How to Simulate Random Stock Price in Black-Scholes Model

Quantitative Strategies
October 2022

Internal

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Goal

To be able to handle more complex trade types than Payments

Trades

Clear

Recalculate all

New

	Trade Name	Expiry	Currency	Principal	Value
×	Payment7715	5/15/2019 12:00:00 AM	GBP	1,657,105,742.00	1657105742.00 (GBP)
×	Payment7583	5/15/2019 12:00:00 AM	GBP	1,628,742,010.00	1628742010.00 (GBP)
×	Payment1421	2/15/2019 12:00:00 AM	USD	305,388,571.00	305388571.00 (USD)
×	Payment2996	3/15/2019 12:00:00 AM	PLN	643,494,503.00	171142155.05 (USD)

Example: European Call Option

- Option is a financial instrument which gives a holder (buyer) a right, but not an obligation, to buy stock at predefined level (strike) and at particular moment in time (maturity)

- Example: Call option with strike $\mathbf{100}$ and maturity $\mathbf{1}$ year
  - Scenario 1: Stock price turns out to be 120.
    Option holder buys a stock at 100 and immediately sells in on the exchange for 120.
    She gains 120–100=20.
  - Scenario 2: Stock price ends up at 80.
    Option holder chooses not to exercise her option because it is cheaper to buy on the exchange.
    Her gain is 0.

Different trade data

- We have Expiry and Currency
- We need also Strike, but do not need Principal

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Clear

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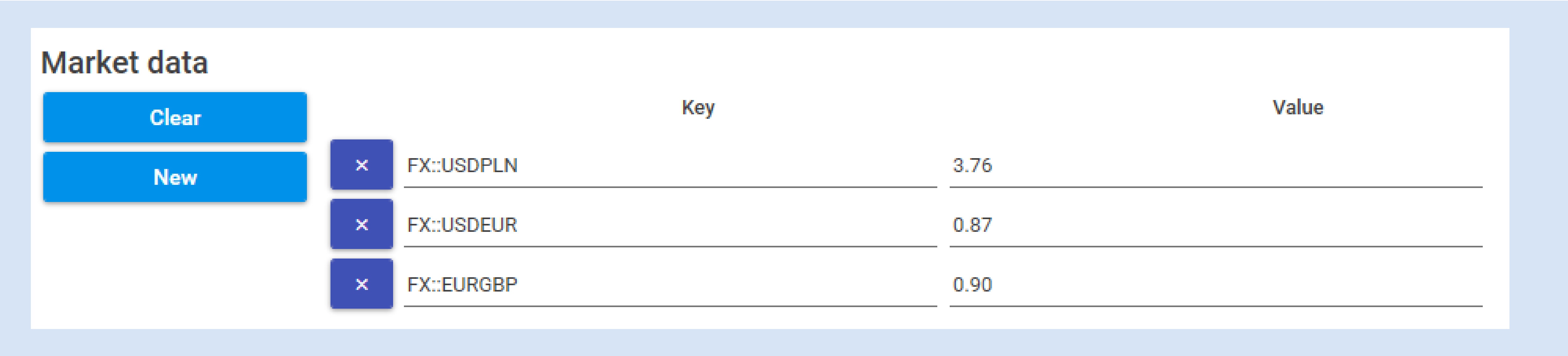
Additional market data

- FX Spot is not enough
- For Options we need e.g. price of the underlying stock



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Basic Concepts

- We will consider portfolio which contains cash and stock
- To understand the behavior of such portfolio we need to understand it for both cash and stock
- Time value of money is the basic concept in finance. It is the principle that a certain currency amount of money today has a different buying power (value) than the same currency amount of money in the future (e.g. interest rates, inflation).

