Electrical Engineering Mathematics

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Prof Scott McCue

Tarang Janawalkar

Contents

Part I
Infinite Series

1 Sequences and Series

1.1 Sequences

A sequence is an **ordered list** of numbers

$$a_1,\ a_2,\ a_3,\ \dots,\ a_n,\ \dots$$

denoted $\{a_n\}_{n=1}^{\infty}$, where n is the index of the sequence. A sequence can be **finite** or **infinite**.

1.2 Limits of Sequences

An infinite sequence $\{a_n\}$ has a limit L if a_n approaches L as n approaches infinity:

$$\lim_{n\to\infty} a_n = L$$

If such a limit exists, the sequence **converges** to L. Otherwise, the sequence **diverges**. Sequences that oscillate between two or more values do not have a limit.

1.3 Series

Given a sequence $\{a_n\}$, we can construct a sequence of **partial sums**,

$$s_n = a_1 + a_2 + \cdots + a_n$$

denoted $\{s_n\}$, such that when $\{s_n\}$ converges to a finite limit L, that is,

$$\lim_{n\to\infty} s_n = L$$

the **infinite series** $\sum_{n=1}^{\infty} a_n$ converges to L. Otherwise, the series $\sum_{n=1}^{\infty} a_n$ diverges.

1.3.1 Common Series

Below are a list of common series that converge to a finite limit:

- **Geometric Series**: A sum of the geometric progression

  $$\sum_{n=0}^{\infty} ar^n$$

  converges when $|r| < 1$, and diverges otherwise. When $|r| < 1$,

  $$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

- **Harmonic Series**: A sum of the reciprocals of natural numbers

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

always diverges.

- p**-Series**: A sum of the reciprocals of p-powers of natural numbers

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges when $p > 1$, and diverges otherwise. This series is closely related to the **Riemann Zeta Function**, and has exact values for even integers p.

1.4 Convergence Tests

There are several tests to determine the convergence of an infinite series. Note that these tests do not determine the value of the limit.

1.4.1 Ratio Test

Given the infinite series $\sum_{n=1}^{\infty} a_n$, with

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

(1) If $\rho < 1$, the series converges.

(2) If $\rho > 1$, the series diverges.

(3) If $\rho = 1$, the test is inconclusive.

1.4.2 Alternating Series Test

Given the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, the alternating series converges if the following conditions are met:

(1) $b_n > 0$ for all n.

(2) $b_{n+1} \leqslant b_n$ for all n.

(3) $\lim_{n\to\infty} b_n = 0$.

2 Taylor Series

2.1 Taylor Polynomials

A Taylor polynomial is a polynomial that approximates a function near a point $x = x_0$. The n-th order Taylor polynomial of an n-times differentiable function $f(x)$ near $x = x_0$ is given by:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Using summation notation, this becomes,

$$f(x) \approx P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

If f is $(n+1)$-times differentiable on an interval including x_0, then the error of this approximation can be bounded by

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1}$$

for some p between x and x_0.

2.2 The Taylor Series

The Taylor polynomials can be extended to Taylor series by taking the limit $n \to \infty$. The Taylor series of an infinitely differentiable function $f(x)$ near $x = x_0$ is defined:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

When $x_0 = 0$, the Taylor series is called the **Maclaurin series**.

2.3 Convergence of Taylor Series

The Taylor series is a form of a **power series**:

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

A power series may converge in one of three ways:

(1) At a single point $x = x_0$, with a radius of convergence $R = 0$.

(2) On a finite open interval $(x_0 - R, x_0 + R)$, with a radius of convergence $R > 0$. The series is not guaranteed to converge at the endpoints of this interval.

(3) Everywhere, with a radius of convergence $R = \infty$.

For elementary functions, the Taylor series converges to the function everywhere within the radius of convergence.

2.4 Common Taylor Series

Below are a list of common Taylor series expansions:

Function	**Taylor Series**	**Convergence**
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$\ln(1-x)$	$-\sum_{n=1}^{\infty} \frac{x^n}{n}$	$-1 \leqslant x < 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$	$-1 < x \leqslant 1$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$

3 Fourier Series

3.1 Periodic Functions

A function $f(t)$ is **periodic** with period T if it satisfies the following condition:

$$f(t+T) = f(t)$$

for all t. As with Taylor polynomials, we wish to build an approximation of $f(t)$ using some basis.

3.2 The Fourier Series

The Fourier series is a representation of a periodic function as the linear combination of sine and cosine functions. For a periodic function $f(t)$ with period T, the Fourier series of $f(t)$ is defined:

$$f_F(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n}{T} t\right) + b_n \sin\left(\frac{2\pi n}{T} t\right) \right).$$

The coefficients a_0, a_n, and b_n are given by:

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t)\,\mathrm{d}t$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos\left(\frac{2\pi n}{T} t\right) \mathrm{d}t$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin\left(\frac{2\pi n}{T} t\right) \mathrm{d}t$$

where t_0 is any value of t, often chosen to be 0 or $-T/2$.

