

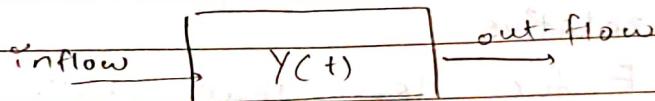
## Compartment Models:



looking at dynamics of compartment as a whole without worrying about individual compartment.

A complex system considered (modelled) as a compartment.

Imagine ~~as~~ atmosphere as a compartment:

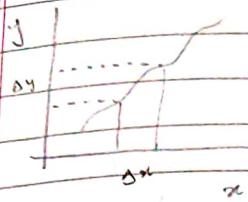


here, inflow and outflow can be of  $\text{CO}_2$

$$\text{Rate of change} = \text{Rate in} - \text{Rate out}$$

Example: - air balloon with air inside it.  
= prick a hole into it - outflow

looking at a system, as a whole, something in it changes due to inflow and/or outflow. The amount changes due to this inflow/outflow.



$$\frac{y(x+\Delta x) - y(x)}{\Delta x}$$

rate of change: change in  $y$  in relation to change in  $x$ .

Example:

$$y(x=0) = 10$$

$$y(x=100) = 1010$$

$$\text{rate of change} = \frac{y(x=100) - y(x=0)}{100 - 0}$$

$$= \frac{1010 - 10}{100}$$

$$= 10$$

variations in the middle of the taken interval are not reflected in above calculation

This is the average rate of change

Solution: we take  $\lim_{\Delta x \rightarrow 0}$

- $\frac{dy}{dx}$
- grow (+ve)
  - no change (=0)
  - decay (-ve)

Rate of change = rate in - rate out

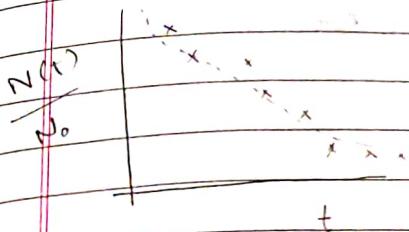
- balance eq:
- rate in = rate out : no change
  - rate in > rate out : grow
  - rate in < rate out : decay

When can we model a system by DEs?

- check if the system can be represented as a compartment

Simple problems modelled like this:

- radioactive decay
  - population of species
  - pollutant in H<sub>2</sub>O bodies
  - Drug delivery / dosage
- } allows you to think in terms of compartments



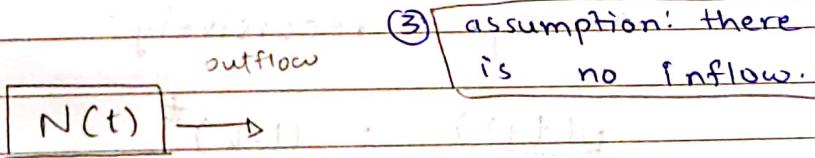
- we know these are particles  
that decay

assumptions! - individual particles are stochastic

① - all particles are independent.

② - number is so large that statistical noise is insignificant

if ① and ② are true, then the system can be modelled as a compartment



$$N(t + \Delta t) - N(t) = ?$$

-ve sign on rhs because we know there is loss.

④ assumption: decrease is proportional to the number of particles present (and linear)

$$N(t + \Delta t) - N(t) = -k N(t) \Delta t$$

$$\frac{dN}{dt} = -k N$$

(Another common assumption is that the distribution of different particles is homogeneous)

→ Solving  $N(t + \Delta t) - N(t) = -k N(t) \Delta t$ :

$$N(t + \Delta t) = (1 - k \Delta t) N(t)$$

set  $\Delta t = 1$

$$N(t+1) = (1-k) N(t)$$

solving recursively:

$$\frac{N(t+1)}{N(t=0)} = (1-k)^t$$

alternatively:

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} = -k N(t)$$

$\Delta t \rightarrow 0$

$$\frac{dN}{dt} = -k N$$

$$\frac{N(t)}{N(t=0)} = e^{-kt}$$

which one is better?

- plug in the values and check!
- or see if they actually differ a lot.

How do you solve a first order differential equation?

$$\frac{dy}{dt} = f(y) \quad \text{linear}$$

$$\frac{d^2y}{dt^2} = f\left(y, \frac{dy}{dt}\right)$$

$$\int \frac{dy}{f(y)} = \int dt$$

- Use Laplace transforms:

$$f(s) = \int_0^\infty e^{-st} f(t) dt = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^\infty e^{-st} \frac{dy}{dt} dt$$

$$= s \tilde{y} - y(0)$$

using

$$\frac{d}{dt} [e^{-st} y] = -se^{-st} y + e^{-st} \frac{dy}{dt}$$

same can be done for any higher order derivative.

$$\begin{aligned} L\left[\frac{d^2y}{dt^2}\right] &= sL\left[\frac{dy}{dt}\right] - y'(0) \\ &= s^2 \tilde{y} - sy(0) - y'(0) \end{aligned}$$

$$\frac{dy}{dt} - a + by \Rightarrow s\tilde{y} - y(0) = \frac{a}{s} + b\tilde{y}$$

$$\therefore (s-b)\tilde{y} = y(0) + \frac{a}{s}$$

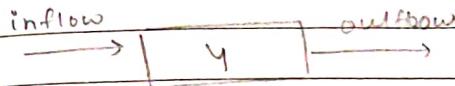
$$\therefore \tilde{y} = \frac{y(0)}{(s-b)} + \frac{a}{s(s-b)}$$

inverse Laplace:

$$y(t) = y(0)e^{-bt} + \frac{a}{b} [e^{bt} - 1]$$

$$\frac{\partial f}{\partial t} + \bar{V} \bar{f} = \bar{0}$$

Compartment Models:



