CCV-18

SC 216 Calculus with Complex Variables

April 9, 2008

POWER SERIES SOLUTIONS AND SPECIAL FUNCTIONS

FUNCTIONS → ELEMENTARY FUNCTIONS (Algebraic, trignometric, inverse trig., exponential, logarithmic or a combination of these)

FUNCTIONS → TRANSCENDENTAL FUNCTIONS (They are solutions of D.E. other than elementary fns)

Algebraic fn.

→ A fn. which satisfies an eqn of the form

$$P_n(x)y^n + P_{n-1}y^{n-1} + \dots + P_1(x)y + P_0(x) = 0$$

where $P_i(x)$ is a polynomial.

Example

$$y'' + y = 0$$

Soln: $y = \sin x$ & $y = \cos x$ (Elementary non algebraic fns.)

Example

$$xy'' + y' + xy = 0$$

We can not solve this eqn in terms of elementary fns.

We look for a soln. in terms of a power series.

Recall

An ∞ series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + \dots$ ---(1)

is called a power series in x.

— Series $\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$ ---(2)

power series in $(x-x_0)$

— Series (1) is convergent if

$\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$ exist.

— (1) is always convergent for $x = 0$.

— $$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots$$

If $\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = L$ (exist)

then from ratio test series is convergent if $L < 1$, diverges if $L > 1$

— For power series $\sum_{n=0}^{\infty} a_n x^n$, if each $a_n \neq 0$ & for a fixed pt. $x \neq 0$

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |x| = L$$

then by ratio test series will be conv. if $L < 1$ & div if $L > 1$

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ is conv. if $|x| < R$
div. if $|x| > R$

where $R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$

(radius of convergence)

Example

$\sum_{n=0}^{\infty} n!\, x^n$ is divergent $\forall\, x \neq 0$ $R = 0$

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ conv. $\forall\, x$ $R = \infty$

$\sum_{n=0}^{\infty} x^n$ is conv. for $|x| < 1$ $\Rightarrow R = 1$

SERIES SOLUTIONS OF FIRST ORDER EQUATIONS

Consider $y' = y$ ---- (1)

Suppose $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$ --- (2)

$|x| < R$, $R > 0$

is a soln. of (1) (assume it is analytic at 0)

$\Rightarrow$ we can differentiate term by term

$\Rightarrow y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + (n+1)a_{n+1} x^n + \dots$ --- (3)

$\because y' = y$ $\Rightarrow$ comparing cofficients gives us

$a_1 = a_0$ $\quad 3a_3 = a_2$

$2a_2 = a_1$ $\quad \vdots$

$(n+1) a_{n+1} = a_n$ ... & so on

$\Rightarrow$ we can express a_n in terms of a_0

$\Rightarrow a_1 = a_0, \; a_2 = \frac{a_1}{2} = \frac{a_0}{2}$

$a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3}, \dots \quad a_n = \frac{a_0}{n!}, \dots$

$\Rightarrow y = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)$ --- (4)

with no conditions on a_0

$\because$ series in (4) conv. $\forall x$ and we know

$y = a_0 e^x$

$\Rightarrow$ this is the required soln.

Example $(1+x) y' = py, \; y(0) = 1$ --- (5)

Show that $y = (1+x)^p$, p is arbitary constt. is a P.S. of (5) using power series method.

(Verify yourself !)

SECOND ORDER LINEAR EQUATIONS

$$y'' + P(x)\,y' + Q(x)\,y = 0 \quad \text{—— (1)}$$

(Homo.)

— When P & Q are const. this eqn can be solved in terms of elementary fns. In most general case this can only be solved by means of power series.

— Behavior of the soln. of (1) near a pt. x_0 depends on the behavior of its coff. fns. $P(x)$ & $Q(x)$ near this pt. x_0.

