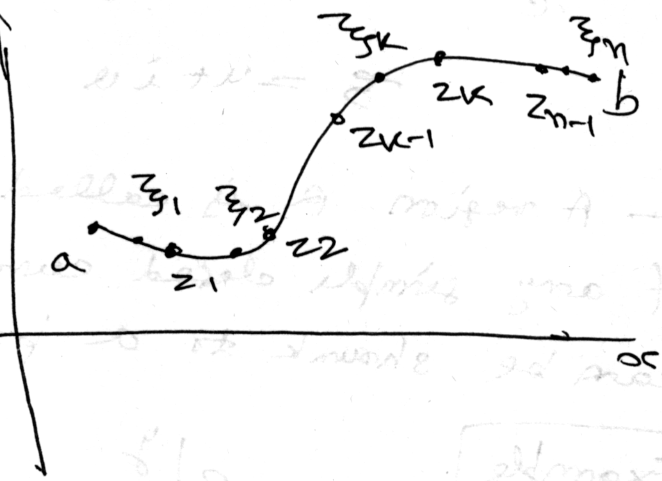
CCV-13

SC 216 Calculus with complex variables

March 10, 2008

Complex Line Integrals



- $f(z)$ cont's at all points of a curve C
- Subdivide C into n parts by $z_1, z_2, \dots, z_{n-1}$ chosen arbitrarily & call $a = z_0$ & $b = z_n$
- On each ~~arc~~ arc joining z_{k-1} to z_k choose a pt ξ_k

$\Delta z_k = z_k - z_{k-1}$

- $S_n = f(\xi_1)(z_1 - a) + f(\xi_2)(z_2 - z_1) + \dots + f(\xi_n)(b - z_{n-1})$

$$\boxed{\int_a^b f(z)\,dz \text{ or } \int_C f(z)\,dz = \lim_{\substack{\text{largest of the chosen length} \\ |\Delta z_k| \to 0}} \sum_{k=1}^{n} f(\xi_k)\,\Delta z_k}$$

Now if $f(z) = u(x,y) + i\,v(x,y) = u + iv$

then

$$\boxed{\begin{aligned}\int_C f(z)\,dz &= \int_C (u + iv)(dx + i\,dy) \\ &= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy\end{aligned}}$$

Properties

(1) $\int_C \{f(z) + g(z)\}\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

(2) $\int_C A\,f(z)\,dz = A\int_C f(z)\,dz$

(3) $\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$

(4) $\int_a^b f(z)\,dz = \int_a^m f(z)\,dz + \int_m^b f(z)\,dz$ a, b, m are pts. on C

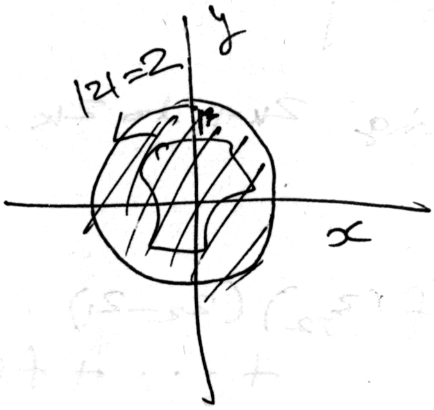
(5) $\left|\int_C f(z)\,dz\right| \le ML$, where $|f(z)| \le M$ L = length of C

$$\int_C f(z)\,dz = \int_{C'} f\{g(\zeta)\}\, g'(\zeta)\, d\zeta$$

$$\zeta = u + i v \quad \& \quad z = g(\zeta)$$

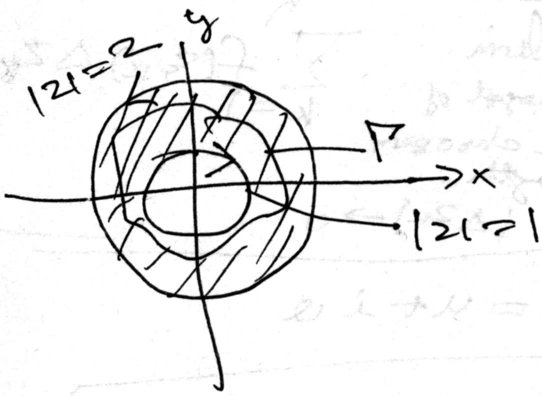
— A region R is called <u>simply-connected</u> if any simple closed curve which lies in R can be shrunk to a point without leaving R.

