Signal Analysis

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Contents

1 Properties of Mathematical Functions

1.1 Even and Odd Functions

Definition 1.1 (Even function)**.** A function $x(t)$ is even if

$$x(-t) = x(t)$$

for all t in the function's domain. Even functions are symmetric about the vertical axis.

Definition 1.2 (Odd function)**.** A function $x(t)$ is odd if

$$x(-t) = -x(t)$$

for all t in the function's domain. Odd functions are symmetric about the origin.

1.1.1 Integrating Even and Odd Functions

When integrating an **even** function $x(t)$ over the domain $[-T, T]$:

$$\int_{-T}^{T} x(t)\,\mathrm{d}t = 2\int_{0}^{T} x(t)\,\mathrm{d}t.$$

Similarly, when integrating an **odd** function $x(t)$ over the domain $[-T, T]$:

$$\int_{-T}^{T} x(t)\,\mathrm{d}t = 0.$$

1.1.2 Products of Even and Odd Functions

1. The product of an **even** function with an **even** function, is an **even** function.

   Proof. Let $f(t)$ and $g(t)$ be even functions, and let $h(t) = f(t)\,g(t)$, then,

   $$h(-t) = f(-t)\,g(-t) = f(t)\,g(t) = h(t).$$

   □

2. The product of an **even** function with an **odd** function, is an **odd** function.

   Proof. Let $f(t)$ be an even function and $g(t)$ be an odd function, and let $h(t) = f(t)\,g(t)$, then,

   $$h(-t) = f(-t)\,g(-t) = (-f(t))\,g(t) = -h(t).$$

   □

3. The product of an **odd** function with an **odd** function, is an **even** function.

   Proof. Let $f(t)$ and $g(t)$ be odd functions, and let $h(t) = f(t)\,g(t)$, then,

   $$h(-t) = f(-t)\,g(-t) = (-f(t))\,(-g(t)) = f(t)\,g(t) = h(t).$$

   □

1.2 Orthogonality

Definition 1.3 (Inner product)**.** An inner product generalises the dot product for general vector spaces. In particular, for the function space $\mathscr{F}([a, b])$, where $t \in [a, b]$, the inner product is defined as the following:

$$\langle f,\ g\rangle = \int_a^b f(t)\, g(t)\, \mathrm{d}t$$

for $f,\ g \in \mathscr{F}([a, b])$.

Definition 1.4 (Orthogonality)**.** Given an inner product space, two vectors are orthogonal iff

$$\langle f,\ g\rangle = 0.$$

1.3 Orthogonality of Trigonometric Functions

Consider the inner product between the sine and cosine functions on the interval $[-T, T]$.

$$\langle \sin(t),\ \cos(t)\rangle = \int_{-T}^{T} \sin(t)\cos(t)\, \mathrm{d}t = 0,$$

as the integrand is an odd function.

1.4 Integrals of Trigonometric Functions

For $n \in \mathbb{Z}$:

$$\begin{aligned}
\int_{t_0}^{t_0+T} \sin(2\pi n f_0 t)\, \mathrm{d}t &= -\frac{1}{2\pi n f_0}\left[\cos(2\pi n f_0 t)\right]_{t_0}^{t_0+T} \\
&= -\frac{1}{2\pi n f_0}\left[\cos\left(\frac{2\pi n}{T}(t_0 + T)\right) - \cos\left(\frac{2\pi n}{T} t_0\right)\right] \\
&= -\frac{1}{2\pi n f_0}\left[\cos\left(\frac{2\pi n}{T} t_0 + 2\pi n\right) - \cos\left(\frac{2\pi n}{T} t_0\right)\right] \\
&= -\frac{1}{2\pi n f_0}\left[\cos\left(\frac{2\pi n}{T} t_0\right) - \cos\left(\frac{2\pi n}{T} t_0\right)\right] \\
&= -\frac{1}{2\pi n f_0}[0] \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\int_{t_0}^{t_0+T} \cos(2\pi n f_0 t)\,\mathrm{d}t &= \frac{1}{2\pi n f_0}\left[\sin(2\pi n f_0 t)\right]_{t_0}^{t_0+T} \\
&= \frac{1}{2\pi n f_0}\left[\sin\left(\frac{2\pi n}{T}(t_0+T)\right) - \sin\left(\frac{2\pi n}{T}t_0\right)\right] \\
&= \frac{1}{2\pi n f_0}\left[\sin\left(\frac{2\pi n}{T}t_0 + 2\pi n\right) - \sin\left(\frac{2\pi n}{T}t_0\right)\right] \\
&= \frac{1}{2\pi n f_0}\left[\sin\left(\frac{2\pi n}{T}t_0\right) - \sin\left(\frac{2\pi n}{T}t_0\right)\right] \\
&= \frac{1}{2\pi n f_0}[0] \\
&= 0.
\end{aligned}$$

1.4.1 Product of Trigonometric Functions

Recall the Werner formulas:

$$\begin{aligned}
2\cos(\alpha)\cos(\beta) &= \cos(\alpha-\beta) + \cos(\alpha+\beta) \\
2\sin(\alpha)\sin(\beta) &= \cos(\alpha-\beta) - \cos(\alpha+\beta) \\
2\sin(\alpha)\cos(\beta) &= \sin(\alpha-\beta) + \sin(\alpha+\beta)
\end{aligned}$$

For $n,\ m \in \mathbb{N}$,
<u>Product of two cosine functions:</u>

$$\int_{t_0}^{t_0+T} \cos(2\pi n f_0 t)\cos(2\pi m f_0 t)\,\mathrm{d}t = \frac{1}{2}\int_{t_0}^{t_0+T} \cos(2\pi(n-m)f_0 t) + \cos(2\pi(n+m)f_0 t)\,\mathrm{d}t$$

When:

- $n = m \implies n - m = 0$ and $(n+m) \in \mathbb{Z}$, so that the integral of the second term is 0, and the integral of the first term results in $\frac{T}{2}$.
- $n \neq m \implies (n-m),\ (n+m) \in \mathbb{Z}$ so that both terms evaluate to 0 when integrated separately.

$$\int_{t_0}^{t_0+T} \cos(2\pi n f_0 t)\cos(2\pi m f_0 t)\,\mathrm{d}t = \begin{cases} \frac{T}{2}, & n = m \\ 0, & n \neq m. \end{cases}$$

<u>Product of two sine functions:</u>

$$\int_{t_0}^{t_0+T} \sin(2\pi n f_0 t)\sin(2\pi m f_0 t)\,\mathrm{d}t = \frac{1}{2}\int_{t_0}^{t_0+T} \cos(2\pi(n-m)f_0 t) - \cos(2\pi(n+m)f_0 t)\,\mathrm{d}t$$

By the same argument as before,

$$\int_{t_0}^{t_0+T} \sin(2\pi n f_0 t)\sin(2\pi m f_0 t)\,\mathrm{d}t = \begin{cases} \frac{T}{2}, & n = m \\ 0, & n \neq m. \end{cases}$$

Product of sine and cosine functions:

$$\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right) \cos\left(2\pi m f_0 t\right) \mathrm{d}t = \frac{1}{2} \int_{t_0}^{t_0+T} \sin\left(2\pi \left(n-m\right) f_0 t\right) + \sin\left(2\pi \left(n+m\right) f_0 t\right) \mathrm{d}t$$

When:

- $n = m \implies n - m = 0$ and $(n+m) \in \mathbb{Z}$, so that the integral reduces to 0.
- $n \neq m \implies (n-m), (n+m) \in \mathbb{Z}$ so that both terms evaluate to 0 when integrated separately.

$$\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right) \cos\left(2\pi m f_0 t\right) \mathrm{d}t = 0.$$

2 Fourier Series

2.1 Fourier Series Expansion

The **Fourier series expansion** of a function $x(t)$ on the interval $[t_0, t_0 + T]$ is given by

$$x_F(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(2\pi n f_0 t\right) + b_n \sin\left(2\pi n f_0 t\right)\right)$$

where $n \in \mathbb{Z}^+$ and $f_0 = \frac{1}{T}$. The coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \,\mathrm{d}t \\ a_n &= \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos\left(2\pi n f_0 t\right) \mathrm{d}t \\ b_n &= \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin\left(2\pi n f_0 t\right) \mathrm{d}t \end{aligned}$$

