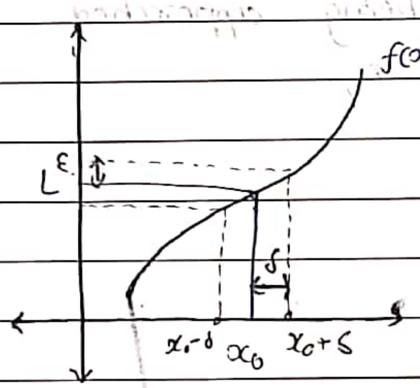
calculus in one vericible.

⇒ suppose $f(x)$ is defined on an open interval about $x_0 \in R$ expect possibly at x_0 itself.

⇒ If $f(x)$ is arbitrarily close to a number L (as close as we like) for all x sufficiently close to x_0, then

⇒ we say $\lim_{x \to x_0} f(x) = L$

$x \neq x_0$



$\delta > 0$

$-\delta < x - x_0 < \delta$

$|x - x_0| < \delta$

$|f(x) - L| \leq \epsilon$

⇒ precige defination of limit :

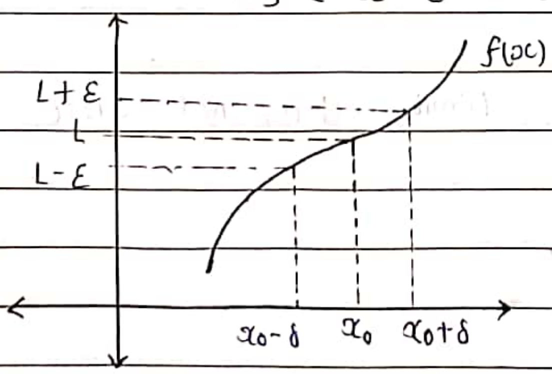
$\lim_{x \to x_0} f(x) = L$

If for every $\epsilon > 0$ (small positive real no.) there exist a $\delta > 0$ such that for all x lie b/w inter $0 < |x - x_0| < \delta$ then $|f(x) - L| < \epsilon$

example: $0 < |x - 2| < 3$

means

$-3 < x - 2 < 3$ except $x = 2$



\# $f(x) = \dfrac{x^2-1}{x-1}$ $\quad x_0 = 1$

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} (x+1) = 2$$

But $f(1)$ does not exist.

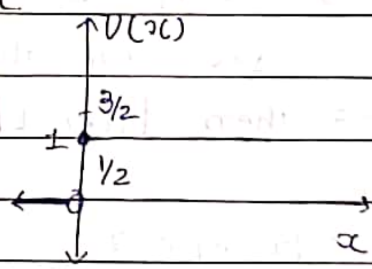
Note: For the existance of limit functions may not to be define at the point being approached

$\Rightarrow$ $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

$\left.\begin{array}{l} \lim_{x\to 1} g(x) = 2 \\ g(1) = 1 \end{array}\right\}$ not equal

If the function is define then it may not be equal to define value.

$\Rightarrow$ $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$



here we can not get value of ε so function is not define.

ex: i) $g(x) = \begin{cases} 1/x & ; x \neq 0 \\ 0 & , x = 0 \end{cases}$

ii) $h(x) = \begin{cases} 0 & ; x \leq 0 \\ \sin 1/x & x > 0 \end{cases}$

Limit does not exist

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__/__/__

Q: $\lim_{x \to 1} (5x-3) = 2$ prove by using defination

$x_0 = 1$, $L = 2$, $f(x) = 5x-3$

for any $\varepsilon > 0$ we have to find a suitable $\delta > 0$ such that if $x \neq 1$ $x \in (1-\delta, 1+\delta)$ and $f(x) \in (2-\varepsilon, 2+\varepsilon)$

let us take any $\varepsilon > 0$

$$|f(x) - L| = |5x-3-2| < \varepsilon$$
$$\Rightarrow |5x-5| < \varepsilon$$
$$\Rightarrow 5|x-1| < \varepsilon$$
$$\Rightarrow |x-1| < \varepsilon/5$$

If we take $\delta = \varepsilon/5$ then $0 < |x-1| < \varepsilon/5$

$$\Rightarrow 0 < |x-1| < \delta$$

Q: $\lim_{x \to x_0} x = x_0$

$f(x) = x$, x_0 , $L = x_0$

so $|f(x) - L| < \varepsilon$

$0 < |x - x_0| < \varepsilon$ $x \neq x_0$

If we choose $\delta = \varepsilon$ then $0 < |x-x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Procedure to find δ- interval.

Step: 1 slove the inequality $|f(x) - L| < \varepsilon$ to find open interval (a, b) cuntaing x_0 on which the above enequality holds for all $x \neq x_0$.

Step: 2 Find a value of $\delta > 0$ that place the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq x_0$ in this δ interval.

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Q: Prove that

$\lim_{x \to 5} \sqrt{x-1} = 2$ Find $\delta > 0$ that works for $\varepsilon = 1$

find $\delta > 0$ such that for all x

$0 < |x-5| < \delta \Rightarrow |\sqrt{x-1} - 2| < 1$

Solution: step: 1 $\Rightarrow |\sqrt{x-1} - 2| < \varepsilon$

$\Rightarrow |\sqrt{x-1} - 2| < 1$ $\qquad |x - x_0| < \delta$

$\Rightarrow -1 < \sqrt{x-1} - 2 < 1$ $\qquad |x-5| < \delta$

$\Rightarrow 1 < \sqrt{x-1} < 3$

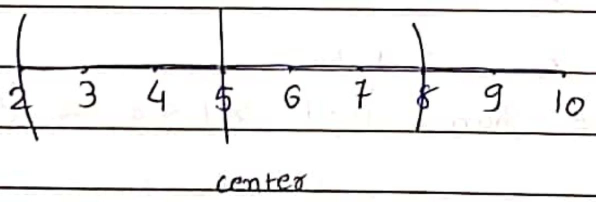
$\Rightarrow 1 < (x-1) < 9$

$\Rightarrow 2 < x < 10$

step: 2 Find a value of $\delta > 0$ to place the interval

$5-\delta < x < 5+\delta$ (centered at 5)

inside the interval $(2, 10)$



so δ ~~=~~ 3.

If we take $\delta = 3$ or any smaller number, then the inequality $0 < |x-5| < \delta = 3 \Rightarrow |\sqrt{x-1} - 2| < 1$

Exam: $f(x) = \begin{cases} x^2 & ; x \neq 2 \\ 1 & ; x = 2 \end{cases}$ Find interval for x in terms of ε and also Find δ.

$\lim_{x \to 2} f(x) = 4$

step: 1 for every x

$|f(x) - L| < \varepsilon$ $\qquad \Rightarrow 4 - \varepsilon < x^2 < \varepsilon + 4$

$\Rightarrow |x^2 - 4| < \varepsilon$ $\qquad \Rightarrow \sqrt{4-\varepsilon} < x < \sqrt{\varepsilon + 4}$

$\Rightarrow -\varepsilon < x^2 - 4 < \varepsilon$

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$\sqrt{4-\varepsilon}$ $\quad$ 2 $\quad$ $\sqrt{4+\varepsilon}$

so $\delta = \min\{2-\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}-2\}$

and for $\varepsilon > 4$ interval is $(0, \sqrt{4+\varepsilon})$

and $\delta = \min\{2, \sqrt{4+\varepsilon}-2\}$

Properties of limit :

⇒ Let $\lim\limits_{x\to c} f(x) = L$ and $\lim\limits_{x\to c} g(x) = M$

• 1) Then $\lim\limits_{x\to c} (f \pm g)(x) = L \pm M$

• 2) $\lim\limits_{x\to c} (\alpha f(x)) = \alpha L$

3) $\lim\limits_{x\to c} (f(x)\cdot g(x)) = L\cdot M$

4) $\lim\limits_{x\to c} f(x)/g(x) = L/M$ provided $M \neq 0$.

5) $\lim\limits_{x\to c} (f(x))^n = L^n$ when n is positive intiger.

6) $\lim\limits_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ n is positive intiger.

7) If $f(x) \le g(x)$ $\forall x$ in an open interval containing c.
(except possible c it self)
then $L \le M$

$x_0-\delta \quad x \quad x_0 \quad x \quad x+\delta$

One sided limit :

⇒ Right hand limit $\lim_{x \to x_0^+} f(x) = L$

If for every $\varepsilon > 0$ there exist a corresponding $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon$$

⇒ Left hand limit $\lim_{x \to x_0^-} f(x) = L$

If for every $\varepsilon > 0$ there exist a correspoding $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$$

If both R.H.L. and L.H.L. exist and are equal then limit $\lim_{x \to x_0} f(x) = L$

Example: $\lim_{x \to 0^+} \sqrt{x} = 0$

Let $\varepsilon > 0$ $\quad x_0 = 0$, $L = 0$, $f(x) = \sqrt{x}$

$|f(x) - L| < \varepsilon$

$|\sqrt{x} - 0| < \varepsilon$ we want to find $\delta > 0$ such that

$$0 < x < \delta \Rightarrow |\sqrt{x} - 0| < \varepsilon$$

$x < \varepsilon^2$

If we take $\varepsilon^2 = \delta$

then $0 < x < \delta \Rightarrow |\sqrt{x} - 0| < \varepsilon$

Continuity :

A function $f(x)$ is continuous at an internal point x_0 of the domain of f if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

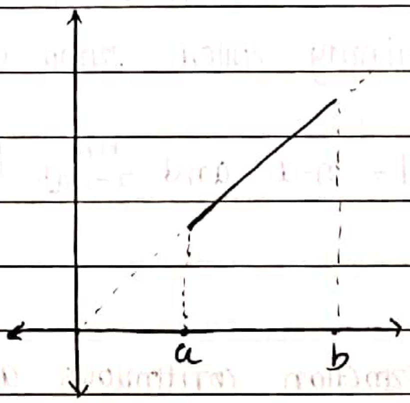
- $\lim_{x \to x_0} f(x)$ exist
- $f(x_0)$ should be define
- and $\lim_{x \to x_0} f(x) = f(x_0)$

precise defination :

f is continuous at x_0 if for $\forall \varepsilon > 0$, there exist a $\delta > 0$ such that for all x in $|x - x_0| < \delta$ and $|f(x) - f(x_0)| < \varepsilon$

