Signal Analysis

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Contents

1 Properties of Mathematical Functions

1.1 Even and Odd Functions

Definition 1.1 (Even function)**.** A function $x(t)$ is even if

$$x(-t) = x(t)$$

for all t in the functions domain. Even functions are symmetric about the vertical axis.

Definition 1.2 (Odd function)**.** A function $x(t)$ is odd if

$$x(-t) = -x(t)$$

for all t in the functions domain. Odd functions are symmetric about the origin.

1.1.1 Integrating Even and Odd Functions

When integrating an **even** function $x(t)$ over the domain $[-T, T]$:

$$\int_{-T}^{T} x(t)\,\mathrm{d}t = 2\int_{0}^{T} x(t)\,\mathrm{d}t.$$

Similarly, when integrating an **odd** function $x(t)$ over the domain $[-T, T]$:

$$\int_{-T}^{T} x(t)\,\mathrm{d}t = 0.$$

1.1.2 Product of Even and Odd Functions

1. The product of an **even** function with an **even** function, is an **even** function.
   Let $f(t)$ and $g(t)$ be even functions, and let $h(t) = f(t)\,g(t)$,
   $$h(-t) = f(-t)\,g(-t) = f(t)\,g(t) = h(t)\,.$$
2. The product of an **even** function with an **odd** function, is an **odd** function.
   Let $f(t)$ be an even function and $g(t)$ be an odd function, and let $h(t) = f(t)\,g(t)$,
   $$h(-t) = f(-t)\,g(-t) = (-f(t))\,g(t) = -h(t)\,.$$
3. The product of an **odd** function with an **odd** function, is an **even** function.
   Let $f(t)$ and $g(t)$ be odd functions, and let $h(t) = f(t)\,g(t)$,
   $$h(-t) = f(-t)\,g(-t) = (-f(t))\,(-g(t)) = f(t)\,g(t) = h(t)\,.$$

1.2 Orthogonality

Definition 1.3 (Inner product)**.** An inner product generalises the dot product in general vector spaces.
In particular, for the function space $\mathscr{F}\left([a,b]\right)$, where $t \in [a,b]$, the inner product is defined as the following:

$$\langle f,\ g\rangle = \int_a^b f\left(t\right) g\left(t\right) \mathrm{d}t$$

for $f,\ g \in \mathscr{F}\left([a,b]\right)$.

Definition 1.4 (Orthogonality)**.** Given an inner product space, two vectors are orthogonal iff

$$\langle f,\ g\rangle = 0.$$

1.3 Orthogonality of Trigonometric Functions

Consider the inner product between the sine and cosine functions on the interval $[-T, T]$.

$$\langle \sin\left(t\right),\ \cos\left(t\right)\rangle = \int_{-T}^{T} \sin\left(t\right)\cos\left(t\right) \mathrm{d}t = 0$$

as the integrand is an odd function.

1.4 Integrals of Trigonometric Functions

For $n \in \mathbb{Z}$:

$$\begin{aligned}
\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right) \mathrm{d}t &= -\frac{1}{2\pi n f_0}\left[\cos\left(2\pi n f_0 t\right)\right]_{t_0}^{t_0+T} \\
&= -\frac{1}{2\pi n f_0}\left[\cos\left(\frac{2\pi n}{T}\left(t_0+T\right)\right) - \cos\left(\frac{2\pi n}{T}t_0\right)\right] \\
&= -\frac{1}{2\pi n f_0}\left[\cos\left(\frac{2\pi n}{T}t_0 + 2\pi n\right) - \cos\left(\frac{2\pi n}{T}t_0\right)\right] \\
&= -\frac{1}{2\pi n f_0}\left[\cos\left(\frac{2\pi n}{T}t_0\right) - \cos\left(\frac{2\pi n}{T}t_0\right)\right] \\
&= -\frac{1}{2\pi n f_0}\left[0\right] \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\int_{t_0}^{t_0+T} \cos\left(2\pi n f_0 t\right) \mathrm{d}t &= \frac{1}{2\pi n f_0} \left[\sin\left(2\pi n f_0 t\right)\right]_{t_0}^{t_0+T} \\
&= \frac{1}{2\pi n f_0} \left[\sin\left(\frac{2\pi n}{T}\left(t_0+T\right)\right) - \sin\left(\frac{2\pi n}{T} t_0\right)\right] \\
&= \frac{1}{2\pi n f_0} \left[\sin\left(\frac{2\pi n}{T} t_0 + 2\pi n\right) - \sin\left(\frac{2\pi n}{T} t_0\right)\right] \\
&= \frac{1}{2\pi n f_0} \left[\sin\left(\frac{2\pi n}{T} t_0\right) - \sin\left(\frac{2\pi n}{T} t_0\right)\right] \\
&= \frac{1}{2\pi n f_0} \left[0\right] \\
&= 0.
\end{aligned}$$

