

# LINEAR INDEPENDENCE OF PERIODS FOR THE SYMMETRIC SQUARE L-FUNCTIONS

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ABSTRACT. Let  $S_k$  denote the space of cuspforms of weight  $k$  for the full modular group. We first introduce periods on  $S_k$  associated to symmetric square  $L$ -functions. We then prove that for a fixed natural number  $n$ , if  $k$  is sufficiently large relative to  $n$ , then any  $n$  such periods are linearly independent. With some extra assumption, we also prove that for  $k \geq e^{12}$ , we can always pick up to  $\frac{\log k}{4}$  arbitrary linearly independent periods.

## 1. INTRODUCTION

In a paper from 1979, Deligne [3] proposed a conjecture on the algebraicity of values of motivic  $L$ -functions at critical points in terms of motivic periods. A special family of examples are the symmetric power  $L$ -functions associated to modular forms. For each even integer  $k \geq 4$ , let  $M_k$  be the space of modular forms of weight  $k$  for the full modular group  $\mathrm{SL}_2(\mathbb{Z})$ , and let  $S_k$  be its subspace of cuspforms. Let  $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz} \in S_k$  be a normalized Hecke eigenform such that  $a(1) = 1$ . For each prime  $p$ , we denote by  $\alpha_p$  and  $\beta_p$  the roots of the polynomial

$$X^2 - a(p)X + p^{k-1}.$$

Let  $m \geq 1$  be an integer. The  $m^{\mathrm{th}}$  symmetric power  $L$ -function of  $f$  is defined as

$$L_{m,f}(s) = \prod_p \prod_{0 \leq i \leq m} (1 - \alpha_p^{m-i} \beta_p^i \cdot p^s)^{-1}, \quad \mathrm{Re}(s) \gg 0$$

and these functions are conjectured [16] to have an analytic continuation to the whole complex plane and to satisfy the following functional equation for the completed  $L$ -function:

$$L_{m,f}^*(s) := \gamma_m(s) \cdot L_{m,f}(s) = \pm L_{m,f}^*((k-1)m+1-s), \quad (1.1)$$

where  $\gamma_m$  is given by

$$\gamma_m(s) := \begin{cases} (2\pi)^{-rs} \prod_{j=0}^{r-1} \Gamma(s - j(k-1)) & m = 2r - 1, \\ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2} - \left\lfloor \frac{r(k-1)}{2} \right\rfloor\right) \gamma_{2r-1}(s) & m = 2r. \end{cases}$$

See also [23, p.177-178] for details.

In particular, if  $m = 1$ , then (1.1) is known as the functional equation of the standard  $L$ -function of the Hecke eigenform  $f$  in  $S_k$ . The case  $m = 2$  was proved by Shimura [18], where the functional equation (1.1) is

$$L_{2,f}^*(s) = 2^{-s} \pi^{-3s/2} \Gamma(s) \Gamma\left(\frac{s-k+2}{2}\right) L_{2,f}(s) = L_{2,f}^*(2k-1-s). \quad (1.2)$$

Recently, Newton and Thorne [12, 13] showed that the analytic continuation holds true for all eigenforms  $f$ . Deligne's conjecture for  $L_{m,f}^*(s)$  at critical points have been considered by various authors. For instance, the case  $m = 1$  was proved by Shimura [19, 20] and Manin [10] (independently), and the case  $m = 2$  was proved by Zagier [23] and Sturm [21].

We can view the values of  $L_{m,f}^*(s)$  at the critical points as linear functionals on  $f \in S_k$ . Let  $\mathcal{H}_k$  be the set of normalized Hecke eigenforms in  $S_k$ . For each  $f = \sum_{g \in \mathcal{H}_k} c_g g \in S_k$  and a critical point  $n$  for  $L_{m,f}^*(s)$ , we define

$$r_{n,\vee^m}(f) := \sum_{g \in \mathcal{H}_k} c_g L_{m,g}^*(n+1), \quad (1.3)$$

which makes  $r_{n,\vee^m}$  an element in the dual space  $S_k^* := \text{Hom}(S_k, \mathbb{C})$ . These linear functionals are expected to carry interesting information. For example, if  $m = 1$ , then

$$r_{n,\vee^1}(f) = \int_0^\infty f(it) t^n dt = \frac{n!}{(2\pi)^{n+1}} L(f, n+1)$$

is the  $n^{\text{th}}$  period of  $f$ , see [6]. These periods are subject to many linear dependence relations, called the Eichler-Shimura relations [10]; more precisely, for  $0 \leq t \leq k-2$ , we have

$$r_t(f) + (-1)^t r_{k-2-t}(f) = 0, \quad (\text{ES1})$$

$$(-1)^t r_t(f) + \sum_{\substack{0 \leq m \leq t \\ m \equiv 0 \pmod{2}}} \binom{t}{m} r_{k-2-t+m}(f) + \sum_{\substack{0 \leq m \leq k-2-t \\ m \equiv t \pmod{2}}} \binom{k-2-t}{m} r_m(f) = 0, \quad (\text{ES2})$$

$$\sum_{\substack{1 \leq m \leq t \\ m \equiv 1 \pmod{2}}} \binom{t}{m} r_{k-2-t+m}(f) + \sum_{\substack{0 \leq m \leq k-2-t \\ m \not\equiv t \pmod{2}}} \binom{k-2-t}{m} r_m(f) = 0, \quad (\text{ES3})$$

where  $r_t(f) = i^{t+1} r_{t,\vee^1}(f)$ . But not much seems to be known about the linear relations among  $r_{n,\vee^m}$  for  $m \geq 2$  since the corresponding motives are mysterious.

Another interesting aspect of these functionals or periods is the linear independence of a subset of  $\{r_{n,\vee^m}\}_n$ . For example, even in the case of  $m = 1$ , it is not simple to know whether a certain set of periods is linearly independent or not by only looking at the period relations (ES1), (ES2) and (ES3). Recently, when  $m = 1$ , Lei et al. [9, 8] have provided some evidence for the linear independence of odd periods and even periods for modular forms, respectively. Furthermore, the linear independence between one odd and one even periods has been addressed in [22]. In this paper, when  $m = 2$ , we shall extend the ideas of [9, 8] to provide some evidence for the linear independence of a subset of the periods  $\{r_{n,\vee^2}\}_n$ . First, note that the critical points for the

symmetric square  $L$ -function  $L_{2,f}^*(s)$  are odd integers in the set  $\{1, 2, \dots, k-1\}$  and even integers in the set  $\{k, k+1, \dots, 2k-2\}$ ; see Hida [5, p. 96]. Thus, via the definition (1.3) and the functional equation (1.2), we will focus on the “critical” periods  $r_{2\ell, \sqrt{2}}$  for integers  $0 \leq \ell \leq (k-2)/2$ . Also, due to Lemma 2.2), we will need to confine ourselves to periods  $r_{2\ell, \sqrt{2}}$  for  $0 \leq \ell \leq (k-4)/2$ .

We now state our results.

**Theorem 1.1.** *Let  $n \geq 1$  be an integer. For sufficiently large  $k$ , if  $0 \leq \ell_1 < \ell_2 < \dots < \ell_n \leq \frac{k-4}{2}$  are integers, then the set of periods  $\{r_{2\ell_i, \sqrt{2}}\}_{i=1}^n$  is linearly independent.*

*Remark 1.1.* (1) In Proposition 5.2, we give an explicit lower bound of  $k$  in Theorem 1.1.  
 (2) We will also prove the following complete result for  $n = 2$ .

**Theorem 1.2.** *Let  $0 \leq \ell_1 < \ell_2 \leq \frac{k-4}{2}$  be integers. If  $\dim S_k \geq 2$ , then  $r_{2\ell_1, \sqrt{2}}$  and  $r_{2\ell_2, \sqrt{2}}$  are linearly independent.*

If we require that the periods  $\{r_{2\ell_i, \sqrt{2}}\}_{i=1}^n$  are “far away” from the “central” period  $r_{\frac{k-4}{2}, \sqrt{2}}$ , for example, if  $0 \leq \ell_1 < \ell_2 < \dots < \ell_n \leq \frac{k-2}{4}$ , then we can pick up to  $\frac{\log k}{4}$  linearly independent periods. More precisely, we proved the following result.

