

1. Find the equation of the tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) on the ellipsoid.

soln:

Consider a scalar function $f(x, y, z)$. Consider a surface over which f is constant, say, $f(x, y, z) = k$. $\vec{\nabla}f$ at a point on this surface is normal to this surface. This normal will also be normal to the tangent plane at this point.

We have $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\vec{\nabla}f = 2 \left(\hat{i} \frac{x}{a^2} + \hat{j} \frac{y}{b^2} + \hat{k} \frac{z}{c^2} \right) \quad (1)$$

At (x_0, y_0, z_0) , $\vec{\nabla}f = 2 \left(\hat{i} \frac{x_0}{a^2} + \hat{j} \frac{y_0}{b^2} + \hat{k} \frac{z_0}{c^2} \right)$.

Every vector on the tangent plane is perpendicular to this vector. So the equation of the tangent plane is given by

$$\begin{aligned} (x - x_0) \frac{x_0}{a^2} + (y - y_0) \frac{y_0}{b^2} + (z - z_0) \frac{z_0}{c^2} &= 0 \\ \text{i.e.} \quad \frac{x_0}{a^2} x + \frac{y_0}{b^2} y + \frac{z_0}{c^2} z &= 1 \end{aligned}$$

2. Prove that for any vector field \vec{A} , $\vec{\nabla} \cdot \vec{A}$ is a scalar.

soln:

Let us denote the coordinates before and after rotation as

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad X' = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}$$

$X' = RX$ where R is the rotation matrix.

$$\therefore x'_i = \sum_j R_{ij} x_j \quad \text{and} \quad x_i = \sum_j (R^T)_{ij} x'_j = \sum_j R_{ji} x'_j \quad (2)$$

Here we have used the fact that $R^{-1} = R^T$ i.e. R is an orthogonal matrix. We have to show that $\vec{\nabla}' \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A}$, i.e.

$$\sum_i \frac{\partial A'_i}{\partial x'_i} = \sum_i \frac{\partial A_i}{\partial x_i}$$

$$\frac{\partial}{\partial x'_i} = \sum_j \frac{\partial}{\partial x_j} \frac{\partial x_j}{\partial x'_i} = \sum_j R_{ij} \frac{\partial}{\partial x_j} \quad \text{from Eq.(2)}$$

$A'_i = \sum_k R_{ik} A_k$ since \vec{A} is a vector.

$$\begin{aligned}
\vec{\nabla}' \cdot \vec{A}' &= \sum_i \frac{\partial A'_i}{\partial x'_i} = \sum_i \sum_j R_{ij} \frac{\partial}{\partial x_j} \left(\sum_k R_{ik} A_k \right) \\
&= \sum_j \sum_k \left(\sum_i R_{ij} R_{ik} \right) \frac{\partial A_k}{\partial x_j}
\end{aligned}$$

Now $\sum_i R_{ij} R_{ik} = \sum_i (R^T)_{ji} R_{ik} = \delta_{jk}$ since $R^T R = \mathbb{I}$.
So only $j = k$ terms survive in the above summation

$$\therefore \vec{\nabla}' \cdot \vec{A}' = \sum_j \frac{\partial A_j}{\partial x_j} = \vec{\nabla} \cdot \vec{A}$$

So the divergence of a vector is invariant under rotation. Hence it is a scalar quantity. This proof is valid for the divergence operator in any dimension since we have only used the orthogonality of R .

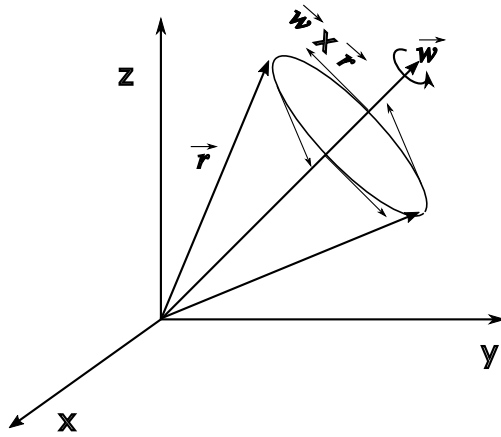
3. Let $\vec{A} = \vec{\omega} \times \vec{r}$ where $\vec{\omega}$ is a fixed vector in space. Find $\vec{\nabla} \times \vec{A}$.

soln:

$$\begin{aligned}
\vec{A} = \vec{\omega} \times \vec{r} &= \hat{i}(\omega_y z - \omega_z y) + \hat{j}(\omega_z x - \omega_x z) + \hat{k}(\omega_x y - \omega_y x) \\
\therefore \vec{\nabla} \times \vec{A} &= \hat{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\
&= \hat{i}(\omega_x - (-\omega_x)) + \hat{j}(\omega_y - (-\omega_y)) + \hat{k}(\omega_z - (-\omega_z)) = 2\vec{\omega}
\end{aligned}$$

Interpretation:

The vector field $\vec{\omega} \times \vec{r}$ curls around the vector $\vec{\omega}$. It is the velocity vector of the particles of a rigid body rotating with angular velocity $\vec{\omega}$.



