1 Variance of Uniform Density Function

The covariance of a function is defined by

$$Cov(X,Y) = E\left[(X - E(X))\,(Y - E(Y))\right] = E[XY] - E[X]E[Y] \tag{1}$$

The variance is given by

$$Var(X) = Cov(X,X) = E[X^2] - (E[X])^2 \tag{2}$$

The uniform distribution is (given that $b > a$)

$$U(x;a,b) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The expected value, $E[X]$, for the uniform distribution can be calculated as follows

$$E_U[X] = \int_{-\infty}^{\infty} xU(x)\mathrm{d}x = \frac{1}{b-a}\int_a^b x\mathrm{d}x = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b+a}{2} \tag{4}$$

The value of $(E[X^2])$ can be calculated as follows

$$E[X^2] = \int_{-\infty}^{+\infty} x^2U(x)\mathrm{d}x = \frac{1}{b-a}\int_a^b x^2\mathrm{d}x = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3} \tag{5}$$

Using equations 4 and 5 to calculate variance according to 2 gives

$$\begin{aligned} \mathrm{Var}_U(X) &= E[X^2] - (E[X])^2 = \left(\tfrac{a^2+ab+b^2}{3}\right) - \left(\tfrac{(b+a)^2}{4}\right) \\ \Rightarrow \mathrm{Var}(X) &= \tfrac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12} = \tfrac{a^2-2ab+b^2}{12} = \tfrac{(b-a)^2}{12} \end{aligned} \tag{6}$$

Therefore, equation 6 proves that

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12} \tag{7}$$