Signal Analysis

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Contents

1 Properties of Mathematical Functions

1.1 Even and Odd Functions

Definition 1.1 (Even function)**.** A function $x(t)$ is even if

$$x(-t) = x(t)$$

for all t in the functions domain. Even functions are symmetric about the vertical axis.

Definition 1.2 (Odd function)**.** A function $x(t)$ is odd if

$$x(-t) = -x(t)$$

for all t in the functions domain. Odd functions are symmetric about the origin.

1.1.1 Integrating Even and Odd Functions

When integrating an **even** function $x(t)$ over the domain $[-T, T]$:

$$\int_{-T}^{T} x(t)\,\mathrm{d}t = 2\int_{0}^{T} x(t)\,\mathrm{d}t.$$

Similarly, when integrating an **odd** function $x(t)$ over the domain $[-T, T]$:

$$\int_{-T}^{T} x(t)\,\mathrm{d}t = 0.$$

1.1.2 Product of Even and Odd Functions

1. The product of an **even** function with an **even** function, is an **even** function.
   Let $f(t)$ and $g(t)$ be even functions, and let $h(t) = f(t)\,g(t)$,
   $$h(-t) = f(-t)\,g(-t) = f(t)\,g(t) = h(t)\,.$$
2. The product of an **even** function with an **odd** function, is an **odd** function.
   Let $f(t)$ be an even function and $g(t)$ be an odd function, and let $h(t) = f(t)\,g(t)$,
   $$h(-t) = f(-t)\,g(-t) = (-f(t))\,g(t) = -h(t)\,.$$
3. The product of an **odd** function with an **odd** function, is an **even** function.
   Let $f(t)$ and $g(t)$ be odd functions, and let $h(t) = f(t)\,g(t)$,
   $$h(-t) = f(-t)\,g(-t) = (-f(t))\,(-g(t)) = f(t)\,g(t) = h(t)\,.$$

1.2 Orthogonality

Definition 1.3 (Inner product)**.** An inner product generalises the dot product in general vector spaces.
In particular, for the function space $\mathscr{F}([a, b])$, where $t \in [a, b]$, the inner product is defined as the following:

$$\langle f,\ g\rangle = \int_a^b f(t)\, g(t)\, \mathrm{d}t$$

for $f,\ g \in \mathscr{F}([a, b])$.

Definition 1.4 (Orthogonality)**.** Given an inner product space, two vectors are orthogonal iff

$$\langle f,\ g\rangle = 0.$$

1.3 Orthogonality of Trigonometric Functions

Consider the inner product between the sine and cosine functions on the interval $[-T, T]$.

$$\langle \sin(t),\ \cos(t)\rangle = \int_{-T}^{T} \sin(t)\cos(t)\, \mathrm{d}t = 0$$

as the integrand is an odd function.

1.4 Integrals of Trigonometric Functions

For $n \in \mathbb{Z}$:

$$\begin{aligned}
\int_{t_0}^{t_0+T} \sin(2n\pi f_0 t)\, \mathrm{d}t &= -\frac{1}{2n\pi f_0}\left[\cos(2n\pi f_0 t)\right]_{t_0}^{t_0+T} \\
&= -\frac{1}{2n\pi f_0}\left[\cos\left(\frac{2n\pi}{T}(t_0 + T)\right) - \cos\left(\frac{2n\pi}{T}t_0\right)\right] \\
&= -\frac{1}{2n\pi f_0}\left[\cos\left(\frac{2n\pi}{T}t_0 + 2n\pi\right) - \cos\left(\frac{2n\pi}{T}t_0\right)\right] \\
&= -\frac{1}{2n\pi f_0}\left[\cos\left(\frac{2n\pi}{T}t_0\right) - \cos\left(\frac{2n\pi}{T}t_0\right)\right] \\
&= -\frac{1}{2n\pi f_0}[0] \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\int_{t_0}^{t_0+T} \cos\left(2n\pi f_0 t\right) \mathrm{d}t &= \frac{1}{2n\pi f_0} \left[\sin\left(2n\pi f_0 t\right)\right]_{t_0}^{t_0+T} \\
&= \frac{1}{2n\pi f_0} \left[\sin\left(\frac{2n\pi}{T}\left(t_0+T\right)\right) - \sin\left(\frac{2n\pi}{T} t_0\right)\right] \\
&= \frac{1}{2n\pi f_0} \left[\sin\left(\frac{2n\pi}{T} t_0 + 2n\pi\right) - \sin\left(\frac{2n\pi}{T} t_0\right)\right] \\
&= \frac{1}{2n\pi f_0} \left[\sin\left(\frac{2n\pi}{T} t_0\right) - \sin\left(\frac{2n\pi}{T} t_0\right)\right] \\
&= \frac{1}{2n\pi f_0} \left[0\right] \\
&= 0.
\end{aligned}$$