— Suppose $P(x)$ & $Q(x)$ are "well behaved" in the sense of being analytic at x_0 this means that each has a power series expansion valid in some neighborhood of this point. We call then x_0 as an ordinary pt. of eqn (1). $\Rightarrow$ Every soln. of (1) is also analytic at this pt. A point which is not an ordinary pt. ~~a~~ of (1) is called a singular point.

Example

$$y'' + y = 0 \quad \text{——— (2)}$$

Here $P(x) = 0$ & $Q(x) = 1$ are analytics at all ~~pts~~.

$\Rightarrow$ P & Q are analytics

We seek a soln. of the form

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

$$\Rightarrow \quad y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + (n+1)a_{n+1}x^n + \dots$$

and

$$y'' = 2a_2 + 2\cdot 3 a_3 x + 3\cdot 4 a_4 x^2 + \dots + (n+1)(n+2)a_{n+2}x^n + \dots$$

$\Rightarrow$ (2) gives

$$(2a_2 + a_0) + (2\cdot 3a_3 + a_1)x + (3\cdot 4a_4 + a_2)x^2 + (4\cdot 5a_5 + a_3)x^3 + \dots + [(n+1)(n+2)a_{n+2} + a_n]x^n + \dots = 0$$

$\Rightarrow$ Equating the ~~co~~ coefficients of successive powers of x gives

$2a_2 + a_0 = 0 \qquad 3\cdot 4\cdot a_4 + a_2 = 0$

$2\cdot 3 a_3 + a_1 = 0 \qquad 4\cdot 5\, a_5 + a_3 = 0$

$\cdots$

$$(n+1)(n+2)\, a_{n+2} + a_n = 0$$

— Express a_n in terms of a_0 or a_1 depending on n is even or odd

we get

$$a_2 = -\frac{a_0}{2}, \quad a_3 = -\frac{a_1}{2\cdot 3}, \quad a_4 = -\frac{a_2}{3\cdot 4} = \frac{a_0}{2\cdot 3\cdot 4}$$

$$a_5 = -\frac{a_3}{4\cdot 5} = \frac{a_1}{2\cdot 3\cdot 4\cdot 5}, \cdots$$

$\Rightarrow$

$$y = a_0 + a_1 x - \frac{a_0}{2}x^2 - \frac{a_1}{2\cdot 3}x^3 + \frac{a_0}{2\cdot 3\cdot 4}x^4 + \frac{a_1}{2\cdot 3\cdot 4\cdot 5}x^5 \cdots$$

$$= a_0 \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}_{y_1(x)\ (\text{say})} + a_1 \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}_{y_2(x)\ (\text{say})}$$

— Note by choosing $a_0 = 1$ & $a_1 = 0$ y_1 satisfies eq^n (2) & by choosing $a_0 = 0$ & $a_1 = 1$

We see y_2 also satisfies the eq^n (2)

— Both the series for y_1 & y_2 conv. $\forall x$

In fact one can see that $y_1 = \cos x$ & $y_2 = \sin x$

$$\Rightarrow \quad y = a_0 \cos x + a_1 \sin x$$

for arbitrary constt. a_0 & a_1

$\blacksquare$

$\boxed{\text{Example}}$ Legendre's Equation

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0 \quad \text{——— (1)}$$

p is a constt.

□ ~~Coff~~ coefficients fns. $P(x) = -\frac{2x}{1-x^2}$ & $Q(x) = \frac{p(p+1)}{1-x^2}$

are analytic at the origin. $\Rightarrow$ $x=0$ is an ordinary pt. & we expect a solution of the form

$$y = \sum a_n x^n \Rightarrow y' = \sum (n+1) a_{n+1} x^n$$

$$\& \quad y'' = \sum (n+1)(n+2) a_{n+2} x^n$$

$$\Rightarrow \quad -x^2 y'' = \sum -(n-1) n a_n x^n$$

$$\Rightarrow \quad -2xy' = \sum -2n a_n x^n$$

$$\Rightarrow \quad p(p+1)y = \sum p(p+1) a_n x^n$$

Putting all this in eqn (1) & so the coefficient of x^n must be zero $\forall n$:

$$(n+1)(n+2)a_{n+2} - (n-1)n a_n - 2n a_n + p(p+1)a_n = 0$$

With some manipulation this becomes

$$a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+1)(n+2)} a_n \quad \text{--- (2)}$$