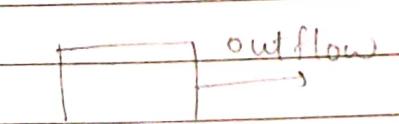
$$\text{Rate of change} = (\text{inflow} - \text{outflow})$$

→ Radioactive decay:

Why not study pendulum? pendulum deterministic while radioactive decay is a probabilistic process.

- ① No. is large (Large enough that any statistical variation is insignificant)
- ② Particles are independent.

Radioactive decay:



$$\frac{dN}{dt} = ?$$

$$= -k?$$

$$= -e^{-n}?$$

You look at what the experiments tell you.

$$\boxed{\frac{dN}{dt} = -kn}$$

What is on the RHS is often the most important job in modelling a system using differential equations.

$$n(t) = n_0 e^{-kt}$$

→ Now we worry about the  $k$

unit : time<sup>-1</sup> : per second

significand : we half life

$$\frac{n(t)}{n_0} = \frac{1}{2}$$

$$k = \frac{0.693}{t_{1/2}}$$

$$\frac{n(t)}{n_0} = e^{-kt}$$

$n_0$  ↼ fraction of radioactive particles that remain at a given instance

$\frac{n(t)}{n_0}$  ← probability of a particle still being left

$$F(t) = 1 - \frac{n(t)}{n_0} = 1 - e^{-kt}$$

$$f(t) = \frac{dF}{dt} = k e^{-kt}$$

$$E_T = \int_0^\infty t f(t) dt = k \int_0^\infty t e^{-kt} dt$$

average time taken for a particle to move out

$$\rightarrow k \left[ - \int_0^{\infty} \frac{d}{dk} e^{-kt} dt \right] = k \int_0^{\infty} t e^{-kt} dt$$

$\because \frac{d}{dk} e^{-kt} = -te^{-kt}$

$$= -k \frac{d}{dk} \int_0^{\infty} e^{-kt} dt$$

$$= -k \frac{d}{dk} \left. \frac{e^{-kt}}{-k} \right|_0^{\infty}$$

$$= -k \frac{d}{dk} \left[ 0 - \left( -\frac{1}{k} \right) \right]$$

$$= -k \frac{d}{dk} \frac{1}{k}$$

$$= -k \left( -\frac{1}{k^2} \right)$$

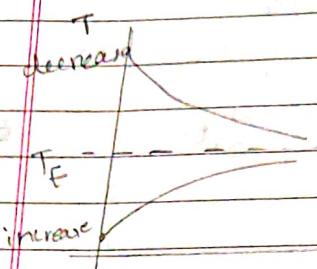
$$= \boxed{\frac{1}{k}}$$

$\rightarrow$  Cooling of coffee

$T$  — Environment temp

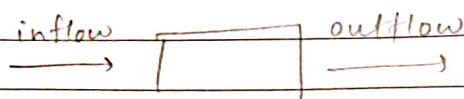
$T$  — Temp of coffee

$$\frac{dT}{dt} \propto (T - T_E)$$



In a system, when gain = loss, system will reach an equilibrium - a steady state

$$\frac{df}{dt} = 0$$



- modelling lake pollution
- Drug dosage problem

MFC - minimum effective concentration

MTC - maximum therapeutic concentration / minimum toxic concentration

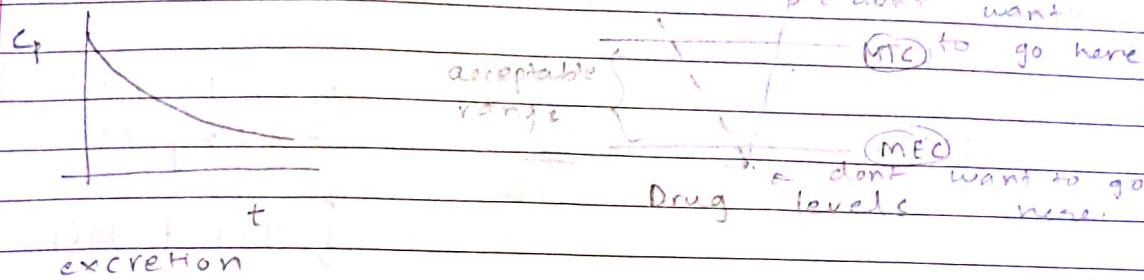
- A Absorption
- D Distribution
- M Metabolism
- E Excretion

One compartment model:



assumption: the entire body is one compartment  
Distribution happens uniformly.

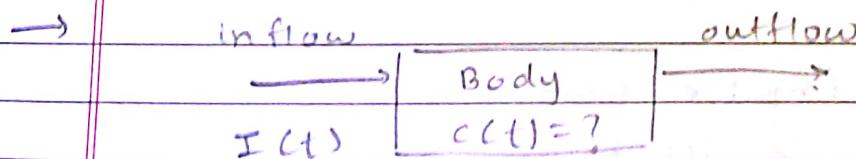
Would blood sample give you a current concentration accurately? - if we assume distribution is uniform.



