SC 216 : Calculus with Complex Variables

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THEOREMS ON LIMITS

Theorem: Suppose that $f(z) = u(x,y) + i\,v(x,y)$, $z_0 = x_0 + i y_0$ and $w_0 = u_0 + i v_0$ then

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{— (1)} \qquad \text{iff} \qquad \left.\begin{array}{l} \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \text{ and} \\ \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 \end{array}\right\} (2)$$

□ ($\Leftarrow$) Suppose (2) holds

$\Rightarrow \forall \epsilon > 0$, $\exists\ \delta_1$ & δ_2 s.t.

$$|u - u_0| < \frac{\epsilon}{2} \quad \text{whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1 \quad \text{---(3)}$$

and

$$|v - v_0| < \frac{\epsilon}{2} \quad \text{whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2 \quad \text{---(4)}$$

Let $\delta = \min(\delta_1, \delta_2)$

Now since $|(u+iv) - (u_0 + i v_0)| = |(u-u_0) + i(v - v_0)|$

$$\le |u-u_0| + |v - v_0|$$

and

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} = |(x-x_0) + i(y-y_0)|$$
$$= |(x+iy) - (x_0 + i y_0)|$$

$\Rightarrow$ Using (3) & (4) we get

$$|(u+iv) - (u_0 + i v_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever $0 < |(x+iy) - (x_0 + i y_0)| < \delta$. Thus limit (1) holds.

Conversely, suppose (1) holds.

$\Rightarrow \forall \epsilon > 0$, $\exists\ \delta$ s.t.

$$|(u+iv) - (u_0 + i v_0)| < \epsilon \quad \text{---(5)}$$

$$|(x+iy) - (x_0 + i y_0)| < \delta \quad \text{---(6)}$$

whenever $0 < |(x+iy) - (x_0 + i y_0)|$

But

$$|u-u_0| \leq |(u-u_0)+i(v-v_0)| = |(u+iv)-(u_0+iv_0)|$$

$$\& \; |v-v_0| \leq |(u-u_0)+i(v-v_0)| = |(u+iv)-(u_0+iv_0)|$$

(Why?)

and

$$|(x+iy)-(x_0+iy_0)| = |(x-x_0)+i(y-y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}$$

$\Rightarrow$ Using (5) & (6)

$$|u-u_0|<\epsilon \;\&\; |v-v_0|<\epsilon \quad \text{whenever}$$

$$0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$$

$\boxed{\text{Theorem}}$ ~~Suppose~~ If z_0 & w_0 are points in the z-plane & w-plane then

① $\lim_{z\to z_0} f(z)=\infty$ iff $\lim_{z\to z_0} \frac{1}{f(z)}=0$

② $\lim_{z\to\infty} f(z)=w_0$ iff $\lim_{z\to 0} f\left(\frac{1}{z}\right)=w_0$

③ $\lim_{z\to\infty} f(z)=\infty$ iff $\lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)}=0$

□ T.P. ① assuming L.H.S. limit ~~ex~~ $\Rightarrow$ ~~gt~~ by def.

$\forall \epsilon>0, \exists \delta$ s.t. $|f(z)|>\frac{1}{\epsilon}$ whenever $0<|z-z_0|<\delta$ ------ ④

i.e. the pt. $w=f(z)$ lies in the ϵ-nbd. $|w|>\frac{1}{\epsilon}$ of ∞ whenever z lies in the deleted nbd. $0<|z-z_0|<\delta$ of z_0. Now (4) can be ~~write~~ written as

$$\left|\frac{1}{f(z)}-0\right|<\epsilon \quad \text{whenever } 0<|z-z_0|<\delta$$

which is R.H.S of ①.

Exercise: prove other two.

CONTINUITY

$f : \mathbb{C} \to \mathbb{C}$ is cont.s at $z_0 \in \mathbb{C}$ if

① $\lim_{z \to z_0} f(z)$ exists

② $f(z_0)$ exists

③ $\lim_{z \to z_0} f(z) = f(z_0)$

In other words $\forall \epsilon > 0$, $\exists \delta$ s.t.

