NE504 - Nuclear fuel cycle analysis
Xenon neutron poison production

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1 Mathematical models

1.1 Iodine

Rate of change of iodine -

$$\frac{dI}{dt} = \gamma_I \Sigma_F \phi_T - \lambda_I I \\ I(0) = 0 \tag{1}$$

1.2 Xenon

Rate of change of xenon -

$$\frac{dX}{dt} = \lambda_I I + \gamma_X \Sigma_F \phi_T - \lambda_X X - \sigma_A \phi_T X \\ X(0) = 0 \tag{2}$$

2 Solutions

2.1 Iodine

Apply laplace transform to eq. 1 -

$$\frac{dI}{dt} = \gamma_I \Sigma_F \phi_T - \lambda_I I \\ s\tilde{I} = \frac{1}{s}\gamma_I \Sigma_F \phi_T - \lambda_I \tilde{I} \tag{3}$$

Rearrange terms to obtain laplace solution -

$$\tilde{I} = \frac{1}{s(s+\lambda_I)}\gamma_I \Sigma_F \phi_T \tag{4}$$

Invert eq. 4 to obtain real time domain solution -

$$I(t) = \frac{\gamma_I \Sigma_F \phi_T}{\lambda_I}(1 - e^{-\lambda_I t}) \tag{5}$$

2.2 Xenon

Substitute the iodine solution in eq. 5 into eq. 2 -

$$\begin{aligned}\frac{dX}{dt} &= \lambda_I I + \gamma_X \Sigma_F \phi_T - \lambda_X X - \sigma_A \phi_T X \\ \frac{dX}{dt} &= \lambda_I \frac{\gamma_I \Sigma_F \phi_T}{\lambda_I}(1 - e^{-\lambda_I t}) + \gamma_X \Sigma_F \phi_T - \lambda_X X - \sigma_A \phi_T X\end{aligned} \tag{6}$$

Rearrange the terms and simplify -

$$\begin{aligned}\frac{dX}{dt} &= \gamma_I \Sigma_F \phi_T (1 - e^{-\lambda_I t}) + \gamma_X \Sigma_F \phi_T - \lambda_X X - \sigma_A \phi_T X \\ \frac{dX}{dt} &= \gamma_I \Sigma_F \phi_T - \gamma_I \Sigma_F \phi_T e^{-\lambda_I t} + \gamma_X \Sigma_F \phi_T - \lambda_X X - \sigma_A \phi_T X\end{aligned} \tag{7}$$

$$\frac{dX}{dt} = \gamma_I \Sigma_F \phi_T - \gamma_I \Sigma_F \phi_T e^{-\lambda_I t} + \gamma_X \Sigma_F \phi_T - (\lambda_X + \sigma_A \phi_T) X \tag{8}$$

Apply laplace transform to eq. 8 -

$$s\tilde{X} = \frac{1}{s}\gamma_I \Sigma_F \phi_T - \frac{1}{s + \lambda_I}\gamma_I \Sigma_F \phi_T + \frac{1}{s}\gamma_X \Sigma_F \phi_T - (\lambda_X + \sigma_A \phi_T)\tilde{X} \tag{9}$$

Rearrange the terms in eq. 9 to obtain the laplace solution -

$$(s + [\lambda_X + \sigma_A \phi_T])\tilde{X} = \frac{1}{s}\gamma_I \Sigma_F \phi_T - \frac{1}{s + \lambda_I}\gamma_I \Sigma_F \phi_T + \frac{1}{s}\gamma_X \Sigma_F \phi_T \tag{10}$$

Then -

$$\tilde{X} = +\frac{1}{s(s + [\lambda_X + \sigma_A \phi_T])}\gamma_I \Sigma_F \phi_T - \frac{1}{(s + \lambda_I)(s + [\lambda_X + \sigma_A \phi_T])}\gamma_I \Sigma_F \phi_T + \frac{1}{s(s + [\lambda_X + \sigma_A \phi_T])}\gamma_X \Sigma_F \phi_T \tag{11}$$

Invert eq. 11 to obtain real time domain solution -

$$X(t) = +\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T}(1 - e^{-(\lambda_X + \sigma_A \phi_T)t}) - \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I}(e^{-\lambda_I t} - e^{-(\lambda_X + \sigma_A \phi_T)t}) + \frac{\gamma_X \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T}(1 - e^{-(\lambda_X + \sigma_A \phi_T)t}) \tag{12}$$

3 Equilibrium time

The equilibrium time for xenon should be when the concentration achieves a steady state; i.e., the rate of change in the concentration is zero.