Proof. Let $m \in \mathbb{N}$. For the coefficient a_0, integrate the function $x(t)$ over the interval $[t_0, t_0 + T]$.

$$\begin{aligned} \int_{t_0}^{t_0+T} x(t) \,\mathrm{d}t &= \int_{t_0}^{t_0+T} a_0 \,\mathrm{d}t + \sum_{n=1}^{\infty} a_n \cancelto{0}{\int_{t_0}^{t_0+T} \cos\left(2\pi n f_0 t\right) \mathrm{d}t} + \sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{t_0}^{t_0+T} \sin\left(2\pi n f_0 t\right) \mathrm{d}t} \\ \int_{t_0}^{t_0+T} x(t) \,\mathrm{d}t &= a_0 T \\ a_0 &= \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \,\mathrm{d}t \end{aligned}$$

so that a_0 represents the average value of x on $[t_0, t_0 + T]$. This coefficient also represents the DC component of a signal. For coefficients a_m, multiply the equation by $\cos(2\pi m f_0 t)$ before integrating.

$$
\begin{aligned}
x(t)\cos(2\pi m f_0 t) &= a_0 \cos(2\pi m f_0 t) \\
&\quad + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t)\cos(2\pi m f_0 t) \\
&\quad + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t)\cos(2\pi m f_0 t) \\
\int_{t_0}^{t_0+T} x(t)\cos(2\pi m f_0 t)\,\mathrm{d}t &= a_0 \cancelto{0}{\int_{t_0}^{t_0+T} \cos(2\pi m f_0 t)\,\mathrm{d}t} \\
&\quad + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos(2\pi n f_0 t)\cos(2\pi m f_0 t)\,\mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{t_0}^{t_0+T} \sin(2\pi n f_0 t)\cos(2\pi m f_0 t)\,\mathrm{d}t} \\
\int_{t_0}^{t_0+T} x(t)\cos(2\pi m f_0 t)\,\mathrm{d}t &= a_m \frac{T}{2} \\
a_m &= \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(2\pi m f_0 t)\,\mathrm{d}t
\end{aligned}
$$

For coefficients b_m, multiply the equation by $\sin(2\pi m f_0 t)$ before integrating.

$$
\begin{aligned}
x(t)\sin(2\pi m f_0 t) = a_0 \sin(2\pi m f_0 t) \\
&+ \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t)\sin(2\pi m f_0 t) \\
&+ \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t)\sin(2\pi m f_0 t) \\
\int_{t_0}^{t_0+T} x(t)\sin(2\pi m f_0 t)\,\mathrm{d}t = a_0 \cancelto{0}{\int_{t_0}^{t_0+T} \sin(2\pi m f_0 t)\,\mathrm{d}t} \\
&+ \sum_{n=1}^{\infty} a_n \cancelto{0}{\int_{t_0}^{t_0+T} \cos(2\pi n f_0 t)\sin(2\pi m f_0 t)\,\mathrm{d}t} \\
&+ \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin(2\pi n f_0 t)\sin(2\pi m f_0 t)\,\mathrm{d}t \\
\int_{t_0}^{t_0+T} x(t)\sin(2\pi m f_0 t)\,\mathrm{d}t = b_m \frac{T}{2} \\
b_m = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(2\pi m f_0 t)\,\mathrm{d}t
\end{aligned}
$$

□

2.1.1 Convergence of a Fourier Series

If $x(t)$ is piecewise smooth on $[t_0, t_0 + L]$, $x_F(t)$ converges to

$$x_F(t) = \lim_{\epsilon \to 0^+} \frac{x(t+\epsilon) + x(t-\epsilon)}{2}$$

that is, $x = x_F$, except at discontinuities, where f_F is equal to the point halfway between the left- and right-handed limits.

2.1.2 Periodicity of a Fourier Series

If x is non-periodic, x_F converges to the periodic extension of x. The endpoints may converge non-uniformly, corresponding to jump discontinuities in the periodic extension of x.

2.2 Fourier Cosine Series

Consider the Fourier series expansion of an even function x on the interval $[-\frac{T}{2}, \frac{T}{2}]$, i.e., $t_0 = -\frac{T}{2}$. In this case,

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(2\pi n f_0 t)\,\mathrm{d}t = 0$$

and the expansion is called the "cosine" or "even" series expansion of x given by:

$$x_c(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t)$$

with coefficients

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,\mathrm{d}t = \frac{2}{T}\int_{0}^{\frac{T}{2}} x(t)\,\mathrm{d}t$$
$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos(2\pi n f_0 t)\,\mathrm{d}t = \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t)\cos(2\pi n f_0 t)\,\mathrm{d}t.$$

2.3 Fourier Sine Series

Consider the Fourier series expansion of an odd function x on the interval $[-\frac{T}{2}, \frac{T}{2}]$. In this case

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(2\pi n f_0 t)\,\mathrm{d}t = 0$$

and the expansion is called the "sine" or "odd" series expansion of x given by:

$$x_s(t) = \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t)$$

with coefficients

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(2\pi n f_0 t)\,\mathrm{d}t = \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t)\sin(2\pi n f_0 t)\,\mathrm{d}t.$$

3 Complex Fourier Series

Definition 3.1. The **complex Fourier series expansion** is a concise form of the Fourier series expansion that uses complex exponentials with a single unknown coefficient c_n:

$$x_C(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

where

$$c_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,e^{-j2\pi n f_0 t}\,\mathrm{d}t$$

for $n \in \mathbb{Z}\setminus\{0\}$ and $f_0 = \frac{1}{T}$. Notice that the mean term remains the same: $c_0 = a_0$.

To determine the complex Fourier series expansion of a function, consider the following trigonometric properties which are derived from Euler's formula:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin(\theta) = -j\frac{e^{j\theta} - e^{-j\theta}}{2}.$$

By substituting these identities into the Fourier series expansion summand, we obtain:

$$\begin{aligned} a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t) &= a_n \frac{e^{j2\pi n f_0 t} + e^{-j2\pi n f_0 t}}{2} - jb_n \frac{e^{j2\pi n f_0 t} - e^{-j2\pi n f_0 t}}{2} \\ &= \frac{a_n - jb_n}{2} e^{j2\pi n f_0 t} + \frac{a_n + jb_n}{2} e^{-j2\pi n f_0 t} \end{aligned}$$

Then, let $c_n = \frac{a_n - jb_n}{2}$ and $c_n^* = \frac{a_n + jb_n}{2}$ (we will see how this simplifies later). Using the definitions for a_n and b_n, the coefficients c_n and c_n^* simplify to:

$$\begin{aligned} c_n &= \frac{1}{2}(a_n - jb_n) \\ &= \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \left(\cos(2\pi n f_0 t) - j \sin(2\pi n f_0 t)\right) \mathrm{d}t \\ &= \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-j2\pi n f_0 t} \, \mathrm{d}t \end{aligned}$$

$$\begin{aligned} c_n^* &= \frac{1}{2}(a_n + jb_n) \\ &= \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \left(\cos(2\pi n f_0 t) + j \sin(2\pi n f_0 t)\right) \mathrm{d}t \\ &= \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{j2\pi n f_0 t} \, \mathrm{d}t \\ &= c_{-n} \end{aligned}$$

Finally, we let $c_0 = a_0$, so that

$$\begin{aligned} x_C(t) &= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)\right) \\ &= c_0 + \sum_{n=1}^{\infty} \left(c_n e^{j2\pi n f_0 t} + c_{-n} e^{-j2\pi n f_0 t}\right) \\ &= c_0 + \sum_{n=1}^{\infty} c_n e^{j2\pi n f_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-j2\pi n f_0 t} \\ &= c_0 + \sum_{n=1}^{\infty} c_n e^{j2\pi n f_0 t} + \sum_{n=-\infty}^{-1} c_n e^{j2\pi n f_0 t} \\ &= \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}. \end{aligned}$$

3.1 Converting between Fourier Series Representations

Given the trigonometric and exponential (complex) Fourier series representations, we can develop a relationship between the coefficients a_n, b_n, and c_n as follows:

$$\begin{aligned} a_0 &= c_0 \\ a_n &= c_n + c_{-n} \\ b_n &= j\left(c_n - c_{-n}\right) \end{aligned}$$

and in the reverse direction:

$$\begin{aligned} c_{-n} &= \frac{a_n + jb_n}{2} \\ c_0 &= a_0 \\ c_n &= \frac{a_n - jb_n}{2}. \end{aligned}$$

3.2 Magnitude and Phase Spectra

As c_n is a complex number, we can consider the polar representation of c_n:

$$c_n = |c_n| e^{j\theta_n}$$

and define the **magnitude spectra** as $|c_n|$ and the **phase spectra** as θ_n. Here, the plot of $|c_n|$ against n is called the "magnitude spectrum" of $x\left(t\right)$ and the plot of θ_n against n is called the "phase spectrum" of $x\left(t\right)$.

