Commutator width of the Grigorchul group

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Abstract

Let G be the Grigorchuk group. In [LMU16] it was shown that the commutator width of G is finite but no explicit bound was given. In the present paper we show that in fact each element of the derived subgroup $g \in G'$ is a product of two commutators. This means that all equations of the form $[x_1, x_2][x_3, x_4]g = 1$ are solvable for $g \in G'$. The computer algebra system [GAP14] is used to derive a series of equations with increasing genus.

Contents

1 Introduction

Theorem 1.1. *The Grigorchuk group G has commutator width 2.*

2 Equations

In this section some standard notations similar to the ones introduced in[JE81] are established. X is a set of *variables*. As it should always be infinite countable it can be assumed to be equal to $\mathbb{N}$. G is some arbitrary group and F_X denotes the free group on the generating set X.
A G-*equation* E is an element of the group $F_X * G$ regarded as reduced word. A G-*homomorphism* from $F_X * G$ to $H * G$ is a homomorphism which is the identity on G. Define:

$$\mathrm{Var}\colon F_X * G \to \mathbb{P}(X), E \mapsto \mathrm{Var}(E), \quad x \in \mathrm{Var}(E) \text{ iff the symbol } x \text{ occurs in } w$$

An *evaluation* is a G-endomorphism $e\colon F_X * G \to G$. A *solution* of an equation E is an evaluation s with $s(E) = 1$. If a solution exists the equation is called *solvable*.
The set of elements $x \in X$ such that $s(x) \neq 1$ is called the *support* of the solution. Often the support of a solution for an equation E is assumed to be minimal and thus a subset of $F_{\mathrm{Var}(E)}$. As the solution is uniquely described by the image of X the data of a minimal solution is equivalent to a map $\mathrm{Var}(E) \to G$. The question of whether an equation E is solvable will be referred to as the *diophantine* problem of E. Any homomorphism $\varphi\colon G \to H$ extends to a homomorphism $\varphi^*\colon F_X * G \to F_X * H$ by extending it as the identity on F_X.

Definition 2.1. Two equations $E, F \in F_X * G$ are equivalent if there is a G-automorphism φ which maps E to F.

Lemma 2.2. *Let φ be a G-homomorphism and E an equation. If $\varphi(E)$ is solvable, then so is E.*

Proof. Let s be the solution of $\varphi(w)$. Write E as $E = \prod_{i=1}^m g_i x_i$ where $g_i \in G$ and $x_i \in X$. Define the evaluation s' by $x \mapsto s(\varphi(x))$. Then

$$s'(E) = \prod_{i=1}^m g_i s'(x_i) = \prod_{i=1}^m g_i s(\varphi(x_i)) = s\left(\prod_{i=1}^m g_i \varphi(x_i)\right) = s(\varphi(E)) = 1.$$

So s' is a solution for E. □

Corollary 2.3. *The diophantine problem is the same for equivalent equations.*

2.1 Quadratic equations

A G-equation E is called *quadratic* if each $x \in \mathrm{Var}(E)$ occurs exactly twice in E regarded as reduced word.
It is called *oriented* if for each variable $x \in \mathrm{Var}(E)$ the number of occurrences with positive and with negative sign coincide. Otherwise the word is called *unoriented*.

Lemma 2.4. *Being oriented or not is invariant under G-automorphisms.*

Proof. Let φ be some G-homomorphism. Fix some $x \in X$. Let $n_{+,y}$ be the number of positive occurrences of x in $\varphi(y)$ and $n_{-,y}$ accordingly. If E is an oriented word then

$$\sum_{y \in \mathrm{Var}(E)} n_{+,y} = \sum_{y \in \mathrm{Var}(E)} n_{-,y^{-1}} = \sum_{y \in \mathrm{Var}(E)} n_{-,y} \ .$$

So $\varphi(E)$ is oriented too. $\square$

2.2 Normal form of quadratic equations

Definition 2.5. For $x_i, y_i, z_i \in F$ and $c_i \in G$ the following two kind of equations are called in *normal form*:

$$O_{n,m}: \qquad [x_1, y_1][x_2, y_2] \cdots [x_n, y_n] c_1^{z_1} \cdots c_{m-1}^{z_{m-1}} c_m \tag{1}$$
$$U_{n,m}: \qquad x_1^2 x_2^2 \cdots x_n^2 c_1^{z_1} \cdots c_{m-1}^{z_{m-1}} c_m \ . \tag{2}$$

The form $O_{n,m}$ is called the oriented case and $U_{n,m}$ for $n > 0$ the unoriented. The parameter n is referred to as *genus* of the normal form of an equation.

We are going to prove the following theorem:

Theorem 2.6 ([JE81])**.** *Each quadratic equation $E \in F_X * G$ is equivalent to an equation in normal form and the isomorphism can be effectively computed.*

Proof. The proof goes through an induction on the number of variables. Starting with the oriented case: If the reduced equation E has no variables then it is already in normal form $O_{0,1}$. If there is a variable $x \in X$ occurring in E then it does also appear with opposite sign. So the equation has the form $E = ux^{-1}vxw$ or can be brought to this form by applying the automorphism $x \mapsto x^{-1}$. Choose $x \in X$ in a way such that $\mathrm{Var}(v)$ is minimal.
We're distinguish between multiple cases:

Case 1.0 $v \in G$. The word uw has less variables then E and can thus be brought into normal form $N \in O_{r,s}$ by G-isomorphism φ. If N ends with a variable we can use the G-isomorphism $\varphi \circ (x \mapsto xw^{-1})$ to map E to the equation $Nv^x \in O_{r,s+1}$.

If N ends with a group constant b, $N = Mb$ we can use the isomorphism $\varphi \circ (x \mapsto xbw^{-1})$ to map E to the equation $Mv^x b \in O_{r,s+1}$.