Let $f: [a, b] \to R$

$f(x) = x$

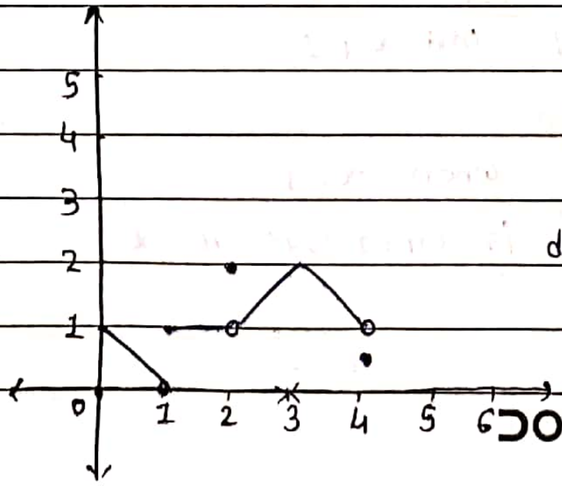


at the end point we do not consider both side limit because function is not define at other point

$f(x)$ is continuous at the left end point a if $\lim_{x \to a^+} f(x) = f(a)$

and $f(x)$ is continuous at the Right end point b if $\lim_{x \to b^-} f(x) = f(b)$

example :



continuous : At $x = 0$ $\lim_{x \to 0^+} f(x) = f(0)$

At $x = 3$ $\lim_{x \to 3} f(x) = f(3) =$

discontinuous : at $x = 1, 2, 4$

At $0 < c < 4$, $c \neq 1$, $c \neq 2$

$\lim_{x \to c} f(x) = f(c)$ function is continuous f is said to be right continuous at C

IF $\lim_{x \to c^+} f(x) = f(c)$

and f is said to be left continuous at c

IF $\lim_{x \to c^-} f(x) = f(c)$

ex: $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$

$\lim_{x \to -2^+} \sqrt{4-x^2} = 0 = f(-2)$

$\lim_{x \to 2^-} \sqrt{4-x^2} = 0 = f(2)$

so function is continuous throught the interval.

example: $f(x) = \lfloor x \rfloor$ greatest integer $\leq x$

this function is discontinuous at all the integer.

and this discontinuity called Jump discontinuity

$\lim_{x \to n^-} \lfloor x \rfloor = n-1$ and $\lim_{x \to n^+} \lfloor x \rfloor = n$ so both are not equal.

example: $f(x) = \frac{x^2-1}{x-1}$ is function continuous at $x=1$

Here $\lim_{x \to 1} f(x) = 2$ (Limit exist) But function is not define. so this type of discontinuity called removeable discontinuty.

and IF we define function

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{and } x \neq 1 \\ 2 & \text{when } x = 1 \end{cases}$$

then function is continuous at $x = 1$

Properties of continuous function:

IF f and g are continuous at $x=c$, then

→ (i) $f+g$, $f-g$, fg are continuous at c.

→ (ii) kf is continuous for any constant k.

→ (iii) f/g is continuous at c, if $g(c) \neq 0$

→ (iv) f^n is continuous when n is positive integer

→ (v) $\sqrt[n]{f}$ is also continuous provided it is defined on an open interval containg c, n is positive integer.

→ (6) if f is continuous at c and g is continuous at $f(c)$ then the composition is also continuous.

example. is $\left|\frac{x\sin x}{x^2+2}\right|$ is continuous is on R

$\sin x$ and x are continuous.

so $x\sin x$ is con. and also $x^2+2 \neq 0$ so $\frac{x\sin x}{x^2+2}$ is also continuous. and $\left|\frac{x\sin x}{x^2+2}\right|$ is also contt.

→ 7) If g is continuous at a point b and $\lim_{x\to c} f(x) = b$, then

$$\lim_{x\to c} g(f(x)) = g\left(\lim_{x\to c} f(x)\right) = g(b)$$

Note: limit can enter into a continuous function.

example: $\lim_{x\to \pi/2} \cos\left(2x + \sin\left(3\pi/2 + x\right)\right)$

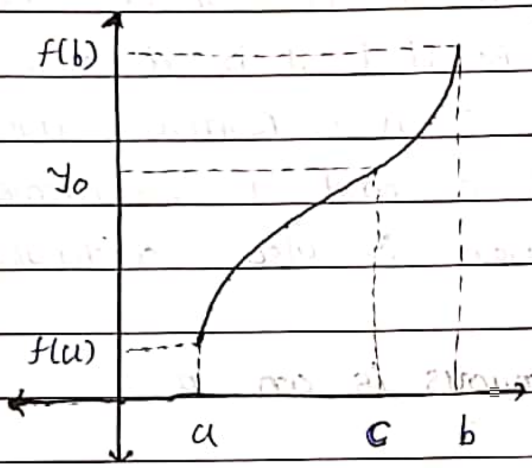
$= \cos\left(\lim_{x\to\pi/2} 2x + \lim_{x\to\pi/2} \sin\left(3\pi/2 + x\right)\right)$

$= \cos(\pi + 0)$

$= \boxed{-1}$

→ 8) Intermediate value Property.

If f is continuous on a closed interval $[a, b]$ and if y_0 is any value between $f(a)$ and $f(b)$ then there exist $c \in (a,b)$ such that $f(c) = y_0$



$f(x) =$

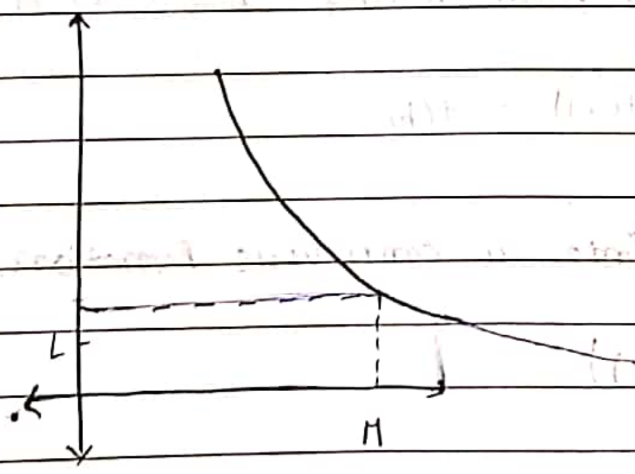
$f(-1) = < 0$

$f(1) > 0$

Limits involving infinity :

$\lim_{x \to \infty} f(x) = L$

If for every $\varepsilon > 0$ there exist a real number $M > 0$ such that for all x, $x > M \Rightarrow |f(x) - L| < \varepsilon$



$\lim_{x \to -\infty} f(x) = L$

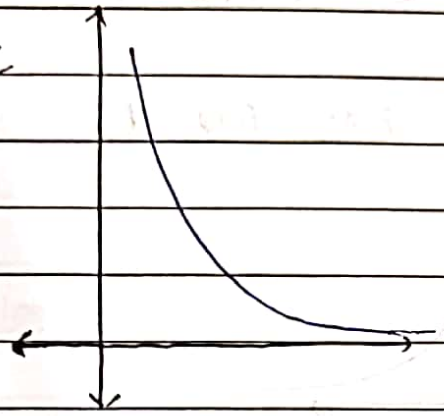
If for every $\varepsilon > 0$, there exist a real number N such that for all x, $x < N \Rightarrow |f(x) - L| < \varepsilon$

Horizontal asymptote

A line $y = b$ is a horizontal asymptote of the garph of the function $y = f(x)$ if either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$

ex

$f(x) = 1/x$



$y=0$ means the x axis is horizontal asymptote.

oblique asymptote:

If the degree of numerater of a ratenal function is one gruter than the degree of denominater, then the graph of the function has an oblique asymptote.

example $f(x) = \dfrac{x^2-3}{2x-4}$

$= \dfrac{x^2-3}{2(x-2)}$

$= \dfrac{(x+2)(x-2)+1}{2(x-2)}$

$= \underbrace{\left(\dfrac{x}{2}+1\right)}_{g(x)} + \left(\dfrac{1}{2x-4}\right)$ Reminder term

$$
\begin{array}{r|l}
 & x+2 \\
\hline
x-2 & x^2-3 \\
 & x^2-2x \\
 & -\quad + \\
\hline
 & 2x-3 \\
 & +2x \mp 4 \\
 & -\quad + \\
\hline
 & 1
\end{array}
$$

when

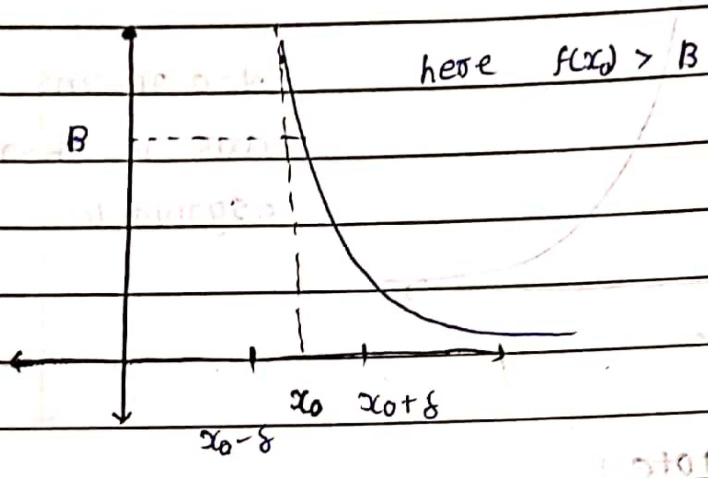
When $x \to \infty$ or $x \to -\infty$ the reminder goes to 0.

$g(x) = \dfrac{x}{2} + 1$ called oblique asymptote for the graph $f(x)$.

