Tutorial-9

Q.1 Show that $f(z)=|z|^2$ is differentiable only at $z=0$; no where else. So it is nowhere analytic.

Sol^n

$$f(z)=|z|^2 = x^2+y^2 \qquad z = x+iy$$
$$\Delta z = \Delta x + i\Delta y$$
$$f(z+\Delta z) = |z+\Delta z|^2 = (x+\Delta x)^2+(y+\Delta y)^2$$

At $z=0$

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$\lim_{\Delta z\to 0}\frac{f(0+\Delta z)-f(0)}{\Delta z}$$

$\Delta z = \Delta$

$$= \lim_{\Delta z\to 0}\frac{|\Delta z|^2-0}{\Delta z} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{(\Delta x)^2+(\Delta y)^2}{\Delta x+i\Delta y}$$

$z\bar{z} = |z|^2$, $\frac{|z|^2}{z} = \bar{z}$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta x - i\Delta y \qquad \left\{ z\bar{z} = |z|^2 \Rightarrow \frac{|z|^2}{z} = \bar{z} \right.$$

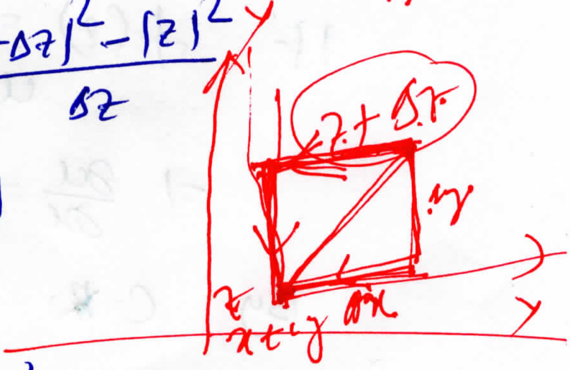
$$= 0$$

$\Rightarrow$ function $f(z)=|z|^2$ is differentiable at $z=0$.

When $z\neq 0$ $\Rightarrow$ ($x\neq 0$ or $y\neq 0$ or both $\neq 0$)

$\Delta z\to 0$

$$f'(z) = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z\to 0}\frac{|z+\Delta z|^2-|z|^2}{\Delta z}$$



$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{(x+\Delta x)^2+(y+\Delta y)^2-(x^2+y^2)}{\Delta x+i\Delta y}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{(\Delta x)^2+2x\Delta x+(\Delta y)^2+2y\Delta y}{\Delta x+i\Delta y}$$

when $\Delta x\to 0$ first then $\Delta y\to 0$,

$$= \lim_{\Delta y\to 0}\frac{(\Delta y)^2+2y\Delta y}{i\Delta y} = \lim_{\Delta y\to 0}\frac{\Delta y+2y}{i} = \frac{2y}{i} = -2iy$$

when $\Delta y \to 0$ first, then $\Delta x \to 0$.

$$f'(z) = \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 2x\Delta x}{\Delta x} = \lim_{\Delta x \to 0} (\Delta x + 2x) = 2x$$

$f'(z) = -2iy = 2x$ this is true only when both $x = y = 0$ otherwise it is not.

Hence function is not differentiable for $z \neq 0$.

$\Rightarrow$ function $f(z) = |z|^2$ is not analytic ~~at any pt~~ at any point z. That is function is nowhere analytic.

Q.2 Prove that an analytic function whose real part is constant is a constant function.

Sol^n Let $f(z)$ is an analytic function whose real part is constant.

If $f(z) = u + iv$

u is constant.

$$\Rightarrow \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 0$$

By C.R. equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = 0$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \Rightarrow v$ is constant.

$\Rightarrow u + iv$ is constant.

$\Rightarrow f(z)$ is constant function.

Q.1 Prove that an analytic function whose modulus is constant is a constant function.

Sol^n Let $f(z)$ be an analytic function whose modulus is constant.

$$f(z) = u + iv$$
$$|f(z)| = u^2 + v^2 = k^2 \quad \text{constant.}$$

Differentiating w.r.t. x

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \quad \text{—— (1)}$$

Differentiating w.r.t. y

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \quad \text{—— (2)}$$

By C.R. equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Putting $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ in (1)

$$u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial x} = 0 \quad \text{—— (3)}$$

Putting $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ in (2)

$$-u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0 \quad \text{—— (4)}$$

(3) $\times u$ + (4) $\times v$

$$u^2\frac{\partial v}{\partial y} + \cancel{uv\frac{\partial v}{\partial x}}$$
$$+ \cancel{-uv\frac{\partial v}{\partial x}} + v^2\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow (u^2+v^2)\frac{\partial v}{\partial y} = 0$$

If $u^2+v^2=0$ then it must be that $u=0, v=0$

$\Rightarrow f = 0 \Rightarrow f$ is constant.

