Probability and Random Processes

Tutorial Sheet-1

Ans 1: Total Cards = 52

(i) P[either a black card or an ace or both]

$$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$$

(ii) P[either an ace of diamond or an ace of hearts]

$$= \frac{1}{52} + \frac{1}{52} = \frac{2}{52} = \frac{1}{26}$$

(iii) P[either a diamond card or an ace or both]

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}.$$

Ans 2: $P(A) = 0.4$, $P(B) = ?$

$P(A \cup B) = 0.7$

(i) If A and B are mutually exclusive

$\Rightarrow P(A \cap B) = 0$

Using, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.7 = 0.4 + P(B)$

$\Rightarrow \boxed{P(B) = 0.3}$

(ii) A and B are independent

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

Using, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.7 = 0.4 + P(B) - 0.4\,P(B)$

$\Rightarrow \quad 0.3 = P(B) \cdot 0.6$

$\Rightarrow \quad \boxed{P(B) = 1/2}$

Ans-3: Conditional Probability:

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad , P(B) \neq 0.$$

Now we have,

$$P(A\cup B|C) = \frac{P((A\cup B)\cap C)}{P(C)} \qquad -(1)$$

Since,

$$\begin{aligned} P((A\cup B)\cap C) &= P((A\cup B)\cap C) \\ &= P[(A\cap C)\cup(B\cap C)] \\ &= P[A\cap C] + P[B\cap C] - P[A\cap B\cap C] \end{aligned}$$

By (1) $$P[A\cup B|C] = \frac{P[A\cap C]}{P[C]} + \frac{P[B\cap C]}{P(C)} - \frac{P[A\cap B\cap C]}{P(C)}$$

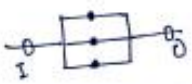
$$P[A\cup B|C] = P[A|C] + P(B|C) - P(A\cap B|C)$$

Ans 4: Let S_i be the event that switch S_i is closed.

$P(S_i) = p \quad , i = 1, 2, 3.$

(a) P[receiving an input signal at output]

$$\begin{aligned} &= 1 - P[\text{all switches are open}] \\ &= 1 - P(\bar{S}_1) \cdot P(\bar{S}_2) \cdot P(\bar{S}_3) \\ &= 1 - (1-p)^3 \\ &= \underline{p^3 - 3p^2 + 3p} \end{aligned}$$



(b) $P[\text{Switch } S_1 \text{ is open} \mid \text{input signal is received at output}]$

$$= \frac{P[(S_1 \text{ is open}) \cap (\text{Input signal is received at output})]}{P[\text{input signal is received at output}]}$$

$$= \frac{(1-p)\left[1-(1-p)^2\right]}{p^3-3p^2+3p}$$

$$= \underline{\frac{(1-p)(2p-p^2)}{p^3-3p^2+3p}}$$

Ans 5:

Let B & C produce x no. of cars. Then A produces $2x$ number of cars.

$P(A) = \frac{2x}{4x} = \frac{1}{2}$, $P(B) = \frac{x}{4x} = \frac{1}{4}$, $P(C) = \frac{x}{4x} = \frac{1}{4}$

Here $P(D|A) = 2\% = \frac{2}{100}$ [D: defective cars]

$P(D|B) = \frac{3}{100}$

$P(D|C) = \frac{4}{100}$

(a) By total probability law

$$P(D) = P(D|A)\cdot P(A) + P(D|B)\cdot P(B) + P(D|C)\cdot P(C)$$

$$= \frac{1}{2} \times \frac{2}{100} + \frac{3}{100} \times \frac{1}{4} + \frac{4}{100} \times \frac{1}{4}$$

$$= 11/400 = \underline{0.0275}$$

(b) $P(A|D) = \frac{P(D|A)\cdot P(A)}{P(D)}$ [using Bayes' theorem]

$$= \frac{2/4 \cdot 2/100}{11/400} = \frac{4}{11} = \underline{0.3636}$$

Ans 6: Let A & B denotes the computer A & B have been marketed respectively.

$P(A) = 0.6$

$P(B) = 0.4$

[Independent events]

$P(A \cup B) = 0.6 + 0.4 - 0.6 \times 0.4$

$= 1.0 - 0.24$

$= 0.76$

$$P(A | A \cup B) = \frac{P[A \cap (A \cup B)]}{P(A \cup B)}$$

$$= \frac{P(A)}{P(A \cup B)} = \frac{0.6}{0.76}$$

$$= 60/76 = \underline{15/19}$$

Ans 7: Since A, B & C are pairwise independent [Given]

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

$P(B \cap C) = P(B) \cdot P(C)$

$P(A \cap C) = P(A) \cdot P(C)$

& A is independent of $B \cup C$ [Given]

$P[A \cap (B \cup C)] = P(A) \cdot P(B \cup C) \quad -(*)$

$\because P(B \cup C) = P(B) + P(C) - P(B \cap C)$

$= P(B) + P(C) - P(B) P(C) \quad -(**)$

By eq(*) & (**)

$P(A \cap (B \cup C)) = P[(A \cap B) \cup (B \cap C)]$

$= P(A) P(B) + P(A) \cdot P(C) - P(A) P(B) P(C)$

$\Rightarrow P(A\cap B) + P(A\cap C) - P(A\cap B\cap C)$

$= P(A)\cdot P(B) + P(A)\cdot P(C) - P(A)\cdot P(B)\cdot P(C)$

$\Rightarrow P(A)\cdot P(B) + P(A)\cdot P(C) - P(A\cap B\cap C)$

$= P(A)\cdot P(B) + P(A)\cdot P(C) - P(A)\cdot P(B)\cdot P(C)$

$\Rightarrow \boxed{P(A\cap B\cap C) = P(A)\cdot P(B)\cdot P(C)}$

Ans 8:

Sum is even if

Case-1) Both numbers are odd (1, 3, 5, 7, 9)

Case-II) Both numbers are even. (2, 4, 6, 8)

S : Sum is even

O : Getting two Odd no's.

E : Getting two even nos'.

P(E) = P(

$$P(O|S) = \frac{P(O\cap S)}{P(S)}$$

$$= \frac{P(O)}{P(S)} = \frac{25/81}{41/81} = \frac{25}{41}$$

Ans 9:

$$\begin{array}{cc} A & B \\ \Downarrow & \Downarrow \\ 6 & 7 \end{array}$$

P[getting sum of 6 on paired dice]

$= 5/36$

$\left\{ \begin{array}{l} \text{Favourable no. of cases:} \\ (1,5), (5,1), (2,4), (4,2), (3,3) \end{array} \right\}$

P[getting sum of 7 on a paired dice]

$= 6/36$

$\left\{ \begin{array}{l} \text{Favourable no. of cases:} \\ (1,6), (6,1), (2,5), (5,2), (3,4)(4,3) \end{array} \right\}$

If A begins the game

P[A wins the game]

$$= P(A) + P(\bar{A}\bar{B}A) + P(\bar{A}\bar{B}\bar{A}\bar{B}A) + \cdots$$

$$= \frac{5}{36} + \frac{31}{36} \cdot \frac{30}{36} \cdot \frac{5}{36} + \frac{31}{36} \cdot \frac{30}{36} \cdot \frac{31}{36} \cdot \frac{30}{36} \cdot \frac{5}{36} + \cdots$$

$$= \frac{5/36}{1 - \left(\frac{31}{36}\right)\left(\frac{30}{36}\right)} = \frac{5/36}{\frac{366}{36 \times 36}} = \frac{36 \times 5}{366}$$

$$= \underline{\underline{30/61}}$$

Ans 10 : No's are : 1 to 50

(i) Total numbers divisible by 3 = 16

Total numbers divisible by 4 = 12

Total numbers divisible by 12 = 4

P[divisible by 3 or 4 or both]

$$= \frac{16}{50} + \frac{12}{50} - \frac{4}{50}$$

$$= \frac{24}{50} = 12/25$$

(ii) Prime numbers less than 37

= 2, 3, 5, 7, 11, 13, 17, 19, 23, 31

P{ getting prime numbers less than 37}

= 11/50

(iii) NO's end in

2 - { 2, 12, 22, 32, 42 }

3 - { 3, 13, 23, 33, 43 }

P[no's end. with 2 or 3]

$$= \frac{5}{50} + \frac{5}{50}$$

$$= \frac{1}{5}$$

Ans-11 :