3.3 Convergence of Fourier Series

If $f(t)$ is piecewise smooth on the interval $[t_0, t_0 + T]$, then the Fourier series converges to $f(t)$ in the interval $[t_0, t_0 + T]$:

$$f_F(t) = \lim_{\epsilon \to 0^+} \frac{f(t+\epsilon) + f(t-\epsilon)}{2},$$

where discontinuous points $\bar{t} \in [t_0, t_0 + T]$ converge to the **average** of their left-hand and right-hand limits. When f is non-periodic, the Fourier series converges to the **periodic extension** of f. The endpoints of the interval may converge non-uniformly, corresponding to jump discontinuities in the periodic extension of f.

3.4 Orthogonality

Both the Taylor series and Fourier series are comprised of an **orthogonal basis**. In this function space, the inner product of two functions $f(t)$ and $g(t)$ is defined:

$$\langle f,\ g \rangle = \int_{t_0}^{t_0+T} f(t)\, g(t)\, \mathrm{d}t$$

on the interval $[t_0, t_0 + T]$. The norm of a function can be defined as $\|f\| = \sqrt{\langle f,\ f \rangle}$. Using this definition, we say that two functions are **orthogonal** if their inner product is zero, and **orthonormal** if their inner product is one.
The Fourier series is defined using an infinite dimensional set of orthogonal basis functions:

$$\left\{1,\ \cos\left(\frac{2\pi n}{T} t\right),\ \sin\left(\frac{2\pi n}{T} t\right)\right\}$$

for all $n \in \mathbb{N}$. The inner products of these basis functions are given by:

$$\begin{aligned}
\left\langle \cos\left(\frac{2\pi n}{T} t\right),\ 1 \right\rangle &= \int_{t_0}^{t_0+T} \cos\left(\frac{2\pi n}{T} t\right) \mathrm{d}t = 0 \\
\left\langle \sin\left(\frac{2\pi n}{T} t\right),\ 1 \right\rangle &= \int_{t_0}^{t_0+T} \sin\left(\frac{2\pi n}{T} t\right) \mathrm{d}t = 0 \\
\left\langle \cos\left(\frac{2\pi m}{T} t\right),\ \sin\left(\frac{2\pi n}{T} t\right) \right\rangle &= \int_{t_0}^{t_0+T} \cos\left(\frac{2\pi m}{T} t\right) \sin\left(\frac{2\pi n}{T} t\right) \mathrm{d}t = 0 \\
\left\langle \cos\left(\frac{2\pi m}{T} t\right),\ \cos\left(\frac{2\pi n}{T} t\right) \right\rangle &= \int_{t_0}^{t_0+T} \cos\left(\frac{2\pi m}{T} t\right) \cos\left(\frac{2\pi n}{T} t\right) \mathrm{d}t = \begin{cases} \frac{T}{2}, & m = n \\ 0 & m \neq n \end{cases} \\
\left\langle \sin\left(\frac{2\pi m}{T} t\right),\ \sin\left(\frac{2\pi n}{T} t\right) \right\rangle &= \int_{t_0}^{t_0+T} \sin\left(\frac{2\pi m}{T} t\right) \sin\left(\frac{2\pi n}{T} t\right) \mathrm{d}t = \begin{cases} \frac{T}{2}, & m = n \\ 0 & m \neq n \end{cases}
\end{aligned}$$

for all m and n not equal to zero. These results allow us to determine the Fourier series coefficients by considering the inner product between $f(t)$ and various basis functions. For the coefficient a_0, consider the inner product of $f(t)$ with the constant function 1:

$$\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\right) \\
\langle f,\ 1\rangle &= a_0\langle 1,\ 1\rangle + \sum_{n=1}^{\infty}\left(a_n\left\langle \cos\left(\frac{2\pi n}{T}t\right),\ 1\right\rangle + b_n\left\langle \sin\left(\frac{2\pi n}{T}t\right),\ 1\right\rangle\right) \\
a_0 &= \frac{\langle f,\ 1\rangle}{\langle 1,\ 1\rangle} = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,\mathrm{d}t
\end{aligned}$$

For the coefficients a_n and b_n, consider the inner product of $f(t)$ with $\cos\left(\frac{2\pi m}{T}t\right)$ and $\sin\left(\frac{2\pi m}{T}t\right)$, respectively. For a_n:

$$\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\right) \\
\left\langle f,\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle &= a_0\left\langle 1,\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle \\
&\quad + \sum_{n=1}^{\infty}\left(a_n\left\langle \cos\left(\frac{2\pi n}{T}t\right),\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle + b_n\left\langle \sin\left(\frac{2\pi n}{T}t\right),\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle\right) \\
a_m &= \frac{\left\langle f,\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle}{\left\langle \cos\left(\frac{2\pi m}{T}t\right),\ \cos\left(\frac{2\pi m}{T}t\right)\right\rangle} = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos\left(\frac{2\pi m}{T}t\right)\mathrm{d}t.
\end{aligned}$$

