NE504 - Nuclear fuel cycle analysis
Two group diffusion

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1 Equation of continuity

The equation of continuity for neutrons describes behavior in a reactor.

[rate of change of neutrons] = [production rate] - [absorption rate] - [leakage rate]

$$\frac{d}{dt}\int_V n dV = \int_V s dV - \int_V \Sigma_A \phi dV - \int_V \underline{J} \cdot \underline{n} dA \tag{1}$$

The terms are -
- $\int_V n dV$ - total number of neutrons

- $\frac{d}{dt}\int_V n dV$ - rate of change

- $\int_V s dV$ - production rate

- $\int_V \Sigma_A \phi dV$ - absorption rate

- $\int_A \underline{J} \cdot \underline{n} dA$ leakage rate

Applying some mathematical relationships -

$$\frac{d}{dt}\int_V n dV = \int_V \frac{\partial n}{\partial t} dV$$
$$\int_A \underline{J} \cdot \underline{n} dA = \int_V \nabla \underline{J} dV \tag{2}$$

Substitute into eq. 1 -

$$\int_V \frac{\partial n}{\partial t} dV = \int_V s dV - \int_V \Sigma_A \phi dV - \int_V \nabla \underline{J} dV \tag{3}$$

Because the control volume is the same -

$$\frac{\partial n}{\partial t} = s - \Sigma_A \phi - \nabla J \tag{4}$$

Apply Fick's law -

$$\frac{\partial n}{\partial t} = D\nabla^2 \phi - \Sigma_A \phi + s$$
$$\frac{1}{v}\frac{\partial \phi}{\partial t} = D\nabla^2 \phi - \Sigma_A \phi + s \tag{5}$$

Assume steady state -

$$D\nabla^2 \phi - \Sigma_A \phi + s = 0 \tag{6}$$

2 Multigroup diffusion theory

The general multigroup diffusion equation based on eq. 6 for an arbitrary group g is -

$$D_g \nabla^2 \phi_g - \Sigma_A^g \phi_g - \sum_{h=g+1}^{N} \Sigma_{g \to h} \phi_g + \sum_{h=1}^{g-1} \Sigma_{h \to g} \phi_h + s_g = 0 \tag{7}$$

In addition to absorption in group g, there is the loss due to scatter from group g to h but also a source terms due to scatter from h to g.

For 2 groups, then; i.e., a fast and thermal group, eq. 7 is -

$$D_1 \nabla^2 \phi_1 - \Sigma_A^1 \phi_1 - \Sigma_{1 \to 2} \phi_1 + \Sigma_{2 \to 1} \phi_2 + s = 0$$
$$D_2 \nabla^2 \phi_2 - \Sigma_A^2 \phi_2 - \Sigma_{2 \to 1} \phi_2 + \Sigma_{1 \to 2} \phi_1 = 0$$

(8)

Neutrons will not scatter from the thermal group (2) to the fast group (1) -

$$D_1 \nabla^2 \phi_1 - \Sigma_A^1 \phi_1 - \Sigma_{1 \to 2} \phi_1 + s = 0$$
$$D_2 \nabla^2 \phi_2 - \Sigma_A^2 \phi_2 + \Sigma_{1 \to 2} \phi_1 = 0$$

(9)

3   Solution

(1)   Assume a one-dimensional, infinite slab.