Case 1.1 $v \in X \cup X^{-1}$. For simplicity let us assume that $v \in X$. In the other case we can apply $v \mapsto v^{-1}$. Now there are two possibilities: Either $v^{-1} \in u$ or $v^{-1} \in w$. In the first case $E = u_1 v^{-1} u_2 x^{-1} v x w$ then the isomorphism $x \mapsto x^{u_1} u_2$, $v \mapsto v^{u_1}$ results in the equation $[v, x] u_1 u_2 w$. In the second case $E = ux^{-1}vxw_1v^{-1}w_2$ is transformed to $[x, v] u w_1 w_2$ by the isomorphism $x \mapsto x^{uw_1} w_1^{-1}$, $v \mapsto v^{-uw_1}$. In both cases $u_1 u_2 w$, resp. $uw_1 w_2$ are of less variable and so composition with the corresponding isomorphism results the normal form.

Case 2 Length$(v) > 1$. Then v is a word consisting of elements $X \cup X^{-1}$ with each symbol occurring at most once as v was chosen with minimal variable set, and some elements of G. If v starts with a constant $b \in G$ we can use the homomorphism $x \mapsto bx$ to achieve that v starts with a variable $y \in X$ by eventually using $y \mapsto y^{-1}$. Like in case 1.1 there are two possibilities either y^{-1} is part of u or part of w. In the first place $E = u_1 y^{-1} u_2 x^{-1} y v_1 x w$ we can use the isomorphism $x \mapsto x^{u_1 v_1} u_2$, $y \mapsto y^{u_1 v_1} v_1^{-1}$ to obtain $[y, x] u_1 v_1 u_2 w$. In the second take the isomorphism

$$x \mapsto x^{u w_1 v_1} v_1^{-1} w_1^{-1}, \qquad y \mapsto y^{-u w_1 v_1} v_1^{-1}$$

to get $[x, y] u w_1 v_1 w_2$. In both cases the second subword has again less variable and can be brought into normal form by induction.

Therefore each oriented equation can be brought to normal form by G-isomorphisms. For the unoriented case decompose the equation into $E = uxvxw$ with again v with a minimal number of variables. The shorter word $uv^{-1}w$ is equivalent by φ to a normal form N by induction.
The G-isomorphism $\varphi \circ (x \mapsto x^u v^{-1})$ maps E to $x^2 N$. If $N \in U_{r,s} \cup O_{0,t}$ for some r, s, t, nothing else is to do. Otherwise $N = [y, z] M$. Then the homomorphism

$$x \mapsto xyz, \qquad y \mapsto z^{-1} y^{-1} x^{-1} yzxyz, \qquad z \mapsto z^{-1} y^{-1} x^{-1} z$$

maps $x^2 N$ to $x^2 y^2 z^2 M$. This homomorphism is indeed an isomorphism as

$$x \mapsto x^2 y^{-1} x^{-1}, \qquad y \mapsto xyx^{-1} z^{-1} x^{-1}, \qquad z \mapsto xz$$

is an inverse homomorphism. If M is still not in $O_{0,s}$ this procedure can be repeated with z instead of x. □

For an quadratic equation E we denote by $\mathfrak{nf}(E) := \mathfrak{nf}_E(E)$ the image of the such constructed isomorphism $\mathfrak{nf}_E$ of E.
From now on we will consider oriented equations $O_{(n,1)}$. For this we will use the abbreviation

$$R_n(x_1, \ldots, x_{2n}) = \prod_{i=1}^{n} [x_{2i-1}, x_{2i}]$$

and often write $R_n = R_n(x_1, \ldots, x_{2n})$ if the x_i are the first generators of F_X.

2.3 Constrainted equations

Definition 2.7 ([LMU16])**.** Given an equation $E \in F_X * G$, a group H, a homomorphism $\pi \colon G \to H$ and a homomorphism $\gamma \colon F_X \to H$ then the pair (E, γ) is called a *constrainted* equation and γ a constraint for the equation E on H.
A solution for (E, γ) is a solution s for E with the additional property, that $s(x)^\pi = \gamma(x)$ for all $x \in F_X$.

3 Grigorchuks Group

Let T_n be an infinite regular rooted n-ary tree and S_n the symmetric group on n symbols. The group $\mathrm{Aut}(T_n)$ consists of all root preserving graph automorphisms of the tree T_n. Note that T_n is isomorphic to any n-ary subtree and therefore there is an isomorphism $\psi\colon \mathrm{Aut}(T_n) \xrightarrow{\sim} \mathrm{Aut}(T) \wr S_n$.
A self similar subgroup of $\mathrm{Aut}(T_n)$ is a group G with $G < \psi(G)$ For the sake of an easy notation we will identify elements with the image of these embedding and will write $g = \langle\!\langle g_1, \ldots, g_n \rangle\!\rangle \pi$ for elements $g \in G$. Furthermore we will call the g_i states of the element g and write $g@i := g_i$.
The Grigorchuk 2-group is a finitely generated self-similar group with finite state generators:

$$a = \langle\!\langle \mathbb{1}, \mathbb{1} \rangle\!\rangle (1,2), \quad b = \langle\!\langle a, c \rangle\!\rangle, \quad c = \langle\!\langle a, d \rangle\!\rangle, \quad d = \langle\!\langle \mathbb{1}, b \rangle\!\rangle \ .$$

Some useful identities are

- $a^2 = b^2 = c^2 = d^2 = \mathbb{1}$
- $b^a = \langle\!\langle c, a \rangle\!\rangle, c^a = \langle\!\langle d, a \rangle\!\rangle, d^a = \langle\!\langle b, \mathbb{1} \rangle\!\rangle$
- $(ad)^4 = (ac)^8 = (ab)^{16} = \mathbb{1}$.

