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Decentralized Consensus for Linear Multi-Agent Systems under General Directed Graphs based on Event-Triggered/Self-Triggered Strategy

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Abstract— **In this paper, we study the event-triggered consensus problem for multi-agent systems with general linear dynamics under a general directed graph. We propose a decentralized event-triggered consensus controller (ETCC) for each agent to achieve consensus, without requiring continuous communication among agents. Each agent only needs to monitor its own state continuously to determine when to trigger an event and broadcast its state to its out-neighbors. The agent updates its controller when it broadcasts its state to its out-neighbors or receives new information from its in-neighbors. The ETCC can be implemented in multiple steps. We prove that under the proposed ETCC there is no Zeno behavior exhibited. To relax the requirement of continuous monitoring of each agent's own state, we further propose a self-triggered consensus controller (STCC). Simulation results are given to illustrate the theoretical analysis and show the advantages of the event-triggered and self-triggered controllers in this paper.**

Index Terms— **Decentralized event-triggering, consensus control, multi-agent systems, general linear dynamics, directed graphs.**

I. Introduction

In the last decade, the consensus problem of continuous-time multi-agent systems (MAS) has been attracting much attention due to its wide applications. Many significant works have been obtained, e.g., see [1]–[6], just to name a few. Note that in the above works the agents need to continuously employ their own and neighbors' states and hence these states need to be obtained continuously. To avoid this disadvantage, some researchers has begun to study the centralized/distributed event-triggered consensus problem [7]–[14]. The event-triggered average-consensus problem was considered for MAS with single-integrator dynamics in [7] and [8]. The event-triggered consensus problem for MAS with general linear dynamics was investigated in [10] and [15]. However, while the controllers in [7], [8], [10], and [15] are updated less often by using the event-triggered algorithms, they still require the agents to communicate with their neighbors continuously.

It is well known that unnecessary communication can lead to a waste of energy. Continuous communication would also cause the communication resource competition among agents. To reduce the communication cost as much as possible, researchers have begun to study the event-triggered consensus without continuous communication, self-triggered consensus, or decentralized event-triggered consensus for continuous-time MAS recently [9], [11]–[14], where there is no need for continuous communication. A periodic event-trigger based consensus algorithm was studied in [9] for single-integrator agents over undirected connected communication topologies. A self-triggered control algorithm for single-integrator agents was given in [11]. In [12], a decentralized event-triggered consensus algorithm was considered for single- and double- integrator agents. However, in [9], [11], and [12], the agents were assumed to be with single- or double- integrator dynamics. For MAS with general linear dynamics, although [13], [14], and [15] have recently solved the event-triggered consensus problem without continuous communication, they all have some limitations. The consensus error in [13] could only converge to a neighborhood around the origin and the communication topology in [14] and [15] was assumed to be undirected. In short, the event-triggered consensus problem for MAS with general linear dynamics under directed graphs has not been addressed.

Motivated by the above discussion, we consider the consensus problem for MAS with general linear dynamics under a general directed graph based on an event-triggered broadcasting scheme. The communication topology among agents is assumed to be a general directed graph containing a directed spanning tree. We propose a decentralized event-triggered consensus controller (ETCC) implemented in multiple steps for each agent to achieve consensus. Under our proposed controller, there is no continuous communication required among agents. We further prove that there is no Zeno behavior exhibited during the control process, that is, the event would not be triggered continuously. Note that under the ETCC, each agent needs to monitor its own state continuously. To relax this limitation, we further propose a self-triggered consensus controller (STCC), where the next triggering instant is predetermined by the agent itself at the previous triggering instant. It should be pointed out that the stability analysis of the closed-loop systems is partly inspired by [5] and [12]. The primary contributions of the paper are summarized as follows.

1) To the best of our knowledge, this is the first paper addressing the event-triggered/self-triggered consensus problem for MAS with general linear dynamics under general directed graphs without continuous communication and monitoring. Most works in the existing literature have some limitations such as agents' dynamics, communication graphs, nonzero final consensus error, and continuous communication. So, the methods proposed in the literature cannot be directly used in

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our paper.

2) The matrix exponential function e^{At} is used in the proposed ETCC to estimate the current states of the agents and exclude the Zeno behavior. Note that, for single-integrator agents, the event-based controller can be obtained directly from the continuous consensus controller. But for double-integrator agents or general linear agents, the continuous controllers cannot be directly implemented in the event-triggered form. Introducing the matrix exponential function e^{At} is an innovative point of our research. The results in [12] dealing with single- and double-integrator dynamics can be regarded a special case of our result. It is worth mentioning that the analysis for convergence and exclusion of Zeno behavior in our framework is nontrivial and there exist significant challenges.

The rest of this paper is organized as follows. Some useful results and the dynamics are introduced in Section II. The event-triggered consensus is investigated in Section III and the self-triggered scheme is discussed in Section IV. A simulation example is given in Section V. Section VI concludes the paper.

