CS7.403 - SMAI: Assignment 2

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1 Eigen Values and Eigen Vectors

Eigen Value Decomposition

A matrix $\mathbf{A}$ can be decomposed into

$$\mathbf{A} = \mathbf{Q}\,\mathbf{\Lambda}\,\mathbf{Q}^{-1} \tag{1}$$

Where the matrix $\mathbf{Q}$ is formed by horizontally stacking eigenvectors of $\mathbf{A}$ as columns and the matrix $\mathbf{\Lambda} = \text{diag}(\lambda)$ where λ is the vector containing the corresponding eigen-values. The relation between $\mathbf{A}$, an eigenvector and corresponding eigen-value is given by

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

Singular Value Decomposition

A matrix $\mathbf{M}$ can be decomposed as

$$\mathbf{M} = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^* \tag{2}$$

Where $\mathbf{\Sigma}$ is a diagonal matrix consisting of singular values (usually in descending order). The columns of $\mathbf{U}$ are formed by left-singular vectors of $\mathbf{M}$, which are the eigenvectors of $\mathbf{M}\,\mathbf{M}^\top$. The columns of $\mathbf{V}$ are formed by right-singular vectors of $\mathbf{M}$, which are the eigenvectors of $\mathbf{M}^\top\,\mathbf{M}$. The matrix $\mathbf{V}^*$ is the conjugate transpose of $\mathbf{V}$.

If $\mathbf{M}$ is real, $\mathbf{U}$ and $\mathbf{V}$ can be guaranteed to be orthogonal matrices and the decomposition can be written as $\mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^\top$.

1.1 A: Generalized to Matrices

Eigenvector decomposition, given by equation 1 can only be done for **square** matrices.
Singular Value Decomposition on the other hand, given by equation 2, can be done for matrices that are not square also.

Hence, *Singular Value Decomposition is more generalizable* to matrices as it can be applied to matrices of any shape (square or not square).

1.2 B: Find SVD of a matrix

Usually, numerical approaches are used to calculate the SVD. This is typically a two-step procedure.

In the first step, the matrix is reduced to a bidiagonal matrix (where the elements in the diagonal and either the diagonal above or the diagonal below are non-zero). This is done because calculating SVD of a bidiagonal matrix is faster.

In the second step, SVD of the resultant bidiagonal matrix is calculated. Here, the left and right eigenvectors can be calculated. This is done using a bounded iterative algorithm like QR Algorithm.

The above is a complex numeric procedure, usually abstracted and available on many platforms. When computing SVD of a given matrix, intuition based methods can be directly used (but they do not generalize or scale well).

SVD of M

The given matrix is

$$M = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}$$

We first calculate the left and right-singular vectors of M, then get the singular values. The left and right matrices are given by

$$M_L = MM^\top = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} \qquad M_R = M^\top M = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix} \tag{3}$$

Calculating the eigen-vectors of M_L. First calculate the eigen-values using $\det(M_L - \lambda I) = 0$ equation

$$\begin{aligned}
&\det(M_L - \lambda \mathrm{I}) = 0 \Rightarrow \det\left(\begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0 \\
&\Rightarrow \det\left(\begin{bmatrix} 80-\lambda & 100 & 40 \\ 100 & 170-\lambda & 140 \\ 40 & 140 & 200-\lambda \end{bmatrix}\right) = 0 \Rightarrow -\lambda^3 + 450\lambda - 32400\lambda = 0 \\
&\Rightarrow \lambda = [0, 90, 360]
\end{aligned}$$

Solving for the eigen-vectors using the following equation

$$\begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 80x + 100y + 40z \\ 100x + 170y + 140z \\ 40x + 140y + 200z \end{bmatrix} = \lambda_i \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4}$$

For different λ values, we get

$$\begin{aligned}
\lambda_1 = 0 \rightarrow \begin{bmatrix} 80x + 100y + 40z \\ 100x + 170y + 140z \\ 40x + 140y + 200z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2z \\ -2z \\ z \end{bmatrix} \\
\lambda_2 = 90 \rightarrow \begin{bmatrix} 80x + 100y + 40z \\ 100x + 170y + 140z \\ 40x + 140y + 200z \end{bmatrix} &= \begin{bmatrix} 90x \\ 90y \\ 90z \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -0.5z \\ z \end{bmatrix} \\
\lambda_3 = 360 \rightarrow \begin{bmatrix} 80x + 100y + 40z \\ 100x + 170y + 140z \\ 40x + 140y + 200z \end{bmatrix} &= \begin{bmatrix} 360x \\ 360y \\ 360z \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.5z \\ z \\ z \end{bmatrix}
\end{aligned}$$

