# Propagation of Gaussian beams

## A comprehensive introduction

**The knowledge of beam propagation is very useful in the application of lasers to material processing and metrology. This paper is a short practical introduction to Gaussian beam propagation in optical systems.**

THE AUTHOR

ALEXANDER HORNBERG

Alexander Hornberg was born in 1961 in Düsseldorf in Germany. He studied physics and mathematics at the University of Karlsruhe. After his industrial job he has been professor of optics engineering and machine vision at University of Applied Sciences of Esslingen since 1992.

Thanks of Author to Bernd Beuter, Rainer Madejsky and Rolf Martin.

Prof. Dr. Alexander Hornberg,
University of Applied Sciences,
Fachhochschule Esslingen,
Robert-Bosch-Strasse 1, 73037 Göppingen
phone: +49 (0) 7161/679-1116
fax: +49 (0) 7161/679-2177
e-mail: alexander.hornberg@fht-esslingen.de

### Matrix approach to paraxial ray tracing and Gaussian optics

In geometrical optics or ray optics, light is assumed to consist of rays that propagate according to FERMAT´s principle. An optical imaging system consists of a series of refracting and/or reflecting surfaces that generally have a common axis of rotational symmetry, called the optical axis. The surface sends the light rays from an object according to the laws of refraction and reflection, thus forming an image.

In paraxial approximation, the angle formed by a ray proceeding from a point object and the optical axis (or the surface normal to the rays) is treated as a being so small that its sine or tangent is the same as the angle itself. The sign of the described angle is positive if it is in a counterclockwise direction. Additionally, in paraxial approximation, the refraction and reflection both occur in a plane that is tangent to the surface of the optical system, and which passes through the vertex which is at the point of intersection of the optical axis and the refracting surface.

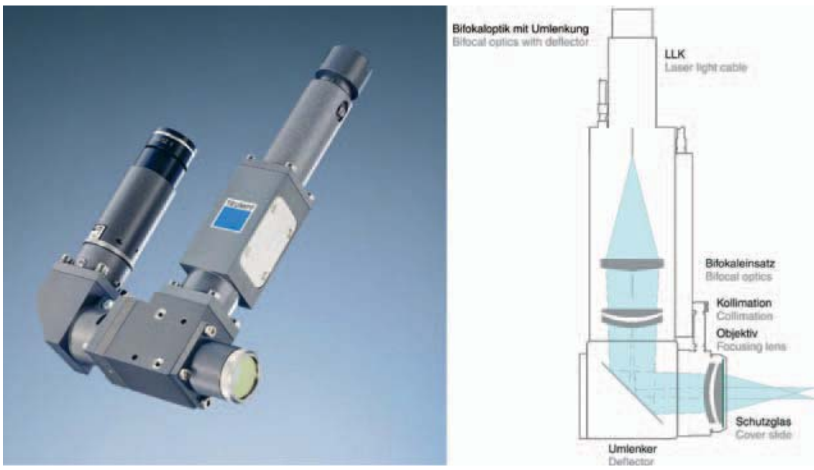
In paraxial approximation, an arbitrary optical system will be bounded by the entrance plane $\mathcal{E}$ and the exit plane $\mathcal{E}'$, as depicted in Fig. 1. The incident and the exit ray will be described by the height $y$, $y'$, and the slope angle $\sigma$, $\sigma'$, related to the optical axis in the entrance plane and exit plane, respectively. All **points** in the object and image space refer to the Cartesian coordinate system at the vertex of the entrance plane $E$, and exit plane $E'$, respectively.

According to the definition given above, the propagation of a ray through a general optical system is determined by the ray propagation matrix (system matrix, ABCD-matrix) and the paraxial ray tracing law

$$\begin{bmatrix} y' \\ n'\sigma' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y \\ n\sigma \end{bmatrix}, \qquad \Delta = AD - BC = 1 \tag{1}$$

The coefficients depend on the particular optical system. In table 1 some important systems are listed [6].

Knowing the coefficients $A$, $B$, $C$, $D$ of the propagation matrix allows one to calculate the cardinal points and cardinal planes (focal points $F$, $F'$ principal points $P$, $P'$, nodal points $N$, $N'$) in the object space and the image space with reference to the coordinates of the object and image space, respectively.




