Electrical Engineering Mathematics

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Contents

Part I
Infinite Series

1 Sequences and Series

1.1 Sequences

A sequence is an **ordered list** of numbers

$$a_1,\ a_2,\ a_3,\ \dots,\ a_n,\ \dots$$

denoted $\{a_n\}_{n=1}^{\infty}$, where n is the index of the sequence. A sequence can be **finite** or **infinite**.

1.2 Limits of Sequences

An infinite sequence $\{a_n\}$ has a limit L if a_n approaches L as n approaches infinity:

$$\lim_{n\to\infty} a_n = L$$

If such a limit exists, the sequence **converges** to L. Otherwise, the sequence **diverges**. Sequences that oscillate between two or more values do not have a limit.

1.3 Series

Given a sequence $\{a_n\}$, we can construct a sequence of **partial sums**,

$$s_n = a_1 + a_2 + \cdots + a_n$$

denoted $\{s_n\}$, such that when $\{s_n\}$ converges to a finite limit L, that is,

$$\lim_{n\to\infty} s_n = L$$

the **infinite series** $\sum_{n=1}^{\infty} a_n$ converges to L. Otherwise, the series $\sum_{n=1}^{\infty} a_n$ diverges.

1.3.1 Common Series

Below are a list of common series that converge to a finite limit:

- **Geometric Series**: A sum of the geometric progression

  $$\sum_{n=0}^{\infty} ar^n$$

  converges when $|r| < 1$, and diverges otherwise. When $|r| < 1$,

  $$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

- **Harmonic Series**: A sum of the reciprocals of natural numbers

  $$\sum_{n=1}^{\infty} \frac{1}{n}$$

  always diverges.

- p**-Series**: A sum of the reciprocals of p-powers of natural numbers

  $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

  converges when $p > 1$, and diverges otherwise. This series is closely related to the **Riemann Zeta Function**, and has exact values for even integers p.

1.4 Convergence Tests

There are several tests to determine the convergence of an infinite series. Note that these tests do not determine the value of the limit.

1.4.1 Ratio Test

Given the infinite series $\sum_{n=1}^{\infty} a_n$, with

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

(1) If $\rho < 1$, the series converges.

(2) If $\rho > 1$, the series diverges.

(3) If $\rho = 1$, the test is inconclusive.

1.4.2 Alternating Series Test

Given the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, the alternating series converges if the following conditions are met:

(1) $b_n > 0$ for all n.

(2) $b_{n+1} \leqslant b_n$ for all n.

(3) $\lim_{n\to\infty} b_n = 0$.

2 Taylor Series

2.1 Taylor Polynomials

A Taylor polynomial is a polynomial that approximates a function near a point $x = x_0$. The n-th order Taylor polynomial of an n-times differentiable function $f(x)$ near $x = x_0$ is given by:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Using summation notation, this becomes,

$$f(x) \approx P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

If f is $(n+1)$-times differentiable on an interval including x_0, then the error of this approximation can be bounded by

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1}$$

for some p between x and x_0.

2.2 The Taylor Series

The Taylor polynomials can be extended to Taylor series by taking the limit $n \to \infty$. The Taylor series of an infinitely differentiable function $f(x)$ near $x = x_0$ is defined:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

When $x_0 = 0$, the Taylor series is called the **Maclaurin series**.

2.3 Convergence of Taylor Series

The Taylor series is a form of a **power series**:

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

A power series may converge in one of three ways:

(1) At a single point $x = x_0$, with a radius of convergence $R = 0$.

(2) On a finite open interval $(x_0 - R, x_0 + R)$, with a radius of convergence $R > 0$. The series is not guaranteed to converge at the endpoints of this interval.

(3) Everywhere, with a radius of convergence $R = \infty$.

For elementary functions, the Taylor series converges to the function everywhere within the radius of convergence.

2.4 Common Taylor Series

Below is a list of common Taylor series expansions:

Function	**Taylor Series**	**Convergence**
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$\ln(1-x)$	$-\sum_{n=1}^{\infty} \frac{x^n}{n}$	$-1 \leqslant x < 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$	$-1 < x \leqslant 1$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$

3 Fourier Series

3.1 Periodic Functions

A function $f(t)$ is **periodic** with period T if it satisfies the following condition:

$$f(t+T) = f(t)$$

for all t. As with Taylor polynomials, we wish to build an approximation of $f(t)$ using some basis.

3.2 The Fourier Series

The Fourier series is a representation of a periodic function as the linear combination of sine and cosine functions. For a periodic function $f(t)$ with period T, the Fourier series of $f(t)$ is defined:

$$f_F(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n}{T} t\right) + b_n \sin\left(\frac{2\pi n}{T} t\right) \right).$$

The coefficients a_0, a_n, and b_n are given by:

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) \, \mathrm{d}t$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos\left(\frac{2\pi n}{T} t\right) \mathrm{d}t$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin\left(\frac{2\pi n}{T} t\right) \mathrm{d}t$$

where t_0 is any value of t, often chosen to be 0 or $-T/2$.

3.3 Convergence of Fourier Series

If $f(t)$ is piecewise smooth on the interval $[t_0, t_0 + T]$, then the Fourier series converges to $f(t)$ in the interval $[t_0, t_0 + T]$:

$$f_F(t) = \lim_{\epsilon \to 0^+} \frac{f(t+\epsilon) + f(t-\epsilon)}{2},$$

where discontinuous points $\bar{t} \in [t_0, t_0 + T]$ converge to the **average** of their left-hand and right-hand limits. When f is non-periodic, the Fourier series converges to the **periodic extension** of f. The endpoints of the interval may converge non-uniformly, corresponding to jump discontinuities in the periodic extension of f.

3.4 Orthogonality

Both the Taylor series and Fourier series are comprised of an **orthogonal basis**. In this function space, the inner product of two functions $f(t)$ and $g(t)$ is defined:

$$\langle f,\ g\rangle = \int_{t_0}^{t_0+T} f(t)\, g(t)\, \mathrm{d}t$$

on the interval $[t_0, t_0 + T]$. The norm of a function can be defined as $\|f\| = \sqrt{\langle f,\ f\rangle}$. Using this definition, we say that two functions are **orthogonal** if their inner product is zero, and **orthonormal** if their inner product is one. The Fourier series is defined using an infinite-dimensional set of orthogonal basis functions:

$$\left\{1,\ \cos\left(\frac{2\pi n}{T}t\right),\ \sin\left(\frac{2\pi n}{T}t\right)\right\}$$

for all $n \in \mathbb{N}$. The inner products of these basis functions are given by:

$$\begin{aligned}
\left\langle \cos\left(\frac{2\pi n}{T}t\right),\ 1\right\rangle &= \int_{t_0}^{t_0+T} \cos\left(\frac{2\pi n}{T}t\right) \mathrm{d}t = 0 \\
\left\langle \sin\left(\frac{2\pi n}{T}t\right),\ 1\right\rangle &= \int_{t_0}^{t_0+T} \sin\left(\frac{2\pi n}{T}t\right) \mathrm{d}t = 0 \\
\left\langle \cos\left(\frac{2\pi m}{T}t\right),\ \sin\left(\frac{2\pi n}{T}t\right)\right\rangle &= \int_{t_0}^{t_0+T} \cos\left(\frac{2\pi m}{T}t\right) \sin\left(\frac{2\pi n}{T}t\right) \mathrm{d}t = 0 \\
\left\langle \cos\left(\frac{2\pi m}{T}t\right),\ \cos\left(\frac{2\pi n}{T}t\right)\right\rangle &= \int_{t_0}^{t_0+T} \cos\left(\frac{2\pi m}{T}t\right) \cos\left(\frac{2\pi n}{T}t\right) \mathrm{d}t = \begin{cases} \frac{T}{2}, & m = n \\ 0 & m \neq n \end{cases} \\
\left\langle \sin\left(\frac{2\pi m}{T}t\right),\ \sin\left(\frac{2\pi n}{T}t\right)\right\rangle &= \int_{t_0}^{t_0+T} \sin\left(\frac{2\pi m}{T}t\right) \sin\left(\frac{2\pi n}{T}t\right) \mathrm{d}t = \begin{cases} \frac{T}{2}, & m = n \\ 0 & m \neq n \end{cases}
\end{aligned}$$

for all m and n not equal to zero. These results allow us to determine the Fourier series coefficients by considering the inner product between $f(t)$ and various basis functions. For the coefficient a_0, consider the inner product of $f(t)$ with the constant function 1:

$$\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\right)\\
\langle f,\ 1\rangle &= a_0\langle 1,\ 1\rangle + \sum_{n=1}^{\infty}\left(a_n\left\langle \cos\left(\frac{2\pi n}{T}t\right),\ 1\right\rangle + b_n\left\langle \sin\left(\frac{2\pi n}{T}t\right),\ 1\right\rangle\right)\\
a_0 &= \frac{\langle f,\ 1\rangle}{\langle 1,\ 1\rangle} = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,\mathrm{d}t
\end{aligned}$$

For the coefficients a_n and b_n, consider the inner product of $f(t)$ with $\cos\left(\frac{2\pi m}{T}t\right)$ and $\sin\left(\frac{2\pi m}{T}t\right)$, respectively. For a_n:

$$\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\right)\\
\left\langle f,\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle &= a_0\left\langle 1,\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle\\
&\quad + \sum_{n=1}^{\infty}\left(a_n\left\langle \cos\left(\frac{2\pi n}{T}t\right),\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle + b_n\left\langle \sin\left(\frac{2\pi n}{T}t\right),\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle\right)\\
a_m &= \frac{\left\langle f,\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle}{\left\langle \cos\left(\frac{2\pi m}{T}t\right),\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle} = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos\left(\frac{2\pi m}{T}t\right)\mathrm{d}t.
\end{aligned}$$

For b_n:

$$\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\right)\\
\left\langle f,\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle &= a_0\left\langle 1,\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle\\
&\quad + \sum_{n=1}^{\infty}\left(a_n\left\langle \cos\left(\frac{2\pi n}{T}t\right),\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle + b_n\left\langle \sin\left(\frac{2\pi n}{T}t\right),\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle\right)\\
b_m &= \frac{\left\langle f,\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle}{\left\langle \sin\left(\frac{2\pi m}{T}t\right),\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle} = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin\left(\frac{2\pi m}{T}t\right)\mathrm{d}t.
\end{aligned}$$