Example



simply connected region

$|z| < 2$



$1 < |z| < 2$

multiply-connected region

— A region which is not simply-connected is called <u>multiply-connected</u>.

Cauchy's Theorem

$\oint_C f(z)\,dz = 0$ if $f(z)$ is analytic with derivative $f'(z)$ which is cont.s at all points inside and on a simple closed curve C.

$\square$ $\because$ $f(z) = u + iv$ is analytic & has a cont.s derivative

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

but C.R. eq^n $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

are cont.s inside & on C.

$$\Rightarrow \oint_C f(z)\,dz = \oint_C (u+iv)\,(dx + i\,dy)$$

$$= \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

using C.R. eqns.

$$= 0$$

Example

(a) $\oint_C dz = 0$

(b) $\oint_C z\,dz = 0$

(c) $\oint_C (z - z_0)\,dz = 0$

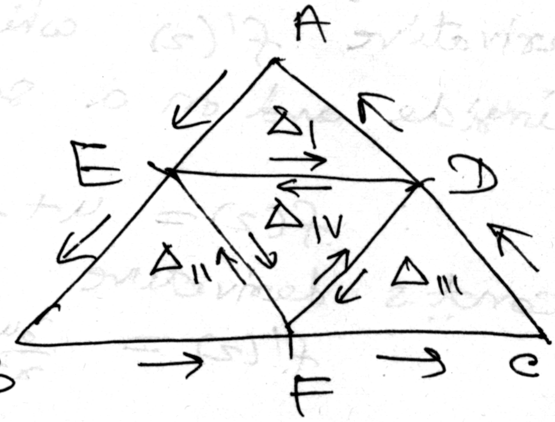
C — ~~simp~~ simple closed curve, z_0 is a constant

$\because$ 1, z & $z - z_0$ are analytic inside C & have cont.s derivatives

Cauchy - Goursat Theorem

This removes the restriction that $f'(z)$ is conti.s

Let us prove it for a triangle.



Consider a triangle ΔABC. Join the mid points D E F to form 4 triangles Δ_I, Δ_{II}, Δ_{III} & Δ_{IV} as shown.

If $f(z)$ is analytic inside and on ΔABC we have (omitting the integrand)

$$\oint_{ABCA} f(z)\,dz = \int_{DAE} + \int_{EBF} + \int_{FCD}$$

$$= \left\{\int_{DAE} + \int_{ED}\right\} + \left\{\int_{EBF} + \int_{FE}\right\}$$

$$+ \left\{\int_{FCD} + \int_{DF}\right\} + \left\{\int_{DE} + \int_{EF} + \int_{FD}\right\}$$

Here we have used

$$\int_{ED} = -\int_{DE}, \quad \int_{FE} = -\int_{EF}$$

$$\& \quad \int_{DF} = -\int_{FD}$$

$$= \int_{DAED} + \int_{EBFE} + \int_{FCDF} + \int_{DEFD}$$

$\Rightarrow$

$$\oint_{ABCA} f(z)\,dz = \int_{\Delta_I} f(z)\,dz + \int_{\Delta_{II}} f(z)\,dz + \int_{\Delta_{III}} f(z)\,dz + \int_{\Delta_{IV}} f(z)\,dz$$

Then

$$\left|\oint_{\Delta} f(z)\,dz\right| \le \left|\oint_{\Delta_{I}} f(z)\,dz\right| + \left|\oint_{\Delta_{II}} f(z)\,dz\right| + \left|\oint_{\Delta_{III}} f(z)\,dz\right| + \left|\oint_{\Delta_{IV}} f(z)\,dz\right|$$

Let Δ_1 be the triangle corresponding to that term on the RHS having largest values (if there are more such term then Δ_1 is any of the associated Δs).

