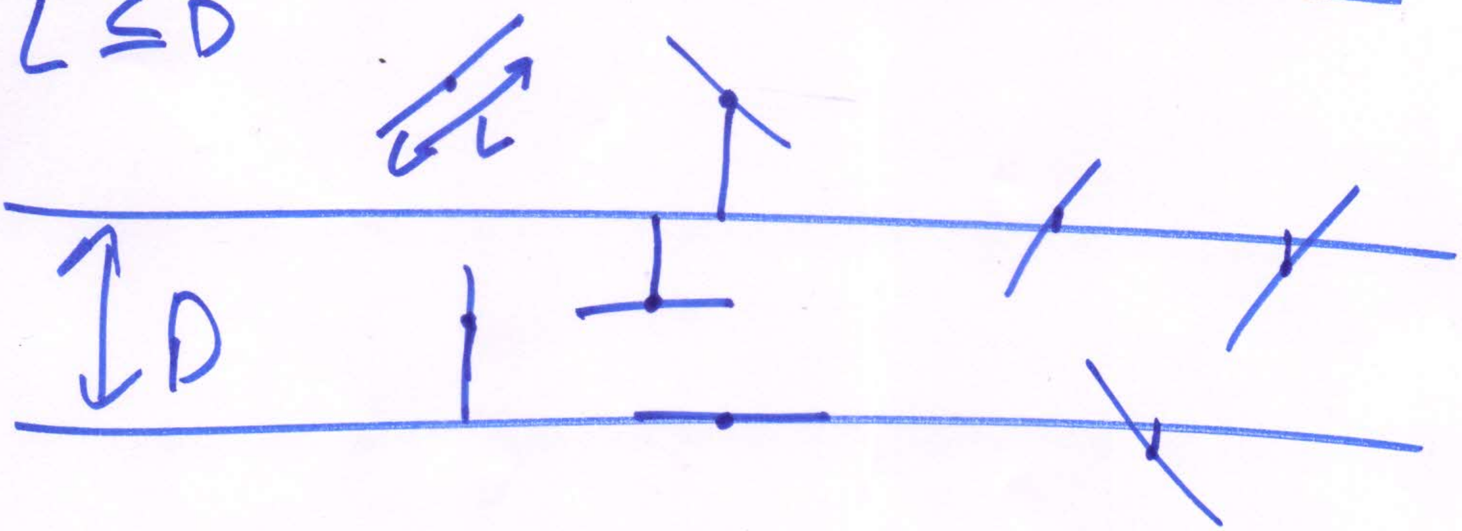
Lecture - 24

P 1

Jointly distributed r.v.