$\underbrace{\text{4 true coins}}_{T}$ & $\underbrace{\text{1 false coin}}_{F}$

P[getting head 5 times]

$$= P(H/T)\,P(T) + P(H/F)\cdot P(F)$$

$$= \left(\tfrac{1}{2}\right)^5 \cdot \left(4/5\right) + 1 \cdot \tfrac{1}{5}$$

$$= \tfrac{1}{5}\left(\tfrac{4}{32} + 1\right)$$

$$= \tfrac{1}{5} \cdot \tfrac{9}{8} = 9/40$$

P[F/getting head 5 times)

$$= \frac{\frac{1}{5}\cdot 1}{9/40} = 8/9$$

Ans 12 :

Let E_1 be the event that letter has come from TATANAGAR

Let E_2 be the event that letter has come from CALCUTTA

A : denotes the event that the two consecutive letter on the envelope are TA

$P(E_1) = 1/2$, $P(E_2) = 1/2$

$P(A|E_1) = 2/8$ {TA, AT, TA, AN, NA, AG, GA, AR}

$P(A|E_2) = 1/7$ {CA, AL, LC, CU, UT, TT, TA}

$$P(E_2|A) = \frac{P(E_2) \cdot P(A|E_2)}{P(A)}$$

$$= \frac{P(E_2) \cdot P(A|E_2)}{P(A|E_1) \cdot P(E_1) + P(A|E_2) \cdot P(E_2)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{7}}{\frac{2}{8} \cdot \frac{1}{2} + \frac{1}{7} \cdot \frac{1}{2}} = \frac{1/14}{22/112}$$

$$= \underline{4/11}$$

Ans 13: P[man reports it was six)

= P(6) · P(man speaks truth)

+ P($\bar{6}$) · P(man speaks false)

$$= \frac{1}{6} \cdot \frac{4}{5} + \frac{5}{6} \cdot \frac{1}{5}$$

$$= \underline{\underline{9/30}}$$

P[it was actually a six | man reports it was a six)

$$= \frac{1/6 \cdot 4/5}{9/30} = \underline{\underline{4/9}}$$

Ans 14 :

Total cases:

$$\{ACBNC,\ ACBCS,\ ANCBC,\ ANCBNCS,\ ANCBNCD\}$$

$$P[\{AC\}] = P[\{ACBNC\} \cup \{ACBCS\}]$$

$$= 1/6$$

$$P[\{BC\}] = P[\{ANCBC\} \cup \{ACBCS\}]$$

$$= 1/8$$

$$P[\{ANCBNCS\}] = 1/525$$

Events AC & BC are Independent.

$$P[\{AC\} \cap \{BC\}] = P[\{ACBCS\}]$$

$$= P[\{AC\}] \cdot P[\{BC\}]$$

$$= \frac{1}{6} \cdot \frac{1}{8} = \frac{1}{48}$$

$$P[\{C\}/\{S\}] = \frac{P[\{C\} \cap \{S\}]}{P[\{S\}]}$$

$$= \frac{P[\{ACBCS\}]}{P[\{ACBCS\} \cup \{ANCBNCS\}]}$$

$$= \frac{P[\{ACBCS\}]}{P[\{ACBCS\}] + P[\{ANCBNCS\}]}$$

$$= \frac{1/48}{\frac{1}{48} + \frac{1}{525}} = \frac{1/48}{575/525 \times 48} = \frac{525}{573} = 0.91623$$

Answers of Tutorialsheet 2

Ans 1 :- (i) Let S be a sample space associated with a given random experiment. A real valued f^n defined on S and taking values in $R(-\infty, \infty)$ is called random variable in one dimensional.

Discrete Random variable :- X has atmost countable number of possible values.

* X represents the sum of two dice

$$X = \{2,3,4,5,6,7,8,9,10,11,12\}$$

Continuous Random variable :- X takes all the values in a given interval of numbers.

* X represents time of arrival of train.
* X represents distance b/w two cities.

(ii) Probability Density Function - Let $f(x)dx$ represents the area bounded by the curve $y = f(x)$, x axis and the ordinates at the points $x - \frac{\Delta x}{2}$ & $x + \frac{\Delta x}{2}$. The f^n $f(x)$ is so defined is known as probability density f^n of the random variable X.

. PDF of a rv X denoted by $f(x)$ has following properties

(i) $f(x) \geq 0 \quad \forall x \in R$

(ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

(iii) $P(a < X < b) = \int_a^b f(x)\, dx$

ex:- $f(x) = \begin{cases} |x|, & -1 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases} \rightarrow (PDF)$

Cumulative Distribution Function - CDF $F(x)$ of a cts random variable X with PDF $f(x)$ is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx, \quad -\infty < x < \infty$$

* If pdf is $f(x) = \begin{cases} |x|, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

then corresponding CDF is

$$F(x) = \begin{cases} -x^2/_2 + 1/_2 & -1 \le x \le 0 \\ 0 & x < -1 \\ \frac{1}{2} - x^2/_2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

Ans 2: (i) $f(x) = kx(1-x), \quad 0 < x < 1$

Here,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} k\,x(1-x)\,dx = 1$$

$$\Rightarrow k\left[\frac{x^2}{2} - x^3/_3\right]_0^1 = 1$$

$$\Rightarrow k\left[\frac{1}{6}\right] = 1$$

$$\Rightarrow \boxed{k = 6}$$

(ii) $P(X < b) = P(X > b) = 1 - P(X \le b)$ $\quad$ X is cts $\left[P[X < b] = P[X \le b]\right]$

$$\Rightarrow P(X < b) = \frac{1}{2}$$

$$\Rightarrow 6\int_0^b (x - x^2)\,dx = 1/_2$$

$$\Rightarrow \left[\frac{x^2}{2} - x^3/_3\right]_0^6 = \frac{1}{12}$$

$$\Rightarrow b^2\left[\frac{1}{2} - b/_3\right] = \frac{1}{12}$$

$$\Rightarrow 6b^2 - 4b^3 - 1 = 0$$

$$\Rightarrow \boxed{b = \frac{1}{2}}$$

Ans 3: $X = |\text{no. of heads} - \text{no. of tails}|$

$X = \{3, 1\}$

(i) PMF :-

$X=x$	1	3	otherwise
$P(X=x)$	$\frac{6}{8} = \frac{3}{4}$	$\frac{2}{8} = \frac{1}{4}$	0

(ii) CDF :-

$$F(x) = \begin{cases} 0 & x < 1 \\ 3/4 & 1 \le x < 3 \\ 1 & x \geq 3 \end{cases}$$

Ans 4 :-

X	1	2	3	4	5	6
P(X)	0.04	0.15	0.37	0.26	0.11	0.07

$P[X\text{-odd}] = 0.04 + 0.37 + 0.11$

$= 0.52$

$P[X < 5] = 1 - 0.18 = 0.82$

(i) $P[X\text{-odd} \mid X < 5] = \dfrac{P[(X\text{-odd}) \cap (X < 5)]}{P[X < 5]} = \dfrac{0.04 + 0.37}{0.82}$

$= \underline{1/2}$

(ii) $P[X < 5 \mid X\text{-odd}] = \dfrac{P[(X<5) \cap (X\text{-odd})]}{P[X\text{-odd}]} = \dfrac{0.41}{0.52} = \underline{41/52}$

(iii) $P[X = 4 \mid X \neq 3]$

$= \dfrac{P[X=4]}{P[X \neq 3]} = \dfrac{0.26}{0.63} = \underline{26/63}$

Ans 5:

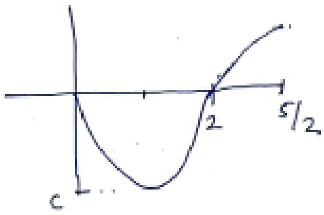
$$f(x) = \begin{cases} C(x^2 - 2x), & 0 < x < 5/2 \\ 0, & \text{elsewhere}. \end{cases}$$

To be pdf:

$C(x^2 - 2x) = C((x-1)^2 - 1)$

(i) $f(x) \geq 0$

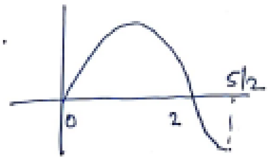
Here for positive value of C



for negative value of C

for some values of x, $f(x)$ is negative.

