Lecture 30

Recap:

The Probabilistic Method

Covariance

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

independence $\Rightarrow Cov(X,Y) = 0$

$\not\Leftarrow$

Variance of Binomial r.v.

Correlation

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\ Var(Y)}}$$

$$= \frac{Cov(X,Y)}{\sigma_X\ \sigma_Y}$$

if X & Y are independent ②

$\rho(X,Y)=0.$

$$\rho(X,Y)=\frac{Cov(X,Y)}{\sigma_X \quad \sigma_Y}$$

$\downarrow$

it is defined when

$\sigma_X \neq 0, \; \sigma_Y \neq 0$

$Var(X) \neq 0$ AND $Var(Y) \neq 0$

when is $Var(X) = 0$?

$\parallel$ X is a constant.

D: $\sum p_i \underbrace{(x_i - \mu)}^2 = 0$

C: $\int \cancel{p} f(x)(x-\mu)^2 dx$

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$\rho(X,Y)$ is defined if X is NOT a constant

AND

Y is NOT a constant

$$-1 \leq \rho(X,Y) \leq +1$$

$$Z = \frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}$$

$$Var(Z) \geq 0$$

$$Var\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) =$$

$$Var\left(\sum X_i\right) = \sum Var(X_i) + 2\sum_{i<j}\sum Cov(X_i, X_j)$$

$$Var\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right)$$

$$= Var\left(\frac{X}{\sigma_X}\right) + Var\left(\frac{Y}{\sigma_Y}\right)$$

$$+ 2\,Cov\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y}\right)$$

$$= Var\left(\frac{1}{\sigma_X} \cdot X\right) + Var\left(\frac{1}{\sigma_Y} \cdot Y\right)$$

$$+ 2\,Cov\left(\frac{1}{\sigma_X} \cdot X, \frac{1}{\sigma_Y} \cdot Y\right)$$

$$= \frac{1}{\sigma_X^2} Var(X) + \frac{1}{\sigma_Y^2} Var(Y) +$$

$$+ 2 \frac{1}{\sigma_X} \cdot \frac{1}{\sigma_Y} Cov(X, Y)$$

$$1 + 1 + 2\rho(X, Y) \geq 0$$

$$2\rho(X, Y) \geq -2$$

$$\rho(X, Y) \geq -1$$

$$\rho(X, Y) \leq 1$$

$$W = \frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}$$

$$Var(W) \geq 0$$

H.W.

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What happens when $\rho = -1$?

When is $\rho = -1$?

