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Bipartite Consensus Tracking for Second-Order Multi-Agent Systems: A Time-Varying Function Based Preset-Time Approach

Boda Ning, *Member, IEEE*, Qing-Long Han, *Fellow, IEEE*, and Zongyu Zuo, *Senior Member, IEEE*

***Abstract*—This paper is concerned with bipartite consensus tracking for second-order multi-agent systems with signed directed graphs. A time-varying function based preset-time approach is proposed to realize the convergence in predetermined time. First, a class of time-varying functions with generalized properties are presented. Second, two time-varying function based auxiliaries and a corresponding manifold are constructed. Under a structurally balanced and strongly connected graph, a time-varying function based controller considering the neighboring state is proposed to guarantee that the system trajectory is constrained on the manifold such that bipartite consensus tracking is achieved in preset-time. Third, for first-order multi-agent systems, a preset-time controller is further developed with simplified design. Finally, numerical examples are provided to demonstrate the effectiveness of the proposed controllers.**

***Index Terms*—Multi-agent systems, bipartite consensus tracking, time-varying functions, preset-time consensus.**

I. INTRODUCTION

WITH recent advancements of micro-controllers and wireless communication, cooperative control of multi-agent systems has received considerable research interest. The cooperative capability enables agents to interact with each other to fulfill complex tasks, e.g., synchronization [1], optimal control in smart grids [2], and formation flying [3]. One fundamental research topic of cooperative control is consensus, which aims to achieve an agreement on states such as position and velocity [4]–[7]. In the presence of a leader, a *consensus tracking* problem is further investigated in [8]–[14], where followers track states of the leader. Note that the most aforementioned consensus tracking results are obtained with cooperative interactions among agents. In other words, the weights in the communication topology are nonnegative. As a result, all agents reach the same state at the end. However, there are counterexamples such as two-party political systems [15], where the interactions may be antagonistic. For example, once a political leader expresses an opinion on a specific incident, the members in his/her own party tend to agree, while the members in the opposite party usually disagree. In this case, both positive and negative interactions exist among different members. In practice, when implementing a traditional consensus tracking controller for multi-robot systems, sign flipping may occur due to communication errors or faulty processes [16]. In this case, it is important to investigate the robustness of consensus tracking in the presence of sign flipping. Following this line, an increasing number of collective dynamics, called *bipartite consensus tracking*, has been investigated in recent years [17]–[19]. For example, in [18], a distributed bipartite controller is designed for linear multi-agent systems and a fully distributed controller is further developed without using global information.

For first-order multi-agent systems, finite-time or fixed-time bipartite consensus is explored in [20]–[22] by taking the convergence time into account. For example, in [20], two classes of finite-time consensus protocols are developed by analyzing the Laplacian potential of signed networks. For second-order multi-agent systems, finite-time bipartite consensus is investigated in [23]–[25]. For example, in [23], a bipartite consensus controller is proposed with a directed signed topology and the homogeneous technique is used to solve the problem of finite-time bipartite consensus. Note that the convergence in [23]–[25] is realized in *finite-time*. In practice, there usually exists a requirement to complete a tracking task in preset-time. However, due to the use of a signum function and (or) a fractional power based function [23]–[25], it is a challenge to realize preset-time bipartite consensus for second-order multi-agent systems. It should be pointed out that the preset-time definition is different from the *fixed-time* one [20], [26]. Although the estimated settling time in fixed-time control does not depend on initial conditions, the estimate may be conservative due to the used fractional power approach. Motivated by the aforementioned observations, in this paper, we aim to solve the problem of bipartite consensus tracking for second-order multi-agent systems by using a time-varying function based *preset-time* approach.

The time-varying function based controllers are developed in the published literature, see [27]–[32]. For example, under a positive graph in [28], pre-specified finite-time leader-following consensus is achieved for high-order multi-agent systems by implementing an observer and a compensator. Note that the convergence is realized either for cooperative multi-agent systems in [28], [30], [32], or for a single system [27], [29], [31], which is different from the convergence achieved for competitive-cooperative multi-agent sys-

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B. Ning is with the Center for Smart Infrastructure and Digital Construction, Swinburne University of Technology, Melbourne, VIC 3122, Australia (e-mail: bodaning@swin.edu.au).

Q.-L. Han is with the the School of Software and Electrical Engineering, Swinburne University of Technology, Melbourne, VIC 3122, Australia (e-mail: qhan@swin.edu.au).

Z. Zuo is with the Seventh Research Division, Beihang University (BUAA), Beijing 100191, China (e-mail: zzybobby@buaa.edu.cn)

tems under a signed graph in this paper.

It should be pointed out that in [30], a class of new controllers are designed by using time-dependent functions. This work solves a consensus problem in preset-time. More importantly, two practical scenarios are considered in [30], i.e., initial states are inaccurate and disturbances are unavoidable. In these practical scenarios, the designed controllers can still perform very well and demonstrate their superiority. Inspired by the work in [29], [30], we present a class of generalized time-varying functions. Using two time-varying functions (including their first and second order derivatives) and neighboring states, under a signed directed graph, a new controller is proposed such that the system trajectory is constrained on a manifold. The manifold is constructed using time-varying function based auxiliaries, on which bipartite consensus tracking of second-order multi-agent systems is achieved in preset-time. Compared to some conventional controllers, the proposed one can achieve accurate bipartite consensus tracking in preset-time.