For b_n:

$$\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\right) \\
\left\langle f,\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle &= a_0\left\langle 1,\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle \\
&\quad + \sum_{n=1}^{\infty}\left(a_n\left\langle \cos\left(\frac{2\pi n}{T}t\right),\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle + b_n\left\langle \sin\left(\frac{2\pi n}{T}t\right),\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle\right) \\
b_m &= \frac{\left\langle f,\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle}{\left\langle \sin\left(\frac{2\pi m}{T}t\right),\ \sin\left(\frac{2\pi m}{T}t\right)\right\rangle} = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin\left(\frac{2\pi m}{T}t\right)\mathrm{d}t.
\end{aligned}$$

3.5 Even and Odd Functions

A function $f(t)$ is **even** if

$$f(-t) = f(t)$$

for all t, and **odd** if

$$f(-t) = -f(t).$$

These functions have a special symmetry property that can be exploited when computing integrals:

$$\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t = \begin{cases} 2\displaystyle\int_0^{T/2} f(t)\,\mathrm{d}t, & \text{if } f(t) \text{ even} \\ 0, & \text{if } f(t) \text{ odd} \end{cases}$$

In the context of the Fourier series expansion, it is important to note that cosine functions are even, and sine functions are odd:

$$\cos(-t) = \cos(t)$$
$$\sin(-t) = -\sin(t)$$

3.6 Fourier Cosine Series

Suppose $f(t)$ is an even function with period T, and let us compute the Fourier series of $f(t)$ on the interval $[-T/2, T/2]$. Consider the coefficients b_n:

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi n}{T}t\right)\mathrm{d}t$$

as $f(t)$ is even, the resulting integrand is odd, and the integral is zero. This results in a series containing only even functions, called the Fourier cosine series expansion of $f(t)$:

$$f_c(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T}t\right)$$

with

$$a_0 = \frac{2}{T}\int_{0}^{T/2} f(t)\,\mathrm{d}t$$
$$a_n = \frac{4}{T}\int_{0}^{T/2} f(t)\cos\left(\frac{2\pi n}{T}t\right)\mathrm{d}t$$

3.7 Fourier Sine Series

Suppose $f(t)$ is an odd function with period T, and let us compute the Fourier series of $f(t)$ on the interval $[-T/2, T/2]$. Consider the coefficients a_0 and a_n:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t$$
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi n}{T}t\right)\mathrm{d}t$$

as $f(t)$ is odd, the resulting integrand is odd for both a_0 and a_n, and the integrals are zero. This results in a series containing only odd functions, called the Fourier sine series expansion of $f(t)$:

$$f_s(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T}t\right)$$

with

$$b_n = \frac{4}{T}\int_{0}^{T/2} f(t)\sin\left(\frac{2\pi n}{T}t\right)\mathrm{d}t$$

3.8 Half-Range Expansions

Suppose a function $f(t)$ is defined on the interval $[0, T]$, that is not necessarily even or odd. We can extend this function onto the interval in one of three ways:

- Fourier series: Extends the function periodically on the interval $[0, T]$, with period T.
- Fourier cosine series: Extends the even expansion of the function on the interval $[-T, T]$, with period $2T$.
- Fourier sine series: Extends the odd expansion of the function on the interval $[-T, T]$, with period $2T$.

Note the period in the even and odd series must be twice the period of the original function. This is illustrated in the figures below:

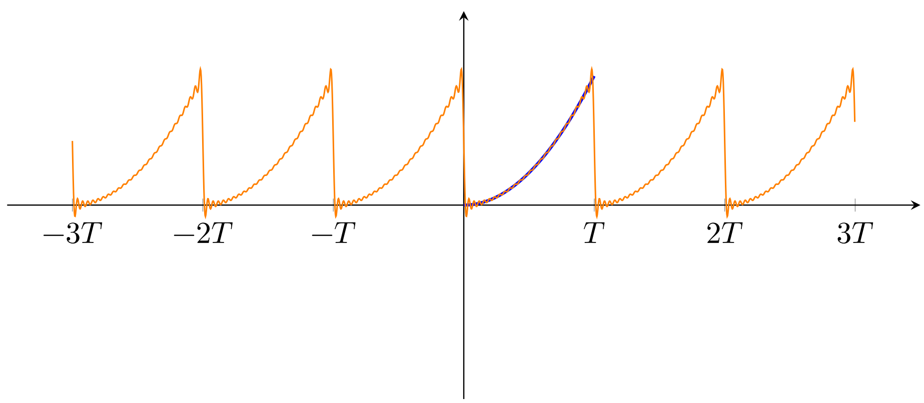

Figure 1: Fourier series expansion of $f(t)$ on the interval $[0, T]$, with the period T.

