Calculus and Differential Equations

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Contents

1 Analysis of Functions

1.1 Functions

Definition 1.1 (Function)**.** A function $f : X \to Y$ is a relation between a set of inputs X and outputs Y, which assigns exactly one output to each input. The set X is called the **domain** of f, and the set Y is called the **codomain** of f. Often, the set of possible values that f can take is restricted to a *subset* of the codomain, called the **image** (or range) of f.

As an example, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. The domain of this function is $\mathbb{R}$, the codomain is $\mathbb{R}$, and the image is $\mathbb{R}^+$, as the square of any real number is always positive.

1.2 Limits

Definition 1.2 (Limit of a function)**.** A limit is the value a function approaches as the input approaches some value. Limits are used to define concepts in calculus and analysis, such as continuity, differentiation, and integration. The limit of the function $f(x)$ is written as

$$\lim_{x \to x_0} f(x) = L,$$

which means that as x approaches x_0, the value of $f(x)$ approaches L. We can alternatively denote this using right arrows:

$$f(x) \to L \text{ as } x \to x_0.$$

Definition 1.3 (ε-δ definition of a limit)**.** The function $f : I \to \mathbb{R}$ tends to L as x tends to x_0 if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x in the domain I, $0 < |x - x_0| < \delta$ implies $|f(x) - L| < \varepsilon$:

$$\lim_{x \to x_0} f(x) = L \iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$$

This is known as the ε-δ definition of a limit, formalised by Cauchy and Weierstrass.

For example,

$$\lim_{x \to 2} 2x = 4$$

because for every $\varepsilon > 0$, we can take $\delta = \varepsilon/2$, so that

$$\left(0 < |x - 2| < \delta \iff |x - 2| < \frac{\varepsilon}{2}\right) \implies |2x - 4| < \varepsilon$$

Definition 1.4 (Left- and right-hand limits)**.** Left- and right-hand limits characterise the behaviour of a function as the input approaches a certain value from either the left or right, respectively. The left-hand limit of the function $f(x)$ as x approaches x_0 is the value of the function as x approaches x_0 from the left, denoted $\lim_{x \to x_0^-} f(x)$. The right-hand limit of the function $f(x)$ as x approaches x_0 is the value of the function as x approaches x_0 from the right, denoted $\lim_{x \to x_0^+} f(x)$.

Theorem 1.2.1 (Existence of a limit)**.** *The function $f(x)$ approaches L as x approaches x_0 if and only if the left- and right-hand limits of this function exist and are equal:*

$$\lim_{x \to x_0} f(x) = L \iff \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L$$

Theorem 1.2.2 (L'Hôpital's Rule)**.** *For two differentiable functions $f(x)$ and $g(x)$. If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ or $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \pm\infty$, then*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

1.3 Continuity

A function is continuous if it does not have any jumps in its graph. In other words, a small variation in the input also results in a small variation in the output. Functions that do not satisfy this property for some value of x are discontinuous at that point.

Theorem 1.3.1 (Continuity at a point)**.** *The function $f(x)$ is continuous at c if and only if*

$$\lim_{x\to c} f(x) = f(c).$$

Theorem 1.3.2 (Continuity over an interval)**.** *The function $f(x)$ is continuous on the interval I if it is continuous for all values of $x \in I$. If the two endpoints of this interval are a and b, then*

- *$f(x)$ is continuous on (a, b) if it is continuous for all $x \in (a, b)$.*
- *$f(x)$ is continuous on $[a, b]$ if it is continuous for all $x \in (a, b)$, but only right-continuous at a and left-continuous at b. That is, the right-hand limit at a and the left-hand limit at b.*

*When $f(x)$ is continuous on $(-\infty, \infty)$, it is said to be continuous **everywhere**.*

Theorem 1.3.3 (Intermediate value theorem)**.** *If the function $f(x)$ is continuous on $I : [a, b]$ and c is any number between $f(a)$ and $f(b)$, inclusive, then there exists an $x \in I$ such that $f(x) = c$.*

1.4 Differentiability

Differentiability is a property of functions that relates to the existence of a derivative at each point on some interval. A valuable starting point for understanding differentiability is the idea of the *average rate of change* of a function $f(x)$ on some small interval $(x_0, x_0 + h)$ (where h is small), which is given by:

$$\left.\frac{\Delta f}{\Delta x}\right|_{x=x_0} = \frac{\text{change in output}}{\text{change in input}} = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0} = \frac{f(x_0+h) - f(x_0)}{h}.$$

The last result is known as a difference quotient.

Definition 1.5 (Differentiablity at a point)**.** A function $f(x)$ is differentiable at $x = x_0$ if the limit

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. When this limit exists, it defines the derivative

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

The derivative can be thought of as the *instantaneous* rate of change of the function $f(x)$ at the point x_0.

Definition 1.6 (Differentiability on an interval)**.** The function $f(x)$ is differentiable on an interval I if $f(x)$ is differentiable for all $x_0 \in I$.

Theorem 1.4.1. *A function that is differentiable at a point is also continuous at that point. In other words, differentiability implies continuity.*

Theorem 1.4.2 (Mean value theorem)**.** *If the function $f(x)$ is differentiable on $I : [a, b]$ (and therefore also continuous on I), then there exists a point $c \in I$ where*

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=c} = \frac{f(b) - f(a)}{b - a}$$

2 Definite Integration

Definite integrals are used to calculate the signed area of a region between a function and the x-axis. Later we will see an important result that relates integration with differentiation allowing us to calculate integrals of functions.

