Signals and Systems (CT 203)

Tutorial Sheet-09 DA-IICT, Gandhinagar.

1. Find the Fourier transform of

   Solution:

   (a) $\delta(t+1)+\delta(t-1)$

   Let $f(t)=\delta(t+1)+\delta(t-1)$

   By using the linearity property of Fourier transform,

   $$F\{f(t)\}=F\{\delta(t+1)\}+F\{\delta(t-1)\}$$

   Shifting property of Fourier transform gives,

   $$F\{f(t-t_0)\}=e^{-j\omega t_0}\cdot F(\omega)$$

   Therefore,

   $$\begin{aligned} F(\omega) &= e^{-j\omega}\delta(\omega)+e^{j\omega}\delta(\omega) \\ &= 2\delta(\omega)\left[\frac{e^{-j\omega}+e^{j\omega}}{2}\right] \\ &= 2\cos(\omega) \end{aligned}$$

   (b) $e^{-2t}u(t)$

   $$f(t)=e^{-2t}u(t)$$

   As we know by definition of the fourier transform,

   $$F\{f(t)=e^{-at}u(t)\}=\frac{1}{a+j\omega}$$

   Hence,

   $$F\{f(t)=e^{-2t}u(t)\}=\frac{1}{2+j\omega}$$

   (c) $e^{-3(t-1)}u(t-1)$

   $$f(t)=e^{-3(t-1)}u(t-1)$$

   Using the definition of the Fourier transform,

   $$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{\infty}\left[e^{-3(t-1)}u(t-1)\right]e^{-j\omega t}\,dt \\ &= \int_{1}^{\infty} e^{-3(t-1)}e^{-j\omega t}\,dt \\ &= \int_{1}^{\infty} e^{-(3+j\omega)t+3}\,dt \end{aligned}$$

$$= \left[\frac{e^{-(3+j\omega)t+3}}{-(3+j\omega)} \right]_1^{\infty}$$

$$= \frac{e^{-j\omega}}{3+j\omega}$$

2. Consider a causal LTI system with frequency response

$$H(j\omega) = \frac{1}{4+j\omega}.$$

For a particular input $x(t)$ this system is observed to produce the output

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t)$$

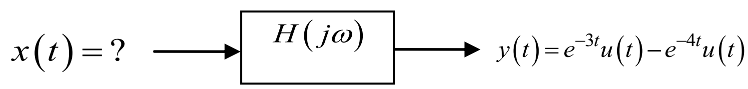

Fig.3. A causal LTI system

Determine $x(t)$.

Solution:

As we know that, frequency response of the system is given by,

$$H(j\omega) = \frac{Y(\omega)}{X(\omega)} \qquad \text{(Due to Convolution Theorem of Fourier Transform)}$$

Therefore,

$$X(\omega) = \frac{Y(\omega)}{H(j\omega)}$$

Here we are given the output y(t) and the Fourier transform of the output is given by the following equation,

$$Y(\omega) = \frac{1}{3+j\omega} - \frac{1}{4+j\omega}$$

$$= \frac{4+j\omega-3-j\omega}{(3+j\omega)(4+j\omega)}$$

$$Y(\omega) = \frac{1}{(3+j\omega)(4+j\omega)}$$

Therefore,

$$X(\omega) = \frac{Y(\omega)}{H(j\omega)} = \frac{1}{(3+j\omega)(4+j\omega)}(4+j\omega)$$

$$X(\omega) = \frac{1}{(3+j\omega)}$$

Taking inverse Fourier transform of the above signal, we get

$$x(t) = F^{-1}(X(\omega)) = e^{-3t}u(t)$$

3. Consider a causal LTI system implemented as the RLC circuit shown in Fig. In this circuit $x(t)$ is the input voltage. The voltage $y(t)$ across the capacitor is considered the system output.

   a) Find the differential equation relating $x(t)$ and $y(t)$
   b) Determine the frequency response using Fourier transform
   c) Determine the impulse response of this electric circuit using inverse Fourier transform and convolution theorem

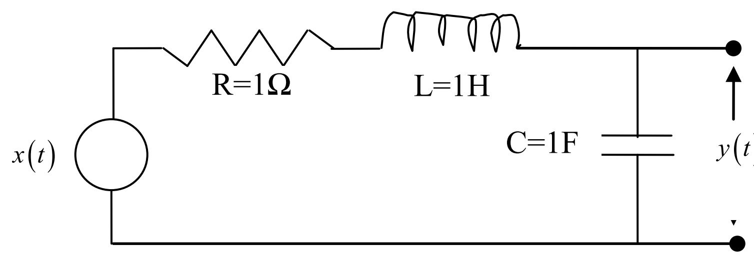

Fig.1. An RLC circuit

Solution:
Applying KVL,

$$y(t) = x(t) - Ri(t) - L\frac{di(t)}{dt}$$

Voltage across the capacitor y(t)

$$y(t) = \tfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

$$i(t) = C\frac{dy(t)}{dt}$$

Here, $R = 1, C = 1, L = 1$

Therefore, $\dfrac{d^2y(t)}{dt^2} + \dfrac{dy(t)}{dt} + y(t) = x(t)$

Applying Fourier transform,

$$(j\omega)^2 Y(\omega) + (j\omega)Y(\omega) + Y(\omega) = X(\omega)$$

$$H(j\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + (j\omega) + 1}$$

Putting $x = j\omega$ we get,

$$\frac{1}{x^2 + x + 1} = \frac{1}{x^2 + x + \frac{1}{4} + \frac{3}{4}} = \frac{1}{(x + \frac{1}{2})^2 + \frac{3}{4}}$$