II. Preliminaries

A. Notation and graph theory

Let $\mathbf{R}^{m\times n}$ and $\mathbf{C}^{m\times n}$ be, respectively, the set of $m\times n$ real and complex matrices. Let $\mathbf{1}_m$ and $\mathbf{0}_m$ denote, respectively, the $m\times 1$ column vector of all ones and all zeros. Let $\mathbf{0}_{m\times n}$ denote the $m\times n$ matrix with all zeros and I_m denote the $m\times m$ identity matrix. The superscript T means the transpose for real matrices. We denote by $\lambda_i(\cdot)$ the ith eigenvalue of a matrix. By $\mathrm{diag}(A_1,\cdots,A_n)$, we denote a block-diagonal matrix with matrices $A_i,\ i=1,\cdots,n$, on its diagonal. A matrix $A\in\mathbf{C}^{m\times m}$ is Hurwitz if all of its eigenvalues have strictly negative real parts. The matrix $A\otimes B$ denotes the Kronecker product of matrices A and B. Let $\|\cdot\|$ denote, respectively, the Euclidean norm for vectors and the induced 2-norm for matrices. Let $\|\cdot\|_F$ denote the Frobenius norm of a matrix. Let $\dim(\cdot)$ describe the dimension of a square matrix. For a complex number, $\mathrm{Re}(\cdot)$ denotes its real part.

A directed graph $\mathcal{G}$ is a pair $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{v_1,\cdots,v_N\}$ is a nonempty finite set of nodes and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is a set of edges, in which an edge is represented by an ordered pair of distinct nodes. An edge (v_i,v_j) means that node v_j can receive information from node v_i or equivalently node v_i can broadcast information to node v_j. Here we call v_i an in-neighbor of v_j and v_j an out-neighbor of v_i. A directed path from node v_{i_1} to node v_{i_l} is a sequence of ordered edges of the form $(v_{i_k},v_{i_{k+1}}),\ k=1,\cdots,l-1$. A directed graph contains a directed spanning tree if there exists a node called the root such that there exist directed paths from this node to every other node. The adjacency matrix $\mathcal{A}=[a_{ij}]\in\mathbf{R}^{N\times N}$ associated with the directed graph $\mathcal{G}$ is defined by $a_{ii}=0,\ a_{ij}>0$ if $(v_j,v_i)\in\mathcal{E}$ and $a_{ij}=0$ otherwise. The Laplacian matrix $\mathcal{L}=[l_{ij}]\in\mathbf{R}^{N\times N}$ is defined as $l_{ii}=\sum_{j=1,j\neq i}^{N}a_{ij}$ and $l_{ij}=-a_{ij},\ i\neq j$. The graph $\mathcal{G}$ is undirected if $a_{ij}=a_{ji},\ \forall i,j=1,\cdots,N$ and directed otherwise.

B. Problem statement and background

Consider a group of N identical agents with general linear dynamics. The dynamics of the ith agent are described by

$$\dot{x}_i(t)=Ax_i(t)+Bu_i(t),\tag{1}$$

where $x_i(t)\in\mathbf{R}^n$ is the state, $u_i(t)\in\mathbf{R}^p$ is the control input, $A\in\mathbf{R}^{n\times n}$, and $B\in\mathbf{R}^{n\times p}$. The communication topology among agents is represented by a general directed graph $\mathcal{G}$. The objective of this paper is to design a distributed event-triggered control law for each agent such that the states of all the agents achieve consensus. We need the following assumption and lemmas to derive our main results.

Assumption 2.1: The matrix pair (A,B) in (1) is stabilizable and the graph $\mathcal{G}$ contains a directed spanning tree.

Lemma 2.1: [16] If $\mathcal{G}$ contains a directed spanning tree, 0 is a simple eigenvalue of the Laplacian matrix $\mathcal{L}$ and all the other eigenvalues have positive real parts. Moreover $\mathbf{1}_N$ is a right eigenvector associated with the zero eigenvalue and there is also a nonnegative left eigenvector associated with the zero eigenvalue.

Lemma 2.2: Suppose that $A\in\mathbf{R}^{n\times n}$ is Hurwitz. Then, for all $t\geq 0$, it holds that $\|e^{At}\|\leq\|P_A\|\|P_A^{-1}\|c_Ae^{a_At}$, where P_A is a nonsingular matrix such that $P_A^{-1}AP_A=J_A$ with J_A being the Jordan canonical form of A, $c_A>0$ is a positive constant determined by A, and $\max_i\mathrm{Re}(\lambda_i(A))<a_A<0$.