Absorption:- instantaneous or slowly diffusing

depends on intravenous goes and sits in gut  
target & almost always and then diffuses  
drug instantaneous slowly

- 1) getting the decrease of concentration
- 2?
- 3) repeated dosage



$C$

$$\frac{dc}{dt} = I(t) - kC$$

outflow

more or less kind of "fixed point rule")

Solve using Laplace transform:

IMP

for exam

here

$$s\tilde{C} - C(0) = \tilde{I} - k\tilde{C}$$

$$(s+k)\tilde{C} = \tilde{I} + C(0)$$

$$\tilde{C} = \frac{C(0)}{s+k} + \frac{\tilde{I}}{s+k}$$

inverse  
laplace

$$C(t) = C(0)e^{-kt} + \int_0^t e^{-k(t-t')} I(t') dt'$$

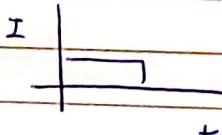
$$\begin{aligned} \mathcal{L}^{-1}[F(s)G(s)] \\ = (f+g)(t) \end{aligned}$$

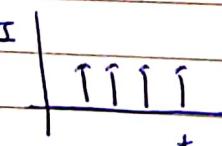
$$\int_0^t f(t-t')g(t') dt'$$

$$(\because \mathcal{L}^{-1}\{\tilde{I}\} = I(t))$$

→ possible input  $I(t)$ :

(i) const  $\delta(t)$

(ii)   $\Theta(t)$   
step function

(iii)  impulse train  
 $\sum_{k=0}^n \delta(t-k)$

(iv) 

ASSUMING  $c(0) = 0$

classmate

Date \_\_\_\_\_

Page \_\_\_\_\_

case 1:

$$I(t) = c_0 \delta(t)$$

$$c(t) = 0 \cdot e^{-kt} + \int_0^t e^{-k(t-t')} c_0 \delta(t') dt'$$

(replacing  $t'$  by zero.)

$$= c_0 e^{-kt}$$

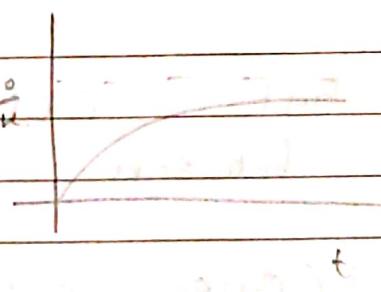
case 2:

$$I(t) = c_0 \theta(t)$$

$$c(t) = c_0 \int_0^t e^{-k(t-t')} \theta(t') dt'$$

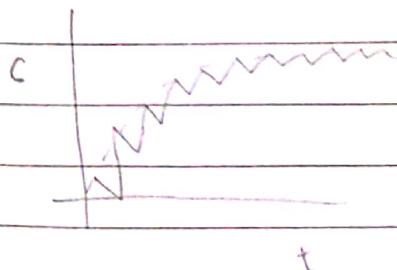
$$= c_0 \int_0^t e^{-ut} dt$$

$$= \frac{c_0}{u} (1 - e^{-ut})$$



case 3:

$$I(t) = \sum_{k=0}^n \delta(t-k)$$



continuing case 2: [assuming  $C(0) \neq 0$ ,  
 $I(t) = C_0 \delta(t)$ ,  $C(0) = C_0$ ]

$$\rightarrow C(t=T^+) = C_0 e^{-kT} + C_0$$

$$\begin{aligned} C(t=2T^+) &= C_0(C_0 e^{-kT} + C_0) + C_0 \\ &= C_0(1 + e^{-kT} + e^{-2kT}) \end{aligned}$$

$$C(t=nT^+) = C_0 \left( \frac{1 - e^{-nkT}}{1 - e^{-kT}} \right)$$

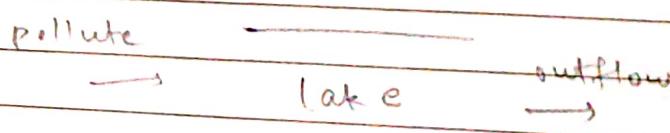
in steady state:

$$C = \frac{C_0}{1 - e^{-kT}} \quad (\text{sum of infinite GP})$$

but  $C(t)$  should always be between  $C_{MFC} < C(t) < C_{MTC}$

(After, our job is to find  $T$  such that this is followed.)