(2)   Compute the derivatives -

$$\frac{d^2 \phi_1}{dx^2} - \frac{1}{K^2} \phi_1 + d = 0 \tag{10}$$

$$\frac{d^2 \phi_2}{dx^2} - \frac{1}{L^2} \phi_2 + \Sigma_{1 \to 2} \phi_1 = 0 \tag{11}$$

Where -

$$L^2 = \frac{D_2}{\Sigma_A^2}$$

$$K^2 = \frac{D_1}{\Sigma_A^1 + \Sigma_{1 \to 2}}$$

$$d = \frac{s}{D_1}$$

Let -

$$\frac{d^2 \phi}{dx^2} \equiv \phi''$$

3.1   Fast group

(1)   Assume the solution is the sum of a homogeneous and a particular solution.

$$\phi_1(x) = \phi_H(x) + \phi_P(x) \tag{12}$$

(2)   Making use of eq. 10 -

$$\frac{d^2 \phi_H}{dx^2} - \frac{1}{K^2} \phi_H = 0 \tag{13}$$

(3)   Assume the homogeneous solution -

$$A e^{-\frac{x}{K}} + C e^{\frac{x}{K}} \tag{14}$$

(4)   Assume the particular solution is an arbitrary constant -

$$\phi_P = B \tag{15}$$

(5)   Substitute eq. 15 in to eq. 10 -

$$0 - \frac{1}{K^2} B + d = 0 \tag{16}$$

(6)   The particular solution is then -

$$\phi_P = dK^2 \tag{17}$$

(7)   The fast flux is then -

$$\phi_1 = A_1 e^{-\frac{x}{K}} + C_1 e^{\frac{x}{K}} + dK^2 \tag{18}$$

2

3.2 Thermal flux

(1) Substitute eq. 18 into eq. 11, using the shorthand notation ($''$) for the second derivative -

$$\phi_2'' - \frac{1}{L^2}\phi_2 + \Sigma_1 A_1 e^{-\frac{x}{K}} + \Sigma_1 C_1 e^{\frac{x}{K}} + \Sigma_1 dK^2 = 0 \tag{19}$$

Where -

$$\Sigma_{1 \to 2} \equiv \Sigma_1$$

(2) Following the same procedure, assume the solution is the sum of a homogeneous and a particular solution.

$$\phi_2 = \phi_H + \phi_P \tag{20}$$

(3) Making use of eq. 11 -

$$\phi_H'' - \frac{1}{L^2}\phi_H = 0 \tag{21}$$

(4) Assume the homogeneous solution -

$$\phi_H = A_2 e^{-\frac{x}{L}} + C_2 e^{\frac{x}{L}} \tag{22}$$

(5) Apply the following theorem for the particular solution -

(a) If $z_1(x)$ is a particular solution of -

$$y'' + p(x)y' + q(x)y = f(x)$$

(b) and $z_2(x)$ is a particular solution of -

$$y'' + p(x)y' + q(x)y = g(x)$$

(c) Then, for any number of forcing functions, $z = z_1(x) + z_2(x)$ is a particular solution of -

$$y'' + p(x)y' + q(x)y = f(x) + g(x)$$

(6) Therefore, because there are three forcing functions in eq. 19 -

$$\phi_P = \phi_F + \phi_G + \phi_I \tag{23}$$

(7) Based on the theorem -

$$\phi_F'' - \frac{1}{L^2}\phi_F + \Sigma_1 A_1 e^{-\frac{x}{K}}$$
$$\phi_G'' - \frac{1}{L^2}\phi_G + \Sigma_1 C_1 e^{\frac{x}{K}} \tag{24}$$
$$\phi_I'' - \frac{1}{L^2}\phi_I + \Sigma_1 dK^2$$

The diffusion length (L^2) does not need to be split because the geometry is the same.

(8) Assume the correponding particular solutions for each forcing function -

$$\phi_F^P = B_1 e^{-\frac{x}{K}}$$
$$\phi_G^P = B_2 e^{\frac{x}{K}} \tag{25}$$
$$\phi_I^P = B_3$$