**Theorem 1.3.** *Let  $n \geq 1$  be an integer. For  $k \geq e^{12}$ , if  $0 \leq \ell_1 < \ell_2 < \dots < \ell_n \leq \frac{k-2}{4}$  are integers and  $n \leq \frac{\log k}{4}$ , then the set of periods  $\{r_{2\ell_i, \sqrt{2}}\}_{i=1}^n$  is linearly independent.*

We sketch the main idea of the proofs. A natural idea is to consider the “dual” of the periods, which are certain modular forms. Thanks to the work of Cohen [2] and Zagier [23], we can find the integral representation of  $r_{2\ell, \sqrt{2}}$  and reduce the linear independence among the periods  $\{r_{2\ell_i, \sqrt{2}}\}_{i=1}^n$  to the linear independence among modular forms  $\{c_{k, k-1-2\ell_i}\}_{i=1}^n$  defined in (2.1). Next, following the idea of [9] and [8], we consider the matrix formed by the Fourier coefficients of  $\{c_{k, k-1-2\ell_i}\}_{i=1}^n$ , which reduces the linear independence among  $\{c_{k, k-1-2\ell_i}\}_{i=1}^n$  to the nonsingularity of the matrix.

The paper is organized as follows. In Section 2, we state the integral representation of the periods and give the equivalence between the linear independence among periods and the linear independence among the corresponding modular forms. In Section 3, we first provide some preparatory lemmas that will be used to estimate the Fourier coefficients of  $c_{k, r}$ . Then we give the asymptotics of the Fourier coefficient of  $c_{k, r}$ . In Section 4, we define the matrix that is formed by the Fourier coefficients of  $\{c_{k, k-1-2\ell_i}\}_{i=1}^n$ . Moreover, we give a criterion for the nonsingularity of this matrix. In Section 5, we prove Theorems 1.1 and 1.3 by showing the corresponding matrices are nonsingular. In the last section, we discuss a potential way to improve Theorems 1.1 and 1.3 and propose a related conjecture.

## 2. INTEGRAL REPRESENTATION OF THE PERIODS

Following Cohen [2] and Zagier [23], we recall the integral representation of the periods  $r_{2\ell, \sqrt{2}}$  (1.3). Let  $k \geq 4$  be an even integer and  $r$  be an odd integer such that  $3 \leq r \leq k-1$ . Define

$$c_{k,r}(z) = \sum_{m=0}^{\infty} \left( \sum_{t \in \mathbb{Z}, t^2 \leq 4m} P_{k,r}(t, m) H(r, 4m - t^2) \right) e^{2\pi i m z}, \quad (2.1)$$

where  $P_{k,r}(t, m)$  is the polynomial defined by [23, p.114]

$$P_{k,r}(t, m) = \text{coefficient of } x^{k-r-1} \text{ in } \frac{1}{(1 - tx + mx^2)^r}.$$

Note that  $P_{k,r}(t, m) = P_{k,r}(-t, m)$ . The generalized Hurwitz class number  $H(r, N)$  is defined as follows (see [2, p.272-273]). Let  $r$  and  $N$  be non negative integer with  $r \geq 1$ . For  $N \geq 1$ , define

$$h(r, N) = \begin{cases} \frac{(-1)^{[r/2]} (r-1)! N^{r-1/2}}{2^{r-1} \pi^r} L(r, \chi_{(-1)^r N}) & (-1)^r N \equiv 0, 1 \pmod{4}, \\ 0 & N \equiv 2, 3 \pmod{4}, \end{cases}$$

where we write  $\chi_D$  for the quadratic character  $\chi_D(d) = \left(\frac{D}{d}\right)$ . For  $N \geq 0$ , we define

$$H(r, N) = \begin{cases} \sum_{d^2 | N} h(r, N/d^2) & (-1)^r N \equiv 0, 1 \pmod{4}, N > 0, \\ \zeta(1-2r) & N = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If we set  $(-1)^r N = Df^2$  with  $D$  a fundamental discriminant (we allow  $D = 1$  as a fundamental discriminant), then

$$\begin{aligned} H(r, N) &= L(1-r, \chi_D) \sum_{d|f} \mu(d) \chi_D(d) d^{r-1} \sigma_{2r-1}(f/d) \\ &= \frac{(-1)^{(r-1)/2} (r-1)!}{2^{r-1} \pi^r} N^{r-1/2} L(r, \chi_D) \sum_{d|f} \mu(d) \chi_D(d) d^{r-1} \sigma_{2r-1}(f/d), \end{aligned}$$

where  $\mu$  is the Möbius function and  $\sigma_{2r-1}(n) = \sum_{d>0, d|n} d^{2r-1}$ .

**Proposition 2.1.** ([2, Theorem 6.2], [23, p.114]) *Let  $k \geq 4$  be an even integer and  $r$  be an odd integer such that  $3 \leq r \leq k-1$ . Then the function  $c_{k,r}$  in (2.1) is a modular form of weight  $k$  for the full modular group. If  $r < k-1$ , it is a cuspform.*

The following lemma gives the integral representation of the periods. It is the reason why we need to exclude the period  $r_{k-2, \sqrt{2}}$  (or equivalently  $r_{k-1, \sqrt{2}}$ ); see the remark before Theorem 1.1.

**Lemma 2.2.** ([23, Theorem 2]) *Let  $r, k$  be integers with  $3 \leq r \leq k-1$ ,  $r$  odd,  $k$  even. The Petersson inner product of the modular form  $c_{k,r}$  with a Hecke eigenform  $f \in S_k$  is given by*

$$(f, c_{k,r}) = -\frac{(r+k-2)!(k-2)!}{(k-r-1)!} \frac{1}{4^{r+k-2} \pi^{2r+k-1}} L_{2,f}(r+k-1). \quad (2.2)$$

Now, we prove the equivalence between the linear independence among the periods and the linear independence among the corresponding modular forms.

**Proposition 2.3.** *Let  $n \geq 1$  be an integer and let  $0 \leq \ell_1 < \ell_2 < \cdots < \ell_n \leq \frac{k-4}{2}$  be integers. The set of periods  $\{r_{2\ell_i, \sqrt{2}}\}_{i=1}^n$  on  $S_k$  is linear independent if and only if the set of modular forms  $\{c_{k, k-1-2\ell_i}\}_{i=1}^n$  in  $S_k$  is linear independent.*

*Proof.* Letting  $r_i = k - 1 - 2\ell_i$  in (2.2), together with (1.2), we get the Petersson inner product

$$\begin{aligned} (f, c_{k, k-1-2\ell_i}) &= A_{k, \ell_i} \cdot L_{2, f}^*(2\ell_i + 1) \\ &= A_{k, \ell_i} \cdot r_{2\ell_i, \sqrt{2}}(f), \end{aligned}$$

where  $f \in S_k$  is a Hecke eigenform, and  $A_{k, \ell_i}$  is some nonzero constant depending only on  $k$  and  $\ell_i$ . Let  $\mathcal{H}_k$  denote the set of normalized Hecke eigenforms in  $S_k$ . Note that

$$\sum_{i=1}^n a_i c_{k, k-1-2\ell_i} = 0 \text{ in } S_k \quad \Leftrightarrow \quad \sum_{i=1}^n \overline{a_i}(f_j, c_{k, k-1-2\ell_i}) = 0 \text{ for all } f_j \in \mathcal{H}_k. \quad (2.3)$$

First, assume that  $\{c_{k, k-1-2\ell_i}\}_{i=1}^n$  is linearly independent. We claim that  $\{r_{2\ell_i, \sqrt{2}}\}_{i=1}^n$  is linearly independent. Suppose that

$$\sum_{i=1}^n b_i r_{2\ell_i, \sqrt{2}} = 0 \in S_k^*.$$

It follows that

$$\sum_{i=1}^n b_i r_{2\ell_i, \sqrt{2}}(f_j) = \sum_{i=1}^n b_i A_{k, \ell_i}^{-1}(f_j, c_{k, k-1-2\ell_i}) = 0 \text{ for all } f_j \in \mathcal{H}_k,$$

which implies that

$$\sum_{i=1}^n \overline{b_i A_{k, \ell_i}^{-1}} c_{k, k-1-2\ell_i} = 0 \quad \text{by (2.3).}$$

Since  $\{c_{k, k-1-2\ell_i}\}_{i=1}^n$  is linearly independent,  $\overline{b_i A_{k, \ell_i}^{-1}} = 0$ , and  $b_i = 0$  for all  $1 \leq i \leq n$ .