$|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$

— A fn. $f : \mathbb{C} \to \mathbb{C}$ is called cont.s in a region R if it is cont.s at each pt. in R.

Theorem If a fn. $f(z)$ is cont.s and nonzero at a pt. z_0, then $f(z) \neq 0$ throughout some nbd. of that point.

Theorem The fn. $f(z) = u(x,y) + i\,v(x,y)$ is cont.s at a pt. $z_0 = (x_0, y_0)$ iff the component fns are cont.s there.

Derivatives Let f be a fn. whose domain of def. contains a nbd. of a pt. $z_0 \in \mathbb{C}$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ provided the limit exists.

— f is differentiable at z_0 if $f'(z_0)$ exists.

— $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

— If $\Delta w = f(z+\Delta z) - f(z)$

then $\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$

$\boxed{\text{Example}}$ $f(z) = z^2$.

At any pt. z $\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$

$= \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$.

$\Rightarrow \frac{dw}{dz} = 2z$ or $f'(z) = 2z$ ▨

$\boxed{\text{Example}}$ $f(z) = |z|^2$

$\frac{\Delta w}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{(z+\Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z}$

$\Rightarrow \frac{\Delta w}{\Delta z} = \bar{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}$

Now ~~the~~ if the $\lim \frac{\Delta w}{\Delta z}$ exist then we can find it by letting the pt. $\Delta z = (\Delta x, \Delta y) \longrightarrow (0,0)$ in any manner

So when $\Delta z \to 0$ through the pts $(\Delta x, 0)$ on the real axis.

$\overline{\Delta z} = \overline{\Delta x + i0} = \Delta x - i0 = \Delta x + i0 = \Delta z$

$\Rightarrow \frac{\Delta w}{\Delta z} = \bar{z} + \overline{\Delta z} + z$ So if this limit exists it must be $= \bar{z} + z$

Now when $\Delta z \to 0$ along $(0, \Delta y)$ on the Im axis

$\overline{\Delta z} = \overline{0 + i\Delta y} = -(0 + \Delta i y) = -\Delta z$

$\Rightarrow \frac{\Delta w}{\Delta z} = \bar{z} + \overline{\Delta z} - z$ so if this limit exist it must be $= \bar{z} - z$ Since limits are ! we must have $\bar{z} + z = \bar{z} - z \Rightarrow z = 0$ if $\frac{dw}{dz}$ to exist.

Now one can ~~see~~ that $\frac{dw}{dz}$ does exist at $z=0$

$\left(\because \frac{\Delta w}{\Delta z} = \overline{\Delta z} \quad \text{when } z=0\right)$

In ~~a~~ fact $\frac{dw}{dz}$ exist only at $z=0$. and $f'(0)=0$

$\therefore$ This is an example of a fn. which is differentiable at o and nowhere ~~~~ else in any nbd. of origin.

Note that for $f(z) = |z|^2$

$$u(x,y) = x^2+y^2 \quad \& \quad v(x,y) = 0 \quad \text{--- (1)}$$

$\Rightarrow$ Real and Imaginary components of a fn. of a complex variable can have cont:s partial derivatives of all orders at a pt. and yet the fn. may not be differentiable there.

— The fn. $f(z) = |z|^2$ is cont:s at each pt. in the plane since (1) are cont:s at each pt.

$\Rightarrow$ continuity of a fn. $\not\Rightarrow$ differentiability

— Existence of the derivative of a fn. at a pt. $\Rightarrow$ continuity of the fn. at that pt.

