Corollary 1

If $X_1, \dots, X_n$, a random sample, comes from a regular exponential class and φ is a function of $Y_1 = \sum_{i=1}^{n} K(X_i)$ such that $\mathbb{E}[\varphi(Y_1)] = \theta$, then $\varphi(Y_1)$ is an uniquely determined MVUE of the parameter θ.

Example 2

$X_1, \dots, X_n$ i.i.d $X_1 \sim f(x, \theta) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\}$, $x \in \mathbb{R}$, $\theta \in \mathbb{R}$, $\sigma^2 > 0$.

Therefore

$$f(x, \theta) = \exp\left\{\frac{\theta}{\sigma^2} x - \frac{x^2}{2\sigma^2} - \log\sqrt{2\pi\sigma^2} - \frac{\theta^2}{2\sigma^2}\right\}$$

is a member of REC. Specifically,

$$p(\theta) = \frac{\theta}{\sigma^2},\ K(x) = x,\ S(x) = \frac{-x^2}{2\sigma^2} - \log\sqrt{2\pi\sigma^2},\ q(\theta) = -\frac{\theta^2}{2\sigma^2}.$$

Thus $Y_1 = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ.

Since $\varphi(Y_1) = \frac{Y_1}{n} = \overline{X}$ is an unbiased estimator of θ, $\varphi(Y_1)$ is a uniquely determined MVUE of θ.

$\overline{X}$ is also a complete sufficient statistic for θ because $Y_1 \xrightarrow{1-1} \overline{X}$.

Example 3

$X \sim Pois(\theta)$, $\theta \in (0, +\infty)$, $S = \{0, 1, 2, \dots\}$

$$f(x, \theta) = e^{-\theta} \frac{\theta^x}{x!} = \exp\left\{(\log\theta) x + \log\left(\frac{1}{x!}\right) + (-\theta)\right\}.\ \text{REC}$$

$p(\theta) = \log\theta$, $K(x) = x$, $S(x) = \log\left(\frac{1}{x!}\right)$, $q(\theta) = -\theta$.

$Y_1 = \sum_{i=1}^{n} X_i$ - complete and sufficient statistic for θ

$\mathbb{E} Y_1 = n\theta$ $\quad$ $\varphi(Y_1) = \overline{X}$ - UD MVUE for θ.

III Theory of Testing Statistical Hypotheses

1. Introduction

Let X be a random variable with the distribution $f(x,\theta)$, $\theta \in \Theta$. Let Θ_0 and Θ_1 (subsets of Θ) be such that $\Theta_0 \cup \Theta_1 = \Theta$ and $\Theta_0 \cap \Theta_1 = \emptyset$.

Definition 1

Supposition $\theta \in \Theta_0$ is called the null hypothesis and is denoted by $H_0: \theta \in \Theta_0$, while supposition $\theta \in \Theta_1$ is called the alternative hypothesis and is denoted by $H_1: \theta \in \Theta_1$.

Definition 2

The testing formulation

$$H_0: \theta \in \Theta_0$$

against

$$H_1: \theta \in \Theta_1$$

is called the testing problem. Cheking statistal hypotheses is called testing (verifying) hypotheses.

Definition 3

If $\#\Theta_0 = 1$ $(\#\Theta_1 = 1)$ the hypothesis H_0 (H_1) is called simple. Otherwise, it is said that the hypothesis H_0 (H_1) is composite.

Let $X_1, \ldots, X_n$ be a sample with $f(x, \theta)$. Consider the testing problem

$$H_0: \theta \in \Theta_0,$$
$$H_1: \theta \in \Theta_1.$$

Let $\mathcal{X} = \{X_1(\omega), \ldots, X_n(\omega) : \omega \in \Omega\}$ be the sample space.

<u>Definition 4</u>

The statistic $T = T(X_1, \ldots, X_n)$ allowing one to assert in the above problem is called the test statistic.

<u>Definition 5</u>

The set $C = \{\underline{x} : \underline{x} = (x_1, \ldots, x_n), \underline{x} \in \mathcal{X}\}$ such that for $\underline{x} \in C$, $T(\underline{x})$ leads to rejection of the null hypothesis is called the critical region.

