NE504 - Nuclear fuel cycle analysis
Xenon neutron poison production

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1 Mathematical models

1.1 Iodine

Rate of change of iodine -

$$\begin{aligned} \frac{dI}{dt} &= \gamma_I \Sigma_F \phi_T - \lambda_I I \\ I(0) &= 0 \end{aligned} \tag{1}$$

1.2 Xenon

Rate of change of xenon -

$$\begin{aligned} \frac{dX}{dt} &= \lambda_I I + \gamma_X \Sigma_F \phi_T - \lambda_X X - \sigma_A \phi_T X \\ X(0) &= 0 \end{aligned} \tag{2}$$

2 Solutions

2.1 Iodine

Apply laplace transform to eq. 51 -

$$\begin{aligned} \frac{dI}{dt} &= \gamma_I \Sigma_F \phi_T - \lambda_I I \\ s\tilde{I} &= \frac{1}{s}\gamma_I \Sigma_F \phi_T - \lambda_I \tilde{I} \end{aligned} \tag{3}$$

Rearrange terms to obtain laplace solution -

$$\tilde{I} = \frac{1}{s(s+\lambda_I)}\gamma_I \Sigma_F \phi_T \tag{4}$$

Invert eq. 4 to obtain real time domain solution -

$$I(t) = \frac{\gamma_I \Sigma_F \phi_T}{\lambda_I}(1 - e^{-\lambda_I t}) \tag{5}$$

2.2 Xenon

Substitute the iodine solution in eq. 14 into eq. 53 -

$$\begin{aligned} \frac{dX}{dt} &= \lambda_I I + \gamma_X \Sigma_F \phi_T - \lambda_X X - \sigma_A \phi_T X \\ \frac{dX}{dt} &= \lambda_I \frac{\gamma_I \Sigma_F \phi_T}{\lambda_I}(1 - e^{-\lambda_I t}) + \gamma_X \Sigma_F \phi_T - \lambda_X X - \sigma_A \phi_T X \end{aligned} \tag{6}$$

Rearrange the terms and simplify -

$$\begin{aligned} \frac{dX}{dt} &= \gamma_I \Sigma_F \phi_T (1 - e^{-\lambda_I t}) + \gamma_X \Sigma_F \phi_T - \lambda_X X - \sigma_A \phi_T X \\ \frac{dX}{dt} &= \gamma_I \Sigma_F \phi_T - \gamma_I \Sigma_F \phi_T e^{-\lambda_I t} + \gamma_X \Sigma_F \phi_T - \lambda_X X - \sigma_A \phi_T X \end{aligned} \tag{7}$$

$$\frac{dX}{dt} = \gamma_I \Sigma_F \phi_T - \gamma_I \Sigma_F \phi_T e^{-\lambda_I t} + \gamma_X \Sigma_F \phi_T - (\lambda_X + \sigma_A \phi_T) X \tag{8}$$

Apply laplace transform to eq. 8 -

$$s\tilde{X} = \frac{1}{s}\gamma_I \Sigma_F \phi_T - \frac{1}{s+\lambda_I}\gamma_I \Sigma_F \phi_T + \frac{1}{s}\gamma_X \Sigma_F \phi_T - (\lambda_X + \sigma_A \phi_T)\tilde{X} \tag{9}$$

Rearrange the terms in eq. 9 to obtain the laplace solution -

$$(s + [\lambda_X + \sigma_A \phi_T])\tilde{X} = \frac{1}{s}\gamma_I \Sigma_F \phi_T - \frac{1}{s+\lambda_I}\gamma_I \Sigma_F \phi_T + \frac{1}{s}\gamma_X \Sigma_F \phi_T \tag{10}$$

Then -

$$\tilde{X} = +\frac{1}{s(s + [\lambda_X + \sigma_A \phi_T])}\gamma_I \Sigma_F \phi_T - \frac{1}{(s+\lambda_I)(s + [\lambda_X + \sigma_A \phi_T])}\gamma_I \Sigma_F \phi_T + \frac{1}{s(s + [\lambda_X + \sigma_A \phi_T])}\gamma_X \Sigma_F \phi_T \tag{11}$$