Lemma 3.1. *The Grigorchuk group is regular branched with branching subgroup $K := \langle (ab)^2, (bada)^2, (abad)^2 \rangle$.
The Quotient $Q := G/K$ is of order 16.*

Lemma 3.2 ([LMU16])**.** *Given $n \in \mathbb{N}$ and any homomorphism $\gamma\colon F_X \to Q$ with $\mathrm{supp}(\gamma) \subset \langle x_1, \ldots x_{2n} \rangle$ there is an element $\varphi \in \mathrm{Stab}(R_n) < \mathrm{Aut}(F_X)$ such that $\mathrm{supp}(\gamma \circ \varphi) \in \langle x_1, \ldots, x_5 \rangle$.*

Remark. This is now implemented in GAP: For this see the attached gap file and the function *ReduceConstraint* which can be called with one argument being either a group homomorphism from a free group to Q or a list of elements of Q. The group Q can be accessed by BranchStructure(GrigorchukGroup).group which comes with the GAP package FR [Bar14].

Lemma 3.3. *Identify the group of homomorphisms $\{\gamma\colon F_X \to Q \mid \mathrm{supp}(\gamma) \subset \langle x_1, \ldots x_6 \rangle\}$ with Q^6. Then*

$$\left| Q^6 / \mathrm{Stab}(R_3) \right| = 90.$$

Proof. This is shown by a GAP calculation. The group $\mathrm{Stab}(R_n)$ is the mapping class group of the surface group of the oriented surface of genus n and can be generated by the following automorphisms of F_{2n}:

$$
\begin{aligned}
\varphi_i\colon \quad & x_i \mapsto x_{i-1}x_i && \text{for } i = 2,4,\ldots,2n\\
\varphi_i\colon \quad & x_i \mapsto x_{i+1}x_i && \text{for } i = 1,3,\ldots,2n-1\\
\psi_i\colon \quad & x_i \mapsto x_{i+1}x_{i+2}^{-1}x_i, &&\\
& x_{i+1} \mapsto x_{i+1}x_{i+2}^{-1}x_{i+1}x_{i+2}x_{i+1}^{-1}, &&\\
& x_{i+2} \mapsto x_{i+1}x_{i+2}^{-1}x_{i+2}x_{i+2}x_{i+1}^{-1}, &&\\
& x_{i+3} \mapsto x_{i+1}x_{i+2}^{-1}x_{i+3} && \text{for } i = 1,3,\ldots,2n-3
\end{aligned}
$$

Now the GAP included standard orbit enumeration algorithm OrbitsDomain can be used to to compute the orbit and verify its size.
To check this the function verifyLemma66orbits from the file *verify.g* can be used. ☐

Definition 3.4. Fix some representative system $\mathfrak{R}$ of the above 90 orbits and for $\gamma\colon F_X \to Q$ with finite support denote by φ_γ the G-homomorphism in $\mathrm{Stab}(R_{\mathbb{N}})$ such that $\gamma \circ \varphi_\gamma \in \mathfrak{R}$.
The element $\gamma \circ \varphi_\gamma$ will be called reduced constraint.

Lemma 3.5. *The solvability of a constrainted equation $(R_n g, \gamma)$ is equivalent to the solvability of $(R_n g, \gamma \circ \varphi_\gamma)$.*

Proof. If s is a solution for $(R_n g, \gamma)$ then $s \circ \varphi_\gamma$ is a solution for $(R_n g, \gamma \circ \varphi_\gamma)$. ☐

Definition 3.6 ([Bar13])**.** A *branch structure* of a group $G \hookrightarrow G \wr P$ consists of

- a branching subgroup $K \trianglelefteq G$ of finite index.
- the corresponding quotient $Q = G/K$ and the factor homomorphism $\pi\colon G \to Q$.
- A group $Q_1 \subset Q \wr P$ such that $\langle\!\langle q_1, q_2\rangle\!\rangle\sigma \in Q_1$ if and only if $\langle\!\langle g_1, g_2\rangle\!\rangle\sigma \in G$ for all $g_i \in \pi^{-1}(q_i)$.
- A map $\omega\colon Q_1 \to Q$ with the following property. If $g = \langle\!\langle g_1, g_2\rangle\!\rangle\sigma \in G$ then $\omega(\langle\!\langle \pi(g_1), \pi(g_2)\rangle\!\rangle\sigma) = \pi(g)$.

Lemma 3.7. *The Grigorchuk group has a branch structure.*

Proof. This can be seen in [Bar13]. In GAP the branch structure can be computed by the method BranchStructure(GrigorchukGroup) which is included in the FR package. ☐

Theorem 3.8 ([LMU16])**.** *The Grigorchuk group has finite commutator width. That is there exists an $N \in \mathbb{N}$ such that for all $g \in G'$ the equation $R_N g$ is solvable.*

Remark. This is not true for all constrainted equations: For example

$$\Big(R_n(ab)^2, (\gamma\colon x_i \mapsto 1\ \forall i)\Big)$$

is not solvable for any n because otherwise it would be $ac, ca \in G'$ which is impossible since ac has activity.

3.1 Good Pairs

The previous remark motivates the following definition.

Definition 3.9. Given $g \in G'$ and $\gamma \in \mathfrak{R}$. The tuple (g, γ) is called a *good pair* if there exists an n such that $(R_n g, \gamma)$ is solvable.

Lemma 3.10. *Denote by*

$$\tau\colon G \to {}^{G}\!/_{K'} \quad \text{and} \quad \varpi\colon {}^{G}\!/_{K'} \to {}^{G/K'}\!/_{K/K'} \simeq {}^{G}\!/_{K}$$

the natural projections.
The pair (g, γ) is a good pair if and only if there is a solution $s\colon F_X \to G/K'$ for $R_3 g^\tau$ with $s(x_i) \in \varpi^{-1}(\gamma(x_i))$.

