# Tute 10 Sol$^n$

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**Sol$^n$ 1**

$X_1, X_2, \ldots, X_n \rightarrow$ Independent geometric R.V.

$X_i \rightarrow$ parameter $= P_i$

for geometric R.V. $\boxed{P(X=n) = (1-p)^{n-1}p}$

for ~~$k \leq n$~~ $k \geq n$

$$P\{S_n = k\} = \sum_{i=1}^{n} p_i q_i^{k-1} \prod_{j \neq i} \frac{p_j}{p_j - p_i}$$

$$S_n = \sum_{i=1}^{n} X_i$$

$$q_i = (1-p_i)$$

Proof with the induction hypothesis. Induction will be on value of $n+k$.

- proposition is true for $n=2, k=2 \rightarrow$ (Induction hypothesis)

Induction Hypothesis

- Assume that this proposition is true for any ~~$n \leq k$~~ $k \geq n$ for which $n+k \leq r$.

- Now suppose $k \geq n$ are such that $n+k = r+1$

$$P\{S_n = k\} = P\{S_n = k \mid X_n = 1\} P\{X_n = 1\} + P\{S_n = k \mid X_n > 1\} P\{X_n > 1\}$$
$$= P\{S_n = k \mid X_n = 1\} p_n + P\{S_n = k \mid X_n > 1\} q_n.$$

now,

$$P\{S_n = k \mid X_n = 1\} = P\{S_{n-1} = k-1 \mid X_n = 1\}$$
$$= P\{S_{n-1} = k-1\} \cdot (\because \text{they are independent})$$
$$\boxed{= \sum_{i=1}^{n-1} p_i q_i^{k-2} \prod_{i \neq j \leq n-1} \frac{p_j}{p_j - p_i}} \quad \text{—— } 1$$

($\because$ Induction hypothesis)

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$$P\{X_i = k \mid X_i > 1\} = P\{X_i = k-1\}$$

→ now, if for a Geometric R.V. X with "p": if it is given that it is greater than 1, is same as 1 + Geometric random variable X. Means first trial is always failure.

$$\therefore P\{S_n = k \mid X_n > 1\} = P\{X_1 + X_2 + \dots + X_n + 1 > k\}$$
$$= P\{S_n > k-1\}$$
$$\boxed{= \sum_{i=1}^{n} p_i q_i^{k-2} \prod_{i \neq j \leq n} \frac{p_j}{p_j - p_i}} \quad \text{—— (2)}$$

↳ ~~(Hypothyses)~~ (Hypothesis)

$$\therefore P\{S_n = k\} = p_n \sum_{i=1}^{n-1} p_i q_i^{k-2} \prod_{i \neq j \leq n-1} \frac{p_j}{p_j - p_i} + q_n \sum_{i=1}^{n} p_i q_i^{k-2} \prod_{i \neq j \leq n} \left(\frac{p_j}{p_j - p_i}\right)$$

$$= p_n \sum_{i=1}^{n-1} p_i q_i^{k-2} \prod_{i \neq j \leq n-1} \frac{p_j}{p_j - p_i} + q_n \sum_{i=1}^{n-1} p_i q_i^{k-2} \prod_{i \neq j \leq n} \left(\frac{p_j}{p_j - p_i}\right)$$

$$+ q_n p_n q_n^{k-2} \prod_{j < n} \frac{p_j}{p_j - p_n}$$

$$= \sum_{i=1}^{n-1} \left( p_i q_i^{k-2} p_n \left(1 + \frac{q_n}{p_n - p_i}\right) \prod_{i \neq j \leq n-1} \frac{p_j}{p_j - p_i} \right) + p_n q_n^{k-1} \prod_{j < n} \frac{p_j}{p_j - p_n}$$

$$1 + \frac{q_n}{p_n - p_i} = \frac{p_n - p_i + q_n}{p_n - p_i} = \frac{1 - p_i}{p_n - p_i} = \boxed{\frac{q_i}{p_n - p_i}}$$

$$= \sum_{i=1}^{n-1} \left( p_i q_i^{k-1} \prod_{i \neq j \leq n} \frac{p_j}{p_j - p_i} \right) + p_n q_n^{k-1} \prod_{j < n} \frac{p_j}{p_j - p_n}$$

$$\boxed{= \sum_{i=1}^{n} p_i q_i^{k-1} \prod_{i \neq j} \frac{p_j}{p_j - p_i}}$$

