Fixed-Time Average Consensus of Nonlinear Delayed MASs Under Switching Topologies: An Event-Based Triggering Approach

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***Abstract*—This article addresses the fixed-time average consensus problem of nonlinear multiagent systems (MASs) subject to input delay, external disturbances, and switching topologies. Different from the finite-time convergence, the convergence time of the fixed-time convergence is independent of initial conditions. Then, an event-based control strategy is presented to reach the fixed-time average consensus under switching topologies and intermittent communication. Because the nonlinear dynamics, external disturbances, switching topologies, and triggering condition for intermittent communication are considered, the fixed-time consensus problem is more challenging under the event-based control than under the continuous-time control. Besides, a new measurement error is designed based on the hyperbolic tangent function to avoid Zeno behavior. Furthermore, an improved triggering function is designed to avoid continuous monitoring. Hence, resource consumption is reduced significantly. Finally, the effectiveness of the algorithms is validated by three simulation examples.**

***Index Terms*—Event-based control, fixed-time average consensus, nonlinear multiagent systems (MASs), switching topologies.**

I. Introduction

In THE past decade, multiagent systems (MASs) have drawn considerable attention, and they can be employed to accomplish some elaborate tasks [1]–[6]. A fundamental research topic of MASs is consensus, and the finite-time consensus has attracted increasing attention [7]–[9]. However, the convergence time of the finite-time convergence depends on the initial conditions [10], [11], which may restrict actual applications.

To address this problem, the fixed-time stability [12] was developed. The corresponding consensus problems of first-order MASs under the fixed topology were addressed in [13], and the ones under the switching topologies were considered in [14]. The nonsingular algorithms of second-order systems were given via the state-feedback control in [15], and via the output-feedback control in [16]–[18]. Moreover, the nonlinear dynamics was considered in [18]. For the high-order systems, the corresponding results were given in [19] and [20]. Moreover, the corresponding results of the nonholonomic chained-form dynamics were developed in [21]. In [22], the practical prescribed time tracking problems were investigated for the nonaffine systems with uncertainties. In [23], the specified-time consensus was obtained.

Due to the nonideal data transmission, time delay frequently occurs in controlled systems [24]–[26]. The existence of delay can degrade the performance of the systems. Hence, it is important to consider the delayed MASs. In [27], the consensus problems of delayed MASs with switching topologies were addressed. The consensus problem of MASs with nonlinear uncertainties and input delay was solved in [28]. To guarantee the faster convergence rate, the finite-time consensus was achieved for the delayed MASs in [29], and the corresponding fixed-time consensus result was developed in [30].

In addition, because of the limited resources of the embedded processors, the event-based triggering approach was presented to solve the consensus problems and the main results are as follows [31]–[41]. The event-based consensus results of the linear systems with continuous communication were given in [31]–[33], and the corresponding result without continuous communication was obtained in [34] and [35]. The time delay was considered in [36]. The consensus problems with the stochastic nonlinear uncertainties were considered in [37]. Based on the state estimates of neighbors, a new distributed event-based communication controller was designed in [38]. In [39], the DoS attacks were considered for the secure consensus problem via an event-based triggering approach. Moreover, the event-based control mechanism was adopted in semi-Markov jump systems to solve the stochastic stabilization problem in [40]. The finite-time event-based consensus was obtained for a particular convergence rate in [42]–[45]. In [42]–[44], the finite-time consensus results were based on

Manuscript received April 15, 2020; revised October 3, 2020; accepted December 29, 2020. This work was supported in part by the National Key Research and Development Program of China under Grant 2018AAA0101400; in part by the National Natural Science Foundation of China under Grant 61921004 , Grant 62033010 and Grant 62033009; in part by the Natural Science Foundation of Jiangsu Province of China under Grant BK20202006; in part by the National Postdoctoral Program for Innovative Talents under Grant BX20200081; in part by the China Postdoctoral Science Foundation under Grant 2020TQ0041; and in part by the Jiangsu Planned Projects for Postdoctoral Research Funds under Grant 2020Z081. This article was recommended by Associate Editor C.-C. Tsai. *(Corresponding authors: Yanling Zhang; Changyin Sun.)*

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Color versions of one or more figures in this article are available at https://doi.org/10.1109/TSMC.2021.3051156.

Digital Object Identifier 10.1109/TSMC.2021.3051156

the first-order systems. The corresponding result of general liner MASs was given in [45], where the Zeno behavior cannot be excluded. Furthermore, the fixed-time event-based consensus problems were tackled in [46] and [47]. It is worth noting that the continuous communication was required in [42]–[47], and continuous communication may be unrealistic in practical applications and can cause high communication cost. To deal with this constraint, some new event-based control strategies are presented in [48]–[55] to avoid continuous communication. In [48]–[52], the finite-time and fixed-time event-based consensus was reached, and the input delay problems were addressed in [54] and [55]. A fixed-time event-based observer was developed in [53]. However, these results did not involve switching topologies. Moreover, [48]–[51] only considered the nonlinear uncertainties and [54] and [55] only considered the input delay.

Motivated by these existing studies, we consider the fixed-time average consensus problem of nonlinear delayed MASs under switching topologies. Besides, two event-based control schemes are present to reduce the resource consumption. The contributions of this article are stated as follows.

1) Different from the fixed-time consensus results [13]–[22], input delay, nonlinear dynamics, and switching topologies are considered simultaneously herein.
2) A new control framework is developed by using the hyperbolic tangent function to avoid Zeno behavior existing in [42], [45], and [46]. Besides, in contrast to [43]–[46], [48]–[52], [54], and [55], the Lyapunov function no longer relies on the communication topology.
3) Compared with the finite/fixed-time event-based consensus [42]–[55], we extend the consensus results to nonlinear MASs with intermittent communication and switching topologies. Moreover, continuous monitoring is no longer required.

Notations: For a variable X, $\operatorname{sign}(X)$ is the sign function, and we can approximate it with the tanh function. Assume that $\operatorname{sign}(X) \approx \tanh(\beta X)$, where $\beta \gg 1$. For a matrix $\boldsymbol{Y}$, $\lambda_2(\boldsymbol{Y})$ and $\lambda_M(\boldsymbol{Y})$ are the second smallest and the largest eigenvalue of $\boldsymbol{Y}$, respectively.

II. Preliminaries

A. Graph Theory

Consider an undirected graph $\mathcal{G}$ with M nodes, and $\mathcal{E}$ is the set of edges. The adjacency matrix $\boldsymbol{\mathcal{A}} = [a_{ij}] \in \mathbb{R}^{M\times M}$, where $a_{ij} = \{ \begin{smallmatrix} 1, \text{ if } (j,i) \in \mathcal{E} \\ 0, \text{ otherwise} \end{smallmatrix}$ and $a_{ij} = 0$ when $i = j$. The degree matrix is $\boldsymbol{\mathcal{D}} = \operatorname{diag}[\hat{d}_1, \ldots, \hat{d}_M]$ with $\hat{d}_i = \sum_{j=1}^{M} a_{ij}$. The Laplacian matrix $\boldsymbol{\mathcal{L}} = [l_{ij}] \in \mathbb{R}^{M\times M}$ of $\mathcal{G}$ is $\boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{D}} - \boldsymbol{\mathcal{A}}$. The graph $\mathcal{G}$ is connected if a path exists between any two nodes, which is used in this article.

B. Definition and Lemma

Assume that the origin is an equilibrium point of the following system:

$$\begin{cases} \dot{x}(t) = f(x(t), t) \\ x(0) = x_0 \end{cases} \tag{1}$$

where $f(x(t), t) : \mathbb{R}^M \times \mathbb{R}^+ \to \mathbb{R}^M$ is an unknown nonlinear function.

Definition 1 [12]: The origin of (1) is globally finite-time stable if it is asymptotically stable and there exists a settling time $T(x_0) > 0$, such that $x(t, x_0)$ can reach the equilibrium in $T(x_0)$. If $\exists T_{\max} > 0$ and settling time $T \leq T_{\max}$ for any initial conditions, it is fixed-time stable.

Lemma 1 [12], [15], [19]: If there exits a Lyapunov function $\mathcal{V}(x(t))$ satisfying

$$\dot{\mathcal{V}}(x(t)) \leq -c_1 \mathcal{V}^{\bar{p}}(x(t)) - c_2 \mathcal{V}^{\bar{q}}(x(t)) \tag{2}$$

for $c_1, c_2 > 0, \bar{p} \in (0, 1)$, $\bar{q} \in (1, \infty)$, the fixed-time stable is obtained and T satisfies

$$T \leq T_{\max} := \frac{1}{c_1(1-\bar{p})} + \frac{1}{c_2(\bar{q}-1)}. \tag{3}$$

Lemma 2 [13], [14], [27]: $\boldsymbol{\mathcal{L}}$ is positive semidefinite under $\mathcal{G}$. If the eigenvalues of $\boldsymbol{\mathcal{L}}$ are $0, \lambda_2, \ldots, \lambda_M$, we have $0 < \lambda_2 \leq \cdots \leq \lambda_M$. Furthermore, if $\mathbf{1}^T x = 0$ with $x = [x_1, x_2, \ldots x_M]^T$, then $x^T \boldsymbol{\mathcal{L}} x \geq \lambda_2 x^T x$, where $x^T \boldsymbol{\mathcal{L}} x = (1/2)\sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}(x_i - x_j)^2$.

Lemma 3 [15]: Let $\zeta_1, \zeta_2, \ldots \zeta_M \geq 0$. Then

$$\sum_{i=1}^{M} \zeta_i^{\hat{p}} \geq \left(\sum_{i=1}^{M} \zeta_i\right)^{\hat{p}}, 0 < \hat{p} \leq 1$$

$$M^{1-\hat{q}}\left(\sum_{i=1}^{M} \zeta_i\right)^{\hat{q}} \leq \sum_{i=1}^{M} \zeta_i^{\hat{q}} \leq \left(\sum_{i=1}^{M} \zeta_i\right)^{\hat{q}}, 1 < \hat{q} \leq \infty.$$

Lemma 4 [43]: For any $y \in \mathbb{R}$, one has

$$0 \leq |y| - y\tanh(\beta y) \leq \frac{\iota}{\beta}$$

where $\beta \gg 1$ and $\iota = 0.2785$.

C. Problem Formulation

Consider the MASs consisting of M agents, the dynamics of each agent is

$$\dot{x}_i(t) = u_i(t-\tau) + f(x_i(t), t) + d_i(x_i(t), t) \tag{4}$$

where $x_i(t)$ and $u_i(t)$ are the state and the control input, τ is the known input delay, and $f(x_i(t), t)$ and $d_i(x_i(t), t)$ are the nonlinear terms and the unknown disturbances. Herein, $\dot{x}_i(t)$ is discontinuous in this article, and the concept of Filippov solutions [56] should be adopted.

There exits a T and a sufficiently small positive constant δ satisfying $lim_{t\to T} |x_i(t) - \bar{x}(t)| \leq \delta$ with $\bar{x}(t) = (1/M)\sum_{j=1}^{M} x_j(t)$, and $|x_i(t) - \bar{x}(t)| \leq \delta$ when $t \geq T$. $\exists T_{\max} > 0$, such that $T \leq T_{\max}$ with arbitrary initial conditions. If $\delta = 0$, it is the fixed-time average consensus; otherwise, it is the practical fixed-time average consensus.

Assumption 1: There are non-negative constants ρ_1 and ρ_2 satisfying

$$\left|f(x_i(t), t) - f\big(x_j(t), t\big)\right| \leq \rho_1 + \rho_2\left|x_i(t) - x_j(t)\right|. \tag{5}$$

Assumption 2: The $|d_i(x_i(t), t)|$ is bounded by a positive constant D, that is

$$|d_i(x_i(t), t)| \leq D. \tag{6}$$

Remark 1: The previous studies [48], [49] only considered the nonlinear uncertainties, and the nonlinear term is a special case of this article.