$\square$ let $f'(z_0)$ exist

consider $\lim\limits_{z \to z_0} [f(z) - f(z_0)]$

$$= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \lim_{z\to z_0} (z-z_0)$$

$$= f'(z_0) \cdot 0 = 0$$

$$\Rightarrow \lim_{z\to z_0} f(z) = f(z_0)$$

~~Exam~~

Exercise. Show that $f'(z)$ does not exist at any pt. z when

(a) $f(z) = \bar{z}$ (b) $f(z) = Re(z)$ (c) $f(z) = Im(z)$

Cauchy-Riemann Equations

Theorem. Suppose $f(z) = u(x,y) + i v(x,y)$ and that ~~f(z)~~ $f'(z)$ exists at a pt. $z_0 = x_0 + i y_0$. Then the first order partial derivatives of u & v must exist at (x_0, y_0) and they must satisfy the Cauchy-Riemann eq^n

$$u_x = v_y \quad \& \quad u_y = -v_x \quad \text{there.} \quad \text{---(1)}$$

Also, we can write

$$f'(z_0) = u_x + i v_x \quad \text{---(2)}$$

where partial derivatives are to be evaluated at (x_0, y_0).

□ $z_0 = x_0 + i y_0$, $\Delta z = \Delta x + i \Delta y$

$$\Delta w = f(z_0 + \Delta z) - f(z_0)$$

$$= [u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)] + i[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)]$$

Assume that $f'(z)$ exist

$$\Updownarrow$$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \text{ exist}$$

$$\Rightarrow f'(z_0) = \lim_{(\Delta x,\Delta y)\to(0,0)} Re\left(\frac{\Delta w}{\Delta z}\right) + i \lim_{(\Delta x,\Delta y)\to(0,0)} Im\left(\frac{\Delta w}{\Delta z}\right) \quad \text{---(3)}$$

(3) is valid as $(\Delta x, \Delta y) \to (0,0)$ in any manner

$\Rightarrow$ As $(\Delta x, \Delta y) \to (0,0)$ along $(\Delta x, 0)$ (i.e. $\Delta y = 0$)

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\,\frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

Thus

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \operatorname{Re}\left(\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$

$$= u_x(x_0, y_0)$$

&

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \operatorname{Im}\left(\frac{\Delta w}{\Delta z}\right) = \lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

$$= v_x(x_0, y_0)$$

$$\Rightarrow \boxed{f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)} \quad \text{—— (A)}$$

Now when we let $\Delta z \to 0$ through the pts $(0, \Delta y)$ i.e. when $\Delta x = 0$

we obtain (verify that you will get this)

$$f'(z_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0)$$

$$\boxed{f'(z_0) = -i\left[u_y(x_0, y_0) + i\, v_y(x_0, y_0)\right]} \quad \text{—— (B)}$$

(A) & (B) gives necessary conditions

$$\boxed{\begin{aligned} u_x(x_0, y_0) &= v_y(x_0, y_0) \\ &\& \\ u_y(x_0, y_0) &= -v_x(x_0, y_0) \end{aligned}}$$ C.R. Eqns.

$\boxed{\text{Example}}$ $f(z) = z^2 = (x^2-y^2) + i\,2xy$

is differentiable everywhere & $f'(z) = 2z$

$\because$ $u(x,y) = x^2-y^2$

$v(x,y) = 2xy$

$\Rightarrow u_x = 2x = v_y$ and $u_y = -2y = -v_x$

$\Rightarrow f'(z) = 2x + i\,2y = 2(x+iy) = 2z$

$\boxed{\text{Example}}$ $f(z) = |z|^2 = \underbrace{(x^2+y^2)}_{u(x,y)} + i\,\underbrace{0}_{v(x,y)}$

If C.R. eq^{ns} are to be hold at (x,y)

$\Rightarrow$ $2x = 0$ & $2y = 0 \Rightarrow x = y = 0$

$\Rightarrow$ $f'(z)$ does not exist at any nonzero pt. (we still need to show that $f'(0)$ exists)

$\boxed{\text{THEOREM}}$ $f(z) = u(x,y) + i\,v(x,y)$ defined throughout some ϵ nbd. of a pt. $z_0 = x_0 + iy_0$ & suppose that u_x & v_y exist ~~ev~~ everywhere in that nbd. If u_x & v_y are conts at (x_0,y_0) and satisfy C-R. eq^n

$$u_x = v_y, \quad u_y = -v_x \quad \text{at}$$

(x_0, y_0) then $f'(z_0)$ exists.

