1 PMF: Finite and Infinite range

Every **PMF** (Probability Mass Function) has to obey the following properties

$$P_X(x_k) \geq 0 \, \forall \, x_k \in R_X = \{x_1, x_2, \cdots\} \tag{1}$$

$$\sum_{x_i \in R_X} P_X(x_i) = 1 \tag{2}$$

Where R_X is the range of values that X, a discrete random variable, can take. Assumptions for this question are as follows

- If R_X has finite elements, the PMF has **finite** range, else if R_X has *countably infinite*[1] elements, the PMF has **infinite** range.

1.1 Finite Range: Bernouli Distribution

A PMF $P_X(x_k) = P(X = x_k)$, with finite range is assumed to have a finite number of elements that the discrete random variable X can take. A simple example is the *Bernouli distribution* where $R_X = \{0, 1\}$.

$$P_X(x; p) = \begin{cases} p & \text{if } x \text{ is } 1 \\ 1 - p & \text{if } x \text{ is } 0 \end{cases} \tag{3}$$

Where $0 \leq p \leq 1$, which also means that $1 \geq 1 - p \geq 0$. This proves 1: as the function P_X can only yield p or $1 - p$, both being in range $[0, 1]$. To prove 2, we can do

$$\sum_{v \in \{0,1\}} P_X(v) = P(X = 0) + P(X = 1) = (1 - p) + p = 1 \tag{4}$$

Hence, a finite range Probability Mass Function is achieved using Bernouli's distribution. Example is a coin toss (with $p = 0.5$).

1.2 Infinite Range: Exponentially decaying distribution

Consider an experiment involving multiple coin tosses. We are interested in calculating the probability of all the outcomes being `HEAD`. The discrete random variable X is the trial number with consecutive heads (for example 1 trial with 1 head, 2 trial with 2 heads, and so on). The range of values for X is the countably infinite set of *natural numbers*, that is $R_X = \mathbb{N}$. Basically, $P(X = i)$ is the probability that we do the coin toss i times and we get heads as the outcome for every trial.

It is clear that $P(X = 1) = 0.5$: probability of head in a single coin toss and $P(X = 2) = 0.5^2 = 0.25$: probability of two coin tosses giving heads. This can be extended to any number of coin tosses, giving the probability function

$$P(X = k) = P_X(k) = (0.5)^k \tag{5}$$

[1] Elements are in one-to-one correspondence with natural numbers

Since every power of 0.5 is positive, property 1 holds good. The proof of property 2 can be done by the sum of infinite geometric progression, the general form is given below

$$S = \sum_{i=0}^{n-1} a\, r^i = \frac{a}{1-r} \tag{6}$$

Consider $a = r = 0.5$, the sum of all probabilities can be calculated as following

$$\sum_{k \in R_X = \mathbb{N}} P_X(k) = \sum_{k=1}^{\infty} 0.5^k = \sum_{k=1}^{\infty} 0.5 \times 0.5^{k-1} = \frac{0.5}{1-0.5} = 1 \tag{7}$$

This proves that P_X is a probability distribution. Its value exponentially decreases (decays) to 0.