Proof: For every A with s distinct eigenvalues $\{\lambda_1(A),\cdots,\lambda_s(A)\}$, there is a nonsingular matrix P_A such that $P_A^{-1}AP_A=J_A=\mathrm{diag}\{J_1,\cdots,J_s\}$ and $\sum_{i=1}^{s}\dim(J_i)=n$. For each eigenvalue $\lambda_i(A),\ i=1,\cdots,s$, the Jordan segment J_i is made up of m_i Jordan blocks, that is, $J_i=\mathrm{diag}\{J_{i1},\cdots,J_{im_i}\}$ with

$$J_{ij}=\begin{bmatrix}\lambda_i(A) & 1 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & \lambda_i(A) & 1\\ 0 & \cdots & 0 & \lambda_i(A)\end{bmatrix},$$

$i=1,\cdots,s,\ j=1,\cdots,m_i$ and $\sum_{j=1}^{m_i}\dim(J_{ij})=\dim(J_i)$.

Note that the matrix exponential function e^{J_At} of the matrix J_A is of a block diagonal form given by $e^{J_At}=\mathrm{diag}\{e^{J_1t},\cdots,e^{J_st}\}$ with $e^{J_it}=\mathrm{diag}\{e^{J_{i1}t},\cdots,e^{J_{im_i}t}\}$. Note that $e^{J_{ij}t}$ has the form

$$e^{J_{ij}t}=\begin{bmatrix}e^{\lambda_i(A)t} & te^{\lambda_i(A)t} & \cdots & \frac{t^{\dim(J_{ij})-1}}{(\dim(J_{ij})-1)!}e^{\lambda_i(A)t}\\ \vdots & \ddots & \ddots & \ddots\\ 0 & \cdots & e^{\lambda_i(A)t} & te^{\lambda_i(A)t}\\ 0 & \cdots & 0 & e^{\lambda_i(A)t}\end{bmatrix}.$$

As the induced 2-norm is always not greater than the Frobenius norm for a matrix, we get $\|e^{J_{ij}t}\|\leq\|e^{J_{ij}t}\|_F\leq$

$|e^{\lambda_i(A)t}|_F \varrho \le e^{\mathrm{Re}(\lambda_i(A))t} \max\{1, t^{\dim(J_{ij})-1}\}$, where

$$\varrho = \left\| \begin{bmatrix} 1 & t & \cdots & \frac{t^{\dim(J_{ij})-1}}{(\dim(J_{ij})-1)!} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & t \\ 0 & \cdots & 0 & 1 \end{bmatrix} \right\|_F .$$

Similarly, for the matrix exponential function e^{At} of the matrix A, it follows that

$$\begin{aligned} \|e^{At}\| &= \|P_A e^{J_A t} P_A^{-1}\| \le \|P_A\| \|e^{J_A t}\| \|P_A^{-1}\| \\ &\le \|P_A\| \|P_A^{-1}\| e^{\max_i \mathrm{Re}(\lambda_i(A))t} \max_{i,j}\{1, t^{\dim(J_{ij})-1}\}. \end{aligned}$$

Since A is Hurwitz, $\mathrm{Re}(\lambda_i(A))$ are all strictly negative. By noting that $\dim(J_{ij}) \le n = \dim(A)$ is finite, there must exist constants $c_A > 0$ and $\max_i \mathrm{Re}(\lambda_i(A)) < a_A < 0$ such that

$$e^{\max_i \mathrm{Re}(\lambda_i(A))t} \max_{i,j}\{1, t^{\dim(J_{ij})-1}\} < c_A e^{a_A t}.$$

Thus, we have $\|e^{At}\| \le \|P_A\| \|P_A^{-1}\| c_A e^{a_A t}$. ■

Zeno behavior is a phenomenon in hybrid systems where an infinite number of discrete transitions occur in a finite time interval.

III. Event-Triggered Consensus Control

In this section, we will propose an event-triggered scheme for MAS with general linear dynamics under a general directed graph and prove that no Zeno behavior is exhibited.

The widely-studied consensus controller for (1) was proposed in [5] as

$$u_i(t) = cK \sum_{j=1}^{N} a_{ij} \left(x_i(t) - x_j(t) \right), \tag{2}$$

where $c > 0$ is the coupling gain, $K \in \mathbf{R}^{p \times n}$ is the feedback gain matrix, and a_{ij} is the ijth entry of the adjacency matrix $\mathcal{A}$. It was proved in [5] that under the assumption that the graph $\mathcal{G}$ contains a directed spanning tree, the controller (2) solves the consensus problem if and only if all matrices $A + c\lambda_i(\mathcal{L})BK$, where $\lambda_i(\mathcal{L}) \ne 0$, are Hurwitz.

In (2) each agent needs to use its in-neighbors' states all the time. Thus continuous communication is needed. To reduce the communication cost among agents, we propose an ETCC that only relies on intermittent communication as

$$u_i(t) = cK \sum_{j=1}^{N} a_{ij} \left(e^{A(t-t^i_{k_i})} x_i(t^i_{k_i}) - e^{A(t-t^j_{k_j})} x_j(t^j_{k_j}) \right), \tag{3}$$

where c, K, and a_{ij} are defined as in (2), $t^i_{k_i}$ is the most recent triggering instant of agent i, $k_i = 1, 2, \cdots$, A is the system matrix of the agents' dynamics, and $x_i(t^i_{k_i})$ is the last broadcast state of agent i.

