

Commutator width of the Grigorchuk group

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Abstract

Let G be the Grigorchuk group. In [LMU13] it was shown that the commutator width of G is finite but no explicit bound was given. In the present paper we show that in fact each element of the derived subgroup $g \in G'$ is a product of two commutators. This means that all equations of the form $[x_1, x_2][x_3, x_4]g = 1$ are solvable for $g \in G'$. The computer algebra system [GAP14] is used to derive a series of equations with increasing genus.

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1 Introduction

Theorem 1.1. *The Grigorchuk group G has commutator width 2.*

2 Equations

In this section some standard notations similar to the ones introduced in [JE81] are established. X is a set of *variables*. As it should always be infinite countable it can be assumed to be equal to \mathbb{N} . G is some arbitrary group and F_X denotes the free group on the generating set X .

A G -*equation* E is an element of the group $F_X * G$ regarded as reduced word. A G -*homomorphism* from $F_X * G$ to $H * G$ is a homomorphism which is the identity on G . Define:

$\text{Var}: F_X * G \rightarrow \mathbb{P}(X), E \mapsto \text{Var}(E), \quad x \in \text{Var}(E) \text{ iff the symbol } x \text{ occurs in } w$

An *evaluation* is a G -endomorphism $e: F_X * G \rightarrow G$. A *solution* of an equation E is an evaluation s with $s(E) = 1$. If a solution exists the equation is called *solvable*.

The set of elements $x \in X$ such that $s(x) \neq 1$ is called the *support* of the solution. Often the support of a solution for an equation E is assumed to be minimal and thus a subset of $F_{\text{Var}(E)}$. As the solution is uniquely described by the image of X the data of a minimal solution is equivalent to a map $\text{Var}(E) \rightarrow G$. The question of whether an equation E is solvable will be referred to as the *diophantine* problem of E . Any homomorphism $\varphi: G \rightarrow H$ extends to a homomorphism $\varphi^*: F_X * G \rightarrow F_X * H$ by extending it as the identity on F_X .

Definition 2.1. Two equations $E, F \in F_X * G$ are equivalent if there is a G -automorphism φ which maps E to F .

Lemma 2.2. Let φ be a G -homomorphism and E an equation. If $\varphi(E)$ is solvable, then so is E .

Proof. Let s be the solution of $\varphi(w)$. Write E as $E = \prod_{i=1}^m g_i x_i$ where $g_i \in G$ and $x_i \in X$. Define the evaluation s' by $x \mapsto s(\varphi(x))$. Then

$$s'(E) = \prod_{i=1}^m g_i s'(x_i) = \prod_{i=1}^m g_i s(\varphi(x_i)) = s \left(\prod_{i=1}^m g_i \varphi(x_i) \right) = s(\varphi(E)) = 1.$$

So s' is a solution for E . □

Corollary 2.3. The diophantine problem is the same for equivalent equations.

2.1 Quadratic equations

A G -equation E is called *quadratic* if each $x \in \text{Var}(E)$ occurs exactly twice in E regarded as reduced word.

It is called *oriented* if for each variable $x \in \text{Var}(E)$ the number of occurrences with positive and with negative sign coincide. Otherwise the word is called *unoriented*.

Lemma 2.4. *Being oriented or not is invariant under G -automorphisms.*

Proof. Let φ be some G -homomorphism. Fix some $x \in X$. Let $n_{+,y}$ be the number of positive occurrences of x in $\varphi(y)$ and $n_{-,y}$ accordingly. If E is an oriented word then

$$\sum_{y \in \text{Var}(E)} n_{+,y} = \sum_{y \in \text{Var}(E)} n_{-,y^{-1}} = \sum_{y \in \text{Var}(E)} n_{-,y}.$$

So $\varphi(E)$ is oriented too. \square

2.2 Normal form of quadratic equations

Definition 2.5. For $x_i, y_i, z_i \in F$ and $c_i \in G$ the following two kind of equations are called in *normal form*:

$$O_{n,m} : [x_1, y_1][x_2, y_2] \cdots [x_n, y_n] c_1^{z_1} \cdots c_{m-1}^{z_{m-1}} c_m \quad (1)$$

$$U_{n,m} : x_1^2 x_2^2 \cdots x_n^2 c_1^{z_1} \cdots c_{m-1}^{z_{m-1}} c_m. \quad (2)$$

The form $O_{n,m}$ is called the oriented case and $U_{n,m}$ for $n > 0$ the unoriented. The parameter n is referred to as *genus* of the normal form of an equation.

We are going to prove the following theorem:

Theorem 2.6 ([JE81]). *Each quadratic equation $E \in F_X * G$ is equivalent to an equation in normal form and the isomorphism can be effectively computed.*

Proof. The proof goes through an induction on the number of variables. Starting with the oriented case: If the reduced equation E has no variables then it is already in normal form $O_{0,1}$. If there is a variable $x \in X$ occurring in E then it does also appear with opposite sign. So the equation has the form $E = ux^{-1}vxw$ or can be brought to this form by applying the automorphism $x \mapsto x^{-1}$. Choose $x \in X$ in a way such that $\text{Var}(v)$ is minimal.

We're distinguish between multiple cases:

Case 1.0 $v \in G$. The word uw has less variables then E and can thus be brought into normal form $N \in O_{r,s}$ by G -isomorphism φ . If N ends with a variable we can use the G -isomorphism $\varphi \circ (x \mapsto xw^{-1})$ to map E to the equation $Nv^x \in O_{r,s+1}$.

If N ends with a group constant b , $N = Mb$ we can use the isomorphism $\varphi \circ (x \mapsto xbw^{-1})$ to map E to the equation $Mv^x b \in O_{r,s+1}$.