Theorem 3.2.1 (Spectra of a real signal)**.** *Given any real function $x\left(t\right)$, the magnitude spectrum is always an even function, and the phase spectrum is always an odd function.*

Proof. Given a real function $x\left(t\right)$, the exponential Fourier series is given by,

$$x_C\left(t\right) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

this is equivalent to

$$x_C\left(t\right) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-j2\pi n f_0 t}.$$

The conjugate of $x_C\left(t\right)$ yields,

$$\overline{x_C\left(t\right)} = \sum_{n=-\infty}^{\infty} \overline{c_n} e^{-j2\pi m f_0 t}$$

As $x\left(t\right) \in \mathbb{R}$, $x_C\left(t\right) = \overline{x_C\left(t\right)}$, so that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} c_{-n} e^{-j2\pi n f_0 t} &= \sum_{n=-\infty}^{\infty} \overline{c_n} e^{-j2\pi m f_0 t} \\ c_{-n} e^{-j2\pi n f_0 t} &= \overline{c_n} e^{-j2\pi n f_0 t} \\ c_{-n} &= \overline{c_n} \end{aligned}$$

Therefore, by representing the above coefficients in polar form, we get,

$$|c_{-n}|e^{j\theta_{-n}} = |c_n|e^{-j\theta_n}$$

as required. □

3.3 Even and Odd Functions

Given an even function $x(t)$, $b_n = 0$, therefore

$$\begin{aligned} 0 &= j(c_n - c_{-n}) \\ c_n &= c_{-n} \end{aligned}$$

so that,

$$\begin{aligned} a_n &= c_n + c_{-n} \\ a_n &= 2c_n \\ c_n &= c_{-n} = \frac{a_n}{2} \end{aligned}$$

Hence c_n and c_{-n} are real coefficients, so that $\theta_n = m\pi$ for some $m \in \mathbb{Z}$. Given an odd function $x(t)$, $a_0 = 0$ and $a_n = 0$, therefore $c_0 = 0$ and

$$\begin{aligned} 0 &= c_n + c_{-n} \\ c_n &= -c_{-n} \end{aligned}$$

so that,

$$\begin{aligned} b_n &= j(c_n - c_{-n}) \\ b_n &= j(c_n + c_n) \\ b_n &= j2c_n \\ c_n &= -j\frac{b_n}{2} \\ c_{-n} &= j\frac{b_n}{2} \end{aligned}$$

Hence c_n and c_{-n} are purely imaginary coefficients, so that $\theta_n = (2m+1)\frac{\pi}{2}$ for some $m \in \mathbb{Z}$.

3.4 Signal Representations

A signal can be represented in various forms depending on the method of measurement. There are three main forms of signal representation:

- Analogue: Continuous in time and continuous in amplitude.
- Discrete: Discrete in time and continuous in amplitude.
- Digital: Discrete in time and discrete in amplitude.

The process of taking discrete time measurements is known as sampling, and taking discrete amplitude measurements is known as quantisation.

Definition 3.2 (Integral transform)**.** An *integral transform* transforms a function through the process of integration, producing a new function of another variable. Two common examples of integral transforms are the Fourier transform and the Laplace transform.

4 Fourier Transform

The Fourier Transform allows us to extend the techniques used in Fourier series representations of functions to non-periodic signals.

Definition 4.1 (Fourier transform)**.** The Fourier transform of a function $x(t)$ is defined by:

$$\mathscr{F}\left\{x(t)\right\} = X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} \,\mathrm{d}t.$$

where $X(f) = \mathscr{F}\left\{x(t)\right\}$ is a function of frequency f.

Definition 4.2 (Inverse Fourier transform)**.** The inverse Fourier transform of a function $X(f)$ is defined by:

$$\mathscr{F}^{-1}\left\{X(f)\right\} = x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} \,\mathrm{d}f.$$

This integral transform is commonly represented as a "Fourier pair" using the following notation:

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(f).$$

We can derive this definition by considering the complex Fourier series representation of $x(t)$ on the interval $[-\frac{T}{2}, \frac{T}{2}]$:

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t} \\ &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-j2\pi n f_0 t} \,\mathrm{d}t\right] e^{j2\pi n f_0 t}. \end{aligned}$$

If we let $f_n = n f_0$, then $\Delta f_n = f_{n+1} - f_n = (n+1) f_0 - n f_0 = f_0 = \frac{1}{T}$:

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-j2\pi f_n t} \,\mathrm{d}t\right] e^{j2\pi f_n t}\, \Delta f_n.$$

By recognition, this is a Riemann sum, so that by taking the limit $T \to \infty$, we find:

$$x(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} \,\mathrm{d}t\right] e^{j2\pi f t} \,\mathrm{d}f$$

where

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} \,\mathrm{d}t.$$

Corollary 4.0.0.1 (Dirichlet conditions)**.** *The Dirichlet conditions provide sufficient conditions for a real-valued function x to be equal to its Fourier Transform X, at each point where x is continuous. The conditions are:*

1. *x has a finite number of maxima and minima over $[-L, L]$.*
2. *x has a finite number of discontinuities, in which the derivative x' exists and does not change sign.*
3. *$\int_{-\infty}^{\infty} |x(t)| \, \mathrm{d}t$ exists.*

4.1 Fourier Transform Properties

4.1.1 Linearity

$$a_1 x_1(t) \pm a_2 x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} a_1 X_1(f) \pm a_2 X_2(f).$$

Due to the linearity of the integral, the Fourier transform is a linear operator.

4.1.2 Complex Conjugate

$$x^*(t) \overset{\mathscr{F}}{\longleftrightarrow} X^*(-f).$$

Proof.

$$\begin{aligned}
\mathscr{F}\{x^*(t)\} &= \int_{-\infty}^{\infty} x^*(t) e^{-j2\pi f t} \, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} x^*(t) \overline{e^{j2\pi f t}} \, \mathrm{d}t^* \\
&= \int_{-\infty}^{\infty} \overline{x(t) e^{j2\pi f t} \, \mathrm{d}t} \\
&= \overline{\int_{-\infty}^{\infty} x(t) e^{j2\pi f t} \, \mathrm{d}t} \\
&= \overline{\int_{-\infty}^{\infty} x(t) e^{-j2\pi(-f)t} \, \mathrm{d}t} \\
&= X^*(-f).
\end{aligned}$$

□

4.1.3 Time Shift

$$x(t - t_0) \overset{\mathscr{F}}{\longleftrightarrow} e^{-j2\pi f t_0} X(f).$$

Proof.

$$\begin{aligned}
x\left(t-t_0\right) &= \int_{-\infty}^{\infty} X\left(f\right) e^{j2\pi f\left(t-t_0\right)} \,\mathrm{d}f \\
&= \int_{-\infty}^{\infty} \left[e^{-j2\pi f t_0} X\left(f\right)\right] e^{-j2\pi f t} \,\mathrm{d}f \\
&= \mathscr{F}^{-1}\left\{e^{-j2\pi f t_0} X\left(f\right)\right\}.
\end{aligned}$$

□

4.1.4 Frequency Shift

$$e^{j2\pi f_0 t} x\left(t\right) \overset{\mathscr{F}}{\longleftrightarrow} X\left(f-f_0\right).$$

Proof.

