Beam propagation and the *ABCD* ray matrices

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We have generalized the *ABCD* propagation law, $Q_2 = (AQ_1 + B)/(CQ_1 + D)$, for an optical system by introducing a generalized complex radius of curvature Q for a general optical beam. The real part of $1/Q$ is related to the mean radius of curvature of the wave front, while the imaginary part is related to the second moment of the amplitude of the beam.

The propagation of a general beam through an optical paraxial *ABCD* system is described by Huygens's integral in the Kirchhoff–Fresnel approximation

$$u_2(x_2) = \left(\frac{j}{B\lambda}\right)^{1/2} \int_{-\infty}^{\infty} u_1(x_1) \times \exp\left[-\frac{j\pi}{\lambda B}(A{x_1}^2 - 2x_1x_2 + D{x_2}^2)\right]\mathrm{d}x_1, \quad (1)$$

where the *ABCD* parameters are the ray's matrix elements of the optical system located between plane 1 and plane 2 (see, e.g., Chap. 20 of Ref. 1). Here we have written the Huygens integral in one dimension only, in order to simplify the mathematical derivation of the following demonstration. However, the result could be generalized for the more usual situation of two-dimensional propagation.

An optical beam is defined as a wave that is slowly diverging while propagating. For practical reasons, in a laboratory one is often interested in gross values of characteristics such as the change in width of the beam and not in the exact beam profile that can usually be obtained by long numerical calculation. The second-order moment of the intensity (or variance) certainly gives good gross information on the beam width. However, we find it more useful to characterize the width of a general beam by its real beam size W^2, which is equal to four times the second-order intensity moment,[2] namely,

$$W^2 = 4\int_{-\infty}^{\infty} x^2\,|u(x)|^2\mathrm{d}x, \quad (2)$$

where it is assumed that the energy in the beam is unity:

$$\int_{-\infty}^{\infty} |u(x)|^2\mathrm{d}x = 1. \quad (3)$$

The propagation rule for this real beam size after its propagation through an optical *ABCD* system can be obtained from the integral equation (1), and the result is

$${W_2}^2 = A^2{W_1}^2 + 2ABV_1 + B^2U_1, \quad (4)$$

where

$$U = \left(\frac{\lambda}{\pi}\right)^2 \int_{-\infty}^{\infty} \left|\frac{\partial u(x)}{\partial x}\right|^2 \mathrm{d}x, \quad (5)$$

$$V = 4\int_{-\infty}^{\infty} x\left[\frac{\partial \Phi(x)}{\partial x}\right]\psi^2(x)\mathrm{d}x. \quad (6)$$

ϕ and ψ are the phase and the amplitude of the beam, respectively, such that

$$u = \psi \exp\left(-j\frac{2\pi}{\lambda}\Phi\right).$$

This result is a generalization of problem 16.7 given by Siegman.[1] The real beam size W_2 at plane 2 is related to the real beam size W_1 at plane 1 by Eq. (4); two additional parameters, U and V, must be calculated for the beam at the entrance plane 1. The same result was derived, using the Wigner distribution for partially coherent light,[3] for the particular case of propagation starting at a waist (here $V_1 = 0$).

This propagation rule is not especially useful since, if one wanted to calculate the real beam size at another location, one would first have to propagate the beam numerically in plane 2 in order to be able to calculate the parameters U_2 and V_2. However, it is possible to derive the propagation rule of these two parameters U and V by the same procedure as before, and the result is

$$V_2 = AC{W_1}^2 + (AD + BC)V_1 + BDU_1, \quad (7)$$

$$U_2 = C^2{W_1}^2 + 2DCV_1 + D^2U_1. \quad (8)$$

Equations (4), (7), and (8) permit one to cascade the calculation of the real beam size W from plane to plane in an optical system. Also using the Wigner distribution, Bastiaans[4] obtained the propagation rule of second-order moments defined in space and in the spatial-frequency domain for partially coherent light. His moments can certainly be related to the present W, V, and U. Here we interpret them in space only, by relating V and U to the phase front of the beam.

The phase of the beam, from plane to plane, is certainly something that we want to evaluate in an optical system. It is not usually possible to define the second-order moment of the phase because this is not generally integrable. However, the parameters V and U contain information on the derivative of the phase, and we show how to use them to define the real radius of a general beam.

First we can show, by using the fundamental rule of the *ABCD* optical system $AD - BC = 1$, the following invariant relation among W, V, and U:

$$W_2{}^2U_2 - V_2{}^2 = W_1{}^2U_1 - V_1{}^2 = \left(\frac{\lambda}{\pi} M^2\right)^2. \tag{9}$$

A direct consequence of this invariance is to reduce the system of three equations, Eqs. (4), (7), and (8), to a system of two independent real equations. Finally, these two real equations can be written as one complex equation, namely,

$$Q_2 = \frac{AQ_1 + B}{CQ_1 + D}, \tag{10}$$

after the following complex radius of curvature is introduced:

$$\frac{1}{Q} = \frac{V}{W^2} - \frac{j\lambda M^2}{\pi W^2}. \tag{11}$$

Equation (10) is therefore a generalization of the propagation rule of rays. The complex radius of curvature Q is itself a generalization of the complex radius of curvature of a Gaussian beam. The real radius of curvature for a general beam is thus defined as

$$\frac{1}{R} = \frac{V}{W^2} = \frac{\int_{-\infty}^{\infty} x\left(\frac{\partial \Phi}{\partial x}\right)\psi^2 \mathrm{d}x}{\int_{-\infty}^{\infty} x^2\psi^2 \mathrm{d}x}. \tag{12}$$