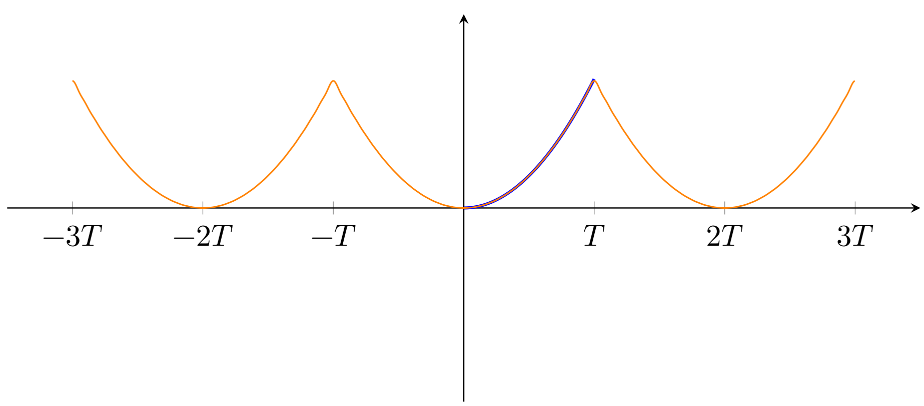

Figure 2: Fourier cosine series expansion of $f(t)$ onto the interval $[-T, T]$, with the period $2T$.

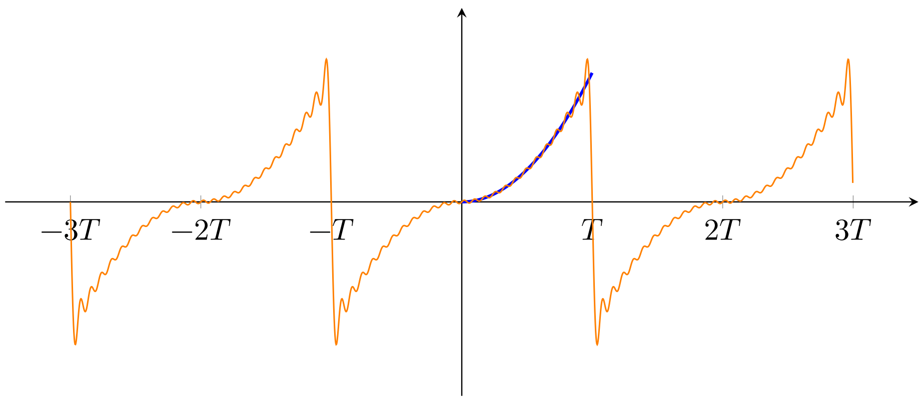

Figure 3: Fourier sine series expansion of $f(t)$ onto the interval $[-T, T]$, with the period $2T$.

Part II
Vector Calculus

4 Scalar Fields

A scalar field is any function $f : \mathbb{R}^n \to \mathbb{R}$, that assigns a scalar value to every vector in $\mathbb{R}^n$.

4.1 Partial Derivatives

The partial derivatives of a scalar field are defined as the derivative of the function with respect to each variable:

$$\frac{\partial f}{\partial x_i} \equiv f_{x_i} = \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

that is, the rate of change of the function in the x_i direction, holding all other variables constant.

4.2 Directional Derivatives

To find the rate of change of a scalar field $f(x_1, \dots, x_n)$ in the direction of a unit vector $\mathbf{u} = [u_1, \dots, u_n]$, we can scale the standard basis vectors by the components of $\mathbf{u}$:

$$D_{\mathbf{u}} f \equiv \frac{\partial f}{\partial \mathbf{u}} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{u_i}{\|\mathbf{u}\|}.$$

This is known as the **directional derivative** of f in the direction of $\mathbf{u}$.

4.3 Gradient

The gradient of a scalar field is an operator grad : $f \to \mathbb{R}^n$ which maps a scalar field f to a vector field:

$$\operatorname{grad} f \equiv \boldsymbol{\nabla} f = \left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right].$$

We can equivalently write the directional derivative as the dot product of the gradient of f with the unit vector $\mathbf{u}$:

$$D_{\mathbf{u}} f = \boldsymbol{\nabla} f \cdot \hat{\mathbf{u}}.$$

4.4 Gradient of a Scalar Field

The gradient of a scalar field f is a vector field that points in the direction of the greatest rate of change of f, with magnitude equal to the rate of change. That is:

- $\boldsymbol{\nabla} f$ points in the direction of greatest increase of f.
- $-\boldsymbol{\nabla} f$ points in the direction of greatest decrease of f.
- $\|\boldsymbol{\nabla} f\|$ is the rate of increase of f in that direction.