1.4.1 Product of Trigonometric Functions

Recall the Werner formulas:

$$\begin{aligned}
2\cos\left(\alpha\right)\cos\left(\beta\right) &= \cos\left(\alpha-\beta\right) + \cos\left(\alpha+\beta\right) \\
2\sin\left(\alpha\right)\sin\left(\beta\right) &= \cos\left(\alpha-\beta\right) - \cos\left(\alpha+\beta\right) \\
2\sin\left(\alpha\right)\cos\left(\beta\right) &= \sin\left(\alpha-\beta\right) + \sin\left(\alpha+\beta\right)
\end{aligned}$$

For $n,\, m \in \mathbb{N}$,
<u>Product of two cosine functions:</u>

$$\int_{t_0}^{t_0+T} \cos\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right) \mathrm{d}t = \frac{1}{2}\int_{t_0}^{t_0+T} \cos\left(2\pi\left(n-m\right) f_0 t\right) + \cos\left(2\pi\left(n+m\right) f_0 t\right) \mathrm{d}t$$

$n = m \implies n - m = 0$ and $(n+m) \in \mathbb{Z}$, so that the integral of the second term is 0, and the integral of the first term results in $\frac{T}{2}$.
$n \neq m \implies (n-m)\,,\,(n+m) \in \mathbb{Z}$ so that both terms evaluate to 0 when integrated separately.

$$\int_{t_0}^{t_0+T} \cos\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right) \mathrm{d}t = \begin{cases} \frac{T}{2}, & n = m \\ 0, & n \neq m. \end{cases}$$

<u>Product of two sine functions:</u>

$$\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right)\sin\left(2\pi m f_0 t\right) \mathrm{d}t = \frac{1}{2}\int_{t_0}^{t_0+T} \cos\left(2\pi\left(n-m\right) f_0 t\right) - \cos\left(2\pi\left(n+m\right) f_0 t\right) \mathrm{d}t$$

By the same argument as before,

$$\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right)\sin\left(2\pi m f_0 t\right) \mathrm{d}t = \begin{cases} \frac{T}{2}, & n = m \\ 0, & n \neq m. \end{cases}$$

<u>Product of sine and cosine functions:</u>

$$\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right) \mathrm{d}t = \frac{1}{2}\int_{t_0}^{t_0+T} \sin\left(2\pi\left(n-m\right) f_0 t\right) + \sin\left(2\pi\left(n+m\right) f_0 t\right) \mathrm{d}t$$

$n = m \implies n - m = 0$ and $(n + m) \in \mathbb{Z}$, so that the integral reduces to 0.
$n \neq m \implies (n - m)\,,\ (n + m) \in \mathbb{Z}$ so that both terms evaluate to 0 when integrated separately.

$$\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right) \cos\left(2\pi m f_0 t\right) \mathrm{d}t = 0.$$

2 Fourier Series

2.1 Fourier Series Expansion

The **Fourier Series Expansion** of a function $x\left(t\right)$ on the interval $[t_0, t_0 + T]$ is given by

$$x_F\left(t\right) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(2\pi n f_0 t\right) + b_n \sin\left(2\pi n f_0 t\right)\right)$$

where $n \in \mathbb{Z}^+$ and $f_0 = \frac{1}{T}$. The coefficients are given by

$$\begin{aligned}
a_0 &= \frac{1}{T} \int_{t_0}^{t_0+T} x\left(t\right) \mathrm{d}t \\
a_n &= \frac{2}{T} \int_{t_0}^{t_0+T} x\left(t\right) \cos\left(2\pi n f_0 t\right) \mathrm{d}t \\
b_n &= \frac{2}{T} \int_{t_0}^{t_0+T} x\left(t\right) \sin\left(2\pi n f_0 t\right) \mathrm{d}t
\end{aligned}$$

Proof. Let $m \in \mathbb{N}$.
For the coefficient a_0, integrate the function $x\left(t\right)$ over the interval $[t_0, t_0 + T]$.

$$\begin{aligned}
\int_{t_0}^{t_0+T} x\left(t\right) \mathrm{d}t &= \int_{t_0}^{t_0+T} a_0 \,\mathrm{d}t + \sum_{n=1}^{\infty} a_n \cancelto{0}{\int_{t_0}^{t_0+T} \cos\left(2\pi n f_0 t\right) \mathrm{d}t} + \sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right) \mathrm{d}t} \\
\int_{t_0}^{t_0+T} x\left(t\right) \mathrm{d}t &= a_0 T \\
a_0 &= \frac{1}{T} \int_{t_0}^{t_0+T} x\left(t\right) \mathrm{d}t
\end{aligned}$$

so that a_0 represents the average value of x on $[t_0, t_0 + T]$.