Case 1.1 $v \in X \cup X^{-1}$. For simplicity let us assume that $v \in X$. In the other case we can apply $v \mapsto v^{-1}$. Now there are two possibilities: Either $v^{-1} \in u$ or $v^{-1} \in w$. In the first case $E = u_1 v^{-1} u_2 x^{-1} v x w$ then the isomorphism $x \mapsto x^{u_1} u_2$, $v \mapsto v^{u_1}$ results in the equation $[v, x] u_1 u_2 w$. In the second case $E = u x^{-1} v x w_1 v^{-1} w_2$ is transformed to $[x, v] u w_1 w_2$ by the isomorphism $x \mapsto x^{u w_1} w_1^{-1}$, $v \mapsto v^{-u w_1}$. In both cases $u_1 u_2 w$, resp. $u w_1 w_2$ are of less variable and so composition with the corresponding isomorphism results the normal form.

Case 2 $\text{Length}(v) > 1$. Then v is a word consisting of elements $X \cup X^{-1}$ with each symbol occurring at most once as v was chosen with minimal variable set, and some elements of G . If v starts with a constant $b \in G$ we can use the homomorphism $x \mapsto bx$ to achieve that v starts with a variable $y \in X$ by eventually using $y \mapsto y^{-1}$. Like in case 1.1 there are two possibilities either y^{-1} is part of u or part of w . In the first place $E = u_1 y^{-1} u_2 x^{-1} y v_1 x w$ we can use the isomorphism $x \mapsto x^{u_1 v_1} u_2$, $y \mapsto y^{u_1 v_1} v_1^{-1}$ to obtain $[y, x] u_1 v_1 u_2 w$. In the second take the isomorphism

$$x \mapsto x^{u w_1 v_1} v_1^{-1} w_1^{-1}, \quad y \mapsto y^{-u w_1 v_1} v_1^{-1}$$

to get $[x, y] u w_1 v_1 w_2$. In both cases the second subword has again less variable and can be brought into normal form by induction.

Therefore each oriented equation can be brought to normal form by G -isomorphisms. For the unoriented case decompose the equation into $E = u x v x w$ with again v with a minimal number of variables. The shorter word $u v^{-1} w$ is equivalent by φ to a normal form N by induction.

The G -isomorphism $\varphi \circ (x \mapsto x^u v^{-1})$ maps E to $x^2 N$. If $N \in U_{r,s} \cup O_{0,t}$ for some r, s, t , nothing else is to do. Otherwise $N = [y, z] M$. Then the homomorphism

$$x \mapsto x y z, \quad y \mapsto z^{-1} y^{-1} x^{-1} y z x y z, \quad z \mapsto z^{-1} y^{-1} x^{-1} z$$

maps $x^2 N$ to $x^2 y^2 z^2 M$. This homomorphism is indeed an isomorphism as

$$x \mapsto x^2 y^{-1} x^{-1}, \quad y \mapsto x y x^{-1} z^{-1} x^{-1}, \quad z \mapsto x z$$

is an inverse homomorphism. If M is still not in $O_{0,s}$ this procedure can be repeated with z instead of x . \square

For an quadratic equation E we denote by $\mathbf{nf}(E) := \mathbf{nf}_E(E)$ the image of the such constructed isomorphism \mathbf{nf}_E of E .

From now on we will consider oriented equations $O_{(n,1)}$. For this we will use the abbreviation

$$R_n(x_1, \dots, x_{2n}) = \prod_{i=1}^n [x_{2i-1}, x_{2i}]$$

and often write $R_n = R_n(x_1, \dots, x_{2n})$ if the x_i are the first generators of F_X .

2.3 Constrained equations

Definition 2.7 ([LMU13]). Given an equation $E \in F_X * G$, a group H , a homomorphism $\pi: G \rightarrow H$ and a homomorphism $\gamma: F_X \rightarrow H$ then the pair (E, γ) is called a *constrained* equation and γ a constraint for the equation E on H .

A solution for (E, γ) is a solution s for E with the additional property, that $s(x)^\pi = \gamma(x)$ for all $x \in F_X$.

3 Grigorchuks Group

Let T_n be an infinite regular rooted n -ary tree. The group $\text{Aut}(T_n)$ consists of all root preserving graph automorphisms of the tree T_n . Note that T_n is isomorphic to any n -ary subtree and therefore $\text{Aut}(T_n) \simeq \text{Aut}(T) \wr S_n$ where S_n is the symmetric group of n symbols.

A self similar subgroup of $\text{Aut}(T_n)$ is a group G with an embedding $G \hookrightarrow G \wr P$ where $P < S_n$. For the sake of an easy notation we will identify elements with the image of these embedding and will write $g = \langle\langle g_1, \dots, g_n \rangle\rangle \pi$ for elements $g \in G$. Furthermore we will call the g_i states of the element g and write $g@i := g_i$

The Grigorchuk 2-group is a finitely generated self-similar group with finite state generators:

$$a = \langle\langle 1, 1 \rangle\rangle(1, 2), \quad b = \langle\langle a, c \rangle\rangle, \quad c = \langle\langle a, d \rangle\rangle, \quad d = \langle\langle 1, b \rangle\rangle.$$

Some useful identities are

- $a^2 = b^2 = c^2 = d^2 = 1$
- $b^a = \langle\langle c, a \rangle\rangle, c^a = \langle\langle d, a \rangle\rangle, d^a = \langle\langle b, 1 \rangle\rangle$
- $(ad)^4 = (ac)^8 = (ab)^{16} = 1.$

Lemma 3.1. *The Grigorchuk group is regular branched with branching subgroup $K := \langle (ab)^2, (bada)^2, (abad)^2 \rangle$.*

The Quotient $Q := G/K$ is of order 16.