4. Find the divergence of the following:

- (a) $\vec{A} = \hat{r}$,
- (b) $\vec{A} = \frac{\hat{r}}{r}$ in 2 dimension
- (c) $\vec{A} = \frac{\hat{r}}{r}$ in 3 dimension
- (d) $\vec{A} = \frac{\hat{r}}{r^2}$ in 3 dimension. Plot this field.
- (e) $\vec{A} = \frac{\hat{r}}{r^3}$ in 3 dimension

soln:

$$\begin{aligned}
 \text{(a)} \quad \vec{A} &= \hat{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}; r = \sqrt{x^2 + y^2 + z^2} \\
 \vec{\nabla} \cdot \vec{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
 \frac{\partial A_x}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{x}{r} \right) = -\frac{x^2}{r^3} + \frac{1}{r} \\
 \text{Similarly } \frac{\partial A_y}{\partial y} &= -\frac{y^2}{r^3} + \frac{1}{r} \text{ and } \frac{\partial A_z}{\partial z} = -\frac{z^2}{r^3} + \frac{1}{r}. \\
 \therefore \vec{\nabla} \cdot \vec{A} &= -\frac{x^2 + y^2 + z^2}{r^3} + 3/r = 2/r.
 \end{aligned}$$

For the remaining parts we evaluate the following:

$$\begin{aligned}
 \vec{\nabla} \cdot (r^n \hat{r}) &= r^n \vec{\nabla} \cdot \hat{r} + \vec{\nabla}(r^n) \cdot \hat{r} \\
 &= r^n \left(\frac{2}{r} \right) + nr^{n-1} \hat{r} \cdot \hat{r} \quad (\text{in 3 dim.}) \\
 &= (n+2)r^{n-1} \\
 \vec{\nabla} \cdot (r^n \hat{r}) &= r^n \left(\frac{1}{r} \right) + nr^{n-1} \hat{r} \cdot \hat{r} \quad (\text{in 2 dim.}) \\
 &= (n+1)r^{n-1}
 \end{aligned}$$

- (b) $\vec{A} = \frac{\hat{r}}{r}$ in 2 dim. ($n = -1$)
 $\vec{\nabla} \cdot \vec{A} = (-1+1)r^{-2} = 0.$
 (This is true only for $r \neq 0$. At $r = 0$, $\vec{\nabla} \cdot \vec{A} \rightarrow \infty$)
- (c) $\vec{A} = \frac{\hat{r}}{r}$ in 3 dim. ($n = -1$)
 $\vec{\nabla} \cdot \vec{A} = (-1+2)r^{-2} = 1/r^2$
- (d) $\vec{A} = \frac{\hat{r}}{r^2}$ in 3 dim. ($n = -2$)
 $\vec{\nabla} \cdot \vec{A} = (-2+2)r^{-3} = 0.$
 (This is true only for $r \neq 0$. At $r = 0$, $\vec{\nabla} \cdot \vec{A} \rightarrow \infty$)
- (e) $\vec{A} = \frac{\hat{r}}{r^3}$ in 3 dim. ($n = -3$)
 $\vec{\nabla} \cdot \vec{A} = (-3+2)r^{-4} = -1/r^4$

All the fields above are radially outward. But the divergence changes from positive to 0 to negative. Explain this.

5. Find the curl of the following:

- (a) $\vec{A} = y\hat{i} - x\hat{j}$
 (b) $\vec{A} = \frac{1}{\sqrt{x^2+y^2}}(y\hat{i} - x\hat{j})$
 (c) $\vec{A} = \frac{1}{x^2+y^2}(y\hat{i} - x\hat{j})$
 (d) $\vec{A} = (x^2 + y^2)\hat{k}$

soln:

(a) $\vec{A} = y\hat{i} - x\hat{j}$

soln

$$\vec{\nabla} \times \vec{A} = \hat{k}(-1 - 1) = -2\hat{k}$$

(b) $\vec{A} = \frac{1}{\sqrt{x^2+y^2}}(y\hat{i} - x\hat{j})$

soln:

We first evaluate $\vec{\nabla} \times [(x^2 + y^2)^n(y\hat{i} - x\hat{j})]$ which will help us work out the other parts.