Definition 2.1 (Definite integration)**.** If the function $f(x)$ is continuous on an interval $I : [a, b]$, then the net signed area A between the graph of $f(x)$ on the interval I is expressed as

$$A = \int_a^b f(x)\,\mathrm{d}x.$$

Properties of Definite Integrals

Suppose that the functions $f(x)$ and $g(x)$ are continuous on the interval $I : [a, b]$, with a, b, $c \in I$, with $a < c < b$, and $k \in \mathbb{R}$ is some constant. Then,

a) $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0.$

b) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$

c) $\displaystyle\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x.$

d) $\displaystyle\int_a^b (f(x) \pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x.$

e) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$

2.1 Riemann Sums

Theorem 2.1.1. *The net signed area A under the function $f(x)$ on the interval $[a, b]$ can be approximated by a Riemann sum:*

$$A \approx \sum_{k=1}^{n} f(x_k)\, \Delta x_k$$

where n is the number of rectangles, x_k is the centre of the rectangle k, and Δx_k is the width of the rectangle k. Note these rectangles do not need to be of equal width. In the limit where the widest rectangle's width approaches zero ($\max \Delta x_k \to 0$), the Riemann sum approaches the definite integral:

$$A = \int_a^b f(x)\, \mathrm{d}x = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k)\, \Delta x_k$$

where, as a consequence, the number of rectangles n also approaches infinity. In the case where every rectangle has the same width, we can express the width of each rectangle as

$$\forall k : \Delta x_k = \frac{b-a}{n}.$$

2.2 Fundamental Theorem of Calculus

The fundamental theorem of calculus provides a logical connection between infinite series (definite integrals) and antiderivatives (indefinite integrals).

Theorem 2.2.1 (The Fundamental Theorem of Calculus: Part 1)**.** *If $f(x)$ is continuous on $[a,\ b]$ and F is any antiderivative of f on $[a,\ b]$ then*

$$\int_a^b f(x)\, \mathrm{d}x = F(b) - F(a)$$

Equivalently

$$\int_a^b \frac{\mathrm{d}F(x)}{\mathrm{d}x}\, \mathrm{d}x = F(b) - F(a) \equiv \left. F(x) \right|_a^b$$

Theorem 2.2.2 (The Fundamental Theorem of Calculus: Part 2)**.** *If $f(x)$ is continuous on I then it has an antiderivative on I. In particular, if $a \in I$, then the function F defined by*

$$F(x) = \int_a^x f(t)\, \mathrm{d}t$$

is an antiderivative of $f(x)$. That is,

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\, \mathrm{d}t\right) = f(x)$$

Theorem 2.2.3. *Differentiation and integration are inverse operations.*

2.3 Taylor and Maclaurin Polynomials

Theorem 2.3.1 (Taylor Polynomials)**.** *If $f(x)$ is an n differentiable function at x_0, then the nth degree Taylor polynomial for $f(x)$ near x_0, is given by*

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Theorem 2.3.2 (Maclaurin Polynomials)**.** *Evaluating a Taylor polynomial near 0, gives the nth degree Maclaurin polynomial for $f(x)$*

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

Theorem 2.3.3 (Error in Approximation)**.** *Let $R_n(x)$ denote the difference between $f(x)$ and its nth Taylor polynomial, that is*

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k = \frac{f^{(n+1)}(s)}{(n+1)!}(x - x_0)^{n+1}$$

where s is between x_0 and x.

3 Taylor and Maclaurin Series

3.1 Infinite Series

Definition 3.1 (Taylor Series)**.** If $f(x)$ has derivatives of all orders at x_0, then the Taylor series for $f(x)$ about $x = x_0$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

Definition 3.2 (Maclaurin Series)**.** If a Taylor series is centred at $x_0 = 0$, it is called a Maclaurin series, defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Definition 3.3 (Power Series)**.** Both Taylor and Maclaurin series are examples of **power series**, which are defined as follows

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

3.2 Convergence

Theorem 3.2.1 (Convergence of a Taylor Series)**.** *The equality*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

holds at a point x iff

$$\lim_{n\to\infty} \left[f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \right] = 0$$
$$\lim_{n\to\infty} R_n(x) = 0$$

Definition 3.4 (Interval of Convergence)**.** The interval of convergence for a power series is the set of x values for which that series converges.

Definition 3.5 (Radius of Convergence)**.** The radius of convergence R is a non-negative real number or ∞ such that a power series converges if

$$|x - a| < R$$

and diverges if

$$|x - a| > R$$

The behaviour of the power series on the boundary, that is, where $|x - a| = R$, can be determined by substituting $x = R + a$ and $x = -R + a$ into the series, for the upper and lower boundaries, respectively.

3.3 Convergence Tests

For any power series of the form $\sum_{i=i_0}^{\infty} a_i x^i$, the following tests can be used to determine convergence.

Alternating Series

Conditions $a_i = (-1)^i b_i$ or $a_i = (-1)^{i+1} b_i$. $b_i > 0$.

$$\text{Is } b_{i+1} \leqslant b_i \ \& \ \lim_{i\to\infty} b_i = 0? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

Ratio Test

$$\text{Is } \lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

The ratio test is inconclusive if $\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = 1$.

3.4 Table of Maclaurin Series

Function	Series	Interval of Convergence
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$\cos(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$\frac{1}{1+x^2}$	$\sum_{n=0}^{\infty} (-1)^n x^{2n}$	$-1 < x < 1$
$\ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$-1 < x \leq 1$

Table 1: Maclaurin series of common functions.

