11

Transmission Lines

Transmission lines are used to transmit electric energy and signals from one point to another, specifically from a source to a load. Examples include the connection between a transmitter and an antenna, connections between computers in a network, or connections between a hydroelectric generating plant and a substation several hundred miles away. Other familiar examples include the interconnects between components of a stereo system and the connection between a cable service provider and your television set. Examples that are less familiar include the connections between devices on a circuit board that are designed to operate at high frequencies.

What all of these examples have in common is that the devices to be connected are separated by distances on the order of a wavelength or much larger, whereas in basic circuit analysis methods, connections between elements are assumed to have negligible length. The latter condition enabled us, for example, to take for granted that the voltage across a resistor on one side of a circuit was exactly in phase with the voltage source on the other side, or, more generally, that the time measured at the source location is precisely the same time as measured at all other points in the circuit. When distances are sufficiently large between source and receiver, time delay effects become appreciable, leading to delay-induced phase differences. In short, we deal with *wave phenomena* on transmission lines, in the same manner that we will find with point-to-point energy propagation in free space or in dielectrics.

The basic elements in a circuit, such as resistors, capacitors, inductors, and the connections between them, are considered *lumped* elements if the time delay in traversing the elements is negligible. On the other hand, if the elements or interconnections are large enough, it may be necessary to consider them as *distributed* elements. This means that their resistive, capacitive, and inductive characteristics must be evaluated on a per-unit-distance basis. Transmission lines have this property in general, and thus they become circuit elements in themselves, possessing impedances that contribute to the circuit problem. The basic rule is that one must consider elements as distributed if the propagation delay across the element dimension is of the order of the shortest time interval of interest. In the time-harmonic case, this condition would lead to a measurable phase difference between each end of the device in question.

In this chapter, we investigate wave phenomena in transmission lines. Our objectives include (1) to understand how to treat transmission lines as circuit elements possessing complex impedances that are functions of line length and frequency, (2) to understand wave propagation on lines, including cases in which losses may occur, (3) to learn methods of combining different transmission lines to accomplish a desired objective, and (4) to understand transient phenomena on lines.

11.1 PHYSICAL DESCRIPTION OF TRANSMISSION LINE PROPAGATION

To obtain a feel for the manner in which waves propagate on transmission lines, the following demonstration may be helpful. Consider a *lossless* line, as shown in Figure 11.1. By lossless, we mean that all power that is launched into the line at the input end eventually arrives at the output end. A battery having voltage V_0 is connected to the input by closing switch S_1 at time $t = 0$. When the switch is closed, the effect is to launch voltage, $V^+ = V_0$. This voltage does not instantaneously appear everywhere on the line, but rather begins to travel from the battery toward the load resistor, R, at a certain velocity. The *wavefront*, represented by the vertical dashed line in Figure 11.1, represents the instantaneous boundary between the section of the line that has been charged to V_0 and the remaining section that is yet to be charged. It also represents the boundary between the section of the line that carries the charging current, I^+, and the remaining section that carries no current. Both current and voltage are discontinuous across the wavefront.

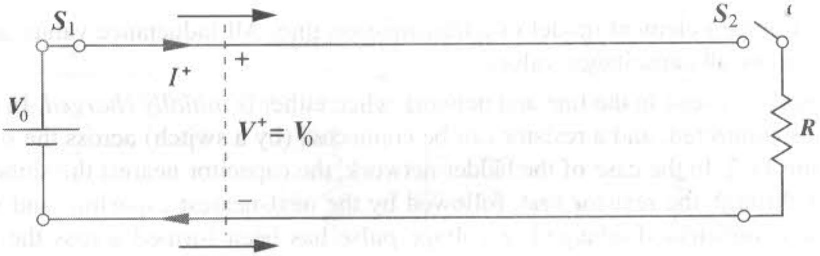

Fig. 11.1 Basic transmission line circuit, showing voltage and current waves initiated by closing switch S_1.

