Example 5

$X_1, \ldots, X_n$ i.i.d. $N(\mu, \sigma^2)$

$\underline{I}(\mu, \sigma^2) = \operatorname{diag}\{\sigma^{-2}, 2\sigma^{-2}\}$

Let $g(\mu, \sigma^2) = \sigma^2$. Then, $\underline{B} = [0, 2\sigma]$ and

$$I(\sigma^2) = \left\{[0, 2\sigma]\begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2}\end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 2\sigma\end{bmatrix}\right\}^{-1} = \left([0 \;\; 2\sigma]\begin{bmatrix} \sigma^2 & 0 \\ 0 & \frac{\sigma^2}{2}\end{bmatrix}\begin{bmatrix} 0 \\ 2\sigma\end{bmatrix}\right)^{-1} =$$

$$\left([0 \;\; \sigma^3]\begin{bmatrix} 0 \\ 2\sigma\end{bmatrix}\right)^{-1} = \left(2\sigma^4\right)^{-1} = \frac{1}{2\sigma^4}.$$

Let $X_1, \ldots, X_n$ be a sample with $f(x, \underline{\theta})$, $\underline{\theta} \in \Theta \subseteq \mathbb{R}^p$, $p > 1$. Let $\Theta = \Theta_0 \cup \Theta_1$, $\Theta_0 \cap \Theta_1 = \phi$. Assume that $\dim \Theta_0 = p - q$, $1 \le q \le p$. Thereby, we actually test the hypothesis on a q-dimensional vector of parameters. Namely,

$$H_0 : \underline{\theta} \in \Theta_0$$

against

$$H_1 : \underline{\theta} \in \Theta_1.$$

The LRT statistic has a form

$$\Lambda = \frac{\sup_{\underline{\theta} \in \Theta_0} L(\underline{\theta})}{\sup_{\underline{\theta} \in \Theta} L(\underline{\theta})} = \frac{\sup_{\underline{\theta} \in \Theta_0} L(\underline{\theta})}{L(\hat{\underline{\theta}})} = \frac{L(\hat{\underline{\theta}}_0)}{L(\hat{\underline{\theta}})},$$

where $\hat{\theta}$ is the MLE of θ, while $\hat{\theta}_0$ is the MLE of θ under the null model. We reject H_0 in favour of H_1 when

$$\Lambda \le c,$$

where

$$\sup_{\underline{\theta} \in \Theta_0} \mathbb{P}_{\underline{\theta}}(\Lambda \le c) = \alpha \quad \text{for } \alpha \in (0, 1).$$

Example 1 (LRT for the mean under normal model)

Let $X_1, \dots, X_n$ be a sample from $N(\mu, \sigma^2)$, σ^2 unknown, distribution.
We test

$$H_0: \mu = \mu_0 \quad \text{against} \quad H_1: \mu \neq \mu_0,$$

where μ_0 is known.

Let $\Theta = \{(\mu, \sigma^2): \mu \in \mathbb{R}, \sigma^2 > 0\}$. Obviously, $\Theta_0 = \{(\mu_0, \sigma^2): \sigma^2 > 0\}$.
Recall that $\underline{\hat{\theta}} = (\hat{\mu}, \hat{\sigma}^2) = (\bar{X}, S_n^2)$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$, $S_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$.
Furthermore, $\underline{\hat{\theta}}_0 = (\mu_0, \hat{\sigma}_0^2)$, where $\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \mu_0)^2$.

We have

$$\Lambda = \frac{L(\underline{\hat{\theta}}_0)}{L(\underline{\hat{\theta}})} = \frac{\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\left(\frac{1}{\hat{\sigma}_0^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{\sum_{i=1}^n (X_i - \mu_0)^2}{2\hat{\sigma}_0^2}\right\}}{\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\left(\frac{1}{\hat{\sigma}^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{2\hat{\sigma}^2}\right\}}$$

$$= \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{\frac{n}{2}} \frac{\exp\{-\frac{n}{2}\}}{\exp\{-\frac{n}{2}\}} = \left(\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sum_{i=1}^n (X_i - \mu_0)^2}\right)^{\frac{n}{2}}.$$

Recall that $\sum_{i=1}^n (X_i - \mu_0)^2 = \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu_0)^2$.