4 Multivariable Calculus

4.1 Multivariable Functions

Definition 4.1. A function is multivariable if its domain consists of several variables. In the reals, these functions are defined

$$f : \mathbb{R}^n \to \mathbb{R}$$

4.2 Level Curves

Definition 4.2. The level curves or *contour curves* of a function of two variables are curves along which the function has a constant value.

$$L_c(f) = \left\{(x,\ y) : f(x,\ y) = c\right\}$$

The level curves of a function can be determined by substituting $z = c$, and solving for y.

4.3 Limits and Continuity

Definition 4.3 (Finite Limit of Multivariable Functions using the ε-δ Definition)**.**

$$\lim_{(x_1,\ \dots,\ x_n)\to(c_1,\ \dots,\ c_n)} f(x_1,\ \dots,\ x_n) = L \iff \forall\varepsilon > 0 : \exists\delta > 0 : \forall(x_1,\ \dots,\ x_n) \in I :$$
$$0 < |x_1 - c_1,\ \dots,\ x_n - c_n| < \delta \implies |f(x_1,\ \dots,\ x_n) - L| < \varepsilon$$

Theorem 4.3.1 (Limits along Smooth Curves)**.** *If $f(x,\ y) \to L$ as $(x,\ y) \to (x_0,\ y_0)$, then $\lim_{(x,\ y)\to(x_0,\ y_0)} = L$ along any smooth curve.*

Theorem 4.3.2 (Existence of a Limit)**.** *If the limit of $f(x,\ y)$ changes along different smooth curves, then $\lim_{(x,\ y)\to(x_0,\ y_0)}$ does not exist.*

Theorem 4.3.3 (Continuity of Multivariable Functions)**.** *A function $f(x_1,\ \dots,\ x_n)$ is continuous at $(c_1,\ \dots,\ c_n)$ iff*

$$\lim_{(x_1,\ \dots,\ x_n)\to(c_1,\ \dots,\ c_n)} f(x_1,\ \dots,\ x_n) = f(c_1,\ \dots,\ c_n)$$

Recognising continuous functions:

- A sum, difference or product of continuous functions is continuous.
- A quotient of continuous functions is continuous except where the denominator is zero.
- A composition of continuous functions is continuous.

4.4 Partial Derivatives

Definition 4.4 (Partial Differentiation)**.** The partial derivative of a multivariable function is its derivative with respect to one of those variables, while the others are held constant.

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1,\ \dots,\ x_{i-1},\ x_i + h,\ x_{i+1},\ \dots,\ x_n) - f(x_1,\ \dots,\ x_n)}{h}$$

4.5 The Gradient Vector

Definition 4.5. Let $\boldsymbol{\nabla}$, pronounced "del", denote the vector differential operator defined as follows

$$\boldsymbol{\nabla} = \begin{bmatrix} \partial x_1 \\ \partial x_2 \\ \vdots \\ \partial x_n \end{bmatrix}$$

4.6 Multivariable Chain Rule

Definition 4.6. Let $f = f(\mathbf{x}(t_1, \ldots, t_n))$ be the composition of f with $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}$, then the partial derivative of f with respect to t_i is given by

$$\frac{\partial f}{\partial t_i} = \boldsymbol{\nabla} f \cdot \frac{\partial \mathbf{x}}{\partial t_i}$$

4.7 Directional Derivatives

Definition 4.7. The directional derivative $\boldsymbol{\nabla}_{\mathbf{u}} f$ is the rate at which the function f changes in the direction $\mathbf{u}$.

$$\begin{aligned} \boldsymbol{\nabla}_{\mathbf{u}} f &= \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h} \\ &= \boldsymbol{\nabla} f \cdot \mathbf{u} \end{aligned}$$

where the slope is given by $\|\boldsymbol{\nabla}_{\mathbf{u}} f\|$.

Remark 1. The directional derivative of f can be denoted in several ways:

$$\boldsymbol{\nabla}_{\mathbf{u}} f = \partial_{\mathbf{u}} f = \frac{\partial f}{\partial \mathbf{u}}$$

Theorem 4.7.1 (Direction of Greatest Ascent)**.** *The direction of greatest ascent is given by*

$$\max_{\|\mathbf{u}\|=1} \boldsymbol{\nabla}_{\mathbf{u}} f = \boldsymbol{\nabla} f$$

where the slope is given by $\|\boldsymbol{\nabla} f\|$.

Theorem 4.7.2 (Direction of Greatest Descent)**.** *The direction of greatest descent is given by*

$$\min_{\|\mathbf{u}\|=1} \boldsymbol{\nabla}_{\mathbf{u}} f = -\boldsymbol{\nabla} f$$

where the slope is $-\|\boldsymbol{\nabla} f\|$.

Proof. Given that $\mathbf{u}$ is a unit vector, the dot product definition gives

$$\begin{aligned} \boldsymbol{\nabla}_{\mathbf{u}} f &= \boldsymbol{\nabla} f \cdot \mathbf{u} \\ &= \|\boldsymbol{\nabla} f\| \|\mathbf{u}\| \cos(\theta) \\ &= \|\boldsymbol{\nabla} f\| \cos(\theta) \qquad (1) \end{aligned}$$

Equation 1 is maximised when $\cos(\theta)$ is maximised. Thus, the maximum slope is

$$\max \boldsymbol{\nabla}_{\mathbf{u}} f = \|\boldsymbol{\nabla} f\|$$

and the direction of greatest ascent is

$$\mathbf{u} = \boldsymbol{\nabla} f$$

$\square$

Theorem 4.7.3. *If $\boldsymbol{\nabla} f(c_1, \dots, c_n) \neq 0$, then $\boldsymbol{\nabla} f(c_1, \dots, c_n)$ is normal to the level curve of f that passes through $(c_1, \dots, c_n)$.*

4.8 Higher-Order Partial Derivatives

Definition 4.8. Higher-order partial derivatives can be denoted using three different notations. The following table shows the mixed partial derivative of $f(x, y)$ w.r.t. x then y.

