INTERPOLATION

Interpolation : Basic Theory

1/. Given a discrete set of values. $[x_i, f(x_i)]$, equally or unequally spaced, find a value between the discrete values provided.

2/. The data set may or may not be monotonic. In simple cases it is monotonic.

3/. To approximate the actual unknown function, by some known function at any x, between x_i ($y_i = f(x_i)$).

4/. Any analytic function can be used. Common approximating functions are polynomials (differentiable and integrable).

5/. Weierstrass Approximation Theorem:
Any continuous (and continuously differentiable) function can be approximated to any order of accuracy by a polynomial of high enough degree.

Known Functions and Taylor Polynomials

In case the continuous function is known, $f(x)$,

$$\boxed{\therefore f(x) = f(x_0) + f'(x)(x-x_0) + \frac{f''(x)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x)}{n!} \times (x-x_0)^n + \cdots}$$

gives the full Taylor expansion upto any order.

A Taylor polynomial is

$$\boxed{P_n(x) = f(x_0) + f'(x)(x-x_0) + \cdots + \frac{f^{(n)}(x)}{n!}(x-x_0)^n}.$$

$$\boxed{\text{Error}(x) = f(x) - P_n(x) \sim \frac{f^{(n+1)}(x)}{(n+1)!}(x-x_0)^{n+1}}.$$

Lagrange Polynomials (for unknown functions)

If a data set comprises two points (x_0, y_0) and (x_1, y_1), in which $\boxed{y_i = f(x_i)}$ and $\boxed{x_0 \neq x_1}$, then the slope of a linear interpolating function is

$$\boxed{m = \frac{y_1 - y_0}{x_1 - x_0}}$$

$\therefore$ The function (linear interpolates) is $\boxed{y = mx + c = x\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + c}$.

$\Rightarrow c = y - x\left(\frac{y_1 - y_0}{x_1 - x_0}\right) \Rightarrow$ when $x = x_0$, and $y = y_0$,

$c = y_0 - x_0\left(\frac{y_1 - y_0}{x_1 - x_0}\right) \Rightarrow c = \frac{y_0 x_1 - \cancel{y_0 x_0} - x_0 y_1 + \cancel{x_0 y_0}}{x_1 - x_0}$

$$\Rightarrow \boxed{C = \frac{y_0 x_1 - x_0 y_1}{x_1 - x_0}}$$

. This result can also be obtained by using $x = x_1$ and $y = y_1$, so that,

$$C = y_1 - \left(\frac{y_1 - y_0}{x_1 - x_0}\right)x_1 = \frac{\cancel{y_1 x_1} - y_1 x_0 - \cancel{y_1 x_1} + x_1 y_0}{x_1 - x_0}$$

$$\Rightarrow \boxed{C = \frac{y_0 x_1 - y_1 x_0}{x_1 - x_0}}$$ as earlier.

$$\therefore \quad y = x\left(\frac{y_1 - y_0}{x_1 - x_0}\right) + \frac{y_0 x_1 - y_1 x_0}{x_1 - x_0}$$

Write $\boxed{y = P_1(x)}$

$$\Rightarrow \boxed{y = \frac{y_0(x_1 - x) + y_1(x - x_0)}{x_1 - x_0} = \left(\frac{x - x_1}{x_0 - x_1}\right)y_0 + \left(\frac{x - x_0}{x_1 - x_0}\right)y_1}$$

<u>Alternatively</u>: $\boxed{\frac{y - y_0}{y_0 - y_1} = \frac{x - x_0}{x_0 - x_1}}$.

$$\Rightarrow y = y_0 + \left(\frac{x - x_0}{x_0 - x_1}\right)(y_0 - y_1) \Rightarrow y = \frac{y_0(x_0 - x_1) + (x - x_0)(y_0 - y_1)}{x_0 - x_1}$$

$$\Rightarrow y = \frac{\cancel{y_0 x_0} - y_0 x_1 + x y_0 - \cancel{x_0 y_0} - x y_1 + x_0 y_1}{x_0 - x_1}$$

$\Rightarrow$ With $\boxed{y = P_1(x)}$

$$\Rightarrow \boxed{y = \frac{(x - x_1)}{x_0 - x_1} y_0 + \left(\frac{x - x_0}{x_1 - x_0}\right) y_1} \rightarrow$$ <u>Lagrange linear interpolation</u>.

