Commutator width of the Grigorchuk Group

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Abstract

Let G be the Grigorchuk group. In [LMU13] it was shown that the commutator width of G is finite but not explicit bound was given. In the present paper we show that in fact each element of the derived subgroup $g \in G'$ is a product of two commutators. This means that all equations of the form $[x_1, x_2][x_3, x_4]g = 1$ are solvable for $g \in G'$. The computer algebra system [GAP14] is used to derive a series of equations with increasing genus.

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1 Equations

In this section some standard notations similar to the ones introduced in[JE81] are established. X is a set of *variables*. As it should always be infinite countable it can be assumed to be equal to $\mathbb{N}$. G is some arbitrary group and F_X denotes the free group on the generating set X.
A G-*equation* E is an element of the group $F_X * G$ regarded as reduced word. A G-*homomorphism* from $F_X * G$ to $H * G$ is a homomorphism which is the

identity on G. Define:

$$\text{Var}\colon F_X * G \to \mathbb{P}(X), E \mapsto \text{Var}(E), \quad x \in \text{Var}(E) \text{ iff the symbol } x \text{ occurs in } w$$

An *evaluation* is a G-endomorphism $e\colon F_X * G \to G$. A *solution* of an equation E is an evaluation s with $s(E) = 1$. If a solution exists the equation is called *solvable.*
The set of elements $x \in X$ such that $s(x) \neq 1$ is called the *support* of the solution. Often the support of a solution for an equation E is assumed to be minimal and thus a subset of $F_{\text{Var}(E)}$. As the solution is uniquely described by the image of X the data of a minimal solution is equivalent to a map $\text{Var}(E) \to G$. The question of whether an equation E is solvable will be referred to as the *diophantine* problem of E. Any homomorphism $\varphi\colon G \to H$ extends to a homomorphism $\varphi^*\colon F_X * G \to F_X * H$ by extending it as the identity on F_X.

Definition 1.1. Two equations $E, F \in F_X * G$ are equivalent if there is a G-automorphism φ which maps E to F.

Lemma 1.2. *Let φ be a G-homomorphism and E an equation. If $\varphi(E)$ is solvable, then so is E.*

Proof. Let s be the solution of $\varphi(w)$. Write E as $E = \prod_{i=1}^m g_i x_i$ where $g_i \in G$ and $x_i \in X$. Define the evaluation s' by $x \mapsto s(\varphi(x))$. Then

$$s'(E) = \prod_{i=1}^m g_i s'(x_i) = \prod_{i=1}^m g_i s(\varphi(x_i)) = s\left(\prod_{i=1}^m g_i \varphi(x_i)\right) = s(\varphi(E)) = 1.$$

So s' is a solution for E. □

Corollary 1.3. *The diophantine problem is the same for equivalent equations.*

1.1 Quadratic equations

A G-equation E is called *quadratic* if each $x \in \text{Var}(E)$ occurs exactly twice in E regarded as reduced word.
It is called *oriented* if for each variable $x \in \text{Var}(E)$ the number of occurrences with positive and with negative sign coincide. Otherwise the word is called *unoriented.*

Lemma 1.4. *Being oriented or not is invariant under G-automorphisms.*

Proof. Let φ be some G-homomorphism. Fix some $x \in X$. Let $n_{+,y}$ be the number of positive occurrences of x in $\varphi(y)$ and $n_{-,y}$ accordingly. If E is an oriented word then

$$\sum_{y \in \text{Var}(E)} n_{+,y} = \sum_{y \in \text{Var}(E)} n_{-,y^{-1}} = \sum_{y \in \text{Var}(E)} n_{-,y} \; .$$

So $\varphi(E)$ is oriented too. □

1.2 Normal form of quadratic equations

Definition 1.5. For $x_i, y_i, z_i \in F$ and $c_i \in G$ the following two kind of equations are called in *normal form*:

$$
\begin{aligned}
O_{n,m}: &\quad [x_1,y_1][x_2,y_2]\cdots[x_n,y_n]c_1^{z_1}\cdots c_{m-1}^{z_{m-1}}c_m && (1)\\
U_{n,m}: &\quad x_1^2x_2^2\cdots x_n^2c_1^{z_1}\cdots c_{m-1}^{z_{m-1}}c_m\ . && (2)
\end{aligned}
$$

The form $O_{n,m}$ is called the oriented case and $U_{n,m}$ for $n>0$ the unoriented. The parameter n is referred to as *genus* of the normal form of an equation.

We are going to prove the following theorem:

Theorem 1.6 ([JE81])**.** *Each quadratic equation $E \in F_X * G$ is equivalent to an equation in normal form and the isomorphism can be effectively computed.*

Proof. The proof goes through an induction on the number of variables. Starting with the oriented case: If the reduced equation E has no variables then it is already in normal form $O_{0,1}$. If there is a variable $x \in X$ occurring in E then it does also appear with opposite sign. So the equation has the form $E = ux^{-1}vxw$ or can be brought to this form by applying the automorphism $x \mapsto x^{-1}$. Choose $x \in X$ in a way such that $\mathrm{Var}(v)$ is minimal.
We're distinguish between multiple cases:

Case 1.0 $v \in G$. The word uw has less variables then E and can thus be brought into normal form $N \in O_{r,s}$ by G-isomorphism φ. If N ends with a variable we can use the G-isomorphism $\varphi \circ (x \mapsto xw^{-1})$ to map E to the equation $Nv^x \in O_{r,s+1}$.

If N ends with a group constant b, $N = Mb$ we can use the isomorphism $\varphi \circ (x \mapsto xbw^{-1})$ to map E to the equation $Mv^xb \in O_{r,s+1}$.