III. Fixed-Time Average Consensus With Input Delay

Herein, the fixed-time average consensus algorithm is proposed with $\rho_2 = 0$. The fixed topology case is given first. Then, the switching topologies case is obtained based on the fixed topology case. The detailed design ideas are presented in the following.

We give the following error:

$$\varsigma_i(t) = \chi_i(t) - \bar{x}(t), \qquad i \in \{1, \ldots, M\} \tag{7}$$

where $\chi_i(t) = x_i(t) + \int_{t-\tau}^{t} u_i(T)dT$.

A. Under Fixed Topology

First, we design the following control input of agent i:

$$\begin{aligned} u_i(t) = &-\kappa_1 \sum_{j=1}^{M} a_{ij}\big(\chi_i(t) - \chi_j(t)\big)^{\mu} \\ &-\kappa_2 \sum_{j=1}^{M} a_{ij}\mathrm{sign}\big(\chi_i(t) - \chi_j(t)\big) \\ &-\kappa_3 \sum_{j=1}^{M} a_{ij}\big(\chi_i(t) - \chi_j(t)\big) \end{aligned} \tag{8}$$

where κ_1, κ_2, and κ_3 are positive constants, and $\mu \in (1, \infty)$ is the ratio of positive odd numbers.

Theorem 1: If the following inequality holds:

$$\kappa_2 \lambda_2(\boldsymbol{\mathcal{L}}_2) > (\rho_1 + D)(2M)^{\frac{1}{2}} \tag{9}$$

the fixed-time average consensus is reached, where $\boldsymbol{\mathcal{L}}_2$ is the Laplacian matrix of graph $\mathcal{G}(\boldsymbol{\mathcal{A}}^2)$.

Proof: Design the following Lyapunov function:

$$V(t) = \frac{1}{2}\sum_{i=1}^{M} \varsigma_i^2(t). \tag{10}$$

Based on the Newton–Leibniz formula, we have

$$\dot{\chi}_i(t) = u_i(t) + f(x_i(t), t) + d_i(x_i(t), t). \tag{11}$$

The derivative of $V(t)$ satisfies

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{M} \varsigma_i(t)\big(u_i(t) + d_i(x_i(t), t) + f(x_i(t), t) - \dot{\bar{x}}(t)\big) \\ &= \sum_{i=1}^{M} \varsigma_i(t)\big(f(x_i(t), t) - \dot{\bar{x}}(t)\big) \\ &\quad + \sum_{i=1}^{M} \varsigma_i(t)\Bigg(-\kappa_1 \sum_{j=1}^{M} a_{ij}\big(\chi_i(t) - \chi_j(t)\big)^{\mu} \\ &\qquad - \kappa_2 \sum_{j=1}^{M} a_{ij}\mathrm{sign}\big(\chi_i(t) - \chi_j(t)\big) \\ &\qquad - \kappa_3 \sum_{j=1}^{M} a_{ij}\big(\chi_i(t) - \chi_j(t)\big) + d_i(x_i(t), t)\Bigg) \\ &\le \sum_{i=1}^{M} \varsigma_i(t)\Bigg(f(x_i(t), t) - f(\bar{x}(t), t) + f(\bar{x}(t), t) \\ &\qquad - \frac{1}{M}\sum_{j=1}^{M} f\big(x_j(t), t\big) - \frac{1}{M}\sum_{j=1}^{M} d_j\big(x_j(t), t\big)\Bigg) \\ &\quad + \sum_{i=1}^{M} |\varsigma_i(t)|D - \kappa_1 \sum_{i=1}^{M} \varsigma_i(t)\left(\sum_{j=1}^{M} a_{ij}\big(\varsigma_i(t) - \varsigma_j(t)\big)\right)^{\mu} \\ &\quad - \kappa_2 \sum_{i=1}^{M} \varsigma_i(t)\left(\sum_{j=1}^{M} a_{ij}\mathrm{sign}\big(\varsigma_i(t) - \varsigma_j(t)\big)\right) \\ &\quad - \kappa_3 \sum_{i=1}^{M} \varsigma_i(t)\left(\sum_{j=1}^{M} a_{ij}\big(\varsigma_i(t) - \varsigma_j(t)\big)\right). \end{aligned} \tag{12}$$

Since $\sum_{i=1}^{M} \varsigma_i(t) = 0$, we have $\sum_{i=1}^{M} \varsigma_i(t)(f(\bar{x}(t), t) - (1/M)\sum_{j=1}^{M} f(x_j(t), t) - (1/M)\sum_{j=1}^{M} d_j(x_j(t), t)) = 0$. Then, according to Assumption 1, one has

$$\begin{aligned} \dot{V}(t) \le &\sum_{i=1}^{M} \varsigma_i(t)(f(x_i(t), t) - f(\bar{x}(t), t)) \\ &+ \sum_{i=1}^{M} |\varsigma_i(t)|D \\ &- \frac{1}{2}\kappa_1 \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\big(\varsigma_i(t) - \varsigma_j(t)\big)\big(\varsigma_i(t) - \varsigma_j(t)\big)^{\mu} \\ &- \frac{1}{2}\kappa_2 \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\big(\varsigma_i(t) - \varsigma_j(t)\big) \\ &* \mathrm{sign}\big(\varsigma_i(t) - \varsigma_j(t)\big) \\ &- \frac{1}{2}\kappa_3 \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\big(\varsigma_i(t) - \varsigma_j(t)\big)\big(\varsigma_i(t) - \varsigma_j(t)\big) \\ \le\ &(\rho_1 + D)\sum_{i=1}^{M} |\varsigma_i(t)| \\ &- \frac{1}{2}\kappa_1 \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\big|\varsigma_i(t) - \varsigma_j(t)\big|^{\mu+1} \\ &- \frac{1}{2}\kappa_2 \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\big|\varsigma_i(t) - \varsigma_j(t)\big| \\ &- \frac{1}{2}\kappa_3 \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\big|\varsigma_i(t) - \varsigma_j(t)\big|^2 \\ \le\ &(\rho_1 + D)M^{\frac{1}{2}}\left(\sum_{i=1}^{M} |\varsigma_i(t)|^2\right)^{\frac{1}{2}} \\ &- \frac{1}{2}\kappa_2\left(\sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}^2\big|\varsigma_i(t) - \varsigma_j(t)\big|^2\right)^{\frac{1}{2}} \end{aligned}$$

$$-\frac{1}{2}\kappa_1 M^{1-\mu}\left(\sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}^{\frac{2}{\mu+1}}\left|\varsigma_i(t)-\varsigma_j(t)\right|^2\right)^{\frac{\mu+1}{2}}$$
$$-\frac{1}{2}\kappa_3\sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\left|\varsigma_i(t)-\varsigma_j(t)\right|^2. \tag{13}$$

Due to

$$\sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\left(\varsigma_i(t)-\varsigma_j(t)\right)^2 = 2\varsigma^T(t)\mathcal{L}\varsigma(t) \tag{14}$$

with $\varsigma(t) = [\varsigma_1(t), \varsigma_2(t), \ldots, \varsigma_M(t)]^T$, we have

$$\sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}^2\left|\varsigma_i(t)-\varsigma_j(t)\right|^2 \geq 4\lambda_2(\mathcal{L}_2)V(t) \tag{15}$$

$$\sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}^{\frac{2}{\mu+1}}\left|\varsigma_i(t)-\varsigma_j(t)\right|^2 \geq 4\lambda_2\left(\mathcal{L}_{2/\mu+1}\right)V(t) \tag{16}$$

where $\mathcal{L}_{2/\mu+1}$ is the Laplacian matrix of graph $\mathcal{G}(\mathcal{A}^{2/\mu+1})$. Hence, we have

$$\begin{aligned}\dot{V}(t) \leq& -\left(\kappa_2\lambda_2(\mathcal{L}_2) - (\rho_1+D)(2M)^{\frac{1}{2}}\right)V(t)^{\frac{1}{2}}\\ &-\frac{1}{2}\kappa_1 M^{1-\mu}\left(4\lambda_2\left(\mathcal{L}_{2/\mu+1}\right)V(t)\right)^{\frac{\mu+1}{2}}\\ &-\kappa_3\lambda_2(\mathcal{L})V(t)\\ \leq& -\alpha_1 V^{\frac{1}{2}}(t) - \alpha_2 V^{\frac{\mu+1}{2}}(t)\end{aligned} \tag{17}$$

where $\alpha_1 = \kappa_2\lambda_2(\mathcal{L}_2) - (\rho_1 + D)(2M)^{(1/2)}$ and $\alpha_2 = (1/2)\kappa_1 M^{1-\mu}(4\lambda_2(\mathcal{L}_{2/\mu+1}))^{(\mu+1)/(2)}$. Based on Lemma 1, we obtain $\lim_{t\to T(\chi)} V(t) = 0$, where

$$T(\chi) \leq T_{\max} = \frac{2}{\alpha_1}+\frac{2}{\alpha_2(\mu-1)} \tag{18}$$

which implies that $\lim_{t\to T(x)} \chi(t) = x(t)$ when $t = T(x) \leq T_{\max} + \tau$. Hence, the average consensus is reached, and $T(x)$ satisfies $T(x) \leq (2)/(\alpha_1)+(2)/(\alpha_2(\mu-1))+\tau$.

B. Under Switching Topologies

In the actual systems, new communication links might appear between the agents due to migration and evolution. These uncertainties may make the edges created or removed. Hence, it is necessary for us to consider the time-varying communication topology.

Different from [43]–[46], [48]–[52], [54], and [55], the Lyapunov function (10) is independent of topology, it provides us with the opportunity of extending the results to switching topologies. First, a switching signal $\eta(t) : [0, +\infty) \to \Xi$ is given, where $\Xi = \{1, 2, \ldots, N\}$ is a finite set. Then, assume that $\mathcal{G}_s = \{\mathcal{E}, \mathcal{A}_{\eta(t)}\}$ is an undirected graph set, and Ξ is the index set of $\mathcal{G}_s$. The graph $\mathcal{G}_{\eta(t)} \in \mathcal{G}_s$ is the corresponding graph at time t and the switching time sequence is $t_0, t_1, \ldots$ Hence, the switching topologies can be described by $\mathcal{G}_{\eta(t)}$, and the $\mathcal{G}_{\eta(t)}$ is connected at any time interval.

Theorem 2: Under the switching topologies $\mathcal{G}_{\eta(t)}$ and the controllers (8), if the following inequality holds:

$$\kappa_2\lambda_2^{\min}(\mathcal{L}_2) > (\rho_1 + D)(2M)^{\frac{1}{2}} \tag{19}$$

the average consensus problem can be solved, where $\lambda_2^{\min}(\mathcal{L}_2) = \min\{\lambda_2(\mathcal{L}_2(t_0)), \lambda_2(\mathcal{L}_2(t_1)), \ldots\}$, $\lambda_2^{\min}(\mathcal{L}) = \min\{\lambda_2(\mathcal{L}(t_0)), \lambda_2(\mathcal{L}(t_1)), \ldots\}$, $\mathcal{L}_2(t)$ and $\mathcal{L}(t)$ are the corresponding Laplacian matrices at time t.