$\Rightarrow$ $f(x)$ can't be pdf



Ans 6: To be a pdf

(i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Here,

$$\int_{-1}^{0} A(1+x)\,dx + \int_{0}^{1} A(1-x)\,dx = 1$$

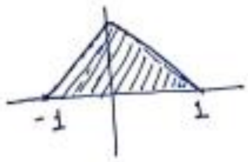
$$\Rightarrow A\left[\left[x + \frac{x^2}{2}\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{1}\right] = 1$$

$$\Rightarrow A\left[\left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)\right] = 1$$

$$\Rightarrow \boxed{A = 1}$$

Here, $f(x) = \begin{cases} 1+x & -1 < x \leq 0 \\ 1-x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(i) Graph of $f(x)$



(ii) Distribution f^n of $f(x)$

$$F(x) = \begin{cases} 0, & x \leq -1 \\ \int_{-1}^{x}(1+x)\,dx = \frac{x^2+2x+1}{2}, & -1 \leq x \leq 0 \\ \int_{-1}^{0}(1+x)\,dx + \int_{0}^{x}(1-x)\,dx = \frac{2x - x^2+1}{2}, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

(iii) $P[X > C] = P[X < C]/2$

$1 - P[X < C] = \frac{1}{2}P[X - C]$

$P[X < C] = 2/3$

If $C \in [-1, 0]$

$$\int_{-1}^{C} (1+x)\,dx = \frac{2}{3}$$

$\Rightarrow \left.\frac{x^2}{2} + x\right]_{-1}^{C} = 2/3$

$\Rightarrow \frac{C^2}{2} + C - \frac{1}{2} + 1 = 2/3$

$\Rightarrow \frac{C^2}{2} + C = \frac{2}{3} - \frac{1}{2} = 1/6$

$\Rightarrow 3C^2 + 6C - 1 = 0$

$\Rightarrow C = -1 \pm \frac{2}{3}\sqrt{3} \notin [-1, 0]$

If $c \in [0,1]$

$$P[X < c] = 2/3$$

$$\int_{-1}^{0} (1+x)\,dx + \int_{0}^{c} (1-x)\,dx = 2/3$$

$$\Rightarrow \left[\frac{x^2}{2} + x\right]_{-1}^{0} + \left[-\frac{x^2}{2} + x\right]_{0}^{c} = 2/3$$

$$\Rightarrow \left(\frac{-1}{2} + 1\right) - \frac{c^2}{2} + c = 2/3$$

$$\Rightarrow \frac{c^2}{2} - c - \frac{1}{2} + \frac{2}{3} = 0$$

$$\Rightarrow c^2 - 2c + 2/6 = 0$$

$$\Rightarrow 3c^2 - 6c + 1 = 0$$

$$\Rightarrow c = 1 \pm \sqrt{2/3}$$

But $c \in [0,1]$

$$\Rightarrow \boxed{c = 1 - \sqrt{\frac{2}{3}}}$$

Ans 7:
$$f(x) = \begin{cases} ax + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Since, $f(x)$ is pdf

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

Here, $\int_{0}^{1} (ax + bx^2)\,dx = 1$

$$\left. a\frac{x^2}{2} + b\frac{x^3}{3} \right]_{0}^{1} = 1$$

$$\Rightarrow \boxed{3a + 2b = 6} \quad \text{—(1)}$$

4 $E(X) = 0.6$ [Given]

$$\int_0^1 x(ax + bx^2)\,dx = 0.6$$

$$= \int_0^1 ax^2 + bx^3\,dx = 0.6$$

$$= \left.\frac{ax^3}{3} + \frac{bx^4}{4}\right]_0^1 = 0.6$$

$$= \boxed{4a + 3b = 7.2} \quad \text{—(2)}$$

By (1) & (2)

$$a = 3.6 \quad \& \quad b = -2.4$$

a) $$P[X < 1/2] = \int_0^{1/2} 3.6x - 2.4x^2\,dx$$

$$= 3.6\left(\frac{1}{8}\right) - 2.4\left(\frac{1}{24}\right)$$

$$= 0.45 - 0.1$$

$$= \underline{\underline{0.35}}$$

b) $$Var(X) = E(X^2) - [E(X)]^2$$

$$= \int_0^1 3.6x^3 - 2.4x^4\,dx - 0.36$$

$$= 3.6\left(\frac{1}{4}\right) - 2.4\left(\frac{1}{5}\right) - 0.36$$

$$= \underline{\underline{0.06}}$$

Ans 8: CDF :- $F(x) = 1 - e^{-2x^2}, \quad x \geq 0$

a) $P(0 < X < 3) = 1 - e^{-2 \cdot 3^2} = 1 - e^{-18}$

b) $P[X > 1] = 1 - P[X \leq 1]$
$= 1 - (1 - e^{-2})$
$= 1/e^2$

c) $P[X = 5] = 0$

Ans 9: Value of the game $= E(X)$
X: Amount in favor of player.

$$X(x) = \begin{cases} \frac{1}{8} & x = 1500 \\ 3/8 & x = 1000 \\ 3/8 & x = 500 \\ 1/8 & x = -2000 \end{cases}$$

$$E(X) = 1500 \times \frac{1}{8} + 1000 \times \frac{3}{8} + 500 \times \frac{3}{8} + \frac{1}{8} \times (-2000)$$
$$= \frac{1}{8}\left[1500 + 3000 + 1500 - 2000\right]$$
$$= \frac{4000}{8} = 500$$

Yes

Ans 10: X: Denotes the no. of defective items drawn.

X	0	1	2	3
f(x)	$\frac{{}^6C_3\, {}^4C_3}{{}^{10}C_3}$	$\frac{{}^6C_2 \cdot {}^4C_1}{{}^{10}C_3}$	$\frac{{}^6C_1 \cdot {}^4C_2}{{}^{10}C_3}$	$\frac{{}^6C_0\, {}^4C_3}{{}^{10}C_3}$
	$= \frac{1}{6}$	$= \frac{1}{2}$	$= \frac{3}{10}$	$= \frac{1}{30}$

$$E(X) = \sum x\, f(x)$$

$$= 0\times\frac{1}{6} + 1\times\frac{1}{2} + 2\times\frac{3}{10} + 3\times\frac{1}{30}$$

$$= \frac{1}{2} + \frac{3}{5} + \frac{1}{10} = \frac{5+6+1}{10} = \underline{\underline{6/5}}$$

Ans 11 :

$$f(x) = \begin{cases} c\,x\,e^{-x/2} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Since $f(x)$ is pdf.

$$c\int_0^\infty x\,e^{-x/2}\,dx = 1$$

$$\Rightarrow \left[\frac{x\,e^{-x/2}}{-1/2} - \int \frac{1\cdot e^{-x/2}}{-1/2}\,dx\right]_0^\infty = \frac{1}{c}$$

$$\Rightarrow \left[-2x\,e^{-x/2} + \frac{2\,e^{-x/2}}{-1/2}\right]_0^\infty = \frac{1}{c}$$

$$\left[\lim_{x\to\infty} x\,e^{-x/2} = 0\right]$$

$$\Rightarrow \left[4e^{-0/2}\right] = \frac{1}{c}$$

$$\Rightarrow \boxed{c = 1/4}$$

$$P[X \ge 5] = 1 - P[X < 5]$$

$$= 1 - \frac{1}{4}\int_0^5 x\,e^{-x/2}\,dx$$

$$= 1 - \frac{1}{4}\left[-2x\,e^{-x/2} - 4e^{-x/2}\right]_0^5$$

$$= 1 - \frac{1}{4}\left[e^{-5/2}[-10-4] + 4\right]$$

$$= 1 - 1 + \frac{e^{-5/2}}{4}\cdot 14$$

$$= \underline{7/2\, e^{-5/2}}$$

Answers for
Tutorial sheet - 3

Probability and Random Processes.