**The left photo and right sketch shows a laser tool for a solid-state laser, the sketch is drawn without the eyepiece. This laser tool is used by twin-spot welding of aluminum. (Source: TRUMPF Group)**

optical power:
$K = -C$ , (refracting, reflecting)

focal lengths:

$$f' = -\frac{n'}{C} = n'K \;,\; f = \frac{n}{C} = -nK$$

focal points:

$$z'_{F'} = -n'\frac{A}{C} = Af' \;,\; z_F = n\frac{D}{C} = -Df' \tag{2}$$

principal points:

$$z'_{P'} = n'\frac{A-1}{C} = z'_{F'} - f' \;,\; z_P = n\frac{D-1}{C} = z_F - f$$

nodal points:

$$z'_{N'} = -n'\frac{A-\frac{n}{n'}}{C} = z'_{F'} + f \;,\; z_N = n\frac{D-\frac{n}{n'}}{C} = z_F + f'$$

having chosen a source point $Q = (x,y,z)$ in the object plane, the image point (or conjugate point of the object) $Q' = (x',y',z')$ can be calculated by using the Gaussian imaging equations

$$x' = -\frac{x}{C\frac{z}{n} - D} \;,\quad y' = -\frac{y}{C\frac{z}{n} - D} \;,\quad \frac{z'}{n'} = -\frac{A\frac{z}{n} - B}{C\frac{z}{n} - D}$$

$$x = \frac{x'}{C\frac{z'}{n'} + A} \;,\; y = \frac{y'}{C\frac{z'}{n'} + A} \;,\; \frac{z}{n} = \frac{D\frac{z'}{n'} + B}{C\frac{z'}{n'} + A} \tag{3}$$

In a later section, we will require the optical path distance $L(P,P')$ between two arbitrary points $P = (x,y,0)$ and $P' = (x',y',0)$ in the entrance and exit plane. Using FERMAT´s principle we find [7].

$$L(P,P') = L(E,E') + \frac{A(x^2+y^2) - 2(xx'+yy') + D(x'^2+y'^2)}{2B} \tag{4}$$

A detailed description of this matrix concept can be found in some modern books on applied optics [3, 5].

## Paraxial wave equation and Gaussian beams

*Paraxial wave equation*

The paraxial wave equation is, in fact, complete enough to describe almost any laser propagation problems which are of practical interest either in or outside of lasers. To have an undertanding of paraxial waves, we begin with an approximation of the spherical wave. If we assume that $\sqrt{x^2+y^2}$ and the free-space wavelength $\lambda_0$ ($k=nk_0$, $k_0=2\pi/\lambda_0$) are small in comparison with $z>0$, we get the *paraxial wave approximation* of the spherical wave

$$\psi(\vec{r},t) = \frac{a_0}{r}\mathrm{e}^{\mathrm{i}(kr-\omega t)} \cong \frac{a_0}{z}\exp\left[\mathrm{i}k\frac{x^2+y^2}{2z}\right]\mathrm{e}^{\mathrm{i}(kz-\omega t)} \tag{5}$$
$$= m(x,y,z)\mathrm{e}^{\mathrm{i}(kz-\omega t)},$$

where $k = nk_0 = n2\pi/\lambda_0$ is the wave number. The plane wave factor $\mathrm{e}^{\mathrm{i}(kz-\omega t)}$ varies very quickly in the $z$-direction in comparison with the first factor $m$ in . Using the partial derivative in the $z$-direction we have the condition

$$\left|\frac{1}{\mathrm{i}k}\frac{\partial}{\partial z}m\right| = |m|\left|\frac{1}{\mathrm{i}kz} + \frac{x^2+y^2}{2z^2}\right| \ll |m| \tag{6}$$

which characterizes this fact. This condition is usually called the *slowly varying envelope approximation*. Therefore, a paraxial wave consists of a plane monochromatic *carrier wave* $\mathrm{e}^{\mathrm{i}(kz-\omega t)}$ with a transverse, modulated amplitude $m$ in the sense of (6).
Substituting an arbitrary wave of the form $\psi(x,y,z,t) = m(x,y,z)\,\mathrm{e}^{\mathrm{i}(kz-\omega t)}$ into the wave equation

$$\partial_t^2\psi(\vec{r},t) - c^2\Delta\psi(\vec{r},t) = 0 \tag{7}$$

yields

$$\Delta m + 2\mathrm{i}k\frac{\partial m}{\partial z} = 0 \tag{8}$$

The condition (6) in the form

$$\left|\frac{1}{\mathrm{i}k}\frac{\partial^2 m}{\partial z^2}\right| << \left|\frac{\partial m}{\partial z}\right|$$

allows one to neglect the term $\partial^2/\partial z^2\, m$ in (7) so we obtain the *paraxial time-independent wave equation*

$$\frac{\partial^2 m}{\partial x^2} + \frac{\partial^2 m}{\partial y^2} + 2\mathrm{i}k\frac{\partial m}{\partial z} = 0 \tag{9}$$

