Taylor Polynomials of Transcendental functions

1/ $\boxed{y = f(x) = e^x}$ $\Rightarrow$ $\boxed{f'(x) = f''(x) = f'''(x) = e^x}$

Order 1: $\boxed{p_1(x) = f(a) + f'(a)(x-a)}$ so that

$\boxed{p_1(a) = f(a)}$ and $\boxed{p_1'(a) = f'(a)}$, which

we get from $\boxed{p_1'(x) = f'(a)}$. We choose $\boxed{a = 0}$
$\Rightarrow f(a) = f'(a) = 1$

$\Rightarrow \boxed{p_1(x) = 1 + x}$ when $x = a = 0$. $p_1(0) = 1 = f(0)$
and $p_1'(x) = 1 \Rightarrow p_1'(0) = 1 = f'(0)$.

Order 2: $\boxed{p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2}$

When $a = 0$, $f(a) = f'(a) = f''(a) = e^a = e^0 = 1$.

$\Rightarrow \boxed{p_2(x) = 1 + x + \frac{x^2}{2!}}$ $\Rightarrow$ $\boxed{p_2'(x) = 1 + x}$, $\boxed{p_2''(x) = 1}$.

Now, $\boxed{p_2(a) = f(a)}$, $\boxed{p_2'(a) = f'(a)}$ & $\boxed{p_2''(a) = f''(a)}$.

$\because \boxed{a = 0}$, $\boxed{p_2(0) = 1 = f(0)}$, $\boxed{p_2'(0) = 1 = f'(0)}$, $\boxed{p_2''(0) = 1 = f''(0)}$.

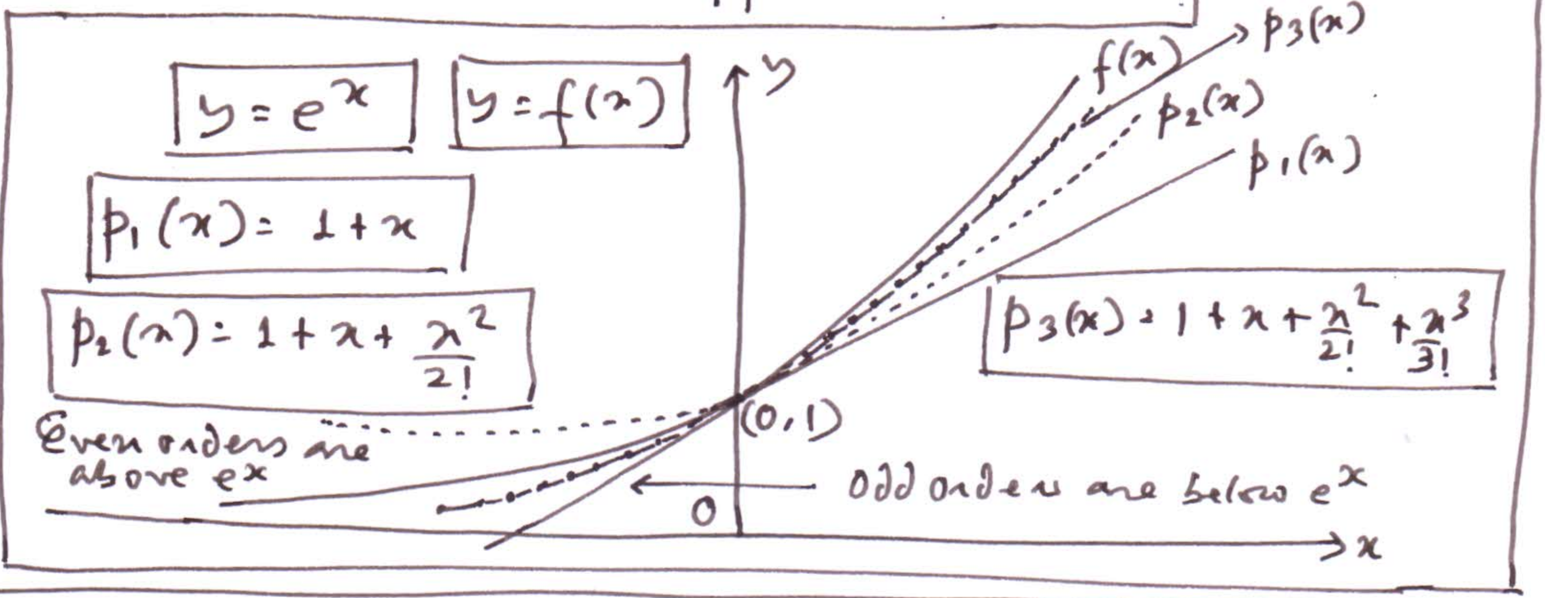
Order 3: $\boxed{p_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3}$

