T-7.

①

Continuous Distributions. [Uniform, Exponential, Erlang, Gamma and Weibull Distributions]

1: A man and women agree to meet at a certain place between 10 a.m. and 11 a.m.. They agree that the one arriving first will have to wait 15 minutes for the other to arrive. Assuming that the arrival times are independent and Uniformly distributed, find the probability that they meet. Ans. (7/16)

Ans: Let $X \rightarrow$ for man (arrival time)
$Y \rightarrow$ arrival time for women

$P[|X-Y|] < \frac{1}{4} \left(\frac{15 \rightarrow \text{minute}}{60 \rightarrow \text{hour}}\right)$

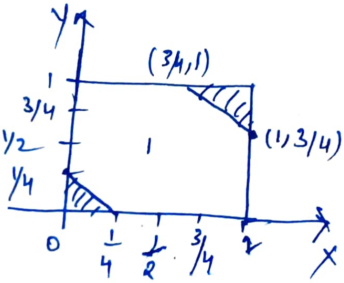
$\because X \sim U(0,1), \quad Y \sim U(0,1)$

pdf of X. $g_X(x) = \begin{cases} \frac{1}{b-a} = 1 & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$

$g_Y(y) = \begin{cases} 1, & \text{if } 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$

Joint pdf is $g_{XY}(x,y) = g_X(x) \cdot g_Y(y)$ ($\because$ both are independent)

$= \begin{cases} 1 & \text{if } 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$



Plotting process:— $\because |x-y| \le 1/4$

$x - y = \frac{1}{4}$ & $y - x = 1/4$

$x = 0, y = 1/4$ & $y = 0, x = 1/4$

$x = 1, y = 3/4$ & $y = 1, x = 3/4$

$P[|X-Y| < \frac{1}{4}] = \frac{\text{Favorable area}}{\text{Total area}} = 1 - \frac{1}{2} \times 3/4 \times 3/4 - \frac{1}{2} \times 3/4 \times 3/4$

$= 1 - 9/16 = \underline{\underline{7/16}}.$

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Q2) If the r.v a is uniformly distributed in the (1, 7) what is the prob that the roots of equation $x^2 + 2ax + (2a+3) = 0$ are real. Ans. (2/3)

Ans:
$a \sim U(1, 7)$

$x^2 + 2ax + (2a+3) = 0 \Rightarrow x = \dfrac{-2a \pm \sqrt{4a^2 - 4(2a+3)}}{2}$

$\Rightarrow x = \dfrac{2(-a \pm \sqrt{a^2 - 2a + 3})}{2} = -a \pm \sqrt{a^2 - 2a + 3}$.

It will have real roots if $a^2 - 2a + 3 \geq 0$.

$\Rightarrow a^2 + a - 3a - 3 \geq 0 \Rightarrow a(a+1) - 3(a+1) \geq 0 \Rightarrow (a-3)(a+1) \geq 0$

i.e. $a - 3 \geq 0$, $a + 1 \geq 0 \Rightarrow a \geq 3$, $a \geq -1$ (not satisfied)

$P(1 \leq a \leq 7) = \int_1^7 \frac{1}{b-a} \cdot da = \frac{1}{6}\int_1^7 da = \frac{1}{6} \times 6 = 1$

$P(a > 3) = \int_3^{\infty} g(a)\, da = \int_3^7 \frac{1}{6}\, da = \frac{1}{6}[7-3] = 2/3$.

Q3) A straight line of length 4 units is given. Two points are taken at random on this line. Find the probability that the distance between them is greater than 3 units.

Ans:
1st cut position $= x$
2nd " " $= y$.

0 ⟝—⟞ 4

$X \sim U(0,4)$, $Y \sim U(0,4)$

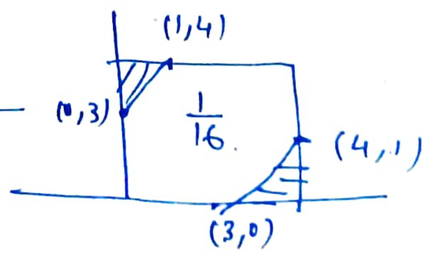
$$f_x(x) = \begin{cases} \frac{1}{b-a} = \frac{1}{4} & \text{if } 0 < x < 4 \\ 0, & \text{otherwise.} \end{cases}$$

$$g_y(y) = \begin{cases} \frac{1}{4} & \text{if } 0 < y < 4 \\ 0, & \text{otherwise.} \end{cases}$$

$$g_{xy}(x,y) = \begin{cases} \frac{1}{16}, & \text{if } 0 < x < 4,\ 0 < y < 4 \\ 0, & \text{otherwise.} \end{cases}$$ ($\because$ both are independent)

$P[|X-Y| > 3] = \dfrac{\text{area of shaded region}}{\text{total area}}$

$= \dfrac{\frac{1}{2} \times 1 + \frac{1}{2} \times 1}{4 \times 4} = \dfrac{1}{16}$.



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The daily consumption of milk in excess of 2000 gallons is approximately exponentially distributed with $\lambda = 1/3000$. The City has a daily stock of 35000 gallons. What is the probability that of 2 days selected at random, the sotck is insufficient for both days. (Ans e^{-5})

Ans: Let r.v. x denotes the daily consumption of milk then r.v $y = x - 20,000$ follows exponential distribution with pdf $g(y) = \lambda e^{-\lambda y}$, $y \geq 0$.

Daily stock is 35,000, so stock will be insufficient for a day if, $P(x > 35000) = P(x - 20000 > 15000)$

$$= P[y > 15000]$$

$$= \int_{15000}^{\infty} \lambda e^{-\lambda y} dy$$

$$= \frac{1}{3000}\left[\frac{e^{-\frac{1}{3000}y}}{-1/3000}\right]_{15000}^{\infty} = \left[-e^{-y/3000}\right]_{15000}^{\infty} = e^{-15000/3000}$$

$$= e^{-5}$$

QN5:- The length of the shower on a tropical is land during raining season has an exponential distribution with parameter 2, time being measured in minutes what is the probability that a shower will last more than 3 minutes? If shower has already lasted for 2 minutes what is the probability that it will last one more minute?

Ans: (i) 0.0025
(II) 0.1353.

Ans 5: Let X be r.v. denoting length of shower:

$\lambda = 2$,

$f(x) = 2e^{-2x}$, $x \geq 0$,

$$P(X > 3) = 2\int_3^{\infty} e^{-2x}\,dx = 2\left[\frac{e^{-2x}}{-2}\right]_3^{\infty} = \left[-e^{-2x}\right]_3^{\infty}$$

$$= 0.00248.$$

$$P[X \geq 3 \mid X \geq 2] = \frac{P[X \geq 3 \cap X \geq 2]}{P[X \geq 2]} = \frac{P[X \geq 3]}{P[X \geq 2]} = \frac{e^{-6}}{e^{-4}}$$

$$= 0.1353.$$

Q6: Suppose that X has an exponential distribution with parameter λ. Compute the probability that X exceeds twice its expected value. (Ans; $1/e^2$)

Ans: $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$, $E(X) = 1/\lambda$.

$$P[X \geq 2/\lambda] = \lambda\int_{2\lambda}^{\infty} e^{-\lambda x}\,dx = \left[\frac{\lambda}{-\lambda}e^{-\lambda x}\right]_{2/\lambda}^{\infty} = \left[-e^{-\lambda x}\right]_{2/\lambda}^{\infty}$$

$$= e^{-\lambda(2/\lambda)} = e^{-2} = 1/e^2.$$

QN.7: If the service life, in hours of a semiconductor is a R.v. having a Weibull distribution with the parameters $\alpha = 0.0375$ and $\beta = 0.55$. (i) How long can such a semiconductor be expected to last?