Example $f(z) = |z|^2$

Now using above result we can say that $|z|^2$ has derivative at $z=0$ and can not have derivative at any $z \neq 0$ since the C.R. eq^ns are not satisfied at such pts. ▨

Example $f(z) = e^z = e^x e^{iy}$

$$= e^x \cos y + i e^x \sin y$$

$$\Rightarrow u(x,y) = e^x \cos y$$

$$v(x,y) = e^x \sin y$$

$\because$ $u_x = v_y$ & $u_y = -v_x$ everywhere

& these are cont.s everywhere

$\Rightarrow f'(z)$ exist everywhere and

$$f'(z) = u_x + i v_x = e^x \cos y + i e^x \sin y$$

$$f'(z) = f(z)$$ ▨

C.R.E. in polar coordinates

Let $f(z) = u(r,\theta) + i v(r,\theta)$ be defined throughout some ϵ-nbd. of a nonzero pt. $z_0 = r_0 \exp(i\theta_0)$ and suppose that u_r and v_θ exist everywhere in that nbd. & are cont.s at (r_0, θ_0) & satisfy the polar C.R.E.

$$r u_r = v_\theta, \quad u_\theta = -r v_r$$

at (r_0, θ_0) then $f'(z_0)$ exists.

and $f'(z_0) = e^{-i\theta}(u_r + i v_r)$

$\boxed{\text{Example}}$ $f(z) = \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$ $(z \neq 0)$

$\Rightarrow u(r,\theta) = \frac{\cos\theta}{r}$ and $v(r,\theta) = -\frac{\sin\theta}{r}$

$\Rightarrow r u_r = v_\theta$ and $u_\theta = -r v_r$ (verify!)

$\Rightarrow f'(z)$ exist when $z \neq 0$

& $f'(z) = e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right) = -\frac{1}{z^2}$

$\boxed{\text{Exercise!}}$ Show that

$$f(z) = \sqrt[3]{r}\, e^{i\theta/3} \quad (r > 0,\ \alpha < \theta < \alpha + 2\pi)$$

has derivative everywhere in its domain of def.

$\boxed{\text{Analytic fns}}$

A fn. $f(z)$ is said to be analytic at a pt. z_0 if it is differentiable at z_0 and also at every pt. in the nbd. of z_0.

$\boxed{\text{Entire fn.}}$ A fn $f(z)$ is said to be entire if it is analytic at each pt. in the entire finite plane.

— Anylatic fn. at z_0 $\Rightarrow$ differentiable at z_0 $\Rightarrow$ cont.s at z_0

Converse is not true.

$\boxed{\text{Example}}$ $f(z) = \dfrac{z^3+4}{(z^2-3)(z^2+1)}$ is analytic

except at $z = \pm\sqrt{3}$ & $z = \pm i$

Example

$f(z) = \cosh x \cos y + i \sinh x \sin y$

is entire fn.

Hint: C.R. eq^ns holds everywhere

THEOREM If $f'(z) = 0$ everywhere in a domain D then $f(z)$ must be constant throughout D.

THEOREM Suppose $f(z) = u(x,y) + i v(x,y)$ & its conjugate $\overline{f(z)} = u(x,y) - i v(x,y)$ are both analytic in a given domain D then $f(z)$ must be a constant fn. throughout D.

□ Let $\overline{f(z)} = U(x,y) + i V(x,y)$

where $U(x,y) = u(x,y)$

$V(x,y) = -v(x,y)$

$\because f(z)$ is analytic, the C.R. eq^n hold in D $u_x = v_y$ & $u_y = -v_x$ —(1)

and $\because \overline{f(z)}$ is also analytic in D we have

$$U_x = V_y \quad \& \quad U_y = -V_x$$

$$\Rightarrow \quad u_x = -v_y \quad \& \quad u_y = v_x \quad \text{—} (2)$$

$\Rightarrow$ (1) & (2) gives $u_x = 0 \Rightarrow v_x = 0$

$$\Rightarrow \quad f'(z) = u_x + i v_x = 0 + i 0 = 0$$

$\Rightarrow f(z)$ is constant.