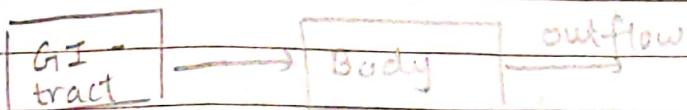
Alternatively,



Volume  $V$

outflow - movement outside  
 reaction with water etc

gastro-intestinal tract



x

y

$$\frac{dx}{dt} = -k_1 x \quad (\text{or } J(t) = -k_1 x).$$

$$\frac{dy}{dt} = k_1 x - k_2 y$$

Solving by Laplace transform:

$$s\tilde{x} - x(0) = -k_1 \tilde{x}$$

$$\tilde{x} = \frac{x(0)}{s+k_1}$$

$$s\tilde{y} = k_1 \tilde{x} - k_2 \tilde{y}$$

$$(s+k_2)\tilde{y} = k_1 \tilde{x}$$

$$\tilde{y} = \left( \frac{k_1}{s+k_2} \right) \left( \frac{x(0)}{s+k_1} \right)$$

$$= \frac{k_1 x(0)}{k_1 - k_2} \left[ \frac{1}{s+k_2} - \frac{1}{s+k_1} \right]$$

Inverse Laplace transform:

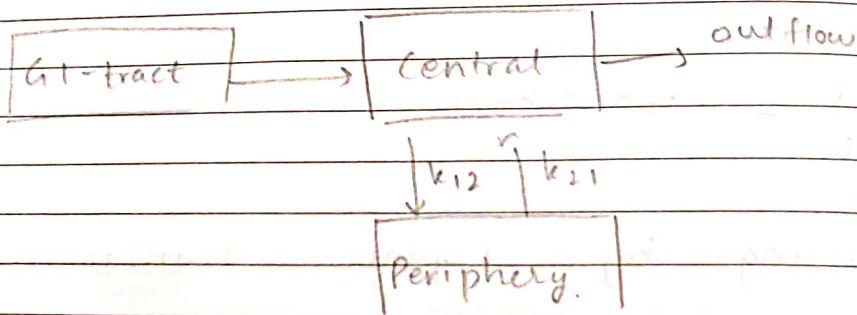
$$y = \frac{k_1}{k_1 - k_2} \times x_0 [e^{-k_2 t} - e^{-k_1 t}]$$

more or less kind of fixed point rule.)

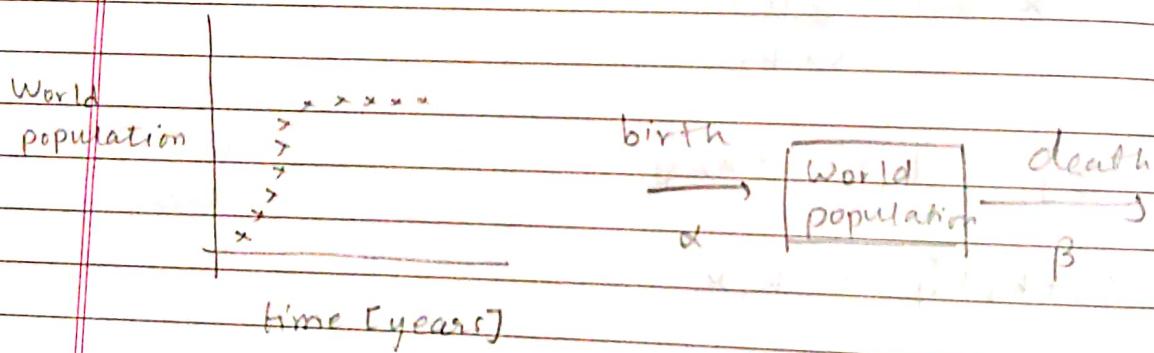
solve using eigenvalues:

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -k_1 & 0 \\ k_1 & -k_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\bar{x}(t) = c_1 e^{\lambda_1 t} \bar{v}_1 + c_2 e^{\lambda_2 t} \bar{v}_2$$



→ How do populations grow?



$$\frac{dx}{dt} = \alpha x - \beta x \leftarrow \text{doesn't work}$$

Malthusian explosion

What would limit them from growing?  
- resources: space, food.

New death rate: we assume to be linear in  $x$ :

$$\text{death rate} : \beta + \gamma x$$

$$\frac{dx}{dt} = \alpha x - (\beta + \gamma x)x$$

$$\frac{dx}{dt} = rx \left(1 - \frac{\gamma}{r}x\right)$$

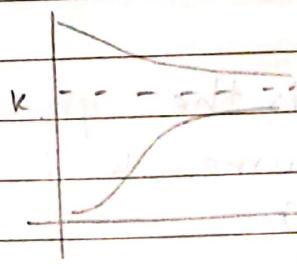
(logistical equation)

here  $r = \alpha - \beta$

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right) \quad k = \frac{r}{\gamma}$$

(i)  $x < k$   $\frac{dx}{dt} \rightarrow +ve$

(ii)  $x > k$   $\frac{dx}{dt} \rightarrow -ve$

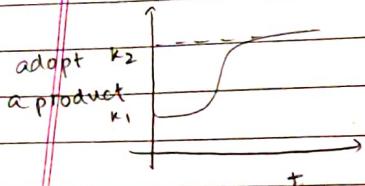


for  $x \ll k$   $\frac{dx}{dt} \approx rx$

20/11/2020

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right)$$

-  $k$  is affected by a lot of factors

Model for  $k=?$ 

$$\frac{dk}{dt} = k(1 - \frac{k}{k_1})$$

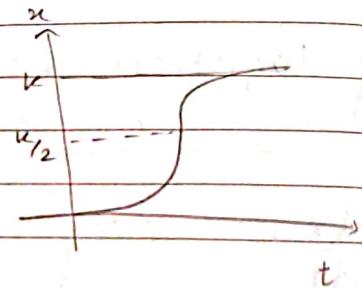
$$\frac{dk}{dt} = ? (k_1, k_2) (1 - \frac{k}{k_1})$$