Infinite limit :

$\lim_{x \to x_0} f(x) \to \infty$

If for every positive number B, there exist $\delta > 0$ such that for all x $0 < |x - x_0| < \delta \Rightarrow f(x) > B$



$\lim_{x \to x_0} f(x) \to -\infty$

If for every negitive number $-B$, there exist $\delta > 0$ such that for all x $0 < |x - x_0| < \delta \Rightarrow f(x) < -B$

Vertical Asymptote :

A line $x = a$ is called a vertical asymptote at the graph $y = f(x)$ of either

$\lim_{x \to a^+} f(x) \to \pm\infty$

$\lim_{x \to a^-} f(x) \to \pm\infty$ so $x = a$ line is vertical asymptote

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Differentiation :

The function $f(x)$ is differentiable at a point x_0 if

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$ exist. which is denoted by $f'(x_0)$

→ The slop of the graph $y = f(x)$ at the point x_0. or slope of the graph tangent.

→ The rate of change of $f(x)$ with respect to x at x_0.

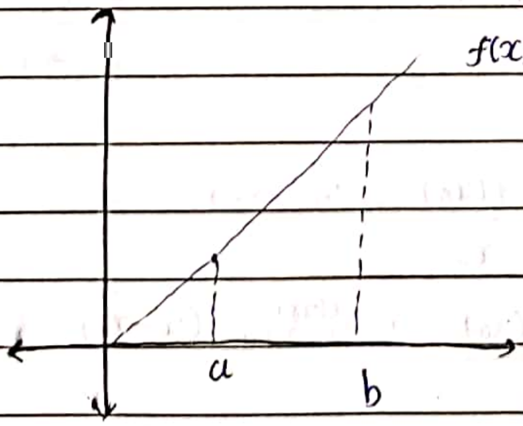
Left hand derivative : (L.H.D.)

$$\lim_{h\to 0-} \frac{f(x_0+h) - f(x_0)}{h}$$

Right hand derivative : (R.H.D.)

$$\lim_{h\to 0+} \frac{f(x_0+h) - f(x_0)}{h}.$$

~ here we assume that x_0. is interior point. then If Both L.H.D. and R.H.D. exist and equal then Function is differentiable at x_a



$f(x) = x$ $\quad f : [a, b] \to R$

$f(x)$ is differentiable at a if

$$\lim_{x\to a+} \frac{f(a+h) - f(a)}{h}$$ exist. and diff. at

b if $$\lim_{x\to b-} \frac{f(a+h) - f(b)}{h}$$. exist.

$x^n|x|$ is n times diffrentiable

$x^n \sin 1/x$ is $(n-1)$ times diffrentiable

example: $f(x) = |x|$ at $x = 0$

$$R.H.D. = \lim_{h\to 0+} \frac{f(0+h) - f(0)}{h} \qquad L.H.D = \lim_{h\to 0-} \frac{|h|}{h} = -1$$

$$= \lim_{h\to 0+} \frac{|h|}{h} = 1$$

so $R.H.D \neq L.H.D.$

so $f(x)$ is not diffrentiable at $x = 0$.

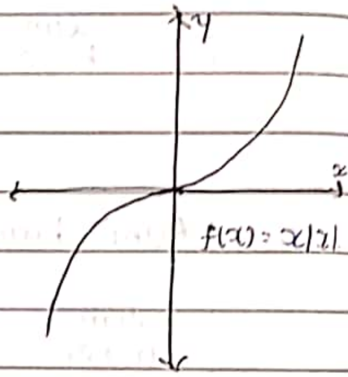
example $f(x) = x|x|$ at $x = 0$

$$R.H.D = \lim_{h\to 0+} \frac{f(h) - f(0)}{h} \qquad L.H.D = \lim_{h\to 0-} \frac{h|h|}{h}$$

$$= \lim_{h\to 0+} \frac{h|h|}{h} \qquad = 0$$

$$= 0$$

so $f(x)$ is diffrentiable at $x = 0$



→ every differential function is continuous but every continuous function is not differetial

→ derivetive of $f(x)$ at x_0

$$f'(x) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exist}$$

$x + h \to t$

$h \to x - t \to 0$

$x \to t$

$$\lim_{x\to x_0} (f(x) - f(x_0)) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \times (x - x_0)$$

$$= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \times \lim_{x\to x_0} (x - x_0)$$

$$= f'(x_0) \times 0$$

$$= 0$$

$$\lim_{x\to x_0} f(x) = f(x_0)$$

means fun is continuous at x_0.

Application OF derivative :

(1) ~ extreme value OF a function

$f: D \rightarrow R$

f has a globle max^m at a point $c \in D$ if $f(x) \leq f(c)$ $\forall x \in D$.

f has a globle min^m at point c if $f(x) \geq f(c)$ $\forall x \in D$.

example. $f(x) = x^2$ Domain $(-\infty, \infty)$

Global minima is zero but there is not global mixima.

if Domain $[0, 2]$

then minima is $= 0$

maxima is $= 4$

if Domain $(0, 2]$

global maxima is 4 but Global minimum Does not exist in the Domain.

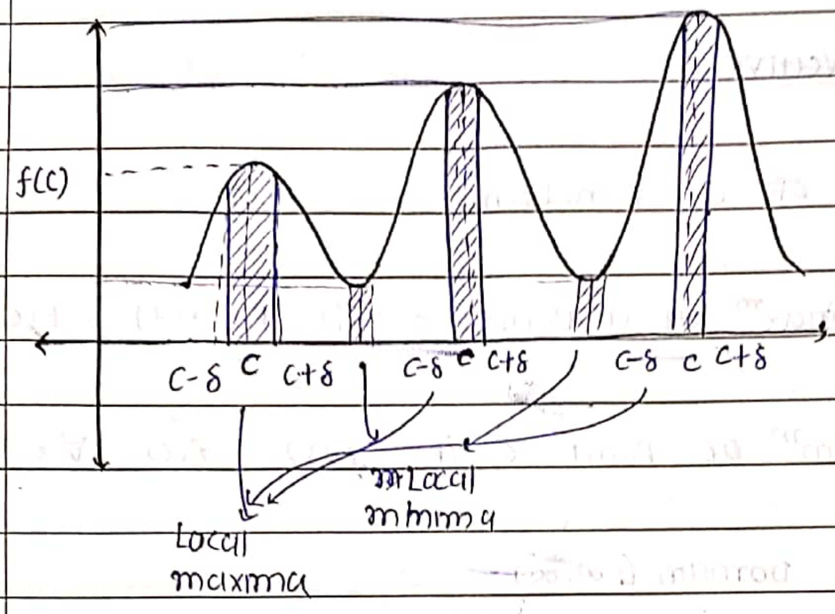
~ Domain $(0, 2)$ no Global maxima, no Global minima

Any continuous function on a closed bounded set has a global mixima and global minima. (exterem value theorem)

local maximum and Local minimum :

A function f has a local maximum at a point $c \in D$ if $f(x)$ is $\leq f(c)$ for all x lying in interval $(c-\delta, c+\delta)$

A function f has a local minimum at a point $c \in D$ if $f(x) \geq f(c)$ $\forall x \in (x-\delta, x+\delta)$



$f : [a,b] \to R$

f has a local maximum at a if $f(x) \le f(a)$ $\forall x \in [a, a+\delta]$

f has a local minima at a if $f(x) \ge f(a)$ $\forall x \in [a, a+\delta)$

similarly at the point b.

The first derivative theorem

If f has a Local max^m or minimum value at an interior point c in it's domain and if f' exist at c, then $f'(c) = 0$.

Proof: If f has a Local max^m at $x = c$, then $f(x) \le f(c)$ $\forall x \in (c-\delta, c+\delta)$ $\Rightarrow$ $f(x) - f(c) \le 0$ $\forall x \in (c-\delta, c+\delta)$

$\lim_{x \to c_+} \frac{f(x) - f(c)}{x - c} \le 0$. as $f(x) - f(c) \le 0$ and $x - c > 0$.

$\lim_{x \to c_-} \frac{f(x) - f(c)}{x - c} \ge 0$ as $f(x) - f(c) \le 0$ and $x - c < 0$.

since $f'(x)$ is exist. $\Rightarrow$ $\lim_{x \to c_+} \frac{f(x) - f(x)}{x - c} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

only possible when both equal to zero.

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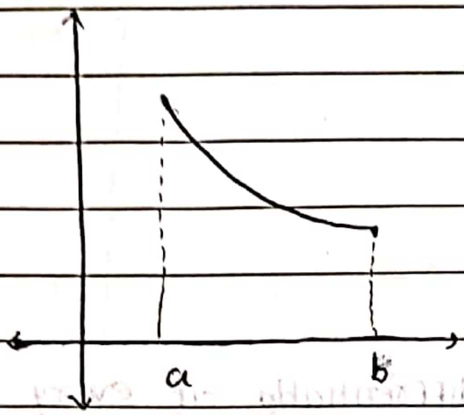
The prove is similar where c is Local minima.

★★ example: $f(x) = |x|$

$f(x) = |x|$ has Local minimum at 0.

But $f'(0)$ does not exist.

exa.



$f : [a,b] \to \mathbb{R}$

f has maximum at a but $f'(x) \neq 0$

and same as point b.

Critical Point :

an interior point of domain of function where $f' = 0$ or f' is not defined is called a critical point of f.

Sufficient condition for Local maxima or Local minima

Let f is differentiable continuously n time and $f'(c) = 0 = f''(c) = \dots f^{(n-1)}(c) = 0$. Let $f^{n}(c) \neq 0$. The First non zero derivative.

Then

(i) c is a local maxima if $f^{n}(c)$ is < 0 and n is even

(ii) c is a local minima if $f^{n}(c) > 0$ and n is even

(iii) if n is odd then c is neighter local maxima nor local minima. where c is called point of inflection.

