_{k_j})$; $\omega_l = \omega_l(t^j_{k_j})$ for $l \in \mathcal{N}_j$.
else $s_j = s_j - \omega_j \cdot (time - t^j_{k_j})$;
end if
end for
update set $\mathcal{S} = \{s_1, s_2, \ldots, s_N\}$; $\mathcal{W} = \{\omega_1, \omega_2, \ldots, \omega_N\}$;
end while
return T_i, $i = 1, \ldots, N$.

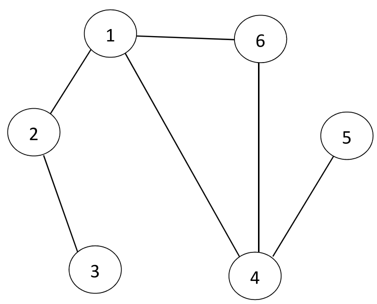

Fig. 1. Communication graph $\mathcal{G}$ of Example 1.

For convenience, we only discuss the case when the multi-agent system is described by (1) and the triggering condition (8) is used. The proposed self-triggered algorithm can also be applied to the multi-agent system (25) and the triggering condition (27). In this case, we only need to set $A = S$, $B = I$, and $\eta_i = \bar{\eta}_i$.

Remark 8: Since the self-triggered algorithm implies (21), the triggering in the self-triggered algorithm occurs always ahead of the triggering determined by the triggering condition (8) if all other conditions are the same. Thus, singular triggering can be avoided for all agents follows from Theorem 2. Besides, according to the self-triggered algorithm, for agent i, the inter-event time can be calculated as $t^i_{k+1} - t^i_k \geq s^i_k/\bar{\omega}^i_k$. It can be seen that the inter-event time is strictly positive when $q_i(t^i_k) \neq 0$. Similar to the proof of Theorem 3, we can discuss the case when $q_i(t^i_k) = 0$ as $k \to \infty$. It can be verified that the limit of $(s^i_k/\bar{\omega}^i_k)$ as $k \to \infty$ is a positive number. Thus, Zeno behavior can also be excluded for all agents.

Remark 9: It is noted that in the self-triggered algorithm, continuous monitoring of measurement errors is not needed any more and thus communication load can be reduced. This is the main advantage of the self-triggered scheme. However, since Algorithm 1 is developed based on (35), which is more conservative than the event-triggering condition (8), the number of triggering times determined by the self-triggered scheme will be in general more than that for the event-triggered scheme. This point can be also observed in Examples 1 and 2 in the next section.

TABLE I
PERFORMANCE COMPARISON FOR EXAMPLE 1

Control scheme	T_s	Triggering numbers for agents					
		1	2	3	4	5	6
Event-triggered	17.48	363	302	266	348	290	303
Self-triggered	17.44	550	475	494	573	604	518

V. EXAMPLES

In this section, we provide two examples to demonstrate the effectiveness of the proposed control schemes.

Example 1: Consider the consensus problem of the multi-agent system (25) with $N = 6$ and

$$S = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & -1 \\ 2 & 2 & 0 \end{bmatrix}.$$

The communication graph is described by Fig. 1.

It is noted that the eigenvalues of S are $0, +2i, -2i$, and thus S is neutrally stable. One solution to $S^T P + PS = 0$ is

$$P = \begin{bmatrix} 1.5 & 0.5 & -0.5 \\ 0.5 & 1.5 & 0.5 \\ -0.5 & 0.5 & 1.5 \end{bmatrix}.$$

Thus, Assumptions 2 and 3 are both satisfied. One can use the event-triggered control scheme or the self-triggered control scheme to solve the consensus problem. Choose $\kappa = \lambda_m/\lambda_M$, and $\sigma_i = 0.999$, then $\bar{\eta}_i$ can be obtained as $\bar{\eta}_i = 0.0625$, for $i = 1, \ldots, 6$. The initial conditions of the closed-loop system are randomly chosen as follows. $x_1(0) = [0.4062\ 8.2218\ 7.5825]$, $x_2(0) = [2.3999\ 5.9046\ 7.3761]$, $x_3(0) = [1.9347\ 6.8045\ 5.4144]$, $x_4(0) = [1.9307\ 8.1729\ 6.1113]$, $x_5(0) = [2.4859\ 7.9111\ 6.1775]$, and $x_6(0) = [-4.2347\ 9.4973\ 12.3484]$.