The invariant coefficient of the beam M^2 is the so-called beam-quality factor.[2,5] M^2 can be related to the divergence of the real beam size in free space by

$$\Theta = \frac{M^2\lambda}{\pi W_0{}^2}, \tag{13}$$

and it can be evaluated in a plane (o) where the beam has a uniform phase by

$$M^4 = W_o{}^2 \int_{-\infty}^{\infty} \left[\frac{\partial \psi_0(x)}{\partial x}\right]^2 \mathrm{d}x. \tag{14}$$

To understand that the real radius of curvature introduced here [Eq. (12)] is a good estimate of the mean radius of curvature of the phase, let us consider the free-space propagation as described by the paraxial wave equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{4\pi j}{\lambda}\frac{\partial u}{\partial z} = 0. \tag{15}$$

Writing the real and the imaginary parts of Eq. (15), we have two differential equations linking the phase and the amplitude of the beam:

$$\frac{\partial^2 \psi}{\partial x^2} - \left(\frac{2\pi}{\lambda}\right)^2\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + 2\frac{\partial \phi}{\partial z}\right]\psi = 0, \tag{16}$$

$$\psi\frac{\partial^2 \phi}{\partial x^2} + 2\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + 2\frac{\partial \psi}{\partial z} = 0. \tag{17}$$

From Eq. (17) we can demonstrate the following identity:

$$n\int_{-\infty}^{\infty} x^{n-1}\psi^2\left(\frac{\partial \phi}{\partial x}\right)\mathrm{d}x = \frac{\partial}{\partial z}\int x^n\psi^2\mathrm{d}x, \qquad n = 0, 1, 2, 3 \ldots. \tag{18}$$

Now, if we write the phase Φ as a power series:

$$\Phi(x) = \Phi_0 + (x^2/2R) + C_4x^4 + C_6x^6 + \ldots,$$

and use this in identity (18), and if we limit the information on the intensity to the second-order moment, the power-series expansion for the phase must be limited to the quadratic term ($1/2R$). We thus conclude that the real radius of curvature [Eq. (12)] is the best-fit radius of curvature of the phase balanced by the second-order intensity moment. We have made numerical calculations for some particular super-Gaussian beam (order 6, 8, 10) and have observed that the real radius of curvature closely approximates the phase where the intensity of the beam is not negligible. In the far field, the real radius of curvature becomes exactly the radius of curvature of the spherical wave inherent to the given beam. In free space we can show with identity (18) that

$$V = \frac{1}{2}\frac{\partial W^2}{\partial z}$$

and from Eq. (16) that

$$U = -8\int_{-\infty}^{\infty}\left(\frac{\partial \phi}{\partial z}\right)\psi^2\mathrm{d}x.$$

Using these two relations and the invariant relation (9), we can show that the on-axis phase shift Φ_0 is given by

$$\Phi_0 = -\frac{\lambda M^2}{4\pi}\tan^{-1}\frac{z}{z_R}, \tag{19}$$

where the Rayleigh distance is

$$z_R = (\pi W_0{}^2/\lambda M^2). \tag{20}$$

We can summarize all these results by concluding that a general beam obeys the same propagation rule in an optical (*ABCD*) system as does a Gaussian beam of the same real beam size, if the wavelength λ is changed to $M^2\lambda$. However, we must understand that this equivalence is exact for the real beam size and the real radius of curvature but that the parabolic phase expansion made here is only approximate. As a consequence of this approximation, the on-axis phase shift predicted [Eq. (19)] here is not exact. For example, at $z = \infty$, Eq. (19) predicts a phase shift ($-2\pi/\lambda\phi_0 = M^2\pi/4$), but, according to Huygens's integral (1), for $z = \infty$ this phase shift must be $(2n + 1)\pi/4$ for any optical beam where $n = 0, 1, 2, \ldots$. The correct result could be obtained if we redefined Eq. (19) to read as

$$\Phi_0 = -\frac{\lambda}{4\pi}[M^2]\tan^{-1}\frac{z}{z_R}, \tag{21}$$

where $[M^2]$ means the integer closest to M^2.

Solving the differential equation (16) for this parabolic approximation of the phase, we obtain the following approximate solution that may be called the embedded Gauss–Hermite beam:

$$u(x) = \frac{\exp\left(-\dfrac{j\pi}{\lambda}\dfrac{x^2}{Q}\right) H_{[M^2]}\left(\sqrt{2}\,\dfrac{Mx}{W}\right)}{(Q)^{\frac{[M^2]}{2}}}.$$

In conclusion, we have shown that it is possible to define a generalized complex radius of curvature Q for a general optical beam, where this parameter Q obeys the same propagation rule as the complex radius of a Gaussian beam propagating in the same arbitrary *ABCD* optical system. The real radius of curvature R is related to the mean radius of curvature of the phase front, while the real beam size W is related to second-order moment of the intensity. The beam-quality factor M^2 multiples the wavelength λ in the defining equation:

$$\frac{1}{Q} = \frac{1}{R} - j\frac{M^2\lambda}{\pi W^2}.$$

This generalization can find many applications in the study and design of optical systems for custom beams.

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