When $a = 0$, $\boxed{p_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}}$ $\Rightarrow$ $\boxed{p_3'(x) = 1 + x + \frac{x^2}{2!}}$

$\boxed{p_3''(x) = 1 + x}$ and $\boxed{p_3'''(x) = 1}$ For $a = 0$, $\boxed{p_3(0) = 1 = f(0)}$

Similarly, $\boxed{p_3'(0) = 1 = f'(0)}$, $\boxed{p_3''(0) = 1 = f''(0)}$

and $\boxed{p_3'''(0) = f'''(0) = 1}$ since $f'''(x) = e^x \Rightarrow f'''(0) = 1$



2/ $\boxed{y = f(x) = \ln x} \Rightarrow \boxed{f'(x) = \frac{1}{x}}$, $\boxed{f''(x) = \frac{-1}{x^2}}$

and $\boxed{f'''(x) = \frac{2}{x^3}}$ We choose $\boxed{a = 1}$.

Order 1: $p_1(x) = f(a) + f'(a)(x-a)$.

$\boxed{f(a) = f(1) = \ln(1) = 0}$, $\boxed{f'(a) = f'(1) = 1}$ $\boxed{f''(1) = -1}$

and $\boxed{f'''(1) = 2} \Rightarrow \boxed{p_1(x) = x - 1} \Rightarrow \boxed{p_1'(x) = 1}$

$\boxed{p_1(a) = p_1(1) = 0 = f(a) = f(1) = 0}$ & $\boxed{p_1'(a) = 1 = f'(1) = 1}$

Order 2: $\boxed{p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2}$.

$\Rightarrow p_2(x) = (x-1) - \frac{1}{2!}(x-1)^2 \Rightarrow \boxed{p_2(x) = (x-1) - \frac{(x-1)^2}{2}}$

$\Rightarrow \boxed{p_2'(x) = 1 - (x-1) = 2 - x}$ and $\boxed{p_2''(x) = -1}$.

$a = 1$

$\boxed{p_2(a) = p_2(1) = 0 = f(1) = 0}$, $\boxed{p_2'(1) = 1 = f'(1) = 1}$

$\boxed{p_2''(1) = -1 = f''(1) = -1}$ All at $\boxed{a = 1}$.

Order 3 : $\boxed{p_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3}$

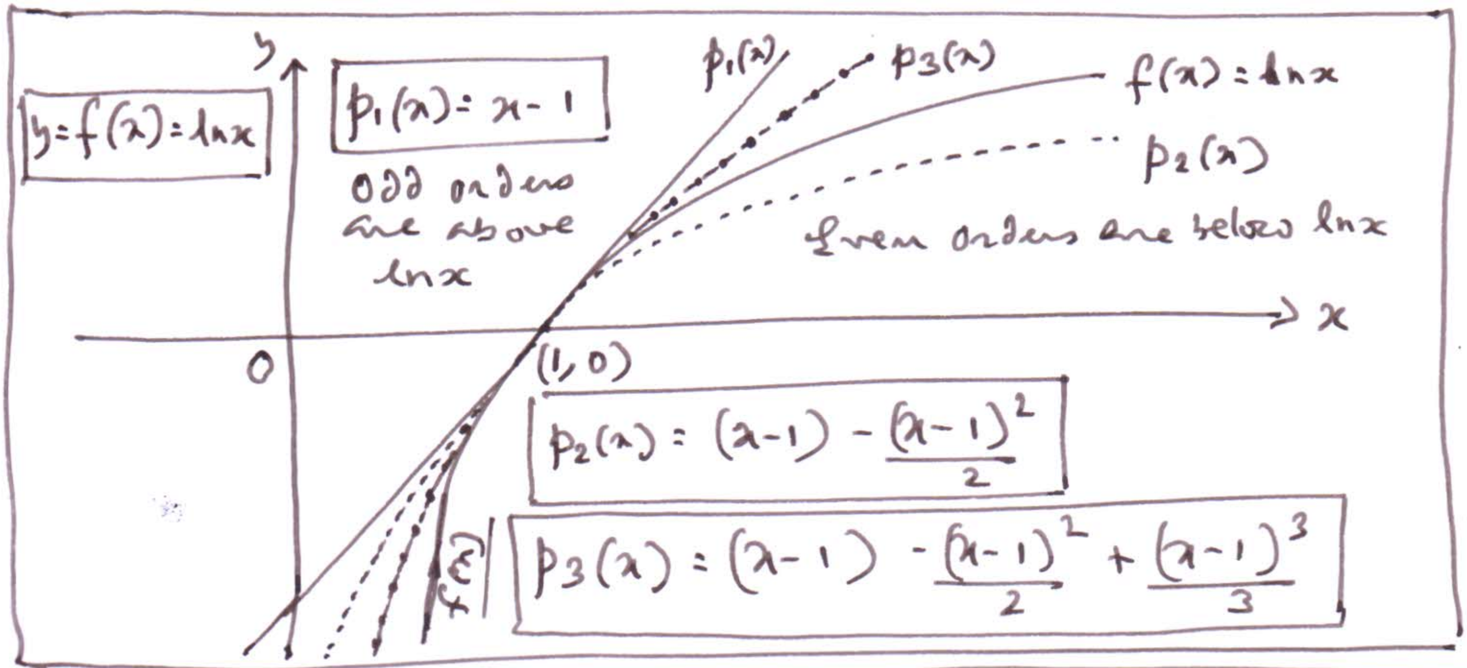
For $\boxed{a = 1}$, $\boxed{p_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}}$