*Gaussian beams*

Lasers can be excited and can radiate in different transverse fields, $m(x,y,z)$, called *modes*. The ground mode $TEM_{00}$ is the simplest and most important mode in most applications. Because the intensity profile of rotational symmetry, perpendicular to the propagation direction, is similar to the Gaussian distribution, the ground mode is also called *Gaussian Beam*.
Substituting a paraxial wave of the form

$$m(x,y,z) = \mathrm{e}^{\mathrm{i}P(z)}\mathrm{e}^{\mathrm{i}Q(z)(x^2+y^2)} \tag{10}$$

where $P(z)$ and $Q(z)$ are arbitrary functions, into (9), yields the Gaussian beam solution [4]

THE INSTITUTE

**FHTE**
**University of Applied Sciences,**
Esslingen, Germany

The University of Applied Sciences in Esslingen (short form FHTE) is one of the oldest engineering schools in Germany with a tradition of more then one hundred years. Regular high standings in different university rankings document the excellent reputation and quality of the scientific training at the FHTE. More information: www.fht-esslingen.de

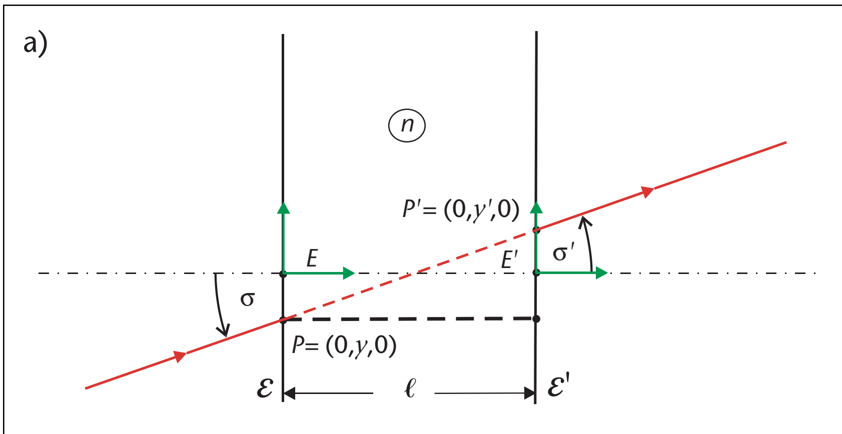
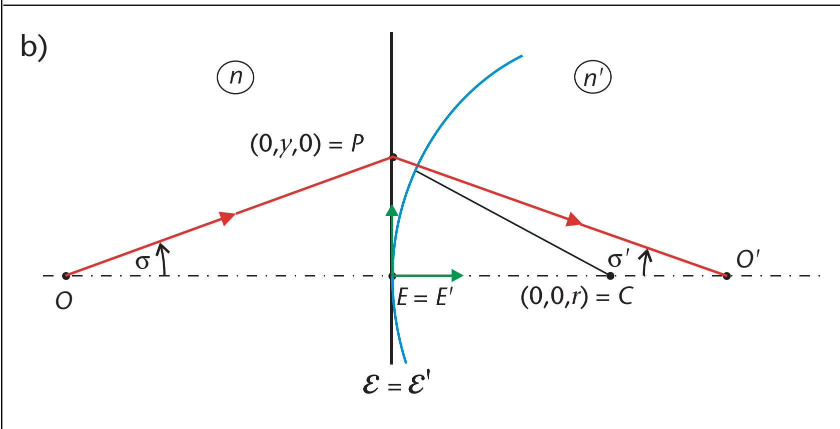
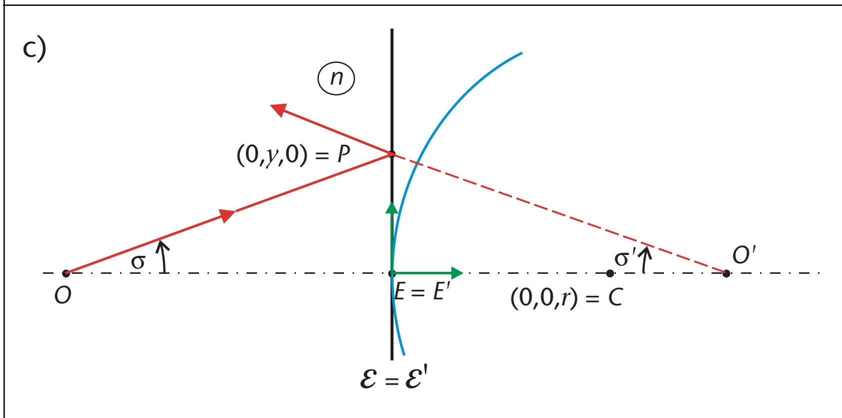
| | |
|---|---|
|   | $T = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ <br> $t = \frac{\ell}{n}$ |
|   | $R = \begin{bmatrix} 1 & 0 \\ -K & 1 \end{bmatrix}$ <br> $K = \frac{n' - n}{r}$ |
|   | $R = \begin{bmatrix} 1 & 0 \\ -K & -1 \end{bmatrix}$ <br> $K = \frac{2}{r}$ |