Finding the roots of the denominator polynomial $(x + \frac{1}{2})^2 + \frac{3}{4} = 0$

$$x + \tfrac{1}{2} = \pm j\tfrac{\sqrt{3}}{2}$$

$$x = -\tfrac{1}{2} \pm j\tfrac{\sqrt{3}}{2}$$

$$H(x)=\frac{1}{x^2+x+1}=\frac{1}{(x+\frac{1}{2}-j\frac{\sqrt{3}}{2})(x+\frac{1}{2}+j\frac{\sqrt{3}}{2})}=\frac{A}{(x+\frac{1}{2}-j\frac{\sqrt{3}}{2})}+\frac{B}{(x+\frac{1}{2}+j\frac{\sqrt{3}}{2})}$$

Solving by using the partial fraction method we get $A=\frac{1}{j\sqrt{3}}, B=-\frac{1}{j\sqrt{3}}$

Putting x= $j\omega$ we get, $H(j\omega)=\frac{-1}{j\sqrt{3}}\left[\frac{-1}{\frac{1}{2}-\frac{\sqrt{3}}{2}j+j\omega}+\frac{1}{\frac{1}{2}+\frac{\sqrt{3}}{2}j+j\omega}\right]$

Taking inverse Fourier transforms,

$$F^{-1}\{H(j\omega\}=h(t)=\frac{-1}{j\sqrt{3}}F^{-1}\left\{\frac{-1}{\frac{1}{2}-\frac{\sqrt{3}}{2}j+j\omega}\right\}+\frac{-1}{j\sqrt{3}}F^{-1}\left\{\frac{-1}{\frac{1}{2}+\frac{\sqrt{3}}{2}j+j\omega}\right\}$$

Hence we get the impulse response of the system is the following

$$h(t)=\frac{-1}{j\sqrt{3}}\left[e^{-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}j\right)t}u(t)+e^{-\left(\frac{1}{2}+\frac{\sqrt{3}}{2}j\right)t}u(t)\right]$$

$$h(t)=\frac{2}{\sqrt{3}}e^{\frac{-t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)u(t)$$

4. Let $x(t)$ and $y(t)$ be two real signals. Then the cross-correlation function of $x(t)$ and $y(t)$ is defined by

$$\phi_{xy}(t)=\int_{-\infty}^{+\infty}x(t+\tau)y(\tau)d\tau$$

and the autocorrelation of $x(t)$ is defined as

$$\phi_{xx}(t)=\int_{-\infty}^{+\infty}x(t+\tau)x(\tau)d\tau$$

Solution:

We have, $\Phi_{xy}(t)=\int_{-\infty}^{\infty}x(t+\tau)y(\tau)d\tau$ ……………………….(1)

(a) What is the relationship between $\Phi_{xy}(\omega)$ and $\Phi_{yx}(\omega)$?

We know that,

$\Phi_{xy}(t)=\Phi_{yx}(-t)$ …………………………..(2)

Taking Fourier transform on both sides

$\Phi_{xy}(\omega)=\Phi_{yx}(-\omega)$ …………………………(3)

Since $\Phi_{yx}(t)$ is real function, using conjugate symmetry property of Fourier transform

$\Phi_{yx}(\omega)=\Phi^*_{yx}(-\omega)$ …………………………(4)

$\Rightarrow\Phi_{yx}(-\omega)=\Phi^*_{yx}(\omega)$

From eq(3) we have

$\Phi^*_{yx}(\omega)=\Phi_{xy}(\omega)$

(b) Find expression for $\Phi_{xy}(\omega)$ in terms of $X(\omega)$ and $Y(\omega)$.

From eq(1) we can have, $\Phi_{xy}(t) = \int_{-\infty}^{\infty} x(t+\tau)y(\tau)d\tau = x(t) * y(-t)$

$\Phi_{xy}(\omega) = F\{x(t) * y(-t)\} = X(\omega)Y(-\omega)$

Since y(t) is real we have

$Y^*(-\omega) = Y(\omega) \Rightarrow Y^*(\omega) = Y(-\omega)$

Therefore, $\Phi_{xy}(\omega) = X(\omega)Y^*(\omega)$ ....(5)

(c) Show that $\Phi_{xx}(\omega)$ is real and nonnegative for every ω

From eq(5) we can write,

$\Phi_{xx}(\omega) = X(\omega)X^*(\omega) = |X(\omega)|^2 \geq 0$ ....(6)

(d) Suppose that $x(t)$ is the input to an LTI system with a real-valued impulse response and with frequency response $H(\omega)$ and that $y(t)$ is the output. Find expressions for $\Phi_{xy}(\omega)$ and $\Phi_{yy}(\omega)$ in terms of $\Phi_{xx}(\omega)$ and $H(\omega)$.

From eq(5) we can have,

$\Phi_{xy}(\omega) = X(\omega)Y^*(\omega) = X(\omega)\left[X(\omega)H(\omega)\right]^*$

$\Phi_{xy}(\omega) = |X(\omega)|^2 H^*(\omega)$

$\Phi_{xy}(\omega) = \Phi_{xx}(\omega)H(\omega)$

$\Phi_{yy}(\omega) = Y(\omega)Y^*(\omega) = \left[X(\omega)H(\omega)\right]\left[X(\omega)H(\omega)\right]^* = |H(\omega)|^2 \Phi_{xx}(\omega)$