Electrical Engineering Mathematics

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Contents

1 Infinite Series

1.1 Sequences

A sequence is an **ordered list** of numbers

$$a_1,\ a_2,\ a_3,\ \dots,\ a_n,\ \dots$$

denoted $\{a_n\}_{n=1}^{\infty}$, where n is the index of the sequence. A sequence can be **finite** or **infinite**.

1.2 Limits of Sequences

An infinite sequence $\{a_n\}$ has a limit L if a_n approaches L as n approaches infinity:

$$\lim_{n\to\infty} a_n = L$$

If such a limit exists, the sequence **converges** to L. Otherwise, the sequence **diverges**. Sequences that oscillate between two or more values do not have a limit.

1.3 Series

Given a sequence $\{a_n\}$, we can construct a sequence of **partial sums**,

$$s_n = a_1 + a_2 + \cdots + a_n$$

denoted $\{s_n\}$, such that when $\{s_n\}$ converges to a finite limit L, that is,

$$\lim_{n\to\infty} s_n = L$$

the **infinite series** $\sum_{n=1}^{\infty} a_n$ converges to L. Otherwise, the series $\sum_{n=1}^{\infty} a_n$ diverges.

1.3.1 Common Series

Below are a list of common series that converge to a finite limit:

- **Geometric Series**: A sum of the geometric progression

  $$\sum_{n=0}^{\infty} ar^n$$

  converges when $|r| < 1$, and diverges otherwise. When $|r| < 1$,

  $$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

- **Harmonic Series**: A sum of the reciprocals of natural numbers

  $$\sum_{n=1}^{\infty} \frac{1}{n}$$

  always diverges.

- p**-Series**: A sum of the reciprocals of p-powers of natural numbers

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges when $p > 1$, and diverges otherwise. This series is closely related to the **Riemann Zeta Function**, and has exact values for even integers p.

1.4 Convergence Tests

There are several tests to determine the convergence of an infinite series. Note that these tests do not determine the value of the limit.

1.4.1 Ratio Test

Given the infinite series $\sum_{n=1}^{\infty} a_n$, with

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

(1) If $\rho < 1$, the series converges.

(2) If $\rho > 1$, the series diverges.

(3) If $\rho = 1$, the test is inconclusive.

1.4.2 Alternating Series Test

Given the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, the alternating series converges if the following conditions are met:

(1) $b_n > 0$ for all n.

(2) $b_{n+1} \leq b_n$ for all n.

(3) $\lim_{n\to\infty} b_n = 0$.

2 Taylor Series

2.1 Taylor Polynomials

A Taylor polynomial is a polynomial that approximates a function near a point $x = x_0$. The n-th order Taylor polynomial of an n-times differentiable function $f(x)$ near $x = x_0$ is given by:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Using summation notation, this becomes,

$$f(x) \approx P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

If f is $(n+1)$-times differentiable on an interval including x_0, then the error of this approximation can be bounded by

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1}$$

for some p between x and x_0.

2.2 The Taylor Series

The Taylor polynomials can be extended to Taylor series by taking the limit $n \to \infty$. The Taylor series of an infinitely differentiable function $f(x)$ near $x = x_0$ is defined:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

When $x_0 = 0$, the Taylor series is called the **Maclaurin series**.

2.3 Convergence of Taylor Series

The Taylor series is a form of a **power series**:

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

A power series may converge in one of three ways:

(1) At a single point $x = x_0$, with a radius of convergence $R = 0$.

(2) On a finite open interval $(x_0 - R, x_0 + R)$, with a radius of convergence $R > 0$. The series is not guaranteed to converge at the endpoints of this interval.

(3) Everywhere, with a radius of convergence $R = \infty$.

For elementary functions, the Taylor series converges to the function everywhere within the radius of convergence.

2.4 Common Taylor Series

Below are a list of common Taylor series expansions:

Function	Taylor Series	Convergence
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$\ln(1-x)$	$-\sum_{n=1}^{\infty} \frac{x^n}{n}$	$-1 \leqslant x < 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$	$-1 < x \leqslant 1$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$

3 Fourier Series

3.1 Periodic Functions

A function $f(t)$ is **periodic** with period T if it satisfies the following condition:

$$f(t+T) = f(t)$$

for all t. As with Taylor polynomials, we wish to build an approximation of $f(t)$ using some basis.