Case 1.1 $v \in X \cup X^{-1}$. For simplicity let us assume that $v \in X$. In the other case we can apply $v \mapsto v^{-1}$. Now there are two possibilities: Either $v^{-1} \in u$ or $v^{-1} \in w$. In the first case $E = u_1v^{-1}u_2x^{-1}vxw$ then the isomorphism $x \mapsto x^{u_1}u_2$, $v \mapsto v^{u_1}$ results in the equation $[v,x]u_1u_2w$. In the second case $E = ux^{-1}vxw_1v^{-1}w_2$ is transformed to $[x,v]uw_1w_2$ by the isomorphism $x \mapsto x^{uw_1}w_1^{-1}$, $v \mapsto v^{-uw_1}$. In both cases u_1u_2w, resp. uw_1w_2 are of less variable and so composition with the corresponding isomorphism results the normal form.

Case 2 $\mathrm{Length}(v) > 1$. Then v is a word consisting of elements $X \cup X^{-1}$ with each symbol occurring at most once as v was chosen with minimal variable set, and some elements of G. If v starts with a constant $b \in G$ we can use the homomorphism $x \mapsto bx$ to achieve that v starts with a variable $y \in X$ by eventually using $y \mapsto y^{-1}$. Like in case 1.1 there are two possibilities either y^{-1} is part of u or part of w. In the first place $E = u_1y^{-1}u_2x^{-1}yv_1xw$ we can use the isomorphism $x \mapsto x^{u_1v_1}u_2$,

$y \mapsto y^{u_1 v_1} v_1^{-1}$ to obtain $[y,x]u_1 v_1 u_2 w$. In the second take the isomorphism

$$x \mapsto x^{uw_1v_1} v_1^{-1} w_1^{-1}, \qquad y \mapsto y^{-uw_1v_1} v_1^{-1}$$

to get $[x,y]uw_1v_1w_2$. In both cases the second subword has again less variable and can be brought into normal form by induction.

Therefore each oriented equation can be brought to normal form by G-isomorphisms. For the unoriented case decompose the equation into $E = uxvxw$ with again v with a minimal number of variables. The shorter word $uv^{-1}w$ is equivalent by φ to a normal form N by induction.
The G-isomorphism $\varphi \circ (x \mapsto x^u v^{-1})$ maps E to $x^2 N$. If $N \in U_{r,s} \cup O_{0,t}$ for some r, s, t, nothing else is to do. Otherwise $N = [y,z]M$. Then the homomorphism

$$x \mapsto xyz, \qquad y \mapsto z^{-1}y^{-1}x^{-1}yzxyz, \qquad z \mapsto z^{-1}y^{-1}x^{-1}z$$

maps x^2N to $x^2y^2z^2M$. This homomorphism is indeed an isomorphism as

$$x \mapsto x^2y^{-1}x^{-1}, \qquad y \mapsto xyx^{-1}z^{-1}x^{-1}, \qquad z \mapsto xz$$

is an inverse homomorphism. If M is still not in $O_{0,s}$ this procedure can be repeated with z instead of x. □

For an quadratic equation E we denote by $\mathfrak{nf}(E) := \mathfrak{nf}_E(E)$ the image of the such constructed isomorphism $\mathfrak{nf}_E$ of E.
From now on we will consider oriented equations $O_{(n,1)}$. For this we will use the abbreviation

$$R_n(x_1, \ldots, x_{2n}) = \prod_{i=1}^{n} [x_{2i-1}, x_{2i}]$$

and often write $R_n = R_n(x_1, \ldots, x_{2n})$ if the x_i are the first generators of F_X.

1.3 Constrainted equations

Definition 1.7 ([LMU13])**.** Given an equation $E \in F_X * G$, a group H, a homomorphism $\pi \colon G \to H$ and a homomorphism $\gamma \colon F_X \to H$ then the pair (E, γ) is called a *constrainted* equation and γ a constraint for the equation E on H.
A solution for (E, γ) is a solution s for E with the additional property, that $s(x)^\pi = \gamma(x)$ for all $x \in F_X$.

2 Grigorchuks Group

Let T_n be an infinite regular rooted n-ary tree. The group $\mathrm{Aut}(T_n)$ consists of all root preserving graph automorphisms of the tree T_n. Note that T_n is

isomorphic to any n-ary subtree and therefore $\mathrm{Aut}(T_n) \simeq \mathrm{Aut}(T) \wr S_n$ where S_n is the symmetric group of n symbols.
A self similar subgroup of $\mathrm{Aut}(T_n)$ is a group G with an embedding $G \hookrightarrow G \wr P$ where $P < S_n$. For the sake of an easy notation we will identify elements with the image of these embedding and will write $g = \langle\!\langle g_1, \ldots, g_n \rangle\!\rangle \pi$ for elements $g \in G$. Furthermore we will call the g_i states of the element g and write $g@i := g_i$
The Grigorchuk 2-group is a finitely generated self-similar group with finite state generators:

$$a = \langle\!\langle \mathbb{1}, \mathbb{1} \rangle\!\rangle (1,2), \quad b = \langle\!\langle a, c \rangle\!\rangle, \quad c = \langle\!\langle a, d \rangle\!\rangle, \quad d = \langle\!\langle \mathbb{1}, b \rangle\!\rangle \ .$$

Some useful identities are

- $a^2 = b^2 = c^2 = d^2 = \mathbb{1}$
- $b^a = \langle\!\langle c, a \rangle\!\rangle, c^a = \langle\!\langle d, a \rangle\!\rangle, d^a = \langle\!\langle b, \mathbb{1} \rangle\!\rangle$
- $(ad)^4 = (ac)^8 = (ab)^{16} = \mathbb{1}$.

Lemma 2.1. *The Grigorchuk group is regular branched with branching subgroup* $K := \langle (ab)^2, (bada)^2, (abad)^2 \rangle$.
The Quotient $Q := G/K$ *is of order* 16.