1.4.1 Product of Trigonometric Functions

Recall the Werner formulas:

$$\begin{aligned}
2\cos\left(\alpha\right)\cos\left(\beta\right) &= \cos\left(\alpha-\beta\right) + \cos\left(\alpha+\beta\right) \\
2\sin\left(\alpha\right)\sin\left(\beta\right) &= \cos\left(\alpha-\beta\right) - \cos\left(\alpha+\beta\right) \\
2\sin\left(\alpha\right)\cos\left(\beta\right) &= \sin\left(\alpha-\beta\right) + \sin\left(\alpha+\beta\right)
\end{aligned}$$

For $n,\ m \in \mathbb{N}$,
<u>Product of two cosine functions:</u>

$$\int_{t_0}^{t_0+T} \cos\left(2n\pi f_0 t\right)\cos\left(2m\pi f_0 t\right) \mathrm{d}t = \frac{1}{2}\int_{t_0}^{t_0+T} \cos\left(2\left(n-m\right)\pi f_0 t\right) + \cos\left(2\left(n+m\right)\pi f_0 t\right) \mathrm{d}t$$

$n = m \implies n - m = 0$ and $(n+m) \in \mathbb{Z}$, so that the integral of the second term is 0, and the integral of the first term results in $\frac{T}{2}$.
$n \neq m \implies (n-m),\ (n+m) \in \mathbb{Z}$ so that both terms evaluate to 0 when integrated separately.

$$\int_{t_0}^{t_0+T} \cos\left(2n\pi f_0 t\right)\cos\left(2m\pi f_0 t\right) \mathrm{d}t = \begin{cases} \frac{T}{2}, & n = m \\ 0, & n \neq m. \end{cases}$$

<u>Product of two sine functions:</u>

$$\int_{t_0}^{t_0+T} \sin\left(2n\pi f_0 t\right)\sin\left(2m\pi f_0 t\right) \mathrm{d}t = \frac{1}{2}\int_{t_0}^{t_0+T} \cos\left(2\left(n-m\right)\pi f_0 t\right) - \cos\left(2\left(n+m\right)\pi f_0 t\right) \mathrm{d}t$$

By the same argument as before,

$$\int_{t_0}^{t_0+T} \sin\left(2n\pi f_0 t\right)\sin\left(2m\pi f_0 t\right) \mathrm{d}t = \begin{cases} \frac{T}{2}, & n = m \\ 0, & n \neq m. \end{cases}$$

<u>Product of sine and cosine functions:</u>

$$\int_{t_0}^{t_0+T} \sin\left(2n\pi f_0 t\right)\cos\left(2m\pi f_0 t\right) \mathrm{d}t = \frac{1}{2}\int_{t_0}^{t_0+T} \sin\left(2\left(n-m\right)\pi f_0 t\right) + \sin\left(2\left(n+m\right)\pi f_0 t\right) \mathrm{d}t$$

$n = m \implies n - m = 0$ and $(n+m) \in \mathbb{Z}$, so that the integral reduces to 0.
$n \neq m \implies (n-m), (n+m) \in \mathbb{Z}$ so that both terms evaluate to 0 when integrated separately.

$$\int_{t_0}^{t_0+T} \sin(2n\pi f_0 t)\cos(2m\pi f_0 t)\,\mathrm{d}t = 0.$$

2 Fourier Series

2.1 Fourier Series Expansion

The **Fourier Series Expansion** of a function $x(t)$ on the interval $[t_0, t_0 + T]$ is given by

$$x_F(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi f_0 t) + \sum_{n=1}^{\infty} b_n \sin(2n\pi f_0 t)$$

where $n \in \mathbb{Z}^+$ and $f_0 = \frac{1}{T}$. The coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t \\ a_n &= \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(2n\pi f_0 t)\,\mathrm{d}t \\ b_n &= \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(2n\pi f_0 t)\,\mathrm{d}t \end{aligned}$$

Proof. Let $m \in \mathbb{N}$.
For the coefficient a_0, integrate the function $x(t)$ over the interval $[t_0, t_0 + T]$.

$$\begin{aligned} \int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t &= \int_{t_0}^{t_0+T} a_0\,\mathrm{d}t + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos(2n\pi f_0 t)\,\mathrm{d}t + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin(2n\pi f_0 t)\,\mathrm{d}t \\ \int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t &= a_0 T \\ a_0 &= \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t \end{aligned}$$

so that a_0 represents the average value of x on $[t_0, t_0 + T]$.

