CREDIT SUISSE

Internal

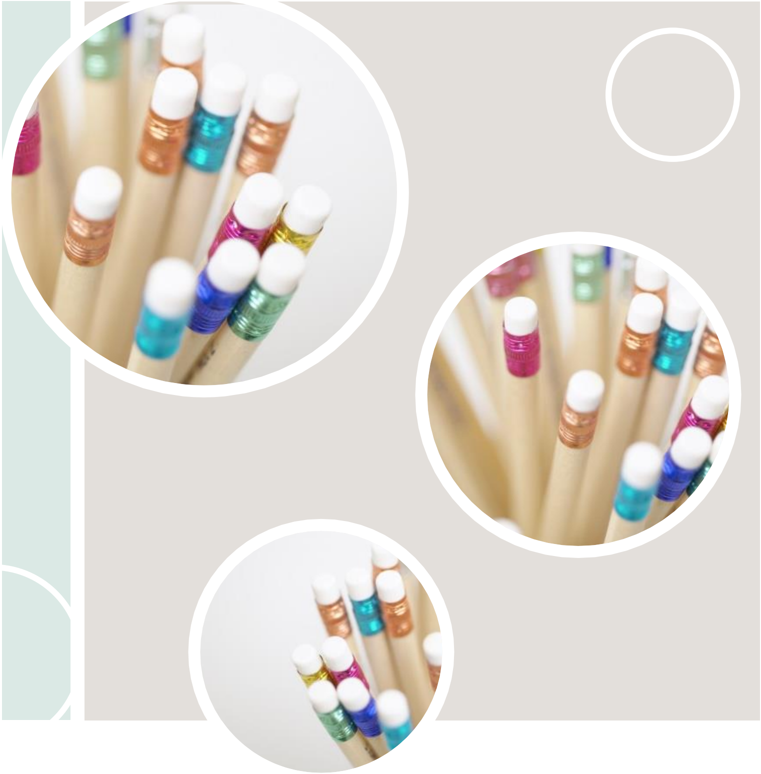
Introduction to Option Pricing

Quantitative Strategies

November 2022

Agenda

Introduction to options



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- Introduction to options
- One step binomial tree model

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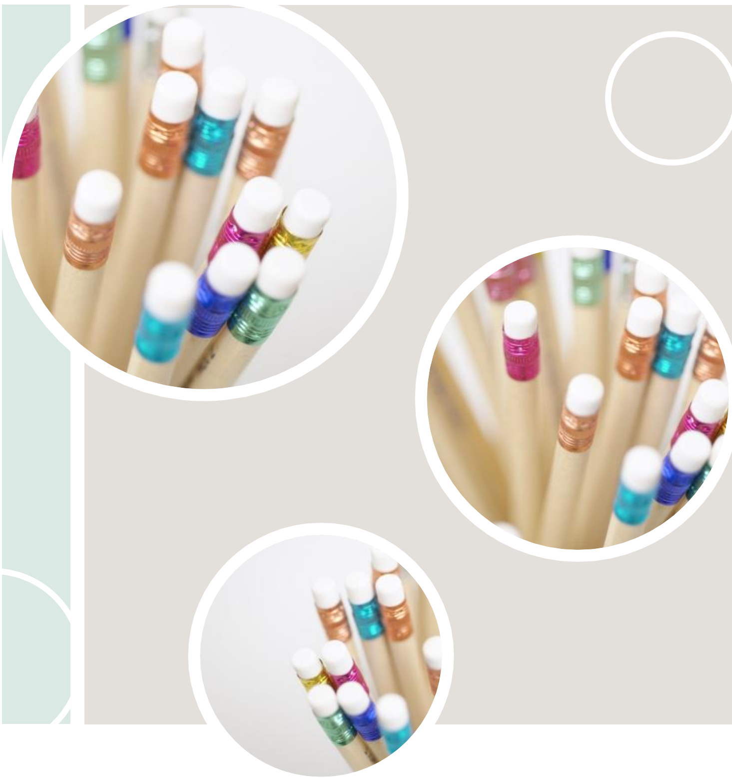
- Multistep binomial tree model



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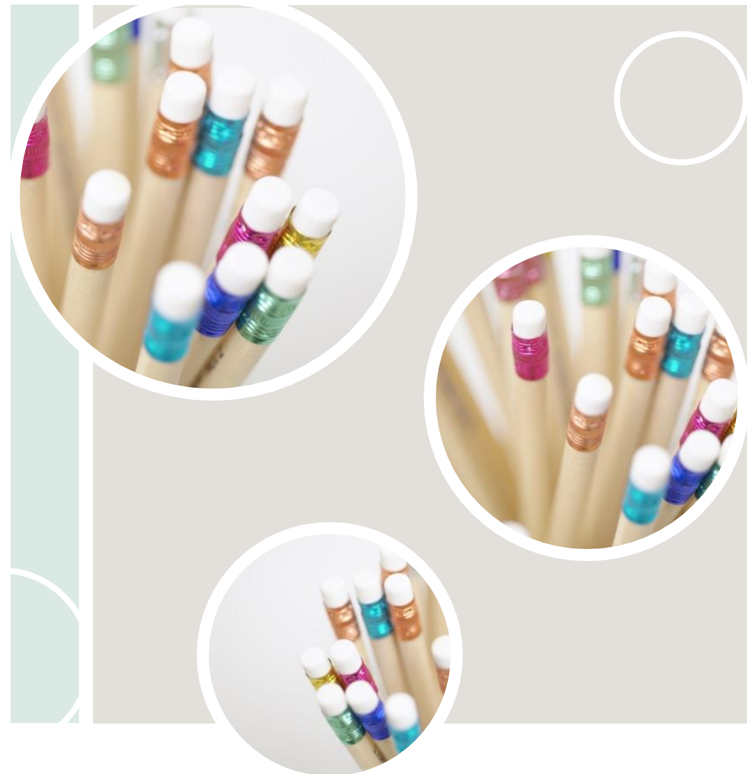

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- Introduction to options
- One step binomial tree model
- Multistep binomial tree model
- Black-Scholes formula
- Greeks in Black Scholes world
- Daily hedging

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- Monte Carlo

Introduction to Options (1/3)

Option is a financial instrument which gives a holder (buyer) a right, but not an obligation, to buy (sell) stock at predefined level (strike) and at particular moment in time (maturity).