$$\begin{aligned}
X\left(f-f_0\right) &= \int_{-\infty}^{\infty} x\left(t\right) e^{-j2\pi\left(f-f_0\right)t} \,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \left[e^{j2\pi f_0 t} x\left(t\right)\right] e^{-j2\pi f t} \,\mathrm{d}t \\
&= \mathscr{F}\left\{e^{j2\pi f_0 t} x\left(t\right)\right\}
\end{aligned}$$

□

4.1.5 Time Reversal

$$x\left(-t\right) \overset{\mathscr{F}}{\longleftrightarrow} X\left(-f\right).$$

Proof. Consider the substitution $u = -t$ so that $\mathrm{d}u = -\,\mathrm{d}t$:

$$\begin{aligned}
\mathscr{F}\left\{x\left(-t\right)\right\} &= \int_{-\infty}^{\infty} x\left(-t\right) e^{-j2\pi f t} \,\mathrm{d}t \\
&= -\int_{\infty}^{-\infty} x\left(u\right) e^{j2\pi f u} \,\mathrm{d}u \\
&= \int_{-\infty}^{\infty} x\left(u\right) e^{-j2\pi\left(-f\right)u} \,\mathrm{d}u \\
&= X\left(-f\right)
\end{aligned}$$

□

4.1.6 Time Scaling

$$x\left(at\right) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{\left|a\right|} X\left(\frac{f}{a}\right).$$

Proof. Consider the substitution $u = at$ so that $\mathrm{d}u = a\,\mathrm{d}t$:

$$\begin{aligned}\mathscr{F}\left\{x\left(at\right)\right\} &= \int_{-\infty}^{\infty} x\left(at\right) e^{-j2\pi f t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} x\left(u\right) e^{-j2\pi f \frac{u}{a}} \frac{1}{a}\,\mathrm{d}u \\ &= \frac{1}{a}\int_{-\infty}^{\infty} x\left(u\right) e^{-j2\pi \frac{f}{a} u}\,\mathrm{d}u \\ &= \frac{1}{a} X\left(\frac{f}{a}\right)\end{aligned}$$

when $a > 0$, $a = |a|$:

$$\mathscr{F}\left\{x\left(at\right)\right\} = \frac{1}{|a|} X\left(\frac{f}{a}\right).$$

When $a < 0$, $a = -|a|$:

$$\begin{aligned}\mathscr{F}\left\{x\left(-|a|t\right)\right\} &= \frac{1}{|a|} X\left(-\frac{f}{|a|}\right) \\ &= \frac{1}{|a|} X\left(\frac{f}{a}\right).\end{aligned}$$

□

4.1.7 Time Differentiation

$$\frac{\mathrm{d}^n x\left(t\right)}{\mathrm{d}t^n} \overset{\mathscr{F}}{\longleftrightarrow} \left(j2\pi f\right)^n X\left(f\right).$$

Proof. Consider the representation of $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ using the inverse Fourier transform:

$$\begin{aligned}\frac{\mathrm{d}x\left(t\right)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} X\left(f\right) e^{j2\pi f t}\,\mathrm{d}f \\ &= \int_{-\infty}^{\infty} X\left(f\right) \frac{\mathrm{d}}{\mathrm{d}t} e^{j2\pi f t}\,\mathrm{d}f \\ &= j2\pi f \int_{-\infty}^{\infty} X\left(f\right) e^{j2\pi f t}\,\mathrm{d}f \\ &= j2\pi f x\left(t\right)\end{aligned}$$

then the Fourier transform of $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ is given by:

$$\begin{aligned}\mathscr{F}\left\{\frac{\mathrm{d}x\left(t\right)}{\mathrm{d}t}\right\} &= \mathscr{F}\left\{j2\pi f x\left(t\right)\right\} \\ &= j2\pi f X\left(f\right)\end{aligned}$$

Repeated differentiation yields the following result:

$$\mathscr{F}\left\{\frac{\mathrm{d}^n x\left(t\right)}{\mathrm{d}t^n}\right\} = \left(j2\pi f\right)^n X\left(f\right)$$

□

4.1.8 Frequency Differentiation

$$\left(\frac{2\pi}{j}\right)^n t^n x(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{\mathrm{d}^n X(f)}{\mathrm{d}f^n}.$$

Proof. Consider the representation of $\frac{\mathrm{d}X(f)}{\mathrm{d}f}$ using the Fourier transform:

$$\begin{aligned}
\frac{\mathrm{d}X(f)}{\mathrm{d}f} &= \frac{\mathrm{d}}{\mathrm{d}f}\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} x(t)\, \frac{\mathrm{d}}{\mathrm{d}f} e^{-j2\pi ft}\,\mathrm{d}t \\
&= -j2\pi t\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\,\mathrm{d}t \\
&= -j2\pi t X(f)
\end{aligned}$$

then the inverse Fourier transform of $\frac{\mathrm{d}X(f)}{\mathrm{d}f}$ is given by:

$$\begin{aligned}
\mathscr{F}^{-1}\left\{\frac{\mathrm{d}X(f)}{\mathrm{d}f}\right\} &= \mathscr{F}^{-1}\left\{-j2\pi t X(f)\right\} \\
&= -j2\pi t \mathscr{F}^{-1}\left\{X(f)\right\} \\
&= \frac{2\pi}{j} t x(t)
\end{aligned}$$

hence

$$\frac{2\pi}{j} t x(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{\mathrm{d}X(f)}{\mathrm{d}f}.$$

Repeated differentiation yields the required result. □

4.1.9 Time Multiplication

$$t^n x(t) \overset{\mathscr{F}}{\longleftrightarrow} \left(\frac{j}{2\pi}\right)^n \frac{\mathrm{d}^n X(f)}{\mathrm{d}f^n}.$$

See the previous section for the proof.

4.1.10 Time Integration

$$\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{j2\pi f} X(f).$$

Proof. Consider the representation of $\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$ using the inverse Fourier transform:

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\,\mathrm{d}f \\
\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau &= \int_{-\infty}^{t} \left[\int_{-\infty}^{\infty} X(f)\, e^{j2\pi f\tau}\,\mathrm{d}f\right]\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} X(f) \left[\int_{-\infty}^{t} e^{j2\pi f\tau}\,\mathrm{d}\tau\right]\mathrm{d}f \\
&= \int_{-\infty}^{\infty} X(f) \left[\frac{1}{j2\pi f} e^{j2\pi f\tau}\right]_{-\infty}^{t}\mathrm{d}f \\
&= \frac{1}{j2\pi f}\int_{-\infty}^{\infty} X(f) \left[e^{j2\pi f\tau}\right]_{-\infty}^{t}\mathrm{d}f \\
&= \frac{1}{j2\pi f}\int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\,\mathrm{d}f \\
&= \frac{1}{j2\pi f} x(t)\,.
\end{aligned}
$$

We can now take the Fourier transform of $\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$ to obtain:

$$
\begin{aligned}
\mathscr{F}\left\{\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau\right\} &= \mathscr{F}\left\{\frac{1}{j2\pi f} x(t)\right\} \\
&= \frac{1}{j2\pi f}\mathscr{F}\{x(t)\} \\
&= \frac{1}{j2\pi f} X(f)\,.
\end{aligned}
$$

□

4.1.11 Duality

$$X(t) \overset{\mathscr{F}}{\longleftrightarrow} x(-f)\,.$$

Proof. Consider the representation of $x(-t)$ using the inverse Fourier transform:

$$
\begin{aligned}
x(-t) &= \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f(-t)}\,\mathrm{d}f \\
&= \int_{-\infty}^{\infty} X(f)\, e^{-j2\pi ft}\,\mathrm{d}t \\
&= \mathscr{F}\{X(f)\}\,.
\end{aligned}
$$

Swapping variables concludes the proof. □

5 Special Functions

5.1 Dirac Delta Function

The Dirac delta (or impulse) function is defined by the following characteristics:

$$\delta\left(t\right) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta\left(t\right) \mathrm{d}t = 1.$$

Likewise, the shifted impulse function can be defined as:

$$\delta\left(t - t_0\right) = \begin{cases} 0, & t_0 \neq 0 \\ \infty, & t_0 = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta\left(t - t_0\right) \mathrm{d}t = 1.$$

Therefore we can infer the following **sifting properties**:

$$\int_{-\infty}^{\infty} f\left(t\right) \delta\left(t - t_0\right) \mathrm{d}t = f\left(t_0\right)$$

and

$$f\left(t\right) \delta\left(t - t_0\right) = f\left(t_0\right) \delta\left(t - t_0\right).$$

The Fourier transform of the Dirac delta function is given by

$$\delta\left(t\right) \overset{\mathscr{F}}{\longleftrightarrow} 1$$

$$\delta\left(t - t_0\right) \overset{\mathscr{F}}{\longleftrightarrow} e^{-j2\pi f t_0}$$

This result is a direct consequence of the definition of the Dirac delta function.