Proof: For the same $V(t)$ of Theorem 1, we have

$$\begin{aligned}\dot{V}(t) \leq& -\left(\kappa_2\lambda_2(\mathcal{L}_2(t)) - (\rho_1+D)(2M)^{\frac{1}{2}}\right)V(t)^{\frac{1}{2}}\\ &-\frac{1}{2}\kappa_1 M^{\frac{1-\mu}{2}}\left(4\lambda_2\left(\mathcal{L}_{2/\mu+1}(t)\right)V(t)\right)^{\frac{\mu+1}{2}}\\ &-\kappa_3\lambda_2(\mathcal{L}(t))V\\ \leq& -\left(\kappa_2\lambda_2^{\min}(\mathcal{L}_2) - (\rho_1+D)(2M)^{\frac{1}{2}}\right)V(t)^{\frac{1}{2}}\\ &-\frac{1}{2}\kappa_1 M^{\frac{1-\mu}{2}}\left(4\lambda_2^{\min}\left(\mathcal{L}_{2/\mu+1}\right)V(t)\right)^{\frac{\mu+1}{2}}\\ &-\kappa_3\lambda_2^{\min}(\mathcal{L})V(t)\\ \leq& -\bar{\alpha}_1 V^{\frac{1}{2}}(t) - \bar{\alpha}_2 V^{\frac{\mu+1}{2}}(t)\end{aligned} \tag{20}$$

where $\bar{\alpha}_1 = \kappa_2\lambda_2^{\min}(\mathcal{L}_2) - (\rho_1 + D)(2M)^{(1/2)}$ and $\bar{\alpha}_2 = (1/2)\kappa_1 M^{(1-\mu/2)}(4\lambda_2^{\min}(\mathcal{L}_{2/\mu+1}))^{(\mu+1)/(2)}$ with $\lambda_2^{\min}(\mathcal{L}_{2/\mu+1}) = \min\{\lambda_2(\mathcal{L}_{2/\mu+1}(t_0)), \lambda_2(\mathcal{L}_{2/\mu+1}(t_1)), \ldots\}$ and $\mathcal{L}_{2/\mu+1}(t)$ is the corresponding Laplacian matrix at time t.

Obviously, $\forall\eta(t) \in \Xi$, inequality (20) holds. Based on Theorem 1, the consensus is reached, and we have $T(x) \leq (2)/(\bar{\alpha}_1)+(2)/(\bar{\alpha}_2(\mu-1)) + \tau$. ■

IV. Fixed-Time Event-Based Average Consensus

For the MASs without input delay ($\tau = 0$), we design the following error of agent i:

$$\xi_i(t) = x_i(t) - \bar{x}(t), \qquad i \in \{1, \ldots, M\}. \tag{21}$$

A. Under Fixed Topology

Under the event-based control framework, the measurement error of agent i is defined as

$$\begin{aligned}e_i(t) =& \kappa_1\sum_{j=1}^{M} a_{ij}\left(x_i\left(t_k^i\right) - x_j\left(t_k^i\right)\right)^{\mu}\\ &+\kappa_2\sum_{j=1}^{M} a_{ij}\mathrm{sign}\left(x_i\left(t_k^i\right) - x_j\left(t_k^i\right)\right)\\ &+\kappa_3\sum_{j=1}^{M} a_{ij}\left(x_i\left(t_k^i\right) - x_j\left(t_k^i\right)\right)\\ &-\kappa_1\sum_{j=1}^{M} a_{ij}\left(x_i(t) - x_j(t)\right)^{\mu}\\ &-\kappa_2\sum_{j=1}^{M} a_{ij}\mathrm{sign}\left(x_i(t) - x_j(t)\right)\\ &-\kappa_3\sum_{j=1}^{M} a_{ij}\left(x_i(t) - x_j(t)\right)\end{aligned} \tag{22}$$

where t_k^i is the latest triggering instant of agent i. Then, the triggering function is that

$$\Gamma_i(t) = |e_i(t)| - \omega\kappa_1 \sum_{j=1}^{M} a_{ij}\big|x_i(t) - x_j(t)\big|^{\mu} - \omega\kappa_2 M - \omega\kappa_3 \sum_{j=1}^{M} a_{ij}\big|x_i(t) - x_j(t)\big| \tag{23}$$

where $\omega \in (0, 1)$ is a constant.

The control input is designed as

$$\begin{aligned} u_i(t) = &-\kappa_1 \sum_{j=1}^{M} a_{ij}\big(x_i(t_k^i) - x_j(t_k^i)\big)^{\mu} \\ &-\kappa_2 \sum_{j=1}^{M} a_{ij}\mathrm{sign}\big(x_i(t_k^i) - x_j(t_k^i)\big) \\ &-\kappa_3 \sum_{j=1}^{M} a_{ij}\big(x_i(t_k^i) - x_j(t_k^i)\big), \; t \in [t_k^i, t_{k+1}^i) \end{aligned} \tag{24}$$

and we can get the next triggering instant t_{k+1}^i by using the triggering condition $t_{k+1}^i = \inf\{t > t_k^i | \Gamma_i(t) \geq 0\}$. The controller of each agent is updated at its own triggering instants $t_0^i, t_1^i, \ldots$

Theorem 3: If the following inequalities holds:

$$\begin{aligned} \kappa_2\sqrt{\lambda_2(\mathcal{L}_2)} &> (\omega\kappa_2 M + \rho_1 + D)(2M)^{\frac{1}{2}} \\ M^{1-\mu}\big(\lambda_2\big(\mathcal{L}_{2/\mu+1}\big)\big)^{\frac{\mu+1}{2}} &> \omega\sqrt{2}M\big(\lambda_M\big(\mathcal{L}_{2/\mu}\big)\big)^{\frac{\mu}{2}} \\ \kappa_3\lambda_2(\mathcal{L}) &> \rho_2 + \omega\kappa_3\sqrt{2\lambda_M(\mathcal{L}_2)}M \end{aligned} \tag{25}$$

the fixed-time average consensus problem can be tackled, where $\mathcal{L}_{2/\mu}$ is the Laplacian matrix of graph $\mathcal{G}(\mathcal{A}^{2/\mu})$.

Proof 1: Design the following Lyapunov function:

$$V(t) = \frac{1}{2}\sum_{i=1}^{M} \xi_i^2(t). \tag{26}$$

Differentiating (26), we have

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{M} \xi_i(t)\big(u_i(t) + d_i(x_i(t), t) + f(x_i(t), t) - \dot{\bar{x}}(t)\big) \\ &= \sum_{i=1}^{M} \xi_i(t)\big(f(x_i(t), t) - \dot{\bar{x}}(t)\big) \\ &\quad + \sum_{i=1}^{M} \xi_i(t)\Bigg(-e_i(t) - \kappa_1 \sum_{j=1}^{M} a_{ij}\big(x_i(t) - x_j(t)\big)^{\mu} \\ &\qquad - \kappa_2 \sum_{j=1}^{M} a_{ij}\mathrm{sign}\big(x_i(t) - x_j(t)\big) \\ &\qquad -\kappa_3 \sum_{j=1}^{M} a_{ij}\big(x_i(t) - x_j(t)\big) + d_i(x_i(t), t)\Bigg) \\ &\leq \sum_{i=1}^{M} \xi_i(t)\Bigg(f(x_i(t), t) - f(\bar{x}(t), t) + f(\bar{x}(t), t) \\ &\qquad - \frac{1}{M}\sum_{j=1}^{M} f\big(x_j(t), t\big) - \frac{1}{M}\sum_{j=1}^{M} d_j\big(x_j(t), t\big)\Bigg) \\ &\quad + \sum_{i=1}^{M} |\xi_i(t)|(|e_i(t)| + D) \\ &\quad - \kappa_1 \sum_{i=1}^{M} \xi_i(t)\Bigg(\sum_{j=1}^{M} a_{ij}\big(\xi_i(t) - \xi_j(t)\big)\Bigg)^{\mu} \\ &\quad - \kappa_2 \sum_{i=1}^{M} \xi_i(t)\Bigg(\sum_{j=1}^{M} a_{ij}\mathrm{sign}\big(\xi_i(t) - \xi_j(t)\big)\Bigg) \\ &\quad - \kappa_3 \sum_{i=1}^{M} \xi_i(t)\Bigg(\sum_{j=1}^{M} a_{ij}\big(\xi_i(t) - \xi_j(t)\big)\Bigg) \\ &\leq \sum_{i=1}^{M} \xi_i(t)(f(x_i(t), t) - f(\bar{x}(t), t)) \\ &\quad + \sum_{i=1}^{M} |\xi_i(t)|\Bigg(\omega\kappa_1 \sum_{j=1}^{M} a_{ij}\big|x_i(t) - x_j(t)\big|^{\mu} \\ &\qquad + \omega\kappa_2 M + \omega\kappa_3 \sum_{j=1}^{M} a_{ij}\big|x_i(t) - x_j(t)\big| + D\Bigg) \\ &\quad - \frac{1}{2}\kappa_1 \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\big(\xi_i(t) - \xi_j(t)\big)\big(\xi_i(t) - \xi_j(t)\big)^{\mu} \\ &\quad - \frac{1}{2}\kappa_2 \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\big(\xi_i(t) - \xi_j(t)\big)*\mathrm{sign}\big(\xi_i(t) - \xi_j(t)\big) \\ &\quad - \frac{1}{2}\kappa_3 \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\big(\xi_i(t) - \xi_j(t)\big)\big(\xi_i(t) - \xi_j(t)\big) \\ &\leq \rho_2 \sum_{i=1}^{M} |\xi_i(t)|^2 + \sum_{i=1}^{M} |\xi_i(t)|(\omega\kappa_2 M + \rho_1 + D) \\ &\quad + \omega\kappa_1 \sum_{i=1}^{M} |\xi_i(t)| \sum_{j=1}^{M} a_{ij}\big|\xi_i(t) - \xi_j(t)\big|^{\mu} \\ &\quad + \omega\kappa_3 \sum_{i=1}^{M} |\xi_i(t)| \sum_{j=1}^{M} a_{ij}\big|\xi_i(t) - \xi_j(t)\big| \\ &\quad - \frac{1}{2}\kappa_1 \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\big|\xi_i(t) - \xi_j(t)\big|^{\mu+1} \\ &\quad - \frac{1}{2}\kappa_2 \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\big|\xi_i(t) - \xi_j(t)\big| \\ &\quad - \frac{1}{2}\kappa_3 \sum_{i=1}^{M}\sum_{j=1}^{M} a_{ij}\big|\xi_i(t) - \xi_j(t)\big|^2 \\ &\leq \rho_2 \sum_{i=1}^{M} |\xi_i(t)|^2 \end{aligned}$$

$$
\begin{aligned}
&+ (\omega\kappa_2 M + \rho_1 + D) M^{\frac{1}{2}} \left(\sum_{i=1}^{M} |\xi_i(t)|^2 \right)^{\frac{1}{2}} \\
&+ \omega\kappa_1 M \left(\sum_{i=1}^{M} |\xi_i(t)|^2 \right)^{\frac{1}{2}} \\
&* \left(\sum_{i=1}^{M} \sum_{j=1}^{M} a_{ij}^{\frac{2}{\mu}} \left|\xi_i(t) - \xi_j(t)\right|^2 \right)^{\frac{\mu}{2}} \\
&+ \omega\kappa_3 M \left(\sum_{i=1}^{M} |\xi_i(t)|^2 \right)^{\frac{1}{2}} \\
&* \left(\sum_{i=1}^{M} \sum_{j=1}^{M} a_{ij}^{2} \left|\xi_i(t) - \xi_j(t)\right|^2 \right)^{\frac{1}{2}} \\
&- \frac{1}{2}\kappa_1 M^{1-\mu} \left(\sum_{i=1}^{M} \sum_{j=1}^{M} a_{ij}^{\frac{2}{\mu+1}} \left|\xi_i(t) - \xi_j(t)\right|^2 \right)^{\frac{\mu+1}{2}} \\
&- \frac{1}{2}\kappa_2 \left(\sum_{i=1}^{M} \sum_{j=1}^{M} a_{ij}^{2} \left|\xi_i(t) - \xi_j(t)\right|^2 \right)^{\frac{1}{2}} \\
&- \frac{1}{2}\kappa_3 \sum_{i=1}^{M} \sum_{j=1}^{M} a_{ij} \left|\xi_i(t) - \xi_j(t)\right|^2 \\
\le& -\Big(\kappa_2 \sqrt{\lambda_2(\mathcal{L}_2)} \\
&\quad - (\omega\kappa_2 M + \rho_1 + D)(2M)^{\frac{1}{2}} \Big) V^{\frac{1}{2}}(t) \\
&- \frac{1}{2}\kappa_1 M^{1-\mu} \big(4\lambda_2(\mathcal{L}_{2/\mu+1}) V(t)\big)^{\frac{\mu+1}{2}} \\
&+ \omega\kappa_1 \sqrt{2} M \big(4\lambda_M(\mathcal{L}_{2/\mu})\big)^{\frac{\mu}{2}} V^{\frac{\mu+1}{2}}(t) \\
&- 2(\kappa_3\lambda_2(\mathcal{L}) - \rho_2) V(t) + \omega\kappa_3 \sqrt{2} M (4\lambda_M(\mathcal{L}_2))^{\frac{1}{2}} V(t) \\
\le& -\alpha_3 V^{\frac{1}{2}}(t) - \alpha_4 V^{\frac{\mu+1}{2}}(t)
\end{aligned} \tag{27}
$$