Conversely, suppose that  $\{r_{2\ell_i, \sqrt{2}}\}_{i=1}^n$  is linearly independent. We show that  $\{c_{k, k-1-2\ell_i}\}_{i=1}^n$  is linear independent by a similar argument. Assume that

$$\sum_{i=1}^n a_i c_{k, k-1-2\ell_i} = 0 \in S_k.$$

It follows that

$$\sum_{i=1}^n \overline{a_i}(f_j, c_{k, k-1-2\ell_i}) = 0 \text{ for all } f_j \in \mathcal{H}_k \quad \Leftrightarrow \quad \sum_{i=1}^n \overline{a_i} A_{k, \ell_i} \cdot r_{2\ell_i, \sqrt{2}}(f_j) = 0 \text{ for all } f_j \in \mathcal{H}_k.$$

Since  $\{r_{2\ell_i, \sqrt{2}}\}_{i=1}^n$  is linearly independent,  $\overline{a_i} A_{k, \ell_i} = 0$ , and  $a_i = 0$  for all  $1 \leq i \leq n$ .  $\square$

## 3. ASYMPTOTICS OF FOURIER COEFFICIENTS

First, we prove some preparatory lemmas that will be used to estimate the Fourier coefficients of  $c_{k,r}$ . The key is that for a perfect square  $m$ , the asymptotics of the  $m^{\text{th}}$  Fourier coefficient of  $c_{k,r}$  is clear. We would like to mention that this observation is inspired by Serre's work [17, Proposition 3], which describes the asymptotics of the trace of the Hecke operator  $T_n$  on  $S_k$ .

**Lemma 3.1.** *For integers  $m \geq 1$  and  $t$  with  $t^2 < 4m$ , we have*

$$|P_{k,r}(t, m)| \leq \binom{k-2}{r-1} \cdot \frac{2^r m^{(k-1)/2}}{(4m - t^2)^{r/2}}.$$

*Proof.* Following Zagier's appendix to [7], we recall that

$$\frac{1}{1 - tx + mx^2} = \sum_{n=0}^{\infty} \frac{\rho^{n+1} - \bar{\rho}^{n+1}}{\rho - \bar{\rho}} x^n,$$

where  $\rho + \bar{\rho} = t$  and  $\rho \cdot \bar{\rho} = m$ . We write  $a_j := (\rho^{j+1} - \bar{\rho}^{j+1})/(\rho - \bar{\rho})$ . Then

$$|a_j| \leq \frac{2|\rho|^{j+1}}{|\rho - \bar{\rho}|} \leq \frac{2m^{(j+1)/2}}{(4m - t^2)^{1/2}}.$$

Since  $P_{k,r}(t, m)$  is the coefficient of  $x^{k-r-1}$  in  $(1 - tx + mx^2)^{-r}$ , we have

$$\begin{aligned} |P_{k,r}(t, m)| &= \left| \sum_{\substack{i_1, \dots, i_r \geq 0 \\ i_1 + \dots + i_r = k-r-1}} a_{i_1} \cdots a_{i_r} \right| \\ &\leq \sum_{\substack{i_1, \dots, i_r \geq 0 \\ i_1 + \dots + i_r = k-r-1}} \prod_{1 \leq j \leq r} \frac{2m^{(i_j+1)/2}}{(4m - t^2)^{1/2}} \\ &= \binom{k-2}{r-1} \cdot \frac{2^r m^{(k-1)/2}}{(4m - t^2)^{r/2}}, \end{aligned}$$

which gives the desired result. □

**Lemma 3.2.** *If  $m$  is a perfect square and  $1 \leq r \leq k-3$ , then*

$$P_{k,r}(2m^{1/2}, m) = \binom{k+r-2}{2r-1} \cdot m^{(k-r-1)/2}.$$

*Proof.* Recall that  $P_{k,r}(2m^{1/2}, m)$  is the coefficient of  $x^{k-r-1}$  in

$$\frac{1}{(1 - 2m^{1/2}x + mx^2)^r} = \frac{1}{(1 - m^{1/2}x)^{2r}}.$$

It is straightforward to see that

$$\frac{1}{(1 - m^{1/2}x)^{2r}} = \frac{m^{-(2r-1)/2}}{(-1)(-2) \cdots (-(2r-1))} \frac{d^{2r-1}}{dx^{2r-1}} \left( \frac{1}{1 - m^{1/2}x} \right)$$

$$\begin{aligned}
 &= \frac{m^{-(2r-1)/2}}{(2r-1)!} \frac{d^{2r-1}}{dx^{2r-1}} \left( \sum_{n=0}^{\infty} (m^{1/2}x)^n \right) \\
 &= \frac{m^{-(2r-1)/2}}{(2r-1)!} \sum_{n=0}^{\infty} m^{(2r-1+n)/2} (2r-1+n) \cdots (n+1) x^n,
 \end{aligned}$$

and letting  $n = k - r - 1$  gives the result.  $\square$

We also need the following estimates of the generalized class number  $H(r, N)$ .

**Lemma 3.3.** *Let  $r \geq 3$  be an odd integer and  $N$  be a positive integer. Then*

$$|H(r, N)| \leq \frac{(r-1)! \zeta(2r-1) \zeta(r)}{2^{r-1} \pi^r} \cdot N^{2r-1/2} \log N.$$

*Proof.* We write  $-N = Df^2$  with  $D$  a fundamental discriminant. First, we give an upper bound of  $L(r, \chi_D)$ . By partial summation (for example, see [11, Theorem 6.8]), we have

$$\sum_{n \leq x} \chi_D(n) n^{-r} = x^{-r} \sum_{n \leq x} \chi_D(n) + r \int_1^x \left( \sum_{n \leq t} \chi_D(n) \right) \frac{dt}{t^{r+1}}.$$

Note that  $\chi_D$  is a primitive Dirichlet character with conductor  $-D$  since  $D$  is a fundamental discriminant. By Pólya's inequality [1, Theorem 8.21],

$$\left| \sum_{n \leq x} \chi_D(n) \right| \leq \sqrt{-D} \log(-D).$$

Letting  $x$  go to infinity gives

$$\begin{aligned}
 |L(r, \chi_D)| &\leq \left( r \int_1^{\infty} \frac{dt}{t^{r+1}} \right) \cdot \sqrt{-D} \log(-D) \\
 &\leq \sqrt{-D} \log(-D).
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 \left| \sum_{d|f} \mu(d) \chi_D(d) d^{r-1} \sigma_{2r-1}(f/d) \right| &\leq \sum_{d|f} d^{r-1} \zeta(2r-1) (f/d)^{2r-1} \\
 &= f^{2r-1} \zeta(2r-1) \sum_{d|f} d^{-r} \\
 &\leq f^{2r-1} \zeta(2r-1) \zeta(r),
 \end{aligned}$$

where we used  $\sigma_{\alpha}(n) \leq \zeta(\alpha) n^{\alpha}$  for  $\alpha > 1$  in the first inequality, see [11, p.245, Exercise 9]. It follows that

$$|H(r, N)| = \left| \frac{(-1)^{(r-1)/2} (r-1)!}{2^{r-1} \pi^r} N^{r-1/2} L(r, \chi_D) \sum_{d|f} \mu(d) \chi_D(d) d^{r-1} \sigma_{2r-1}(f/d) \right|$$

