is called the likelihood ratio test in the problem (H_0, H_1). If $P_{\theta_0}(\Lambda \leq c) = \alpha$, the test has a size α.

Example 1

$X_1,\dots,X_n$ i.i.d $X_i \sim f(x,\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$, $x,\theta > 0$.

Then
$$L(\theta) = \theta^{-n}\exp\left\{-\frac{n}{\theta}\bar{X}\right\}.$$

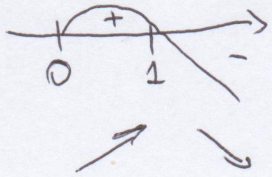
Furthermore, $\hat{\theta} = \bar{X}$ is the MLE of θ. We have
$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\theta_0^{-n}\exp\left\{-\frac{n}{\theta_0}\hat{\theta}\right\}}{\hat{\theta}^{-n}\exp\left\{-\frac{n}{\hat{\theta}}\hat{\theta}\right\}} = \left(\frac{\hat{\theta}}{\theta_0}\right)^n \exp\left\{-\frac{n}{\theta_0}\hat{\theta} + n\right\}.$$

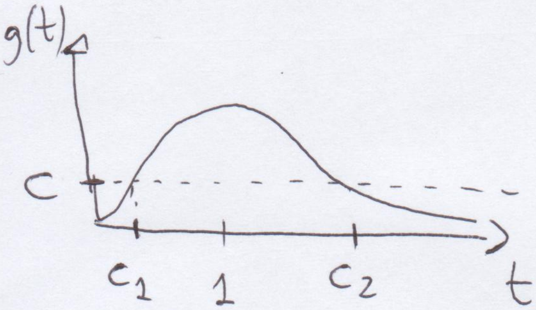
A critical region has the form
$$\Lambda \leq c,$$
where $\Lambda = g(t) = t^n e^{-nt}$, while $t = \frac{\bar{X}}{\theta_0}$.

Moreover, $g'(t) = nt^{n-1}e^{-nt} + t^n e^{-nt}(-n) = ne^{-nt}(t^{n-1} - t^n) = 0$

$ne^{-t} \cdot t^{n-1}(1-t)$

$\iff t = 1,\ t = 0$





Thereby,

$\Lambda \leq c \iff g(t) \leq c \iff t \leq c_1$ or $t \geq c_2$

So,
$$\frac{\bar{X}}{\theta_0} \leq c_1 \text{ or } \frac{\bar{X}}{\theta_0} \geq c_2.$$

Under H_0, the statistic $\frac{2}{\theta_0}\sum_{i=1}^n X_i \sim \chi^2(2n)$. As a resut, the critical region of the α-size LRT has the form
$$C = \left\{\underline{x} : \frac{2}{\theta_0}\sum_{i=1}^n x_i \leq q_{\chi^2_{2n}}\left(\frac{\alpha}{2}\right)\right\} \cup \left\{\underline{x} : \frac{2}{\theta_0}\sum_{i=1}^n x_i \geq q_{\chi^2_{2n}}\left(1-\frac{\alpha}{2}\right)\right\}.$$
where $q_{\chi^2_{2n}}(\alpha)$ is the α-quantile of the chi-square distr. with 2 d.of.f.

Example 2

$X_1, \dots, X_n$ i.i.d, $X_1 \sim N(\theta, \sigma^2)$, $\theta \in \mathbb{R}$, $\sigma^2 > 0$ and known.

We verify

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta = \theta_0 ,$$

where θ_0 is fixed. We have

$$L(\theta) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \theta)^2\right\}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right\} \exp\left\{-\frac{1}{2\sigma^2} n(\bar{x} - \theta)^2\right\}.$$

Furthermore, $\hat{\theta} = \bar{X}$ is the MLE of θ. Thus,

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \exp\left\{-\frac{1}{2\sigma^2} n(\bar{x} - \theta_0)^2\right\},$$

and $\Lambda \leq c$ is equivalent to $-2\log\Lambda \geq -2\log c$.

Under H_0, $-2\log\Lambda = \left(\frac{\bar{X} - \theta_0}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim \chi^2(1)$.

We reject H_0 in favour of H_1, if

$$-2\log\Lambda \geq q_{\chi^2(1)}(1-\alpha).$$

Theorem 1

Let $X_1, \dots, X_n$ be a sample with $f(x, \theta_0)$, $\theta_0 \in \Theta$ satisfying the regularity conditions (R0) - (R5).