Then

$$\Lambda \leq c \iff \Lambda^{-\frac{2}{n}} \geq c^{-\frac{2}{n}} = c' \iff$$

$$\frac{\sum_{i=1}^n (X_i - \mu_0)^2}{\sum_{i=1}^n (X_i - \bar{X})^2} = \frac{\sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu_0)^2}{\sum_{i=1}^n (X_i - \bar{X})^2} = 1 + \frac{n(\bar{X} - \mu_0)^2}{\sum (X_i - \bar{X})^2} \geq c' \iff$$

$$\left(\frac{\sqrt{n}(\bar{X} - \mu_0)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2}}\right)^2 \geq (c' - 1)(n-1) = c'' \iff T^2 \geq c'' \iff |T| \geq \sqrt{c''} = c'''.$$

Under H_0, the statistic T has the t-Student distribution with $n-1$ degrees of freedom. Taking $c'' = q_{t(n-1)}\left(1-\frac{\alpha}{2}\right)$, we obtain the α-size test.

Remark 1

If σ^2 is known, the LRT statistic has the form

$$U = \frac{\sqrt{n}\,(\bar{X}-\mu_0)}{\sigma}.$$

Under H_0, $U \sim \mathcal{N}(0,1)$. We reject H_0 when $|U| > \Phi^{-1}\left(1-\frac{\alpha}{2}\right)$.

Remark 2

Let $X_1, \dots, X_m$ be a sample from $\mathcal{N}(\mu_1, \sigma_1^2)$ distribution, and $Y_1, \dots, Y_n$ be an independent from X_i's sample from $\mathcal{N}(\mu_2, \sigma_2^2)$ distribution. We test

$$H_0: \sigma_1^2 = \sigma_2^2,$$
$$H_1: \sigma_1^2 \neq \sigma_2^2.$$

(i) If μ_1, μ_2 are not known, the LRT statistic has the form

$$F = \frac{\frac{1}{m-1}\sum_{i=1}^{m}(X_i-\bar{X})^2}{\frac{1}{n-1}\sum_{j=1}^{n}(Y_j-\bar{Y})^2}.$$

Under H_0, $F \sim F(m-1, n-1)$. We reject H_0 when

$$F < q_{F(m-1,n-1)}\left(\frac{\alpha}{2}\right) \quad \text{or} \quad F > q_{F(m-1,n-1)}\left(1-\frac{\alpha}{2}\right).$$

(ii) If μ_1, μ_2 are known, the LRT statistic has the form

$$F = \frac{\frac{1}{m}\sum_{i=1}^{m}(X_i-\mu_1)^2}{\frac{1}{n}\sum_{j=1}^{n}(Y_j-\mu_2)^2}.$$

Under H_0, $F \sim F(m,n)$. We reject H_0 when

$$F < q_{F(m,n)}\left(\frac{\alpha}{2}\right) \quad \text{or} \quad F > q_{F(m,n)}\left(1-\frac{\alpha}{2}\right).$$

Remark 3

Let $X_1, \dots, X_m$ be a sample from $\mathcal{N}(\mu_1, \sigma^2)$ distribution.
Let $Y_1, \dots, Y_n$ be — " — — " — $\mathcal{N}(\mu_2, \sigma^2)$ — " —.
~~Both~~ The samples are independent, while σ^2 is unknown.

We verify

$$H_0: \mu_1 = \mu_2,$$
$$H_1: \mu \neq \mu_2.$$

The LRT statistic has the form

$$T = \sqrt{\frac{mn}{m+n}} \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{1}{m+n-2}\left[\sum_{i=1}^{m}(X_i - \bar{X})^2 + \sum_{j=1}^{n}(Y_j - \bar{Y})^2\right]}}.$$

Under H_0, ~~the~~ T statistic has the t-Student distribution with $m+n-2$ degrees of freedom. We reject H_0 when

$$|T| > q_{t(m+n-2)}\left(1-\frac{\alpha}{2}\right).$$

I Inferences about Normal Models

1. Quadratic Forms

Definition 1

A homogeneous polynomial of degree 2 in n variables is called a quadratic form in those variables. If both the variables and the coefficients are real, the form is called a real quadratic form.