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- **Call option with strike 100 and maturity 1 year**
  In 1 year time holder will have a right to buy a stock for 100:
  – *Scenario 1: Stock price turns out to be 120.*

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  - *Scenario 2: Stock price ends up at* 80*.*
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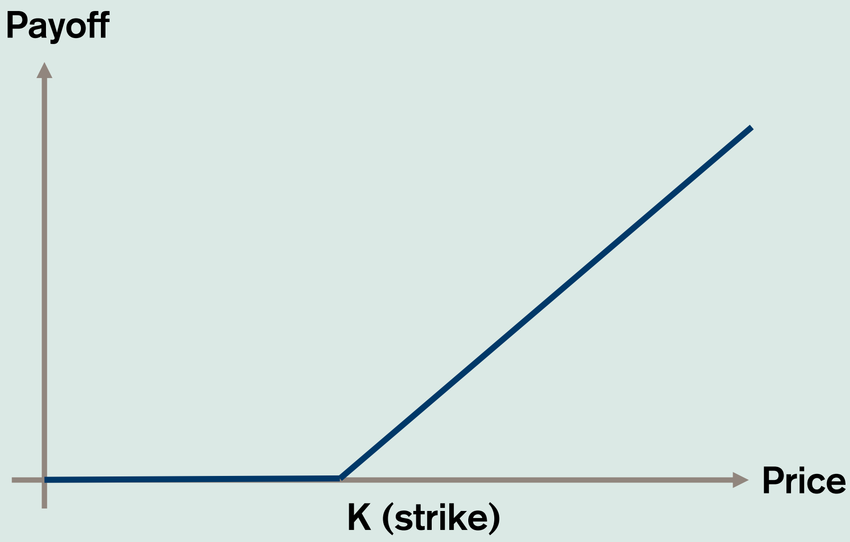
- **Put option with strike 120 and maturity 1 year**
  In 1 year time holder will have a right to sell a stock for 120. Situation is reverse: if stock finishes at 80, gain is 40, if stock finishes at 140 gain is 0.

Introduction to Options (2/3)

Call and put payoff plots

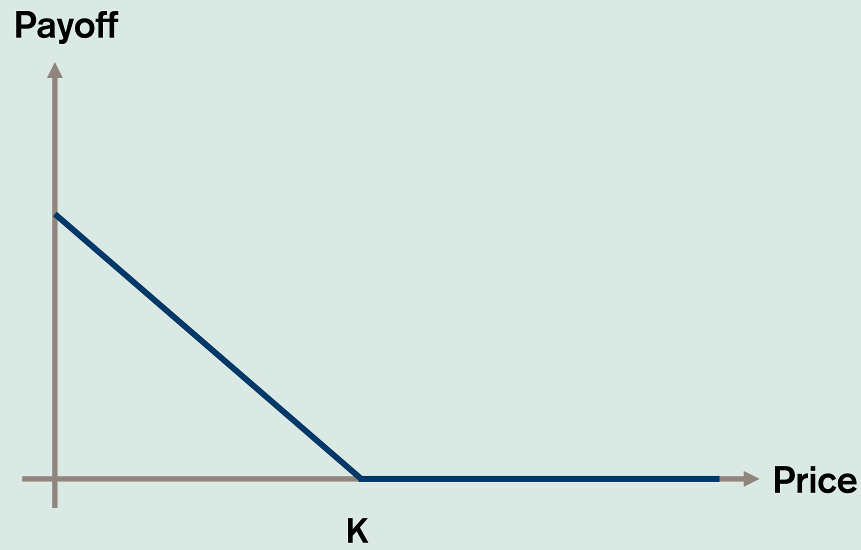
Call option payoff is given by the formula:

$$payoff \; = \max(price - \; strike, 0) \stackrel{\text{def}}{=} (price - strike)^{+}$$



Put option payoff is given by the formula:

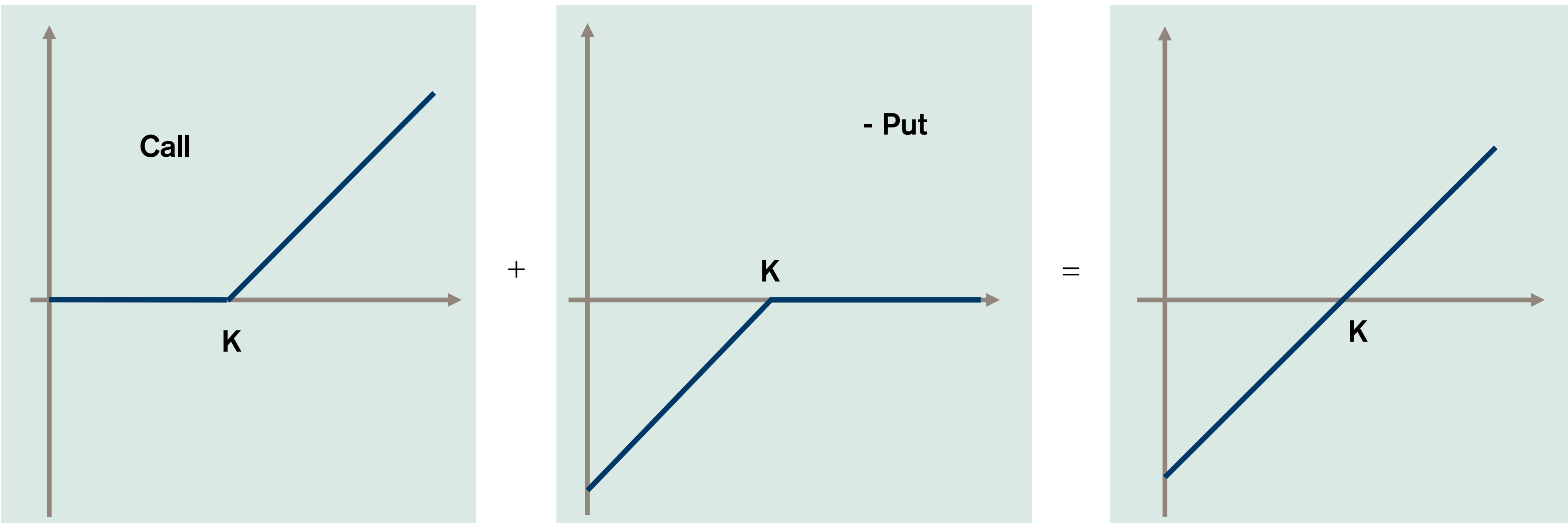
$$payoff \; = \max(strike - price, 0) \stackrel{\text{def}}{=} (strike - price)^{+}$$



Lack of arbitrage assumption: option payoff is nonnegative therefore it have to cost money (paid upfront)

Introduction to Options (3/3)

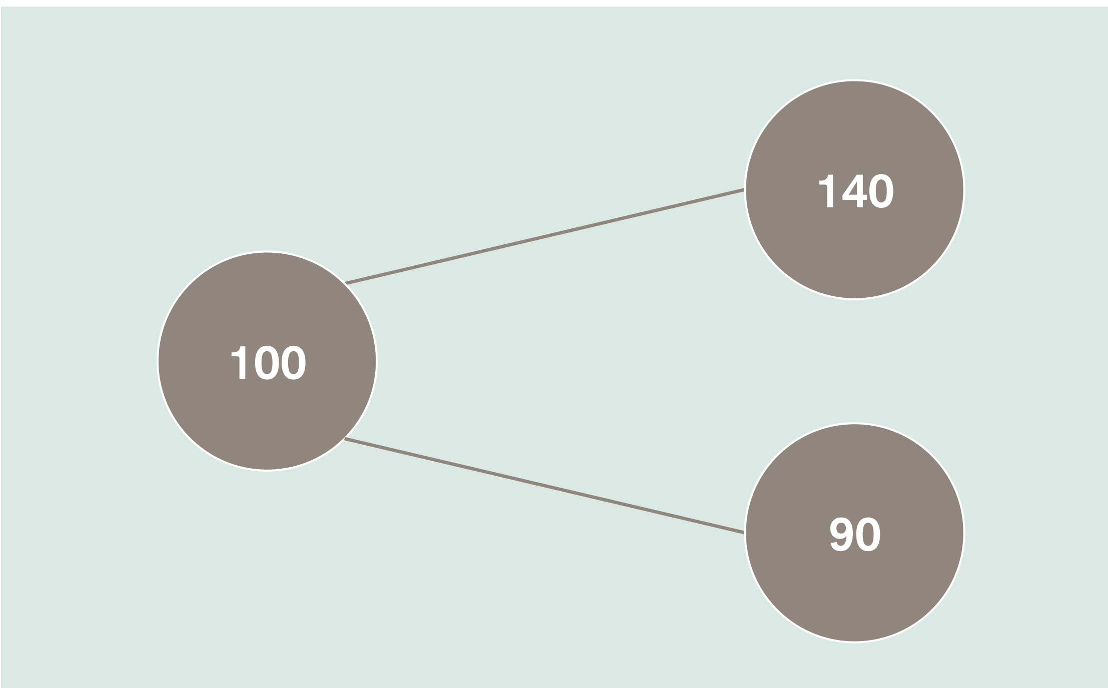
Call-Put parity: call (long) and put (short) with the same strike give forward contract



One Step Binomial Tree Model (1/5)