when $Var(Z) = 0$

$$Var\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) = 0$$

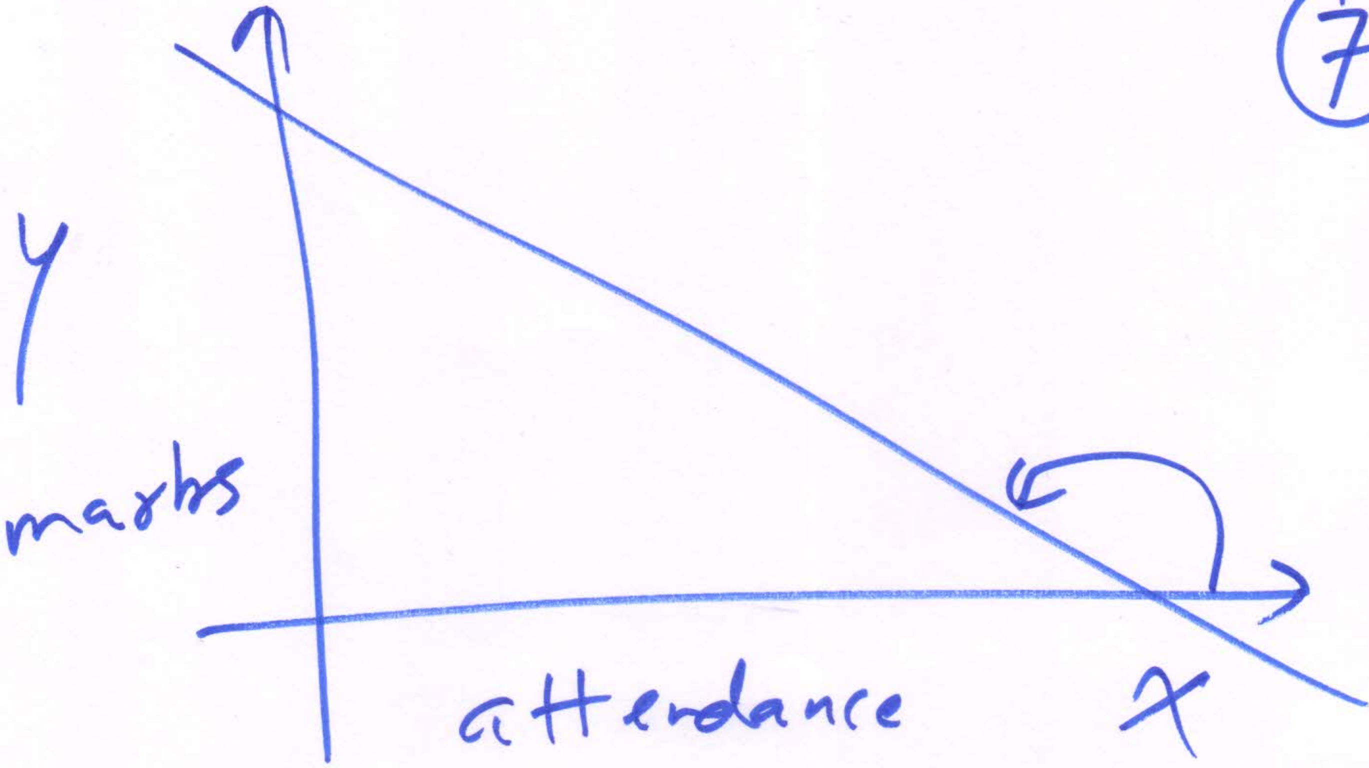
$$\Updownarrow$$

$$\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y} = c$$

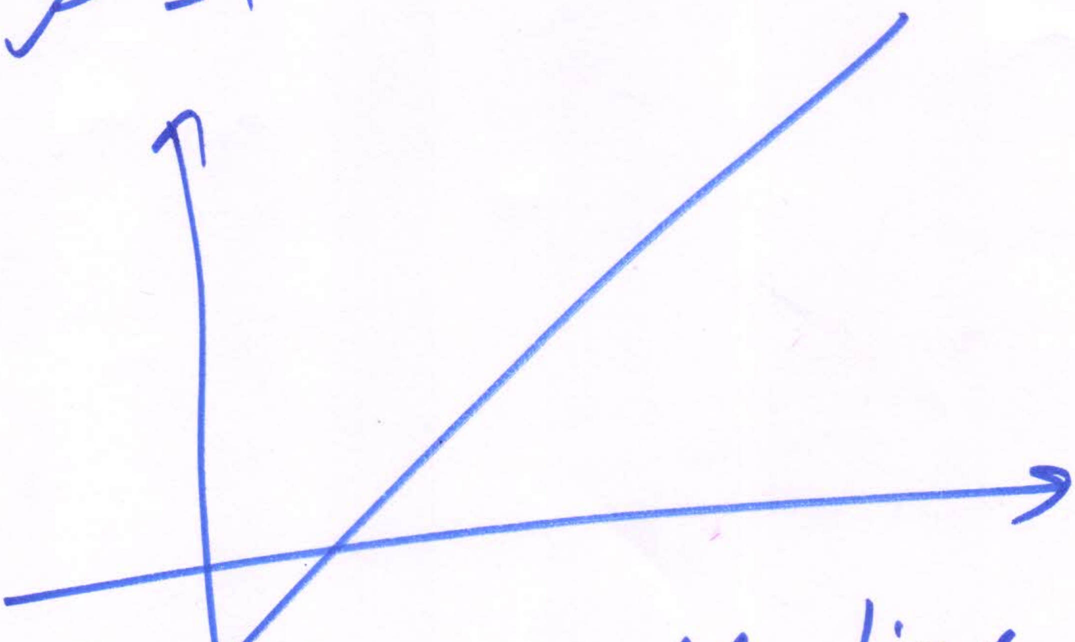
$$\frac{Y}{\sigma_Y} = c - \frac{X}{\sigma_X}$$

$$Y = c \cdot \sigma_Y - \frac{\sigma_Y}{\sigma_X} \cdot X$$

Y is linear in X. Slope is negative.



