Signal Analysis

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Contents

1 Properties of Mathematical Functions

1.1 Even and Odd Functions

Definition 1.1 (Even function)**.** A function $x(t)$ is even if

$$x(-t) = x(t)$$

for all t in the functions domain. Even functions are symmetric about the vertical axis.

Definition 1.2 (Odd function)**.** A function $x(t)$ is odd if

$$x(-t) = -x(t)$$

for all t in the functions domain. Odd functions are symmetric about the origin.

1.1.1 Integrating Even and Odd Functions

When integrating an **even** function $x(t)$ over the domain $[-T, T]$:

$$\int_{-T}^{T} x(t)\,\mathrm{d}t = 2\int_{0}^{T} x(t)\,\mathrm{d}t.$$

Similarly, when integrating an **odd** function $x(t)$ over the domain $[-T, T]$:

$$\int_{-T}^{T} x(t)\,\mathrm{d}t = 0.$$

1.1.2 Product of Even and Odd Functions

1. The product of an **even** function with an **even** function, is an **even** function.
   Let $f(t)$ and $g(t)$ be even functions, and let $h(t) = f(t)\,g(t)$,
   $$h(-t) = f(-t)\,g(-t) = f(t)\,g(t) = h(t).$$
2. The product of an **even** function with an **odd** function, is an **odd** function.
   Let $f(t)$ be an even function and $g(t)$ be an odd function, and let $h(t) = f(t)\,g(t)$,
   $$h(-t) = f(-t)\,g(-t) = (-f(t))\,g(t) = -h(t).$$
3. The product of an **odd** function with an **odd** function, is an **even** function.
   Let $f(t)$ and $g(t)$ be odd functions, and let $h(t) = f(t)\,g(t)$,
   $$h(-t) = f(-t)\,g(-t) = (-f(t))\,(-g(t)) = f(t)\,g(t) = h(t).$$

1.2 Orthogonality

Definition 1.3 (Inner product)**.** An inner product generalises the dot product in general vector spaces.
In particular, for the function space $\mathscr{F}([a, b])$, where $t \in [a, b]$, the inner product is defined as the following:

$$\langle f,\ g\rangle = \int_a^b f(t)\, g(t)\, \mathrm{d}t$$

for $f,\ g \in \mathscr{F}([a, b])$.

Definition 1.4 (Orthogonality)**.** Given an inner product space, two vectors are orthogonal iff

$$\langle f,\ g\rangle = 0.$$

1.3 Orthogonality of Trigonometric Functions

Consider the inner product between the sine and cosine functions on the interval $[-T, T]$.

$$\langle \sin(t),\ \cos(t)\rangle = \int_{-T}^{T} \sin(t)\cos(t)\, \mathrm{d}t = 0$$

as the integrand is an odd function.

1.4 Integrals of Trigonometric Functions

For $n \in \mathbb{Z}$:

$$\begin{aligned}
\int_{t_0}^{t_0+T} \sin(2\pi n f_0 t)\, \mathrm{d}t &= -\frac{1}{2\pi n f_0}\left[\cos(2\pi n f_0 t)\right]_{t_0}^{t_0+T} \\
&= -\frac{1}{2\pi n f_0}\left[\cos\left(\frac{2\pi n}{T}(t_0 + T)\right) - \cos\left(\frac{2\pi n}{T} t_0\right)\right] \\
&= -\frac{1}{2\pi n f_0}\left[\cos\left(\frac{2\pi n}{T} t_0 + 2\pi n\right) - \cos\left(\frac{2\pi n}{T} t_0\right)\right] \\
&= -\frac{1}{2\pi n f_0}\left[\cos\left(\frac{2\pi n}{T} t_0\right) - \cos\left(\frac{2\pi n}{T} t_0\right)\right] \\
&= -\frac{1}{2\pi n f_0}[0] \\
&= 0.
\end{aligned}$$

$$
\begin{aligned}
\int_{t_0}^{t_0+T} \cos\left(2\pi n f_0 t\right) \mathrm{d}t &= \frac{1}{2\pi n f_0} \left[\sin\left(2\pi n f_0 t\right)\right]_{t_0}^{t_0+T} \\
&= \frac{1}{2\pi n f_0} \left[\sin\left(\frac{2\pi n}{T}\left(t_0 + T\right)\right) - \sin\left(\frac{2\pi n}{T} t_0\right)\right] \\
&= \frac{1}{2\pi n f_0} \left[\sin\left(\frac{2\pi n}{T} t_0 + 2\pi n\right) - \sin\left(\frac{2\pi n}{T} t_0\right)\right] \\
&= \frac{1}{2\pi n f_0} \left[\sin\left(\frac{2\pi n}{T} t_0\right) - \sin\left(\frac{2\pi n}{T} t_0\right)\right] \\
&= \frac{1}{2\pi n f_0} \left[0\right] \\
&= 0.
\end{aligned}
$$