In the literature, there exists some other related work in achieving preset-time or prescribed-time control [28], [33]–[35]. In [28], an observer-based controller is proposed by using sophisticated time-varying functions to realize leader-following consensus successfully. By contrast, without using an observer, two concise time-varying functions are exploited in this paper to design a preset-time controller under a signed directed graph to achieve bipartite consensus tracking. In [33] and [34], the appointed-time consensus is achieved by using a motion-planning approach without a leader, which is different from the investigated work in this paper in the presence of a dynamic leader. In [35], time-varying gains are exploited to achieve consensus for second-order multi-agent systems, and the proposed time-varying gains suit for a predefined time frame. In other words, the gains are not well defined when the time is beyond the frame. Instead, time-varying functions in this paper are well defined on the whole time frame.

II. Preliminaries and Problem Formulation

A. Graph Theory of Signed Graphs

A network can be described by a signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ is a vertex set indexed by an associated agent set $\mathcal{N} = \{1, 2, \ldots, N\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} = \{(v_i, v_j) \mid v_i, v_j \in \mathcal{V}\}$ is the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix of the signed weights of $\mathcal{G}$ such that $a_{ij} \neq 0$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. It is assumed that no self-loops exist, i.e., $(v_i, v_i) \notin \mathcal{E}$, thus $a_{ii} = 0$. Note that $(v_j, v_i) \in \mathcal{E}$ indicates agent i can received information from agent j, but not necessarily vice versa. A graph with nonnegative weights is called a positive graph, which is a special case of the signed graph.

The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ of a signed graph $\mathcal{G}$ is defined by [20]

$$l_{ij} = \begin{cases} \sum_{j=1, j\neq i}^{N} |a_{ij}|, & j = i; \\ -a_{ij}, & j \neq i. \end{cases}$$

The signed graph $\mathcal{G}$ is said to be *structurally balanced* if there exists a bipartition $\{\mathcal{V}_1, \mathcal{V}_2\}$ of the vertices, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$, such that $a_{ij} \geq 0$ for $\forall v_i, v_j \in \mathcal{V}_q$ $(q \in \{1, 2\})$ and $a_{ij} \leq 0$ for $\forall v_i \in \mathcal{V}_q, v_j \in \mathcal{V}_{3-q}$ $(q \in \{1, 2\})$, and $\mathcal{G}$ is structurally unbalanced otherwise [23].

A directed path in the signed graph $\mathcal{G}$ is a finite sequence $v_{i_1}, v_{i_2}, \ldots, v_{i_j}$ of distinct vertices such that $(v_{i_k}, v_{i_{k+1}}) \in \mathcal{E}$, $\forall k = 1, 2, \ldots, j-1$. The signed graph $\mathcal{G}$ is said to be *strongly connected* if there exists a directed path between any two vertices. A directed cycle is a directed path starting from and ending at the same vertex. A directed cycle is positive (negative, respectively) if the product of all its signed edge weights is positive (negative, respectively).

The information of a reference trajectory for N followers can be treated as a command produced from a leader with index 0, which can be directly accessed by some followers. It is supposed that, the leader only sends out information and receives no information from any other followers. When the ith follower has direct information access from the leader, a notation, b_i, is set to be a positive number, and to be zero otherwise. Denote $\mathcal{B} = \mathrm{diag}\{b_1, b_2, \ldots, b_N\}$.

B. Problem Formulation

Consider a dynamical network consisting of N followers, of which the dynamics of the ith follower, $i \in \mathcal{N}$, is described by

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = u_i(t) + \epsilon_i(t), \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}^m$ and $v_i(t) \in \mathbb{R}^m$ represent the states of the ith follower, $u_i(t) \in \mathbb{R}^m$ is the control input to be designed, and $\epsilon_i(t) \in \mathbb{R}^m$ denotes some external disturbances.

Besides N followers, there exists a leader whose dynamics is described by

$$\begin{cases} \dot{x}_0(t) = v_0(t), \\ \dot{v}_0(t) = u_0(t), \end{cases} \tag{2}$$

where $x_0(t) \in \mathbb{R}^m$ and $v_0(t) \in \mathbb{R}^m$ represent the states of the leader, and $u_0(t) \in \mathbb{R}^m$ is the control input of the leader. In this paper, it is assumed that $u_0(t)$ is bounded.

Definition 1: The system described by (1)-(2) is said to achieve *preset-time* bipartite consensus tracking, if the following conditions of

$$\begin{cases} \lim_{t\to t_f} \|x_i(t) - x_0(t)\| = 0, & \forall i \in \mathcal{V}_1, \\ \|x_i(t) - x_0(t)\| = 0, \forall t > t_{\mathrm{f}}, & \forall i \in \mathcal{V}_1, \\ \lim_{t\to t_f} \|x_i(t) + x_0(t)\| = 0, & \forall i \in \mathcal{V}_2, \\ \|x_i(t) + x_0(t)\| = 0, \forall t > t_{\mathrm{f}}, & \forall i \in \mathcal{V}_2, \end{cases} \tag{3}$$

and

$$\begin{cases} \lim_{t\to t_f} \|v_i(t) - v_0(t)\| = 0, & \forall i \in \mathcal{V}_1, \\ \|v_i(t) - v_0(t)\| = 0, \forall t > t_{\mathrm{f}}, & \forall i \in \mathcal{V}_1, \\ \lim_{t\to t_f} \|v_i(t) + v_0(t)\| = 0, & \forall i \in \mathcal{V}_2, \\ \|v_i(t) + v_0(t)\| = 0, \forall t > t_{\mathrm{f}}, & \forall i \in \mathcal{V}_2, \end{cases} \tag{4}$$

are satisfied, where t_{f} can be predetermined.