Then

$$\left|\oint_{\Delta} f(z)\,dz\right| \le 4\left|\oint_{\Delta_1} f(z)\,dz\right|$$

By joining mid points of Δ_1 we obtain similarly a triangle Δ_2 s.t.

$$\left|\oint_{\Delta_1} f(z)\,dz\right| \le 4\left|\oint_{\Delta_2} f(z)\,dz\right|$$

$$\Rightarrow \left|\oint_{\Delta} f(z)\,dz\right| \le 4^2\left|\oint_{\Delta_2} f(z)\,dz\right|$$

After n steps we obtain a triangle

$$\left|\oint_{\Delta} f(z)\,dz\right| \le 4^n\left|\oint_{\Delta_n} f(z)\,dz\right|$$

$\Delta, \Delta_1, \Delta_2, \Delta_3 \ldots$ is a seq. of triangles each of which is contained in the preceding (i.e. a seq. of nested triangles) and $\exists$ a pt. z_0 which lies in every tringle of the seq.

$\therefore$ z_0 lies inside or on the boundary of Δ $\Rightarrow$ $f(z)$ is analytic at z_0

$$\Rightarrow f(z) = f(z_0) + f'(z_0)(z-z_0) + \eta(z-z_0) \quad \text{—— (1)}$$

where for any $\epsilon > 0$ we can find δ s.t.

$|\eta| < \epsilon$ whenever $|z-z_0| < \delta$

integrating both sides of (1)

$$\oint_{\Delta_n} f(z)\,dz = \oint_{\Delta_n} \eta(z-z_0)\,dz \quad \text{—— (2)}$$

If P is the perimeter of Δ then perimeter of Δ_n is $P_n = \dfrac{P}{2^n}$

If z is any pt on Δ_n

then $|z-z_0| < \dfrac{P}{2^n} < \delta$

$$\Rightarrow \left|\oint_{\Delta_n} f(z)\,dz\right| = \left|\oint_{\Delta_n} \eta(z-z_0)\,dz\right|$$

$$\le \epsilon \cdot \frac{P}{2^n} \cdot \frac{P}{2^n} = \frac{\epsilon \cdot P^2}{4^n}$$

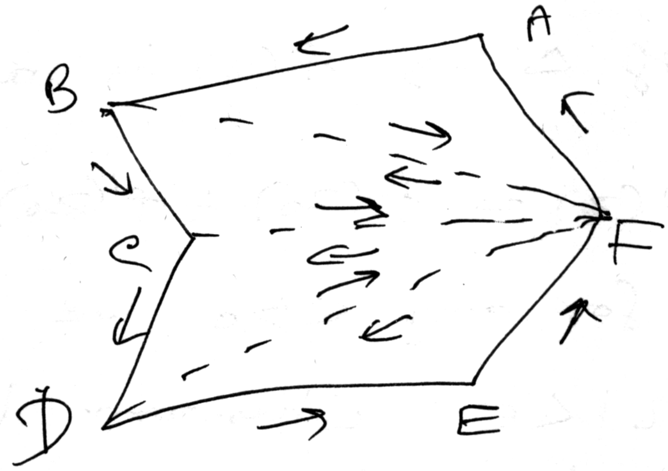
$$\Rightarrow \left|\oint_{\Delta} f(z)\,dz\right| \le 4^n \cdot \frac{\epsilon \cdot P^2}{4^n} = \epsilon \cdot P^2$$