5 Vector Fields

A vector field is any function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, that assigns a vector to every vector in $\mathbb{R}^n$.

5.1 Partial Derivatives

The partial derivatives of a vector field are defined as the partial derivatives of each component of the vector field:

$$\frac{\partial \mathbf{F}}{\partial x_i} = \mathbf{F}_{x_i} = \left[\frac{\partial F_1}{\partial x_i}, \dots, \frac{\partial F_n}{\partial x_i}\right]$$

5.2 Divergence

The divergence of a vector field is an operator div : $\mathbf{F} \to \mathbb{R}$, which maps a vector field $\mathbf{F}$ to a scalar:

$$\operatorname{div} \mathbf{F} = \boldsymbol{\nabla} \cdot \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}.$$

The divergence of a vector field measures the rate at which the vector field flows out of a point P.

- When $\operatorname{div} \mathbf{F} > 0$, the vector field tends to flows away from P (source).
- When $\operatorname{div} \mathbf{F} < 0$, the vector field tends to flows towards P (sink).
- When $\operatorname{div} \mathbf{F} = 0$, the net flow of the vector field at P is zero (conservative).

5.3 Curl

The curl of a vector field is an operator curl : $\mathbf{F} \to \mathbf{G}$, which maps a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ to another vector field $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\operatorname{curl}\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

The curl may also be defined for vector fields in $\mathbb{R}^2$, where $F_3 = 0$. The curl of a vector field measures the rotation of the vector field at a point P.

- When $\operatorname{curl}\mathbf{F} > 0$, the vector field tends to rotate counterclockwise around P.
- When $\operatorname{curl}\mathbf{F} < 0$, the vector field tends to rotate clockwise around P.
- When $\operatorname{curl}\mathbf{F} = 0$, the net rotation of the vector field around P is zero.

6 Multiple Integrals

Scalar functions can be integrated over regions in $\mathbb{R}^n$ through multiple integrals.

6.1 Double Integrals

When integrating over some region R in $\mathbb{R}^2$, consider the small subregion R_{ij} with area $\Delta A_i = \Delta x_i \Delta y_i$, so that the double integral of a function $f(x,\ y)$ over R is defined as the contribution of each subregion:

$$\iint_R f(x,\ y)\,\mathrm{d}A = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i,\ y_i)\,\Delta A_i.$$

To compute this integral, we must bound the region by two functions g and h in either the x- or y-direction.

- In the y-direction, the region is bounded by the curves:

  $$\begin{aligned} g(x) &\leqslant y \leqslant h(x) \\ a &\leqslant x \leqslant b \end{aligned}$$

  for some functions $g(x)$ and $h(x)$ so that

  $$\iint_R f(x,\ y)\,\mathrm{d}A = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,\ y)\,\mathrm{d}y\right] \mathrm{d}x.$$

  Here we are adding up vertical strips of width $\mathrm{d}x$, where each strips height is given by the distance between $g(x)$ and $h(x)$, weighted by the function $f(x,\ y)$.

- In the x-direction, the region is bounded by the curves:

$$\begin{aligned} c &\leqslant x \leqslant d \\ g(y) &\leqslant x \leqslant h(y) \end{aligned}$$

  for some functions $g(y)$ and $h(y)$ so that

$$\iint_R f(x,\ y)\,\mathrm{d}A = \int_c^d \left[\int_{g(y)}^{h(y)} f(x,\ y)\,\mathrm{d}x\right] \mathrm{d}y.$$

  Here we are adding up horizontal strips of width $\mathrm{d}y$, where each strips height is given by the distance between $g(y)$ and $h(y)$, weighted by the function $f(x,\ y)$.

6.2 Order of Integration

By Fubini's theorem, any permutation of the order of integration of an iterated integral is equivalent if the function being integrated is integrable, that is if:

$$\int_R |f(\mathbf{x})|\,\mathrm{d}\mathbf{x} < \infty.$$

When applying Fubini's theorem, we must appropriately modify the bounds of integration to account for the region R. For example, if the region is bounded by the curves:

$$R = \{(x,\ y) : a \leqslant x \leqslant b,\ g(x) \leqslant y \leqslant h(x)\},$$

where g and h are invertible on the interval $[a, b]$, and the integral of a function $f(x,\ y)$ over R is given by:

$$\iint_R f(x,\ y)\,\mathrm{d}A = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,\ y)\,\mathrm{d}y\right] \mathrm{d}x,$$

we can equivalently integrate over the region R by reversing the order of integration:

$$\iint_R f(x,\ y)\,\mathrm{d}A = \int_{g(a)}^{h(b)} \left[\int_{h^{-1}(y)}^{g^{-1}(y)} f(x,\ y)\,\mathrm{d}x\right] \mathrm{d}y.$$

