The *phasor* voltage is then formed by dropping the $e^{j\omega t}$ factor from the complex instantaneous form:

$$\boxed{V_s(z) = V_0 e^{\pm j\beta z}} \tag{36}$$

The phasor voltage can be defined provided we have *sinusoidal steady-state* conditions—meaning that V_0 is independent of time. This has in fact been our assumption all along, because a time-varying amplitude would imply the existence of other frequency components in our signal. Again, we are treating only a single-frequency wave. The significance of the phasor voltage is that we are effectively letting time stand still and observing the stationary wave in space at $t = 0$. The processes of evaluating relative phases between various line positions and of combining multiple waves is made much simpler in phasor form. Again, this works only if all waves under consideration have the same frequency. With the definitions in (35) and (36), the real instantaneous voltage can be constructed using (34):

$$\mathcal{V}(z,t) = |V_0|\cos[\omega t \pm \beta z + \phi] = \text{Re}[V_c(z,t)] = \frac{1}{2}V_c + c.c. \tag{37a}$$

Or, in terms of the phasor voltage:

$$\boxed{\mathcal{V}(z,t) = |V_0|\cos[\omega t \pm \beta z + \phi] = \text{Re}[V_s(z)e^{j\omega t}] = \frac{1}{2}V_s(z)e^{j\omega t} + c.c.} \tag{37b}$$

In words, we may obtain our real sinusoidal voltage wave by multiplying the phasor voltage by $e^{j\omega t}$ (reincorporating the time dependence) and then taking the real part of the resulting expression. It is imperative that one becomes familiar with these relations and their meaning before proceeding further.

EXAMPLE 11.1

Two voltage waves having equal frequencies and amplitudes propagate in opposite directions on a lossless transmission line. Determine the total voltage as a function of time and position.

Solution. Since the waves have the same frequency, we can write their combination using their phasor forms. Assuming phase constant, β, and real amplitude, V_0, the two wave voltages combine in this way:

$$V_{sT}(z) = V_0 e^{-j\beta z} + V_0 e^{+j\beta z} = 2V_0\cos(\beta z)$$

In real instantaneous form, this becomes

$$\mathcal{V}(z,t) = \text{Re}[2V_0\cos(\beta z)e^{j\omega t}] = 2V_0\cos(\beta z)\cos(\omega t)$$

We recognize this as a *standing wave,* in which the amplitude varies as $\cos(\beta z)$, and oscillates in time as $\cos(\omega t)$. Zeros in the amplitude (nulls) occur at fixed locations, $z_n = (m\pi)/(2\beta)$ where m is an odd integer. We extend the concept in Section 11.10, where we explore the *voltage standing wave ratio* as a measurement technique.

11.6 TRANSMISSION LINE EQUATIONS AND THEIR SOLUTIONS IN PHASOR FORM

We now apply our results of the previous section to the transmission line equations, beginning with the general wave equation, (11). This is rewritten as follows, for the real instantaneous voltage, $\mathcal{V}(z,t)$:

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} = LC\frac{\partial^2 \mathcal{V}}{\partial t^2} + (LG + RC)\frac{\partial \mathcal{V}}{\partial t} + RG\mathcal{V} \tag{38}$$

We next substitute $\mathcal{V}(z,t)$ as given by the far right-hand side of (37*b*), noting that the complex conjugate term ($c.c.$) will form a separate redundant equation. We also use the fact that the operator $\partial/\partial t$, when applied to the complex form, is equivalent to multiplying by a factor of $j\omega$. After substitution, and after all time derivatives are taken, the factor $e^{j\omega t}$ divides out. We are left with the wave equation in terms of the phasor voltage:

$$\frac{d^2 V_s}{dz^2} = -\omega^2 LCV_s + j\omega(LG + RC)V_s + RGV_s \tag{39}$$

