The $3n+3^k$ problem

The Gypsy Cossack

Abstract

The Collatz conjecture is the $3n+3^k$ problem when $k = 0$. The full statement is: Every $3n+3^k$ sequence, $\forall k \in \mathbb{N}_0$ contains the term 3^k. In Claim 1, there is a simple formula to construct every $3n+1$ sequence from its corresponding $3n+3$ sequence. In Claim 2, another formula can construct every $3n+3^r$ sequence from any other corresponding $3n+3^k$ sequence where $0 \leq k < r$, $\forall\ k, r \in \mathbb{N}_0$. We prove all $3n+d$ cases.

Preliminaries and Definitions

Definition 0.1. *A $3n+3^k$ sequence, $\forall k \in \mathbb{N}_0$ can be defined as $f : \mathbb{N} \longrightarrow \mathbb{N}$*

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n + 3^k & \text{if } n \text{ is odd} \end{cases}$$

Then, given any positive integer m, define a sequence such that $a_1 = m$ and for $i \geq 1$, $a(i+1) = f(a_i)$.

Similarly,

Definition 0.2. *A $3n+d$ sequence, $\forall d \in \mathbb{N}_{odd}$ can be defined as $f : \mathbb{N} \longrightarrow \mathbb{N}$*

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n + d & \text{if } n \text{ is odd} \end{cases}$$

We call any sequence constructed as defined above, a c-sequence. We shall denote three terms for a c-sequence. For simplicity, we'll choose the first odd term to be the initial term of a sequence, denoted $\mathbf{m} \in \mathbb{N}_{odd}$. A term that is not the first term and is odd by N_i. And an even term that is of the form: $3(N_i)+d$ (where d could be 3^k), by M_i. Note well that the indexing is not meant to track *steps* in a c-sequence, but rather for association only. It will be seen that roughly two thirds of the numbers generated in a sequence, are not relevant. That is, we are only interested in the numbers defined as M_i.

Definition 0.3. *A $\mathcal{C}$-sequence is the sequence of all the even numbers from a c-sequence of the form:* $3(N_i)+d = M_i \ \forall i \in \mathbb{N}_{>0}$ *where* $N_i \in \mathbb{N}_{odd}$.

Given a $\mathcal{C}$-sequence, its c-sequence from M_1, may be constructed with just divisions by 2.

Claim 1: Let $M_i = 3(N_i) + 3$ and $M_i' = 3(N_i') + 1$ be for $3n+3$ and $3n+1$, respectively. Further, let $\mathbf{m}$ and $\mathbf{m}'$ be the first odd terms for the respective sequences where $\mathbf{m} = 2(\mathbf{m}') - 1$. Then $3(M_{i-1}') = M_i \quad \forall\ \mathbf{m}, \mathbf{m}' \in \mathbb{N}_{odd}$.

Proof: Given a pair, $\mathbf{m}$, $\mathbf{m}'$ we have $\mathbf{m} = 2(\mathbf{m}') - 1$. Let $M_1' = 3(\mathbf{m}') + 1$. Then by the claim, $M_2 = 9(\mathbf{m}') + 3$.

We also have that: $M_1 = 3(\mathbf{m}) + 3 = 3(2\mathbf{m}' - 1) + 3 = 6(\mathbf{m}') - 3 + 3 = 6(\mathbf{m}')$. So, $N_2 = \frac{6(\mathbf{m}')}{2} = 3(\mathbf{m}')$ where $3(N_2) + 3 = M_2 = 9(\mathbf{m}') + 3$. It follows that $3(M_{i-1}') = M_i \ \forall i \in \mathbb{N}_{>0}$. This is equivalent to $\frac{M_i}{2^k}+1 = N_{i+1}+1 = M_i'$ where k is the number of 2 multiples of M_{i-1}' and of M_i.

In Appendix 1, we show the $3n+1$ c-sequence for $\mathbf{m}' = 27$. Below that, we show its associated $\mathcal{C}$-sequence ($M_i' = 3(N_i')+1$), where the terms are coloured wrt. their *desig* [1]. Followed by the conversion of the $\mathcal{C}$-sequence for the $3n+3$ sequence with $\mathbf{m} = 53$ to the $\mathcal{C}$-sequence for the $3n+1$ sequence with $\mathbf{m}' = 27$.

[1] a *desig* is one of three possible assignments for any even number, see Law 1.

Claim 2: Let $M_i = 3(N_i) + 3^r$ and $M_i' = 3(N_i') + 3^k$ be for $3n{+}3^r$ and $3n{+}3^k$ respectively, where $0 \leq k < r$. Further, let $\mathbf{m}$ and $\mathbf{m'}$ be the first odd terms for the respective sequences where $\mathbf{m} = 3^{r-k}(\mathbf{m'})$. Then $3^{r-k}(M_i' = 2^a 3^b R) = M_i = 2^a 3^{b+r-k} R \quad \forall\ \mathbf{m}, \mathbf{m'}, R \in \mathbb{N}_{odd}$ and $a, b \geq 1$.

Proof: Given a pair, $\mathbf{m}$, $\mathbf{m'}$ we have $\mathbf{m} = 3^{r-k}(\mathbf{m'})$. Let $M_1' = 3(\mathbf{m'}) + 3^k$ and $M_1 = 3(\mathbf{m}) + 3^r$. It follows that $M_1 = 3(3^{r-k}\mathbf{m'}) + 3^r$. Then by dividing by 3^{r-k}, we have $\frac{M_1}{3^{r-k}} = 3\mathbf{m'} + \frac{3^r}{3^{r-k}}$. This reduces to, $\frac{M_1}{3^{r-k}} = 3\mathbf{m'} + \frac{1}{3^{-k}}$ giving, $\frac{M_1}{3^{r-k}} = 3\mathbf{m'} + 3^k = M_1'$. And so, $M_1 = 3^{r-k}(M_1')$. It follows that $M_1 = 3^{r-k}(2^a 3^b R) = 2^a 3^{b+r-k} R$. The argument is the same for every pair N_i and N_i'.

Observe that the correspondence between sequences in Claim 1 is just wrt. $3n{+}1$ and $3n{+}3$. Whereas the correspondence between sequences in Claim 2 is for all $3n{+}3^k$ sequences, $\forall k \in \mathbb{N}_0$. What Claims 1 and 2 establish is that the Collatz conjecture is true if and only if the $3n{+}3^k$ conjecture is true.

Definition 0.4. *A $\mathcal{D}$-sequence is constructed from a $\mathcal{C}$-sequence for a $3n{+}d$ sequence by $M_i{+}2(d)$ $\forall i \in \mathbb{N}_{>0}$. The terms of a $\mathcal{D}$-sequence are denoted by t_i.*

1 Law 1: M_i terms of $3n{+}d$ sequence generation

For any $3n{+}d$ sequence, an M_i can have only one of three *designations* with respect to $+2(d)$ and $-2(d)$, as follows:

i) If M_i is divisible by 2, two or more times, then it's $\overset{\bullet}{M}_i$. It will always be such that $M_i{+}2(d)$ and $M_i{-}2(d)$ will be divisible by 2 exactly once, in this case.

ii) If M_i is divisible by 2 exactly once and $M_i{-}2(d)$ is divisible by 2, three or more times, then it's $\overset{\blacksquare}{M}_i$. It will always be such that $M_i{+}2(d)$ will be divisible by 2 exactly twice, in this case.

iii) If M_i is divisible by 2 exactly once and $M_i{+}2(d)$ is divisible by 2, three or more times, then it's $\overset{\blacktriangle}{M}_i$. It will always be such that $M_i{-}2(d)$ will be divisible by 2 exactly twice, in this case.

Every M_i must be one of the three *designations* or *desig*, listed above. The *desig* cases are proved in Appendix 2 (the law applies to all even $\mathbb{N}$). Note well that we use the *desig* coloured circle, square and triangle with the terms, t_i of the $\mathcal{D}$-sequence and to the left of down pointing arrows between an M_i and its associated t_i in our example for $\mathbf{m} = 53$, in Appendix 4. It's to be understood that the *desig* is always for M_i and that it's just an association with $t_i = M_i + 2(d)$.

Definition 1.1. *An R-subsequence or R-sub begins with a* $\overset{\blacktriangle}{t}_i$ *if we have* $\overset{\bullet}{t}_{i-1}$ *or* $i = 1$*, and ends with the last term* $\overset{\bullet}{t}_k$ *preceding* $\overset{\blacktriangle}{t}_{k+1}$ *or the* $\mathcal{C}$*-sequence terminated at* $\overset{\bullet}{M}_k$*. An R-sub has two parts, called the head and the tail. The head contains only* $\overset{\blacktriangle}{t}_i$ *terms and the tail contains only* $\overset{\blacksquare}{t}_j$ *and* $\overset{\bullet}{t}_k$ *terms.*

R-subs are always contiguous and we index as: R-sub$_1$, R-sub$_2$, R-sub$_3$, ... For example, a $\mathcal{D}$-sequence could have initially, terms: $\overset{\bullet}{t}_1$, $\overset{\bullet}{t}_2$, $\overset{\blacksquare}{t}_3$, $\overset{\bullet}{t}_4$, $\overset{\blacktriangle}{t}_5$. Its *associated* $\mathcal{C}$-sequence is of course: $\overset{\bullet}{M}_1$, $\overset{\bullet}{M}_2$, $\overset{\blacksquare}{M}_3$, $\overset{\bullet}{M}_4$, $\overset{\blacktriangle}{M}_5$. In which case the first R-sub begins with $\overset{\blacktriangle}{t}_5$. Some $\mathcal{C}$-sequences may have no $\overset{\blacktriangle}{M}_i$ terms and would simply be a tail only sequence.