**TABELLE 1:** **Optical system matrices of a) free space propagation, b) refracting and c) reflecting surface, *r* radius of curvature in the vertex of the surface, *K* optical power of the system.**

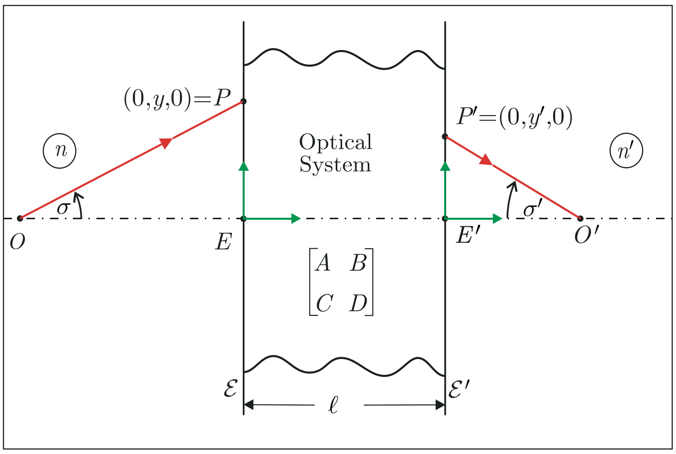



FIGURE 1: **An optical system in paraxial approximation will be bounded by an entrance and exit plane. The ABCD Matrix describes the ray propagation between the boundaries in the optical system. The object space has the refraction index *n,* and the image space *n′*, respectively.**

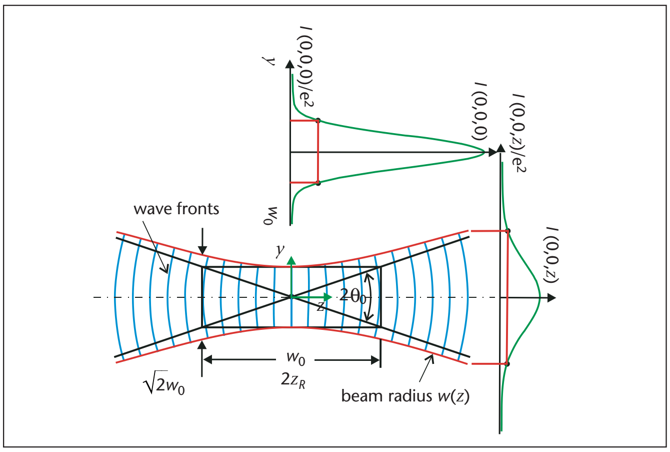



FIGURE 2. **Properties of Gaussian Beam: The red lines mark the shape of the beam, the blue lines the wave fronts with the distance of one wave length. The two plots show the beam profile at the waist and an arbitrary position, which can be measured by a line detector.**

$$\psi(x,y,z,t) = a(x,y,z)\mathrm{e}^{\mathrm{i}(k_0 S(x,y,z)-\omega t)} = m(x,y,z)\mathrm{e}^{\mathrm{i}(k_0 nz-\omega t)}$$

with the parameters:
RAYLEIGH-length $z_R > 0$, radius of curvature

$$R(z) = z + \frac{z_R^2}{z} \tag{11}$$

in the vertex of the wave front

$$S(x,y,z) = \frac{x^2+y^2}{2R(z)} + z \tag{12}$$

the beam waist $w_0 = \sqrt{2z_R/k}$ and the beam radius

$$w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2} \tag{13}$$

the amplitude

$$a(x,y,z) = a_0\frac{w_0}{w(z)}\exp\left[-\frac{x^2+y^2}{w^2(z)}\right]\mathrm{e}^{-\mathrm{i}\arctan(z/z_R)} \quad . \tag{14}$$

Keeping in mind that, in the limit $z_R \to 0$, we obtain the paraxial spherical wave. The far-field pattern is described by the divergence angle $\tan\theta_0 = w_0/z_R$, which is relatively small.
You will find more details in the book [6].