(II) What is the probability that such a semiconductor will still be in operating condition after 4000h?

{Ans: 667 hr, 0.0276

$E(x) = \alpha^{-1/\beta}\,\overline{|1+\beta} = (0.0375)^{-\frac{1}{0.55}}\,\overline{|(1+\frac{1}{0.55})}$

$= 391.445\,\overline{|2.82} = 39.445 \times 1.705$

$= 667.414$

$P[X > 4000] = \int_{4000}^{\infty} 0.0375 \times 0.55\, e^{-0.0375 x^{0.55}} \cdot x^{-0.45}\, dx.$

$= 0.0206 \int_{4000}^{\infty} e^{-0.0375 x^{0.55}}\, x^{-0.45}\, dx.$

Let $0.0375\, x^{0.55} = t$

$\Rightarrow 0.0375 \times 0.55\, x^{-0.45} dx = dt \Rightarrow 0.0206\, x^{-0.45}\, dx = dt$

$\therefore P[X > 4000] = \int_{0.375(4000)^{0.55}}^{\infty} e^{-t}\, dt = \left[-e^{-t}\right]_{0.0975 \times (4000)^{0.55}}^{\infty}$

$= -\left[0 - e^{-0.375 \times [4000]^{0.55}}\right] = e^{-3.591}$
$= 0.0276$

QN 8:- If the life in years of a certain type of taxi has a Weibull distribution with the parameter $\beta = 2$, find the value of the parameter α, given the probability that the life of the taxi exceeds 6 years is $e^{-0.36}$. For these values of α and β, find the mean and variance.

Ans: $e^{-0.36} = P(X > 6) = 2\alpha \int_6^{\infty} e^{-\alpha x^2} x\, dx.$

$\because \alpha x^2 = t \Rightarrow 2\alpha x\, dx = dt.$

$e^{-0.36} = \int_{36\alpha}^{\infty} e^{-t} dt = \left[-e^{-t}\right]_{36\alpha}^{\infty}$

$= e^{-36\alpha}$

By comparison

$0.36 = 36\alpha$

$\Rightarrow \alpha = 0.01.$

$E(x) = \alpha^{-1/\beta}\,\overline{|(1 + 1/\beta)} = (0.01)^{-1/2}\,\overline{|(1 + 1/2)}$

$= (0.01)^{-1/2} \cdot \Gamma(3/2) = \left(\frac{1}{100}\right)^{-1/2} \frac{1}{2}\sqrt{\pi}$

$= \sqrt{100}\, \frac{1}{2} \sqrt{\pi} = 5\sqrt{\pi}.$

Variance $= \alpha^{-2/\beta}\left[\overline{|1 + 2/\beta} - \left(\overline{|1 + 1/\beta}\right)^2\right]$

$= (0.01)^{-1}\left[\overline{|(2)} - \left(\frac{1}{2}\sqrt{\pi}\right)^2\right]$

$= 100\left[1 - \pi/4\right].$

T=7

QN-9: The life (in months) of a certain bacteria ... Erlang distribution with parameters $K=3$ and $\lambda = \frac{1}{2}$. ... the probability that this bacteria will survive at m...

Ans: In gamma distribution

pdf is $\frac{1}{\Gamma K \, \lambda^K} x^{K-1} \cdot e^{-x/\lambda}$

$E(X) = K\lambda$

$V(X) = K\lambda^2$

C.DF is $\frac{1}{\Gamma K} \gamma(k, x/\lambda) \Rightarrow k=2, \lambda = 3.$

$$f(x) = \frac{1}{\Gamma 2 \cdot 3^2} \cdot x e^{-x/3} = \frac{1}{9} x e^{-x/3}.$$

$$P[1 \le x \le 2] = \frac{1}{9} \int_{1}^{2} x e^{-x/3} dx$$

$$= \frac{1}{9} \left[\left(\frac{x e^{-x/3}}{-1/3} \right)_1^2 - \int_1^2 \frac{3 \cdot e^{-x/3}}{-1/3} dx \right]$$

$$= \frac{1}{9} \left[-3 \left(2 e^{-2/3} - e^{-1/3} \right) + \frac{3}{-1/3} \left(e^{-x/3} \right)_1^2 \right].$$

$$= \frac{1}{9} \left[-6 e^{-2/3} + 3 e^{-1/3} - 9 \left(e^{-2/3} - e^{-1/3} \right) \right].$$

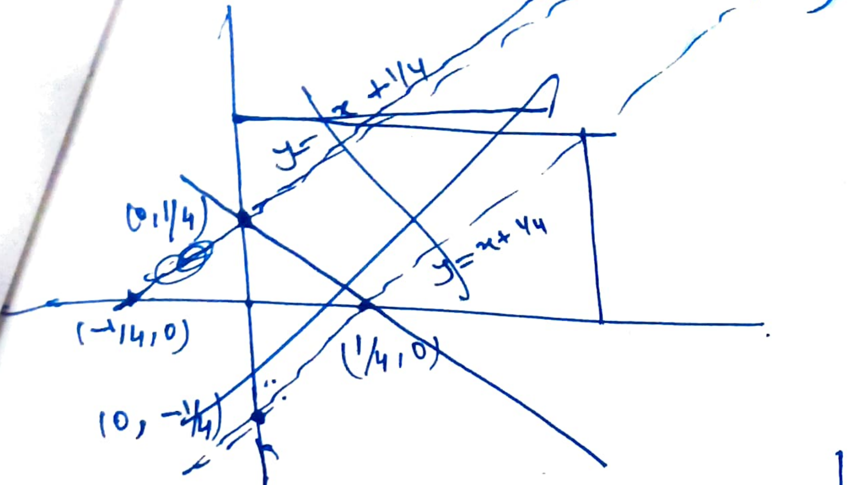
$$= \frac{1}{9} \left[-15 e^{-2/3} + 12 e^{-1/3} \right]$$

$$= \frac{4 e^{-1/3} - 5 e^{-2/3}}{3} = \underline{0.0997}$$

Prob. & r.p.

T-7

$P[\,|x-y| \le \frac{1}{4}] \Rightarrow$



$|x-y| < \frac{1}{4}$

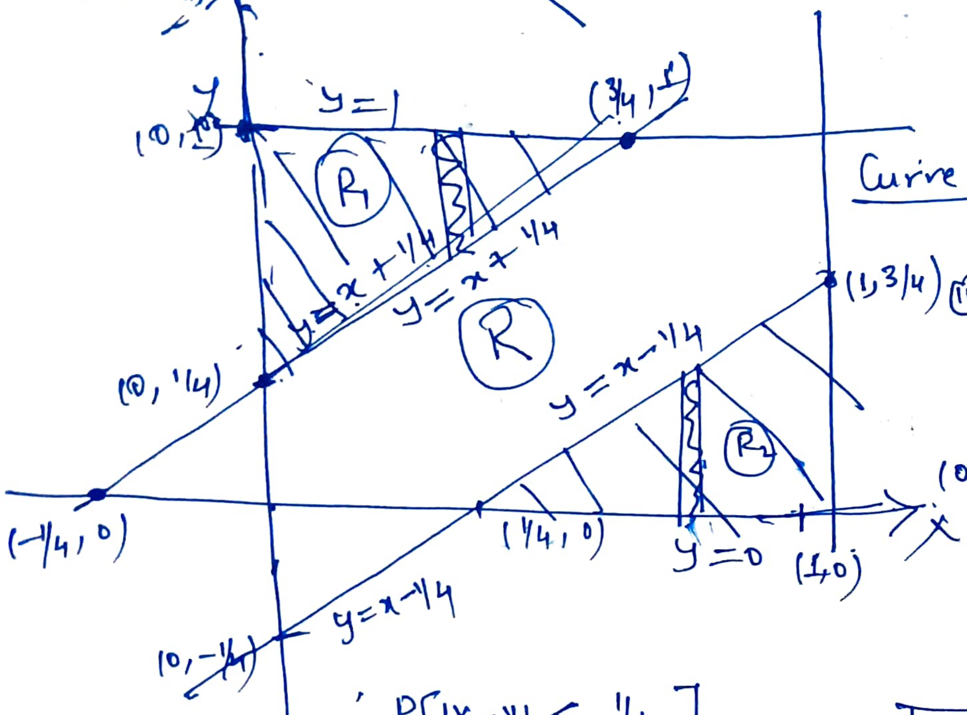
$\Rightarrow -\frac{1}{4} \le (x-y) \le \frac{1}{4}$

$\Rightarrow x-y \ge -\frac{1}{4}$

$\Rightarrow \boxed{y \le x+\frac{1}{4}}$ — (i).