Leibniz	**Euler**	**Legendre**
$\dfrac{\partial^2 f}{\partial y\, \partial x}$	$\partial_y\, \partial_x f$	f_{xy}

Table 2: Mixed Partial Derivative Notation

For partial derivatives w.r.t. the same variable, a superscript can be used in Leibniz and Euler notation.

Leibniz	**Euler**	**Legendre**
$\dfrac{\partial^2 f}{\partial x^2}$	$\partial_x^2 f$	f_{xx}

Table 3: Second-Order Partial Derivative Notation

4.9 Hessian Matrix

Definition 4.9. Let the Hessian matrix $\mathbf{H}$ be the matrix of second-order partial derivative operators as shown below

$$\mathbf{H} = \begin{bmatrix} \partial^2 x_1 & \cdots & \partial x_n\, \partial x_1 \\ \vdots & \ddots & \vdots \\ \partial x_1\, \partial x_n & \cdots & \partial^2 x_n \end{bmatrix}$$

Operating on the function $f(x, y)$ gives

$$\mathbf{H}_f = \begin{bmatrix} \partial_{x_1}^2 f & \cdots & \partial_{x_n}\, \partial_{x_1} f \\ \vdots & \ddots & \vdots \\ \partial_{x_1}\, \partial_{x_n} f & \cdots & \partial_{x_n}^2 f \end{bmatrix}$$

4.10 Critical Points

For the function $f(x,\ y)$, the point $(x_0,\ y_0)$ is a critical point if

$$\boldsymbol{\nabla} f(x_0,\ y_0) = 0$$

or if $\boldsymbol{\nabla} f(x_0,\ y_0)$ is undefined.

4.11 Classification of Critical Points

The nature of a critical point can be classified using the second derivative test:

- if $\det(\mathbf{H}_f)\big|_{(x_0,\, y_0)} > 0$, then the point is a local minima or maxima
  - if $f_{xx}(x_0,\ y_0) < 0$, then the point is a local maxima
  - if $f_{xx}(x_0,\ y_0) > 0$, then the point is a local minima[1]
- if $\det(\mathbf{H}_f)\big|_{(x_0,\, y_0)} < 0$, then the point is a saddle point
- if $\det(\mathbf{H}_f)\big|_{(x_0,\, y_0)} = 0$, then the test is inconclusive.

[1] For the local minima/maxima, the second derivative can also be taken w.r.t. y.

5 Double and Triple Integrals

5.1 Volume under a Two Variable Function

Definition 5.1. If f is a function of two variables that is continuous and non-negative on a region Ω in the xy-plane, then the volume of the solid enclosed between the surface $z = f(x,\ y)$ and the region Ω is defined by

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*,\ y_k^*)\,\Delta A_k \tag{1}$$

Proof. Using lines parallel to the coordinate axes, the region Ω can be divided into n rectangles, where any rectangles outside Ω are discarded. The area of the kth remaining rectangle at the arbitrary point $(x_k^*,\ y_k^*)$ is given by ΔA_k. Thus, the product $f(x_k^*,\ y_k^*)\,\Delta A_k$ is the volume of the kth rectangular parallelepiped, and the sum of all n volumes over the region Ω approximate the volume V of the entire solid. □

5.2 Double Integral

Definition 5.2. By extension of the definite integral of a single variable function expressed in Theorem 2.1.1, the sums in Equation 1 are also called Riemann sums, and the limit is denoted as

$$\iint\limits_{\Omega} f(x,\ y)\,\mathrm{d}A = \sum_{k=1}^{\infty} f(x_k^*,\ y_k^*)\,\Delta A_k$$

Properties of Double Integrals

Theorem 5.2.1. *Suppose that $f(x,\ y)$ and $g(x,\ y)$ are continuous on Ω, and Ω can be subdivided into Ω_1 and Ω_2, then*

a) $\displaystyle\iint\limits_{\Omega} kf(x,\ y)\,\mathrm{d}A = k\iint\limits_{\Omega} f(x,\ y)\,\mathrm{d}A.$

b) $\displaystyle\iint\limits_{\Omega} f(x,\ y) + g(x,\ y)\,\mathrm{d}A = \iint\limits_{\Omega} f(x,\ y)\,\mathrm{d}A + \iint\limits_{\Omega} g(x,\ y)\,\mathrm{d}A.$

c) $\displaystyle\iint\limits_{\Omega} f(x,\ y)\,\mathrm{d}A = \iint\limits_{\Omega_1} f(x,\ y)\,\mathrm{d}A + \iint\limits_{\Omega_2} f(x,\ y)\,\mathrm{d}A.$

5.3 Rectangular Regions

If Ω is a region bounded by $a \le x \le b$ and $c \le y \le d$, then

$$\iint\limits_{\Omega} f(x,\ y)\,\mathrm{d}A = \int_c^d \int_a^b f(x,\ y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_c^d f(x,\ y)\,\mathrm{d}y\,\mathrm{d}x$$

5.4 Non-rectangular Regions

If the limits of integration depend on the variable x or y, then the region may be classified as Type I or Type II.