## Beam propagation

*Wave propagation in the paraxial approximation*

We know that a beam is a paraxial wave. Thus we must describe the propagation using the model of paraxial wave propagation. Using the stationary phase approximation and the angle spectrum, Walther [8] derived the solution of wave propagation in a very general way and found a fundamental connection between the wave propagation kernel and the eikonals of ray optics. Therefore, we can use the system matrix of ray optics to calculate the wave propagation in a paraxial approximation.
We now proceed in a more elementary way and consider the propagation of waves in an optical system between the entrance plane $\mathcal{E} = \{(x,y,z) \in \mathbb{R}^3 \mid z = z_0\}$ and the exit plane $\mathcal{E}' = \{(x,y,z) \in \mathbb{R}^3 \mid z = z_0 + \ell\}$ (see Fig. 1). This linear propagation problem must have a solution of the form

$$u(x,y,z) = \iint_{\varepsilon} K(x,y,z;\xi,\eta,z_0)u_0(\xi,\eta,z_0)d(\xi,\eta) \tag{15}$$

In order to gain some insight, a wave of the type

$$u_0(\xi,\eta,z_0) = \delta(\xi - x_0)\delta(\eta - y_0)$$

originates from a point source at the point in the entrance plane and results in an output wave

$$u(x,y,z) = K(x,y,z;x_0,y_0,z_0) \quad .$$

This is why $K$ is called an (coherent paraxial) *point spread function*. Separating the point spread function

$$PSF(x,y,z;\xi,\eta,z_0) = A(x,y,z;\xi,\eta,z_0)\mathrm{e}^{\mathrm{i}k_0 L(x,y,z;\xi,\eta,z_0)} \ .$$

with an amplitude $A$ and phase function $L$, we recognize that $L$ corresponds to the paraxial optical path distance (4). To derive the amplitude, much more work needs to be done, but we do not really need it in our analysis here. The conservation of energy yields the magnitude of the amplitude and comparison with the known FRESNEL-diffraction integral yields the phase of the amplitude. Finally one obtains the BAUES-COLLINS-integral kernel [1],[2]

$$K(x,y,z;\xi,\eta,z_0) = \frac{k_0}{2\pi i}\sqrt{\frac{n}{n'}}\frac{\mathrm{e}^{\mathrm{i}k_0 L_0}}{\mathrm{B}}\exp \mathrm{i}k_0\frac{A(\xi^2+\eta^2)-2(\xi x+\eta y)+D(x^2+y^2)}{2B} \tag{16}$$

The coefficients $A$, $B$, $C$ and $D$ are given by the paraxial ray propagation matrix (1).

**Example 1:** The ray-tracing matrix of free-space light propagation is

$$M = \underset{\varepsilon \to \varepsilon'}{\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}} \quad .$$

Now we obtain the integral kernel ($L_0 = z$)

$$K(x,y,z;\xi,\eta,0) = \frac{k_0}{2\pi\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}k_0 z}}{z}\exp\left\{\mathrm{i}k_0\frac{(x-\xi)^2+(y-\eta)^2}{2z}\right\}$$

and the paraxial wave

$$u(x,y,z) = \frac{k_0}{2\pi\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}k_0 z}}{z}\iint_{\varepsilon} u(\xi,\eta,0)\exp\left\{\mathrm{i}k_0\frac{(\xi-x)^2+(\eta-y)^2}{2z}\right\}d(\xi,\eta) \ . \tag{17}$$

This integral is usually called the *FRESNEL diffraction integral.*

**Example 2:** Another special case of the general paraxial diffraction integral is the *Fraunhofer diffraction integral*. The exit plane is the back focal plane of the optical imaging system, for example, a thin lens. Now we have the ray-tracing matrix

$$\begin{bmatrix} 1 & f' \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/f' & 1 \end{bmatrix}\begin{bmatrix} 1 & -z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & f' \\ -1/f' & 1+z/f' \end{bmatrix}$$