$\boxed{p_3'(x) = 1 - (x-1) + (x-1)^2}$, $\boxed{p_3''(x) = -1 + 2(x-1) = -1 + 2x - 2 = 2x - 3}$

and $\boxed{p_3'''(x) = 2}$. Hence, $\boxed{p_3(1) = 0 = f(1) = 0}$.

$\boxed{p_3'(1) = 1 = f'(1) = 1}$, $\boxed{p_3''(1) = -1 = f''(1) = -1}$.

$\boxed{p_3'''(1) = 2 = f'''(1) = 2}$ all valid for $\boxed{a = 1}$.



At $a = 1$ $f(x) = \ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$

Write $\boxed{z = x - 1}$ $\Rightarrow$ $\boxed{x = 1 + z}$. This gives us the transcendental series of $\ln(1+z)$ as,

$\boxed{\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \cdots\cdots}$ or equivalently,

Resubstituting variables

$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots}$ (The series has the argument $1+x$)

3/. $\boxed{y = f(x) = \sin x}$ $\boxed{f'(x) = \cos x}$ $\boxed{f''(x) = -\sin x}$
and $\boxed{f'''(x) = -\cos x}$

<u>Order 1:</u> $\boxed{p_1(x) = f(a) + f'(a)(x-a)}$. Choose $\boxed{a = 0}$.

<u>Even orders = 0</u>

$\boxed{f(0) = 0}$, $\boxed{f'(0) = 1}$. $\boxed{f''(0) = 0}$ $\boxed{f'''(0) = -1}$

<u>Even orders vanish</u>

Hence, $\boxed{p_1(x) = x}$ $\Rightarrow$ $\boxed{p_1'(x) = 1}$ $\therefore$ $\boxed{p_1(0) = 0 = f(0)}$

and $\boxed{p_1'(0) = 1 = f'(0) = 1}$ at $\boxed{a = 0}$.

<u>Order 2:</u> $\boxed{p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2}$

At $a = 0$, $\boxed{p_2(x) = 0 + x + 0 = x}$ $\therefore$ $\boxed{p_2(x) = p_1(x)}$

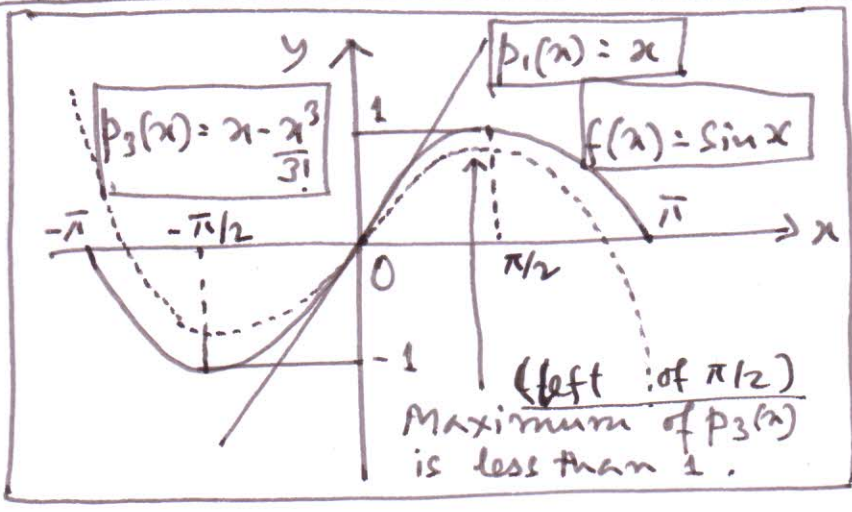
<u>Order 3:</u> $\boxed{p_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3}$

