Department of Mathematics

Probability and Random Processes **15B11MA301**

Tutorial Sheet 14 **B.Tech. Core**

Poisson Random Process

Q.1: Define a Poisson process with suitable example. State and prove all the properties of Poisson process.

Q.2: The particles are emitted from a radioactive source at the rate of 40 per hour. Find the probability that exactly 6 particles are emitted during a 25 minutes period.

Solution:

$\lambda = 40$ per hour $\Rightarrow \lambda = \frac{2}{3}$ per min.

$t = 25$, $n = 6$

$$P\{X(t) = 6\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \frac{e^{-\frac{2}{3}\times 25}\left(\frac{2}{3}\times 25\right)^6}{6!}$$

$$= \frac{e^{-50/3}\left(\frac{50}{3}\right)^6}{6!}$$

$$= 0.00171996$$

Q.3: Customers arrive at the complaint department of a store at the rate of 5 per hour for male customers and 10 per hour for female customers. If arrivals in each case follow Poisson process, calculate the probabilities that

(a) at most 4 male customers,

Solution:

3:- (a) λ_1 (male) $= 5$ per hour, λ_2 (female) $= 10$ per hour

$n = 4$

$$P\{X(t) \le 4\} = \frac{e^{-5t}(5t)^0}{0!} + \frac{e^{-5t}(5t)^1}{1!} + \frac{e^{-5t}(5t)^2}{2!} + \frac{e^{-5t}(5t)^3}{3!} + \frac{e^{-5t}(5t)^4}{4!}$$

$$= e^{-5t}\left[1 + 5t + \frac{25}{2}t^2 + \frac{125}{6}t^3 + \frac{625}{24}t^4\right]$$

(b) at most 4 female customers will arrive in a 30-minute period

Solution:

(b) $n = 4$, $\lambda_2 = 10$ per hour , $t = 0.5$ hour

$$P[X_2(t) \le 4] = e^{-10t} + \frac{e^{-10t}(10t)^1}{1!} + \frac{e^{-10t}(10t)^2}{2!} + \frac{e^{-10t}(10t)^3}{3!} + \frac{e^{-10t}(10t)^4}{4!}$$

$$= e^{-10t}\left[1 + 10t + \frac{100t^2}{2} + \frac{1000t^3}{6} + \frac{10000}{24}t^4\right]$$

$$= e^{-5}\left[1 + 5 + 12.5 + 2.8333 + 26.041667\right]$$

$$= e^{-5} \times 65.374997$$

$$= 0.4405 \text{ Ans}$$

(c) the inter arrival time for male candidates exceeds 15 minutes.

Solution:

(c) by the property (4) for poisson process follow exponential distribution with mean $\frac{1}{\lambda}$.

$$f(t) = \lambda e^{-\lambda t} \quad (t \geq 0)$$

$$P\left(T > \frac{15}{60}\right) = ?$$

Given $t = \frac{1}{4}$, $\lambda_1 = 5$

i.e. $$P\left(T > \frac{1}{4}\right) = \int_{1/4}^{\infty} \lambda_1 e^{-\lambda_1 t}\, dt$$

$$= \int_{1/4}^{\infty} 5e^{-5t}\, dt$$

$$= 5\left[\frac{e^{-5t}}{-5}\right]_{1/4}^{\infty}$$

$$= 5\left[0 + \frac{1}{5}e^{-5/4}\right]$$

$$= e^{-5/4}$$

Q.4: If customers arrive at a service counter in accordance with a Poisson process with a mean rate of 5 per minute, find the probability that the interval between 2 successive arrivals is

(i) more than 3 minutes,

Solution: Given

$\lambda = 5$ per min

$f(x) = \lambda e^{-\lambda t} \quad (t \geq 0)$, $T \sim exp(\lambda)$

i) $P(T>3) = \int_3^{\infty} 5e^{-5t}\, dt$

$= e^{-15}$

(ii) between 4 to 7 minutes and

ii) $P(4<T<7) = 5\int_4^7 e^{-5t}\, dt$

$= -\frac{5}{5} e^{-5t} \Big|_4^7$

$= e^{-20} - e^{-35}$

$= 0.2 \times 10^{-8}$

(iii) less than 6 minutes.

iii) $P(T<6) = 5\int_0^6 e^{-5t}\, dt$

$= -e^{-30} + 1 \sim 1$

Q.5: The number of accidents in a city follows a Poisson process with a mean of 2 per day and the number X_i of people involved in the i^{th} accident has the distribution (independent)

$P\{X_i = k\} = \frac{1}{2^k}$ $(k \geq 1)$. Find the mean and variance of the number of people involved in accidents per week.

Solution:

Solution The mean and variance of the distribution $P\{X_i = k\} = \frac{1}{2^k}$, $k = 1, 2, 3, \ldots, \infty$ can be obtained as 2 and 2.

Let the number of accidents on any day be assumed as n.

The numbers of people involved in these accidents be $X_1, X_2, \ldots, X_n$.

$X_1, X_2, \ldots, X_n$ are independent and identically distributed RVs with mean 2 and variance 2.

Therefore, by central limit theorem, $(X_1 + X_2 + \ldots + X_n)$ follows a normal distribution with mean $2n$ and variance $2n$, i.e., the total number of people involved in all the accidents on a day with n accidents = $2n$.

If N denotes the number of people involved in accidents on any day, then

$$P\{N = 2n\} = P\{X(t) = n\} \text{ [where } X(t) \text{ is the number of accidents]}$$

$$= \frac{e^{-2t}(2t)^n}{\lfloor n} \text{ (by data)}$$

$$\therefore \quad E\{N\} = \sum_{n=0}^{\infty} \frac{2n\, e^{-2t}(2t)^n}{\lfloor n}$$

$$= 2E\{X(t)\} = 4t$$

$$\text{Var}\{N\} = E\{N^2\} - E^2(N)$$

$$= \sum_{n=0}^{\infty} \frac{4n^2\, e^{-2t}(2t)^n}{\lfloor n} - 16t^2$$

$$= 4E\{X^2(t)\} - 16t^2$$

$$= 4[\text{Var}(X(t)\} + E^2\{X(t)\}] - 16t^2$$

$$= 4[2t + 4t^2] - 16t^2 = 8t$$

Therefore, mean and variance of the number of people involved in accidents per week are 28 and 56 respectively.