Ans 1 (a) Two dimensional Random Variable -

Let S be a sample space associated with the random experiment E. Let $X = X(s)$ and $Y = Y(s)$ be two functions, each assigning a real number to each outcome $s \in S$ of the random experiment, then (X, Y) is called the two dimensional random variable,

(b) Marginal Probability Distribution -

• Let (X,Y) be a 2-dim discrete R.V. Then marginal probability distribution fn of r.v. X is defined as

$X = \{x_i, i = 1, 2, \ldots\}$
$Y = \{y_j, j = 1, 2, \ldots\}$

$$P(X = x_i) = \sum_{j=1}^{m} p_{ij} = P_{i*}$$

$$P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = P_{*j}$$

• When (X, Y) is a two dimensional cts random variable, then the marginal density fn of r.v X is defined as

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx.$$

Conditional Probability Distribution -

Let (X,Y) be a two dimensional discrete random variable, then

$$P(X = x_i | Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)} = \frac{P_{ij}}{P_{*j}}$$

If (X, Y) is 2 dim cts random variable, then

$$f(x|y) = \frac{f(x,y)}{f(y)}$$ is called conditional probability fn of X given by Y.

Ans 2: X: No. of kings, Y: No. of aces

(i) Joint PMF :-

Marginal distribution

Y \ X	0	1	2	Marginal $f_Y(y)$
0	$\frac{^{44}C_2}{^{52}C_2} = \frac{473}{663}$	$\frac{^{44}C_1 \cdot ^{4}C_1}{^{52}C_2} = \frac{88}{663}$	$\frac{^{4}C_2}{^{52}C_2} = \frac{3}{663}$	$\frac{564}{663}$
1	$\frac{^{44}C_1 \cdot ^{4}C_1}{^{52}C_2} = \frac{88}{663}$	$\frac{^{4}C_1 \, ^{4}C_1}{^{52}C_2} = \frac{8}{663}$	0	$\frac{96}{663}$
2	$\frac{^{4}C_2}{^{52}C_2} = \frac{3}{663}$	0	0	$\frac{3}{663}$
$f_X(x)$	$\frac{188}{221}$	$\frac{33}{221}$	$1/221$	1

(ii) Marginal distribution

(iii) $$P[X=2 \mid Y=1] = \frac{P[X=2, Y=1]}{P[Y=1]}$$

$$= 0$$

(iv) $$P[X<2 \mid 0<Y<2] = \frac{P[X=0,1 \cap Y=1]}{P[Y=1]}$$

$$= \frac{\frac{88}{663} + \frac{8}{663}}{96/663} = 1.$$

(v) $$P[1 \le X \le 2 \mid Y=0,2) = \frac{P[X=1,2 \cap Y=0,2)}{P[Y=0,2]}$$

$$= \frac{91/663}{567/663} = 13/81$$

Ans-3: $f(x,y) = k(xy+y^2)$, $0 \le x \le 2$, $0 \le y \le 1$

(i) $k\int_0^2\int_0^1 xy+y^2 \, dy \, dx = 1$

$\Rightarrow k\int_0^2 \left[\frac{x}{2}y^2 + y^3/3\right]_0^1 dx = 1$

$\Rightarrow k\int_0^2 \left(x/2 + 1/3\right) dx = 1$

$\Rightarrow k\left[x^2/4 + x/3\right]_0^2 = 1$

$\Rightarrow \boxed{k = 3/5}$

(ii) $P(X>1)$

$= P(X>1, 0 \le Y \le 1)$

$= \int_0^1\int_1^2 3/5\,(xy+y^2)\,dx\,dy$

$= \frac{3}{5}\int_0^1 \left(x^2/2\, y + xy^2\right]_1^2 dy$

$= \frac{3}{5}\int_0^1 \left(2y+2y^2 - y/2 - y^2\right) dy$

$= \frac{3}{5}\left[y^3/3 + 3y^2/4\right]_0^1$

$= \frac{3}{5}\left[\frac{1}{3}+\frac{3}{4}\right] = \frac{3}{12\cdot 5}(13) = \frac{13}{20}$

(iii) $P(X+Y<1) = \int_0^1\int_0^{1-x} \frac{3}{5}(xy+y^2)\,dy\,dx$

$= \frac{3}{5}\int_0^1 \left(\frac{x}{2}y^2 + y^3/3\right)_0^{1-x} dx$

$= \frac{3}{5}\left(\int_0^1 \frac{x}{2}(1-x)^2 + \frac{(1-x)^3}{3} dx\right)$

$$= \frac{3}{5}\left[\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}\right)-\left(\frac{1}{3}\cdot\frac{2}{3}\right)+\frac{1}{3}\left(1-\frac{3}{2}+1-\frac{1}{4}\right)\right]$$

$$= \frac{3}{5}\left[\frac{1}{24}+\frac{1}{12}\right] = \underline{\underline{3/40}}$$

(iv) $P[X<1,\ Y>1/2]$

$$= \frac{3}{5}\int_0^1\int_{1/2}^1 (xy+y^2)\,dy\,dx$$

$$= \frac{3}{5}\int_0^1 \left[\frac{x}{2}y^2+\frac{y^3}{3}\right]_{1/2}^1 dx$$

$$= \frac{3}{5}\int_0^1 \left(\frac{3}{8}x+\frac{7}{24}\right)dx$$

$$= \frac{3}{5}\left[\frac{3}{16}+\frac{7}{24}\right] = \underline{\underline{23/80}}$$

v) $$f_X(x) = \frac{3}{5}\int_0^1 (xy+y^2)\,dy = \frac{3}{5}\left[\frac{x}{2}y^2+\frac{y^3}{3}\right]_0^1$$

$$= \frac{3}{5}\left[\frac{x}{2}+\frac{1}{3}\right] = \frac{3x+2}{10}, \quad 0\le x\le 2.$$

$$f_Y(y) = \frac{3}{5}\int_0^2 (xy+y^2)\,dx$$

$$= \frac{3}{5}\left[\frac{y}{2}x^2+y^2x\right]_0^2 = \frac{3}{2}\left[2y+2y^2\right]$$

$$= \frac{6(y+y^2)}{5}, \quad 0\le y\le 1$$

To check the Independence :-

$$f_X(x)\,f_Y(y) = \left(\frac{3x+2}{10}\right)\left(\frac{6y+6y^2}{5}\right)$$

$$= \frac{1}{50}\left[18xy+18xy^2+12y+12y^2\right]$$

$$\ne f(x,y) \qquad \text{Not independent.}$$

Ans 4 :-

$$f(x,y) = \frac{1}{4} e^{-|x|-|y|} \qquad -\infty < x < \infty, \; -\infty < y < \infty$$

$$= \begin{cases} \frac{1}{4} e^{x+y} & -\infty < x \le 0, -\infty < y \le 0 \\ \frac{1}{4} x-y & -\infty < x \le 0, 0 \le y < \infty \\ \frac{1}{4} e^{-x+y} & 0 \le x < \infty, -\infty < y \le 0 \\ \frac{1}{4} e^{-x-y} & 0 \le x < \infty, 0 \le y < \infty \end{cases}$$