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Sol$^n$ 2

$X$ & $Y$ $\rightarrow$ Independent Poisson R.V.
$\downarrow$ $\downarrow$
$\lambda_1$ & $\lambda_2$ Parameters

[convolution $f^n$]

$P\{X=k \mid X+Y=n\}$

• Here $P\{X+Y=n\} = \sum_{k=0}^{n} P(\cancel{k}\cdot X=k, Y=n-k)$

$$= \sum_{k=0}^{n} P(X=k)\, P(Y=n-k)$$

$$= \sum_{k=0}^{n} e^{-\lambda_1} e^{-\lambda_2} \frac{\lambda_1^k}{k!} \cdot \frac{\lambda_2^{n-k}}{(n-k)!} \cdot \frac{n!}{n!}$$

$$= \frac{e^{-\lambda_1-\lambda_2}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k}$$

$$\boxed{= \frac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^n}{n!}} \quad \text{—— (1)}$$

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$X+Y \rightarrow$ Poisson R.V. with parameters: $\lambda_1+\lambda_2$.

now, $P\{X=k \mid X+Y=n\} = \dfrac{P\{X=k, X+Y=n\}}{P\{X+Y=n\}}$

$$= \frac{P\{X=k, Y=n-k\}}{P\{X+Y=n\}}$$

$$= \frac{P\{X=k\}\, P\{Y=n-k\}}{P\{X+Y=n\}}$$

$$= \frac{e^{-\lambda_1}\lambda_1^k}{k!} \; \frac{e^{-\lambda_2}\lambda_2^{(n-k)}}{(n-k)!} \left[\frac{n!}{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}\right]$$

$$\boxed{= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}}$$

$Sol^n 3$ Multinomial distribution.

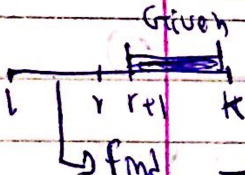
$$P\{X_i = n_i,\ i=1,2,\dots,k\} = \frac{n!}{n_1!\,n_2!\dots n_k!}\, p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

$$\sum_{i=1}^{k} p_i = 1 \qquad\qquad n = \sum_{i=1}^{k} n_i$$

- Given that $n_j$ of trials are resulted in outcome $j$. $j = r+1, \dots, k$. $\sum_{j=r+1}^{k} n_j = m \le n$

$X_i,\ i=1,2,\dots,k$ shows number of trials resulted in outcome $i$.

$\Rightarrow$ $n-m$ trials must have resulted in one of the ~~trials~~ results ~~$X_1,\dots,X_r$~~ trials $1,2,\dots,r$.




$\rightarrow$ This is same as the conditional distribution of $X_1,\dots,X_r$ on $n-m$ trials with respective outcome probabilities

$$\boxed{F_r = \sum_{i=1}^{r} p_i}$$

$$P\{X_1 = n_1, \dots, X_r = n_r \mid X_{r+1} = n_{r+1}, \dots, X_k = n_k\}$$

$$= \frac{P\{X_1 = n_1, \dots, X_k = n_k\}}{P\{X_{r+1} = n_{r+1} \dots X_k = n_k\}}$$

$$= \frac{\dfrac{n!}{n_1!\,n_2!\dots n_k!}\, p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}\, p_{r+1}^{n_{r+1}} \dots p_k^{n_k}}{\dfrac{n!}{(n-m)!\,n_{r+1}!\dots n_k!}\, F_r^{n-m}\, p_{r+1}^{n_{r+1}} \dots p_k^{n_k}} \quad \text{—(1)}$$

Here $P\{X_{r+1} = n_{r+1}, \dots, X_k = n_k\}$ is obtained by making outcomes $1,2,\dots,r$ as a single outcome with the probability $F_r$. Thus this probability is the same

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as the multinomial probability on 'n' trials with outcome probability $F_r, P_{r+1}, \ldots, P_k$. $\sum_{i=r}^{p} n_i = n-m$.