Similarly, if the region is bounded by the curves:

$$R = \{(x,\ y) : c \leqslant y \leqslant d,\ g(y) \leqslant x \leqslant h(y)\},$$

we can integrate over the region R by reversing the order of integration:

$$\iint_R f(x,\ y)\,\mathrm{d}A = \int_{g(c)}^{h(d)} \left[\int_{h^{-1}(x)}^{g^{-1}(x)} f(x,\ y)\,\mathrm{d}y\right] \mathrm{d}x.$$

6.3 Triple Integrals

When integrating over some volume V in $\mathbb{R}^3$, consider the small subregion V_{ijk} with volume $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$, so that the triple integral of a function $f(x,\ y,\ z)$ over V is defined as the contribution of each subregion:

$$\iiint_V f(x,\ y,\ z)\,\mathrm{d}V = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i,\ y_i,\ z_i)\,\Delta V_i.$$

To compute this integral, we require three intervals for each variable x, y, and z, that enclose the volume V. As we introduce another dimension, the function bounding the innermost integral may depend on both the outer variables. This integral may take the form:

$$\iiint_V f(x,\ y,\ z)\,\mathrm{d}V = \int_a^b \left[\int_c^d \left[\int_g^h f(x,\ y,\ z)\,\mathrm{d}z\right] \mathrm{d}y\right] \mathrm{d}x.$$

for the volume enclosed by:

$$V = \{(x,\ y,\ z) : a \leqslant x \leqslant b,\ c(x) \leqslant y \leqslant d(x)\,,\ g(x,\ y) \leqslant z \leqslant h(x,\ y)\}\,.$$

Note that the bounds of any integral must not include any variables that appear inside that integral. When modifying the order of integration, we must ensure the same region is enclosed by the new bounds.

6.4 Transformation of Coordinates

In single variable calculus, we used a change of variables to simplify integration by considering a transformation $u = S(x)$, to rewrite an integral in terms of u, with the differential:

$$\mathrm{d}x = \frac{1}{\frac{\mathrm{d}S(x)}{\mathrm{d}x}}\,\mathrm{d}u = \frac{\mathrm{d}S^{-1}(u)}{\mathrm{d}u}\,\mathrm{d}u$$

where $x = S^{-1}(u)$ is the inverse transformation. This concept can be extended to integrals with multiple variables by considering the inverse transformation $x = T(u) = S^{-1}(u)$, where we can use the chain rule to find the differential:

$$\mathrm{d}x = \frac{\mathrm{d}T(u)}{\mathrm{d}u}\,\mathrm{d}u.$$

To transform the coordinates in a multivariable integral, we must consider a matrix of all the partial derivatives of a transformation. For the transformation $\mathbf{x} = \mathbf{T}(\mathbf{u})$, consider the Jacobian matrix of partial derivatives:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x_1}{\partial u_1} & \cdots & \dfrac{\partial x_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x_n}{\partial u_1} & \cdots & \dfrac{\partial x_n}{\partial u_n} \end{bmatrix}.$$

The determinant of this matrix is known as the Jacobian of a transformation, and it gives us the factor by which the volume of the region is scaled under the transformation, giving us the new differential:

$$\mathrm{d}\mathbf{x} = |\det \mathbf{J}|\,\mathrm{d}\mathbf{u}.$$

Therefore, given a bijective transformation $T : \Omega \subset \mathbb{R}^n \to \Omega' \subset \mathbb{R}^n$, where T has continuous partial derivatives, an integral in $\mathbf{x}$ can be transformed to an integral in $\mathbf{u}$ by:

$$\int_{\Omega} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\Omega'} f(\mathbf{T}(\mathbf{u}))\,|\det \mathbf{J}|\,\mathrm{d}\mathbf{u}.$$

6.4.1 Polar Coordinates (2D)

To transform a Cartesian coordinate system to polar coordinates, consider the transformation:

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta & \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}$$

for $r \geqslant 0$ and $0 \leqslant \theta \leqslant 2\pi$. This transformation has the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

with the determinant:

$$|\det \mathbf{J}| = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, the differential in polar coordinates is:

$$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta,$$

giving the integral transformation:

$$\iint_R f(x,\ y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{R'} f(r,\ \theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta.$$

The gradient of this transformation can be computed by applying the chain rule:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} \end{aligned}$$

The partial derivatives in these expressions are given by:

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\theta & \frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} = -\frac{1}{r}\sin\theta \\ \frac{\partial r}{\partial y} &= \frac{y}{r} = \sin\theta & \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} = \frac{1}{r}\cos\theta \end{aligned}$$

so that the gradient in polar coordinates is defined:

$$\begin{aligned}
\boldsymbol{\nabla} &= \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} \\
&= \mathbf{e}_x \left[\cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}\right] + \mathbf{e}_y \left[\sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}\right] \\
&= \left[\cos\theta \mathbf{e}_x + \sin\theta \mathbf{e}_y\right] \frac{\partial}{\partial r} + \left[\cos\theta \mathbf{e}_y - \sin\theta \mathbf{e}_x\right] \frac{1}{r}\frac{\partial}{\partial \theta} \\
&= \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}
\end{aligned}$$

giving the transformed basis vectors:

$$\begin{aligned}
\mathbf{e}_r &= \cos\theta \mathbf{e}_x + \sin\theta \mathbf{e}_y \\
\mathbf{e}_\theta &= \cos\theta \mathbf{e}_y - \sin\theta \mathbf{e}_x
\end{aligned}$$

6.4.2 Cylindrical Coordinates (3D)

To transform a Cartesian coordinate system to cylindrical coordinates, consider the transformation:

$$\begin{aligned}
x &= r\cos\theta & r &= \sqrt{x^2 + y^2} \\
y &= r\sin\theta & \theta &= \arctan\left(\frac{y}{x}\right) \\
z &= z & z &= z
\end{aligned}$$

for $r \geqslant 0$, $0 \leqslant \theta \leqslant 2\pi$, and $-\infty < z < \infty$. This transformation has the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with the determinant:

$$|\det \mathbf{J}| = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, the differential in cylindrical coordinates is:

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z,$$

giving the integral transformation:

$$\iiint_V f(x,\ y,\ z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{V'} f(r,\ \theta,\ z)\,r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

The gradient of this transformation can be computed by applying the chain rule:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial}{\partial z} \\
\frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial z}{\partial z}\frac{\partial}{\partial z}
\end{aligned}$$

The partial derivatives in these expressions are given by:

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\theta & \frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} = -\frac{1}{r}\sin\theta & \frac{\partial z}{\partial x} &= 0 \\
\frac{\partial r}{\partial y} &= \frac{y}{r} = \sin\theta & \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} = \frac{1}{r}\cos\theta & \frac{\partial z}{\partial y} &= 0 \\
\frac{\partial r}{\partial z} &= 0 & \frac{\partial \theta}{\partial z} &= 0 & \frac{\partial z}{\partial z} &= 1
\end{aligned}$$

so that the gradient in cylindrical coordinates is defined:

$$\begin{aligned}
\boldsymbol{\nabla} &= \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} \\
&= \mathbf{e}_x \left[\cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}\right] + \mathbf{e}_y \left[\sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}\right] + \mathbf{e}_z \frac{\partial}{\partial z} \\
&= \left[\cos\theta \mathbf{e}_x + \sin\theta \mathbf{e}_y\right] \frac{\partial}{\partial r} + \left[\cos\theta \mathbf{e}_y - \sin\theta \mathbf{e}_x\right] \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \\
&= \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}
\end{aligned}$$

giving the transformed basis vectors:

$$\begin{aligned}
\mathbf{e}_r &= \cos\theta \mathbf{e}_x + \sin\theta \mathbf{e}_y \\
\mathbf{e}_\theta &= \cos\theta \mathbf{e}_y - \sin\theta \mathbf{e}_x \\
\mathbf{e}_z &= \mathbf{e}_z
\end{aligned}$$

6.4.3 Spherical Coordinates (3D)

To transform a Cartesian coordinate system to spherical coordinates, consider the transformation:

$$\begin{aligned}
x &= r\cos\theta\sin\phi & r &= \sqrt{x^2+y^2+z^2} \\
y &= r\sin\theta\sin\phi & \theta &= \arctan\left(\frac{y}{x}\right) \\
z &= r\cos\phi & \phi &= \arccos\left(\frac{z}{r}\right)
\end{aligned}$$

for $r \geqslant 0$, $0 \leqslant \theta \leqslant 2\pi$, and $0 \leqslant \phi \leqslant \pi$. This transformation has the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\phi & -r\sin\phi & 0 \end{bmatrix}$$

with the determinant:

$$|\det \mathbf{J}| = r^2\sin\phi.$$

Therefore, the differential in spherical coordinates is:

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2 \sin\phi\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta,$$

giving the integral transformation:

$$\iiint_V f\left(x,\ y,\ z\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{V'} f\left(r,\ \theta,\ \phi\right) r^2 \sin\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

The gradient of this transformation can be computed by applying the chain rule:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta}
\end{aligned}$$