Marginal

$$f_X(x) = \frac{1}{4}\left[\int_{-\infty}^{0} e^{x+y}\,dy + \int_{0}^{\infty} e^{x-y}\,dy \right] \qquad -\infty < x \le 0$$

$$= \frac{1}{4}\left[\left[e^{x+y}\right]_{-\infty}^{0} - \left[e^{x-y}\right]_{0}^{\infty} \right]$$

$$= \frac{1}{4}\left[e^{x} - 0 - (0 - e^{x}) \right] = e^{x}/2$$

$$f_X(x) = \frac{1}{4}\left[\int_{-\infty}^{0} e^{-x+y}\,dy + \int_{0}^{\infty} e^{-x-y}\,dy \right], \qquad 0 \le x < \infty$$

$$= \frac{1}{4}\left[\left(e^{-x+y}\right)_{-\infty}^{0} - \left(e^{-x-y}\right)_{0}^{\infty} \right]$$

$$= \frac{1}{4}\left[e^{-x} - (0 - e^{-x}) \right] = e^{-x}/2$$

$$f_X(x) = \begin{cases} e^{x}/2, & -\infty < x \le 0 \\ e^{-x}/2, & 0 \le x < \infty \end{cases}$$

Similarly,

$$f_Y(y) = \begin{cases} e^{y}/2, & -\infty < y \le 0 \\ e^{-y}/2, & 0 \le y < \infty \end{cases}$$

Here $\boxed{f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y)} \rightarrow$ Independent.

(iii) $P(X \le 1, Y \le 0)$

$$= \frac{1}{4}\left[\int_{-\infty}^{0}\int_{-\infty}^{0} e^{x}e^{y}\,dy\,dx + \int_{0}^{1}\int_{-\infty}^{0} e^{-x}e^{y}\,dy\,dx\right]$$

$$= \frac{1}{4}\left[\int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{1} e^{-x}\,dx\right]$$

$$= \frac{1}{4}\left[\left(1-e^{-x}\right)_0^1\right] = \frac{1}{4}\left(1-\left(\frac{1}{e}-1\right)\right) = \frac{1}{2}-\frac{1}{4e}$$

Ans 5:

• Joint PMF

$$f(x,y) = \frac{2x+y}{27}, \quad x, y = 0, 1, 2$$

X \ Y	0	1	2	$f_X(x)$
0	0	1/27	2/27	3/27
1	2/27	3/27	4/27	9/27
2	4/27	5/27	6/27	15/27
$f_Y(y)$	6/27	9/27	12/27	1

• Conditional Probability distribution - $P_{Y|X}(Y=y|X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$

X \ Y	0	1	2
0	0	1/3	2/3
1	2/9	1/3	4/9
2	4/15	1/3	6/15

Ans 6 : x_1: 1 2 3 4 5 6
x_2: 1 2 3 4 5 6

$Y = \max(x_1, x_2)$

y \ x_1	1	2	3	4	5	6	$f_Y(y)$
1	1/36	0	0	0	0	0	1/36
2	1/36	2/36	0	0	0	0	3/36
3	1/36	1/36	3/36	0	0	0	5/36
4	1/36	1/36	1/36	4/36	0	0	7/36
5	1/36	1/36	1/36	1/36	5/36	0	9/36
6	1/36	1/36	1/36	1/36	1/36	6/36	11/36
$f_{X_1}(x_1)$	1/6	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

(b) $E(Y)$ + $\operatorname{var}(Y)$

$$E(Y) = \Sigma\, y\, f_Y(y)$$

$$= 1\left(\tfrac{1}{36}\right) + 2\left(\tfrac{3}{36}\right) + 3\left(\tfrac{5}{36}\right) + 4\left(\tfrac{7}{36}\right) + 5\left(\tfrac{9}{36}\right) + 6\left(\tfrac{11}{36}\right)$$

$$= \frac{1+6+15+28+45+66}{36}$$

$$= \frac{161}{36}$$

$$E(Y^2) = \Sigma y^2 f_Y(y)$$

$$= 1\left(\tfrac{1}{36}\right) + 4\left(\tfrac{3}{36}\right) + 9\left(\tfrac{5}{36}\right) + 16\left(\tfrac{7}{36}\right) + 25\left(\tfrac{9}{36}\right) + 36\left(\tfrac{11}{36}\right)$$

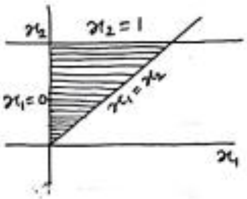
$$= \frac{1+12+45+112+225+396}{36}$$

$$= \frac{791}{36} \quad A$$

$$Var(Y) = E(Y^2) - [E(Y)]^2$$
$$= \frac{791}{36} - \left(\frac{161}{36}\right)^2$$
$$= \frac{2555}{1296}$$

Ans 7:-
$$f(x_1, x_2) = \begin{cases} 21 x_1^2 x_2^3, & 0 < x_1 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

conditional mean $f_{x_1/x_2}(x_1/x_2) = \frac{f(x_1, x_2)}{f_{x_2}(x_2)}$



first find

$$f_{x_2}(x_2) = \int_0^{x_2} 21 x_1^2 x_2^3 \, dx_1, \quad 0 < x_2 < 1$$
$$= \left[7 x_2^3 \cdot x_1^3\right]_0^{x_2}, \quad 0 < x_2 < 1$$
$$= 7 x_2^3 [x_2^3], \quad 0 < x_2 < 1$$

$$f_{x_1/x_2}(x_1/x_2) = \frac{f(x_1, x_2)}{f_{x_2}(x_2)}$$
$$= \frac{21 x_1^2 x_2^3}{7 x_2^6}, \quad 0 < x_1 < x_2 < 1$$
$$= \frac{3x_1^2}{x_2^3}$$

$$E(x_1/x_2) = \int_0^{x_2} x_1 \left(\frac{3x_1^2}{x_2^3}\right) dx_1$$
$$= \frac{3}{x_2^3}\left[\frac{1}{4} x_1^4\right]_0^{x_2}$$
$$= \frac{3}{4x_2^3}(x_2^4)$$
$$= \frac{3x_2}{4}, \quad 0 < x_2 < 1$$

$$E(x_1^2/x_2) = \int_0^{x_2} x_1^2 \left(\frac{3x_1^2}{x_2^3}\right) dx_1$$

$$= \frac{3}{x_2^3} \cdot \left[\frac{x_1^5}{5} \right]_0^{x_2}$$

$$= \frac{3}{5} \cdot \frac{1}{x_2^3} (x_2^5)$$

$$= \frac{3}{5} x_2^2 \quad , \quad 0 < x_2 < 1$$

$$\text{var}(x_1 / x_2) = \frac{3}{5} x_2^2 - \left(\frac{3}{4}\right)^2 x_2^2$$

$$= \left(\frac{3}{5} - \frac{9}{16}\right) x_2^2$$

$$= \frac{3}{80} x_2^2 \quad , \quad 0 < x_2 < 1$$

<u>Ans 8</u>:- $S = \{ (1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$
$(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$
$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6)$
$(4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6)$
$(5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6)$
$(6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6) \}$

