Bairstow's Method (Sir Leonard Bairstow, 1880–1963)

How to find roots of a polynomial $P(x)$ with real coefficients *without using complex arithmetic.*

The idea: search for a quadratic divisor of P. If the degree of the quotient Q is greater than 2, reiterate with Q.

Given real numbers B and C, we have:

$$P(x) = (x^2 + Bx + C)\ Q(x) + (Rx + S),$$

This defines a mapping:

$$\mathcal{F} : \begin{pmatrix} B \\ C \end{pmatrix} \mapsto \begin{pmatrix} R \\ S \end{pmatrix}.$$

Cancel $\mathcal{F}(B, C)$ using Newton's method.

The beauty of this method: How to compute partial derivatives of $\mathcal{F}$? This is nice:

Once P as been divided by $(x^2 + Bx + C)$, that is:

$$P(x) = (x^2 + Bx + C)\ Q(x) + (Rx + S),$$

just cancel the first variations of P. This gives for C:

$$0 = \frac{\partial P}{\partial C} = (x^2 + Bx + C)\ \frac{\partial Q}{\partial C} + Q(x) + x\frac{\partial R}{\partial C} + \frac{\partial S}{\partial C}.$$

Then, we just need to divide $-Q$ by the same quadratic, and get the partial derivatives:

$$-Q(x) = (x^2 + Bx + C)\ \frac{\partial Q}{\partial C} + x\frac{\partial R}{\partial C} + \frac{\partial S}{\partial C}.$$

For B, we get:

$$-xQ(x) = (x^2 + Bx + C)\ \frac{\partial Q}{\partial B} + x\frac{\partial R}{\partial B} + \frac{\partial S}{\partial B}.$$

That's all.