Proof. If (g, γ) is a good pair and s a solution for $R_n g, \gamma$ then $s(x_i) \in K$ for $i \geq 6$, so $s(R_n) = s(R_3) \cdot k'$ for some $k' \in K'$. Therefore there is a solution $\tau \circ s$ for $R_3 g^\tau$ with $s(x_i) = \gamma(x_i)$.
On the other hand if there is a solution $s\colon F_X \to G/K'$ for $R_3 g^\tau$ with for each $s(x_i) \in \varpi^{-1}(\gamma(x_i))$ then for $g_i \in \tau^{-1}(s(x_i))$ there is some $k' \in K'$ such that $R_n(x_1, \ldots, x_6) g k' = 1$ and so (g, γ) is a good pair. □

The previous lemma shows that the question if (g, γ) is a good pair depends only on the image $q = g^\tau$ in G/K' . So (q, γ) will be called a good pair if (g, γ) is a good pair for one (and hence all) preimages of q under τ.

Corollary 3.11. *The following are equivalent:*

a) *K is of finite commutator width.*

b) *There is a $N \in \mathbb{N}$ uniform for all good pairs $(g, \gamma), g \in G', \gamma \in \mathfrak{R}$ such that $(R_N g, \gamma)$ is solvable.*

Proof. First the easy direction: If $k \in K'$ then $(k, \mathbb{1})$ is a good pair. So $(R_n k, \mathbb{1})$ is solvable in G for an $n \leq N$ but the constraints ensures that it is solvable in K. Therefore the commutator width of K is at most N.
If (g, γ) is a good pair there is an $m \in \mathbb{N}$ and a solution s for $R_m g, \gamma$. As $s(x_i)^\pi = 1$ for all $i \geq 6$ there is some $k \in K'$ such that s is a solution for $R_3 k g, \gamma$. By a) there is an N such that all $k \in K'$ can be written as product of N commutators of elements of K and therefore there is a solution for $(R_{N+3} g, \gamma)$. □

This motivates to study K' and G/K' further.

Lemma 3.12. *Denote by* $k_1 := (ab)^2, k_2 := \langle\!\langle 1, k_1 \rangle\!\rangle = (abad)^2$ *and* $k_3 := \langle\!\langle k_1, 1 \rangle\!\rangle = (bada)^2$ *then*

$$\begin{aligned}
G' &= \left\langle k_1, k_2, k_3, (ad)^2 \right\rangle, \\
K &= \langle k_1, k_2, k_3 \rangle, \\
K \times K &= \{\langle\!\langle k, k' \rangle\!\rangle \mid k, k' \in K\} \\
&= \left\langle k_2, k_3, k_2 k_1^{-1} k_2^{-1} k_1, (k_2 k_1^{-1} k_2^{-1} k_1)^a, k_2 k_1 k_2 k_1^{-1}, (k_2 k_1 k_2 k_1^{-1})^a \right\rangle, \\
K' &= \langle [k_1, k_2] \rangle^G \\
&= \Big\langle (dacabaca)^2 (baca)^4, ((ca)^2 baca)^2, (dacabaca)^2 c(acab)^3 acad, \\
&\qquad ((ac)^3 ab)^2, bacadacab(ac)^2 (acab)^3, (acadacab)^2 (acab)^4 \Big\rangle^{1,a}.
\end{aligned}$$

Furthermore we have this chain of indices:

$$[G : G'] = 8, \quad [G' : K] = 2, \quad [K : K \times K] = 4, \quad [K \times K : K'] = 16.$$

Proof. This is shown in [BGS03] and can be verified using the GAP standard methods NormalClosure and Index. $\square$

3.2 Succesing pairs

Definition 3.13. We define the activity of an element $q \in Q$ as the activity of an arbitrary element of $\pi^{-1}(q)$. This is well defined as $K < \mathrm{Stab}(1)$.
Consider a constraint $\gamma \colon F_X \to Q$. Define $\mathcal{A}\mathbf{ct}(\gamma) := x \mapsto \mathcal{A}\mathbf{ct}(\gamma(x))$.
Denote by $\mathfrak{R}_{act}$ the reduced constraints which have a nontrivial activity.

Lemma 3.14. *For each* $q \in G'/K'$ *there is* $\gamma \in \mathfrak{R}_{act}$ *such that* (q, γ) *is a good pair.*

Proof. This is a finite problem which can be checked in GAP with the function verifyLemmaExistGoodGammas. $\square$

Denote by S the set $\{1, a, b, c, d, ab, ad, ba\} \subset G$. We will define a map Γ^q which maps a constraint $\gamma \colon F_{2n} \to Q$ with nontrivial activity to a a set of constraints $\gamma' \colon F_{4n-1} \to Q$ with the following properties:

- There is a generator $y_{\gamma'}$ in F_{4n-1} and $x \in S$ such that $\gamma'(y_{\gamma'}) = x^\pi$. Denote by F_{4n-2} the subgroup of F_{4n-1} without the generator $y_{\gamma'}$.
- The solvability of $(R_{2n-1}(g@2)^x \cdot g@1, \gamma'|_{F_{4n-2}})$ implies the solvability of $(R_n g, \gamma)$.

We will define this map in several steps and afterwards show that for all good pairs (q, γ) and all g such that $g^\tau = q$ there is some constraint $\gamma' \in \Gamma^q(\gamma)$ such that $((g@2)^x \cdot g@1, \gamma')$ is a good pair.