(9) Compute the second derivative of each particular solution -

$$\phi_F'' = \frac{B_1}{K^2} e^{-\frac{x}{K}}$$

$$\phi_G'' = \frac{B_2}{K^2} e^{\frac{x}{K}} \tag{26}$$

$$\phi_H'' = 0$$

(10) Substitute eq. 26 and 25 into eq. 24 -

$$\frac{B_1}{K^2} e^{-\frac{x}{K}} - \frac{B_1}{L^2} e^{-\frac{x}{K}} + \Sigma_1 A_1 e^{-\frac{x}{K}} = 0$$

$$\frac{B_2}{K^2} e^{\frac{x}{K}} - \frac{B_2}{L^2} e^{\frac{x}{K}} + \Sigma_1 C_1 e^{\frac{x}{K}} = 0 \tag{27}$$

$$0 - \frac{B_3}{L^2} + \Sigma_1 dK^2 = 0$$

(11) Simplfying -

$$\frac{B_1}{K^2} - \frac{B_1}{L^2} + \Sigma_1 A_1 = 0$$

$$\frac{B_2}{K^2} - \frac{B_2}{L^2} + \Sigma_1 C_1 = 0 \tag{28}$$

$$\frac{B_3}{L^2} - \Sigma_1 dK^2 = 0$$

(12) Solve for the constants -

$$B_1 = \frac{\Sigma_1 A_1 K^2 L^2}{K^2 - L^2}$$

$$B_2 = \frac{\Sigma_1 C_1 K^2 L^2}{K^2 - L^2} \tag{29}$$

$$B_3 = \Sigma_1 dK^2 L^2$$

(13) Substitute the constants in eq. 29 into eq. 25 -

$$\phi_F^P = \frac{\Sigma_1 A_1 K^2 L^2}{K^2 - L^2} e^{-\frac{x}{K}}$$

$$\phi_G^P = \frac{\Sigma_1 C_1 K^2 L^2}{K^2 - L^2} e^{\frac{x}{K}} \tag{30}$$

$$\phi_I^P = \Sigma_1 dK^2 L^2$$

(14) Finally, substitute the particular solutions in eq. 30 and the homogeneous solution in eq. 22 into eq. 20 to obtain the termal flux -

$$\phi_2 = A_2 e^{-\frac{x}{L}} + C_2 e^{\frac{x}{L}} + \frac{\Sigma_1 A_1 K^2 L^2}{K^2 - L^2} e^{-\frac{x}{K}} + \frac{\Sigma_1 C_1 K^2 L^2}{K^2 - L^2} e^{\frac{x}{K}} + \Sigma_1 dK^2 L^2 \tag{31}$$

4 Solution check

4.1 Fast flux

(1) The fast flux is -

$$\phi_1 = A_1 e^{-\frac{x}{K}} + C_1 e^{\frac{x}{K}} + dK^2 \tag{32}$$

(2) And the solution must satisfy -

$$\frac{d^2\phi_1}{dx^2} - \frac{1}{K^2}\phi_1 + d = 0 \tag{33}$$

(3) Compute the second derivative of eq. 32 -

$$\phi_1'' = \frac{A_1}{K^2}e^{-\frac{x}{K}} + \frac{C_1}{K^2}e^{\frac{x}{K}} \tag{34}$$

(4) Substitute the result into eq. 33 and simplify -

$$(\frac{A_1}{K^2}e^{-\frac{x}{K}} + \frac{C_1}{K^2}e^{\frac{x}{K}}) - \frac{1}{K^2}[A_1 e^{-\frac{x}{K}} + C_1 e^{\frac{x}{K}} + dK^2] + d = 0$$

$$\frac{A_1}{K^2}e^{-\frac{x}{K}} + \frac{C_1}{K^2}e^{\frac{x}{K}} - \frac{A_1}{K^2}e^{-\frac{x}{K}} - \frac{C_1}{K^2}e^{\frac{x}{K}} - d + d = 0$$

Check satisfied.