$f'(x) = 0$

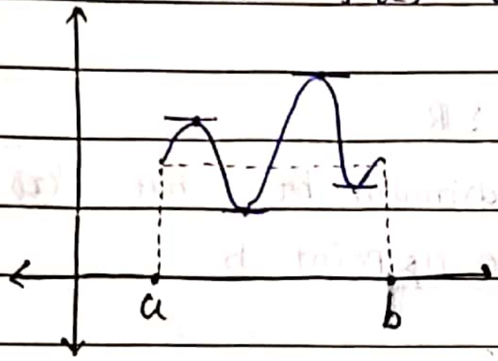
$n=2$ $f''(x) < 0$ max^{n}

$f'''(x)$

mean value theorem :

(i) Rolle's theorem :

Suppose $f(x)$ is continuous in $[a,b]$ and differentiable on (a,b) and $f(a) = f(b)$ then there exist $c \in (a,b)$ such that $f'(c) = 0$.



Proof: It is given that f is differentiable at every interior point of (a,b). f is continuous on a close interval $[a,b]$. so max^m and min^m exist

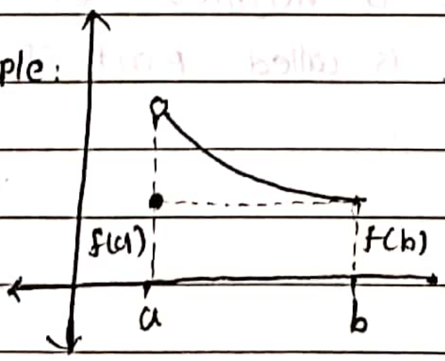
if either the max^m or min^m occures at the interior point c, then By First derivative theorem $f'(c) = 0$.

If Both the maximum and minimum occures at the end points and since $f(a) = f(b)$ means f is a constant function means $f'(x) = 0$ for all $x \in (a,b)$

so there exist at least one $c \in (a,b)$ such that $f'(c) = 0$.

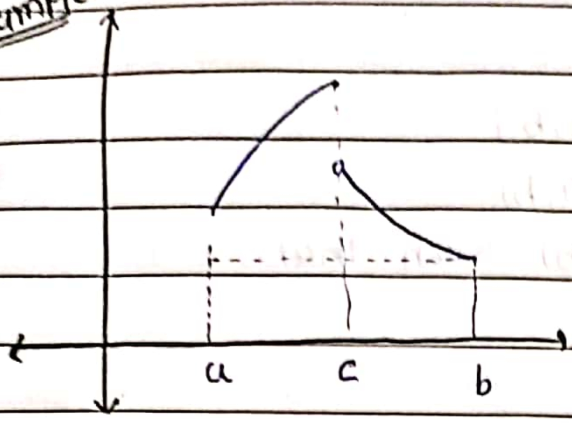
$\boxed{\text{If the hypothesis fail at one point then the graph of the function may not have a horizontal tangent.}}$

example:



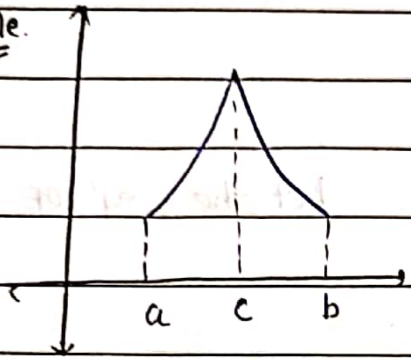
function is not continuous at point a. so there exist no point where $f'(c) = 0$.

example.



at Point c Function is dis continuous means $f'(c)$ does not exist. so we can not apply rot theorem

example.



function is not differential at point c.

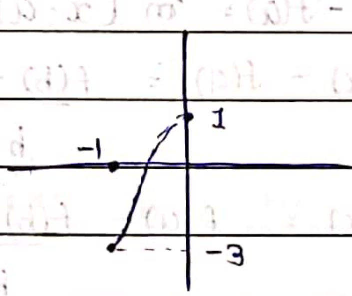
problem: Show that eq^n $x^3 + 3x + 1 = 0$ has exactly one real root.

$f(x) = x^3 + 3x + 1$ is continuous Function

$f(-1) = -3 < 0$

$f(0) = 1 > 0$

By intermediate value property it must cross the x-axis



$\Rightarrow$ There is at least one real root.

suppose there are two real root say a and b

$f(a) = f(b) = 0.$

By Roll's theorem there exist $c \in (a, b)$ such that $f'(c) = 0$.

$f'(x) = 3x^2 + 3 = 3(x^2 + 1)$

But $f'(x)$ has no real root.

so this contrast our assumption.

So there exist only one real root.

ii) The mean value theorem:

suppose $y = f(x)$

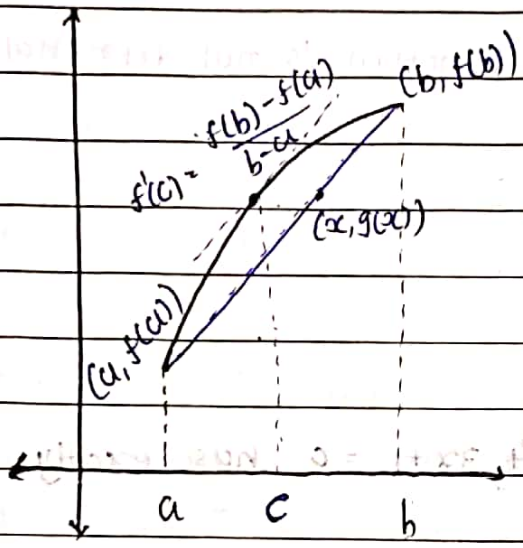
(i) is continuous on $[a,b]$

(ii) is differential on (a,b)

then there exist $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Proof



Let the eqn of cord $g(x)$

eqn of a cord joining $(a, f(a))$ and $(x, g(x))$

$g(x) - f(a) = m(x-a)$

$\Rightarrow g(x) - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$

$\Rightarrow g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$

Let $h(x)$ be the vertical difference b/w f and g.

$h(x) = f(x) - g(x)$

$= f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$

$h(x)$ is continuous function $[a, b]$

$h(x)$ is also differentiable of (a, b)

$h(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a)$

$= 0$

and $h(b) = 0$

so $h(a) = h(b)$.

so by roles theoram there exist $c \in (a,b)$ such that

$h'(c) = 0$

$$h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

At soon point $c \in (a,b)$ the slope of the chord Joining $(a, f(a))$ and $(b, f(b))$ is same as the slope of the tangent to the curve $f(x)$ at c.

Incresing and decresing functions.

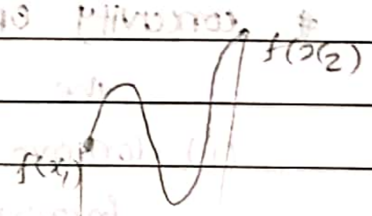
suppose f is continuous on $[a,b]$ and differential on (a,b). if $f'(c) > 0$ at each point $x \in (a,b)$ then f is incresing on $[a,b]$

If $f'(c) < 0$ at each point $x \in (a,b)$ then f is decresing at $[a,b]$

Proof Let x_1, x_2 be any two points in $[a,b]$ with $x_1 < x_2$

then apply MVT on $[x_1, x_2]$

there exist $c \in (x_1, x_2)$ such that



$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

if $f'(x) > 0$ for $x \in (a,b) \Rightarrow f(c) > 0$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

$$\Rightarrow f(x_2) > f(x_1)$$

means function is incresing.

if $f(x) < 0 \Rightarrow f'(c) < 0$

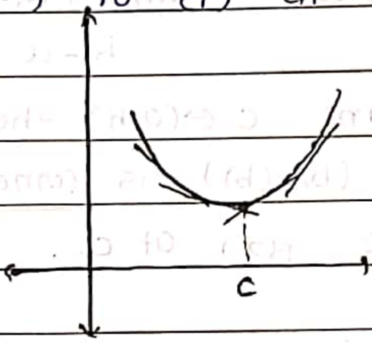
$\Rightarrow f(x_2) < f(x_1)$

means function is decresing.

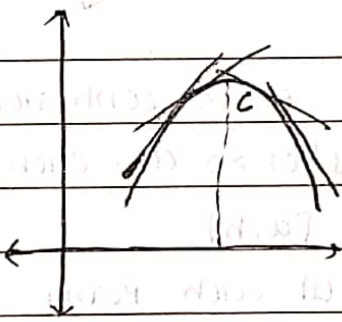
First derivative test for Local maxima and minima

suppose c is a critical point of continuous function f is differentiable at every point in some open interval containing c. except possibly at c itself.

(1) if $f'(x)$ changes to $(-)$ to $(+)$ at c, then f has Local minima at c.



(2) if $f'(x)$ changes to $(+)$ to $(-)$ at c, then f has Local maxima at c.

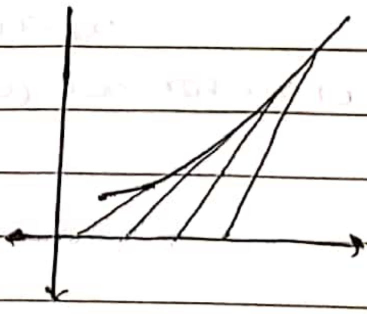


(3) if $f'(x)$ does not change sign near c, than it is neighter maxima nor minima.

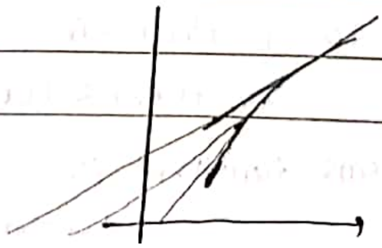
concavity of curve.

the graph of a differentiable function $y = f(x)$ is

(i) concave up on a open interval I if f' is incresing on I.



(ii) concave down on the open interval I if f' is decresing on I.



suppose $y = f(x)$ has second derivative

then (i) f' is incresing if $f''(x) > 0$ on I.

(ii) f' is decresing if $f''(x) < 0$ on I.