Lemma 3.2 ([LMU13]). *Given $n \in \mathbb{N}$ and any homomorphism $\gamma: F_X \rightarrow Q$ with $\text{supp}(\gamma) \subset \langle x_1, \dots, x_{2n} \rangle$ there is an element $\varphi \in \text{Stab}(R_n) < \text{Aut}(F_X)$ such that $\text{supp}(\gamma \circ \varphi) \in \langle x_1, \dots, x_5 \rangle$.*

Remark. This is now implemented in GAP: For this see the attached gap file and the function *ReduceConstraint* which can be called with one argument being either a group homomorphism from a free group to Q or a list of elements of Q . The group Q can be accessed by `BranchStructure(GrigorchukGroup).group` which comes with the GAP package FR [Bar14].

Lemma 3.3. *Identify the group of homomorphisms $\{\gamma: F_X \rightarrow Q \mid \text{supp}(\gamma) \subset \langle x_1, \dots, x_6 \rangle\}$ with Q^6 . Then*

$$\left| Q^6 / \text{Stab}(R_3) \right| = 90.$$

Proof. This is shown by a GAP calculation. The group $\text{Stab}(R_n)$ is the mapping class group of the surface group of the oriented surface of genus n and

can be generated by the following automorphisms of F_{2n} :

$$\begin{aligned}
\varphi_i: \quad & x_i \mapsto x_{i-1}x_i && \text{for } i = 2, 4, \dots, 2n \\
\varphi_i: \quad & x_i \mapsto x_{i+1}x_i && \text{for } i = 1, 3, \dots, 2n-1 \\
\psi_i: \quad & x_i \mapsto x_{i+1}x_{i+2}^{-1}x_i, \\
& x_{i+1} \mapsto x_{i+1}x_{i+2}^{-1}x_{i+1}x_{i+2}x_{i+1}^{-1}, \\
& x_{i+2} \mapsto x_{i+1}x_{i+2}^{-1}x_{i+2}x_{i+1}^{-1}x_{i+2}, \\
& x_{i+3} \mapsto x_{i+1}x_{i+2}^{-1}x_{i+3} && \text{for } i = 1, 3, \dots, 2n-3
\end{aligned}$$

Now the GAP included standard orbit enumeration algorithm `OrbitsDomain` can be used to compute the orbit and verify its size.

To check this the function `verifyLemma90orbits` from the file *verify.g* can be used. With argument `true` it uses the precomputed orbits and just checks their length. Otherwise it recomputes the orbits which can take up to 24 h and about 7.5 GB. of ram. \square

Definition 3.4. Fix some representative system \mathfrak{R} of the above 90 orbits and for $\gamma: F_X \rightarrow Q$ with finite support denote by φ_γ the G -homomorphism in $\text{Stab}(R_{\mathbb{N}})$ such that $\gamma \circ \varphi_\gamma \in \mathfrak{R}$.

The element $\gamma \circ \varphi_\gamma$ will be called reduced constraint.

Lemma 3.5. *The solvability of a constrained equation $(R_n g, \gamma)$ is equivalent to the solvability of $(R_n g, \gamma \circ \varphi_\gamma)$.*

Proof. If s is a solution for $(R_n g, \gamma)$ then $s \circ \varphi_\gamma$ is a solution for $(R_n g, \gamma \circ \varphi_\gamma)$. \square

Definition 3.6 ([BGS03]). A *branch structure* of a group $G \hookrightarrow G \wr P$ consists of

- a branching subgroup $K \trianglelefteq G$ of finite index.
- the corresponding quotient $Q = G/K$ and the factor homomorphism $\pi: G \rightarrow Q$.
- A group $Q_1 \subset Q \wr P$ such that $\langle\langle g_1, g_2 \rangle\rangle \sigma \in Q_1$ if and only if $\langle\langle g_1, g_2 \rangle\rangle \sigma \in G$ for all $g_i \in \pi^{-1}(q_i)$.
- A map $\omega: Q_1 \rightarrow Q$ with the following property. If $g = \langle\langle g_1, g_2 \rangle\rangle \sigma \in G$ then $\omega(\langle\langle \pi(g_1), \pi(g_2) \rangle\rangle \sigma) = \pi(g)$.

Lemma 3.7. *The Grigorchuk group has a branch structure.*

Proof. This can be seen in [BGS03]. In GAP the branch structure can be computed by the method `BranchStructure(GrigorchukGroup)` which is included in the FR package. \square

Theorem 3.8 ([LMU13]). *The Grigorchuk group has finite commutator width. That is there exists an $N \in \mathbb{N}$ such that for all $g \in G'$ the equation $R_N g$ is solvable.*

Remark. This is not true for all constrained equations: For example

$$(R_n(ab)^2, (\gamma: x_i \mapsto 1 \ \forall i))$$

is not solvable for any n because otherwise it would be $ac, ca \in G'$ which is impossible since ac has activity.

3.1 Good Pairs

The previous remark motivates the following definition.

Definition 3.9. Given $g \in G'$ and $\gamma \in \mathfrak{R}$. The tuple (g, γ) is called a *good pair* if there exists an n such that $(R_n g, \gamma)$ is solvable.

Lemma 3.10. Denote by

$$\tau: G \rightarrow G/K' \quad \text{and} \quad \varpi: G/K' \rightarrow G/K' / K/K' \simeq G/K$$

the natural projections.

The pair (g, γ) is a good pair if and only if there is a solution $s: F_X \rightarrow G/K'$ for $R_3 g^\tau$ with $s(x_i) \in \varpi^{-1}(\gamma(x_i))$.