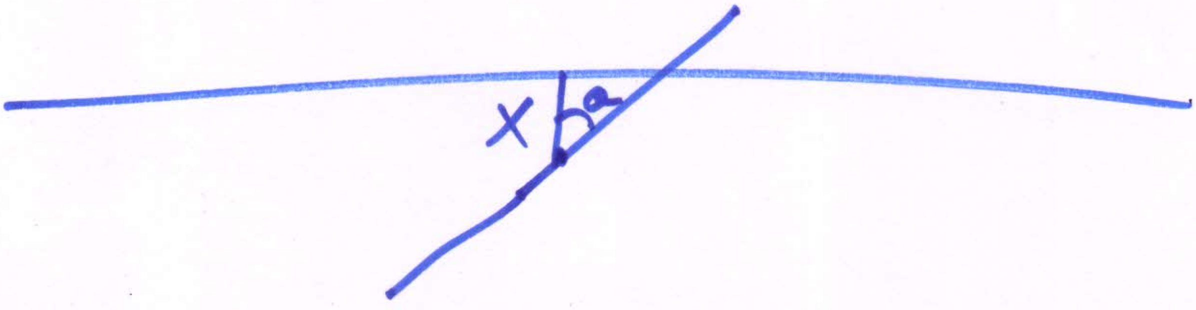
Buffon's Needle Problem

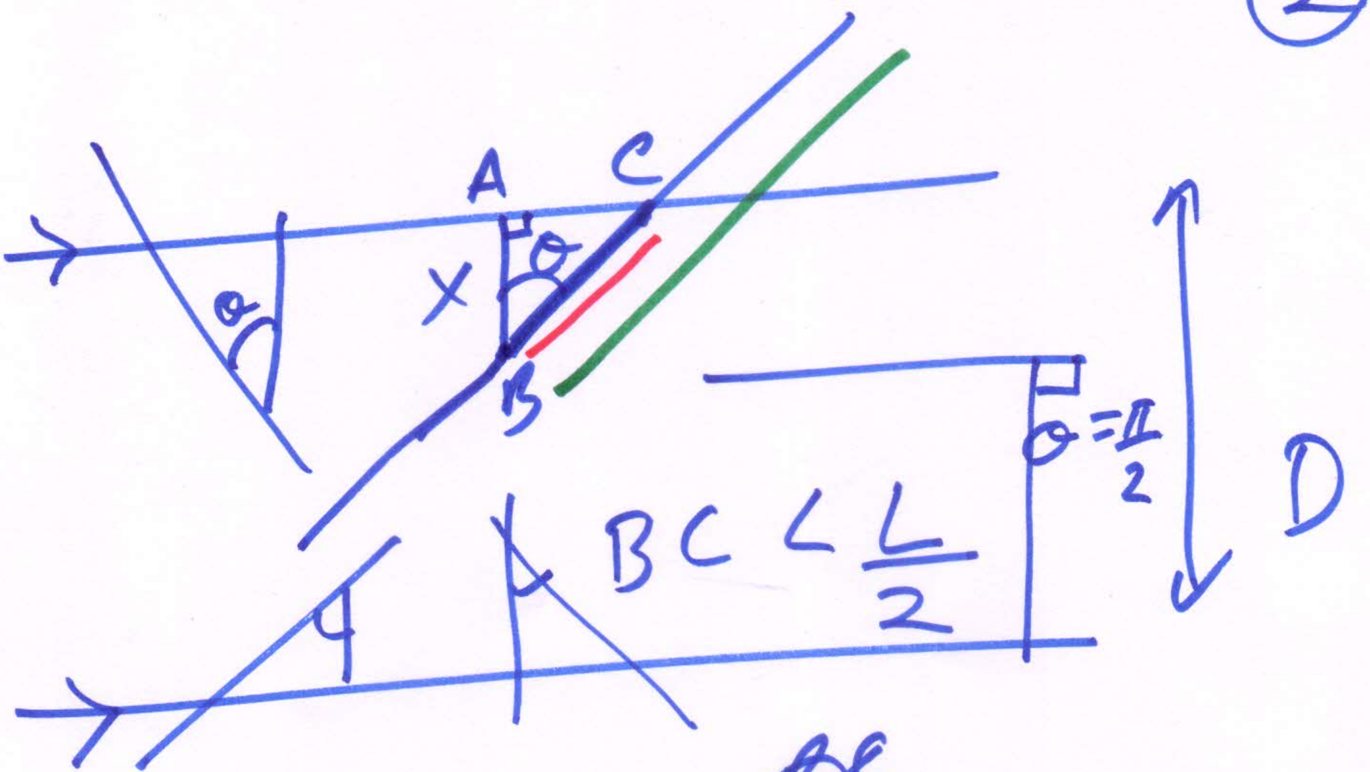
$L \leq D$



X: distance between the center of the needle and the nearer line

θ: angle that the $\perp$ makes with the nearer line





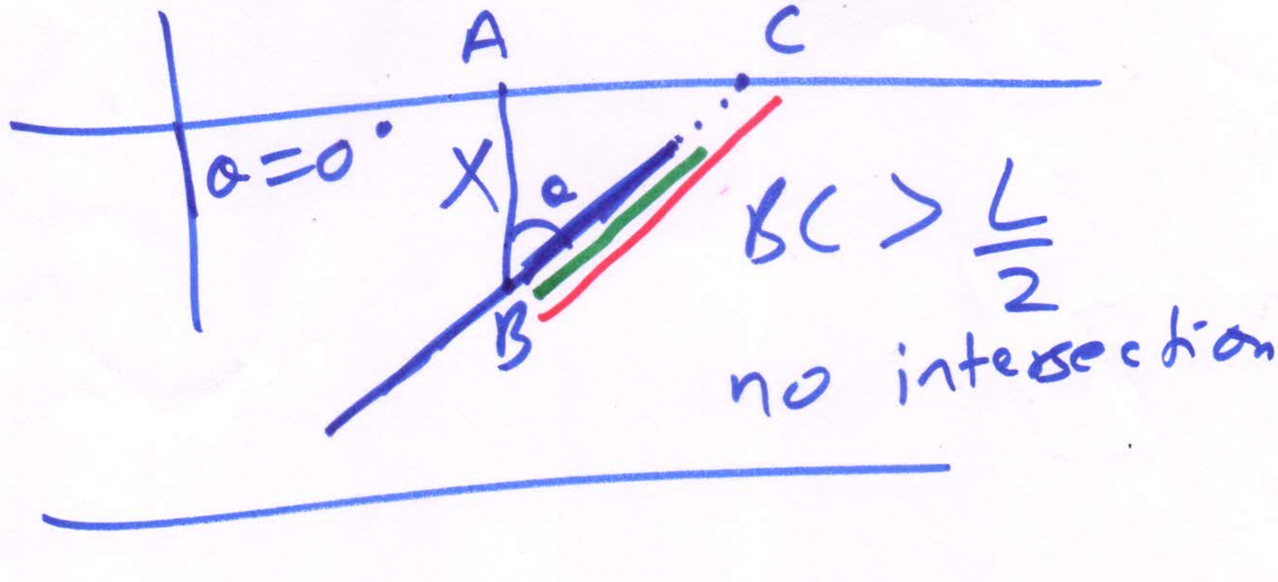
~~BC~~

$X : [0, D/2]$

$\theta : [0, \pi/2]$

$$\frac{X}{BC} = \cos\theta$$

$$BC = \frac{X}{\cos\theta} < \frac{L}{2}$$



$$P\left(\begin{array}{l}\text{needle intersects}\\ \text{a line}\end{array}\right)$$

$$= P\left(\frac{X}{\cos\theta} < \frac{L}{2} \right)$$

$$= P\left(X < \frac{L}{2} \cos\theta \right)$$

independent.

$$= \iint_{X < \frac{L}{2}\cos\theta} \underbrace{f_X(x)\, f_\theta(\theta)}_{f_{X,\theta}(x,\theta)}\, dx\, d\theta$$

$$\frac{2}{D} \cdot \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{\frac{L}{2}\cos\theta} dx\, d\theta$$

$$\frac{4}{\pi D}\int_0^{\frac{\pi}{2}}\int_0^{\frac{L}{2}\cos\theta} dx\, d\theta$$

$$\frac{4}{\pi D}\int_0^{\frac{\pi}{2}} \frac{L}{2}\cos\theta\, d\theta$$

$$\frac{2L}{\pi D}\left(\sin\theta\right)_0^{\pi/2}$$

$$= \frac{2L}{\pi D}$$

e.g.

X, Y, Z : independent

uniform over $(0,1)$

$P(X \geq YZ)$.

$$= \iiint_{X \geq YZ} f_{X,Y,Z}(x,y,z)\, dx\, dy\, dz$$

$$f_{X,Y,Z}(x,y,z) = \underbrace{f_X(x)}_{=1}\, \underbrace{f_Y(y)}_{=1}\, \underbrace{f_Z(z)}_{=1}$$

$$= \int_0^1 \int_0^1 \int_{yz}^1 dx\, dy\, dz = \frac{3}{4}$$

Sums of independent random variables.