Time Value of Money (Discount Factor)

We will assume constant continuously compounded interest rates with intensity r.
This means that USD 1 today is worth:

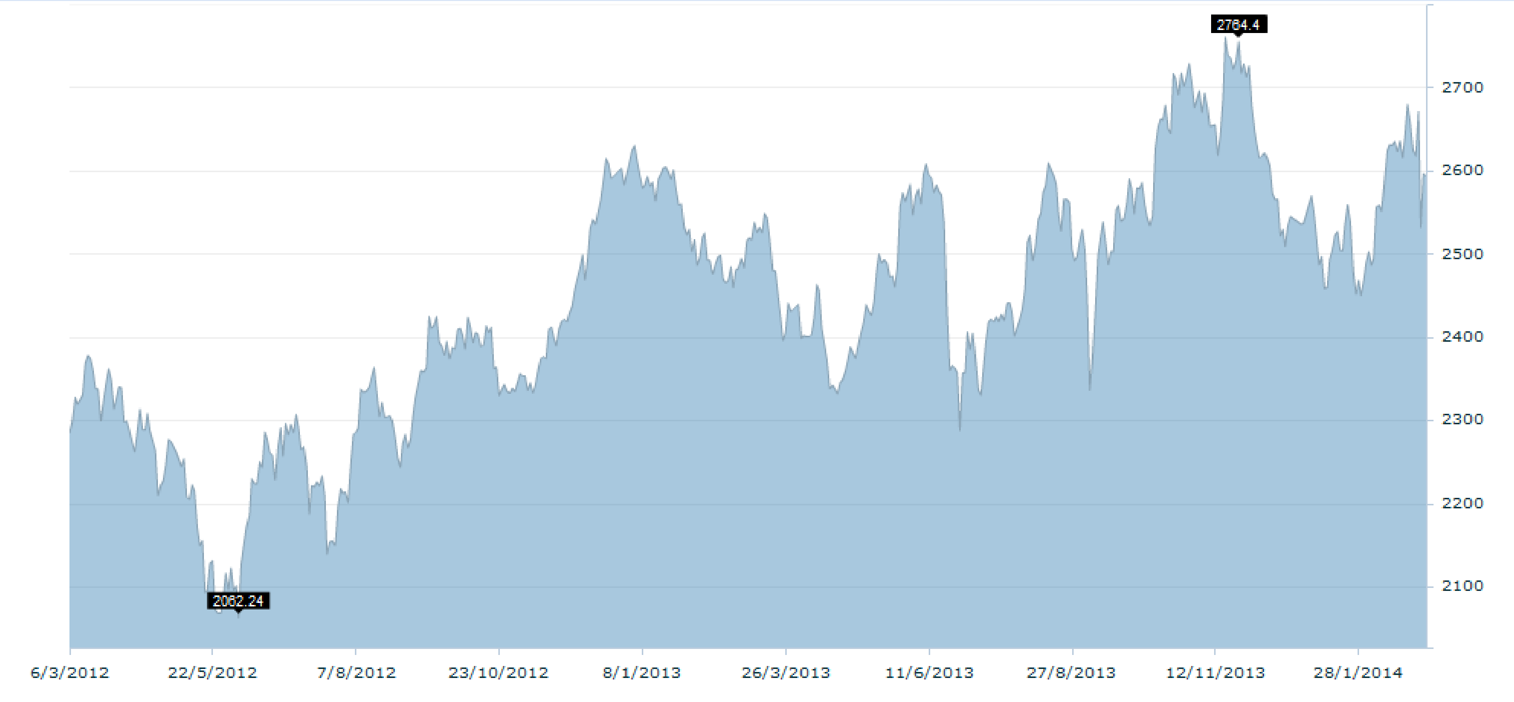
$$e^{rT}$$

At time T (T=1 denotes one year).
This is equivalent to the statement that USD 1 at time T is worth today:

$$e^{-rT}$$

Stock Price as a Random Variable (1/3)

The behavior of the quoted prices of stock is far from being predictable. In below figures we can see behavior of WIG30 (Warsaw stock index) and KGHM over the period February 2012 to February 2014.



