1.

$X \backslash Y$	0	1	2	$p(x_i)$
0	1/8	1/8	0	1/4
1	1/8	2/8	1/8	1/2
2	0	1/8	1/8	1/4
$p(y_j)$	1/4	1/2	1/4	

HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

$$E(X) = \sum_{i=0}^{2} x_i \, p(x_i)$$

$$= 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4}$$

$$= 1$$

lly $E(Y) = 1$

$$\sigma_x^2 = E(x^2) - [E(x)]^2$$

$$= 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 4 \times \frac{1}{4} - 1$$

$$= \frac{1}{2}$$

lly $\sigma_y^2 = \frac{1}{2}$

$$\text{COV}(XY) = E(XY) - E(X).E(Y)$$

$$= 0 \times \frac{1}{8} + 0 \times \frac{1}{8} + 0 \times 0$$

$$+ 0 \times \frac{1}{8} + 1 \times \frac{2}{8} + 2 \times \frac{1}{8}$$

$$+ 0 \times 0 + 2 \times \frac{1}{8} + 4 \times \frac{1}{8}$$

$$= \frac{5}{4}$$

$$\rho_{xy} = \frac{5/4}{1/2} = \frac{1}{2}$$

Tutorial - 5

2

a) $$\int_0^1\int_0^2 cx^2 + \frac{xy}{3}\, dy\, dx = 1$$

$$\Rightarrow \int_0^1 cx^2[y]_0^2 + \frac{x}{3}\left[\frac{y^2}{2}\right]_0^2 dx = 1$$

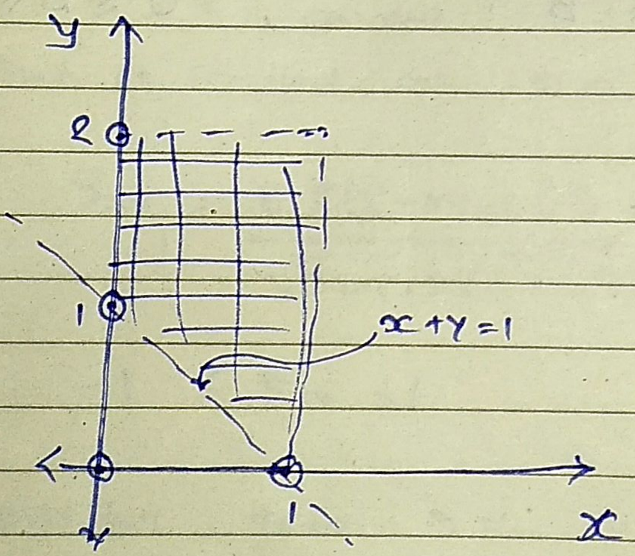
$$\Rightarrow \int_0^1 2cx^2 + \frac{2x}{3}\, dx = 1$$

$$\Rightarrow 2c\left[\frac{x^3}{3}\right]_0^1 + \frac{2}{3}\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$\Rightarrow \frac{2c}{3} + \frac{1}{3} = 1$$

$$\Rightarrow \boxed{c = 1}$$

b)



$P[X+Y \geq 1]$

$$= \int_0^1\int_{1-x}^2 x^2 + \frac{xy}{3}\, dy\, dx$$

$$= \frac{65}{72}$$

c) $$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$= \int_0^2 x^2 + \frac{xy}{3}\, dy$$

$$= 2x^2 + \frac{2x}{3}, \quad 0 \leq x \leq 1$$

$$f_Y(y) = \int_0^1 x^2 + \frac{xy}{3}\, dx$$

$$= \frac{1}{3} + \frac{y}{6}, \quad 0 \leq y \leq 2$$

d) ~~d)~~ No, X & Y are not independent.

e)

$$\mathrm{COV}(X,Y) = E(XY) - E(X).E(Y)$$

$$E(XY) = \int_0^1 \int_0^2 xy \underbrace{\left(x^2 + \frac{xy}{3}\right)}_{f_{X,Y}(x,y)} dy.dx$$

$$= \frac{26}{27}$$

$$E[X] = \int_0^1 x.f_X(x)\,dx = \int_0^1 2x^3 + \frac{2x^2}{3}\,dx = \frac{13}{18}$$

$$E[Y] = \int_0^2 y.f_Y(y)\,dy = \int_0^2 \frac{y}{3} + \frac{y^2}{6}\,dy = \frac{10}{9}$$

$$\mathrm{COV}(X,Y) = \frac{26}{27} - \frac{13}{18}.\frac{10}{9} = -\frac{13}{27}$$

3.

a) $|A - \lambda I| = 0$

$$\therefore \left| \begin{bmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & \frac{1}{4} \\ \frac{1}{4} & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 - \frac{1}{16} = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 - \frac{1}{16} = 0$$

$$\Rightarrow \lambda - 2\lambda + \frac{15}{16} = 0$$

$$\Rightarrow \lambda_1 = 1.25, \quad \lambda_2 = 0.75$$

$\lambda = 1.25$

$$\begin{bmatrix} -0.25 & 0.25 \\ 0.25 & -0.25 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + y = 0$$

$$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = 0.75$

$$\begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x + y = 0$$

$$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$P = U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$C_x = V \lambda U$$

$$= \cancel{\begin{bmatrix} 1.25 & 0 \\ 0 & 0.75 \end{bmatrix}}\Big[$$

$$P^T C_x P = P$$

3) $E(XY) = \int_0^\infty \int_0^\infty xy \cdot f_{XY}(x,y)\, dy\, dx$

$= \int_0^\infty \int_0^\infty 2xy e^{-x} \cdot e^{-y}\, dy \cdot dx$

$= 1$

$E(X) = \int_0^\infty x \cdot f_X(x) \cdot dx$

Here, $f_X(x) = \int_0^\infty f_{XY}(x,y)\, dy = 2e^{-x}(1-e^{-x})$

$f_Y(y) = \int_0^\infty f_{XY}(x,y) \cdot dx = 2e^{-2y}$

$E(X) = \int_0^\infty 2x e^{-x}(1-e^{-x})\, dx$

$= 2\int_0^\infty xe^{-x} - xe^{-2x}\, dx$

Log
Inves
Algebric
Trigno.
Exponent

$= 3/2$

$E(Y) = 1/2$

$\operatorname{cov}(XY) = 1 - \frac{3}{2} \cdot \frac{1}{2} = 1/4$

$\rho(XY = \frac{\operatorname{Cov}(XY)}{\sigma_X \sigma_Y} = \frac{1/4}{\sqrt{5/4}\sqrt{1/4}} = \frac{1}{\sqrt{5}}$