$\therefore P\{X_{r+1}=n_{r+1}, \ldots, X_k=n_k\} = \dfrac{n!}{(n-m)!\,n_{r+1}!\ldots n_k!} F_r^{n-m} P_{r+1}^{n_{r+1}} \ldots P_k^{n_k}$

$\therefore P\{X_1=n_1, \ldots, X_r=n_r \mid X_{r+1}=n_{r+1}, \ldots, X_k=n_k\}$

$= \dfrac{(n-m)!}{n_1!\,n_2!\ldots n_r!} \left(\dfrac{P_1}{F_r}\right)^{n_1} \ldots \left(\dfrac{P_r}{F_r}\right)^{n_r}$

Sol$^n$ 4 N different types of coupons.

→ Each time one obtains a coupon, it is equally likey to be any one of the N types.

X = Number of coupons collected before a complete set is attained.

$E[X] = ?$

$X_i$ = Number of additional coupons that need to be obtained after i distinct types have been collected in order to obtain another distinct type.

$\therefore X = \sum_{i=0}^{N-1} X_i$ (Because innitialy we will have no coupons and at the end we'll have n-1 coupons and we will try to get last coupons)

$X_i$ → Geometric random variable with

$P_i = \dfrac{N-i}{N}$ (we have to get another type of coupon after i types have been obtained)

$$\therefore P\{X_i = k\} = \frac{N-i}{N}\left(1 - \frac{N-i}{N}\right)^{k-1}$$

$$\boxed{= \frac{N-i}{N}\left(\frac{i}{N}\right)^{k-1}}$$

Now $E[X_i] = \frac{1}{p_i} = \boxed{\frac{N}{N-i}}$

Since $X = \sum_{i=0}^{N-1} X_i$

$$\therefore E[X] = \sum_{i=0}^{N-1} E[X_i]$$

$$= \sum_{i=0}^{N-1} \frac{N}{N-i}$$

$$= \cancel{N}\left[\cancel{1} + \cancel{\frac{1}{N-1}} + \dots + \cancel{1}\right]$$

$$\therefore \boxed{E[X] = N\left[1 + \frac{1}{2} + \dots + \frac{1}{N-1} + \frac{1}{N}\right]}$$

**Sol^n 5** n Elements $\to 1, 2, \dots, n$, must be stored in a computer to form an ordered list.

- Each unit of time, a request will be made for one of these elements.

$P(i)$ = Probability that element $i$ will get requested it is independent of past.

$\sum_i P(i) = 1 \to$ known.

$\to$ What ordering minimizes the average position in the of the element requested?

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→ first off all let's number the elements in terms of its probability, & As probabilities are known let's assume that $P(1) \geq P(2) \geq \dots \geq P(n)$. So the optimal ordering will be $1, 2, \dots, n$, to prove that → $X$ = Position of the requested element.

$O \to$ ordering $O = i_1, i_2, \dots, i_n \to$ order in terms of probability (random order)

$$\therefore P_O\{X \geq k\} = \sum_{j=k}^{n} P(i_j)$$

$$\geq \sum_{j=k}^{n} P(j) = \boxed{P_{1,2,3,\dots,n}\{X \geq k\}}$$

order in terms of probability

$$(\text{because } P(i_j) \geq P(j))$$

$$\Rightarrow E_O[X] \geq E_{1,2,3,\dots,n}[X]$$

→ Thus by ordering the elements in the ~~discrr~~ decrising order of the probability that they are requested we can minimize the expected position of the elements requested.

Sol$^n$ 6

An urn contains $n+m$ balls

$n \to$ special balls

$m \to$ ordinary balls.

$X$ = Number of balls that needs be withdrawn until a total $r$ special balls have been removed.