Example 1

$X_1^2 + X_1X_2 + X_2^2$ is a quadratic form in the two variables X_1 and X_2

$X_1^2 + X_2^2 + X_3^2 - 2X_1X_2$ is a quadratic form in the three variables X_1, X_2, and X_3.

$(X_1-1)^2 + (X_2-2)^2 = X_1^2 + X_2^2 - 2X_1 - 4X_2 + 5$ is not quadratic form in X_1 and X_2, although it is a quadratic form in the variables $X_1 - 1$ and $X_2 - 2$.

$$(n-1)S^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2 = \sum_{i=1}^{n}\left(X_i - \frac{X_1 + \ldots + X_n}{n}\right)^2 =$$

$$\frac{n-1}{n}\left(X_1^2 + X_2^2 + \ldots + X_n^2\right) - \frac{2}{n}\left(X_1X_2 + \ldots + X_1X_n + \ldots + X_{n-1}X_n\right)$$

is a quadratic form in the n variables $X_1, X_2, \ldots, X_n$.

Theorem 1

Let $Q = Q_1 + Q_2 + \ldots + Q_{k-1} + Q_k$, where $Q, Q_1, \ldots, Q_k$ are $k+1$ random variables that are real quadratic forms in n independent random variables which are normally distributed with common mean and variance $\mu^{=0}$ and σ^2, respectively. Let $Q/\sigma^2, Q_1/\sigma^2, \ldots, Q_{k-1}/\sigma^2$ have chi-square distributions with degrees of freedom $r, r_1, \ldots, r_{k-1}$, respectively. Let Q_k be nonnegative. Then:

a) $Q_1, \dots, Q_k$ are independent, and hence

b) Q_k/σ^2 has a chi-square distribution with $r - (r_1 + \dots + r_{k-1}) = r_k$ degrees of freedom.

<u>Proof</u> (later)

<u>Example 2</u>

Let the random variable X have a distribution that is $N(\mu, \sigma^2)$. Let a and b denote positive integers greater than 1 and let $n = ab$. Consider a random sample of size n from this normal distribution

$$
\begin{array}{l}
X_{11}, X_{12}, \dots, X_{1j}, \dots, X_{1b} \\
X_{21}, X_{22}, \dots, X_{2j}, \dots, X_{2b} \\
\vdots \\
X_{i1}, X_{i2}, \dots, X_{ij}, \dots, X_{ib} \\
\vdots \\
X_{a1}, X_{a2}, \dots, X_{aj}, \dots, X_{ab}.
\end{array}
$$

We now define $a+b+1$ statistics. They are

$$\overline{X}_{..} = \frac{X_{11} + \dots + X_{1b} + \dots X_{a1} + \dots + X_{ab}}{ab} = \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} X_{ij}}{ab},$$

$$\overline{X}_{i.} = \quad \dots \quad = \frac{\sum_{j=1}^{b} X_{ij}}{b}, \quad i = 1, 2, \dots, a,$$

and

$$\overline{X}_{.j} = \quad \dots \quad = \frac{\sum_{i=1}^{a} X_{ij}}{a}, \quad j = 1, 2, \dots, b.$$

Consider the variance S^2 of the random sample of size $n = ab$. We have the algebraic identity

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$$(ab-1)S^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}(X_{ij} - \bar{X}_{..})^2$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b}\left[(X_{ij} - \bar{X}_{i.}) + (\bar{X}_{i.} - \bar{X}_{..})\right]^2$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b}(X_{ij} - \bar{X}_{i.})^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{X}_{i.} - \bar{X}_{..})^2$$

$$+ 2\sum_{i=1}^{a}\sum_{j=1}^{b}(X_{ij} - \bar{X}_{i.})(\bar{X}_{i.} - \bar{X}_{..}).$$

Furthermore,

$$2\sum_{i=1}^{a}\left[(\bar{X}_{i.} - \bar{X}_{..})\sum_{j=1}^{b}(X_{ij} - \bar{X}_{i.})\right] = 2\sum_{i=1}^{a}\left[(\bar{X}_{i.} - \bar{X}_{..})(b\bar{X}_{i.} - b\bar{X}_{i.})\right] = 0,$$

and

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{X}_{i.} - \bar{X}_{..})^2 = b\sum_{i=1}^{a}(\bar{X}_{i.} - \bar{X}_{..})^2.$$