Making $\epsilon \to 0$

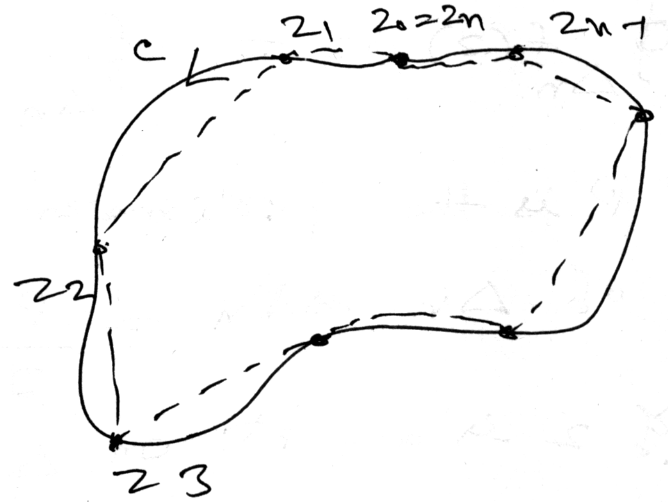
$$\oint_{\Delta} f(z)\,dz = 0$$

∎

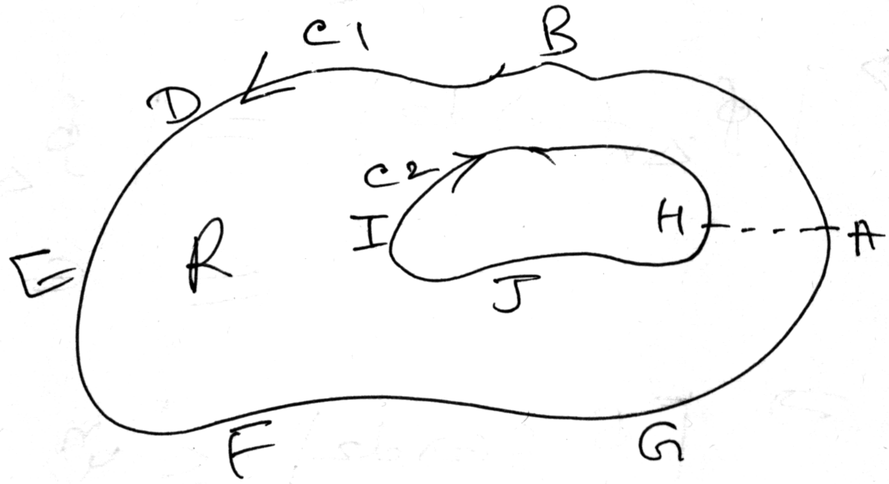
— Cauchy-Goursat theorem for any closed polygon



— Cauchy-Goursat theorem for any simple closed curve C.



— Cauchy-Goursat theorem for multiply-connected regions.



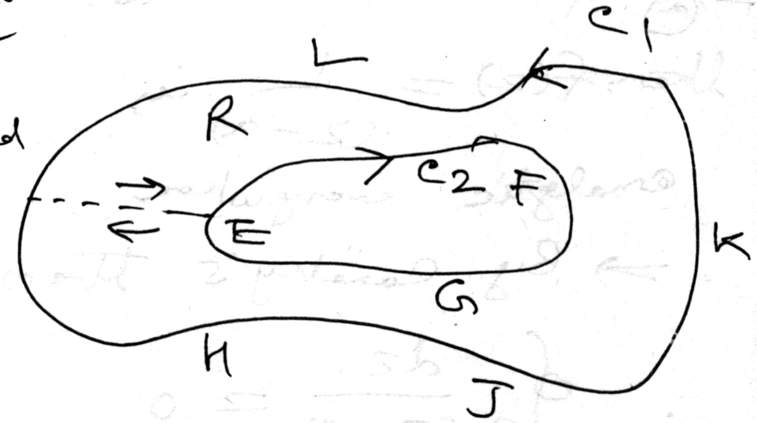
Consequences of Cauchy's Theorem

— If $f(z)$ is analytic in a simply connected region R. prove that $\int_a^b f(z)\,dz$ is independent of the path in R joining any two points a & b in R.

— If $f(z)$ is analytic in a region R bounded by two simple closed curves c_1 & c_2 & also on c_1 & c_2 show that

$$\oint_{c_1} f(z)\,dz = \oint_{c_2} f(z)\,dz$$

where c_1 & c_2 are traversed in the +ve sense.



Construct a cross cut DE then $\because f(z)$ is analytic in R we have cauchy's theorem

$$\int_{DEFGEDHJKLD} f(z)\,dz = 0$$

or

$$\int_{DE} + \int_{EFGE} + \int_{ED} + \int_{DHJKLD} = 0$$

$$\because \int_{DE} = -\int_{ED}$$

$$\Rightarrow \int_{DHJKLD} = -\int_{EFGE} = \int_{EGFE}$$

or $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$ ▨

Example

$$\oint_C \frac{dz}{(z-a)}$$ C is simple closed curve

ⓐ $z = a$ is outside C
ⓑ $z = a$ is inside C.

□ ⓐ If a is outside C

then $f(z) = \frac{1}{(z-a)}$ is analytic everywhere

⇒ By Cauchy's theorem

$$\oint_C \frac{dz}{(z-a)} = 0$$

ⓑ $z = a$ is inside C & Γ be a circle of radius ε with center at $z = a$

~~then~~ $$\oint_C \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a}$$

Now on Γ, $|z-a| = \epsilon \Rightarrow z-a = \epsilon e^{i\theta}$

$\Rightarrow dz = i\epsilon e^{i\theta}\,d\theta \qquad 0 \le \theta < 2\pi$

$\Rightarrow$

$$\text{R.H.S.} = \int_{\theta=0}^{2\pi} \frac{i\epsilon e^{i\theta}\,d\theta}{\epsilon e^{i\theta}} = i\int_0^{2\pi} d\theta$$

$$= 2\pi i$$

Evaluate $\oint_C \frac{dz}{(z-a)^n}$ $n=2,3,4,\dots$ where $z=a$ is inside the simple closed curve C.

