1 Mean and Variance of continuous normal PDF

The normal probability density function for a continuous random variable X is given by

$$N(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{?? revisited}$$

Note that the expectation of a function of a continuous random variable is given by

$$E_{x\sim X}[f(X)] = \int_x f(x)p(x)\,\mathrm{d}x \tag{1}$$

Where $p(x)$ is the probability density function of X (in this case, it's described by $N(x;\mu,\sigma^2)$ in **??** revisited). We shall also use a commonly known identity

$$\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi} \tag{2}$$

This is proven as **??** in Appendix **??**. Another known identity we will use is

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2} \tag{3}$$

This is proven in Appendix **??**. We can proceed to prove the mean of $N(x;\mu,\sigma^2)$ first.

1.1 Proving Mean of Normal PDF

The mean value of a PDF is given as $E(X)$, for the normal PDF, we get

$$\begin{aligned} E_{X\sim N}[X] &= \int_{-\infty}^{\infty} x\,N(x;\mu,\sigma^2)\mathrm{d}x = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}}\, e^{-\left(\frac{(x-\mu)}{\sigma\sqrt{2}}\right)^2}\mathrm{d}x \end{aligned} \tag{4}$$

We can solve 4 by substituting a function of x for v (use the equations below)

$$\begin{aligned} \frac{x-\mu}{\sigma\sqrt{2}} = v \Rightarrow x &= \left(\sigma\sqrt{2}\right)v + \mu \\ \Rightarrow \mathrm{d}x &= \sigma\sqrt{2}\,\mathrm{d}v \\ x \to (-\infty,\infty) &\Rightarrow v \to (-\infty,\infty) \end{aligned} \tag{5}$$

Substituting this in 4, we get

$$\begin{aligned} E_{X\sim N}[X] &= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}}\, e^{-\left(\frac{(x-\mu)}{\sigma\sqrt{2}}\right)^2}\mathrm{d}x = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sigma\sqrt{2}\,v + \mu\right) e^{-v^2}\sigma\sqrt{2}\,\mathrm{d}v \\ &= \frac{1}{\sqrt{\pi}}\left[\sigma\sqrt{2}\int_{-\infty}^{\infty} v e^{-v^2}\mathrm{d}v + \mu\int_{-\infty}^{\infty} e^{-v^2}\mathrm{d}v\right] = \frac{\mu}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-v^2}\mathrm{d}v \\ &= \frac{\mu}{\sqrt{\pi}}\sqrt{\pi} = \mu \end{aligned} \tag{6}$$

Note that $\int_{-\infty}^{\infty} ve^{-v^2}\mathrm{d}v = 0$ as $f_1(v) = ve^{-v^2}$ is an odd function ($f_1(-v) = -f_1(v)$), so the integral over $(-\infty, \infty)$ is 0 (all values cancel out). The equation 6 proves that indeed for a normal distribution, the mean is μ, that is

$$E_{X\sim N}[X] = \mu \tag{7}$$

We can now prove the variance for a normal distribution

1.2 Proving Variance of Normal PDF

The variance of a Normal Probability Density Function for a continuous random variable is given by (consider $E[X] = \mu$ as proven in subsection 1.1)

$$Var(X) = E\left[(X - E[X])^2\right] = E_N\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 N(x;\mu,\sigma^2)dx \tag{8}$$

To solve this, we make the same assumption as in subsection 1.1.

$$\begin{aligned} \frac{x-\mu}{\sigma\sqrt{2}} = v \Rightarrow x = \left(\sigma\sqrt{2}\right)v + \mu \Rightarrow (x-\mu)^2 = 2\sigma^2 v^2 \\ \rightarrow \mathrm{d}x = \sigma\sqrt{2}\,\mathrm{d}v \\ x \rightarrow (-\infty,\infty) \Rightarrow v \rightarrow (-\infty,\infty) \end{aligned} \tag{9}$$

Now, solving for variance becomes

$$\begin{aligned} Var(X) &= \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{(x-\mu)}{\sigma\sqrt{2}}\right)^2} \mathrm{d}x = \int_{-\infty}^{\infty} \frac{2\sigma^2 v^2}{\sqrt{2\pi\sigma^2}} e^{-v^2} \sigma\sqrt{2}\,\mathrm{d}v \\ &= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} v^2 e^{-v^2}\,\mathrm{d}v = \frac{2\sigma^2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = \sigma^2 \end{aligned} \tag{10}$$

The result $\int_{-\infty}^{\infty} v^2 e^{-v^2}\,\mathrm{d}v$ is proved in Appendix **??**. The above result proves that for a normal distribution, the variance is σ^2, that is

$$Var(X) = E_N\left[(X - E_N[X])^2\right] = \sigma^2 \tag{11}$$