3.5 Even and Odd Functions

A function $f(t)$ is **even** if

$$f(-t) = f(t)$$

for all t, and **odd** if

$$f(-t) = -f(t).$$

These functions have a special symmetry property that can be exploited when computing integrals:

$$\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t = \begin{cases} 2\displaystyle\int_0^{T/2} f(t)\,\mathrm{d}t, & \text{if } f(t) \text{ even}\\ 0, & \text{if } f(t) \text{ odd}\end{cases}$$

In the context of the Fourier series expansion, it is important to note that cosine functions are even, and sine functions are odd:

$$\cos(-t) = \cos(t)$$
$$\sin(-t) = -\sin(t)$$

3.6 Fourier Cosine Series

Suppose $f(t)$ is an even function with period T, and let us compute the Fourier series of $f(t)$ on the interval $[-T/2, T/2]$. Consider the coefficients b_n:

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi n}{T}t\right)\mathrm{d}t$$

as $f(t)$ is even, the resulting integrand is odd, and the integral is zero. This results in a series containing only even functions, called the Fourier cosine series expansion of $f(t)$:

$$f_c(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T}t\right)$$

with

$$a_0 = \frac{2}{T}\int_{0}^{T/2} f(t)\,\mathrm{d}t$$
$$a_n = \frac{4}{T}\int_{0}^{T/2} f(t)\cos\left(\frac{2\pi n}{T}t\right)\mathrm{d}t$$

3.7 Fourier Sine Series

Suppose $f(t)$ is an odd function with period T, and let us compute the Fourier series of $f(t)$ on the interval $[-T/2, T/2]$. Consider the coefficients a_0 and a_n:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t$$
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi n}{T}t\right)\mathrm{d}t$$

as $f(t)$ is odd, the resulting integrand is odd for both a_0 and a_n, and the integrals are zero. This results in a series containing only odd functions, called the Fourier sine series expansion of $f(t)$:

$$f_s(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T}t\right)$$

with

$$b_n = \frac{4}{T}\int_{0}^{T/2} f(t)\sin\left(\frac{2\pi n}{T}t\right)\mathrm{d}t$$

3.8 Half-Range Expansions

Suppose a function $f(t)$ is defined on the interval $[0, T]$, that is not necessarily even or odd. We can extend this function onto the interval in one of three ways:

- Fourier series: Extends the function periodically on the interval $[0, T]$, with period T.
- Fourier cosine series: Extends the even expansion of the function on the interval $[-T, T]$, with period $2T$.
- Fourier sine series: Extends the odd expansion of the function on the interval $[-T, T]$, with period $2T$.

Note the period in the even and odd series must be twice the period of the original function. This is illustrated in the figures below for the function $f(t) = t^2$ on the interval $[0, T]$:

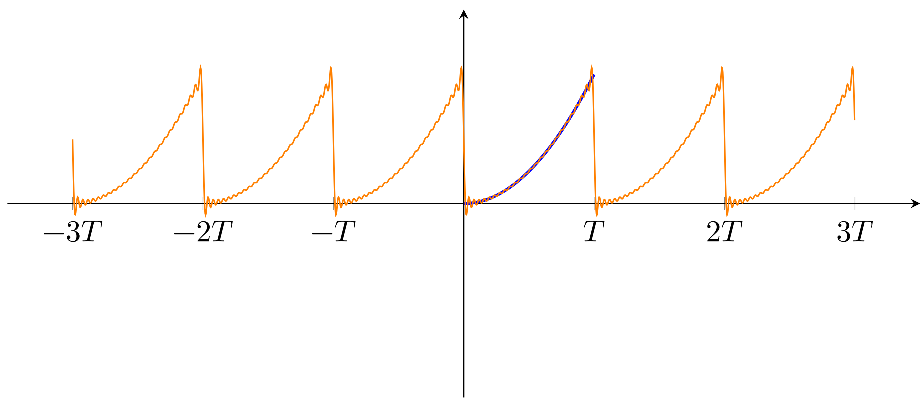

Figure 1: Fourier series expansion of $f(t)$ on the interval $[0, T]$, with the period T.

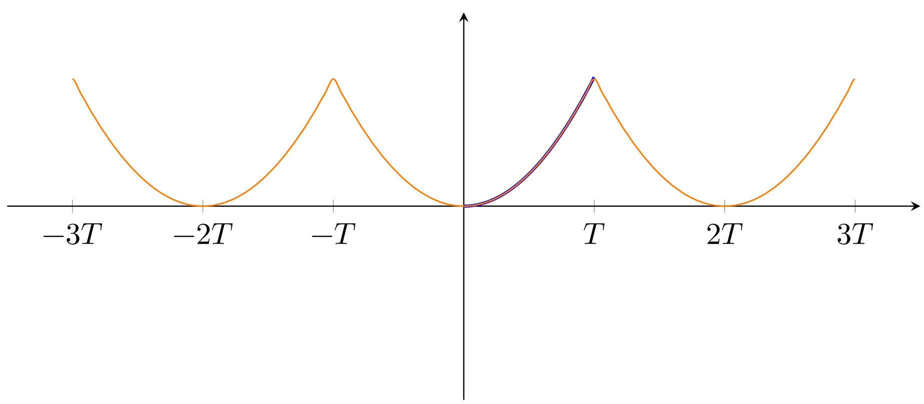

Figure 2: Fourier cosine series expansion of $f(t)$ onto the interval $[-T, T]$, with the period $2T$.

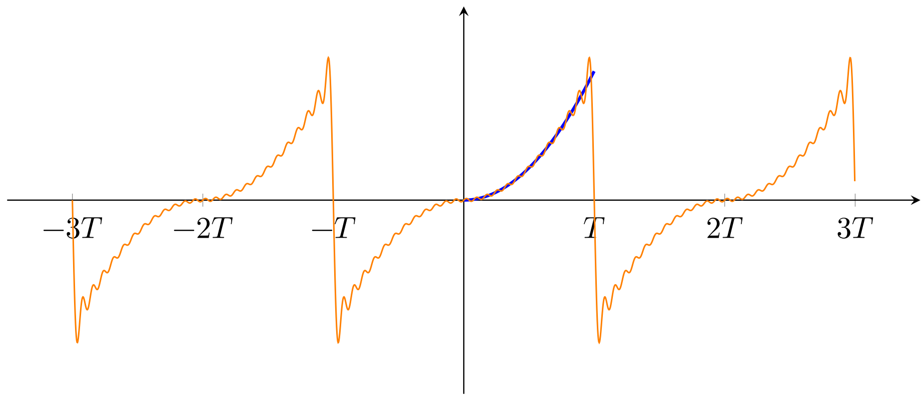

Figure 3: Fourier sine series expansion of $f(t)$ onto the interval $[-T, T]$, with the period $2T$.

Part II
Vector Calculus

4 Scalar Fields

A scalar field is any function $f : \mathbb{R}^n \to \mathbb{R}$, that assigns a scalar value to every vector in $\mathbb{R}^n$.

4.1 Partial Derivatives

The partial derivatives of a scalar field are defined as the derivative of the function with respect to each variable:

$$\frac{\partial f}{\partial x_i} \equiv f_{x_i} = \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

that is, the rate of change of the function in the x_i direction, holding all other variables constant.

4.2 Directional Derivatives

To find the rate of change of a scalar field $f(x_1, \dots, x_n)$ in the direction of a unit vector $\mathbf{u} = [u_1, \dots, u_n]$, we can scale the standard basis vectors by the components of $\mathbf{u}$:

$$D_{\mathbf{u}} f \equiv \frac{\partial f}{\partial \mathbf{u}} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{u_i}{\|\mathbf{u}\|}.$$

This is known as the **directional derivative** of f in the direction of $\mathbf{u}$.

4.3 Gradient

The gradient of a scalar field is an operator grad : $f \to \mathbb{R}^n$ which maps a scalar field f to a vector field:

$$\operatorname{grad} f \equiv \boldsymbol{\nabla} f = \left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right].$$

We can equivalently write the directional derivative as the dot product of the gradient of f with the unit vector $\mathbf{u}$:

$$D_{\mathbf{u}} f = \boldsymbol{\nabla} f \cdot \hat{\mathbf{u}}.$$

4.4 Gradient of a Scalar Field

The gradient of a scalar field f is a vector field that points in the direction of the greatest rate of change of f, with magnitude equal to the rate of change. That is:

- $\boldsymbol{\nabla} f$ points in the direction of greatest increase of f.
- $-\boldsymbol{\nabla} f$ points in the direction of greatest decrease of f.
- $\|\boldsymbol{\nabla} f\|$ is the rate of increase of f in that direction.

5 Vector Fields

A vector field is any function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, that assigns a vector to every vector in $\mathbb{R}^n$.

5.1 Partial Derivatives

The partial derivatives of a vector field are defined as the partial derivatives of each component of the vector field:

$$\frac{\partial \mathbf{F}}{\partial x_i} = \mathbf{F}_{x_i} = \left[\frac{\partial F_1}{\partial x_i}, \ldots, \frac{\partial F_n}{\partial x_i}\right]$$

5.2 Divergence

The divergence of a vector field is an operator div : $\mathbf{F} \to \mathbb{R}$, which maps a vector field $\mathbf{F}$ to a scalar:

$$\operatorname{div} \mathbf{F} = \boldsymbol{\nabla} \cdot \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}.$$

The divergence of a vector field measures the rate at which the vector field flows out of a point P.

- When $\operatorname{div} \mathbf{F} > 0$, the vector field tends to flow away from P (source).
- When $\operatorname{div} \mathbf{F} < 0$, the vector field tends to flows towards P (sink).
- When $\operatorname{div} \mathbf{F} = 0$, the net flow of the vector field at P is zero (conservative).

5.3 Curl

The curl of a vector field is an operator curl : $\mathbf{F} \to \mathbf{G}$, which maps a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ to another vector field $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\operatorname{curl} \mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

The curl may also be defined for vector fields in $\mathbb{R}^2$, where $F_3 = 0$. The curl of a vector field measures the rotation of the vector field at a point P.

- When $\operatorname{curl} \mathbf{F} > 0$, the vector field tends to rotate counterclockwise around P.
- When $\operatorname{curl} \mathbf{F} < 0$, the vector field tends to rotate clockwise around P.
- When $\operatorname{curl} \mathbf{F} = 0$, the net rotation of the vector field around P is zero.