Thus

$$(ab-1)S^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}(X_{ij} - \bar{X}_{i.})^2 + b\sum_{i=1}^{a}(\bar{X}_{i.} - \bar{X}_{..})^2,$$

or, for brevity,

$$Q = Q_1 + Q_2.$$

Since $X_{ij} \sim N(\mu, \sigma^2)$, we have $(ab-1)S^2/\sigma^2 = Q/\sigma^2 \sim \chi^2(ab-1)$. For each fixed value of i, $\sum_{j=1}^{b}(X_{ij} - \bar{X}_{i.})^2/\sigma^2 \sim \chi^2(b-1)$ and $\sum_{j=1}^{b}(X_{1j} - \bar{X}_{1.})^2/\sigma^2, \ldots, \sum_{j=1}^{b}(X_{aj} - \bar{X}_{a.})^2/\sigma^2$ are independent. Therefore,

$$\sum_{i=1}^{a}\left[\sum_{j=1}^{b}(X_{ij} - \bar{X}_{i.})^2/\sigma^2\right] = Q_1/\sigma^2 \sim \chi^2(a[b-1]).$$ Now $Q_2 = b\sum_{i=1}^{a}(\bar{X}_{i.} - \bar{X}_{..})^2 \geq 0$.

Theorem 1 implies that Q_1 and Q_2 are independent and $\frac{Q_2}{\sigma^2} \sim \chi^2(\cancel{a(b-1)}\; ab-1-a(b-1) = a-1)$

Similarly, replacing $X_{ij}-\bar{X}_{..}$ by $(X_{ij}-\bar{X}_{.j})+(\bar{X}_{.j}-\bar{X}_{..})$, in $(ab-1)S^2$,

we ~~obtain~~ get

$$(ab-1)S^2 = \sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ij}-\bar{X}_{.j})^2 + a\sum_{j=1}^{b}(\bar{X}_{.j}-\bar{X}_{..})^2,$$

or, for ~~brevity~~ short,

$$Q = Q_3 + Q_4.$$

As a result $Q_3/\sigma^2 \sim \chi^2(b[a-1])$ and $Q_4/\sigma^2 \sim \chi^2(b-1)$ and are independent.

In like manner, in $(ab-1)S^2$, replacing $X_{ij}-\bar{X}_{..}$ by $(\bar{X}_{i.}-\bar{X}_{..})+(\bar{X}_{.j}-\bar{X}_{..})+(X_{ij}-\bar{X}_{i.}-\bar{X}_{.j}+\bar{X}_{..})$, we ~~get~~ obtain

$$(ab-1)S^2 = b\sum_{i=1}^{a}(\bar{X}_{i.}-\bar{X}_{..})^2 + a\sum_{j=1}^{b}(\bar{X}_{.j}-\bar{X}_{..})^2 + \sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ij}-\bar{X}_{i.}-\bar{X}_{.j}+\bar{X}_{..})^2,$$

or, for clarity,

$$Q = Q_2 + Q_4 + Q_5.$$

Moreover,

$$\frac{Q}{\sigma^2} \sim \chi^2(ab-1),\ \frac{Q_2}{\sigma^2} \sim \chi^2(a-1),\ \frac{Q_4}{\sigma^2} \sim \chi^2(b-1).$$

Since $Q_5 \geq 0$, the theorem asserts that, Q_2, Q_4, and Q_5 are independent and that $\frac{Q_5}{\sigma^2} \sim \chi^2([a-1][b-1])$.

Finally,

$$\frac{\frac{Q_4}{\sigma^2(b-1)}}{\frac{Q_3}{\sigma^2 b(a-1)}} \sim F(b-1, a-1), \text{ and}$$

$$\frac{\frac{Q_4}{\sigma^2(b-1)}}{\frac{Q_5}{\sigma^2(a-1)(b-1)}} \sim F(b-1, [a-1][b-1]).$$