Proof. If (g, γ) is a good pair and s a solution for $R_n g, \gamma$ then $s(x_i) \in K$ for $i \geq 6$, so $s(R_n) = s(R_3) \cdot k'$ for some $k' \in K'$. Therefore there is a solution $\tau \circ s$ for $R_3 g^\tau$ with $s(x_i) = \gamma(x_i)$.

On the other hand if there is a solution $s: F_X \rightarrow G/K'$ for $R_3 g^\tau$ with for each $s(x_i) \in \varpi^{-1}(\gamma(x_i))$ then for $g_i \in \tau^{-1}(s(x_i))$ there is some $k' \in K'$ such that $R_n(x_1, \dots, x_6)gk' = 1$ and so (g, γ) is a good pair. \square

The previous lemma shows that the question if (g, γ) is a good pair depends only on the image $q = g^\tau$ in G/K' . So (q, γ) will be called a good pair if (g, γ) is a good pair for one (and hence all) preimages of q under τ .

Corollary 3.11. The following are equivalent:

- a) K is of finite commutator width.
- b) There is a $N \in \mathbb{N}$ uniform for all good pairs $(g, \gamma), g \in G', \gamma \in \mathfrak{R}$ such that $(R_N g, \gamma)$ is solvable.

Proof. First the easy direction: If $k \in K'$ then $(k, 1)$ is a good pair. So $(R_n k, 1)$ is solvable in G for an $n \leq N$ but the constraints ensures that it is solvable in K . Therefore the commutator width of K is at most N .

If (g, γ) is a good pair there is an $m \in \mathbb{N}$ and a solution s for $R_m g, \gamma$. As $s(x_i)^\pi = 1$ for all $i \geq 6$ there is some $k \in K'$ such that s is a solution for $R_3 k g, \gamma$. By a) there is an N such that all $k \in K'$ can be written as product of N commutators of elements of K and therefore there is a solution for $(R_{N+3} g, \gamma)$. \square

This motivates to study K' and G/K' further.

Lemma 3.12. *Denote by $k_1 := (ab)^2$, $k_2 := \langle\langle 1, k_1 \rangle\rangle = (abad)^2$ and $k_3 := \langle\langle k_1, 1 \rangle\rangle = (bada)^2$ then*

$$\begin{aligned}
G' &= \langle k_1, k_2, k_3, (ad)^2 \rangle, \\
K &= \langle k_1, k_2, k_3 \rangle, \\
K \times K &= \{ \langle\langle k, k' \rangle\rangle \mid k, k' \in K \} \\
&= \langle k_2, k_3, k_2 k_1^{-1} k_2^{-1} k_1, (k_2 k_1^{-1} k_2^{-1} k_1)^a, k_2 k_1 k_2 k_1^{-1}, (k_2 k_1 k_2 k_1^{-1})^a \rangle, \\
K' &= \langle [k_1, k_2] \rangle^G \\
&= \langle (dacabaca)^2 (baca)^4, ((ca)^2 baca)^2, (dacabaca)^2 c (acab)^3 acad, \\
&\quad ((ac)^3 ab)^2, bacadacab (ac)^2 (acab)^3, (acadacab)^2 (acab)^4 \rangle^{1,a}.
\end{aligned}$$

Furthermore we have this chain of indices:

$$[G : G'] = 8, \quad [G' : K] = 2, \quad [K : K \times K] = 4, \quad [K \times K : K'] = 16.$$

Proof. This is shown in [BGS03] and can be verified using the GAP standard methods NormalClosure and Index. \square

3.2 Main proposition

Definition 3.13. We define the activity of an element $q \in Q$ as the activity of an arbitrary element of $\pi^{-1}(q)$. This is well defined as $K < \text{Stab}(1)$. Consider a constraint $\gamma: F_X \rightarrow Q$. Define $\mathcal{Act}(\gamma) := x \mapsto \mathcal{Act}(\gamma(x))$. Denote by \mathfrak{R}_{act} the reduced constraints which have a nontrivial activity.

Lemma 3.14. *For each $q \in G'/K'$ there is $\gamma \in \mathfrak{R}_{act}$ such that (q, γ) is a good pair.*

Proof. This is a finite problem which can be checked in GAP with the function verifyLemmaExistGoodGammas. If this function is called with the argument true then it uses the pre-computed list of good pairs. Otherwise it computes the list of good pairs first, which will take about 1 min. \square

Proposition 3.15. *For each good pair (q, γ) with $q \in G'/K'$ and $\gamma \in \mathfrak{R}_{act}$ there is a pair (γ', x) , with $\gamma' \in \mathfrak{R}_{act}$ and*

$$x \in \{1, a, b, c, d, ab, ad, ba\}$$

such that for all g with $g^\tau = q$ the following holds

- $((g@2)^x \cdot g@1, \gamma')$ is a good pair.
- The solvability of $(R_{2n-1}(g@2)^x \cdot g@1, \gamma')$ implies the solvability of $(R_n g, \gamma)$.

This pair (γ', x) can be effectively computed.

For fixed $g \in G'/K'$ and $\gamma \in \mathfrak{R}_{act}$ such that (g, γ) is a good pair denote by $(g_k, \gamma_k)_{k \in \mathbb{N}}$ the following sequence of pairs: First set $(g_1, \gamma_1) = (g, \gamma)$ then define (g_k, γ_k) recursively by applying Proposition 3.15 to the pair (g_{k-1}, γ_{k-1}) and fix a pair (γ', x) with the described properties. Then set $\gamma_k := \gamma'$ and $g_k := (g_{k-1} \otimes 2)^x \cdot g_{k-1} \otimes 1$.