At $\boxed{a = 0}$, $p_3(x) = 0 + x + 0 - \frac{1}{3!}x^3 \Rightarrow \boxed{p_3(x) = x - \frac{x^3}{3!}}$

With $\boxed{p_3(x) = x - \frac{x^3}{3!}}$ $\boxed{p_3'(x) = 1 - \frac{x^2}{2}}$, $\boxed{p_3''(x) = -x}$

and $\boxed{p_3'''(x) = -1}$. $\boxed{p_3(0) = f(0) = 0}$. $\boxed{p_3'(0) = 1 = f'(0)}$

$\boxed{p_3''(0) = 0 = f''(0)}$ and $\boxed{p_3'''(0) = -1 = f'''(0)}$ at $\boxed{a = 0}$.



$\boxed{p_3(x) = x - \frac{x^3}{3!}}$ Turning points ↓

$p_3'(x) = 1 - \frac{x^2}{2} = 0 \Rightarrow \boxed{x = \pm\sqrt{2}}$

$\boxed{p_3''(x) = -(\pm\sqrt{2})}$ $\boxed{x = +\sqrt{2}}$ is the maximum.

$\Rightarrow \boxed{p_3(\sqrt{2}) = \sqrt{2} - \frac{2\sqrt{2}}{3 \times 2} = \sqrt{2} \cdot \frac{2}{3} < 1}$

$\boxed{p_3(\sqrt{2}) < 1}$ $\therefore$ The maximum of $p_3(x)$ is below 1.

47. $\boxed{y = f(x) = \cos x}$ $\boxed{f'(x) = -\sin x}$ $\boxed{f''(x) = -\cos x}$

and $\boxed{f'''(x) = \sin x}$. Choose $\boxed{a = 0}$.

Order 1: $\boxed{p_1(x) = f(a) + f'(a)(x-a)}$ Odd orders vanish

$\boxed{f(0) = 1}$, $\boxed{f'(0) = 0}$, $\boxed{f''(0) = -1}$, $\boxed{f'''(0) = 0}$

At $\boxed{a=0}$, $\boxed{p_1(x) = 1}$ $\boxed{p_1(0) = 1 = f(0) = 1}$

Order 2: $\boxed{p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2}$.

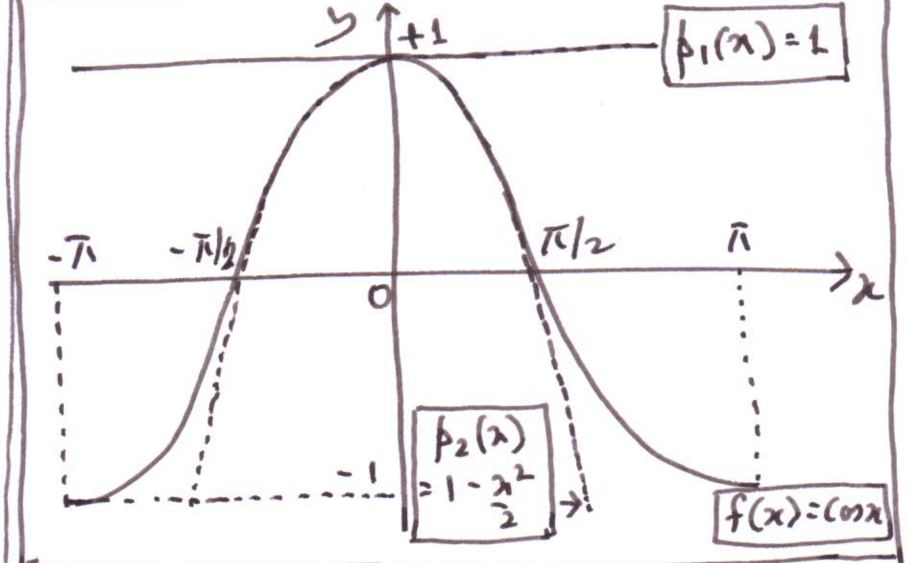
At $\boxed{a=0}$, $\boxed{p_2(x) = 1 - \frac{x^2}{2}}$ $\Rightarrow \boxed{p_2'(x) = -x}$ $\boxed{p_2''(x) = -1}$