3.2 The Fourier Series

The Fourier series is a representation of a periodic function as the linear combination of sine and cosine functions. For a periodic function $f(t)$ with period T, the Fourier series of $f(t)$ is defined:

$$f_F(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n}{T} t\right) + b_n \sin\left(\frac{2\pi n}{T} t\right) \right).$$

The coefficients a_0, a_n, and b_n are given by:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{t_0}^{t_0+T} f(t) \,\mathrm{d}t \\ a_n &= \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos\left(\frac{2\pi n}{T} t\right) \mathrm{d}t \\ b_n &= \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin\left(\frac{2\pi n}{T} t\right) \mathrm{d}t \end{aligned}$$

where t_0 is any value of t, often chosen to be 0 or $-T/2$.

3.3 Convergence of Fourier Series

If $f(t)$ is piecewise smooth on the interval $[t_0, t_0+T]$, then the Fourier series converges to $f(t)$ in the interval $[t_0, t_0+T]$:

$$f_F(t) = \lim_{\epsilon \to 0^+} \frac{f(t+\epsilon) + f(t-\epsilon)}{2},$$

where discontinuous points $\bar{t} \in [t_0, t_0+T]$ converge to the **average** of their left-hand and right-hand limits. When f is non-periodic, the Fourier series converges to the **periodic extension** of f. The endpoints of the interval may converge non-uniformly, corresponding to jump discontinuities in the periodic extension of f.

3.4 Orthogonality

Both the Taylor series and Fourier series are comprised of an **orthogonal basis**. In this function space, the inner product of two functions $f(t)$ and $g(t)$ is defined:

$$\langle f,\, g\rangle = \int_{t_0}^{t_0+T} f(t)\, g(t)\, \mathrm{d}t$$

on the interval $[t_0, t_0+T]$. The norm of a function can be defined as $\|f\| = \sqrt{\langle f,\, f\rangle}$. Using this definition, we say that two functions are **orthogonal** if their inner product is zero, and **orthonormal** if their inner product is one.
The Fourier series is defined using an infinite dimensional set of orthogonal basis functions:

$$\left\{1,\, \cos\left(\frac{2\pi n}{T}t\right),\, \sin\left(\frac{2\pi n}{T}t\right)\right\}$$

for all $n \in \mathbb{N}$. The inner products of these basis functions are given by:

$$\begin{aligned}
\left\langle \cos\left(\frac{2\pi n}{T}t\right),\, 1\right\rangle &= \int_{t_0}^{t_0+T} \cos\left(\frac{2\pi n}{T}t\right) \mathrm{d}t = 0 \\
\left\langle \sin\left(\frac{2\pi n}{T}t\right),\, 1\right\rangle &= \int_{t_0}^{t_0+T} \sin\left(\frac{2\pi n}{T}t\right) \mathrm{d}t = 0 \\
\left\langle \cos\left(\frac{2\pi m}{T}t\right),\, \sin\left(\frac{2\pi n}{T}t\right)\right\rangle &= \int_{t_0}^{t_0+T} \cos\left(\frac{2\pi m}{T}t\right) \sin\left(\frac{2\pi n}{T}t\right) \mathrm{d}t = 0 \\
\left\langle \cos\left(\frac{2\pi m}{T}t\right),\, \cos\left(\frac{2\pi n}{T}t\right)\right\rangle &= \int_{t_0}^{t_0+T} \cos\left(\frac{2\pi m}{T}t\right) \cos\left(\frac{2\pi n}{T}t\right) \mathrm{d}t = \begin{cases} \frac{T}{2}, & m = n \\ 0 & m \neq n \end{cases} \\
\left\langle \sin\left(\frac{2\pi m}{T}t\right),\, \sin\left(\frac{2\pi n}{T}t\right)\right\rangle &= \int_{t_0}^{t_0+T} \sin\left(\frac{2\pi m}{T}t\right) \sin\left(\frac{2\pi n}{T}t\right) \mathrm{d}t = \begin{cases} \frac{T}{2}, & m = n \\ 0 & m \neq n \end{cases}
\end{aligned}$$

for all m and n not equal to zero. These results allow us to determine the Fourier series coefficients by considering the inner product between $f(t)$ and various basis functions. For the coefficient a_0, consider the inner product of $f(t)$ with the constant function 1:

$$\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\right)\\
\langle f,\ 1\rangle &= a_0\langle 1,\ 1\rangle + \sum_{n=1}^{\infty}\left(a_n\left\langle \cos\left(\frac{2\pi n}{T}t\right),\ 1\right\rangle + b_n\left\langle \sin\left(\frac{2\pi n}{T}t\right),\ 1\right\rangle\right)\\
a_0 &= \frac{\langle f,\ 1\rangle}{\langle 1,\ 1\rangle} = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,\mathrm{d}t
\end{aligned}$$