Rearranging terms leads to the simplified form:

$$\boxed{\frac{d^2 V_s}{dz^2} = \underbrace{(R + j\omega L)}_{Z}\underbrace{(G + j\omega C)}_{Y} V_s = \gamma^2 V_s} \tag{40}$$

where Z and Y, as indicated, are respectively the *net series impedance* and the *net shunt admittance* in the transmission line—both as per-unit-distance measures. The *propagation constant* in the line is defined as

$$\boxed{\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{ZY} = \alpha + j\beta} \tag{41}$$

The significance of the term will be explained in Section 11.7. For our immediate purposes, the solution of (40) will be

$$\boxed{V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}} \tag{42a}$$

The wave equation for current will be identical in form to (40). We therefore expect the phasor current to be in the form:

$$\boxed{I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}} \tag{42b}$$

The relation between the current and voltage waves is now found, as before, through the telegraphist's equations, (5) and (8). In a manner consistent with Eq. (37*b*), we write the sinusoidal current as

$$\mathcal{I}(z,t) = |I_0|\cos(\omega t \pm \beta z + \xi) = \frac{1}{2}\underbrace{(|I_0|e^{j\xi})}_{I_0} e^{\pm j\beta z} e^{j\omega t} + c.c. = \frac{1}{2}I_s(z)e^{j\omega t} + c.c. \tag{43}$$

Substituting the far right-hand sides of (37*b*) and (43) into (5) and (8) transforms the latter equations as follows:

$$\frac{\partial \mathcal{V}}{\partial z} = -\left(R\mathcal{I} + L\frac{\partial \mathcal{I}}{\partial t}\right) \Rightarrow \boxed{\frac{dV_s}{dz} = -(R + j\omega L)I_s = -ZI_s} \tag{44a}$$

and

$$\frac{\partial \mathcal{I}}{\partial z} = -\left(G\mathcal{V} + C\frac{\partial \mathcal{V}}{\partial t}\right) \Rightarrow \boxed{\frac{dI_s}{dz} = -(G + j\omega C)V_s = -YV_s} \tag{44b}$$

We can now substitute (42*a*) and (42*b*) into either (44*a*) or (44*b*) [we will use (44a)] to find:

$$-\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z} = -Z(I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}) \tag{45}$$

Next, equating coefficients of $e^{-\gamma z}$ and $e^{\gamma z}$, we find the general expression for the line characteristic impedance:

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{Z}{\gamma} = \frac{Z}{\sqrt{ZY}} = \sqrt{\frac{Z}{Y}} \tag{46}$$

Incorporating the expressions for Z and Y, we find the characteristic impedance in terms of our known line parameters:

$$\boxed{Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = |Z_0|e^{j\theta}} \tag{47}$$

Note that with the voltage and current as given in (37*b*) and (43), we would identify the phase of the characteristic impedance, $\theta = \phi - \xi$.

EXAMPLE 11.2

A lossless transmission line is 80 cm long and operates at a frequency of 600 MHz. The line parameters are $L = 0.25$ μH/m and $C = 100$ pF/m. Find the characteristic impedance, the phase constant, and the phase velocity.

Solution. Since the line is lossless, both R and G are zero. The characteristic impedance is

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.25 \times 10^{-6}}{100 \times 10^{-12}}} = 50\ \Omega$$

Since $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} = j\omega\sqrt{LC}$, we see that

$$\beta = \omega\sqrt{LC} = 2\pi(600 \times 10^6)\sqrt{(0.25 \times 10^{-6})(100 \times 10^{-12})} = 18.85 \text{ rad/m}$$

Also,

$$\nu_p = \frac{\omega}{\beta} = \frac{2\pi(600 \times 10^6)}{18.85} = 2 \times 10^8 \text{ m/s}$$

11.7 LOSSLESS AND LOW-LOSS PROPAGATION

Having obtained the phasor forms of voltage and current in a general transmission line [Eqs. (42a) and (42b)], we can now look more closely at the significance of these results. First we incorporate (41) into (42a) to obtain

$$V_s(z) = V_0^+ e^{-\alpha z} e^{-j\beta z} + V_0^- e^{\alpha z} e^{j\beta z} \tag{48}$$