For coefficients a_m, multiply the equation by $\cos(2m\pi f_0 t)$ before integrating.

$$\begin{aligned}
x(t)\cos(2m\pi f_0 t) &= a_0 \cos(2m\pi f_0 t) \\
&\quad + \sum_{n=1}^{\infty} a_n \cos(2n\pi f_0 t)\cos(2m\pi f_0 t) \\
&\quad + \sum_{n=1}^{\infty} b_n \sin(2n\pi f_0 t)\cos(2m\pi f_0 t) \\
\int_{t_0}^{t_0+T} x(t)\cos(2m\pi f_0 t)\,\mathrm{d}t &= a_0 \cancelto{0}{\int_{t_0}^{t_0+T} \cos(2m\pi f_0 t)\,\mathrm{d}t} \\
&\quad + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos(2n\pi f_0 t)\cos(2m\pi f_0 t)\,\mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{t_0}^{t_0+T} \sin(2n\pi f_0 t)\cos(2m\pi f_0 t)\,\mathrm{d}t} \\
\int_{t_0}^{t_0+T} x(t)\cos(2m\pi f_0 t)\,\mathrm{d}t &= a_m \frac{T}{2} \\
a_m &= \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(2m\pi f_0 t)\,\mathrm{d}t
\end{aligned}$$

For coefficients b_m, multiply the equation by $\sin(2m\pi f_0 t)$ before integrating.

$$\begin{aligned}
x(t)\sin(2m\pi f_0 t) &= a_0 \sin(2m\pi f_0 t) \\
&\quad + \sum_{n=1}^{\infty} a_n \cos(2n\pi f_0 t)\sin(2m\pi f_0 t) \\
&\quad + \sum_{n=1}^{\infty} b_n \sin(2n\pi f_0 t)\sin(2m\pi f_0 t) \\
\int_{t_0}^{t_0+T} x(t)\sin(2m\pi f_0 t)\,\mathrm{d}t &= a_0 \cancelto{0}{\int_{t_0}^{t_0+T} \sin(2m\pi f_0 t)\,\mathrm{d}t} \\
&\quad + \sum_{n=1}^{\infty} a_n \cancelto{0}{\int_{t_0}^{t_0+T} \cos(2n\pi f_0 t)\sin(2m\pi f_0 t)\,\mathrm{d}t} \\
&\quad + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin(2n\pi f_0 t)\sin(2m\pi f_0 t)\,\mathrm{d}t \\
\int_{t_0}^{t_0+T} x(t)\sin(2m\pi f_0 t)\,\mathrm{d}t &= b_m \frac{T}{2} \\
b_m &= \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(2m\pi f_0 t)\,\mathrm{d}t
\end{aligned}$$

To summarise,

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x\left(t\right)\mathrm{d}t$$
$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x\left(t\right)\cos\left(2n\pi f_0 t\right)\mathrm{d}t$$
$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x\left(t\right)\sin\left(2n\pi f_0 t\right)\mathrm{d}t$$

□

2.1.1 Convergence of a Fourier Series

If $x\left(t\right)$ is piecewise smooth on $[t_0, t_0 + L]$, $x_F\left(t\right)$ converges to

$$x_F\left(t\right) = \lim_{\epsilon\to 0^+} \frac{x\left(t+\epsilon\right) + x\left(t-\epsilon\right)}{2}$$

that is, $x = x_F$, except at discontinuities, where f_F is equal to the point halfway between the left- and right-handed limits.

2.1.2 Periodicity of a Fourier Series

If x is non-periodic, x_F converges to the periodic extension of x. The endpoints may converge non-uniformly, corresponding to jump discontinuities in the periodic extension of x.

2.2 Fourier Cosine Series

Consider the Fourier series expansion of an even function x on the interval $[-\frac{T}{2}, \frac{T}{2}]$, i.e., $t_0 = -\frac{T}{2}$. In this case,

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x\left(t\right)\sin\left(2n\pi f_0 t\right)\mathrm{d}t = 0$$

and the Fourier series is a "Fourier cosine series", given by:

$$x_C\left(t\right) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(2n\pi f_0 t\right)$$

with coefficients

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x\left(t\right)\mathrm{d}t = \frac{2}{T}\int_{0}^{\frac{T}{2}} x\left(t\right)\mathrm{d}t$$
$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x\left(t\right)\cos\left(2n\pi f_0 t\right)\mathrm{d}t = \frac{4}{T}\int_{0}^{\frac{T}{2}} x\left(t\right)\cos\left(2n\pi f_0 t\right)\mathrm{d}t.$$

2.3 Fourier Sine Series

Consider the Fourier series expansion of an odd function x on the interval $[-\frac{T}{2}, \frac{T}{2}]$. In this case

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x\left(t\right) \sin\left(2n\pi f_0 t\right) \mathrm{d}t = 0$$

and the Fourier series is a “Fourier sine series”, given by:

$$x_S\left(t\right) = \sum_{n=1}^{\infty} b_n \sin\left(2n\pi f_0 t\right)$$

with coefficients

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x\left(t\right) \sin\left(2n\pi f_0 t\right) \mathrm{d}t = \frac{4}{T} \int_{0}^{\frac{T}{2}} x\left(t\right) \sin\left(2n\pi f_0 t\right) \mathrm{d}t.$$