$(x-y) \le \frac{1}{4}$

$\Rightarrow y \ge x-\frac{1}{4}$ — (ii)



Curve (i): $y = x + 1/4$.

$(0, 1/4)$, $(-1/4, 0)$

$y = x - 1/4$.

$(0, -1/4)$, $(1/4, 0)$

$y = x + \frac{1}{4}$

$(0, \frac{1}{4})$, $(-1/4, 0)$

$y = x - 1/4$

$(0, -1/4)$, $(1/4, 0)$

$\therefore P[|x-y| < 1/4]$

$= \dfrac{\text{Favorable area}}{\text{Total Area}}$

$|x-y| < 1/4$

$\Rightarrow x-y < 1/4$ or $y-x < 1/4$

In R1:

$x+1/4 \le y \le 1$, $0 \le x \le 3/4$

In R2:

$0 \le y \le x-1/4$, $\frac{1}{4} \le x \le 1$

Area: $\displaystyle\int_0^{3/4}\int_{x+1/4}^{1} dy\cdot dx + \int_{1/4}^{1}\int_0^{x-1/4} dy\cdot dx.$

$= \displaystyle\int_0^{3/4} (1-x-1/4)\,dx + \int_{1/4}^{1} (x-1/4)\,dx$

$= -\frac{x^2}{2} - \frac{5}{4}x\Big|_0^{3/4} + \left(\frac{x^2}{2} - \frac{1}{4}x\right)_{1/4}^{1}$

$= -\frac{9}{16\times 2} - \frac{18}{16} + \frac{1}{2} - \frac{1}{4} - \left(\frac{1}{16\times 2} - \frac{1}{16}\right)$

~~Required Prob = P[|x-y| < 1/4] = 1 - 9/16~~

$\frac{-9-30}{16\times 2} - \frac{3}{4} + \frac{1}{32}$

$= \frac{-39+1}{32} - \frac{3}{4} = \frac{19}{16} - 3/4$

$\frac{19-12}{16}$

$= \frac{7}{16}$

$-\frac{38}{32}$ $\frac{19}{16}$

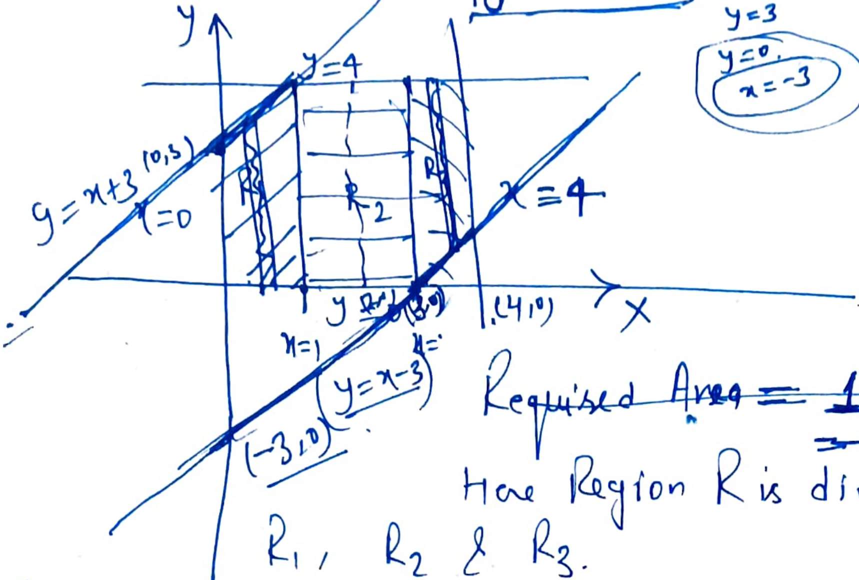
Q2: $P\{|X-Y| < 3\} = P\{-3 < X-Y <$ ... $0 \le x$, $0 \le y$ uniform or

$-3 < X - Y \Rightarrow Y < X+3$, $X - Y < 3 \Rightarrow Y$ ...

or $\boxed{y = x+3}$

$x=0$
$y=3$
$y=0$, $x=-3$

$y = x$ ...
$x=0$
$y=-3$



~~Required Area = $1 - \frac{1}{2}\times1\times1 - \frac{1}{2}\times1\times1$~~

Here Region R is divided into 3 regions R_1, R_2 & R_3.

In Region R_1; $0 \le x \le 1$, $0 \le y < x+3$,

R_2; $1 \le x \le 3$, $0 \le y \le 4$

R_3; $3 \le x \le 4$, $x-3 \le y \le 4$

Area of $R_2 = 2 \times 4 = 8$.

Area of $R_1 = \int_0^1 \int_0^{x+3} dy\, dx = 7/2$

" " $R_3 = \int_3^4 \int_{x-3}^{4} dy\, dx = 7/2$

Therefore $P(AB < 3) = \frac{1}{16}(7/2 + 8 + 7/2) = \frac{15}{16}$

Hence $P(AB > 3) = 1 - \frac{15}{16} = (1/16)$

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or: $P(|X-Y| > 3) = \dfrac{\text{area of the shaded region}}{\text{total area.}}$

$= \dfrac{\frac{1}{2}\times1 + \frac{1}{2}\times1}{4\times4} = \frac{1}{16}$

Continuous Distribution from Book:

Uniform or Rectangular distribution:

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

Distribution function: $F(x) = \begin{cases} 0, & \text{if } -\infty < x < a. \\ \frac{x-a}{x-b}. & a \leq x \leq b. \\ 1 & b < x < \infty. \end{cases}$

T-8-②

It is given that X and Y are indepen. normal variates and $X \sim N(1,4)$, $Y \sim N(3,16)$. Find the value of K such that $P(2X+Y \le K) = P(4X-Y \ge 2K)$.

Ans. $K = \frac{5\sqrt{5}+\sqrt{2}}{\sqrt{5}+2\sqrt{2}}$

Ans: $X \sim N(1,4)$, $Y \sim N(3,16)$

$E(X) = 1$, $E(Y) = 3$, $V(X) = 4$, $V(Y) = 16$.