Next, multiplying (48) by $e^{j\omega t}$ and taking the real part gives the real instantaneous voltage:

$$\mathcal{V}(z,t) = V_0^+ e^{-\alpha z}\cos(\omega t - \beta z) + V_0^- e^{\alpha z}\cos(\omega t + \beta z) \tag{49}$$

In this exercise, we have assigned V_0^+ and V_0^- to be real. Eq. (49) is recognized as describing forward- and backward-propagating waves that diminish in amplitude with distance according to $e^{-\alpha z}$ for the forward wave, and $e^{\alpha z}$ for the backward wave. Both waves are said to *attenuate* with propagation distance at a rate determined by the *attenuation coefficient*, α, expressed in units of nepers/m [Np/m].[2]

The phase constant, β, found by taking the imaginary part of (41), is likely to be a somewhat complicated function, and will in general depend on R and G. Nevertheless, β is still defined as the ratio ω/ν_p, and the wavelength is still defined as the distance that provides a phase shift of 2π rad, so that $\lambda = 2\pi/\beta$. By inspecting (41), we observe that losses in propagation are avoided (or $\alpha = 0$) only when $R = G = 0$. In that case, (41) gives $\gamma = j\beta = j\omega\sqrt{LC}$, and so $\nu_p = 1/\sqrt{LC}$, as we found before.

Expressions for α and β when losses are small can be readily obtained from (41). In the *low-loss approximation,* we require $R \ll \omega L$ and $G \ll \omega C$, a condition that is often true in practice. Before we apply these conditions, Eq. (41) can be written in the form:

$$\begin{aligned}\gamma = \alpha + j\beta &= [(R + j\omega L)(G + j\omega C)]^{1/2} \\ &= j\omega\sqrt{LC}\left[\left(1 + \frac{R}{j\omega L}\right)^{1/2}\left(1 + \frac{G}{j\omega C}\right)^{1/2}\right]\end{aligned} \tag{50}$$

The low-loss approximation then allows us to use the first three terms in the binomial series:

$$\sqrt{1+x} \doteq 1 + \frac{x}{2} - \frac{x^2}{8} \quad (x \ll 1) \tag{51}$$

We use (51) to expand the terms in large parentheses in (50), obtaining

$$\gamma \doteq j\omega\sqrt{LC}\left[\left(1 + \frac{R}{j2\omega L} + \frac{R^2}{8\omega^2 L^2}\right)\left(1 + \frac{G}{j2\omega C} + \frac{G^2}{8\omega^2 C^2}\right)\right] \tag{52}$$

All products in (52) are then carried out, neglecting the terms involving RG^2, R^2G, and R^2G^2, as these will be negligible compared to all others. The result is

$$\gamma = \alpha + j\beta \doteq j\omega\sqrt{LC}\left[1 + \frac{1}{j2\omega}\left(\frac{R}{L} + \frac{G}{C}\right) + \frac{1}{8\omega^2}\left(\frac{R^2}{L^2} - \frac{2RG}{LC} + \frac{G^2}{C^2}\right)\right] \tag{53}$$

[2] The term *neper* was selected (by some poor speller) to honor John Napier, a Scottish mathematician who first proposed the use of logarithms.

Now, separating real and imaginary parts of (53) yields α and β:

$$\boxed{\alpha \doteq \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)} \tag{54a}$$

and

$$\boxed{\beta \doteq \omega\sqrt{LC}\left[1 + \frac{1}{8}\left(\frac{G}{\omega C} - \frac{R}{\omega L}\right)^2\right]} \tag{54b}$$

We note that α scales in direct proportion to R and G, as would be expected. We also note that the terms in (54b) that involve R and G lead to a phase velocity, $\nu_p = \omega/\beta$, that is frequency-dependent. Moreover, the *group velocity*, $\nu_g = d\omega/d\beta$, will also depend on frequency, and will lead to signal distortion, as we will explore in Chapter 13. Note that with nonzero R and G, phase and group velocities that are constant with frequency can be obtained when $R/L = G/C$, known as *Heaviside's condition*. In this case, (54b) becomes $\beta \doteq \omega\sqrt{LC}$, and the line is said to be *distortionless*. Further complications occur when accounting for possible frequency dependencies within R, G, L, and C. Consequently, conditions of low-loss or distortion-free propagation will usually occur over limited frequency ranges. As a rule, loss increases with increasing frequency, mostly because of the increase in R with frequency. The nature of this latter effect, known as *skin effect* loss, requires field theory to understand and quantify. We will study this in Chapter 12, and we will apply it to transmission line structures in Chapter 14.