Q-3 ~~Contd~~ Continued

If $u^2+v^2 \neq 0$, then $\frac{\partial v}{\partial y} = 0$

By C.R. equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$

Similarly

$u^2\frac{\partial v}{\partial y} + v^2\frac{\partial v}{\partial x}$

$-u^2\frac{\partial v}{\partial x} + v^2$

Putting $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ in equation (1) and (2)

we get $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$

$\Rightarrow$ u and v are constant.

$\Rightarrow f = u+iv$ is constant.

Q.4 Find the principal value of the argument for the following

(I) $1-i$ (IV) $-5+5i$

(II) $3+4i$

(III) $-\pi-\pi i$

Sol (I) $z = 1-i$ $x=1,\ y=-1$

$x>0$.

$\operatorname{Arg} z = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1) = -\pi/4$

(II) $z = 3+4i$, $x=3,\ y=4$, $x>0$

$\operatorname{Arg} z = \tan^{-1}(y/x) = \tan^{-1}(4/3) = 0.9273$

(III) $z = -\pi-\pi i$, $x=-\pi,\ y=-\pi$

$x<0,\ y<0$

$\operatorname{Arg}(z) = -\pi + \tan^{-1}(y/x) = -\pi + \tan^{-1}\left(\frac{-\pi}{-\pi}\right)$

$= -\pi + \tan^{-1}(1) = -\pi + \pi/4 = -3\pi/4$

(IV) $z = -5+5i$, $x=-5<0,\ y=5>0$

$\operatorname{Arg}(z) = \pi + \tan^{-1}(y/x) = \pi + \tan^{-1}\left(\frac{5}{-5}\right)$

$= \pi + \tan^{-1}(-1) = \pi + (-\pi/4) = \frac{3\pi}{4}$

$\tan^{-1}(\infty) = \pi/2$

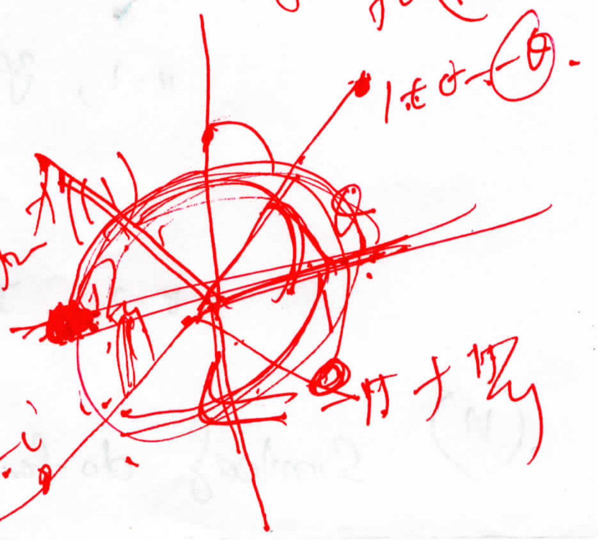
$\tan^{-1}(-1)$

1st or $-\theta$.

$\tan^{-1}(1)$

$\tan^{-1}(y/x) = \tan^{-1}(1)$

$= \tan^{-1}\frac{1}{-1}$



$-\pi + \pi/4$

$-1-i$

Q.5 Find (i) $\cancel{\sqrt[4]{-7+24i}}$ $\sqrt[3]{1+i}$

(ii) $\sqrt[4]{-4}$

Sol^n $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$

$k = 0, 2, \dots, n-1$

r is modulus of z

θ is principal value of ~~the~~ argument of z.