Specifically, if the i^{th} term is $\overset{\blacktriangle}{t}_i$ then the $i{+}1^{th}$ term can only be another $\overset{\blacktriangle}{t}_{i+1}$ or $\overset{\blacksquare}{t}_{i+1}$. And the $i{-}1^{th}$ term can only be another $\overset{\blacktriangle}{t}_{i-1}$ or $\overset{\bullet}{t}_{i-1}$. If the i^{th} term is $\overset{\blacksquare}{t}_i$ then the $i{+}1^{th}$ term can only be $\overset{\bullet}{t}_{i+1}$. And the $i{-}1^{th}$ term can only be $\overset{\blacktriangle}{t}_{i-1}$ or $\overset{\bullet}{t}_{i-1}$. Lastly, if the i^{th} term is $\overset{\bullet}{t}_i$ then the $i{+}1^{th}$ term can be another $\overset{\bullet}{t}_{i+1}$ or $\overset{\blacksquare}{t}_{i+1}$ or $\overset{\blacktriangle}{t}_i$. The $i{-}1^{th}$ term can only be another $\overset{\bullet}{t}_{i-1}$ or $\overset{\blacksquare}{t}_{i-1}$. (proofs in Appendix 2)

2 Law 2: $2^a 3^b R$ structure for $3n+d$ sequences

All $3n+d$ sequences are such that every t_i from their *associated* $\mathcal{D}$-sequences, are divisible by at least one 2 and one 3. The $\mathcal{D}$-sequences reveal distinct structure. More precisely, every t_i can be written as: $2^a 3^b R$, where $a, b, R \in \mathbb{N}_{>0}$. R is the remainder, always odd, not divisible by 3 and not necessarily prime. Moreover, if $t_i = 2^a 3^b R$, where $a > 1$, then t_{i+1} must be

$2^{a-1}3^{b+1}R$. That is, if $M_i+2(d) = t_i = 2^a3^bR$ then $M_{i+1}+2(d) = t_{i+1} = 2^{a-1}3^{b+1}R$, if $a > 1$. We will distinguish a different R with R'.

In general, we have: $2^a3^bR \longrightarrow 2^{a-1}3^{b+1}R$ where $a, b,\ a-1,\ b+1 \in \mathbb{N}_{>0}$. In fact, every 2^a3^bR where $a \geq 2$ must continue until we have: $2^13^{b+a-1}R = t_k$ where M_k is always $\overset{\bullet}{M}_k$. If t_i is $\overset{\blacktriangle}{t}_i = M_i+2(d)$, where $\overset{\blacktriangle}{t}_i = 2^33^bR$, then t_{i+1} is $\overset{\blacksquare}{t}_{i+1} = M_{i+1}+2(d) = 2^23^{b+1}R$. It must follow that t_{i+2} is $\overset{\bullet}{t}_{i+2} = M_{i+2}+2(d) = 2^13^{b+2}R$. This implies that there are no consecutive terms as: $\overset{\blacksquare}{t}_i$, $\overset{\blacksquare}{t}_{i+1}$. A tail can be at best, an alternating sequence as: $\overset{\blacksquare}{t}_i$, $\overset{\bullet}{t}_{i+1}$, $\overset{\blacksquare}{t}_{i+2}$, $\overset{\bullet}{t}_{i+3}$, $\ldots$ (proofs in Appendix 2). Since $3n+d$ sequence generation can be associated to 2^a3^bR representations for every M_i, we are able to plainly see that sequences have a particular structure. In Appendix 4, we have an example showing the $\mathcal{C}$-sequence and its associated $\mathcal{D}$-sequence for $\mathbf{m} = 53$ where $M_1 = 3(53)+3 = 162$. Observe that the sequence begins with an R-sub and in fact there are eight contiguous, in all. They are explicitly itemized in Appendix 3.

Every $\mathcal{C}$-sequence for a $3n+d$ sequence has an *associated* $\mathcal{D}$-sequence. It will be shown in this section that no $3n+d$ sequence is divergent. Thus, all $3n+d$ sequences will have a cycle, meaning that terms would be repeated.

Definition 2.1. *A* $\boldsymbol{3n+d}$ *sequence starting with* $\boldsymbol{m}$ *has a trivial cycle if* $\boldsymbol{d}$ *is one of the terms. Otherwise, the sequence has a non-trivial cycle.*

We will refer to the number of terms in a c-sequence, as the number of *steps*.

Definition 2.2. *Given an R-sub, there exists a first (f), term* $\overset{\blacktriangle}{M}_f$ *and a last (l), term* $\overset{\bullet}{M}_l$*. Then let* $X \in \mathbb{N}_{odd}$ *be* $\overset{\blacktriangle}{M}_f$ *divided by* 2 *(there is only one multiple of* 2 *by Law 1). And let* $Y \in \mathbb{N}_{odd}$ *be* $\overset{\bullet}{M}_l$ *divided by* 2^r*, where* $r \geq 2$ *(by Law 1). Then an* R^X*-sub is an R-sub such that* $\frac{2X-d}{3} < Y$*. And an* R^Y*-sub is an R-sub such that* $Y < \frac{2X-d}{3}$*. We denote an* R^X*-sub*$_i$ *as:* $(h, t \mid \overset{\blacktriangle}{t}_f = 2^a3^bR \mid \overset{\bullet}{t}_l = 2^c3^dR' or R)$ *is the* i^{th} *R-sub such that* $\frac{2X-d}{3} < Y$*, where h (head), is the number of* $\overset{\blacktriangle}{t}$ *terms and t (tail), is the number of* $\overset{\blacksquare}{t}$ *and* $\overset{\bullet}{t}$ *terms for* *this R-sub. Similarly, we denote an R^Y-sub$_i$ as:* $(h, t \mid \overset{\blacktriangle}{t}_f = 2^a 3^b R \mid \overset{\bullet}{t}_l = 2^c 3^d R' or R)$ *is the* i^{th} *R-sub such that* $Y < \frac{2X-d}{3}$. *If* $Y = \frac{2X-d}{3}$ *then it's just an R-sub.*

Example i) Below, is the $\mathcal{C}$-sequence and its associated $\mathcal{D}$-sequence for $3n+5$ with $\mathbf{m} = 11$. It has one R^X-sub, since $Y = 19 > 11 = \frac{2X-d}{3}$. It has a non-trivial cycle which is required for such a special case. ie. a lone R^X-sub.

$$\begin{tikzcd}
\mathcal{C}\rightarrow & 38 \arrow[r] \arrow[d, "\blacktriangle"'] & 62 \arrow[r] \arrow[d, "\blacktriangle"'] & 98 \arrow[r] \arrow[d, "\blacksquare"'] & 152 \arrow[r] \arrow[d, "\bullet"'] & 62 \arrow[d, "\blacksquare"'] \\
\mathcal{D}\rightarrow & 2^4 3^1 \mathbf{1} & 2^3 3^2 \mathbf{1} & 2^2 3^3 \mathbf{1} & 2^1 3^4 \mathbf{1} & 2^3 3^2 \mathbf{1}
\end{tikzcd}$$

Example ii) Below, is the $\mathcal{C}$-sequence and its associated $\mathcal{D}$-sequence for $3n+5$ with $\mathbf{m} = 19$. It has one R-sub which is neither X or Y, since $Y = \frac{2X-d}{3} = 19$. It has a non-trivial cycle which would never occur for any $3n+3^k$ sequence.

$$\begin{tikzcd}
\mathcal{C}\rightarrow & 62 \arrow[r] \arrow[d, "\blacktriangle"'] & 98 \arrow[r] \arrow[d, "\blacksquare"'] & 152 \arrow[r] \arrow[d, "\bullet"'] & 62 \arrow[d, "\blacksquare"'] \\
\mathcal{D}\rightarrow & 2^3 3^2 \mathbf{1} & 2^2 3^3 \mathbf{1} & 2^1 3^4 \mathbf{1} & 2^3 3^2 \mathbf{1}
\end{tikzcd}$$

Law 2: R-sub tails' associated M_i terms

Every consecutive pair of $\overset{\blacksquare}{M} \rightarrow \overset{\bullet}{M}$ terms have a property that distinguishes such pairs from one another. If N_{i+1} of $\overset{\blacksquare}{M}_i = X$ and N_{i+2} of $\overset{\bullet}{M}_{i+1} = Y$ and $|X - Y|$ is divisible by 2, exactly once, the term M_{i+2} must be $\overset{\blacktriangle}{M}_{i+2}$ or $\overset{\blacksquare}{M}_{i+2} > \overset{\blacksquare}{M}_i$. Or, if $|X - Y|$ is divisible by 4 or more, the term M_{i+2} must be $\overset{\bullet}{M}_{i+2}$. If N_{i+1} of $\overset{\bullet}{M}_i = X$ and N_{i+2} of $\overset{\bullet}{M}_{i+1} = Y$ and $|X - Y|$ is divisible by 2, exactly once, the term M_{i+2} must be $\overset{\blacktriangle}{M}_{i+2}$ or $\overset{\blacksquare}{M}_{i+2}$. Or, if $|X - Y|$ is divisible by 4 or more, the term M_{i+2} must be $\overset{\bullet}{M}_{i+2}$.

If N_{i+1} of $\overset{\blacksquare}{M}_i = X$ and N_{j+1} of $\overset{\bullet}{M}_j = Y$, $j > i+1$, and $|X - Y|$ is divisible by 2, exactly once, the term M_{j+1} must be $\overset{\blacktriangle}{M}_{j+1}$ or $\overset{\blacksquare}{M}_{j+1}$. If $|X - Y|$ is divisible by 4 or more, the term M_{j+1} must be $\overset{\bullet}{M}_{j+1}$.

If N_{i+1} of $\overset{\bullet}{M}_i = X$ and N_{j+1} of $\overset{\bullet}{M}_j = Y$, $j > i+1$, and $|X-Y|$ is divisible by 2, exactly once, the term M_{j+1} must be $\overset{\blacktriangle}{M}_{j+1}$ or $\overset{\bullet}{M}_{j+1}$. If $|X-Y|$ is divisible by 4 or more, the term M_{j+1} must be $\overset{\bullet}{M}_{j+1}$ or $\overset{\blacktriangle}{M}_{j+1}$ or $\overset{\blacksquare}{M}_{j+1}$. Finally, for any $\overset{\blacksquare}{M}_i \cdots \overset{\blacksquare}{M}_j \cdots \overset{\blacksquare}{M}_k$ the limit for an alternating portion is N_{t+1} of $\overset{\bullet}{M}_t < N_{i+1}$ $\forall t$, $i+1 \le t \le k+1$. Now, if we let the first associated term of a *tail* be $\overset{\blacksquare}{M}_i$ where $N_{i+1} = X$ and let $\overset{\bullet}{M}_j$ where $N_{j+1} = Y$, be another associated term of the *tail*, it's always the case that $X > Y$. Thus, $X > Y$, is a general result of Law 2.