$\rho = 1$



straight line,
with +ive slope.

if $\rho = 0$, we say that X & Y are uncorrelated.

Properties of Conditional Expectation.

$$E[E[X|Y]] = E[X]$$

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e.g. $f(x,y) = \dfrac{e^{-x/y}\, e^{-y}}{y}$

$0 < x < \infty$

$0 < y < \infty$

Compute

$E[X | Y = y] = ?$

Continuous case

$$E[X|Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) \, dx$$

?

Discrete case

$$E[X|Y=y] = \sum_{x} x \, p_{X|Y}(x|y)$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$ $\rightarrow$ given

?

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx$$

$f_{X|Y}(x|y)$

$$= \frac{1}{y} e^{-x/y}$$

exponential

$$E[X|Y=y] = y$$

$$E[X] = E[E[X|Y]]$$

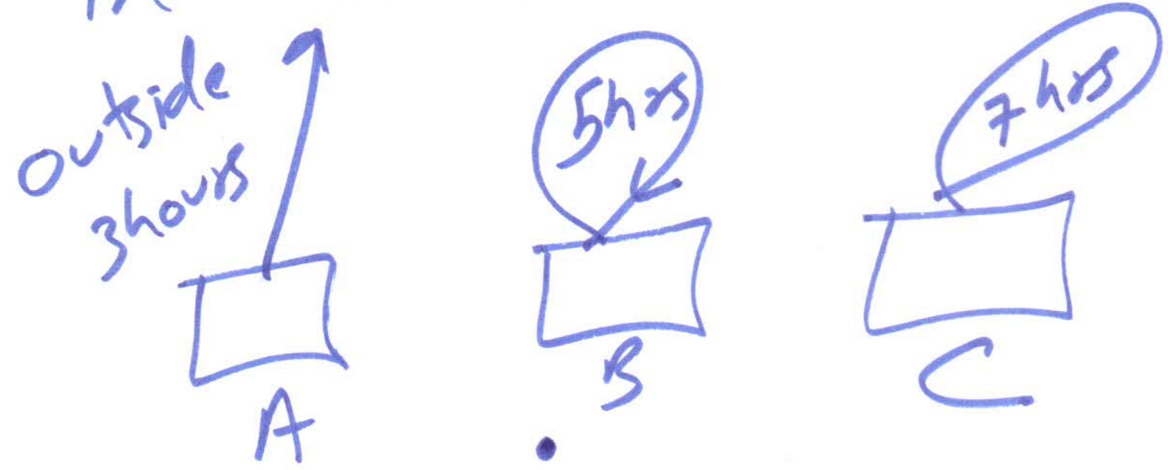
$$D: E[X] = \sum_y \underbrace{E[X|Y=y]} P(Y=y)$$

$$C: E[X] = \int_{-\infty}^{\infty} \underbrace{E[X|Y=y]} f_Y(y)\, dy$$

(considering $E[X|Y]$ as a random variable, which is a function of Y.

e.g.

A miner is trapped in a coal mine.



X = no. of hours before he gets out

$$Y = \{\underset{A}{1}, \underset{B}{2}, \underset{C}{3}\}$$

$$E[X] = E[E[X|Y]]$$

$$= E[X|Y=1]P(Y=1) + E[X|Y=2]P(Y=2) + E[X|Y=3]P(Y=3)$$

$P(Y=1) = P(Y=2) = P(Y=3) = \frac{1}{3}$ 12

$E[X \mid Y=1] = 3$

$E[X \mid Y=2] = 5 + E[X]$

$E[X \mid Y=3] = 7 + E[X]$