The Cube Roots of Unity

The cube roots of unity, three in number, are extracted from $\boxed{x^3 - 1 = 0}$.

Factorisation: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\Rightarrow a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$\Rightarrow a^3 + b^3 = (a+b)\left[(a+b)^2 - 3ab\right]$

$\Rightarrow a^3 + b^3 = (a+b)\left[a^2 + 2ab + b^2 - 3ab\right]$

$\Rightarrow a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Equating $\boxed{a = x}$ and $\boxed{b = -1}$, we get $\boxed{x^3 - 1 = (x-1)(x^2 + x + 1) = 0}$

The solutions are $\boxed{x = 1}$ and $\boxed{x^2 + x + 1 = 0}$

Solving the quadratic, $x = \dfrac{-1 \pm \sqrt{1 - 4}}{2}$

$\Rightarrow \boxed{x = \dfrac{-1 \pm \sqrt{3}\, i}{2}}$ as well as $\boxed{x = 1}$.

Hence unity has one real root and two complex roots.

From the theory of quadratic equations, for $\boxed{ax^2 + bx + c = 0}$, there are two roots, α and β.

The roots are $\boxed{x = \alpha, \beta = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$.

$$\alpha + \beta = -\frac{b}{2a} + \cancel{\sqrt{b^2-4ac}} - \frac{b}{2a} - \cancel{\sqrt{b^2-4c}} = -\frac{b}{a}.$$

And $\alpha\beta = \dfrac{b^2 - (b^2 - 4ac)}{4a^2} = \dfrac{4ac}{4a^2} = \dfrac{c}{a}$.

$\therefore$ $\boxed{\alpha + \beta = -\dfrac{b}{a}}$ and $\boxed{\alpha\beta = \dfrac{c}{a}}$.

For the equation, $\boxed{x^2 + x + 1 = 0}$,

$\boxed{a = b = c = 1}$. $\Rightarrow$ $\boxed{\alpha + \beta = -1}$ & $\boxed{\alpha\beta = 1}$

Now $\boxed{\alpha^3 = 1}$ and $\boxed{\beta^3 = 1}$, since both α and β are cube roots of unity.

Since, $\alpha\beta = 1 \Rightarrow \alpha^3\beta = \alpha^2 \Rightarrow \boxed{\beta = \alpha^2}$

or $\alpha\beta^3 = \beta^2 \Rightarrow \boxed{\alpha = \beta^2}$. $\Rightarrow$ Either complex root of unity is the square of the other.

It is customary to write the roots as $\boxed{1, \omega \text{ and } \omega^2}$ in which $\boxed{\omega^* = \omega^2}$, where <u>$\omega^*$ is the complex conjugate of ω.</u>

Further, from $x^2 + x + 1 = 0$, we see that $\boxed{1 + \omega + \omega^2 = 0}$, i.e. <u>the sum of the three roots is zero.</u>

Again $\boxed{\omega\cdot\omega^2 = \omega^3 = 1} \Rightarrow \boxed{\omega = \frac{1}{\omega^2}}$,
i.e. the product of the two complex roots is unity and each is the reciprocal of the other. Also $\boxed{1\cdot\omega\cdot\omega^2 = 1}$, i.e. the product of all the three roots is unity.

Check: Let $\boxed{\omega = \frac{-1+\sqrt{3}i}{2}}$, $\boxed{\omega^* = \frac{-1-\sqrt{3}i}{2}}$

$$\therefore \frac{1}{\omega} = \frac{2}{-1+\sqrt{3}i} = \frac{2(-1-\sqrt{3}i)}{(-1+\sqrt{3}i)(-1-\sqrt{3}i)} = \frac{-2(1+\sqrt{3}i)}{1-(3)(-1)}$$

$$\Rightarrow \frac{1}{\omega} = -\frac{2(1+\sqrt{3}i)}{4} = -\frac{(1+\sqrt{3}i)}{2} = \frac{-1-\sqrt{3}i}{2} = \omega^*$$