Definition 2.3. *An H-form R-sub is denoted by* $(h, 2 \mid 2^a \overset{\blacktriangle}{3}{}^b R \mid 2^1 \overset{\bullet}{3}{}^{a+b-1} R)$, *a tail of length 2, and an L-form R-sub is denoted by* $(1, t \mid 2^3 \overset{\blacktriangle}{3}{}^b R \mid 2^1 \overset{\bullet}{3}{}^d R')$, *a head of length 1. A maximum* $\overset{max}{H}$*-form is denoted by* $(h, 2 \mid 2^a \overset{\blacktriangle}{3}{}^1 R \mid 2^1 \overset{\bullet}{3}{}^a R)$. *ie.* 3^b *where* b=1.

The L-form and the H-form could be viewed as the two extremes of R-subs since L-forms have the shortest *head*, of length one with $\overset{\blacktriangle}{t}_f = 2^3 3^b R$ and $\overset{\bullet}{t}_l = 2^1 3^d R'$ and H-forms have the shortest *tail*, of length two. In other words, they can play a role with respect to the bounds for $3n$+d generation.

Next, the R-subs are partitioned or grouped in similar fashion as the $\overset{\blacktriangle}{t}$, $\overset{\blacksquare}{t}$ and $\overset{\bullet}{t}$ terms for the R-subs were wrt. the distinction between head and tail.

Definition 2.4. *An **S**-string is a collection of contiguous R-subs: R-sub$_i$ → R-sub$_{i+1}$ → R-sub$_{i+2}$ → ⋯ → R-sub$_j$ → R-sub$_{j+1}$ → R-sub$_{j+2}$ → ⋯ → R-sub$_k$ where the R-sub$_i$, R-sub$_{i+1}$, R-sub$_{i+2}$, … , R-subs must be R^X-subs and the R-sub$_j$, R-sub$_{j+1}$, R-sub$_{j+2}$, … R-sub$_k$, R-subs must be R^Y-subs.*

The R^X-subs followed by the R^Y-subs of an **S**-string shall be the *head* and *tail*, respectively. **S**-strings are contiguous and are indexed as: **S**-string$_1$, **S**-string$_2$, **S**-string$_3$, ... See Appendix 3 for examples.

Definition 2.5. *Given an* ***S****-string there exists a first term* $\overset{\blacktriangle}{M}_f$ *and a last term* $\overset{\bullet}{M}_l$. *Then let* $X \in \mathbb{N}_{odd}$ *be* $\overset{\blacktriangle}{M}_f$ *divided by* 2. *And let* $Y \in \mathbb{N}_{odd}$ *be* $\overset{\bullet}{M}_l$ *divided by* 2^r, *where* $r \geq 2$. *Then an* $\boldsymbol{S}^X$*-string is an* ***S****-string such that* $\frac{2X-d}{3} < Y$. *And an* $\boldsymbol{S}^Y$*-string is an* ***S****-string such that* $Y < \frac{2X-d}{3}$. *We denote an* $\boldsymbol{S}^X$*-string*$_i$ *as the* i^{th} ***S****-string such that* $\frac{2X-d}{3} < Y$. *Similarly, we denote an* $\boldsymbol{S}^Y$*-string*$_i$ *as the* i^{th} ***S****-string such that* $Y < \frac{2X-d}{3}$. *If* $Y = \frac{2X-d}{3}$ *then it's just an* ***S****-string.*

Definition 2.6. *For* $3n+d$, *if* N_i *(odd), is such that* $2(N_i) - d$ *is not divisible by* 3 *and* $4(N_i) - d$ *is not divisible by* 3, *then* N_i *is a dead-ender.*

A *dead−ender* cannot appear in a c-sequence as any N_i, $i > 1$, because it cannot be generated, by which we mean no N_{i-1} exists. It follows that the $\overset{\blacktriangle}{M}_f$ term for any $\overset{\textbf{max}}{H}$-form is derived from a *dead−ender*. That is, a *dead−ender* can only be N_1 where $3(N_1)+d = \overset{\blacktriangle}{M}_f$, of an R-sub$_1$. This implies that such an $\overset{\blacktriangle}{M}_f$, cannot be an M_i of a cycle.

Definition 2.7. *A dead-ender's cap* O, *is an even number of the form:* $2^k N_i$ *where* $k \geq 1$ *and* N_i *is a dead-ender.*

Note that an O is not of the form $3(N_i)+d$, so it's not part of a $\mathcal{C}$-sequence.

Even number distribution of $\mathbb{N}$

Define the set $\mathbb{N}_{even}$ with the standard $<$, so the set is ordered. Further, let every even number be of the form: $2^a S_i$ where $a \in \mathbb{N}_1$ and $S_i \in \mathbb{N}_{odd}$.

We shall establish that every $3n+d$ sequence is due solely to the distribution of the even numbers wrt. the form $2^a S_i$. Moreover, the distribution guarantees that every $3n+d$ sequence converges.

Let $e_i \in \mathbb{N}_{even}$ where $e_1 = 2$. Then wrt. a chosen desig convention (the other would simply be reversing $\blacksquare$ and $\blacktriangle$, with each other), we have: $\overset{\blacktriangle}{e}_1$, $\overset{\bullet}{e}_2$, $\overset{\blacksquare}{e}_3$, $\overset{\bullet}{e}_4$, $\overset{\blacktriangle}{e}_5$, $\overset{\bullet}{e}_6$, $\overset{\blacksquare}{e}_7$, $\overset{\bullet}{e}_8$, $\overset{\blacktriangle}{e}_9$, $\overset{\bullet}{e}_{10}$, $\overset{\blacksquare}{e}_{11}$, $\overset{\bullet}{e}_{12}$, $\ldots$. Each $3n+d$ sequence is distinguished by whether their d is divisible by 3 or not. If a $3n+d$ sequence is such that d is divisible by 3 then every N_i for the sequence must also be divisible by 3. If not, it's a *dead−ender* and can only be N_1, the only exception. If a $3n+d$ sequence is such that d is not divisible by 3 then every N_i for the sequence must also not be divisible by 3. Otherwise, it's a *dead−ender* and can only be N_1. What this means is that $3n+d$ sequences whose d is divisible by 3 use different numbers from $\mathbb{N}_{even}$ than the $3n+d$ sequences whose d is not divisible by 3. Our interest is that we need to know if the distribution of the even numbers wrt. their desigs are similar. We are also interested in the particular desig outcome from any $\overset{\bullet}{M}_i$ not derived from a *dead−ender*. ie. N_i not to be a *dead−ender*. Above, we have an alternating pattern wrt. $\overset{\bullet}{e}_i$ and $\overset{\blacksquare}{e}_j$ or $\overset{\blacktriangle}{e}_k$. It follows that half the numbers are $\overset{\bullet}{e}_i$ numbers for any given non-trivial range of consecutive even numbers with a consistent alternating distribution throughout $\mathbb{N}_{even}$.

An M_i for a $3n+d$ sequence whose $d \in \mathbb{N}_{odd \geq 3}$ is divisible by 3 and is not derived from a *dead−ender* could be: $\overset{\bullet}{e}_i$, $\overset{\blacksquare}{e}_{i+9}$, $e_{i+2(9)}$, $\overset{\blacktriangle}{e}_{i+3(9)}$, $\overset{\bullet}{e}_{i+4(9)}$, $\overset{\blacksquare}{e}_{i+5(9)}$, $\overset{\bullet}{e}_{i+6(9)}$, $\overset{\blacktriangle}{e}_{i+7(9)}$, $\overset{\bullet}{e}_{i+8(9)}$, $\overset{\blacksquare}{e}_{i+9(9)}$, $\overset{\bullet}{e}_{i+10(9)}$, $\overset{\blacktriangle}{e}_{i+11(9)}$, $\ldots$. ie. every ninth number of $\mathbb{N}_{even}$ starting at $3(d)+d$. Observe that they have the same alternating distribution as $\mathbb{N}_{even}$. This implies that half the numbers are $\overset{\bullet}{e}_i$ numbers in any given non-trivial range and that the alternating distribution remains consistent throughout this subset of $\mathbb{N}_{even}$. Note that the even numbers between e_j and e_{j+9}, namely, e_{j+3} and e_{j+6} are derived from *dead−enders* for a particular value of d wrt. $3(n)+d$. And the even numbers e_{j+1}, e_{j+2}, e_{j+4}, e_{j+5}, e_{j+7} and e_{j+8} cannot be produced for a particular value of d wrt. $3(n)+d$.