Harmonic fn.

$H(x,y)$ is said to be Harmonic in D if it has cont.s partial derivatives of 1st & 2nd order and they satisfy Laplace eqn.

$$H_{xx}(x,y) + H_{yy}(x,y) = 0$$
$$\text{or } \nabla^2 H = 0$$

THEOREM If a fn. $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain D then its component fns. u & v are harmonic in D.

$\Box$ $\because$ $f(z)$ is analytic $\Rightarrow$ ~~they~~ u & v satisfy C.R. eqn

$$\Rightarrow \quad u_x = v_y \quad \& \quad u_y = -v_x$$

differentiating both sides w.r. to x

$$u_{xx} = v_{yx} \quad \& \quad u_{yx} = -v_{xx}$$

& diff. w.r. to y gives

$$u_{xy} = v_{yy} \quad \& \quad u_{yy} = -v_{xy}$$

But $u_{yx} = u_{xy}$ & $v_{yx} = v_{xy}$ (Why?)

it follows that

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0$$

$\Rightarrow$ u & v are harmonic fn.

Harmonic conjugate

So if $f(z) = u + iv$ is analytic in D then u & v are necessarily harmonic in D. Now suppose $u(x,y)$ is a real fn. you found which is harmonic in D. If you can find another fn. $v(x,y)$ (harmonic fn.) so that u & v satisfy the C.R. eqn then v is called the harmonic conjugate of u.

$\Rightarrow$ the fn. $u(x,y) + i v(x,y)$ will be an analytic fn. in D.

So we arrive at

THEOREM A fn. $f(z) = u + iv$ is analytic in D iff v is a harmonic conjugate of u.

Example Suppose

$$u(x,y) = x^2 - y^2$$

$$\& \quad v(x,y) = 2xy$$

$\because$ $Re(z^2) = u$ & $Im(z^2) = v$

we know v is a harmonic conjugate of u throughout the plane

but u can not be a harmonic conjugate of v. $\Rightarrow$ the fn. $2xy + i(x^2 - y^2)$ is not analytic anywhere.

Example Let $u(x,y) = y^3 - 3x^2 y$

find the harmonic conjugate of u.

$\square$ Note that $u(x,y)$ is harmonic fn.

$\because$ $\nabla^2 u(x,y) = 0$

Now a harmonic conjugate $v(x,y)$ is ~~reated~~ related to $u(x,y)$ by C.R. eqn we have

$$u_x = v_y \quad \text{—(1)} \qquad \& \qquad u_y = -v_x \quad \text{—(2)}$$

(1) gives $\quad v_y(x,y) = -6xy$

Integrating w.r. to y (keeping x constant) we get

$$u(x,y) = -3xy^2 + \phi(x)$$ → arb. fn of x

using (2) eqn we get

$$3y^2 - 3x^2 = 3y^2 - \phi'(x)$$

$$\Rightarrow \phi'(x) = 3x^2 \Rightarrow \phi(x) = x^3 + c$$

$c \in \mathbb{R}$

$$\Rightarrow u(x,y) = -3xy^2 + x^3 + c$$

is the required harmonic conjugate.

⇒ The corresponding analytic fn. is

$$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3 + c)$$

$$f(z) = i(z^3 + c)$$

$\boxed{\text{Example}}$ $f(z) = e^{-y}\sin x - i e^{-y}\cos x$ is an entire fn. (Verify!)

$\boxed{\text{Example}}$ $f(z) = \dfrac{i}{z^2}$ $(z \neq 0)$ is analytic

(Verify!)

$\boxed{\text{Exercise}}$ Verify that $u(x,y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane. and find the harmonic conjugate fn. of u.