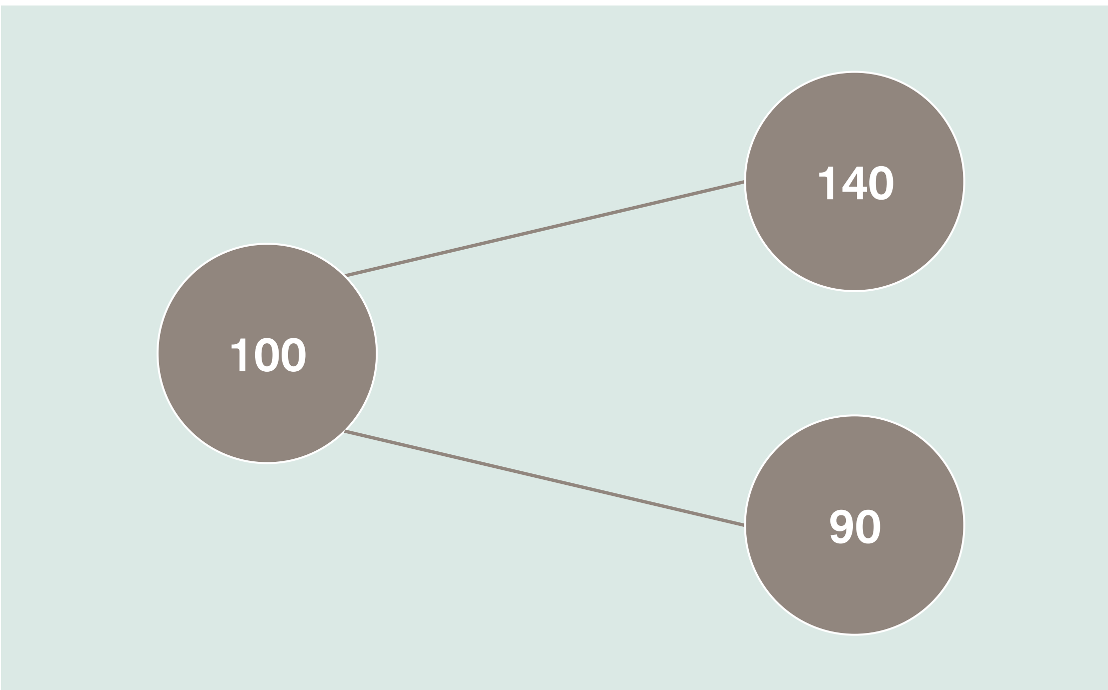
Model assumptions

- Price can only evolve into 2 possible future states
- No interest rates (for simplicity of presentation, we could easily include them in the model)
- Example:
  - Today's price is 100
  - Possible future prices are 140 and 90



One Step Binomial Tree Model (2/5)

Call option with strike 110



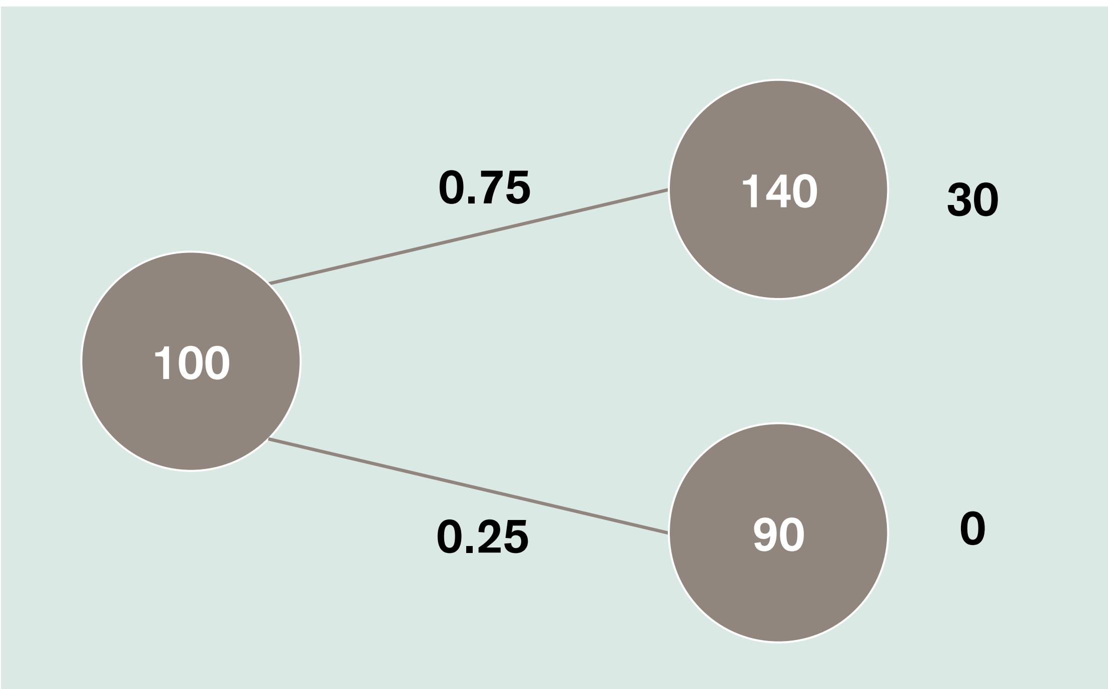
Payoff:

$$30 = (140 - 110)^{+}$$

$$0 = (90 - 110)^{+}$$

One Step Binomial Tree Model (3/5)

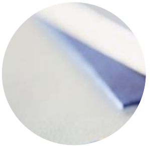
- How can we calculate the price of such an option?
- The answer is simple: use probabilities and calculate the expectation
- We can estimate these probabilities from asset price history



$$30 \cdot 0.75 + 0 \cdot 0.25 = 22.5$$

One Step Binomial Tree Model (4/5)

We can do better!



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- Hence we have equations

$$\begin{cases} 140 \cdot Q + C = 30 \\ 90 \cdot Q + C = 0 \end{cases} \Rightarrow \begin{cases} Q = 0.6 \\ C = -54 \end{cases}$$