An M_i for a $3n+d$ sequence whose $d \in \mathbb{N}_{odd}$ is not divisible by 3 and is not derived from a *dead−ender*, starting at $3(d)+d$ as in the cases of $3n+5$ or

$3n$+11, could be: $\overset{\bullet}{e}_i$, $\overset{\blacktriangle}{e}_{i+3}$, $\overset{\blacksquare}{e}_{i+9}$, $\overset{\bullet}{e}_{i+12}$, $\overset{\bullet}{e}_{i+18}$, $\overset{\blacksquare}{e}_{i+21}$, $\overset{\blacktriangle}{e}_{i+27}$, $\overset{\bullet}{e}_{i+30}$, $\overset{\bullet}{e}_{i+36}$, $\overset{\blacktriangle}{e}_{i+39}$, $\overset{\blacksquare}{e}_{i+45}$, $\overset{\bullet}{e}_{i+48}$, $\ldots$. Or again, starting at $3(d)+d$ as in the cases of $3n+1$ or $3n+7$, could be: $\overset{\bullet}{e}_i$, $\overset{\bullet}{e}_{i+6}$, $\overset{\blacksquare}{e}_{i+9}$, $\overset{\blacktriangle}{e}_{i+15}$, $\overset{\bullet}{e}_{i+18}$, $\overset{\bullet}{e}_{i+24}$, $\overset{\blacktriangle}{e}_{i+27}$, $\overset{\blacksquare}{e}_{i+33}$, $\overset{\bullet}{e}_{i+36}$, $\overset{\bullet}{e}_{i+42}$, $\overset{\blacksquare}{e}_{i+45}$, $\overset{\blacktriangle}{e}_{i+51}$, $\ldots$. In all cases, there is a consistent alternating pattern wrt. the pairs $\overset{\bullet}{e}_k$, $\overset{\blacksquare}{e}_{k+3}$ with the pairs $\overset{\blacktriangle}{e}_{k+9}$, $\overset{\bullet}{e}_{k+12}$, such that for any given non-trivial range, the alternating distribution remains consistent throughout this subset of $\mathbb{N}_{even}$, where again, half the numbers are $\overset{\bullet}{e}_i$ numbers. Note that the even number between e_j and e_{j+9}, namely, e_{j+6} is derived from a *dead−ender* for a particular value of d wrt. $3(n)+d$, for $d = 5, 11, 17, \ldots$. And the even number between e_j and e_{j+9}, namely, e_{j+3} is derived from a *dead−ender* for a particular value of d wrt. $3(n)+d$, for $d = 1, 7, 13, \ldots$. Lastly, the even numbers e_{j+1}, e_{j+2}, e_{j+4}, e_{j+5}, e_{j+7} and e_{j+8} cannot be produced for a particular value of d wrt. $3(n)+d$.

For $\mathbb{N}_{even}$, we have that $2^r S_i = e_k$ is st. $e_{k+2^{r-1}} = 2^s S_j$ $\forall s > r$. And $2^r S_i = e_k$ is st. $e_{k+2^r} = 2^r S_j$.

For the subset of $\mathbb{N}_{even}$ used by $3n+d$ sequences whose $d \in \mathbb{N}_{odd \geq 3}$ is divisible by 3 and is not derived from a *dead−ender*, we have that $2^r S_i = e_k$ is st. $e_{k+36} = 2^s S_j$ where $s \neq r$ $\forall t, s \geq 3$. And $2^r S_i = e_k$ is st. $e_{k+9(2^r)} = 2^r S_j$.

For the two subsets of $\mathbb{N}_{even}$ used by $3n+d$ sequences whose $d \in \mathbb{N}_{odd}$ is not divisible by 3 and is not derived from a *dead−ender*, we have that $2^r S_i = e_k$ is st. $e_{k+12} = 2^s S_j$ $\forall s \geq 3$. Or, we have that $2^r S_i = e_k$ is st. $e_{k+24} = 2^s S_j$ $\forall s \geq 3$, always alternating between +12 and +24. Additionally: $2^1 S_i = e_k$ is st. $e_{k+6} = 2^1 S_j$ and $e_{k+18} = 2^1 S_u$ and $e_{k+24} = 2^1 S_v$ $\ldots$, alternating between +6 and +12. Lastly, $2^2 S_i = e_k$ is st. $e_{k+12} = 2^2 S_j$ and $e_{k+36} = 2^2 S_u$ and $e_{k+48} = 2^2 S_v$ $\ldots$, alternating between +12 and +24.

Note well: Consider the set $\mathbb{N}_{even}$ and the subsets of $\mathbb{N}_{even}$ used by the $3n+d$ sequences whose $d \in \mathbb{N}_{odd}$ is and is not divisible by 3, respectively, and is not derived from a *dead−ender*. Then wrt. a non-trivial range of $\mathbb{N}_{even}$, the number of $2^1 S_i$ even numbers equals the sum of all the $2^t S_i$ even numbers $\forall t \geq 2$, that's in the given range. And the number of $2^2 S_i$ even numbers equals the sum of all the $2^t S_i$ even numbers $\forall t \geq 3$, that's in the given range, and so on, until some value of t is reached, where the summation equality does not hold, for a few remaining values of t. Recall that every $\mathcal{C}$-sequence for a

$3n+d$ sequence has an associated $\mathcal{D}$-sequence. This implies that every $3n+d$ sequence obeys Law 2. Moreover, by the analysis above, $3n+d$ sequences will be similar regardless of the value for d. Their similarity in more precise terms is outlined in the sections to follow for Laws 3 and 4.

$\mathbf{S}^X$-strings and $\mathbf{S}^Y$-strings

It is significant wrt. the first and last c-sequence numbers of an associated *tail* that $X > Y$, for any R-sub because it means that the numbers associated to a *tail* do not produce an overall increase in the size of the numbers for the c-sequence. In other words, if there was an overall increase in the size of the c-sequence numbers as the number of *steps* increase, it was because of the numbers associated to *head* terms. For example, the second last R^Y-sub$_7$ of the $3n+3$ sequence, $\mathbf{m} = 53$ (Appendix 4), contains the largest number of the entire c-sequence, namely, $\overset{\bullet}{M}_{34} = 27,696$. But overall, it does not contribute to the increase in the size of the numbers to follow in the c-sequence. Moreover, no R^Y-sub associated number contributes to the overall increase in the size of the numbers. That is, only the numbers associated to the *head* terms of R^X-subs from $\mathbf{S}^X$-strings (should one or more exist), increase the overall size of the numbers for a c-sequence.

It's worth noting that an R^X-sub having just one *head* term, is very nearly an R^Y-sub because $\frac{2X-d}{3}$ wouldn't be much smaller than Y, so that even if one *tail* term $\overset{\bullet}{M}$ had been divisible by 2 once more, it could have been $\frac{2X-d}{3} > Y$.

The determined part of a c-sequence wrt. its $\mathcal{C}$-sequence that's associated to an R-sub is: $M_{i-1} \to \overset{\blacktriangle}{M}_i \cdots \overset{\blacktriangle}{M}_{j-1} \to \overset{\blacksquare}{M}_j \to \overset{\bullet}{M}_{j+1}$, where i to $j-1$ are all head terms.

The determined part of a c-sequence wrt. its $\mathcal{C}$-sequence that's associated to an R-sub that may exist more than once is: $\overset{\blacksquare}{M}_k \to \overset{\bullet}{M}_{k+1}$, where $k > j+1$. Note that M_{i-1} could be $\overset{\bullet}{M}_{i-1}$ or another head term. What we have left to explain for a $3n+d$ sequence with first term $\mathbf{m}$, is which outcome for $\overset{\bullet}{M}_k \to M_{k+1}$, is produced. By Law 2, the first associated *head* term $t_i = M_i+2d =$

2^a3^bR for any $3n{+}d$ sequence is such that $a \geq 3$. Recall that the number of *head* terms for any R-sub equals $a{-}2$, so a of 2^a3^bR must be 3 or greater. It follows that the size of a for the first *head* term is what determines any possible increase in size of the numbers of the c-sequence, overall.

For notational convenience, let ${\color{green}\bullet} \to {\color{green}\bullet}$ represent any two consecutive $\overset{{\color{green}\bullet}}{M}$ terms. And let ${\color{green}\bullet} \to {\color{blue}\blacksquare}$ represent $\overset{{\color{green}\bullet}}{M}_t \to \overset{{\color{blue}\blacksquare}}{M}_{t+1}$, where $t \geq j{+}1$.

Further, let ${\color{green}\bullet} \to {\color{red}\blacktriangle}$ represent $M_{i-1} \to \overset{{\color{red}\blacktriangle}}{M}_i$.

The counting rules

It is the number distribution of $\mathbb{N}_{even}$ wrt. the form: 2^aS_i, that produces the somewhat limited variations of $3n{+}d$ sequences, which is best defined by the count of the scenarios described just above.

Let the number of ${\color{green}\bullet} \to {\color{blue}\blacksquare}$ plus the number of ${\color{green}\bullet} \to {\color{green}\bullet}$ = $\mathbf{A}$. Note that there is a one to one correspondence between ${\color{green}\bullet} \to {\color{blue}\blacksquare}$ and ${\color{blue}\blacksquare} \to {\color{green}\bullet}$. And let the number of ${\color{green}\bullet} \to {\color{red}\blacktriangle}$ = $\mathbf{B}$, which is the counting of the first head terms.

There are specific rules to counting the various $\overset{{\color{green}\bullet}}{M}_s \to M_{s+1}$ scenarios whether they belong to an R^X-sub or an R^Y-sub, an $\mathbf{S}^X$-string or an $\mathbf{S}^Y$-string.

They are as follows if at least one $\mathbf{S}$-string exists:

R^X**-subs :** The ${\color{green}\bullet} \to {\color{red}\blacksquare}$ scenarios are counted. But each L-form R^X-sub is counted as 0.5. All ${\color{green}\bullet} \to {\color{green}\bullet}$ scenarios are counted. All ${\color{green}\bullet} \to {\color{blue}\blacksquare}$ scenarios are counted.

R^Y**-subs :** No ${\color{green}\bullet} \to {\color{red}\blacktriangle}$ scenarios are counted, in all $\mathbf{S}$-strings. Additionally, for each H-form R^Y-sub, a +1 is assigned to its $\mathbf{S}$-string's $\mathbf{A}$ value. All ${\color{green}\bullet} \to {\color{green}\bullet}$ scenarios are counted. All ${\color{green}\bullet} \to {\color{blue}\blacksquare}$ scenarios are counted.

$\mathbf{S}^X$-strings : The $\bullet \to \blacktriangle$ scenarios are counted just for R^X-subs, where all contiguous H-form R^X-subs are counted once only and each L-form R^X-sub is counted as 0.5. All $\bullet \to \bullet$ scenarios are counted. All $\bullet \to \blacksquare$ scenarios are counted. (see Appendix 3 regarding how to count R^X-subs of both forms)

$\mathbf{S}^Y$-strings : No $\bullet \to \blacktriangle$ scenarios are counted, for both its R^Y-subs and R^X-subs. ie. $\mathbf{B} = 0$. All $\bullet \to \bullet$ scenarios are counted. All $\bullet \to \blacksquare$ scenarios are counted.