e.g. X : $(0,1)$ uniform

Y : $(0,1)$ uniform

$x+y=a$

$(0,2)$ $Z = X+Y$

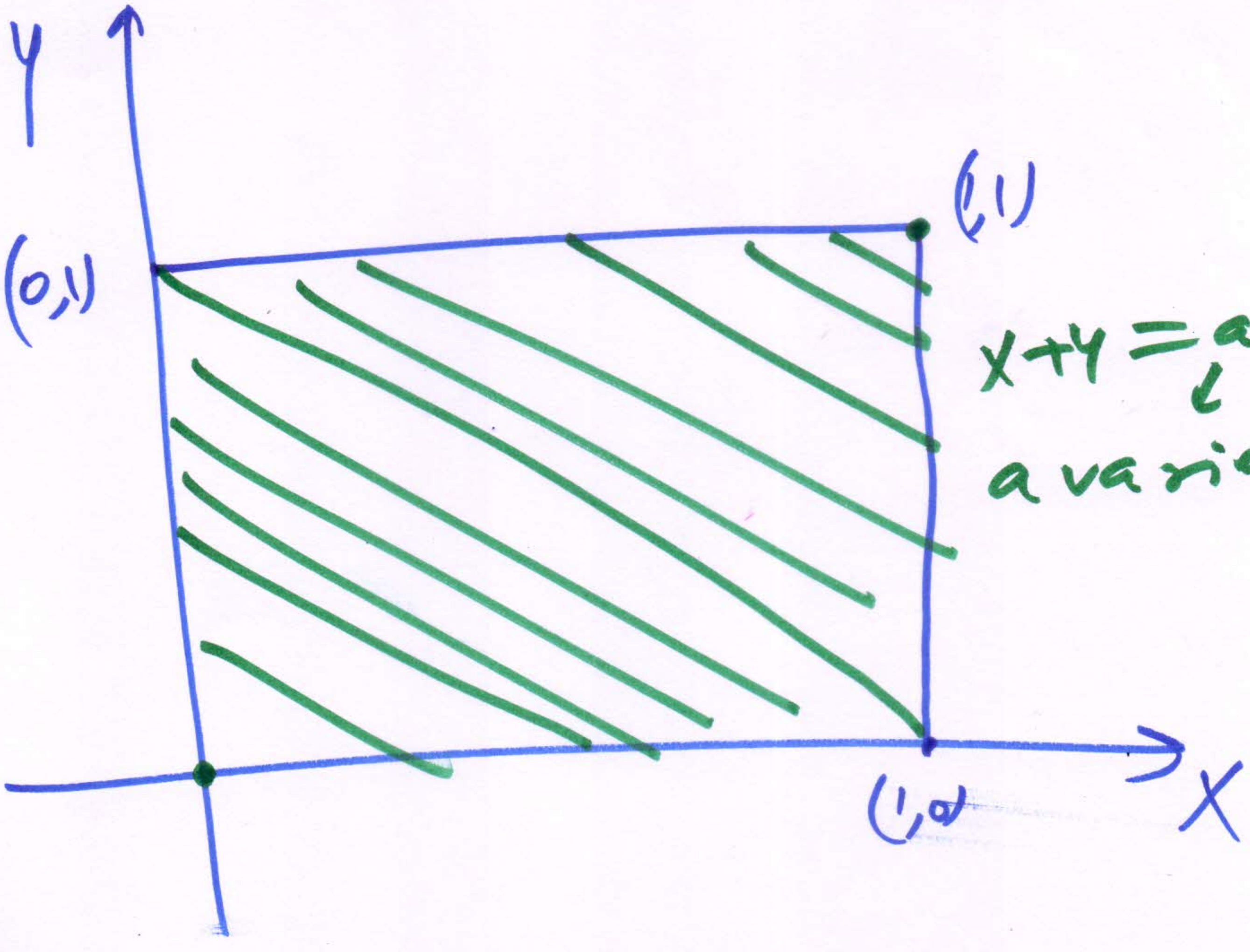