Invert eq. 11 to obtain real time domain solution -

$$X(t) = \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T}(1 - e^{-(\lambda_X + \sigma_A \phi_T)t}) - \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I}(e^{-\lambda_I t} - e^{-(\lambda_X + \sigma_A \phi_T)t}) + \frac{\gamma_X \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T}(1 - e^{-(\lambda_X + \sigma_A \phi_T)t}) \tag{12}$$

3 Derivatives

3.1 Iodine

$$I(t) = \frac{\gamma_I \Sigma_F \phi_T}{\lambda_I}(1 - e^{-\lambda_I t}) \tag{13}$$

Expand -

$$I(t) = \frac{\gamma_I \Sigma_F \phi_T}{\lambda_I} - \frac{\gamma_I \Sigma_F \phi_T}{\lambda_I} e^{-\lambda_I t} \tag{14}$$

Compute derivative -

$$\begin{aligned} \frac{dI}{dt} &= \frac{d}{dt}[\frac{\gamma_I \Sigma_F \phi_T}{\lambda_I}] - \frac{d}{dt}[\frac{\gamma_I \Sigma_F \phi_T}{\lambda_I} e^{-\lambda_I t}] \\ \frac{dI}{dt} &= 0 - (-\lambda_I)(\frac{\gamma_I \Sigma_F \phi_T}{\lambda_I} e^{-\lambda_I t}) \end{aligned} \tag{15}$$

Therefore -

$$\frac{dI}{dt} = \gamma_I \Sigma_F \phi_T e^{-\lambda_I t} \tag{16}$$

3.2 Xenon

$$X(t) = \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T}(1 - e^{-(\lambda_X + \sigma_A \phi_T)t}) - \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I}(e^{-\lambda_I t} - e^{-(\lambda_X + \sigma_A \phi_T)t}) + \frac{\gamma_X \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T}(1 - e^{-(\lambda_X + \sigma_A \phi_T)t}) \tag{17}$$

Rearrange to make computing the derivative easier -

$$\begin{aligned} X(t) = & \\ & (\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} - \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} e^{-(\lambda_X + \sigma_A \phi_T)t}) \\ & - (\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} - \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-(\lambda_X + \sigma_A \phi_T)t}) \\ & + (\frac{\gamma_X \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} - \frac{\gamma_X \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} e^{-(\lambda_X + \sigma_A \phi_T)t}) \end{aligned} \tag{18}$$

$$\begin{aligned}X(t) = &\\ &(\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} - \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} e^{-(\lambda_X + \sigma_A \phi_T)t}) \\ &- (\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} - \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-(\lambda_X + \sigma_A \phi_T)t}) \\ &+ (\frac{\gamma_X \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} - \frac{\gamma_X \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} e^{-(\lambda_X + \sigma_A \phi_T)t})\end{aligned} \tag{19}$$

$$\begin{aligned}X(t) = &\\ &\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} - \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} e^{-(\lambda_X + \sigma_A \phi_T)t} \\ &- \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} + \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-(\lambda_X + \sigma_A \phi_T)t} \\ &+ \frac{\gamma_X \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} - \frac{\gamma_X \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} e^{-(\lambda_X + \sigma_A \phi_T)t}\end{aligned} \tag{20}$$

Group the terms by exponent -

$$\begin{aligned}X(t) = &\\ &\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} + \frac{\gamma_X \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} \\ &- \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} \\ &+ \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-(\lambda_X + \sigma_A \phi_T)t} - \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} e^{-(\lambda_X + \sigma_A \phi_T)t} - \frac{\gamma_X \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} e^{-(\lambda_X + \sigma_A \phi_T)t}\end{aligned} \tag{21}$$

$$X(t) = \frac{(\gamma_I + \gamma_X) \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} - \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} + \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-(\lambda_X + \sigma_A \phi_T)t} - \frac{(\gamma_I + \gamma_X) \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} e^{-(\lambda_X + \sigma_A \phi_T)t} \tag{22}$$

$$X(t) = \frac{(\gamma_I + \gamma_X) \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} - \frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} + (\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} - \frac{(\gamma_I + \gamma_X) \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T}) e^{-(\lambda_X + \sigma_A \phi_T)t} \tag{23}$$