X	0	1	2
$P(x)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

$$\text{MGF } [M_X(t)] = \Sigma\, e^{tx} P(x)$$

$$= e^0 \cdot \frac{25}{36} + e^t \cdot \frac{10}{36} + e^{2t} \cdot \frac{1}{36}$$

$$= \frac{25}{36} + \frac{10}{36} e^t + \frac{1}{36} e^{2t}$$

$$= \frac{25}{36} + \frac{10}{36} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$

$$+ \frac{1}{36} \left(1 + 2t + \frac{4t^2}{2!} + \cdots\right)$$

$$E(X) = \left.\frac{d M_X(t)}{dt}\right|_{t=0} = \left[\frac{10}{36} e^t + \frac{2}{36} e^{2t} \right]_{t=0}$$

$$= \frac{12}{36} = \frac{1}{3}$$

$$E(X^2) = \frac{d^2 M_X(t)}{dt}$$

$$= \left[\frac{10}{36} e^{t} + \frac{4}{36} e^{2t}\right]_{t=0}$$

$$= \frac{14}{36}$$

$$= \frac{7}{18}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{7}{18} - \left(\frac{1}{3}\right)^2$$

$$= \frac{5}{18}$$

Ans 9:-

$$f(x) = k \frac{e^{-|x|}}{5}, \quad -\infty < x < \infty$$

$$= \begin{cases} \frac{k}{5} e^{x} & , -\infty < x < 0 \\ \frac{k}{5} e^{-x} & , 0 \le x < \infty \end{cases}$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \frac{k}{5} \int_{-\infty}^{0} e^{tx} e^{x} dx + \frac{k}{5} \int_{0}^{\infty} e^{tx} e^{-x} dx$$

$$= \frac{k}{5} \left[\left(\frac{e^{(t+1)x}}{(t+1)} \right)_{-\infty}^{0} + \left(\frac{e^{(t-1)x}}{(t-1)} \right)_{0}^{\infty} \right]$$

$$t > -1 \qquad\qquad t < -1$$

$$= \frac{k}{5}\left[\frac{1}{t+1} - 0\right] + \frac{k}{5}\left[0 - \frac{1}{t-1}\right]$$

$$= \frac{k}{5}\left[\frac{1}{t+1} - \frac{1}{t-1}\right]$$

$$= \frac{k}{5}\left[\frac{t-1-t-1}{t^2-1}\right]$$

$$= \frac{k}{5}\left(\frac{-2}{t^2-1}\right)$$

$$= \frac{-2k}{5(t^2-1)}$$

$$= \frac{2k}{5}(1-t^2)^{-1}$$

$$= \frac{2k}{5}(1+t^2+t^4+t^6+\cdots) \quad , \quad -1<t<1$$

Coeff. of $t = 0$

$\therefore$ Mean $E(X) = 0$

Coeff. of $\frac{t^2}{2!} = \frac{4k}{5}$

$\Rightarrow E(X^2) = \frac{4}{5}\times\frac{5}{2} \Rightarrow E(X^2) = \frac{4}{2} = 2$

Coeff. of $\frac{t^3}{3!} = 0 = E(X^3)$

find k?

$$\frac{k}{5}\left[\int_{-\infty}^{0} e^{x}dx + \int_{0}^{\infty} e^{-x}dx\right] = 1$$

$$\frac{k}{5}[(1-0)-(0-1)] = 1$$

$$\frac{2k}{5} = 1$$

$$\Rightarrow \boxed{k=\frac{5}{2}}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 2 \text{ Ans}$$

Ans 10 :-

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & , \text{ if } 0\le x, y\le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \frac{3}{2}\int_0^1 (x^2+y^2)\,dy$$

$$= \frac{3}{2}\left[\left(x^2+\frac{y^3}{3}\right)_0^1\right]$$

$$= \frac{3}{2}\left(x^2+\frac{1}{3}\right)$$

$$= \frac{3x^2+1}{2} \quad , \quad 0 \le x \le 1$$

$$f_Y(y) = \frac{3}{2} \int_0^1 (x^2+y^2)\, dx$$

$$= \frac{3y^2+1}{2} \quad , \quad 0 \le y \le 1$$

$$E(X) = \frac{1}{2} \int_0^1 x\,(3x^2+1)\, dx$$

$$= \frac{1}{2} \left[\frac{3}{4}x^4 + \frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{3}{4} + \frac{1}{2} \right]$$

$$= \frac{5}{8}$$

$$E(Y) = \frac{1}{2} \int_0^1 y\,(3y^2+1)\, dy = \frac{5}{8}$$

$$E(XY) = \frac{3}{2} \int_0^1 \int_0^1 xy\,(x^2+y^2)\, dx\,dy$$

$$= \frac{3}{2} \int_0^1 \left[\frac{x^4 y}{4} + \frac{x^2 y^3}{2} \right]_0^1 dy$$

$$= \frac{3}{2} \int_0^1 \left(\frac{y}{4} + \frac{y^3}{2} \right) dy$$

$$= \frac{3}{2} \left(\frac{y^2}{8} + \frac{y^4}{8} \right)_0^1$$

$$= \frac{3}{8}$$

$$C_{XY} = E(XY) - E(X)\,E(Y)$$

$$= \frac{3}{8} - \frac{5}{8} \times \frac{5}{8}$$

$$= \frac{-1}{64}$$

$\rho_{XY} = \dfrac{C_{XY}}{\sigma_X \sigma_Y}$, first find σ_X^2, σ_Y^2

$$E(X^2) = \frac{1}{2}\int_0^1 x^2(3x^2+1)\,dx$$

$$= \frac{1}{2}\left[\frac{3}{5}x^5 + \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{3}{5}+\frac{1}{3}\right]$$

$$= \frac{7}{15}$$

$$E(Y^2) = \frac{7}{15}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{7}{15} - \frac{25}{64}$$

$$= \frac{73}{960}$$

$$Var(Y) = E(Y^2) - [E(Y)]^2$$

$$= \frac{7}{15} - \frac{25}{64}$$

$$= \frac{73}{960}$$

$$\therefore \quad \rho_{XY} = \frac{C_{XY}}{\sigma_X \sigma_Y}$$

$$= \frac{-1}{64} \Big/ \sqrt{\frac{73}{960}} \cdot \sqrt{\frac{73}{960}}$$

$$= \frac{-1}{64} \times \frac{960}{73}$$

$$= \frac{-15}{73}$$

Ans 11:- $P_X(k) = \frac{1}{k!} e^{-2} 2^k$, $P_Y(k) = \frac{1}{k!} e^{-3} 3^k$

MGF of $Z = 2X + 3Y$?

$$M_Z(t) = E\left(e^{t(2X+3Y)}\right)$$

$$= E\left(e^{2tX} \cdot e^{3tY}\right)$$

$$= E\left(e^{2tX}\right) E\left(e^{3tY}\right)$$

$$= M_X(2t) \; M_Y(3t)$$

$$= e^{2(e^{2t}-1)} \; e^{3(e^{3t}-1)}$$

$$= e^{2e^{2t} + 3e^{3t} - 5}$$

Ans 12:-

$$P_{XY}(k,l) = \begin{cases} 1/3 & , k = l = 0 \\ 1/6 & , k = \pm 1, l = 0 \\ 1/6 & , k = l = \pm 1 \\ 0 & , \text{else} \end{cases}$$

$$\phi_{X,Y}(\omega_1, \omega_2) = E\left(e^{i(\omega_1 X + \omega_2 Y)}\right)$$

X/Y	−1	0	1
−1	1/6	1/6	0
0	0	1/3	0
1	0	1/6	1/6

$$\phi_{XY}(\omega_1, \omega_2) = \Sigma \, e^{i(\omega_1 X + \omega_2 Y)} \; p(x)$$

$$= \frac{1}{6} e^{-i(\omega_1 - \omega_2)} + \frac{1}{6} e^{i(-\omega_1)} + \frac{1}{3} e^{i(0)} + \frac{1}{6} e^{i(\omega_1)} + \frac{1}{6} e^{i(\omega_1 + \omega_2)}$$

$$= \frac{1}{3} \cos(\omega_1 + \omega_2) + \frac{1}{3} \cos\omega_1 + \frac{1}{3}$$