$$\begin{aligned} \vec{\nabla} \times [(x^2 + y^2)^n(y\hat{i} - x\hat{j})] &= (x^2 + y^2)^n \vec{\nabla} \times (y\hat{i} - x\hat{j}) + \vec{\nabla}(x^2 + y^2)^n \times (y\hat{i} - x\hat{j}) \quad \text{product rules} \\ &= (x^2 + y^2)^n(-2\hat{k}) + n(x^2 + y^2)^{n-1}(2x\hat{i} + 2y\hat{j}) \times (y\hat{i} - x\hat{j}) \\ &= -2(x^2 + y^2)^n\hat{k} + n(x^2 + y^2)^{n-1}(-2x^2 - 2y^2)\hat{k} \\ &= -2(x^2 + y^2)^n(1 + n)\hat{k} \end{aligned}$$

For $\vec{A} = (y\hat{i} - x\hat{j})/\sqrt{x^2 + y^2}$, $n = -1/2$.

$$\vec{\nabla} \times \vec{A} = -\frac{1}{\sqrt{x^2 + y^2}}\hat{k}$$

This is not differentiable at $x = y = 0$, i.e along the z axis.

(c) $\vec{A} = \frac{1}{x^2+y^2}(y\hat{i} - x\hat{j})$

soln

Here $n = -1$.

$$\therefore \vec{\nabla} \times \vec{A} = 0$$

This is not valid along the z axis.

(d) $\vec{A} = (x^2 + y^2)\hat{k}$

soln

$$\begin{aligned} \vec{\nabla} \times \vec{A} &= \hat{i} \left(\frac{\partial A_z}{\partial y} \right) + \hat{j} \left(-\frac{\partial A_z}{\partial x} \right) \\ &= 2y\hat{i} - 2x\hat{j} = 2(y\hat{i} - x\hat{j}) \end{aligned}$$

6. For any vector field \vec{A} and any scalar field F show that

- (i) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$; (ii) $\vec{\nabla} \times (\vec{\nabla} F) = 0$.

soln:

$$\begin{aligned}
\text{(i)} \quad \vec{\nabla} \times \vec{A} &= \hat{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\
\therefore \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) &= \frac{\partial}{\partial x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) = 0. \\
\text{(ii)} \quad \vec{\nabla} \times \vec{\nabla} F &= \hat{i} \left(\frac{\partial^2 F}{\partial y \partial x} - \frac{\partial^2 F}{\partial x \partial y} \right) + \dots = 0
\end{aligned}$$

7. Can we find a scalar function F such that $\vec{\nabla} F = y\hat{i} - x\hat{j}$?
What about $\vec{\nabla} F = \frac{1}{x^2+y^2}(y\hat{i} - x\hat{j})$?

soln

$$\vec{\nabla} \times \vec{\nabla} F = \vec{\nabla} \times (y\hat{i} - x\hat{j}) = -2\hat{k}$$

But curl of the gradient of any scalar field must be zero. So there exist no such F such that $\vec{\nabla} F = y\hat{i} - x\hat{j}$

If $\vec{\nabla} F = \frac{1}{x^2+y^2}(y\hat{i} - x\hat{j})$ it can be shown that $\vec{\nabla} \times (\vec{\nabla} F) = 0$ at almost all places but the result is not applicable when $x = y = 0$ which is the z axis. In fact it can be shown that the $\vec{\nabla} \times (\vec{\nabla} F) \neq 0$ along the z axis. The scalar function F , that we find, steadily increases along $\vec{\nabla} F$. Hence along a circle around the origin, the plot of given $\vec{\nabla} F$ shows that F steadily increases. After completing a full circle when we come to the same point we arrive at a different value of F . This makes F discontinuous, and hence not differentiable. Let us find a F and demonstrate this phenomenon. We have the following system of differential equations

$$\frac{\partial F}{\partial x} = \frac{y}{x^2 + y^2} \quad ; \quad \frac{\partial F}{\partial y} = \frac{-x}{x^2 + y^2}$$

The first one gives $F(x, y) = \tan^{-1}(x/y) + g(y)$ while the second equation gives $F(x, y) = -\tan^{-1}(y/x) + h(x) = -(\pi/2 - \tan^{-1}(x/y)) + h(x)$.

For consistency we need $g(y)$ and $h(x)$ to be constants, say, c_1 and c_2 .

$c_2 - c_1 = \pi/2$. We can take $c_1 = 0$ and $c_2 = \pi/2$. This gives

$$F(x, y) = \tan^{-1}(x/y).$$

So it appears that This function is not continuous. To see this we reparametrize $x = r \sin \theta$, $y = r \cos \theta$. Then we get $F(x, y) = \theta$. As we go around a circle of radius 1, starting at $(0, 1)$ and come back to the same point the value of the function changes from 0 to 2π . So it is discontinuous at $(0, 1)$, hence, not differentiable at this point which lies on the y axis. In fact it is not differentiable at any point on the y axis. Hence the given vector field cannot be the gradient of a scalar function.