$\therefore k$  changes from  $k_1$  to  $k_2$

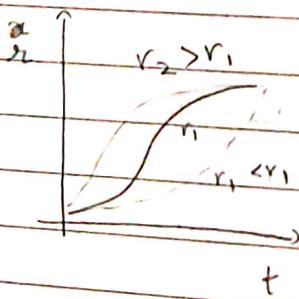
$$\frac{d^2x}{dt^2} = 0 \Rightarrow x = \frac{k}{2}$$

very IMP

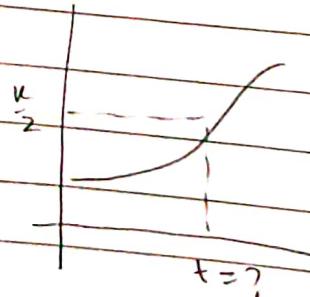
$$x(t) = \frac{kx_0}{x_0 + (k-x_0)e^{-\frac{rt}{k}}}$$



We can think of this as a two compartment model where first compartment corresponds to  $k$  and second compartment corresponds to population.



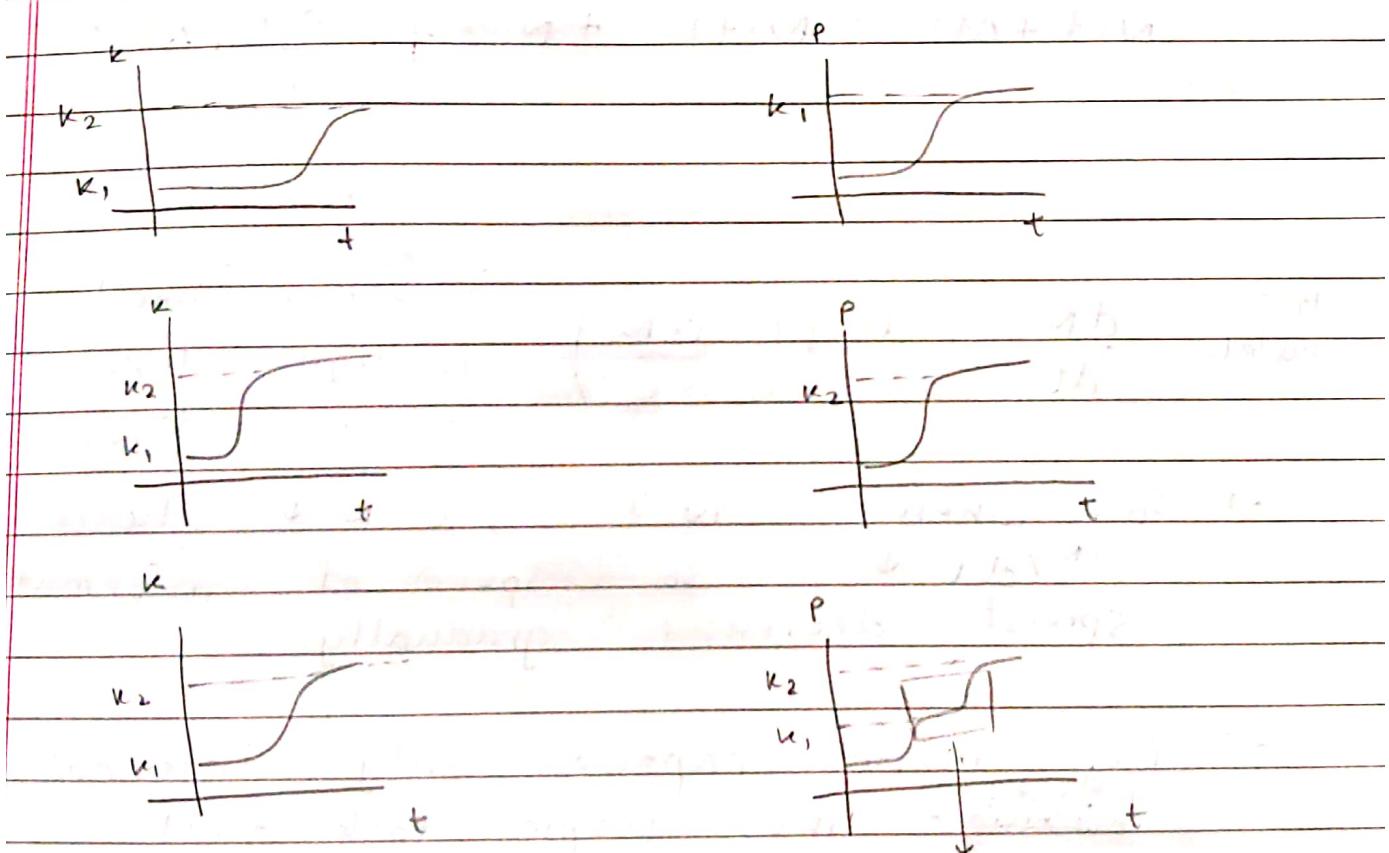
here,  $r$  is the growth rate. (assume  $k$  is constant)

Find the  $x$ 

What is important is that growth rate in  $k$  is compared to rate of change in population.