$\boxed{p_2(0) = 1 = f(0) = 1}$ and $\boxed{p_2'(0) = 0 = f'(0) = 0}$ $\boxed{p_2''(0) = f''(0) = -1}$

Order 3: $\boxed{p_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3}$

At $\boxed{a=0}$ $p_3(x) = 1 + 0 - \frac{x^2}{2} + 0 \Rightarrow \boxed{p_3(x) = 1 - \frac{x^2}{2}}$

Hence $\boxed{p_3(x) = p_2(x)}$ $\therefore$ $\boxed{p_3'''(x) = 0 = p_3'''(0) = f'''(0)}$



$\boxed{p_2(x) = 1 - \frac{x^2}{2}}$ When $p_2(x) = 0$

$\boxed{x = \pm\sqrt{2}}$. $\boxed{\cos x = 0}$

$\Rightarrow \boxed{x = \pm \pi/2}$ Now $\sqrt{2} \simeq 1.4$ and $\pi/2 \simeq 1.5$

Hence $p_2(x)$ matches $\cos x$ very closely within $-\pi/2 < x < \pi/2$. $\therefore$ Over a half-cycle there is a good match. (Sin x has a better match with $p_3(x)$ over a full cycle).

Theory of Equations (Polynomial)

1/. $\boxed{f(x) = g(x)(x-x_1) + R}$. If $f(x)$ is exactly divisible by $x-x_1$, then $\boxed{R=0}$.

$\Rightarrow \boxed{f(x_1) = 0}$. $\therefore$ x_1 is a root of the polynomial equation.

2/. $\boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_2 x^2 + a_1 x + a_0}$.

If $x = x_1$ is a root, then $\boxed{f(x) = \phi_1(x)(x-x_1)}$.

Again if $x = x_2$ is a root, then $\boxed{f(x) = \phi_2(x)(x-x_1) \times (x-x_2)}$

Likewise, $\boxed{f(x) = a_n(x-x_1)(x-x_2)\cdots(x-x_n)}$.

If $x \neq x_1, x_2, x_3 \ldots, x_n$ then $f(x) \neq 0$. Hence, there can be no roots other than $\{x_1, x_2, \cdots x_n\}$.

$\boxed{f(x) = 0}$ has got only n roots.

3/. $\boxed{f(x) = 0}$ is a polynomial equation with real coefficients, and it has an imaginary root $\boxed{a+ib}$ $(a, b \neq 0)$.

We consider a quadratic factor $\boxed{(x-a)^2 + b^2}$ and write $\boxed{f(x) = g\left[(x-a)^2 + b^2\right] + (R_1 x + R_0)}$.

We know $\boxed{f(a+ib) = 0} \Rightarrow \boxed{R_1(a+ib) + R_0 = 0}$.

3/. (continued)

$\Rightarrow$ $\boxed{R_1 a + R_0 = 0}$ and $\boxed{R_1 b = 0}$.

$\because$ $b \neq 0$, $\Rightarrow$ $\boxed{R_1 = 0}$ $\Rightarrow$ $\boxed{R_0 = 0}$ No remainder.

$\therefore$ $f(x)$ is exactly divisible by $\boxed{(x-a)^2 + b^2}$.

But $(x-a)^2 + b^2 = (x - a + ib)(x - a - ib)$.

$\Rightarrow$ If $x = a + ib$ is a root then $x = a - ib$ is also a root. Complex roots occur in conjugate pairs.

4/. i) If the coefficients are all positive, the equation has No positive root.

Eg. $\boxed{x^5 + x^2 + 2x + 1 = 0}$ has No positive root.

ii.) If the coefficients of the even powers of x are all of one sign, and the coefficients of the odd powers are all of the contrary sign, the equation has No negative root.

Eg. $\boxed{x^7 + x^5 - 2x^4 + x^3 - 3x^2 + 7x - 5 = 0}$ has No negative root

iii.) If the equation contains only even powers of x and the coefficients are all of the same sign, the equation has no real root.