$\Rightarrow$ we can express a_n in terms of a_0 or a_1 according as n is even or odd.

$$a_2 = -\frac{p(p+1)}{1.2} a_0$$

$$a_3 = -\frac{(p-1)(p+2)}{2\cdot 3} a_1$$

$$a_4 = -\frac{(p-2)(p+3)}{3\cdot 4} a_2 = \frac{p(p-1)(p+1)(p+3)}{4!} a_0$$

$$a_5 = \frac{(p-3)(p+4)}{4\cdot 5} a_3 = \frac{(p-1)(p-3)(p+2)(p+4)}{5!} a_1$$

& so on.

$$\Rightarrow y = a_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p-2)(p+1)(p+3)}{4!}x^4 - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!}x^6 + \cdots\right]$$

$$+ a_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5 - \frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!}x^7 + \cdots\right]$$

as our formal solution. ---(3)

- When p is not an integer each series in brackets has radius of convergence $R=1$. We can prove this using recursion formula (2).

Replace n by $2n$ gives

$$\left|\frac{a_{2n+2}\,x^{2n+2}}{a_{2n}\,x^{2n}}\right| = \left|\frac{-(p-2n)(p+2n+1)}{(2n+1)(2n+2)}\right| |x|^2$$

$$\to |x|^2 \text{ as } n \to \infty$$

Similarly for second series $\Rightarrow R=1$

- One can also see that both the series are l.i.

Thus for arbitory const. a_0 & a_1 (3) is a general soln. for $|x|<1$.

$\boxed{\text{THEOREM}}$ Let x_0 be an ordinary point of the D.E.

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{----(4)}$$

and let a_0 & a_1 be arbitrary constants. Then $\exists$ a $!$ fn. $y(x)$ that is analytic at x_0, is a soln. of (4) in a certain nbd. of this pt. and satisfying the initial conditions $y(x_0) = a_0$ & $y'(x_0) = a_1$. Further, if the power series expansion of $P(x)$ & $Q(x)$ are valid on an interval $|x - x_0| < R$, $R > 0$. then the power series soln. of y is also valid on the same interval.

☐ Try yourself.

Hints:

① Take $x_0 = 0$ (W.L.O.G.)

② $P(x) = \sum_{n=0}^{\infty} p_n x^n$ & $Q(x) = \sum_{n=0}^{\infty} q_n x^n$ for $|x| < R$, $R > 0$

③ Seek the solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

with R as radius of conv. (at least)

Substitute in (4) & solve.

Suppose $y'' + P(x)y' + Q(x)y = 0$ ---(1) **REGULAR SINGULAR POINT**

— A pt. x_0 is a singular pt. of the D.E. if one or ~~the~~ the other (or both) of the coefficient fns. $P(x)$ & $Q(x)$ fails to be analytic at x_0.

$\boxed{\text{Example}}$ $y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0$

$x = 0$ is a singular pt.

— A singular pt. x_0 of eq^n (1) is said to be regular if the fns. $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ are analytic and irregular otherwise.

$\boxed{\text{Example}}$ $y'' - \frac{2x}{(1-x^2)}y' + \frac{p(p+1)}{(1-x^2)}y = 0$

has $x = 1$ & $x = -1$ as singular pts.
$\downarrow$ regular $\quad$ $\downarrow$ regular. $\quad$ (Legendre's eq^n)

$\boxed{\text{Example}}$ Bessel's eq^n of order p ($p \geq 0$ const.)

$x^2 y'' + xy' + (x^2 - p^2)y = 0$

$\Rightarrow y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0$

$\Rightarrow x = 0$ is a regular singular pt.

$\because xP(x) = 1$ & $x^2 Q(x) = x^2 - p^2$.