Remark 2,

Y_2 - estimate of θ, $EY_2 = \theta$, Y_1 - sufficient statistic

$\varphi(Y_1) = \mathbb{E}[Y_2 | Y_1]$, $\operatorname{var} \varphi(Y_1) \leq \operatorname{var} Y_2$.

Y_3 - estimate of θ, $EY_3 = \theta$, Y_3 - is not a suff. stat.

$\psi(Y_3) = \mathbb{E}[\varphi(Y_1) | Y_3]$, $\mathbb{E}\psi(Y_3) = \theta$ and $\operatorname{Var} \psi(Y_3) \leq \operatorname{Var} \varphi(Y_1)$

Since Y_3 is not a sufficient statistic, the conditional distribution of Y_1 given Y_3 depends upon θ. Thus $\psi(Y_3)$ is not a statistic. because $\psi(Y_3)$ depends on θ.

Example 2,

X_1, X_2, X_3 i.i.d. $\operatorname{Exp}(\theta)$, $\theta > 0$.

$$(X_1, X_2, X_3) \sim \frac{1}{\theta^3} e^{-(x_1+x_2+x_3)/\theta}, \quad x_i > 0, \ i = 1, 2, 3.$$

The factorization theorem implies that $Y_1 = X_1 + X_2 + X_3$ is a sufficient statistic for θ.

$\mathbb{E}Y_1 = \mathbb{E}(X_1 + X_2 + X_3) = 3\theta$. Thus $\mathbb{E}\left[\frac{Y_1}{3}\right] = \theta$. $\bar{X} = \varphi(Y_1) = \frac{Y_1}{3}$

Let $Y_2 = X_2 + X_3$, $Y_3 = X_3$. The one-to-one transformation

$$x_1 = y_1 - y_2, \quad x_2 = y_2 - y_3, \quad x_3 = y_3,$$

has Jacobian

$$J = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1,$$

and the joint distribution of (Y_1, Y_2, Y_3) is

$$g(y_1, y_2, y_3) = \frac{1}{\theta^3} e^{-y_1/\theta} \mathbb{1}(0 < y_3 < y_2 < y_1 < +\infty)$$

The joint distribution of (Y_1, Y_3)

$$g_{13}(y_1, y_3, \theta) = \int_{y_3}^{y_1} g(y_1, y_2, y_3)\,dy_2 = \int_{y_3}^{y_1} \left(\frac{1}{\theta}\right)^3 e^{-y_1/\theta}\,dy_2 =$$

$$= \frac{1}{\theta^3} e^{-y_1/\theta} (y_1 - y_3), \quad 0 < y_3 < y_1 < +\infty.$$

Obviously, $g_3(y_3 | \theta) = \frac{1}{\theta} e^{-y_3/\theta}, \quad 0 < y_3 < +\infty$.

The condition distribution of Y_1 given $Y_3 = y_3$, is

$$g_{1|3}(y_1 | y_3) = \frac{g_{13}(y_1, y_3 | \theta)}{g_3(y_3, \theta)} = \frac{\frac{1}{\theta^3} e^{-y_1/\theta}(y_1 - y_3)}{\frac{1}{\theta} e^{-y_3/\theta}} =$$

$$\frac{1}{\theta^2}(y_1 - y_3) e^{-(y_1 - y_3)/\theta}, \quad 0 < y_3 < y_1 < +\infty$$

We have

$$\mathbb{E}\left[\frac{Y_1}{3} \Big| Y_3\right] = \mathbb{E}\left[\frac{Y_1 - Y_3}{3} \Big| Y_3\right] + \mathbb{E}\left[\frac{Y_3}{3} \Big| Y_3\right] =$$

$$\frac{1}{3}\int_{y_3}^{+\infty} \frac{1}{\theta^2}(y_1 - y_3)^2 e^{-(y_1 - y_3)/\theta}\,dy_1 + \frac{Y_3}{3} = \left\{\begin{array}{l} z = y_1 - y_3 \\ dz = dy_1 \end{array}\right\} =$$