Hence $\boxed{\omega^* = 1/\omega}$ or $\boxed{\omega^*\omega = 1}$

Also, $\omega^2 = \left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \frac{1-2\sqrt{3}i-3}{4}$

$$\Rightarrow \omega^2 = \frac{-2-2\sqrt{3}i}{4} = \frac{-1-\sqrt{3}i}{2} = \omega^*$$

$\therefore \boxed{\omega^* = \omega^2}$ verifying $\boxed{\omega^* = \frac{1}{\omega} = \omega^2}$

Powers of ω: $\boxed{\omega^n}$ with n being an integer.

i.) $\boxed{n = 3m}$ (m is an integer). $\Rightarrow \omega^{3m} = 1^m = 1$

ii.) $\boxed{n = 3m+1} \Rightarrow \omega^{3m+1} = 1\cdot\omega^1 = \omega$

iii.) $\boxed{n = 3m+2} \Rightarrow \omega^{3m+2} = 1\cdot\omega^2 = \omega^2$

Since $\omega^3 = 1$

The cube roots of a number $\boxed{a = a \times 1}$ are $\boxed{\sqrt[3]{a},\ \sqrt[3]{a}\,\omega \text{ and } \sqrt[3]{a}\,\omega^2}$ in which $\sqrt[3]{a}$ is the <u>arithmetical cube root.</u>

Representations on the Argand plane

The three cube roots of unity are 1, $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$, i.e, $1, \omega, \omega^*$ (or ω^2)

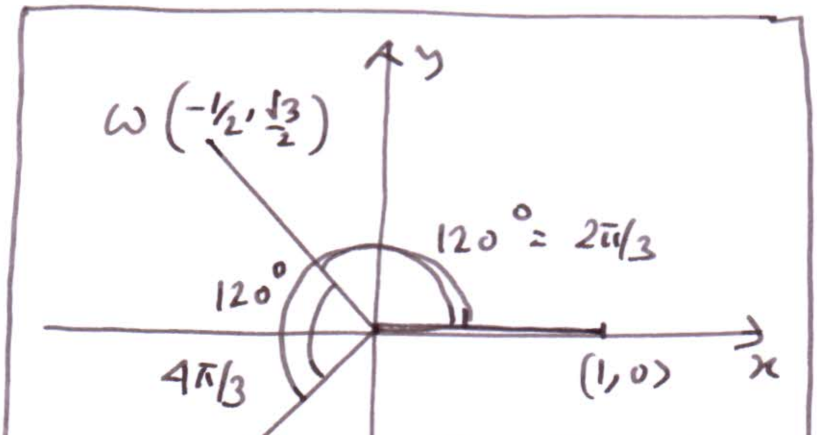
We write $\boxed{\omega = \cos\theta + i\sin\theta}$ $\Rightarrow \cos\theta = -1/2$ and $\sin\theta = \frac{\sqrt{3}}{2}$.

Similarly $\boxed{\omega^* = \cos\theta + i\sin\theta}$ $\Rightarrow \cos\theta = -1/2$ and $\sin\theta = \sqrt{3}/2$.

$\therefore \omega = \cos 120^\circ + i\sin 120^\circ = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$\omega^* = \cos(240^\circ) + i\sin(240^\circ) = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$.

Using $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$, $\boxed{\omega = e^{i2\pi/3}}$ and $\boxed{\omega^* = e^{i4\pi/3}}$.



The Argand (complex) plane.

Now $\boxed{\omega^3 = 1} \Rightarrow \boxed{\omega = 1/\omega^2}$

But $\boxed{\omega^* = \omega^2}$

$\Rightarrow \boxed{\omega = e^{i2\pi/3} = e^{-i4\pi/3}}$

and $\boxed{\omega^2 = e^{i4\pi/3} = e^{-i2\pi/3}}$

$\therefore \omega^3 = \omega^2\omega = e^{i2\pi/3 + i\frac{4\pi}{3}}$

$\Rightarrow \omega^3 = e^{i2\pi} = \cos(2\pi) + i\sin(2\pi)$

$\Rightarrow \omega^3 = 1$