This gives the potential candidates for U. For V, we get the eigen-vectors of M_R. First calculating eigen-values using

$$\begin{aligned}
&\det(M_R - \lambda \mathrm{I}) = 0 \Rightarrow \det\left(\begin{bmatrix} 333-\lambda & 81 \\ 81 & 117-\lambda \end{bmatrix}\right) = \lambda^2 - 450\lambda + 32400 = 0 \\
&\Rightarrow \lambda = [90, 360]
\end{aligned}$$

Solving for eigen-vectors using the following equation

$$\begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 333x + 81y \\ 81x + 117y \end{bmatrix} = \lambda_i \begin{bmatrix} x \\ y \end{bmatrix} \tag{5}$$

For different λ values, we get

$$\begin{aligned}
\lambda_1 = 90 \rightarrow \begin{bmatrix} 333x + 81y \\ 81x + 117y \end{bmatrix} &= \begin{bmatrix} 90x \\ 90y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y/3 \\ y \end{bmatrix} \\
\lambda_2 = 360 \rightarrow \begin{bmatrix} 333x + 81y \\ 81x + 117y \end{bmatrix} &= \begin{bmatrix} 360x \\ 360y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3y \\ y \end{bmatrix}
\end{aligned}$$

All the above eigen-vectors can be assumed to be unit vectors (so that the matrices U and V become orthogonal). The resultant matrix is given by

$$\begin{aligned}
M = U\Sigma V^\top \Rightarrow \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} &= \begin{bmatrix} 0.5z_1 & -z_2 & 2z_3 \\ z_1 & -0.5z_2 & -2z_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3y_1 & -y_2/3 \\ y_1 & y_2 \end{bmatrix}^\top \\
\Rightarrow M = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} &= \begin{bmatrix} 1.5\sigma_1 y_1 z_1 + \frac{\sigma_2 y_2 z_2}{3} & 0.5\sigma_1 y_1 z_1 - \sigma_2 y_2 z_2 \\ 3\sigma_1 y_1 z_1 + \frac{5\sigma_2 y_2 z_2}{30} & \sigma_1 y_1 z_1 - 0.5\sigma_2 y_2 z_2 \\ 3\sigma_1 y_1 z_1 - \frac{\sigma_2 y_2 z_2}{3} & \sigma_1 y_1 z_1 + \sigma_2 y_2 z_2 \end{bmatrix}
\end{aligned}$$

Since vectors in U are unit vectors: $z_1 = \pm 2/3$, $z_2 = \pm 2/3$, $z_3 = \pm 1/3$. Since vectors in V are also unit vectors: $y_1 = \pm 1/\sqrt{10}$, $y_2 = \pm 3/\sqrt{10}$. Using only the +ve values, we get the following

$$M = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} = \begin{bmatrix} 0.316227\sigma_1 + 0.210818\sigma_2 & 0.105409\sigma_1 - 0.632455\sigma_2 \\ 0.632455\sigma_1 + 0.105409\sigma_2 & 0.210818\sigma_1 - 0.316227\sigma_2 \\ 0.632455\sigma_1 - 0.210818\sigma_2 & 0.210818\sigma_1 + 0.632455\sigma_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} 18.973665 \\ -9.486832 \end{bmatrix}$$

But since singular values **have** to be positive, we can change the sign of z_2 and thereby consider $z_2 = -2/3$. This changes the above equation as

$$M = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} = \begin{bmatrix} 0.316227\sigma_1 - 0.210818\sigma_2 & 0.105409\sigma_1 + 0.632455\sigma_2 \\ 0.632455\sigma_1 - 0.105409\sigma_2 & 0.210818\sigma_1 + 0.316227\sigma_2 \\ 0.632455\sigma_1 + 0.210818\sigma_2 & 0.210818\sigma_1 - 0.632455\sigma_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} 18.973665 \\ 9.486832 \end{bmatrix}$$

The above equations give

$$U = \begin{bmatrix} 0.\bar{3} & 0.\bar{6} & 0.\bar{6} \\ 0.\bar{6} & 0.\bar{3} & -0.\bar{6} \\ 0.\bar{6} & -0.\bar{6} & 0.\bar{3} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 18.973665 & 0 \\ 0 & 9.486832 \\ 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} 0.948683 & -0.316227 \\ 0.316227 & 0.948683 \end{bmatrix} \tag{6}$$

Hence, the singular value decomposition of M is given by

$$M = U\Sigma V^\top = \begin{bmatrix} 0.\bar{3} & 0.\bar{6} & 0.\bar{6} \\ 0.\bar{6} & 0.\bar{3} & -0.\bar{6} \\ 0.\bar{6} & -0.\bar{6} & 0.\bar{3} \end{bmatrix} \begin{bmatrix} 18.973665 & 0 \\ 0 & 9.486832 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.948683 & 0.316227 \\ -0.316227 & 0.948683 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}$$

The equation 6 gives the SVD of M.