If  $k$  changes very slowly, we almost treat as constant and population saturates at  $\approx k$ .

If  $k$  changes very fast, it reaches  $k_2$  very quickly and population saturates at  $k_2$ .



the amount of time spent here depends on how  $k$  is changing

### Information spread

→  $N(t)$  - Number of people with information at Time  $t$

→  $c$  - Total population (insensitive to the time span  $t$  we take)

→  $f$  - fraction of people who receive info from one person

$$N(t + \Delta t) = N(t) + N \times f \left( \frac{c-N}{c} \right) \Delta t.$$

↑  
added to balance dimensions

$c \rightarrow$  total size

$f$  unit  $\rightarrow$  time $^{-1}$

$N$  large :  $dN/dt \approx f \frac{N}{c}$

$N$  small :  $dN/dt \approx fN$

Bass model:

$$\frac{dN}{dt} = N f \left( \frac{c-N}{c} \right)$$

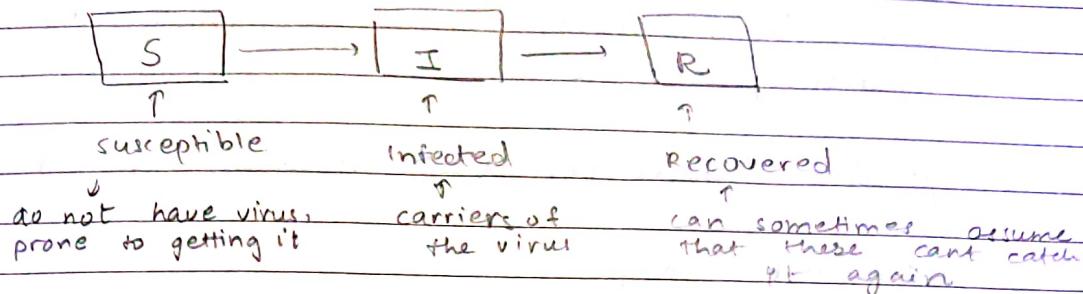
→ So when  $N \uparrow$ ,  $c-N \downarrow$ , hence  $dN/dt \downarrow$ , so speed of information spread decreases gradually.

→ Bass model captures only internal mixing - like people talking, it neglects external effects like ads.

23/1/2020

## Epidemiology

Kermack  
McKendrick (1927)



- Death doesn't happen
- Population does not mix
- Each individual has same probability of interacting with any kind of human (homogeneous population spread)
- Number is large.

C - number of contacts per unit time that each individual makes

p - fraction of susceptibles that gets infected out of all the susceptibles an infected person meets

$$\frac{dS}{dt} = - \frac{S \times C \times p \times I}{N} = -\beta S I$$

$(\beta = CP/N, CP = P_f, \beta = P_f/N)$

$$\frac{dI}{dt} = \beta S I - \gamma I = \frac{C S p I}{N} - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

To minimize kind of fixed point rule.)

- Incubation period - time that it takes for symptoms to be visible
- Latent period - time that it takes for a person to start infecting others after catching the virus

typically, latent period < incubation period

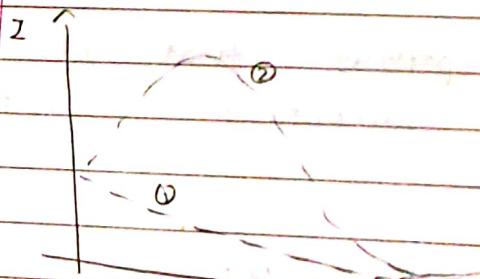
$$\boxed{S + I + R = N}$$

$S, I \}$  also possible models  
 $S, I, S$

①  $S \rightarrow$  bound

②  $R \rightarrow$  bound (cannot exceed  $N$ )

from ① & ②, we can say that long time value of  $I$  is zero.



$$\frac{dI}{dt} = \gamma I \left( \frac{\beta_f s_0}{\gamma N} - 1 \right)$$

$$\gamma > 0, \beta_f > 0, s_0 > 0, N > 0$$

if  $\frac{\beta_f s_0}{\gamma N} \geq 1$  initially, then the

spread becomes an epidemic (2<sup>nd</sup> graph)

- This gives the critical value of parameters

Initially  $\frac{S_0}{N} \approx 1$  ( $\because S_0 \approx N$  initially)

$$\therefore \frac{\beta_f}{\gamma} > 1 \Rightarrow \frac{\beta_f}{\gamma} = R_0 \leftarrow \text{basic reproduction number}$$

$$\frac{\beta_f S_0}{\gamma N} = R_e \leftarrow \text{effective reproduction number}$$

- $R_e = \frac{R_0 / \gamma P}{C P \frac{S}{N}}$   $\underbrace{C P \frac{S}{N}}_{\gamma}$  number of susceptibles an infected will infect per unit time

- $\frac{1}{\gamma}$   $\leftarrow$  avg time for an infected to remain infected.

- $C P \frac{S}{N} \times \frac{1}{\gamma}$   $\leftarrow$  number of susceptibles an (one) infected infects during the time it was infected

How to decrease  $\frac{C P \frac{S}{N}}{\gamma}$  ?