6 Multiple Integrals

Scalar functions can be integrated over regions in $\mathbb{R}^n$ through multiple integrals.

6.1 Double Integrals

When integrating over some region R in $\mathbb{R}^2$, consider the small subregion R_{ij} with area $\Delta A_i = \Delta x_i \Delta y_i$, so that the double integral of a function $f(x,\ y)$ over R is defined as the contribution of each subregion:

$$\iint_R f(x,\ y)\,\mathrm{d}A = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i,\ y_i)\,\Delta A_i.$$

To compute this integral, we must bound the region by two functions g and h in either the x- or y-direction.

- In the y-direction, the region is bounded by the curves:

$$\begin{aligned} g(x) \ &\leqslant\ y \ \leqslant\ h(x) \\ a \ &\leqslant\ x \ \leqslant\ b \end{aligned}$$

  for some functions $g(x)$ and $h(x)$ so that

$$\iint_R f(x,\ y)\,\mathrm{d}A = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,\ y)\,\mathrm{d}y\right] \mathrm{d}x.$$

  Here we are adding up vertical strips of width $\mathrm{d}x$, where each strips height is given by the distance between $g(x)$ and $h(x)$, weighted by the function $f(x,\ y)$.

- In the x-direction, the region is bounded by the curves:

$$\begin{aligned} c \ &\leqslant \ x \ \leqslant \ d \\ g(y) \ &\leqslant \ x \ \leqslant \ h(y) \end{aligned}$$

for some functions $g(y)$ and $h(y)$ so that

$$\iint_R f(x,\ y)\, \mathrm{d}A = \int_c^d \left[\int_{g(y)}^{h(y)} f(x,\ y)\, \mathrm{d}x \right] \mathrm{d}y.$$

Here we are adding up horizontal strips of width $\mathrm{d}y$, where each strips height is given by the distance between $g(y)$ and $h(y)$, weighted by the function $f(x,\ y)$.

6.2 Order of Integration

By Fubini's theorem, any permutation of the order of integration of an iterated integral is equivalent if the function being integrated is integrable, that is if:

$$\int_R |f(\mathbf{x})|\, \mathrm{d}\mathbf{x} < \infty.$$

When applying Fubini's theorem, we must appropriately modify the bounds of integration to account for the region R. For example, if the region is bounded by the curves:

$$R = \{(x,\ y) : a \leqslant x \leqslant b,\ g(x) \leqslant y \leqslant h(x)\},$$

where g and h are invertible on the interval $[a, b]$, and the integral of a function $f(x,\ y)$ over R is given by:

$$\iint_R f(x,\ y)\, \mathrm{d}A = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,\ y)\, \mathrm{d}y \right] \mathrm{d}x,$$

we can equivalently integrate over the region R by reversing the order of integration:

$$\iint_R f(x,\ y)\, \mathrm{d}A = \int_{g(a)}^{h(b)} \left[\int_{h^{-1}(y)}^{g^{-1}(y)} f(x,\ y)\, \mathrm{d}x \right] \mathrm{d}y.$$

Similarly, if the region is bounded by the curves:

$$R = \{(x,\ y) : c \leqslant y \leqslant d,\ g(y) \leqslant x \leqslant h(y)\},$$

we can integrate over the region R by reversing the order of integration:

$$\iint_R f(x,\ y)\, \mathrm{d}A = \int_{g(c)}^{h(d)} \left[\int_{h^{-1}(x)}^{g^{-1}(x)} f(x,\ y)\, \mathrm{d}y \right] \mathrm{d}x.$$

6.3 Triple Integrals

When integrating over some volume V in $\mathbb{R}^3$, consider the small subregion V_{ijk} with volume $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$, so that the triple integral of a function $f(x,\ y,\ z)$ over V is defined as the contribution of each subregion:

$$\iiint_V f(x,\ y,\ z)\,\mathrm{d}V = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i,\ y_i,\ z_i)\,\Delta V_i.$$

To compute this integral, we require three intervals for each variable x, y, and z, that enclose the volume V. As we introduce another dimension, the function bounding the innermost integral may depend on both the outer variables. This integral may take the form:

$$\iiint_V f(x,\ y,\ z)\,\mathrm{d}V = \int_a^b \left[\int_c^d \left[\int_g^h f(x,\ y,\ z)\,\mathrm{d}z\right]\mathrm{d}y\right]\mathrm{d}x$$

for the volume enclosed by:

$$V = \{(x,\ y,\ z) : a \leqslant x \leqslant b,\ c(x) \leqslant y \leqslant d(x),\ g(x,\ y) \leqslant z \leqslant h(x,\ y)\}.$$

Note that the bounds of any integral must not include any variables that appear inside that integral. When modifying the order of integration, we must ensure the same region is enclosed by the new bounds.

6.4 Transformation of Coordinates

In single variable calculus, we used a change of variables to simplify integration by considering a continuously differentiable transformation $u = S(x)$, to rewrite an integral in terms of u. We did this by considering the derivative of u with respect to x:

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\mathrm{d}S(x)}{\mathrm{d}x}\\ \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x &= \frac{\mathrm{d}S(x)}{\mathrm{d}x}\,\mathrm{d}x\\ \mathrm{d}u &= \frac{\mathrm{d}S(x)}{\mathrm{d}x}\,\mathrm{d}x.\end{aligned}$$

This gave us the differential $\mathrm{d}x$:

$$\mathrm{d}x = \frac{1}{\frac{\mathrm{d}S(x)}{\mathrm{d}x}}\,\mathrm{d}u.$$

Here we can use the inverse function theorem to show that:

$$\frac{1}{\frac{\mathrm{d}S(x)}{\mathrm{d}x}} = \frac{\mathrm{d}x}{\mathrm{d}S(x)} = \frac{\mathrm{d}x}{\mathrm{d}S(S^{-1}(u))} = \frac{\mathrm{d}S^{-1}(u)}{\mathrm{d}u},$$

where $x = S^{-1}(u)$ is the inverse transformation. Therefore,

$$\mathrm{d}x = \frac{\mathrm{d}S^{-1}(u)}{\mathrm{d}u}\,\mathrm{d}u.$$

This transformed an integral in x to an integral in u:

$$\int_a^b f(x)\,\mathrm{d}x = \int_{u(a)}^{u(b)} f\left(S^{-1}(u)\right)\frac{\mathrm{d}S^{-1}(u)}{\mathrm{d}u}\,\mathrm{d}u.$$

This concept can be extended to integrals with multiple variables by instead considering the *inverse* transformation $T(u) = x = S^{-1}(u)$, where we can use the chain rule to find the same differential we found above:

$$\mathrm{d}x = \frac{\mathrm{d}T(u)}{\mathrm{d}u}\,\mathrm{d}u.$$

To transform the coordinates in a multivariable integral, we must consider a matrix of all the partial derivatives of a transformation. For the transformation $\mathbf{x} = \mathbf{T}(\mathbf{u})$, consider the Jacobian matrix of partial derivatives:

$$\mathbf{J} = \frac{\partial \mathbf{T}}{\partial \mathbf{u}} = \begin{bmatrix} \dfrac{\partial x_1}{\partial u_1} & \cdots & \dfrac{\partial x_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x_n}{\partial u_1} & \cdots & \dfrac{\partial x_n}{\partial u_n} \end{bmatrix}.$$

The determinant of this matrix is known as the Jacobian of a transformation, and it gives us the factor by which the measure (area/volume) of the region is scaled under the transformation, giving us a generalised differential:

$$\mathrm{d}\mathbf{x} = \left|\det\left(\frac{\partial \mathbf{T}}{\partial \mathbf{u}}\right)\right|\mathrm{d}\mathbf{u} = \left|\det \mathbf{J}\right|\mathrm{d}\mathbf{u}.$$

Therefore, given a bijective transformation $T : \Omega \subset \mathbb{R}^n \to \Omega' \subset \mathbb{R}^n$, where T has continuous partial derivatives, an integral in $\mathbf{x}$ can be transformed to an integral in $\mathbf{u}$ by:

$$\int_\Omega f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\Omega'} f(\mathbf{T}(\mathbf{u}))\left|\det\left(\frac{\partial \mathbf{T}}{\partial \mathbf{u}}\right)\right|\mathrm{d}\mathbf{u} = \int_{\Omega'} f(\mathbf{T}(\mathbf{u}))\left|\det \mathbf{J}\right|\mathrm{d}\mathbf{u}.$$

6.4.1 Cartesian Coordinates (3D)

The Cartesian coordinate system is the identity transformation

$$\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= z \end{aligned}$$

for 3 mutually orthogonal components $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Here, the inverse transformation function is also given by the identity function:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{T}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and has a Jacobian matrix equal to the 3×3 identity matrix:

$$\mathbf{J} = \mathbf{I}_3$$

with the determinant:

$$|\det \mathbf{J}| = 1.$$

The basis vectors in this coordinate system are given by the standard/canonical basis vectors:

$$\mathbf{e}_x = \mathbf{e}_1 = \hat{\mathbf{i}}$$
$$\mathbf{e}_y = \mathbf{e}_2 = \hat{\mathbf{j}}$$
$$\mathbf{e}_z = \mathbf{e}_3 = \hat{\mathbf{k}}$$

and the gradient vector is given by:

$$\boldsymbol{\nabla} = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}.$$

The divergence and curl of a vector field $\mathbf{F}$ with components F_x, F_y, and F_z are given by

$$\operatorname{div} \mathbf{F} = \boldsymbol{\nabla} \cdot \mathbf{F} = \left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \cdot \left(F_x \mathbf{e}_x + F_y \mathbf{e}_y + F_z \mathbf{e}_z\right) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\operatorname{curl} \mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F} = \left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \times \left(F_x \mathbf{e}_x + F_y \mathbf{e}_y + F_z \mathbf{e}_z\right) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

6.4.2 Polar Coordinates (2D)

To transform a Cartesian coordinate system to polar coordinates, consider the transformation:

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

for a radius $r \geqslant 0$ from the origin and an azimuthal angle $0 \leqslant \theta \leqslant 2\pi$ around the origin. Here, the inverse transformation function is given by:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{T}\left(\begin{bmatrix} r \\ \theta \end{bmatrix}\right) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

This transformation has the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

with the determinant:

$$|\det \mathbf{J}| = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, the differential in polar coordinates is:

$$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta,$$

giving the integral transformation:

$$\iint_R f(x,\ y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{R'} f(r,\ \theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta.$$

