The Trapezoidal Method

$\boxed{Y'(x) = f(x, Y(x))}$ $\boxed{Y(x_0) = Y_0}$ $\underline{x_{n+1} = x_n + h}$

$$\Rightarrow \boxed{Y(x_{n+1}) = Y(x_n) \int_{x_n}^{x_{n+1}} f(x, Y(x))\, dx}$$

Substitute the integral by the trapezoidal formula.

$$\Rightarrow Y(x_{n+1}) \simeq Y(x_n) + \frac{h}{2}\Big[f(x_n, Y(x_n)) + f(x_{n+1}, Y(x_{n+1}))\Big]$$

$$\Rightarrow \boxed{y_{n+1} = y_n + \frac{h}{2}\Big[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\Big]} \quad \begin{array}{l} y_0 = Y_0 \\ x \geq 0 \end{array}$$

This is an implicit method. To make it explicit we substitute y_{n+1} on the right hand side by the Euler formula,

$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$ to get,

$$\boxed{y_{n+1} = y_n + \frac{h}{2}\Big[f(x_n, y_n) + f(x_{n+1}, y_n + h f(x_n, y_n))\Big]}$$

This is Heun's method, of second-order accuracy.

The Taylor Method

On the initial-value problem $\boxed{Y'(x) = f(x, Y(x))}$,

Euler's method: $Y'(x) \simeq \dfrac{Y(x_n + h) - Y(x_n)}{h}$

$$\Rightarrow Y(x_n + h) \simeq Y(x_n) + Y'(x_n)h \simeq Y(x_n) + h f(x_n, Y(x_n))$$

Euler's formula is <u>Taylor expansion</u> of the <u>first order</u> (linear order).

<u>Second order:</u> $\boxed{Y(x_{n+1}) = Y(x_n) + Y'(x_n)h + \frac{Y''(x_n)h^2}{2}}$

The <u>truncation error</u> T_{n+1} is of the <u>third order</u>, $\underline{Y'''(x_n)h^3/6}$.

<u>Example:</u> $\boxed{Y'(x) = -Y(x) + 2\cos x}$ $\underline{Y(0) = 1}$

$\boxed{Y'(x) = a\,Y(x) + b(x)}$ $a = -1$, $b = 2\cos x$.

$\Rightarrow Y(x) = \left[c + 2\int e^x \cos x \, dx\right] e^{-x}$

$I = \int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

$\Rightarrow I = e^x \sin x - \left[-(\cos x)e^x - \int (-\cos x)\, e^x \, dx\right]$

$\Rightarrow I = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$

$\Rightarrow I = e^x(\sin x + \cos x) - I$

$\Rightarrow 2I = e^x(\sin x + \cos x)$.

$\Rightarrow Y(x) = ce^{-x} + \sin x + \cos x$.

When $x = 0$, $Y = 1$. $\Rightarrow 1 = c + 0 + 1$

$\Rightarrow c = 0$ $\Rightarrow$ $\boxed{Y(x) = \sin x + \cos x}$

(P.T.O.)

The Euler formula is

$$Y(x_{n+1}) \simeq Y(x_n) + h\, Y'(x_n)$$

$$\Rightarrow Y(x_{n+1}) \simeq Y(x_n) + h\left[-Y(x_n) + 2\cos(x_n)\right]$$

$$\Rightarrow \boxed{y_{n+1} = y_n + h\left[-y_n + 2\cos(x_n)\right]}$$

The second-order Taylor formula is

$$Y(x_{n+1}) = Y(x_n) + Y'(x_n)h + \frac{Y''(x_n)}{2}h^2$$

Now $\boxed{Y''(x_n) = -Y'(x) - 2\sin x}$

$\because \left(Y''(x) = Y(x) - 2\cos x - 2\sin x \right)$

$$\Rightarrow \boxed{\begin{aligned} y_{n+1} = y_n &+ h\left[-y_n + 2\cos(x_n)\right] \\ &+ \frac{h^2}{2}\left[y_n - 2\cos(x_n) - 2\sin(x_n)\right] \end{aligned}}$$

In general, for an initial-value problem.

$\boxed{Y'(x) = f(x, Y(x))}$, $\boxed{Y(x_0) = Y_0}$.

$$Y(x_{n+1}) \simeq Y(x_n) + h\,Y'(x_n) + \dots + \frac{Y^{(p)}(x_n)\,h^p}{p!}$$

The truncation error is $\boxed{T_{n+1} \simeq \frac{h^{p+1}}{(p+1)!} y^{(p+1)}(x)}$

It is $O(h^{p+1})$.