For the first step take the branching structure (K, Q, π, Q_1, ω) of the Grigorchuk group as before and complement the set S to a transversal S' of G/K and denote by $\mathrm{rep}\colon Q \to S'$ the map such that $\mathrm{rep}(q)^\pi = q$.
Denote $x_i \in X$ such that $\mathrm{supp}(\gamma) \subset \langle x_1, \ldots, x_{2n} \rangle$ and choose some other $y_1, \ldots, y_{4n} \in X \setminus \{x_1, \ldots, x_{2n}\}$. Now define

$$\Gamma_1(\gamma) = \{\gamma'\colon \langle y_1, \ldots, y_{4n} \rangle \to Q \mid \langle\!\langle \gamma'(y_{2i-1}), \gamma'(y_{2i}) \rangle\!\rangle \in w^{-1}(\gamma(x_i)), i = 1 \ldots 2n\}.$$

Let $F_1 = \langle g \rangle, F_2 = \langle g_1, g_2 \rangle$ be free groups. Now define a homomorphism

$$\begin{aligned} \Phi_\gamma\colon F_X * F_1 &\to (F_X * F_2) \wr C_2 \\ g &\mapsto \langle\!\langle g_1, g_2 \rangle\!\rangle, \\ x_i \qquad &\mapsto \langle\!\langle x_i^{(1)}, x_i^{(2)} \rangle\!\rangle \mathcal{A}\mathbf{ct}(\gamma(x_i)). \end{aligned}$$

Take $q_1, q_2 \in Q, n \geq 3 \in \mathbb{N}$ arbitrary and define

$$\Gamma_2^{q_1,q_2,n}(\gamma) = \left\{ \gamma' \in \Gamma_1(\gamma) \,\middle|^{\pi\colon F_2 \to Q}_{\substack{g_1 \mapsto q_1 \\ g_2 \mapsto q_2}}, (\gamma' * \pi)^2(\Phi_\gamma(R_n g)) = \langle\!\langle 1, 1 \rangle\!\rangle \right\}.$$

Denote by $v, w = v_n, w_n$ the elements such that $\Phi_\gamma(R_n g) = \langle\!\langle v, w \rangle\!\rangle \langle\!\langle g_1, g_2 \rangle\!\rangle$. By the following Lemma 3.17 there is $x_0 \in X \cup X^{-1}$ such that $v = v_1 x_0 v_2$ and $w = w_1 x_0^{-1} w_2$. Then the homomorphism

$$l_{x_0}\colon F_X * F_2 \to F_X * F_2, x_i \mapsto \begin{cases} x_i & \text{if } x_i \neq x_0 \\ w_2 g_2 w_1 & \text{if } x_i = x_0 \end{cases}$$

maps $v g_1 \mapsto v_1 w_2 g_2 w_1 v_1 g_1$ and $w g_2 \mapsto 1$. For $\gamma' \in \Gamma_2^{q_1,q_2,n}(\gamma)$ it is $(\gamma' * \pi)(x_0) = (\gamma' * \pi)(w_2 g_2 w_1)$ so with $X' = X \setminus x_0$ there is no loss of information if we consider $\gamma'|_{F_{X'}}$ instead of γ'. From section 2.2 remember the normalization automorphism $\mathfrak{nf}_{\gamma,n,x_0} := \mathfrak{nf}_{v_1 w_2 g_2 w_1 v_1 g_1}\colon F_{X'} * F_2 \to F_{X'} * F_2$ and note that $\mathfrak{nf}_{\gamma,n,x_0}(l_{x_0}(v)) = R_{2n-1} g_2^{y_{\gamma'}} g_1$ for some generator $y_{\gamma'} \in X$. This leads to the following definition.

$$\Gamma_3^{q_1,q_2,n,x_0}(\gamma) = \left\{ \gamma'|_{X'} \circ \mathfrak{nf}_{\gamma,n,x_0} \,\middle|\, \gamma' \in \Gamma_2^{q_1,q_2,n} \right\}.$$

A solution for the the constrainted equation

$$(R_{2n-1}(g@2)^{\mathrm{rep}(\gamma''(y_{\gamma''}))}(g@1), \gamma'') \text{ for } \gamma'' \in \Gamma_3^{(g@1)^\pi,(g@2)^\pi,n,x_0}$$

can be extended by the map $y_{\gamma''} \mapsto \mathrm{rep}(\gamma''(y_{\gamma''}))$ to a solution s' of the equation $(R_{2n-1}(g@2)^{y_{\gamma'}} g@1, \gamma'')$. The map $s' \circ \mathfrak{nf}_{\gamma,n,x_0}^{-1}$ is a solution for the constrainted equation $(v_1 w_2 g_2 w_1 v_1 g_1, \gamma'|_{X'} := \gamma'' \circ \mathfrak{nf}_{\gamma,n,x_0}^{-1})$. Which can be extended by the mapping $x_0 \mapsto w_2(g@2)w_1$ to a solution s of $(\Phi_\gamma(R_n g), \gamma')$. By definition of ω it is $t_i := \langle\!\langle s(y_{2i-1}), s(y_{2i}) \rangle\!\rangle \mathcal{A}\mathbf{ct}(\gamma(x_i)) \in G$ for all i. So the mapping $x_i \mapsto t_i$ is a solution for $(R_n g, \gamma)$.
The map $\Gamma_3^{q_1,q_2,n,x_0}$ does not depend on the value of n: Assume $m < n$ then choose fitting v, w such that $\Phi_\gamma(R_m g) = \langle\!\langle v, w \rangle\!\rangle \langle\!\langle g_1, g_2 \rangle\!\rangle$ then $\Phi_\gamma(R_n g) =$