Now compute the derivative -

$$\frac{dX}{dt} = \frac{d}{dt}[\frac{(\gamma_I + \gamma_X) \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T}] - \frac{d}{dt}[\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t}] + \frac{d}{dt}[(\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T - \lambda_I} - \frac{(\gamma_I + \gamma_X) \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T}) e^{-(\lambda_X + \sigma_A \phi_T)t}] \tag{24}$$

$$\frac{dX}{dt} = 0 + (\lambda_I)\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A\phi_T - \lambda_I}e^{-\lambda_I t} - (\lambda_X + \sigma_A\phi_T)(\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A\phi_T - \lambda_I} - \frac{(\gamma_I + \gamma_X)\Sigma_F\phi_T}{\lambda_X + \sigma_A\phi_T})e^{-(\lambda_X+\sigma_A\phi_T)t} \tag{25}$$

$$\frac{dX}{dt} = +(\lambda_I)\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A\phi_T - \lambda_I}e^{-\lambda_I t} - (\lambda_X + \sigma_A\phi_T)(\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A\phi_T - \lambda_I} - \frac{(\gamma_I + \gamma_X)\Sigma_F\phi_T}{\lambda_X + \sigma_A\phi_T})e^{-(\lambda_X+\sigma_A\phi_T)t} \tag{26}$$

$$\frac{dX}{dt} = \frac{\gamma_I \lambda_I \Sigma_F \phi_T}{\lambda_X + \sigma_A\phi_T - \lambda_I}e^{-\lambda_I t} - (\lambda_X + \sigma_A\phi_T)(\frac{\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A\phi_T - \lambda_I} - \frac{(\gamma_I + \gamma_X)\Sigma_F\phi_T}{\lambda_X + \sigma_A\phi_T})e^{-(\lambda_X+\sigma_A\phi_T)t} \tag{27}$$

$$\frac{dX}{dt} = \frac{\gamma_I \lambda_I \Sigma_F \phi_T}{\lambda_X + \sigma_A\phi_T - \lambda_I}e^{-\lambda_I t} - (\frac{(\lambda_X + \sigma_A\phi_T)\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A\phi_T - \lambda_I} - \frac{(\lambda_X + \sigma_A\phi_T)(\gamma_I + \gamma_X)\Sigma_F\phi_T}{\lambda_X + \sigma_A\phi_T})e^{-(\lambda_X+\sigma_A\phi_T)t} \tag{28}$$

$$\frac{dX}{dt} = \frac{\gamma_I \lambda_I \Sigma_F \phi_T}{\lambda_X + \sigma_A\phi_T - \lambda_I}e^{-\lambda_I t} - (\frac{(\lambda_X + \sigma_A\phi_T)\gamma_I \Sigma_F \phi_T}{\lambda_X + \sigma_A\phi_T - \lambda_I} - (\gamma_I + \gamma_X)\Sigma_F\phi_T)e^{-(\lambda_X+\sigma_A\phi_T)t} \tag{29}$$

$$\frac{dX}{dt} = \frac{\gamma_I \lambda_I \Sigma_F \phi_T}{\lambda_X + \sigma_A\phi_T - \lambda_I}e^{-\lambda_I t} - \Sigma_F\phi_T(\frac{(\lambda_X + \sigma_A\phi_T)\gamma_I}{\lambda_X + \sigma_A\phi_T - \lambda_I} - (\gamma_I + \gamma_X))e^{-(\lambda_X+\sigma_A\phi_T)t} \tag{30}$$

$$\frac{dX}{dt} = \Sigma_F\phi_T\frac{\gamma_I \lambda_I}{\lambda_X + \sigma_A\phi_T - \lambda_I}e^{-\lambda_I t} - \Sigma_F\phi_T(\frac{(\lambda_X + \sigma_A\phi_T)\gamma_I}{\lambda_X + \sigma_A\phi_T - \lambda_I} - (\gamma_I + \gamma_X))e^{-(\lambda_X+\sigma_A\phi_T)t} \tag{31}$$