1.4.1 Product of Trigonometric Functions

Recall the Werner formulas:

$$
\begin{aligned}
2\cos\left(\alpha\right)\cos\left(\beta\right) &= \cos\left(\alpha - \beta\right) + \cos\left(\alpha + \beta\right) \\
2\sin\left(\alpha\right)\sin\left(\beta\right) &= \cos\left(\alpha - \beta\right) - \cos\left(\alpha + \beta\right) \\
2\sin\left(\alpha\right)\cos\left(\beta\right) &= \sin\left(\alpha - \beta\right) + \sin\left(\alpha + \beta\right)
\end{aligned}
$$

For $n,\ m \in \mathbb{N}$,
<u>Product of two cosine functions:</u>

$$
\int_{t_0}^{t_0+T} \cos\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right) \mathrm{d}t = \frac{1}{2}\int_{t_0}^{t_0+T} \cos\left(2\pi\left(n - m\right) f_0 t\right) + \cos\left(2\pi\left(n + m\right) f_0 t\right) \mathrm{d}t
$$

$n = m \implies n - m = 0$ and $\left(n + m\right) \in \mathbb{Z}$, so that the integral of the second term is 0, and the integral of the first term results in $\frac{T}{2}$.
$n \neq m \implies \left(n - m\right),\ \left(n + m\right) \in \mathbb{Z}$ so that both terms evaluate to 0 when integrated separately.

$$
\int_{t_0}^{t_0+T} \cos\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right) \mathrm{d}t = \begin{cases} \frac{T}{2}, & n = m \\ 0, & n \neq m. \end{cases}
$$

<u>Product of two sine functions:</u>

$$
\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right)\sin\left(2\pi m f_0 t\right) \mathrm{d}t = \frac{1}{2}\int_{t_0}^{t_0+T} \cos\left(2\pi\left(n - m\right) f_0 t\right) - \cos\left(2\pi\left(n + m\right) f_0 t\right) \mathrm{d}t
$$

By the same argument as before,

$$
\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right)\sin\left(2\pi m f_0 t\right) \mathrm{d}t = \begin{cases} \frac{T}{2}, & n = m \\ 0, & n \neq m. \end{cases}
$$

<u>Product of sine and cosine functions:</u>

$$
\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right) \mathrm{d}t = \frac{1}{2}\int_{t_0}^{t_0+T} \sin\left(2\pi\left(n - m\right) f_0 t\right) + \sin\left(2\pi\left(n + m\right) f_0 t\right) \mathrm{d}t
$$

$n = m \implies n - m = 0$ and $(n+m) \in \mathbb{Z}$, so that the integral reduces to 0.
$n \neq m \implies (n-m)\,,\ (n+m) \in \mathbb{Z}$ so that both terms evaluate to 0 when integrated separately.

$$\int_{t_0}^{t_0+T} \sin(2\pi n f_0 t)\cos(2\pi m f_0 t)\,\mathrm{d}t = 0.$$

2 Fourier Series

2.1 Fourier Series Expansion

The **Fourier Series Expansion** of a function $x(t)$ on the interval $[t_0, t_0+T]$ is given by

$$x_F(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)\right)$$

where $n \in \mathbb{Z}^+$ and $f_0 = \frac{1}{T}$. The coefficients are given by

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t \\
a_n &= \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(2\pi n f_0 t)\,\mathrm{d}t \\
b_n &= \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(2\pi n f_0 t)\,\mathrm{d}t
\end{aligned}$$

Proof. Let $m \in \mathbb{N}$.
For the coefficient a_0, integrate the function $x(t)$ over the interval $[t_0, t_0+T]$.

$$\begin{aligned}
\int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t &= \int_{t_0}^{t_0+T} a_0\,\mathrm{d}t + \sum_{n=1}^{\infty} a_n \cancelto{0}{\int_{t_0}^{t_0+T} \cos(2\pi n f_0 t)\,\mathrm{d}t} + \sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{t_0}^{t_0+T} \sin(2\pi n f_0 t)\,\mathrm{d}t} \\
\int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t &= a_0 T \\
a_0 &= \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t
\end{aligned}$$

so that a_0 represents the average value of x on $[t_0, t_0+T]$. This coefficient also represents the DC component of a signal.