For coefficients a_m, multiply the equation by $\cos\left(2\pi m f_0 t\right)$ before integrating.

$$\begin{aligned}
x\left(t\right)\cos\left(2\pi m f_0 t\right) &= a_0 \cos\left(2\pi m f_0 t\right) \\
&\quad + \sum_{n=1}^{\infty} a_n \cos\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right) \\
&\quad + \sum_{n=1}^{\infty} b_n \sin\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right) \\
\int_{t_0}^{t_0+T} x\left(t\right)\cos\left(2\pi m f_0 t\right)\mathrm{d}t &= a_0 \cancelto{0}{\int_{t_0}^{t_0+T} \cos\left(2\pi m f_0 t\right)\mathrm{d}t} \\
&\quad + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right)\mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right)\cos\left(2\pi m f_0 t\right)\mathrm{d}t} \\
\int_{t_0}^{t_0+T} x\left(t\right)\cos\left(2\pi m f_0 t\right)\mathrm{d}t &= a_m \frac{T}{2} \\
a_m &= \frac{2}{T}\int_{t_0}^{t_0+T} x\left(t\right)\cos\left(2\pi m f_0 t\right)\mathrm{d}t
\end{aligned}$$

For coefficients b_m, multiply the equation by $\sin\left(2\pi m f_0 t\right)$ before integrating.

$$\begin{aligned}
x\left(t\right)\sin\left(2\pi m f_0 t\right) &= a_0 \sin\left(2\pi m f_0 t\right) \\
&\quad + \sum_{n=1}^{\infty} a_n \cos\left(2\pi n f_0 t\right)\sin\left(2\pi m f_0 t\right) \\
&\quad + \sum_{n=1}^{\infty} b_n \sin\left(2\pi n f_0 t\right)\sin\left(2\pi m f_0 t\right) \\
\int_{t_0}^{t_0+T} x\left(t\right)\sin\left(2\pi m f_0 t\right)\mathrm{d}t &= a_0 \cancelto{0}{\int_{t_0}^{t_0+T} \sin\left(2\pi m f_0 t\right)\mathrm{d}t} \\
&\quad + \sum_{n=1}^{\infty} a_n \cancelto{0}{\int_{t_0}^{t_0+T} \cos\left(2\pi n f_0 t\right)\sin\left(2\pi m f_0 t\right)\mathrm{d}t} \\
&\quad + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right)\sin\left(2\pi m f_0 t\right)\mathrm{d}t \\
\int_{t_0}^{t_0+T} x\left(t\right)\sin\left(2\pi m f_0 t\right)\mathrm{d}t &= b_m \frac{T}{2} \\
b_m &= \frac{2}{T}\int_{t_0}^{t_0+T} x\left(t\right)\sin\left(2\pi m f_0 t\right)\mathrm{d}t
\end{aligned}$$

□

2.1.1 Convergence of a Fourier Series

If $x(t)$ is piecewise smooth on $[t_0, t_0 + L]$, $x_F(t)$ converges to

$$x_F(t) = \lim_{\epsilon \to 0^+} \frac{x(t+\epsilon) + x(t-\epsilon)}{2}$$

that is, $x = x_F$, except at discontinuities, where f_F is equal to the point halfway between the left- and right-handed limits.

2.1.2 Periodicity of a Fourier Series

If x is non-periodic, x_F converges to the periodic extension of x. The endpoints may converge non-uniformly, corresponding to jump discontinuities in the periodic extension of x.