$\langle\!\langle v,w\rangle\!\rangle\langle\!\langle R_{n-m}g_1, R'_{n-m}g_2\rangle\!\rangle$ then after applying the homomorphism l_{x_0} the word which needs to be normalized is $v_1w_2R_{n-m}g_2w_1v_1R'_{n-m}g_1$. The automorphisms

$$\begin{array}{llll} \psi_1\colon F_X * F_2 \to F_X * F_2, & \psi_2\colon F_X * F_2 \to F_X * F_2 & & \\ \qquad y \mapsto y^{g_1^{-1}}, & \qquad y \mapsto y^{g_2^{x11}g_1}, & \text{for } y \in \mathrm{Var}(R'_{n-m}) \\ \qquad z \mapsto z^{(g_2w_1v_1g_1)^{-1}} & \qquad z \mapsto z^{g_2^{x11}g_1} & \text{for } z \in \mathrm{Var}(R_{n-m}) \\ \qquad x \mapsto x & & \text{for all other generators} \end{array}$$

have the property that $\mathfrak{nf}_{v_1w_2R_{n-m}g_2w_1v_1R'_{n-m}g_1} = \psi_2 \circ \mathfrak{nf}_{v_1w_2g_2w_1v_1g_1} \circ \psi_1$ and $\gamma' \circ \psi_i = \gamma'$. Hence $\Gamma_3^{q_1,q_2,n,x_0}$ is independent of n.
The map $\Gamma_3^{q_1,q_2,n,x_0}$ does depend on x_0, therefore we take the union of all of them and define

$$\Gamma_4^{q_1,q_2}(\gamma) \in \bigcup_{x_0 \in \mathrm{Var}(v)\cap\mathrm{Var}(w)} \Gamma_3^{q_1,q_2,n,x_0}(\gamma).$$

Note now that $q_1, q_2 \in Q$ are determined by $q \in G'/K'$ in the sense that there is a map $\bar{@}i\colon G'/K' \to Q$ such that if $g^\tau = q$ and $g_i = g@i$ then $q_i = q\bar{@}i$ (Lemma 3.20). So we can write $\Gamma_4^{q_1,q_2}$ as Γ_4^q instead and filter those constraints out which doesn't fulfill the requested properties and therefore finally define

$$\Gamma^q(\gamma) := \left\{\gamma' \in \Gamma_4^q(\gamma) \mid \mathcal{A}\mathbf{ct}(\gamma') \neq 1, \gamma'(y_{\gamma'}) \in S^\pi\right\}.$$

Proposition 3.15. *For each good pair (q,γ) with $q \in G'/K'$ and $\gamma \in \mathfrak{R}_{act}$ the set $\Gamma^q(\gamma)$ contains some γ' with special generator $y_{\gamma'}$ such that for all g with $g^\tau = q$ the pair $\left((g@2)^{\mathrm{rep}(\gamma'(y_{\gamma'})} \cdot g@1, \gamma'\right)$ is a good pair.*

Proof. In the construction before it is clear that the sets Γ_3^{q,x_0} are nonempty and for the finitely many $\gamma \in \mathfrak{R}_{act}$ it is easy to check whether some of the finitely many $\gamma' \in \bigcup_{x_0}\Gamma_3^{q,x_0}(\gamma)$ fulfill $\gamma'(y_\gamma) \in S^\pi$ and $\mathcal{A}\mathbf{ct}(\gamma') \neq 1$.
Define for $h \in G$ maps $p_h\colon G \to G$ by $g \mapsto (g@2)^h \cdot g@1$ this maps are in general not homomorphisms but by Lemma 3.18 we see that for $g \in G'$ that $p_h(g) \in G'$ for all $h \in G$ thus there is a chance that these elements form good pairs with the correct choices of $\gamma' \in \Gamma^q(\gamma)$.
By Lemma 3.19 we can define the map $\bar{p}_h\colon G'/K' \to G'/(K \times K)$ and the natural homomorphism

$$\varpi'\colon {G'}/{K'} \to {\left(G'/K'\right)}/{\left(K \times K/K'\right)} \simeq {G'}/{K \times K}$$

and now only need to show that there is a $\gamma' \in \Gamma^q(\gamma)$ such that all preimages of $\bar{p}_{\mathrm{rep}(\gamma'(y_{\gamma'})}(q)$ under ϖ' form good pairs with γ'. In formulas what needs to be checked is: Let $\mathcal{G}$ be the predicate of being a good pair.

$$\forall q \in {G'}/{K'}\ \forall\gamma \in \mathfrak{R}_{act}\exists\gamma' \in \Gamma^q(\gamma)\forall r \in \varpi'^{-1}(\bar{p}_{\mathrm{rep}(\gamma'(y_{\gamma'}))}) : \mathcal{G}(q,\gamma) \Rightarrow \mathcal{G}(r,\gamma').$$

This last formula quantifies only over finite sets and is implemented in GAP and can be verified there by calling verifyPropExistsSuccessor. □

Definition 3.16. For each fixed $q \in G'/K'$ and $\gamma \in \mathfrak{R}_{act}$ such that (q, γ) is a good pair fix a constraint $\gamma' \in \Gamma^q(\gamma)$ and the element $x = \text{rep}\gamma'(y_{\gamma'}) \in S$ which exist by the previous proposition.
With Lemma 3.5 we can assume that $\gamma'|_{F_{X \setminus y_{\gamma'}}}$ can be replaced by a reduced constraint γ'_r. For a good pair $(g, \gamma) \in G' \times \mathfrak{R}_{act}$ the *succesing pair* is defined as $((g@2)^x g@1, \gamma'_r)$. Moreover by applying this iteratively we get the *succesing sequence* (g_k, γ_k) of (g, γ).