The problem to be addressed in this paper is stated as: for the system described by (1)-(2), design a controller $u_i(t)$, $i \in \mathcal{N}$, to achieve *preset-time* bipartite consensus tracking.

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C. Preliminaries

First, an assumption is made about the interaction graph.

Assumption 1: The signed directed graph $\mathcal{G}$ corresponding to followers is strongly connected and structurally balanced. At least one follower has the direct information access from the leader.

Since the requirement of a structurally balanced signed graph is important to derive the main results in Section III, we provide the following lemma.

Lemma 1 ([17]): A strongly connected signed directed graph $\mathcal{G}$ is structurally balanced if and only if any of the following condition holds:

1) all directed cycles of $\mathcal{G}$ are positive;
2) there exists a diagonal matrix $D = \mathrm{diag}\{d_1, d_2, \ldots, d_N\}$ with $d_i \in \{\pm 1\}$ for all $i \in \mathcal{N}$ such that $D\mathcal{A}D$ has all nonnegative entries; and
3) 0 is an eigenvalue of $\mathcal{L}$, and $\mathrm{rank}(\mathcal{L}) = N - 1$.

Lemma 2 ([36]): Let $Z = [z_{ij}] \in \mathbb{R}^{N\times N}$ be a diagonally dominant matrix and

$$J = \left\{ i \in \{1, 2, \ldots, N\} : |z_{ii}| > \sum_{j=1, j\neq i} |z_{ij}| \right\} \neq \varnothing.$$

If for each $i \notin J$, there exists a sequence of nonzero elements of Z, i.e., $z_{ii_1}, z_{i_1 i_2}, \ldots, z_{i_r j}$ with $j \in J$, then Z is nonsingular.

Then, a new matrix is defined as $\mathcal{M} := \mathcal{L} + \mathcal{B}$. We have the following lemma.

Lemma 3: Under Assumption 1, $\mathcal{M}$ is nonsingular.

Proof: It is obvious that $\mathcal{L}$ is a diagonal dominant matrix. Due to the assumption that at least one follower has the direct information access from the leader and $\mathcal{G}$ is strongly connected, one obtains that there always exists a directed path from that follower(s) to all other followers, which implies that $\mathcal{M}$ is nonsingular by using Lemma 2. ■

Assumption 2: The external disturbance ϵ_i is bounded, $i \in \mathcal{N}$, satisfying

$$\|\epsilon_i\| \leq \bar{\epsilon}, \tag{5}$$

where $\bar{\epsilon}$ is a known positive constant.

III. Bipartite Consensus Tracking for Second-Order Multi-Agent Systems

In this section, by using a time-varying function based preset-time approach, we investigate the bipartite consensus tracking problem for second-order multi-agent systems.

A. Time-Varying Function

Before presenting our main results, two time-varying functions of $h_1(t)$ and $h_2(t)$ are presented, which satisfy the following generalized properties:

- $h_1(t)$ and $h_2(t)$ are at least C^2 on $(0, +\infty)$;
- $h_1(t)$ is continuous from an initial value $h_1(0) = 1$ to a terminal value $h_1(t_\mathrm{f}) = 0$, where $t_\mathrm{f} < +\infty$ is a prescribed time instant;
- $h_2(t)$ is continuous from an initial value $h_2(0) = 0$ to a terminal value $h_2(t_\mathrm{f}) = 0$;
- $\dot{h}_1(0) = \dot{h}_1(t_\mathrm{f}) = 0$. When $t > t_\mathrm{f}$, $h_1(t) = 0$ and thus $\dot{h}_1(t) = 0$; and
- $\dot{h}_2(0) = 1$ and $\dot{h}_2(t_\mathrm{f}) = 0$. When $t > t_\mathrm{f}$, $h_2(t) = 0$ and thus $\dot{h}_2(t) = 0$.

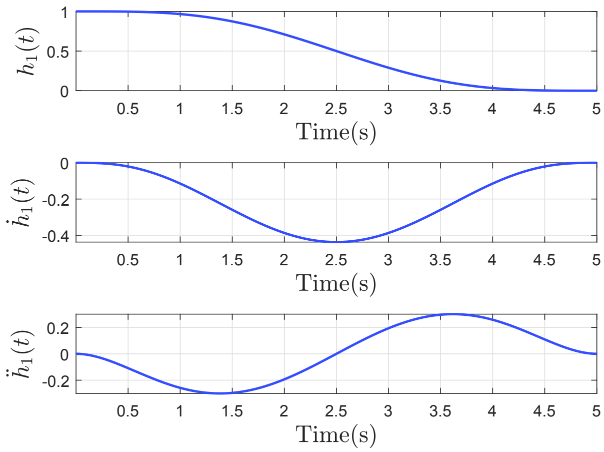

Fig. 1: An example of h_1.

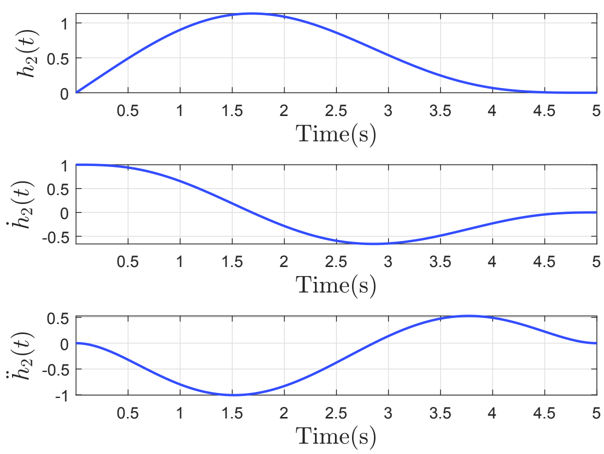

Fig. 2: An example of h_2.