As the line charges, the wavefront moves from left to right at velocity ν, which is to be determined. On reaching the far end, all or a fraction of the wave voltage and current will reflect, depending on what the line is attached to. For example, if the resistor at the far end is left disconnected (switch S_2 is open), then all of the wavefront voltage will be reflected. If the resistor is connected, then some fraction of the incident voltage will reflect. The details of this will be treated in Section 11.9. Of interest at the moment are the factors that determine the wave velocity. The key to understanding and quantifying this is to note that the conducting transmission line will possess capacitance and inductance that are expressed on a per-unit-length basis. We have already derived expressions for these and evaluated them in Chapters 6 and 9 for certain transmission line geometries. Knowing these line characteristics, we can construct a model for the transmission line using lumped capacitors and inductors, as shown in Figure 11.2. The ladder network thus formed is referred to as a *pulse-forming network*, for reasons that will soon become clear.[1]

Consider now what happens when we connect the same switched voltage source to the network. Referring to Figure 11.2, on closing the switch at the battery location, current begins to increase in L_1, allowing C_1 to charge. As C_1 approaches full charge, current in L_2 begins to increase, allowing C_2 to charge next. This progressive charging process continues down the network, until all three capacitors are fully charged. In the network, a "wavefront" location can be identified as the point

[1] Designs and applications of pulse-forming networks are discussed in Reference 1.

between two adjacent capacitors that exhibit the most difference between their charge levels. As the charging process continues, the wavefront moves from left to right. Its speed depends on how fast each inductor can reach its full-current state, and simultaneously by how fast each capacitor is able to charge to full voltage. The wave is faster if the values of L_i and C_i are lower. We therefore expect the wave velocity to be inversely proportional to a function involving the product of inductance and capacitance. In the lossless transmission line, it turns out (as will be shown) that the wave velocity is given by $\nu = 1/\sqrt{LC}$, where L and C are specified per unit length.

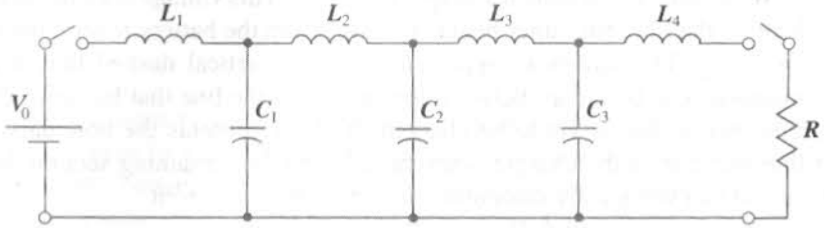

Fig. 11.2 Lumped-element model of a transmission line. All inductance values are equal, as are all capacitance values.

Similar behavior is seen in the line and network when either is *initially charged*. In this case, the battery remains connected, and a resistor can be connected (by a switch) across the output end, as shown in Figure 11.2. In the case of the ladder network, the capacitor nearest the shunted end (C_3) will discharge through the resistor first, followed by the next-nearest capacitor, and so on. When the network is completely discharged, a voltage pulse has been formed across the resistor, and so we see why this ladder configuration is called a pulse-forming network. Essentially identical behavior is seen in a charged transmission line when connecting a resistor between conductors at the output end. The switched voltage exercises, as used in these discussions, are examples of transient problems on transmission lines. Transients will be treated in detail in Section 11.14. In the beginning, line responses to sinusoidal signals are emphasized.

Finally, we surmise that the existence of voltage and current across and within the transmission line conductors implies the existence of electric and magnetic fields in the space around the conductors, and which are associated with the voltage and current. Consequently, we have two possible approaches to the analysis of transmission lines: (1) We can solve Maxwell's equations subject to the line configuration to obtain the fields, and with these find general expressions for the wave power, velocity, and other parameters of interest. (2) Or we can (for now) avoid the fields and solve for the voltage and current using an appropriate circuit model. It is the latter approach that we use in this chapter; the contribution of field theory is solely in the prior (and assumed) evaluation of the inductance and capacitance parameters. We will find, however, that circuit models become inconvenient or useless when losses in transmission lines are to be fully characterized, or when analyzing more complicated wave behavior (i.e., moding) which may occur as frequencies get high. The loss issues will be taken up in Section 11.5. Moding phenomena will be considered in Chapter 14.

11.2 THE TRANSMISSION LINE EQUATIONS

Our first goal is to obtain the differential equations, known as the *wave equations,* which the voltage or current must satisfy on a uniform transmission line. To do this, we construct a circuit model for an incremental length of line, write two circuit equations, and use these to obtain the wave equations.

Our circuit model contains the *primary constants* of the transmission line. These include the inductance, L, and capacitance, C, as well as the shunt conductance, G, and series resistance, R—all of which have values that are specified *per unit length*. The shunt conductance is used to model leakage current through the dielectric that may occur throughout the line length; the assumption is that the dielectric may possess conductivity, σ_d, in addition to a dielectric constant, ϵ_r where the latter affects the capacitance. The series resistance is associated with any finite conductivity, σ_c, in the conductors. Either one of the latter parameters, R and G, will be responsible for power loss in transmission. In general, both are functions of frequency. Knowing the frequency and the dimensions, we can determine the values of R, G, L, and C by using formulas developed in earlier chapters.