For the coefficients a_n and b_n, consider the inner product of $f(t)$ with $\cos\left(\frac{2\pi m}{T}t\right)$ and $\sin\left(\frac{2\pi m}{T}t\right)$, respectively. For a_n:

$$\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\right)\\
\left\langle f,\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle &= a_0\left\langle 1,\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle\\
&\quad + \sum_{n=1}^{\infty}\left(a_n\left\langle \cos\left(\frac{2\pi n}{T}t\right),\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle + b_n\left\langle \sin\left(\frac{2\pi n}{T}t\right),\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle\right)\\
a_m &= \frac{\left\langle f,\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle}{\left\langle \cos\left(\frac{2\pi m}{T}t\right),\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle} = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos\left(\frac{2\pi m}{T}t\right)\mathrm{d}t.
\end{aligned}$$

For b_n:

$$\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\right)\\
\left\langle f,\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle &= a_0\left\langle 1,\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle\\
&\quad + \sum_{n=1}^{\infty}\left(a_n\left\langle \cos\left(\frac{2\pi n}{T}t\right),\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle + b_n\left\langle \sin\left(\frac{2\pi n}{T}t\right),\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle\right)\\
b_m &= \frac{\left\langle f,\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle}{\left\langle \sin\left(\frac{2\pi m}{T}t\right),\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle} = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin\left(\frac{2\pi m}{T}t\right)\mathrm{d}t.
\end{aligned}$$

3.5 Even and Odd Functions

A function $f(t)$ is **even** if

$$f(-t) = f(t)$$

for all t, and **odd** if

$$f(-t) = -f(t).$$

These functions have a special symmetry property that can be exploited when computing integrals:

$$\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t = \begin{cases} 2\displaystyle\int_{0}^{T/2} f(t)\,\mathrm{d}t, & \text{if } f(t) \text{ even}\\ 0, & \text{if } f(t) \text{ odd}\end{cases}$$

In the context of the Fourier series expansion, it is important to note that cosine functions are even, and sine functions are odd:

$$\cos(-t) = \cos(t)$$
$$\sin(-t) = -\sin(t)$$

3.6 Fourier Cosine Series

Suppose $f(t)$ is an even function with period T, and let us compute the Fourier series of $f(t)$ on the interval $[-T/2, T/2]$. Consider the coefficients b_n:

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi n}{T}t\right)\mathrm{d}t$$

as $f(t)$ is even, the resulting integrand is odd, and the integral is zero. This results in a series containing only even functions, called the Fourier cosine series expansion of $f(t)$:

$$f_c(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T}t\right)$$

with

$$a_0 = \frac{2}{T}\int_{0}^{T/2} f(t)\,\mathrm{d}t$$
$$a_n = \frac{4}{T}\int_{0}^{T/2} f(t)\cos\left(\frac{2\pi n}{T}t\right)\mathrm{d}t$$

3.7 Fourier Sine Series

Suppose $f(t)$ is an odd function with period T, and let us compute the Fourier series of $f(t)$ on the interval $[-T/2, T/2]$. Consider the coefficients a_0 and a_n:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t$$
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi n}{T}t\right)\mathrm{d}t$$

as $f(t)$ is odd, the resulting integrand is odd for both a_0 and a_n, and the integrals are zero. This results in a series containing only odd functions, called the Fourier sine series expansion of $f(t)$:

$$f_s(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T}t\right)$$

with

$$b_n = \frac{4}{T}\int_{0}^{T/2} f(t)\sin\left(\frac{2\pi n}{T}t\right)\mathrm{d}t$$

3.8 Half-Range Expansions

Suppose a function $f(t)$ is defined on the interval $[0, T]$, that is not necessarily even or odd. We can extend this function onto the interval in one of three ways:

- Fourier series: Extends the function periodically on the interval $[0, T]$, with period T.
- Fourier cosine series: Extends the even expansion of the function on the interval $[-T, T]$, with period $2T$.
- Fourier sine series: Extends the odd expansion of the function on the interval $[-T, T]$, with period $2T$.

