Next, we write

$$\Delta I = \frac{\partial I}{\partial z}\Delta z \quad \text{and} \quad \Delta V = \frac{\partial V}{\partial z}\Delta z \tag{3}$$

which are then substituted into (2) to result in

$$\frac{\partial V}{\partial z} = -\left(1 + \frac{\Delta z}{2}\frac{\partial}{\partial z}\right)\left(RI + L\frac{\partial I}{\partial t}\right) \tag{4}$$

Now, in the limit as $\Delta z$ approaches zero (or a value small enough to be negligible), (4) simplifies to the final form:

$$\boxed{\frac{\partial V}{\partial z} = -\left(RI + L\frac{\partial I}{\partial t}\right)} \tag{5}$$

Equation (5) is the first of the two equations that we are looking for. To find the second equation, we apply KCL to the upper central node in the circuit of Figure 11.3, noting from the symmetry that the voltage at the node will be $V + \Delta V/2$:

$$\begin{aligned} I = I_g + I_c + (I + \Delta I) = G\Delta z\left(V + \frac{\Delta V}{2}\right) \\ + C\Delta z\frac{\partial}{\partial t}\left(V + \frac{\Delta V}{2}\right) + (I + \Delta I) \end{aligned} \tag{6}$$

Then, using (3) and simplifying, we obtain

$$\frac{\partial I}{\partial z} = -\left(1 + \frac{\Delta z}{2}\frac{\partial}{\partial z}\right)\left(GV + C\frac{\partial V}{\partial t}\right) \tag{7}$$

Again, we obtain the final form by allowing $\Delta z$ to be reduced to a negligible magnitude. The result is

$$\boxed{\frac{\partial I}{\partial z} = -\left(GV + C\frac{\partial V}{\partial t}\right)} \tag{8}$$

The coupled differential equations, (5) and (8), describe the evolution of current and voltage in any transmission line. Historically, they have been referred to as the *telegraphist's equations*. Their solution leads to the wave equation for the transmission line, which we now undertake. We begin by differentiating Eq. (5) with respect to $z$ and Eq. (8) with respect to $t$, obtaining

$$\frac{\partial^2 V}{\partial z^2} = -R\frac{\partial I}{\partial z} - L\frac{\partial^2 I}{\partial t \partial z} \tag{9}$$

and

$$\frac{\partial I}{\partial z \partial t} = -G\frac{\partial V}{\partial t} - C\frac{\partial^2 V}{\partial t^2} \tag{10}$$

Next, Eqs. (8) and (10) are substituted into (9). After rearranging terms, the result is

$$\boxed{\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2} + (LG + RC)\frac{\partial V}{\partial t} + RGV} \tag{11}$$

An analogous procedure involves differentiating Eq. (5) with respect to $t$ and Eq. (8) with respect to $z$. Then, Eq. (5) and its derivative are substituted into the derivative of (8) to obtain an equation for the current that is in identical form to that of (11):

$$\boxed{\frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2} + (LG + RC)\frac{\partial I}{\partial t} + RGI} \tag{12}$$

Equations (11) and (12) are the *general wave equations* for the transmission line. Their solutions under various conditions form a major part of our study.

## 11.3 LOSSLESS PROPAGATION

Lossless propagation means that power is not dissipated or otherwise deviated as the wave travels down the transmission line; all power at the input end eventually reaches the output end. More realistically, any mechanisms that would cause losses to occur have negligible effect. In our model, lossless propagation occurs when $R = G = 0$. Under this condition, only the first term on the right-hand side of either Eq. (11) or Eq. (12) survives. Equation (11), for example, becomes

$$\boxed{\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}} \tag{13}$$

In considering the voltage function that will satisfy (13), it is most expedient to simply state the solution, and then show that it is correct. The solution of (13) is of the form:

$$\boxed{V(z,t) = f_1\left(t - \frac{z}{\nu}\right) + f_2\left(t + \frac{z}{\nu}\right) = V^+ + V^-} \tag{14}$$

where $\nu$, the *wave velocity*, is a constant. The expressions $(t \pm z/\nu)$ are the arguments of functions $f_1$ and $f_2$. The identities of the functions themselves are not critical to the solution of (13). Therefore $f_1$ and $f_2$ can be *any* function.

Animations

The arguments of $f_1$ and $f_2$ indicate, respectively, travel of the functions in the forward and backward $z$ directions. We assign the symbols $V^+$ and $V^-$ to identify the forward and backward voltage wave components. To understand the behavior, consider for example the value of $f_1$ (whatever this might be) at the zero value of its argument, occurring when $z = t = 0$. Now, as time increases to positive values (as it must), and if we are to keep track of $f_1(0)$, then the value of $z$ must also increase to keep the argument $(t - z/\nu)$ equal to zero. The function $f_1$ therefore moves (or propagates) in the positive $z$ direction. Using similar reasoning, the function $f_2$ will propagate in the *negative* $z$ direction, as $z$ in the argument $(t + z/\nu)$ must *decrease* to offset the increase in $t$. Therefore we associate the argument $(t - z/\nu)$ with *forward* $z$ propagation, and the argument $(t + z/\nu)$ with *backward* $z$ travel. This behavior occurs irrespective of what $f_1$ and $f_2$ are. As is evident in the argument forms, the propagation velocity is $\nu$ in both cases.

We next verify that functions having the argument forms expressed in (14) are solutions to (13). First, we take partial derivatives of $f_1$, for example with respect to $z$ and $t$. Using the chain rule,