Eg. $\boxed{2x^8 + 3x^4 + x^2 + 7 = 0}$ has No real root.

iv.) If the equation has only odd powers of x, with all coefficients of the same sign, there is No real root except $\boxed{x = 0}$. Eg. $\boxed{x^9 + 2x^5 + 3x^3 + x = 0}$ has no real root except $x = 0$.

5/. $\boxed{f(x) = 0}$ cannot have more positive roots than there are changes of sign in $f(x)$, and cannot have more negative roots than there are changes of sign in $f(-x)$.
(Descartes' <u>RULE OF SIGNS</u>).

Eg. $\boxed{f(x) = x^9 + 5x^8 - x^3 + 7x + 2 = 0}$ Maximum 2 positive roots.

$\boxed{f(-x) = -x^9 + 5x^8 + x^3 - 7x + 2 = 0}$ Maximum 3 negative roots.

$\therefore$ There are at least 4 complex roots.

6/. If x changes continuously from $x = a$ to $x = b$, then $f(x)$ will also change continuously.

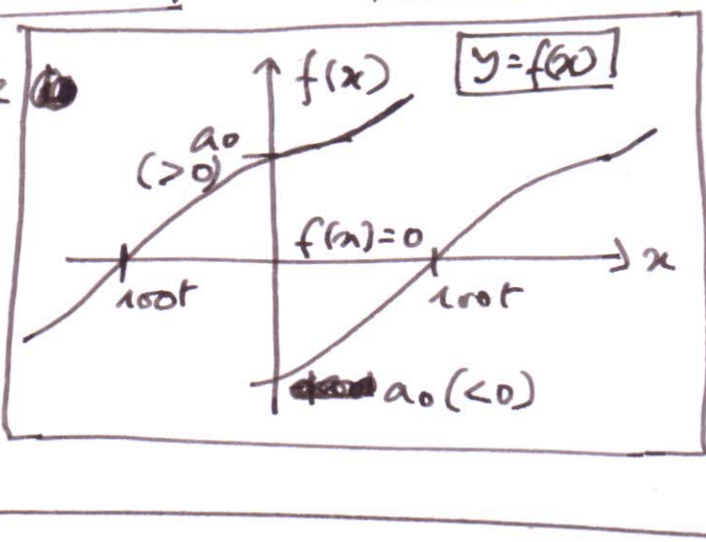
7/. If $f(a)$ and $f(b)$ are of contrary signs, then between $x = a$ and $x = b$ there is <u>at least one</u> root of $f(x) = 0$. (<u>The Bisection Principle</u>)

8/. Every equation of an <u>odd degree</u> must have at least one real root whose sign is opposite to that of its last term.

$\boxed{f(\infty) = \infty}$, $\boxed{f(-\infty) = -\infty}$ and $\boxed{f(0) = a_0}$.

i.) If $a_0 > 0$, then there is a root between $-\infty < x < 0$.

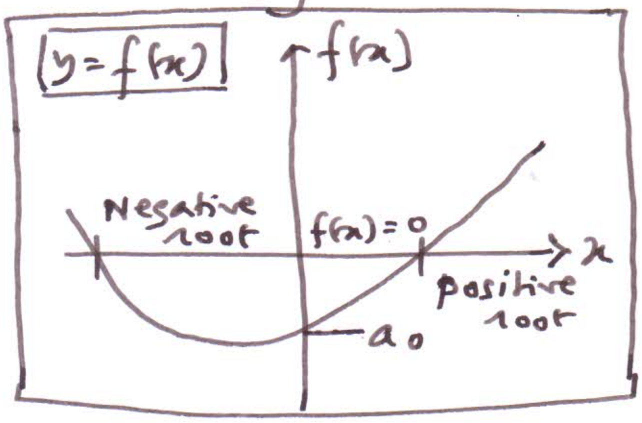
ii.) If $a_0 < 0$, then there is a root between $0 < x < \infty$.