$E(2X+Y) = 2E(X) + E(Y) = 5$

$V(2X+Y) = 4V(X) + V(Y) = 4\times4+16 = 32.$

Now $E(4X-Y) = 4E(X) - E(Y) = 4-3 = 1$

$V(4X-Y) = 16V(X) + V(Y) = 16\times4 + 16 = 80.$

Now, Given that $P(2X+Y \le K) = P(4X-Y \ge 2K)$

$\Rightarrow P\left(Z \le \frac{K-5}{\sqrt{32}}\right) = P\left(Z \ge \frac{2K-1}{\sqrt{80}}\right)$

$\Rightarrow \frac{K-5}{\sqrt{32}} = \frac{2K-1}{\sqrt{80}} \Rightarrow \frac{K-5}{4\sqrt{2}} = \frac{1-2K}{4\sqrt{5}}$

$\Rightarrow \sqrt{5}(K-5) = \sqrt{2} - 2\sqrt{2}K \Rightarrow K(\sqrt{5}+2\sqrt{2}) = \sqrt{2}+5\sqrt{5}$

$\Rightarrow K = \frac{\sqrt{2}+5\sqrt{5}}{\sqrt{5}+2\sqrt{2}}.$

Q4:- Variable X is a normal random variable with S.D. 3. If the prob. that X is less than 16 is 0.84, then the expected value of X is approximately? (Ans 13)

Ans:- $\sigma = 0.3$, $P(X<16) = 0.84$, $\mu = ?$

$\because Z = \frac{X-\mu}{\sigma} \Rightarrow$ By Table. $P(Z<1) = 0.84$ (approxi)

$\Rightarrow 1 = \frac{16-\mu}{3} \Rightarrow \mu = 16-3 = 13.$

Q5:- If $P\{-3 < Z < -2\} = P\{2 < Z < x\}$, then find x?

Ans:- $P\{-3 < Z < -2\} = P\{2 < Z < x\}$.

Now $P\{-3 < Z < -2\} = P(-3 < Z < 0) - P(-2 < Z < 0)$

$= P(0 < Z < 3) - P(0 < Z < 2)$

$= 0.4987 - 0.4772 = 0.0215$

$\Rightarrow P(2 < Z < x) = 0.0215$

$\Rightarrow P(0 < Z < x) - P(0 < Z < 2) = 0.0215$

$\Rightarrow P(0 < Z < x) = 0.0215 + P(0 < Z < 2) = 0.0215 + 0.4772$

$= 0.4987$

By Table: $P(0 < Z < 3.02) = 0.4987 \Rightarrow \boxed{x = 3.02}$

T-8. Normal Distribution

Q1: In a normal population with mean 12 and ... know that 750 observations exceed 15. Find the ... of observations in the population.

Ans: mean $\mu = 12$, $\sigma = 4$, Let N be the total no. of observations of the population

$N \cdot P(X > 15) = 750$ $\quad \left[\because Z = \frac{X-\mu}{\sigma} = \frac{15-12}{4} = 0.75 \right]$

$\Rightarrow N \cdot P(Z > 0.75) = 750$

$\Rightarrow N \cdot [0.5 - P(Z < 0.5)] = 750.$

$\Rightarrow N[0.5 - 0.2734) = 750 \Rightarrow N \times 0.2266 = 750$

$\Rightarrow N = 3310$ (approx).

Q No. 2: At a certain examination 10% of the students who appeared for the paper in advanced Mathematics got less than 30 marks and 97% of the students got less than 62 marks. Assuming the distribution is normal, find the mean and the S.D. of the distribution.

Ans: $P(X < 30) = 0.10$, $P(X < 62) = 0.97$.

$\Rightarrow P\left(Z < \frac{30-\mu}{\sigma}\right) = 0.10$

$\Rightarrow P(Z < -1.28) \Rightarrow \frac{30-\mu}{\sigma} = -1.28.$

$\Rightarrow \mu - 1.28\sigma = 30$ — (1)

Similarly $P(X < 62) = 0.97 \Rightarrow P\left(Z < \frac{62-\mu}{\sigma}\right) = 0.97.$

$\Rightarrow P(Z < 1.88)$

$\Rightarrow \frac{62-\mu}{\sigma} = 1.88 \Rightarrow \mu + 1.88\sigma = 62$ — (2)

On solving eqn (1) & (2), we get:

$\mu = 42.97$, $\sigma = 10.13$.

If $P\{-a < Z < a\} = 2P\{Z < a\} - 1$, then find a

$P\{-a < Z < a\} = 2P\{Z < a\} - 1$

$\Rightarrow 2P\{0 < Z < a\} = 2P(Z < a) - 1$

$\Rightarrow 1 + 2P\{0 < Z < a\} = 2P\{Z < a\}$

$\Rightarrow 0.5 + P\{0 < Z < a\} = P\{Z < a\}$.

$\because P(-a < Z < a) = P(-a < Z < 0) + P(0 < Z < a) = P(0 < Z < a) + P(0 < Z < a) = 2P(0 < Z < a)$

L.H.S. ensures that a is on the R.H.S of zero.

$\Rightarrow 0.5 + P\{0 < Z < a\} = 0.5 + P\{0 < Z < a\}$.

which is true for all values of a. $\boxed{a \in (-\infty, \infty)}$

Q 7: Find x (using three type of standard normal table) if

(i) $P\{Z > x\} = 0.05$ (ii) $P\{Z > x\} = 0.95$ (iii) $P\{Z < x\} = 0.66$.

(iv) $P\{Z < x\} = 0.40$ (v) $P\{|Z| < x\} = 0.99$ (vi) $P\{|Z| < x\} = 0.1$

(vii) $P\{|Z| > x\} = 0.8$.

Ans: (i) $P(Z > x) = 0.05 \Rightarrow 0.5 - P(0 < Z < x) = 0.05$.

$\Rightarrow P(0 < Z < x) = 0.45 = P(0 < Z < 1.65)$

Hence $x = 1.65$.

(ii) $P(Z > x) = 0.95 \Rightarrow 0.5 + P(x < Z < 0) = 0.95$.

$\Rightarrow 0.5 + P(0 < Z < -x) = 0.95 \Rightarrow P(0 < Z < -x) = 0.45$.

$\Rightarrow -x = 1.65 \Rightarrow \boxed{x = -1.65}$.

(iii). $P(Z < x) = 0.66 \Rightarrow 0.5 + P(0 < Z < x) = 0.66$.

$\Rightarrow P(0 < Z < x) = 0.66 - 0.5 = 0.16. = P(0 < Z < 0.42)$

$\Rightarrow \boxed{x = 0.42}$.

(iv); $P(Z < x) = 0.40 \Rightarrow 0.5 - P(0 < Z < -x) = 0.40$.

$\Rightarrow P(0 < Z < -x) = 0.5 - 0.40 = 0.10$.

$\Rightarrow -x = 0.25 \Rightarrow \boxed{x = -0.25.}$

(v). $P\{|Z| < x\} = 0.99 \Rightarrow P\{-x < Z < x\} = 0.99$

$P(-x < Z < 0) + P(0 < Z < x) = 0.99$

$\Rightarrow P(-x < Z < 0) + P(0 < Z < x) = 0.99 \Rightarrow P(0 < Z < x) = \frac{0.99}{2}$

$\Rightarrow 2P(0 < Z < x) = 0.99$

$\Rightarrow P(0 < Z < x) = 0.485 = P(0 < Z < 2.17)$ (By Table)

$\Rightarrow \boxed{x = 2.17.}$

(VI): $P\{|z| < x\} = 0.1$

$\Rightarrow 2P\{0 < z < x\} = 0.1 \Rightarrow P\{0 < z < x\} = \dots$

$P\{0 < z < x\} = P(0 < z < 0.13)$

$\Rightarrow \boxed{x = 0.13.}$

(VII): $P\{|z| > x\} = 0.9$

$\Rightarrow 1 - P\{|z| < x\} = 0.9.$

$\Rightarrow 1 - 2P(0 < z < x) = 0.9 \Rightarrow 2P(0 < z < x) = 0.1$

$\Rightarrow P(0 < z < x) = 0.05 \Rightarrow x = -1.64$ (By Table)

$(VIII)$ $P\{|z| > x\} = 0.8 \Rightarrow 1 - P(|z| < x) = 0.8.$

$\Rightarrow 1 - 2P(0 < z < x) = 0.8 \Rightarrow 2P(0 < z < x) = 1 - 0.8 = 0.2$

$\Rightarrow P(0 < z < x) = 0.2/2 = 0.1$

$\Rightarrow x = 0.26$ (By Table)

Prob and r.p.