$$\frac{1}{3}\int_0^{+\infty} \frac{1}{\theta^2} z^2 e^{-z/\theta}\,dz + \frac{Y_3}{3} = \frac{1}{3}\int_0^{+\infty} \frac{1}{\theta^2} \underbrace{\frac{\Gamma(3)\theta^3}{\Gamma(3)\theta^3} z^2 e^{-z/\theta}}_{\Gamma(3,\theta)}\,dz + \frac{Y_3}{3} =$$

$$= \frac{1}{3}\Gamma(3)\theta + \frac{Y_3}{3} = \frac{2}{3}\theta + \frac{Y_3}{3}.$$

Put $\psi(Y_3) = \mathbb{E}\left[\frac{Y_1}{3} \Big| Y_3\right] = \frac{2}{3}\theta + \frac{Y_3}{3}$.

$\mathbb{E}\psi(Y_3) = \theta$ and $Var\,\psi(Y_3) \leq Var\left(\frac{Y_1}{3}\right)$,

but $\psi(Y_3)$ is not a statistic. (depends upon the parameter θ).

4. Complete statistics and uniqueness of MVUEs.

Let $X_1, \dots, X_n$ be a random sample from the Poisson distribution (23)

$$f(x, \theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, 2, \dots, \quad \theta > 0.$$

Recall that $Y_1 = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ and

$$g_1(y_1, \theta) = \frac{(n\theta)^{y_1} e^{-n\theta}}{y_1!}, \quad y_1 = 0, 1, \dots,.$$

Consider the family $\{ g_1(y_1, \theta) : \theta > 0 \}$.

Suppose that the function $u(Y_1)$ of Y_1 is such that $E_\theta[u(Y_1)] = 0$ for every $\theta > 0$. We shall show that then

$$u(y_1) = 0 \quad \text{for } y_1 = 0, 1, 2, \dots$$

We have, for all $\theta > 0$,

$$0 = E[u(Y_1)] = \sum_{y_1=0}^{\infty} u(y_1) \frac{(n\theta)^{y_1} e^{-n\theta}}{y_1!} =$$

$$e^{-n\theta} \left[u(0) + u(1) \frac{n\theta}{1!} + u(2) \frac{(n\theta)^2}{2!} + \dots \right].$$

Since $e^{-n\theta} > 0$, ~~the~~ we have

$$u(0) + u(1) \frac{n\theta}{1!} + u(2) \frac{(n\theta)^2}{2!} + \dots = 0. \quad (\text{polynomial } (\theta))$$

(Polynomial vanishes, then coefficients as well)

$$u(0) = 0, \quad u(1) \frac{n\theta}{1!} = 0, \quad \frac{n^2\theta^2}{2!} u(2) = 0, \dots$$

Thereby

$$u(0) = 0, \quad u(1) = 0, \quad u(2) = 0, \dots.$$

Definition 1

Let Z be a random variable with "the density" from the family $\{h(z, \theta) : \theta \in \Theta\}$.

If
$$\bigwedge_{\theta \in \Theta} \bigwedge_{u: \mathbb{R} \to \mathbb{R}} \mathbb{E}_\theta[u(Z)] = 0 \implies u(z) \equiv 0 \text{ almost surely},$$
then ~~the family~~ $\{h(z, \theta) : \theta \in \Theta\}$ is called a complete family while Z is called a complete statistic.

Let $X_1, \dots, X_n$ be a sample with $f(x, \theta), \theta \in \Theta$.
Let $Y_1 = u_1(X_1, \dots, X_n)$ be a sufficient statistic for θ, while $f_{Y_1}(y_1, \theta)$ is the density of Y_1. If Y_2 is ~~un~~ an unbiased estimator of θ which is not a function of Y_1, then $\varphi(Y_1) = \mathbb{E}[Y_2 | Y_1]$ is also the unbiased estimator of θ. Suppose, there is another function $\psi \neq \varphi$ of Y_1 such that $\mathbb{E}[\psi(Y_1)] = \theta$ for any $\theta \in \Theta$.