Greater accuracy with higher orders and smaller step size h.

Implicit Derivatives

$\boxed{y'(x) = f(x, y(x))}$. $\quad$ $z \equiv y$ gives $f \equiv f(x,z)$

$\therefore \; y''(x) = \dfrac{df}{dx}$ $\quad$ Since $\boxed{f \equiv f(x, y(x))}$

Now $df = \dfrac{\partial f}{\partial x} dx + \dfrac{\partial f}{\partial z} dz$. But $\boxed{\dfrac{dz}{dx} = y' = f}$

$$\Rightarrow \boxed{\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{dz}{dx} = f_x + f_z f},$$

$\boxed{f_x \equiv \dfrac{\partial f}{\partial x}}$ and $\boxed{f_z \equiv \dfrac{\partial f}{\partial z}}$ $\Rightarrow \boxed{y''(x) = f_x + f_z f}$

$\boxed{y'''(x) = \dfrac{d}{dx}\left(\dfrac{df}{dx}\right)}$ Now $d\left(\dfrac{df}{dx}\right) =$

$$\frac{\partial f_x}{\partial x} dx + \frac{\partial f_x}{\partial z} dz + d(f_z f)$$

Also $d(f_z f) = f_z\, df + f\, df_z$

$$\Rightarrow d(f_z f) = f_z\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial z} dz\right) + f\left[\frac{\partial f_z}{\partial x} dx + \frac{\partial f_z}{\partial z} dz\right]$$

$$\Rightarrow d\left(\frac{df}{dx}\right) = \frac{\partial f_x}{\partial x} dx + \frac{\partial f_x}{\partial z} dz + f_z(f_x + f_z f)\,dx$$
$$+ f\left[\frac{\partial f_z}{\partial x} dx + \frac{\partial f_z}{\partial z} dz\right].$$

$$\Rightarrow \frac{d^2 f}{dx^2} = f_{xx} + f_{xz} f + f_z(f_x + f_z f)$$
$$+ f f_{zx} + f^2 f_{zz} \qquad (f_{zx} = f_{xz})$$

$$\Rightarrow \boxed{y'''(x) = f_{xx} + 2 f_{xz} f + f_z(f_x + f_z f) + f_{zz} f^2}$$

The Runge - Kutta Method

The difficulty of the Taylor method is that it requires evaluating long higher-order Derivatives in the Taylor expansion,

$$\boxed{y_{n+1} = y_n + h y_n' + \frac{h^2}{2} y_n'' + \dots + \frac{h^p}{p!} y_n^{(p)}}.$$

The Runge-Kutta method is based on

$$\boxed{y_{n+1} = y_n + h F(x_n, y_n; h)} \quad \underline{n \geq 0} \quad \underline{y_0 = Y_0}$$

in which $F(x_n, y_n; h)$ is an average slope that captures the effect of the higher orders. However, it has the simplicity of the linear form. In the linear order the slope in the interval $[x_n, x_{n+1}]$ is $f(x_n, y_n)$. In the Runge-Kutta method it is replaced by $\underline{F(x_n, y_n; h)}$.

For methods of order 2, we choose

$$\boxed{F(x,y;h) = \gamma_1 f(x,y) + \gamma_2 f(x + \alpha h, y + \beta h f(x,y))}$$

The constants α, β, γ_1 and γ_2 are chosen in such a way that $y(x_{n+1}) - [y_n + h F(x_n, y_n; h)]$ gives a truncation error of $O(h^3)$ like the Taylor method of order 2. This will give an accuracy upto the second order.

~~Note~~ Noting that $f \equiv f(x,z)$, we write a Taylor expansion upto the first order as

$$\boxed{f(x+\alpha h,\, y+\beta h f(x,y)) = f(x,y) + f_x \alpha h + f_z \beta h f + O(h^2)}.$$

Now, $\boxed{y''(x) = f_x + f_z f}$, which we use in

$$y(x+h) = y(x) + y'(x)h + y''(x)\frac{h^2}{2} + O(h^3)$$

$$\Rightarrow y(x+h) = y(x) + hf + \frac{h^2}{2}(f_x + f_z f) + O(h^3)$$

Hence $y(x+h) - [y(x) + hF(x, y(x); h)]$

$$= \cancel{y} + hf + \frac{h^2}{2}(f_x + f_z f) - [\cancel{y} + h\gamma_1 f + h\gamma_2(f + \alpha h f_x + \beta h f_z f)] + O(h^3)$$

$$= hf(1-\gamma_1-\gamma_2) + \frac{h^2}{2}(f_x + f_z f - 2\alpha\gamma_2 f_x - 2\gamma_2\beta f_z f) + O(h^3)$$

$$= hf(1-\gamma_1-\gamma_2) + \frac{h^2}{2}\left[(1-2\alpha\gamma_2)f_x + (1-2\beta\gamma_2)f_z f\right] + O(h^3)$$