$$\begin{aligned}
&\leq \frac{(r-1)!}{2^{r-1}\pi^r} N^{r-1/2} \cdot (\sqrt{-D} \log(-D)) \cdot f^{2r-1} \zeta(2r-1) \zeta(r) \\
&= \frac{(r-1)! \zeta(2r-1) \zeta(r)}{2^{r-1}\pi^r} N^{r-1/2} (\sqrt{-D} \log(-D)) \cdot \left( \frac{N}{-D} \right)^{r-1/2} \\
&\leq \frac{(r-1)! \zeta(2r-1) \zeta(r)}{2^{r-1}\pi^r} N^{r-1/2} (\sqrt{N} \log N) \cdot N^{r-1/2} \\
&\leq \frac{(r-1)! \zeta(2r-1) \zeta(r)}{2^{r-1}\pi^r} \cdot N^{2r-1/2} \log N,
\end{aligned}$$

as desired.  $\square$

**Lemma 3.4.** *For odd integer  $r \geq 1$ , we have*

$$|H(r, 0)| = \frac{2(2r-1)! \zeta(2r)}{(2\pi)^{2r}}.$$

*Proof.* Recall the functional equation of  $\zeta(s)$  (for example, see [1, Theorem 12.7]),

$$\zeta(1-s) = 2(2\pi)^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s). \quad (3.1)$$

Letting  $s = 2r$  in (3.1) gives the result.  $\square$

To simplify the calculation, we normalize  $c_{k,r}$  so that its first Fourier coefficient is 1. For even integer  $k$  and odd integer  $r$  with  $3 \leq r \leq k-3$ , we write

$$f_{k,r} := \frac{c_{k,r}}{\sum_{t^2 \leq 4} P_{k,r}(t, 1) H(r, 4-t^2)} \quad (3.2)$$

and denote by  $a_{k,r}(m)$  the  $m^{\text{th}}$  Fourier coefficient of  $f_{k,r}$ . Moreover, we define

$$\delta_{k,m,r} := \frac{a_{k,r}(m)}{m^{(k-r-1)/2}} - 1.$$

The next proposition describes the asymptotics of  $a_{k,r}(m)$ .

**Proposition 3.5.** *Let  $m \geq 1$  be a perfect square. If  $k \geq 776$  then*

$$|\delta_{k,m,r}| \leq 5 \left( \frac{32\pi}{k-1} \right)^r m^{2r} \log 4m.$$

*Proof.* By the definition of  $c_{k,r}$  and Lemma 3.2, we have

$$\begin{aligned}
\frac{a_{k,r}(m)}{m^{(k-r-1)/2}} &= m^{-(k-r-1)/2} \cdot \frac{P_{k,r}(2m^{1/2}, m) H(r, 0) + \sum_{t^2 < 4m} P_{k,r}(t, m) H(r, 4m-t^2)}{P_{k,r}(2, 1) H(r, 0) + \sum_{t^2 < 4} P_{k,r}(t, 1) H(r, 4-t^2)} \\
&= \frac{\binom{k+r-2}{2r-1} H(r, 0) + m^{-(k-r-1)/2} \sum_{t^2 < 4m} P_{k,r}(t, m) H(r, 4m-t^2)}{\binom{k+r-2}{2r-1} H(r, 0) + \sum_{t^2 < 4} P_{k,r}(t, 1) H(r, 4-t^2)} \\
&= \frac{1 + A_1}{1 + A_2},
\end{aligned}$$



where  $A_1$  and  $A_2$  are given by

$$A_1 := \frac{\sum_{t^2 < 4m} P_{k,r}(t, m) H(r, 4m - t^2)}{\binom{k+r-2}{2r-1} H(r, 0) m^{(k-r-1)/2}},$$

$$A_2 := \frac{\sum_{t^2 < 4} P_{k,r}(t, 1) H(r, 4 - t^2)}{\binom{k+r-2}{2r-1} H(r, 0)}.$$

It follows that when  $|A_2| < 1$ , we have

$$|\delta_{k,m,r}| = \left| \frac{A_1 - A_2}{1 + A_2} \right| \leq \frac{|A_1| + |A_2|}{1 - |A_2|}. \quad (3.3)$$

We first estimate  $A_1$ . By Lemma 3.1 and 3.3, we have

$$\begin{aligned} |A_1| &\leq \frac{\sum_{t^2 < 4m} \binom{k-2}{r-1} \frac{2^r m^{(k-1)/2}}{(4m-t^2)^{r/2}} \cdot \frac{(r-1)! \zeta(2r-1) \zeta(r)}{2^{r-1} \pi^r} \cdot (4m - t^2)^{2r-1/2} \log(4m - t^2)}{\binom{k+r-2}{2r-1} |H(r, 0)| m^{(k-r-1)/2}} \\ &= \frac{(r-1)! \binom{k-2}{r-1}}{\binom{k+r-2}{2r-1} |H(r, 0)|} \cdot \frac{2\zeta(2r-1) \zeta(r) m^{r/2}}{\pi^r} \sum_{t^2 < 4m} (4m - t^2)^{(3r-1)/2} \log(4m - t^2) \\ &\leq \frac{(r-1)! \binom{k-2}{r-1}}{\binom{k+r-2}{2r-1} |H(r, 0)|} \cdot \frac{2\zeta(5) \zeta(3) m^{r/2}}{\pi^r} \cdot 2(4m)^{1/2} (4m)^{(3r-1)/2} \log 4m \\ &= \frac{(r-1)! \binom{k-2}{r-1}}{\binom{k+r-2}{2r-1} |H(r, 0)|} \cdot \frac{4\zeta(5) \zeta(3) 2^{3r} m^{2r} \log 4m}{\pi^r}. \end{aligned}$$

By Lemma 3.4, we have

$$\begin{aligned} \frac{(r-1)! \binom{k-2}{r-1}}{\binom{k+r-2}{2r-1} |H(r, 0)|} &= \frac{(k-2)!(2r-1)!}{(k+r-2)!} \cdot \frac{(2\pi)^{2r}}{2(2r-1)! \zeta(2r)} \\ &= \frac{(k-2)(k-3) \cdots 1}{(k+r-2) \cdots (k-1)(k-2) \cdots 1} \cdot \frac{(2\pi)^{2r}}{2\zeta(2r)} \\ &\leq \frac{1}{(k-1)^r} \cdot \frac{(2\pi)^{2r}}{2}. \end{aligned}$$

It follows that

$$\begin{aligned} |A_1| &\leq \frac{(2\pi)^{2r}}{2(k-1)^r} \cdot \frac{4\zeta(5) \zeta(3) 2^{3r} m^{2r} \log 4m}{\pi^r} \\ &= 2\zeta(5) \zeta(3) \left( \frac{32\pi}{k-1} \right)^r m^{2r} \log 4m, \end{aligned} \quad (3.4)$$

Note that the above argument in particular implies that

$$|A_2| \leq 2\zeta(5) \zeta(3) \left( \frac{32\pi}{k-1} \right)^r \log 4 \leq 2\zeta(5) \zeta(3) \left( \frac{32\pi}{k-1} \right)^r m^{2r} \log 4m. \quad (3.5)$$