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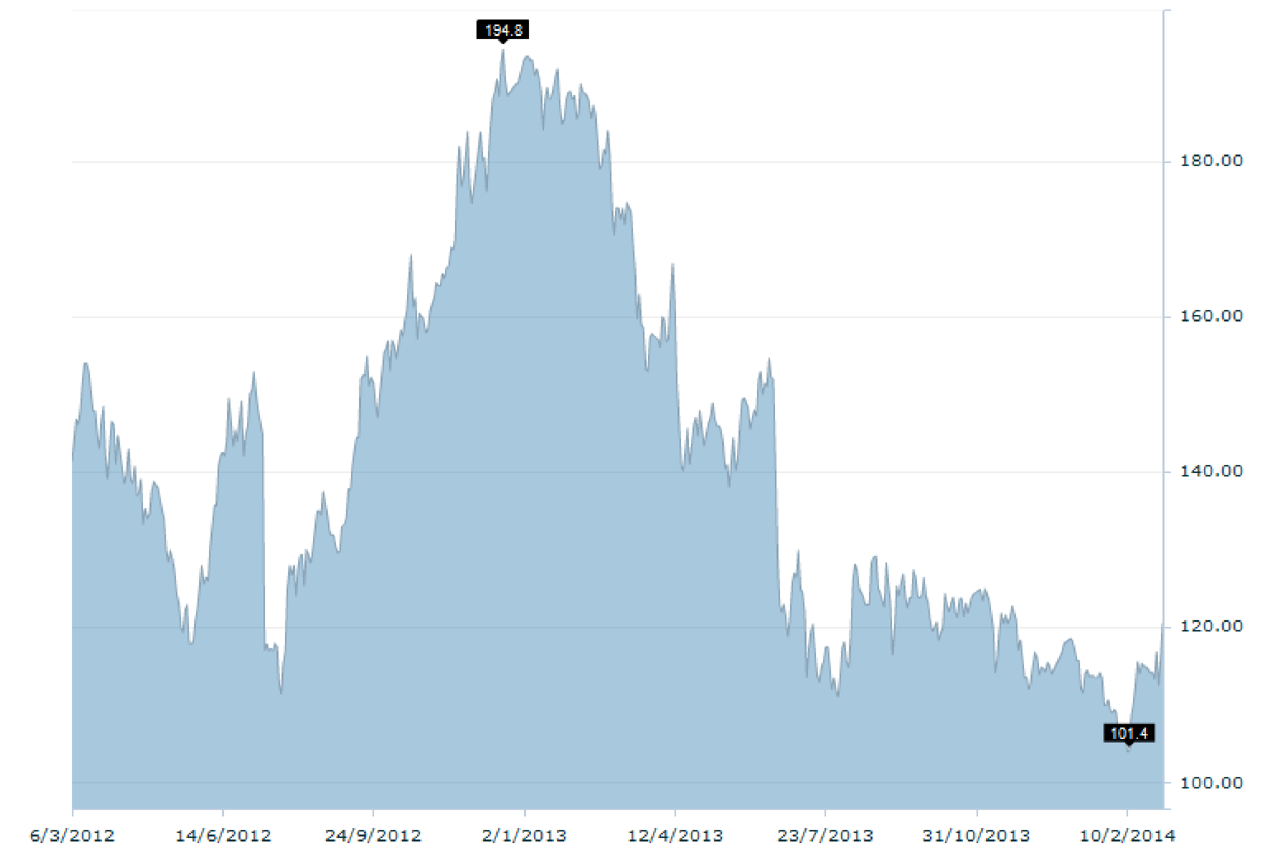
Stock Price as a Random Variable (2/3)



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Stock Price as a Random Variable (3/3)

- A variable whose value is unknown or a function that assigns values to each of an experiment's outcomes
- Random variables are often designated by letters and can be classified as discrete, which are variables that have specific values, or continuous, which are variables that can have any values within a continuous range

Binomial Distribution
Definition

In general, if the random variable X follows the binomial distribution with parameters n and p, we write X~$B(n,p)$.

The probability of getting exactly k successes in n trials is given by the probability mass function:

$$f(k;n,p) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

For k=0,1, 2, …, n where:

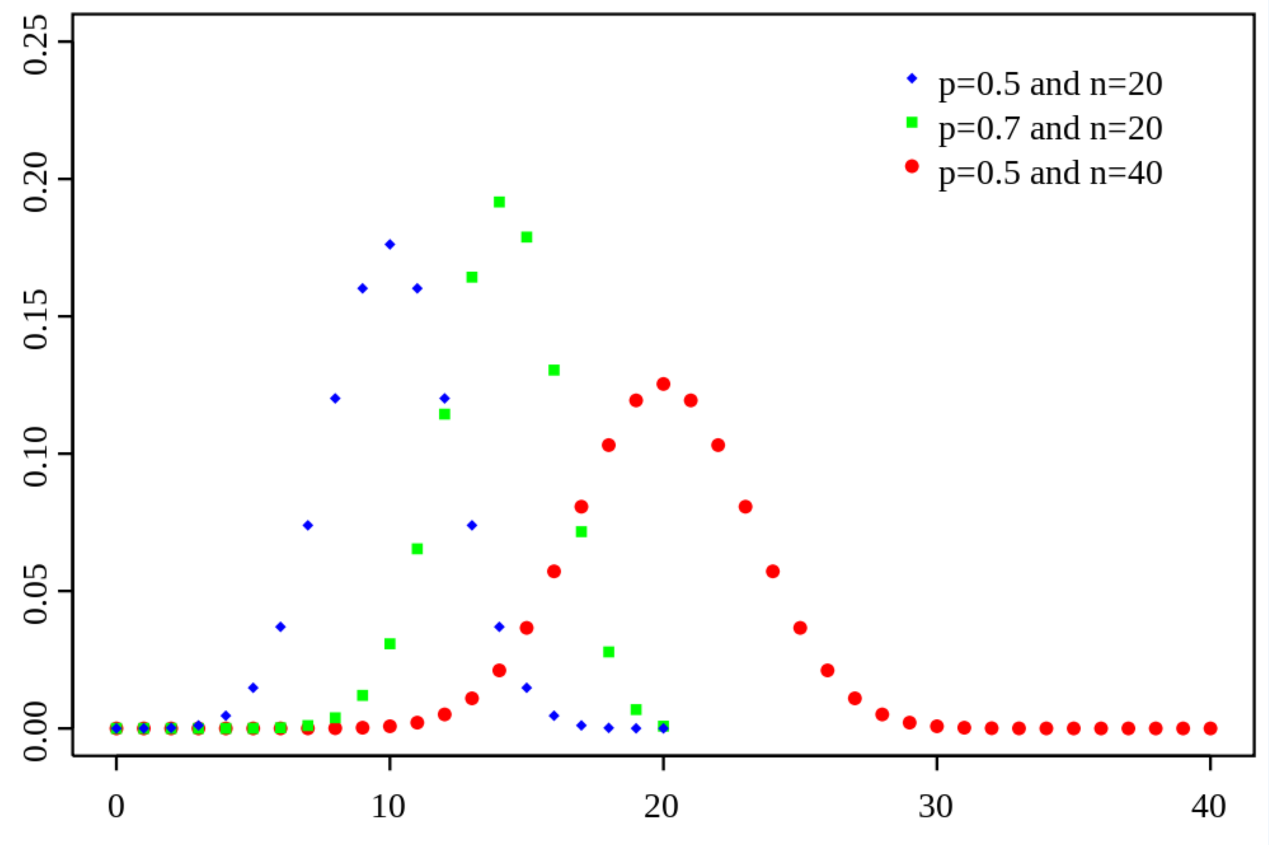
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Moreover mean and variance are equal respectively:

$$E(X) = np$$
$$Var(X) = E(X - EX)^2 = n(p - p^2)$$

Binomial Distribution

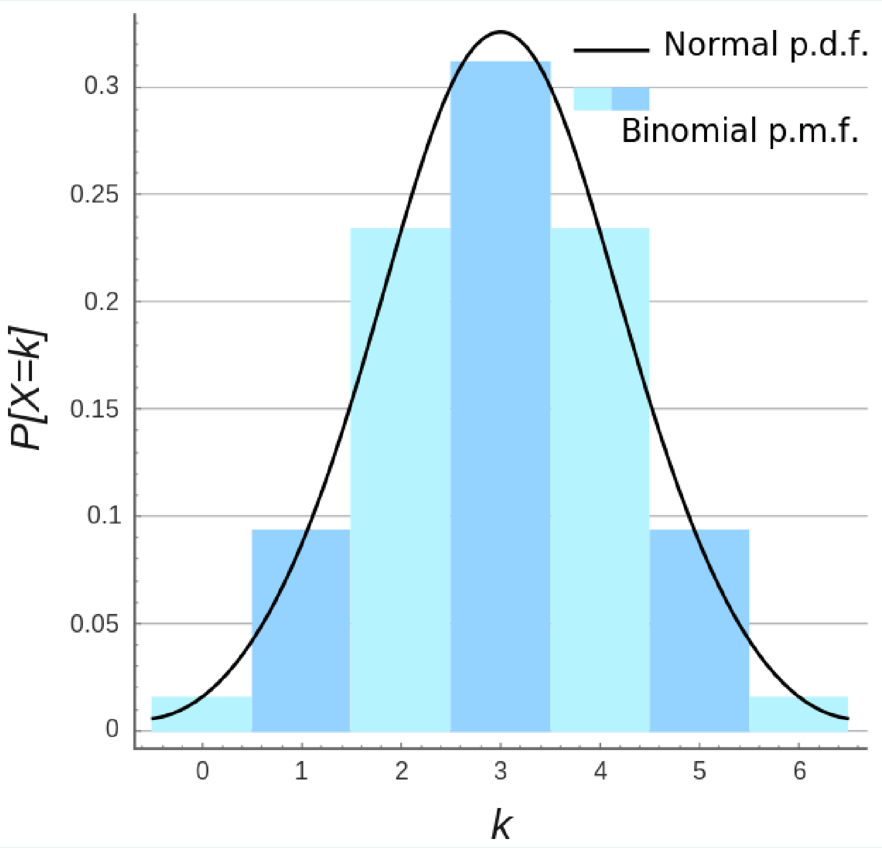
Probability Mass Function



Standard Normal Distribution 1

- Normal distribution is a very commonly occurring continuous probability distribution – a function that tells the probability that an observation in some context will fall between any two real numbers
- Normal distributions are extremely important in statistics and are often used in the natural and social sciences for real-valued random variables whose distributions are not known

Binomial Distribution
Normal Approximation



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Standard Normal Distribution 2

- The normal distribution is immensely useful because of the central limit theorem, which states that, under mild conditions, the mean of many random variables independently drawn from the same distribution is distributed approximately normally

- Irrespective of the form of the original distribution: physical quantities that are expected to be the sum of many independent processes often have a distribution very close to the normal

- Moreover, many results and methods can be derived analytically in explicit form when the relevant variables are normally distributed

Cumulative Distribution Function

The cumulative distribution function of a real-valued random variable X is the function given by:

$$F_X(x) = P(X \leq x)$$

where the right-hand side represents the probability that the random variable X takes on a value less than or equal to x.

In case of continuous distribution we also define probability density function f which is equal to the first derivative of the cumulative distribution function with respect to x:

$$F_X(x) = \int_{-\infty}^{x} f(s)ds$$

Standard Normal Distribution 3

The cumulative distribution function of the standard normal distribution, usually denoted with the capital Greek letter Φ, is the integral:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}}\, ds$$

And probability density function:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Normal Distribution

The family of normal distribution is closed to the linear transformation (i.e., You can multiply it by real number and also add a real number to it and it still will be normal random variable, but with different parameters).

If:

$$X \sim N(0,1)$$

Then:

$$\mu + \sigma X \sim N(\mu, \sigma^2)$$

Brownian Motion

- In mathematics, the Wiener process is a continuous-time stochastic process named in honor of Norbert Wiener
- It is often called standard Brownian motion, after Robert Brown
- It occurs frequently in pure and applied mathematics, economics, quantitative finance and physics

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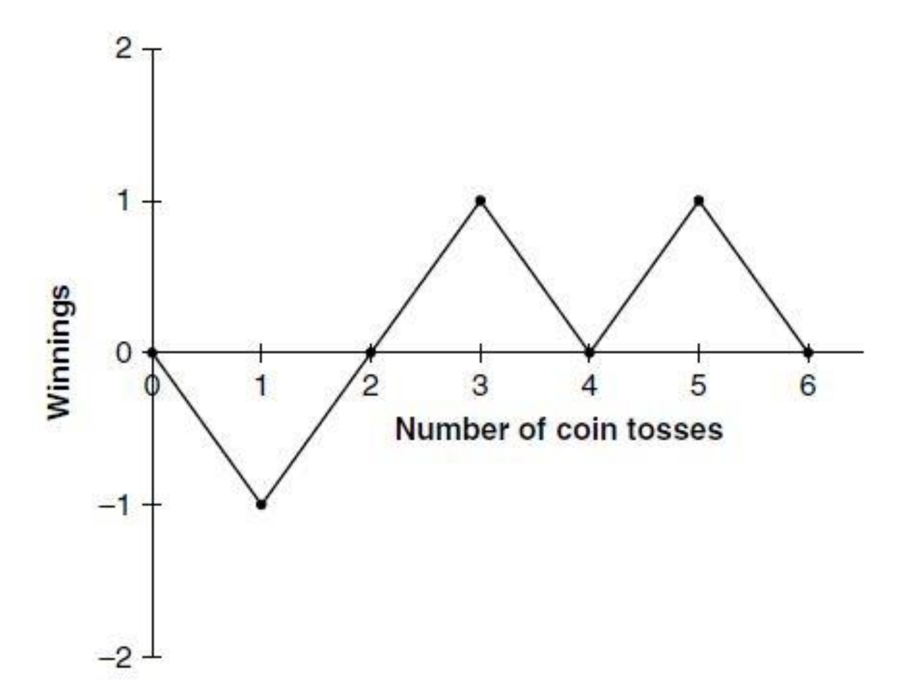
Random Walk (1/3)