We assume propagation in the $\mathbf{a}_z$ direction. Our model consists of a line section of length Δz containing resistance $R\Delta z$, inductance $L\Delta z$, conductance $G\Delta z$, and capacitance $C\Delta z$, as indicated in Figure 11.3. Since the section of the line looks the same from either end, we divide the series elements in half to produce a symmetrical network. We could equally well have placed half the conductance and half the capacitance at each end.

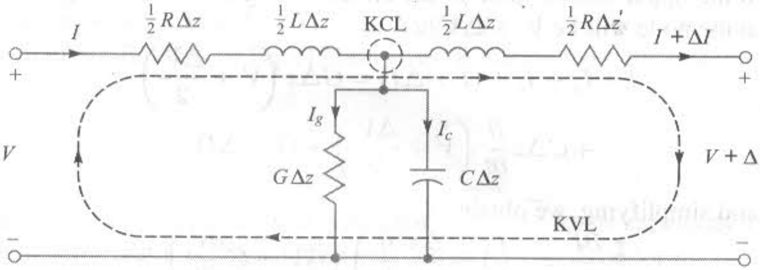

Fig. 11.3 Lumped-element model of a short transmission line section with losses. The length of the section is Δz. Analysis involves applying Kirchoff's voltage and current laws (KVL and KCL) to the indicated loop and node respectively.

Our objective is to determine the manner and extent to which the output voltage and current are changed from their input values in the limit as the length approaches a very small value. We will consequently obtain a pair of differential equations that describe the rates of change of voltage and current with respect to z. In Figure 11.3, the input and output voltages and currents differ respectively by quantities ΔV and ΔI, which are to be determined. The two equations are obtained by successive applications of Kirchoff's voltage law (KVL) and Kirchoff's current law (KCL).

First, KVL is applied to the loop that encompasses the entire section length, as shown in Figure 11.3:

$$V = \frac{1}{2}RI\Delta z + \frac{1}{2}L\frac{\partial I}{\partial t}\Delta z + \frac{1}{2}L\left(\frac{\partial I}{\partial t} + \frac{\partial \Delta I}{\partial t}\right)\Delta z + \frac{1}{2}R(I+\Delta I)\Delta z + (V+\Delta V) \tag{1}$$

We can solve Eq. (1) for the ratio, $\Delta V/\Delta z$, obtaining

$$\frac{\Delta V}{\Delta z} = -\left(RI + L\frac{\partial I}{\partial t} + \frac{1}{2}L\frac{\partial \Delta I}{\partial t} + \frac{1}{2}R\Delta I\right) \tag{2}$$

the z partial derivative is

$$\frac{\partial f_1}{\partial z} = \frac{\partial f_1}{\partial(t - z/\nu)}\frac{\partial(t - z/\nu)}{\partial z} = -\frac{1}{\nu}f_1' \tag{15}$$

where it is apparent that the primed function, f_1', denotes the derivative of f_1 with respect to its argument. The partial derivative with respect to time is

$$\frac{\partial f_1}{\partial t} = \frac{\partial f_1}{\partial(t - z/\nu)}\frac{\partial(t - z/\nu)}{\partial t} = f_1' \tag{16}$$

Next, the second partial derivatives with respect to z and t can be taken using similar reasoning:

$$\frac{\partial^2 f_1}{\partial z^2} = \frac{1}{\nu^2}f_1'' \qquad \text{and} \qquad \frac{\partial^2 f_1}{\partial t^2} = f_1'' \tag{17}$$

where f_1'' is the second derivative of f_1 with respect to its argument. The results in (17) can now be substituted into (13), obtaining

$$\frac{1}{\nu^2}f_1'' = LCf_1'' \tag{18}$$

We now identify the wave velocity for lossless propagation, which is the condition for equality in (18):

$$\boxed{\nu = \frac{1}{\sqrt{LC}}} \tag{19}$$

Performing the same procedure using f_2 (and its argument) leads to the same expression for ν.

The form of ν as expressed in Eq. (19) confirms our original expectation that the wave velocity would be in some inverse proportion to L and C. The same result will be true for current, since Eq. (12) under lossless conditions would lead to a solution of the form identical to that of (14), with velocity given by (19). What is not known yet, however, is the relation *between* voltage and current.