For each agent i, we define the measurement error

$$e_i(t) = e^{A(t-t^i_{k_i})} x_i(t^i_{k_i}) - x_i(t). \tag{4}$$

The triggering function for each agent i is given by

$$f_i(t, e_i(t)) = \|e_i(t)\| - c_1 e^{-\alpha t}, \tag{5}$$

where $c_1 > 0$ and α is a positive constant to be determined. Under the ETCC, the controller of agent i monitors its own state continuously. When the measurement error of agent i exceeds a certain given threshold, that is, $f_i(t, e_i(t)) \ge 0$, an event is triggered for agent i. Agent i updates its controller using its current state and broadcast its current state to its out-neighbors at the same time. Meanwhile, the measurement error of agent i is reset to zero. When agent i receives new states broadcast by its in-neighbors (equivalently, its in-neighbors' events are triggered), the agent also updates its controller immediately. If the measurement error is less than the threshold, there is no communication needed until the next event is triggered.

Remark 1: The matrix exponential function e^{At} in (3) is used to estimate the current states of the agents. Introducing the matrix exponential function e^{At} is an innovative point of our research. We extend the results in [12] to the agents with general linear dynamics. In fact, the main results in [12] can be considered as special cases of our results. Theorem 3.2 (Theorem 5.2) in [12] for agents with single-integrator (double-integrator) dynamics is a special case of Theorem 3.1 in our paper.

For the whole system with N agents, let t^* denote the latest triggering instant. With the stack vectors $x(t) = \left[x_1^T(t), \cdots, x_N^T(t)\right]^T$, $x(t^*) = [x_1^T(t^1_{k_1}), \cdots, x_N^T(t^N_{k_N})]^T$, and $e(t) = \left[e_1^T(t), \cdots, e_N^T(t)\right]^T$, the closed-loop system of (1) using (3) can be written as

$$\begin{aligned} \dot{x}(t) &= (I_N \otimes A)\, x(t) + (c\mathcal{L} \otimes BK)\, e^{(I_N \otimes A)(t-t^*)} x(t^*) \\ &= (I_N \otimes A + c\mathcal{L} \otimes BK)\, x(t) + (c\mathcal{L} \otimes BK)\, e(t). \end{aligned} \tag{6}$$

In order to analyze the stability of (6), we define the disagreement vector

$$\delta(t) = x(t) - \left(\mathbf{1}_N r^T \otimes I_n\right) x(t), \tag{7}$$

where $r = [r_1, \cdots, r_N]^T$ is the nonnegative left eigenvector of the Laplacian matrix $\mathcal{L}$ associated with the zero eigenvalue satisfying $\sum_{j=1}^{N} r_j = 1$. Note that 0 is a simple eigenvalue of $I_N - \mathbf{1}_N r^T$ with $\mathbf{1}_N$ being the right eigenvector and 1 is the eigenvalue with algebraic multiplicity $N-1$. From (7), we know that $\delta(t) = \mathbf{0}_{Nn}$ if and only if $x_1(t) = \cdots = x_N(t)$. So the consensus problem can be converted to the stability problem of $\delta(t)$ under the ETCC (3).

Now we are ready to present our main result.

Theorem 3.1: Consider the MAS (1) satisfying Assumption 2.1. Suppose the triggering function (5) with $c_1 > 0$ and $0 < \alpha < -\max_i \mathrm{Re}(\lambda_i(\Pi))$, where Π is defined after (9). Then, with the ETCC (3) and the triggering function (5), the disagreement vector $\delta(t)$ of the closed-loop system (6) asymptotically converges to zero for all initial conditions if and only if all matrices $A + c\lambda_i(\mathcal{L})BK$, where $\lambda_i(\mathcal{L}) \ne 0$, are Hurwitz. Moreover, the closed-loop system (6) does not exhibit Zeno behavior under the ETCC.

Proof: (Sufficiency) We define a new vector

$$\varepsilon(t) = \left(T^{-1} \otimes I_n\right) \delta(t) = \left[\varepsilon_1^T(t), \varepsilon_{2-N}^T(t)\right]^T, \tag{8}$$

where $\varepsilon_1(t) \in \mathbf{C}^n$ and $\varepsilon_{2-N}(t) \in \mathbf{C}^{(N-1)n}$. Using similar derivations in [5], it follows that $\varepsilon_1(t) \equiv \mathbf{0}_n$ and the vector $\varepsilon_{2-N}(t)$ satisfies

$$\dot{\varepsilon}_{2-N}(t) = \Pi\varepsilon_{2-N}(t) + (c\Delta W \otimes BK)e_{2-N}(t), \quad (9)$$

where $\Pi \triangleq I_{N-1} \otimes A + c\Delta \otimes BK \in \mathbf{C}^{(N-1)n\times(N-1)n}$ and $e_{2-N}(t) \triangleq [e_2^T(t), \cdots, e_N^T(t)]^T$.