Instead of using a specific form of $h_1(t)$ and $h_2(t)$, in this paper we utilize the above generalized properties to denote a class of time-varying functions. Detailed examples of $h_1(t)$ and $h_2(t)$ are provided as

$$h_1(t) = \begin{cases} \frac{20}{5^7}t^7 - \frac{70}{5^6}t^6 + \frac{84}{5^5}t^5 - \frac{35}{5^4}t^4 + 1, & 0 \leq t \leq t_\mathrm{f}; \\ 0, & t > t_\mathrm{f}, \end{cases} \tag{6}$$

and

$$h_2(t) = \begin{cases} \frac{10}{5^6}t^7 - \frac{36}{5^5}t^6 + \frac{45}{5^4}t^5 - \frac{20}{5^3}t^4 + t, & 0 \leq t \leq t_\mathrm{f}; \\ 0, & t > t_\mathrm{f}, \end{cases} \tag{7}$$

where $t_\mathrm{f} = 5$s. The Matlab plot of (6) and (7) together with their first-order and second-order derivatives is shown in Figs. 1 and 2.

Remark 1: It should be pointed out that a more general class of time-varying functions are designed in [29], using which predefined-time convergence has been successfully achieved for a single high-order system. From this sense, the presented $h_1(t)$ and $h_2(t)$ in this paper can be regarded as a special case of the functions designed in [29]. It is worth mentioning that there is a $h_3(t)$ presented in Section IV for first-order multi-agent systems. More detail is provided in Remark 3.

B. Preset-Time Bipartite Consensus Tracking Without Disturbances

In this section, we investigate the preset-time bipartite consensus tracking problem without considering disturbances, i.e., $\epsilon_i(t) = 0$ in (1). The following two error auxiliaries for the ith agent are defined:

$$\begin{cases} e_i^x(t) = \sum_{j=1}^{N} |a_{ij}|(x_i(t) - \mathrm{sign}(a_{ij})x_j(t)) \\ \qquad +b_i(x_i(t) - d_i x_0(t)), \\ e_i^v(t) = \sum_{j=1}^{N} |a_{ij}|(v_i(t) - \mathrm{sign}(a_{ij})v_j(t)) \\ \qquad +b_i(v_i(t) - d_i v_0(t)). \end{cases} \tag{8}$$

Based on the time-varying functions $h_1(t)$ and $h_2(t)$ in Section III-A, a controller for the ith agent, $i \in \mathcal{N}$, is designed in the form of

$$\begin{aligned} u_i(t) = \left(\sum_{j=1}^{N} |a_{ij}| + b_i\right)^{-1} \Bigg(\sum_{j=1}^{N} |a_{ij}|\mathrm{sign}(a_{ij})u_j(t) \\ + b_i d_i u_0(t) + \ddot{h}_1(t)e_i^x(0) + \ddot{h}_2(t)e_i^v(0) \\ - \lambda(e_i^v(t) - \dot{h}_1(t)e_i^x(0) - \dot{h}_2(t)e_i^v(0))\Bigg). \end{aligned} \tag{9}$$

We now state and establish the following theorem.

Theorem 1: Under Assumption 1, with $\epsilon_i(t) = 0$, the second-order system described by (1)-(2) realizes preset-time bipartite consensus tracking by using controller (9).

Proof: Two auxiliaries are defined as

$$\begin{cases} \xi_i^x(t) = e_i^x(t) - h_1(t)e_i^x(0) - h_2(t)e_i^v(0), \\ \xi_i^v(t) = e_i^v(t) - \dot{h}_1(t)e_i^x(0) - \dot{h}_2(t)e_i^v(0). \end{cases} \tag{10}$$

Next, construct a variable of

$$s_i(t) = \lambda \xi_i^x(t) + \xi_i^v(t), \tag{11}$$

where λ is a positive constant. Correspondingly, construct a manifold of $S = \{(x_1, x_2, \ldots, x_N, v_1, v_2, \ldots, v_N) \mid s_i(t) = 0, \forall i = 1, 2, \ldots, N\}$.

The derivative of $s_i(t)$ is calculated as

$$\begin{aligned} \dot{s}_i(t) = \lambda\Big(e_i^v(t) - \dot{h}_1(t)e_i^x(0) - \dot{h}_2(t)e_i^v(0)\Big) \\ + \dot{e}_i^v(t) - \ddot{h}_1(t)e_i^x(0) - \ddot{h}_2(t)e_i^v(0). \end{aligned} \tag{12}$$

Note that

$$\begin{aligned} \dot{e}_i^v(t) &= \sum_{j=1}^{N} |a_{ij}|(\dot{v}_i(t) - \mathrm{sign}(a_{ij})\dot{v}_j(t)) \\ &\quad + b_i(\dot{v}_i(t) - d_i\dot{v}_0(t)) \\ &= \sum_{j=1}^{N} |a_{ij}|(u_i(t) - \mathrm{sign}(a_{ij})u_j(t)) \\ &\quad + b_i(u_i(t) - d_i u_0(t)) \\ &= \left(\sum_{j=1}^{N} |a_{ij}| + b_i\right)u_i(t) - \sum_{j=1}^{N} |a_{ij}|\mathrm{sign}(a_{ij})u_j(t) \\ &\quad - b_i d_i u_0(t). \end{aligned} \tag{13}$$

