The pdf of $Y_1 = \min\{X_1, \dots, X_n\}$ is

$$f_{Y_1}(y_1, \theta) = n e^{-n(y_1 - \theta)} \mathbb{1}_{(\theta, \infty)}(y_1).$$

We have

$$\forall_i \quad \theta < x_i < +\infty$$

$$\frac{\prod_{i=1}^{n} f(x_i, \theta)}{f_{Y_1}(y_1, \theta)} = \frac{\prod_{i=1}^{n} e^{-(x_i - \theta)} \mathbb{1}_{(\theta, +\infty)}(x_i)}{n e^{-n(y_1 - \theta)} \mathbb{1}_{(\theta, \infty)}(y_1)} =$$

$$\frac{e^{-\sum_{i=1}^{n} x_i + n\theta} \mathbb{1}_{(\theta, +\infty)}(x_{(1)}) \mathbb{1}_{(-\infty, +\infty)}(x_{(n)})}{n e^{-n x_{(1)} + n\theta} \mathbb{1}_{(\theta, +\infty)}(x_{(1)})} =$$

$$\frac{e^{-\sum_{i=1}^{n} x_i}}{n e^{-n x_{(1)}}} \quad \text{does not depend on } \theta.$$

Y_1 - sufficient statistic for θ.

<u>Theorem 1 (factorization theorem)</u>

Let $X_1, \dots, X_n$ denote a random sample from a distribution $f(x, \theta)$, $\theta \in \Theta$. The statistic $Y_1 = u_1(X_1, \dots, X_n)$ is a sufficient statistic for θ if and only if there are two nonnegative functions k_1 and k_2, such that

$$\prod_{i=1}^{n} f(x_i, \theta) = k_1(u_1(x_1, \dots, x_n), \theta) \cdot k_2(x_1, \dots, x_n),$$

where $k_2(x_1, \dots, x_n)$ does not depend upon θ.

Example 4

$X_1, \dots, X_n$ i.i.d $X_1 \sim N(\theta, \sigma^2)$, σ^2 - known

Let $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. We have

$$\sum_{i=1}^{n}(x_i - \theta)^2 = \sum_{i=1}^{n}\left[(x_i - \bar{x}) + (\bar{x} - \theta)\right]^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - \theta)^2$$

because

$$2\sum_{i=1}^{n}(x_i - \bar{x})(\bar{x} - \theta) = 2(\bar{x} - \theta)\sum_{i=1}^{n}(x_i - \bar{x}) = 0.$$

The joint density of $X_1, \dots, X_n$ has the form

$$\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left[-\sum_{i=1}^{n}(x_i - \theta)^2 / 2\sigma^2\right] = \cancel{\exp}$$

$$\exp\left[-n(\bar{x} - \theta)^2 / 2\sigma^2\right] \left\{ \frac{\exp\left[-\sum_{i=1}^{n}(x_i - \bar{x})^2 / 2\sigma^2\right]}{(\sqrt{2\pi}\,\sigma)^n} \right\}$$

Thus, $\bar{X}$ is ~~the~~ a sufficient statistic for the parameter θ.

Example 5

$X_1, \dots, X_n$ i.i.d $X_i \sim f(x, \theta) = \theta x^{\theta - 1} \mathbb{1}_{(0,1)}(x)$, $\theta > 0$.

The joint density of $X_1, \dots, X_n$

$$\theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta - 1} = \underbrace{\left[\theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta}\right]}_{k_1(u_1(x_1, \dots, x_n), \theta)} \underbrace{\frac{1}{\prod_{i=1}^{n} x_i}}_{k_2(x_1, \dots, x_n)}$$

$\prod_{i=1}^{n} X_i$ - sufficient statistic for θ.

3. Properties of a Sufficient Statistic

Suppose $X_1, \dots, X_n$ is a random sample with a "density" $f(x, \theta)$, $\theta \in \Theta$

Remark 1

A sufficient statistic is not unique

Proof

Let $Y_1 = u_1(X_1, \dots, X_n)$ be a sufficient statistic for θ.

Let $Y_2 = g(Y_1)$, where $g: \mathbb{R} \xrightarrow{1-1} \mathbb{R}$. Then,

$$\prod_{i=1}^{n} f(x_i, \theta) = k_1[u_1(x_1, \dots, x_n), \theta] \cdot k_2(x_1, \dots, x_n)$$
$$= k_1(y_1, \theta) k_2(\underline{x}) = k_1(g^{-1}(y_2), \theta) k_2(\underline{x}).$$

By the factorization theorem Y_2 is also a sufficient statistic.