Numerous simulations are conducted and some results are shown here. The state error between agents i and j is defined as $e_{ij} = \mathrm{col}(e^1_{ij}, e^2_{ij}, e^3_{ij})$, $i, j = 1, \ldots, 6$. The time responses of e^1_{ij} and e^3_{ij} via the event-triggered and self-triggered control schemes are shown in Figs. 2 and 3, respectively. It can be clearly observed that consensus is achieved asymptotically.

We further define the settling time as a minimum time T_s(sec), such that, when $t \geq T_s$, $\|x_i(t) - x_j(t)\| \leq 10^{-4}$, for $i, j = 1, \ldots, 6$. Then the settling time and the numbers of triggering times by using the event-triggered and self-triggered control schemes, respectively, are recorded in Table I. It can be concluded that the self-triggered scheme needs more triggering times to reach the consensus compared with the event-triggered control scheme, while they both have a similar settling time.

Example 2: In this example, we will apply both event-triggered control scheme and self-triggered control scheme to solve a spacecraft formation flying problem in the low Earth orbit [1], [30].

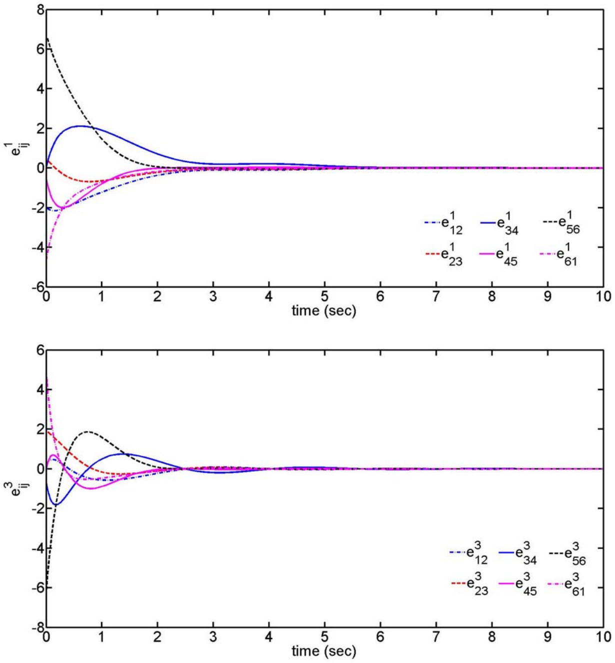

Fig. 2. System responses of all agents via the event-triggered control scheme for Example 1.

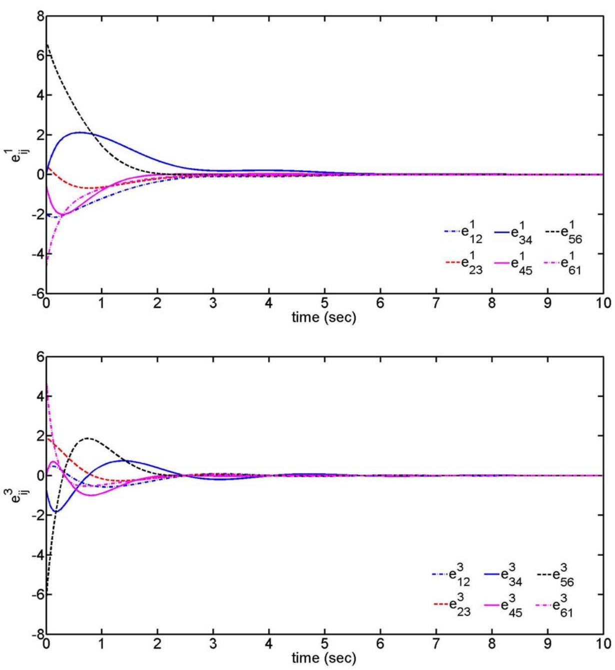

Fig. 3. System responses of all agents via the self-triggered control scheme for Example 1.