For coefficients a_m, multiply the equation by $\cos\left(2\pi m f_0 t\right)$ before integrating.

$$\begin{aligned}
x\left(t\right)\cos\left(2\pi m f_0 t\right) &= a_0 \cos\left(2\pi m f_0 t\right) \\
&\quad + \sum_{n=1}^{\infty} a_n \cos\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right) \\
&\quad + \sum_{n=1}^{\infty} b_n \sin\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right) \\
\int_{t_0}^{t_0+T} x\left(t\right)\cos\left(2\pi m f_0 t\right)\mathrm{d}t &= a_0 \cancelto{0}{\int_{t_0}^{t_0+T} \cos\left(2\pi m f_0 t\right)\mathrm{d}t} \\
&\quad + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right)\mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right)\mathrm{d}t} \\
\int_{t_0}^{t_0+T} x\left(t\right)\cos\left(2\pi m f_0 t\right)\mathrm{d}t &= a_m \frac{T}{2} \\
a_m &= \frac{2}{T}\int_{t_0}^{t_0+T} x\left(t\right)\cos\left(2\pi m f_0 t\right)\mathrm{d}t
\end{aligned}$$

For coefficients b_m, multiply the equation by $\sin\left(2\pi m f_0 t\right)$ before integrating.

$$\begin{aligned}
x\left(t\right)\sin\left(2\pi m f_0 t\right) &= a_0 \sin\left(2\pi m f_0 t\right) \\
&\quad + \sum_{n=1}^{\infty} a_n \cos\left(2\pi n f_0 t\right)\sin\left(2\pi m f_0 t\right) \\
&\quad + \sum_{n=1}^{\infty} b_n \sin\left(2\pi n f_0 t\right)\sin\left(2\pi m f_0 t\right) \\
\int_{t_0}^{t_0+T} x\left(t\right)\sin\left(2\pi m f_0 t\right)\mathrm{d}t &= a_0 \cancelto{0}{\int_{t_0}^{t_0+T} \sin\left(2\pi m f_0 t\right)\mathrm{d}t} \\
&\quad + \sum_{n=1}^{\infty} a_n \cancelto{0}{\int_{t_0}^{t_0+T} \cos\left(2\pi n f_0 t\right)\sin\left(2\pi m f_0 t\right)\mathrm{d}t} \\
&\quad + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right)\sin\left(2\pi m f_0 t\right)\mathrm{d}t \\
\int_{t_0}^{t_0+T} x\left(t\right)\sin\left(2\pi m f_0 t\right)\mathrm{d}t &= b_m \frac{T}{2} \\
b_m &= \frac{2}{T}\int_{t_0}^{t_0+T} x\left(t\right)\sin\left(2\pi m f_0 t\right)\mathrm{d}t
\end{aligned}$$

□

2.1.1 Convergence of a Fourier Series

If $x(t)$ is piecewise smooth on $[t_0, t_0 + L]$, $x_F(t)$ converges to

$$x_F(t) = \lim_{\epsilon \to 0^+} \frac{x(t+\epsilon) + x(t-\epsilon)}{2}$$

that is, $x = x_F$, except at discontinuities, where f_F is equal to the point halfway between the left- and right-handed limits.

2.1.2 Periodicity of a Fourier Series

If x is non-periodic, x_F converges to the periodic extension of x. The endpoints may converge non-uniformly, corresponding to jump discontinuities in the periodic extension of x.