Lemma 2.2 ([LMU13])**.** *Given* $n \in \mathbb{N}$ *and any homomorphism* $\gamma \colon F_X \to Q$ *with* $\mathrm{supp}(\gamma) \subset \langle x_1, \ldots x_{2n} \rangle$ *there is an element* $\varphi \in \mathrm{Stab}(R_n) < \mathrm{Aut}(F_X)$ *such that* $\mathrm{supp}(\gamma \circ \varphi) \in \langle x_1, \ldots, x_5 \rangle$.

Remark. Implemented in GAP: *GammaSimplify.*

Lemma 2.3. *Identify the group of homomorphisms* $\{\gamma \colon F_X \to Q \mid \mathrm{supp}(\gamma) \subset \langle x_1, \ldots x_6 \rangle\}$ *with* Q^6. *Then*

$$\left| Q^6 \big/ \mathrm{Stab}(R_3) \right| = 90.$$

Proof. Shown by GAP calculation. □

Definition 2.4. Fix some representative system $\mathfrak{R}$ of the above 90 orbits and for $\gamma \colon F_X \to Q$ with finite support denote by φ_γ the G-homomorphism in $\mathrm{Stab}(R_\mathbb{N})$ such that $\gamma \circ \varphi_\gamma \in \mathfrak{R}$.
The element $\gamma \circ \varphi_\gamma$ will be called reduced constraint.

Corollary 2.5. *The solvability of a constrainted equation* $(R_n g, \gamma)$ *is equivalent to the solvability of* $(R_n g, \gamma \circ \varphi_\gamma)$.

Proof. If s is a solution for $(R_n g, \gamma)$ then $s \circ \varphi_\gamma$ is a solution for $(R_n g, \gamma \circ \varphi_\gamma)$. □

Definition 2.6 ([BGS03])**.** A *branch structure* of a group $G \hookrightarrow G \wr P$ consists of

- a branching subgroup $K \trianglelefteq G$ of finite index.
- the corresponding Quotient $Q = {}^{G}\!/_{K}$ and the factor homomorphism $\pi\colon G \to Q$.
- A group $Q_1 \subset Q \wr P$ such that $\langle\!\langle q_1, q_2 \rangle\!\rangle \sigma \in Q_1$ if and only if $\langle\!\langle g_1, g_2 \rangle\!\rangle \sigma \in G$ for all $g_i \in \pi^{-1}(q_i)$.
- A map $\omega\colon Q_1 \to Q$ with the following property. If $g = \langle\!\langle g_1, g_2 \rangle\!\rangle \sigma \in G$ then $\omega(\langle\!\langle \pi(g_1), \pi(g_2) \rangle\!\rangle \sigma) = \pi(g)$.

Lemma 2.7. *The Grigorchuk Group has a branch structure.*

Theorem 2.8 ([LMU13])**.** *The Grigorchuk group has finite commutator width. That is there exists an $N \in \mathbb{N}$ such that for all $g \in G'$ the equation $R_N g$ is solvable.*

Remark. This is not true for constrainted equations: For example

$$R_n((ab)^2), \gamma\colon x_i \mapsto 1 \ \forall i$$

is not solvable for any n because otherwise it would be $ac, ca \in G'$.

2.1 Good Pairs

Definition 2.9. Given $g \in G'$ and $\gamma \in \mathfrak{R}$. The tuple (g, γ) is called a *good pair* if there exists an n such that $(R_n g, \gamma)$ is solvable.

Lemma 2.10. *Denote by*

$$\tau\colon G \to {}^{G}\!/_{K'} \qquad \textit{and} \qquad \varpi\colon {}^{G}\!/_{K'} \to {}^{G/K'}\!/_{K/K'} \simeq {}^{G}\!/_{K}$$

the natural projections.
The pair (g, γ) is a good pair if and only if there is a solution $s\colon F_X \to {}^{G}\!/_{K'}$ for $R_3 g^\tau$ with $s(x_i) \in \varpi^{-1}(\gamma(x_i))$.

Proof. If (g, γ) is a good pair, s a solution for $R_n g, \gamma$ then $s(x_i) \in K$ for $i \geq 6$, so $s(R_n) = s(R_3) \cdot k'$ for some $k' \in K'$ therefore there is a solution $\tau \circ s$ for $R_3 g^\tau$ with $s(x_i) = \gamma(x_i)$.
On the other hand if there is a solution $s\colon F_X \to {}^{G}\!/_{K'}$ for $R_3 g^\tau$ with for each $s(x_i) \in \varpi^{-1}(\gamma(x_i))$ then for $g_i \in \tau^{-1}(s(x_i))$ there is some $k' \in K'$ such that $R_n(x_1, \ldots, x_6) g k' = 1$ and so (g, γ) is a good pair. ☐

The previous lemma shows that the question if (g, γ) is a good pair depends only on the image $q = g^\tau$ in ${}^{G}\!/_{K'}$. So (q, γ) will be called a good pair if (g, γ) is a good pair for one (and hence all) preimages of q under τ.

Corollary 2.11. *The following are equivalent:*

a) *K is of finite commutator width.*

b) *There is a $N \in \mathbb{N}$ uniform for all good pairs $(g, \gamma), g \in G', \gamma \in \mathfrak{R}$ such that $(R_N g, \gamma)$ is solvable.*

Proof. First the easy direction: If $k \in K'$ then $(k, \mathbb{1})$ is a good pair. So $(R_n k, \mathbb{1})$ is solvable in G for an $n \leq N$ but the constraints ensures that it is solvable in K. Therefore the commutator width of K is at most N.
If (g, γ) is a good pair there is an $m \in \mathbb{N}$ and a solution s for $R_m g, \gamma$. As $s(x_i)^\pi = 1$ for all $i \geq 6$ there is some $k \in K'$ such that s is a solution for $R_3 k g, \gamma$. By a) there is an N such that all $k \in K'$ can be written as product of N commutators of elements of K and therefore there is a solution for $(R_{N+3} g, \gamma)$. ☐

This motivates to study K' and G/K' further.

