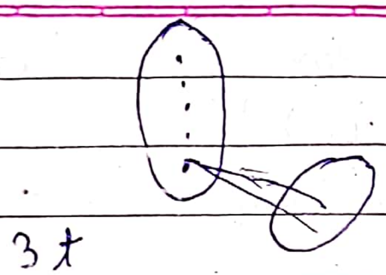


3 t

Cuts and connectivity

removal increases no. of connected components

→ CUT vertex

cut edge

vertex cut — set of vertices whose removal disconnects the graph.

edge cut

vertex connectivity

edge connectivity

k- (vertex) connected

k - edge connected.

local connectivity

~~Se~~ global connectivity.

→ Vertex cut - subset of vertices whose removal results in disconnected graph. Similarly edge cut can be defined. ~~See LH~~

→ A graph is k- connected if cannot be ~~removal~~ disconnected by removal of ~~atleast~~ k vertices.

→ Connectivity is the smallest number that removal of vertex set of that size leads to disconnected graph or single vertex.

→ No. of connected components ≤ ~~vertex~~ dg.

edge connectivity.

→ K, K' and δ are equal for hypercubes and Harary graphs

→ ~~Hk~~ $H_{k,n}$

→ Draw n vertices in a circle.

→ Each ~~vertice~~ vertex ~~in~~ is connected to $\frac{(K-1)}{2}$ vertices on both vertices and to opposite vertex if K is odd.

→ Cycle is $H_{2,n}$. Complete graph is $H_{n-1,n}$.

→ All Harary Graphs are Hamiltonian.

→ ~~Harary graphs~~ $K(Q_{k-1}) = \cancel{K(Q_k)}\ k-1 \Rightarrow K(Q_k) = k$

→ Block- Maximal two-connected subgraph

→ Block cutpoint tree.

Whitney's theorem on 2-connected graphs:

→ A graph is 2-connected if and only if it can be formed by starting with a cycle and repeatedly adding open-ears ~~~~ ~~every pair of vertices~~.

→ ear- Adding path between ~~two~~ distinct existing vertices

→ Open ears: A path added between 2 distinct vertices of a graph through new vertices.

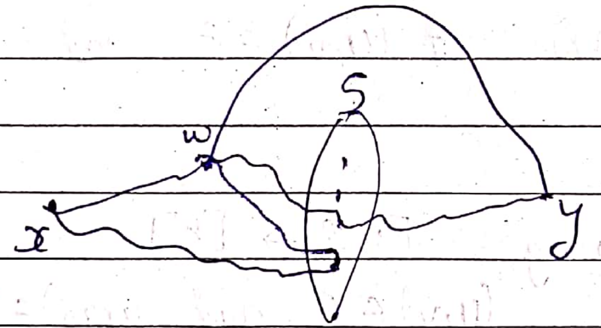
→ Menger's theorem: No of internally disjoint paths between a pair of vertices x and y is less than or equal to λ_{xy}.

→ It is possible to construct λ_{xy} ~~k~~ internally ~~vertex~~ disjoint paths.

→ Let S be the set which disconnects x and y. ~~set vertex cut between x and y.~~

→ Vertices not on x to y path are irrelevant.

→ Thus, every vertex is either before S or after S. Violation gives us a path which bypasses the barrier.



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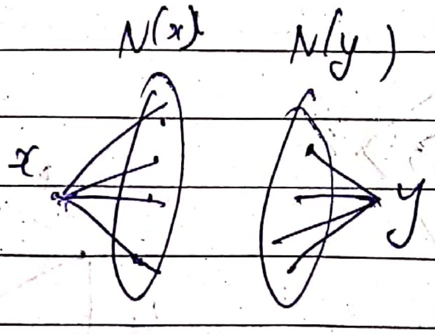


Graph colouring (Vertex coloring and edge coloring)

Line graphs

→ Line graphs is a ~~tool~~ rule? to transform ~~edge~~ ~~vertex~~ edge of a graph vertex ~~edge~~

colouring problem to ~~vertex~~ ~~edg~~ colouring problem, on a different graph

→ Proper colouring

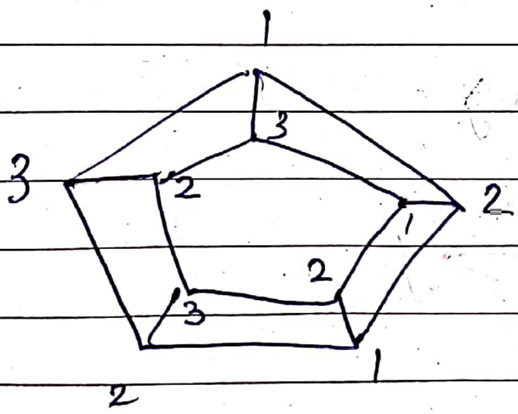
Vertex colouring: Its a function with domain V and codomain [k]