The reasoning for counting contiguous H-form R^X-subs in $\mathbf{S}^X$-strings, just once with wrt. $\bullet \to \blacktriangle$ scenarios is because they can be viewed collectively as one *extended* head. That is, for this counting, the number of head terms is not relevant. What is relevant is when there exists an alternating pattern of H-form R^X-subs with non H-form R^X-subs in an $\mathbf{S}^X$-string, where all non-contiguous H-form R^X-subs will be counted. This would be the alternating of the heads for an $\mathbf{S}^X$-string. And just like the alternating of tail terms in R-subs, which obeys $X > Y$, there exists a limit wrt. alternating heads, as well. By the same reasoning, not counting $\bullet \to \blacktriangle$ scenarios for all R^Y-subs (which are always contiguous), is because they can be viewed collectively as one *extended* tail. L-form R^X-subs are very close to being R^Y-subs, so they are assigned a value of just 0.5 each.

Most $3n{+}d$ sequences are such that the majority of their $\dot{M}_i$ terms are divisible by 2, at most twice. This implies that there is no extreme reliance necessarily, on $\dot{M}_i$ terms to be divisible by 2, three or more times. However, if an H-form R^Y-sub exists, it was necessary that its one $\dot{M}_i$ term was divisible by 2, at least three times. This is why for each H-form R^Y-sub, a +1 is assigned to its $\mathbf{S}$-string's $\mathbf{A}$ value. It's an acknowledgement to a special case, namely, an H-form R^Y-sub. With these considerations for the counting of $\dot{M}_k \to M_{k+1}$ scenarios, we have that for every $\mathbf{S}^X$-string and $\mathbf{S}^Y$-string, $\mathbf{A} > \mathbf{B}$. Lastly, for a given $3n{+}d$ sequence with first term $\mathbf{m}$, let the number of $\mathbf{S}^Y$-strings = $\mathbf{C}$ and let the number of $\mathbf{S}^X$-strings = $\mathbf{D}$. Then $\mathbf{C} \geq \mathbf{D}$.

These relationships could only be preserved because Law 2 is obeyed by every $3n{+}d$ sequence for every d and $\mathbf{m}$. More plainly, there can be no divergence if Law 2 is obeyed by a $3n{+}d$ sequence, which is the case.

In summary, each $\mathbf{S}^X$-string and $\mathbf{S}^Y$-string reflects the distribution of $\mathbb{N}_{even}$ and this precludes any possibility of a $3n{+}d$ sequence to diverge. It follows that all $3n{+}d$ sequences converge with either a trivial cycle or a non-trivial cycle. What remains to establish is which $3n{+}d$ sequences have a trivial cycle and those $3n{+}d$ sequences having a non-trivial cycle. This is completely determined by just $d = 3^k$ or by d and $\mathbf{m}$ for a given $3n{+}d$ sequence.

3 Law 3: Correspondence of $3n{+}d$ and $3n{+}f$ sequences

Definition 3.1. *We say that two sequences $3n{+}d$ with first term $\boldsymbol{m}$ and $3n{+}f$ with first term $\boldsymbol{m}'$, where $d, f \in \mathbb{N}_{odd}$, correspond if there exists a number x where $x \in \mathbb{N}_{odd>1}$ is such that multiplying d and $\boldsymbol{m}$ by x produces $3n{+}f$ with first term $\boldsymbol{m}'$ or multiplying f and $\boldsymbol{m}'$ by x produces $3n{+}d$ with first term $\boldsymbol{m}$.*

Corresponding c-sequences will have the same number of *steps*, the same type and size of cycle and their respective $\mathcal{C}$-sequence and $\mathcal{D}$-sequence terms will only differ as M_i, $x(M_i)$ and t_i, $x(t_i)$ for each $i \in \mathbb{N}_{>0}$. It is shown in Appendix 5, the plotted graphs for the c-sequences of $3n{+}1$ and $3n{+}3$, starting at $\mathbf{m}' = 27$ and $\mathbf{m} = 81$, respectively. Observe that they are the same graph, ignoring vertical scale.

Law 3 is the generalization of Claim 2 for d in general, where we use d, $x(d) = f$, $x(\mathbf{m}) = \mathbf{m}'$ or we use f, $x(f) = d$, $x(\mathbf{m}') = \mathbf{m}$, for any $3n{+}d$ sequence and its corresponding $3n{+}f$ sequence. (proof in Appendix 2)

Definition 3.2. *Given a $3n{+}d$ sequence with first term $\boldsymbol{m}$, we can construct other sequences of the form: $3n{+}3^k d$ with $\boldsymbol{m}$, where $k \in \mathbb{N}_1$. We call such constructions the augmentation of d with 3^k for a $3n{+}d$ sequence.*

A $3n{+}d$ sequence with $\mathbf{m}$, and any *augmentation* $3n{+}3^k d$ with $\mathbf{m}$, is a special case of Law 3. Specifically, two sequences $3n{+}d$ and $3n{+}3^k d$, both with first term $\mathbf{m}$, where $d, 3^k d \in \mathbb{N}_{odd}$, may have corresponding cycles so that multiplying every cycle term of $3n{+}d$ by 3^k, produces every cycle term of $3n{+}3^k d$. All corresponding sequences and some of the special cases, have corresponding cycles. ie. some special cases may not have corresponding non-trivial cycles.

4 Law 4: $3n+d$, m, trivial and non-trivial cycles

Let two positive odd integers, $d < \mathbf{m}$ have prime decompositions: $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \ldots \cdot p_r^{\alpha_r}$ and $p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot p_3^{\beta_3} \ldots \cdot p_s^{\beta_s}$, respectively. We'll use the notation $d \subset \mathbf{m}$ and $d \not\subset \mathbf{m}$ to mean that $\mathbf{m}$'s prime decomposition is divisible by the prime decomposition of d, or it's not. We chose this notation instead of $\mathbf{m}/d$ or $\mathbf{m}$ *div* d.

Lemma 4.1. *Let d have no factor of 3^k for any $k \in \mathbb{N}_1$. Then a $3n+d$ sequence with first term $\boldsymbol{m}$ has a non-trivial cycle if and only if $d \not\subset \boldsymbol{m}$.*

Proof. We can induct on the size of $\mathbf{m}$ and d, considering up to $\mathbf{m} \geq d$, to be true. Note that if $\mathbf{m} = d$, then the $3n+d$ sequence is just three *steps* and has a trivial cycle. Let the base case be $d = 5$ and $\mathbf{m} = 7$, since we consider no 3^k for any $k \in \mathbb{N}_1$, and if $d = 1$, every $\mathbf{m}$ would be divisible by d.

A non-trivial cycle for a $3n+d$ sequence $\implies d \not\subset \mathbf{m}$.

Suppose to the contrary that $d \subset \mathbf{m}$. Then by Law 3, we have that $\mathbf{m}$ is divisible by d and $d/d = 1$. This gives the corresponding sequence $3n+1$ with first term $\mathbf{m}/d$. Since $\mathbf{m}/d < \mathbf{m}$ and $d = 1$, we know that this sequence has a trivial cycle which implies that $3n+d$ has a trivial cycle as well, by Law 3. Thus, a non-trivial cycle for a $3n+d$ sequence implies that $d \not\subset \mathbf{m}$.

If $d \not\subset \mathbf{m} \implies$ a non-trivial cycle for a $3n+d$ sequence.

We know that if it were the case that $d \subset \mathbf{m}$, then every term in the $3n+d$ sequence would also be divisible by d. Consequently, every $3n+d$ sequence where $d \subset \mathbf{m}$ has a corresponding sequence $3n+1$ with $\mathbf{m}/d$. And this implies that the cycle for such a sequence must be trivial. However, in this case, none of the terms of the sequence are divisible by d, thereby d does not appear as a term, so by definition, the sequence has a non-trivial cycle.

It follows that a $3n+1$ sequence with $\mathbf{m}$, must have a trivial cycle since $d \subset \mathbf{m}$. Moreover, all values of $d \leq \mathbf{m}$ where $d \subset \mathbf{m}$, will have trivial cycles. Therefore, the inductive step is valid and so, the result follows.

∎

The only exception to lemma 4.1 is the special cases of Law 3. The reason is that it could be the case that $d \subset \mathbf{m}$ while $3^k \not\subset \mathbf{m}$, yet the $3n{+}3^k d$ sequence will have a trivial cycle. Of course, if $d \not\subset \mathbf{m}$ and $3^k \not\subset \mathbf{m}$ or $3^k \subset \mathbf{m}$, then the $3n{+}3^k d$ sequence will have a non-trivial cycle. Since for all special cases of Law 3 where the $3n{+}d$ sequence has a trivial cycle implies that its corresponding $3n{+}3^k d$ sequence will also have a trivial cycle, then it follows that every $3n{+}3^k$ sequence ($d = 1$, so only trivial cycles), $\forall k \in \mathbb{N}_0$ has a trivial cycle, hence contains the term 3^k.

It should be clear that if $f > \mathbf{m}$, then a $3n{+}f$ sequence necessarily has a non-trivial cycle, unless $f = 3^k d$ and $d \subset \mathbf{m}$, in which case, the $3n{+}f$ sequence has a trivial cycle. Moreover, all laws are obeyed even if $f > \mathbf{m}$.

Finally, all $3n{+}d$ sequences are convergent and if the $\gcd(\mathbf{m}, d) = d$, then just copies of d are being produced and reduced, so that the only possible outcome is that d appears in the cycle. If the $\gcd(\mathbf{m}, d) \neq d$ (if d has no factor 3^k), for a $3n{+}d$ sequence, then it must necessarily have a non-trivial cycle.

5 Final comments

When a $3n{+}d$ sequence starting with $\mathbf{m}$ is such that $d \subset \mathbf{m}$ then Collatz sequence generation could be described simply as an inefficient scheme to calculate the $\gcd(\mathbf{m}, d) = d$. And when $d \not\subset \mathbf{m}$ (if d has no factor 3^k), we have a number generator of unknown value. However, this is a story about numbers 2 and 3 and their relationship with each other.