T-9 — 1

1. A household appliances is advertised as having more than a 10-year life. If it's pdf is given by

$f(t) = 0.1\,(1+0.050t)^{-3}, \quad t \geq 0.$

(a) Determine its reliability for the next 10 years. If it has survived a 1-year warranty period.

(b) What is MTTF before the warranty period.

(c) What is MTTF after the warranty period assuming that it has still survived?

Ans: $\because$ $R(t) = \int_t^{\infty} f(t)\,dt = \int_t^{\infty} 0.1\,(1+0.05t)^{-3}\,t$

$= (1+0.05t)^{-2}$ — (1)

(a) $R(t/t_0) = \dfrac{R(t+T_0)}{R(t_0)} = \dfrac{R(11)}{R(1)} = \dfrac{(1+0.05\times 11)^{-2}}{(1+0.05)^{-2}} \approx 0.4589$

≈ 0.46 — (2)

(b) MTTF before the warranty period

$MTTF = \int_0^1 t\,f(t)\,dt = \int_0^1 t\,(1+0.05t)^{-3}.(0.1)\,dt$

$= -\dfrac{800\,(t+10)}{(t+20)^2}\Big|_0^1 = 0.0453$ year.

(II) MTTF after warranty period.

$MTTF = \int_1^{\infty} t\,f(t)\,dt = 19.955$ years.

Q2: A component has the following hazard rate, where t is in years. $\lambda(t) = 0.44t, \; t \geq 0.$

(a) find $R(t)$ (b) Determine the prob of the component failing within the first month of its operation.

(c) What is the design life if a reliability of 0.95 is desired.

Ans: $\lambda(t) = 0.44t, \; t \geq 0,$

(a) $R(t) = \exp\left[-\int_0^t \lambda(t)\,dt\right] = e^{-\int_0^t 0.44t\,dt}$

$= e^{-0.22t^2} \approx e^{-t^2/5}$

(b) P(component failing in 1st mont) $= P(T < 1/_{12})$

$= 1 - P(T > 1/_{12}) = 1 - R(1/_{12})$

$= 1 - \exp(-1/5 \cdot 1/_{12^2}) = 0.00138 \approx 0.0014$

Ans2) ∵ Given $R(t_D) = 0.95$

$$e^{-t_D^2/5} = 0.95 \Rightarrow t_D^2 = -5 \log_{10}$$

$$t_D^2 = 0.25646 \Rightarrow t = 0.5064 \text{ (in year)}$$

Q3: The pdf of the time to failure of a system given by $f(t) = 0.01,\ 0 \le t \le 100$ days. Find
(a) $R(t)$ (b) the hazard rate function (c) The MTTF
(d) The standard deviation.

Ans: $f(t) = 0.01, \quad 0 \le t \le 100$ days.

(a) $R(t) = 1 - \int_0^t f(t)\,dt = 1 - [0.01t]_0^t = 1 - 0.01t$
$= \frac{100-t}{100}$.

(b) $\lambda(t) = \frac{f(t)}{R(t)} = \frac{0.01}{1-0.01t} = \frac{1}{100-t}$.

(c) $MTTF = \int_0^{100} t f(t)\,dt = \left[0.01 \frac{t^2}{2}\right]_0^{100} = 50.$

(d) Standard deviation

$$\sigma^2 = \left[\int_0^{100} t^2 f(t)\,dt - (MTTF)^2\right] = \left[\frac{t^3}{3}(0.01)\Big|_0^{100} - (50)^2\right]$$

$$= \frac{10000}{3} - (50)^2 \Rightarrow \boxed{\sigma = 28.86}$$

Q4: Experience shows that the failure rate of a certain electrical component is a linear function. Suppose that after two full days of operation, the failure rate is 10% per hour and after three full days of operation it is 15% per ~~day~~ hour.
(a) Find the prob that the component operates for at least 30 hours.
(b) Suppose that the component has been operating for 30 hours. What is the prob that it fails within the next hour?

$\lambda(t$ After 2 After

Q4: After

ns 4: Let Linear function of failure rate be:

$\lambda(t) = a + bt$.

After 2 full days, $0.1 = a + b(24\times2)$ (in hours)

$\Rightarrow 0.1 = a + 48b$ — ①.

After 3 full days; $0.15 = a + 72b$ — ②.

from ① & ② $0.05 = 24b \Rightarrow b = \frac{1}{480}$.

$\therefore \lambda(t) = \frac{1}{480}t$.

$\Rightarrow R(t) = e^{-\int_0^t \lambda(t)\,dt} = e^{-\int_0^t \frac{t}{480}dt} \Rightarrow R(t) = e^{-t^2/960}$.

(a) $R(30) = e^{-\frac{900}{960}} = 0.3916$.

(b) $P(t < 30+1 / t > 30) = P(t < 31 / t > 30)$

$= \frac{P(30 < t < 31)}{R(30)} = \frac{e^{-\int_{30}^{31} \frac{t}{480}dt}}{R(30)}$

$= 0.06156$.

The reliability of a communication channel is 0.40. How many channels should be placed in paralled redudancy so as to achive the reliability of reciving the information is 0.80. If these channels are used to configure high level and low level redundant systems, what are the corresponding system reliablities?

For parallel series. $R = 1-(1-R)^n$
For series only. $R = R^n$.

Ans: Let no. of channels $= n$.
As per given information., $R(t) = 0.80$.

$\Rightarrow 1-(1-R(t))^n = 0.8 \Rightarrow 0.2 = (0.6)^n \Rightarrow n = 3.15 \approx 4$

Now $n = m = 2 = ?$

Case (I): High redudant System.

$R(t) = 1 - [1-(R(t))^n]^m = 1-(1-R^n)^m$

$= 1-(1-(0.4)^2)^2$

$= 1-(1-0.16)^2 = 0.2944$

Case (II) ~~R_{low}~~ For low redudant System $R_{low} = (1-(1-R)^n)^m$

$R_{low} = [1-(1-0.4)^2]^2 = (1-0.36)^2 = 0.4096.$

Q2: Which of the following systems has the higher reliability at the end of 100 hrs of operation?

(i) Two constant failure rates redudant components each having MTTF of 1000 hrs.

(II) A Weibull component with shape parameter of 2 and a characteristic life of 10000 hrs in series with a constant failure rate of 0.00005. failure rates components with

Given $\boxed{n=2}$

For expo. distn. $R(t) = e^{-\lambda t}$
MTTF $= 1/\lambda$
For weibull distn.
MTTF $= \theta\,\Gamma(\frac{1}{\beta}+1)$
$R(t) = e^{-(t/\theta)^\beta}$
For const failure rate λt
$R_p(t) = 1-(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})$

Ans: MTTF $= 1000$, $\Rightarrow \lambda = \frac{1}{1000}$ & $t = 100$.

$R_p(t) = 1-(1-R_1(t))(1-R_2(t))$, $R(t) = e^{-\lambda t}$

$= 1-(1-e^{-\frac{1}{1000}\times 100})^2$

$R_p(t) = 1-(1-e^{-0.1})^2 = 0.9909$

Q2. (II): Weibull components:

Given $\beta=2$, $\theta = 10000$, $\lambda = 0.00005$

$R(t) = \int_t^{\infty} f(t)\,dt = \int_t^{\infty} \frac{\beta}{\theta}\cdot\left(\frac{t}{\theta}\right)^{\beta-1}\cdot e^{-(t/\theta)^{\beta}}\,dt$

$= \int_t^{\infty} \left(\frac{2}{\theta}\right)\cdot\left(\frac{t}{\theta}\right)^{1} e^{-(t/\theta)^2}\,dt = \int_t^{\theta} \frac{2t}{\theta^2}\,e$

Let (t/θ)

$2\left(\frac{t}{\theta}\right)\cdot\frac{1}{\theta}\cdot dt = dx$

$= \left[\frac{e^{-(t/\theta)^2}}{-1}\right]_t^{\infty} = e^{-(t/\theta)^2}$ Ans (II)

at $\theta = 10000$, $t = 100$.