Let $P \in \mathbf{C}^{(N-1)n\times(N-1)n}$ and $P^{-1} \in \mathbf{C}^{(N-1)n\times(N-1)n}$ be the matrices such that $P^{-1}\Pi P = J_\Pi$, where J_Π is the Jordan canonical form of the matrix Π. From the definition of Π after (9), it is obvious that if all matrices $A + c\lambda_i(\mathcal{L})BK$, where $\lambda_i(\mathcal{L}) \neq 0$, are Hurwitz, the matrix Π is surely Hurwitz and all $\mathrm{Re}(\lambda_i(\Pi)) < 0$. Since the triggering function $f_i(t, e_i(t))$ for agent i is reset to zero when an event is triggered. Before the next event is triggered, $f_i(t, e_i(t))$ will not cross zero, that is, $\|e_i(t)\| < c_1 e^{-\alpha t}$ is satisfied until the next event is triggered. Hence $\|e_{2-N}(t)\| < \sqrt{N-1}c_1 e^{-\alpha t}$ and $\|e_{2-N}(t)\| \to 0$, as $t \to \infty$. It follows from (9) and the input-to-state stability argument that $\varepsilon_{2-N}(t)$ approaches zero. Then, it follows that the disagreement vector $\delta(t)$ of the closed-loop system (6) asymptotically converges to zero for all initial conditions, that is, the ETCC (3) solves the event-triggered consensus problem.

Next, we will show that under the ETCC (3), the closed-loop system (6) does not exhibit the Zeno behavior. The solution of $\varepsilon_{2-N}(t)$ can be obtained as

$$\begin{aligned}\varepsilon_{2-N}(t) &= e^{\Pi t}\varepsilon_{2-N}(0) \\ &\quad + \int_0^t e^{\Pi(t-s)}\left(c\Delta W \otimes BK\right)e_{2-N}(s)\mathrm{d}s. \quad (10)\end{aligned}$$

It follows from Lemma 2.2 that for $0 \le s \le t$,

$$\begin{aligned}&\|e^{\Pi(t-s)}\left(c\Delta W \otimes BK\right)e_{2-N}(s)\| \\ &\le c_\Pi c_1\sqrt{N-1}\|P\|\|P^{-1}\|\|c\Delta W \otimes BK\|e^{a_\Pi(t-s)}e^{-\alpha s}, \quad (11)\end{aligned}$$

where c_Π is a positive constant with respect to Π and $\max_i \mathrm{Re}(\lambda_i(\Pi)) < a_\Pi < 0$.

Let $a_1 = c_\Pi\|P\|\|P^{-1}\|\|\varepsilon_{2-N}(0)\|$ and $a_2 = c_\Pi c_1\sqrt{N-1}\ \|P\|\|P^{-1}\|\|c\Delta W \otimes BK\|$. It follows from (10), (11), and Lemma 2.2 that

$$\|\varepsilon(t)\| = \|\varepsilon_{2-N}(t)\| \le \left(a_1 + \frac{a_2}{|a_\Pi + \alpha|}\right)e^{a_\Pi t} + \frac{a_2}{|a_\Pi + \alpha|}e^{-\alpha t}.$$

Then it follows from (8) that $\|\delta(t)\|$ satisfies

$$\|\delta(t)\| \le \|T \otimes I_n\|\|\varepsilon(t)\| \le k_1 e^{a_\Pi t} + k_2 e^{-\alpha t},$$

where $k_1 = \|T\|\left(a_1 + \frac{a_2}{|a_\Pi+\alpha|}\right)$ and $k_2 = \|T\|\frac{a_2}{|a_\Pi+\alpha|}$.

Let $u(t)$ be the column stack vector of $u_i(t)$. Using the property: $\mathcal{L}\mathbf{1}_N \equiv \mathbf{0}_N$, we conclude that

$$\begin{aligned}(I_N \otimes B)u(t) &= (c\mathcal{L} \otimes BK)(x(t) + e(t)) \\ &\quad -(c\mathcal{L} \otimes BK)\left((\mathbf{1}_N\mathbf{r}^T \otimes I_n)x(t)\right) \\ &= (c\mathcal{L} \otimes BK)(\delta(t) + e(t)).\end{aligned}$$

Similarly, $\|(I_N \otimes B)u(t)\|$ is upper bounded by

$$\begin{aligned}\|(I_N \otimes B)u(t)\| &\le \|c\mathcal{L} \otimes BK\|(\|\delta(t)\| + \|e(t)\|) \\ &= b_1 e^{a_\Pi t} + b_2 e^{-\alpha t}, \quad (12)\end{aligned}$$

where $b_1 = \|c\mathcal{L} \otimes BK\|k_1$ and $b_2 = \|c\mathcal{L} \otimes BK\|(k_2 + \sqrt{N}c_1)$.