We have already found that voltage and current are related through the telegraphist's equations, (5) and (8). These, under lossless conditions ($R = G = 0$) become

$$\boxed{\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}} \tag{20}$$

$$\boxed{\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}} \tag{21}$$

Using the voltage function, we can substitute (14) into (20) and use the methods demonstrated in (15) to write

$$\frac{\partial I}{\partial t} = -\frac{1}{L}\frac{\partial V}{\partial z} = \frac{1}{L\nu}(f_1' - f_2') \tag{22}$$

We next integrate (22) over time, obtaining the current in terms of its forward and backward propagating components:

$$\boxed{I(z,t) = \frac{1}{L\nu}\left[f_1\left(t - \frac{z}{\nu}\right) - f_2\left(t + \frac{z}{\nu}\right)\right] = I^+ + I^-} \tag{23}$$

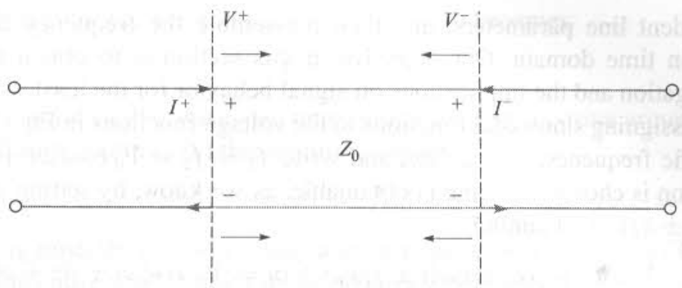

Fig. 11.4 Current directions in waves having positive voltage polarity.

In performing this integration, all integration constants are set to zero. The reason for this, as demonstrated by (20) and (21), is that a time-varying voltage must lead to a time-varying current, with the reverse also true. The factor $1/L\nu$ appearing in (23) multiplies voltage to obtain current, and so we identify the product $L\nu$ as the *characteristic impedance*, Z_0, of the lossless line. Z_0 is defined as the ratio of the voltage to the current in a single propagating wave. Using (19), we write the characteristic impedance as

$$\boxed{Z_0 = L\nu = \sqrt{\frac{L}{C}}} \tag{24}$$

By inspecting (14) and (23), we now note that

$$\boxed{V^+ = Z_0 I^+} \tag{25a}$$

and

$$\boxed{V^- = -Z_0 I^-} \tag{25b}$$

The significance of the preceding relations can be seen in Figure 11.4. The figure shows forward- and backward-propagating voltage waves, V^+ and V^-, both of which have positive polarity. The currents that are associated with these voltages will flow in opposite directions. We define *positive current* as having a *clockwise* flow in the line, and *negative current* as having a *counterclockwise* flow. The minus sign in (25*b*) thus assures that negative current will be associated with a backward-propagating wave that has positive polarity. This is a general convention, applying to lines with losses also. Propagation with losses is studied by solving (11) under the assumption that either R or G (or both) are not zero. We will do this in Section 11.7 under the special case of sinusoidal voltages and currents. Sinusoids in lossless transmission lines are considered in Section 11.4.

11.4 LOSSLESS PROPAGATION OF SINUSOIDAL VOLTAGES

An understanding of sinusoidal waves on transmission lines is important because any signal that is transmitted in practice can be decomposed into a discrete or continuous summation of sinusoids. This is the basis of *frequency domain* analysis of signals on lines. In such studies, the effect of the transmission line on any signal can be determined by noting the effects on the frequency components. This means that one can effectively propagate the spectrum of a given signal, using frequency-dependent line parameters, and then reassemble the frequency components into the resultant signal in time domain. Our objective in this section is to obtain an understanding of sinusoidal propagation and the implications on signal behavior for the lossless line case.

We begin by assigning sinusoidal functions to the voltage functions in Eq. (14). Specifically, we consider a specific frequency, $f = \omega/2\pi$, and write $f_1 = f_2 = V_0 \cos(\omega t + \phi)$. By convention, the cosine function is chosen; the sine is obtainable, as we know, by setting $\phi = -\pi/2$. We next replace t with $(t \pm z/\nu_p)$, obtaining

$$\mathcal{V}(z,t) = |V_0| \cos[\omega(t \pm z/\nu_p) + \phi] = |V_0| \cos[\omega t \pm \beta z + \phi] \tag{26}$$

where we have assigned a new notation to the velocity, which is now called the *phase velocity*, ν_p. This is applicable to a pure sinusoid (having a single frequency) and will be found to depend on frequency in some cases. Choosing, for the moment, $\phi = 0$, we obtain the two possibilities of forward or backward z travel by choosing the minus or plus sign in (26). The two cases are

$$\boxed{\mathcal{V}_f(z,t) = |V_0| \cos(\omega t - \beta z)} \qquad \text{(forward } z \text{ propagation)} \tag{27a}$$