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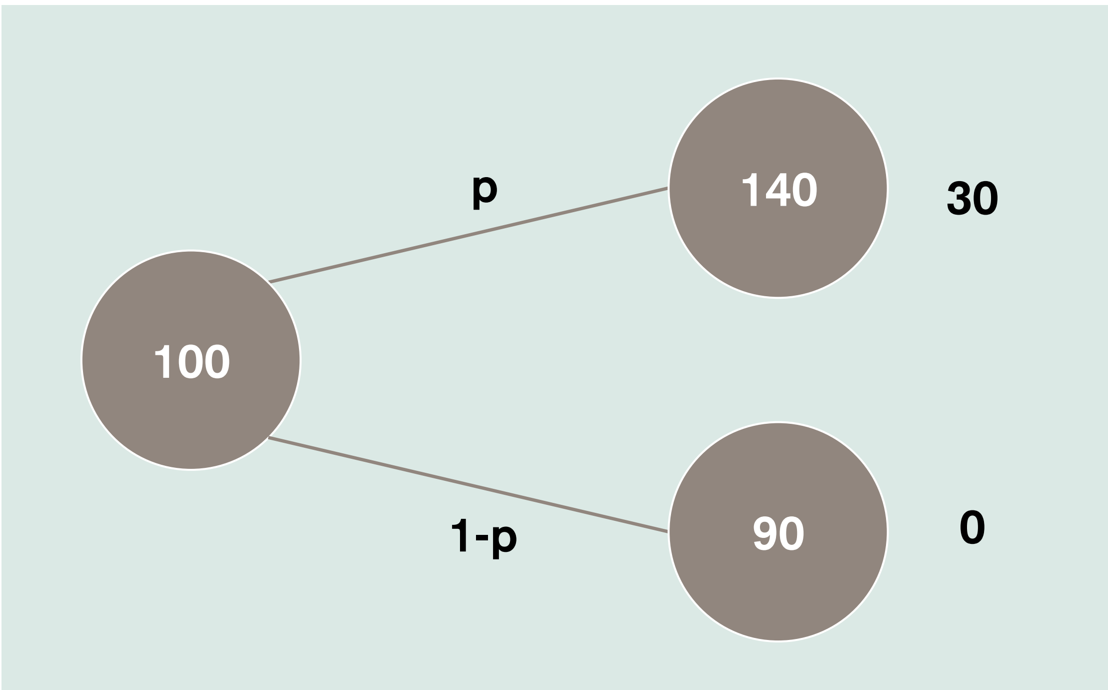
- So the price of the option needs to be euqal to 6 since the option is equivalent to our portfolio!

One Step Binomial Tree Model (5/5)

We can think of the solution as an expectation with respect to different probabilities



$$\mathbf{30} \cdot \boldsymbol{p} + \ \mathbf{0} \cdot (\mathbf{1} - \boldsymbol{p}) = \mathbf{6}$$

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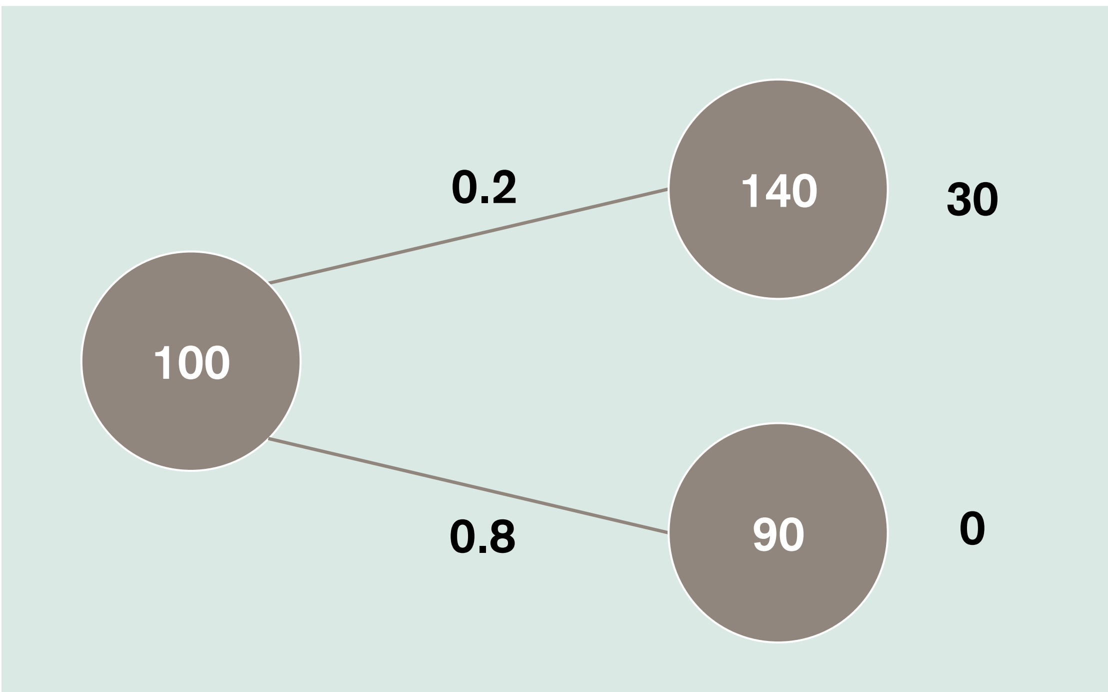