5.2 Signum Function

The signum function is defined as:

$$\operatorname{sgn}\left(t\right) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

The signum function is related to the Dirac delta function by the following property:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\operatorname{sgn}\left(t\right)}{2} = \delta\left(t\right).$$

Using the integration property of the Fourier transform:

$$\begin{aligned}\mathscr{F}\left\{\int_{-\infty}^{t}\delta\left(\tau\right)\mathrm{d}\tau\right\} &= \frac{1}{j2\pi f}\mathscr{F}\left\{\delta\left(t\right)\right\}\\ \mathscr{F}\left\{\frac{\operatorname{sgn}\left(t\right)}{2}\right\} &= \frac{1}{j2\pi f}\\ \mathscr{F}\left\{\operatorname{sgn}\left(t\right)\right\} &= \frac{1}{j\pi f}\end{aligned}$$

5.3 Unit Step Function

The unit step (or Heaviside) function is defined as:

$$u\left(t\right) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

It follows that the derivative of the unit step function is precisely the delta function:

$$\frac{\mathrm{d}u\left(t\right)}{\mathrm{d}t} = \delta\left(t\right).$$

To determine the Fourier transform of the unit step function, we must use an alternative definition of the unit step function using the signum function:

$$u\left(t\right) = \frac{1}{2}\left(1 + \operatorname{sgn}\left(t\right)\right).$$

Therefore, the Fourier transform of the unit step function is given by:

$$\begin{aligned}\mathscr{F}\left\{u\left(t\right)\right\} &= \mathscr{F}\left\{\frac{1}{2}\left(1 + \operatorname{sgn}\left(t\right)\right)\right\}\\ &= \frac{1}{2}\mathscr{F}\left\{1\right\} + \frac{1}{2}\operatorname{sgn}\left(t\right)\\ &= \frac{1}{2}\delta\left(t\right) + \frac{1}{j2\pi f}\end{aligned}$$

5.4 Gate Function

The gate (or rectangle) function is defined as:

$$\Pi_T\left(t\right) = \operatorname{rect}\left(\frac{t}{T}\right) = \begin{cases} 1 & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

The Fourier transform of the gate function can be determined using first principles:

$$\begin{aligned}
\mathscr{F}\left\{\Pi_T\left(t\right)\right\} &= \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi ft}\,\mathrm{d}t \\
&= \frac{1}{-j2\pi f}\left[e^{-j2\pi ft}\right]_{-\frac{T}{2}}^{\frac{T}{2}} \\
&= \frac{1}{-j2\pi f}\left[e^{-j\pi fT} - e^{j\pi fT}\right] \\
&= \frac{1}{\pi f}\frac{e^{j\pi fT} - e^{-j\pi fT}}{j2} \\
&= \frac{1}{\pi f}\sin\left(\pi fT\right) \\
&= T\frac{\sin\left(\pi fT\right)}{\pi fT} \\
&= T\operatorname{sinc}\left(fT\right)
\end{aligned}$$

The function $\operatorname{sinc}\left(\theta\right) = \frac{\sin\left(\pi\theta\right)}{\pi\theta}$ is known as the normalised sinc function.

5.5 Triangular Function

The triangular function is defined as:

$$\Lambda_T\left(t\right) = \begin{cases} 1 - \frac{|t|}{T} & |t| < T \\ 0 & \text{otherwise} \end{cases}$$

To determine the Fourier transform of the triangular function, we will use the convolution property of the rectangular function, as described in the next section:

$$\begin{aligned}
\mathscr{F}\left\{\Lambda_T\left(t\right)\right\} &= \frac{1}{T}\mathscr{F}\left\{\Pi_T\left(t\right) * \Pi_T\left(t\right)\right\} \\
&= \frac{1}{T}\mathscr{F}\left\{\Pi_T\left(t\right)\right\}\mathscr{F}\left\{\Pi_T\left(t\right)\right\} \\
&= \frac{1}{T}T\operatorname{sinc}\left(fT\right) \times T\operatorname{sinc}\left(fT\right) \\
&= T\operatorname{sinc}^2\left(fT\right)
\end{aligned}$$

we must scale by $1/T$ as the convolution of two rectangular functions has a maximum value of T not 1.

6 Control Systems

A control system is a system in which a given initial state will always produce the same output. The system H maps the input signal $x(t)$ to the output $y(t)$.

6.1 Linear Time-Invariance

A system H is **linear time-invariant** (LTI) if it satisfies the following properties:

1. The system is **linear** iff it satisfies the superposition principle.

$$\begin{aligned} \text{Input: } & x(t) = a_1 x_1(t) + a_2 x_2(t) \\ \text{Output: } & y(t) = a_1 y_1(t) + a_2 y_2(t) \end{aligned}$$

   Given an input x which consists of a linear combination of signals, the output y of the system is the linear combination of the system's responses to each signal separately.

2. The system is **time-invariant** if a delay in the input produces a delay in the output.

$$\begin{aligned} \text{Input: } & x(t) = x(t - t_0) \\ \text{Output: } & y(t) = y(t - t_0) \end{aligned}$$

6.2 Signal Approximation

Consider the pulse function p such that

$$p(t) = \begin{cases} \frac{1}{\Delta t} & 0 < t < \Delta t \\ 0 & \text{otherwise} \end{cases}$$
$$p(t - t_0) = \begin{cases} \frac{1}{\Delta t} & t_0 < t < t_0 + \Delta t \\ 0 & \text{otherwise} \end{cases}$$

Given a signal $x(t)$ sampled at the rate $1/\Delta t$ (Hz), we can approximate $x(t)$ by:

$$x(t) \approx \sum_{k=-\infty}^{\infty} x(k\,\Delta t)\, p(t - k\,\Delta t)\, \Delta t$$

where $t = k\,\Delta t$ is the time at the start of each interval where the signal $x(t)$ is sampled. If we consider the limit as $\Delta t \to 0$, we obtain the following equality:

$$\begin{aligned} x(t) &= \lim_{\Delta t \to 0} \sum_{k=-\infty}^{\infty} x(k\,\Delta t)\, p(t - k\,\Delta t)\, \Delta t \\ &= \int_{-\infty}^{\infty} x(\tau)\, \delta(t - \tau)\, \mathrm{d}\tau \end{aligned}$$

where $\Delta t \to \mathrm{d}\tau$, and $k\,\Delta t \to \tau$. This equivalent form represents $x(t)$ as a linear combination of shifted impulses $\delta(t - \tau)$ with weights $x(\tau)$.

6.3 Impulse Response

A system's **impulse response** $h(t)$ is defined as

$$\begin{aligned}\text{Input: } x(t) &= \delta(t)\\ \text{Output: } y(t) &= h(t)\end{aligned}$$

that is, the response to an impulse input signal. Passing the signal $x(t)$ into this system gives the following result:

$$\begin{aligned}\text{Input: } x(t) &= \lim_{\Delta t\to 0}\sum_{k=-\infty}^{\infty} x(k\,\Delta t)\,p(t-k\,\Delta t)\,\Delta t\\ \text{Output: } y(t) &= \lim_{\Delta t\to 0}\sum_{k=-\infty}^{\infty} x(k\,\Delta t)\,h_p(t-k\,\Delta t)\,\Delta t\\ &= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,\mathrm{d}\tau.\end{aligned}$$

6.4 Convolution

The integral seen in the previous section is known as the **convolution** of x with h, denoted by an asterisk ($*$).