Lemma 3.17. *If γ is a constraint with nontrivial activity, and $\Phi_\gamma(R_n g) = \langle\!\langle w_1, w_2 \rangle\!\rangle$ then $\text{Var}(w_1) \cap \text{Var}(w_2) \neq \emptyset$.*

Proof. Let x be generator of F_X with non vanishing constraint activity. Then R_n contains either a factor $[x, y]$ or $[y, x]$ for another generator y. Assume without loss of generality the first case. Let further be $\Phi_\gamma(x) = \langle\!\langle x_1, x_2 \rangle\!\rangle(1, 2)$ and $\Phi_\gamma(y) = \langle\!\langle y_1, y_2 \rangle\!\rangle\sigma$. Then $\Phi_\gamma(R_n g)$ contains a factor

$$[\langle\!\langle x_1, x_2 \rangle\!\rangle(1, 2), \langle\!\langle y_1, y_2 \rangle\!\rangle\sigma] = \begin{cases} \langle\!\langle x_2^{-1} y_2^{-1} x_2 y_1, x_1^{-1} y_1^{-1} x_1 y_2 \rangle\!\rangle & \text{if } \sigma = \mathbb{1} \\ \langle\!\langle x_2^{-1} y_1^{-1} x_1 y_2, x_1^{-1} y_2^{-1} x_2 y_1 \rangle\!\rangle & \text{if } \sigma = (1, 2). \end{cases}$$

So in both cases $y_1, y_2 \in \text{Var}(w_1) \cap \text{Var}(w_2)$. □

Lemma 3.18. *Let $h \in G$ and $p_h \colon G \to G$ be the map $g \mapsto \left((g@2)^h \cdot g@1, \gamma'''\right)$. It holds that $p_h(G') \subset G'$ for all $h \in G$ and $p_1(K) \subset K$.*

Proof. Denote first by $p := p_1$ then each element $g \in G'$ is a word in generators $w((ab)^2, (abad)^2, (bada)^2, (ad)^2)$. The generators have the following form:

$$(ab)^2 = \langle\!\langle ca, ac \rangle\!\rangle, (abad)^2 = \langle\!\langle 1, (ab)^2 \rangle\!\rangle, (bada)^2 = \langle\!\langle (ab)^2, 1 \rangle\!\rangle, (ad)^2 = \langle\!\langle b, b \rangle\!\rangle.$$

Therefore it is

$$\begin{aligned} p(g) &= w(ac, (ab)^2, 1, b) \cdot w(ca, 1, (ab)^2, b) \\ &\equiv w(ac, 1, 1, 1) \cdot w(ca, 1, 1, 1) \cdot w(1, 1, 1, b)^2 \equiv 1 \mod G'. \end{aligned}$$

For $h \in G$ it is $p_h(g) = [h, (g@2)^{-1}]p(g)$ and therefore $p_h(g) \in G'$ for all $g \in G'$.
An element $g \in K$ is a word $w((ab)^2, (abad)^2, (bada)^2,)$ and therefore

$$\begin{aligned} p(g) &= w(ac, (ab)^2, 1) \cdot w(ca, 1, (ab)^2) \\ &\equiv w(ac, 1, 1) \cdot w(ca, 1, 1) \equiv 1 \mod K. \end{aligned}$$

□

Lemma 3.19. *The map*

$$\begin{aligned} \bar{p}_h \colon G'/K' &\to G'/K \times K \\ gK' &\mapsto \left(g@2)^h \cdot g@1\right) K \times K \end{aligned}$$

is well defined.

Proof. It's easy to verify by GAP that $k@i \in K \times K$ for $i = 1, 2$ and $k \in K'$ using Lemma 3.12. For check this call the function verifyLemmaStatesOfKPinKxK of the attached GAP file. Then for $k \in K'$ it is

$$p_h(gk) = ((gk)@2)^h \cdot (gk)@1 = (g@2)^h \cdot (k@2)^h \cdot g@1 \cdot k@1 \in (g@2)^h \cdot g@1K \times K.$$

$\square$

Lemma 3.20. *The maps* $@i \colon G \to G, g \mapsto g@i$ *induce well defined maps* $\bar{@}i \colon G/K' \to G/K$

Proof. As before it is $k'@i \in K \times K < K$. $\square$

3.3 Product of 3 commutators

We will prove that every element $g \in G'$ is a product of three commutators by proving that all sequences (g_k, γ_k) as defined after Proposition 3.28 are finite. For this purpose remember the map $p_x \colon g \mapsto (g@2)^x g@1$ from the proof of Proposition 3.28. We will show that for each $g \in G'$ the sequence of sets

$$\mathrm{Suc}_1^g = \{g\},\ \mathrm{Suc}_n^g = \{p_x(h) \mid h \in \mathrm{Suc}_{n-1}^g, x \in S\}$$

stagnates in a finite set.
In [Bar98] there is a choice of weights on generators which result in a length on G with good properties.

Lemma 3.21 ([Bar98])**.** *Let* $\eta \approx 0.811$ *be the real root of* $x^3 + x^2 + x - 2$ *and set the weights*

$$\omega(a) = 1 - \eta^3 \qquad \omega(c) = 1 - \eta^2$$
$$\omega(b) = \eta^3 \qquad \omega(d) = 1 - \eta$$

then

$$\begin{aligned} \eta(\omega(b) + \omega(a)) &= \omega(c) + \omega(a) \\ \eta(\omega(c) + \omega(a)) &= \omega(d) + \omega(a) \\ \eta(\omega(d) + \omega(a)) &= \omega(b). \end{aligned}$$

The next lemma is a small variation of a lemma in [Bar98].