Substituting (13) into (12), one gets

$$\begin{aligned} \dot{s}_i(t) = \lambda\Big(e_i^v(t) - \dot{h}_1(t)e_i^x(0) - \dot{h}_2(t)e_i^v(0)\Big) \\ + \left(\sum_{j=1}^{N} |a_{ij}| + b_i\right)u_i(t) - \sum_{j=1}^{N} |a_{ij}|\mathrm{sign}(a_{ij})u_j(t) \\ - b_i d_i u_0(t) - \ddot{h}_1(t)e_i^x(0) - \ddot{h}_2(t)e_i^v(0). \end{aligned} \tag{14}$$

Substituting (9) into (14), one obtains

$$\dot{s}_i(t) = 0,$$

which implies that $s_i(t)$ is constrained to $s_i(0)$. In (11), it can be calculated that $s_i(0) = 0$ due to the fact that $\xi_i^x(0) = 0$ and $\xi_i^v(0) = 0$. Therefore, the system trajectory is constrained on the manifold S from initial time. Then, (11) can be rewritten as

$$\dot{\xi}_i^x(t) = -\lambda \xi_i^x(t). \tag{15}$$

Since $\xi_i^x(0) = 0$, $\xi_i^x(t) = 0$ holds for $t \geq 0$, which further indicates that $\xi_i^v(t) = 0$ for $t \geq 0$. Then, from (10), one obtains that $e_i^x(t) = 0$ and $e_i^v(t) = 0$ for $t \geq t_{\mathrm{f}}$.

In what follows, we prove that $e_i^x(t) = 0$ ($e_i^v(t) = 0$, respectively) equivalents to $\psi_i(t) := x_i(t) - d_i x_0 = 0$ ($\varphi_i(t) := v_i(t) - d_i v_0 = 0$, respectively).

Let $e^x(t) = [(e_1^x(t))^T, (e_2^x(t))^T, \ldots, (e_N^x(t))^T]^T$, $e^v(t) = [(e_1^v(t))^T, (e_2^v(t))^T, \ldots, (e_N^v(t))^T]^T$, $\psi(t) = [\psi_1^T(t), \psi_2^T(t), \ldots, \psi_N^T(t)]^T$, and $\varphi(t) = [\varphi_1^T(t), \varphi_2^T(t), \ldots, \varphi_N^T(t)]^T$. Stacking the two error auxiliaries in (8), one obtains the corresponding vector forms of

$$\begin{cases} e^x(t) = \mathcal{M}\psi(t), \\ e^v(t) = \mathcal{M}\varphi(t). \end{cases} \tag{16}$$

By using Lemma 3, $\mathcal{M}$ is nonsingular. Since $e^x(t) = e^v(t) = 0$ for $t \geq t_{\mathrm{f}}$, it can be obtained that $\psi(t) = \varphi(t) = 0$ for $t \geq t_{\mathrm{f}}$. Therefore, the preset-time bipartite consensus tracking is achieved. ■

Remark 2: In some existing work [6], [12], [13], the fixed-time bound can only be adjusted through changing design parameters. Instead, t_{f} in this paper can be preset without direct dependance on design parameters. Although controller (9) is dependent on initial states, i.e., $e_i^x(0)$ and $e_i^v(0)$, it does not conflict with the focus of this paper according to Definition 1, which is to obtain a preset settling time. In fact, by using controller (9), the obtained t_{f} itself is not directly dependent on initial states and design parameters. It should be pointed out that the approach in this paper may not achieve preset-time convergence if there exist difficulties in getting accurate initial states. How to realize preset-time bipartite consensus tracking without using accurate initial states deserves further investigation in future work.

C. Preset-Time Bipartite Consensus Tracking With Disturbances

In this section, in the presence of disturbances, i.e., $\epsilon_i(t) \neq 0$ in (1), we propose a controller for the ith agent, $i \in \mathcal{N}$, in the form of

$$\begin{aligned} u_i(t) = \Big(\sum_{j=1}^{N}|a_{ij}| + b_i\Big)^{-1}\Big(&\sum_{j=1}^{N}|a_{ij}|\text{sign}(a_{ij})u_j(t) \\ &+ b_i d_i u_0(t) + \ddot{h}_1(t)e_i^x(0) + \ddot{h}_2(t)e_i^v(0) \\ &- \lambda(e_i^v(t) - \dot{h}_1(t)e_i^x(0) - \dot{h}_2(t)e_i^v(0)) \\ &- \vartheta_i \text{sign}(s_i(t))\Big), \end{aligned} \tag{17}$$

where $\vartheta_i \geq \big(2\sum_{j=1}^{N}|a_{ij}|+b_i\big)\bar{\epsilon}$. Note that a_{ij} and b_i are local information available to agent i. We now present the following theorem.

Theorem 2: Under Assumptions 1 and 2, the second-order system described by (1)-(2) realizes preset-time bipartite consensus tracking by using controller (17).