Then, compute the derivative of the xenon concentration model in eq. 12 -

$$\begin{aligned}
\frac{dX}{dt} =& \\
&+ [\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T}]((\lambda_X + \sigma_A \phi_T)e^{-(\lambda_X + \sigma_A \phi_T)t}) \\
&- [\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I}]((\lambda_X + \sigma_A \phi_T)e^{-(\lambda_X + \sigma_A \phi_T)t} - \lambda_I e^{-\lambda_I t}) \\
&+ [\frac{\gamma_X \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T}]((\lambda_X + \sigma_A \phi_T)e^{-(\lambda_X + \sigma_A \phi_T)t})
\end{aligned} \tag{13}$$

Multiply out all the terms -

$$\begin{aligned}
\frac{dX}{dt} =& \\
&+ \gamma_I \Sigma_F \phi_T e^{-(\lambda_X + \sigma_A \phi_T)t} \\
&- \frac{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} e^{-(\lambda_X + \sigma_A \phi_T)t} \\
&+ \frac{(\gamma_I \Sigma_F \phi_T)(\lambda_I)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} e^{-\lambda_I t} \\
&+ \gamma_X \Sigma_F \phi_T e^{-(\lambda_X + \sigma_A \phi_T)t}
\end{aligned} \tag{14}$$

Let $\frac{dX}{dt} = 0$ and group the terms by exponential -

$$(\gamma_I \Sigma_F \phi_T + \gamma_X \Sigma_F \phi_T - \frac{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)}) e^{-(\lambda_X + \sigma_A \phi_T)t} + \frac{(\gamma_I \Sigma_F \phi_T)(\lambda_I)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} e^{-\lambda_I t} = 0 \tag{15}$$

The next series of steps is just some algebra -

$$\begin{aligned} -(\gamma_I \Sigma_F \phi_T + \gamma_X \Sigma_F \phi_T - \frac{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)}) e^{-(\lambda_X + \sigma_A \phi_T)t} &= \frac{(\gamma_I \Sigma_F \phi_T)(\lambda_I)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} e^{-\lambda_I t} \\ (-\gamma_I \Sigma_F \phi_T - \gamma_X \Sigma_F \phi_T + \frac{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)}) e^{-(\lambda_X + \sigma_A \phi_T)t} &= \frac{(\gamma_I \Sigma_F \phi_T)(\lambda_I)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} e^{-\lambda_I t} \end{aligned} \tag{16}$$

$$(\frac{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} - \gamma_I \Sigma_F \phi_T - \gamma_X \Sigma_F \phi_T) e^{-(\lambda_X + \sigma_A \phi_T)t} = \frac{(\gamma_I \Sigma_F \phi_T)(\lambda_I)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} e^{-\lambda_I t} \tag{17}$$

Divide through eq. 17 by $e^{-\lambda_I t}$ and $(\frac{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} - \gamma_I \Sigma_F \phi_T - \gamma_X \Sigma_F \phi_T)$ -

$$\frac{e^{-(\lambda_X + \sigma_A \phi_T)t}}{e^{-\lambda_I t}} = \frac{\frac{(\gamma_I \Sigma_F \phi_T)(\lambda_I)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)}}{(\frac{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} - \gamma_I \Sigma_F \phi_T - \gamma_X \Sigma_F \phi_T)} \tag{18}$$

Compute the exponential division -

$$e^{(\lambda_I - (\lambda_X + \sigma_A \phi_T))t} = \frac{\frac{(\gamma_I \Sigma_F \phi_T)(\lambda_I)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)}}{(\frac{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} - \gamma_I \Sigma_F \phi_T - \gamma_X \Sigma_F \phi_T)} \tag{19}$$