$\boxed{\text{Answer} = 0}$

□ $$\oint_C \frac{dz}{(z-a)^n} = \oint_\Gamma \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{i\epsilon e^{i\theta}\,d\theta}{\epsilon^n e^{in\theta}}$$

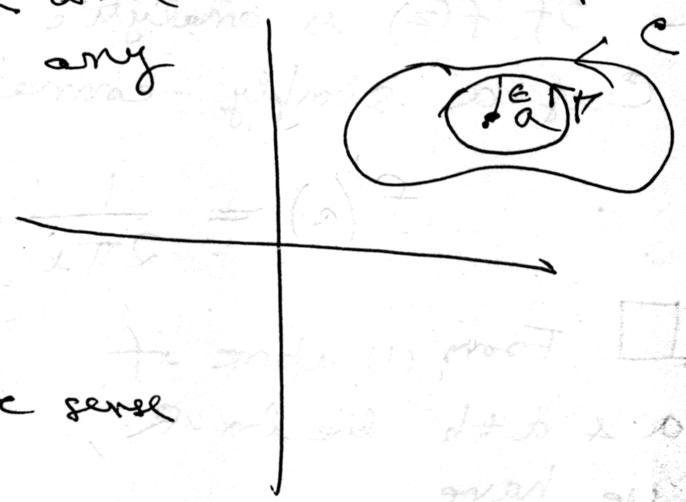
$$= \frac{i}{\epsilon^{n-1}} \int_0^{2\pi} e^{(1-n)i\theta}\,d\theta = \frac{i}{\epsilon^{n-1}} \left.\frac{e^{(1-n)i\theta}}{(1-n)i}\right|_0^{2\pi}$$

$$= \frac{1}{(1-n)\epsilon^{n-1}}\left[e^{2(1-n)\pi i} - 1\right] = 0 \quad (n \neq 1)$$

■

Cauchy's Integral formulae

If $f(z)$ is analytic inside and on a simple closed curve C and a is any pt. inside C then



$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)}\,dz$$

C is traversed in +ve sense

also

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \quad n=1,2,3,\dots$$

□ $\frac{f(z)}{z-a}$ is analytic inside & on C except at pt. $z=a$ $\Rightarrow$

$$\oint_C \frac{f(z)}{(z-a)}\,dz = \oint_\Gamma \frac{f(z)}{(z-a)}\,dz$$

$\Gamma: |z-a| = \epsilon$
$\Rightarrow$
$z = a + \epsilon e^{i\theta}$
$0 \le \theta < 2\pi$

$$\Rightarrow \oint_{\Gamma} \frac{f(z)}{(z-a)} dz = \int_0^{2\pi} \frac{f(a+\epsilon e^{i\theta})\, i\, \epsilon e^{i\theta}}{\epsilon\, e^{i\theta}} d\theta$$

$$= i \int_0^{2\pi} f(a+\epsilon e^{i\theta})\, d\theta$$

$$\Rightarrow \oint_{c} \frac{f(z)}{(z-a)} dz = i \int_0^{2\pi} f(a+\epsilon e^{i\theta})\, d\theta$$

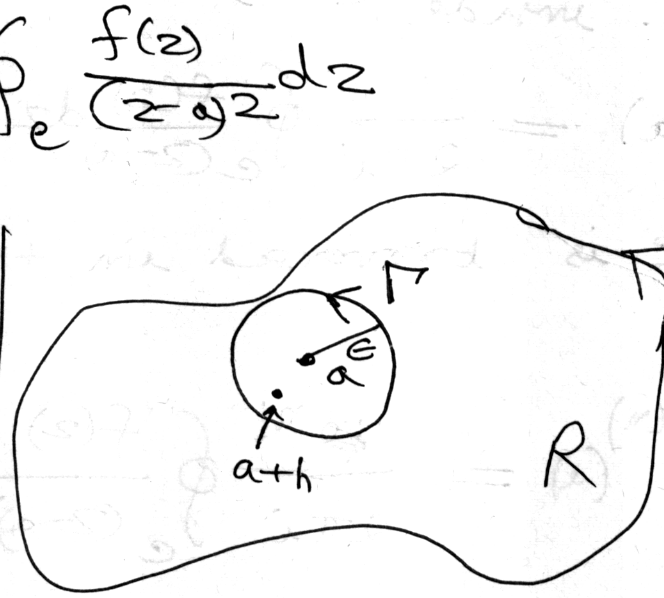
Taking lim $\epsilon \to 0$ ($\because f(z)$ is conts)