• $X \to$ Hypergeometric R.V. → $X = k$

⇒ first $k-1$ withdrawals consist of $r-1$ special and $k-r$ ordinary balls.

⇒ $k$th ball is special.

$$\therefore P\{X = k\} = \frac{\binom{n}{r-1}\binom{m}{k-r}}{\binom{n+m}{k-1}}$$

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$\therefore P\{X=k\} = \dfrac{n-r+1}{n+m-k+1}$

- Now say $X$ = number of ordinary balls that are withdrawn before a total of $r$ special balls have been removed.

$m$ ordinary balls $\rightarrow O_1, O_2, \ldots, O_m$.

$$A_i = \begin{cases} 1 & \text{if } O_i \text{ if withdrawn before } r \text{ special balls are removed.} \\ 0 & \text{otherwise.} \end{cases}$$

$$\therefore X = \sum_{i=1}^{m} A_i \qquad \boxed{Y = r + X}$$

$$\therefore E[X] = \sum_{i=1}^{m} E[A_i] = \sum_{i=1}^{m} P[A_i]$$

- To find $P(A_i)$ let's consider the permutation of $n$ special and one ordinary ball. Here all the balls are identical (by type)

$\Rightarrow$ Total permutations $= \dfrac{(n+1)!}{n! \cdot 1!} = \boxed{n+1}$

$\rightarrow$ Now among this $n+1$ permutation the ordinary ball comes before $r^{th}$ special ball in $r$ ways. That is the position of ordinary ball is either $1, 2, \ldots$ or $r$

$\therefore P(A_i) = \dfrac{\#(O_i \text{ before } r^{th} \text{ special})}{\text{total ways}}$

$$\boxed{= \frac{r}{n+1}}$$

(Note that we are ignoring other ordinary balls as they won't matter. Just see old tutorials)

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Now, $Y = r + X$

$$\therefore E[Y] = E[r+X]$$
$$= r + E[X]$$
$$= r + \sum_{i=1}^{m} P(A_i)$$
$$= r + \sum_{i=1}^{m} \frac{r}{n+1}$$
$$= r + \frac{mr}{n+1}$$
$$\boxed{= \frac{r(n+m+1)}{n+1}}$$

Since $Y = r + X$ $(r = \text{constant})$

$$\boxed{Var(Y) = Var(X)}$$

$$Var[X] = E[X^2] - (E[X])^2$$
$$E[X^2] = E[X] + 2\sum_{i<j}\sum P(A_i A_j) \quad 0 < i < j < m$$

$A_i A_j \rightarrow$ can be seen as $n$ special balls and 2 ordinary balls $\Rightarrow$ total $n+2$ balls

$$\therefore \boxed{P(A_i A_j) = \left(\frac{r+1}{n+2}\right)\left(\frac{r}{n+1}\right)}$$

$$\therefore E[X^2] = E[X] + 2\sum_{i<j}\sum \frac{r(r+1)}{(n+1)(n+2)}$$
$$= \sum_{i=1}^{m} \frac{r}{n+1} + \binom{m}{2} \cdot 2 \times \frac{r(r+1)}{(n+1)(n+2)}$$
$$\boxed{= \frac{mr}{n+1} + \frac{m(m-1)\,r(r+1)}{(n+1)(n+2)}}$$

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$$\therefore Var[X] = E[X^2] - (E(X))^2$$

$$= \frac{(m-1)m \cdot r(r+1)}{(n+1)(n+2)} + \frac{mr}{n+1} - \frac{m^2 r^2}{(n+1)^2}$$

$$= \frac{m(m-1)r(r+1)(n+1) + mr(n+1)(n+2) - m^2r^2(n+2)}{(n+1)^2(n+2)}$$

$$= \frac{mr[(m-1)(r+1)(n+1) + (n+1)(n+2) - mr(n+2)]}{(n+1)^2(n+2)}$$

$$\therefore \boxed{Var[Y]} \rightarrow \boxed{= \frac{mr(n+1-r)(n+m+1)}{(n+1)^2(n+2)}}$$