Proof. We will define for each good pair (q, γ) a set $\Gamma^q(\gamma) \subset \mathfrak{R}_{act} \times Q$ where the elements fulfill the asked properties by replacing Q by a fixed transversal in G containing the set $S = \{1, a, b, c, d, ab, ad, ba\}$. Then GAP is used to show that for all choices $q \in G'/K', \gamma \in \mathfrak{R}_{act}$ the constructed sets are never empty.

Take the branching structure (K, Q, π, Q_1, ω) of the Grigorchuk group as before and denote by $\text{rep}: Q \rightarrow S$ the map such that $\text{rep}(q)^\pi = q$.

Denote $x_i \in X$ such that $\text{supp}(\gamma) \subset \langle x_1, \dots, x_6 \rangle$ and choose $y_1, \dots, y_{12} \in X \setminus \{x_1, \dots, x_6\}$.

$$\Gamma_1(\gamma) = \{\gamma': \langle y_1, \dots, y_{12} \rangle \rightarrow Q \mid \langle \gamma'(y_{2i-1}), \gamma'(y_{2i}) \rangle \in w^{-1}(\gamma(x_i))\}$$

Let $F_1 = \langle g \rangle, F_2 = \langle g_1, g_2 \rangle$ be free groups. Now define a homomorphism

$$\begin{aligned} \Phi_\gamma: F_X * F_1 &\rightarrow (F_X * F_2) \wr C_2 \\ g &\mapsto \langle\langle g_1, g_2 \rangle\rangle, \\ x_i &\mapsto \langle\langle x_i^{(1)}, x_i^{(2)} \rangle\rangle \text{Act}(\gamma(x_i)). \end{aligned}$$

Take $q_1, q_2 \in Q, n \geq 3 \in \mathbb{N}$ arbitrary and define

$$\Gamma_2^{q_1, q_2, n}(\gamma) = \left\{ \gamma' \in \Gamma_1(\gamma) \mid \begin{array}{l} \pi: F_2 \rightarrow Q \\ g_1 \mapsto q_1 \\ g_2 \mapsto q_2 \end{array}, (\gamma' * \pi)^2(\Phi_\gamma(R_n g)) = \langle\langle 1, 1 \rangle\rangle \right\}.$$

Denote by $v, w = v_n, w_n$ the elements such that $\Phi_\gamma(R_n g) = \langle\langle v, w \rangle\rangle \langle\langle g_1, g_2 \rangle\rangle$. By the following Lemma 3.16 there is $x_0 \in X \cup X^{-1}$ such that $v = v_1 x_0 v_2$ and $w = w_1 x_0^{-1} w_2$. Then the homomorphism

$$l_{x_0}: F_X * F_2 \rightarrow F_X * F_2, x_i \mapsto \begin{cases} x_i & \text{if } x_i \neq x_0 \\ w_2 g_2 w_1 & \text{if } x_i = x_0 \end{cases}$$

maps $v g_1 \mapsto v_1 w_2 g_2 w_1 v_1 g_1$ and $w g_2 \mapsto 1$. For $\gamma' \in \Gamma_2^{q_1, q_2, n}(\gamma)$ it is $(\gamma' * \pi)(x_0) = (\gamma' * \pi)(w_2 g_2 w_1)$ so with $X' = X \setminus x_0$ there is no loss of information if we consider $\gamma'|_{F_{X'}}$ instead of γ' . From section 2.2 remember the normalization automorphism $\mathbf{nf}_{\gamma, n, x_0} := \mathbf{nf}_{v_1 w_2 g_2 w_1 v_1 g_1}: F_{X'} * F_2 \rightarrow F_{X'} * F_2$ and note that $\mathbf{nf}_{\gamma, n, x_0}(l_x(v)) = R_{2n-1} g_2^{x_{4n-1}} g_1$. This leads to the following definition.

$$\Gamma_3^{q_1, q_2, n, x_0}(\gamma) = \left\{ (\gamma''|_{X \setminus \{x_{4n-1}\}}, \gamma''(x_{4n-1})) \mid \gamma'' = \gamma'|_{X'} \circ \mathbf{nf}_{\gamma, n, x_0}, \gamma' \in \Gamma_2^{q_1, q_2, n} \right\}.$$

A solution for the the constrained equation

$$(R_{2n-1}(g@2)^{\text{rep}(y)}(g@1), \gamma''') \text{ for } (\gamma''', y) \in \Gamma_3^{(g@1)^\pi, (g@2)^\pi, n, x_0}$$

can be extended by sending $x_{4n-1} \mapsto \text{rep}(y)$ to a solution s' of the equation $(R_{2n-1}g@2^{x_{4n-1}}g@1, \gamma'')$. The map $s' \circ \mathbf{nf}_{\gamma, n, x_0}^{-1}$ is a solution for the constrained equation $(v_1 w_2 g_2 w_1 v_1 g_1, \gamma'|_{X'})$. Which can be extended by the mapping $x_0 \mapsto w_2(g@2)w_1$ to a solution s of $(\Phi_\gamma(R_n g), \gamma')$. By definition of ω it is $t_i := \langle\langle s(y_{2i-1}), s(y_{2i}) \rangle\rangle \text{Act}(\gamma(x_i)) \in G$ for all i . So the mapping $x_i \mapsto t_i$ is a solution for $(R_n g, \gamma)$.