5.4.1 Type I Regions

If the region is:

Bounded on the left & right by: $x = a$ and $x = b$

Bounded below & above by: $y = g_1(x)$ and $y = g_2(x)$

where $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$, then

$$\iint\limits_{\Omega} f(x,\ y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,\ y)\,\mathrm{d}y\,\mathrm{d}x.$$

5.4.2 Type II Regions

If the region is:

Bounded on the left & right by: $x = h_1(y)$ and $x = h_2(y)$

Bounded below & above by: $y = c$ and $y = d$

where $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$, then

$$\iint\limits_{\Omega} f(x,\ y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,\ y)\,\mathrm{d}x\,\mathrm{d}y.$$

5.5 Polar Coordinates

To determine the area of a region defined using polar coordinates, the function can be integrated w.r.t. the radius r, and the angle θ.

$$\iint\limits_{\Omega} f(r,\ \theta)\,\mathrm{d}A = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\ \theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta.$$

5.6 Volume of a Three Variable Function

Definition 5.3. If f is a function of three variables that is continuous and non-negative on a region Ω in the xyz-space, then the volume enclosed by $f(x,\ y,\ z)$ and the region Ω is defined by

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*,\ y_k^*,\ z_k^*)\,\Delta V_k \tag{1}$$

Proof. Using planes parallel to the coordinate planes, the region Ω can be divided into n boxes, where boxes containing points outside Ω are discarded. The volume of the kth remaining box at the arbitrary point $(x_k^*,\ y_k^*,\ z_k^*)$ is ΔV_k. Thus, the product $f(x_k^*,\ y_k^*,\ z_k^*)\,\Delta V_k$ is the volume of the kth box, and the sum of all n volumes over the region Ω approximate the volume V of the entire solid. □

5.7 Triple Integrals

Definition 5.4. The triple integral of a function is the net signed volume defined over a finite closed solid region Ω, and the sums in Equation 1 are also called Riemann sums, and the limit is denoted as

$$\iiint_{\Omega} f(x,\, y,\, z)\,\mathrm{d}V = \sum_{k=1}^{\infty} f(x_k^*,\, y_k^*,\, z_k^*)\,\Delta V_k$$

Properties of Triple Integrals

Theorem 5.7.1. *Suppose that $f(x,\, y,\, z)$ and $g(x,\, y,\, z)$ are continuous on Ω, and Ω can be subdivided into Ω_1 and Ω_2 then*

a) $\displaystyle\iiint_{\Omega} kf(x,\, y,\, z)\,\mathrm{d}V = k\iiint_{\Omega} f(x,\, y,\, z)\,\mathrm{d}V.$

b) $\displaystyle\iiint_{\Omega} f(x,\, y,\, z) + g(x,\, y,\, z)\,\mathrm{d}V = \iiint_{\Omega} f(x,\, y,\, z)\,\mathrm{d}V + \iiint_{\Omega} g(x,\, y,\, z)\,\mathrm{d}V.$

c) $\displaystyle\iiint_{\Omega} f(x,\, y,\, z)\,\mathrm{d}V = \iiint_{\Omega_1} f(x,\, y,\, z)\,\mathrm{d}V + \iiint_{\Omega_2} f(x,\, y,\, z)\,\mathrm{d}V.$

6 Vector-Valued Functions

Definition 6.1. A vector-valued function (VVF) is some function $\mathbf{r}$ with domain $\mathbb{R}$ and codomain $\mathbb{R}^n$. For example,

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^3 : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

is a VVF where x, y, $z : \mathbb{R} \to \mathbb{R}$.

Theorem 6.0.1. *The domain of* $\mathbf{r}(t)$ *is the intersection of the domains of its components.*

Definition 6.2 (Orientation)**.** The orientation of $\mathbf{r}(t)$ is the direction of motion along the curve as the value of the parameter increases.

6.1 Limits and Continuity

Theorem 6.1.1 (Limits of VVFs)**.** *The limit of a VVF is the vector of the limits of its components.*

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \right\rangle$$

Theorem 6.1.2 (Continuity of VVFs)**.** *The VVF* $\mathbf{r}(t)$ *is continuous at* $t = a$ *iff*

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a).$$

This follows that a VVF is continuous if each of its components are also continuous.

6.2 Calculus with VVFs

Theorem 6.2.1 (Derivatives of VVFs)**.** *The derivative of a VVF is the vector of the derivatives of its components.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t), \frac{\mathrm{d}}{\mathrm{d}t}y(t), \frac{\mathrm{d}}{\mathrm{d}t}z(t)$$

Theorem 6.2.2 (Integration of VVFs)**.** *The integral of a VVF is the vector of the integrals of its components.*

$$\int \mathbf{r}(t)\,\mathrm{d}t = \left\langle \int x(t)\,\mathrm{d}t, \int y(t)\,\mathrm{d}t, \int z(t)\,\mathrm{d}t \right\rangle$$

Remark 1. When integrating a VVF, each component has its own constant of integration.

6.3 Parametrising Lines with VVFs

Definition 6.3 (Equation for a Line)**.** A line can be expressed as

$$\mathbf{l}(t) = \mathbf{P}_0 + t\mathbf{v}$$

where the line $\mathbf{l}(t)$ passes through the point $\mathbf{P}_0$, and is parallel to the vector $\mathbf{v}$.

Definition 6.4 (Tangent Lines)**.** If a VVF $\mathbf{r}(t)$ is differentiable at t_0 and $\mathbf{r}'(t_0) \neq \mathbf{0}$, the tangent line at $t = t_0$ is given by

$$\mathbf{l}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0).$$

Remark 1. Higher-order approximations can be determined using Taylor's formula.