Under $H_0 : \theta = \theta_0$,

$$-2\log\Lambda \xrightarrow{D} \chi^2(1).$$

Remark 1

If there is a problem with finding an exact form of the statistic Λ, we can apply the test based on the statistic $\chi^2_L = -2\log\Lambda$ at the asymptotic significance level α rejecting H_0 in favour of H_1 when

$$\chi^2_L \geqslant q_{\chi^2(1)}(1-\alpha).$$

Definition 2

In the testing problem $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ the test based on the statistic

$$\chi^2_W = \left\{\sqrt{nI(\hat\theta)}\,(\hat\theta - \theta_0)\right\}^2$$

is called the Wald test. We reject H_0 at the asymptotic significance level α, when

$$\chi^2_W \geqslant q_{\chi^2(1)}(1-\alpha).$$

Definition 3

In the problem of verifying $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ the test based on the statistic

$$\chi^2_R = \left\{\frac{\ell'(\theta_0)}{\sqrt{nI(\theta_0)}}\right\}^2$$

is called the Rao-score test. We reject H_0, at the asymptotic significance level α, when

$$\chi^2_R \geqslant q_{\chi^2(1)}(1-\alpha).$$

Example 3

$X_1, \ldots, X_n$ i.i.d , $X_i \sim B(\theta, 1)$

We test

$$H_0 : \theta = 1 \quad \text{against} \quad H_1 : \theta \neq 1 .$$

Under H_0, $X_i \sim \mathcal{U}(0,1)$,

Moreover, $\hat{\theta} = \dfrac{-n}{\sum_{i=1} \log X_i}$ - EN MLE of θ.

We have,

$$f(x, \theta) = \frac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma(1)} x^{\theta-1}(1-x)^{1-1} = \theta x^{\theta-1} \mathbb{1}_{(0,1)}(x)$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}, \quad L(1) = 1$$

$$L(\hat{\theta}) = \left(\frac{-n}{\sum \log X_i}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{-\frac{n}{\sum \log X_i} - 1} =$$

$$n^n \left(-\sum \log X_i\right)^{-n} \exp\left(\log \left(\prod_{i=1}^{n} x_i\right)^{-\frac{n}{\sum \log X_i} - 1}\right) =$$

$$n^n \left(-\sum \log X_i\right)^{-n} \exp\left(\left[-\frac{n}{\sum \log X_i} - 1\right] \log \prod_{i=1}^{n} X_i\right) =$$

$$\cancel{\exp[n(\log n)]} \quad \exp(n \log n)\left(-\sum \log X_i\right)^{-n} \exp\left[-n - \sum \log X_i\right] =$$

$$\left(-\sum_{i=1}^{n} \log X_i\right)^{-n} \exp\left(-\sum_{i=1}^{n} \log X_i\right) \exp\left[n(\log n - 1)\right]$$

Thus

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{1}{L(\hat{\theta})}. \quad \text{Therefore,}$$

$$\chi_L^2 = -2 \log \Lambda = -2 \left\{ -n \log\left(-\sum_{i=1}^{n} \log X_i\right) - \sum_{i=1}^{n} \log X_i + n(\log n - 1) \right\}$$

Recall that $I(\theta) = \theta^2$. As a result

$$\chi_W^2 = \left\{\sqrt{nI(\hat\theta)}\,(\hat\theta - \theta_0)\right\}^2 = \left\{\sqrt{\frac{n}{\hat\theta^2}}\,(\hat\theta - 1)\right\}^2 = n\left(1 - \frac{1}{\hat\theta}\right)^2 =$$

$$= n\left(1 + \frac{\sum_{i=1}^n \log X_i}{n}\right)^2.$$

Furthermore,

$$\ell'(1) = \ell'(\theta_0) = \sum_{i=1}^n \frac{\partial \log f(X_i,\theta)}{\partial\theta}\Bigg|_{\theta=\theta_0} = \sum_{i=1}^n \frac{\partial \log(\theta X_i^{\theta-1})}{\partial\theta}\Bigg|_{\theta=\theta_0}$$

$$= \sum_{i=1}^n \frac{\partial[\log\theta + (\theta-1)\log X_i]}{\partial\theta}\Bigg|_{\theta=\theta_0} = \sum_{i=1}^n (1 + \log X_i) = n + \sum_{i=1}^n \log X_i$$

Finally,

$$\chi_R^2 = \left\{\frac{\ell'(\theta_0)}{\sqrt{nI(\theta_0)}}\right\}^2 = \left(\frac{\sum_i \log X_i + n}{\sqrt{n}}\right)^2 = n\left(1 + \frac{\sum_{i=1}^n \log X_i}{n}\right)^2$$

Example 4

Consider ~~the~~ shift model

$$X_i = \theta + e_i, \quad i = 1,\dots,n,$$

where ~~e_i~~ $f(x) = \frac{1}{2}e^{-|x|}$.

We test

$$H_0: \theta = \theta_0,$$
$$H_1: \theta \neq \theta_0.$$

MLE of θ is $\hat\theta = med\{X_1,\dots,X_n\}$, $X_i \sim f(x,\theta) = \frac{1}{2}\exp\{-|x-\theta|\}$

$$L(\theta_0) = 2^{-n}\exp\left\{-\sum_{i=1}^n |X_i - \theta_0|\right\}$$

$$L(\hat\theta) = 2^{-n}\exp\left\{-\sum_{i=1}^n |X_i - \hat\theta|\right\}$$

So

$$-2\log\Lambda = -2\log\frac{L(\theta_0)}{L(\hat\theta)} = 2\left[\sum_{i=1}^n |X_i - \theta_0| - \sum_{i=1}^n |X_i - \hat\theta|\right]$$