On the other hand, the upper bound of  $|A_2|$  can be obtained by

$$|A_2| = \left| \frac{P_{k,r}(0, 1) + 2P_{k,r}(1, 1) H(r, 3)}{\binom{k+r-2}{2r-1} H(r, 0)} \right|$$

$$\begin{aligned}
&\leq \left| \frac{P_{k,r}(0,1)}{\binom{k+r-2}{2r-1}H(r,0)} \right| + \left| \frac{2P_{k,r}(1,1)H(r,3)}{\binom{k+r-2}{2r-1}H(r,0)} \right| \\
&\leq \frac{\binom{k-2}{r-1} \frac{2^r}{4^{r/2}} \frac{(r-1)!\zeta(2r-1)\zeta(r)}{2^{r-1}\pi^r} 4^{2r-1/2} \log 4}{\binom{k+r-2}{2r-1}|H(r,0)|} + 2 \frac{\binom{k-2}{r-1} \frac{2^r}{3^{r/2}} \frac{(r-1)!\zeta(2r-1)\zeta(r)}{2^{r-1}\pi^r} 3^{2r-1/2} \log 3}{\binom{k+r-2}{2r-1}|H(r,0)|} \\
&\leq \frac{(r-1)!\binom{k-2}{r-1}}{\binom{k+r-2}{2r-1}|H(r,0)|} \left( \frac{2^r \zeta(2r-1)\zeta(r) 4^{2r-1/2} \log 4}{4^{r/2} 2^{r-1} \pi^r} + 2 \frac{2^r \zeta(2r-1)\zeta(r) 3^{2r-1/2} \log 3}{3^{r/2} 2^{r-1} \pi^r} \right) \\
&\leq \frac{1}{(k-1)^r} \cdot \frac{(2\pi)^{2r}}{2} \left( \frac{\zeta(2r-1)\zeta(r)(\log 4) 4^{3r/2}}{\pi^r} + \frac{4}{\sqrt{3}} \frac{\zeta(2r-1)\zeta(r)(\log 3) 3^{3r/2}}{\pi^r} \right) \\
&\leq \frac{\zeta(5)\zeta(3) \log 4}{2} \left( \frac{32\pi}{k-1} \right)^r + \frac{2\zeta(5)\zeta(3) \log 3}{\sqrt{3}} \left( \frac{12\sqrt{3}\pi}{k-1} \right)^r \tag{3.6}
\end{aligned}$$

Since  $k \geq 776$ , we have

$$|A_2| \leq \frac{\zeta(5)\zeta(3) \log 4}{2} \left( \frac{32\pi}{776-1} \right)^3 + \frac{2\zeta(5)\zeta(3) \log 3}{\sqrt{3}} \left( \frac{12\sqrt{3}\pi}{776-1} \right)^3 < 1.$$

Plugging the bounds (3.4), (3.5) and (3.6) into (3.3) gives

$$\begin{aligned}
|\delta_{k,m,r}| &\leq 2 \cdot \frac{2\zeta(5)\zeta(3) \left( \frac{32\pi}{k-1} \right)^r m^{2r} \log 4m}{1 - \frac{\zeta(5)\zeta(3) \log 4}{2} \left( \frac{32\pi}{k-1} \right)^r - \frac{2\zeta(5)\zeta(3) \log 3}{\sqrt{3}} \left( \frac{12\sqrt{3}\pi}{k-1} \right)^r} \\
&\leq \frac{4\zeta(5)\zeta(3)}{1 - \frac{\zeta(5)\zeta(3) \log 4}{2} \left( \frac{32\pi}{776-1} \right)^3 - \frac{2\zeta(5)\zeta(3) \log 3}{\sqrt{3}} \left( \frac{12\sqrt{3}\pi}{776-1} \right)^3} \cdot \left( \frac{32\pi}{k-1} \right)^r m^{2r} \log 4m \\
&\leq 4.999942 \left( \frac{32\pi}{k-1} \right)^r m^{2r} \log 4m,
\end{aligned}$$

as desired.  $\square$

#### 4. CRITERION FOR THE LINEAR INDEPENDENCE

Following the idea of [9] and [8], we define matrices that are used to study the linear independence of  $\{c_{k,k-1-2\ell_i}\}_{i=1}^n$ , see (2.1) and Proposition 2.3. Let  $k \geq 4$  be an even integer and  $\ell$  be an integer with  $0 \leq \ell \leq \frac{k-4}{2}$ . Define a  $(n, n)$ -matrix:

$$\begin{aligned}
A_{k;\ell_1,\dots,\ell_n} &:= [a_{k,k-1-2\ell_i}(4^{j-1})]_{i=1,\dots,n;j=1,\dots,n} \\
&= [4^{\ell_i(j-1)}(1 + \delta_{k,4^{j-1},k-1-2\ell_i})]_{i=1,\dots,n;j=1,\dots,n}. \tag{4.1}
\end{aligned}$$

We write  $\varepsilon_{k,i,j} := \delta_{k,4^{j-1},k-1-2\ell_i}$ . It is obvious that

$$A_{k;\ell_1,\dots,\ell_n} \text{ is nonsingular} \Rightarrow \{c_{k,k-1-2\ell_i}\}_{i=1}^n \text{ is linearly independent.}$$

The rest part of this section is devoted to giving a criterion for the nonsingularity of  $A_{k;\ell_1,\dots,\ell_n}$ . We consider the “limit” matrix of  $A_{k;\ell_1,\dots,\ell_n}$ , that is, ignoring the error terms  $\varepsilon_{k,i,j}$  and defining

$$\mathcal{A}_{k;\ell_1,\dots,\ell_n} := \begin{bmatrix} 1 & 4^{\ell_1} & (4^{\ell_1})^2 & \dots & (4^{\ell_1})^{n-1} \\ 1 & 4^{\ell_2} & (4^{\ell_2})^2 & \dots & (4^{\ell_2})^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 4^{\ell_{n-1}} & (4^{\ell_{n-1}})^2 & \dots & (4^{\ell_{n-1}})^{n-1} \\ 1 & 4^{\ell_n} & (4^{\ell_n})^2 & \dots & (4^{\ell_n})^{n-1} \end{bmatrix}$$

which is a Vondermonde matrix. Thus, similar to [9, (2.8)], we have

$$\begin{aligned} \det \mathcal{A}_{k;\ell_1,\dots,\ell_n} &= \prod_{1 \leq i < j \leq n} (4^{\ell_j} - 4^{\ell_i}) \\ &= \prod_{1 \leq i < j \leq n} 4^{\ell_j} (1 - 4^{\ell_i - \ell_j}) \\ &\geq \prod_{1 \leq i < j \leq n} \frac{3}{4} \cdot 4^{\ell_j} \\ &= \left(\frac{3}{4}\right)^{n(n-1)/2} \prod_{j=1}^n 4^{\ell_j(j-1)}. \end{aligned} \tag{4.2}$$

We have the following observation, see also [9, Lemma 2.4] and [9, Corollary 2.5].

**Lemma 4.1.** *Let  $S_n$  denote the permutation group on the set  $\{1, \dots, n\}$  and  $\tau \in S_n$ . Then*

$$\prod_{1 \leq i \leq n} 4^{\ell_i(\tau(i)-1)}$$

*attains maximum when  $\tau = \mathbf{1}$  in  $S_n$ .*

*Proof.* It suffices to show that for integers  $\ell_2 > \ell_1 \geq 0$  and  $n_2 > n_1 \geq 1$  we have

$$4^{\ell_2(n_2-1)} \cdot 4^{\ell_1(n_1-1)} > 4^{\ell_2(n_1-1)} \cdot 4^{\ell_1(n_2-1)},$$

which is clear from

$$\ell_2(n_2 - 1) + \ell_1(n_1 - 1) - \ell_2(n_1 - 1) - \ell_1(n_2 - 1) = (\ell_2 - \ell_1)(n_2 - n_1) > 0.$$

This finishes the proof. □

Combining the inequality (4.2) and Lemma 4.1, we have the following.

**Corollary 4.1.** *For any  $\tau \in S_n$ , we have*

$$\prod_{i=1}^n 4^{\ell_i(\tau(i)-1)} \leq \left(\frac{4}{3}\right)^{n(n-1)/2} \cdot \det \mathcal{A}_n. \tag{4.3}$$

Now, we give the criterion for the nonsingularity of  $A_{k;\ell_1,\dots,\ell_n}$ .