Lemma 2.12. *Denote by $k_1 := (ab)^2, k_2 := \langle\!\langle 1, k_1 \rangle\!\rangle = (abad)^2$ and $k_3 := \langle\!\langle k_1, 1 \rangle\!\rangle = (bada)^2$ then*

$$\begin{aligned}
G' &= \left\langle k_1, k_2, k_3, (ad)^2 \right\rangle, \\
K &= \langle k_1, k_2, k_3 \rangle, \\
K \times K &= \{\langle\!\langle k, k' \rangle\!\rangle \mid k, k' \in K\} \\
&= \left\langle k_2, k_3, k_2 k_1^{-1} k_2^{-1} k_1, (k_2 k_1^{-1} k_2^{-1} k_1)^a, k_2 k_1 k_2 k_1^{-1}, (k_2 k_1 k_2 k_1^{-1})^a \right\rangle, \\
K' &= \langle [k_1, k_2] \rangle^G \\
&= \Big\langle (dacabaca)^2 (baca)^4, ((ca)^2 baca)^2, (dacabaca)^2 c (acab)^3 acad, \\
&\qquad ((ac)^3 ab)^2, bacadacab (ac)^2 (acab)^3, (acadacab)^2 (acab)^4 \Big\rangle^{1,a} .
\end{aligned}$$

Furthermore we have this chain of indices:

$$[G : G'] = 8, \quad [G' : K] = 2, \quad [K : K \times K] = 4, \quad [K \times K : K'] = 16.$$

2.2 Main proposition

Definition 2.13. We define the activity of an element $q \in Q$ as the activity of an arbitrary element of $\pi^{-1}(q)$. This is well defined as $K < \mathrm{Stab}(1)$.
Consider a constraint $\gamma \colon F_X \to Q$. Define $\mathcal{A}\mathbf{ct}(\gamma) := x \mapsto \mathcal{A}\mathbf{ct}(\gamma(x))$.
Denote by $\mathfrak{R}_{act}$ the reduced constraints which have a nontrivial activity.

Lemma 2.14. *For each $q \in G'/K'$ there is $\gamma \in \mathfrak{R}_{act}$ such that (q, γ) is a good pair.*

Proof. Assert(ForAll(GPmodKP,q->ForAny(AGPnontrivial,L->q in L))) ☐

Proposition 2.15. *For each good pair (q, γ) with $q \in G'/K'$ and $\gamma \in \mathfrak{R}_{act}$ there is a pair (γ', x), with $\gamma' \in \mathfrak{R}_{act}$ and*

$$x \in \{1, a, b, c, d, ab, ad, ba\}$$

such that for all g with $g^\tau = q$ the following holds

- $(\gamma', (g@2)^x \cdot g@1)$ *is a good pair.*
- *The solvability of* $(R_{2n-1}(g@2)^x \cdot g@1, \gamma')$ *implies the solvability of* $(R_n g, \gamma)$.

This pair (γ', x) can be effectively computed.

For fixed $g \in G'/K'$ and $\gamma \in \mathfrak{R}_{act}$ such that (g, γ) is a good pair denote by $(g_k, \gamma_k)_{k\in\mathbb{N}}$ the following sequence of pairs: First set $(g_1, \gamma_1) = (g, \gamma)$ then define (g_k, γ_k) recursively by apply Proposition 2.15 to the pair (g_{k-1}, γ_{k-1}) and fix a pair (γ', x) with the described properties. Then set $\gamma_k := \gamma'$ and $g_k := (g_{k-1}@2)^x \cdot g_{k-1}@1$.

Corollary 2.16. *If K has finite commutator width, this implies that the commutator width of G is at most 3.*

Proof. Starting with some element $g \in G'$ there is a $\gamma \in \mathfrak{R}_{act}$ such that (g, γ) is a good pair. (Lemma 2.14).
By Proposition 2.15 there exist good pairs $(g_k, \gamma_k)_{k\in\mathbb{N}}$ such that $(R_3 g, \gamma)$ is solvable if one (and hence all) of the constrainted equations $(R_{2^k+1} g_k, \gamma_k)$ is solvable.
If K is of finite commutator width then by Corollary 2.11 there is an $N \in \mathbb{N}$ such that all for all good pairs (h, γ) and $n \geq N$ the constrainted equations $(R_n g, \gamma)$ are solvable. Therefore the sequence of good pairs (g_k, γ_k) is a sequence of solvable equations $(R_{2^k+1} g_k, \gamma_k)$. □

Corollary 2.17. *If for each $g \in G'$ the set $\{h \in G' \mid \exists \gamma' : (h, \gamma') \in (g_k, \gamma_k)_{k\in\mathbb{N}}\}$ is finite, then the commutator width of G is at most 3.*

Proof. If the sequence g_k is finite then as $\mathfrak{R}_{act}$ is finite there is a circle in the sequence of (g_k, γ_k) say $(g_k, \gamma_k) = (g_l, \gamma_l)$ for some $k \neq l$. Thus the solvablility of $(R_n g_k, \gamma_k)$ is equivalent to the solvability of $(R_m g_k, \gamma_k)$ for arbitrary large m but the latter is solvable since (g_k, γ_k) is a good pair. □

Remark. If the "x" in Proposition 2.15 would be always trivial then the assumption of the previous lemma would be true as the sequence $g_1 = g, g_k = g_{k-1}@2 \cdot g_{k-1}@1$ is finite for all g.

Remark. Computer experiments of "random" elements of G' (words of at most 100 generators) did always result in finite sequences (g_k, γ_k) of size at most 30.

