

# NE504 - Nuclear fuel cycle analysis

## Two group reactor equation

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## 1 Equation of continuity

The equation of continuity for neutrons describes behavior in a reactor.

[rate of change of neutrons] = [production rate] - [absorption rate] - [leakage rate]

$$\frac{d}{dt} \int_V n dV = \int_V s dV - \int_V \Sigma_A \phi dV - \int_V \underline{J} \cdot \underline{n} dA \quad (1)$$

The terms are -

- $\int_V n dV$  - total number of neutrons
- $\frac{d}{dt} \int_V n dV$  - rate of change
- $\int_V s dV$  - production rate
- $\int_V \Sigma_A \phi dV$  - absorption rate
- $\int_A \underline{J} \cdot \underline{n} dA$  leakage rate

Applying some mathematical relationships -

$$\begin{aligned} \frac{d}{dt} \int_V n dV &= \int_V \frac{\partial n}{\partial t} dV \\ \int_A \underline{J} \cdot \underline{n} dA &= \int_V \nabla \cdot \underline{J} dV \end{aligned} \quad (2)$$

Substitute into eq. 1 -

$$\int_V \frac{\partial n}{\partial t} dV = \int_V s dV - \int_V \Sigma_A \phi dV - \int_V \nabla \cdot \underline{J} dV \quad (3)$$

Because the control volume is the same -

$$\frac{\partial n}{\partial t} = s - \Sigma_A \phi - \nabla \cdot \underline{J} \quad (4)$$

Apply Fick's law -

$$\begin{aligned} \frac{\partial n}{\partial t} &= D \nabla^2 \phi - \Sigma_A \phi + s \\ \frac{1}{v} \frac{\partial \phi}{\partial t} &= D \nabla^2 \phi - \Sigma_A \phi + s \end{aligned} \quad (5)$$

Assume steady state -

$$D \nabla^2 \phi - \Sigma_A \phi + s = 0 \quad (6)$$

## 2 Multigroup diffusion theory

The general multigroup diffusion equation based on eq. 6 for an arbitrary group  $g$  is -

$$D_g \nabla^2 \phi_g - \Sigma_A^g \phi_g - \sum_{h=g+1}^N \Sigma_{g \rightarrow h} \phi_h + \sum_{h=1}^{g-1} \Sigma_{h \rightarrow g} \phi_h + s_g = 0 \quad (7)$$

In addition to absorption in group  $g$ , there is the loss due to scatter from group  $g$  to  $h$  but also a source terms due to scatter from  $h$  to  $g$ .

For 2 groups, then; i.e., a fast and thermal group, eq. 7 is -

$$\begin{aligned} D_1 \nabla^2 \phi_1 - \Sigma_A^1 \phi_1 - \Sigma_{1 \rightarrow 2} \phi_1 + \Sigma_{2 \rightarrow 1} \phi_2 + s_1 &= 0 \\ D_T \nabla^2 \phi_T - \Sigma_A^T \phi_T - \Sigma_{T \rightarrow 1} \phi_T + s_T &= 0 \end{aligned} \quad (8)$$

Neutrons will not scatter from the thermal group ( $T$ ) to the fast group (1) -

$$\begin{aligned} D_1 \nabla^2 \phi_1 - \Sigma_A^1 \phi_1 - \Sigma_{1 \rightarrow 2} \phi_1 + s_1 &= 0 \\ D_T \nabla^2 \phi_T - \Sigma_A^T \phi_T + s_T &= 0 \end{aligned} \quad (9)$$

### 3 Solution

#### 3.1 Source

(1) Assume the bulk of fissions are induced by thermal neutrons.

$$s_1 = \eta_T f \epsilon \Sigma_A^T \phi_T = \frac{k_\infty}{p} \Sigma_A^T \phi_T \quad (10)$$

(2) Fast neutrons scattering into the thermal region must survive through the resonances.

$$s_T = p \Sigma_{1 \rightarrow T} \phi_1 \quad (11)$$

#### 3.2 Reactor equations

(1) Assume a one-dimensional, infinite slab.

(2) Compute the derivatives -

Let -

$$\frac{d^2 \phi}{dx^2} \equiv \phi''$$

(3) Substitute the sources in eqns. 10 and 11 into eq. 9 -

$$D_1 \phi_1'' - \Sigma_A^1 \phi_1 - \Sigma_{1 \rightarrow T} \phi_1 + \frac{k_\infty}{p} \Sigma_A^T \phi_T = 0 \quad (12)$$

$$D_T \phi_T'' - \Sigma_A^T \phi_T + p \Sigma_{1 \rightarrow T} \phi_1 = 0 \quad (13)$$

(4) Then -

$$\phi_1'' - \frac{1}{K^2} \phi_1 + f \phi_T = 0 \quad (14)$$

$$\phi_T'' + g \phi_1 - \frac{1}{L^2} \phi_T = 0 \quad (15)$$

Where -

$$K^2 = \frac{D_1}{\Sigma_{1 \rightarrow T} + \Sigma_A^1}$$

$$L^2 = \frac{D_T}{\Sigma_A^T}$$

$$f = \frac{k \Sigma_A^T}{p D_1}$$

$$g = \frac{p \Sigma_{1 \rightarrow T}}{D_T}$$

### 3.3 Eigenvalue approach

(1) Put in matrix form -

$$\underline{\phi}'' = \underline{A} \cdot \underline{\phi} \quad (16)$$

$$\begin{bmatrix} \phi_1'' \\ \phi_T'' \end{bmatrix} = \begin{bmatrix} \frac{1}{K^2} & -f \\ -g & \frac{1}{L^2} \end{bmatrix} \cdot \begin{bmatrix} \phi_1 \\ \phi_T \end{bmatrix} \quad (17)$$

(2) Find eigenvalues -

$$\underline{A} \cdot \underline{x} = \lambda^2 \underline{x}$$

$$(\underline{A} - \lambda^2 \underline{I}) \underline{x} = 0$$

$$\det(\underline{A} - \lambda^2 \underline{I}) = 0$$

(3) Applying eqns. 14 and 15 -

$$\det \begin{bmatrix} \frac{1}{K^2} - \lambda^2 & -f \\ -g & \frac{1}{L^2} - \lambda^2 \end{bmatrix} = 0 \quad (18)$$

$$\frac{1}{K^2} \left( \frac{1}{L^2} - \lambda^2 \right) + \lambda^4 - \lambda^2 \frac{1}{L^2} + fg = 0 \quad (19)$$

(4) Then -

$$K^2 L^2 \lambda^4 - (L^2 + K^2) \lambda^2 + fg K^2 L^2 + 1 = 0 \quad (20)$$

Eq. 20 is quadratic in  $\lambda^2$

(5) The general solution (p585) is -

$$\underline{\phi} = \sum_{j=1}^n \left( a_j \cos \omega_j t + b_j \frac{\sin \omega_j t}{\omega_j} \right) \underline{x}_j \quad (21)$$

Where the eigenvalues ( $\lambda$ ) relate as -

$$\lambda_j = -\omega_j^2$$

Eq. 21 assumes all eigenvalues of  $\underline{A}$  are negative or zero.