4.2 Thermal flux

(1) The thermal flux is -

$$\phi_2 = A_2 e^{-\frac{x}{L}} + C_2 e^{\frac{x}{L}} + \frac{\Sigma_1 A_1 K^2 L^2}{K^2 - L^2}e^{-\frac{x}{K}} + \frac{\Sigma_1 C_1 K^2 L^2}{K^2 - L^2}e^{\frac{x}{K}} + \Sigma_1 dK^2 L^2 \tag{35}$$

(2) And the solution must satisfy -

$$\frac{d^2\phi_2}{dx^2} - \frac{1}{L^2}\phi_2 + \Sigma_1\phi_1 = 0 \tag{36}$$

(3) Compute the second derivative of eq. 35 -

$$\phi_2'' = \frac{A_2}{L^2}e^{-\frac{x}{L}} + \frac{C_2}{L^2}e^{\frac{x}{L}} + \frac{\Sigma_1 A_1 K^2 L^2}{(K^2 - L^2)K^2}e^{-\frac{x}{K}} + \frac{\Sigma_1 C_1 K^2 L^2}{(K^2 - L^2)K^2}e^{\frac{x}{K}} \tag{37}$$

(4) Substitute the result into eq. 36 and simplify -

$$0 =$$

$$+ [\frac{A_2}{L^2}e^{-\frac{x}{L}} + \frac{C_2}{L^2}e^{\frac{x}{L}} + \frac{\Sigma_1 A_1 K^2 L^2}{(K^2 - L^2)K^2}e^{-\frac{x}{K}} + \frac{\Sigma_1 C_1 K^2 L^2}{(K^2 - L^2)K^2}e^{\frac{x}{K}}]$$

$$- [\frac{A_2}{L^2}e^{-\frac{x}{L}} + \frac{C_2}{L^2}e^{\frac{x}{L}} + \frac{\Sigma_1 A_1 K^2 L^2}{(K^2 - L^2)L^2}e^{-\frac{x}{K}} + \frac{\Sigma_1 C_1 K^2 L^2}{(K^2 - L^2)L^2}e^{\frac{x}{K}} + \Sigma_1 dK^2]$$

$$+ [\Sigma_1 A_1 e^{-\frac{x}{K}} + \Sigma_1 C_1 e^{\frac{x}{K}} + \Sigma_1 dK^2]$$

$$0 =$$

$$+ (\frac{A_2}{L^2} - \frac{A_2}{L^2})e^{-\frac{x}{L}} + (\frac{C_2}{L^2} - \frac{C_2}{L^2})e^{\frac{x}{L}} + (\Sigma_1 dK^2 - \Sigma_1 dK^2)$$

$$+ [\frac{\Sigma_1 A_1 K^2 L^2}{(K^2 - L^2)K^2} - \frac{\Sigma_1 A_1 K^2 L^2}{(K^2 - L^2)L^2} + \Sigma_1 A_1]e^{-\frac{x}{K}}$$

$$+ [\frac{\Sigma_1 C_1 K^2 L^2}{(K^2 - L^2)K^2} - \frac{\Sigma_1 C_1 K^2 L^2}{(K^2 - L^2)L^2} + \Sigma_1 C_1]e^{\frac{x}{K}}$$

$$0 =$$

$$+ (\frac{A_2}{L^2} - \frac{A_2}{L^2})e^{-\frac{x}{L}} + (\frac{C_2}{L^2} - \frac{C_2}{L^2})e^{\frac{x}{L}} + (\Sigma_1 dK^2 - \Sigma_1 dK^2)$$

$$+ [\frac{\Sigma_1 A_1 L^2}{K^2 - L^2} - \frac{\Sigma_1 A_1 K^2}{K^2 - L^2} + \frac{\Sigma_1 A_1 (K^2 - L^2)}{K^2 - L^2}]e^{-\frac{x}{K}}$$

$$+ [\frac{\Sigma_1 C_1 L^2}{K^2 - L^2} - \frac{\Sigma_1 C_1 K^2}{K^2 - L^2} + \frac{\Sigma_1 C_1 (K^2 - L^2)}{K^2 - L^2}]e^{\frac{x}{K}}$$

Check satisfied.