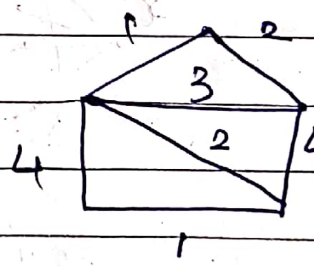
$F: V \to [k]$ such that $\forall (u,v) \in E$ $F(u) \neq F(v)$

→ Complete multipartite graphs are only graphs where F exists, such that $F(u) \neq F(v)$ if $(u,v) \in E$ and $F(u) = F(v)$ if $(u,v) \notin E$

Edge colouring: $F: E \to [k]$

$(u,v) \in E$ and $(u,w) \in E \Rightarrow F((u,v)) \neq F((u,w))$





→ No. of colours in edge colouring $\geq \Delta$

~~Maxm~~ (Maximum degree)

→ Colouring is a partition of vertex set into independent sets.

→ Partition into matchings for edge colouring.

→ χ - chromatic number

χ' - chromatic ~~id~~ index (edge chromatic number)

→ Chromatic number: Minimum number of colours required to vertex colour the graph.

→ Chromatic index: Minimum number of colours required to edge colour the graph.

Upper and lower bound on number of colours required

→ G is a graph, G_c is a properly coloured version of G.

→ H is a subgraph of G, H_c is a restriction of H

→ Restriction - subgraph of coloured graph (colours should remain same).

→ $$\chi(G) \geq \chi(H)$$

$$\chi'(G) \geq \chi'(H)$$

→ Critical graph: Removal of even a single edge (no matter which) drops chromatic number or chromatic index.

→ Every bipartite graph is Δ-edge colourable.

→ $\chi(G) \geq \omega(G)$

→ $\chi(G) \leq n$

→ Every graph can be viewed as union of stars connected at each ~~ea~~ vertex.

→ $\chi'(G) \geq \Delta$

$\chi'(G) \leq m$

→ Colour class: set of all vertices or edges getting same colour.

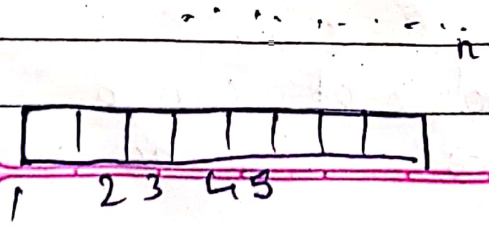
→ $\chi(G) \geq \left\lceil \frac{n}{\alpha} \right\rceil$

$\chi'(G) \geq \left\lceil \frac{m}{\alpha'} \right\rceil$

→ $\sum_{v \in V} d_v \geq \frac{\chi(\chi-1)}{\cancel{2}}$

$\therefore E \geq \frac{\chi(\chi-1)}{2}$

→ Greedy colouring $\Delta+1$ colours



→ Gives $\Delta+1$ colours for complete graphs and odd cycles

$\chi - \omega \geq k \rightarrow$ positive integer

→ Two non-adjacent vertices with same neighbourhood can get same colour

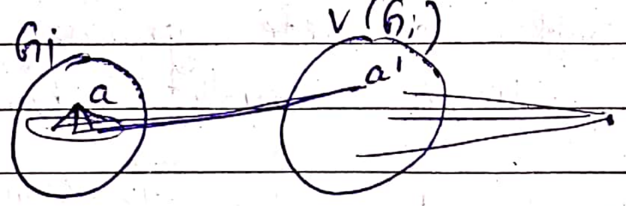
→ Myclelski's construction.

→ Construct G_{i+1} from G_i such that $\chi(G_{i+1}) - \omega(G_{i+1}) = \chi(G_i) - \omega(G_i) + 1$

<u>How to Construct G_{i+1}?</u>

→ Take G_i, $V(G_i)$ and 1 other vertex.

$\therefore |V(G_{i+1})| = 2|V(G_i)| + 1$



→ ~~Take~~ ~~Connect~~ Connect a' with $N(a)$. Vertices in $V(G_i)$ are independent and can be coloured by one colour. Also, a' and a can be assigned same colour.