The map $\Gamma_3^{q_1, q_2, n, x_0}$ does not depend on the value of n : Choose fitting v, w such that $\Phi_\gamma(R_3 g) = \langle\langle v, w \rangle\rangle \langle\langle g_1, g_2 \rangle\rangle$ then $\Phi_\gamma(R_n g) = \langle\langle v, w \rangle\rangle \langle\langle R_{n-3} g_1, R'_{n-3} g_2 \rangle\rangle$ then after applying the homomorphism l_{x_0} the word which needs to be normalized is $v_1 w_2 R_{n-3} g_2 w_1 v_1 R'_{n-3} g_1$. The automorphisms

$$\begin{aligned} \psi_1: F_X * F_2 &\rightarrow F_X * F_2, & \psi_2: F_X * F_2 &\rightarrow F_X * F_2 \\ y &\mapsto y^{g_1^{-1}}, & y &\mapsto y^{g_2^{x_{11}} g_1}, \quad \text{for } y \in \text{Var}(R'_{n-3}) \\ z &\mapsto z^{(g_2 w_1 v_1 g_1)^{-1}}, & z &\mapsto z^{g_2^{x_{11}} g_1} \quad \text{for } z \in \text{Var}(R_{n-3}) \\ x &\mapsto x & & \text{for all other generators} \end{aligned}$$

have the property that $\mathbf{nf}_{v_1 w_2 R_{n-3} g_2 w_1 v_1 R'_{n-3} g_1} = \psi_2 \circ \mathbf{nf}_{v_1 w_2 g_2 w_1 v_1 g_1} \circ \psi_1$ and $\gamma' \circ \psi_i = \gamma'$. The map $\Gamma_3^{q_1, q_2, n, x_0}$ does depend on x_0 , therefore we take the union of all of them and define

$$\Gamma_4^{q_1, q_2}(\gamma) := \left\{ (\gamma''' = \gamma'' \circ \varphi_{\gamma''}, y) \mid (\gamma'', y) \in \bigcup_{x_0 \in \text{Var}(v) \cap \text{Var}(w)} \Gamma_3^{q_1, q_2, n, x_0}(\gamma) \right\}.$$

Note now that $q_1, q_2 \in Q$ are determined by $q \in G'/K'$ in the sense that there is a map $\bar{\omega}: G'/K' \rightarrow Q$ such that if $g^\tau = q$ and $g_i = g@i$ then $q_i = q\bar{\omega}i$ (Lemma 3.19). So we can write $\Gamma_4^{q_1, q_2}$ as Γ_4^q instead and finally define

$$\Gamma^q(\gamma) := \{\gamma' \in \Gamma_4^q(\gamma) \mid \text{Act}(\gamma') \neq 1\}.$$

As a next step we want to make sure that for all preimages g of q under τ there are good pairs among the resulting pairs $((g@2)^{\text{Rep}(y)} \cdot g@1, \gamma''')$.

Define for $h \in G$ maps $p_h: G \rightarrow G$ by $g \mapsto ((g@2)^h \cdot g@1, \gamma''')$ this maps are in general not homomorphisms but by Lemma 3.17 we see that for $g \in G'$ that $p_h(g) \in G'$ for all $h \in G$ thus there is a chance that these elements form good pairs with the correct choices of γ .

We can show even better: For each fixed $q \in G'/K'$ and fixed $\gamma \in \mathfrak{R}_{act}$ there is $(\gamma', x) \in \Gamma^q(\gamma)$ such that for all g such that $g^\tau = q$ the element $(p_{\text{rep}(x)}(g), \gamma')$ is a good pair.

For this purpose we need to reduce this to a finite number of checks. By Lemma 3.18 we can define the map $\bar{p}_h: G'/K' \rightarrow G'/(K \times K)$ and the natural homomorphism

$$\varpi': G'/K' \rightarrow (G'/K') / (K \times K/K') \simeq G'/K \times K$$

and now only need to show that there is a $(\gamma', x) \in \Gamma^q(\gamma)$ such that all preimages of $\bar{p}_{\text{rep}(x)}(q)$ under ϖ' form good pairs with γ' . In formulas what needs to be checked is:

$$\forall q \in G'/K' \quad \forall \gamma \in \mathfrak{R}_{act} \exists (\gamma', x) \in \Gamma^q(\gamma) \forall r \in \varpi'^{-1}(\bar{p}_{\text{rep}(x)}) : \mathcal{G}(q, \gamma) \Rightarrow \mathcal{G}(r, \gamma').$$

Here \mathcal{G} is the predicate of being a good pair. This last formula quantifies only over finite sets and is implemented in GAP and can be verified there by calling `verifyPropExistsSuccessor`. \square

Lemma 3.16. *If γ is a constraint with nontrivial activity, and $\Phi_\gamma(R_n g) = \langle\langle w_1, w_2 \rangle\rangle$ then $\text{Var}(w_1) \cap \text{Var}(w_2) \neq \emptyset$.*

Proof. Let x be generator of F_X with non vanishing constraint activity. Then R_n contains either a factor $[x, y]$ or $[y, x]$ for another generator y . Assume without loss of generality the first case. Let further be $\Phi_\gamma(x) = \langle\langle x_1, x_2 \rangle\rangle(1, 2)$ and $\Phi_\gamma(y) = \langle\langle y_1, y_2 \rangle\rangle\sigma$. Then $\Phi_\gamma(R_n g)$ contains a factor

$$[\langle\langle x_1, x_2 \rangle\rangle(1, 2), \langle\langle y_1, y_2 \rangle\rangle\sigma] = \begin{cases} \langle\langle x_2^{-1}y_2^{-1}x_2y_1, x_1^{-1}y_1^{-1}x_1y_2 \rangle\rangle & \text{if } \sigma = \mathbb{1} \\ \langle\langle x_2^{-1}y_1^{-1}x_1y_2, x_1^{-1}y_2^{-1}x_2y_1 \rangle\rangle & \text{if } \sigma = (1, 2). \end{cases}$$