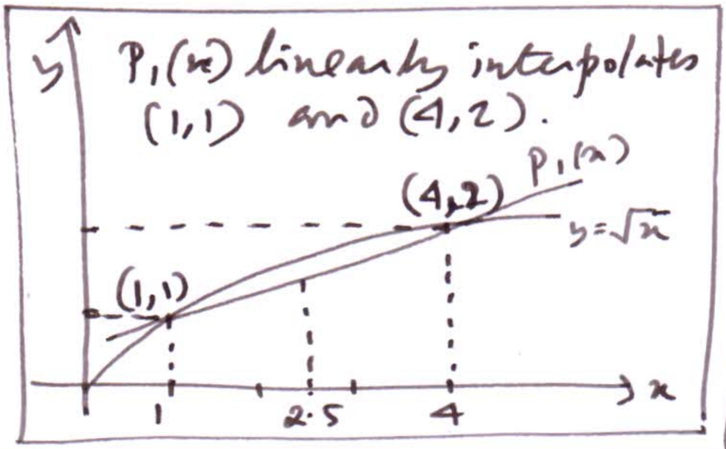
<u>Example</u>: Data points are $(1,1)$, $(4,2)$.

$\therefore \boxed{x_0 = 1,\ y_0 = 1,\ x_1 = 4,\ y_1 = 2}$. These data points are a part of the function $\boxed{y = f(x) = \sqrt{x}}$.

The linear interpolation function $P_1(x)$ is

$$P_1(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) y_0 + \left(\frac{x - x_0}{x_1 - x_0}\right) y_1 = \left(\frac{x-4}{1-4}\right) 1 + \left(\frac{x-1}{4-1}\right) 2$$

$$\Rightarrow \boxed{P_1(x) = \frac{x-4}{-3} + \frac{(x-1)2}{3} = \frac{(4-x) + 2(x-1)}{3}}.$$



At $\boxed{x = 2.5}$ $\quad y = \sqrt{2.5} \simeq 1.58$

$$P_1(x) = \frac{1.5 + 2 \times 1.5}{3} = \frac{4.5}{3}$$

$$P_1(x) = 1.5.$$

$\Rightarrow y - P_1(x) \simeq 1.58 - 1.5 = 0.08$
(The error due to linearisation)

Example: $\boxed{\begin{matrix} x_0 = 0.82, & y_0 = 2.270500 \\ x_1 = 0.83, & y_1 = 2.293319 \end{matrix}}$ $\quad \boxed{y = e^x}$.

Interpolate linearly for $\boxed{x = 0.826}$.

$$\boxed{P_1(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) y_0 + \left(\frac{x - x_0}{x_1 - x_0}\right) y_1} \Rightarrow P_1(0.826) = \left(\frac{0.826 - 0.83}{0.82 - 0.83}\right) \times 2.270500 + \text{(next line)}$$

(continued from the previous line) $\left(\frac{0.826 - 0.820}{0.83 - 0.82}\right) \times 2.293319.$ $\quad$ Now $\boxed{\left(\frac{0.826 - 0.83}{0.82 - 0.83}\right) = 0.004}$

and $\boxed{\left(\frac{0.826 - 0.82}{0.83 - 0.82}\right) = 0.006}$ $\Rightarrow P_1(0.826) = \frac{0.004}{0.01} \times 2.270500 + \frac{0.006}{0.01} \times 2.293319.$

$\Rightarrow \boxed{P_1(0.826) = 2.2841914}$ whereas $\boxed{e^{0.826} = 2.2841638}$.
(close matching)

Lagrange Quadratic Interpolation

Given three data points (x_0, y_0), (x_1, y_1), (x_2, y_2), or (x_i, y_i) with $i = 0, 1, 2$, there is only ONE unique quadratic polynomial.

Proof: Let there be two polynomials of at MOST the quadratic order, $P_2(x)$ and $Q_2(x)$, which interpolate x_i. $\therefore$ $\boxed{P_2(x_i) = Q_2(x_i)}$.

Define $\boxed{R(x) = P_2(x) - Q_2(x)}$. Now both $P_2(x)$ and $Q_2(x)$ are at most of degree 2 (≤ 2)

But $\boxed{R(x_i) = P_2(x_i) - Q_2(x_i) = 0}$ is satisfied at three value of x_i ($i = 1, 2, 3$). This can only be possible if $\boxed{R(x) = 0}$, i.e. it is a zero polynomial with all coefficients zero, since with $P_2(x)$ and $Q_2(x)$ being at most of degree 2 (quadratic), $R(x)$ can be of no higher degree than quadratic. Hence, it will be impossible for $R(x)$ to have three roots.