Every time you throw a head I give you USD 1, every time you throw a tail you give me USD 1. Below figure shows how much money you have after six tosses. In this experiment the sequence was THHTHT, and we finished even.



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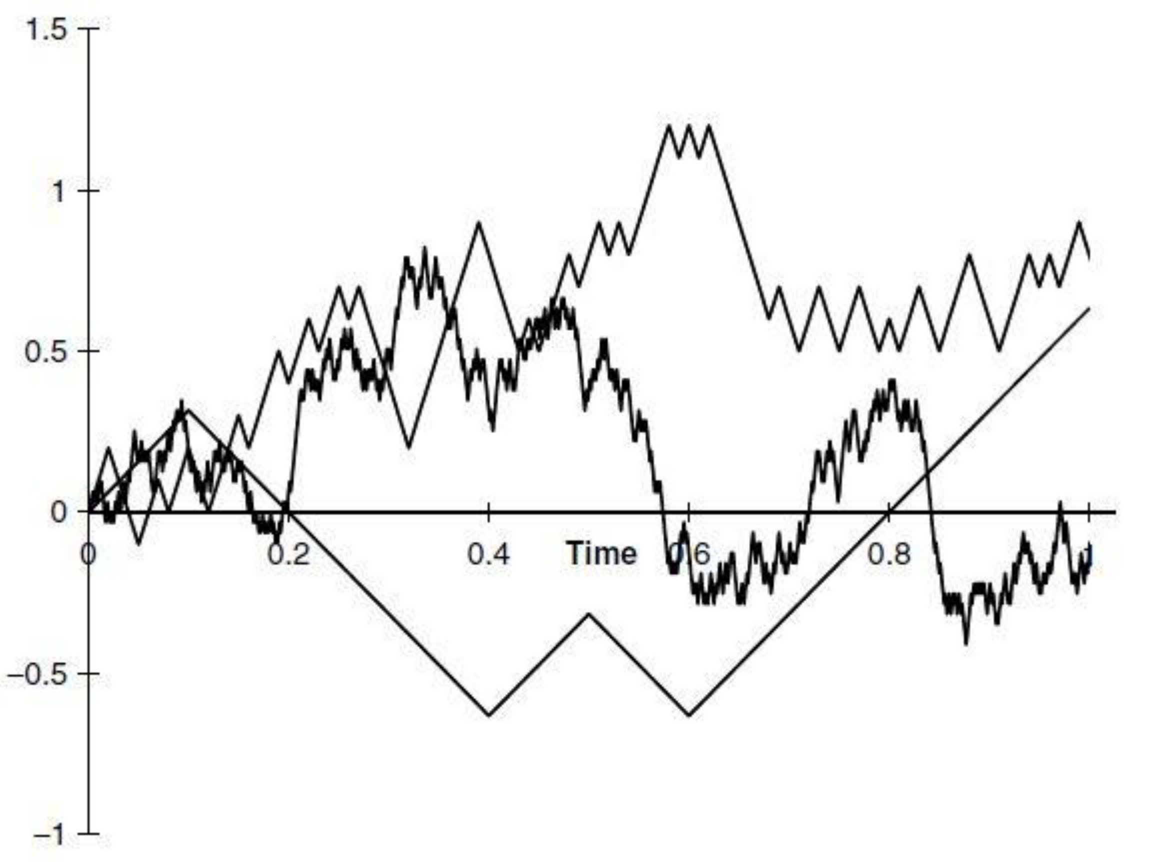
Random Walk (2/3)

- We are going to change the rules of our coin-tossing experiment
- First of all we are going to restrict the time allowed for the n tosses to a period t, so each toss will take a time t/n
- Second, the size of the bet will not be USD 1 but $\sqrt{\frac{t}{n}}$ where n is number of steps (tosses).