- $\frac{1}{\gamma}$  - medicine

- $P$  - hand-washing

- $S/N$  - vaccination

- $C$  - isolation

To alternating kind of doesn't count - <sup>mainly</sup>  
fixed point rule.)

24/1/2020

$$\frac{dn}{dt} = rn \left(1 - \frac{n}{K}\right) = \frac{n^2}{1+n^2}$$

N  
0  
n  
l  
i  
n  
e  
a  
r

$$\gamma = \frac{rA}{B}$$

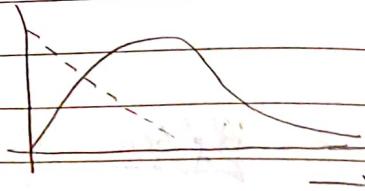
$$K = \frac{K}{A}$$

$$n = \frac{N}{A}$$

$$\tau = t \frac{A}{B}$$

$n^* = 0$  unstable

$$\gamma \left(1 - \frac{n^*}{K}\right) = \frac{n^*}{1+n^{*2}}$$



24/1/2020

$$\frac{ds}{dt} = -\beta SI$$

$$\beta = \frac{\beta_F}{N} = \frac{C \dot{p}}{N}$$

$$\frac{dI}{dt} = \beta SI - \alpha I$$

$$\frac{dR}{dt} = \alpha I$$

$\rightarrow \frac{\beta_F S}{\alpha} > 1$  epidemic threshold

$$\left(\frac{\beta_F}{\alpha}\right) \frac{S}{N} \leftarrow \text{effective reproduction number}$$

basic reproduction number =  $R_0$

$$\frac{1}{\alpha} \leftarrow CP \times \frac{s_0}{N}$$

↓ vaccination  
medicine      ↓ isolation  
marks

### → Vaccination (herd immunization)

$s \rightarrow$  fraction of people that are vaccinated

$$\frac{R_0}{N} (1-s)(s_0) < 1$$

make sure that the epidemic threshold is not crossed by decreasing this

basic reproduction number

$$1-s < \frac{N}{R_0 s_0}$$

(assuming  $N \approx s_0$ ) initially  $s$

$$1-s < \frac{1}{R_0}$$

$$s > 1 - \frac{1}{R_0}$$

→ take  $R_0 = 1.5$ , then we have to vaccinate more than  $\frac{1}{3}$ rd of the population. [because  $s = 1 - \frac{1}{R_0} = 1 - \frac{1}{1.5} = \frac{1}{3}$ ]

→ In case of small pox  $R_0 = 5$

How to estimate  $\alpha$ ? How to  
estimate  $B_f$ ?

$\Delta t$  = average time for which people are sick

$$\rightarrow \frac{ds}{dR} = -\frac{\beta SI}{\alpha I} \quad \frac{ds}{dR} = \frac{ds}{dt} / \frac{dR}{dt}$$

$$= -\frac{\beta}{\alpha} S \quad [R_0 = \frac{B_f}{\alpha}, \quad \frac{R_0}{N} = \frac{\beta}{\alpha}]$$

$$= -\frac{R_0}{N} S$$

$$\ln S|_{S_0} = -\frac{R_0}{N} [R - R_0]$$

$$\frac{s(t)}{s(0)} = e^{-\frac{R_0}{N} (R(t) - R_0)}$$

$$\lim_{t \rightarrow \infty} \frac{s(\infty)}{s(0)} = e^{-\frac{R_0}{N} (R(\infty) - R_0)} \quad \begin{matrix} \text{initially no-one} \\ \text{recovered, so zero} \end{matrix}$$

initially all susceptible,  
 $s_0 = N$

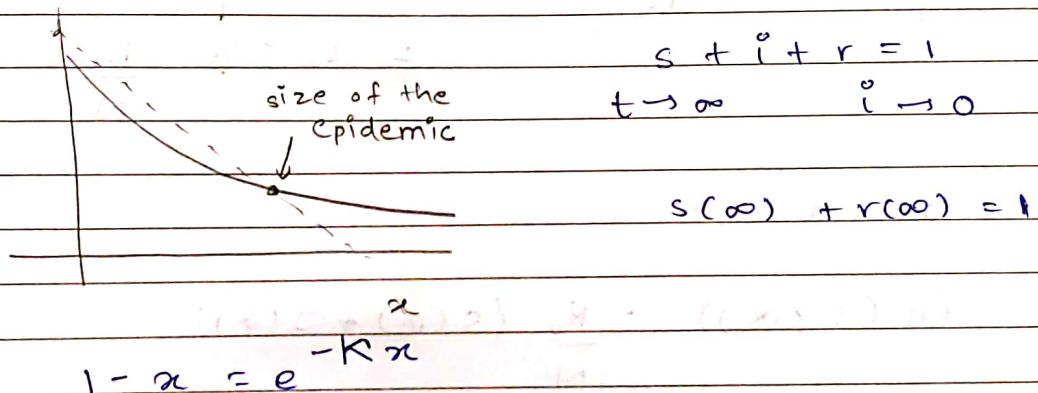
$$\frac{s(\infty)}{N} = e^{-\frac{R_0 R(\infty)}{N}}$$

$$\left. \begin{array}{l} N = S + I + R \\ I(\infty) = 0 \end{array} \right\} \begin{matrix} (\text{no birth, no death}) \end{matrix}$$

$$\therefore \frac{N - R(\infty)}{N} = e^{-\frac{R_0 R(\infty)}{\alpha}}$$

$$\therefore 1 - \alpha(\infty) = e^{-\frac{R_0 R(\infty)}{\alpha}}$$

here.  $R(\infty)$  is the maximum number of people infected at peak of outbreak. because we are assuming that all infected people eventually recover.