Note that the states of each agent $x_i(t_{k_i}^i)$, $i = 1, \cdots, N$, remain constant since the latest triggering instant t^*. It follows from (4) that $\dot{e}(t) = \frac{d}{dt}e^{(I_N\otimes A)(t-t^*)}x(t^*) - \dot{x}(t) = (I_N \otimes A)e(t) - (I_N \otimes B)u(t)$. Moreover, with (12) and the fact that $\|e(t)\| \le \sqrt{N}c_1 e^{-\alpha t}$ before the next event is triggered, we can get the upper bound of $\|\dot{e}(t)\|$ between the two triggered events as $\|\dot{e}(t) \le b_1 e^{a_\Pi t} + d_2 e^{-\alpha t} \triangleq g(t)$, where $d_2 = \|I_N \otimes A\|\sqrt{N}c_1 + b_2$. Note that b_1 and d_2 are both positive constants here.

Since the latest triggering instant, it follows that $\|e(t)\| = \left\|\int_{t^*}^t \dot{e}(s)\mathrm{d}s\right\| \le \int_{t^*}^t g(s)\mathrm{d}s$. From the definition of the triggering function (5), we know that the next event will not be triggered before $f_i(t, e_i(t)) = 0$ or equivalently $\|e_i(t)\| = c_1 e^{-\alpha t}$. Hence the next event will not be triggered before $\int_{t^*}^t g(s)\mathrm{d}s = \sqrt{N}c_1 e^{-\alpha t}$. Since $t \ge t^*$ and both a_Π and $-\alpha$ are negative, we have $e^{a_\Pi t} \le e^{a_\Pi t^*}$ and $e^{-\alpha t} \le e^{-\alpha t^*}$. Let $\tau = t - t^*$ be the time-interval between the two triggered events. So τ is greater than or equal to the solution of the implicit equation $\left(b_1 e^{a_\Pi t^*} + d_2 e^{-\alpha t^*}\right)\tilde{\tau} = \sqrt{N}c_1 e^{-\alpha(t^*+\tilde{\tau})}$, which is equivalent to $\left(b_1 e^{(\alpha+a_\Pi)t^*} + d_2\right)\tilde{\tau} = \sqrt{N}c_1 e^{-\alpha\tilde{\tau}}$. Noting that from the condition $\alpha < -\max_i \mathrm{Re}(\lambda_i(\Pi))$ given in Theorem 3.1, there must exist a negative constant a_Π such that $\max_i \mathrm{Re}(\lambda_i(\Pi)) < a_\Pi < -\alpha < 0$. As $\alpha < -a_\Pi$, we know the term $b_1 e^{(\alpha+a_\Pi))t^*} + d_2$ is upper bounded by $b_1 + d_2$. So the solution of the implicit equation is greater than or equal to the solution of $(b_1 + d_2)\bar{\tau} = \sqrt{N}c_1 e^{-\alpha\bar{\tau}}$, which is strictly positive. It means that if the coefficients c_1 and α in (5) satisfy $c_1 > 0$ and $0 < \alpha < -\mathrm{Re}(\lambda_1(\Pi))$, there is a positive lower bound $\bar{\tau}$ on the inter-event times. So, the event-triggered consensus problem of the general linear MAS is solved with no Zeno behavior exhibited.

(Necessity) The necessity is obvious. Note that the initial measurement error might not be zero. If at least one matrix $A + c\lambda_i(\mathcal{L})BK$ is not Hurwitz, where $\lambda_i(\mathcal{L}) \neq 0$, $\varepsilon_{2-N}(t)$ will go to infinity as $t \to \infty$ and so will $\delta(t)$. Then, the states of the N agents will not reach consensus for all initial conditions. ■

Remark 2: In the triggering function (5), the designed parameter α plays an important role in the convergence rate of the measurement error vector. It is noted from (10) that $-\mathrm{Re}(\lambda_i(\Pi))$ could be understood as the convergence rate of the closed-loop system (6). Actually, the convergence rate α of the measurement error's threshold should be smaller than the convergence rate of the closed-loop system $-\mathrm{Re}(\lambda_i(\Pi))$. Otherwise, there will be the Zero behavior exhibited. That also explains why we need α to be smaller than $-\max_i \mathrm{Re}(\lambda_i(\Pi))$.

Motivated by [5], we now present a multi-step event-triggered consensus control algorithm.

Algorithm 1: Given (A, B) that is stabilizable, an event-triggered algorithm in the form of (3) and (5) solving the consensus problem of general linear MAS can be constructed according to the following steps.

1) Solve the following linear matrix inequality
$$AP + PA^T - 2BB^T < 0 \tag{13}$$
to get one symmetric positive-definite solution P. Then, choose the feedback gain matrix $K = -B^T P^{-1}$.
2) Select the coupling gain c in (3) given by $c > \frac{1}{\min_i \mathrm{Re}(\lambda_i(\mathcal{L}))}$, where $\lambda_i(\mathcal{L}) \neq 0$, denote the nonzero eigenvalues of Laplacian matrix $\mathcal{L}$.
3) Choose the constants in the function (5) to satisfy $c_1 > 0$ and $0 < \alpha < -\max_i \mathrm{Re}(\lambda_i(\Pi))$, where Π is defined after (9).