One final comment is that number theory appears to be enslaved to all things involving the odd primes. However, the one prime that's even, is a part of every other number in the set $\mathbb{N}$, seems to be all but ignored. And yet, Collatz sequences are the expression of how the even numbers play a non-trivial role as seen when viewed in their $2^r S_i$ form. Moreover, all the other primes $\in \mathbb{N}_{odd>3}$, have no relevance wrt. any $3n{+}d$ sequence. ie. 2 is a special prime.

6 Appendix 1

The $3n+1$ sequence for $\mathbf{m'} = 27 \rightarrow 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322,$ $161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700,$ $350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251,$ $754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079,$ $3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077,$ $9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61,$ $184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1$

The c-sequence above, contains the $\mathcal{C}$-sequence $\rightarrow 82, 124, 94, 142, 214, 322, 484,$ $364, 274, 412, 310, 466, 700, 526, 790, 1186, 1780, 1336, 502, 754, 1132, 850, 1276,$ $958, 1438, 2158, 3238, 4858, 7288, 2734, 4102, 6154, 9232, 1732, 1300, 976, 184,$ $70, 106, 160, 16$

From the $3n+3$ $\mathcal{C}$-sequence for $\mathbf{m} = 2(27) - 1 = 53$, we obtain the $3n+1$ $\mathcal{C}$-sequence for $\mathbf{m'} = 27$ by $\frac{M_i}{2^k}+1 = N_{i+1}+1 = M_i'$ as follows:

$\mathbf{m} = 53 \rightarrow \frac{162}{2^1} + 1 = 82,\ \frac{246}{2^1} + 1 = 124,\ \frac{372}{2^2} + 1 = 94,\ \frac{282}{2^1} + 1 = 142,$
$\frac{426}{2^1} + 1 = 214,\ \frac{642}{2^1} + 1 = 322,\ \frac{966}{2^1} + 1 = 484,\ \frac{1452}{2^2} + 1 = 364,\ \frac{1092}{2^2} + 1 = 274,$
$\frac{822}{2^1} + 1 = 412,\ \frac{1236}{2^2} + 1 = 310,\ \frac{930}{2^1} + 1 = 466,\ \frac{1398}{2^1} + 1 = 700,\ \frac{2100}{2^2} + 1 = 526,$
$\frac{1578}{2^1} + 1 = 790,\ \frac{2370}{2^1} + 1 = 1186,\ \frac{3558}{2^1} + 1 = 1780,\ \frac{5340}{2^2} + 1 = 1336,$
$\frac{4008}{2^3} + 1 = 502,\ \frac{1506}{2^1} + 1 = 754,\ \frac{2262}{2^1} + 1 = 1132,\ \frac{3396}{2^2} + 1 = 850,$
$\frac{2550}{2^1} + 1 = 1276,\ \frac{3828}{2^2} + 1 = 958,\ \frac{2874}{2^1} + 1 = 1438,\ \frac{4314}{2^1} + 1 = 2158,$
$\frac{6474}{2^1} + 1 = 3238,\ \frac{9714}{2^1} + 1 = 4858,\ \frac{14574}{2^1} + 1 = 7288,\ \frac{21864}{2^3} + 1 = 2734,$
$\frac{8202}{2^1} + 1 = 4102,\ \frac{12306}{2^1} + 1 = 6154,\ \frac{18462}{2^1} + 1 = 9232,\ \frac{27696}{2^4} + 1 = 1732,$
$\frac{5196}{2^2} + 1 = 1300,\ \frac{3900}{2^2} + 1 = 976,\ \frac{2928}{2^4} + 1 = 184,\ \frac{552}{2^3} + 1 = 70,$
$\frac{210}{2^1} + 1 = 106,\ \frac{318}{2^1} + 1 = 160,\ \frac{480}{2^5} + 1 = 16$

7 Appendix 2

Proof for Law 1

Let $M_i = 3(N_i){+}d$ where N_i is odd. Suppose M_i is divisible by 2 exactly once. Then, let $3(N_i){+}d = 2R$ where R, R' and $R'' \in \mathbb{N}_{odd}$. Then, $M_i{+}2d = 2R + 2d = 2(R{+}d)$. But $R{+}d$ must be even, so let it be $2R'$. Then, $M_i{+}2d = 2^2R'$ must be divisible by at least 4. Suppose it's exactly 4. Now, we have that $M_i{-}2d = 2R{-}2d = 2(R{-}d)$. But $R{-}d$ must be even, so let it be $2R''$. Then, $M_i{-}2d = 2^2R''$ must be divisible by at least 4. Suppose it's exactly 4 as well. Then, $M_i{-}2d = 2^2R''$. But then we have that $2^2R'' + 4d = 2^2R'$. And this is equivalent to $R'' + d = R' \longrightarrow$ odd + odd = odd, a contradiction by the assumption that $M_i{-}2d$ was divisible by exactly 4 (assuming $M_i{+}2d$ was). Therefore, M_i must be $\overset{\blacksquare}{M}_i$. Observe that if $M_i{+}2d$ and $M_i{-}2d$ are swapped in the argument above, then the contradiction would imply that M_i is $\overset{\blacktriangle}{M}_i$ instead. Now suppose M_i is divisible by 4. Then M_i is $\overset{\bullet}{M}_i$ by definition. Let $M_i = 2^2R$. It follows that $M_i{+}2d = 2^2R + 2d = 2(2R{+}d)$. But $2R{+}d$ is odd. Let $R' = 2R{+}d$, so $M_i{+}2d = 2R'$, thus divisible by 2 exactly once. Similarly, $M_i{-}2d = 2^2R - 2d = 2(2R{-}d)$. But $2R{-}d$ is odd. Let $R'' = 2R{-}d$, so $M_i{-}2d = 2R''$, thus divisible by 2 exactly once. Since the arguments here, would be unchanged for any $3n{+}d$ sequence, $d \in \mathbb{N}_{odd}$ then the law applies to all $\mathbb{N}_{even}$.

Proof for $2^a3^bR \longrightarrow 2^{a-1}3^{b+1}R$ where $a \geq 3$ and for $\overset{\blacktriangle}{t}_i \longrightarrow \overset{\bullet}{t}_{i+1}$

Let $M_i{+}2d = 2^a3^bR$ where $a \geq 3$. Then $M_i = 2^a3^bR - 2d = 2(2^{a-1}3^bR - 2d)$. But M_i is divisible by 2 exactly once (by Law 1). It follows that $N_{i+1} = 2^{a-1}3^bR - d$.

So, $3(N_{i+1}) + d = 3(2^{a-1}3^bR - d) + d = 2^{a-1}3^{b+1}R - 3d + d = 2^{a-1}3^{b+1}R - 2d = M_{i+1}$. Thus, $M_{i+1}{+}2d = 2^{a-1}3^{b+1}R = t_{i+1}$. It follows immediately that if $t_i = 2^33^bR$ then $2^23^{b+1}R = t_{i+1}$, where $t_{i+1} = M_{i+1}{+}2d = 2^23^{b+1}R$ is divisible by exactly 4, so M_{i+1} is $\overset{\blacktriangle}{M}_{i+1}$ by Law 1. Moreover, it follows that $t_{i+2} = M_{i+2}{+}2d = 2^13^{b+2}R$ is divisible by 2 exactly once. Then M_{i+2} is $\overset{\bullet}{M}_{i+2}$ by Law 1.

Suppose now that some $\overset{\blacktriangle}{t}_i$ does not follow a $\overset{\blacksquare}{t}_{i-1}$. Then again, $t_i = M_i{+}2d = 2^2 3^b R$ is divisible by exactly 4. M_i is divisible by 2 exactly once and it follows that $\frac{M_i}{2} = N_{i+1} = \frac{2^2 3^b R - 2d}{2} = 2^1 3^b R - d$. Then $M_{i+1} = 3(2^1 3^b R{-}d) + d = 2^1 3^{b+1} R - 3d + d = 2^1 3^{b+1} R - 2d$. So, $M_{i+1}{+}2d = 2^1 3^{b+1} R$ and by Law 1, M_{i+1} is $\overset{\bullet}{M}_{i+1}$

Proof for $\overset{\blacksquare}{t}_i = 2^2 3^b R$.

Suppose t_{i-1} is $\overset{\blacktriangle}{t}_{i-1} = \overset{\blacktriangle}{t}_f$ is $2^3 3^1 R$ (the least it could be), then $\overset{\blacksquare}{t}_{f+1}$ is $2^2 3^2 R$. If t_{i-1} is $\overset{\bullet}{t}_{i-1}$, then $\overset{\bullet}{M}_{i-1} = 2^r 3^1 S$ where $r \geq 2$ and S is odd and not divisible by 3. We have that $N_i = 3^1 S$, so $\overset{\blacksquare}{M}_i = 3(3^1 S) + d$. It follows that $\overset{\blacksquare}{M}_i{+}2d = 3(3^1 S) + 3d = 3(3^1 S + d) = 2^a 3^2 S'$ where $a = 2$ by Law 1, and $S' \in \mathbb{N}_{odd}$.

Proof for Law 3

Let $M_i = 3(N_i) + d$ and $M_i' = 3(N_i') + f$ be for $3n{+}d$ and $3n{+}f$ respectively. Further, let $\mathbf{m}$ and $\mathbf{m'}$ be the first odd terms for the respective sequences where $\mathbf{m'} = x(\mathbf{m})$ and $f = x(d)$ with $x \in \mathbb{N}_{odd>1}$.

Proof: We have the relations, $\mathbf{m'} = x(\mathbf{m})$ and $f = x(d)$. Let $M_1' = 3(\mathbf{m'}) + f$ and $M_1 = 3(\mathbf{m}) + d$. By substitution $M_1' = 3x(\mathbf{m}) + x(d)$. And so, $M_1' = x(M_1)$. The argument is the same for every pair N_i and N_i'.

8 Appendix 3

For $\mathbf{m} = 2(27) - 1 = 53$ there are 2 **S**-strings having 4 R-subs each.