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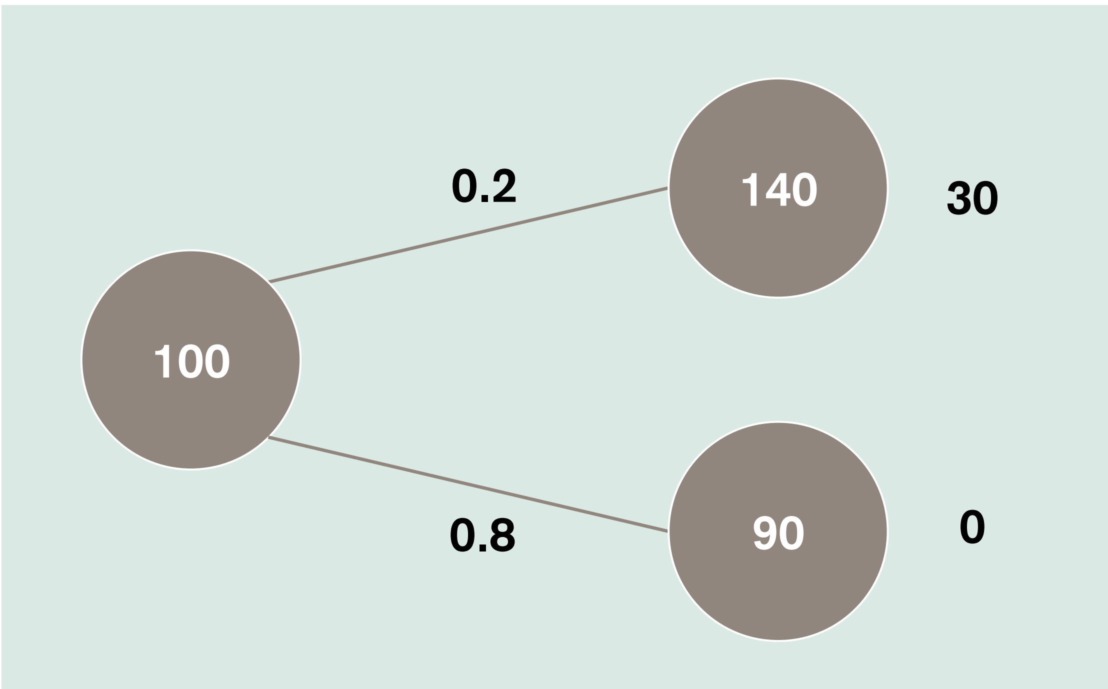

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Side note: these probabilities make expectation of stock value in the future equal to the current value (martingale)

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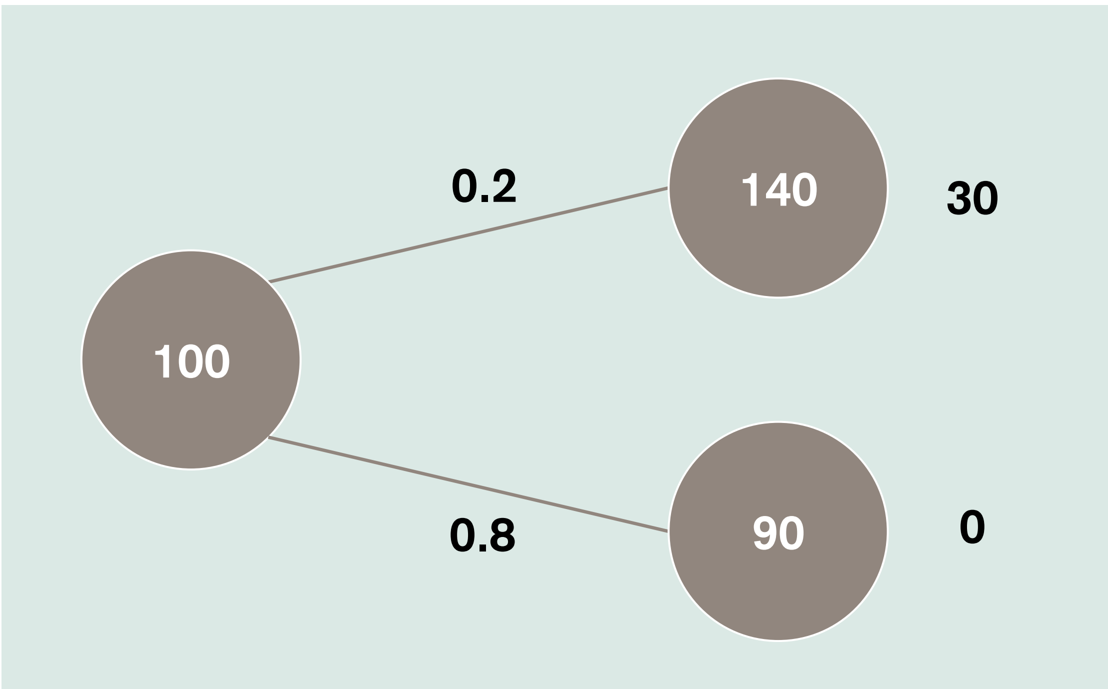

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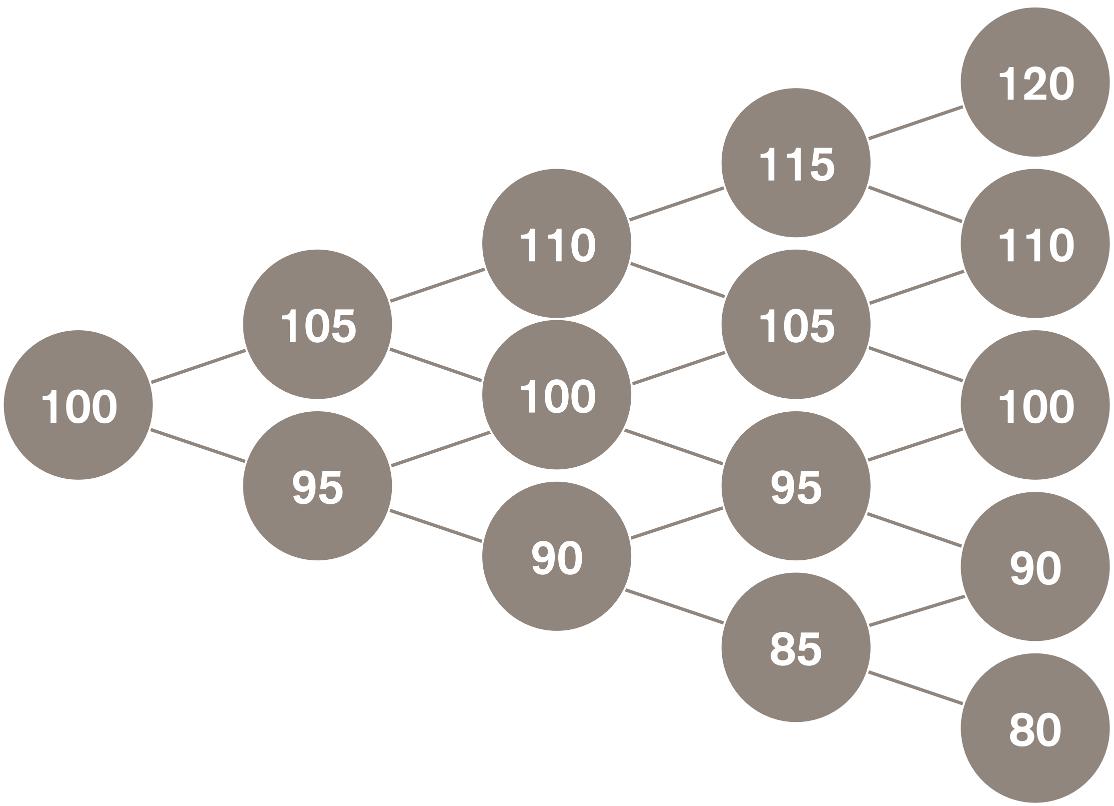
$$140 \cdot 0.2 + 90 \cdot 0.8 = 100$$