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,\mathrm{d}\tau.$$

Therefore,

$$\begin{aligned}\text{Input: } x(t) &= x(t) * \delta(t)\\ \text{Output: } y(t) &= x(t) * h(t).\end{aligned}$$

It represents the influence of the impulse response $h(t)$ on the input signal $x(t)$.
The convolution between two general signals $x(t)$ and $y(t)$ is given by:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)\,\mathrm{d}\tau.$$

In the frequency domain we can express the convolution as the product of the Fourier transforms of the signals:

$$\begin{aligned}\mathscr{F}(x(t) * y(t)) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)\,\mathrm{d}\tau\right] e^{-j2\pi ft}\,\mathrm{d}t\\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)\,e^{-j2\pi ft}\,\mathrm{d}t\right]\mathrm{d}\tau\\ &= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} y(t-\tau)\,e^{-j2\pi ft}\,\mathrm{d}t\right]\mathrm{d}\tau.\end{aligned}$$

Let $u = t - \tau$, so that $\mathrm{d}u = \mathrm{d}t$ and the limits of integration are $-\infty \leq u \leq \infty$. Then

$$\begin{aligned}
\mathscr{F}\left\{x(t) * y(t)\right\} &= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} y(u)\, e^{-j2\pi f(u+\tau)}\,\mathrm{d}u\right]\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f\tau}\left[\int_{-\infty}^{\infty} y(u)\, e^{-j2\pi fu}\,\mathrm{d}u\right]\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f\tau}\,\mathrm{d}\tau \int_{-\infty}^{\infty} y(u)\, e^{-j2\pi fu}\,\mathrm{d}u \\
&= X(f)\, Y(f)\,.
\end{aligned}$$

6.5 Discrete Time Convolution

The convolution of two sequences (sampled waveforms) $x[n]$ and $h[n]$, for an integer n is given by:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

If the sequence $x[n]$ is of length L and $h[n]$ is of length N, then $y[n]$ is of length $L + N - 1$.

6.6 Stability and Causality

An LTI system is Bounded Input Bounded Output (BIBO) stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\,\mathrm{d}t < \infty$$

Causality means that the output at time t does not depend on future values of the input signal. This means that $h(t) = 0$ for $t < 0$.

6.7 Transfer Functions

The Fourier Transform of the impulse response is the **transfer function** or the frequency response of the system. At any frequency, the transfer function $H(f)$ can be measured by taking the ratio of the output and input:

$$H(f) = \frac{Y(f)}{X(f)}$$

As the Fourier transform of the impulse function is uniformly equal to one, the impulse is the ideal signal used to find $H(f)$. For sinusoidal steady state analysis, the complex transfer function of the system is the ratio of the output phasor to the input phasor. If the input is a current flowing between two terminals and the output is the voltage across the terminals, then the transfer function is the ratio of the output voltage to the input current, or the complex impedance between the two terminals.

6.8 Filters

Filters are used to attenuate ranges of frequencies in a signal.

- Low pass — only allows low frequencies to pass un-attenuated
- High pass — only allows high frequencies to pass un-attenuated
- Band pass — only allows a range of frequencies to pass un-attenuated
- Band stop — attenuates a range of frequencies

The region where frequencies are **un-attenuated** is called the **pass band**. The region where frequencies are **attenuated** is called the **stop band**.

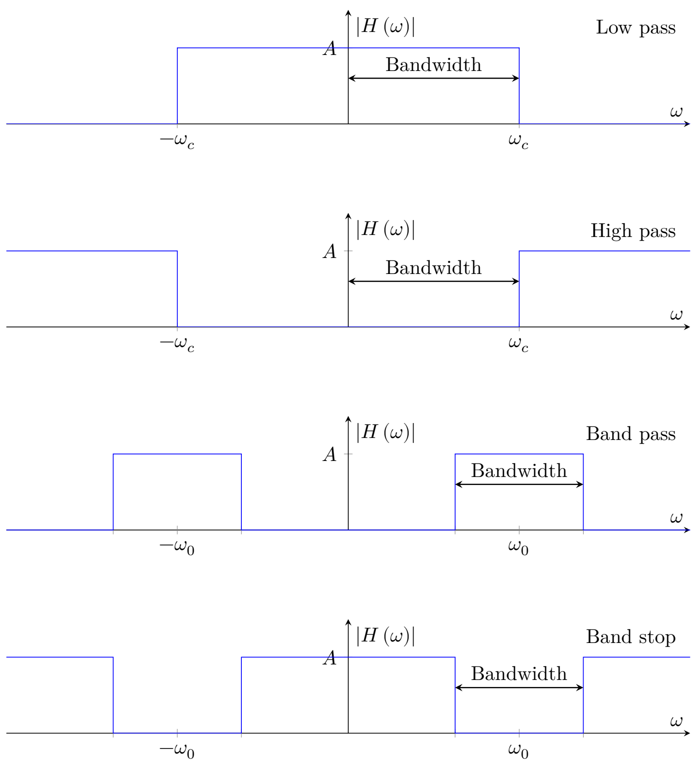

Figure 1: Magnitude spectra of various ideal filters.

6.9 Ideal Filters

An ideal low/high pass filter only allows frequencies below/above a **cut-off frequency** f_c to pass. This is achieved by setting the transfer function to zero for frequencies above/below the cut-off frequency. An ideal band pass/stop filter only allows frequencies within a range of frequencies to pass/stop. These frequencies are centred at a **centre frequency** f_0 and have a **bandwidth** which is the width of the pass/stop regions. Ideal filters attenuate completely in their stop bands, and pass signals completely in their pass bands, and have no effect on the phase of the signal.

6.10 Practical Filters

In practice, ideal filters are not realisable because they have infinite attenuation in the stop band. An ideal filter usually has a **transition region** between the pass and stop bands. As this roll-off is not instantaneous, the cut-off frequency and bandwidth of the filter are determined using the 3 dB point where the attenuation is $\frac{1}{\sqrt{2}}$:

$$|H(\omega)| = \frac{1}{\sqrt{2}}$$

where $\omega = 2\pi f$ is the angular frequency. Consider the second order parallel RLC circuit with transfer function:

$$H(\omega) = \frac{1}{\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)}$$

The **resonance frequency** is the frequency at which the transfer function is at its maximum:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and the bandwidth is given by:

$$B = \frac{1}{RC}$$

The **Q factor** is the ratio of the resonance frequency to the bandwidth:

$$Q = \frac{\omega_0}{B}$$

which describes the sharpness of the peak in the magnitude response. For the parallel RLC circuit, the Q factor is given by:

$$Q = R\sqrt{\frac{C}{L}}$$

so that increasing R increases the Q factor.

7 Laplace Transform

When signals are neither periodic nor sinusoidal, the Fourier transform cannot be used. Any sudden change in a signal is regarded as a transient which disturbs the steady-state operation of a system. The Laplace transform is a generalisation of the Fourier transform used to determine the response of a system under both transient and steady-state conditions.

7.1 Definition

The two-sided Laplace transform of a function $f(t)$ is given by:

$$F(s) = \mathscr{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, \mathrm{d}t$$

where $s = \sigma + j\omega$ is a complex variable for $\sigma, \omega \in \mathbb{R}$. For causal systems, the lower limit of integration is zero. The inverse Laplace transform is given by the contour integral:

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\, e^{st}\, \mathrm{d}s$$

where we define a contour along the vertical line $\Re(s) = \sigma$ in the complex plane such that σ is greater than all poles of $F(s)$. For a causal system, σ is to the right of the pole with the largest real part, whereas for a non-causal system it is in the vertical strip between two poles. Note that we can typically use tables and partial fraction decomposition to find the inverse Laplace transform of real-valued functions.

7.2 Conditions for Existence

The Laplace transform converges when

$$\int_{-\infty}^{\infty} \left|f(t)\, e^{-\sigma t}\right| \mathrm{d}t < \infty$$

for finite σ. Or equivalently when,

$$\lim_{t\to\infty} f(t)\, e^{-st} = 0.$$

7.3 Laplace Transform Properties

Like the Fourier transform, many properties of the Laplace transform can be derived from its definition. Notably, the Laplace transform is linear and also satisfies the convolution theorem.

7.4 Zeros and Poles

The zeros and poles of a function $f(t)$ are the roots of the numerator and denominator of its Laplace transform.

7.5 Circuit Analysis

Two common circuits are the RC and RL circuits.