6.4 Applications of VVFs

Theorem 6.4.1 (Curve of Intersection)**.** *A VVF can be used to determine the curve of intersection between two surfaces. The method is to choose one of the variables (commonly the first) as the parameter, and express the remaining variables in terms of that parameter. If the intersection is bounded between two points, the domain can be calculated using the component which was parametrised. For example, the curve of intersection between*

$$y = 2x - 4 \quad \textit{and} \quad z = 3x - 1$$

between the points

$$\mathbf{P}_1 = (2,\ 0,\ 7) \quad \textit{and} \quad \mathbf{P}_2 = (3,\ 2,\ 10)$$

is given by

$$\mathbf{r}(t) = \langle t,\ 2t - 4,\ 3t - 1 \rangle : 2 \leq t \leq 3.$$

Definition 6.5 (Arc Length)**.** The arc length S of a smooth continuous VVF $\mathbf{r}(t)$, is the distance along $\mathbf{r}(t)$ between $t = a$ and $t = b$, defined by

$$S = \int_a^b \|\mathbf{r}'(t)\| \, \mathrm{d}t$$

7 Differential Equations

Definition 7.1 (Differential Equations)**.** A differential equation (DE) is an equation which involves the derivatives of one or more unknown functions (called dependent variables), that are with respect to one or more independent variables.

Definition 7.2 (Ordinary Differential Equations)**.** An ordinary differential equation (ODE) is a differential equation with derivatives with respect to a single variable.

Definition 7.3 (Partial Differential Equations)**.** A partial differential equation (PDE) is a differential equation with derivatives with respect to multiple variables.

Definition 7.4 (Order of Differential Equations)**.** The order of a differential equation is the highest derivative in the equation.

Definition 7.5 (Autonomous Differential Equations)**.** An autonomous differential equation does not depend explicitly on the independent variable.

Definition 7.6 (Linear Differential Equations)**.** A linear differential equation does not have any products of the dependent variable with itself or its derivatives. The general form of a linear ODE of order n is

$$a_n\left(x\right)y^{(n)}+a_{n-1}\left(x\right)y^{(n-1)}+\cdots+a_1\left(x\right)y'+a_0\left(x\right)y=F\left(x\right).$$

The dependent variable cannot be composed in another function.

7.1 Qualitative Analysis

With qualitative analysis we aim to understand the behaviour of solutions to the ODE. By computing fixed points, we can draw a phase line diagram, and sketch solution curves. For the autonomous differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t}=f\left(y\right)$$

Definition 7.7 (Fixed Point)**.** A fixed point is the value of y for which $f\left(y\right)=0$.

Definition 7.8 (Stability)**.** By analysing the behaviour of $f\left(y\right)$ given a perturbation near fixed points, we can determine the stability of those fixed points.

Behaviour in Positive/Negative Directions	**Stability**
Both toward fixed point	stable
Both away from fixed point	unstable
One toward and one away from fixed point	semi-stable

Definition 7.9 (Phase Plane)**.** Using the information about the behaviour of $f\left(y\right)$ around fixed points, $f\left(y\right)$ can be plotted against y to construct a phase plane diagram.

Definition 7.10 (Phase Line)**.** A phase line is the one-dimensional form of a phase plane, that shows the limiting behaviour of y as $t\to\infty$.

Definition 7.11 (Sample Solutions)**.** Using a phase line, we sketch the behaviour of sample solutions of $f\left(y\right)$ where the curves asymptote toward stable (also semi-stable) fixed points, and diverge from unstable (also semi-stable) fixed points.

8 First-Order Differential Equations

8.1 Directly Integrable ODEs

For a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(x\right)$$

$$y\left(x\right) = \int f\left(x\right) \mathrm{d}x.$$

8.2 Separable ODEs

For a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p\left(x\right) q\left(y\right),$$

a separation of variables followed by an integration w.r.t., x yields an implicit solution.

$$\int \frac{1}{q\left(y\right)} \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x = \int p\left(x\right) \mathrm{d}x.$$

8.3 Linear ODEs

For a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p\left(x\right) y = q\left(x\right)$$

we can use the integrating factor

$$I\left(x\right) = e^{\int p(x)\,\mathrm{d}x}$$

so that

$$y\left(x\right) = \frac{1}{I\left(x\right)} \int I\left(x\right) q\left(x\right) \mathrm{d}x$$

8.4 Exact ODEs

A differential equation of the form

$$P\left(x,\ y\right) + Q\left(x,\ y\right) \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

has the solution

$$\Psi\left(x,\ y\right) = c,$$

iff it is exact, namely when

$$P_y = Q_x$$

where $P = \Psi_x$ and $Q = \Psi_y$. Then

$$\Psi(x,\ y) = \int P(x,\ y)\,\mathrm{d}x + f(y)$$
$$\Psi(x,\ y) = \int Q(x,\ y)\,\mathrm{d}y + g(x)$$

and $f(y)$ and $g(x)$ can be determined by solving these equations simultaneously.

9 Second-Order Differential Equations

A linear second-order differential equation is of the form

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = F(x)\,.$$

- If $F(x) = 0$, then the equation is homogeneous.
- If $F(x) \neq 0$, then the equation is nonhomogeneous.

Definition 9.1 (Initial Value Problem)**.** An initial value problem specifies the value for y and its derivative at a single value of the independent variable:

$$y(x_0) = y_0, \quad y'(x_0) = y_1.$$

Definition 9.2 (Boundary Value Problem)**.** A boundary value problem specifies the value for y at two different values of the independent variable:

$$y(x_0) = y_0, \quad y(x_1) = y_1.$$

Theorem 9.0.1 (Superposition Principle)**.** *Consider the linear homogeneous ODE*

$$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \cdots + a_1(x)\,y' + a_0(x)\,y = 0.$$

If $y_1(x)\,,\ y_2(x)\,,\ \ldots,\ y_n(x)$ are solutions of the differential equation, then the linear combination of these solutions

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

also satisfies the ODE.