where $\alpha_3 = \kappa_2\sqrt{\lambda_2(\mathcal{L}_2)} - (\omega\kappa_2 M + \rho_1 + D)(2M)^{(1/2)}$ and $\alpha_4 = (1/2)\kappa_1 M^{1-\mu}(4\lambda_2(\mathcal{L}_{2/\mu+1}))^{(\mu+1)/(2)} - \omega\kappa_1\sqrt{2}M(4\lambda_M(\mathcal{L}_{2/\mu}))^{(\mu/2)}$.

Therefore, the fixed-time average consensus is reached and one has

$$
T(x) \le T_{\max} = \frac{2}{\alpha_3} + \frac{2}{\alpha_4(\mu-1)}. \tag{28}
$$

In the following, a new $e_i(t)$ is given. Based on the new $e_i(t)$, we can analyze the Zeno behavior

$$
\begin{aligned}
e_i(t) =& \kappa_1 \sum_{j=1}^{M} a_{ij} \big(x_i(t_k^i) - x_j(t_k^i)\big)^{\mu} \\
&+ \kappa_2 \sum_{j=1}^{M} a_{ij} \operatorname{sign}\big(x_i(t_k^i) - x_j(t_k^i)\big) \\
&+ \kappa_3 \sum_{j=1}^{M} a_{ij} \big(x_i(t_k^i) - x_j(t_k^i)\big) \\
&- \kappa_1 \sum_{j=1}^{M} a_{ij} \big(x_i(t) - x_j(t)\big)^{\mu} \\
&- \kappa_2 \sum_{j=1}^{M} a_{ij} \tanh\big(\beta\big(x_i(t) - x_j(t)\big)\big) \\
&- \kappa_3 \sum_{j=1}^{M} a_{ij} \big(x_i(t) - x_j(t)\big).
\end{aligned} \tag{29}
$$

According to Lemma 4 and (27), $\dot{V}(t)$ can be computed as

$$
\begin{aligned}
\dot{V}(t) =& \sum_{i=1}^{M} \xi_i(t) \big(f(x_i(t), t) - \dot{\bar{x}}(t)\big) \\
&+ \sum_{i=1}^{M} \xi_i(t) \Bigg(-e_i(t) - \kappa_1 \sum_{j=1}^{M} a_{ij} \big(x_i(t) - x_j(t)\big)^{\mu} \\
&\quad - \kappa_2 \sum_{j=1}^{M} a_{ij} \tanh\big(\beta\big(x_i(t) - x_j(t)\big)\big) \\
&\quad - \kappa_3 \sum_{j=1}^{M} a_{ij} \big(x_i(t) - x_j(t)\big) + d_i(x_i(t), t) \Bigg) \\
\le& \sum_{i=1}^{M} \xi_i(t) (f(x_i(t), t) - f(\bar{x}(t), t)) \\
&+ \sum_{i=1}^{M} |\xi_i(t)| \Bigg(\omega\kappa_1 \sum_{j=1}^{M} a_{ij} \left|x_i(t) - x_j(t)\right|^{\mu} \\
&+ \omega\kappa_2 M + \omega\kappa_3 \sum_{j=1}^{M} a_{ij} \left|x_i(t) - x_j(t)\right| + D \Bigg) \\
&- \frac{1}{2}\kappa_1 \sum_{i=1}^{M} \sum_{j=1}^{M} a_{ij} \big(\xi_i(t) - \xi_j(t)\big) \big(\xi_i(t) - \xi_j(t)\big)^{\mu} \\
&- \frac{1}{2}\kappa_2 \sum_{i=1}^{M} \sum_{j=1}^{M} a_{ij} \big(\xi_i(t) - \xi_j(t)\big) \\
&* \tanh\big(\beta\big(\xi_i(t) - \xi_j(t)\big)\big) \\
&- \frac{1}{2}\kappa_3 \sum_{i=1}^{M} \sum_{j=1}^{M} a_{ij} \big(\xi_i(t) - \xi_j(t)\big) \big(\xi_i(t) - \xi_j(t)\big) \\
\le& \rho_2 \sum_{i=1}^{M} |\xi_i(t)|^2 + \sum_{i=1}^{M} |\xi_i(t)| (\omega\kappa_2 M + \rho_1 + D) \\
&+ \omega\kappa_1 \sum_{i=1}^{M} |\xi_i(t)| \sum_{j=1}^{M} a_{ij} \left|\xi_i(t) - \xi_j(t)\right|^{\mu} \\
&+ \omega\kappa_3 \sum_{i=1}^{M} |\xi_i(t)| \sum_{j=1}^{M} a_{ij} \left|\xi_i(t) - \xi_j(t)\right| \\
&- \frac{1}{2}\kappa_1 \sum_{i=1}^{M} \sum_{j=1}^{M} a_{ij} \left|\xi_i(t) - \xi_j(t)\right|^{\mu+1} \\
&- \frac{1}{2}\kappa_2 \sum_{i=1}^{M} \sum_{j=1}^{M} a_{ij} \left|\xi_i(t) - \xi_j(t)\right| + \frac{\kappa_2 \iota M^2}{2\beta}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\kappa_3\sum_{i=1}^{M}\sum_{j=1}^{M}a_{ij}\left|\xi_i(t)-\xi_j(t)\right|^2 \\
&\le -\alpha_3 V^{\frac{1}{2}}(t)-\alpha_4 V^{\frac{\mu+1}{2}}(t)+\frac{\kappa_2\iota M^2}{2\beta}. \qquad (30)
\end{aligned}
$$

Thus, $V(t)$ converges into a very small area within a fixed time. Because $\beta \gg 1$ can be chosen as we need, the small area can be made as small as desired.

Theorem 4: Under the measurement error (29), the Zeno behavior can be excluded.

Proof: Based on (29), one has

$$
\begin{aligned}
D^{+}|e_i(t)| &\le |\dot{e}_i(t)| \\
&= \left|\left(-\kappa_1\sum_{j=1}^{M}a_{ij}\left(x_i(t)-x_j(t)\right)^{\mu}\right.\right. \\
&\quad -\kappa_2\sum_{j=1}^{M}a_{ij}\tanh\left(\beta\left(x_i(t)-x_j(t)\right)\right) \\
&\quad \left.\left.-\kappa_3\sum_{j=1}^{M}a_{ij}\left(x_i(t)-x_j(t)\right)\right)'\right| \\
&\le \left|\sum_{j=1}^{M}a_{ij}\left(\kappa_1\mu\left(x_i(t)-x_j(t)\right)^{\mu-1}\right.\right. \\
&\quad \left.+\kappa_2\beta\left(1-\tanh^2\left(\beta\left(x_i(t)-x_j(t)\right)\right)\right)+\kappa_3\right) \\
&\quad *(u_i(t)+d_i(x_i(t),t)+f(x_i(t),t) \\
&\quad \left.-u_j(t)-d_j(x_i(t),t)-f\left(x_j(t),t\right)\right)\Big| \\
&\le \sum_{j=1}^{M}a_{ij}\left(\kappa_1\mu\left|\xi_i(t)-\xi_j(t)\right|^{\mu-1}+\kappa_2\beta+\kappa_3\right) \\
&\quad *\left(\left|u_i(t)-u_j(t)\right|+\rho_2\left|x_i(t)-x_j(t)\right|\right. \\
&\quad \left.+\rho_1+2D\right) \\
&\le \kappa_1\mu\rho_2\sum_{j=1}^{M}a_{ij}\left|\xi_i(t)-\xi_j(t)\right|^{\mu} \\
&\quad +(\kappa_2\beta+\kappa_3)\rho_2\sum_{j=1}^{M}a_{ij}\left|\xi_i(t)-\xi_j(t)\right| \\
&\quad +\sum_{j=1}^{M}a_{ij}\left(\kappa_1\mu\left|\xi_i(t)-\xi_j(t)\right|^{\mu-1}+\kappa_2\beta+\kappa_3\right) \\
&\quad *\left(\left|u_i(t)-u_j(t)\right|+\rho_1+2D\right) \\
&\le \kappa_1\mu\rho_2\left(4\lambda_M\left(\mathcal{L}_{2/\mu}\right)V(0)\right)^{\frac{\mu}{2}} \\
&\quad +(\kappa_2\beta+\kappa_3)\rho_2 M^{\frac{1}{2}}\left(4\lambda_M(\mathcal{L}_2)V(0)\right)^{\frac{1}{2}} \\
&\quad +\left(\kappa_1\mu\left(4\lambda_M\left(\mathcal{L}_{2/\mu-1}\right)V(0)\right)^{\frac{\mu-1}{2}}+M(\kappa_2\beta+\kappa_3)\right) \\
&\quad *\left(M\left|u_i\left(t_k^i\right)\right|+\left|\sum_{j=1}^{M}u_j\left(t_{k'}^j\right)\right|+\rho_1+2D\right) \\
&\le \vartheta_1\left(t_k^i\right)+\vartheta_2\left(t_{k'}^j\right) \qquad (31)
\end{aligned}
$$

where

$$
\begin{aligned}
\vartheta_1\left(t_k^i\right) &= \kappa_1\mu\rho_2\left(4\lambda_M\left(\mathcal{L}_{2/\mu}\right)V(0)\right)^{\frac{\mu}{2}} \\
&\quad +(\kappa_2\beta+\kappa_3)\rho_2 M^{\frac{1}{2}}\left(4\lambda_M(\mathcal{L}_2)V(0)\right)^{\frac{1}{2}} \\
&\quad +M\left(\kappa_1\mu\left(4\lambda_M\left(\mathcal{L}_{2/\mu-1}\right)V(0)\right)^{\frac{\mu-1}{2}}\right. \\
&\quad \left.+M(\kappa_2\beta+\kappa_3)\right)\left|u_i\left(t_k^i\right)\right| \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_2\left(t_{k'}^j\right) &= \left(\kappa_1\mu\left(4\lambda_M\left(\mathcal{L}_{2/\mu-1}\right)V(0)\right)^{\frac{\mu-1}{2}}\right. \\
&\quad \left.+M(\kappa_2\beta+\kappa_3)\right)\left(\left|\sum_{j=1}^{M}u_j\left(t_{k'}^j\right)\right|+\rho_1+2D\right) \qquad (33)
\end{aligned}
$$