Extending this argument, we can say that a polynomial of degree n, passing through $n+1$ discrete points is UNIQUE.

For (x_0, y_0), (x_1, y_1) and (x_2, y_2), the quadratic polynomial ~~[illegible]~~ is written by extending $P_1(x)$ as

$$\boxed{P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2}$$

Example: Extending the exercise on $\boxed{y = e^x}$ by an extra point, $\boxed{x_2 = 0.84,\ y_2 = 2.316367}$ we get

$$P_2(x) = 2.270500\left[\frac{(0.826-0.83)(0.826-0.84)}{(0.82-0.83)(0.82-0.84)}\right]$$

$$+ 2.293319\left[\frac{(0.826-0.82)(0.826-0.84)}{(0.83-0.82)(0.83-0.84)}\right]$$

$$+ 2.316367\left[\frac{(0.826-0.82)(0.826-0.83)}{(0.84-0.82)(0.84-0.83)}\right]$$

$$\Rightarrow P_2(x) = 2.270500\times 0.28 + 2.293319\times 0.84 + 2.316367\times(-0.12)$$

$\Rightarrow \boxed{P_2(x) = 2.2841639}$. $\because$ $P_1(x) = 2.2841414$

$P_2(x)$ is closer to $e^{0.826} = 2.2841638$.

Cubic-Order Lagrange Polynomials

x	3.35	3.40	3.50	3.60
$f(x)$	0.298507	0.294118	0.285714	0.277778

$\boxed{y = f(x) = 1/x}$

The table above contains four data points, with which a cubic Lagrange polynomial can be formed.

I/. Linear Interpolation for $\boxed{x = 3.44}$ $\boxed{y = 1/x}$

$= \frac{1}{3.44}$

$= 0.290698$

The two closest points in the table are $x_0 = 3.40$, $x_1 = 3.50$.

$$\therefore \boxed{P_1(x) = y_0\left(\frac{x - x_1}{x_0 - x_1}\right) + y_1\left(\frac{x - x_0}{x_1 - x_0}\right)}$$

$$\Rightarrow P_1(3.44) = \left(\frac{3.44 - 3.50}{3.40 - 3.50}\right) \times 0.294118 + \left(\frac{3.44 - 3.40}{3.50 - 3.40}\right) \times 0.285714$$

$$\Rightarrow P_1(3.44) = 0.6 \times 0.294118 + 0.4 \times 0.285714 = 0.290756$$

II/. Quadratic Interpolation for ~~3.44~~ $\boxed{x = 3.44}$

The three closest points are $x_0 = 3.35$, $x_1 = 3.40$, $x_2 = 3.50$.

$$\boxed{P_2(x) = y_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}}$$

$$\therefore P_2(x) = 0.298507 \times \left[\frac{(3.44-3.40)(3.44-3.50)}{(3.35-3.40)(3.35-3.50)}\right]$$
$$+ 0.294118 \times \left[\frac{(3.44-3.35)(3.44-3.50)}{(3.40-3.35)(3.40-3.50)}\right]$$
$$+ 0.285714 \times \left[\frac{(3.44-3.35)(3.44-3.40)}{(3.50-3.35)(3.50-3.40)}\right]$$

$$= 0.298507 \times (-0.32) + 0.294118 \times 1.08 + 0.285714 \times 0.24$$

$\Rightarrow$ $P_2(x) = 0.290697$ (improving on the linear value)

III. Cubic Interpolation for $\boxed{x = 3.44}$

$$\boxed{\begin{aligned} P_3(x) &= y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ &+ y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \end{aligned}}$$

$$P_3(x) = 0.298507 \left[\frac{(3.44-3.40)(3.44-3.50)(3.44-3.60)}{(3.35-3.40)(3.35-3.50)(3.35-3.60)}\right]$$

$$+ 0.294118 \left[\frac{(3.44-3.35)(3.44-3.50)(3.44-3.60)}{(3.40-3.35)(3.40-3.50)(3.40-3.60)}\right]$$

$$+ 0.285714 \left[\frac{(3.44-3.35)(3.44-3.40)(3.44-3.60)}{(3.50-3.35)(3.50-3.40)(3.50-3.60)}\right]$$

$$+ 0.27778 \left[\frac{(3.44-3.35)(3.44-3.40)(3.44-3.50)}{(3.60-3.35)(3.60-3.40)(3.60-3.50)}\right]$$

$$\Rightarrow P_3(x) = 0.298507 \times (-0.2048) + 0.294118 \times 0.864$$
$$+ 0.285714 \times (0.384) + 0.27778 \times (-0.0432)$$