**Proposition 4.2.** [9, Lemma 2.7] *Let  $A_{k;\ell_1,\dots,\ell_n}$  be as in (4.1). Define a function*

$$f(x_1, \dots, x_n) := \prod_{1 \leq i \leq n} (1 + x_i) - 1. \quad (4.4)$$

*Suppose that*

$$\sup_{\tau \in S_n} |f(\varepsilon_{k,1,\tau(1)}, \dots, \varepsilon_{k,n,\tau(n)})| < \left(\frac{3}{4}\right)^{n(n-1)/2} \cdot (n!)^{-1}. \quad (4.5)$$

*Then  $A_{k;\ell_1,\dots,\ell_n}$  is nonsingular.*

*Proof.* Note that

$$\begin{aligned} \det A_{k;\ell_1,\dots,\ell_n} &= \sum_{\tau \in S_n} (-1)^{\text{sign}(\tau)} \prod_{i=1}^n 4^{\ell_i(\tau(i)-1)} (1 + \varepsilon_{k;i,\tau(i)}) \\ &= \det \mathcal{A}_{k;\ell_1,\dots,\ell_n} + \sum_{\tau \in S_n} (-1)^{\text{sign}(\tau)} f(\varepsilon_{k,1,\tau(1)}, \dots, \varepsilon_{k,n,\tau(n)}) \prod_{i=1}^n 4^{\ell_i(\tau(i)-1)} \\ &\geq \det \mathcal{A}_{k;\ell_1,\dots,\ell_n} \left( 1 - n! \cdot \left(\frac{4}{3}\right)^{n(n-1)/2} \cdot \sup_{\tau \in S_n} |f(\varepsilon_{k,1,\tau(1)}, \dots, \varepsilon_{k,n,\tau(n)})| \right), \end{aligned}$$

where the inequality comes from (4.3). Thus, (4.5) implies that  $\det A_{k;\ell_1,\dots,\ell_n} > 0$ .  $\square$

## 5. PROOF OF RESULTS

In this section, we prove Theorems 1.1, 1.2 and 1.3. It suffices to show the nonsingularity of  $A_{k;\ell_1,\dots,\ell_n}$  in each case. The key is to manipulate  $\sup_{\tau \in S_n} |f(\varepsilon_{k,1,\tau(1)}, \dots, \varepsilon_{k,n,\tau(n)})|$  so that the condition (4.5) of Proposition 4.2 is satisfied. First, we prove an elementary inequality.

**Lemma 5.1.** *Let  $f$  be defined as in (4.4). Suppose there is an  $M > 0$  such that  $|x_i| \leq M$  for all  $1 \leq i \leq n$ . Then*

$$|f(x_1, \dots, x_n)| \leq (1 + M)^n - 1.$$

*Proof.* Note that

$$\begin{aligned} |f(x_1, \dots, x_n)| &= \left| \prod_{1 \leq i \leq n} (1 + x_i) - 1 \right| \\ &= \left| \sum_{i=1}^n x_i + \sum_{1 \leq i < j \leq n} x_i x_j + \dots + x_1 x_2 \dots x_n \right| \\ &\leq \binom{n}{1} M + \binom{n}{2} M^2 + \dots + \binom{n}{n} M^n \\ &= (1 + M)^n - 1, \end{aligned} \quad (5.1)$$

as desired.  $\square$

Theorem 1.1 follows immediately from the result below.

**Proposition 5.2.** *Let  $n \geq 1$  be an integer and  $k$  be an even integer with*

$$k > \max \left( 776, \frac{32\pi \sqrt[3]{20} \cdot 4^{13(n-1)/6}}{\sqrt[3]{n \sqrt{\left(\frac{3}{4}\right)^{(n-1)n/2} (n!)^{-1} + 1 - 1}}} + 1 \right) \quad (5.2)$$

*If  $0 \leq \ell_1 < \ell_2 < \dots < \ell_n \leq \frac{k-4}{2}$ , then  $A_{k;\ell_1,\dots,\ell_n}$  is nonsingular.*

*Proof.* Let  $m = 4^{j-1}$  and  $r = k - 1 - 2\ell_i$ , where  $1 \leq i, j \leq n$ . The assumption (5.2) on  $k$  allows us to use Proposition 3.5, which implies that

$$\begin{aligned} |\delta_{k,m,r}| &\leq 5 \left( \frac{32\pi}{k-1} \right)^r m^{2r} \log 4m \\ &\leq 5 \left( \frac{32\pi}{k-1} \right)^r m^{2r} 2(4m)^{1/2} \\ &\leq 20 \left( \frac{32\pi}{k-1} \right)^r m^{2r+r/6} \\ &= 20 \left( \frac{32\pi \cdot 4^{13(j-1)/6}}{k-1} \right)^r \\ &\leq 20 \left( \frac{32\pi \cdot 4^{13(n-1)/6}}{k-1} \right)^r. \end{aligned}$$

Note that (5.2) in particular implies that

$$\left( \frac{32\pi \cdot 4^{13(n-1)/6}}{k-1} \right)^r \leq \left( \frac{32\pi \cdot 4^{13(n-1)/6} \sqrt[3]{\sqrt{2}-1}}{32\pi \sqrt[3]{20} \cdot 4^{13(n-1)/6}} \right)^r < 1.$$

Then by Lemma 5.1

$$\begin{aligned} &\sup_{\tau \in S_n} |f(\varepsilon_{k,1,\tau(1)}, \dots, \varepsilon_{k,n,\tau(n)})| \\ &< \left( 1 + \left( \frac{32\pi \sqrt[3]{20} \cdot 4^{13(n-1)/6} \cdot \sqrt[3]{n \sqrt{\left(\frac{3}{4}\right)^{(n-1)n/2} (n!)^{-1} + 1 - 1}}}{32\pi \sqrt[3]{20} \cdot 4^{13(n-1)/6}} \right)^3 \right)^n - 1 \\ &= \left( \frac{3}{4} \right)^{n(n-1)/2} \cdot (n!)^{-1}. \end{aligned}$$

Hence  $A_{k;\ell_1,\dots,\ell_n}$  is nonsingular by Proposition 4.2. □

We next present the proof of Theorem 1.2.

*Proof of Theorem 1.2.* Furthermore, note that Theorem 1.1 or Proposition 5.2 implies that for even integer  $k$  with

$$\begin{aligned} k &> \frac{32\pi \sqrt[3]{20} \cdot 4^{13(2-1)/6}}{\sqrt[3]{\sqrt[2]{\left(\frac{3}{4}\right)^{(2-1)2/2} (2!)^{-1} + 1 - 1}}} + 1 \\ &\approx 9880.98 \\ &\geq 9880 \end{aligned}$$

if  $\ell_1$  and  $\ell_2$  are integers such that  $0 \leq \ell_1 < \ell_2 \leq \frac{k-4}{2}$ , then the periods  $r_{2\ell_1, \vee^2}$  and  $r_{2\ell_2, \vee^2}$  are linearly independent. Denote by  $c_{k,r}(m)$  the  $m^{\text{th}}$  Fourier coefficient of  $c_{k,r}$ . We verified the nonsingularity of the matrix

$$C_{k;\ell_1,\ell_2} := \begin{bmatrix} c_{k,k-1-2\ell_1}(1) & c_{k,k-1-2\ell_1}(4) \\ c_{k,k-1-2\ell_2}(1) & c_{k,k-1-2\ell_2}(4) \end{bmatrix} \quad (5.3)$$

for  $12 \leq k \leq 9882, k \neq 14$ , and  $0 \leq \ell_1 < \ell_2 \leq \frac{k-4}{2}$  by **SageMath**, see [14] for the codes. Thus, the proof of Theorem 1.2 is complete.  $\square$

We last prove Theorem 1.3.