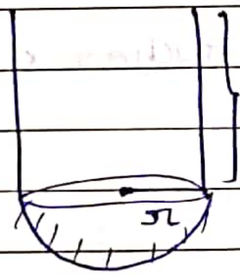
so we can say

(i) If $f'' > 0$ on I then the graph of f is concave up.

(ii) If $f'' < 0$ on I then the graph of f is concave down

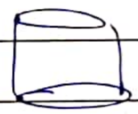
A point where the graph of a function has tangent line and where the concavity changes is called a point of a inflection.

Problem: we want to design an one litre can shaped like right circular cylineder. what dimention will use the least amount of material to construct the can.



Let r be the redious of bese and height of the can.

volume $= \pi r^2 h = 1000\ cm^3$



surface area $= 2\pi r^2 + 2\pi r h$

given that $\pi r^2 h = 1000$

$$h = \frac{1000}{\pi r^2}$$

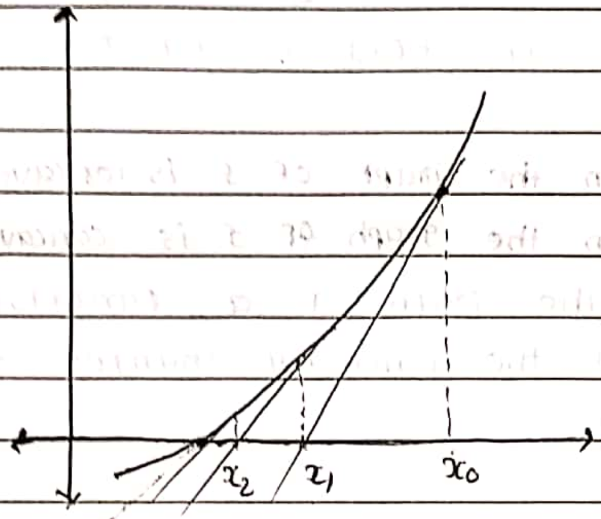
$$A(r) = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$$

$$A'(r) = 4\pi r - \frac{2000}{r^2} = 0$$

$$4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{500}{\pi} \Rightarrow r =$$

Application to find root of an eq^n



equation of the tangent at passing through $(x_m, f(x_m))$

$$y - f(x_m) = f'(x_m)(x - x_m)$$

$$y = f(x_m) + f'(x_m)(x - x_m)$$

So we want to find point when it touches x-axis

$y = 0$

$$f(x_m) + f'(x_m)(x - x_m) = 0$$

$$x - x_m = -\frac{f(x_m)}{f'(x_m)}$$

$$x = x_m - \frac{f(x_m)}{f'(x_m)}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$ Newton's methos.

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Intigration.

⇒ Definite integration

$$\int_a^b f(x)\, dx$$

~ Partenten $[a,b]$

$a = x_0 < x_1 < x_2 < \dots < x_n = b$

$P = \{x_0, x_1, \dots, x_n\}$

P devides the $[a,b]$ in to n partes

$[x_0, x_1]$, $[x_1, x_2]$, ... $[x_{n-1}, x_n]$

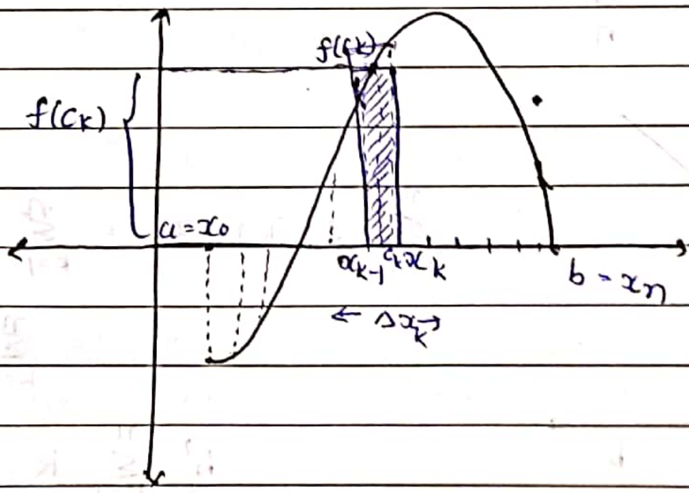
Let k_{th} substitution $[x_{k-1}, x_k]$

lenth of kth substiterval $x_k - x_{k-1} = \Delta x_k$

~ In the k^{th} sub interval we select some point c_k

the function value at c_k is $f(c_k)$

consider the sum $S_P = \sum_{i=1}^{n} f(c_k)\, \Delta x_k$ → sum of the area of all the rectangle.



$f(c_k) \cdot \Delta x_k$ is the area of this rectangle.

~ so S_P is called the Riemann sum.

~ so As $n \to \infty$ $\lim_{n\to\infty} \sum_{k=1}^{n} f(x) \cdot \Delta x = \sum_{k=1}^{\infty} f(c_k) \cdot \Delta x_k$ or

~ $\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\, x_n$

↳ P is the partition.

This quantity exactly match the area bounded by the curve $y = f(x)$ and x axis in the interval $[a,b]$.

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{\infty} f(c_k)\,\Delta x \text{ or } \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

compute

example $\int_0^b x\,dx$ and Find the area A under $y=x$ over the interval over the $[0,b]$, $b>0$

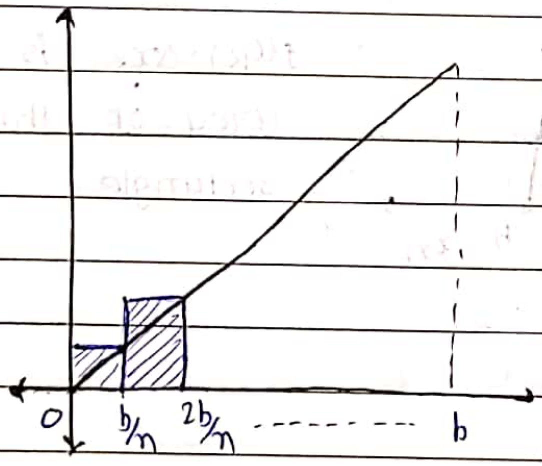
sol^n: we chooze the partition P. It does not metter that how we choose the partition or the points c_k as long as $\|P\|\to 0$.

→ All choice gives exactly same limit.

→ consider the partition $P = \left\{0, \frac{b}{n}, \frac{2b}{n}, \frac{nb}{n}\right\}$

$$\Delta x = \frac{b}{n}$$

and choose $c_k = \frac{kb}{n}$



$A = \frac{1}{2} b \times b$

$\frac{b^2}{2}$

$$\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} \frac{kb}{n}\cdot\frac{b}{n}$$

$$= \sum_{k=1}^{n} \frac{kb^2}{n^2}$$

$$= \frac{b^2}{n^2}\cdot\sum_{k=1}^{n} k$$

$$= \frac{b^2}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= \frac{b^2}{2}\left(1+\frac{1}{n}\right)$$

AS $n\to\infty$ or $\|P\|\to 0$. then sum

$\lim_{n\to\infty} \frac{b^2}{2}(1+1/n)$ gose to : $b^2/2$

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example: $f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$

function is discontinuous at every point. so we can note calculate the area of function

sol^n: Pick a partition P on $[0,1]$ to choose c_k to be the point giving maximum value for f on $[x_{k-1}, x_k]$.

since each subinterval $[x_{k-1}, x_k]$ content a rational number

so $f(c_k) = 1$

Then $\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = 1$

As $n \to \infty$ $\sum_{k=1}^{n} f(x_k)\,\Delta x_n = 1.$

on the other interval if we choose c_k to be the point giving minimum value to F on $[x_{k-1}, x_k]$

$$\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k = \sum_{k=1}^{n} 0 \times \Delta x_k = 0$$

So $n \to \infty$ $\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k = 0$

so this function is not integrable

1) if a function is continuous over the interval $[a,b]$, then it is integrable over $[a,b]$

2) IF $f(x)$ has at many finite many jump discontinuty then $\int_a^b f(x)\,dx$ exist.

→ Mean value theorem of definite integral.

If f is continuous on $[a,b]$ then there exist $c \in (a,b)$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx \qquad dg = f$$

$$= \frac{g(b) - g(a)}{g - a} = g'(c) = f(c)$$

First Fundamental theorum of calculus.

If f is continuous of $[a,b]$ then $F(x) = \int_a^x f(t)\, dt$

is continuous of $[a,b]$ and differentiable of $[a,b]$ and $F'(x) = f(x)$.

example $y = \int_a^x (1+t^3)\, dt$

$$\frac{dy}{dx} = 1 + x^3$$

example: $y = \int_x^5 3t \sin t\, dt$ Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = -3x\sin x$$

example $y = \int_1^{x^2} \cos t\, dt$

$$\frac{dy}{dx} = \cos x^2 \cdot 2x - 0$$

$$= 2x\cos x^2$$

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second fundumental theorum

IF f is continuous at every point in [a,b] at F is any antiderivative of f on [a,b], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Application of Integrals.

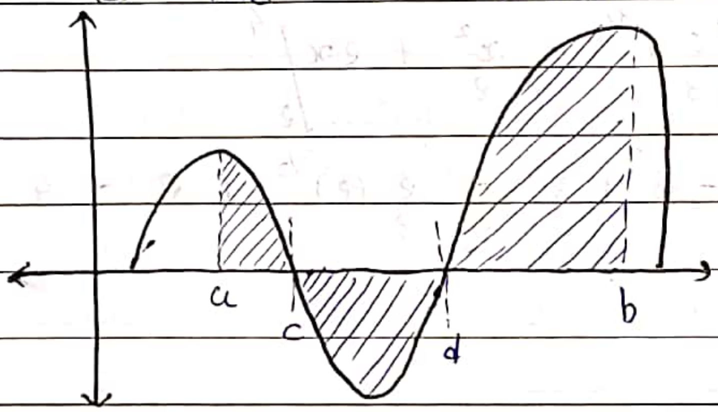
1) Area:

$y = f(x)$ to Find the area of $f(x)$ and the x axis over the interval [a, b]

(i) subdivide [a, b] at the roots of f.