4 Equilibrium time

The equilibrium time for xenon should be when the concentration achieves a steady state; i.e., the rate of change in the concentration is zero.

$$\frac{dI}{dt} = \gamma_I \Sigma_F \phi_T e^{-\lambda_I t} \tag{32}$$

$$\frac{dX}{dt} = \Sigma_F \phi_T \frac{\gamma_I \lambda_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} - \Sigma_F \phi_T \left(\frac{(\lambda_X + \sigma_A \phi_T)\gamma_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} - (\gamma_I + \gamma_X)\right) e^{-(\lambda_X + \sigma_A \phi_T)t} \tag{33}$$

4.1 Iodine

$$\gamma_I \Sigma_F \phi_T e^{-\lambda_I t} = 0 \tag{34}$$

Here, the constants drop out -

$$e^{-\lambda_I t} = 0 \tag{35}$$

4.2 Xenon

$$\Sigma_F \phi_T \frac{\gamma_I \lambda_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} - \Sigma_F \phi_T \left(\frac{(\lambda_X + \sigma_A \phi_T)\gamma_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} - (\gamma_I + \gamma_X)\right) e^{-(\lambda_X + \sigma_A \phi_T)t} = 0 \tag{36}$$

With this equation, $\Sigma_F \phi_T$ drops out -

$$\frac{\gamma_I \lambda_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} - \left(\frac{(\lambda_X + \sigma_A \phi_T)\gamma_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} - (\gamma_I + \gamma_X)\right) e^{-(\lambda_X + \sigma_A \phi_T)t} = 0 \tag{37}$$

4.3 Solution

However, neither equation for iodine or xenon cannot be solved explicitly for t because $\ln[0]$ is not defined. There are a variety of techniques that can be applied to obtain the equilibrium time.

5 Approximation methods

5.1 Newton's method

For $f(t) = 0$ -

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$$

5.1.1 Iodine

$$e^{-\lambda_I t} = 0 \tag{38}$$

$$f_I(t) = e^{-\lambda_I t} \tag{39}$$

$$f_I'(t) = -\lambda_I e^{-\lambda_I t} \tag{40}$$

5.1.2 Xenon

$$\frac{\gamma_I \lambda_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} - (\frac{(\lambda_X + \sigma_A \phi_T)\gamma_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} - (\gamma_I + \gamma_X))e^{-(\lambda_X + \sigma_A \phi_T)t} = 0 \tag{41}$$

$$f_X(t) = \frac{\gamma_I \lambda_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} - (\frac{(\lambda_X + \sigma_A \phi_T)\gamma_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} - (\gamma_I + \gamma_X))e^{-(\lambda_X + \sigma_A \phi_T)t} \tag{42}$$

$$f_X'(t) = -\frac{\gamma_I \lambda_I \lambda_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} + (\lambda_X + \sigma_A \phi_T)(\frac{(\lambda_X + \sigma_A \phi_T)\gamma_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} - (\gamma_I + \gamma_X))e^{-(\lambda_X + \sigma_A \phi_T)t} \tag{43}$$

$$f_X'(t) = -\frac{\gamma_I \lambda_I \lambda_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} + (\lambda_X + \sigma_A \phi_T)(\frac{(\lambda_X + \sigma_A \phi_T)\gamma_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} - \gamma_I - \gamma_X)e^{-(\lambda_X + \sigma_A \phi_T)t} \tag{44}$$

$$f_X'(t) = -\frac{\gamma_I \lambda_I \lambda_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} + (\lambda_X + \sigma_A \phi_T)(\frac{\gamma_I(\lambda_X + \sigma_A \phi_T)}{\lambda_X + \sigma_A \phi_T - \lambda_I} - \frac{\gamma_I(\lambda_X + \sigma_A \phi_T - \lambda_I)}{\lambda_X + \sigma_A \phi_T - \lambda_I} - \gamma_X)e^{-(\lambda_X + \sigma_A \phi_T)t} \tag{45}$$