Lemma 3.22. *Denote by* ∂_ω *the length on* G *induced by the weight* ω*. Then* $\partial_\omega(p_x(g)) \leq \delta\partial_\omega(g)$ *for all* $x \in S, g \in G$ *with* $\partial_\omega(g) > C$ *some constant* $C \in \mathbb{N}, \delta < 1$*.*

Corollary 3.23. *The sequences of sets*

$$\mathrm{Suc}_1^g = \{g\},\ \mathrm{Suc}_n^g = \{p_x(h) \mid h \in \mathrm{Suc}_{n-1}^g, x \in S\}$$

stagnates at a finite step for all $g \in G$*.*

Proof of Lemma.([Bar98]). Each element $g \in G$ can be written in a word of minimal length of the form $g = a^{\varepsilon} x_1 a x_2 a \dots x_n a^{\delta}$ where $x_i \in \{b, c, d\}$ and $\varepsilon, \delta \in \{0, 1\}$. Denote by n_b, n_c, n_d the number of occurrences of b, c, d accordingly. Then

$$\begin{aligned}
\partial_\omega(g) &= (n - 1 + \varepsilon + \delta)\omega(a) + n_b\omega(b) + n_c\omega(c) + n_d\omega(d) \\
\partial_\omega(p_x(g)) &\leq (n_b + n_c)\omega(a) + n_b\omega(c) + n_c\omega(d) + n_d\omega(b) + 2\partial_\omega(x) \\
&= \eta\left((n_b + n_c + n_d)\omega(a) + n_b\omega(b) + n_c\omega(c) + n_d\omega(d)\right) + 2\partial_\omega(x) \\
&= \eta(\partial_\omega(g) + (1 - \varepsilon - \delta)\omega(a)) + 2\partial_\omega(x) \\
&\leq \eta(\partial_\omega(g) + \omega(a)) + 2(\omega(a) + \omega(b)) \\
&= \eta(\partial_\omega(g) + \omega(a)) + 2.
\end{aligned}$$

Thus the length of $p_x(g)$ growths with a linear factor smaller 1 in terms of the length of g. Therefore the claim holds. For instance one could take $\delta = 0.86$ and $C = 50$ or $\delta = 0.96$ and $C = 16$. □

This completes the proof of the following proposition

Proposition 3.24. *If $n \geq 3$ and (g, γ) is a good pair with active constraint γ with $\operatorname{supp}(\gamma) \subset \{x_1, \dots, x_{2n}\}$ then the constrainted equation $(R_n(x_1, \dots, x_{2n})g, \gamma)$ is solvable.*

Corollary 3.25. *The Grigorchuk group G has commutator width at most 3.*

Proof. This is a direct consequence of the proposition and Lemma 3.14. □

Corollary 3.26. *K has commutator width 3.*

Proof. To show that K has commutator width 3 it is sufficient, to show that the constrainted equations $(R_3 g, \mathbb{1})$ have solutions for all $g \in K'$. Since $\mathbb{1}$ has trivial activity one cannot directly apply Proposition 3.24. But one can check that all pairs $(h, \gamma_1), (f, \gamma_2)$ such that $g = \langle\!\langle h, f \rangle\!\rangle$ and $\gamma_1, \gamma_2 = (1, 1, 1, 1, (bad)^\pi, 1)$, $\gamma_2 = (1, 1, 1, 1, 1, (ca)^\pi)$ are good pairs with active constraints and hence the equations $(R_3 g, \mathbb{1})$ have solutions for all $g \in K'$. This check is implemented in the GAP function verifyCorollaryFiniteCWK. □

3.4 Product of 2 commutators

The case of products of two commutators can be reduced to the case of three commutators by using the same method as before.
We can compute the orbits of $\operatorname{Aut}(F\_4)/\!\operatorname{Stab}(R_2)$ and take a representative system denoted by $\mathfrak{R}^4$. It turns out that there are 86 orbits and we can check that there are again enough active constraints:

Lemma 3.27. *For each $q \in G'/K'$ there is $\gamma \in \mathfrak{R}^4_{act}$ such that (q, γ) is a good pair.*

Proof. This can be checked in GAP with the function verifyLemmaExistGoodGammasForRed4
. ☐

Analog to Proposition 3.28 we can formulate the following proposition.

Proposition 3.28. *For each good pair* (q, γ) *with* $q \in G'/K'$ *and* $\gamma \in \mathfrak{R}^4_{act}$ *the set* $\Gamma^q(\gamma)$ *contains some* γ' *with special generator* $y_{\gamma'}$ *such that for all* g *with* $g^\tau = q$ *the pair* $\left((g@2)^{\mathrm{rep}(\gamma'(y_{\gamma'})} \cdot g@1, \gamma'\right)$ *is a good pair.*

Proof. ☐

The resulting succesing pairs are now equations of genus 3 with an active constraint. Those are already shown to be solvable by 3.24. So together with 3.27 this proves the following corollary.

Corollary 3.29. *The Grigorchuk group* G *has commutator width at most* 2*.*

3.5 Not every element is a commutator

The previous procedure can not be used to prove that each element is a commutator since for equations of genus 1 the genus does not increase by passing to a succesing pair.
In fact not every element $g \in G'$ is a commutator. This can be seen by passing to finite quotients. If every element would be a commutator then it would be a commutator in the quotient group. For example the element $d(ac)^2ada$ is not a commutator in the third level.

Corollary 3.30. *All elements* $g \in G'$ *are products of two commutators.*

4 Implementation in GAP

4.1 Usage of the attached files

Together with this document there come some files which contain the algorithms used for some proofs. The file *verify.g* is meant to be the starting point. The file contains the main methods to explore the results. After reading this file the following functions can be used:

- ReducedConstraint: Given a group homomorphism $\varphi\colon F_{2n} \to Q$ this function returns a reduced constraint. Example:

```
f1 := Q.1; f2 := Q.2;
gamma:= [f1,f1,f1,f1,f1,f1];
constr := ReducedConstraint(gamma);;
Print(constr.constraint);
gamma := GroupHomomorphismByImages(FreeGroup(6),Q,[f2,f2,
    f2,f2,f2,f2]);
constr := ReducedConstraint(gamma);;
Print(constr.constraint);
```

4.2 Implementation details

Lemma 3.20

Lemma ??

References

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