$\mathcal{L}_{2/\mu-1}$ is the Laplacian matrix of graph $\mathcal{G}(\mathcal{A}^{2/\mu-1})$, and $t_{k'}^j$ is the last triggering instant of agent j. Since $e_i(t_k^i)=0$, taking (31) into account, one has

$$
\begin{aligned}
|e_i(t)| &\le \int_{t_k^i}^{t}\left|\dot{E}_i(s)\right|ds \\
&\le \int_{t_k^i}^{t}\left(\vartheta_1\left(t_k^i\right)+\vartheta_2\left(t_{k'}^j\right)\right)ds. \qquad (34)
\end{aligned}
$$

Based on the triggering condition and (31), we can find that

$$
\begin{aligned}
\left|e_i\left(t_{k+1}^i\right)\right| &= \omega\kappa_1\left|\sum_{j=1}^{M}a_{ij}x_i\left(t_{k+1}^i\right)-x_j\left(t_{k+1}^i\right)\right|^{\mu} \\
&\quad +\omega\kappa_2+\omega\kappa_3\left|\sum_{j=1}^{M}a_{ij}x_i\left(t_{k+1}^i\right)-x_j\left(t_{k+1}^i\right)\right| \\
&\le \int_{t_k^i}^{t_{k+1}^i}\left(\vartheta_1\left(t_k^i\right)+\vartheta_2\left(t_{k'}^j\right)\right)ds \\
&\le \int_{t_k^i}^{t_{k+1}^i}\left(\bar{\vartheta}_1+\bar{\vartheta}_2\right)ds \qquad (35)
\end{aligned}
$$

where $\bar{\vartheta}_1 = \max\{\vartheta_1(t_0^i),\vartheta_1(t_1^i),\ldots\}$ and $\bar{\vartheta}_2 = \max\{\vartheta_1(t_0^j),\vartheta_1(t_1^j),\ldots\}$. From (35), we have $t_{k+1}^i - t_k^i \ge (\omega\kappa_2)/(\bar{\vartheta}_1+\bar{\vartheta}_2)$, which implies that there is no Zeno behavior. ■

Remark 2: We can avoid the Zeno behavior by utilizing the tanh function, and the error $\xi_i(t)$ converges to a very small neighborhood of the origin as required, implying that the practical average consensus is reached.

B. Under Switching Topologies

Theorem 5: Under Assumptions 1 and 2, the switching topologies $\mathcal{G}_{\eta(t)}$, the controllers (24), and the triggering function (23), if the following inequalities holds:

$$
\begin{aligned}
\kappa_2\sqrt{\lambda_2^{\min}(\mathcal{L}_2)} &> (\omega\kappa_2 M+\rho_1+D)(2M)^{\frac{1}{2}} \\
M^{1-\mu}\left(\lambda_2^{\min}\left(\mathcal{L}_{2/\mu+1}\right)\right)^{\frac{\mu+1}{2}} &> \omega\sqrt{2}M\left(\lambda_M^{\max}\left(\mathcal{L}_{2/\mu}\right)\right)^{\frac{\mu}{2}} \\
\kappa_3\lambda_2^{\min}(\mathcal{L}) &> \rho_2+\omega\kappa_3\sqrt{2\lambda_M^{\max}(\mathcal{L}_2)}M \qquad (36)
\end{aligned}
$$

the fixed-time average consensus problem can be tackled, where $\lambda_2^{\min}(\mathcal{L}_2) = \min\{\lambda_2(\mathcal{L}_2(t_0)), \lambda_2(\mathcal{L}_2(t_1)), \ldots\}$, $\lambda_2^{\min}(\mathcal{L}_{2/\mu+1}) = \min\{\lambda_2(\mathcal{L}_{2/\mu+1}(t_0)), \lambda_2(\mathcal{L}_{2/\mu+1}(t_1)), \ldots\}$, $\lambda_M^{\max}(\mathcal{L}_{2/\mu}) = \max\{\lambda_M(\mathcal{L}_{2/\mu}(t_0)), \lambda_M(\mathcal{L}_{2/\mu}(t_1)), \ldots\}$, $\lambda_2^{\min}(\mathcal{L}) = \min\{\lambda_2(\mathcal{L}(t_0)), \lambda_2(\mathcal{L}(t_1)), \ldots\}$, $\lambda_M^{\max}(\mathcal{L}_2) = \max\{\lambda_M(\mathcal{L}_2(t_0)), \lambda_M(\mathcal{L}_2(t_1)), \ldots\}$, $\mathcal{L}_{2/\mu}(t)$ is the corresponding Laplacian matrix at time t.

Proof: For the same $V(t)$ of Theorem 3, we have

$$\begin{aligned}\dot{V}(t) \le & -\Big(\kappa_2\sqrt{\lambda_2(\mathcal{L}_2(t))} \\ & -(\omega\kappa_2 M + \rho_1 + D)(2M)^{\frac{1}{2}}\Big)V^{\frac{1}{2}}(t) \\ & -\frac{1}{2}\kappa_1 M^{1-\mu}\big(4\lambda_2\big(\mathcal{L}_{2/\mu+1}(t)\big)V(t)\big)^{\frac{\mu+1}{2}} \\ & +\omega\kappa_1\sqrt{2}M\big(4\lambda_M\big(\mathcal{L}_{2/\mu}(t)\big)\big)^{\frac{\mu}{2}}V^{\frac{\mu+1}{2}}(t) \\ & -2(\kappa_3\lambda_2(\mathcal{L}(t)) - \rho_2)V(t) \\ & +\omega\kappa_3\sqrt{2}M(4\lambda_M(\mathcal{L}_2(t)))^{\frac{1}{2}}V(t) \\ \le & -\Big(\kappa_2\sqrt{\lambda_2^{\min}(\mathcal{L}_2)} \\ & -(\omega\kappa_2 M + \rho_1 + D)(2M)^{\frac{1}{2}}\Big)V^{\frac{1}{2}}(t) \\ & -\frac{1}{2}\kappa_1 M^{1-\mu}\Big(4\lambda_2^{\min}\big(\mathcal{L}_{2/\mu+1}\big)V(t)\Big)^{\frac{\mu+1}{2}} \\ & +\omega\kappa_1\sqrt{2}M\big(4\lambda_M^{\max}\big(\mathcal{L}_{2/\mu}\big)\big)^{\frac{\mu}{2}}V^{\frac{\mu+1}{2}}(t) \\ & -2\Big(\kappa_3\lambda_2^{\min}(\mathcal{L}) - \rho_2\Big)V(t) \\ & +\omega\kappa_3\sqrt{2}M\big(4\lambda_M^{\max}(\mathcal{L}_2)\big)^{\frac{1}{2}}V(t) \\ \le & -\bar{\alpha}_3 V^{\frac{1}{2}}(t) - \bar{\alpha}_4 V^{\frac{\mu+1}{2}}(t)\end{aligned} \tag{37}$$

where $\bar{\alpha}_3 = \kappa_2\sqrt{\lambda_2^{\min}(\mathcal{L}_2)} - (\omega\kappa_2 + \rho_1 + D)(2M)^{(1/2)}$ and $\bar{\alpha}_4 = (1/2)\kappa_1 M^{1-\mu}(4\lambda_2^{\min}(\mathcal{L}_{2/\mu+1}))^{(\mu+1/2)} - \omega\kappa_1\sqrt{2}M(4\lambda_M^{\max}(\mathcal{L}_{2/\mu}))^{(\mu/2)}$.

Obviously, $\forall\eta(t) \in \Xi$, inequality (37) holds. Based on Theorems 2 and 3, the average consensus is reached, and we have

$$T(x) \le \frac{2}{\bar{\alpha}_3} + \frac{2}{\bar{\alpha}_4(\mu - 1)}. \tag{38}$$

■

Remark 3: Compared with the existing results [46], [47], [49]–[51], [54], [55], we extend the results to nonlinear MASs with switching topologies and intermittent communication. Moreover, the corresponding results obtained in [42]–[44] and [48] are a special case of this article.

C. Fixed-Time Average Consensus via Improved Triggering Function

The aforementioned consensus algorithms are developed under an assumption that each agent is required to get continuous information from its neighbors. The improved algorithms without continuous communication are proposed herein.

Under the improved event-based control, the measurement error is redesigned as

$$\Phi_i(t) = \int_{t_k^i}^{t}\Big(\vartheta_1\big(t_k^i\big) + \vartheta_2\Big(t_{k'}^j\Big)\Big)ds \tag{39}$$

and the triggering function is that

$$\Upsilon_i(t) = \Phi_i(t) - \Delta\big(t_k^i\big) \tag{40}$$

where

$$\begin{aligned}\Delta\big(t_k^i\big) = \frac{\omega}{1+\omega}\Bigg| & \kappa_1\sum_{j=1}^{M} a_{ij}\big(x_i\big(t_k^i\big) - x_j\big(t_k^i\big)\big)^{\mu} \\ & +\kappa_2\sum_{j=1}^{M} a_{ij}\mathrm{sign}\big(x_i\big(t_k^i\big) - x_j\big(t_k^i\big)\big) \\ & +\kappa_3\sum_{j=1}^{M} a_{ij}\big(x_i\big(t_k^i\big) - x_j\big(t_k^i\big)\big)\Bigg|.\end{aligned}$$

Then, the control protocol is the same as (24), and we can get t_{k+1}^i based on the triggering condition $t_{k+1}^i = \inf\{t > t_k^i | \Upsilon_i(t) \ge 0\}$.

Theorem 6: Under Assumptions 1 and 2, the controllers (24), and the triggering function (40), if the following inequalities holds:

$$\begin{aligned}\kappa_2\sqrt{\lambda_2(\mathcal{L}_2)} &> (\omega\kappa_2 M + \rho_1 + D)(2M)^{\frac{1}{2}} \\ M^{1-\mu}\big(\lambda_2\big(\mathcal{L}_{2/\mu+1}\big)\big)^{\frac{\mu+1}{2}} &> \omega\sqrt{2}M\big(\lambda_M\big(\mathcal{L}_{2/\mu}\big)\big)^{\frac{\mu}{2}} \\ \kappa_3\lambda_2(\mathcal{L}) &> \rho_2 + \omega\kappa_3\sqrt{2\lambda_M(\mathcal{L}_2)}M\end{aligned} \tag{41}$$

the practical event-based average consensus with intermittent communication is obtained under the fixed topology.

Proof: According to (34) and (39), we have

$$|e_i(t)| \le \Phi_i(t), \quad t \in [t_k^i, t_{k+1}^i). \tag{42}$$

Furthermore, the triggering function (40) enforces

$$\Phi_i(t) \le \Delta\big(t_k^i\big). \tag{43}$$

Substituting (43) into (42) then gives

$$|e_i(t)| \le \Delta\big(t_k^i\big). \tag{44}$$

Utilizing (29), we have that $|e_i(t)| \le \Delta(t_k^i)$ is a sufficient condition for

$$\begin{aligned}|e_i(t)| \le & \,\omega\kappa_1\Bigg|\sum_{j=1}^{M} a_{ij}\big(x_i(t) - x_j(t)\big)^{\mu}\Bigg| \\ & +\omega\kappa_2\Bigg|\sum_{j=1}^{M} a_{ij}\mathrm{sign}\big(x_i(t) - x_j(t)\big)\Bigg| \\ & +\omega\kappa_3\Bigg|\sum_{j=1}^{M} a_{ij}\big(x_i(t) - x_j(t)\big)\Bigg| \\ \le & \,\omega\kappa_1\sum_{j=1}^{M} a_{ij}\big|x_i(t) - x_j(t)\big|^{\mu} \\ & +\omega\kappa_2 M + \omega\kappa_3\sum_{j=1}^{M} a_{ij}\big|x_i(t) - x_j(t)\big|.\end{aligned} \tag{45}$$

According to (45), we can find that the improved triggering condition can guarantee the inequality $\Gamma_i(t) \le 0$ holds, and $\Gamma_i(t)$ is the triggering function (23). Hence, based on

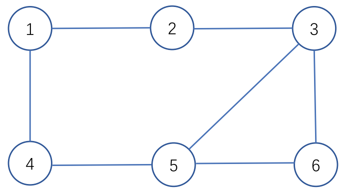

Fig. 1. Communication graph.