$\Rightarrow P_3(x) = 0.290698$ (equal to $\frac{1}{3.44}$ upto 6 decimal places)

Generalisation to n-Degree Polynomial.

$$\boxed{P_n(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x)}.$$

where $\boxed{L_i(x) = \dfrac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}}$.

Divided Differences

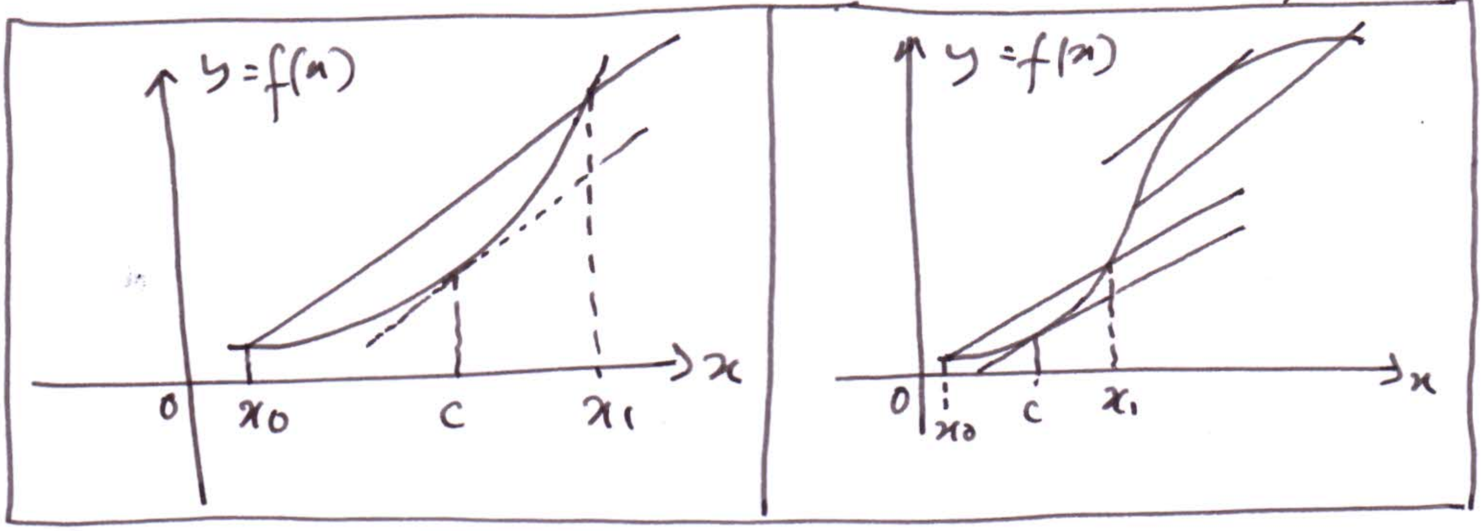
Let $\boxed{y = f(x)}$. For $x = x_0$ and $x = x_1$, a discrete derivative is $\boxed{f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}$.

It is the first-order divided difference of x.

Mean-Value Theorem

If $f(x)$ is differentiable on an interval that contains x_0 and x_1, then by the mean value theorem $\boxed{f[x_0, x_1] = f'(c)}$ where

c lies between x_0 and x_1. (at least one such point).



Further, if x_0 and x_1 are close together,

$$\boxed{f[x_0, x_1] \simeq f'\left(\frac{x_0 + x}{2}\right)} \quad \text{(approximately)}$$

Proof: Let $\boxed{z = \dfrac{x_1 + x_0}{2}}$ and $\boxed{h = \dfrac{x_1 - x_0}{2}}$.

