Example 1

$X_1, \dots, X_n$ i.i.d $X_i \sim f(x, \theta) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-\theta)^2}{2}\right\}$, $x \in \mathbb{R}$.

We verify

$$H_0: \theta = 0,$$
$$H_1: \theta = 1.$$

We have

$$\frac{L(1, \underline{x})}{L(0, \underline{x})} = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(\frac{-\sum_{i=1}^n (x_i - 1)^2}{2}\right)}{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{\sum_{i=1}^n x_i^2}{2}\right)} = \exp\left(\sum_{i=1}^n x_i - \frac{n}{2}\right) > k_1.$$

Thus

$$\sum_{i=1}^n x_i - \frac{n}{2} > \log k_1 \iff \sum_{i=1}^n x_i > \log k_1 + \frac{n}{2} = k.$$

Thereby, the critical region of the UMP test has the form

$$C = \left\{\underline{x} : \sum_{i=1}^n x_i > k\right\},$$

while the constant k satifies the condition $P_0(\underline{X} \in C) = \alpha$

So, $P_0\left(\sum_{i=1}^n X_i > k\right) = \alpha$. Since, under H_0, $\sum_{i=1}^n X_i \sim N(0, n)$, we have $P_0\left(\sum_{i=1}^n X_i > k\right) = P_0\left(\frac{\sum_{i=1}^n X_i}{\sqrt{n}} > \frac{k}{\sqrt{n}}\right) =$

$1 - \Phi\left(\frac{k}{\sqrt{n}}\right) = \alpha$. As a result, $\frac{k}{\sqrt{n}} = \Phi^{-1}(1-\alpha)$ and $k = \sqrt{n}\,\Phi^{-1}(1-\alpha)$.

On the other hand, under H_1, $\sum_{i=1}^n X_i \sim N(n, n)$, and

$$\gamma(\theta_1) = \gamma(1) = P_1\left(\sum_{i=1}^n X_i > k\right) = P_1\left(\frac{\sum_{i=1}^n X_i - n}{\sqrt{n}} > \frac{k-n}{\sqrt{n}}\right) =$$

$$1 - \Phi\left(\frac{k-n}{\sqrt{n}}\right) = 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{n}\right).$$

3. The UMP for a composite ~~hyp~~ alternative

Example 1

$X_1, \dots, X_n$ i.i.d , $X_1 \sim N(0, \theta)$, $\theta > 0$.

We test

$$H_0: \theta = \theta_0,$$
$$A: \theta > \theta_0.$$

We find the UMP α-level test.

We have

$$L(\theta) = L(\theta, \underline{x}) = \left(\frac{1}{2\pi\theta}\right)^{n/2} \exp\left\{-\frac{1}{2\theta}\sum_{i=1}^{n} x_i^2\right\}.$$

Let $\theta_1 > \theta_0$, and $k_1 > 0$. Then,

$$\frac{L(\theta_1)}{L(\theta_0)} \geq k_1 \iff \left(\frac{\theta_0}{\theta_1}\right)^{n} \exp\left\{-\frac{\theta_1 - \theta_0}{2\theta_0\theta_1} \sum_{i=1}^{n} x_i^2\right\} \geq k_1 \iff$$

$$\sum_{i=1}^{n} x_i^2 \geq k.$$

The critical region has a form

$$C = \left\{(x_1, \dots, x_n) : \sum_{i=1}^{n} x_i^2 \geq k\right\}$$

and correspods the the UMP test u the problem

$$H_0: \theta = \theta_0,$$
$$H_1: \theta = \theta_1,$$

whee the constant k satisfies the condition $P_0\left(\sum_{i=1}^{n} X_i^2 \geq k\right) = \alpha$.

Since $\sum_{i=1}^{n} X_i^2 / \theta \sim \chi^2_n$, we have

$$P_0\left(\sum_{i=1}^{n} X_i^2 \geq k\right) = P_0\left(\sum_{i=1}^{n} X_i^2 / \theta_0 \geq \frac{k}{\theta_0}\right) = 1 - F_{\chi^2(n)}\left(\frac{k}{\theta_0}\right).$$

As a result, $\frac{k}{\theta_0} = q_{\chi^2(n)}(0.95)$.