2.2 Fourier Cosine Series

Consider the Fourier series expansion of an even function x on the interval $[-\frac{T}{2}, \frac{T}{2}]$, i.e., $t_0 = -\frac{T}{2}$. In this case,

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin(2\pi n f_0 t) \,\mathrm{d}t = 0$$

and the Fourier series is a "Fourier cosine series", given by:

$$x_c(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t)$$

with coefficients

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \,\mathrm{d}t = \frac{2}{T} \int_0^{\frac{T}{2}} x(t) \,\mathrm{d}t \\ a_n &= \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos(2\pi n f_0 t) \,\mathrm{d}t = \frac{4}{T} \int_0^{\frac{T}{2}} x(t) \cos(2\pi n f_0 t) \,\mathrm{d}t. \end{aligned}$$

2.3 Fourier Sine Series

Consider the Fourier series expansion of an odd function x on the interval $[-\frac{T}{2}, \frac{T}{2}]$. In this case

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin(2\pi n f_0 t) \,\mathrm{d}t = 0$$

and the Fourier series is a "Fourier sine series", given by:

$$x_s(t) = \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t)$$

with coefficients

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x\left(t\right)\sin\left(2\pi n f_0 t\right)\mathrm{d}t = \frac{4}{T}\int_{0}^{\frac{T}{2}} x\left(t\right)\sin\left(2\pi n f_0 t\right)\mathrm{d}t.$$

3 Complex Fourier Series

Definition 3.1. The **Complex Fourier Series Expansion** is a concise form of the Fourier series expansion that uses complex exponentials with a single unknown coefficient.

$$x_C\left(t\right) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

where

$$c_n = \frac{1}{T}\int_{t_0}^{t_0+T} x\left(t\right) e^{-j2\pi n f_0 t}\,\mathrm{d}t.$$

for $n \in \mathbb{Z}$ and $f_0 = \frac{1}{T}$.

To determine the complex Fourier series expansion consider the following identities:

$$\begin{aligned}\cos\left(\theta\right) &= \frac{e^{j\theta}+e^{-j\theta}}{2}\\ \sin\left(\theta\right) &= -j\frac{e^{j\theta}-e^{-j\theta}}{2}.\end{aligned}$$

By substituting these identities into the Fourier series expansion summand, we obtain:

$$\begin{aligned}a_n\cos\left(2\pi n f_0 t\right) + b_n\sin\left(2\pi n f_0 t\right) &= a_n\frac{e^{j2\pi n f_0 t}+e^{-j2\pi n f_0 t}}{2} - jb_n\frac{e^{j2\pi n f_0 t}-e^{-j2\pi n f_0 t}}{2}\\ &= \frac{a_n - jb_n}{2}e^{j2\pi n f_0 t} + \frac{a_n + jb_n}{2}e^{-j2\pi n f_0 t}\end{aligned}$$

Let $c_n = \frac{a_n - jb_n}{2}$ and $c_n^* = \frac{a_n + jb_n}{2}$ (we will see how this simplifies later). Using the definitions for a_n and b_n:

$$\begin{aligned}c_n &= \frac{1}{2}\left(a_n - jb_n\right)\\ &= \frac{1}{T}\int_{t_0}^{t_0+T} x\left(t\right)\left(\cos\left(2\pi n f_0 t\right) - j\sin\left(2\pi n f_0 t\right)\right)\mathrm{d}t\\ &= \frac{1}{T}\int_{t_0}^{t_0+T} x\left(t\right)e^{-j2\pi n f_0 t}\,\mathrm{d}t\end{aligned}$$

$$\begin{aligned}
c_n^* &= \frac{1}{2}\left(a_n + jb_n\right) \\
&= \frac{1}{T}\int_{t_0}^{t_0+T} x\left(t\right)\left(\cos\left(2\pi n f_0 t\right) + j\sin\left(2\pi n f_0 t\right)\right)\mathrm{d}t \\
&= \frac{1}{T}\int_{t_0}^{t_0+T} x\left(t\right)e^{j2\pi n f_0 t}\,\mathrm{d}t \\
&= c_{-n}
\end{aligned}$$

Let $c_0 = a_0$, so that

$$\begin{aligned}
x_C\left(t\right) &= a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(2\pi n f_0 t\right) + b_n\sin\left(2\pi n f_0 t\right)\right) \\
&= c_0 + \sum_{n=1}^{\infty}\left(c_n e^{j2\pi n f_0 t} + c_{-n}e^{-j2\pi n f_0 t}\right) \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{j2\pi n f_0 t} + \sum_{n=1}^{\infty} c_{-n}e^{-j2\pi n f_0 t} \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{j2\pi n f_0 t} + \sum_{n=-\infty}^{-1} c_n e^{j2\pi n f_0 t} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}.
\end{aligned}$$

3.1 Converting between Fourier Series Representations

Given the Trigonometric and Exponential (Complex) Fourier Series Representations (FSR), we can develop a relationship between the coefficients a_n, b_n, and c_n by:

$$\begin{aligned}
a_0 &= c_0 \\
a_n &= c_n + c_{-n} \\
b_n &= j\left(c_n - c_{-n}\right).
\end{aligned}$$

3.2 Magnitude and Phase Spectra

As c_n is a complex number, consider the polar representation of c_n:

$$c_n = |c_n|e^{j\theta_n}$$

so that the **magnitude spectra** is given by $|c_n|$ and the **phase spectra** is given by θ_n.
The plot of $|c_n|$ against n is called the "magnitude spectrum" of $x\left(t\right)$ and the plot of θ_n against n is called the "phase spectrum" of $x\left(t\right)$.