<u>Remark 1</u>

The critical region C of the form

(i) $\{\underline{x} : T(\underline{x}) > c_1\}$ for some $c_1 \in \mathbb{R}$ is called the right-tailed critical region.

(ii) $\{\underline{x} : T(\underline{x}) < c_2\}$ for some $c_2 \in \mathbb{R}$ is called the left-tailed critical region.

(iii) $\{\underline{x} : T(\underline{x}) > c_3\} \cup \{\underline{x} : T(\underline{x}) < c_4\}$ for some $c_3, c_4 \in \mathbb{R}$ is called two-tailed critical region

(iv) In general the critical region (i) or (ii) is called a one-tailed critical region.

Definition 6

An error relying on rejection of a true null hypothesis H_0 is called the Type I error (the error of the first kind).

An error relying on acceptance a false null hypothesis H_0 is called the Type II error (the error of the second kind).

Illustration

		Decision	
		H_0	H_1
Truth	H_0	X	Type I error
	H_1	Type II error	X

Definition 7

Let C be a critical region. The measurable function of the form $\mathbb{1}_C(\underline{x})$ is called a (non-randomized) test of the hypothesis H_0 against the alternative H_1 and is denoted by $\varphi(\underline{x})$ or φ, for short.

Definition 8

A number $\alpha \in (0,1)$ is called the significance level.

Remark 2.

Usually, $\alpha = 0.01$, $\alpha = 0.05$ $\alpha = 0.1$.

Definition 9

Let $\alpha \in (0,1)$. It is called that the test φ is at the significance level α, if (and only if)

$$\sup_{\theta \in \Theta_0} \mathbb{E}_\theta[\varphi(\underline{X})] = \sup_{\theta \in \Theta_0} \mathbb{P}_\theta(\underline{X} \in C) \leq \alpha .$$

If

$$\sup_{\theta \in \Theta_0} \mathbb{E}_\theta[\varphi(\underline{X})] = \alpha ,$$

it is said that the test φ has the size α.

Definition 10

The function $\gamma : \Theta \to [0,1]$ defined as follows:

$$\gamma(\theta) = P_\theta(\underline{X} \in C) = \mathbb{E}_\theta[\varphi(\underline{X})] \quad \text{for } \theta \in \Theta$$

is called ~~the~~ the power function of the test φ.

The number $\gamma(\theta)$ for $\theta \in \Theta_1$ is called the power of the test φ under the alternative θ.

Remark 3

Statistical tests are constructed in such a manner in order to minimize the probability of making the Type II error under ~~give~~ fixed probability of making the Type I error equals α.

2. Neyman - Pearson Lemma

Definition 1

It is said that the test φ_0 is ~~the~~ uniformly most powerful at the significance level α, if for any another test φ at the same significance level

$$\mathbb{E}_\theta[\varphi(\underline{X})] \leq \mathbb{E}_\theta[\varphi_0(\underline{X})] \quad \text{for } \cancel{\text{each}} \text{ any } \theta \in \Theta_1.$$

Theorem 1 (Neyman Pearson Lemma)

Let $X_1, \ldots, X_n$ be a sample with $f(x, \theta)$.
Consider the testing problem

$$H_0: \theta \in \Theta_0,$$
$$H_1: \theta = \theta_1,$$

and the α-size φ_0 test of the form

$$\varphi_0(\underline{x}) = \begin{cases} 1, & \text{if } \prod_{i=1}^n f(x_i, \theta_0) < k \prod_{i=1}^n f(x_i, \theta_1), \\ \gamma, & \text{if } \quad = \quad , \\ 0, & \text{if } \quad > \quad , \end{cases}$$

where the constants k and γ ~~are~~ satisfing the condition $\mathbb{E}_{\theta_0}[\varphi(\underline{X})] = \alpha$. Then, φ_0 is the UMP test in the problem (H_0, H_1).

Corollary 1

Under the conditions of Theorem 1, $\gamma_{\varphi_0}(\theta_1) \geq \alpha$.