Ans 13:-

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-tix}\,\phi(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\cdot e^{-\frac{\sigma^2 t^2}{2}}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\sigma^2 t^2 + \frac{2itx\sigma}{\sigma}\right)}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\left(\sigma t + \frac{ix}{\sigma}\right)^2 - \frac{i^2x^2}{\sigma^2}\right]}\,dt$$

$$= \frac{1}{2\pi}\, e^{\frac{-x^2}{2\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\sigma t + \frac{ix}{\sigma}\right)^2}\,dt$$

taking $\sigma t + \frac{ix}{\sigma} = \xi$

$\sigma dt = d\xi$

$$= \frac{1}{2\pi}\, e^{\frac{-x^2}{2\sigma^2}}\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\xi^2}}{\sigma}\,d\xi$$

$$= \frac{1}{2\pi\sigma}\, e^{-x^2/2\sigma^2}\cdot 2\int_{0}^{\infty} e^{-\xi^2/2}\,d\xi$$

$$= \frac{1}{\sigma\pi}\, e^{-x^2/2\sigma^2}\cdot\frac{\sqrt{\pi}}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2\sigma^2} \quad \text{Ans}$$

Note:-

$\because \int_0^\infty e^{-\xi^2/2}\,d\xi$

Put $\xi^2/2 = y$, $2\xi\, d\xi/2 = dy$

$\Rightarrow \int_0^\infty \frac{e^{-y}}{\sqrt{2y}}\,dy$

$\Rightarrow \frac{1}{\sqrt{2}}\int_0^\infty y^{-1/2} e^{-y}\,dy \qquad \because \int_0^\infty e^{-x} x^{n-1}\,dx = \frac{1}{2}\Gamma n$

$\Rightarrow \frac{1}{2}\Gamma\frac{1}{2}$

$\Rightarrow \sqrt{\frac{\pi}{2}}$ Ans

Q.3:- System operate when one half of its component function

P(3-com system will operate effectively)

$$= {}^3C_2 p^2 q + {}^3C_3 p^3$$

$$= {}^3C_2 p^2 (1-p) + {}^3C_3 p^3$$

$$= 3p^2(1-p) + p^3$$

P(5-com system will operate effectively)

$$= {}^5C_3 p^3 (1-p)^2 + {}^5C_4 (1-p) p^4 + {}^5C_5 p^5$$

Now according to question

$$\Rightarrow 10p^3(1-p)^2 + 5p^4(1-p) + p^5 > 3p^2(1-p) + p^3$$

$$\Rightarrow 10p^3(1-p)^2 + 5p^4(1-p) > 3p^2(1-p) + p^3(1-p^2)$$

$$\Rightarrow 10p^3(1-p) + 5p^4 > 3p^2 + p^3(1+p)$$

$$\Rightarrow 9p^3 > 6p^4 + 3p^2$$

$$\Rightarrow 3p > 2p^2 + 1$$

$$\Rightarrow 2p^2 - 3p + 1 < 0$$

$$\Rightarrow 2p^2 - 2p - p + 1 < 0$$

$$\Rightarrow (2p-1)(p-1) < 0$$

$$\Rightarrow p = 1, \tfrac{1}{2}$$

Check for $p = \frac{1}{4} \Rightarrow \left(\frac{1}{2} - 1\right)\left(\frac{1}{4} - 1\right) > 0$

for $p = \frac{3}{4} \Rightarrow \left(\frac{3}{2} - 1\right)\left(\frac{3}{4} - 1\right) = \frac{1}{2} \cdot \left(-\frac{1}{4}\right) < 0 \quad \therefore \frac{1}{2} < p < 1$

Q.4:- $n = 100,000$, $p \to 0$ follow distribution

$P(\text{one com. being defective}) = \frac{2}{100,000}$

$P(\text{mission will be in danger}) = P(x \geq 5)$

$= 1 - P(x < 5)$

$p = 2 \times 10^{-5}$

$\because \lambda = np = 2 \times 10^{-5} \times 10^{5} = 2$

i.e. $\boxed{\lambda = 2}$

$P(x \geq 5) = 1 - P(x<5)$

$= 1 - \sum_{x=0}^{4} \frac{e^{-2} \cdot 2^{x}}{x!}$

Q.5:- In 100 tape recorders, 25 are defective & 75 are non defective.

$N = 100$

$r = 25$, $N - r = 75$, $n = 10$, $x = 2$

i.e. ${}^{10}C_2\, p^2 q^8 = 0.28157$

Q.6:- Average no. of error per page $= \frac{390}{520} = \frac{3}{4} = \lambda$

prob. of x error per page

$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$

$= \frac{e^{-3/4} \left(\frac{3}{4}\right)^{x}}{x!}$, $x = 0, 1, 2, \ldots$

$P(X=0) = \frac{e^{-0.75} (0.75)^{0}}{0!} = e^{-0.75}$

Prob. of no error on a page = $P(X=0) = e^{-0.75}$

Prob. of no error on 5 page = $(e^{-0.75})^5 = e^{-3.75}$

Sol^n 7 :- P(hitting the target) = $0.05 = \frac{1}{20}$

P(his 10^{th} throw is his 5^{th} hit)

= P(getting 4 hits in 9 throws).(hit in 10^{th} throw)

$= {}^9C_4 \left(\frac{1}{20}\right)^4 \cdot \left(\frac{19}{20}\right)^5 \times \frac{1}{20}$

$= 126 \times \frac{(19)^{15}}{(20)^{10}}$

P(getting x hits in 9 throws)

$= \binom{9}{x}\left(\frac{1}{20}\right)^x \left(\frac{19}{20}\right)^{9-x}$

Sol^n 8 :- p(positive reaction) = $0.4 = \frac{2}{5} = p$

p(negative reaction) = $\frac{3}{5} = q$

$P(x<5)$, $r=1$

$= pq^0 + qp + q^2p + q^3p + q^4p$

$= \frac{2}{5} + \frac{3}{5}\left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)^2 \cdot \frac{2}{5} + \left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)$

$= \frac{2}{5} + \frac{6}{25} + \frac{18}{125} + \frac{54}{625} + \frac{162}{3125}$

$= 0.4 + 0.24 + 0.144 + 0.0864 + 0.05184$

$= 0.92224$

Sol^n 9 :- $P(X=x)$ = Getting x failure until first success is achived in $(x+1)$ trials

$P(X=0) = q^0 p = p$ (1 trial)

$P(X=2) = q^2 p$ (3 trial)

$P(X=2n) = q^{2n} p$ in $2n+1$ trials

Req. Prob. is $\Sigma' pq^x$

$= p[q^0 + q^2 + q^4 + \dots]$

$= \dfrac{p}{1-q^2}$

$= \dfrac{p}{(1-q)(1+q)}$ $\because p+q=1$

$= \dfrac{1-q}{(1-q)(1+q)}$

Now $0.6 = \dfrac{1}{1+q} \Rightarrow \dfrac{3}{5} = \dfrac{1}{1+q}$

$\boxed{q = \frac{1}{3}}$

$\underline{\text{Sol}^n \text{ (10)}}$:- $\lambda = 2$

$$P(X = 1,2 / X \geq 1) = \frac{P(X = 1,2 \cap X \geq 1)}{P(X \geq 1)}$$

$$P(Y=1) + P(Y=2) = \frac{P(X = 1,2)}{P(X \geq 1)}$$

$$= \frac{P(X=1) + P(X=2)}{1 - P(X=0)}$$

$$= \frac{\frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!}}{1 - \frac{e^{-\lambda}\lambda^0}{0!}}$$

$$= \frac{e^{-2}\left(2 + \frac{4}{2}\right)}{1 - e^{-2}}$$

$$= \frac{4e^{-2}}{1-e^{-2}}$$

$$= 0.626$$

$$E(Y) = P(Y=1) + 2P(Y=2) + 3P(Y=3) + \cdots$$

$$= \frac{P(X=1)}{P(X \geq 1)} + 2\frac{P(X=2)}{P(X \geq 1)} + 3\frac{P(X=3)}{P(X \geq 1)} + \cdots$$

$$= \frac{E(X)}{P(X \geq 1)}$$

$$= \frac{2}{1-e^{-2}}$$

$$\simeq 2.3$$

Soln (11):- $P(\text{defective}) = \frac{5}{100}$

$P(\text{non defective}) = \frac{95}{100}$

Requried Probability

$$= 1 - P(X=3) - P(X=2)$$

$$= 1 - \left(\frac{5}{100}\right)^3 \left(\frac{95}{100}\right)^0 - {}^3C_2 \left(\frac{5}{100}\right)^2 \left(\frac{95}{100}\right)$$