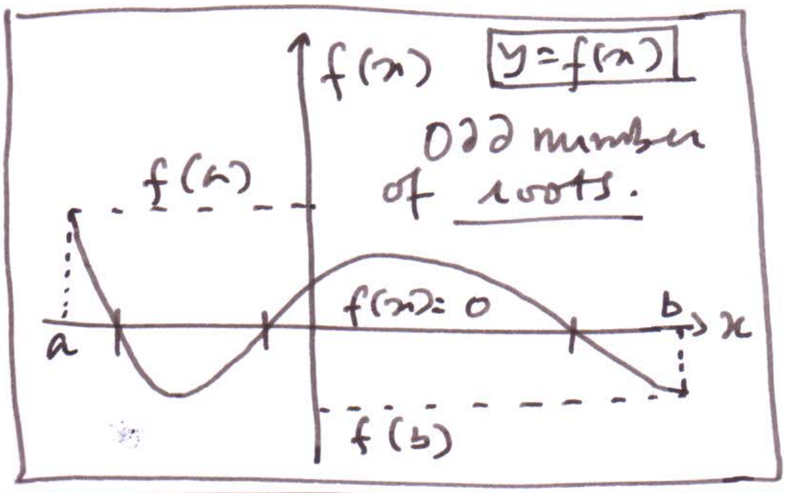
9/. Every equation of even degree with a negative last term has at least two real roots, one positive and one negative.

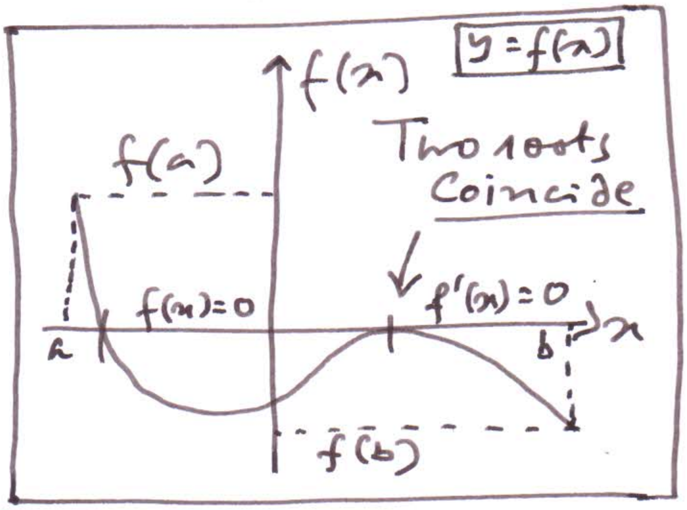
$\boxed{f(\pm\infty) = \infty}$, $\boxed{f(0) = a_0}$

with $a_0 < 0$. There are at least two real roots of $f(x) = 0$, one positive and the other negative.

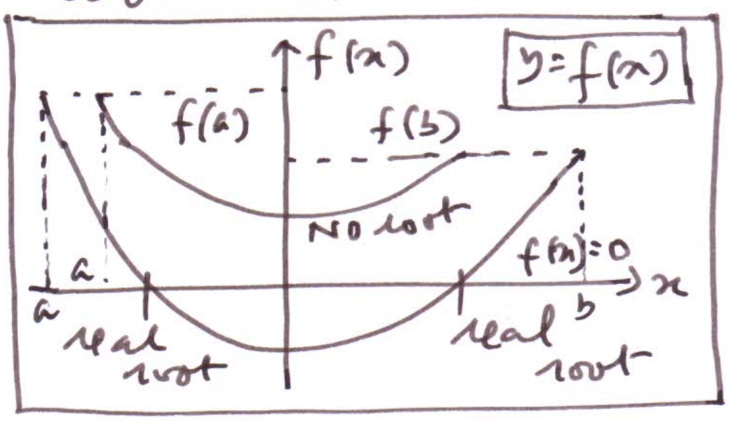


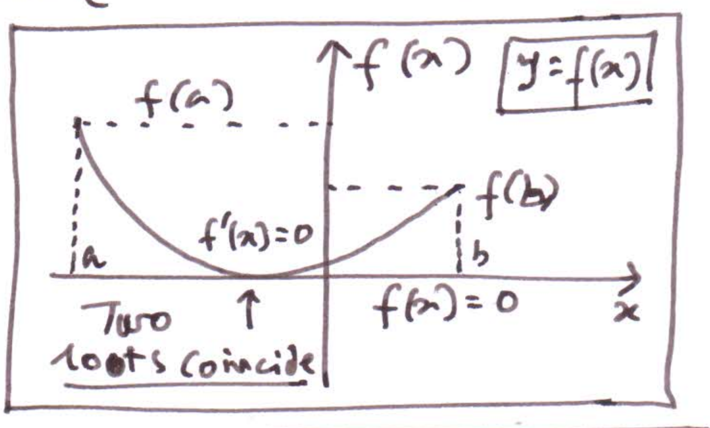
10/. If $f(a)$ and $f(b)$ have contrary signs, an odd number of roots exists between $a < x < b$. (Counting coinciding roots)





11/. If $f(a)$ and $f(b)$ have same signs either there are NO roots or an even number of roots between $a < x < b$ (with coinciding roots).