→ Connect isolated vertex with every vertex in $V(G_i)$.

→ Hence, χ increased by 1.

→ 3-clique free graph: Neighbourhood of every vertex forms an independent set.

→ A graph is triangle-free if the subgraph induced by neighbourhood of every ~~~~ vertex forms an independent set.

→ If G_i is triangle free, then G_{i+1} ~~does~~ has same ω.

→ You can get a graph with unbounded difference between χ and ω.

→ ....

Minimize maximum value of right degree.

$$\chi(G) \leq \text{Degeneracy} + 1$$

Brook's theorem

→ Odd cycles and complete graphs are the only graphs were equality holds.

~~Δ colouring~~

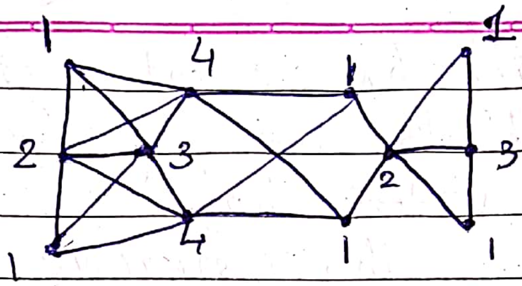
→ Take root as vertex of ~~minimum~~ submaximum degree.

→ Grow a spanning tree.

→ Now, every node has degree less than delta(Δ).

→ A node is coloured only if all its neighbours in the tree are coloured.

→ Every vertex in tree has degree at most $\Delta-1$ (except parents) and has at least one and at most $N(a)$ neighbours coloured.



$\chi'(G) \leq \Delta(G) + 1$

$\chi'(G) \geq \Delta$

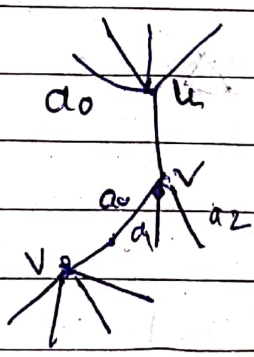
Vizing's theorem :

Class I – Bipartite graphs – Δ edge colourable

class II – regular graphs on odd number of vertices, $\Delta+1$ edge colourable.

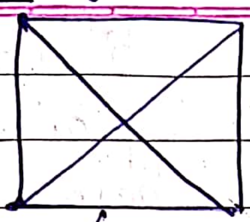
→ Kempe chain: Maximal path in a 2-edge coloured ~~path~~ graph.

→ Kempe chain swap:

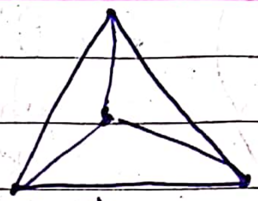


Planar graphs

Both are Planar graphs

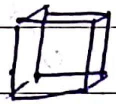
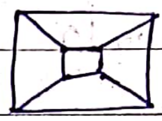


This is not plane ~~graph~~ drawing



Plane graph drawing

→

→  

→ Non planarity is closed under supergraph operation.

→ Planarity is closed under ~~super~~ subgraph operation. (contrapositive of above).

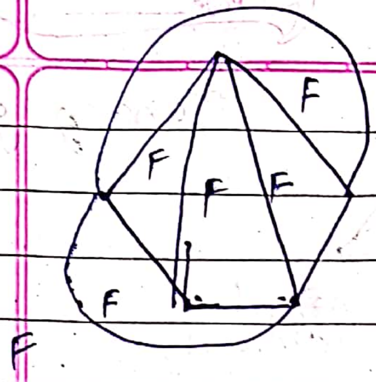
→ Parallel edges and self loops do not violate planarity.

→ Subdivision of edge does not violate planarity.

Attributes irrelevant for planarity:

(1) Parallel edges
(2) Self loops
(3) Subdivision.
(4) Opposite of subdivision

→ If K_n is planar, then no graph on n vertices in non-planar.



→ Biggest 5 vertex graph which is planar is $K_5 - e$. where e is any edge

F → Face → Regions made by edges
→ Equivalence classes of connected plane by graph.