Finally, we can apply the low-loss approximation to the characteristic impedance, Eq. (47). Using (51), we find

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{j\omega L\left(1 + \frac{R}{j\omega L}\right)}{j\omega C\left(1 + \frac{G}{j\omega C}\right)}} \doteq \sqrt{\frac{L}{C}}\left[\frac{\left(1 + \frac{R}{j2\omega L} + \frac{R^2}{8\omega^2 L^2}\right)}{\left(1 + \frac{G}{j2\omega C} + \frac{G^2}{8\omega^2 C^2}\right)}\right] \tag{55}$$

Next, we multiply (55) by a factor of 1, in the form of the complex conjugate of the denominator of (55) divided by itself. The resulting expression is simplified by neglecting all terms on the order of R^2G, G^2R, and higher. Additionally, the approximation, $1/(1+x) \doteq 1 - x$, where $x \ll 1$ is used. The result is

$$Z_0 \doteq \sqrt{\frac{L}{C}}\left\{1 + \frac{1}{2\omega^2}\left[\frac{1}{4}\left(\frac{R}{L} + \frac{G}{C}\right)^2 - \frac{G^2}{C^2}\right] + \frac{j}{2\omega}\left(\frac{G}{C} - \frac{R}{L}\right)\right\} \tag{56}$$

Note that when Heaviside's condition (again, $R/L = G/C$) holds, Z_0 simplifies to just $\sqrt{L/C}$, as is true when both R and G are zero.

EXAMPLE 11.3

Suppose in a certain transmission line $G = 0$, but R is finite-valued and satisfies the low-loss requirement, $R \ll \omega L$. Use Eq. (56) to write the approximate magnitude and phase of Z_0.

Solution. With $G = 0$, the imaginary part of (56) is much greater than the second term in the real part [proportional to $(R/\omega L)^2$].

Therefore, the characteristic impedance becomes

$$Z_0(G=0) \doteq \sqrt{\frac{L}{C}}\left(1 - j\frac{R}{2\omega L}\right) = |Z_0|e^{j\theta}$$

where $|Z_0| \doteq \sqrt{L/C}$, and $\theta = \tan^{-1}(-R/2\omega L)$.

D11.1 At an operating radian frequency of 500 Mrad/s, typical circuit values for a certain transmission line are: $R = 0.2\ \Omega$/m, $L = 0.25\ \mu$H/m, $G = 10\ \mu$S/m, and $C = 100$ pF/m. Find: (*a*) α; (*b*) β; (*c*) λ; (*d*) ν_p; (*e*) Z_0.

Ans. 2.25 mNp/m; 2.50 rad/m; 2.51 m; 2×10^8 m/sec; $50.0 - j0.0350\ \Omega$

11.8 POWER TRANSMISSION AND LOSS CHARACTERIZATION

Having found the sinusoidal voltage and current in a lossy transmission line, we next evaluate the power transmitted over a specified distance as a function of voltage and current amplitudes. We start with the *instantaneous* power, given simply as the product of the real voltage and current. Consider the forward-propagating term in (49), where again, the amplitude, $V_0^+ = |V_0|$, is taken to be real. The current waveform will be similar, but will generally be shifted in phase. Both current and voltage attenuate according to the factor $e^{-\alpha z}$. The instantaneous power therefore becomes

$$\mathcal{P}(z,t) = \mathcal{V}(z,t)\mathcal{I}(z,t) = |V_0||I_0|e^{-2\alpha z}\cos(\omega t - \beta z)\cos(\omega t - \beta z + \theta) \tag{57}$$