Proof: In the presence of disturbances, the derivative of $e_i^v(t)$ is calculated as

$$\begin{aligned} \dot{e}_i^v(t) = &\Big(\sum_{j=1}^{N}|a_{ij}| + b_i\Big)u_i(t) - \sum_{j=1}^{N}|a_{ij}|\text{sign}(a_{ij})u_j(t) \\ &+ \Big(\sum_{j=1}^{N}|a_{ij}| + b_i\Big)\epsilon_i(t) - \sum_{j=1}^{N}|a_{ij}|\text{sign}(a_{ij})\epsilon_j(t) \\ &- b_i d_i u_0(t). \end{aligned} \tag{18}$$

Substituting (18) into (12), one gets

$$\begin{aligned} \dot{s}_i(t) = &\lambda\Big(e_i^v(t) - \dot{h}_1(t)e_i^x(0) - \dot{h}_2(t)e_i^v(0)\Big) \\ &+ \Big(\sum_{j=1}^{N}|a_{ij}| + b_i\Big)u_i(t) - \sum_{j=1}^{N}|a_{ij}|\text{sign}(a_{ij})u_j(t) \\ &+ \Big(\sum_{j=1}^{N}|a_{ij}| + b_i\Big)\epsilon_i(t) - \sum_{j=1}^{N}|a_{ij}|\text{sign}(a_{ij})\epsilon_j(t) \\ &- b_i d_i u_0(t) - \ddot{h}_1(t)e_i^x(0) - \ddot{h}_2(t)e_i^v(0). \end{aligned} \tag{19}$$

Substituting (17) into (19), one obtains

$$\begin{aligned} \dot{s}_i(t) = &\Big(\sum_{j=1}^{N}|a_{ij}| + b_i\Big)\epsilon_i(t) - \sum_{j=1}^{N}|a_{ij}|\text{sign}(a_{ij})\epsilon_j(t) \\ &- \vartheta_i \text{sign}(s_i(t)). \end{aligned} \tag{20}$$

Construct a Lyapunov function candidate

$$V(t) = \frac{1}{2}s_i^T(t)s_i(t).$$

The derivative of $V(t)$ along (20) is

$$\begin{aligned} \dot{V}(t) = &\Big(\sum_{j=1}^{N}|a_{ij}| + b_i\Big)\epsilon_i^T(t)s_i(t) \\ &- \sum_{j=1}^{N}|a_{ij}|\text{sign}(a_{ij})\epsilon_j^T(t)s_i(t) - \vartheta_i\|s_i(t)\|_1 \\ \leq &\Big(2\sum_{j=1}^{N}|a_{ij}| + b_i\Big)\|\epsilon_i(t)\|\|s_i(t)\| - \vartheta_i\|s_i(t)\| \\ \leq &-\Big(\vartheta_i - (2\sum_{j=1}^{N}|a_{ij}| + b_i)\bar{\epsilon}\Big)\|s_i(t)\| \leq 0. \end{aligned} \tag{21}$$

Since $s_i(0) = 0$, the system trajectory is constrained on the manifold S from initial time. The remaining steps follow that in Theorem 1. ■

IV. Bipartite Consensus Tracking for First-Order Multi-Agent Systems

In this section, we investigate the bipartite consensus tracking problem for first-order multi-agent systems. The dynamics of the ith follower, $i \in \mathcal{N}$, is described by

$$\dot{x}_i(t) = u_i(t), \tag{22}$$

where $x_i(t) \in \mathbb{R}^m$ represents the states of the ith follower and $u_i(t) \in \mathbb{R}^m$ is the control input to be designed.

The dynamics of the leader is described by

$$\dot{x}_0(t) = u_0(t), \tag{23}$$

where $x_0(t) \in \mathbb{R}^m$ represents the states of the leader and $u_0(t) \in \mathbb{R}^m$ is a bounded control input of the leader. The system described by (22)-(23) is said to achieve preset-time bipartite consensus tracking if condition (3) is satisfied.

An auxiliary for the ith agent is defined:

$$\phi_i^x(t) = e_i^x(t) - h_3(t)e_i^x(0), \tag{24}$$

where $e_i^x(t)$ is defined in (8), and $h_3(t)$ is a time-varying function satisfying: i) $h_3(t)$ is at least C^1 on $(0, +\infty)$; ii) $h_3(t)$ is continuous from an initial value $h_3(0) = 1$ to a terminal value $h_3(t_\text{f}) = 0$; and iii) when $t > t_\text{f}$, $h_3(t) = 0$.

The derivative of $\phi_i^x(t)$ is calculated as

$$\begin{aligned} \dot{\phi}_i^x(t) = &\sum_{j=1}^{N}|a_{ij}|(\dot{x}_i(t) - \text{sign}(a_{ij})\dot{x}_j(t)) \\ &+ b_i(\dot{x}_i(t) - d_i\dot{x}_0(t)) - \dot{h}_3(t)e_i^x(0) \\ = &\Big(\sum_{j=1}^{N}|a_{ij}| + b_i\Big)u_i(t) - \sum_{j=1}^{N}|a_{ij}|\text{sign}(a_{ij})u_j(t) \\ &- b_i d_i u_0(t) - \dot{h}_3(t)e_i^x(0). \end{aligned} \tag{25}$$

A controller for the ith agent, $i \in \mathcal{N}$, is designed in the form of

$$\begin{aligned} u_i(t) = \Big(\sum_{j=1}^{N}|a_{ij}| + b_i\Big)^{-1}\Big(&\sum_{j=1}^{N}|a_{ij}|\text{sign}(a_{ij})u_j(t) \\ &+ b_i d_i u_0(t) + \dot{h}_3(t)e_i^x(0)\Big). \end{aligned} \tag{26}$$

Substituting (26) into (25), one obtains that $\dot{\phi}_i^x(t) = 0$. Since $\phi_i^x(0) = 0$ due to the fact that $h_3(0) = 1$, one concludes that $\phi_i^x(t) = 0$ for $t \geq 0$. As a result, from (24), one obtains that $e_i^x(t) = 0$ for $t \geq t_\text{f}$ due to the fact that $h_3(t) = 0$ for $t \geq t_\text{f}$. Since $e^x(t) = \mathcal{M}\psi(t)$, where $\mathcal{M}$ is nonsingular, it can be obtained that $\psi(t) = 0$ for $t \geq t_\text{f}$. Therefore, condition (3) is satisfied. In other words, the preset-time bipartite consensus tracking is achieved. The results of this section can be summarized as the following theorem.