Lemma 1

If X_1 and X_2 are random variables such that ~~$Var X_1$~~ the variance of X_2 exist, then

$$\mathbb{E}X_2 = \mathbb{E}[\mathbb{E}[X_2 | X_1]] \quad \text{and} \quad Var X_2 \geqslant Var[\mathbb{E}[X_2 | X_1]].$$

Y_1 - sufficient statistic for θ

Y_2 - unbiased estimator of θ

$$\mathbb{E}[Y_2 | Y_1] = \varphi(Y_1),$$
$$\theta = \mathbb{E}[Y_2] = \mathbb{E}[\varphi(Y_1)],$$
$$Var\, Y_2 \geqslant Var[\varphi(Y_1)].$$

Theorem 1 (Rao-Blackwell)

Let $X_1, \dots, X_n$ be a random sample with the "density" $f(x, \theta)$, $\theta \in \Theta$. Let $Y_1 = u_1(X_1, \dots, X_n)$ be a sufficient statistic for θ, and let $Y_2 = u_2(X_1, \dots, X_n)$, not a function of Y_1, be an unbiased estimator of θ. Then $\mathbb{E}[Y_2 | Y_1] = \varphi(Y_1)$ (a function of the sufficient statistic) is an unbiased estimator of θ, and its variance is smaller than ~~or equal to~~ the variance of Y_2.

ok

Theorem 2

Let $X_1, \dots, X_n$ be a random sample with $f(x, \theta)$, $\theta \in \Theta$. If a sufficient statistic $Y_1 = u_1(X_1, \dots, X_n)$ for θ exists and a maximum likelihood estimator $\hat{\theta}$ of $\hat{\theta}$ exists and is unique, then $\hat{\theta}$ is a function of Y_1.

Proof

1. $f_{Y_1}(y_1, \theta)$ - density of Y_1.
2. We have

$$L(\theta) = L(\theta, x_1, \dots, x_n) = \prod_{i=1}^{n} f(x_i, \theta) = f_{Y_1}(y_1, \theta) \cdot H(x_1, \dots, x_n) =$$

$$f_{Y_1}[u_1(x_1, \dots, x_n); \theta] \cdot H(x_1, \dots, x_n), \text{ where } H(x_1, \dots, x_n)$$

does not depend on θ.

3. Thus

L and f_{Y_1} as functions of θ are maximized simultaneously.

4. Since $\hat{\theta}$ is unique, it also maximized f_{Y_1} and must

be a function of $u_1(x_1, .., x_n)$.

Example 1

$X_1, .., X_n$ i.i.d $\quad X_i \sim f(x, \theta) = \theta e^{-\theta x} \mathbb{1}_{(0,+\infty)}(x), \ \theta > 0$

We want to find the MVUE of θ.

$$L(\theta) = \theta^n e^{-\theta \sum_{i=1}^n x_i}$$

$$Y_1 = \sum_{i=1}^n X_i \ \text{- sufficient statistic}$$

$$\ell(\theta) = \log L(\theta) = n \log\theta - \theta \sum_{i=1}^n x_i$$

$$\ell'(\theta) = \frac{n}{\theta} - \sum_{i=1}^n x_i = 0 \Rightarrow \theta = \frac{1}{\bar{x}}$$

$$\ell''(\theta) = -\frac{n}{\theta^2} < 0 \Rightarrow \hat{\theta} = \frac{1}{\bar{X}} \ \text{MLE of } \theta.$$

Y_2 - asymptotically unbiased

$X_1 \sim \Gamma(1, \frac{1}{\theta})$, $Y_1 \sim \Gamma(n, \frac{1}{\theta})$, and

$$\mathbb{E}\left[\frac{1}{\bar{X}}\right] = n\,\mathbb{E}\left[\frac{1}{Y_1}\right] = n\int_0^\infty \frac{1}{x} \frac{\theta^n}{\Gamma(n)} x^{n-1} e^{-\theta x} dx =$$

$$n\int_0^\infty \frac{\theta^n}{\Gamma(n)} x^{n-2} e^{-\theta x} dx = n \frac{\Gamma(n-1)\,\theta}{\Gamma(n)} \int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)} x^{n-2} e^{-\theta x} dx =$$

$$\frac{n}{n-1}\theta.$$

Thus $\dfrac{n-1}{\sum_{i=1}^n X_i}$ is the MVUE of the parameter θ.