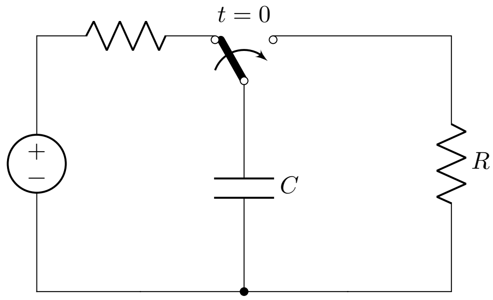

Figure 2: RC circuit.

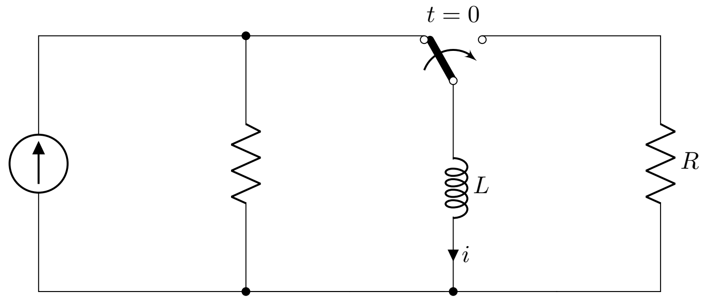

Figure 3: RL circuit.

For an RC circuit:

$$v(t) = v(0)\, e^{-t/\tau}$$

with $\tau = RC$.

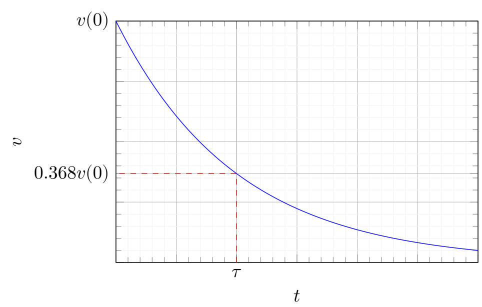

Figure 4: Natural response of RC circuit.

For an RL circuit:

$$i(t) = i(0)\, e^{-t/\tau}$$

with $\tau = \frac{1}{R} L$.

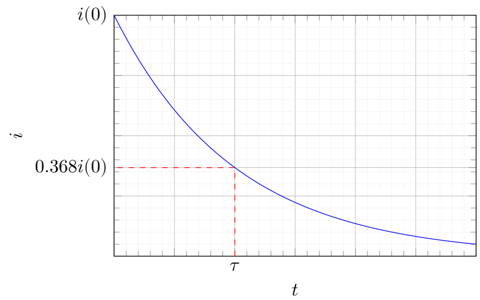

Figure 5: Natural response of RL circuit.

τ is the time constant that describes the time taken for the response to decay to $1/e$ of its initial value. For the step responses, consider the following circuits:

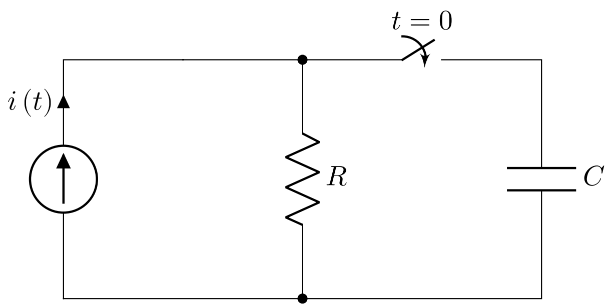

Figure 6: RC circuit.

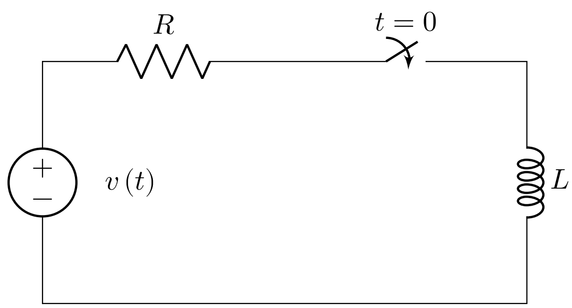

Figure 7: RL circuit.

For an RC circuit:

$$v(t) = v(\infty) + \left[v(0^+) - v(\infty)\right] e^{-t/\tau}$$

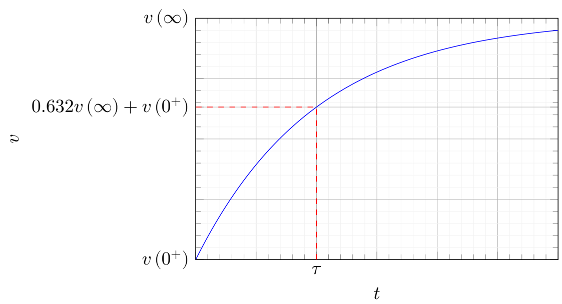

Figure 8: Step response of an RC circuit.

For an RL circuit:

$$i(t) = i(\infty) + [i(0^+) - i(\infty)]\, e^{-t/\tau}$$

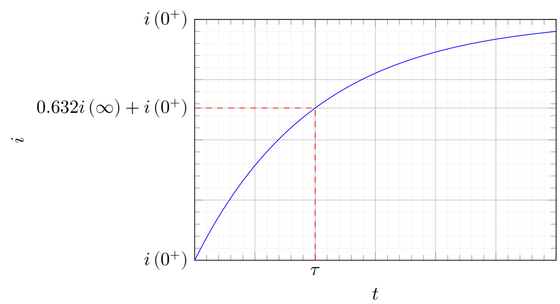

Figure 9: Step response of an RL circuit.

where $v(\infty)$ and $i(\infty)$ are the steady-state values of the voltage and current respectively. To determine the initial and steady-state values of the voltage and current, we can use the Laplace transform.

7.6 Initial and Final Value Theorems

If the poles of $F(s)$ are in the left-hand side of the s-plane, the system is **stable**. Given a stable system,

$$f(0^+) = \lim_{s\to\infty} sF(s)$$
$$f(\infty) = \lim_{s\to 0} sF(s)$$

7.7 Impedance

The impedance of an element is the ratio of the voltage to the current. Consider the capacitor and inductor definitions:

$$i(t) = C\frac{\mathrm{d}v}{\mathrm{d}t}$$
$$v(t) = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

taking the Laplace transform gives:

$$I(s) = sCV(s)$$
$$V(s) = sLI(s)$$

where we assume that the initial conditions are zero. This gives us an s-domain definition of impedance for reactive elements:

$$Z_C = \frac{1}{sC}$$
$$Z_L = sL$$

These definitions allow us to analyse circuits by first applying the Laplace transform to all elements and solving the circuit via Kirchhoff's laws either through mesh or nodal analysis.

7.8 Common Filters

Using the above definitions, we can derive the transfer functions of common RC/RL filters. Here $\omega_c = 1/\tau$ is the cut-off (angular) frequency.

7.8.1 RC Low Pass Filter

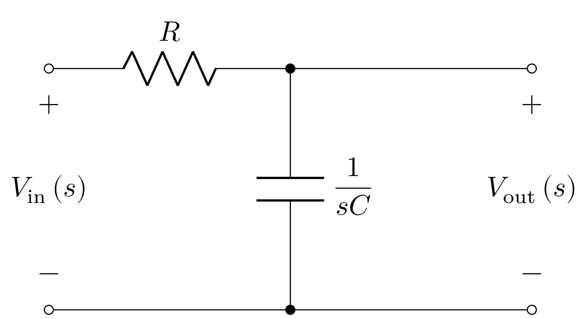

Figure 10: RC low pass filter.

The transfer function of the RC low pass filter is:

$$\begin{aligned} H(s) &= \frac{\frac{1}{sC}}{R + \frac{1}{sC}} && \left(\times \frac{s/R}{s/R}\right) \\ &= \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \\ &= \frac{\omega_c}{s + \omega_c} \end{aligned}$$

with $\omega_c = \frac{1}{RC}$.

7.8.2 RC High Pass Filter

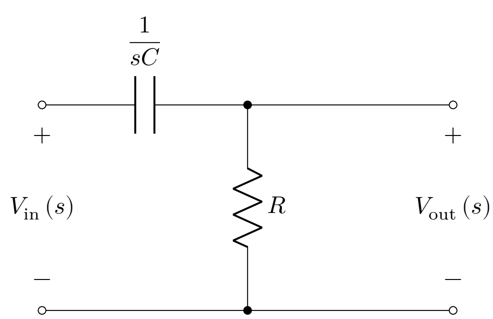

Figure 11: RC high pass filter.