(2) Integrate f over each sub. interval

(3) Add the absolute value all the integrals

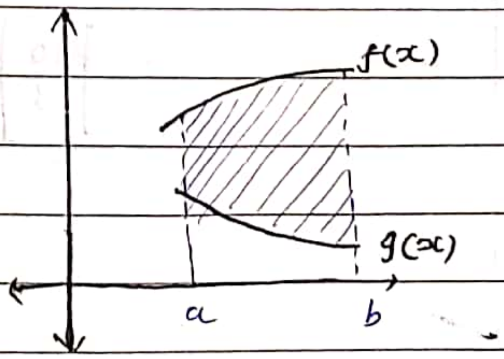


$$\left|\int_a^c f(x)\,dx\right| + \left|\int_c^d f(x)\,dx\right| + \left|\int_d^b f(x)\,dx\right|$$

Area between two curves.

IF f and g are continuous when $f(x) \geq g(x)$ through out [a,b] then the area of the rigion between $y = f(x)$ and $y = g(x)$ from a to b is

$$A = \int_a^b (f(x) - g(x))\,dx$$



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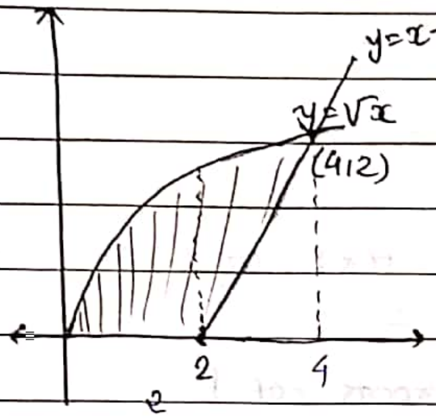
example:

Find the area of the region in the I^{st} quadrant that is bounded above by $y = \sqrt{x}$ and bellow by the x axis and the line $y = x - 2$



$x = y^2 \quad x = y + 2$

$$\text{Area} = \int_0^2 \sqrt{x}\, dx + \int_2^4 (\sqrt{x} - x + 2)\, dx$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \cancel{\frac{2}{3}(2)^{3/2}} + \frac{16}{3} - \cancel{8} + \cancel{8} - \cancel{\frac{2}{3}(2)^{3/2}} + 2 - 4$$

$$= \frac{16}{3} - 2 = \frac{10}{3}$$

Method II

$$\int_0^2 (y + 2 - y^2)\, dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2$$

$$= 2 + 4 - \frac{8}{3}$$

$$= \boxed{\frac{10}{3}}$$

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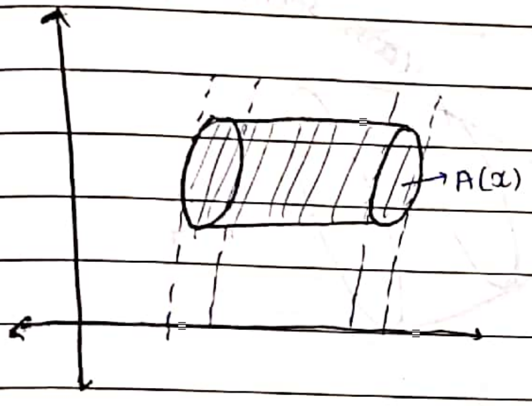
Libnits rule:

IF f is continuous on $[a,b]$ and if $u(x)$ and $v(x)$ are differentiable function of x where value lie in $[a,b]$. then

$$\frac{d}{dx}\left(\int_{u(x)}^{v(x)} f(t)\,dt\right) = f(v(x))\cdot\frac{dv}{dx} - f(u(x))\cdot\frac{du}{dx}.$$

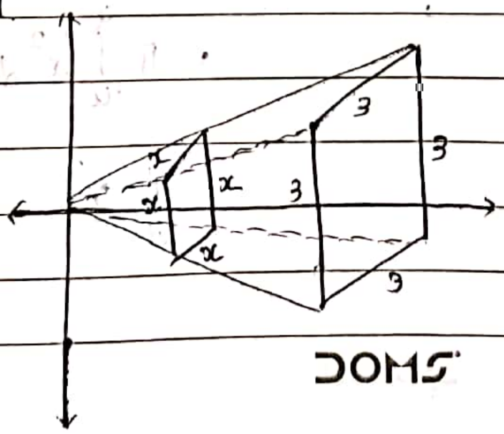
volume using cross-section.



1) sketch the solid and a typical cross-section
2) Find the formula for $A(x)$
3) find the limit of integration
4) integrate $A(x)$ to find the value

$$V(x) = \int_0^a A(x)\,dx$$

example : A pyramid of 3 m hight has a square base that is 3 metere size. the cross section of the pyramid is ⊥r to the altitude x meter down from the vertex is a square x meter on each side. Find the volume of pyramid



$$A(x) = x^2$$

$$V = \int_0^3 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^3 = 9\ m^3$$

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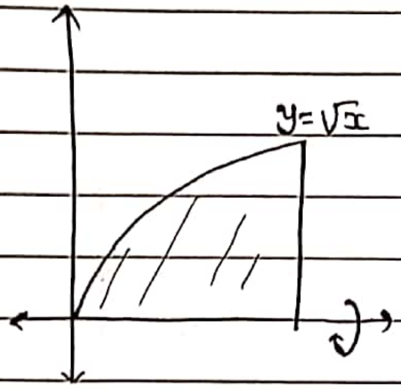
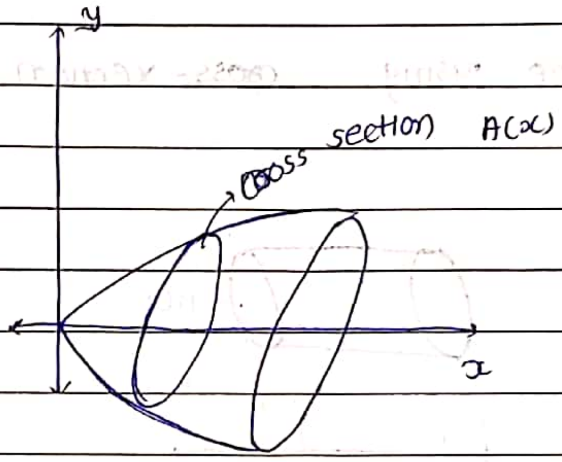
solids of revolution

The solid generated by rotating a plane region about an axis in its plane is called solid of revolution





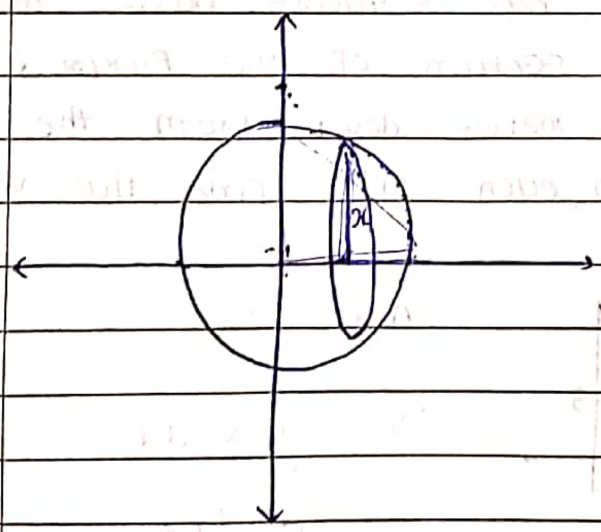
Radious of disck is $= \sqrt{x}$

Area $A(x)$ of cross-sectinal disc $= \pi(\sqrt{x})^2 = \pi x$

so volume $= \int_0^4 \pi x \, dx$

$= \frac{\pi}{2}(16)\ m^3 = 8\pi\ m^3$

Problem: the circle $x^2+y^2=a^2$ is rotated about x-axis ~~to~~ generate a sphere find it's volume



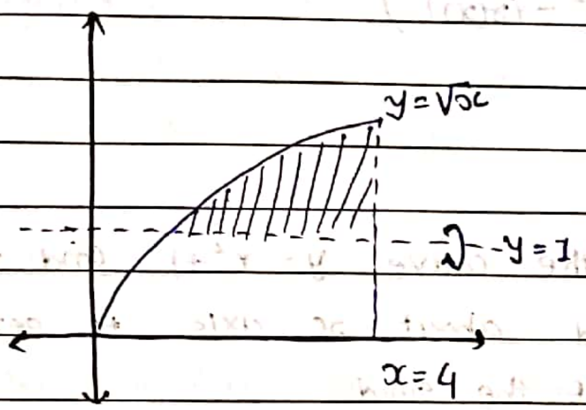
$A(x) = \pi x^2$

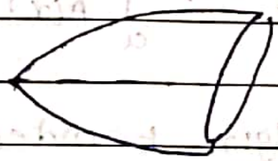
$V(x) = \int_{-a}^{a} \pi x^2 \qquad = 2\pi \int_0^a x^2$

$= \pi \int_{-a}^{a} x^2 dx \qquad 2\pi \left[\frac{x^3}{3}\right]_0^a$

$\frac{\pi}{3} x^3$

example: Find the volume of the solid generated by revolving the regin bounded by $y = \sqrt{x}$ and the lines $y=1$, $x=4$, about the line $y=1$.





radius of the disc is $\sqrt{x} - 1$

Area $= \pi(\sqrt{x}-1)^2 = \pi(x - 2\sqrt{x} + 1)$

$$V = \int_1^4 \pi(x+1-2\sqrt{x})\,dx$$

$$= \pi\left[\frac{x^2}{2} + x - \frac{4}{3}x^{3/2}\right]_1^4$$

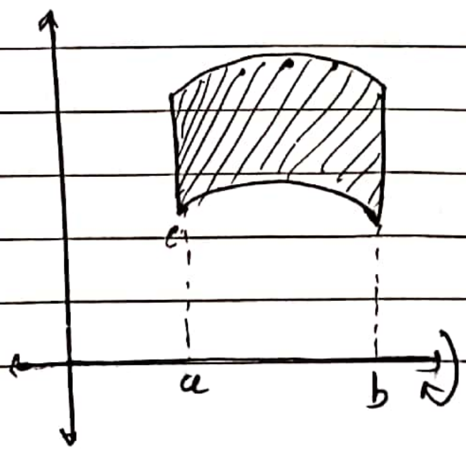
$$= \frac{7}{6}\pi$$

volume of solid by rotating a region about y-axis

$$V = \int_c^d A(y)\,dy = \int_c^d \pi(R(y))^2\,dy$$

washer method:

IF the region we revole to generate a solid does not border on or cross the axis of revolution, then the solid generated has a hole in-side.



outer radious $R(x)$

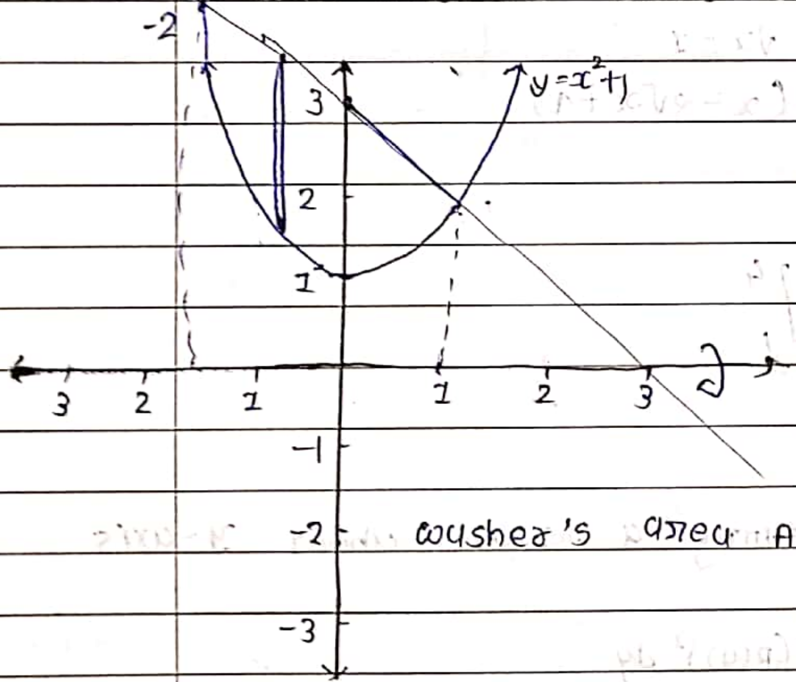
Inner radious $r(x)$

washer's area $A(x)$

$$= \pi \left((R(x))^2 - (r(x))^2 \right)$$

$$\text{volume} = \int_a^b A(x)\, dx$$

ex: The rigin bounded by the curve $y = x^2+1$ and the line $Y = -x+3$ is revoluted about x axis to generate a solid. find the volume of the solid.



$R(x) = -x+3$

$r(x) = x^2+1$

washer's area $A(x) = \pi \left(-x+3 - x^2 - 1 \right)$

$= \pi \left(-x + 2 - x^2 \right)$

$$\text{Volume} = \int_{-2}^{1} \pi \left(-x + 2 - x^2 \right) dx$$

$$\pi \left[\frac{-x^2}{2} + 2x - \frac{x^3}{3} \right]_{-2}^{1}$$

$$= \frac{117}{5} \pi$$

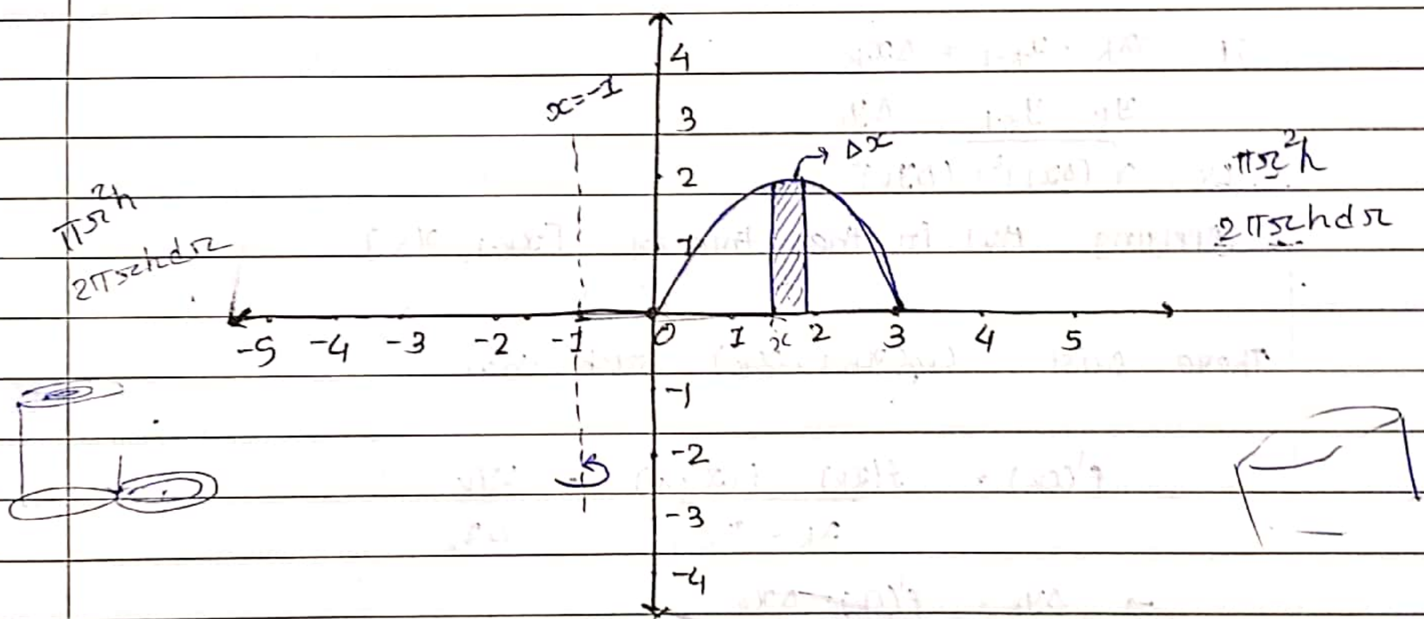
Volume of a solid using cylinderal shells.

suppose we slice through the solid using circular cylinder of incresing radii.

we slice straight down through the solid so that the axis of each cylinder is Parallel to y axis.

The vertical axis of each cyliner is the same but the raddii of the cylinder increse with each slice.

Ex: The region enclosed by the x axis and the parabola $y = 3x - x^2$ is revoluted about the verticle line $x = -1$ to generate a solid find the volume of the solid

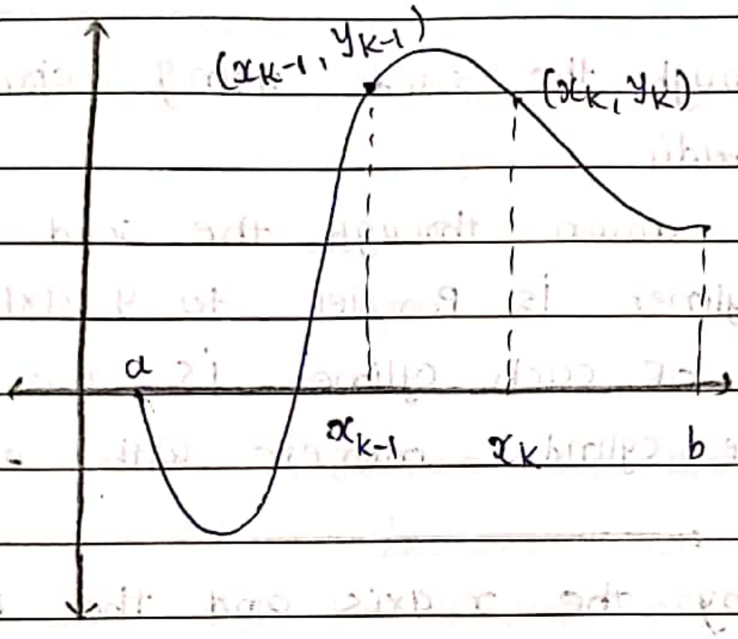


A cylindrical shell of height y_x is obtained by rotating a vertical strip of thickness Δx_k about the line $x = -1$.

$$V = \int_0^3 2\pi (1+x)(3x - x^2)\, dx$$

$$= \int_a^b 2\pi (\text{shell radius})(\text{shell height})\, dx$$

Lenth of curve



$$l_k = \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$$

IF $\quad x_k - x_{k-1} = \Delta x_k$

$\quad\quad y_k - y_{k-1} = \Delta y_k$

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

Applying MVT in the interval $[x_{k-1}, x_k]$

There exist $c_k \in (x_{k-1}, x_k)$ such that

$$f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{\Delta y_k}{\Delta x_k}$$

$$\Rightarrow \Delta y_k = f'(c_k) \cdot \Delta x_k$$

$L_k =$

$$L_k = \sqrt{(\Delta x_k)^2 + (f'(c_k) \cdot (\Delta x_k))^2}$$

$$= \Delta x_k \sqrt{1 + f'(c_k)}$$

taking the sum of all these lenth of the cords

$$\sum_{k=1}^{n} = \sum_{k=1}^{n} \sqrt{1+(f'(c_k)^2)} \cdot \Delta x_k$$

taking $n \to \infty$
we will get

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1+(f'(c_k))^2} \cdot \Delta x_k$$