  In below figure we show a series of experiments, each lasting for a time 1, with increasing number of tosses per experiment.

Random Walk (3/3)



Random Walk (3/3)

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- The limiting process for this random walk as the time steps go to zero is called **Brownian motion** or Wiener process

Brownian Motion

Definition

The Wiener process W_t is characterized by four facts:

- $W_0 = 0$
- W_t is almost surely continuous
- W_t has independent increments
- $W_t - W_s \sim N(0, t-s)$ for $0 \leq s \leq t$.

Where $N\ (\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2. The condition that it has independent increments means that if $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$ then $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independent random variables.

Brownian Motion
Basic Properties

These are some basic properties of Brownian motion W_t

- BM has lack of memory property
- BM is symetric
- BM is a „fair game" (martingale)
- Reflection principle states that $P(\max_{s\leq t} W_s \geq x) = 2P(W_t \geq x)$
- Covariance of W_t and W_s is equal to $\min(s,t)$
- W_t has unbounded variation on every interval
- W_t is nowhere differentiable but everywhere continuous

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Geometric Brownian Motion (1/4)

- Unfortunately Brownian motion can become negative with non zero probability and this is obvious disadvantage while simulating stock price
- That is why we use geometric Brownian motion
- A geometric Brownian motion (GBM) (also known as exponential Brownian motion) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (also called a Wiener process) with drift
- It is used in mathematical finance to model stock prices in the Black-Scholes model:

$$S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t}$$

Note that:

$$E[S_t] = \ ??$$

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Note that:

$$E[S_t] = S_0 e^{rt}$$

Geometric Brownian Motion (2/4)

Some of the arguments for using GBM to model stock prices are:

- The expected returns of GBM are independent of the value of the process (stock price), which agrees with what we would expect in reality
- A GBM process only assumes positive values, just like real stock prices
- A GBM process shows the same kind of "roughness" in its paths as we see in real stock prices
- Calculations with GBM processes are relatively easy

Geometric Brownian Motion (3/4)

However, GBM is not a completely realistic model, in particular it falls short of reality in the following points:

- In real stock prices, volatility changes over time (possibly stochastically), but in GBM, volatility is assumed constant.

- In real stock prices, returns are usually not normally distributed (real stock returns have higher kurtosis ("fatter tails"), which means there is a higher chance of large price changes. In addition, returns have negative skewness).

For example if we take a look at logarithm returns for Perez Companc we see that it doesn't suit normal distribution perfectly:

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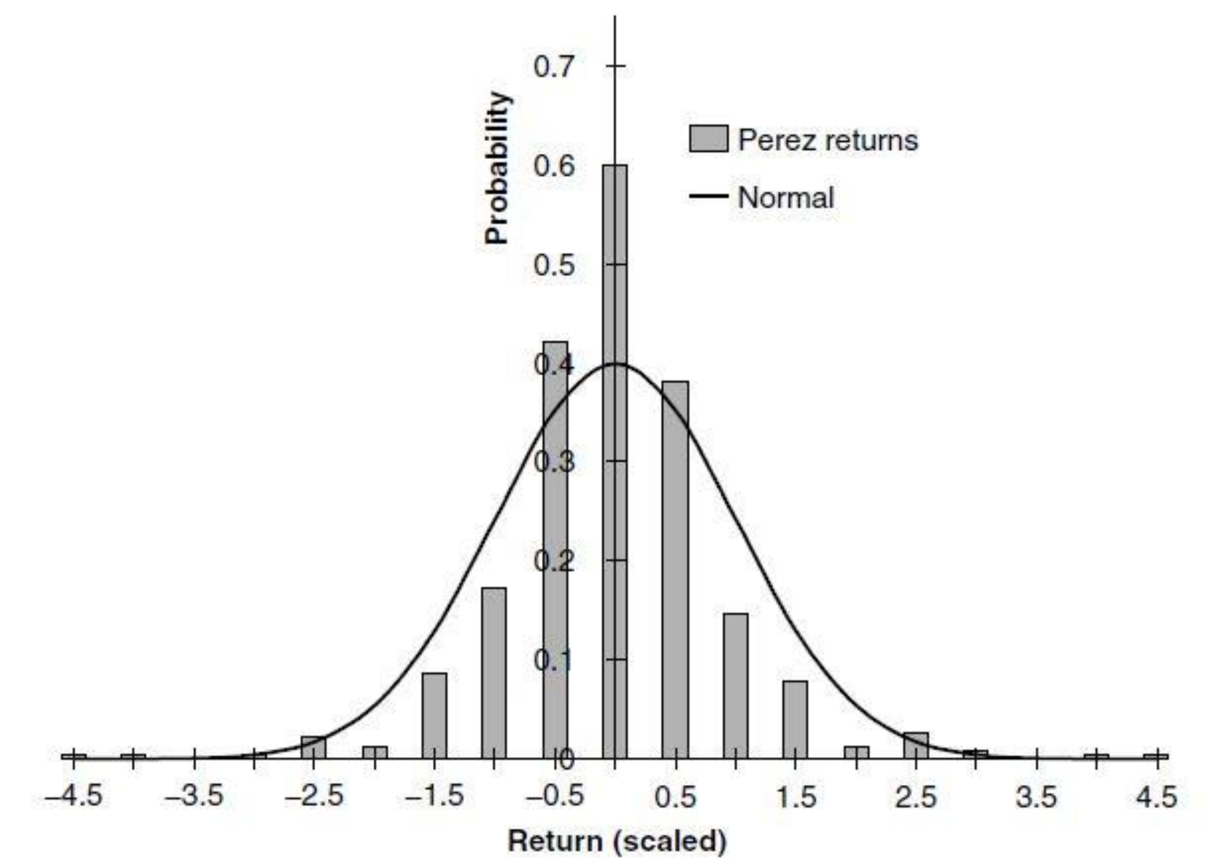
Geometric Brownian Motion (4/4)