2.2 Fourier Cosine Series

Consider the Fourier series expansion of an even function x on the interval $[-\frac{T}{2}, \frac{T}{2}]$, i.e., $t_0 = -\frac{T}{2}$. In this case,

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin(2\pi n f_0 t) \, \mathrm{d}t = 0$$

and the Fourier series is a "Fourier cosine series", given by:

$$x_c(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t)$$

with coefficients

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \, \mathrm{d}t = \frac{2}{T} \int_{0}^{\frac{T}{2}} x(t) \, \mathrm{d}t \\ a_n &= \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos(2\pi n f_0 t) \, \mathrm{d}t = \frac{4}{T} \int_{0}^{\frac{T}{2}} x(t) \cos(2\pi n f_0 t) \, \mathrm{d}t. \end{aligned}$$

2.3 Fourier Sine Series

Consider the Fourier series expansion of an odd function x on the interval $[-\frac{T}{2}, \frac{T}{2}]$. In this case

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin(2\pi n f_0 t) \, \mathrm{d}t = 0$$

and the Fourier series is a "Fourier sine series", given by:

$$x_s(t) = \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t)$$

with coefficients

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(2\pi n f_0 t)\,\mathrm{d}t = \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t)\sin(2\pi n f_0 t)\,\mathrm{d}t.$$

3 Complex Fourier Series

Definition 3.1. The **Complex Fourier Series Expansion** is a concise form of the Fourier series expansion that uses complex exponentials with a single unknown coefficient.

$$x_C(t) = \sum_{n=-\infty}^{\infty} c_n e^{-j2\pi n f_0 t}$$

where

$$c_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-j2\pi n f_0 t}\,\mathrm{d}t.$$

for $n \in \mathbb{Z}$ and $f_0 = \frac{1}{T}$.

To determine the complex Fourier series expansion consider the following identities:

$$\begin{aligned}\cos(\theta) &= \frac{e^{j\theta}+e^{-j\theta}}{2}\\ \sin(\theta) &= -j\frac{e^{j\theta}-e^{-j\theta}}{2}.\end{aligned}$$

By substituting these identities into the Fourier series expansion summand, we obtain:

$$\begin{aligned}a_n\cos(2\pi n f_0 t) + b_n\sin(2\pi n f_0 t) &= a_n\frac{e^{j2\pi n f_0 t}+e^{-j2\pi n f_0 t}}{2} - jb_n\frac{e^{j2\pi n f_0 t}-e^{-j2\pi n f_0 t}}{2}\\ &= \frac{a_n - jb_n}{2}e^{j2\pi n f_0 t} + \frac{a_n + jb_n}{2}e^{-j2\pi n f_0 t}\end{aligned}$$

Let $c_n = \frac{a_n - jb_n}{2}$ and $c_n^* = \frac{a_n + jb_n}{2}$ (we will see how this simplifies later). Using the definitions for a_n and b_n:

$$\begin{aligned}c_n &= \frac{1}{2}(a_n - jb_n)\\ &= \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\left(\cos(2\pi n f_0 t) - j\sin(2\pi n f_0 t)\right)\mathrm{d}t\\ &= \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-j2\pi n f_0 t}\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
c_n^* &= \frac{1}{2}\left(a_n + jb_n\right) \\
&= \frac{1}{T}\int_{t_0}^{t_0+T} x\left(t\right)\left(\cos\left(2\pi n f_0 t\right) + j\sin\left(2\pi n f_0 t\right)\right)\mathrm{d}t \\
&= \frac{1}{T}\int_{t_0}^{t_0+T} x\left(t\right)e^{j2\pi n f_0 t}\,\mathrm{d}t \\
&= c_{-n}
\end{aligned}
$$

Let $c_0 = a_0$, so that

$$
\begin{aligned}
x_C\left(t\right) &= a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(2\pi n f_0 t\right) + b_n\sin\left(2\pi n f_0 t\right)\right) \\
&= c_0 + \sum_{n=1}^{\infty}\left(c_n e^{j2\pi n f_0 t} + c_{-n}e^{-j2\pi n f_0 t}\right) \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{j2\pi n f_0 t} + \sum_{n=1}^{\infty} c_{-n}e^{-j2\pi n f_0 t} \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{j2\pi n f_0 t} + \sum_{n=-\infty}^{-1} c_n e^{j2\pi n f_0 t} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}.
\end{aligned}
$$

3.1 Converting between Fourier Series Representations

Given the Trigonometric and Exponential Fourier Series Representations (FSR), we can develop a relationship between the coefficients a_n, b_n, and c_n by:

$$
\begin{aligned}
a_0 &= c_0 \\
a_n &= c_n + c_{-n} \\
b_n &= j\left(c_0 - c_{-n}\right).
\end{aligned}
$$

3.2 Magnitude and Phase Spectra

As c_n is a complex number, consider the polar representation of c_n:

$$c_n = |c_n|e^{j\theta_n}$$

where $|c_n|$ is the magnitude spectra and θ_n is the phase spectra of $x\left(t\right)$.
The plot of $|c_n|$ against n is called the "magnitude spectrum" and the plot of θ_n against n is called the "phase spectrum" of $x\left(t\right)$.