Theorem 3.2.1 (Spectra of a real signal)**.** *Given any real function $x\left(t\right)$, the magnitude spectrum is always an even function, and the phase spectrum is always an odd function.*

Proof. Given a real function $x(t)$, the exponential Fourier series is given by,

$$x_C(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

this is equivalent to

$$x_C(t) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-j2\pi n f_0 t}.$$

The conjugate of $x_C(t)$ yields,

$$\overline{x_C(t)} = \sum_{n=-\infty}^{\infty} \overline{c_n} e^{-j2\pi m f_0 t}$$

As $x(t) \in \mathbb{R}$, $x_C(t) = \overline{x_C(t)}$, so that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} c_{-n} e^{-j2\pi n f_0 t} &= \sum_{n=-\infty}^{\infty} \overline{c_n} e^{-j2\pi m f_0 t} \\ c_{-n} e^{-j2\pi n f_0 t} &= \overline{c_n} e^{-j2\pi n f_0 t} \\ c_{-n} &= \overline{c_n}\end{aligned}$$

Therefore by representing the coefficients above in polar form we get,

$$|c_{-n}| e^{j\theta_{-n}} = |c_n| e^{-j\theta_n}$$

as required. □

3.3 Even and Odd Functions

Given an even function $x(t)$, $b_n = 0$, therefore

$$\begin{aligned}0 &= j(c_n - c_{-n}) \\ c_n &= c_{-n}\end{aligned}$$

so that,

$$\begin{aligned}a_n &= c_n + c_{-n} \\ a_n &= 2c_n \\ c_n &= c_{-n} = \frac{a_n}{2}\end{aligned}$$

Hence c_n and c_{-n} are real coefficients, so that $|c_n|$ is an even function, and $\theta_n = m\pi$ for some $m \in \mathbb{Z}$.
Given an odd function $x(t)$, $a_0 = 0$ and $a_n = 0$, therefore $c_0 = 0$ and

$$\begin{aligned}0 &= c_n + c_{-n} \\ c_n &= -c_{-n}\end{aligned}$$

so that,

$$\begin{aligned}
b_n &= j\left(c_n - c_{-n}\right) \\
b_n &= j\left(c_n + c_n\right) \\
b_n &= j2c_n \\
c_n &= -j\frac{b_n}{2} \\
c_{-n} &= j\frac{b_n}{2}
\end{aligned}$$

Hence c_n and c_{-n} are purely imaginary coefficients, so that $|c_n|$ is an odd function, and $\theta_n = (2m+1)\frac{\pi}{2}$ for some $m \in \mathbb{Z}$.

3.4 Signal Representations

A signal can be represented in various forms depending on the method of measurement. There are three main forms of signal representation:

- Analogue: Continuous in time and continuous in amplitude.
- Discrete: Discrete in time and continuous in amplitude.
- Digital: Discrete in time and discrete in amplitude.

The process of taking discrete time measurements is known as sampling, and taking discrete amplitude measurements is known as quantisation.

Definition 3.2 (Integral transform)**.** An *integral transform* transforms a function through the process of integration, producing a new function of another variable. Two common examples of integral transforms are the Fourier transform and the Laplace transform.

4 Fourier Transform

The Fourier Transform allows us to extend the techniques used in Fourier series representations of functions to non-periodic signals.

Definition 4.1 (Fourier transform)**.** The Fourier transform of a function $x(t)$ is defined by:

$$\mathscr{F}\left\{x(t)\right\} = X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\,\mathrm{d}t.$$

where $X(f) = \mathscr{F}\left\{x(t)\right\}$ is a function of frequency f.