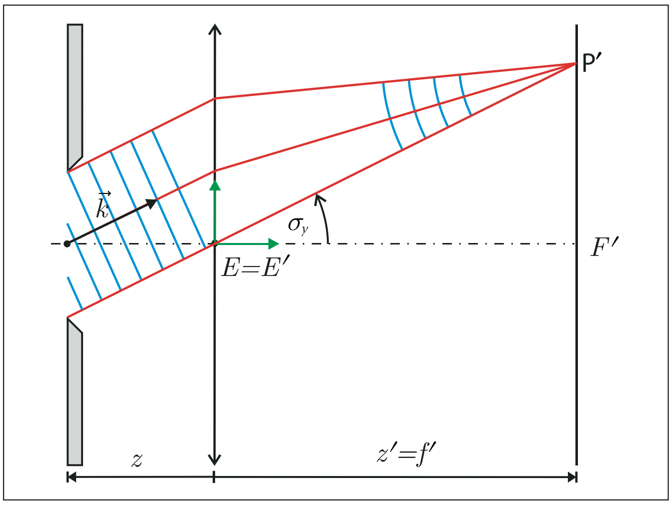



**FIGURE 3: Arrangement of Fraunhofer-diffraction: The image of the aperture will be observed in the back focal plane of the lens. Then the image is the Fourier transform of the wave at the aperture plane. That is the reason why the plane is also called Fourier plane.**

and

$$u(x,y,f') = \frac{1}{i\lambda_0}\frac{e^{-ik_0(z-f')}}{f'}\exp\left[ik_0\left(1+\frac{z}{f'}\right)\frac{x^2+y^2}{2f'}\right] \iint_{\varepsilon} u(\xi,\eta,z)e^{-ik_0(x\xi+y\eta)/f'}d(\xi,\eta) \quad . \tag{18}$$

The integral is the angle spectrum of $u$ in the entrance plane if we define $k_x = k_0x/f'$, $k_y = k_0y/f'$ as spatial frequencies.

A very simple representation is obtained, when the entrance plane coincides with the object-side focal plane of the imaging system, this means $z = -f'$. This arrangement is sometimes called the $2f'$-arrangement.

*Propagation of Gaussian beams*

To obtain the shape of the Gaussian beam in the image space behind the optical system, we have to integrate the BAUES-COLLINS diffraction integral (15) and (16) over the Gaussian beam profile in the entrance plane, $z = 0$. It is useful to write the Gaussian beams using the $q$-parameter

$$q(z) = z - z_W - iz_R \quad , q_0 = q(0) \tag{19}$$

where $z_W$ is the position of the beam waist, and the reciprocal

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{2}{ik}\frac{1}{w^2(z)} . \tag{20}$$

Substituting (20) in (12) and (14), we get the complex representation of the Gaussian beam

$$u(x,y,z) = a_0\frac{q_0}{q(z)}\exp\left[ik\frac{x^2+y^2}{2q(z)}\right]e^{ikz}$$

in the object space in front of the optical system. Using the integral

$$\int_{-\infty}^{\infty} e^{-ax^2-2bx}dx = \sqrt{\frac{\pi}{a}}e^{b^2/a} \; , \; \mathrm{Re}\{a\} > 0$$

and the relation $AD - CB = 1$ the integration process yields the Gaussian beam at the exit plane ($z = \ell$)

$$u(x,y,\ell) = \sqrt{\frac{n}{n'}}\frac{a_0q_0e^{ik_0L_0}}{Aq_0+B}\exp\left[i\frac{k_0}{2}\frac{Cq_0+D}{Aq_0+B}(x^2+y^2)\right], \tag{21}$$

with the $q$-parameter

$$q'(\ell) = \frac{Aq(0)+B}{Cq(0)+D} \tag{22}$$

of the Gaussian beam in the image space.