Thus
$$\mathbb{E}[\varphi(Y_1) - \psi(Y_1)] = 0 \quad \text{for } \theta \in \Theta.$$

If the ~~sa~~ family $\{f_{Y_1}(y_1, \theta) : \theta \in \Theta\}$ is complete, $\varphi(y_1) - \psi(y_1) = 0$ except on a set of points that has probability zero. Equivalently $\varphi(y_1) = \psi(y_1)$ a.s.
Therefore, $\varphi(Y_1)$ is the unique function of Y_1 such that $\mathbb{E}[\varphi(Y_1)] = \theta$. As a result, the Rao-Blackwell

implies that $\varphi(Y_1)$ is uniquely determined MVUE of θ.

Theorem 1 (Lehmann-Scheffé)

Let $X_1, \ldots, X_n$ be a sample for $f(x,\theta)$, $\theta \in \Theta$.
Let $Y_1 = u_1(X_1, \ldots, X_n)$ be a sufficient statistic for θ, and let the family $\{f_{Y_1}(y_1, \theta) : \theta \in \Theta\}$ be complete.
If there is a function φ of Y_1 that is $\varphi(Y_1)$, an unbiased estimator of θ, then ~~this function~~ $\varphi(Y_1)$ is uniquely determined MVUE of θ.

5. Exponential Class of Distributions

Consider a family of distributions $\{f(x,\theta) : \theta \in \Theta\}$, where $\Theta = \{\theta : \gamma < \theta < \delta\}$ while γ and δ are known constants (they may be $\pm\infty$), and

$$f(x,\theta) = \begin{cases} \exp[p(\theta)K(x) + S(x) + q(\theta)], & x \in \mathcal{S}, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\mathcal{S}$ is the support of X.

Definition 1

It is said that $f(x,\theta)$ is a member of the regular exponential class if

(i) $\mathcal{S}$ does not depend upon θ,

(ii) $p(\theta)$ is a nontrivial continuous function of θ

(iii) - if X is continuous, $K'(x) \not\equiv 0$ and $S(x)$ is a continuous function of $x \in \mathcal{S}$
- if X is discrete, $K(x)$ is a nontrivial function of $x \in \mathcal{S}$.

Example 1

(i) The family $\{f(x|\theta): 0<\theta<+\infty\}$, where

$$f(x|\theta)=\frac{1}{\sqrt{2\pi}\,\theta}e^{-\frac{x^2}{2\theta}}=\exp\left[-\frac{1}{2\theta}x^2-\log\sqrt{2\pi\theta}\right],\quad x\in\mathbb{R}$$

is a regular exponential class of the continuous type.

(ii) The family $\{f(x|\theta): 0<\theta<+\infty\}$, where

$$f(x,\theta)=\frac{1}{\theta}\mathbb{1}_{(0,\theta)}(x)=\exp\{-\log\theta\}\mathbb{1}_{(0,\theta)}(x)$$

is not a regular exponential class.

Let $X_1,\dots,X_n$ denote a random sample from a distribution being a regular exponential class. The joint distribution has the form

$$\exp\left[p(\theta)\sum_{i=1}^{n}K(x_i)+\sum_{i=1}^{n}S(x_i)+nq(\theta)\right]\quad\text{for } x_i\in S,\ i=1,\dots,n.$$

Equivalently

$$\exp\left[p(\theta)\sum_{i=1}^{n}K(x_i)+nq(\theta)\right]\cdot\exp\left[\sum_{i=1}^{n}S(x_i)\right].$$

The factorization theorem implies that $Y_1=\sum_{i=1}^{n}K(X_i)$ is a sufficient statistic for θ.

Theorem 2

Let $f(x|\theta)$, $\gamma<\theta<\delta$ be the distribution of a random variable X being a member of a regular exponential class. If $X_1,\dots,X_n$ is a random sample from $f(x|\theta)$, the statistic $Y_1=\sum_{i=1}^{n}K(X_i)$ is a complete sufficient stat for θ.