Proof of Proposition 2.15. The proof constructs for each good pair (q, γ) a set $\Gamma^q(\gamma) \subset \mathfrak{R}_{act} \times Q$ where the elements fulfill the asked properties by replacing Q by a fixed representative system in G. Then GAP is used to show that for all choices $q \in G'/K'$,$\gamma \in \mathfrak{R}_{act}$ the constructed sets are never empty.
Take the branching structure (K, Q, π, Q_1, ω) of the Grigorchuk group as before and the representative system

$$S = \{1, a, b, c, d, ab, ad, ba\}$$

for Q in G. Denote by rep: $Q \to S$ the map such that $\mathrm{rep}(q)^\pi = q$.
Denote $x_i \in X$ such that $\mathrm{supp}(\gamma) \subset \langle x_1, \ldots, x_6\rangle$ and choose $y_1, \ldots, y_{12} \in X \setminus \{x_1, \ldots, x_6\}$.

$$\Gamma_1(\gamma) = \{\gamma' \colon \langle y_1, \ldots, y_{12}\rangle \to Q \mid \langle\!\langle \gamma'(y_{2i-1}), \gamma'(y_{2i})\rangle\!\rangle \in w^{-1}(\gamma(x_i)\}$$

Let $F_1 = \langle g\rangle, F_2 = \langle g_1, g_2\rangle$ be free groups. Now define a homomorphism

$$\begin{aligned}\Phi_\gamma \colon F_X * F_1 &\to (F_X * F_2) \wr C_2\\ g &\mapsto \langle\!\langle g_1, g_2\rangle\!\rangle,\\ x_i &\mapsto \langle\!\langle x_i^{(1)}, x_i^{(2)}\rangle\!\rangle \mathcal{A}\mathbf{ct}(\gamma(x_i)).\end{aligned}$$

Take $q_1, q_2 \in Q, n > 3 \in \mathbb{N}$ arbitrary and define

$$\Gamma_2^{q_1,q_2,n}(\gamma) = \left\{\gamma' \in \Gamma_1(\gamma) \,\middle|\, \substack{\pi \colon F_2 \to Q\\ g_1 \mapsto q_1\\ g_2 \mapsto q_2}, (\gamma' * \pi)^2(\Phi_\gamma(R_n g)) = \langle\!\langle 1, 1\rangle\!\rangle\right\}.$$

Denote by $v, w = v_n, w_n$ the elements such that $\Phi_\gamma(R_n g) = \langle\!\langle v, w\rangle\!\rangle\langle\!\langle g_1, g_2\rangle\!\rangle$. By the following Lemma 2.18 there is $x \in X \cup X^{-1}$ such that $v = v_1 x v_2$ and $w = w_1 x^{-1} w_2$. Then the homomorphism

$$l_x \colon F_X * F_2 \to F_X * F_2, x_i \mapsto \begin{cases} x_i & \text{if } x_i \neq x\\ w_2 g_2 w_1 & \text{if } x_i = x\end{cases}$$

maps $v g_1 \mapsto v_1 w_2 g_2 w_1 v_1 g_1$ and $w g_2 \mapsto 1$. For $\gamma' \in \Gamma_2^{q_1,q_2,n}(\gamma)$ it is $x^{\gamma' * \pi} = (w_2 g_2 w_1)^{\gamma' * \pi}$ so with $X' = X \setminus x$ there is no loss of information if we consider $\gamma'|_{F_{X'}}$ instead of γ'. From section 1.2 remember the normalization automorphism $\mathfrak{nf}_{\gamma,n,x} := \mathfrak{nf}_{v_1 w_2 g_2 w_1 v_1 g_1} \colon F_{X'} * F_2 \to F_{X'} * F_2$ and note that $\mathfrak{nf}_{\gamma,n,x}(l_x(v)) = R_{2n-1} g_2^{x_{4n-1}} g_1$. This leads to the following definition.

$$\Gamma_3^{q_1,q_2,n,x}(\gamma) = \left\{(\gamma''|_{X\setminus\{x_{4n-1}\}}, \gamma''(x_{4n-1})) \,\middle|\, \gamma'' = \gamma'|_{X'} \circ \mathfrak{nf}_{\gamma,n,x}, \gamma' \in \Gamma_2^{q_1,q_2,n}\right\}.$$

A solution for the the constrainted equation

$$(R_{2n-1}(g@2)^{\mathrm{rep}(y)}(g@1), \gamma''') \text{ for } (\gamma''', y) \in \Gamma_3^{(g@1)^\pi,(g@2)^\pi,n,x}$$

can be extended by sending $x_{4n-1} \mapsto \mathrm{rep}(y)$ to a solution s' of the equation $(R_{2n-1} g@2^{x_{4n-1}} g@1, \gamma'')$. The map $s' \circ \mathfrak{nf}_{\gamma,n,x}^{-1}$ is a solution for the constrainted

equation $(v_1w_2g_2w_1v_1g_1, \gamma'|_{X'})$. Which can be extended by the mapping $x \mapsto w_2(g@2)w_1$ to a solution s of $(\Phi_\gamma(R_ng), \gamma')$. By definition of ω it is $t_i := \langle\!\langle s(y_{2i-1}), s(y_{2i})\rangle\!\rangle \mathcal{A}\mathbf{ct}(\gamma(x_i)) \in G$ for all i. So the mapping $x_i \mapsto t_i$ is a solution for (R_ng, γ).
The map $\Gamma_3^{q_1,q_2,n,x}$ does not depend on the value of n: Choose fitting v, w such that $\Phi_\gamma(R_3g) = \langle\!\langle v, w\rangle\!\rangle\langle\!\langle g_1, g_2\rangle\!\rangle$ then $\Phi_\gamma(R_ng) = \langle\!\langle v, w\rangle\!\rangle\langle\!\langle R_{n-3}g_1, R'_{n-3}g_2\rangle\!\rangle$ then after applying the homomorphism l_x the word which needs to be normalized is $v_1w_2R_{n-3}g_2w_1v_1R'_{n-3}g_1$. The automorphisms