Since the above holds for any $\theta_1 > \theta_0$,
the UMP test in the problem (H_0, A) has the form $\square_C$.

Example 2.

$X_1, \dots, X_n$ i.i.d, $X_i \sim N(\theta, 1)$

We verify

$$H_0: \theta = \theta_0,$$
$$H_1: \theta \neq \theta_0.$$

It will be shown that there is no UMP test in the above problem. We have $\Theta = \mathbb{R}$. Let $\theta_1 \in \Theta$ and $\theta_1 \neq \theta_0$. Consider

$$\frac{L(\theta_1)}{L(\theta_0)} = \frac{\left(\frac{1}{2\pi}\right)^{n/2} \exp\left[-\frac{1}{2}\sum_{i=1}^n (x_i - \theta_1)^2\right]}{\left(\frac{1}{2\pi}\right)^{n/2} \exp\left[-\frac{1}{2}\sum_{i=1}^n (x_i - \theta_0)^2\right]} \geq k.$$

Equivalently,

$$\exp\left[(\theta_1 - \theta_0)\sum_{i=1}^n x_i - \frac{n}{2}(\theta_1^2 - \theta_0^2)\right] \geq k$$

or

$$(\theta_1 - \theta_0)\sum x_i \geq \frac{n}{2}(\theta_1^2 - \theta_0^2) + \log k,$$

so,

$$\sum_{i=1}^n x_i \geq \frac{n}{2}(\theta_1 + \theta_0) + \frac{\log k}{\theta_1 - \theta_0} \quad \text{provided that } \theta_1 > \theta_0$$

or

$$\sum_{i=1}^n x_i \leq \frac{n}{2}(\theta_1 + \theta_0) + \frac{\log k}{\theta_1 - \theta_0} \quad \text{provided that } \theta_1 > \theta_0.$$

The first expression defines the critical region of the UMP test in the problem of testing $H_0: \theta = \theta_0$ against $H_1^+: \theta = \theta_1$ provided that $\theta_1 > \theta_0$, while the second expression defines the critical region of the UMP test ... $H_0: \theta = \theta_0$

$H_1^-: \theta = \theta_1$ ~ in $\theta_1 < \theta_0$.

The UMP test does not exist because these regions are different.

Remark 1

Under the assumptions of Example 1, there are UMP tests in the problems

$$H_0: \theta = \theta_0, \quad H_1^+: \theta > \theta_0, \qquad \text{and} \qquad H_0: \theta = \theta_0, \quad H_1^-: \theta < \theta_0.$$

Let $X_1, \dots, X_n$ be ~~th~~ sample from $f(x, \theta)$, $\theta \in \Theta$, and $Y = u(X_1, \dots, X_n)$ a sufficient statistic for θ.

Factorization theorem implies

$L(\theta, \underline{x}) = k_1[u(\underline{x}), \theta] \cdot k_2(\underline{x})$, where $k(\underline{x})$ does not depend on θ. Thereby, for $\theta_0, \theta_1 \in \Theta$

$$\frac{L(\theta_1, \underline{x})}{L(\theta_0, \underline{x})} = \frac{k_1[u(\underline{x}), \theta_1]}{k_1[u(\underline{x}), \theta_0]}$$

depends on $x_1, \dots, x_n$ only by $u(\underline{x})$.

Corollary 1

The UMP test is a function of a sufficient statistic.

Definition 1

It is said that the family of distributions ~~likelihood~~ $\{L(\theta, x) : \theta \in \Theta\}$ has monotone likelihood ratio with respect to a statistic $Y = u(\underline{X})$ if for all $\theta_0, \theta_1 \in \Theta$ such that $\theta_0 < \theta_1$ the ratio

$$\frac{L(\theta_1, \underline{x})}{L(\theta_0, \underline{x})}$$

is a nondecreasing function of $y = u(\underline{x})$.