$x+y=0$

$x+y=1$

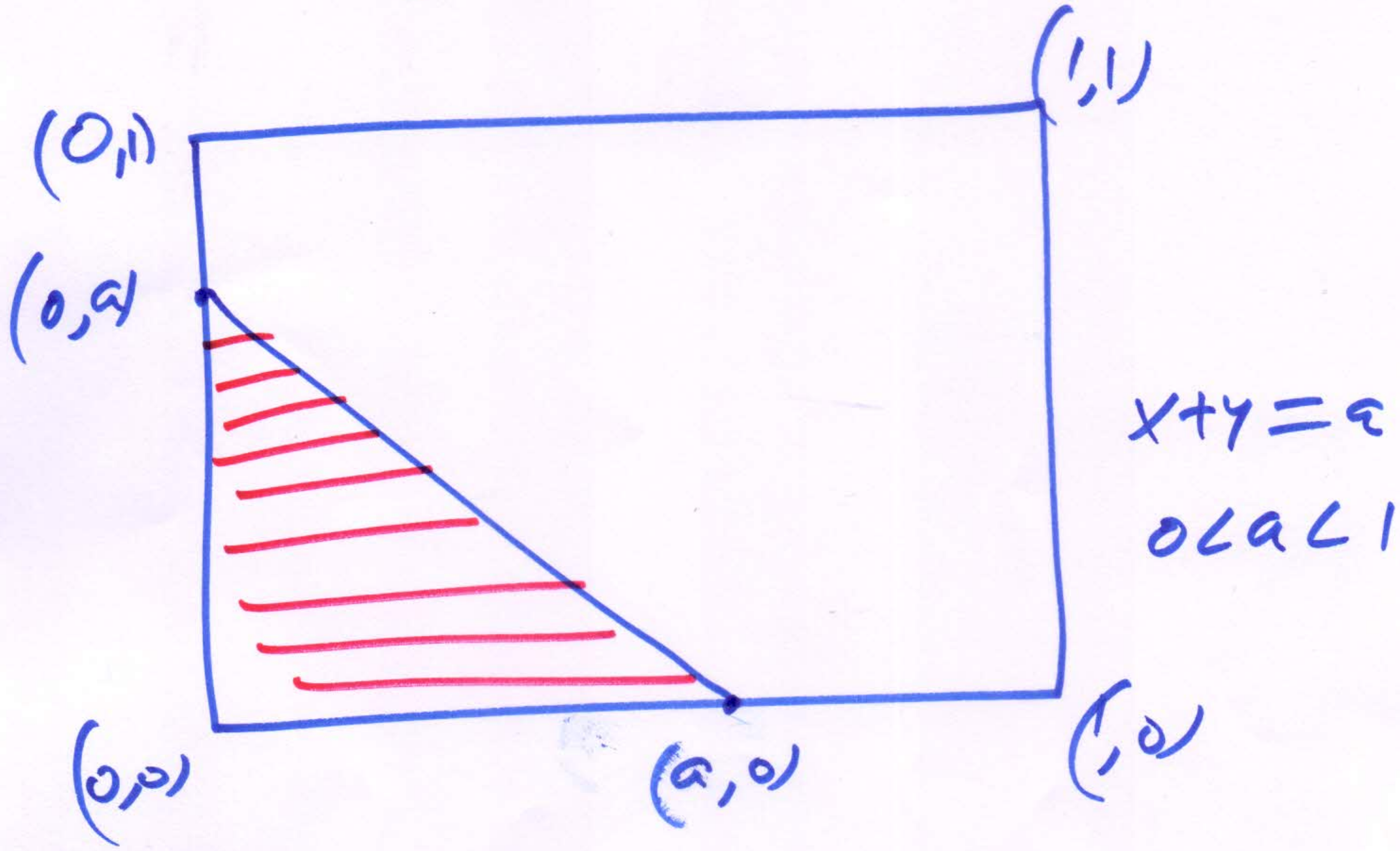
$Z \to$ density?