The gradient of this transformation can be computed by applying the chain rule:

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}\end{aligned}$$

where the partial derivatives in these expressions are given by:

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\theta & \frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} = -\frac{1}{r}\sin\theta\\ \frac{\partial r}{\partial y} &= \frac{y}{r} = \sin\theta & \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} = \frac{1}{r}\cos\theta\end{aligned}$$

so that the gradient in polar coordinates is defined:

$$\begin{aligned}\boldsymbol{\nabla} &= \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y}\\ &= \mathbf{e}_x\left[\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}\right] + \mathbf{e}_y\left[\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta}\right]\\ &= \left[\cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y\right]\frac{\partial}{\partial r} + \left[\cos\theta\mathbf{e}_y - \sin\theta\mathbf{e}_x\right]\frac{1}{r}\frac{\partial}{\partial \theta}\\ &= \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\end{aligned}$$

giving the transformed basis vectors:

$$\begin{aligned}\mathbf{e}_r &= \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y\\ \mathbf{e}_\theta &= \cos\theta\mathbf{e}_y - \sin\theta\mathbf{e}_x\end{aligned}$$

The divergence and curl of a vector field $\mathbf{F}$ with components F_r and F_θ are given by

$$\begin{aligned}\operatorname{div}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\right)\cdot\left(rF_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta\right) = \frac{\partial rF_r}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\\ \operatorname{curl}\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{F} &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\right)\times\left(rF_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta\right) = \left|\mathbf{i}\right|\end{aligned}$$

6.4.3 Cylindrical Coordinates (3D)

To transform a Cartesian coordinate system to cylindrical coordinates, consider the transformation:

$$\begin{aligned}x &= r\cos\theta & r &= \sqrt{x^2+y^2}\\ y &= r\sin\theta & \theta &= \arctan\left(\frac{y}{x}\right)\\ z &= z & z &= z\end{aligned}$$

for a radius $r \geqslant 0$ perpendicular to the z-axis, an azimuthal angle $0 \leqslant \theta \leqslant 2\pi$ around the z-axis, and a distance $-\infty < z < \infty$ along the z-axis. Here, the inverse transformation function is given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{T}\left(\begin{bmatrix} r \\ \theta \\ z \end{bmatrix}\right) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}$$

This transformation has the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with the determinant:

$$|\det \mathbf{J}| = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, the differential in cylindrical coordinates is:

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z,$$

giving the integral transformation:

$$\iiint_V f\left(x,\ y,\ z\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{V'} f\left(r,\ \theta,\ z\right) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

The gradient of this transformation can be computed by applying the chain rule:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial z}{\partial z}\frac{\partial}{\partial z} \end{aligned}$$

The partial derivatives in these expressions are given by:

$$\begin{aligned} &\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta && \frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{1}{r}\sin\theta && \frac{\partial z}{\partial x} = 0 \\ &\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta && \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{1}{r}\cos\theta && \frac{\partial z}{\partial y} = 0 \\ &\frac{\partial r}{\partial z} = 0 && \frac{\partial \theta}{\partial z} = 0 && \frac{\partial z}{\partial z} = 1 \end{aligned}$$

so that the gradient in cylindrical coordinates is defined:

$$\begin{aligned}
\boldsymbol{\nabla} &= \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} \\
&= \mathbf{e}_x \left[\cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}\right] + \mathbf{e}_y \left[\sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}\right] + \mathbf{e}_z \frac{\partial}{\partial z} \\
&= \left[\cos\theta \mathbf{e}_x + \sin\theta \mathbf{e}_y\right] \frac{\partial}{\partial r} + \left[\cos\theta \mathbf{e}_y - \sin\theta \mathbf{e}_x\right] \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \\
&= \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}
\end{aligned}$$

giving the transformed basis vectors:

$$\begin{aligned}
\mathbf{e}_r &= \cos\theta \mathbf{e}_x + \sin\theta \mathbf{e}_y \\
\mathbf{e}_\theta &= \cos\theta \mathbf{e}_y - \sin\theta \mathbf{e}_x \\
\mathbf{e}_z &= \mathbf{e}_z
\end{aligned}$$

6.4.4 Spherical Coordinates (3D)

To transform a Cartesian coordinate system to spherical coordinates, consider the transformation:

$$\begin{aligned}
x &= r\cos\theta\sin\phi & r &= \sqrt{x^2+y^2+z^2} \\
y &= r\sin\theta\sin\phi & \theta &= \arctan\left(\frac{y}{x}\right) \\
z &= r\cos\phi & \phi &= \arccos\left(\frac{z}{r}\right)
\end{aligned}$$

for a radius $r \geqslant 0$ from the origin, an azimuthal angle $0 \leqslant \theta \leqslant 2\pi$ around the z-axis, and a polar angle $0 \leqslant \phi \leqslant \pi$ measured downwards from the positive z-axis. Here, the inverse transformation function is given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{T}\left(\begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix}\right) = \begin{bmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{bmatrix}$$

This transformation has the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\phi & -r\sin\phi & 0 \end{bmatrix}$$

with the determinant:

$$|\det \mathbf{J}| = r^2\sin\phi.$$

Therefore, the differential in spherical coordinates is:

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2\sin\phi\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta,$$

giving the integral transformation:

$$\iiint_V f\left(x,\ y,\ z\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iiint_{V'} f\left(r,\ \theta,\ \phi\right) r^2 \sin\phi\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi.$$

The gradient of this transformation can be computed by applying the chain rule:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta}
\end{aligned}$$

The partial derivatives in these expressions are given by:

$$\begin{aligned}
&\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta\sin\phi && \frac{\partial \phi}{\partial x} = -\frac{z\left(-xr^{-3}\right)}{\sqrt{1-\frac{z^2}{r^2}}} = \frac{\cos\theta\cos\phi}{r} && \frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r\sin\phi} \\
&\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\phi && \frac{\partial \phi}{\partial y} = -\frac{z\left(-yr^{-3}\right)}{\sqrt{1-\frac{z^2}{r^2}}} = \frac{\sin\theta\cos\phi}{r} && \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r\sin\phi} \\
&\frac{\partial r}{\partial z} = \frac{z}{r} = \cos\phi && \frac{\partial \phi}{\partial z} = -\frac{\left(r-z^2r^{-1}\right)r^{-2}}{\sqrt{1-\frac{z^2}{r^2}}} = -\frac{\sin\phi}{r} && \frac{\partial \theta}{\partial z} = 0
\end{aligned}$$

so that the gradient in spherical coordinates is defined:

$$\begin{aligned}
\boldsymbol{\nabla} &= \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z} \\
&= \mathbf{e}_x\left[\cos\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial}{\partial \theta}\right] \\
&\quad +\mathbf{e}_y\left[\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\sin\theta\cos\phi}{r}\frac{\partial}{\partial \phi} + \frac{\cos\theta}{r\sin\phi}\frac{\partial}{\partial \theta}\right] \\
&\quad +\mathbf{e}_z\left[\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\right] \\
&= \left[\cos\theta\sin\phi\mathbf{e}_x + \sin\theta\sin\phi\mathbf{e}_y + \cos\phi\mathbf{e}_z\right]\frac{\partial}{\partial r} \\
&\quad + \left[\cos\theta\cos\phi\mathbf{e}_x + \sin\theta\cos\phi\mathbf{e}_y - \sin\phi\mathbf{e}_z\right]\frac{1}{r}\frac{\partial}{\partial \phi} \\
&\quad + \left[\cos\theta\mathbf{e}_y - \sin\theta\mathbf{e}_x\right]\frac{1}{r\sin\phi}\frac{\partial}{\partial \theta} \\
&= \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\phi\frac{1}{r}\frac{\partial}{\partial \phi} + \mathbf{e}_\theta\frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}
\end{aligned}$$

giving the transformed basis vectors:

$$\begin{aligned}\mathbf{e}_r &= \cos\theta\sin\phi\mathbf{e}_x + \sin\theta\sin\phi\mathbf{e}_y + \cos\phi\mathbf{e}_z \\ \mathbf{e}_\phi &= \cos\theta\cos\phi\mathbf{e}_x + \sin\theta\cos\phi\mathbf{e}_y - \sin\phi\mathbf{e}_z \\ \mathbf{e}_\theta &= \cos\theta\mathbf{e}_y - \sin\theta\mathbf{e}_x\end{aligned}$$

6.5 Physical Interpretation of Integrals

Integrals can be used to represent various physical quantities.

6.5.1 Measures

The measure of a region $R \in \mathbb{R}^n$ is given by the integral of the unit density function $\rho(\mathbf{x}) = 1$ over R:

$$\mu = \int_R \mathrm{d}\mathbf{x}.$$

- In 1D, the measure represents the length of R.
- In 2D, the measure represents the area of R.
- In 3D, the measure represents the volume of R.

6.5.2 Mass

The mass of a region $R \in \mathbb{R}^n$ is given by the integral of the density function $\rho(\mathbf{x})$ over R:

$$M = \int_R \rho(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

6.5.3 Centroid

The centroid (average position) of a region $R \in \mathbb{R}^n$ with uniform density $\rho(\mathbf{x}) = 1$ is given by:

$$\mathbf{c} = \frac{1}{\mu}\int_R \mathbf{x}\,\mathrm{d}\mathbf{x}.$$

6.5.4 Centre of Mass

The centre of mass of a region $R \in \mathbb{R}^n$ with density function $\rho(\mathbf{x})$ is given by:

$$\mathbf{c}_\rho = \frac{1}{M}\int_R \rho(\mathbf{x})\,\mathbf{x}\,\mathrm{d}\mathbf{x}.$$

6.5.5 Average Value

The average value of a function $f(\mathbf{x})$ over a region $R \in \mathbb{R}^n$ is given by:

$$\bar{f} = \frac{1}{\mu}\int_R f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

7 Line Integrals

7.1 Parametric Curves

A path is a continuous function $\mathbf{r}(t)$ that maps a parameter t to a point in $\mathbb{R}^n$:

$$\mathbf{r}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

A curve $\mathscr{C}$ in $\mathbb{R}^n$ is the set of points corresponding to the range of the path $\mathbf{r}(t)$, where $a \leqslant t \leqslant b$:

$$\mathscr{C} = \{\mathbf{r}(t) : a \leqslant t \leqslant b\}.$$

A path is closed if $\mathbf{r}(a) = \mathbf{r}(b)$. The velocity of a path is given by the derivative of the path:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

This allows us to define the speed of the path as the magnitude of the velocity:

$$\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{x_1'^2 + x_2'^2 + \cdots + x_n'^2}.$$

Integrals along paths are called line integrals.