Usually, the *time-averaged* power, $\langle\mathcal{P}\rangle$, is of interest. We find this through:

$$\langle\mathcal{P}\rangle = \frac{1}{T}\int_0^T |V_0||I_0|e^{-2\alpha z}\cos(\omega t - \beta z)\cos(\omega t - \beta z + \theta)dt \tag{58}$$

where $T = 2\pi/\omega$ is the time period for one oscillation cycle. Using a trigonometric identity, the product of cosines in the integrand can be written as the sum of individual cosines at the sum and difference frequencies:

$$\langle\mathcal{P}\rangle = \frac{1}{T}\int_0^T \frac{1}{2}|V_0||I_0|[\cos(2\omega t - 2\beta z + \theta) + \cos(\theta)]\,dt \tag{59}$$

The first cosine term integrates to zero, leaving the $\cos\theta$ term. The remaining integral easily evaluates as

$$\langle\mathcal{P}\rangle = \frac{1}{2}|V_0||I_0|e^{-2\alpha z}\cos\theta = \frac{1}{2}\frac{|V_0|^2}{|Z_0|}e^{-2\alpha z}\cos\theta \text{ [W]} \tag{60}$$

The same result can be obtained directly from the phasor voltage and current. We begin with these, expressed as

$$V_s(z) = V_0e^{-\alpha z}e^{-j\beta z} \tag{61}$$

and

$$I_s(z) = I_0e^{-\alpha z}e^{-j\beta z} = \frac{V_0}{Z_0}e^{-\alpha z}e^{-j\beta z} \tag{62}$$

where $Z_0 = |Z_0|e^{j\theta}$. We now note that the time-averaged power as expressed in (60) can be obtained from the phasor forms through

$$\boxed{\langle \mathcal{P} \rangle = \frac{1}{2}\mathrm{Re}\{V_s I_s^*\}} \tag{63}$$

where again, the asterisk (*) denotes the complex conjugate (applied in this case to the current phasor only). Using (61) and (62) in (63), it is found that

$$\begin{aligned} \langle \mathcal{P} \rangle &= \frac{1}{2}\mathrm{Re}\left\{ V_0 e^{-\alpha z} e^{-j\beta z} \frac{V_0^*}{|Z_0|e^{-j\theta}} e^{-\alpha z} e^{+j\beta z} \right\} \\ &= \frac{1}{2}\mathrm{Re}\left\{ \frac{V_0 V_0^*}{|Z_0|} e^{-2\alpha z} e^{j\theta} \right\} = \frac{1}{2}\frac{|V_0|^2}{|Z_0|} e^{-2\alpha z}\cos\theta \end{aligned} \tag{64}$$

which we note is identical to the time-integrated result in (60). Equation (63) applies to any single-frequency wave.

An important result of the preceding exercise is that power attenuates as $e^{-2\alpha z}$, or

$$\boxed{\langle \mathcal{P}(z) \rangle = \langle \mathcal{P}(0) \rangle e^{-2\alpha z}} \tag{65}$$

Power drops at twice the exponential rate with distance as either voltage or current.

A convenient measure of power loss is in *decibel* units. This is based on expressing the power decrease as a power of 10. Specifically, we write

$$\frac{\langle \mathcal{P}(z) \rangle}{\langle \mathcal{P}(0) \rangle} = e^{-2\alpha z} = 10^{-\kappa\alpha z} \tag{66}$$

where the constant, κ, is to be determined. Setting $\alpha z = 1$, we find

$$e^{-2} = 10^{-\kappa} \quad \Rightarrow \quad \kappa = \log_{10}(e^2) = 0.869 \tag{67}$$