Theorem 9.0.2 (Fundamental Set of Solutions)**.** *The nth order linear homogeneous ODE with continuous coefficients on an open interval I, has n non-trivial linearly independent solutions that form a fundamental set of solutions on I.*

9.1 Reduction of Order

Reduction of order is a method for finding a second solution to an ODE, given a known solution. The second solution is of the form

$$y_2(x) = v(x)\,y_1(x)\,.$$

$v(x)$ can be determined by substituting y_2 into the ODE.

9.2 Homogeneous ODEs

A second-order constant-coefficient homogeneous ODE

$$a_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0$$

has solutions of the form

$$y(x) = e^{\lambda x}.$$

9.3 Characteristic Equation

Substituting this form into the ODE gives the characteristic equation

$$a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

This equation has three distinct cases.

Real Distinct Roots. If $a_1^2 > 4a_0a_2$.

Real Repeated Roots. If $a_1^2 = 4a_0a_2$.

Complex Conjugate Roots. If $a_1^2 < 4a_0a_2$.

9.3.1 Real Distinct Roots

For two real and distinct roots, λ_1 and λ_2, the general solution is

$$y(x) = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$$

9.3.2 Real Repeated Roots

For the real repeated root, λ, the general solution is

$$y(x) = c_1e^{\lambda x} + c_2te^{\lambda x}$$

9.3.3 Complex Conjugate Roots

For two complex conjugate roots, $\lambda = \alpha \pm \beta i$, the general solution is

$$y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$$

10 Nonhomogeneous Differential Equations

A second-order constant-coefficient nonhomogeneous ODE

$$a_2y'' + a_1y' + a_0y = F(x)$$

has a general solution of the form

$$y(x) = y_H(x) + y_P(x)$$

where $y_H(x)$ satisfies

$$a_2\frac{\mathrm{d}^2y_H}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}y_H}{\mathrm{d}x} + a_0y_H = 0$$

and $y_P(x)$ satisfies

$$a_2\frac{\mathrm{d}^2y_P}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}y_P}{\mathrm{d}x} + a_0y_P = F(x)$$

10.1 Method of Undetermined Coefficients

To find the particular solution y_P, we must choose a likely form that it would take. The following table summarises appropriate forms of y_P based on F.

$F(x)$ form	$y_P(x)$ guess
a constant	A
a polynomial of degree n	$\sum_{i=0}^{n} A_i x^i$
e^{kx}	Ae^{kx}
$\cos(\omega x)$ or $\sin(\omega x)$	$A_0 \cos(\omega x) + A_1 \sin(\omega x)$

Table 4: Particular solutions for undetermined coefficients.

Once the form of the particular solution has been determined, it can be substituted into the nonhomogeneous ODE, to determine the undetermined coefficients.

10.2 Special Forms

10.2.1 Product of Forms

If $F(x)$ is a product of the functions shown above, then the particular solutions are also multiplied together and any constants are simplified.

10.2.2 Sum of Forms

If $F(x)$ is a sum of the functions shown above, then the particular solutions are also added together.

10.2.3 Linearly Dependent Forms

If $F(x)$ is similar to any homogeneous solution, then by *definition* of a homogeneous solution, the solution will be 0. Hence, y_P must be multiplied by x to ensure that the particular solution is linearly independent to the homogeneous solutions, in order to form a *fundamental set of solutions.*

10.3 Finding the General Solution

1. Solve y_H
2. Find an appropriate form for y_P
3. Ensure that y_P is linearly independent to the homogeneous solutions
4. Substitute y_P into the nonhomogeneous ODE and solve for the undetermined coefficients
5. Find the general solution $y = y_H + y_P$
6. Apply any initial or boundary conditions

10.4 Applications of Second-Order ODEs

10.4.1 Spring and Mass Systems

$$F = \text{spring force } (F_s) + \text{damping force } (F_d) + \text{external force } (f(t))$$

Newton's Law $F = my''$

Spring force. $F_s = -ky$

Damping force. $F_d = -\gamma y'$

$$my'' + \gamma y' + ky = f(t)$$

10.4.2 Electrical Circuits

Current i is defined as the rate of change of charge q

$$i = \frac{\mathrm{d}q}{\mathrm{d}t}$$

The voltage drop across various elements is given below:

Voltage drop across a resistor: iR

Voltage drop across a capacitor: q/C

Voltage drop across an inductor: $L\frac{\mathrm{d}i}{\mathrm{d}t}$

where R is the resistance measured in Ohms ( ), C is the capacitance measured in Farads (F), and L is the inductance measured in Henrys (H). Kirchhoff's Voltage law states that the sum of voltages around a loop equals 0. Therefore, in an RLC circuit, with a voltage source supplying $v(t)$ V,

$$\begin{aligned} v(t) - iR - L\frac{\mathrm{d}i}{\mathrm{d}t} - \frac{q}{C} &= 0 \\ L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q &= v(t). \end{aligned}$$

A Integration Techniques

A.1 Table of Derivatives

Let $f(x)$ be a function, and $a \in \mathbb{R}$ be a constant.