The steps 1 and 2 are borrowed from [5] to ensure that $A + c\lambda_i(\mathcal{L})BK$, $i = 2, \cdots, N$ are Hurwitz. Since Assumption 2.1 holds, there must exist a matrix P satisfying (13).

IV. Self-Triggered Consensus Control

The event-triggered scheme proposed in the last section needs each agent to monitor its own states continuously to check the triggering function. In this section, we extend the result to the self-triggered scheme, where the continuous self-state monitoring is relaxed. The next triggering instant $t^i_{k_i+1}$ for agent i is predetermined at the previous triggering instant $t^i_{k_i}$. No monitoring is required between two triggering events.

For agent i, since the k_ith triggering instant $t^i_{k_i}$, its state can be calculated by $x_i(t) = e^{A(t-t^i_{k_i})}x_i(t^i_{k_i}) + \int_{t^i_{k_i}}^{t} e^{A(t-s)}Bu(s)\mathrm{d}s$, where $u(s) = cK\sum_{j=1}^{N} a_{ij}(e^{A(s-t^i_{k_i})}x_i(t^i_{k_i}) - e^{A(s-t^j_{k_j})}x_j(t^j_{k_j}))$. Note that $e_i(t) = e^{A(t-t^i_{k_i})}x_i(t^i_k) - x_i(t) = -\int_{t^i_{k_i}}^{t} e^{A(t-s)}Bu(s)\mathrm{d}s$ and $t^i_{k_i}, x_i(t^i_{k_i}), t^j_{k_j}$, and $x_j(t^j_{k_j})$ are known to agent i at the triggering instant $t^i_{k_i}$. Recalling (5) and using the notation $\xi_i = t - t^i_{k_i}$, the self-triggering function is written as
$$\begin{aligned} f_i(t, e_i(t)) = &\left\| \int_{t^i_{k_i}}^{t^i_{k_i}+\xi_i} e^{A(t^i_{k_i}+\xi_i-s)}Bu(s)\mathrm{d}s \right\| - c_1 e^{-\alpha(t^i_{k_i}+\xi_i)} \\ = &\,0, \end{aligned} \tag{14}$$
where ξ_i can be decided by agent i at time $t^i_{k_i}$.

Based on the above observation, the self-triggering policy to determine the next triggering instant for agent i at time $t^i_{k_i}$ is defined as follows: assume the solution of the implicit equation (14) is $\bar{\xi}_i$, then the next triggering instant $t^i_{k_i+1}$ takes place at most $\bar{\xi}_i$ time units after $t^i_{k_i}$, i.e., $t^i_{k_i+1} \leq t^i_{k_i} + \bar{\xi}_i$. For agent i and all $t \in [t^i_{k_i}, t^i_{k_i} + \bar{\xi}_i]$, if there is an event triggered in one of its in-neighbors, i.e., some new state is broadcast to agent i, agent i re-check the self-triggering function (14) with the new information. Otherwise, agent i waits until its predetermined triggering instant $t^i_{k_i+1}$ to re-compute the condition (14). Similar to the last section, the time interval between two consecutive triggered events is strictly positive for agent i. Note that there is no continuous monitoring required for each agent. The triggering instant is predetermined by agent i itself at the previous triggering instant. Now we present a multi-step STCC algorithm.

Algorithm 2: Given (A, B) that is stabilizable, a self-trigger based algorithm in the form of (3) and (14) solving the consensus problem of general linear MAS can be constructed according to the following steps.

1) Same as Algorithm 1.
2) Same as Algorithm 1.
3) Choose the constants in the function (14) to satisfy $c_1 > 0$ and $0 < \alpha < -\max_i \mathrm{Re}(\lambda_i(\Pi))$.
4) At time $t^i_{k_i}$, solve the self-triggering function (14) to get a solution $\bar{\xi}_i$.
5) Predetermine the next triggering instant at $t^i_{k_i+1}$ which is at most $\bar{\xi}_i$ after $t^i_{k_i}$, i.e., $t^i_{k_i+1} \leq t^i_{k_i} + \bar{\xi}_i$.
6) For all $t \in [t^i_{k_i}, t^i_{k_i} + \bar{\xi}_i]$, if there is an event triggered in one of its in-neighbors, re-check (14) using the new information received. Otherwise, agent i waits until $t^i_{k_i+1}$.

The preceding analysis, along with Theorem 3.1, yields the following result.

