1 Simplified variance for discrete random variables

Reiterating the equation which has to be proven

$$Var(X) = Cov(X,X) = E\left[(X - E[X])^2\right] = E[X^2] - (E[X])^2 \tag{\textbf{??} revisited}$$

Starting from the left side of **??** revisited, we get

$$Var(X) = Cov(X,X) = E\left[(X - E[X])^2\right] \tag{1}$$

The expectation for a function of a discrete random variable X (whose possible values are given by R_X), is given by

$$E_{x\sim X}\left[f(X)\right] = \sum_{x\in R_X} f(x)P(X = x) \tag{2}$$

Simplifying the equation 1 using the following known conditions

1. The value of $E[X]$ is the mean of the discrete random variable given by

$$E[X] = \sum_{x\in R_X} xP_X(x) = \mu \tag{3}$$

2. μ is a constant for a given probability distribution P_X
3. $\sum_{x\in R_X} P_X(x) = 1$ because P_X is a probability mass function

we get

$$\begin{aligned} E\left[(X - E[X])^2\right] &= E\left[(X-\mu)^2\right] = \sum_{x\in R_X} (x-\mu)^2 P_X(x) = \sum_{x\in R_X} \left(x^2 + \mu^2 - 2x\mu\right) P_X(x) \\ &= \sum_{x\in R_X} x^2 P_X(x) + \mu^2 \sum_{x\in R_X} P_X(x) - 2\mu \sum_{x\in R_X} xP_X(x) \\ &= E_{x\sim X}[X^2] + \mu^2 \cdot 1 - 2\mu\cdot\mu = E[X^2] + \mu^2 - 2\mu^2 = E[X^2] - \mu^2 \\ &= E[X^2] - (E[X])^2 \end{aligned} \tag{4}$$

The equation 4 proves **??** revisited, we therefore obtain

$$Var(X) = E[X^2] - (E[X])^2 \tag{5}$$