and

$$\boxed{\mathcal{V}_b(z,t) = |V_0| \cos(\omega t + \beta z)} \qquad \text{(backward } z \text{ propagation)} \tag{27b}$$

where the magnitude factor, $|V_0|$, is the value of $\mathcal{V}$ at $z = 0, t = 0$. We define the *phase constant* β, obtained from (26), as

$$\boxed{\beta \equiv \frac{\omega}{\nu_p}} \tag{28}$$

We refer to the solutions expressed in (27a) and (27b) as the *real instantaneous* forms of the transmission-line voltage. They are the mathematical representations of what one would experimentally measure. The terms ωt and βz, appearing in these equations, have units of angle and are usually expressed in radians. We know that ω is the radian time frequency, measuring phase shift *per unit time,* and it has units of rad/s. In a similar way, we see that β will be interpreted as a *spatial* frequency, which in the present case measures the phase shift *per unit distance* along the z direction. Its units are rad/m. If we were to fix the time at $t = 0$, Eqs. (27a) and (27b) would become

$$\mathcal{V}_f(z,0) = \mathcal{V}_b(z,0) = |V_0| \cos(\beta z) \tag{29}$$

which we identify as a simple periodic function that repeats every incremental distance λ, known as the *wavelength*. The requirement is that $\beta\lambda = 2\pi$, and so

$$\boxed{\lambda = \frac{2\pi}{\beta} = \frac{\nu_p}{f}} \tag{30}$$

We next consider a point (such as a wave crest) on the cosine function of Eq. (27a), the occurrence of which requires the argument of the cosine to be an integer multiple of 2π.

Considering the mth crest of the wave, the condition at $t = 0$ becomes

$$\beta z = 2m\pi$$

To keep track of this point on the wave, we require that the entire cosine argument be the same multiple of 2π for all time. From (27a) the condition becomes

$$\omega t - \beta z = \omega(t - z/\nu_p) = 2m\pi \tag{31}$$

Again, with increasing time, the position z must also increase in order to satisfy (31). Consequently the wave crest (and the entire wave) travels in the positive z direction at velocity ν_p. Equation (27b), having cosine argument $(\omega t + \beta z)$, describes a wave that travels in the *negative* z direction, since as time increases, z must now *decrease* to keep the argument constant. Similar behavior is found for the wave current, but complications arise from line-dependent phase differences that occur between current and voltage. These issues are best addressed once we are familiar with complex analysis of sinusoidal signals.

11.5 COMPLEX ANALYSIS OF SINUSOIDAL WAVES

Expressing sinusoidal waves as complex functions is useful (and essentially indispensable) because it greatly eases the evaluation and visualization of phase that will be found to accumulate by way of many mechanisms. In addition, we will find many cases in which two or more sinusoidal waves must be combined to form a resultant wave—a task made much easier if complex analysis is used.

Expressing sinusoidal functions in complex form is based on the Euler identity:

$$e^{\pm jx} = \cos(x) \pm j\sin(x) \tag{32}$$

from which we may write the cosine and sine, respectively, as the real and imaginary parts of the complex exponent:

$$\cos(x) = \mathrm{Re}[e^{\pm jx}] = \frac{1}{2}(e^{jx} + e^{-jx}) = \frac{1}{2}e^{jx} + c.c. \tag{33a}$$

$$\sin(x) = \pm\mathrm{Im}[e^{\pm jx}] = \frac{1}{2j}(e^{jx} - e^{-jx}) = \frac{1}{2j}e^{jx} + c.c. \tag{33b}$$

where $j \equiv \sqrt{-1}$, and where $c.c.$ denotes the complex conjugate of the preceding term. The conjugate is formed by changing the sign of j wherever it appears in the complex expression.

We may next apply (33a) to our voltage wave function, Eq. (26):

$$\mathcal{V}(z,t) = |V_0|\cos[\omega t \pm \beta z + \phi] = \frac{1}{2}\underbrace{(|V_0|e^{j\phi})}_{V_0} e^{\pm j\beta z}e^{j\omega t} + c.c. \tag{34}$$

Note that we have arranged the phases in (34) such that we identify the *complex amplitude* of the wave as $V_0 = (|V_0|e^{j\phi})$. In future usage, a single symbol (V_0 in the present example) will usually be used for the voltage or current amplitudes, with the understanding that these will generally be complex (having magnitude and phase).

Two additional definitions follow from Eq. (34). First, we define the *complex instantaneous* voltage as

$$\boxed{V_c(z,t) = V_0 e^{\pm j\beta z}e^{j\omega t}} \tag{35}$$