Soln (12):- (a) pmf of X. $p_X(x) = \binom{x-1}{9} \left(\frac{4}{5}\right)^{10} \left(\frac{1}{5}\right)^{x-10}$

$x = 10, 11, 12, \ldots$

$$= \frac{(x-1)!}{9!\,(x-10)!} \quad \frac{4^{10}}{5^x}$$

$$= \binom{y+r-1}{r-1} p^r q^y \quad , \; y = 0,1,2$$

(b) Mean $= 10 \left(\frac{1}{0.8}\right) = 12.5$ $\left(\text{from } \frac{r}{p}\right)$

variance $= \frac{10\,(0.2)}{(0.8)^2} = 3.125$ $\left(\frac{r(1-p)}{p^2}\right)$

(c) $p(X=12) = \binom{11}{9} \left(\frac{4}{5}\right)^{10} \left(\frac{1}{5}\right)^2$

$$= 0.2362$$

$\underline{Sol^n}$ ①:- $X \to$ for man (arrival time)

$Y \to$ arrival time for woman

$P[|X-Y|] < \frac{1}{4} \left(\frac{15}{60}\right)$

$\because$ $X \sim U(0,1)$, $Y \sim U(0,1)$

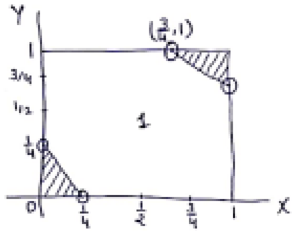
pdf of X. $g_X(x) = \begin{cases} \frac{1}{b-a} = 1 & \text{if } 0<x<1 \\ 0 & \text{, otherwise} \end{cases}$

$g_Y(y) = \begin{cases} 1 & \text{if } 0<y<1 \\ 0 & \text{, otherwise} \end{cases}$

Joint pdf is

$g_{X,Y}(x,y) = g_X(x)\, g_Y(y)$ $\because$ both are independent

$= \begin{cases} 1 & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{, otherwise} \end{cases}$



Plotting Process:-

$X-Y = \frac{1}{4} \Rightarrow Y-X = \frac{1}{4}$

$X=0, Y=\frac{1}{4}$ + $Y=0, X=\frac{1}{4}$

$X=1, Y=\frac{3}{4}$ + $Y=1, X=\frac{3}{4}$

$P[|X-Y| < \frac{1}{4}] = \frac{\text{Favorable area}}{\text{Total area}}$

$= 1 - \frac{1}{2} \times \frac{3}{4} \times \frac{3}{4} - \frac{1}{2} \times \frac{3}{4} \times \frac{3}{4}$

$= 1 - \frac{9}{16}$

$= \frac{7}{16}$

Sol^n ②:- $a \sim U(1,7)$

$$x^2 + 2ax + (2a+3) = 0$$

$$x = \frac{-2a \pm \sqrt{4a^2 - 4(2a+3)}}{2}$$

$$\Rightarrow x = \frac{2(-a \pm \sqrt{a^2 - 2a + 3})}{2}$$

$$\Rightarrow x = -a \pm \sqrt{a^2 - 2a + 3}$$

It will have real roots if $a^2 - 2a + 3 \geqslant 0$

i.e. $a^2 + a - 3a - 3 \geqslant 0$

$a(a+1) - 3(a+1) \geqslant 0$

$(a-3)(a+1) \geqslant 0$

i.e. $a - 3 \geqslant 0$, $a + 1 \geqslant 0$

$a \geqslant 3$ $a \geqslant -1$ (not satisfied)

$$P(1 \leq a \leq 7) = \int_1^7 \frac{1}{b-a}\, da$$

$$= \frac{1}{6} \int_1^7 da$$

$$= \frac{1}{6} \times 6$$

$$= 1$$

$$P(a > 3) = \int_3^\infty g(a)\, da$$

$$= \int_3^7 \frac{1}{6}\, da$$

$$= \frac{1}{6}[7-3]$$

$$= \frac{2}{3}$$

Sol^n ③ :- First cut position = X

2^{nd} Cut position = Y   0 ———— 4

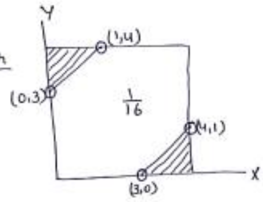
$X \sim U(0,4)$, $Y \sim U(0,4)$

$$g_X(x) = \begin{cases} \frac{1}{b-a} = \frac{1}{4} & \text{if } 0<X<4 \\ 0 & \text{, otherwise} \end{cases}$$

$$g_Y(y) = \begin{cases} \frac{1}{4} & \text{if } 0<Y<4 \\ 0 & \text{otherwise} \end{cases}$$

$$g_{X,Y}(x,y) = \begin{cases} \frac{1}{16} & \text{if } 0<X<4,\ 0<Y<4 \\ 0 & \text{otherwise} \end{cases}$$ $\because$ both are Ind.

$$P(|X-Y|>3) = \frac{\text{area of shaded region}}{\text{total area}}$$



$$= \frac{\frac{1}{2}\times 1 + \frac{1}{2}\times 1}{4\times 4}$$

$$= \frac{1}{16} \quad \text{Ans}$$

Sol^n ④ i:-

Let r.v X denotes the daily consumption of milk then r.v $Y = X - 20,000$ follows exponential distribution with pdf

$$g(y) = \lambda e^{-\lambda y} \quad , \; y \geq 0$$

Daily stock is 35,000. So stock will be insufficient for a day if

$$P(X > 35000) = P(X - 20,000 > 15000)$$

$$= P(Y > 15000)$$

$$= \int_{15000}^{\infty} \lambda e^{-\lambda y} \, dy$$

$$= \frac{1}{3000} \left[\frac{e^{-1/3000 y}}{-1/3000} \right]_{15000}^{\infty}$$

$$= \left[-e^{-y/3000} \right]_{15000}^{\infty}$$

$$= 0 + e^{-15000/3000}$$

$$= e^{-5} \quad \text{Ans}$$

Sol^n (5) :-

X be r.v denoting length of shower in min.

$\lambda = 2$

$f(x) = 2e^{-2x}, \quad x \geq 0$

$$P(X>3) = 2\int_3^{\infty} e^{-2x}\,dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_3^{\infty}$$

$$= \left[-e^{-2x}\right]_3^{\infty}$$

$$= 0 + e^{-6}$$

$$= 0.00248$$

$$P(X \geq 3 / X \geq 2) = \frac{P(X \geq 3 \cap X \geq 2)}{P(X \geq 2)}$$

$$= \frac{P(X \geq 3)}{P(X \geq 2)}$$

$$= \frac{e^{-6}}{e^{-4}}$$

$$= 0.1353 \quad \text{An}$$

Sol^n ⑥ :-

$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

$E(X) = \frac{1}{\lambda}$

$P(X \geq \frac{2}{\lambda}) = \lambda \int_{2\lambda}^{\infty} e^{-\lambda x} \, dx$

$= \left[\frac{\lambda}{-\lambda} e^{-\lambda x} \right]_{2/\lambda}^{\infty}$

$= \left[-e^{-\lambda x} \right]_{2/\lambda}^{\infty}$

$= 0 + e^{-\lambda (2/\lambda)}$

$= e^{-2}$

$= \frac{1}{e^2}$ Ans