The partial derivatives in these expressions are given by:

$$\begin{aligned}
&\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta\sin\phi &&\frac{\partial \phi}{\partial x} = -\frac{z\left(-xr^{-3}\right)}{\sqrt{1-\frac{z^2}{r^2}}} = \frac{\cos\theta\cos\phi}{r} &&\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r\sin\phi} \\
&\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\phi &&\frac{\partial \phi}{\partial y} = -\frac{z\left(-yr^{-3}\right)}{\sqrt{1-\frac{z^2}{r^2}}} = \frac{\sin\theta\cos\phi}{r} &&\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r\sin\phi} \\
&\frac{\partial r}{\partial z} = \frac{z}{r} = \cos\phi &&\frac{\partial \phi}{\partial z} = -\frac{\left(r - z^2r^{-1}\right)r^{-2}}{\sqrt{1-\frac{z^2}{r^2}}} = -\frac{\sin\phi}{r} &&\frac{\partial \theta}{\partial z} = 0
\end{aligned}$$

so that the gradient in spherical coordinates is defined:

$$\begin{aligned}
\boldsymbol{\nabla} = \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}& \\
= \quad & \mathbf{e}_x\left[\cos\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial}{\partial \theta}\right] \\
& +\mathbf{e}_y\left[\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\sin\theta\cos\phi}{r}\frac{\partial}{\partial \phi} + \frac{\cos\theta}{r\sin\phi}\frac{\partial}{\partial \theta}\right] \\
& +\mathbf{e}_z\left[\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\right] \\
= \quad & \left[\cos\theta\sin\phi\mathbf{e}_x + \sin\theta\sin\phi\mathbf{e}_y + \cos\phi\mathbf{e}_z\right]\frac{\partial}{\partial r} \\
& + \left[\cos\theta\cos\phi\mathbf{e}_x + \sin\theta\cos\phi\mathbf{e}_y - \sin\phi\mathbf{e}_z\right]\frac{1}{r}\frac{\partial}{\partial \phi} \\
& + \left[\cos\theta\mathbf{e}_y - \sin\theta\mathbf{e}_x\right]\frac{1}{r\sin\phi}\frac{\partial}{\partial \theta} \\
= \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\phi\frac{1}{r}\frac{\partial}{\partial \phi} + \mathbf{e}_\theta\frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}&
\end{aligned}$$

giving the transformed basis vectors:

$$\begin{aligned}
\mathbf{e}_r &= \cos\theta\sin\phi\mathbf{e}_x + \sin\theta\sin\phi\mathbf{e}_y + \cos\phi\mathbf{e}_z \\
\mathbf{e}_\phi &= \cos\theta\cos\phi\mathbf{e}_x + \sin\theta\cos\phi\mathbf{e}_y - \sin\phi\mathbf{e}_z \\
\mathbf{e}_\theta &= \cos\theta\mathbf{e}_y - \sin\theta\mathbf{e}_x
\end{aligned}$$

6.5 Physical Interpretation of Integrals

7 Line and Surface Integrals

For a curve $\mathscr{C}$ defined by the parametric function $\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ on the interval $a \leqslant t \leqslant b$:

- **Arc length**

$$s(t) = \int_a^t \|\mathbf{r}'(\tau)\| \,\mathrm{d}\tau.$$

- **Line Integral over Scalar Field f**

$$\int_{\mathscr{C}} f \,\mathrm{d}s = \int_{\mathscr{C}} f \frac{\mathrm{d}s}{\mathrm{d}t} \,\mathrm{d}t = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \,\mathrm{d}t.$$

- **Line Integral over Vector Field F**

$$\int_{\mathscr{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_{\mathscr{C}} \mathbf{F} \cdot \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} \,\mathrm{d}t = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) \,\mathrm{d}t.$$

For a surface $\mathscr{S}$ defined by the parametric function $\mathbf{r}(s,\ t) = \begin{bmatrix} x(s,\ t) \\ y(s,\ t) \\ z(s,\ t) \end{bmatrix}$ on the intervals $a \leqslant s \leqslant b$ and $c \leqslant t \leqslant d$:

- **Surface Area**

$$A = \iint_{\mathscr{S}} \mathrm{d}\sigma = \int_c^d \int_a^b \|\mathbf{r}_s \times \mathbf{r}_t\| \,\mathrm{d}s \,\mathrm{d}t.$$

- **Surface Integral over Scalar Field f**

$$\iint_{\mathscr{S}} f \,\mathrm{d}\sigma = \int_c^d \int_a^b f\|\mathbf{r}_s \times \mathbf{r}_t\| \,\mathrm{d}s \,\mathrm{d}t.$$

- **Surface Integral over Vector Field F**

$$\iint_{\mathscr{S}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\sigma} = \int_c^d \int_a^b \mathbf{F} \cdot (\mathbf{r}_s \times \mathbf{r}_t) \,\mathrm{d}s \,\mathrm{d}t.$$

where $\mathbf{n} = \mathbf{r}_s \times \mathbf{r}_t$ is a normal vector to the surface.