Now, by definition, the power loss in decibels (dB) is

$$\boxed{\text{Power Loss (dB)} = 10\log_{10}\left[\frac{\langle \mathcal{P}(0) \rangle}{\langle \mathcal{P}(z) \rangle}\right] = 8.69\alpha z} \tag{68}$$

where we note that inverting the power ratio in the argument of the log function [as compared to the ratio in (66)] yields a positive number for the dB loss. Also, noting that $\langle \mathcal{P} \rangle \propto |V_0|^2$, we may write, equivalently:

$$\boxed{\text{Power Loss (dB)} = 10\log_{10}\left[\frac{\langle \mathcal{P}(0) \rangle}{\langle \mathcal{P}(z) \rangle}\right] = 20\log_{10}\left[\frac{|V_0(0)|}{|V_0(z)|}\right]} \tag{69}$$

where $|V_0(z)| = |V_0(0)|e^{-\alpha z}$.

EXAMPLE 11.4

A piece of coaxial line having characteristic impedance of 50 Ω is shown in the following Figure 11.5. The spacing between the inner and the outer conductor of this line is filled with a dielectric, which can be considered lossless for all practical purposes. However, the conductors are not perfect resulting into the attenuation along the line at the rate of 20 dB/km at the frequency of 500 MHz. The line inductance is given as $L = 238.1\ \text{nH m}^{-1}$. Determine (a) the resistance R of this line per length; (b) the capacitance C of the line per unit length; (c) the velocity of the propagation along the line, and (d) the dielectric constant of the material which is filled between the two conductors given in the Figure 11.5.

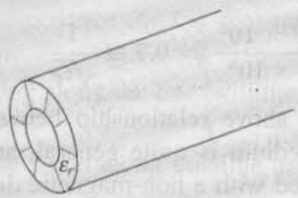

Fig. 11.5 A piece of coaxial line.

Solution.

First of all, the attenuation constant is to be converted into Np/m from dB/km. This can be done with the help of the following equation:

$$\text{Attenuation in Np} \equiv \text{Attenuation in dB}/8.69$$

Hence, the attenuation constant in Np/m is computed as follows:

$$\alpha = 20\,\text{dB/km} \equiv \frac{20}{1000}\,\text{dB/m} \equiv \frac{20}{8.69 \times 1000}\,\text{Np/m} \equiv 2.302 \times 10^{-3}[\text{Np/m}]$$

(a) It is given that the dielectric loss can be considered zero for all practical purposes. The dielectric loss in Figure 11.3 is represented by the shunt conductance G, and hence in the present situation, $G = 0$. The series resistance R in the present case can be computed using the formula for the attenuation constant of low-loss line given by (54a)

$$\alpha = \frac{1}{2} R \sqrt{\frac{C}{L}} \cong \frac{R}{2\,Z_0} = 2.302 \times 10^{-3}\ \text{Np/m}$$

where $G = 0$ is substituted, and the expression for the characteristic impedance $Z_0 \cong \sqrt{L/C}$ is used, which is valid for the low-loss line as per Eq. (56) under the condition that $G = 0$ and $R \ll \omega L$. Now, the characteristic impedance of the line $Z_0 = 50\ \Omega$ is given, which can be substituted in the above expression to obtain the series resistance of the line

$$R = 2\alpha Z_0 = 2 \times 2.302 \times 10^{-3}\ \text{Np/m} \times 50 = 0.2302\ \Omega\,\text{m}^{-1}$$

It is to be noted here that at 500 MHz, $\omega L \equiv 2 \times \pi \times 500 \times 10^6 \times 238.1 \times 10^{-9} = 748.01 \gg 0.2302$, which verifies our assumption that $R \ll \omega L$, and it is also the main reason to define the characteristic impedance as a purely real quantity in most of the practical situations at RF and microwave frequencies.