Since the error is $O(h^3)$, all coefficients of h and h^2 must vanish. This gives $\boxed{1-\gamma_1-\gamma_2 = 0}$ from the h term.

Likewise, from the h^2 term we get,

$\boxed{1-2\gamma_2\alpha = 0}$ and $\boxed{1-2\beta\gamma_2 = 0}$. There are three equations with four unknown values.

$\therefore$ $\boxed{\gamma_1 = 1 - \gamma_2}$ and $\boxed{\alpha = \beta = \frac{1}{2\gamma_2}}$.

$\gamma_2 \neq 0$. Choose $\boxed{\gamma_2 = 1/2}$. This gives.

$$\boxed{F(x_n, y_n; h) = \frac{1}{2} f(x_n, y_n) + \frac{1}{2} f(x_n + h,\ y + h f(x_n, y_n))}$$

Using this we get Heun's method, $(n \geq 0)$

$$\boxed{y_{n+1} = y_n + \frac{1}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n)) \right]}.$$

<u>The fourth-Order Runge-Kutta Method</u>

$$\boxed{\begin{array}{l} v_1 = f(x_n, y_n), \quad v_2 = f(x_n + \frac{h}{2},\ y_n + \frac{h}{2} v_1) \\ v_3 = f(x_n + \frac{h}{2},\ y_n + \frac{h}{2} v_2), \quad v_4 = f(x_n + h,\ y_n + h v_3) \end{array}}$$

and $\boxed{y_{n+1} = y_n + \frac{h}{6}\left[v_1 + 2v_2 + 2v_3 + v_4 \right]}$.

<u>The truncation error is $O(h^5)$.</u>

<u>Special Case:</u> $\boxed{Y'(x) = f(x)}$

$\therefore v_1 = f(x_n)$, $v_2 = f(x_n + \frac{h}{2})$, $v_3 = v_2$

$v_4 = f(x_n + h)$ (<u>Ignore the second argument in f</u>)

$$\therefore y_{n+1} = y_n + \frac{h}{6}\left[f(x_n) + 4f(x_n + \tfrac{h}{2}) + f(x_n + h) \right]$$

Now $\boxed{x_n + h = x_{n+1}}$ $\therefore$ $\boxed{x_n + \frac{h}{2} = \frac{x_n + x_{n+1}}{2}}$.

Also $\boxed{2h = x_{n+1} - x_n}$ (redefining the value of h)

$\Rightarrow h \to 2h$. With these we get <u>Simpson's method</u>.

Systems of Differential Equations

The general form of a system of two first-order differential equation is

$$\boxed{\begin{aligned} Y_1'(x) &= f_1(x, Y_1(x), Y_2(x)) \\ Y_2'(x) &= f_2(x, Y_1(x), Y_2(x)) \end{aligned}} \quad \boxed{\begin{aligned} Y_1(x_0) &= Y_{1,0} \\ Y_2(x_0) &= Y_{2,0} \end{aligned}}$$

This gives the initial-value problem.

The general form of m first-order (differential) equations

$$\boxed{\begin{array}{ll} Y_1'(x) = f_1(x, Y_1, \dots, Y_m) & Y_1(x_0) = Y_{1,0} \\ Y_2'(x) = f_2(x, Y_1, \dots, Y_m) & Y_2(x_0) = Y_{2,0} \\ \vdots & \vdots \\ Y_m'(x) = f_m(x, Y_1, \dots, Y_m) & Y_m(x_0) = Y_{m,0} \end{array}}$$

Defining column vectors as follows:

$$\vec{Y}(x) = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix}, \quad \vec{Y}_0 = \begin{bmatrix} Y_{1,0} \\ Y_{2,0} \\ \vdots \\ Y_{m,0} \end{bmatrix}, \quad \vec{f}(x, \vec{z}) = \begin{bmatrix} f_1(x, z_1, \dots, z_m) \\ f_2(x, z_1, \dots, z_m) \\ \vdots \\ f_m(x, z_1, \dots, z_m) \end{bmatrix}$$