$P(Z \le a)$ $x+y=2$

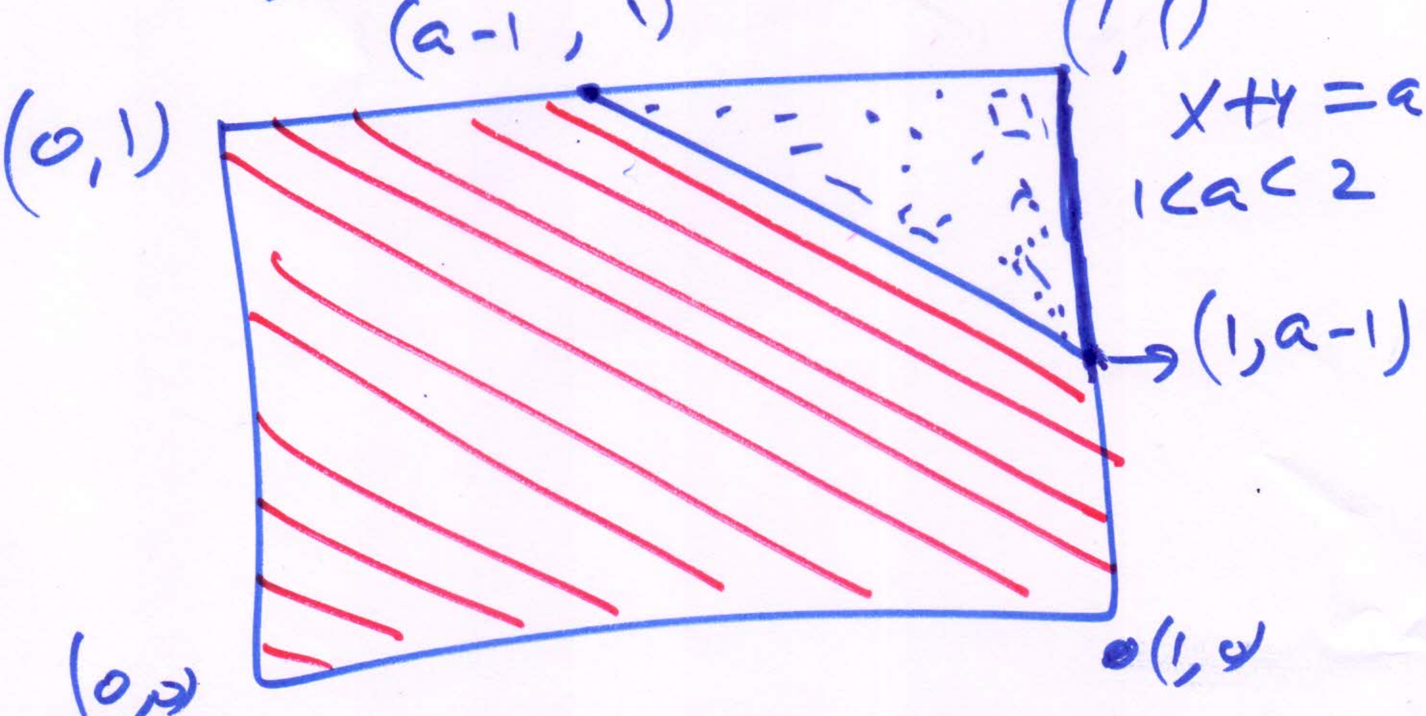
$P(X+Y \le a)$



$$P(\underset{0<a<1}{X+Y \leq a}) = \frac{a^2}{2}$$



$$\frac{1}{2} \cdot a \cdot a = \frac{a^2}{2}$$



$P(X+Y \leq a)$

$= 1 - \text{area of } \Delta$

$= 1 - \frac{1}{2} \cdot (2-a)^2$

$= 1 - \frac{1}{2}(4 + a^2 - 4a)$

$= 1 - 2 - \frac{a^2}{2} + 2a$

$= 2a - \frac{a^2}{2} - 1$

$= P(X+Y \leq a), \quad 1 < a < 2$

$0 < a < 1$

$$f_Z(a) = \frac{d}{da}\left(\frac{a^2}{2}\right)$$

$$= a$$

$1 < a < 2$

$$f_Z(a) = \frac{d}{da}\left(2a - \frac{a^2}{2} - 1\right)$$

$$= 2 - a$$

Triangular distribution