In the following, the quantities of wave optics in the image space are indicated by a prime, and those in the object space without a prime, respectively, in a similar way to ray optics. Substituting (19) into (22), we obtain

$$\ell - z'_w - iz'_R = \frac{A(-z_w - iz_R)+B}{C(-z_w - iz_R)+D}$$

Relating the sizes of the image space to the coordinate system in the exit plane at $z = \ell$ according to optics, we obtain the new waist coordinates related to the new coordinate systems, $z'_0 = z'_W - \ell$, $z_0 = z_W$, and further, we obtain the transformation law

$$-z'_0 - iz'_R = \frac{A(-z_0 - iz_R)+B}{C(-z_0 - iz_R)+D} . \tag{23}$$

Finally, the separation into real and imaginary parts results in the imaging equation of the Gaussian beam for the waist position

$$z'_0 = -\frac{(Az_0 - B)(Cz_0 - D) + ACz_R^2}{(Cz_0 - D)^2 + C^2z_R^2} , \tag{24}$$

and for the RAYLEIGH length

$$z'_R = \frac{z_R}{(Cz_0 - D)^2 + C^2z_R^2} \tag{25}$$

of the transformed Gaussian beam. The transformation law of the

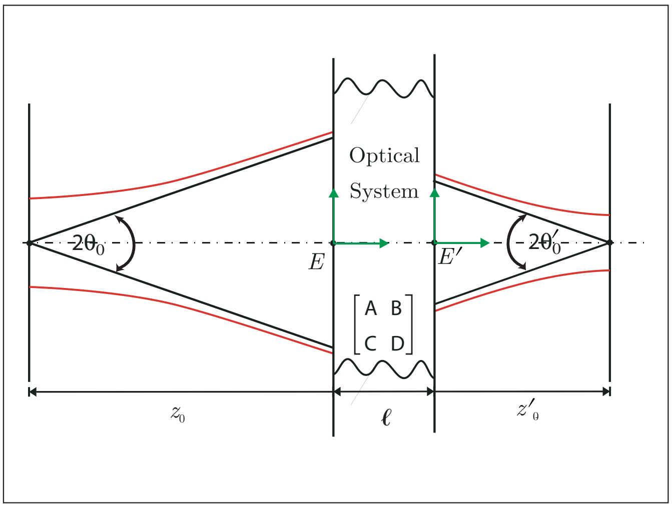



FIGURE 4: **Gaussian beam propagation through optical systems: The left and the right plane marks the waist of the incoming and the outgoing gaussian beam, respectively, The z-coordinate of the beam waist is referenced to the entrance and exit plane, respectively.**

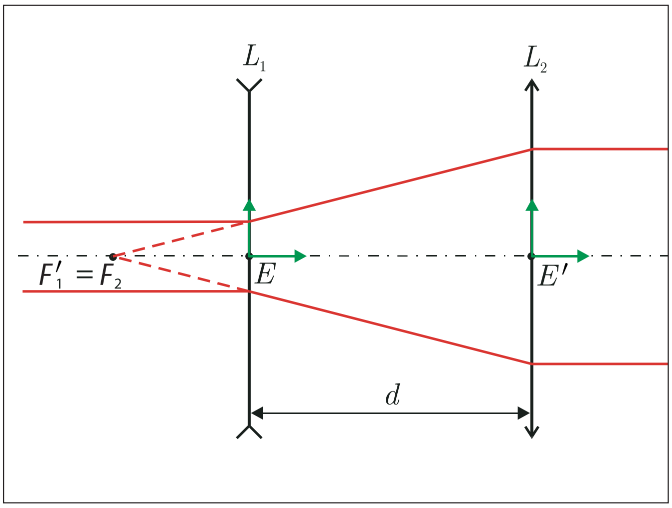



FIGURE 5: **Ray propagation in a Galilean beam expander.**

waist radius and the divergence angle follow directly as

$$\frac{w'_0}{w_0} = \sqrt{\frac{z'_R}{z_R}}\,,\ \frac{\theta'_0}{\theta_0} = \sqrt{\frac{z_R}{z'_R}}\,, \tag{26}$$

with the fundamental relation, $w'_0\theta'_0 = w_0\theta_0$.

### Application: beam expander

In many cases a reversed telescope, a so-called *beam expander*, is used to increase the spot size. The most common type is derived from the Galilean telescope, which has in the simplest arrangement, two lenses: one negative input lens, $f'_1 < 0$, and one positive output lens, $f'_2 > 0$, as shown in Fig. 5. For low magnifications or expansion ratios (1.3 – 20×) the Galilean expander is frequently used due to its simplicity, small package size and low cost.