The transfer function of the RC high pass filter is:

$$\begin{aligned} H(s) &= \frac{R}{\frac{1}{sC} + R} && \left(\times \frac{s/R}{s/R}\right) \\ &= \frac{s}{\frac{1}{RC} + s} \\ &= \frac{s}{\omega_c + s} \end{aligned}$$

with $\omega_c = \frac{1}{RC}$.

7.8.3 RL Low Pass Filter

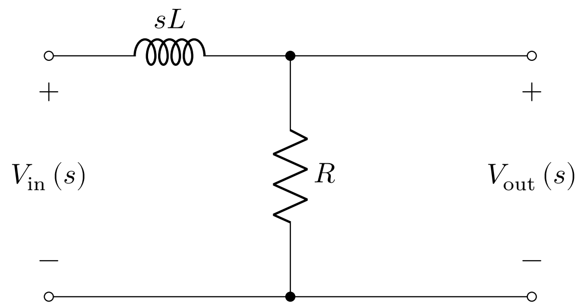

Figure 12: RL low pass filter.

The transfer function of the RL low pass filter is:

$$
\begin{aligned}
H(s) &= \frac{R}{sL+R} && \left(\times \frac{1/R}{1/R}\right) \\
&= \frac{1}{s\frac{1}{R}L+1} \\
&= \frac{1}{s\omega_c+1}
\end{aligned}
$$

with $\omega_c = \frac{1}{\frac{1}{R}L}$.

7.8.4 RL High Pass Filter

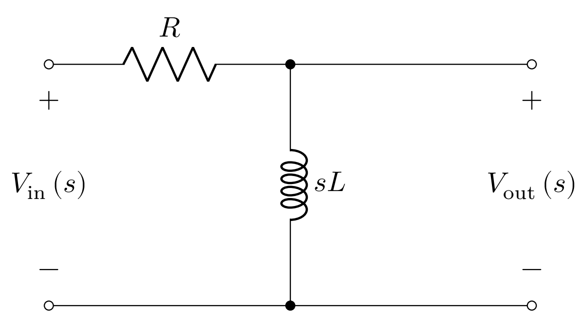

Figure 13: RL high pass filter.

The transfer function of the RL high pass filter is:

$$\begin{aligned} H(s) &= \frac{sL}{R+sL} & \left(\times\frac{1/R}{1/R}\right) \\ &= \frac{s\frac{1}{R}L}{1+s\frac{1}{R}L} \\ &= \frac{s}{1+s\omega_c} \end{aligned}$$

with $\omega_c = \frac{1}{\frac{1}{R}L}$.

8 Sampling

Analogue to digital conversion is performed in two steps:

1. Sampling
2. Quantisation

where sampling produces a discrete-time signal.

8.1 Sampling Theorem

Given a signal $x(t)$ with bandwidth f_m, the sampling theorem states that the signal can be perfectly reconstructed from a sequence of samples if the sampling frequency f_s is at least twice the bandwidth:

$$f_s \geq 2f_m$$

where f_s has units of samples per second. When the sampling frequency is exactly twice the bandwidth, the signal is sampled at the **Nyquist rate (or frequency)**. The Nyquist rate ensures that the signal does not **alias**, i.e., the copies of the signal in the frequency domain do not overlap.

8.2 Anti-Aliasing Filter

When the maximum frequency of the signal is greater than half the sampling frequency, the signal will alias. To prevent aliasing, an **anti-aliasing (low-pass) filter** is used to eliminate spectral overlap before sampling. While this process removes high frequency information, it ensures that the signal is reconstructed correctly.

8.3 Sampling a Signal

Recall the pulse function $p(t)$ centred at $t = 0$ with width T_s:

$$p(t) = \begin{cases} \frac{1}{T_s} & -\frac{T_s}{2} < t < \frac{T_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$p(t-t_0) = \begin{cases} \frac{1}{T_s} & t_0 - \frac{T_s}{2} < t < t_0 + \frac{T_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

When the signal $x(t)$ is sampled at a rate f_s, it can be written in terms of it's sampled signal values $x[n]$:

$$x_s(t) = T_s \sum_{k=-\infty}^{\infty} x(kT_s)\, p(t - kT_s) \rightarrow x[n] = x_s(nT_s)$$

where the signal is sampled at times $t = nT_s$ and $T_s = \frac{1}{f_s}$ is the sampling period.

8.4 Spectra of Sampled Signals

Consider the Fourier series of a train of impulses centred at $t = 0$ with period T_s:

$$\delta_{\text{train}}(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_s t)$$

where $a_0 = \frac{1}{T_s}$ and $a_n = \frac{2}{T_s}$ so that,

$$\delta_{\text{train}}(t) = \frac{1}{T_s} + \frac{2}{T_s} \sum_{n=1}^{\infty} \cos(2\pi n f_s t)$$

If we multiply the signal $x(t)$ by this train of impulses, we get:

$$\begin{aligned} x_s(t) &= x(t)\, \delta_{\text{train}}(t) \\ &= \frac{1}{T_s} x(t) + \frac{2}{T_s} \sum_{n=1}^{\infty} x(t) \cos(2\pi n f_s t) \\ &= \frac{1}{T_s} \left[x(t) + 2 \sum_{n=1}^{\infty} x(t) \cos(2\pi n f_s t) \right] \end{aligned}$$

Now if we take the Fourier transform of this sampled signal, we get:

$$\begin{aligned} X_s(f) &= \frac{1}{T_s} \left[X(f) + 2 \sum_{n=1}^{\infty} \frac{1}{2} \left[X(f - nf_s) + X(f + nf_s) \right] \right] \\ &= \frac{1}{T_s} \left[X(f) + \sum_{n=1}^{\infty} \left[X(f - nf_s) + X(f + nf_s) \right] \right] \\ &= \frac{1}{T_s} \left[X(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} X(f - nf_s) \right] \\ &= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(f - nf_s) \end{aligned}$$

so that the spectrum of the sampled signal contains copies of the original signal at frequencies nf_s which can be filtered out by using a low-pass filter with gain T_s to rescale the spectra to its original magnitude.

8.5 Sample and Hold

As an impulse train is not practical, we can use a **sample and hold** circuit where each sample is held constant for a duration $T \leq T_s$. This circuit is defined:

$$h_0(t) = \operatorname{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$$
$$H_0(f) = e^{-j\pi f T} T \operatorname{sinc}(Tf).$$

If we take the convolution of the sampled signal with h_0, we can "hold" each sample for a duration T:

$$x_h(t) = x_s(t) * h_0(t)$$
$$X_h(f) = e^{-j\pi f T} T \operatorname{sinc}(Tf) \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

This method still produces copies at frequencies nf_s, but the magnitude spectra is now multiplied by the sinc function and the phase is shifted by $T/2$ due to the exponential term.

8.6 Equaliser

To recover the original magnitude spectra of the signal, we can use the **equaliser** $H_0^{-1}(f)$ (reciprocal of H_0) to remove the spectral distortion caused by the sample and hold circuit:

$$H_0^{-1}(f) = \frac{e^{j\pi f T}}{T \operatorname{sinc}(Tf)}.$$

This equaliser is applied after the additional copies have been removed by the low-pass filter.

9 Quantisation

Digital signals are discrete in both time and amplitude. A **quantiser** transforms each sampled value to take one of L distinct levels. The L levels are allocated over the entire dynamic range of the analog signal

$$x_{\min} \leq x(t) \leq x_{\max}.$$

This procedure is non-invertible.

9.1 Uniform Quantisers

A uniform quantiser divides the **dynamic range** $(x_{\max} - x_{\min})$ into L equal intervals, known as **quantisation levels**. The step size Δ between quantisation levels is calculated by

$$\Delta = \frac{x_{\max} - x_{\min}}{L}$$

Many applications have $x_{\max} = -x_{\min}$, so the step size is

$$\Delta = \frac{2x_{\max}}{L}$$

The number of levels L is usually a power of two,

$$L = 2^n$$

where n is the number of bits to encode L levels.