Electrical Engineering Mathematics

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Contents

Contents **1**

1 Infinite Series **2**
1.1 Sequences . . . 2
1.2 Limits of Sequences . . . 2
1.3 Series . . . 2
1.3.1 Common Series . . . 2
1.4 Convergence Tests . . . 3
1.4.1 Ratio Test . . . 3
1.4.2 Alternating Series Test . . . 3

2 Series Approximations **3**
2.1 Taylor Polynomials . . . 3
2.2 Taylor Series . . . 4
2.3 Convergence of Taylor Series . . . 4

1 Infinite Series

1.1 Sequences

A sequence is an **ordered list** of numbers

$$a_1,\ a_2,\ a_3,\ \dots,\ a_n,\ \dots$$

denoted $\{a_n\}_{n=1}^{\infty}$, where n is the index of the sequence. A sequence can be **finite** or **infinite**.

1.2 Limits of Sequences

An infinite sequence $\{a_n\}$ has a limit L if a_n approaches L as n approaches infinity:

$$\lim_{n\to\infty} a_n = L$$

If such a limit exists, the sequence **converges** to L. Otherwise, the sequence **diverges**. Sequences that oscillate between two or more values do not have a limit.

1.3 Series

Given a sequence $\{a_n\}$, we can construct a sequence of **partial sums**,

$$s_n = a_1 + a_2 + \cdots + a_n$$

denoted $\{s_n\}$, such that when $\{s_n\}$ converges to a finite limit L, that is,

$$\lim_{n\to\infty} s_n = L$$

the **infinite series** $\sum_{n=1}^{\infty} a_n$ converges to L. Otherwise, the series $\sum_{n=1}^{\infty} a_n$ diverges.

1.3.1 Common Series

Below are a list of common series that converge to a finite limit:

- **Geometric Series**: A sum of the geometric progression

  $$\sum_{n=0}^{\infty} ar^n$$

  converges when $|r| < 1$, and diverges otherwise. When $|r| < 1$,

  $$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

- **Harmonic Series**: A sum of the reciprocals of natural numbers

  $$\sum_{n=1}^{\infty} \frac{1}{n}$$

  always diverges.

- p-**Series**: A sum of the reciprocals of p-powers of natural numbers

  $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

  converges when $p > 1$, and diverges otherwise. This series is closely related to the **Riemann Zeta Function**, and has exact values for even integers p.

1.4 Convergence Tests

There are several tests to determine the convergence of an infinite series. Note that these tests do not determine the value of the limit.

1.4.1 Ratio Test

Given the infinite series $\sum_{n=1}^{\infty} a_n$, with

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

(1) If $\rho < 1$, the series converges.

(2) If $\rho > 1$, the series diverges.

(3) If $\rho = 1$, the test is inconclusive.

1.4.2 Alternating Series Test

Given the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, the alternating series converges if the following conditions are met:

(1) $b_n > 0$ for all n.

(2) $b_{n+1} \leq b_n$ for all n.

(3) $\lim_{n\to\infty} b_n = 0$.

2 Series Approximations

2.1 Taylor Polynomials

A Taylor polynomial is a polynomial that approximates a function near a point $x = x_0$. The n-th order Taylor polynomial of an n-times differentiable function $f(x)$ near $x = x_0$ is given by:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Using summation notation, this becomes,

$$f(x) \approx P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

If f is $(n+1)$-times differentiable on an interval including x_0, then the error of this approximation can be bounded by

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(p)}{(n+1)!}(x-x_0)^{n+1}$$

for some p between x and x_0.

2.2 Taylor Series

The Taylor polynomials can be extended to Taylor series by taking the limit $n \to \infty$. The Taylor series of an infinitely differentiable function $f(x)$ near $x = x_0$ is defined:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

When $x_0 = 0$, the Taylor series is called the **Maclaurin series**.

2.3 Convergence of Taylor Series

The Taylor series is a form of a **power series**:

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n.$$

A power series may converge in one of three ways:

(1) At a single point $x = x_0$, with a radius of convergence $R = 0$.

(2) On a finite open interval $(x_0 - R, x_0 + R)$, with a radius of convergence $R > 0$. The series is not guaranteed to converge at the endpoints of this interval.

(3) Everywhere, with a radius of convergence $R = \infty$.

For elementary functions, the Taylor series converges to the function everywhere within the radius of convergence.