7.2 Line Integrals of Scalar Fields

Line integrals of scalar fields have the form:

$$\int_{\mathscr{C}} f \, \mathrm{d}r$$

These integrals represent a weighted sum over the scalar field f along a curve parametrised by the path $\mathbf{r}(t)$. To compute the differential element $\mathrm{d}r$, consider its relationship with the canonical Euclidean differential elements:

$$\mathrm{d}r^2 = \sum_{i=1}^{n} \mathrm{d}x_i^2 \implies \mathrm{d}r = \sqrt{\sum_{i=1}^{n} \mathrm{d}x_i^2} = \sqrt{\sum_{i=1}^{n} \left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)^2} \, \mathrm{d}t = \|\mathbf{r}'(t)\| \, \mathrm{d}t.$$

Using the chain rule, we can also represent the differential element $\mathrm{d}r$ using the form:

$$\mathrm{d}r = \frac{\mathrm{d}r}{\mathrm{d}t} \, \mathrm{d}t$$

so that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \|\mathbf{r}'(t)\|$$

represents the speed of the path. Therefore, the line integral of f along the curve $\mathscr{C}$ is given by:

$$\int_{\mathscr{C}} f \, \mathrm{d}r = \int_a^b f(\mathbf{r}(t)) \, \|\mathbf{r}'(t)\| \, \mathrm{d}t.$$

7.2.1 Arc Length

The arc length of the curve $\mathscr{C}$ is a function which measures the length of the path $\mathbf{r}(t)$ from a to τ. It is defined as the line integral of a uniform field:

$$s(\tau) = \int_a^\tau \mathrm{d}r = \int_a^\tau \|\mathbf{r}'(t)\| \,\mathrm{d}t.$$

The total length of the curve $\mathscr{C}$ is therefore:

$$L = s(b) = \int_{\mathscr{C}} \mathrm{d}r = \int_a^b \|\mathbf{r}'(t)\| \,\mathrm{d}t.$$

7.3 Line Integrals of Vector Fields

Line integrals of vector fields have the form:

$$W = \int_{\mathscr{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

These integrals represent the work done by the vector field $\mathbf{F}$ along a curve parametrised by the path $\mathbf{r}(t)$. The differential element $\mathrm{d}\mathbf{r}$ can be computed using the chain rule:

$$\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \,\mathrm{d}t = \mathbf{r}'(t) \,\mathrm{d}t$$

where $\mathbf{r}'(t)$ is the velocity of the path. Therefore, the line integral of $\mathbf{F}$ along the curve $\mathscr{C}$ is given by:

$$\int_{\mathscr{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \,\mathrm{d}t.$$

7.3.1 Circulation

When the path $\mathbf{r}(t)$ is closed, line integrals of vector fields along $\mathscr{C}$ can be denoted as:

$$\Gamma = \oint_{\mathscr{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where Γ is the circulation of the vector field $\mathbf{F}$ along $\mathscr{C}$.

7.3.2 Conservative Fields

The vector field $\mathbf{F}$ is called conservative when

$$\mathbf{F} = \boldsymbol{\nabla}\phi$$

for some scalar function ϕ. When $\mathbf{F}$ is conservative, the line integral along $\mathscr{C}$ is path independent:

$$\int_{\mathscr{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{C}} \boldsymbol{\nabla}\phi \cdot \mathrm{d}\mathbf{r} = \int_a^b \boldsymbol{\nabla}\phi(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \,\mathrm{d}t = \int_a^b \frac{\mathrm{d}\phi(\mathbf{r}(t))}{\mathrm{d}t} \,\mathrm{d}t = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)).$$

When $\mathbf{r}(t)$ is a closed path, the circulation is zero:

$$\Gamma = \oint_{\mathscr{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)) = 0.$$

8 Surface Integrals

8.1 Parametric Surfaces

Consider the parametric function $\mathbf{r}(s,\ t)$ that maps the parameters s and t to a point in $\mathbb{R}^3$:

$$\mathbf{r}(s,\ t) = \begin{bmatrix} x(s,\ t) \\ y(s,\ t) \\ z(s,\ t) \end{bmatrix}$$

A surface $\mathcal{S}$ in $\mathbb{R}^3$ is the set of points corresponding to the range of the parametric function $\mathbf{r}(s,\ t)$, where $a \leqslant s \leqslant b$ and $c \leqslant t \leqslant d$:

$$\mathcal{S} = \{\mathbf{r}(s,\ t) : a \leqslant s \leqslant b,\ c \leqslant t \leqslant d\}.$$

When the partial derivatives of $\mathbf{r}$ are linearly independent, we can define the following normal vector to the surface:

$$\hat{\mathbf{n}} = \frac{\mathbf{r}_s \times \mathbf{r}_t}{\|\mathbf{r}_s \times \mathbf{r}_t\|},$$

where $\mathbf{r}_s$ and $\mathbf{r}_t$ are the two partial derivatives of $\mathbf{r}$. Integrals over such surfaces are called surface integrals.

8.2 Surface Integrals of Scalar Fields

Surface integrals of scalar fields have the form:

$$\iint_{\mathcal{S}} f \,\mathrm{d}\sigma$$

These integrals represent the weighted area over the scalar field f over the surface parametrised by $\mathbf{r}(s,\ t)$. The differential element $\mathrm{d}\sigma$ is given by

$$\mathrm{d}\sigma = \|\mathbf{r}_s \times \mathbf{r}_t\| \,\mathrm{d}s\,\mathrm{d}t$$

where $\|\mathbf{r}_s \times \mathbf{r}_t\|$ represents the area of the parallelogram spanned by $\mathbf{r}_s$ and $\mathbf{r}_t$. Therefore, the surface integral of f over the surface $\mathcal{S}$ is given by:

$$\iint_{\mathcal{S}} f \,\mathrm{d}\sigma = \int_c^d \int_a^b f(\mathbf{r}(s,\ t)) \|\mathbf{r}_s \times \mathbf{r}_t\| \,\mathrm{d}s\,\mathrm{d}t.$$

8.2.1 Surface Area

The surface area of the surface $\mathcal{S}$ is defined as the surface integral of a uniform field:

$$A = \iint_{\mathcal{S}} \mathrm{d}\sigma = \int_c^d \int_a^b \|\mathbf{r}_s \times \mathbf{r}_t\| \,\mathrm{d}s\,\mathrm{d}t.$$

8.2.2 Explicit Surfaces

When the surface $\mathcal{S}$ can be expressed explicitly by the function

$$z = g(x,\ y),$$

we can use the parametrisation:

$$\mathbf{r}(x,\ y) = \begin{bmatrix} x \\ y \\ g(x,\ y) \end{bmatrix}$$

where we choose $s = x$ and $t = y$. This simplifies the surface integral over f to:

$$\begin{aligned} \iint_{\mathcal{S}} f \,\mathrm{d}\sigma &= \iint_{\mathcal{S}} f(\mathbf{r}(x,\ y)) \left\|\mathbf{r}_x \times \mathbf{r}_y\right\| \mathrm{d}x\,\mathrm{d}y \\ &= \iint_{\mathcal{S}} f(x,\ y,\ g(x,\ y)) \left\| \begin{bmatrix} 1 \\ 0 \\ g_x \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ g_y \end{bmatrix} \right\| \mathrm{d}x\,\mathrm{d}y \\ &= \iint_{\mathcal{S}} f(x,\ y,\ g(x,\ y)) \left\| \begin{bmatrix} -g_x \\ -g_y \\ 1 \end{bmatrix} \right\| \mathrm{d}x\,\mathrm{d}y \\ &= \iint_{\mathcal{S}} f(x,\ y,\ g(x,\ y)) \sqrt{g_x^2 + g_y^2 + 1}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

8.3 Surface Integrals of Vector Fields

Surface integrals of vector fields have the form:

$$\Phi = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\sigma}$$

These integrals represent the flux of the vector field $\mathbf{F}$ through the surface parametrised by $\mathbf{r}(s,\ t)$. The differential element $\mathrm{d}\boldsymbol{\sigma}$ is given by

$$\mathrm{d}\boldsymbol{\sigma} = \hat{\mathbf{n}}\,\mathrm{d}\sigma = (\mathbf{r}_s \times \mathbf{r}_t)\,\mathrm{d}s\,\mathrm{d}t$$

where $\mathbf{r}_s \times \mathbf{r}_t$ represents a normal vector to the surface $\mathcal{S}$. Therefore, the surface integral of $\mathbf{F}$ over the surface $\mathcal{S}$ is given by:

$$\Phi = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\sigma} = \iint_{\mathcal{S}} (\mathbf{F} \cdot \hat{\mathbf{n}})\,\mathrm{d}\sigma = \int_c^d \int_a^b \mathbf{F}(\mathbf{r}(s,\ t)) \cdot (\mathbf{r}_s \times \mathbf{r}_t)\,\mathrm{d}s\,\mathrm{d}t.$$

Note that this integral represents the outward or inward flow of $\mathbf{F}$ through $\mathcal{S}$, where the direction of flow depends on the sign of $\mathbf{n}$.

8.3.1 Explicit Surfaces

When the surface $\mathcal{S}$ can be expressed explicitly by the function

$$z = g(x,\ y),$$

we can use the parametrisation:

$$\mathbf{r}(x,\ y) = \begin{bmatrix} x \\ y \\ g(x,\ y) \end{bmatrix}$$

where we choose $s = x$ and $t = y$. This simplifies the surface integral of $\mathbf{F}$ over $\mathscr{S}$ to:

$$\begin{aligned}
\Phi = \iint_{\mathscr{S}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\sigma} = \iint_{\mathscr{S}} (\mathbf{F} \cdot \hat{\mathbf{n}})\,\mathrm{d}\sigma &= \iint_{\mathscr{S}} \mathbf{F}(\mathbf{r}(x,\ y)) \cdot \left(\mathbf{r}_x \times \mathbf{r}_y\right) \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\mathscr{S}} \mathbf{F}(x,\ y,\ g(x,\ y)) \cdot \left(\begin{bmatrix} 1 \\ 0 \\ g_x \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ g_y \end{bmatrix} \right) \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\mathscr{S}} \mathbf{F}(x,\ y,\ g(x,\ y)) \cdot \begin{bmatrix} -g_x \\ -g_y \\ 1 \end{bmatrix} \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\mathscr{S}} \left(-F_1 g_x - F_2 g_y + F_3\right) \mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

9 Fundamental Theorems of Calculus

The three vector calculus operators have corresponding theorems which generalise the Fundamental Theorem of Calculus to higher dimensions. These results are commonly used to simplify line and surface integrals.