→ Euler formula for planar graphs (regardless of plane drawing)

$$|F| + |V| - |E| = 2$$

→ All trees are planar. No of faces for trees = 1

Proof by structural induction

Case 1: Trees $|F| = 1$ (Unbounded face)

$$|E| = |V| - 1$$

Case 2:

→ Cycles are planar. $|V| = |E|$, $|F| = 2$

→ Add a tree to cycle. It does not ~~ad~~ ~~increase~~ alter no. of faces. In this case, change in no. of vertices = change in no. of edges. $|V| = |E|$ (One vertex is old)

→ Add open ear. It increases the no. of faces by one. One disappears and two appear. $|V| - |E| = -1$ (Two are old)

→ Add closed ear. $|V| = |E| = -1$ (~~One~~ cycle and one old vertex)

→ Thus, for all ~~graphs~~ connected planar graphs, this formula holds.

For disconnected planar graphs (k components)

$$\cancel{|V| - |E| + |F| - K + 1 = 2}$$

$$\sum_{i=1}^{k} |V_i| + \sum_{i=1}^{k} |E_i| + \sum_{i=1}^{k} |F_i| - K + 1 = 2$$

→ Maximal planar simple graph on fixed number of vertices (no more edges can be added)

→ In a maximal planar graph, every face has size 3. (3 edges and 3 vertices).

→

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Upper bound on the number of edges in a fixed vertex set : edge maximal simple planar graphs using Euler's formula for connected planar graphs.

→ There cannot be leaves in such a graph.
→ Every edge is in exactly two faces.

$E \leq 3V - 6$

$V - E + F = 2$

$E = \frac{3F}{2}$

$V - E + \frac{2E}{3} = 2$

$V - \frac{E}{3} = 2$

$\therefore V - 2 = \frac{E}{3}$

$\therefore \boxed{E = 3V - 6}$

Total degree $\leq 6V - 12$
↳ Converse is not true.

Average degree $\leq 6 - \frac{12}{V}$

Planar graphs are 5- degenerate.
~~Chromaticity can be atmost 6~~

Degeneracy of planar graphs ≤ 5

$$\chi(G) \leq 6.$$

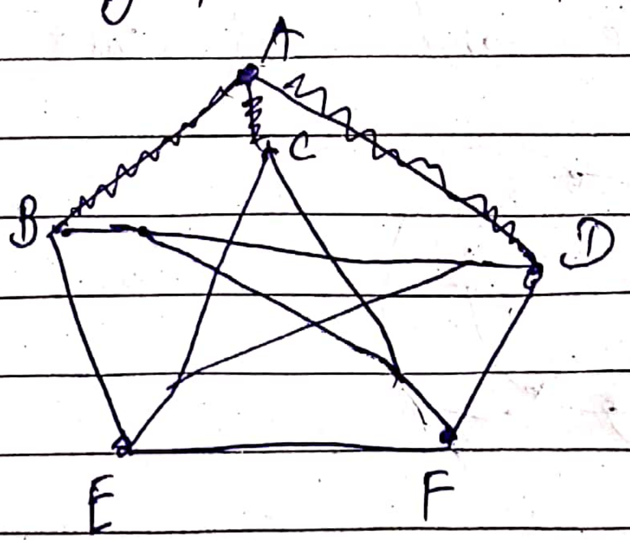
Chromatic number of planar graphs ≤ 4.

Kuratovski's theorem:

Two graphs which are non-planar: K_5 and $K_{3,3}$

Every graph which contains atleast one of K_5 ~~or~~ and $K_{3,3}$ or its subdivisions as subgraph is non-planar. ~~Otherw~~ Elese, it is planar.

Petersen graph has $K_{3,3}$ subdivision.



AEF, BCD

→ Every planar graph is 5-degenerate. The converse is true.

K_5 is non-planar:

Proof! Assume planar. Apply Euler's formula. 7 faces. Every faces has 3 edges

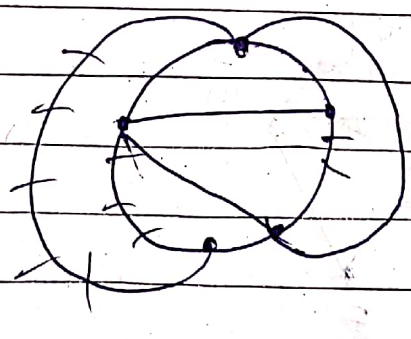
∴ No. of edges = $\frac{7 \times 3}{2} = \frac{21}{2}$

No. of edges ≥ 11

But K_5 has 10 edges. Thus, K_5 is non-planar.

Similar for $K_{3,3}$. Every face has 4 edges.

Assume a cycle and at least 5 vertices.



→ 5- colour theorem

Every planar graph is 5- colourable.

Create an ordering of the vertices with maximum right degree ≤ 5.

Flip the colour classes for a given ~~goo~~ vertices.

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