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Simulation of Normal r.v.

To get both Brownian motion and geometric Brownian motion we need to be able to draw standard normal random variables. If we don't have normal random number generator we can use one of following methods:

- Draw uniform random variable U from the interval [0,1]. Then $\Phi^{-1}(U)$ is standard normal random variable.

- Box-Muller transform: draw two independent uniformly distributed random variables U_1 and U_2 from the interval $[0,1]$. Then $\mathrm{N}_1 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$ and $\mathrm{N}_2 = \sqrt{-2\ln U_1}\cos(2\pi U_2)$ are two independent standard normal random variables.

Simulation of Standard Brownian Motion

The algorithm of simulation of the Brownian motion on the interval $[0,t]$ at n points is following:

- Set $W_0 = 0$
- Draw n independent standard normal random variables $Z_1, Z_2, \ldots, Z_n$
- For $1 \leq i \leq n$ we set $W_{i\frac{t}{n}} = W_{(i-1)\frac{t}{n}} + \sqrt{\frac{t}{n}} Z_i$

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Simulation of Geometric Brownian Motion

The algorithm of simulation of geometric Brownian motion on the interval $[0, t]$ at n points.
We denote interest rate intensity by r and volatility σ, then:

- Set $S_0 = S(0) = initial\ piece\ of\ the\ stock$

- Draw n independent standard normal random variables $Z_1, Z_2, \dots, Z_n$

- For $1 \leq i \leq n$ we set $S_{i\frac{t}{n}} = S_{(i-1)\frac{t}{n}} * e^{\left(r-\frac{\sigma^2}{2}\right)\frac{t}{n}+\sigma\sqrt{\frac{t}{n}}Z_i}$

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Historical Volatility (1/2)

We can calculate historical (realized) volatility of the stock looking at the logarithm of the returns:

$$S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma Wt}$$

$$S_{t+\Delta t} = S_t e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t+\sigma(W_{t+\Delta t}-W_t)}$$

$$\ln\frac{S_{t+\Delta t}}{S_t} = \left(r-\frac{\sigma^2}{2}\right)\Delta t + \sigma(W_{t+\Delta t}-W_t) \sim N((r-\frac{\sigma^2}{2})\Delta t, \sigma^2\Delta t)$$

Historical Volatility (2/2)

Hence we can get estimation of the volatility:

$$R_i = \ln \frac{S_{(i+1)\frac{t}{n}}}{S_{i\frac{t}{n}}}$$

$$\bar{R} = \frac{1}{n} \sum_{i=1}^{n} R_i$$

$$\hat{\sigma}^2 = \frac{n}{t(n-1)} \sum_{i=1}^{n} (R_i - \bar{R})^2$$