We can write the m first-order equations as

$$\boxed{\vec{Y}(x) = \vec{f}(x, \vec{Y}(x))}, \quad \boxed{\vec{Y}(x_0) = \vec{Y}_0}$$

Higher-Order Ordinary Differential Equations

$$\boxed{y''(x) = f(x, y(x), y'(x))} \qquad \boxed{\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_0' \end{aligned}}$$

Define $\boxed{y_1(x) = y(x)}$ and $\boxed{y_2(x) = y'(x)}$.

$$\Rightarrow \boxed{\begin{aligned} y_1'(x) &= y'(x) = y_2 \\ y_2'(x) &= y''(x) = f(x, y_1, y_2) \end{aligned}} \qquad \boxed{\begin{aligned} y_1(x_0) &= y_0 \\ y_2(x_0) &= y_0' \end{aligned}}$$

A General m-Order Differential Equation

$$\boxed{\frac{d^m y}{dx^m} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{m-1} y}{dx^{m-1}}\right)}$$

$$y(x_0) = y_0, \quad y'(x_0) = y_0', \quad y^{(m-1)}(x_0) = y_0^{(m-1)}$$

$$y_1 = y, \quad y_2 = y', \quad \ldots, \quad y_m = y^{(m-1)}$$

$$\boxed{\begin{array}{ll} y_1' = y_2 & y_1(x_0) = y_0 \\ \vdots & \vdots \\ y_{m-1}' = y_m & y_{m-1}(x_0) = y_0^{(m-2)} \\ y_m' = f(x, y_1, y_2, \ldots, y_m) & y_m(x_0) = y_0^{(m-1)} \end{array}}$$

Numerical Methods for Systems (Second-Order)

$$y_1(x_{n+1}) = y_1(x_n) + h f_1(x_n, y_1(x_n), y_2(x_n)) + O(h^2)$$

$$y_2(x_{n+1}) = y_2(x_n) + h f_2(x_n, y(x_n), y_2(x_n)) + O(h^2)$$

(P.T.O.)

$$\Rightarrow \quad y_{1,n+1} = y_{1,n} + h f_1(x_n, y_{1,n}, y_{2,n})$$

$$y_{2,n+1} = y_{2,n} + h f_2(x_n, y_{1,n}, y_{2,n})$$

Examples: **The Predator-Prey Model**

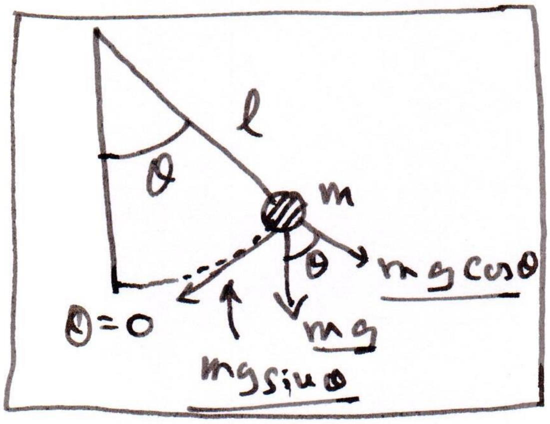
$$\boxed{\frac{dY_1}{dx} = A Y_1 (1 - B Y_2)}, \quad \boxed{\frac{dY_2}{dx} = -C Y_2 (1 - D Y_1)}$$

Initial value: $Y_1(0) = Y_{1,0}$, $Y_2(0) = Y_{2,0}$

Example: **Competitive Exclusion**

$$\boxed{\frac{dY_1}{dx} = AY - BY_1^2 - C Y_1 Y_2}, \quad \boxed{\frac{dY_2}{dx} = D Y_2 - E Y_2^2 - F Y_1 Y_2}$$

Example: **Simple Harmonic Oscillator**



$$m \frac{d^2(l\theta)}{dt^2} = -mg \sin\theta$$

$$\Rightarrow \boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta}$$

$t \to x$, $\theta \to Y_1$, $\theta' \to Y_2$

$$\therefore \boxed{Y_1(x) = \theta(t)}, \quad \boxed{Y_2(x) = \theta'(t)}.$$

$$\Rightarrow \boxed{\frac{dY_1}{dx} = Y_2} \quad \text{and} \quad \boxed{\frac{dY_2}{dx} = -\frac{g}{l}\sin(Y_1)}$$

This is a coupled system. The dynamics of Y_1 is influenced by Y_2, and vice versa.