~~(i)~~ $\cancel{\sqrt[4]{-7+24i}}$

~~Here~~ $\cancel{z = -7+24i}$

$\cancel{r = |z| = \sqrt{(-7)^2+(24)^2} = \sqrt{49+576} = \sqrt{625} = 25}$

$\cancel{\theta = \operatorname{Arg}(z) = \pi + \tan^{-1}(y/x) = \pi + \tan^{-1}\left(\frac{24}{-7}\right)}$

(i) $\sqrt[3]{1+i}$ Here $z = 1+i$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \operatorname{Arg}(z) = \tan^{-1}(y/x) = \tan^{-1}(1) = \pi/4$

$$\sqrt[3]{1+i} = (\sqrt{2})^{1/3}\left(\cos\frac{\pi/4+2k\pi}{3} + i\sin\frac{\pi/4+2k\pi}{3}\right)$$

$k = 0, 1, 2$

For $k=0$, $\sqrt[3]{1+i} = (\sqrt{2})^{1/3}\left(\cos\pi/12 + i\sin\pi/12\right)$

$k=1$, $\sqrt[3]{1+i} = (\sqrt{2})^{1/3}\left(\cos\frac{\pi/4+2\pi}{3} + i\sin\frac{\pi/4+2\pi}{3}\right)$

$= (\sqrt{2})^{1/3}\left(\cos\frac{9\pi}{12} + i\sin 9\pi/12\right)$

$k=2$, $\sqrt[3]{1+i} = (\sqrt{2})^{1/3}\left(\cos\frac{\pi/4+4\pi}{3} + i\sin\frac{\pi/4+4\pi}{3}\right)$

$= (\sqrt{2})^{1/3}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$

(ii) Similarly do for $\sqrt[4]{-4}$ | ~~Ans~~

Q-6 Show that an analytic function is independent of $\bar{z}$.

Sol Let $f(z) = u + iv$ is an analytic function

$z = x + iy$

We have $x = \frac{z+\bar{z}}{2}$ $y = \frac{z-\bar{z}}{2i}$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$= \frac{\partial f}{\partial x}\cdot\frac{1}{2} + \frac{\partial f}{\partial y}\cdot\frac{1}{2i} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y}\right]$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial f}{\partial x}\cdot\frac{1}{2} + \frac{\partial f}{\partial y}\cdot\left(-\frac{1}{2i}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right]$$

As $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y}$

$$= \frac{1}{2}\left[\cancel{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}}^{\,0} + i\left(\cancel{\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}}^{\,0}\right)\right]$$

$$= 0$$

$\therefore \frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow f$ is independent of $\bar{z}$.

Q.2 Show that $f(z) = |\operatorname{Re} z \operatorname{Im} z|^{\frac{1}{2}}$
satisfies the C.R. equation at the origin,
but is not differentiable at origin.

Kajib
Friday
Lab
GR-5

Sol^n $\quad z = x+iy$

$f(z) = u+iv \quad \operatorname{Re} z = x$

$= |xy|^{\frac{1}{2}} \quad \operatorname{Im} z = y$

$u(x,y) = |xy|^{\frac{1}{2}}, \quad v(x,y) = 0.$

$$\frac{\partial u}{\partial x}(0,0) = \lim_{h\to 0} \frac{u(0+h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

Similarly $\frac{\partial v}{\partial x}(0,0) = 0$

$\frac{\partial u}{\partial y}(0,0) = 0 \qquad \frac{\partial v}{\partial y}(0,0) = 0.$

So $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ at the origin.

$\Rightarrow$ C.R. equations are satisfied at the origin.

But $f(z)$ is not differentiable at $z=0$.

Reason
Putting into polar form $x = r\cos\theta$, $y = r\sin\theta$.

$$f(z) = |xy|^{\frac{1}{2}} = |r\cos\theta \cdot r\sin\theta|^{\frac{1}{2}}$$
$$= (r^2)^{\frac{1}{2}} \cdot |\cos\theta\sin\theta|^{\frac{1}{2}}$$
$$= r\frac{|\sin 2\theta|^{\frac{1}{2}}}{\sqrt{2}}$$

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{r\to 0} \frac{\frac{\cancel{r}|\sin 2\theta|^{\frac{1}{2}}}{\sqrt{2}} - 0}{\cancel{r}(\cos\theta + i\sin\theta)} = \frac{e^{-i\theta}|\sin 2\theta|^{\frac{1}{2}}}{\sqrt{2}}$$

which is different for different values of θ.

So $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist $\Rightarrow$ f is not differentiable at ~~z=0~~