**Lemma 5.3.** *Let  $n \geq 1$  and  $k \geq e^{12}$ . If  $0 \leq \ell_1 < \dots < \ell_n \leq \frac{k-2}{4}$  are integers and  $n \leq \frac{\log k}{4}$ , then*

$$\sup_{\tau \in S_n} |f(\varepsilon_{k,1,\tau(1)}, \dots, \varepsilon_{k,n,\tau(n)})| \leq k^{\left[ \frac{\log(k-1) - \log(32\pi \cdot 4^{-13/6})}{\log k} - \frac{13 \log 4}{24} - \frac{2(\log(5(\log k)^2) - \log 4)}{k \log k} \right] \cdot \frac{-k}{2}}.$$

*Proof.* First, we find a uniform bound for  $\varepsilon_{k,i,j}$  ( $1 \leq i, j \leq n$ ). Let  $m = 4^{j-1}$  and  $r = k - 1 - 2\ell_i$ . By Proposition 3.5, we have

$$\begin{aligned} |\varepsilon_{k,i,j}| &\leq 5 \left( \frac{32\pi}{k-1} \right)^r m^{2r} \log 4m \\ &\leq 5 \left( \frac{32\pi}{k-1} \right)^r m^{2r} 2(4m)^{1/2} \\ &\leq 20 \left( \frac{32\pi}{k-1} \right)^r m^{2r} \cdot m^{r/6} \\ &= 20 \left( \frac{32\pi \cdot 4^{13(j-1)/6}}{k-1} \right)^{k-1-2\ell_i}. \end{aligned} \quad (5.4)$$

Since  $j \leq \frac{\log k}{4}$ , we have

$$\begin{aligned} \frac{32\pi \cdot 4^{13(j-1)/6}}{k-1} &= \frac{32\pi \cdot 4^{-13/6} \cdot 4^{13j/6}}{k-1} \\ &\leq \frac{32\pi \cdot 4^{-13/6} \cdot 4^{13 \log k / 24}}{k-1} \\ &= k^{-\frac{\log(k-1) - \log(32\pi \cdot 4^{-13/6})}{\log k} + \frac{13 \log 4}{24}}. \end{aligned} \quad (5.5)$$

Note that in particular,

$$k^{-\frac{\log(k-1)-\log(32\pi\cdot 4^{-13/6})}{\log k} + \frac{13\log 4}{24}} \leq k^{-\frac{\log(e^{12}-1)-\log(32\pi\cdot 4^{-13/6})}{\log e^{12}} + \frac{13\log 4}{24}} \leq k^{-0.115} < 1. \quad (5.6)$$

As  $k-1-2\ell_i \geq \frac{k}{2}$ , (5.4) and (5.5) imply that

$$|\varepsilon_{k,i,j}| \leq 20k^{\left[\frac{\log(k-1)-\log(32\pi\cdot 4^{-13/6})}{\log k} - \frac{13\log 4}{24}\right] \cdot \frac{-k}{2}} \quad (5.7)$$

We now give the bound of  $\sup_{\tau \in S_n} |f(\varepsilon_{k,1,\tau(1)}, \dots, \varepsilon_{k,n,\tau(n)})|$ . To simplify the proof, we will use a different bound of  $f$  defined in (4.4). By a similar argument as (5.1), if  $|x_i| \leq M$  for all  $1 \leq i \leq n$  and  $nM \leq 1$  then

$$|f(x_1, \dots, x_n)| \leq \binom{n}{1}M + \binom{n}{2}M^2 + \dots + \binom{n}{n}M^n \leq \sum_{i=1}^n (nM)^i \leq n^2M. \quad (5.8)$$

Note that (5.7) and (5.6) imply that

$$|\varepsilon_{k,i,j}| \leq 20 \left( \frac{1}{k^{0.115}} \right)^{k/2} \leq \frac{4}{\log k} \leq \frac{1}{n}.$$

Hence by (5.8) and (5.7),

$$\begin{aligned} \sup_{\tau \in S_n} |f(\varepsilon_{k,1,\tau(1)}, \dots, \varepsilon_{k,n,\tau(n)})| &\leq n^2 \cdot 20k^{\left[\frac{\log(k-1)-\log(32\pi\cdot 4^{-13/6})}{\log k} - \frac{13\log 4}{24}\right] \cdot \frac{-k}{2}} \\ &\leq \frac{5(\log k)^2}{4} \cdot k^{\left[\frac{\log(k-1)-\log(32\pi\cdot 4^{-13/6})}{\log k} - \frac{13\log 4}{24}\right] \cdot \frac{-k}{2}} \\ &= k^{\left[\frac{\log(k-1)-\log(32\pi\cdot 4^{-13/6})}{\log k} - \frac{13\log 4}{24} - \frac{2(\log(5(\log k)^2)-\log 4)}{k \log k}\right] \cdot \frac{-k}{2}}, \end{aligned}$$

as desired.  $\square$

Theorem 1.3 follows from the next result.

**Proposition 5.4.** *Let  $n \geq 1$  and  $k \geq e^{12}$ . If  $0 \leq \ell_1 < \ell_2 < \dots < \ell_n \leq \frac{k-2}{4}$  and  $n \leq \frac{\log k}{4}$ , then  $A_{k;\ell_1,\dots,\ell_n}$  is nonsingular.*

*Proof.* First, we give a lower bound of the right hand side of (4.5). Recall the non-asymptotic Stirling's approximation [15], which states that for all  $n \geq 1$

$$\sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{1}{12n}}.$$

In particular, we have

$$\sqrt{2\pi} < \frac{n!e^n}{n^{n+\frac{1}{2}}} \leq e. \quad (5.9)$$

Using (5.9) and the fact that  $(e/x)^x$  decreases for  $x > 0$ , we get

$$\left( \frac{3}{4} \right)^{n(n-1)/2} \cdot (n!)^{-1} \geq \left( \frac{3}{4} \right)^{n(n-1)/2} \frac{1}{e\sqrt{2\pi n}} \left( \frac{e}{n} \right)^n$$

$$\begin{aligned}
&\geq \left(\frac{3}{4}\right)^{\frac{\log k}{4} \left(\frac{\log k}{4} - 1\right)/2} \frac{2}{e\sqrt{2\pi \log k}} \left(\frac{4e}{\log k}\right)^{\frac{\log k}{4}} \\
&= k^{\frac{-\log(4/3)}{4} \cdot \left(\frac{\log k}{4} - 1\right)/2} \frac{2}{e\sqrt{2\pi \log k}} \left(\frac{4e}{\log k}\right)^{\frac{\log k}{4}}.
\end{aligned}$$

It is not difficult to see that for all  $k \geq e^{12}$

$$\frac{2}{e\sqrt{2\pi \log k}} \geq \frac{1}{k^{\log(4/3)}} \quad \text{and} \quad \left(\frac{4e}{\log k}\right)^{\frac{\log k}{4}} \geq k^{\frac{-\log(4/3)}{4} \cdot \frac{\log k}{4}},$$

which implies that

$$\begin{aligned}
\left(\frac{3}{4}\right)^{n(n-1)/2} \cdot (n!)^{-1} &\geq k^{\frac{-\log(4/3)}{4} \cdot \left[\left(\frac{\log k}{4} - 1\right)/2 + 4 + \frac{\log k}{4}\right]} \\
&= k^{\frac{\log(4/3)}{4} \cdot \left(-\frac{3 \log k}{8} - \frac{7}{2}\right)} \\
&\geq k^{\frac{\log(4/3)}{4} \cdot \frac{-k}{2}}.
\end{aligned}$$

On the other hand,

$$\sup_{\tau \in S_n} |f(\varepsilon_{k,1,\tau(1)}, \dots, \varepsilon_{k,n,\tau(n)})| \leq k^{\left[ \frac{\log(k-1) - \log(32\pi \cdot 4^{-13/6})}{\log k} - \frac{13 \log 4}{24} - \frac{2(\log(5(\log k)^2) - \log 4)}{k \log k} \right] \cdot \frac{-k}{2}}.$$

Note that for  $k \geq e^{12}$  we have

$$\frac{\log(k-1) - \log(32\pi \cdot 4^{-13/6})}{\log k} - \frac{13 \log 4}{24} - \frac{2(\log(5(\log k)^2) - \log 4)}{k \log k} \geq 0.115 > \frac{\log(4/3)}{4},$$

which implies that

$$\sup_{\tau \in S_n} |f(\varepsilon_{k,1,\tau(1)}, \dots, \varepsilon_{k,n,\tau(n)})| \leq k^{0.115 \cdot \frac{-k}{2}} < k^{\frac{\log(4/3)}{4} \cdot \frac{-k}{2}} \leq \left(\frac{3}{4}\right)^{n(n-1)/2} \cdot (n!)^{-1}$$