(*b*) The capacitance C of the line per unit length can be calculated as

$$Z_0 \cong \sqrt{\frac{L}{C}} \quad \Rightarrow \ C \cong \frac{L}{Z_0^2} = \frac{238.1 \times 10^{-9}}{(50)^2} = 95.2 \text{ pF m}^{-1}$$

(*c*) The phase velocity can be calculated with the help of (28) and (54b) by applying the conditio[n] that $G = 0$ and $R \ll \omega L$, which results into

$$v_p = \frac{\omega}{\beta}; \ \beta \cong \omega\sqrt{LC} \ \Rightarrow \ v_p \cong \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{238.1 \times 10^{-9} \times 95.2 \times 10^{-12}}} = 2.1 \times 10^8 \text{ m/sec}$$

(*d*) The dielectric constant of the material filled between the conductors can be computed b[y] comparing the actual velocity along the line with the velocity in the free space or the air fille[d] medium. This ratio is usually called the velocity factor, and for the non-magnetic dielectrics can be written as

$$\frac{v_p}{c} = \frac{2.1 \times 10^8}{3 \times 10^8} = 0.7 \equiv \cdot \frac{1}{\sqrt{\epsilon_r}} \quad \Rightarrow \epsilon_r = 2.04$$

You should remember that the above relationship between the velocity in free space an[d] the velocity in the dielectric medium is quite general, and is usually valid for all types o[f] transmission lines which are filled with a non-magnetic dielectric medium other than air. Th[e] basis of the above relationship is that the capacitance per unit length between the two conductor[s] is directly proportional to the dielectric constant of the material which is filled inside the regio[n] between these conductors.

EXAMPLE 11.5

A 20 m length of transmission line is known to produce a 2.0 dB drop in power from end to end. (*a*) What fraction of the input power reaches the output? (*b*) What fraction of the input power reaches the midpoint of the line? (*c*) What exponential attenuation coefficient, α, does this represent?

Solution.

(*a*) The power fraction will be

$$\frac{\langle \mathcal{P}(20) \rangle}{\langle \mathcal{P}(0) \rangle} = 10^{-0.2} = 0.63$$

(*b*) 2 dB in 20 m implies a loss rating of 0.1 dB/m. So, over a 10-metre span, the loss is 1.0 dB. Thi[s] represents the power fraction, $10^{-0.1} = 0.79$.

(*c*) The exponential attenuation coefficient is found through

$$\alpha = \frac{2.0 \text{ dB}}{(8.69 \text{ dB/Np})(20 \text{ m})} = 0.012 \,[\text{Np/m}]$$

A final point addresses the question: Why use decibels? The most compelling reason is that whe[n] evaluating the accumulated loss for several lines and devices that are all end-to-end connected, th[e] net loss in dB for the entire span is just the sum of the dB losses of the individual elements.

D11.2 Two transmission lines are to be joined end-to-end. Line 1 is 30 m long and is rated at 0.1 dB/m. Line 2 is 45 m long and is rated at 0.15 dB/m. The joint is not done well and imparts a 3 dB loss. What percentage of the input power reaches the output of the combination?
Ans. 5.3%

EXAMPLE 11.6

A distortionless transmission line operating at 250 MHz has $R = 30\ \Omega/\text{m}$, $L = 200$ nH/m, and $C = 80$ pF/m.

(*a*) Determine the characteristic impedance Z_0, the propagation constant γ, and the velocity of propagation v_p along the line.

(*b*) After how many metres traveling along the line, will the voltage wave get reduced to 30% of its initial value?

(*c*) How far should a voltage wave travel along this line in order to undergo a phase shift of 20°?

Solution. For a distortionless line, $\dfrac{R}{L} = \dfrac{G}{C}$ (i)

(*a*) If the above condition is applied to Eq. (56), it results into

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{200 \times 10^{-9}}{80 \times 10^{-12}}} = 50\,\Omega \tag{ii}$$