Theorem 4.1: Consider the MAS (1) satisfying Assumption 2.1. Suppose that in the self-triggering function (14) $c_1 > 0$ and $0 < \alpha < -\max_i \mathrm{Re}(\lambda_i(\Pi))$, where Π is defined after (9). Assume that the next triggering instant is chosen according to Algorithm 2. Then, with the controller (3), the disagreement vector $\delta(t)$ of the closed-loop system (6) asymptotically converges to zero for all initial conditions if and only if all matrices $A + c\lambda_i(\mathcal{L})BK$, where $\lambda_i(\mathcal{L}) \neq 0$, are Hurwitz. Moreover, the closed-loop system (6) does not exhibit the Zeno behavior.

V. Simulation

In this section, we illustrate the above theoretical results by simulation. Consider a group of 6 agents with general linear dynamics described by (1), where $A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. We choose the feedback gain matrix $K = [-1\ 2]$ so that $A + BK$ is Hurwitz. The communication topology among agents is shown in Fig. 1, which is a directed graph containing a directed spanning tree. The Laplacian matrix of the communication graph is
$$\mathcal{L} = \begin{bmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$
Obviously, the nonzero eigenvalues of the Laplacian matrix $\mathcal{L}$ are $1, 1.3376 \pm 0.5623i, 2, 3.3247$.

We first illustrate the ETCC algorithm (Algorithm 1). Here by step 2 of Algorithm 1, we choose $c = 1.1$. The eigenvalues of Π defined after (9). are -1, -1, -1, -1, -1, -1.1, $-1.4714 \pm 0.6185i$, -2.2, -3.6572. So, according to the conditions required in step 3 of Algorithm 1, we choose $c_1 = 0.5$ and $\alpha = 0.9$. The initial states are given by $x_1(0) = [0.4; 0.3]$, $x_2(0) = [0.5; 0.2]$, $x_3(0) = [0.6; 0.1]$, $x_4(0) = [0.7; 0]$, $x_5(0) = [0.8; -0.1]$, and $x_6(0) = [0.4; -0.2]$. The state trajectories are presented in Fig. 2. The convergent time under the ETCC and the controller (2) in [5] is almost the

TABLE I

THE NUMBER OF COMMUNICATIONS.

Agent	Traditional Controller	ETCC	STCC
1	continuous communication	9	9
2	continuous communication	6	6
3	continuous communication	7	7
4	continuous communication	5	5
5	continuous communication	6	6
6	continuous communication	7	7

same. But the dynamic performance under the ETCC is a little worse. This is due to the tradeoff between performance and communication cost. The measurement errors and their thresholds of agents are shown in Fig. 3. The bottom of Fig. 2 presents the triggering instants of Algorithm 1, where we can see that the communication among agents is discrete and the Zeno behavior is avoided. Compared with the continuous communication, the ETCC can reduce the communication cost and save much communication source.

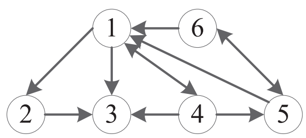

Fig. 1. The communication graph among agents.

We also consider the STCC algorithm. Table I compares the communication under the ETCC and the STCC during the consensus process. The communication times are the same for both controllers. Because we consider the normal model for the agents without disturbances, the performances of the ETCC and STCC strategies are identical, which may not be achieved in general model with uncertainties.

VI. CONCLUSION

This paper considered the event-triggered consensus problem for MAS with general linear dynamics under general directed graphs. We proposed a decentralized event-triggered broadcasting algorithm for each agent to achieve consensus, without requiring continuous communication among agents. It is proved that under the proposed event-triggered control algorithm implemented in multiple steps, there is no Zeno behavior exhibited. We further proposed a self-triggered control algorithm to relax the requirement of continuous self-monitoring for each agent, under which the next triggering instant is predetermined by each agent itself at the previous triggering instant. This paper extended the existing results on event-triggered consensus control without continuous communication for single-integrator and double-integrator systems to the case of agents with general linear dynamics. Delay event-triggered consensus for MAS with general linear dynamics and event-triggered consensus for discrete-time linear MAS are future topics to be discussed.

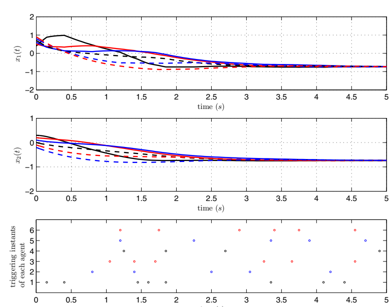

Fig. 2. The states and the triggering instants of each agent under the ETCC.

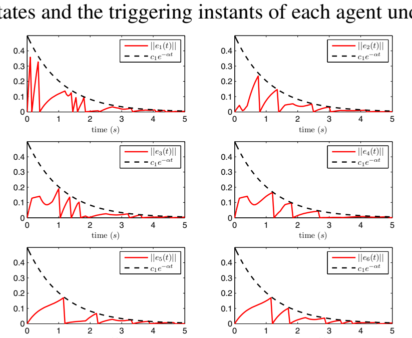

Fig. 3. The errors and the thresholds of the errors of each agent under the ETCC.

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