$$\Rightarrow \boxed{x_1 = z+h} \text{ and } \boxed{x_0 = z-h}.$$

Hence, $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{f(z+h) - f(z-h)}{2h}$

$$\Rightarrow f[x_0,x_1] = \frac{\left[\cancel{f(z)} + f'(z)h + \cancel{f''(z)h^2/2!} + f'''(z)h^3/3! + \cdots\right] - \left[\cancel{f(z)} - f'(z)h + \cancel{f''(z)h^2/2!} - f'''(z)h^3/3! + \cdots\right]}{2h}$$

$$\Rightarrow f[x_0,x_1] \simeq \frac{2f'(z)h + 2f'''(z)h^3/3!}{2h}$$

$$\Rightarrow \boxed{f[x_0,x_1] \simeq f'(z) + \frac{f'''(z)h^2}{6}}$$

$(h \to 0)$. $\because$ h is very small when x_0 and x_1 are close.

$$\Rightarrow \boxed{f[x_0,x_1] \simeq f'(z) \simeq f'\left(\frac{x_0+x_1}{2}\right)}$$

Example: $\boxed{f(x) = \cos x}$, $\boxed{x_0 = 0.2}$, $\boxed{x_1 = 0.3}$ (Unit $\to$ radian)

$$\therefore f[x_0, x_1] = \frac{\cos(0.3) - \cos(0.2)}{0.1} = -0.2473009.$$

$f'(x) = -\sin x \Rightarrow f[x_0,x_1] \simeq f'\left(\dfrac{x_0+x_1}{2}\right) = f'(0.25).$

$\therefore f'(0.25) = -\sin(0.25) = -0.2474040.$

The Second-Order Divided Difference

Let x_0, x_1 and x_2 be distinct real numbers,

$$\Rightarrow \boxed{f[x_0,x_1,x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0}} \to \text{Second-order Divided Difference.}$$

The Third-Order Divided Difference

For x_0, x_1, x_2 and x_3 distinct values,

$$\boxed{f[x_0,x_1,x_2,x_3] = \frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}}$$ is

the Third-Order Divided Difference.

The General n-Order Divided Difference

For $x_0, x_1, \ldots x_n$ being $n+1$ distinct numbers,

$$\boxed{f[x_0,x_1,\ldots x_n] = \frac{f[x_1,x_2,\ldots,x_n]-f[x_0,x_1,\ldots,x_{n-1}]}{x_n-x_0}}$$

is the Newton Divided Difference of n Order.

Let $n \geq 1$, and $f(x)$ is n times continuously differentiable on some interval $\alpha \leq x \leq \beta$. Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct numbers in $[\alpha, \beta]$. Then $\boxed{f[x_0,x_1,\ldots,x_n] = \frac{1}{n!} f^{(n)}(c)}$ for some unknown point c between the minimum and maximum of $x_0, x_1, \ldots, x_n$.

Example: For $\boxed{f(x) = \cos x}$, $x_0 = 0.2$, $x_1 = 0.3$, $x_2 = 0.4$,

$f[x_0,x_1] = -0.2473009$. $f[x_1,x_2] = \dfrac{\cos(0.4)-\cos(0.3)}{0.4-0.3}$

$\Rightarrow f[x_1,x_2] = -0.3427550$. $f[x_0,x_1,x_2] = \dfrac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$.

$\therefore f[x_0,x_1,x_2] = -0.4772705$. For $\boxed{c \simeq x_1}$ $f''(x_1)/2! = -\cos(0.3)/2$

$= -0.4776682$

A Property of Divided Differences

If the order of $x_0, x_1, \cdots, x_n$ is exchanged or permutated, the $f[x_0, x_1, \cdots, x_n]$ does not change in value.

I/. $f[x_1, x_0] = \dfrac{f(x_0) - f(x_1)}{x_0 - x_1} = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$.

II/. $f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

$$\Rightarrow f[x_0, x_1, x_2] = \frac{1}{x_2 - x_0}\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right]$$

$$= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_0)}{(x_2 - x_0)(x_1 - x_0)} - \frac{f(x_1)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1)}{(x_2 - x_0)(x_1 - x_0)}$$

$$= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_0)}{(x_2 - x_0)(x_1 - x_0)} - \frac{f(x_1)}{(x_2 - x_0)}\left[\frac{1}{x_2 - x_1} + \frac{1}{x_1 - x_0}\right]$$

Now $\dfrac{1}{x_2 - x_1} + \dfrac{1}{x_1 - x_0} = \dfrac{\cancel{x_1} - x_0 + x_2 - \cancel{x_1}}{(x_2 - x_1)(x_1 - x_0)} = \dfrac{x_2 - x_0}{(x_2 - x_1)(x_1 - x_0)}$

$$\therefore -\frac{f(x_1)}{\cancel{(x_2 - x_0)}} \cdot \frac{\cancel{x_2 - x_0}}{(x_2 - x_1)(x_1 - x_0)} = \frac{-f(x_1)}{(x_2 - x_1)(x_1 - x_0)} = \frac{f(x_1)}{(x_1 - x_2)(x_1 - x_0)}$$

$$\therefore \boxed{f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}}$$

Exchange x_0 and x_1, The first two terms are exchanged. Similarly for exchanging x_1 and x_2, x_2 and x_0.