Theorem 3: Under Assumption 1, the first-order system described by (22)-(23) realizes preset-time bipartite consensus tracking by using controller (26).

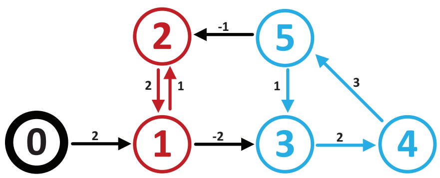

Fig. 3: The interaction topology.

Remark 3: Note that $h_3(t)$ is different from $h_1(t)$ due to $h_3(t)$ is at least C^1 on $(0,+\infty)$, while $h_1(t)$ is required to be at least C^2. In this paper, all agents use the same $h_1(t)$, $h_2(t)$, or $h_3(t)$. In fact, nonidentical time-varying functions can be implemented for different agents. For example, $h_{1i}(t)$, $i \in \mathcal{N}$, can be used for the ith agent as long as $h_{11}(t)$, $h_{12}(t)$, ..., $h_{1N}(t)$ hold the generalized properties presented in Section III-A with the same t_f.

Remark 4: For second-order multi-agent systems in Section III-B, two time-varying functions of $h_1(t)$ and $h_2(t)$ are exploited. On top of the two time-varying functions, a manifold of S is constructed to realize the preset-time bipartite consensus tracking. By contrast, for first-order multi-agent systems in Section IV, a single time-varying function of $h_3(t)$ is exploited without involving the manifold. In the presence of disturbances (i.e., the dynamics of the ith follower, $i \in \mathcal{N}$, is described by $\dot{x}_i(t) = u_i(t) + \epsilon_i(t)$), the controller design for first-order multi-agent systems follows similar steps in Section III-C.

Remark 5: In [32], a practical consensus controller is proposed for first-order multi-agent systems by using a time base generator. Note that the results obtained in Section IV are different from that in [32]. First, in [32] the tracking error converges to a neighborhood of the origin. In contrast, as shown in Theorem 3, agents achieve the accurate convergence. Compared to the practical convergence, the zero-error convergence is more favorable. For example, due to some cyber attacks, allies can become enemies in an unmanned aerial vehicle network, and the enemies aim to steal the accurate information of consensus state to launch a further attack in order to destroy the whole network. On the other hand, remaining cooperative allies tend to achieve the accurate convergence to complete allocated tasks such as tracking a military target. Second, in [32] consensus tracking is achieved with conventional positive graphs, while in this section the bipartite consensus tracking is realized with signed graphs.

Remark 6: From the theoretical point of view, the proposed time-varying controllers in this paper can guarantee system stability. In practice, when implementing these controllers, there exists a communication loop problem due to the decoupling design. One possible solution would be adding a communication buffer. Then, each agent can use neighbors' input at the last sampling time instant. As long as the sampling period is small, system stability can be guaranteed.

V. Numerical Examples

In this section, two examples are provided to demonstrate the effectiveness of the proposed controllers. We consider a

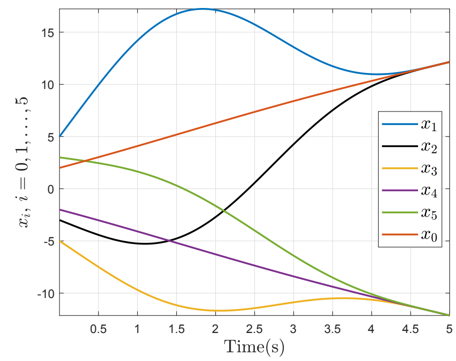

Fig. 4: Second-order multi-agent system–x_i.

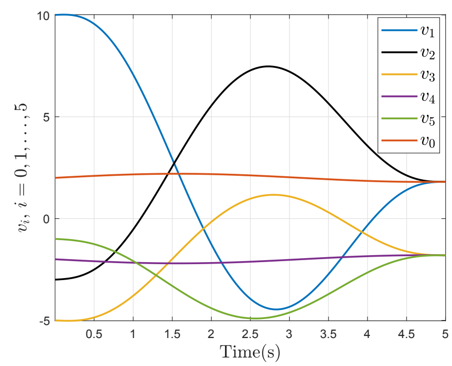

Fig. 5: Second-order multi-agent system–v_i.

multi-agent systems with five followers ($N = 5$) and one leader. A signed directed graph shown in Fig. 3 is used as the interaction topology, and its associated matrix $\mathcal{L}$ is denoted as

$$\mathcal{L} = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 3 & 0 & -1 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & -3 & 3 \end{bmatrix}.$$

Using condition 2) in Lemma 1, it can be obtained that $d_1 = d_2 = 1$, $d_3 = d_4 = d_5 = -1$. The input for the leader is $u_0(t) = 0.2\cos(t)$.

Example 1: Bipartite consensus tracking for second-order multi-agent systems.