$$\begin{aligned}
&\psi_1\colon F_X * F_2 \to F_X * F_2, && \psi_2\colon F_X * F_2 \to F_X * F_2 && \\
&\qquad y \mapsto y^{g_1^{-1}}, && \qquad y \mapsto y^{g_2^{x_{11}} g_1}, && \text{for } y \in \mathrm{Var}(R'_{n-3}) \\
&\qquad z \mapsto z^{(g_2w_1v_1g_1)^{-1}} && \qquad z \mapsto z^{g_2^{x_{11}} g_1} && \text{for } z \in \mathrm{Var}(R_{n-3}) \\
&\qquad x \mapsto x && && \text{for all other generators}
\end{aligned}$$

have the property that $\mathfrak{nf}_{v_1w_2R_{n-3}g_2w_1v_1R'_{n-3}g_1} = \psi_2 \circ \mathfrak{nf}_{v_1w_2g_2w_1v_1g_1} \circ \psi_1$ and $\gamma' \circ \psi_i = \gamma'$. The map $\Gamma_3^{q_1,q_2,n,x}$ does depend on x, therefore we take the union of all of them and define

$$\Gamma_4^{q_1,q_2}(\gamma) := \left\{ (\gamma''' = \gamma'' \circ \varphi_{\gamma''}, y) \,\middle|\, (\gamma'', y) \in \bigcup_{x \in \mathrm{Var}(v) \cap \mathrm{Var}(w)} \Gamma_3^{q_1,q_2,n}(\gamma) \right\}.$$

Note now that $q_1, q_2 \in Q$ are determined by $q \in G'/K'$ in the sense that there is a map $\bar{@}i\colon G'/K' \to Q$ such that if $g^\tau = q$ and $g_i = g@i$ then $q_i = q\bar{@}i$ (Lemma 2.21). So we can write $\Gamma_4^{q_1,q_2}$ as Γ_4^q instead and finally define

$$\Gamma^q(\gamma) := \{\gamma' \in \Gamma_4^q(\gamma) \mid \mathcal{A}\mathbf{ct}(\gamma') \neq 1\}.$$

As a next step we want to make sure that for all preimages g of q under τ there are good pairs among the resulting pairs $\big((g@2)^{\mathrm{Rep}(y)} \cdot g@1, \gamma'''\big)$.
Define for $h \in G$ maps $p_h\colon G \to G$ by $g \mapsto \big((g@2)^h \cdot g@1, \gamma'''\big)$ this maps are in general not homomorphisms but by Lemma 2.19 we see that for $g \in G'$ that $p_h(g) \in G'$ for all $h \in G$ thus there is a chance that these elements form good pairs with the correct choices of γ.
We can show even better: For each fixed $q \in G'/K'$ and fixed $\gamma \in \mathfrak{R}_{act}$ there is $(\gamma', x) \in \Gamma^q(\gamma)$ such that for all g such that $g^\tau = q$ the element $\big(p_{\mathrm{rep}(x)}(g), \gamma'\big)$ is a good pair.
For this purpose we need to reduce this to a finite number of checks. By Lemma 2.20 we can define the map $\bar{p}_h\colon G'/K' \to G'/(K \times K)$ and the natural homomorphism

$$\varpi'\colon {G'}/{K'} \to {\left(G'/K'\right)}/{\left(K \times K/K'\right)} \simeq {G'}/{K \times K}$$

and now only need to show that there is a $(\gamma', x) \in \Gamma^q(\gamma)$ such that all preimages of $\bar{p}_{\mathrm{rep}(x)}(q)$ under ϖ' form good pairs with γ'. In formulas what needs to be checked is:

$$\forall q \in {G'}/{K'}\ \forall \gamma \in \mathfrak{R}_{act} \exists (\gamma', x) \in \Gamma^q(\gamma) \forall r \in \varpi'^{-1}(\bar{p}_{\mathrm{rep}(x)}) : (r, \gamma') \text{ is a good pair.}$$

This last formula quantifies only over finite sets and is implemented in GAP and can be verified there. $\square$

Lemma 2.18. *If γ is a constraint with nontrivial activity, and $\Phi_\gamma(R_n g) = \langle\!\langle w_1, w_2 \rangle\!\rangle$ then $\mathrm{Var}(w_1) \cap \mathrm{Var}(w_2) \neq \emptyset$.*

Proof. Let x be generator of F_X with non vanishing constraint activity. Then R_n contains either a factor $[x, y]$ or $[y, x]$ for another generator y. Assume without loss of generality the first case. Let further be $\Phi_\gamma(x) = \langle\!\langle x_1, x_2 \rangle\!\rangle(1,2)$ and $\Phi_\gamma(y) = \langle\!\langle y_1, y_2 \rangle\!\rangle\sigma$. Then $\Phi_\gamma(R_n g)$ contains a factor

$$[\langle\!\langle x_1, x_2 \rangle\!\rangle(1,2), \langle\!\langle y_1, y_2 \rangle\!\rangle\sigma] = \begin{cases} \langle\!\langle x_2^{-1} y_2^{-1} x_2 y_1, x_1^{-1} y_1^{-1} x_1 y_2 \rangle\!\rangle & \text{if } \sigma = \mathbb{1} \\ \langle\!\langle x_2^{-1} y_1^{-1} x_1 y_2, x_1^{-1} y_2^{-1} x_2 y_1 \rangle\!\rangle & \text{if } \sigma = (1,2). \end{cases}$$

So in both cases $y_1, y_2 \in \mathrm{Var}(w_1) \cap \mathrm{Var}(w_2)$. $\square$