Compute the natural logarithm -

$$(\lambda_I - (\lambda_X + \sigma_A \phi_T))t = ln[\frac{\frac{(\gamma_I \Sigma_F \phi_T)(\lambda_I)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)}}{(\frac{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} - \gamma_I \Sigma_F \phi_T - \gamma_X \Sigma_F \phi_T)}] \tag{20}$$

Solve for time (t) -

$$t = \frac{1}{\lambda_I - (\lambda_X + \sigma_A \phi_T)} \cdot ln[\frac{\frac{(\gamma_I \Sigma_F \phi_T)(\lambda_I)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)}}{(\frac{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} - \gamma_I \Sigma_F \phi_T - \gamma_X \Sigma_F \phi_T)}] \tag{21}$$

Simplify $\ln[\cdot]$ -

$$\frac{\frac{(\gamma_I \Sigma_F \phi_T)(\lambda_I)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} \cdot (\lambda_X + \sigma_A \phi_T - \lambda_I)}{(\frac{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T)}{(\lambda_X + \sigma_A \phi_T - \lambda_I)} \cdot (\lambda_X + \sigma_A \phi_T - \lambda_I) - \gamma_I \Sigma_F \phi_T (\lambda_X + \sigma_A \phi_T - \lambda_I) - \gamma_X \Sigma_F \phi_T (\lambda_X + \sigma_A \phi_T - \lambda_I))} \tag{22}$$

$$\frac{\gamma_I \lambda_I \Sigma_F \phi_T}{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T) - \gamma_I \Sigma_F \phi_T (\lambda_X + \sigma_A \phi_T - \lambda_I) - \gamma_X \Sigma_F \phi_T (\lambda_X + \sigma_A \phi_T - \lambda_I)} \tag{23}$$

$$\frac{\gamma_I \lambda_I \Sigma_F \phi_T}{(\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T) - (\gamma_I \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T - \lambda_I) - (\gamma_X \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T - \lambda_I)} \tag{24}$$

$$\frac{\gamma_I \lambda_I \Sigma_F \phi_T}{(\gamma_I \Sigma_F \phi_T)((\lambda_X + \sigma_A \phi_T) - (\lambda_X + \sigma_A \phi_T - \lambda_I)) - (\gamma_X \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T - \lambda_I)} \tag{25}$$

$$\frac{\gamma_I \lambda_I \Sigma_F \phi_T}{(\gamma_I \Sigma_F \phi_T)(\gamma_X + \sigma_A \phi_T - \gamma_X - \sigma_A \phi_T + \lambda_I) - (\gamma_X \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T - \lambda_I)} \tag{26}$$

$$\frac{\gamma_I \lambda_I \Sigma_F \phi_T}{\gamma_I \lambda_I \Sigma_F \phi_T - (\gamma_X \Sigma_F \phi_T)(\lambda_X + \sigma_A \phi_T - \lambda_I)} \tag{27}$$

$$\frac{\gamma_I \lambda_I \Sigma_F(\phi_T)}{\gamma_I \lambda_I \Sigma_F(\phi_T) - (\gamma_X \Sigma_F(\phi_T))(\lambda_X + \sigma_A \phi_T - \lambda_I)} \tag{28}$$

$$\frac{\gamma_I \lambda_I \Sigma_F}{\gamma_I \lambda_I \Sigma_F - \gamma_X \Sigma_F(\lambda_X + \sigma_A \phi_T - \lambda_I)} \tag{29}$$

Finally, substitute eq. 29 back into eq. 30 -

$$t = \frac{1}{\lambda_I - (\lambda_X + \sigma_A \phi_T)} \cdot ln[\frac{\gamma_I \lambda_I \Sigma_F}{\gamma_I \lambda_I \Sigma_F - \gamma_X \Sigma_F(\lambda_X + \sigma_A \phi_T - \lambda_I)}] \tag{30}$$