Definition 4.2 (Inverse Fourier transform)**.** The inverse Fourier transform of a function $X(f)$ is defined by:

$$\mathscr{F}^{-1}\left\{X(f)\right\} = x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\,\mathrm{d}f.$$

This integral transform is commonly represented as a "Fourier pair" using the following notation:

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(f).$$

We can derive this definition by considering the complex Fourier series representation of $x(t)$ on the interval $[-\frac{T}{2}, \frac{T}{2}]$:

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t} \\ &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi n f_0 t} \, \mathrm{d}t \right] e^{j2\pi n f_0 t}. \end{aligned}$$

If we let $f_n = n f_0$, then $\Delta f_n = f_{n+1} - f_n = (n+1) f_0 - n f_0 = f_0 = \frac{1}{T}$:

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi f_n t} \, \mathrm{d}t \right] e^{j2\pi f_n t} \, \Delta f_n.$$

By recognition, this is a Riemann sum, so that by taking the limit $T \to \infty$ we get:

$$x(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, \mathrm{d}t \right] e^{j2\pi f t} \, \mathrm{d}f$$

where

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, \mathrm{d}t.$$

Corollary 4.0.0.1 (Dirichlet conditions)**.** *The Dirichlet conditions provide sufficient conditions for a real-valued function x to be equal to its Fourier Transform X, at each point where x is continuous. The conditions are:*

1. *x has a finite number of maxima and minima over $[-L, L]$.*
2. *x has a finite number of discontinuities, in each of which the derivative x' exists and does not change sign.*
3. *$\int_{-\infty}^{\infty} |x(t)| \, \mathrm{d}t$ exists.*

4.1 Fourier Transform Properties

4.1.1 Linearity

$$a_1 x_1(t) \pm a_2 x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} a_1 X_1(f) \pm a_2 X_2(f).$$

Due to the linearity of the integral, the Fourier transform is a linear operator.

4.1.2 Complex Conjugate

$$x^*(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X^*(-f).$$

Proof.

$$\begin{aligned}
\mathscr{F}\{x^*(t)\} &= \int_{-\infty}^{\infty} x^*(t)\, e^{-j2\pi ft}\, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} x^*(t)\, \overline{e^{j2\pi ft}}\, \mathrm{d}t^* \\
&= \int_{-\infty}^{\infty} \overline{x(t)\, e^{j2\pi ft}\, \mathrm{d}t} \\
&= \overline{\int_{-\infty}^{\infty} x(t)\, e^{j2\pi ft}\, \mathrm{d}t} \\
&= \overline{\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(-f)t}\, \mathrm{d}t} \\
&= X^*(-f).
\end{aligned}$$

□

4.1.3 Time Shift

$$x(t-t_0) \stackrel{\mathscr{F}}{\longleftrightarrow} e^{-j2\pi ft_0} X(f).$$

Proof.

$$\begin{aligned}
x(t-t_0) &= \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f(t-t_0)}\, \mathrm{d}f \\
&= \int_{-\infty}^{\infty} \left[e^{-j2\pi ft_0} X(f)\right] e^{-j2\pi ft}\, \mathrm{d}f \\
&= \mathscr{F}^{-1}\big(e^{-j2\pi ft_0} X(f)\big).
\end{aligned}$$

□

4.2 Frequency Shift

$$e^{j2\pi f_0 t} x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(f-f_0).$$

Proof.

$$\begin{aligned}
X(f-f_0) &= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f-f_0)t}\, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} \left[e^{j2\pi f_0 t} x(t)\right] e^{-j2\pi ft}\, \mathrm{d}t \\
&= \mathscr{F}\left\{e^{j2\pi f_0 t} x(t)\right\}
\end{aligned}$$

□

4.3 Time Reversal

$$x(-t) \overset{\mathscr{F}}{\longleftrightarrow} X(-f).$$

Proof. Consider the substitution $u = -t$ so that $\mathrm{d}u = -\,\mathrm{d}t$:

$$\begin{aligned}
\mathscr{F}\{x(-t)\} &= \int_{-\infty}^{\infty} x(-t)\, e^{-j2\pi f t}\,\mathrm{d}t \\
&= -\int_{\infty}^{-\infty} x(u)\, e^{j2\pi f u}\,\mathrm{d}u \\
&= \int_{-\infty}^{\infty} x(u)\, e^{-j2\pi(-f)u}\,\mathrm{d}u \\
&= X(-f)
\end{aligned}$$