$R_1(t) = e^{-\left(\frac{100}{10000}\right)^2}$, $R_2(t) = e^{-\lambda t} = e^{-0.00005\times 100} = e^{-0.005}$.

$\because$ They are in series:

$\therefore$ $R_s(t) = R_1(t)\times R_2(t) = e^{-0.005}\times e^{-\left(\frac{t}{100}\right)^2} = \underline{\underline{0.9949}}$

Q3 → Specifications for a power unit consisting of 3 independent and serially connected components require a design life of 5 years with 0.95 ~~reliability~~ reliability.

(i) If the constant failure rates $\lambda_1, \lambda_2, \lambda_3$ are such that $\frac{\lambda_1}{2} = \frac{\lambda_2}{1} = \frac{\lambda_3}{3}$, what should be MTTF of each component of the system?

(II) If 2 identical power units are placed in parallel, what is the system reliability at 5 years and what is the system MTTF?

Ans → (i) For seriel configuration

$R_s = R_1\cdot R_2\cdot R_3 = e^{-\lambda_1 t}\cdot e^{-\lambda_2 t}\cdot e^{-\lambda_3 t}$

$= e^{-(\lambda_1+\lambda_2+\lambda_3)t}$ — ①

$\because$ $\frac{\lambda_1}{2} = \frac{\lambda_2}{1} = \frac{\lambda_3}{3} = \lambda$ (let)

$\Rightarrow \lambda_1 = 2\lambda,\ \lambda_2 = \lambda,\ \lambda_3 = 3\lambda$

$\therefore$ $R_s = e^{-6\lambda t} = e^{-6\lambda\cdot 5}$

$\Rightarrow 0.95 = e^{-6\lambda\cdot 5} = e^{-30\lambda}$

on solving $\boxed{\lambda = 0.00171}$

Theory

Const failure rate: Series

$R(t) = e^{-\lambda t}$, $MTTF = \frac{1}{\lambda}$

$R_s(t) = e^{-(\lambda_1+\lambda_2+\lambda_3)t}$.

design life $= t_D$

$MTTF = E(T) = \int_0^{\infty} R(t)\,dt$

$= \frac{1}{\lambda}$

Parallel

$R_p = 1-(1-R_1)(1-R_2)(1-R_3)$,

$MTTF = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}$

T-10-3

3.)

$\because$ $MTTF = \frac{1}{\lambda}$.

$\therefore$ $MTTF(1) = \frac{1}{\lambda_1} = \frac{1}{2\lambda} \doteq 292$

$MTTF(2) = \frac{1}{\lambda_2} = \frac{1}{\lambda} = 585$.

$MTTF(3) = \frac{1}{\lambda_3} = \frac{1}{3\lambda} = 195$.

Ans (II) For parallel configuration. $R_p = 1-(1-R_1)(1-R_2)(1-R_3)$

$R_p = 1 - [1 - \text{Reliability of Power Unit}]^2$

$= 1 - (1-0.95)^2 = 0.9975$

$\because$ $R(t) = e^{-\lambda t}$.

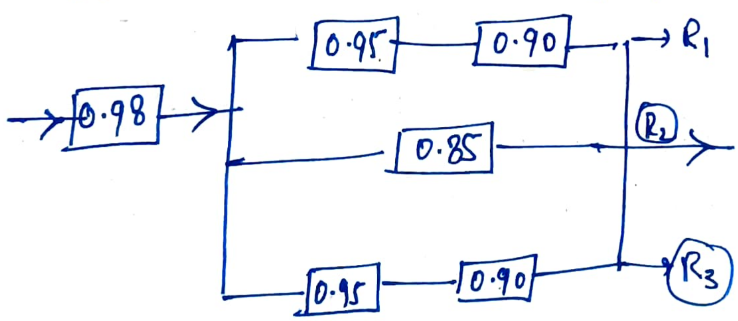
$\Rightarrow 0.95 = e^{-\lambda \cdot 5} \Rightarrow \lambda = 0.01026$.

$\because$ $MTTF = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}$, Since $\lambda_1 = \lambda_2$

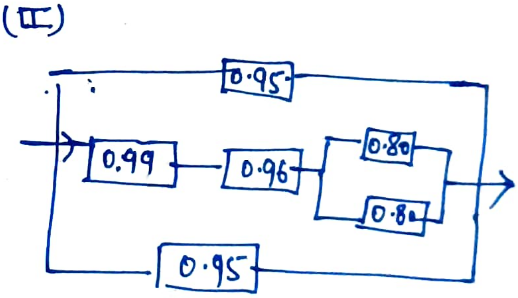
$\therefore$ $MTTF = \frac{2}{\lambda} - \frac{1}{2\lambda} = 3/2\lambda. = \frac{3}{2 \times 0.01026} = 147$ years.

Q4: Calculate the reliability of following systems.

(i)



(II)



Ans (a) 0.9769
(b) 0.9998.

Ans (i):

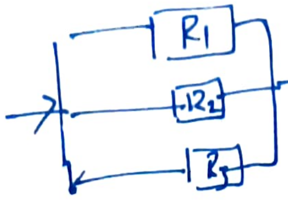
$R_{system} = (0.98)[1-(1-R_1)(1-R_2)(1-R_3)]$

$= 0.98[1-(1-(0.95\times0.90))(1-0.85)(1-(0.95)(0.90))]$

$= 0.98[1-(1-0.855)(1-0.850)(1-0.855)]$

$= 0.98[1-(0.145)^2(0.15)] = 0.9988 \times 0.98$

$R_{syst} = 0.9768$.

(II) System $\approx$  where $R_1 = 0.9$

$R_2 = 0.99 \times 0.96 \times \{1-$

$= 0.9124.$

$R_3 = 0.95.$

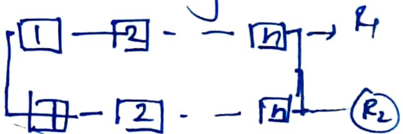
Define a rand
A random va
to every ou
process is
outcome
for exam

$$\therefore R_{system} = 1-(1-R_1)(1-R_2)(1-R_3)$$
$$= 1-(0.05)(1-0.9124)(0.05)$$
$$= \underline{0.999781}$$

Q5: 2n identical constant failure rate components are used to configure redudant systems either with two subsystems in parallel or with n subsystems in series. Which system will give higher reliability.

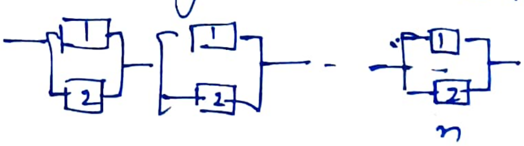
Ans (n subsystems in series)

Ans: (i) If two systems are in parallel (each subsystem will contain n components).



$$R_{system} = 1-(1-R_1)\cdot(1-R_2)$$
$$= 1-(1-R^n)(1-R^n)$$
$$= 1-(1-R^n)^2 = 1-(1+R^{2n}-2R^n) = R^n(2-R^n) \quad \text{—} (1)$$

Case 2; n subsystem in series (each subsystem will contain 2 components)



$$\therefore R_{system} = [1-(1-R)^2]^n$$
$$= [1-(1+R^2-2R)]^n = R^n(2-R)^n \quad \text{—} (2)$$

from (1) and (2), $R^n(2-R)^n > R^n(2-R^n)$

$\Rightarrow$ Case 2 gives high reliability

Probability and Random Process.

Define a random process and classify them with suitable Example.