Theorems 3 and 4, the practical fixed-time event-based average consensus is obtained.

Moreover, based on Theorems 5 and 6, the corresponding results under the switching topologies can also be obtained as follows.

Theorem 7: Under Assumptions 1 and 2, the switching topologies $\mathcal{G}_{\eta(t)}$, the controllers (24), and the triggering function (40), if the following inequalities holds:

$$\kappa_2\sqrt{\lambda_2^{\min}(\mathcal{L}_2)} > (\omega\kappa_2 M + \rho_1 + D)(2M)^{\frac{1}{2}}$$
$$M^{1-\mu}\left(\lambda_2^{\min}(\mathcal{L}_{2/\mu+1})\right)^{\frac{\mu+1}{2}} > \omega\sqrt{2}M\left(\lambda_M^{\max}(\mathcal{L}_{2/\mu})\right)^{\frac{\mu}{2}}$$
$$\kappa_3\lambda_2^{\min}(\mathcal{L}) > \rho_2 + \omega\kappa_3\sqrt{2\lambda_M^{\max}(\mathcal{L}_2)M} \quad (46)$$

the practical consensus with intermittent communication is obtained under the switching topologies.

Remark 4: By the definition of $\Upsilon_i(t)$, we just use $\vartheta_1(t_k^i)$, $\vartheta_2(t_{k'}^j)$, $x_i(t_k^i)$, and $x_j(t_k^i)$ to determine t_{k+1}^i. Therefore, continuous communication is not required.

V. Simulation Results

The effectiveness of the algorithms is validated by three simulation examples. Example 1 shows the results of Section III. Example 2 shows the results of Section IV. Example 3 considers a practical system.

Example 1: For the delayed MASs without nonlinear terms, there are six agents, and $\tau = 0.06$, $d_i(x_i(t), t) = 0.2\cos(x_i(t))$. Hence, we have $D = 0.2$. We assume $x(0) = [-5 \;\; 0 \;\; 4 \;\; 9 \;\; -3 \;\; 2]^T$. For the fixed topology shown in Fig. 1, we obtain

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

and we can get $\lambda_2(\mathcal{L}) = \lambda_2(\mathcal{L}_2) = 1$.

Under the controllers (8), we set $\kappa_1 = 0.5$, $\kappa_2 = 3$, $\kappa_3 = 4$, $\mu = 7/5$, and $\beta = 100$, and these parameters satisfy the inequalities (9). Then, we have $T(x) \leq (2)/(\alpha_1)+(2)/(\alpha_2(\mu-1)) + \tau = 9.06$. Fig. 2 indicates the state evolution of the six agents, and the consensus is obtained within 0.5 s, which satisfies $T(x) \leq 9.06$. The control input evolution is shown in Fig. 3.

Consider the switching topologies in Fig. 4, and we can get $\lambda_2^{\min}(\mathcal{L}) = \lambda_2^{\min}(\mathcal{L}_2) = 0.885$. Note that the three graphs are connected. The system (4) begins from topology (1) and switches to topology (2) at $t = 0.1$ s, then switches to topology (3) at $t = 0.2$ s. Under the fixed-time consensus controllers (8), we set $\kappa_1 = 0.5$, $\kappa_2 = 3$, $\kappa_3 = 4$, $\mu = 7/5$, and $\beta = 100$, and these parameters satisfy the inequalities (19). Then, we have $T(x) \leq (2)/(\bar{\alpha}_1)+(2)/(\bar{\alpha}_2(\mu-1)) + \tau = 10.62$. Fig. 5 indicates the state evolution, and the consensus is obtained within 0.5 s, which satisfies $T(x) \leq 10.62$. The control input evolution is presented in Fig. 6.

Comparing Fig. 2 with Fig. 5, one has that the consensus can be reached while communication topology is switching. Compared with Fig. 3, Fig. 6 indicates the topologies switches at $t = 0.1$ s and $t = 0.2$ s.

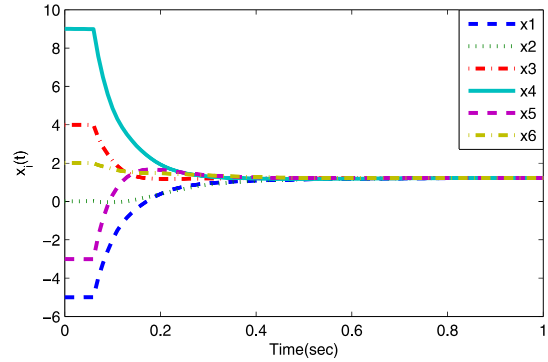

Fig. 2. State evolution of the six agents under the fixed topology.

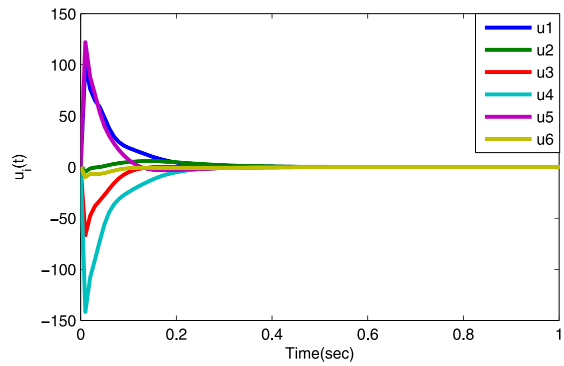

Fig. 3. Control input evolution of each agent under the fixed topology.

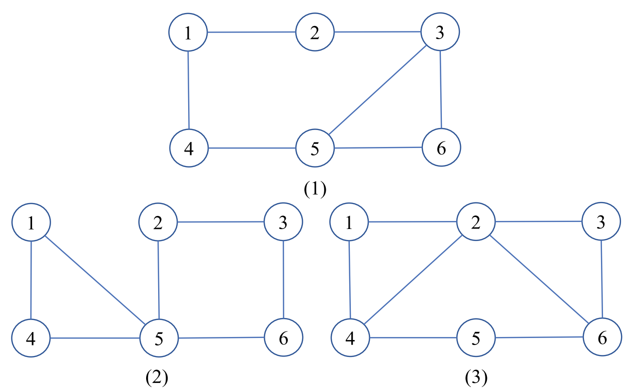

Fig. 4. Switching topologies.

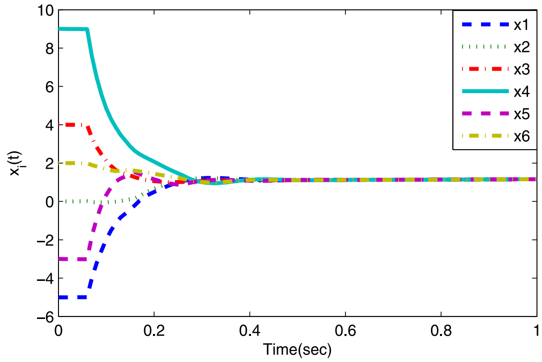

Fig. 5. State evolution of the six agents under the switching topologies.

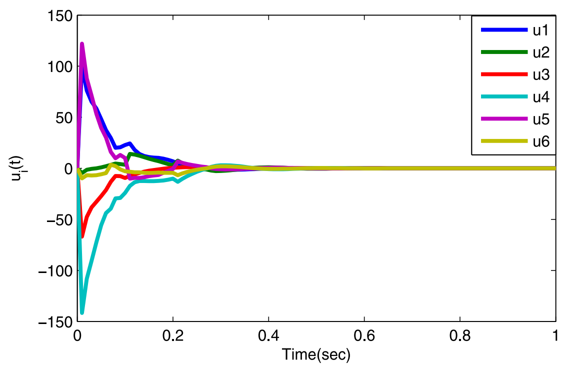

Fig. 6. Control input evolution of each agent under the switching topologies.

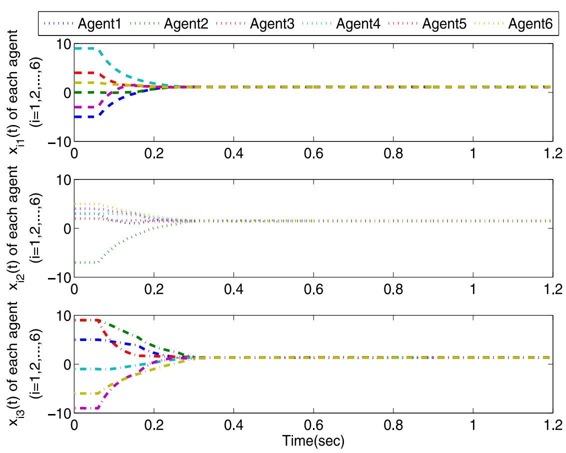

Fig. 7. State evolution of each agent with 3-D variables under the triggering function (23) and the switching topologies.

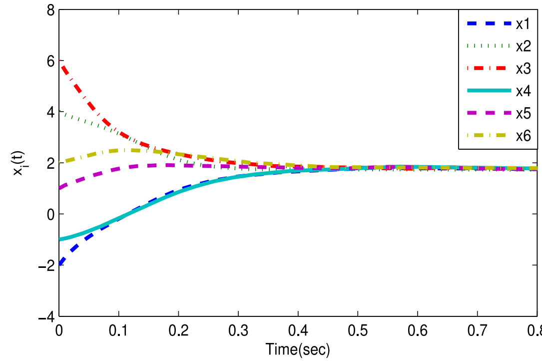

Fig. 8. State evolution of each agent under the triggering function (23) and the fixed topology.

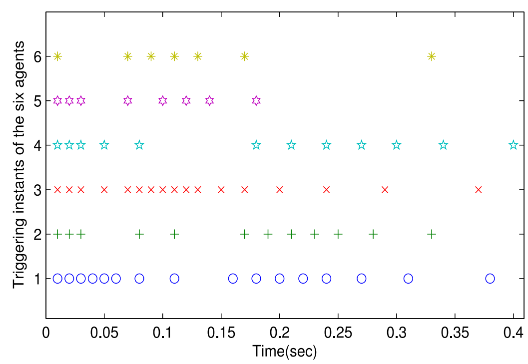

Fig. 9. Triggering instants of the six agents under the triggering function (23) and the fixed topology.

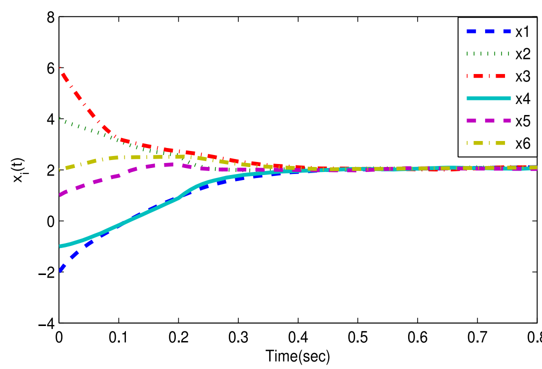

Fig. 10. State evolution of each agent under the triggering function (23) and the switching topologies.

