2. One-way ANOVA

Consider

$$X_{11}, \dots, X_{1b};$$
$$\vdots \quad \ddots \quad \vdots$$
$$X_{a1}, \dots, X_{ab},$$

~~a ra~~ independent identically distributed (iid) random variables, where $X_{ij} \sim N(\mu_j, \sigma^2)$, $i = 1, .., a$, $j = 1, .., b$, and all parameters are unknown.

The appropriate model for the observations is as follows

$$X_{ij} = \mu_j + e_{ij};\ i = 1, .., a,\ j = 1, .., b,$$

where e_{ij} are iid $N(0, \sigma^2)$.

Suppose that it is desired to test the composite hypothesis

$$H_0: \mu_1 = \mu_2 = \dots = \mu_b = \mu,$$

μ unspecified, against

$$H_1: \sim H_0.$$

A likelihood ratio test will be used.

Remark 1.

The problem is often summarized that we have one factor at b levels. The model is called a one-way model. As we will see, the likelihood ratio test can be thought of in terms of estimates of variance. Hence, this is an example of an analysis of variance (ANOVA). In short, we say that this example is a one-way ANOVA problem.

The total parameter space is

$$\Omega = \{(\mu_1, \mu_2, \ldots, \mu_b, \sigma^2) : -\infty < \mu_j < \infty,\ 0 < \sigma^2 < +\infty\}$$

and

$$\omega = \{(\mu_1, \ldots, \mu_b, \sigma^2) : -\infty < \mu_1 = \mu_2 = \ldots = \mu_b = \mu < \infty,\ 0 < \sigma^2 < +\infty\}.$$

The likelihood functions, denoted by $L(\omega)$ and $L(\Omega)$ are, respectively,

$$L(\omega) = \left(\frac{1}{2\pi\sigma^2}\right)^{ab/2} \exp\left[-\frac{1}{2\sigma^2}\sum_{j=1}^{b}\sum_{i=1}^{a}(x_{ij}-\mu)^2\right]$$

and

$$L(\Omega) = \left(\frac{1}{2\pi\sigma^2}\right)^{ab/2} \exp\left[-\frac{1}{2\sigma^2}\sum_{j=1}^{b}\sum_{i=1}^{a}(x_{ij}-\mu_j)^2\right].$$

Now

$$\frac{\partial \log L(\omega)}{\partial \mu} = \sigma^{-2}\sum_{j=1}^{b}\sum_{i=1}^{a}(x_{ij}-\mu)$$

and

$$\frac{\partial \log L(\omega)}{\partial (\sigma^2)} = -\frac{ab}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{j=1}^{b}\sum_{i=1}^{a}(x_{ij}-\mu)^2.$$

Solving $\dfrac{\partial \log L(\omega)}{\partial \mu} = 0$ and $\dfrac{\partial \log L(\omega)}{\partial (\sigma^2)} = 0$, we obtain

$$\hat{\mu} = \bar{x}_{..} = \frac{1}{ab}\sum_{j=1}^{b}\sum_{i=1}^{a}x_{ij},$$

$$\hat{\sigma}_0^2 = v = \frac{1}{ab}\sum_{j=1}^{b}\sum_{i=1}^{a}(x_{ij}-\bar{x}_{..})^2,$$

Sufficient condition!

and these values maximize $L(\omega)$.

Furthermore,

$$\frac{\partial \log L(\Omega)}{\partial \mu_j} = \sigma^{-2} \sum_{i=1}^{a} (x_{ij} - \mu_j), \quad j = 1, 2, \dots, b,$$

and

$$\frac{\partial \log L(\Omega)}{\partial (\sigma^2)} = -\frac{ab}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{j=1}^{b} \sum_{i=1}^{a} (x_{ij} - \mu_j)^2.$$

~~then~~ As a result

$$\hat{\mu}_j = \bar{x}_{.j} = \frac{1}{a} \sum_{i=1}^{a} x_{ij}, \quad j = 1, 2, \dots, b$$

sufficient condition!

$$\hat{\sigma}_1^2 = w = \frac{1}{ab} \sum_{j=1}^{b} \sum_{i=1}^{a} (x_{ij} - \bar{x}_{.j})^2$$

maximize $L(\Omega)$. These maxima are, respectively,

$$L(\hat{\omega}) = \left[\frac{ab}{2\pi \sum_{j=1}^{b} \sum_{i=1}^{a} (x_{ij} - \bar{x}_{..})^2} \right]^{ab/2} \exp\left[- \frac{ab \sum_{j=1}^{b} \sum_{i=1}^{a} (x_{ij} - \bar{x}_{..})^2}{2 \sum_{j=1}^{b} \sum_{i=1}^{a} (x_{ij} - \bar{x}_{..})^2} \right]$$

$$= \left[\frac{ab}{2\pi \sum_{j=1}^{b} \sum_{i=1}^{a} (x_{ij} - \bar{x}_{..})^2} \right]^{ab/2} \exp\left[-\frac{ab}{2} \right]$$

and

$$L(\hat{\Omega}) = \left[\frac{ab}{2\pi \sum_{j=1}^{b} \sum_{i=1}^{a} (x_{ij} - \bar{x}_{.j})^2} \right]^{ab/2} \exp\left[-\frac{ab}{2} \right].$$

Finally,

$$\Lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \left[\frac{\sum_{j=1}^{b} \sum_{i=1}^{a} (x_{ij} - \bar{x}_{.j})^2}{\sum_{j=1}^{b} \sum_{i=1}^{a} (x_{ij} - \bar{x}_{..})^2} \right]^{ab/2} := \left[\frac{Q_3}{Q} \right]^{\frac{ab}{2}}.$$

We reject the hypothesis H_0 if $\Lambda \leqslant \lambda_0$.
We find λ_0.

We have

$$\frac{Q_3}{Q} = \frac{Q_3}{Q_3 + Q_4} = \frac{1}{1 + \frac{Q_4}{Q_3}}.$$

~~The significance level of the test of H_0 is~~ Therefore,

$$\alpha = P_{H_0}\left[\frac{1}{1 + Q_4/Q_3} \leq \lambda_0^{\frac{2}{ab}}\right] \overset{!}{=} P_{H_0}\left[\frac{\frac{Q_4}{b-1}}{\frac{Q_3}{b(a-1)}} \geq c\right],$$

where

$$c = \frac{b(a-1)}{b-1}\left(\lambda_0^{-2/ab} - 1\right).$$

But

$$F = \frac{\frac{Q_4}{\sigma^2(b-1)}}{\frac{Q_3}{\sigma^2 b(a-1)}} = \frac{\frac{Q_4}{b-1}}{\frac{Q_3}{b(a-1)}}$$

has an F-distribution with $b-1$ and $b(a-1)$ degrees of freedom.

As a result, ~~The constant c~~ is so selected ~~as to~~ yield the desi~~red~~ value of ~~α i.e~~ $c = q_{F(b-1,\, b(a-1))}(1-\alpha)$.

Remark 2

The samples may be of different sizes, for instance,

$a_1, a_2, \dots, a_b$.