A random variable (RV) is a rule that assigns a real number to every outcome of a random experiment; while a random process is a rule that assigns a time function to every outcome of a random experiment.

For example: Consider the random experiment of tossing a dice and observing the number on the top face. For each outcome of the experiment.

Outcome:	1	2	3	4	5	6

fun of time: $x_1(t) = -4t$, $x_2(t) = 2t^2$, $x_3(t) = \sin t$, $x_4(t) = \cos t$, $x_5(t) = \sin 2t$, $x_6(t) = 4$.

then the set of functions $\{x_1(t), x_2(t), \ldots x_6(t)\}$. represents a random process.

Q2:- In an experiment of two fair dice, the process $\{X(t)\}$ is defined as $X(t) = \sin \pi t$, if the experiment shows a prime sum and $X(t) = 2t+1$, otherwise. Find the mean of the process. Is the process stationary?

Ans:-
$$X(t) = \begin{cases} \sin \pi t, & \text{experiment shows prime Sum} \\ 2t+1, & \text{otherwise.} \end{cases}$$

Mean of $X(t)$
$= \mu_X(t) = E[X(t)]$.

$S = \{\boxed{(1,1)}, \boxed{(1,2)}, (1,3), \boxed{(1,4)}, (1,5), \boxed{(1,6)}$
$\boxed{(2,1)}, (2,2), \boxed{(2,3)}, (2,4), \boxed{(2,5)}, (2,6)$
$(3,1), \boxed{(3,2)}, (3,3), \boxed{(3,4)}, (3,5), (3,6)$
$\boxed{(4,1)}, (4,2), \boxed{(4,3)}, (4,4), (4,5), (4,6)$
$(5,1), \boxed{(5,2)}, (5,3), (5,4), (5,5), \boxed{(5,6)}$
$\boxed{(6,1)}, (6,2), (6,3), (6,4), \boxed{(6,5)}, (6,6)\}$.

$P[\text{Sum is prime}] = 15/36 = 5/12$

$P[\text{Sum is not prime}] = 1 - 5/12 = 7/12$

Now $E[X(t)] = P\{X(t) = \sin \pi t\}. \sin \pi t + P\{X(t) = 2t+1\}.(2t+1)$

$= \frac{5}{12} . \sin \pi t + \frac{7}{12} .(2t+1)$

Here mean depends on t.

$\Rightarrow$ Process is not stationary.

Q3: Let $X(t) = A\cos\lambda t + B\sin\lambda t$, with random va[riable A] taking values 1 and 3 with equal probabilities and [B] taking values -1 and 1 with prob. $1/4$ and $3/4$ resp. [Test] the process $\{X(t)\}$ for stationarity.

Ans: $X(t) = A\cos\lambda t + B\sin\lambda t$.

$E[X(t)] = E\{A\cos\lambda t + B\sin\lambda t\} = \cos\lambda t \cdot E[A] + \sin\lambda t\, E[B]$

$\because E[A] = 1\times\frac{1}{2} + 3\times\frac{1}{2} = \frac{1}{2} + \frac{3}{2} = 2.$

$E[B] = -1\times\frac{1}{4} + 1\times 3/4 = 1/2$

$\therefore E[X(t)] = \cos\lambda t \cdot 2 + \frac{1}{2}\sin\lambda t.$

$\because E[X(t)]$ depends on t, Hence it is not stationary.

Q4: Test the random processes $\{X(t)\}$ and $\{Y(t)\}$ for WSS when

(i) $X(t) = \cos(\lambda t + Y)$, where λ is constant and Y is Uniform in $(0, 2\pi)$ [Ans WSS]

(II) $Y(t) = X\sin(\lambda t)$, where λ is constant and X is Uniform in $(-1, 1)$, (Ans not WSS)

Ans: (i) $X(t) = \cos(\lambda t + Y)$

$\because Y \sim f_Y(y) = \frac{1}{2\pi}, \quad 0 \le y \le 2\pi$

Now. $E[X(t)] = E\cos(\lambda t + Y)$

$= E[\cos\lambda t \cdot \cos Y - \sin\lambda t \cdot \sin Y]$

$= \cos\lambda t\, E[\cos Y] - \sin\lambda t\, E[\sin Y]$. —— ①

Now. $E[\cos y] = \int_0^{2\pi} \frac{1}{2\pi}\cos y\, dy = -\frac{1}{2\pi}[\sin y]_0^{2\pi} = 0$

$E[\sin y] = \int_0^{2\pi} \frac{1}{2\pi}\sin y\, dy = \frac{1}{2\pi}[\cos y]_0^{2\pi} = 0.$

$\therefore E[X(t)] = 0.$

$E[X(t_1)\cdot X(t_2)] = E[\cos(\lambda t_1 + y)\cdot\cos(\lambda t_2 + y)].$

$= \frac{E}{2}[\cos(\lambda t_1 + y + \lambda t_2 + y) + \cos(\lambda t_1 + \cancel{y} - \lambda t_2 - \cancel{y})]$

$= \frac{E}{2}[\cos(\lambda(t_1 + t_2) + 2y) + \cos\lambda(t_1 - t_2)].$

$= \frac{1}{2\pi}\cdot\frac{1}{2}\left[\int_0^{2\pi}[\cos(\lambda(t_1 + t_2) + 2y) + \cos\lambda(t_1 - t_2)]\, dy\right.$

$= \frac{1}{4\pi}\left[\frac{\sin[\lambda(t_1 + t_2) + 2y]}{2} + \cos\lambda(t_1 - t_2)\cdot y\right]_0^{2\pi}$

$= \frac{1}{4\pi}\left[\frac{\sin(\lambda(t_1 + t_2) + 4\pi) - \sin(\lambda(t_1 + t_2))}{2} + 2\pi\cos(\lambda(t_1 - t_2))\right]$

$= \frac{1}{2}\cos\lambda(t_1 - t_2)$ which is fun. of $(t_1 - t_2)$

$\Rightarrow \{X(t)\}$ is WSS process.

WSS.

$E\{X(t)\} = \mu = \text{Const}.$

and

$E\{X(t)\cdot X(t - \tau)\} = R(\tau)$

$\Rightarrow$ mean is Const &

auto Correlation depends only on time difference.

$R_X(t_1, t_2) = E[X(t_1)\cdot X(t_2)]$

$R_X(t, s) = E[X(t), X(s)]$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f_X(x_1, x_2, t_1, t_2)\, dx_1\, dx_2$

4(II): $Y(t)$ ... and X ... $x \sim f_X$ ... $E[Y(t)] =$ ... $R(t_1, t_2)$ ... $\therefore R(t$ ...

4 (II): $Y(t) = X\sin(\lambda t)$, where λ is constant.
and X is Uniform in $(-1, 1)$.

$\therefore X \sim f_X(x) = \frac{1}{2}, \ -1 \le x \le 1$

$E[Y(t)] = \int_{-1}^{1} x \sin \lambda t \cdot \frac{1}{2} dx = \frac{1}{2} \sin \lambda t \left(\frac{x^2}{2}\right)_{-1}^{1} = 0.$

$R(t_1, t_2) = E[Y(t_1) \cdot Y(t_2)] = E[x^2 \sin \lambda t_1 \cdot \sin \lambda t_2]$.

$[\because \cos(a-b) - \cos(a+b) = 2\sin a \sin b.]$

$\therefore R(t_1, t_2) = E\left[\frac{x^2}{2}[\cos\lambda(t_1 - t_2) - \cos(\lambda(t_1+t_2))]\right]$.