To find the system matrix of the beam expander, we have to multiply the ray propagation matrices of the subsystem

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -1/f'_2 & 1 \end{bmatrix}}_{\text{lens 2}} \underbrace{\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}}_{\text{gap}} \underbrace{\begin{bmatrix} 1 & 0 \\ -1/f'_1 & 1 \end{bmatrix}}_{\text{lens 1}} \tag{27}$$

$$= \begin{bmatrix} 1 - d/f'_1 & d \\ -(f'_1 + f'_2 - d)/(f'_1 f'_2) & 1 - d/f'_2 \end{bmatrix} .$$

The system matrix represents an afocal system, if the optical power is zero, that means $d - f'_1 - f'_2 = 0$. Then the matrix of a beam expander reduces to

$$\begin{bmatrix} -f'_2/f'_1 & f'_1 + f'_2 \\ 0 & -f'_1/f'_2 \end{bmatrix}$$

Using the coefficients of (27) we get the output Rayleigh length

$$\frac{z'_R}{z_R} = \frac{1}{D^2} = \left(\frac{f'_2}{f'_1}\right)^2 > 1\,, \tag{28}$$

the *beam waist magnification (expansion power)*

$$\beta'_0 = \frac{w'_0}{w_0} = \sqrt{\frac{z'_R}{z_R}} = \left|\frac{f'_2}{f'_1}\right| > 1\,, \tag{29}$$

and the *divergence angle magnification*

$$\gamma'_0 = \frac{\theta'_0}{\theta_0} = \sqrt{\frac{z_R}{z'_R}} = \left|\frac{f'_1}{f'_2}\right| < 1. \tag{30}$$

From the conservation of energy, we have $\beta'_0\gamma'_0 = 1$. The position of the beam waist is obtained from

$$z'_0 = \frac{Az_0 - B}{D} = f'_2 + \beta'_0\left(z_0 + f'_0\right) \tag{31}$$

by using (24). Note that the image of $z_0 = f'_1$ is $z'_0 = f'_2$.

In cases where either larger expansion ratios or spatial filtering is required, the Keplerian beam expander is preferred. The Keplerian telescope has a positive input lens representing a real beam waist to the focus. In addition, spatial filtering can be implemented by placing a pinhole at the focus of the first lens (see example 2).

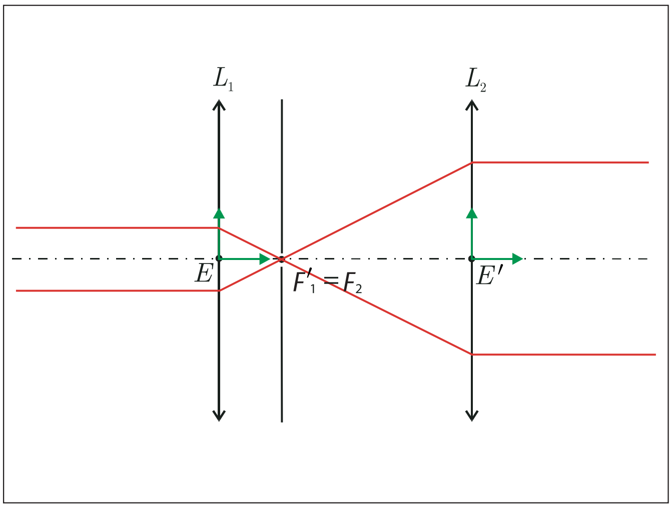



FIGURE 6: **Ray propagation in Keplerian expander with pinhole in the Fourier-plane of the first lens for spatial filtering.**

### References


[1] *P. Baues,* Huygens´ principle in inhomogeneous isotropic media and a general integral equation applicable to optical resonators, Opto-Electronics **1**, 37–44 (1969).
[2] *Collins Jr.,* Lens-Systems Diffraction Written in Terms of Matrix Optics, J. Optical Society of America **46**, 51 1168–1177 (1970).
[3] *R. Ditteon,* Modern Geometrical Optics, ch. 3 (John Wiley & Sons, 1998).
[4] *R. D. Guenther,* Modern Optics, ch. 9 (John Wiley & Sons, 1990).
[5] *V. N. Mahajan,* Optical Imaging and Aberrations, Part I: Ray Geometrical Optics, ch. 2 (SPIE Optical Engineering Press, 1998).
[6] *B. E. A. Saleh, M. C. Teich,* Fundamentals of Photonics, ch. 1–3 (John Wiley & Sons, 1991).
[7] *A. E. Siegman,* Lasers, ch. 20 (University Science Books, 1986).
[8] *A. Walther,* The Ray and Wave Theory of Lenses (Cambridge University Press, 1995).