$$\boxed{\oint_{c} \frac{f(z)}{(z-a)} dz = 2\pi i f(a)} \quad \text{---}(1)$$

– If $f(z)$ is analytic inside and on the boundary C of a simply-connected region R prove that

$$f'(a) = \frac{1}{2\pi i} \oint_{c} \frac{f(z)}{(z-a)^2} dz$$

☐ From (1) above if a & $a+h$ lie in R, we have



$$\frac{f(a+h)-f(a)}{h} = f'(a)$$

$$= \frac{1}{2\pi i} \oint_{c} \frac{1}{h} \left\{ \frac{1}{z-(a+h)} - \frac{1}{(z-a)} \right\} f(z)\, dz$$

$$\Rightarrow f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)}$$

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^2} + \frac{h}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)^2}$$

Now if we show that as $h \to 0$

$$\frac{h}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)^2} \to 0$$ we are done.

~~Now~~ If Γ is a circle of radius ϵ & center a which lies entirely in R then

$$\frac{h}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)^2} = \frac{h}{2\pi i}\oint_\Gamma \frac{f(z)\,dz}{(z-a-h)(z-a)^2}$$

Choosing h so small in absolute value so that $a+h$ lies in Γ & $|h| < \frac{\epsilon}{2}$

then $|z-a-h| \geq |z-a| - |h| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$

Also since $f(z)$ is analytic in R we can find a no. M s.t. $|f(z)| < M$ (Why?)

& length of Γ is $2\pi\epsilon$ we have

$$\left|\frac{h}{2\pi i}\oint_\Gamma \frac{f(z)\,dz}{(z-a-h)(z-a)^2}\right| \leq \frac{|h|\,M(2\pi\epsilon)}{2\pi(\epsilon/2)(\epsilon^2)}$$

$$= \frac{2|h|\cdot M}{\epsilon^2}$$

$$\to 0$$

as $h \to 0$

In general, we will have

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

$n = 0, 1, 2, 3, \ldots$

Exercise: prove it for $n = 2$!

— Note that

$$\frac{d}{da} f(a) = \frac{d}{da}\left\{ \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)} dz \right\}$$

$$= \frac{1}{2\pi i} \oint_C \frac{\partial}{\partial a}\left\{ \frac{f(z)}{z-a} \right\} dz$$

etc.

$\boxed{\text{Example}}$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = ?$$

$C: |z| = 3$

□ Note $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

We can apply Cauchy Integral formulae

$\because$ for $a = 2$ & $a = 1$

$$\oint_C \frac{f(z)\,dz}{(z-2)} = 2\pi i \left\{ \sin \pi (2)^2 + \cos \pi (2)^2 \right\} = 2\pi i$$

Similarly $\oint_C \dfrac{f(z)}{z-1} dz = -2\pi i$

$\because$ $z = 1$ & 2 are inside C and

$f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic inside C.

$$\Rightarrow \oint_c \frac{f(z)}{(z-1)(z-2)}\,dz = 2\pi i - (-2\pi i) = 4\pi i \quad ▨$$

Example

$$\oint_c \frac{e^{2z}}{(z+1)^4}\,dz = ?$$

$C: |z| = 3$

□ $f(z) = e^{2z}$ & $a = -1$ & $n = 3$

$f'''(z) = 8e^{2z}$ & $f'''(-1) = 8e^{-2}$

$$\Rightarrow \quad 8e^{-2} = \frac{3!}{2\pi i} \oint_c \frac{e^{2z}}{(z+1)^4}\,dz$$

$$\therefore \text{Ans.} = \frac{8\pi i e^{-2}}{3} \quad ▨$$

— Cauchy's Integrals formula also holds for multiply-connected regions.