Consider the relative dynamics of the ith satellite with respect to a virtual satellite in the following linearized form:

$$\begin{aligned} \ddot{\widetilde{x}}_i - 2\omega_0\dot{\widetilde{y}}_i &= u_i^x \\ \ddot{\widetilde{y}}_i + 2\omega_0\dot{\widetilde{x}}_i - 3\omega_0^2\widetilde{y}_i &= u_i^y \\ \ddot{\widetilde{z}}_i + \omega_0^2\widetilde{z}_i &= u_i^z \end{aligned} \tag{48}$$

where $\widetilde{x}_i$, $\widetilde{y}_i$, and $\widetilde{z}_i$ represent the position of the ith satellite in the rotating coordinates, u_i^x, u_i^y, and u_i^z are the control inputs, and ω_0 is the angular rate of the virtual satellite.

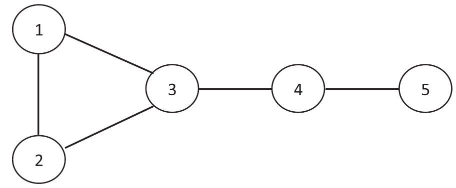

Fig. 4. Communication graph $\mathcal{G}$ of Example 2.

Rewrite the position components in vector form as $r_i = \mathrm{col}(\widetilde{x}_i, \widetilde{y}_i, \widetilde{z}_i)$, velocity vector as $\dot{r}_i = \mathrm{col}(v_i^x, v_i^y, v_i^z)$, and the control vector as $u_i = \mathrm{col}(u_i^x, u_i^y, u_i^z)$. Satellite formation flying is said to be reached if the velocity vectors of all satellites converge to the same value and the satellites keep a prescribed distance from each other, that is, $r_i - h_i \to r_j - h_j$, $\dot{r}_i \to \dot{r}_j$, $\forall i, j \in \mathcal{N}$ as $t \to \infty$, where $h_i = \mathrm{col}(h_i^x, h_i^y, h_i^z)$, and $h_i - h_j \in R^3$ is the desired constant distance between satellite i and j. Define $\bar{r}_i = r_i - h_i$ and $x_i = \mathrm{col}(\bar{r}_i, \dot{r}_i)$, then (48) can be rewritten as

$$\dot{x}_i = \begin{bmatrix} 0 & I_3 \\ A_1 & A_2 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ I_3 \end{bmatrix} \bar{u}_i \tag{49}$$

where $A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3\omega_0^2 & 0 \\ 0 & 0 & -\omega_0^2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 2\omega_0 & 0 \\ -2\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\bar{u}_i = u_i + A_1 h_i$. It can be seen that (49) is in the form of (1). Therefore, the satellites formation problem can be viewed as a consensus problem.

Consider the scenario that the formation flying model consists of five satellites with a communication topology described by Fig. 4, and $\omega_0 = 0.001$. It can be verified that the conditions of Theorems 1 and 4 are all satisfied, and thus the problem can be solved by utilizing the two proposed control schemes, respectively.