Lemma 2.19. *Let $h \in G$ and $p_h \colon G \to G$ be the map $g \mapsto \big((g@2)^h \cdot g@1, \gamma'''\big)$. It holds that $p_h(G') \subset G'$ for all $h \in G$ and $p_1(K) \subset K$.*

Proof. Denote first by $p := p_1$ then each element $g \in G'$ is a word in generators $w((ab)^2, (abad)^2, (bada)^2, (ad)^2)$. The generators have the following form:

$$(ab)^2 = \langle\!\langle ca, ac \rangle\!\rangle, (abad)^2 = \langle\!\langle 1, (ab)^2 \rangle\!\rangle, (bada)^2 = \langle\!\langle (ab)^2, 1 \rangle\!\rangle, (ad)^2 = \langle\!\langle b, b \rangle\!\rangle.$$

Therefore it is

$$\begin{aligned} p(g) &= w(ac, (ab)^2, 1, b) \cdot w(ca, 1, (ab)^2, b) \\ &\equiv w(ac, 1, 1, 1) \cdot w(ca, 1, 1, 1) \cdot w(1, 1, 1, b)^2 \equiv 1 \mod G'. \end{aligned}$$

For $h \in G$ it is $p_h(g) = [h, (g@2)^{-1}]p(g)$ and therefore $p_h(g) \in G'$ for all $g \in G'$. An element $g \in K$ is a word $w((ab)^2, (abad)^2, (bada)^2,)$ and therefore

$$\begin{aligned} p(g) &= w(ac, (ab)^2, 1) \cdot w(ca, 1, (ab)^2) \\ &\equiv w(ac, 1, 1) \cdot w(ca, 1, 1) \equiv 1 \mod K. \end{aligned}$$

$\square$

Lemma 2.20. *The map*

$$\begin{aligned} \bar{p}_h \colon {}^{G'}\!/_{K'} &\to {}^{G'}\!/_{K \times K} \\ gK' &\mapsto \big((g@2)^h \cdot g@1\big) K \times K \end{aligned}$$

is well defined.

Proof. It's easy to verify by GAP that $k@i \in K \times K$ for $i = 1, 2$ and $k \in K'$ using Lemma 2.12. Then for $k \in K'$ it is

$$p_h(gk) = ((gk)@2)^h \cdot (gk)@1 = (g@2)^h \cdot (k@2)^h \cdot g@1 \cdot k@1 \in (g@2)^h \cdot g@1K \times K.$$

$\square$

Lemma 2.21. *The maps* $@i\colon G \to G, g \mapsto g@i$ *induce well defined maps* $\bar{@}i\colon G/K' \to G/K$

Proof. Either by same brute-force argument as before that $k'@i \in K \times K < K$ or:
Note that $@i|_{G'}$ is a group homomorphism then consider the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & K/K' \arrow[r] \arrow[d, "\varphi"'] & G'/K' \arrow[r] & G'/K \arrow[r] \arrow[d, "="'] & 1 \\
1 \arrow[r] & K/K \times K \arrow[r] & G'/K \times K \arrow[r] & G'/K \arrow[r] & 1
\end{tikzcd}$$

Where the map φ exists because the group $K/K \times K$ has order 4 and hence is abelian. So there need to be a homomorphism $G'/K' \to G'/K \times K$ which makes all cells commute. □

2.2.1 Product of 3 commutators

We will prove that every element $g \in G'$ is a product of three commutators by proving that the assumptions of Corollary 2.17 are always satisfied. For this purpose remember the map $p_x\colon g \mapsto (g@2)^x g@1$ from the proof of Proposition 2.15. We will show that for each $g \in G'$ the sequence of sets

$$\mathrm{Suc}_1^g = \{g\},\ \mathrm{Suc}_n^g = \{p_x(h) \mid h \in \mathrm{Suc}_{n-1}^g, x \in S\}$$

stagnates in a finite set.
In [Bar98] there is a choice of weights on generators which result in a length on G with good properties.

Lemma 2.22 ([Bar98])**.** *Let* $\eta \approx 0.811$ *be the real root of* $x^3 + x^2 + x - 2$ *and set the weights*

$$\omega(a) = 1 - \eta^3 \qquad \omega(c) = 1 - \eta^2$$
$$\omega(b) = \eta^3 \qquad \omega(d) = 1 - \eta$$

then

$$\begin{aligned}
\eta(\omega(b) + \omega(a)) &= \omega(c) + \omega(a) \\
\eta(\omega(c) + \omega(a)) &= \omega(d) + \omega(a) \\
\eta(\omega(d) + \omega(a)) &= \omega(b).
\end{aligned}$$

The next lemma is a small variation of a lemma in [Bar98].

Lemma 2.23. *Denote by* ∂_ω *the length on* G *induced by the weight* ω*. Then* $\partial_\omega(p_x(g)) \leq \delta\partial_\omega(g)$ *for all* $x \in S, g \in G$ *with* $\partial_\omega(g) > C$ *some constant* $C \in \mathbb{N}, \delta < 1$.

Corollary 2.24. *The sequences of sets*

$$\mathrm{Suc}_1^g = \{g\},\ \mathrm{Suc}_n^g = \{p_x(h) \mid h \in \mathrm{Suc}_{n-1}^g, x \in S\}$$

stagnates at a finite step for all $g \in G$.