For the delayed MASs with three-dimensional (3-D) dynamics, we assume that the initial states are as follows:

$$x_1(0)=\begin{bmatrix}-5\\3\\5\end{bmatrix}, x_2(0)=\begin{bmatrix}0\\-7\\9\end{bmatrix}, x_3(0)=\begin{bmatrix}4\\2\\9\end{bmatrix}$$
$$x_4(0)=\begin{bmatrix}9\\3\\-1\end{bmatrix}, x_5(0)=\begin{bmatrix}-3\\4\\9\end{bmatrix}, x_6(0)=\begin{bmatrix}2\\5\\-6\end{bmatrix}.$$

Choose the same parameters above. Fig. 7 indicates the state evolution of the six agents with 3-D variables under the switching topologies.

Example 2: For the MASs without input delay, there are six agents, and $f(x_i(t),t) = 0.2x_i(t) - 0.1\sin(x_i(t))$, and $d_i(x_i(t),t) = 0.2\cos(x_i(t))$. Hence, we have $\rho_1 = \rho_2 = D = 0.2$. We assume $x(0) = [-2 \quad 4 \quad 6 \quad -1 \quad 1 \quad 2]^T$. The fixed topology is shown in Fig. 1, and we can get $\lambda_2(\mathcal{L}_{2/\mu+1}) = 1$ and $\lambda_M(\mathcal{L}_{2/\mu}) = \lambda_M(\mathcal{L}_2) = 4.414$.

It is noteworthy that the parameters given in Theorems 4 and 6 may be conservative. Hence, we set $\kappa_1 = 0.5$, $\kappa_2 = 4$, $\kappa_3 = 3$, $\mu = 7/5$, $\beta = 100$, and $\omega = 0.05$. Fig. 8 indicates the state evolution of the six agents. The triggering instants of the six agents are demonstrated in Fig. 9.

Consider the switching topologies in Fig. 4, and we can get $\lambda_2^{\min}(\mathcal{L}_{2/\mu+1}) = 0.885$ and $\lambda_M^{\max}(\mathcal{L}_{2/\mu}) = \lambda_M^{\max}(\mathcal{L}_2) =$

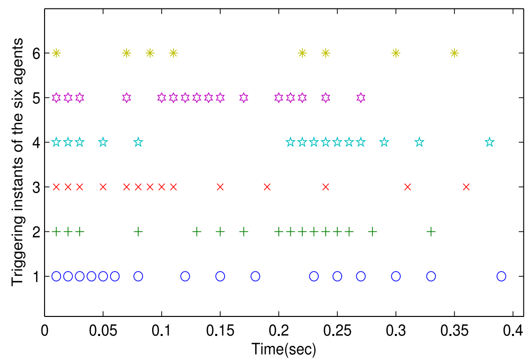

Fig. 11. Triggering instants of the six agents under the triggering function (23) and the switching topologies.

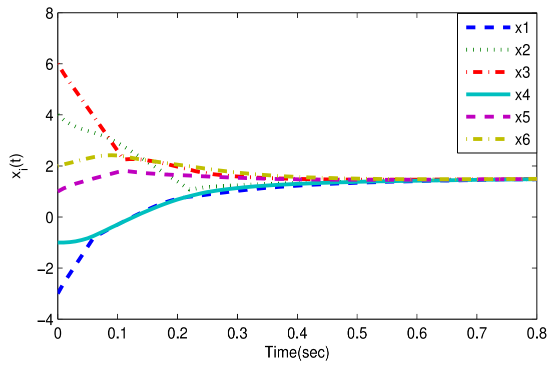

Fig. 12. State evolution of each agent under the triggering function (40) and the fixed topology.

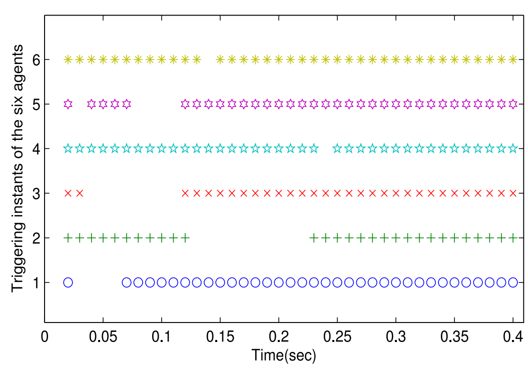

Fig. 13. Triggering instants of the six agents under the triggering function (40) and the fixed topology.

5.302. Note that the three graphs are connected. The system (4) begins from topology (1) and switches to topology (2) at $t = 0.1$ s, then switches to topology (3) at $t = 0.2$ s. Similarly, we set $\kappa_1 = 0.5$, $\kappa_2 = 4$, $\kappa_3 = 3$, $\mu = 7/5$, $\beta = 100$, and $\omega = 0.05$. Fig. 10 indicates the state evolution of the six agents. The triggering instants of the six agents are demonstrated in Fig. 11.

For the improved event-based function, we adopt the same controllers (24) and the fixed topology with $x(0) = [-3 \quad 4 \quad 6 \quad -1 \quad 1 \quad 2]^T$. Fig. 12 indicates the state evolution of the six agents. The triggering instants of the six agents are demonstrated in Fig. 13. The parameters in Theorem 3 may be conservative, and we can choose relaxed parameters to obtain good average consensus properties.

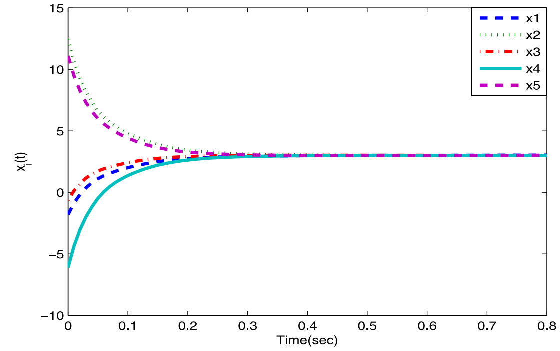

Fig. 14. State evolution of the five agents with the same initial value, communication topology, and control gains in [43].

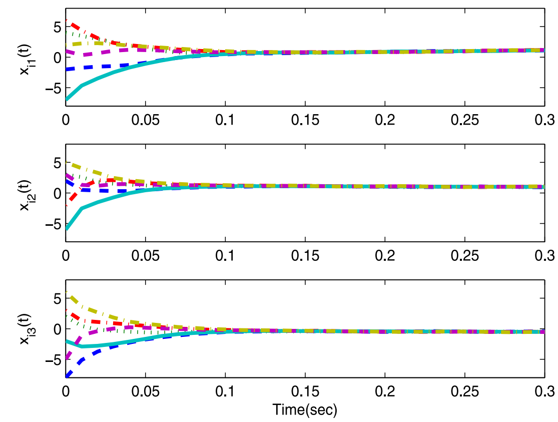

Fig. 15. State evolution of each agent under the triggering function (23).

Finally, we also make performance comparisons between the proposed method and some existing methods in [43]. Nair *et al.* [43] proposed the finite-time event-based consensus algorithms for linear MASs. To guarantee the reliability of the comparison, we adopted the same initial value, communication topology, and control gains. The corresponding simulation result based on our proposed method is shown in Fig. 14. Based on the simulation results, one has that we have a faster convergence rate. Moreover, we also consider the nonlinear terms and the switching topologies.

Example 3: Herein, each agent is assumed to be Chua's circuit [13] under a fixed topology. The dynamics of each agent is that

$$\dot{x}_i(t) = u_i(t) + f(x_i(t), t) + d_i(x_i(t), t) \tag{47}$$

where $f(x_i(t), t) = [\hat{\alpha}_1(x_{i2}(t) - x_{i1}(t) - \Lambda(x_{i1}(t))), x_{i1}(t) - x_{i2}(t) + x_{i3}(t), -\hat{\alpha}_2 x_{i2}(t) - \hat{\alpha}_3 x_{i3}(t)]^T$, $d_i(x_i(t), t) = [\sin(x_{i1}(t)), \cos(x_{i2}(t)), \sin(x_{i3}(t))]^T$, $x_i(t) = [x_{i1}(t), x_{i2}(t), x_{i3}(t)]^T \in \mathbb{R}^3$, and $\Lambda(x_{i1}(t)) = \hat{\beta}_1 x_{i1}(t) + 0.5(\hat{\beta}_1 - \hat{\beta}_2)(|x_{i1}(t) + 1| - |x_{i1}(t) - 1|)$. For Chua's circuit, we have $\hat{\alpha}_1 = -1.3018$, $\hat{\alpha}_2 = -0.0135$, $\hat{\alpha}_3 = -0.0297$, $\hat{\beta}_1 = -0.57$, $\hat{\beta}_2 = 0.1091$, $\rho_1 = 0$, $\rho_2 = 3$, and $D = 1$. Herein, we set $\kappa_1 = 3$, $\kappa_2 = 18$, $\kappa_3 = 12$, $\mu = 7/5$, $\beta = 100$, and $\omega = 0.05$. We assume that the initial states are

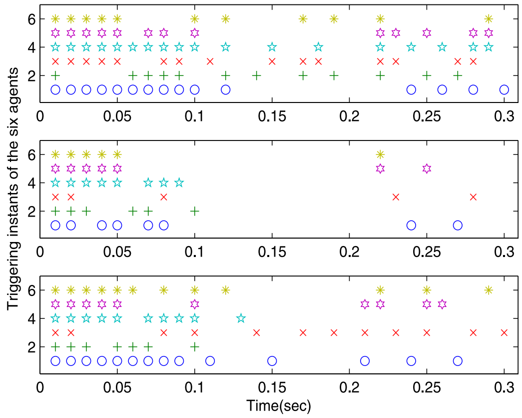

Fig. 16. Triggering instants under the triggering function (23).

as follows:

$$x_1(0)=\begin{bmatrix}-2\\2\\-8\end{bmatrix}, x_2(0)=\begin{bmatrix}4\\3\\2\end{bmatrix}, x_3(0)=\begin{bmatrix}6\\-2\\3\end{bmatrix}$$
$$x_4(0)=\begin{bmatrix}-7\\-6\\-2\end{bmatrix}, x_5(0)=\begin{bmatrix}1\\3\\-5\end{bmatrix}, x_6(0)=\begin{bmatrix}2\\5\\6\end{bmatrix}.$$

Fig. 15 indicates the state evolution of the six agents. The triggering instants are demonstrated in Fig. 16.

VI. Conclusion

The fixed-time average consensus controllers are presented for nonlinear MASs with input delay and switching topologies. The input delay, nonlinear dynamics, and switching topologies are considered simultaneously. Moreover, an event-based control scheme is presented to reach the fixed-time average consensus of nonlinear MASs with switching topologies, and no Zeno behavior occurs. Furthermore, an improved triggering function is designed, which can remove the assumption that agents need to broadcast continuously.

References

[1] X. Dong and G. Hu, "Time-varying formation tracking for linear multiagent systems with multiple leaders," *IEEE Trans. Autom. Control*, vol. 62, no. 7, pp. 3658–3664, Jul. 2017.

[2] Y. Zhao, Y. Liu, Z. Li, and Z. Duan, "Distributed average tracking for multiple signals generated by linear dynamical systems: An edge-based framework," *Automatica*, vol. 75, pp. 158–166, Jan. 2017.

[3] Z. Wang and D. Liu, "Data-based controllability and observability analysis of linear discrete-time systems," *IEEE Trans. Neural Netw.*, vol. 22, no. 12, pp. 2388–2392, Dec. 2011.

[4] R. Lu, W. Yu, J. Lü, and A. Xue, "Synchronization on complex networks of networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 11, pp. 2110–2118, Nov. 2014.

[5] Z. Ji, H. Lin, S. Cao, Q. Qi, and H. Ma, "The complexity in complete graphic characterizations of multiagent controllability," *IEEE Trans. Cybern.*, vol. 51, no. 1, pp. 64–76, Jan. 2021.

[6] Y. Wu, S. Zhuang, C. K. Ahn, and W. Li, "Aperiodically intermittent discrete-time state observation noise for consensus of multiagent systems," *IEEE Trans. Syst., Man, Cybern., Syst.*, early access, Sep. 10, 2020, doi: 10.1109/TSMC.2020.3018156. .