$R^X\text{-sub}_1 = (1, 2 \mid 2^3 3^1 7 \mid 2^1 3^3 7)$ — $\overset{\textbf{max}}{H}$-form

$R^X\text{-sub}_2 = (3, 5 \mid 2^5 3^2 1 \mid 2^1 3^3 23)$

$R^X\text{-sub}_3 = (1, 2 \mid 2^3 3^2 13 \mid 2^1 3^4 13)$ — H-form

$R^Y\text{-sub}_4 = (2, 3 \mid 2^4 3^2 11 \mid 2^1 3^2 223)$

$\mathbf{S}^X\text{-string}_1$.

$R^X\text{-sub}_5 = (1, 4 \mid 2^3 3^3 7 \mid 2^1 3^3 71)$ — L-form

$R^X\text{-sub}_6 = (4, 2 \mid 2^6 3^2 5 \mid 2^1 3^7 5)$ — H-form

$R^Y\text{-sub}_7 = (2, 6 \mid 2^4 3^3 19 \mid 2^1 3^2 31)$

$R^Y\text{-sub}_8 = (1, 3 \mid 2^3 3^3 1 \mid 2^1 3^3 1)$ — L-form

$\mathbf{S}^Y\text{-string}_2$.

Observe that both $R^X\text{-sub}_1$ and $R^X\text{-sub}_3$ above, are also L-forms.

Altogether, we have: $\mathbf{S}^X\text{-string}_1 \to \mathbf{S}^Y\text{-string}_2$.

Note: If there exists contiguous H-form R^X-subs where some or all are also L-form R^X-subs in an $\mathbf{S}^X$-string, then count the contiguous H-form R^X-subs first and then subtract 0.5 for each of the L-form R^X-subs.

For $\mathbf{m} = 2(63,728,127)-1 = 127,456,253$ there are 14 $\mathbf{S}$-strings and 54 R-subs.

$R^X\text{-sub}_1 = (8,2 \mid 2^{10}3^{1}124469 \mid 2^{1}3^{10}124469)$ — $\overset{\mathbf{max}}{H}$-form

$R^X\text{-sub}_2 = (5,2 \mid 2^{7}3^{2}9570013 \mid 2^{1}3^{8}9570013)$ — H-form

$R^X\text{-sub}_3 = (6,2 \mid 2^{8}3^{2}40878161 \mid 2^{1}3^{9}40878161)$ — H-form

$R^X\text{-sub}_4 = (4,2 \mid 2^{6}3^{3}87305213 \mid 2^{1}3^{8}87305213)$ — H-form

$R^Y\text{-sub}_5 = (2,6 \mid 2^{4}3^{2}5966765651 \mid 2^{1}3^{3}33982594997)$

$R^Y\text{-sub}_6 = (1,5 \mid 2^{3}3^{3}3185868281 \mid 2^{1}3^{3}6048171815)$ — L-form

$\mathbf{S}^X\text{-string}_1$.

$R^X\text{-sub}_7 = (2,5 \mid 2^{4}3^{2}1701048323 \mid 2^{1}3^{3}12917335703)$

$R^Y\text{-sub}_8 = (1,8 \mid 2^{3}3^{2}7266001333 \mid 2^{1}3^{3}7759152791)$ — L-form

$R^Y\text{-sub}_9 = (1,7 \mid 2^{3}3^{2}4364523445 \mid 2^{1}3^{2}2330374213)$ — L-form

$\mathbf{S}^Y\text{-string}_2$.

$R^X\text{-sub}_{10} = (1,2 \mid 2^{3}3^{2}436945165 \mid 2^{1}3^{4}436945165)$ — H-form

$R^X\text{-sub}_{11} = (2,3 \mid 2^{4}3^{2}368672483 \mid 2^{1}3^{2}7465617781)$

$R^X\text{-sub}_{12} = (2,3 \mid 2^{4}3^{2}699901667 \mid 2^{1}3^{2}14173008757)$

$R^X\text{-sub}_{13} = (2,4 \mid 2^{4}3^{2}1328719571 \mid 2^{1}3^{2}20179928485)$

$R^Y\text{-sub}_{14} = (1,3 \mid 2^{3}3^{2}3783736591 \mid 2^{1}3^{4}4256703665)$ — L-form

$\mathbf{S}^X\text{-string}_3$.

Observe that $R^X\text{-sub}_{10}$ is also an L-form.

$R^X\text{-sub}_{15} = (1, 3 \mid 2^3 3^2 598598953 \mid 2^1 3^2 4040542933)$ — L-form

$R^Y\text{-sub}_{16} = (4, 5 \mid 2^6 3^2 94700225 \mid 2^1 3^3 809021063)$

$R^Y\text{-sub}_{17} = (3, 2 \mid 2^5 3^2 113768587 \mid 2^1 3^6 113768587)$ — H-form

$\mathbf{S}^Y$-string$_4$.

$R^X\text{-sub}_{18} = (4, 6 \mid 2^6 3^3 8999273 \mid 2^1 3^2 2075773717)$

$R^X\text{-sub}_{19} = (3, 2 \mid 2^5 3^2 97301893 \mid 2^1 3^6 97301893)$ — H-form

$R^X\text{-sub}_{20} = (3, 2 \mid 2^5 3^2 369443125 \mid 2^1 3^6 369443125)$ — H-form

$R^X\text{-sub}_{21} = (1, 8 \mid 2^3 3^2 5610917461 \mid 2^1 3^2 35950419397)$ — L-form

$R^X\text{-sub}_{22} = (1, 4 \mid 2^3 3^2 6740703637 \mid 2^1 3^3 22749874775)$ — L-form

$R^Y\text{-sub}_{23} = (1, 6 \mid 2^3 3^2 12796804561 \mid 2^1 3^3 6073483415)$ — L-form

$R^Y\text{-sub}_{24} = (1, 8 \mid 2^3 3^2 3416334421 \mid 2^1 3^3 3648204775)$ — L-form

$R^Y\text{-sub}_{25} = (2, 3 \mid 2^4 3^2 1026057593 \mid 2^1 3^2 5194416565)$

$R^Y\text{-sub}_{26} = (2, 2 \mid 2^4 3^2 486976553 \mid 2^1 3^5 486976553)$ — H-form

$\mathbf{S}^X$-string$_5$.

$R^X\text{-sub}_{27} = (3, 4 \mid 2^5 3^2 154082425 \mid 2^1 3^3 1170063415)$

$R^Y\text{-sub}_{28} = (1, 7 \mid 2^3 3^2 658160671 \mid 2^1 3^3 29284615)$ — L-form

$\mathbf{S}^Y$-string$_6$.

$R^X\text{-sub}_{29} = (3, 2 \mid 2^5 3^2 4118149 \mid 2^1 3^6 4118149)$ — H-form

$R^Y\text{-sub}_{30} = (2, 9 \mid 2^4 3^3 15636097 \mid 2^1 3^2 338121055)$

$R^{Y}\text{-sub}_{31} = (1, 3 \mid 2^{3}3^{3}10566283 \mid 2^{1}3^{2}213967231)$ — L-form

$R^{Y}\text{-sub}_{32} = (3, 3 \mid 2^{5}3^{3}1671619 \mid 2^{1}3^{2}304652563)$

$\mathbf{S}^{Y}$-string$_{7}$.

$R^{X}\text{-sub}_{33} = (1, 2 \mid 2^{3}3^{2}14280589 \mid 2^{1}3^{4}14280589)$ — H-form

$R^{X}\text{-sub}_{34} = (2, 2 \mid 2^{4}3^{2}12049247 \mid 2^{1}3^{5}12049247)$ — H-form

$R^{Y}\text{-sub}_{35} = (1, 3 \mid 2^{3}3^{2}60999313 \mid 2^{1}3^{2}411745363)$ — L-form

$\mathbf{S}^{X}$-string$_{8}$.

Observe that R^{X}-sub$_{33}$ is also an L-form.

$R^{X}\text{-sub}_{36} = (3, 2 \mid 2^{5}3^{2}4825141 \mid 2^{1}3^{6}4825141)$ — H-form

$R^{Y}\text{-sub}_{37} = (1, 13 \mid 2^{3}3^{2}73281829 \mid 2^{1}3^{2}13927807)$ — L-form

$\mathbf{S}^{Y}$-string$_{9}$.

$R^{X}\text{-sub}_{38} = (3, 3 \mid 2^{5}3^{3}108811 \mid 2^{1}3^{2}19830805)$

$R^{X}\text{-sub}_{39} = (3, 3 \mid 2^{5}3^{2}929569 \mid 2^{1}3^{2}56471317)$

$R^{X}\text{-sub}_{40} = (3, 2 \mid 2^{5}3^{2}2647093 \mid 2^{1}3^{6}2647093)$ — H-form

$R^{Y}\text{-sub}_{41} = (1, 7 \mid 2^{3}3^{2}40202725 \mid 2^{1}3^{2}167701)$ — L-form

$\mathbf{S}^{Y}$-string$_{10}$.

$R^{X}\text{-sub}_{42} = (3, 2 \mid 2^{5}3^{2}7861 \mid 2^{1}3^{6}7861)$ — H-form

$R^{X}\text{-sub}_{43} = (1, 2 \mid 2^{3}3^{2}119389 \mid 2^{1}3^{4}119389)$ — H-form

$R^X\text{-sub}_{44} = (1, 2 \mid 2^3 3^2 201469 \mid 2^1 3^4 201469)$ — H-form

$R^Y\text{-sub}_{45} = (1, 3 \mid 2^3 3^2 339979 \mid 2^1 3^2 573715)$ — L-form

$\mathbf{S}^Y$-string$_{11}$.

Observe that both R^X-sub$_{43}$ and R^X-sub$_{44}$ above, are also L-forms.

$R^X\text{-sub}_{46} = (1, 2 \mid 2^3 3^2 26893 \mid 2^1 3^4 26893)$ — H-form

$R^X\text{-sub}_{47} = (2, 3 \mid 2^4 3^2 22691 \mid 2^1 3^2 459493)$

$R^Y\text{-sub}_{48} = (1, 3 \mid 2^3 3^2 86155 \mid 2^1 3^2 145387)$ — L-form

$\mathbf{S}^Y$-string$_{12}$.

