SC 216 Calculus with Complex Variables

April 3, 2008

Complete eqn

The general second order linear D.E. is

$$\boxed{\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = R(x) \quad \text{---} (1)}$$

In general we do not know the explicit soln.

So we want to know existence & uniqueness

Theorem Let $P(x)$, $Q(x)$ & $R(x)$ be conts fns. on a closed interval. If $x_0 \in [a,b]$ & if y_0 & y_0' are any numbers whatever then (1) has one and only one soln. $y(x)$ on the entire interval s.t. $y(x_0) = y_0$ & $y'(x_0) = y_0'$.

Example Find the soln. of the IVP (Initial Value problem)

$$y'' + y = 0 \quad , \; y(0) = 0 \;\; \& \;\; y'(0) = 1$$

$\square$ We know $y = \sin x$ & $y = \cos x$

in general $y = c_1 \sin x + c_2 \cos x$ are all solns. of this D.E. But only $y = \sin x$ is the soln. that satisfies the initial conditions

Simly, $y = \cos x$ is the only soln. of the following IVP

$$y'' + y = 0 \quad , \; y(0) = 1 \;\; \& \;\; y'(0) = 0.$$

B.V.P. If in eqn (1) we put the conditions of the form $y(x_0) = y_0$ & $y(x_1) = y_1$ for $x_0, x_1 \in [a,b]$ such a problem is called Boundary Value problem (B.V.P.)

Reduced $\swarrow$ eq^n

— Eq^{n}s of the type

$$\boxed{y'' + P(x)y' + Q(x)y = 0} \quad \text{--- (2)}$$

are called Homogeneous eq^{n}s.

— If $R(x) \neq 0$ in (1) we call it a non Homo. eq^n.

— $\boxed{\text{Theorem}}$

If $y_g(x, c_1, c_2)$ is the general soln (G.S.) of the reduced eq^n (2) & y_p is ~~the~~ any Particular Soln. (P.S.) of the complete eq^n (1) then $y_g + y_p$ is the general soln. of (1).

☐ ~~Note that $y(x) - y_p(x)$~~

Suppose $y(x)$ in any soln. of (1) then note that $y(x) - y_p(x)$ is a soln. of (2)

For ~~y_p~~ $(y - y_p)'' + P(x)(y - y_p)' + Q(x)(y - y_p)$

$$= \boxed{y'' + P(x)y' + Q(x)y} - [y_p'' + P(x)y_p' + Q(x)y_p]$$

$$= R(x) - R(x) = 0$$

~~$\Rightarrow$~~ $\because$ $y_g(x, c_1, c_2)$ is the G.S. of (2)

$$\Rightarrow y(x) - y_p(x) = y_g(x, c_1, c_2)$$

$$\Rightarrow y(x) = y_g(x, c_1, c_2) + y_p(x)$$

$\boxed{\text{Theorem}}$ If $y_1(x)$ & $y_2(x)$ are two solutions of (2) then $c_1 y_1(x) + c_2 y_2(x)$ is also a soln. for any consts. c_1 & c_2

☐ Prove it your self.

Remark The above theorem tells us that the linear combination of two solutions of homo. eqn is also a soln.

— The soln. given by above theorem will be general if y_1 & y_2 are l.i. (linearly independent) soln. (Recall your Algebraic Str class)

i.e. if $\nexists$ a const. λ s.t. $y_1 = \lambda y_2$)
(does not exist)

Recall Wronskian of two functions!

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$y'' + P(x) y' + Q(x) y = 0 \quad (*)$$

Lemma 1 If $y_1(x)$ & $y_2(x)$ are any two solns. of $(*)$ on $[a, b]$ then their Wronskian $W = W(y_1, y_2)$ is either identically zero or never zero on $[a, b]$.

$\square$ $\because W = y_1 y_2' - y_2 y_1'$

$$\Rightarrow W' = y_1 y_2'' + y_1' y_2' - y_2 y_1'' - y_2' y_1'$$
$$= y_1 y_2'' - y_2 y_1''$$

Now both y_1 & y_2 are solns. of (1) we have

$$y_1'' + P y_1' + Q y_1 = 0$$
$$\& \; y_2'' + P y_2' + Q y_2 = 0$$

Multiply first by y_2 & second by y_1 & subtract to get

$$(y_1 y_2'' - y_2 y_1'') + P(y_1 y_2' - y_2 y_1') = 0$$

or $$\frac{dW}{dx} + PW = 0$$

$\Rightarrow$ G.S. of this first order eqn is

$$W = c \cdot e^{-\int P dx} \neq 0 \Rightarrow W \equiv 0 \text{ if } c = 0$$
$$\& \neq 0 \text{ if } c \neq 0 \quad \blacksquare$$

$\boxed{\text{Lemma 2}}$ If $y_1(x)$ & $y_2(x)$ are two solutions of $y'' + P(x)y' + Q(x)y = 0$ on $[a,b)$ then they are l.d. on $[a,b]$ iff their Wronskian $W(y_1, y_2) = y_1 y_2' - y_2 y_1' \equiv 0$.

□ Proof is left as an exercise!

(Hint: Assume $y_2 = c y_1$)

$\boxed{\text{Theorem}}$ Let $y_1(x)$ & $y_2(x)$ be l.i. solns of the homo eq^n $y'' + P(x)y' + Q(x)y = 0$ on $[a,b]$. Then $c_1 y_1(x) + c_2 y_2(x)$ is the general soln. in the sense that every soln can be obtained from this by suitable choice of arbitrary const. c_1 & c_2.

$\boxed{\text{Example}}$ Show that $y = c_1 \sin x + c_2 \cos x$ is the G.S. of $y'' + y = 0$ on any interval & find the P.S. for which $y(0) = 2$ & $y'(0) = 3$

□ $y_1 = \sin x$ & $y_2 = \cos x$ is a soln. (Verify!)

and $W(y_1, y_2) = -1 \neq 0$

$\Rightarrow$ y_1 & y_2 are l.i.

$\Rightarrow$ $y = c_1 \sin x + c_2 \cos x$ is a G.S.

To find the P.S. put $y(0) = 2$ & $y'(0) = 3$

gives

$$c_1 \sin 0 + c_2 \cos 0 = 2$$
$$c_1 \cos 0 - c_2 \sin 0 = 3$$

$\Rightarrow c_2 = 2$ & $c_1 = 3$

$\Rightarrow y = 3 \sin x + 2 \cos x$ is a P.S.