The propagation constant is given by

$$\gamma \equiv \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \equiv \sqrt{(R + j\omega L)\left(\frac{RC}{L} + j\omega C\right)}$$

$$\equiv \sqrt{(R + j\omega L)\frac{C}{L}(R + j\omega L)} = \left(R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC}\right) \tag{iii}$$

$$\equiv \left(30\sqrt{\frac{80 \times 10^{-12}}{200 \times 10^{-9}}} + j \times 2 \times \pi \times 250 \times 10^6 \times \sqrt{200 \times 10^{-9} \times 80 \times 10^{-12}}\right)$$

$$\equiv (0.60 + j\,6.2832)\ \text{m}^{-1} \tag{iv}$$

where, we have made use of the relationship given by (i). The above equation can be separated into real and imaginary parts to define the value of attenuation constant and the phase constant as follows

$$\alpha = 0.6\,\text{Np/m}; \quad \beta = 6.2832\,\text{rad/m} \tag{v}$$

The velocity of propagation along the line is given by

$$\Rightarrow \quad v_p = \frac{\omega}{\beta} \equiv \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{200 \times 10^{-9} \times 80 \times 10^{-12}}} = 2.5 \times 10^8\ \text{m/sec}$$

where (iii) is used to define the propagation constant β.

(*b*) Let us assume that the wave is travelling along the z-direction, and let the magnitude of the voltage wave be V_0 at $z = 0$. The magnitude of this voltage wave at any arbitrary position along the line would then be given by $V_0 \exp(-\alpha z)$. Now, we have to find the position '$z = d$' at which the magnitude of this wave gets reduced to 30% of its initial value, i.e.,

$$V_0 \exp(-\alpha d) \equiv 0.3\,V_0 \Rightarrow \exp(\alpha d) \equiv \frac{1}{0.3}$$

$$\Rightarrow d \equiv \frac{1}{\alpha}\ln\left(\frac{1}{0.3}\right) = \frac{1}{0.6}\ln\left(\frac{1}{0.3}\right) = 2\ \text{m}$$

(*c*) As the wave travels along the line, it would continuously undergo a phase shift. If it is assumed that the wave is travelling along the +ve z-direction and at the initial position this wave is expressed by $V_0 \exp(-j0)$ then this wave can be expressed at any arbitrary position along the line by $V_0 \exp(-j\beta z)$. Hence, the phase shift after traversing a distance 'l' would be represented by βl. Now, it is given that

$$\beta l = 20\,\text{deg} \equiv 20 \times \frac{\pi}{180}\ \text{rad}$$

$$\Rightarrow l \equiv \frac{20}{\beta} \times \frac{\pi}{180}\ m = \frac{20}{6.2832} \times \frac{\pi}{180}\ m = 5.56\ \text{cm}$$

where the value of the phase constant β is substituted from (v).

11.9 WAVE REFLECTION AT DISCONTINUITIES

The concept of wave reflection was introduced in Section 11.1. As implied there, the need for a reflected wave originates from the necessity to satisfy all voltage and current boundary conditions at the ends of transmission lines and at locations at which two dissimilar lines are connected to each other. The consequences of reflected waves are usually less than desirable, in that some of the power that was intended to be transmitted to a load, for example, reflects and propagates back to the source. Conditions for achieving *no* reflected waves are therefore important to understand.

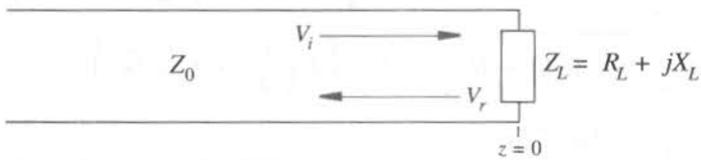

Fig. 11.6 Voltage wave reflection from a complex load impedance.

The basic reflection problem is illustrated in Figure 11.6. In it, a transmission line of characteristic impedance Z_0 is terminated by a load having complex impedance, $Z_L = R_L + jX_L$. If the line is lossy, then we know that Z_0 will also be complex. For convenience, we assign coordinates such that the load is at location $z = 0$. Therefore, the line occupies the region $z < 0$. A voltage wave is presumed to be incident on the load, and is expressed in phasor form for all z:

$$V_i(z) = V_{0i}e^{-\alpha z}e^{-j\beta z} \tag{70a}$$

When the wave reaches the load, a reflected wave is generated that back-propagates:

$$V_r(z) = V_{0r}e^{+\alpha z}e^{+j\beta z} \tag{70b}$$