Observe that R^X-sub$_{46}$ is also an L-form.

$R^X\text{-sub}_{49} = (3, 2 \mid 2^5 3^3 71 \mid 2^1 3^7 71)$ — H-form

$R^Y\text{-sub}_{50} = (1, 4 \mid 2^3 3^2 3235 \mid 2^1 3^4 455)$ — L-form

$\mathbf{S}^Y$-string$_{13}$.

$R^X\text{-sub}_{51} = (8, 3 \mid 2^{10} 3^3 1 \mid 2^1 3^2 44287)$

$R^X\text{-sub}_{52} = (4, 2 \mid 2^6 3^3 173 \mid 2^1 3^8 173)$ — H-form

$R^Y\text{-sub}_{53} = (1, 22 \mid 2^3 3^2 23647 \mid 2^1 3^2 85)$ — L-form

$R^Y\text{-sub}_{54} = (5, 10 \mid 2^7 3^2 1 \mid 2^1 3^3 1)$

$\mathbf{S}^Y$-string$_{14}$.

Altogether, we have: $\mathbf{S}^X$-string$_1$ $\to$ $\mathbf{S}^Y$-string$_2$ $\to$ $\mathbf{S}^X$-string$_3$ $\to$ $\mathbf{S}^Y$-string$_4$ $\to$ $\mathbf{S}^X$-string$_5$ $\to$ $\mathbf{S}^Y$-string$_6$ $\to$ $\mathbf{S}^Y$-string$_7$ $\to$ $\mathbf{S}^X$-string$_8$ $\to$ $\mathbf{S}^Y$-string$_9$ $\to$ $\mathbf{S}^Y$-string$_{10}$ $\to$ $\mathbf{S}^Y$-string$_{11}$ $\to$ $\mathbf{S}^Y$-string$_{12}$ $\to$ $\mathbf{S}^Y$-string$_{13}$ $\to$. $\mathbf{S}^Y$-string$_{14}$

9 Appendix 4

Example: $\mathbf{m} = 53$, $M_1 = 3(53)+3 = 162$

$$\begin{tikzcd}
\mathcal{C}\rightarrow 162 \arrow[r] \arrow[d, "\blacktriangle"'] & 246 \arrow[r] \arrow[d, "\blacksquare"'] & 372 \arrow[r] \arrow[d, "\bullet"'] & 282 \arrow[r] \arrow[d, "\blacktriangle"'] & 426 \arrow[r] \arrow[d, "\blacktriangle"'] & 642 \arrow[r] \arrow[d, "\blacktriangle"'] & 966 \arrow[r] \arrow[d, "\blacksquare"'] & 1452 \arrow[r] \arrow[d, "\bullet"'] & {} \\
\mathcal{D}\rightarrow 2^3 3^1 \mathbf{7} & 2^2 3^2 \mathbf{7} & 2^1 3^3 \mathbf{7} & 2^5 3^2 \mathbf{1} & 2^4 3^3 \mathbf{1} & 2^3 3^4 \mathbf{1} & 2^2 3^5 \mathbf{1} & 2^1 3^6 \mathbf{1} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}\rightarrow 1092 \arrow[r] \arrow[d, "\bullet"'] & 822 \arrow[r] \arrow[d, "\blacksquare"'] & 1236 \arrow[r] \arrow[d, "\bullet"'] & 930 \arrow[r] \arrow[d, "\blacktriangle"'] & 1398 \arrow[r] \arrow[d, "\blacksquare"'] & 2100 \arrow[r] \arrow[d, "\bullet"'] & 1578 \arrow[r] \arrow[d, "\blacktriangle"'] & 2370 \arrow[r] \arrow[d, "\blacktriangle"'] & {} \\
\mathcal{D}\rightarrow 2^1 3^2 \mathbf{61} & 2^2 3^2 \mathbf{23} & 2^1 3^3 \mathbf{23} & 2^3 3^2 \mathbf{13} & 2^2 3^3 \mathbf{13} & 2^1 3^4 \mathbf{13} & 2^4 3^2 \mathbf{11} & 2^3 3^3 \mathbf{11} &
\end{tikzcd}$$

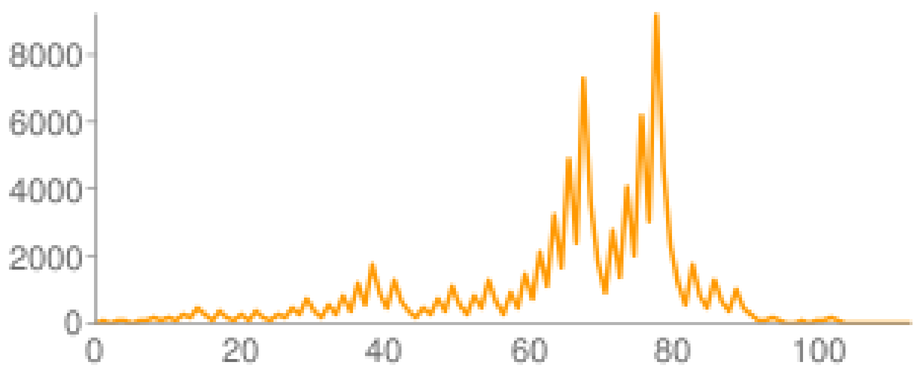
$$\begin{tikzcd}
\mathcal{C}\rightarrow 3558 \arrow[r] \arrow[d, "\blacksquare"'] & 5340 \arrow[r] \arrow[d, "\bullet"'] & 4008 \arrow[r] \arrow[d, "\bullet"'] & 1506 \arrow[r] \arrow[d, "\blacktriangle"'] & 2262 \arrow[r] \arrow[d, "\blacksquare"'] & 3396 \arrow[r] \arrow[d, "\bullet"'] & 2550 \arrow[r] \arrow[d, "\blacksquare"'] & 3828 \arrow[r] \arrow[d, "\bullet"'] & {} \\
\mathcal{D}\rightarrow 2^2 3^4 \mathbf{11} & 2^1 3^5 \mathbf{11} & 2^1 3^2 \mathbf{223} & 2^3 3^3 \mathbf{7} & 2^2 3^4 \mathbf{7} & 2^1 3^5 \mathbf{7} & 2^2 3^2 \mathbf{71} & 2^1 3^3 \mathbf{71} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}\rightarrow 2874 \arrow[r] \arrow[d, "\blacktriangle"'] & 4314 \arrow[r] \arrow[d, "\blacktriangle"'] & 6474 \arrow[r] \arrow[d, "\blacktriangle"'] & 9714 \arrow[r] \arrow[d, "\blacktriangle"'] & 14574 \arrow[r] \arrow[d, "\blacksquare"'] & 21864 \arrow[r] \arrow[d, "\bullet"'] & 8202 \arrow[r] \arrow[d, "\blacktriangle"'] & {} \\
\mathcal{D}\rightarrow 2^6 3^2 \mathbf{5} & 2^5 3^3 \mathbf{5} & 2^4 3^4 \mathbf{5} & 2^3 3^5 \mathbf{5} & 2^2 3^6 \mathbf{5} & 2^1 3^7 \mathbf{5} & 2^4 3^3 \mathbf{19} &
\end{tikzcd}$$

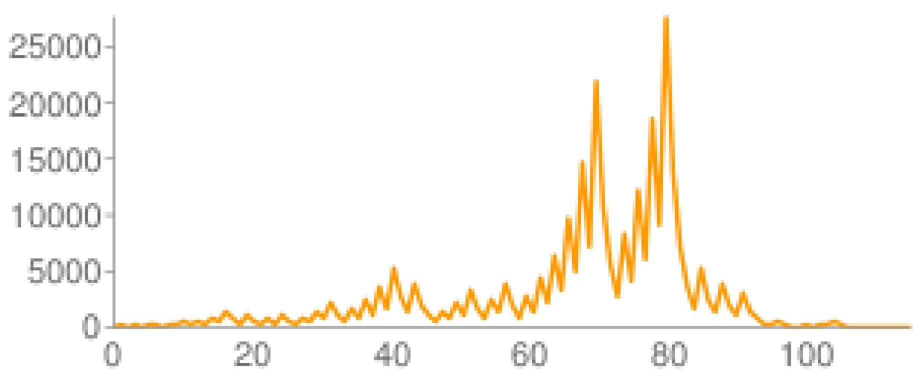
$$\begin{tikzcd}
\mathcal{C}\rightarrow 12306 \arrow[r] \arrow[d, "\blacktriangle"'] & 18462 \arrow[r] \arrow[d, "\blacksquare"'] & 27696 \arrow[r] \arrow[d, "\bullet"'] & 5196 \arrow[r] \arrow[d, "\bullet"'] & 3900 \arrow[r] \arrow[d, "\bullet"'] & 2928 \arrow[r] \arrow[d, "\bullet"'] & 552 \arrow[r] \arrow[d, "\bullet"'] & {} \\
\mathcal{D}\rightarrow 2^3 3^4 \mathbf{19} & 2^2 3^5 \mathbf{19} & 2^1 3^6 \mathbf{19} & 2^1 3^2 \mathbf{289} & 2^1 3^2 \mathbf{217} & 2^1 3^2 \mathbf{163} & 2^1 3^2 \mathbf{31} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}\rightarrow 210 \arrow[r] \arrow[d, "\blacktriangle"'] & 318 \arrow[r] \arrow[d, "\blacksquare"'] & 480 \arrow[r] \arrow[d, "\bullet"'] & 48 \arrow[d, "\bullet"'] \\
\mathcal{D}\rightarrow 2^3 3^3 \mathbf{1} & 2^2 3^4 \mathbf{1} & 2^1 3^5 \mathbf{1} & 2^1 3^3 \mathbf{1}
\end{tikzcd}$$

10 Appendix 5



The $3n$+1 sequence for $\mathbf{m'}$ = 27



The $3n$+3 sequence for $\mathbf{m}$ = 81

11 Resources

The graphs were generated at https://www.dcode.fr/collatz-conjecture