So in both cases $y_1, y_2 \in \text{Var}(w_1) \cap \text{Var}(w_2)$. \square

Lemma 3.17. *Let $h \in G$ and $p_h: G \rightarrow G$ be the map $g \mapsto ((g@2)^h \cdot g@1, \gamma''')$. It holds that $p_h(G') \subset G'$ for all $h \in G$ and $p_1(K) \subset K$.*

Proof. Denote first by $p := p_1$ then each element $g \in G'$ is a word in generators $w((ab)^2, (abad)^2, (bada)^2, (ad)^2)$. The generators have the following form:

$$(ab)^2 = \langle\langle ca, ac \rangle\rangle, (abad)^2 = \langle\langle 1, (ab)^2 \rangle\rangle, (bada)^2 = \langle\langle (ab)^2, 1 \rangle\rangle, (ad)^2 = \langle\langle b, b \rangle\rangle.$$

Therefore it is

$$\begin{aligned} p(g) &= w(ac, (ab)^2, 1, b) \cdot w(ca, 1, (ab)^2, b) \\ &\equiv w(ac, 1, 1, 1) \cdot w(ca, 1, 1, 1) \cdot w(1, 1, 1, b)^2 \equiv 1 \pmod{G'}. \end{aligned}$$

For $h \in G$ it is $p_h(g) = [h, (g@2)^{-1}]p(g)$ and therefore $p_h(g) \in G'$ for all $g \in G'$. An element $g \in K$ is a word $w((ab)^2, (abad)^2, (bada)^2,)$ and therefore

$$\begin{aligned} p(g) &= w(ac, (ab)^2, 1) \cdot w(ca, 1, (ab)^2) \\ &\equiv w(ac, 1, 1) \cdot w(ca, 1, 1) \equiv 1 \pmod{K}. \end{aligned}$$

\square

Lemma 3.18. *The map*

$$\begin{aligned} \bar{p}_h: G'/K' &\rightarrow G'/K \times K \\ gK' &\mapsto (g@2)^h \cdot g@1 \quad K \times K \end{aligned}$$

is well defined.

Proof. It's easy to verify by GAP that $k@i \in K \times K$ for $i = 1, 2$ and $k \in K'$ using Lemma 3.12. For check this call the function `verifyLemmaStatesOfKPinKxK` of the attached GAP file. Then for $k \in K'$ it is

$$p_h(gk) = ((gk)@2)^h \cdot (gk)@1 = (g@2)^h \cdot (k@2)^h \cdot g@1 \cdot k@1 \in (g@2)^h \cdot g@1K \times K.$$

□

Lemma 3.19. *The maps $@i: G \rightarrow G, g \mapsto g@i$ induce well defined maps $\bar{@}i: G/K' \rightarrow G/K$*

Proof. As before it is $k'@i \in K \times K < K$. □

3.3 Product of 3 commutators

We will prove that every element $g \in G'$ is a product of three commutators by proving that all sequences (g_k, γ_k) as defined after Proposition 3.15 are finite. For this purpose remember the map $p_x: g \mapsto (g@2)^x g@1$ from the proof of Proposition 3.15. We will show that for each $g \in G'$ the sequence of sets

$$\text{Suc}_1^g = \{g\}, \text{Suc}_n^g = \{p_x(h) \mid h \in \text{Suc}_{n-1}^g, x \in S\}$$

stagnates in a finite set.

In [Bar98] there is a choice of weights on generators which result in a length on G with good properties.

Lemma 3.20 ([Bar98]). *Let $\eta \approx 0.811$ be the real root of $x^3 + x^2 + x - 2$ and set the weights*

$$\begin{aligned} \omega(a) &= 1 - \eta^3 & \omega(c) &= 1 - \eta^2 \\ \omega(b) &= \eta^3 & \omega(d) &= 1 - \eta \end{aligned}$$

then

$$\begin{aligned} \eta(\omega(b) + \omega(a)) &= \omega(c) + \omega(a) \\ \eta(\omega(c) + \omega(a)) &= \omega(d) + \omega(a) \\ \eta(\omega(d) + \omega(a)) &= \omega(b). \end{aligned}$$

The next lemma is a small variation of a lemma in [Bar98].

Lemma 3.21. *Denote by ∂_ω the length on G induced by the weight ω . Then $\partial_\omega(p_x(g)) \leq \delta \partial_\omega(g)$ for all $x \in S, g \in G$ with $\partial_\omega(g) > C$ some constant $C \in \mathbb{N}, \delta < 1$.*

Corollary 3.22. *The sequences of sets*

$$\text{Suc}_1^g = \{g\}, \text{Suc}_n^g = \{p_x(h) \mid h \in \text{Suc}_{n-1}^g, x \in S\}$$

stagnates at a finite step for all $g \in G$.