□

4.4 Time Scaling

$$x(at) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{f}{a}\right).$$

Proof. Consider the substitution $u = at$ so that $\mathrm{d}u = a\,\mathrm{d}t$:

$$\begin{aligned}
\mathscr{F}\{x(at)\} &= \int_{-\infty}^{\infty} x(at)\, e^{-j2\pi f t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} x(u)\, e^{-j2\pi f \frac{u}{a}} \frac{1}{a}\,\mathrm{d}u \\
&= \frac{1}{a}\int_{-\infty}^{\infty} x(u)\, e^{-j2\pi \frac{f}{a} u}\,\mathrm{d}u \\
&= \frac{1}{a} X\left(\frac{f}{a}\right)
\end{aligned}$$

when $a > 0$, $a = |a|$:

$$\mathscr{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{f}{a}\right).$$

When $a < 0$, $a = -|a|$:

$$\begin{aligned}
\mathscr{F}\{x(-|a|t)\} &= \frac{1}{|a|} X\left(-\frac{f}{|a|}\right) \\
&= \frac{1}{|a|} X\left(\frac{f}{a}\right).
\end{aligned}$$

□

4.5 Time Differentiation

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} \overset{\mathscr{F}}{\longleftrightarrow} (j2\pi f)^n X(f).$$

Proof. Consider the representation of $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ using the inverse Fourier transform:

$$\begin{aligned}
\frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\,\mathrm{d}f \\
&= \int_{-\infty}^{\infty} X(f)\,\frac{\mathrm{d}}{\mathrm{d}t} e^{j2\pi ft}\,\mathrm{d}f \\
&= j2\pi f \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\,\mathrm{d}f \\
&= j2\pi f x(t)
\end{aligned}$$

then the Fourier transform of $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ is given by:

$$\begin{aligned}
\mathscr{F}\left\{\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right\} &= \mathscr{F}\left\{j2\pi f x(t)\right\} \\
&= j2\pi f X(f)
\end{aligned}$$

Repeated differentiation yields the following result:

$$\mathscr{F}\left\{\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}\right\} = (j2\pi f)^n X(f)$$

□

4.6 Frequency Differentiation

$$\left(\frac{2\pi}{j}\right)^n t^n x(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{\mathrm{d}^n X(f)}{\mathrm{d}f^n}.$$

Proof. Consider the representation of $\frac{\mathrm{d}X(f)}{\mathrm{d}f}$ using the Fourier transform:

$$\begin{aligned}
\frac{\mathrm{d}X(f)}{\mathrm{d}f} &= \frac{\mathrm{d}}{\mathrm{d}f}\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} x(t)\,\frac{\mathrm{d}}{\mathrm{d}f} e^{-j2\pi ft}\,\mathrm{d}t \\
&= -j2\pi t \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\,\mathrm{d}t \\
&= -j2\pi t X(f)
\end{aligned}$$

then the inverse Fourier transform of $\frac{\mathrm{d}X(f)}{\mathrm{d}f}$ is given by:

$$\begin{aligned}\mathscr{F}^{-1}\left\{\frac{\mathrm{d}X\left(f\right)}{\mathrm{d}f}\right\} &= \mathscr{F}^{-1}\left\{-j2\pi t X\left(f\right)\right\} \\ &= -j2\pi t\mathscr{F}^{-1}\left\{X\left(f\right)\right\} \\ &= \frac{2\pi}{j}tx\left(t\right)\end{aligned}$$

hence

$$\frac{2\pi}{j}tx\left(t\right) \overset{\mathscr{F}}{\longleftrightarrow} \frac{\mathrm{d}X\left(f\right)}{\mathrm{d}f}.$$

Repeated differentiation yields the required result. □

4.6.1 Time Multiplication

$$t^n x\left(t\right) \overset{\mathscr{F}}{\longleftrightarrow} \left(\frac{j}{2\pi}\right)^n \frac{\mathrm{d}^n X\left(f\right)}{\mathrm{d}f^n}.$$

See the previous section for the proof.