Why does  $s(\infty) + r(\infty) = 1$ ? Does every susceptible get infected? Why not?

→ Because no more infected remain.

Disease does not stop spreading due to absence of susceptible. It stops due to absence of infected.

To understand kind of disease count - due

$$\rightarrow \frac{ds}{dI} = \frac{\beta SI}{\alpha I - \beta SI}$$

$$\frac{ds}{dI} = \frac{\beta/\alpha s}{1 - \beta/\alpha s}$$

$$\int ds = \frac{1 - \frac{R_0/N}{s}}{s} = \frac{R_0}{N} \int dI$$

$$\boxed{\ln s - \frac{R_0}{N} (s + I) = \text{const}}$$

$$= \ln s(0) - \frac{R_0}{N} [s(0) + I(0)]$$

$\rightarrow$  at  $t \rightarrow \infty$

$$\ln(s(\infty)) - \frac{R_0}{N} (s(\infty) + I(\infty))$$

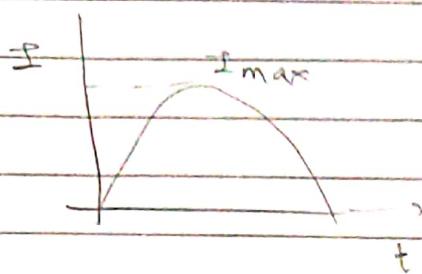
$$= \ln(s(0)) - \frac{R_0}{N} [s(0) + I(0)]$$

$$(I(\infty) = 0 \quad s(0) + I(0) = N)$$

$$\ln \frac{s(\infty)}{s(0)} = R_0 \left( \frac{s(\infty)}{N} - 1 \right)$$

$$\boxed{R_0 = \frac{\ln \left( \frac{s(\infty)}{s(0)} \right)}{\frac{s(\infty)}{N} - 1}}$$

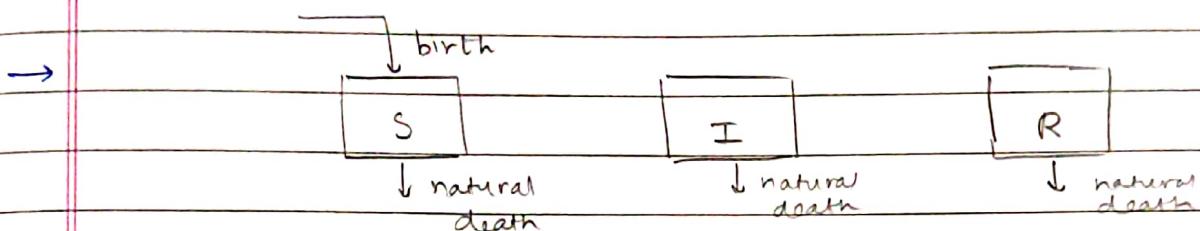
$$R_0 = \frac{\beta f}{\alpha}$$



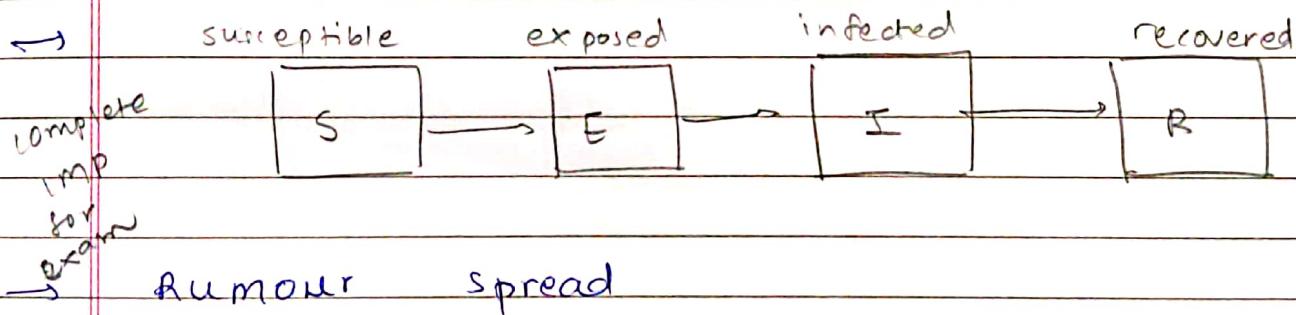
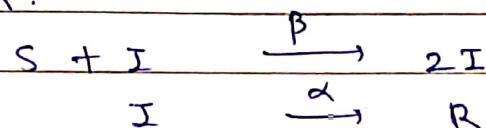
$\frac{dI}{dt} = 0$  at the time when  $I = I_{\max}$

$$\beta SI - \alpha I = 0$$

$$S = \frac{\alpha}{\beta} = \frac{N}{R_0}$$



notation:



I - ignorant

S - spreader

R - ~~stifler~~ stifler