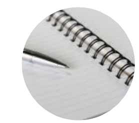
They are called risk neutral probabilities/risk neutral measure. They are independent of option payoff.

Multistep Binomial Tree Model (1/2)

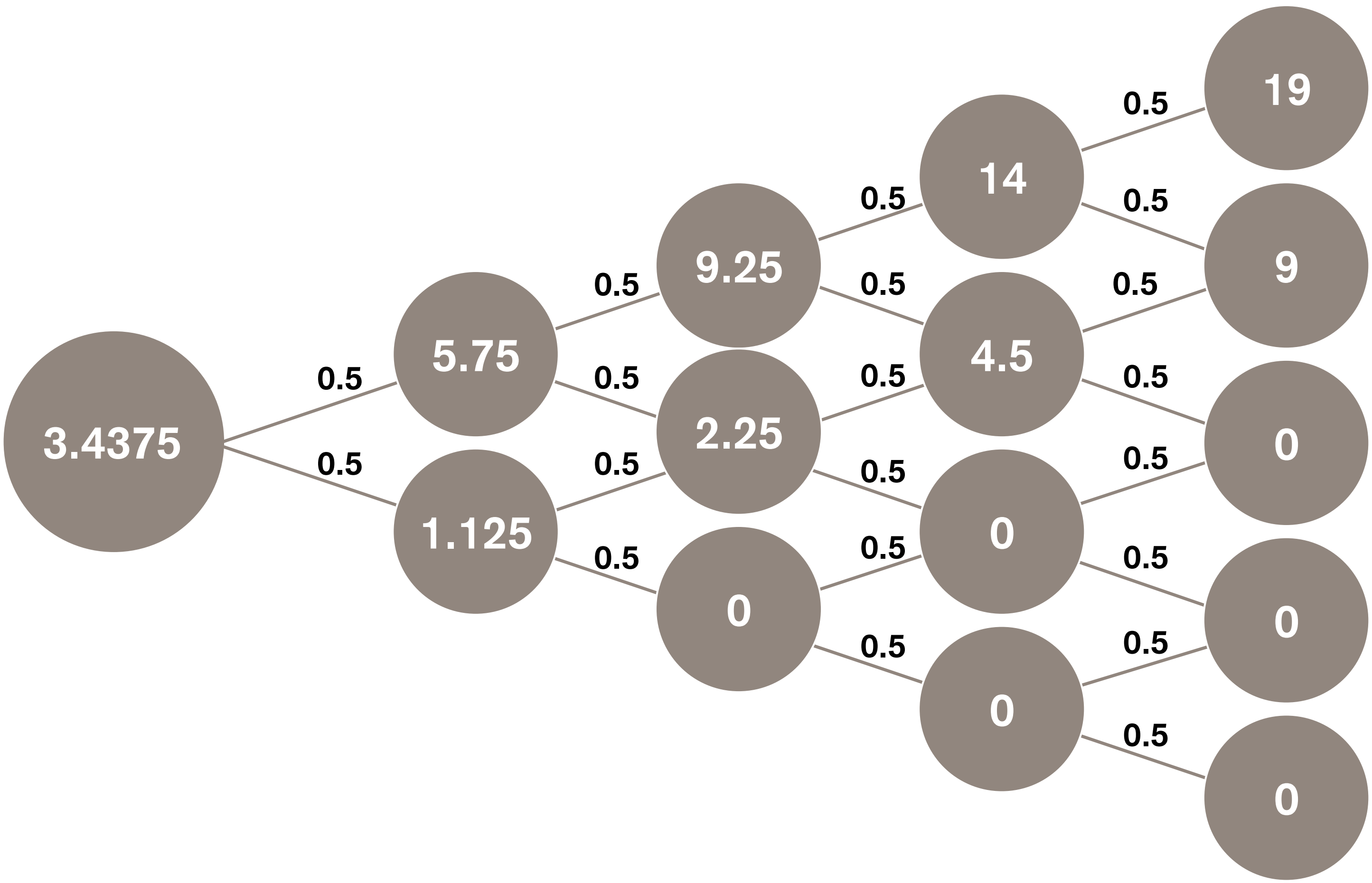
- One step is unrealistic
- We can do more but smaller steps



Multistep Binomial Tree Model (2/2)



Work out solution just like in one step model from back (leaves) to the beginning (root):



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Expectation of call option payoff