For all $i \in \mathcal{N}$, choose $\mu = 0.5$, $\sigma_i = 0.999$, and $\kappa = 1/\rho < 2/\rho$. It can be calculated from (7) that $K = \mu B^T P = \begin{bmatrix} 1.4176 & -0.0013 & 0.0000 & 2.1776 & -0.0000 & 0.0000 \\ 0.0013 & 1.4176 & -0.0000 & -0.0000 & 2.1776 & -0.0000 \\ 0.0000 & 0.0000 & 1.4176 & 0.0000 & 0.0000 & 2.1776 \end{bmatrix}$ and $\eta_i = 0.037$, $i \in \mathcal{N}$. It is noted that the initial conditions of the closed-loop system are randomly chosen. One set of initial conditions is given as follows:

$$\begin{aligned} x_1(0) &= [9.5050\ 9.8570\ 2.4490\ 5.7340\ 3.1180\ 9.4920]^T \\ x_2(0) &= [4.6060\ 6.1520\ 10.8340\ 9.2940\ 5.1810\ 4.5810]^T \\ x_3(0) &= [10.1010\ 5.9770\ 9.3120\ 3.6180\ 6.6590\ 5.5630]^T \\ x_4(0) &= [8.2400\ 10.7970\ 8.0370\ 2.8030\ 5.0320\ 6.5300]^T \\ x_5(0) &= [7.4430\ 8.9090\ 8.0530\ 3.2360\ 7.0900\ 4.5170]^T. \end{aligned}$$

We define position error as $\bar{r}_i - \bar{r}_j = \mathrm{col}(e_{ij}^x, e_{ij}^y, e_{ij}^z)$, with e_{ij}^x representing the position error on x-axis between agents i and j. Let $\bar{u}_i = \mathrm{col}(\bar{u}_i^x, \bar{u}_i^y, \bar{u}_i^z)$, where $\bar{u}_i^x$ represents the control input on x-axis for agent i in (49). The time responses of the satellites' position error and velocity on x-axis via the event-triggered scheme and the self-triggered algorithm are shown in Figs. 5 and 6, respectively. The figures show that consensus is achieved asymptotically with both control schemes. To better demonstrate the triggering situations

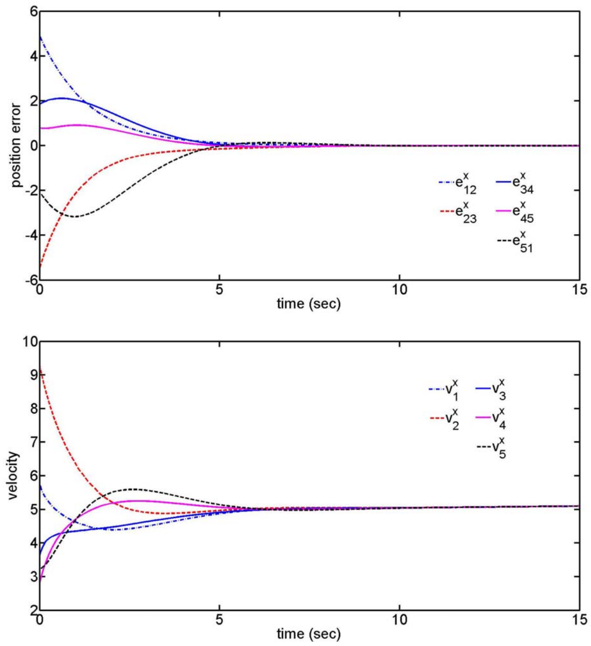

Fig. 5. System responses on x-axis via the event-triggered control scheme for Example 2.