Root-Finding by the Bisection Method

1/. Given $\boxed{y = f(x)}$, if $\boxed{f(a)f(b) < 0}$, then there exists at least one root of $f(x) = 0$ at $x = \alpha$, in $\boxed{a < x < b}$.

2/. To approach the root at $\boxed{x = \alpha}$ where $\boxed{f(\alpha) = 0}$, bisect the length from $x = a$ to $x = b$ at $\boxed{x = c = \frac{a+b}{2}}$.

Alternatively $\boxed{c = a + \frac{b-a}{2} = \frac{a+b}{2}}$.

3/. If $\boxed{f(a)f(c) < 0}$, then the root lies between $x = a$ and $x = c$. Assign $b = c$.

4/. If $\boxed{f(a)f(c) > 0}$, then the root lies between $x = c$ and $x = b$. Assign $a = c$.

5/. Apply an error tolerance, ϵ.

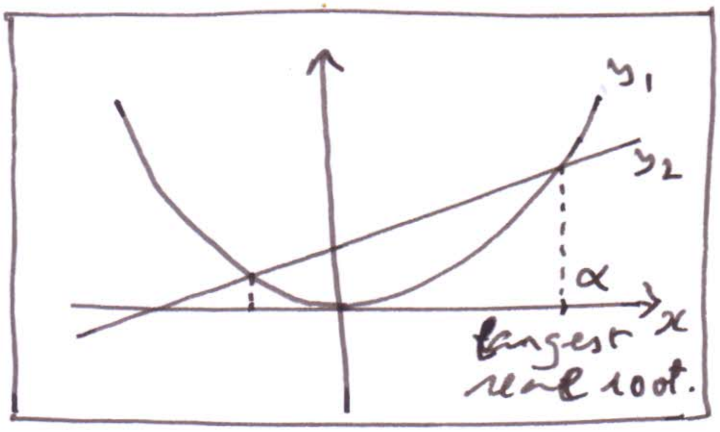
6/. If $b - c$ or $c - a \leq \epsilon$, then accept c as the root and stop.

Example: Find the largest root of

$\boxed{f(x) \equiv x^6 - x - 1 = 0}$. Write $\boxed{y_1 = x^6}$ and $\boxed{y_2 = x + 1}$.

$\boxed{\epsilon = 0.001}$

A root exists when $\boxed{y_1 = y_2}$ (at $f(\alpha) = 0$). From the graph there are only 2 such real roots. $f(x)$ has 6 roots in total. There are 4 complex roots.



Guess value: $\boxed{f(1) = -1}$, $\boxed{f(2) = 61}$ $\boxed{a = 1}$, $\boxed{b = 2}$.

a	b	$c = \frac{a+b}{2}$	$f(c)$	$b-c = c-a$	Assign	$f(a) \times f(c)$
1	2	1.5	8.8906	0.5	Set b = c	$f(a)f(c) < 0$
1	1.5	1.25	1.5647	0.25	Set b = c	$f(a)f(c) < 0$
1	1.25	1.125	-0.0977	0.125	Set a = c	$f(a)f(c) > 0$
1.125	1.25	1.1875	0.6167	0.0625	Set b = c	$f(a)f(c) < 0$
1.125	1.1875	1.1563	0.2333	0.0312	Set b = c	$f(a)f(c) < 0$
1.125	1.1563	1.1407	0.0616	0.0156	Set b = c	$f(a)f(c) < 0$
1.125	1.1407	1.1328	-0.0196	0.0079	Set a = c	$f(a)f(c) > 0$
1.1328	1.1407	1.1368	0.0204	0.0039	Set b = c	$f(a)f(c) < 0$
1.1328	1.1368	1.1348	0.00080	0.0020	Set b = c	$f(a)f(c) < 0$
1.1328	1.1348	1.1338	-0.0095	0.0010	Set b = c	$f(a)f(c) < 0$

$\because$ $b - c = c - a \leq \epsilon = 0.001$ we accept c as the root α. $\Rightarrow$ $\boxed{\alpha = c = 1.1338}$.