Two time-varying functions $h_1(t)$ and $h_2(t)$ in the form of (6) and (7) with $t_f = 5$s are used in this example. The initial conditions are set as $x_1(0) = 5$, $x_2(0) = -3$, $x_3(0) = -5$, $x_4(0) = -2$, $x_5(0) = 3$, $x_0(0) = 2$; $v_1(0) = 10$, $v_2(0) = -3$, $v_3(0) = -5$, $v_4(0) = -2$, $v_5(0) = -1$, $v_0(0) = 2$. Implementing controller (9) with $N = 5$, the results are shown in Figs. 4 and 5, where the bipartite consensus tracking is achieved at 5s. Then, change the initial conditions to $x_1(0) = 20$, $x_2(0) = 25$, $x_3(0) = 15$, $x_4(0) = 30$, $x_5(0) = -25$, $x_0(0) = -20$; $v_1(0) = -5$, $v_2(0) = 9$, $v_3(0) = 20$, $v_4(0) = 4$, $v_5(0) = -15$, $v_0(0) = -2$. Moreover, disturbances are added with $\epsilon_1 = \epsilon_3 = \epsilon_5 = 2\cos(t)$ and $\epsilon_2 = \epsilon_4 = 2\sin(t)$. Due to $\vartheta_i \geq \left(2\sum_{j=1}^{N}|a_{ij}| + b_i\right)\bar{\epsilon}$, we select $\vartheta_1 = \vartheta_3 = \vartheta_5 = 12$, and $\vartheta_2 = \vartheta_4 = 8$. Implementing controller (17) with $N = 5$, the results are shown in Figs. 6 and 7. It can be found that

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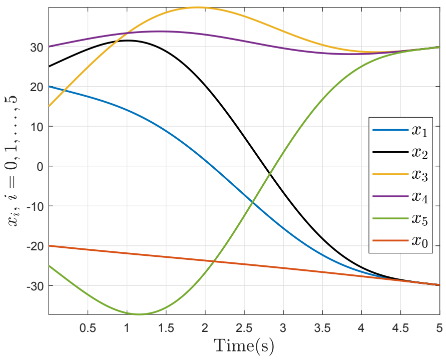

Fig. 6: Second-order multi-agent system–x_i.

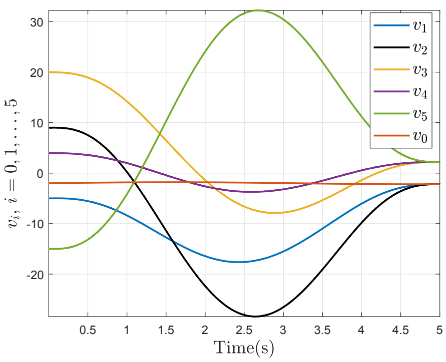

Fig. 7: Second-order multi-agent system–v_i.

the increased initial tracking error and occurred disturbances do not affect the preset-time feature by using the proposed controller.

Example 2: Bipartite consensus tracking for first-order multi-agent systems.

A time-varying function $h_3(t)$ in the form of (6) with $t_\mathrm{f} = 5$s is used in this example. The initial conditions are set as $x_1(0) = 4$, $x_2(0) = -13$, $x_3(0) = -15$, $x_4(0) = 13$, $x_5(0) = 7$, $x_0(0) = 11$. Implementing controller (26) with $N = 5$, the results are shown in Fig. 8, where the bipartite consensus tracking is achieved at 5s. Then, change the initial conditions to $x_1(0) = -90$, $x_2(0) = 40$, $x_3(0) = 30$, $x_4(0) = 60$, $x_5(0) = -60$, $x_0(0) = 50$. The results are shown in Fig. 9, which shows the increased initial tracking error does not affect the preset-time bipartite consensus tracking.

VI. Conclusion

This paper investigates the problem of bipartite consensus tracking for second-order multi-agent systems with signed directed graphs. A time-varying function based approach is proposed to realize the bipartite convergence in preset-time. A class of time-varying functions with generalized properties are presented. Then, a proper manifold is constructed, on which bipartite consensus tracking is achieved in preset-time. Under a structurally balanced and strongly connected graph, a time-varying function based controller is proposed such that the system trajectory is constrained on the manifold. For first-order multi-agent systems, a time-varying gain based controller is further developed without constructing a manifold, which simplifies the design procedure. Finally, numerical examples are provided to demonstrate the effectiveness of the proposed controllers.

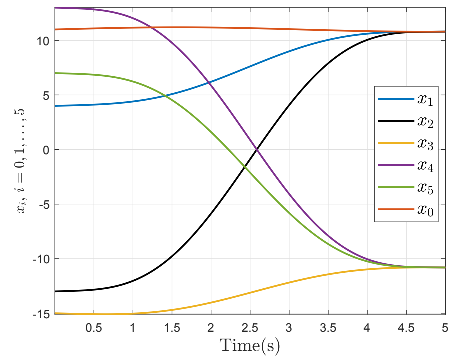

Fig. 8: First-order multi-agent system–x_i.

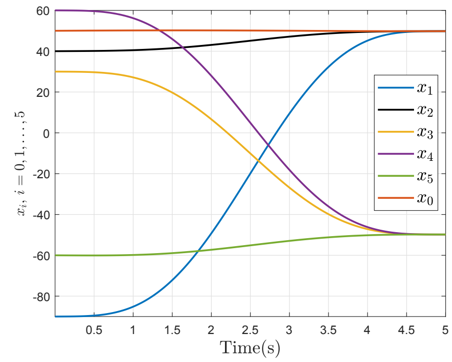

Fig. 9: First-order multi-agent system–x_i.

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