Proof of Lemma.([Bar98]). Each element $g \in G$ can be written in a word of minimal length of the form $g = a^{\varepsilon} x_1 a x_2 a \dots x_n a^{\delta}$ where $x_i \in \{b, c, d\}$ and $\varepsilon, \delta \in \{0, 1\}$. Denote by n_b, n_c, n_d the number of occurrences of b, c, d accordingly. Then

$$\begin{aligned}
\partial_\omega(g) &= (n - 1 + \varepsilon + \delta)\omega(a) + n_b\omega(b) + n_c\omega(c) + n_d\omega(d) \\
\partial_\omega(p_x(g)) &\leq (n_b + n_c)\omega(a) + n_b\omega(c) + n_c\omega(d) + n_d\omega(b) + 2\partial_\omega(x) \\
&= \eta\left((n_b + n_c + n_d)\omega(a) + n_b\omega(b) + n_c\omega(c) + n_d\omega(d)\right) + 2\partial_\omega(x) \\
&= \eta(\partial_\omega(g) + (1 - \varepsilon - \delta)\omega(a)) + 2\partial_\omega(x) \\
&\leq \eta(\partial_\omega(g) + \omega(a)) + 2(\omega(a) + \omega(b)) \\
&= \eta(\partial_\omega(g) + \omega(a)) + 2.
\end{aligned}$$

Thus the length of $p_x(g)$ growths with a linear factor smaller 1 in terms of the length of g. Therefore the claim holds. For instance one could take $\delta = 0.86$ and $C = 50$ or $\delta = 0.96$ and $C = 16$. $\square$

Corollary 2.25. *K has commutator width 3.*

Proof. To show that K has commutator width 3 it is sufficient, to show that the constrainted equations $(R_3 g, \mathbb{1})$ have solutions for all $g \in K'$. Since $\mathbb{1}$ has trivial activity one cannot simply apply Proposition 2.15. But one can check that all pairs $(h, \gamma_1), (f, \gamma_2)$ such that $g = \langle\!\langle h, f \rangle\!\rangle$ and $\gamma_1, \gamma_2 = (1, 1, 1, 1, (bad)^\pi, 1)$, $\gamma_2 = (1, 1, 1, 1, 1, (ca)^\pi)$ are good pairs with active constraints and hence the equations $(R_3 g, \mathbb{1})$ have solutions for all $g \in K'$. $\square$

2.3 Product of 2 commutators

If it were true that for each $q \in G'/K'$ there is a $\gamma \in \mathfrak{R}_{act}$ with (q, γ) a good pair and $\gamma(x_i) = 1$ for $i \geq 4$ then it would follow immediately that under the assumptions of either Corollary 2.16 or Corollary 2.17 that each $g \in G'$ is a product of two commutators. But unfortunately this isn't the case.
In fact there are 8 elements of G'/K' which are no good pairs for active γ's with trivial activity in the 5^{th} component. G'/K' is generated by the following elements $q_1 = ((ab)^2)^\tau, q_2 = ((bada)^2)^\tau, q_3 = ((abad)^2)^\tau, q_4 = ((ad)^2)^\tau$.
Fortunately there are inactive γ's with support inside the first 4 coordinates such that all descendant problems are in none of these problematic cases.
For instance the table below shows all problematic $q \in G'/K'$ and a choice of γ such that (q, γ) is a good pair and $(g@i)^\tau, \gamma_i)$ are good pairs with active constraints for $i = 1, 2$ and all g such that $g^\tau = q$. Furthermore the solvability of $(R_2 g@i, \gamma_i)$ implies the solvability of $(R_2 g, \gamma)$ and the latter is solvable by the previous section.

q	γ	γ_1	γ_2
$q_2q_1^2$	$(1,1,b^\pi,dada^\pi,1,1)$	$(1,bad^\pi,a^\pi,ad^\pi,1,1)$	$(1,ca^\pi,c^\pi,da^\pi,1,1)$
$q_2^{-1}q_4q_1^{-2}q_4$	$(1,1,b^\pi,dada^\pi,1,1)$	$(1,bad^\pi,a^\pi,ad^\pi,1,1)$	$(1,ca^\pi,c^\pi,da^\pi,1,1)$
$q_3q_1^2$	$(1,1,c^\pi,b^\pi,1,1)$	$(1,1,a,^\pi bada^\pi,1,1)$	$(1,1,d^\pi,dad^\pi,1,1)$
$q_2^{-1}(q_1^{-1}q_4)^2$	$(1,1,c^\pi,b^\pi,1,1)$	$(1,1,a^\pi,bada^\pi,1,1)$	$(1,1,d^\pi,dad^\pi,1,1)$
$q_1^{-2}q_2$	$(1,1,b,dada^\pi,1,1)$	$(1,bad^\pi,a^\pi,ad^\pi,1,1)$	$(1,ca^\pi,c^\pi,da^\pi,1,1)$
$q_1^{-2}q_2^{-1}$	$(1,1,b^\pi,dada^\pi,1,1)$	$(1,bad^\pi,a^\pi,ad^\pi,1,1)$	$(1,ca^\pi,c^\pi,da^\pi,1,1)$
$q_1^{-2}q_3$	$(1,1,c^\pi,b^\pi,1,1)$	$(1,1,a^\pi,bada^\pi,1,1)$	$(1,1,d^\pi,dad^\pi,1,1)$
$q_1^{-2}q_3^{-1}$	$(1,1,c^\pi,b^\pi,1,1)$	$(1,1,a^\pi,bada^\pi,1,1)$	$(1,1,d^\pi,dad^\pi,1,1)$

Corollary 2.26. *All elements $g \in G'$ are products of two commutators.*

References

[Bar98] Laurent Bartholdi, *The growth of grigorchuk's torsion group*, IMRN **20** (1998), 1049–1054.

[BGS03] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Sunik, *Branch groups*, Handbook of algebra **3** (2003), 989–1112.

[GAP14] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.7.5*, 2014.

[JE81] Leo P Comerford Jr. and Charles C Edmunds, *Quadratic equations over free groups and free products*, Journal of Algebra **68** (1981), no. 2, 276 – 297.

[LMU13] Igor Lysenok, Alexei Miasnikov, and Alexander Ushakov, *Quadratic equations in the Grigorchuk group*, 1304.5579.