9.1 Fundamental Theorem of Calculus Part II

Consider the continuously differentiable function $F : [a, b] \to \mathbb{R}$, then

$$\int_a^b \frac{\mathrm{d}F}{\mathrm{d}x}\,\mathrm{d}x = F(b) - F(a).$$

This theorem states that the integral of the derivative of F is equal to the difference of the values of the function at the endpoints.

9.2 Fundamental Theorem of Line Integrals (Gradient Theorem)

Consider the differentiable scalar function $\phi : \mathbb{R}^n \to \mathbb{R}$. Let $\mathscr{C}$ be a curve parametrised by $\mathbf{r}(t)$, where $a \leq t \leq b$, then

$$\int_{\mathscr{C}} \boldsymbol{\nabla}\phi \cdot \mathrm{d}\mathbf{r} = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)).$$

This theorem states that line integrals through conservative fields $\mathbf{F} = \boldsymbol{\nabla}\phi$ are path independent.

9.3 Gauss's Theorem (Divergence Theorem)

Consider the closed region $R \in \mathbb{R}^n$ with boundary ∂R and let $\mathbf{F} : R \to \mathbb{R}^n$ be a continuously differentiable vector field in R. Then,

$$\underbrace{\int \cdots \int_R}_{n} (\boldsymbol{\nabla} \cdot \mathbf{F})\, \mathrm{d}\mathbf{x} = \underbrace{\oint \cdots \oint_{\partial R}}_{n-1} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\sigma}.$$

This theorem states that the volume integral of the divergence of a vector field $\mathbf{F}$ over the region R is equal to the surface integral of $\mathbf{F}$ over the boundary ∂R.

9.4 Stoke's Theorem (Curl Theorem)

Consider the surface $\mathcal{S}$ parametrised by $\mathbf{r}(s,\, t)$, with boundary $\partial\mathcal{S}$ oriented positively with respect to the normal vector $\mathbf{n}$, and let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable vector field in $\mathcal{S}$. Then,

$$\iint_{\mathcal{S}} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathrm{d}\boldsymbol{\sigma} = \oint_{\partial\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

This theorem states that the surface integral of the curl of a vector field over the surface $\mathcal{S}$ is equal to the line integral of $\mathbf{F}$ along the boundary $\partial\mathcal{S}$.

9.5 Green's Theorem

Consider the bounded region $R \in \mathbb{R}^2$ with boundary ∂R oriented positively (traversed in the counterclockwise direction) and let $\mathbf{F} : R \to \mathbb{R}^2$ be a continuously differentiable vector field in $\mathbb{R}$. Then,

$$\iint_R (\boldsymbol{\nabla} \times \mathbf{F})\, \mathrm{d}A = \oint_{\partial R} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

This expands to

$$\iint_R \left(\frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y = \oint_{\partial R} \mathbf{F}_1\, \mathrm{d}x + \mathbf{F}_2\, \mathrm{d}y.$$

This theorem states that the area integral of the curl of a vector field $\mathbf{F}$ is equal to the line integral of $\mathbf{F}$ along the boundary ∂R. Green's theorem is a special case of both the divergence and curl theorems in 2 dimensions.

Part III
Ordinary Differential Equations

10 Laplace Transform

Consider a forced linear constant-coefficient differential equation of the form:

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = f(t)$$

with initial conditions $y(0) = y_0$, $y'(0) = y_1$, ..., $y^{(n-1)}(0) = y_{n-1}$. Certain choices of $f(t)$ can make the solution to this differential equation difficult to determine using methods such as undetermined coefficients or variation of parameters, due to the complexity of the integrals involved. Here we introduce the Laplace transform, which allows us to transform a differential equation into an algebraic equation. The Laplace transform of a function $f(t)$ is defined as:

$$\mathscr{L}\{f(t)\} \equiv F(s) = \int_0^\infty f(t)\, e^{-st}\, \mathrm{d}t,$$

where s is a complex number. Consider the Laplace transform of $\dfrac{\mathrm{d}y(t)}{\mathrm{d}t}$:

$$\begin{aligned}\mathscr{L}\left\{\frac{\mathrm{d}y(t)}{\mathrm{d}t}\right\} &= \int_0^\infty \frac{\mathrm{d}y(t)}{\mathrm{d}t} e^{-st}\, \mathrm{d}t \\ &= e^{-st}y(t)\big|_0^\infty + s\int_0^\infty e^{-st}y(t)\, \mathrm{d}t \\ &= sY(s) - y(0).\end{aligned}$$

Now consider the Laplace transform of $\dfrac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}$:

$$\begin{aligned}\mathscr{L}\left\{\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}\right\} &= \int_0^\infty \frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} e^{-st}\, \mathrm{d}t \\ &= e^{-st}y'(t)\big|_0^\infty + se^{-st}y(t)\big|_0^\infty + s^2\int_0^\infty e^{-st}y(t)\, \mathrm{d}t \\ &= s^2Y(s) - sy(0) - y'(0).\end{aligned}$$

Therefore, we can deduce that the Laplace transform of $\dfrac{\mathrm{d}^ny(t)}{\mathrm{d}t^n}$ is given by:

$$\begin{aligned}\mathscr{L}\left\{\frac{\mathrm{d}^ny(t)}{\mathrm{d}t^n}\right\} &= s^nY(s) - \left[s^{n-1}y(0) + s^{n-2}y'(0) + \cdots + y^{(n-1)}(0)\right] \\ &= s^nY(s) - \sum_{k=0}^{n-1} s^{n-1-k}y^{(k)}(0).\end{aligned}$$

Using this property, we can transform the above differential equation into the s-domain by taking the Laplace transform of both sides, allowing us to solve for $Y(s)$ algebraically.

10.1 Existence of the Laplace Transform

A sufficient condition for the existence of the Laplace transform of $f(t)$ is that $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order for some time $t > T$. A function $f(t)$ is said to be of exponential order if after $t = T$, there exist constants M and a such that $|f(t)| \leqslant Me^{at}$ for all $t > T$. Such a function satisfies the limit:

$$\lim_{t\to\infty} e^{at}f(t) = 0,$$

and the Laplace transform exists for $\Re(s) > a$.

10.2 Inverse Laplace Transform

We can recover the original function $f(t)$ from its Laplace transform $F(s)$ by taking the inverse Laplace transform which is defined by the following line integral in the complex plane:

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, e^{st}\,\mathrm{d}s,$$

where c is a real number such that the region of integration is to the right of all singularities of $F(s)$. However, for many simple functions, the inverse Laplace transform can be determined using partial fraction decomposition and Laplace transform tables. This is shown below.

10.2.1 Partial Fraction Decomposition

Given the rational function

$$\frac{P(s)}{Q(s)}$$

where $P(s)$ and $Q(s)$ are polynomials and the degree of $P(s)$ is less than the degree of $Q(s)$, we can decompose it into a sum of partial fractions by factoring $Q(s)$ into real linear and irreducible quadratic terms.

1. For every real linear factor $(s-q)^m$ of $Q(s)$, we must include terms of the form:

$$\frac{A_1}{s-q} + \frac{A_2}{(s-q)^2} + \cdots + \frac{A_m}{(s-q)^m} = \sum_{i=1}^{m} \frac{A_i}{(s-q)^i}$$

   where A_i are constants.

2. For every irreducible quadratic factor $(as^2+bs+c)^m$ of $Q(s)$, we must include terms of the form:

$$\frac{B_1 s + C_1}{as^2+bs+c} + \frac{B_2 s + C_2}{(as^2+bs+c)^2} + \cdots + \frac{B_m s + C_m}{(as^2+bs+c)^m} = \sum_{i=1}^{m} \frac{B_i s + C_i}{(as^2+bs+c)^i}$$

   where B_i and C_i are constants.

The constants A_i, B_i and C_i can be determined by first multiplying both sides of the equation by $Q(s)$ to obtain a polynomial equation, and then substituting suitable values of s to eliminate terms. These values will typically be the roots of $Q(s)$, and for repeated roots, we can use any other value of s that does not eliminate the term we are trying to determine.

10.3 Heaviside Step Function

The Heaviside step function is defined as:

$$u(t-a) = \begin{cases} 0, & 0 \leqslant t \leqslant a \\ 1, & t > a \end{cases}$$

This function is used to model the behaviour of systems with instantaneous responses to a change in input at time a.

10.4 Dirac Delta Function

To motivate the Dirac delta function, consider the pulse function:

$$p\left(t-a\right)=\begin{cases}0, & t<a-\Delta t\\ \dfrac{1}{2\,\Delta t}, & a-\Delta t\leqslant t\leqslant a+\Delta t\\ 0, & t>a+\Delta t\end{cases}$$

whose area is equal to 1. The Dirac delta function can then be defined as the limit of the pulse function as $\Delta t\to 0$:

$$\delta\left(t-a\right)=\lim_{\Delta t\to 0}p\left(t-a\right).$$

This leads to the following properties of the Dirac delta function:

$$\delta\left(t\right)\simeq\begin{cases}0, & t\neq a\\ \infty, & t=a\end{cases}$$

$$\int_{-\infty}^{\infty}\delta\left(t-a\right)\mathrm{d}t=1.$$

This implies that

$$\int_{-\infty}^{\infty}f\left(t\right)\delta\left(t-a\right)\mathrm{d}t=f\left(a\right).$$

The Dirac delta function is used to model impulses in systems. The Laplace transform of the Dirac delta function is given by:

$$\mathscr{L}\left\{\delta\left(t-a\right)\right\}=e^{-as}.$$

10.5 Shift Theorems

The first shift theorem states that:

$$\mathscr{L}\left\{e^{at}f\left(t\right)\right\}=F\left(s-a\right).$$

The second shift theorem states that:

$$\mathscr{L}\left\{u\left(t-a\right)f\left(t-a\right)\right\}=e^{-as}F\left(s\right).$$

10.6 Convolution Theorem

The convolution of two functions $f\left(t\right)$ and $g\left(t\right)$ is defined as:

$$\left(f*g\right)\left(t\right)=\int_{-\infty}^{\infty}f\left(\tau\right)g\left(t-\tau\right)\mathrm{d}\tau.$$

The convolution theorem allows us to rewrite the convolution of two functions in the time as a product of their Laplace transforms:

$$\mathscr{L}\left\{\left(f*g\right)\left(t\right)\right\}=F\left(s\right)G\left(s\right).$$

11 Nonlinear ODEs

Often nonlinear ordinary differential equations cannot be solved analytically and require numerical methods to approximate the solution. In such cases, we instead consider qualitative methods to understand the behaviour of the solution using stability analysis.