4.6.2 Time Integration

$$\int_{-\infty}^{t} x\left(\tau\right)\mathrm{d}\tau \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{j2\pi f}X\left(f\right).$$

Proof. Consider the representation of $\int_{-\infty}^{t} x\left(\tau\right)\mathrm{d}\tau$ using the inverse Fourier transform:

$$\begin{aligned}x\left(t\right) &= \int_{-\infty}^{\infty} X\left(f\right)e^{j2\pi f t}\,\mathrm{d}f \\ \int_{-\infty}^{t} x\left(\tau\right)\mathrm{d}\tau &= \int_{-\infty}^{t}\left[\int_{-\infty}^{\infty} X\left(f\right)e^{j2\pi f\tau}\,\mathrm{d}f\right]\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} X\left(f\right)\left[\int_{-\infty}^{t} e^{j2\pi f\tau}\,\mathrm{d}\tau\right]\mathrm{d}f \\ &= \int_{-\infty}^{\infty} X\left(f\right)\left[\frac{1}{j2\pi f}e^{j2\pi f\tau}\right]_{-\infty}^{t}\mathrm{d}f \\ &= \frac{1}{j2\pi f}\int_{-\infty}^{\infty} X\left(f\right)\left[e^{j2\pi f\tau}\right]_{-\infty}^{t}\mathrm{d}f \\ &= \frac{1}{j2\pi f}\int_{-\infty}^{\infty} X\left(f\right)e^{j2\pi f t}\,\mathrm{d}f \\ &= \frac{1}{j2\pi f}x\left(t\right).\end{aligned}$$

We can now take the Fourier transform of $\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$ to obtain:

$$\begin{aligned}
\mathscr{F}\left\{\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau\right\} &= \mathscr{F}\left\{\frac{1}{j2\pi f}x(t)\right\} \\
&= \frac{1}{j2\pi f}\mathscr{F}\{x(t)\} \\
&= \frac{1}{j2\pi f}X(f)\,.
\end{aligned}$$

□

To summarise:

$$\begin{aligned}
x(t) &\overset{\mathscr{F}}{\leftrightarrow} X(f) && \\
x^*(t) &\overset{\mathscr{F}}{\longleftrightarrow} X^*(-f) && \text{(complex-conjugate)} \\
x(t-t_0) &\overset{\mathscr{F}}{\leftrightarrow} e^{-j2\pi n f t_0}X(f) && \text{(time-shift)} \\
e^{j2\pi n f_0 t}x(t) &\overset{\mathscr{F}}{\leftrightarrow} X(f-f_0) && \text{(frequency-shift)} \\
x(-t) &\overset{\mathscr{F}}{\leftrightarrow} X(-f) && \text{(time reversing)} \\
x(at) &\overset{\mathscr{F}}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{f}{a}\right) && \text{(time scaling)} \\
tx(t) &\overset{\mathscr{F}}{\leftrightarrow} \frac{j}{2\pi}\frac{\mathrm{d}X(f)}{\mathrm{d}f} && \\
e^{at}u(t) &\overset{\mathscr{F}}{\leftrightarrow} \frac{1}{a+j2\pi f} && \\
x(t)\cos(2\pi f_0 t) &\overset{\mathscr{F}}{\leftrightarrow} \frac{1}{2}\left[X(f-f_0)+X(f+f_0)\right] && \text{(time modulation)}
\end{aligned}$$

5 Special Functions

5.1 Dirac Delta Function

The Dirac delta (or impulse) function is defined by the following characteristics:

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1.$$

Likewise, the shifted impulse function can be defined as:

$$\delta\left(t-t_0\right)=\begin{cases}0, & t_0\neq 0\\ \infty, & t_0=0\end{cases}$$

$$\int_{-\infty}^{\infty}\delta\left(t-t_0\right)\mathrm{d}t=1.$$

Therefore we can infer the following *shifting properties*:

$$\int_{-\infty}^{\infty}f\left(t\right)\delta\left(t-t_0\right)\mathrm{d}t=f\left(t_0\right)$$

and

$$f\left(t\right)\delta\left(t-t_0\right)=f\left(t_0\right)\delta\left(t-t_0\right).$$

5.2 Signum Function

The signum function is defined as:

$$\operatorname{sgn}\left(t\right)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$$

5.3 Unit Step Function

The unit step (or Heaviside) function is defined as:

$$u_s\left(t\right)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

5.4 Triangular Function

The triangular function is defined as:

$$\Lambda_T\left(t\right)=\begin{cases}1-\frac{|t|}{T} & |t|<T\\ 0 & \text{otherwise}\end{cases}$$

5.5 Gate Function

The gate (or rectangle) function is defined as:

$${}_T\left(t\right)=\operatorname{rect}\left(\frac{t}{T}\right)=\begin{cases}1 & -\frac{T}{2}\leq t\leq\frac{T}{2}\\ 0 & \text{otherwise}\end{cases}$$