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Black-Scholes Formula 2

Let's calculate the second integral

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$$\int_{\ln\frac{K}{S_0}}^{\infty} K\frac{1}{\sigma\sqrt{2\pi T}} e^{\frac{-\left(y-\left(r-\frac{\sigma^2}{2}\right)T\right)^2}{2\sigma^2 T}} dy = K(1-\Phi\left(\frac{\ln\frac{K}{S_0}-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = \mathrm{K}\Phi\left(\frac{\ln\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

To calculate the first integral let's consider exponent and define define $\mu = \left(r-\frac{\sigma^2}{2}\right)T$ and $\alpha^2 = \sigma^2 T$ for short

$$-\frac{\left(y-\left(r-\frac{\sigma^2}{2}\right)T\right)^2}{2\sigma^2 T} + y = -\frac{(y-\mu)^2 - 2y\alpha^2}{2\alpha^2} = -\frac{\left(y-(\mu+\alpha^2)\right)^2 - 2\mu\alpha^2 - \alpha^4}{2\alpha^2} =$$

$$= -\frac{\left(y-\left(\left(r-\frac{\sigma^2}{2}\right)+\sigma^2 T\right)\right)^2}{2\sigma^2 T} + \left(r-\frac{\sigma^2}{2}\right)T + \frac{\sigma^2 T}{2}$$

Black-Scholes Formula 3

Hence we can substitute exponent in the first integral to obtain

$$e^{\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma^2 T}{2}}\int_{\ln\frac{K}{S_0}}^{\infty}\frac{1}{\sigma\sqrt{2\pi T}}e^{\frac{-\left(y-\left(\left(r-\frac{\sigma^2}{2}\right)+\sigma^2 T\right)\right)^2}{2\sigma^2 T}}dy = e^{\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma^2 T}{2}}\left(1-\Phi\left(\frac{\ln\frac{K}{S_0}-\left(\left(r-\frac{\sigma^2}{2}\right)T+\sigma^2 T\right)}{\sigma\sqrt{T}}\right)\right)=$$

$$= e^{rT}\Phi\left(\frac{\ln\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

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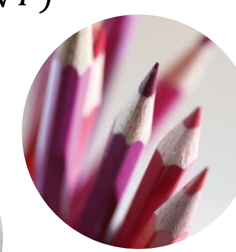
Putting it all together multiplying by discount factor e^{-rT} we obtain Black-Scholes formula for call option price

$$BS_{call} = S_0\Phi\left(\frac{\ln\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\ln\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_1-\sigma\sqrt{T})$$

$$d_1 = \frac{\ln\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$



Black-Scholes Formula 4

Using Call-Put parity we can easily obtain formula for put option price

$$BS_{put} = Ke^{-rT}\Phi\left(\frac{\ln\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - S_0\Phi\left(\frac{\ln\frac{K}{S_0} - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = Ke^{-rT}\Phi\left(\sigma\sqrt{T} - d_1\right) - S_0\Phi(-d_1)$$

Black-Scholes Formula

History
First derivation in 1973 by Fischer Black and Myron Scholes
1977 enhancements by Robert Merton
Merton and Scholes receive Nobel prize (Black died in 1995)
Original derivation is different from described earlier, it uses stochastic differential equations
Mathematician and hedge fund manager Ed Thorp derived and used this formula in 1969 to make himself very rich

Greeks in Black Scholes World (1/2)

We know call option price. But how to construct replicating porftolio?

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Both previous terms are the same with opposite signs so

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- We call the first derivative delta. It measures sensitivity of portfolio to stock price movements.
- Amount of cash needed for our replicating portfolio can be easily obtain when have option price and delta
- Sensitivity to other values are traditionally referred by the name of Greeks. They are used to risk manage complex portfolios.

Daily Hedging

Since price of the stock changes in every moment hence our strategy needs to do that same. In practice it's not possible to hedge every millisecond and it would create huge cost due to performing market operations. In practice one wants to rehedge less frequently i.e. once a day.

Algorithm

- Buy delta stocks and C of cash at time t_0
- In t_1: stock moved, calculate new delta, buy (sell) to have new delta of stocks using some of available cash
- Proceed similarly in t_2, t_3, t_4, ..., T until maturity
- At maturity pay the buyer option payoff
- Work out how much money you have left or lack. This is your PnL (profit and loss) for that particular market evolution scenario.
- One can simulate lots of such paths to obtain distribution of PnL

Pros	Cons
We don't loose money due to frequent rehedging	We don't have exact, replicating strategy – potential losses

Monte Carlo (1/3)

Asian options

- Averaged strike, stock
- Different kinds of averaging (arithmetic, geometric)
- Different monitor frequency (continuous, discrete)

$$Payoff = \left(S_T - AVG(S_t)\right)^+$$

$$Payoff = (AVG(S_t) - K)^+$$

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No closed formula for Asian options with arithmetic averaging

What can we do about it?

Monte Carlo (2/3)

Since option price is an expectation we can approximate it using law of the large numbers

$$E[X] \approx \frac{X_1 + X_2 + \cdots + X_n}{n}$$

For discrete monitored Asian option the Monte Carlo algorithm will look like

- Simulate stock at t_1, t_2, t_3, ..., t_n (in risk neutral measure)
- Take average
- Calculate payoff
- Repeat
- After large number of simulations spot and take avarage of all recorded results
- This is approximately the price of the option

Monte Carlo (3/3)

Monte Carlo

Pros

- Applicability: It is useful for more complicated instruments when close form formula is not known/ not possible to calculate

Cons

- Calculation burden: it much more involving in terms of number of calculations → slower
- Error: It’s not exact. But can be controlled by increasing number of simulation and other more advanced techniques as variation reduction.