Note the period in the even and odd series must be twice the period of the original function. This is illustrated in the figures below:

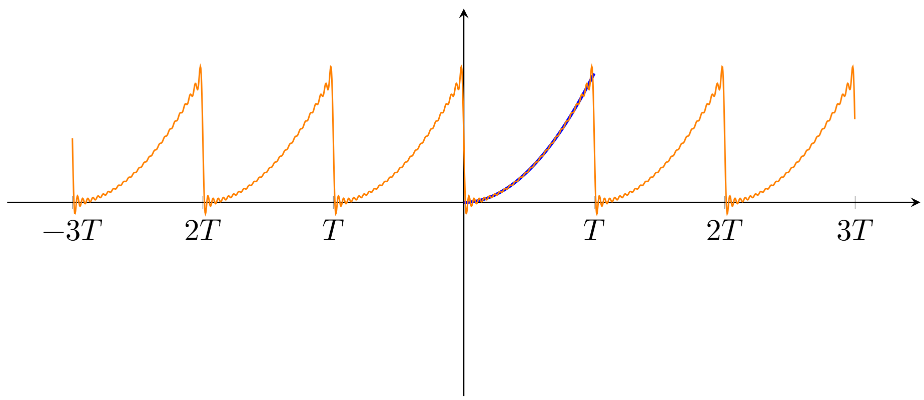

Figure 1: Fourier series expansion of $f(t)$ on the interval $[0, T]$, with the period T.

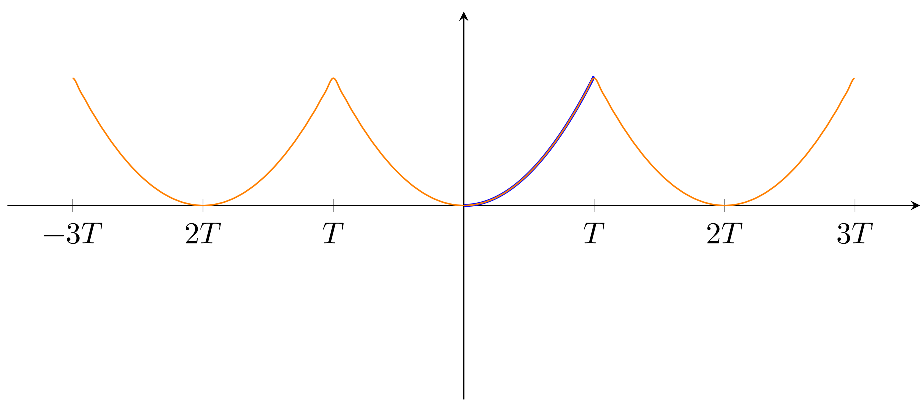

Figure 2: Fourier cosine series expansion of $f(t)$ onto the interval $[-T, T]$, with the period $2T$.

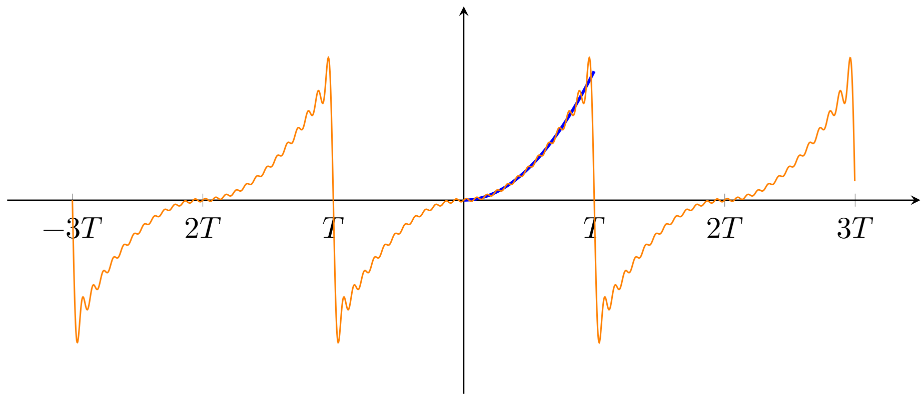

Figure 3: Fourier sine series expansion of $f(t)$ onto the interval $[-T, T]$, with the period $2T$.