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
x^a	ax^{a-1}
$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$
a^x	$\ln(a)a^x$
e^x	e^x
$\log_a(x),\ a \in \mathbb{R}\setminus\{0\}$	$\dfrac{1}{x\ln(a)}$
$\ln(x)$	$\dfrac{1}{x}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
a	0
x	1
$a_1 u(x) \pm a_2 v(x)$	$a_1 \dfrac{\mathrm{d}u}{\mathrm{d}x} \pm a_2 \dfrac{\mathrm{d}v}{\mathrm{d}x}$
$u(x)\,v(x)$	$\dfrac{\mathrm{d}u}{\mathrm{d}x} v + u \dfrac{\mathrm{d}v}{\mathrm{d}x}$
$\dfrac{u(x)}{v(x)}$	$\dfrac{\frac{\mathrm{d}u}{\mathrm{d}x} v - u \frac{\mathrm{d}v}{\mathrm{d}x}}{v(x)^2}$
$u(v(x))$	$\dfrac{\mathrm{d}u}{\mathrm{d}v} \dfrac{\mathrm{d}v}{\mathrm{d}x}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\csc^2(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\csc(ax)$	$-a\csc(ax)\cot(ax)$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\arcsin(ax)$	$\dfrac{a}{\sqrt{1-a^2x^2}}$
$\arccos(ax)$	$-\dfrac{a}{\sqrt{1-a^2x^2}}$
$\arctan(ax)$	$\dfrac{a}{1+a^2x^2}$
$\operatorname{arccot}(ax)$	$-\dfrac{a}{1+a^2x^2}$
$\operatorname{arcsec}(ax)$	$\dfrac{1}{x\sqrt{a^2x^2-1}}$
$\operatorname{arccsc}(ax)$	$-\dfrac{1}{x\sqrt{a^2x^2-1}}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\sinh(ax)$	$a\cosh(ax)$
$\cosh(ax)$	$a\sinh(ax)$
$\tanh(ax)$	$a\,\mathrm{sech}^2(ax)$
$\coth(ax)$	$-a\,\mathrm{csch}^2(ax)$
$\mathrm{sech}(ax)$	$-a\,\mathrm{sech}(ax)\tan(ax)$
$\mathrm{csch}(ax)$	$-a\,\mathrm{csch}(ax)\cot(ax)$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\mathrm{arcsinh}(ax)$	$\frac{a}{\sqrt{1+a^2x^2}}$
$\mathrm{arccosh}(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$
$\mathrm{arctanh}(ax)$	$\frac{a}{1-a^2x^2}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\mathrm{arccoth}(ax)$	$\frac{a}{1-a^2x^2}$
$\mathrm{arcsech}(ax)$	$-\frac{1}{a(1+ax)\sqrt{\frac{1-ax}{1+ax}}}$
$\mathrm{arccsch}(ax)$	$-\frac{1}{ax^2\sqrt{1+\frac{1}{a^2x^2}}}$

Table 5: Derivatives of Elementary Functions

A.2 Trigonometric Identities

A.2.1 Pythagorean Identities

$$\sin^2(x)+\cos^2(x)=1$$

Dividing by either the sine or cosine function gives:

$$\tan^2(x)+1=\sec^2(x)$$
$$1+\cot^2(x)=\csc^2(x)$$

A.2.2 Double-Angle Identities

$$\sin(2x)=2\sin(x)\cos(x) \qquad \csc(2x)=\frac{\sec(x)\csc(x)}{2}$$

$$\cos(2x)=\cos^2(x)-\sin^2(x) \qquad \sec(2x)=\frac{\sec^2(x)\csc^2(x)}{\csc^2(x)-\sec^2(x)}$$

$$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)} \qquad \cot(2x)=\frac{\cot^2(x)-1}{2\cot(x)}$$

A.2.3 Power Reducing Identities

$$\sin^2(x)=\frac{1-\cos(2x)}{2} \qquad \csc^2(x)=\frac{2}{1-\cos(2x)}$$

$$\cos^2(x)=\frac{1+\cos(2x)}{2} \qquad \sec^2(x)=\frac{2}{1+\cos(2x)}$$

$$\tan^2(x)=\frac{1-\cos(2x)}{1+\cos(2x)} \qquad \cot^2(x)=\frac{1+\cos(2x)}{1-\cos(2x)}$$

A.3 Partial Fractions

Definition A.1 (Partial Fraction Decomposition)**. Partial fraction decomposition** is a method where a rational function $\frac{P(x)}{Q(x)}$ is rewritten as a sum of fraction.

Factor in denominator	**Term in partial fraction decomposition**
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k,\ k \in \mathbb{N}$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{A}{ax^2+bx+c}$
$(ax^2+bx+c)^k,\ k \in \mathbb{N}$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Table 6: Partial Fraction Forms

A.4 Integration by Parts

Theorem A.4.1.

$$\int u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\,\mathrm{d}x = u(x)\,v(x) - \int v(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\,\mathrm{d}x \implies \int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

A.5 Integration by Substitution

Theorem A.5.1.

$$\int f(g(x))\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x = \int f(u)\,\mathrm{d}u,\ \textit{where } u = g(x)$$

A.6 Trigonometric Substitutions

Form	**Substitution**	**Result**	**Domain**
$(a^2-b^2x^2)^n$	$x = \frac{a}{b}\sin(\theta)$	$a^2\cos^2(\theta)$	$\theta \in [-\frac{\pi}{2},\ \frac{\pi}{2}]$
$(a^2+b^2x^2)^n$	$x = \frac{a}{b}\tan(\theta)$	$a^2\sec^2(\theta)$	$\theta \in (-\frac{\pi}{2},\ \frac{\pi}{2})$
$(b^2x^2-a^2)^n$	$x = \frac{a}{b}\sec(\theta)$	$a^2\tan^2(\theta)$	$\theta \in [0,\ \frac{\pi}{2}) \cup (\frac{\pi}{2},\ \pi]$

Table 7: Trigonometric substitutions for various forms.