11.1 Stability Analysis

For an autonomous differential equation of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g\left(x\right),$$

we can define an **equilibrium point** x_e as one which satisfies $g\left(x_e\right) = 0$. This point represents a state from which the system does not change over time. We can classify such points as:

- **stable** if the solution converges to the equilibrium point. Small perturbations in the solution eventually cause the system to return to the equilibrium point.
- **unstable** if the solution diverges from the equilibrium point. Small perturbations in the solution eventually cause the system to diverge from the equilibrium point.
- **semi-stable** if the solution converges to the equilibrium point from one side, but diverges from the equilibrium point from the other side.

11.2 Phase Line Analysis

We can visualise this behaviour on a **phase line**, marking regions between equilibrium points with arrows indicating the direction of the system over time. In this graph, we draw a vertical line representing the solution x, marking equilibrium points along this line. The direction of the system is calculated by considering the sign of $\frac{\mathrm{d}x}{\mathrm{d}t}$ in each region between equilibrium points:

- If $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$, then x is increasing in that region.
- If $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$, then x is decreasing in that region.

We can also determine this visually by constructing a **phase portrait** of the system by plotting $g\left(x\right)$ against x. We can then determine the sign of $\frac{\mathrm{d}x}{\mathrm{d}t}$ in each region by looking at when the function $g\left(x\right)$ is positive or negative. A phase portrait and phase line of the system $\frac{\mathrm{d}x}{\mathrm{d}t} = -x^4 - x^3 - x^2 - x = -x\left(x+1\right)\left(x-1\right)^2$ is shown below:

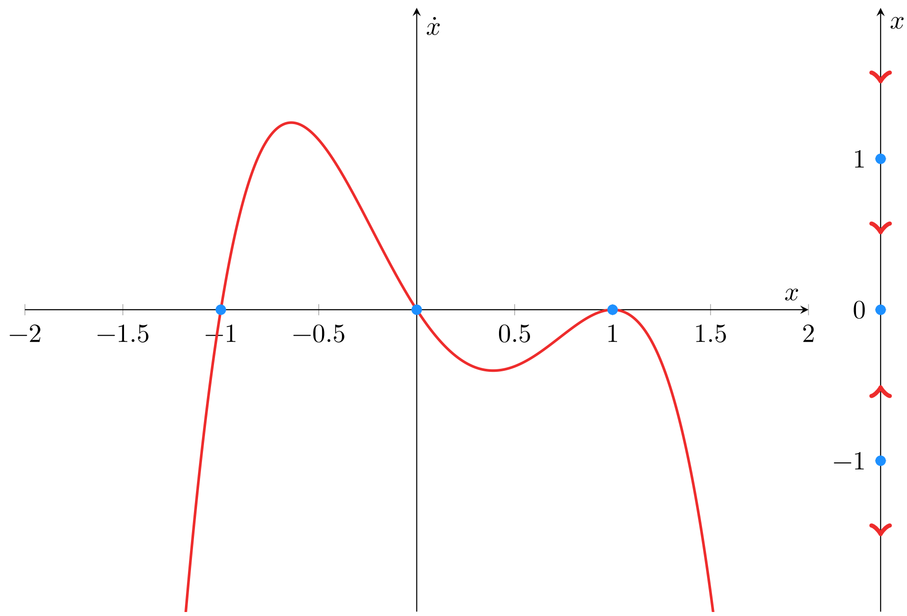

Figure 4: Phase line analysis of the system $\frac{\mathrm{d}x}{\mathrm{d}t} = -x^4 - x^3 - x^2 - x$. $x_e = 1$ is a semi-stable equilibrium point, $x_e = 0$ is a stable equilibrium point, and $x_e = -1$ is an unstable equilibrium point.

The solution to this system is an implicit function:

$$g(x) = x(x+1)^{-1/4}(x-1)^{-3/4}\exp\left(-\frac{1}{2(x-1)}\right) = Ae^{-t},$$

where A solves the initial condition $x(0) = x_0$:

$$A = x_0(x_0+1)^{-1/4}(x_0-1)^{-3/4}\exp\left(-\frac{1}{2(x_0-1)}\right).$$

11.3 Solution Curves

This analysis allows us to construct **solution curves** for a system, which are curves in the x-t plane that represent the solution x as a function of time t. Here we mark equilibrium points on the vertical axis, and draw trajectories to represent the possible behaviours of the system around all equilibrium points over time. A system has **finite-time blow up** if one of these trajectories approaches infinity in finite time. This time often depends on the initial condition. The solution curves for the system $\frac{\mathrm{d}x}{\mathrm{d}t} = -x^4 - x^3 - x^2 - x$ are shown below:

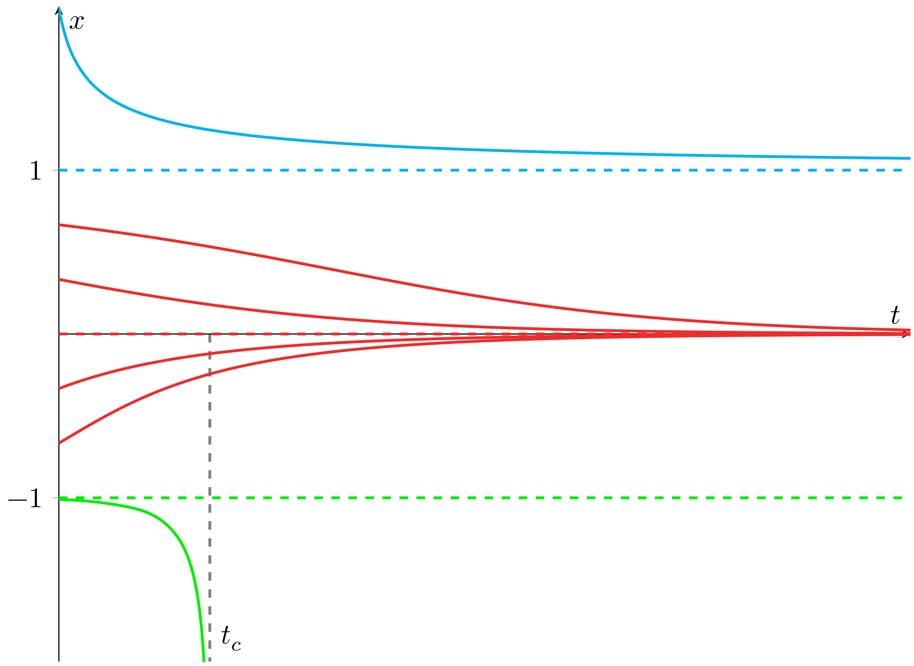

Figure 5: Solution curves of the system $\frac{\mathrm{d}x}{\mathrm{d}t} = -x^4 - x^3 - x^2 - x$. The system experiences finite-time blow up for the time t_c at which $g\left(x\right)$ is singular.

11.4 Bifurcation Analysis

Consider a parametrised differential equation of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g\left(x,\ \lambda\right).$$

Here we consider the behaviour of the function $g\left(x,\ \lambda\right)$ as a function of the parameter λ. We make note of special values of λ at which the equilibrium points x_e change, for example, $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. This lets us draw a **bifurcation diagram** for the system by plotting the contour $g\left(x,\ \lambda\right) = 0$ on an x-λ plane, which shows how equilibrium points evolve as the parameter λ is varied. We can depict the stability of these contours by drawing vertical arrows above and below lines, or by using solid and dashed lines. Here, solid lines represent stable equilibrium points, while dashed lines represent unstable equilibrium points.

11.5 Stability Analysis Example

Consider the following nonlinear parametrised differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g\left(x,\ \lambda\right) = \lambda x + x^2 - x^3.$$

We can find equilibrium points x_e by solving:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \implies \lambda x + x^2 - x^3 = 0.$$

This gives us three equilibrium points:

$$x_{e1} = 0,\ x_{e2} = \frac{1-\sqrt{1+4\lambda}}{2},\ x_{e3} = \frac{1+\sqrt{1+4\lambda}}{2}.$$

To classify the stability of these points, we must consider how the function $g(x,\ \lambda)$ behaves for various values of λ. For this analysis, we will consider the following cases:

- $\lambda < -1/4$: The discriminant $1+4\lambda$ is negative, so the system only has one equilibrium point $x_e = 0$.
- $\lambda = -1/4$: The discriminant $1+4\lambda$ is zero, so the system has two equilibrium points $x_{e1} = 0$ and $x_{e2} = 1/2$ with multiplicity 2.
- $-1/4 < \lambda < 0$: The discriminant $1+4\lambda$ is positive, and the system has three equilibrium points.
- $\lambda = 0$: The discriminant $1+4\lambda$ is positive, but one of the roots is zero, so the system has two equilibrium points $x_{e1} = 0$ with multiplicity 2 and $x_{e2} = 1/2$.
- $0 < \lambda < 1/4$: The discriminant $1+4\lambda$ is positive, and the system has three equilibrium points.