Newton's Divided Difference Interpolation Formula

Let $P_n(x)$ denote the polynomial interpolating $f(x_i)$ at x_i for $i = 0, 1, 2, \ldots, n$.

$\therefore$ $\boxed{P_n(x_i) = f(x_i)}$ and $\boxed{\text{degree}(P_n) \le n}$.

$$P_1(x) = f(x_0) + (x - x_0) f[x_0, x_1]$$

$$P_2(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) \times f[x_0, x_1, x_2].$$

$\vdots$

$$\boxed{P_n(x) = f(x_0) + (x - x_0) f[x_0, x_1] + \cdots \cdots + (x - x_0) \cdots (x - x_{n-1}) f[x_0, x_1, \ldots, x_n]}$$

$$\Rightarrow \boxed{P_{k+1}(x) = P_k(x) + (x - x_0) \cdots (x - x_k) f[x_0, x_1, \ldots, x_{k+1}]}$$

Check: 1/. When $x = x_0$, $P_1(x_0) = f(x_0)$.

2/. When $x = x_1$, $P_1(x_1) = f(x_0) + \cancel{(x_1 - x_0)} \dfrac{f(x_1) - f(x_0)}{\cancel{x_1 - x_0}}$
$= f(x_1)$.

3/. When $x = x_0$, $P_2(x_0) = f(x_0) = P_1(x_0)$.

4/. When $x = x_1$, $P_2(x_1) = f(x_1) = P_1(x_1)$.

5/. When $x = x_2$, $P_2(x_2) = f(x_0) + (x_2 - x_0) f[x_0, x_1] + (x_2 - x_0)(x_2 - x_1) f[x_0, x_1, x_2]$.

P. T. O.

$$\Rightarrow P_2(x_2) = f(x_0) + (x_2 - x_0)\frac{f(x_1) - f(x_0)}{x_1 - x_0} + \cancel{(x_2 - x_0)}(x_2 - x_1)\frac{1}{\cancel{x_2 - x_0}}\left\{f[x_1, x_2] - f[x_0, x_1]\right\}$$

$$\Rightarrow P_2(x_2) = f(x_0) + (x_2 - x_0)\frac{f(x_1) - f(x_0)}{x_1 - x_0} + (x_2 - x_1)\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right]$$

$$\Rightarrow P_2(x_2) = f(x_0) + \frac{(x_2 - x_0)}{(x_1 - x_0)}\left[f(x_1) - f(x_0)\right] + f(x_2) - f(x_1) - \frac{(x_2 - x_1)}{(x_1 - x_0)}\left[f(x_1) - f(x_0)\right]$$

$$\Rightarrow P_2(x_2) = f(x_0) + f(x_2) - f(x_1) + \left[f(x_1) - f(x_0)\right] \times \left[\frac{\cancel{x_2} - x_0 - \cancel{x_2} + x_1}{x_1 - x_0}\right].$$

$$\Rightarrow P_2(x_2) = f(x_0) + f(x_2) - f(x_1) + \left[f(x_1) - f(x_0)\right] \times \left[\frac{\cancel{x_1 - x_0}}{\cancel{x_1 - x_0}}\right]$$

$$\Rightarrow P_2(x_2) = \cancel{f(x_0)} + f(x_2) - \cancel{f(x_1)} + \cancel{f(x_1)} - \cancel{f(x_0)}$$

$$\Rightarrow \boxed{P_2(x_2) = f(x_2)}$$ Hence $P_2(x_0) = f(x_0)$.

$P_2(x_1) = f(x_1)$ and $P_2(x_2) = f(x_2)$, i.e., for $i = 0, 1, 2$, $\boxed{P_2(x_i) = f(x_i)}$. This is a unique quadratic polynomial for interpolating $f(x)$ at x_0, x_1, x_2.