[7] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 950–955, Apr. 2010.

[8] S. Khoo, L. Xie, and Z. Man, "Robust finite-time consensus tracking algorithm for multirobot systems," *IEEE/ASME Trans. Mechatronics*, vol. 14, no. 2, pp. 219–228, Apr. 2009.

[9] S. Li, H. Du, and X. Lin, "Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics," *Automatica*, vol. 47, no. 8, pp. 1706–1712, 2011.

[10] Z. Wang, Y. Xu, R. Lu, and H. Peng, "Finite-time state estimation for coupled Markovian neural networks with sensor nonlinearities," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 28, no. 3, pp. 630–638, Mar. 2017.

[11] Y. Li, K. Li, and S. Tong, "Adaptive neural network finite-time control for multi-input and multi-output nonlinear systems with positive powers of odd rational numbers," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 31, no. 7, pp. 2532–2543, Jul. 2020.

[12] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Trans. Autom. Control*, vol. 57, no. 8, pp. 2106–2110, Aug. 2012.

[13] H. Hong, W. Yu, G. Wen, and X. Yu, "Distributed robust fixed-time consensus for nonlinear and disturbed multiagent systems," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 7, pp. 1464–1473, Jul. 2017.

[14] B. Ning, J. Jin, and J. Zheng, "Fixed-time consensus for multi-agent systems with discontinuous inherent dynamics over switching topology," *Int. J. Syst. Sci*, vol. 48, no. 10, pp. 2023–2032, 2017.

[15] Z. Zuo, "Nonsingular fixed-time consensus tracking for second-order multi-agent networks," *Automatica*, vol. 54, pp. 305–309, Apr. 2015.

[16] J. Fu and J. Wang, "Fixed-time coordinated tracking for second-order multi-agent systems with bounded input uncertainties," *Syst. Control Lett.*, vol. 93, pp. 1–12, Jul. 2016.

[17] B. Tian, H. Lu, Z. Zuo, and W. Yang, "Fixed-time leader-follower output feedback consensus for second-order multiagent systems," *IEEE Trans. Cybern.*, vol. 49, no. 4, pp. 1545–1550, Apr. 2019.

[18] Y. Huang and Y. Jia, "Fixed-time consensus tracking control of second-order multi-agent systems with inherent nonlinear dynamics via output feedback," *Nonlinear Dyn.*, vol. 91, no. 2, pp. 1289–1306, 2018.

[19] Z. Zuo, B. Tian, M. Defoort, and Z. Ding, "Fixed-time consensus tracking for multiagent systems with high-order integrator dynamics," *IEEE Trans. Autom. Control*, vol. 63, no. 2, pp. 563–570, Feb. 2018.

[20] Y. Wang and Y. Song, "Leader-following control of high-order multi-agent systems under directed graphs: Pre-specified finite time approach," *Automatica*, vol. 87, pp. 113–120, Jan. 2018.

[21] B. Ning and Q.-L. Han, "Prescribed finite-time consensus tracking for multiagent systems with nonholonomic chained-form dynamics," *IEEE Trans. Autom. Control*, vol. 64, no. 4, pp. 1686–1693, Apr. 2019.

[22] K. Zhao, Y. Song, and Y. Wang, "Regular error feedback based adaptive practical prescribed time tracking control of normal-form nonaffine systems," *J. Franklin Inst.*, vol. 356, no. 5, pp. 2759–2779, 2019.

[23] Y. Zhao, Y. Liu, G. Wen, W. Ren, and G. Chen, "Designing distributed specified-time consensus protocols for linear multiagent systems over directed graphs," *IEEE Trans. Autom. Control*, vol. 64, no. 7, pp. 2945–2952, Jul. 2019.

[24] H. Rezaee and F. Abdollahi, "Stationary consensus control of a class of high-order uncertain nonlinear agents with communication delays," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 49, no. 6, pp. 1285–1290, Jun. 2019.

[25] C. P. Chen, G.-X. Wen, Y.-J. Liu, and F.-Y. Wang, "Adaptive consensus control for a class of nonlinear multiagent time-delay systems using neural networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 6, pp. 1217–1226, Jun. 2014.

[26] Y. Li, F. Qu, and S. Tong, "Observer-based fuzzy adaptive finite-time containment control of nonlinear multiagent systems with input delay," *IEEE Trans. Cybern.*, vol. 51, no. 1, pp. 126–137, Jan. 2021.

[27] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[28] C. Wang, Z. Zuo, Z. Lin, and Z. Ding, "A truncated prediction approach to consensus control of Lipschitz nonlinear multi-agent systems with input delay," *IEEE Trans. Control Netw. Syst.*, vol. 4, no. 4, pp. 716–724, Dec. 2017.

[29] X. Li, X. Luo, J. Wang, and X. Guan, "Finite-time consensus for nonlinear multi-agent systems with time-varying delay: An auxiliary system approach," *J. Franklin Inst.*, vol. 355, no. 5, pp. 2703–2719, 2018.

[30] J. Ni, L. Liu, C. Liu, and J. Liu, "Fixed-time leader-following consensus for second-order multiagent systems with input delay," *IEEE Trans. Ind. Electron.*, vol. 64, no. 11, pp. 8635–8646, Nov. 2017.

[31] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *IEEE Trans. Autom. Control*, vol. 57, no. 5, pp. 1291–1297, May 2012.

[32] W. Hu, L. Liu, and G. Feng, "Consensus of linear multi-agent systems by distributed event-triggered strategy," *IEEE Trans. Cybern.*, vol. 46, no. 1, pp. 148–157, Jan. 2016.

[33] H. Li, X. Liao, T. Huang, and W. Zhu, "Event-triggering sampling based leader-following consensus in second-order multi-agent systems," *IEEE Trans. Autom. Control*, vol. 60, no. 7, pp. 1998–2003, Jul. 2015.

[34] W. Zhu, Z.-P. Jiang, and G. Feng, "Event-based consensus of multi-agent systems with general linear models," *Automatica*, vol. 50, no. 2, pp. 552–558, 2014.

[35] K. Liu, Z. Ji, and X. Zhang, "Periodic event-triggered consensus of multi-agent systems under directed topology," *Neurocomputing*, vol. 385, pp. 33–41, Apr. 2020.

[36] F. Xiao, T. Chen, and H. Gao, "Synchronous hybrid event-and time-driven consensus in multiagent networks with time delays," *IEEE Trans. Cybern.*, vol. 46, no. 5, pp. 1165–1174, May 2016.

[37] W. Zou, P. Shi, Z. Xiang, and Y. Shi, "Consensus tracking control of switched stochastic nonlinear multiagent systems via event-triggered strategy," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 31, no. 3, pp. 1036–1045, Mar. 2020.

[38] Z.-G. Wu, Y. Xu, R. Lu, Y. Wu, and T. Huang, "Event-triggered control for consensus of multiagent systems with fixed/switching topologies," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 48, no. 10, pp. 1736–1746, Oct. 2018.

[39] Y. Xu, M. Fang, Y.-J. Pan, K. Shi, and Z.-G. Wu, "Event-triggered output synchronization for nonhomogeneous agent systems with periodic denial-of-service attacks," *Int. J. Robust Nonlinear Control*, to be published.

[40] X. Wu and X. Mu, "Event-triggered control for networked nonlinear semi-Markovian jump systems with randomly occurring uncertainties and transmission delay," *Inf. Sci.*, vol. 487, pp. 84–96, Jun. 2019.

[41] Y. Xu and Z.-G. Wu, "Distributed adaptive event-triggered fault-tolerant synchronization for multiagent systems," *IEEE Trans. Ind. Electron.*, vol. 68, no. 2, pp. 1537–1547, Feb. 2021.

[42] H. Zhang, D. Yue, X. Yin, S. Hu, and C. Dou, "Finite-time distributed event-triggered consensus control for multi-agent systems," *Inf. Sci.*, vol. 339, pp. 132–142, Apr. 2016.

[43] R. R. Nair, L. Behera, and S. Kumar, "Event-triggered finite-time integral sliding mode controller for consensus-based formation of multirobot systems with disturbances," *IEEE Trans. Control Syst. Technol.*, vol. 27, no. 1, pp. 39–47, Jan. 2019.

[44] B. Hu, Z.-H. Guan, and M. Fu, "Distributed event-driven control for finite-time consensus," *Automatica*, vol. 103, pp. 88–95, May 2019.

[45] Z. Cao, C. Li, X. Wang, and T. Huang, "Finite-time consensus of linear multi-agent system via distributed event-triggered strategy," *J. Franklin Inst.*, vol. 355, no. 3, pp. 1338–1350, 2018.

[46] J. Liu, Y. Yu, J. Sun, and C. Sun, "Distributed event-triggered fixed-time consensus for leader–follower multiagent systems with nonlinear dynamics and uncertain disturbances," *Int. J. Robust Nonlinear Control*, vol. 28, no. 11, pp. 3543–3559, 2018.

[47] J. Liu, X. Chai, Y. Zhang, Y. Yu, and C. Sun, "Fixed-time event-triggered average consensus of nonlinear MASs with external disturbances and switching topologies," in *Proc. Chin. Control Conf. (CCC)*, 2019, pp. 6007–6012.

[48] Y. Dong and J. G. Xian, "Finite-time event-triggered consensus for non-linear multi-agent networks under directed network topology," *IET Control Theory Appl.*, vol. 11, no. 15, pp. 2458–2464, 2017.

[49] J. Liu, Y. Zhang, Y. Yu, and C. Sun, "Fixed-time event-triggered consensus for nonlinear multiagent systems without continuous communications," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 49, no. 11, pp. 2221–2229, Nov. 2019.

[50] T. Xu, G. Lv, Z. Duan, Z. Sun, and J. Yu, "Distributed fixed-time triggering-based containment control for networked nonlinear agents under directed graphs," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 67, no. 10, pp. 3541–3552, Oct. 2020.

[51] J. Liu, Y. Zhang, Y. Yu, and C. Sun, "Fixed-time leader-follower consensus of networked nonlinear systems via event/self-triggered control," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 31, no. 11, pp. 5029–5037, Nov. 2020.

[52] J. Liu, Y. Yu, H. He, and C. Sun, "Team-triggered practical fixed-time consensus of double-integrator agents with uncertain disturbance," *IEEE Trans. Cybern.*, early access, Jun. 25, 2020, doi: 10.1109/TCYB.2020.2999199.

[53] J. Ni, P. Shi, Y. Zhao, Q. Pan, and S. Wang, "Fixed-time event-triggered output consensus tracking of high-order multiagent systems under directed interaction graphs," *IEEE Trans. Cybern.*, early access, Nov. 25, 2020, doi: 10.1109/TCYB.2020.3034013.

[54] G. Ran, J. Liu, C. Li, L. Chen, and D. Li, "Event-based finite-time consensus control of second-order delayed multi-agent systems," *IEEE Trans. Circuits Syst. II Exp. Briefs*, vol. 68, no. 1, pp. 276–280, Jan. 2021.

[55] J. Liu, Y. Zhang, C. Sun, and Y. Yu, "Fixed-time consensus of multi-agent systems with input delay and uncertain disturbances via event-triggered control," *Inf. Sci.*, vol. 480, pp. 261–272, Apr. 2019.

[56] A. Filippov, *Differential Equations With Discontinuous Righthand Sides*. Dordrecht, The Netherlands: Kluwer Acad., 1988.

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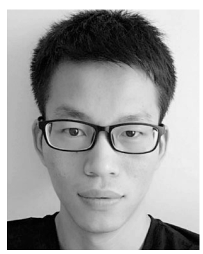

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