Theorem 1 (Karlin-Rubin) (continuous version)

Consider the testing problem

$$H_0: \theta \leq \theta_0,$$
$$H_1: \theta > \theta_0.$$

If the family of distributions $\{L(\theta, \underline{x}) : \theta \in \Theta\}$ has monotone likelihood ratio with respect to a statistic $Y = u(\underline{X})$, then, in the problem (H_0, H_1), there is UMP test with the critical region

$$C = \{\underline{x} : u(\underline{x}) > k\},$$

where the constant satisfies the condition $P_{\theta_0}(Y > k) = P_{\theta_0}(u(\underline{X}) > k) = \alpha$.

Example 3

$X_1, \dots, X_n$ i.i.d $X_i \sim b(1, p)$, $p = \theta$, $0 < \theta < 1$. Let $\theta_1 > \theta_0$ and consider the ratio

$$\frac{L(\theta_1, \underline{x})}{L(\theta_0, \underline{x})} = \frac{\theta_1^{\sum_{i=1}^n x_i}(1-\theta_1)^{n-\sum_i x_i}}{\theta_0^{\sum x_i}(1-\theta_0)^{n-\sum x_i}} = \left[\frac{\theta_1(1-\theta_0)}{\theta_0(1-\theta_1)}\right]^{\sum_{i=1}^n x_i} \left(\frac{1-\theta_1}{1-\theta_0}\right)^n$$

which is an increasing function of $y = u(\underline{x}) = \sum_{i=1}^n x_i$.

As a result, the family of distributions $\{L(\theta, \underline{x}) : \theta \in (0,1)\}$ has monotone likelihood ratio with respect to $Y = \sum_{i=1}^n X_i$.

Consider the testing problem

$$H_0: \theta \leq \theta_0$$

against

$$H_1: \theta > \theta_0.$$

The test $\Box_C$, where $C = \{\underline{x} : \sum_{i=1}^n x_i \geq k\}$, is the UMP test in the problem (H_0, H_1) at the significance level α if $P_{\theta_0}(\sum X_i \geq k) = \alpha$.

Let $X_1,\dots,X_n$ be the sample from $f(x,\theta)$, $\theta\in\Theta$,
where $f(x,\theta) = \exp[p(\theta)K(x) + S(x) + q(\theta)]$, $x\in\mathcal{S}$, while $\mathcal{S}$ is the support of X_1 independent from θ.

Assume that $p(\theta)$ is an increasing function of θ.

Then, for $\theta_1 > \theta_0$, the ratio

$$\frac{L(\theta_1)}{L(\theta_0)} = \exp\left\{[p(\theta_1)-p(\theta_0)]\sum_{i=1}^{n}K(x_i) + n[q(\theta_1)-q(\theta_0)]\right\}$$

is a nondecreasing function of $Y=\sum_{i=1}^{n}K(X_i)$.

Thus, in the testing problem

$$H_0: \theta \le \theta_0,$$
$$H_1: \theta > \theta_0,$$

there exists the UMP test of the form $\mathbb{1}_C$,
where $C=\{\underline{x} : \sum_{i=1}^{n}K(x_i) \geq k\}$ while k satisfies the condition $\alpha = P_{\theta_0}(\underline{X}\in C)$

4. Likelihood Ratio Tests

Let $X_1,\dots,X_n$ be a sample with $f(x,\theta)$, $\theta\in\Theta$.
We test

$$H_0: \theta=\theta_0 \quad \text{against} \quad H_1: \theta=\theta_0,$$

where θ_0 is a fixed constant.

Let $\hat{\theta}$ be MLE of the parameter θ.

Def 1

The test with critical region $C=\{\Lambda \le c\}$, where

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\sup_{\theta\in\Theta_0} L(\theta)}{\sup_{\theta\in\Theta} L(\theta)}$$