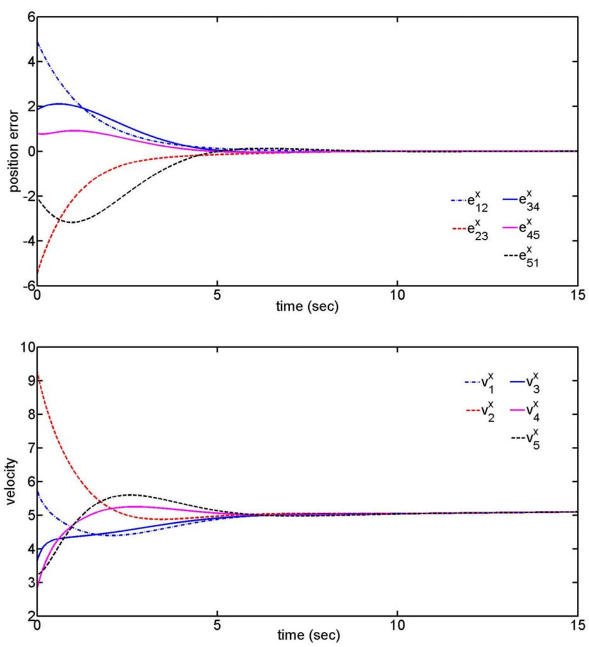

Fig. 6. System responses on x-axis via the self-triggered control scheme for Example 2.

for each agent, we further present two figures describing the control inputs on x-axis for all agents with the event-triggered and self-triggered control schemes applied respectively, as shown in Figs. 7 and 8.

We also investigate how the choices of different values of parameter μ affect the convergence rate for consensus and the number of triggering times. The settling time is used to measure the convergence rate for consensus, and the same definition is employed as in Example 1. Then, the numbers of triggering times by using the event-triggered and self-triggered control schemes with different μ, respectively, are recorded in Table II.

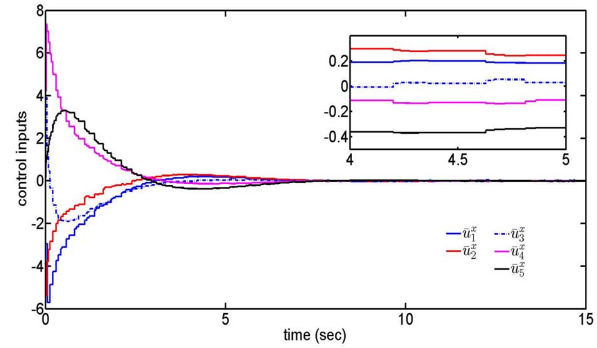

Fig. 7. Control inputs on x-axis via the event-triggered control scheme for Example 2.

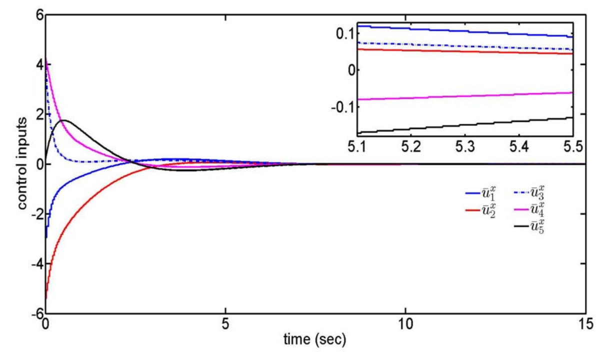

Fig. 8. Control inputs on x-axis via the self-triggered control scheme for Example 2.

TABLE II
PERFORMANCE COMPARISON WITH DIFFERENT μ FOR EXAMPLE 2

Control scheme	μ	T_s	Triggering numbers for agents				
			1	2	3	4	5
Event-triggered	2	13.15	518	536	589	592	599
	1	14.18	340	338	354	365	374
	0.5	19.43	309	317	322	333	334
Self-triggered	2	13.11	330	330	324	331	342
	1	14.20	373	375	366	373	401
	0.5	19.26	571	586	561	634	683

VI. CONCLUSION

In this paper, a novel event-triggered control scheme for the consensus problem of linear multi-agent systems is proposed. It is shown that with this event-triggered control scheme, consensus can be reached asymptotically, and singular triggering and Zeno behavior can be both excluded. A self-triggered algorithm is further developed, where the next triggering time instant of an agent can be determined by its previous one, without checking the triggering condition continuously. As a result, the communication load among all agents can be significantly reduced.

Future research topics could include considering more realistic communication topologies such as time-varying topology or directed topology.

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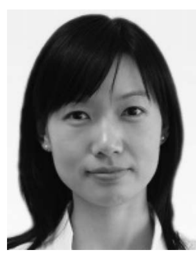

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