$= \int_{-1}^{1} \frac{1}{2} \cdot \frac{x^2}{2} \cdot [\cos\lambda(t_1-t_2) - \cos\lambda(t_1+t_2)] \, dx$

$= \frac{\cos\lambda(t_1-t_2) - \cos\lambda(t_1+t_2)}{4\times 3} [x^3]_{-1}^{1} = \frac{\cos\lambda(t_1-t_2) - \cos\lambda(t_1+t_2)}{6}$

$\neq$ fun. of $(t_1 - t_2)$
So it is not WSS Process

Q5 H Find auto correlation functions of the processes $\{X(t)\}$ and $\{Y(t)\}$. Such that $X(t) = A\cos\lambda t + B\sin\lambda t$ and $Y(t) = B\cos\lambda t - A\sin\lambda t$. where A and B are uncorrelated random variable taking value -4 and 4 with equal probabilities. Prove that $\{X(t)\}$ and $\{Y(t)\}$ are jointly WSS.

Auto Correlation fun.
$R_X(t_1, t_2) = E[X(t_1) X(t_2)]$

Ans:
$X(t) = A\cos\lambda t + B\sin\lambda t$.
$Y(t) = B\cos\lambda t - A\sin\lambda t$.
where A, B are uncorrelated variables.

Now ~~$E[A] = E[A\cos\lambda t + B\sin\lambda t]$~~ ~~$E[B]$~~.
$=$ ~~$\cos\lambda t \, E[A] + \sin\lambda t$~~

$E[A] = -4 \times \frac{1}{2} + 4 \times \frac{1}{2} = 0, \ E[B] = 0.$

$E(A^2) = 16 \times \frac{1}{2} + 16 \times \frac{1}{2}$
$= 16 = Var(A)$
$E[B^2] = 16 = Var(B)$

$\Rightarrow E[AB] = E[A] \cdot E[B] = 0.$

Now $R_{XX}(t_1, t_2) = E[X(t_1) \, X(t_2)]$
$= E[(A\cos\lambda t_1 + B\sin\lambda t_1) \cdot (A\cos\lambda t_2 + B\sin\lambda t_2)]$.

$= E[A^2 \cos\lambda t_1 \cdot \cos\lambda t_2 + AB\cos\lambda t_1 \cdot \sin\lambda t_2 + BA\sin\lambda t_1 \cdot \cos\lambda t_2 + B^2 \sin\lambda t_1 \cdot \sin\lambda t_2]$

$= 16 \cdot \cos\lambda t_1 \cdot \cos\lambda t_2 + 0 + 0 + 16 \cdot \sin\lambda t_1 \cdot \sin\lambda t_2$

$\cos A\cos B + \sin A\sin B = \cos(A-B)$

$= 16 \cdot \cos\lambda(t_1 - t_2)$

$R_{yy}(t_1, t_2) = E[Y(t_1) \cdot Y(t_2)]$

$= E[(B\cos\lambda t_1 - A\sin\lambda t_1) \cdot (B\cos\lambda t_2 - A\sin\lambda t_2)$

$= 16(\cos\lambda t_1 \cos\lambda t_2 + \sin\lambda t_1 \sin\lambda t_2)$

$= 16\cos\lambda(t_1 - t_2) \Rightarrow$ fun of $(t_1 - t_2)$

$R_{xy}(t_1, t_2) = E[X(t_1) \cdot Y(t_2)]$

$= E[(A\cos\lambda t_1 + B\sin\lambda t_1) \cdot (B\cos\lambda t_2 - A\sin\lambda t_2)]$

$= E[AB\cos\lambda t_1 \cos\lambda t_2 - A^2 \sin\lambda t_2 \cdot \cos\lambda t_1 - AB\sin\lambda t_1 \cdot \sin\lambda t_2$
$+ B^2 \sin\lambda t_1 \cdot \cos\lambda t_2]$

$= 16[\sin\lambda t_1 \cdot \cos\lambda t_2 - \sin\lambda t_2 \cdot \cos\lambda t_1]$

$= 16 \cdot \sin\lambda(t_1 - t_2) \Rightarrow$ fun of $(t_1 - t_2)$

$\Rightarrow X(t)$ and $Y(t)$ are jointly WSS.

Q NO 6:- If $X(t) = A\sin w(wt + \theta)$ where A and w are constants and θ is r.v. uniformly distributed over $(-\pi, \pi)$. Find the autocorrelation of $\{Y(t)\}$ where $Y(t) = X^2(t)$.

[Ans. $R(t_1, t_2) = \frac{A^4}{8}\{2 + \cos 2w(t_1 - t_2)\}$]

Ans 6: $X(t) = A\sin w(wt + \theta)$, where A and w are constants & θ is a r.v. s.t. $\theta \sim f(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \le \theta \le \pi \\ 0, & \text{otherwise.} \end{cases}$

Find $R_y(t_1, t_2)$ where $Y(t) = X^2(t)$

$R_y(t_1, t_2) = E[Y(t_1) \cdot Y(t_2)] = E[X^2(t_1) \cdot X^2(t_2)]$

$= E[A^2 \sin^2 w(wt_1 + \theta) \cdot A^2 \sin^2 w(wt_2 + \theta)]$

$= A^4 \cdot E[\sin^2 w(wt_1 + \theta) \cdot \sin^2 w(wt_2 + \theta)]$

$= A^4 E[\frac{1}{4}\{\cos(w(wt_1 + \theta) - w(wt_2 + \theta)) - \cos[w(wt_1 + \theta) + w(wt_2 + \theta)]\}^2]$ (2)

$= \frac{A^4}{4} E[\{\cos w[w(t_1 + t_2) - \cos w[w(t_1 + t_2) + 2\theta\}^2]$

1) $\cos 2x = 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$.

2) $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

3) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

4) $2\cos A \cos B = \cos(A + B) + \cos(A - B)$

$R_{xx}(t_1, t_2) = \frac{A^4}{4} E\left[\cos^2 w[w(t_1-t_2)] + \cos^2 w[w(t_1+t_2)+2\theta]\right.$

$\left. - 2\cos w[w(t_1-t_2)] \cdot \cos w[w(t_1+t_2)+2\theta]\right]$

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$\frac{A^4}{4}\cos^2 w[w(t_1-t_2)] \cdot \frac{(\pi+\pi)}{2\pi} + \frac{A^4}{4} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos^2 w[w(t_1+t_2)+2\theta]\right.$

$\left. - 2\cos w[w(t_1-t_2)] \cos w[w(t_1+t_2)+2\theta]\right] d\theta.$

$= \frac{A^4}{4}\left[\cos^2[w(t_1-t_2)] + \frac{1}{2}\right]$

$= \frac{A^4}{4}\left[\frac{\cos 2w^2(t_1-t_2)}{2} + 1\right] = \frac{A^4}{8}\left[2 + \cos 2w^2(t_1-t_2)\right]$

Q7:- If $\{X(t)\}$ is a WSS process with $E\{X(t)\} = 2$ and $R_{xx}(T) = 4 + e^{-|T|/10}$, find the variance of $X(1)$, $X(2)$, and $X(3)$. Also compute the second order moment about origin of $X(1) + X(2) + X(3)$.

Ans:- Here $\{X(t)\}$ is WSS process.

$\therefore E\{X(t)\} = 2.$

$R_X(T) = R_{xx}(t_1, t_2) = 4 + e^{-|T|/10}$

$= 4 + e^{-|t_1 - t_2|/10}$

$E\{X(t) \cdot X(t-T)\} = R(T)$

$R_X(T) = \mu_X^2$

Find: $Var(X(1))$, $Var(X(2))$, $Var(X(3))$.

If $Y = X(1) + X(2) + X(3)$ find $E[Y^2]$.

$\because Var[X(t)] = E[X^2(t)] - (E[X(t)])^2$

$E[X^2(t)] = R_{xx}(t_1, t_1) = 4 + e^0 = 5$

$\therefore Var(X(1)) = Var(X(2)) = Var(X(3)) = 5 - 4 = 1.$