Proof of Lemma. ([Bar98]). Each element $g \in G$ can be written in a word of minimal length of the form $g = a^\varepsilon x_1 a x_2 a \dots x_n a^\delta$ where $x_i \in \{b, c, d\}$ and $\varepsilon, \delta \in \{0, 1\}$. Denote by n_b, n_c, n_d the number of occurrences of b, c, d accordingly. Then

$$\begin{aligned}
\partial_\omega(g) &= (n - 1 + \varepsilon + \delta)\omega(a) + n_b\omega(b) + n_c\omega(c) + n_d\omega(d) \\
\partial_\omega(p_x(g)) &\leq (n_b + n_c)\omega(a) + n_b\omega(c) + n_c\omega(d) + n_d\omega(b) + 2\partial_\omega(x) \\
&= \eta((n_b + n_c + n_d)\omega(a) + n_b\omega(b) + n_c\omega(c) + n_d\omega(d)) + 2\partial_\omega(x) \\
&= \eta(\partial_\omega(g) + (1 - \varepsilon - \delta)\omega(a)) + 2\partial_\omega(x) \\
&\leq \eta(\partial_\omega(g) + \omega(a)) + 2(\omega(a) + \omega(b)) \\
&= \eta(\partial_\omega(g) + \omega(a)) + 2.
\end{aligned}$$

Thus the length of $p_x(g)$ grows with a linear factor smaller 1 in terms of the length of g . Therefore the claim holds. For instance one could take $\delta = 0.86$ and $C = 50$ or $\delta = 0.96$ and $C = 16$. \square

This completes the proof of the following proposition

Proposition 3.23. *If (g, γ) is a good pair with active constraint γ and with $\text{supp}(\gamma) \subset \{x_1, \dots, x_{2n}\}$ then the equation $R_n(x_1, \dots, x_{2n})g, \gamma$ is solvable.*

Corollary 3.24. *The Grigorchuk group G has commutator width at most 3.*

Proof. This is a direct consequence of the proposition and Lemma 3.14. \square

Corollary 3.25. *K has commutator width 3.*

Proof. To show that K has commutator width 3 it is sufficient, to show that the constrained equations $(R_3g, \mathbb{1})$ have solutions for all $g \in K'$. Since $\mathbb{1}$ has trivial activity one cannot directly apply Proposition 3.23. But one can check that all pairs $(h, \gamma_1), (f, \gamma_2)$ such that $g = \langle\langle h, f \rangle\rangle$ and $\gamma_1, \gamma_2 = (1, 1, 1, 1, (bad)^\pi, 1)$, $\gamma_2 = (1, 1, 1, 1, 1, (ca)^\pi)$ are good pairs with active constraints and hence the equations $(R_3g, \mathbb{1})$ have solutions for all $g \in K'$. This check is implemented in the GAP function `verifyCorollaryFiniteCWK`. \square

3.4 Product of 2 commutators

Denote by \mathfrak{R}_{act}^4 those active constraints $\gamma \in \mathfrak{R}_{act}$ with $\gamma(x_i) = 1$ for $i \geq 4$. If it were true that for each $q \in G'/K'$ there is a $\gamma \in \mathfrak{R}_{act}^4$ with (q, γ) then it would follow immediately that each $g \in G'$ is a product of two commutators. But unfortunately this isn't the case.

In fact there are 8 elements of G'/K' which are no good pairs for active $\gamma \in \mathfrak{R}_{act}^4$. In the GAP file they can be accessed by G'/K' is generated by the following elements $q_1 = ((ab)^2)^\tau, q_2 = ((bada)^2)^\tau, q_3 = ((abad)^2)^\tau, q_4 = ((ad)^2)^\tau$.

Fortunately there are inactive γ 's with support inside the first 4 coordinates such that all descendant problems are in none of these problematic cases.

For instance the table below shows all problematic $q \in G'/K'$ and a choice of γ such that (q, γ) is a good pair and $(g@i)^\tau, \gamma_i)$ are good pairs with active constraints for $i = 1, 2$ and all g such that $g^\tau = q$. Furthermore the solvability of $(R_2g@i, \gamma_i)$ implies the solvability of (R_2g, γ) and the latter is solvable by the previous section.

q	γ	γ_1	γ_2
$q_2q_1^2$	$(1, 1, b^\pi, dada^\pi, 1, 1)$	$(1, bad^\pi, a^\pi, ad^\pi, 1, 1)$	$(1, ca^\pi, c^\pi, da^\pi, 1, 1)$
$q_2^{-1}q_4q_1^{-2}q_4$	$(1, 1, b^\pi, dada^\pi, 1, 1)$	$(1, bad^\pi, a^\pi, ad^\pi, 1, 1)$	$(1, ca^\pi, c^\pi, da^\pi, 1, 1)$
$q_3q_1^2$	$(1, 1, c^\pi, b^\pi, 1, 1)$	$(1, 1, a^\pi, bada^\pi, 1, 1)$	$(1, 1, d^\pi, dad^\pi, 1, 1)$
$q_2^{-1}(q_1^{-1}q_4)^2$	$(1, 1, c^\pi, b^\pi, 1, 1)$	$(1, 1, a^\pi, bada^\pi, 1, 1)$	$(1, 1, d^\pi, dad^\pi, 1, 1)$
$q_1^{-2}q_2$	$(1, 1, b, dada^\pi, 1, 1)$	$(1, bad^\pi, a^\pi, ad^\pi, 1, 1)$	$(1, ca^\pi, c^\pi, da^\pi, 1, 1)$
$q_1^{-2}q_2^{-1}$	$(1, 1, b^\pi, dada^\pi, 1, 1)$	$(1, bad^\pi, a^\pi, ad^\pi, 1, 1)$	$(1, ca^\pi, c^\pi, da^\pi, 1, 1)$
$q_1^{-2}q_3$	$(1, 1, c^\pi, b^\pi, 1, 1)$	$(1, 1, a^\pi, bada^\pi, 1, 1)$	$(1, 1, d^\pi, dad^\pi, 1, 1)$
$q_1^{-2}q_3^{-1}$	$(1, 1, c^\pi, b^\pi, 1, 1)$	$(1, 1, a^\pi, bada^\pi, 1, 1)$	$(1, 1, d^\pi, dad^\pi, 1, 1)$

Corollary 3.26. *All elements $g \in G'$ are products of two commutators.*

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