Therefore,  $A_{k;\ell_1,\dots,\ell_n}$  is nonsingular by Proposition 4.2.  $\square$

## 6. DISCUSSIONS

Since the lower bound for  $k$  in Proposition 5.2 is gigantic, a natural question is how much we can improve it. As mentioned in [9, Remark 2.9], there is another natural choice of matrix to investigate the linear independence of the periods. Recall that  $a_{k,r}(m)$  denotes the  $m^{\text{th}}$  Fourier coefficient of  $f_r$  defined by (3.2). We define

$$\begin{aligned}
B_{k;\ell_1,\dots,\ell_n} &:= [a_{k-1-2\ell_i}(j^2)]_{i=1,\dots,n;j=1,\dots,n} \\
&= [j^{2\ell_i}(1 + \eta_{k,i,j})]_{i=1,\dots,n;j=1,\dots,n},
\end{aligned}$$



where  $\eta_{k,i,j}$  goes to zero as  $k$  tends to infinity. Then the “limit” matrix becomes

$$\mathcal{B}_{k;\ell_1,\dots,\ell_n} := \begin{bmatrix} 1 & 2^{2\ell_1} & 3^{2\ell_1} & \dots & n^{2\ell_1} \\ 1 & 2^{2\ell_2} & 3^{2\ell_2} & \dots & n^{2\ell_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{2\ell_{n-1}} & 3^{2\ell_{n-1}} & \dots & n^{2\ell_{n-1}} \\ 1 & 2^{2\ell_n} & 3^{2\ell_n} & \dots & n^{2\ell_n} \end{bmatrix}, \quad (6.1)$$

which is a generalized Vandermonde matrix [4, Vol.2, p.99]. If one can get a good result about the lower bound of the determinant of (6.1), then the upper bound for  $\sup_{\tau \in S_n} |f(\varepsilon_{k,1,\tau(1)}, \dots, \varepsilon_{k,n,\tau(n)})|$  in (4.5) becomes bigger, which will lead to a smaller lower bound for  $k$  in Proposition 5.2. It is reasonable to propose the following conjecture.

**Conjecture 6.1.** Let  $0 \leq \ell_1 < \ell_2 < \dots < \ell_n \leq \frac{k-4}{2}$ . If  $n \leq \dim S_k$ , then the set of periods  $\{r_{2\ell_i, \sqrt{2}}\}_{i=1}^n$  is linearly independent.

Note that Theorem 1.2 provides evidence for Conjecture 6.1 when  $n = 2$ . The case  $n = 1$  is also true since we verified the nonsingularity of the matrix  $C_{k,\ell_1,\ell_2}$  defined in (5.3) for  $12 \leq k \leq 9882, k \neq 14$ . There is another question related to the set of periods  $\{r_{2\ell, \sqrt{2}}\}_{\ell=0}^{(k-4)/2}$ , that is, can this set span the whole space of  $S_k^*$ ? We believe that the answer is affirmative, which also follows from Conjecture 6.1, but we are only able to show the following corollary of Theorem 1.3.

**Corollary 6.1.** Suppose  $k \geq e^{12}$  is even. Then

$$\dim \text{Span} \left( \{r_{2\ell, \sqrt{2}}\}_{\ell=0}^{(k-4)/2} \right) \geq \left\lfloor \frac{\log k}{4} \right\rfloor.$$

*Proof.* This is a direct corollary of Theorem 1.3. □

Another natural question about the periods  $r_{2\ell, \sqrt{2}}$  is whether we can find Eichler-Shimura type linear relations (ES2) and (ES3) among them. We hope to take it up in a future work.

## REFERENCES

- [1] Tom M. Apostol. *Introduction to analytic number theory*. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976.
- [2] Henri Cohen. Sums involving the values at negative integers of  $L$ -functions of quadratic characters. *Math. Ann.*, 217(3):271–285, 1975.
- [3] P. Deligne. Valeurs de fonctions  $L$  et périodes d’intégrales. In *Automorphic forms, representations and  $L$ -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 313–346, ., 1979. With an appendix by N. Koblitz and A. Ogus.
- [4] F. R. Gantmacher. *The theory of matrices. Vols. 1, 2*. Chelsea Publishing Co., New York, 1959. Translated by K. A. Hirsch.

- [5] Haruzo Hida.  $p$ -adic  $L$ -functions for base change lifts of  $GL_2$  to  $GL_3$ . In *Automorphic forms, Shimura varieties, and  $L$ -functions, Vol. II (Ann Arbor, MI, 1988)*, volume 11 of *Perspect. Math.*, pages 93–142. Academic Press, Boston, MA, 1990.
- [6] Winfried Kohnen and Don Zagier. Modular forms with rational periods. In *Modular forms (Durham, 1983)*, Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., pages 197–249. Horwood, Chichester, 1984.
- [7] Serge Lang. *Introduction to modular forms*, volume 222 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1995. With appendixes by D. Zagier and Walter Feit, Corrected reprint of the 1976 original.
- [8] Austin Lei, Tianyu Ni, and Hui Xue. Linear independence of even periods of modular forms. *J. Number Theory*, 248:120—139, 2023.
- [9] Austin Lei, Tianyu Ni, and Hui Xue. Linear independence of odd periods of modular forms. *Res. Number Theory*, 9(2):Paper No. 33, 20, 2023.
- [10] Ju. I. Manin. Periods of cusp forms, and  $p$ -adic Hecke series. *Mat. Sb. (N.S.)*, 92(134):378–401, 503, 1973.
- [11] Melvyn B. Nathanson. *Elementary methods in number theory*, volume 195 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.
- [12] James Newton and Jack A. Thorne. Symmetric power functoriality for holomorphic modular forms, II. *Publ. Math. Inst. Hautes Études Sci.*, 134:117–152, 2021.
- [13] James Newton and Jack A. Thorne. Symmetric power functoriality for holomorphic modular forms, II. *Publ. Math. Inst. Hautes Études Sci.*, 134:117–152, 2021.
- [14] Tianyu Ni. Sage code for computations of the coefficient matrix. <https://github.com/TianyuNi1994math/LINEAR-INDEPENDENCE-OF-PERIODS-FOR-THE-SECOND-SYMMETRIC-POWER-OF-L-FUNCTIONS>.
- [15] Herbert Robbins. A remark on Stirling’s formula. *Amer. Math. Monthly*, 62:26–29, 1955.
- [16] Jean-Pierre Serre. Une interprétation des congruences relatives à la fonction  $\tau$  de Ramanujan. *Séminaire Delange-Pisot-Poitou. Théorie des nombres*, 9(1), 1967-1968. talk:14.
- [17] Jean-Pierre Serre. Répartition asymptotique des valeurs propres de l’opérateur de Hecke  $T_p$ . *J. Amer. Math. Soc.*, 10(1):75–102, 1997.
- [18] Goro Shimura. On the holomorphy of certain Dirichlet series. *Proc. London Math. Soc. (3)*, 31(1):79–98, 1975.
- [19] Goro Shimura. The special values of the zeta functions associated with cusp forms. *Comm. Pure Appl. Math.*, 29(6):783–804, 1976.
- [20] Goro Shimura. On the periods of modular forms. *Math. Ann.*, 229(3):211–221, 1977.
- [21] Jacob Sturm. Special values of zeta functions, and Eisenstein series of half integral weight. *Amer. J. Math.*, 102(2):219–240, 1980.
- [22] Hui Xue. Linear independence between odd and even periods of modular forms. Submitted 2023. Available at <https://www.math.clemson.edu/numbertheory/wp-content/uploads/2023/10/Mixedperiods.pdf>.
- [23] Don Zagier. Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields. In *Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, pages 105–169. Lecture Notes in Math., Vol. 627, 1977.

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