$$f_X'(t) = -\frac{\gamma_I \lambda_I \lambda_I}{\lambda_X + \sigma_A \phi_T - \lambda_I} e^{-\lambda_I t} + (\frac{\gamma_I(\lambda_X + \sigma_A \phi_T)(\lambda_X + \sigma_A \phi_T)}{\lambda_X + \sigma_A \phi_T - \lambda_I} - \frac{\gamma_I(\lambda_X + \sigma_A \phi_T)(\lambda_X + \sigma_A \phi_T - \lambda_I)}{\lambda_X + \sigma_A \phi_T - \lambda_I} - \gamma_X(\lambda_X + \sigma_A \phi_T))e^{-(\lambda_X + \sigma_A \phi_T)t} \tag{46}$$

$$f'_X(t) = -\frac{\gamma_I\lambda_I\lambda_I}{\lambda_X+\sigma_A\phi_T-\lambda_I}e^{-\lambda_I t} + (\frac{\gamma_I(\lambda_X+\sigma_A\phi_T)(\lambda_X+\sigma_A\phi_T) - \gamma_I(\lambda_X+\sigma_A\phi_T)(\lambda_X+\sigma_A\phi_T-\lambda_I)}{\lambda_X+\sigma_A\phi_T-\lambda_I} - \gamma_X(\lambda_X+\sigma_A\phi_T))e^{-(\lambda_X+\sigma_A\phi_T)t} \tag{47}$$

$$f'_X(t) = -\frac{\gamma_I\lambda_I\lambda_I}{\lambda_X+\sigma_A\phi_T-\lambda_I}e^{-\lambda_I t} + (\frac{\gamma_I(\lambda_X+\sigma_A\phi_T)(\lambda_X+\sigma_A\phi_T-\lambda_X-\sigma_A\phi_T+\lambda_I)}{\lambda_X+\sigma_A\phi_T-\lambda_I} - \gamma_X(\lambda_X+\sigma_A\phi_T))e^{-(\lambda_X+\sigma_A\phi_T)t} \tag{48}$$

$$f'_X(t) = -\frac{\gamma_I\lambda_I\lambda_I}{\lambda_X+\sigma_A\phi_T-\lambda_I}e^{-\lambda_I t} + (\frac{\gamma_I\lambda_I(\lambda_X+\sigma_A\phi_T)}{\lambda_X+\sigma_A\phi_T-\lambda_I} - \gamma_X(\lambda_X+\sigma_A\phi_T))e^{-(\lambda_X+\sigma_A\phi_T)t} \tag{49}$$

6 Steady state concentrations

The concentration of I and Xe at steady state can be obtained by placing the derivatives equal to zero; $\frac{dI}{dt} = 0$ and $\frac{dX}{dt} = 0$.

Iodine -

$$\frac{dI}{dt} = \gamma_I \Sigma_F \phi_T - \lambda_I I \tag{50}$$

$$\begin{aligned} \gamma_I \Sigma_F \phi_T - \lambda_I I &= 0 \\ \lambda_I I &= \gamma_I \Sigma_F \phi_T \\ I_\infty &= \frac{\gamma_I \Sigma_F \phi_T}{\lambda_I} \end{aligned} \tag{51}$$

$$\frac{dX}{dt} = \lambda_I I + \gamma_X \Sigma_F \phi_T - \lambda_X X - \sigma_A \phi_T X \tag{52}$$

$$\begin{aligned} \lambda_I I + \gamma_X \Sigma_F \phi_T - \lambda_X X - \sigma_A \phi_T X &= 0 \\ \lambda_I (\frac{\gamma_I \Sigma_F \phi_T}{\lambda_I}) + \gamma_X \Sigma_F \phi_T - (\lambda_X + \sigma_A \phi_T) X &= 0 \\ (\lambda_X + \sigma_A \phi_T) X &= \lambda_I (\frac{\gamma_I \Sigma_F \phi_T}{\lambda_I}) + \gamma_X \Sigma_F \phi_T \\ (\lambda_X + \sigma_A \phi_T) X &= \gamma_I \Sigma_F \phi_T + \gamma_X \Sigma_F \phi_T \\ X_\infty &= \frac{(\gamma_I + \gamma_X) \Sigma_F \phi_T}{\lambda_X + \sigma_A \phi_T} \end{aligned} \tag{53}$$