A plot of each of these cases is shown below:

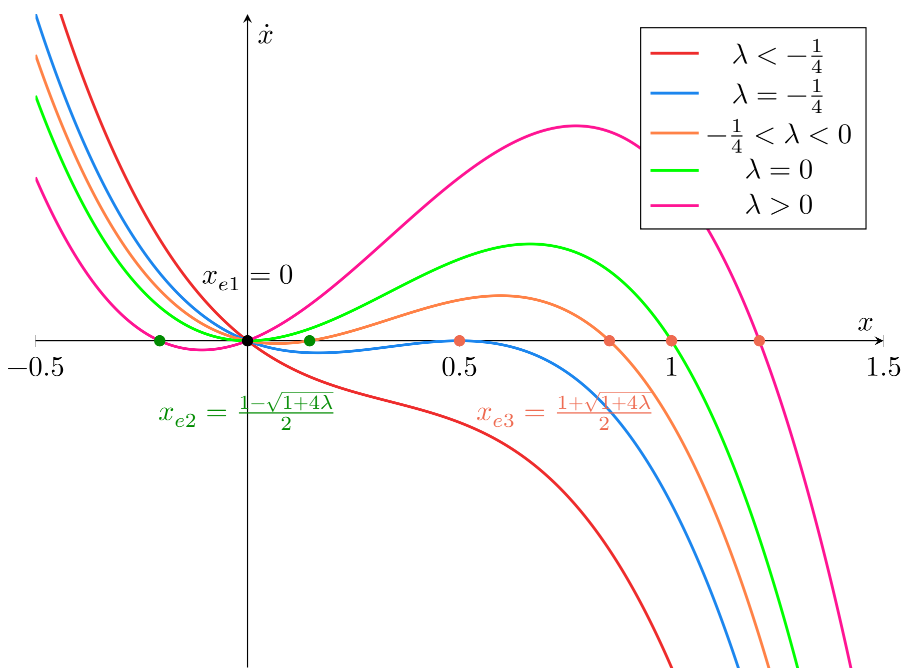

Figure 6: Behaviour of the system $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda x + x^2 - x^3$ for various values of λ.

From these plots, we can identify the stability of each region on 5 phase lines or draw these directly onto a bifurcation diagram.

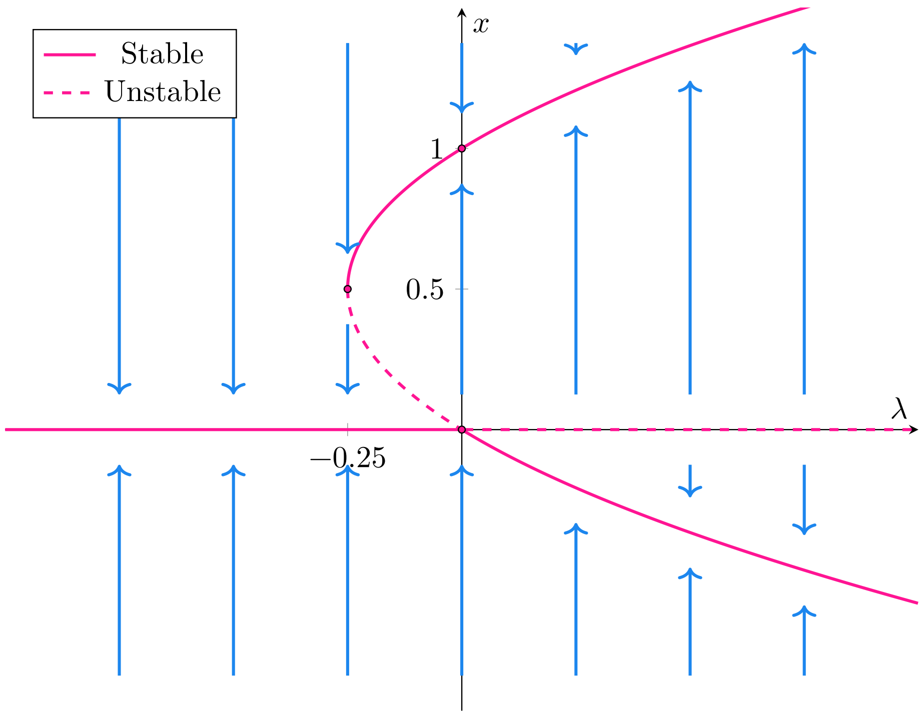

Figure 7: Bifurcation diagram of the system $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda x + x^2 - x^3$.

12 System of Differential Equations

Consider a two-dimensional system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,\ y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(x,\ y)\,.$$

If we restrict f and g to be linear functions of x and y, then we can express this as a system of linear equations:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

where $\mathbf{x} = (x(t)\,,\ y(t))$. As with first-order ODEs, let us first consider the homogeneous part of this equation and assume the solution form:

$$\mathbf{x} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{bmatrix} = \mathbf{v} e^{\lambda t}$$

for constants v_i and λ. If we substitute this back into the original system, we find the eigenvalue problem:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v}e^{\lambda t}\right) &= \mathbf{A}\mathbf{v}e^{\lambda t}\\ \lambda\mathbf{v}e^{\lambda t} &= \mathbf{A}\mathbf{v}e^{\lambda t}\\ \mathbf{A}\mathbf{v} &= \lambda\mathbf{v}.\end{aligned}$$

Therefore, by solving for the eigenvalues and eigenvectors of $\mathbf{A}$, we can determine the general solution to this system of differential equations.

12.1 Real Distinct Eigenvalues

If the eigenvalues of $\mathbf{A}$ are real and distinct, then the general solution to the system of differential equations is given by:

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues λ_1 and λ_2, and c_1 and c_2 are constants determined by the initial conditions.

- When $\lambda_1, \lambda_2 < 0$, the system has a **stable node** at the origin. The trajectories of the system approach this node as $t \to \infty$.
- When $\lambda_1, \lambda_2 > 0$ with $\lambda_1 > \lambda_2$, the system has an **unstable node** at the origin. In general $\mathbf{x} \sim c_1\mathbf{v}_1 e^{\lambda_1 t}$ as $t \to \infty$.
- When $\lambda_1 < 0 < \lambda_2$, the system has a **saddle point** at the origin. In general, $\mathbf{x} \sim c_2\mathbf{v}_2 e^{\lambda_2 t}$ as $t \to \infty$, but certain initial conditions can cause trajectories to approach the origin.

12.2 Real Repeated Eigenvalues

If the eigenvalues of $\mathbf{A}$ are real and repeated, then the general solution to the system of differential equations depends on the number of linearly independent eigenvectors corresponding to the repeated eigenvalue λ:

- If the system has two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then the general solution is given by:
$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda t} + c_2\mathbf{v}_2 e^{\lambda t}.$$
- If the system has only one linearly independent eigenvector $\mathbf{v}$, then the general solution is given by:
$$\mathbf{x} = c_1\mathbf{v}e^{\lambda t} + c_2\left(\mathbf{v}t + \mathbf{w}\right)e^{\lambda t}$$
where $\mathbf{w}$ is a generalised eigenvector that satisfies:
$$\left(\mathbf{A} - \lambda\mathbf{I}\right)\mathbf{w} = \mathbf{v}.$$

In both cases, the system has a **degenerate node** at the origin:

- When $\lambda < 0$, we have a **degenerate stable node** at the origin. Trajectories of the system move towards the origin.
- When $\lambda > 0$, we have a **degenerate unstable node** at the origin. Trajectories of the system move away from the origin.

12.3 Complex Eigenvalues

If the eigenvalues of $\mathbf{A}$ are complex, then the general solution to the system of differential equations is given by:

$$\mathbf{x} = c_1 \left[\mathbf{w}_1 \cos(\beta t) - \mathbf{w}_2 \sin(\beta t)\right] e^{\alpha t} + c_2 \left[\mathbf{w}_2 \cos(\beta t) + \mathbf{w}_1 \sin(\beta t)\right] e^{\alpha t}$$

where α and β are the real and imaginary parts of the complex eigenvalues $\lambda = \alpha \pm i\beta$, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, which we have used to defined $\mathbf{w}_1 = (\mathbf{v}_1 + \mathbf{v}_2)/2$ and $\mathbf{w}_2 = (\mathbf{v}_1 - \mathbf{v}_2)/2i$.

- When $\alpha < 0$, the system has a **stable spiral** (source) at the origin and trajectories of the system spiral inwards towards. In this case, the solution oscillates and decays as $t \to \infty$.
- When $\alpha > 0$, the system has an **unstable spiral** (sink) at the origin and trajectories of the system spiral outwards. In this case, the solution oscillates and grows exponentially as $t \to \infty$.
- When $\alpha = 0$, the system has a **centre** at the origin and trajectories of the system spiral around the origin without growth or decay. In this case, the solution oscillates indefinitely.

The orientation of this spiral is determined by finding the direction of the system near the origin by evaluating $\mathbf{A}\mathbf{x}_0$ for some small vector $\mathbf{x}_0$.

12.4 Nonlinear Systems

As in the case of a single nonlinear differential equation, we can use qualitative methods to understand the behaviour of solutions to a nonlinear system of differential equations. Consider the two-dimensional system of differential equations introduced earlier:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= f(x,\, y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g(x,\, y).\end{aligned}$$

Equilibrium points for this system are defined as points $(x_e,\, y_e)$ that satisfy $f(x_e,\, y_e) = 0$ and $g(x_e,\, y_e) = 0$. If we consider the small region around an equilibrium point $(x_e,\, y_e)$, we can analyse the local behaviour of the system by linearising the system about this point. This involves using a first-order Taylor series approximation of the system near the equilibrium point:

$$\begin{aligned}f(x,\, y) &\approx f(x_e,\, y_e) + \frac{\partial f}{\partial x}(x - x_e) + \frac{\partial f}{\partial y}(y - y_e) \\ g(x,\, y) &\approx g(x_e,\, y_e) + \frac{\partial g}{\partial x}(x - x_e) + \frac{\partial g}{\partial y}(y - y_e).\end{aligned}$$

Substituting this back into the system, we obtain the linearised system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x - x_e \\ y - y_e \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix}_{(x_e,\, y_e)} \begin{bmatrix} x - x_e \\ y - y_e \end{bmatrix} \iff \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x} - \mathbf{x}_e) = \mathbf{J}(\mathbf{x} - \mathbf{x}_e)$$

where $\mathbf{J}$ is the Jacobian matrix of the system evaluated at the equilibrium point $(x_e,\, y_e)$. We can then find the local behaviour of the system by finding the eigenvalues of $\mathbf{J}$ for each equilibrium point.

12.5 Phase Plane Analysis

Using the above information, we can draw a **phase plane** to visualise the behaviour of the system over time. Here the horizontal axis represents the change in x and the vertical axis represents the change in y. We mark equilibrium points on the phase plane and draw eigenvectors from these points to represent the trajectory of the system near equilibrium points. The direction of trajectories along eigenvectors is determined by the sign of the corresponding eigenvalues.
We can also draw **nullclines** on the phase plane, which are curves where one of f or g is equal to zero.