But interval is very small

$$= \int_a^b \sqrt{1+(f'(x))^2}\, dx$$

Defination: If $f'(x)$ is continuous on $[a,b]$ than the arc Lenth of the curve $y = f(x)$ From the point $A = (a, f(a))$ to $B = (b, f(b))$ is

$$\int_a^b \sqrt{1+(f'(x))^2}\, dx.$$

example Find the length of the curve

$$f(x) = \frac{x^3}{12} + \frac{1}{x} \qquad 1 \le x \le 4$$

$$\Rightarrow f'(x) = \frac{3x^2}{12} - \frac{1}{x^2}$$

$$= \frac{x^2}{4} - \frac{1}{x^2}$$

$\Rightarrow$ length of the curve $= \int_1^4 \sqrt{1+\left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2}\, dx$

$$= \int_1^4 \sqrt{1+\left(\frac{x^4-4}{4x^2}\right)^2}\, dx$$

$$= \int_1^4 \sqrt{\frac{16x^4 + x^8 - 8x^4 + 16}{(4x^2)^2}}\, dx$$

$$= \int_1^4 \frac{x^4+4}{4x^2} = 6$$

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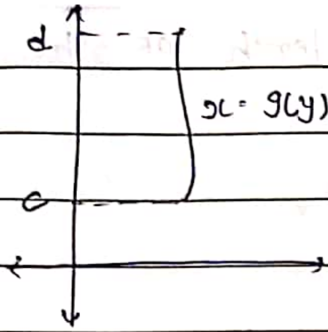
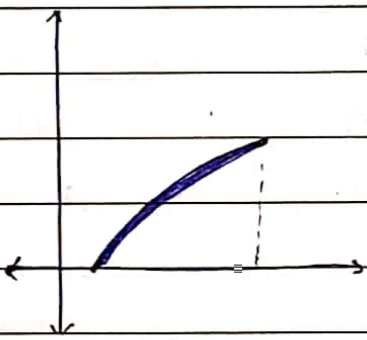


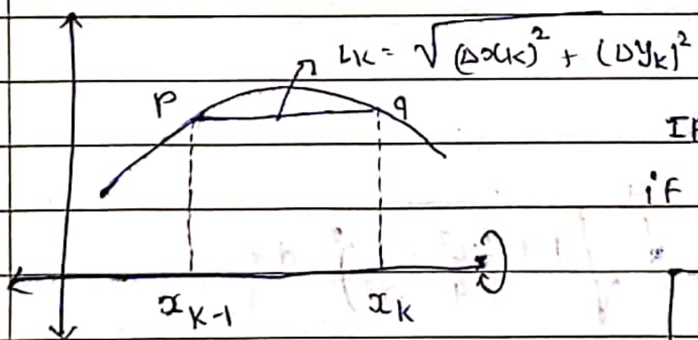
If the curve $x = g(y)$, $c \le y \le d$. If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from $A = (c, g(c))$, to $B = (d, g(d))$

is
$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dx = \int_c^d \sqrt{1 + (g'(x))^2}\, dx$$

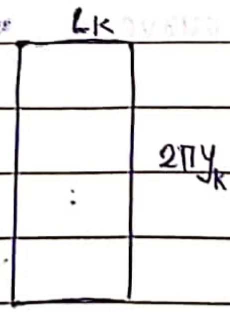
Areas of surfaces of revolution:



If you revolve only the boundary curve, it does not sweep out any interior volume. But rather a surface that surrounds the solid and forms a part of it's boundary.



It will be like rubber band if we cut it



surface area is $2\pi y_K L_K$

$V = 2\pi$

$$\text{Area} = 2\pi \cdot \frac{f(x_{k-1}) \cdot f(x_k)}{2} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

summing all this surfce are

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \pi \; f(x_{k-1}) + f(x_k) \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$= \sum_{k=1}^{n} \pi \cdot \left[f(x_{k-1}) + f(x_k)\right] \Delta x_k \sqrt{1 + (f'(x))^2}$$

taking $\lim_{n \to \infty}$

$$\lim_{n \to \infty} \sum_{k=1}^{n} S_k$$

$$= \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \; dx$$

Definition: IF the function $f(x) \geq 0$ is continuous and diffrentiable on (a,b), then the area of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \; dx$$

revolution about y axis

IF $x = g(y) \geq 0$ is continuous and diffrentiable on $[c,d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis

$$S = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} \; dx$$

$x^{1/2}$
$x^{-1/2}$
$\frac{1}{2} \cdot x^{-1/2}$

/ /

example: Find the area of the surface by revolving curve $y = 2\sqrt{x}$ $1 \le x \le 2$ about x axis.

$$\Rightarrow \text{Area} = \int_1^2 2\pi \, 2\sqrt{x} \left(\sqrt{1 + \left(\tfrac{1}{\sqrt{x}}\right)^2}\right) dx$$

$$= \int_1^2 2\pi \cdot \cancel{2\sqrt{x}} \, \frac{\sqrt{x+1}}{\cancel{\sqrt{x}}} \, dx$$

$$= 4\pi \int \cancel{\left[\frac{1}{2\sqrt{x+1}}\right]} \, dx$$

$$= \left[4\pi \cdot \frac{2(x+1)^{3/2}}{3}\right]_1^2$$

$$= \frac{8\pi}{3} \left[(3)^{3/2} - (2)^{3/2} \right) = \frac{8\pi}{3} \left(3\sqrt{3} - 2\sqrt{2} \right)$$

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functions of several variables :

$f: R \to R$

$f(x_1, x_2, x_3, \dots x_n)$

→ suppose we want to find volume of cone

$V = \frac{1}{3}\pi r^2 h = f(r,h)$

r and h are indipendent veriable

V is dependent variabe

→ radius of sphere $= \sqrt{x^2+y^2+z^2} = f(x,y,z)$

Functions of two variable :

→ Interior Point

A point (x_0, y_0) in a rigion R in the x-y plane is said to be an interior point of the region R if it is a centere of a disk of positvie radious that lies entiry in R

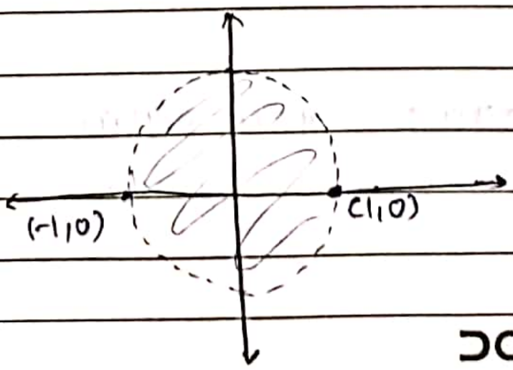
→ Boundry Point :

A point (x_0, y_0) is Boundry point of the rigion R if every disk centered at (x_0, y_0) contains points that lies out side of R as well as it contains points inside R. or in R

open rigion :

$\{(x,y) : x^2+y^2 < 1\}$

A region said to be open if all of it's points are interior Point.



close rigion :

If it contains all of it's Boundry points.

$R = \{(x,y) : x^2+y^2 = 1\}$ ~ it is close rigion

~ $R \cdot \{(x,y) : x^2+y^2 \leq 1\}$ closed point rigion

~ $\{(x,y) : x^2+y^2 < 1\}$ open rigion.

A rigion in the plane is bounded if it lies inside a disk of fixed radius.

~ A rigion is unbounded if it is not Bounded

Leavel curves :

The set of point in the plane where a function $f(x,y)$ has a constant value $f(x,y) = c$ is called a leavel function

example: $f(x,y) = x^2+y^2$

$\left.\begin{array}{l} x^2+y^2=1 \\ x^2+y^2=4 \end{array}\right\}$ Leavel function.

3 variable case.

Leuvel surface :

The set of point (x,y,z) in space where a function of 3 indipendent variable has a constent value $f(x,y,z) = c$ is called a Leavel surface.

$f(x,y,z) = x^2+y^2+z^2$

$\left.\begin{array}{l} x^2+y^2+z^2=1 \\ x^2+y^2+z^2=2 \end{array}\right\}$ Leavel surfaces

similarly we can define open rigion.
close rigion, ...

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Limits of a function of two variable:

If the values of $f(x,y)$ lie arbitrarily close to a fixed real number L for all points (x,y) sufficiently close to a point (x_0,y_0), we say that f approches to limit L as (x,y) approches (x_0,y_0)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

Precize defination:

If for $\varepsilon>0$ there exist a correspoint $\delta>0$, such that for all (x,y) in the domain of $|f(x,y)-L|<\varepsilon$ whenever $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$

so that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

example $\lim_{(x,y)\to(x_0,y_0)} x = x_0$

$f(x,y) = x$ and $L = x_0$

suppose there exists $\varepsilon>0$,

$|f(x,y)-L|<\varepsilon$ and $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta=\varepsilon$

$|x-x_0|<\varepsilon$

$\Rightarrow \sqrt{(x-x_0)^2}<\delta$

$\Rightarrow |x-x_0|<\delta$

so $\boxed{\varepsilon=\delta}$

example: find $\lim_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2}$ if it exist:

suppose we approch $(0,0)$ throught x axis.

then $y=0$ then $\lim_{(x,y)\to(0,0)} f(x) = 0$.

if we approch through y axis then $\lim_{(x,y)\to(0,0)} = 0$

Let $\varepsilon > 0$ be given

we want to find $\delta > 0$, such that

$$\left| \frac{4xy^2}{x^2+y^2} - 0 \right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta$$

$$\frac{4|x|y^2}{x^2+y^2} < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta$$

since $y^2 < x^2+y^2$

so we write $\dfrac{4|x|y^2}{x^2+y^2} < \dfrac{4|x|\,(x^2+y^2)}{x^2+y^2} = 4|x| =$

$= 4\sqrt{x^2}$

$< 4\sqrt{x^2+y^2} < \varepsilon$

$$\delta = \varepsilon/4$$

so that $0 < \sqrt{x^2+y^2} < \delta \Rightarrow \dfrac{4xy^2}{x^2+y^2} < 4\sqrt{x^2+y^2} = \varepsilon)$

Properties of limit :

IF $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$

then

→ 1) $\lim_{(x,y)\to(x_0,y_0)} \left(f(x,y) \pm g(x,y) \right) = L \pm M$

→ 2) $\lim_{(x,y)\to(x_0,y_0)} (k \cdot f(x,y)) = kL$

k is a real constant

→ 3) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)\, g(x,y) = LM$

→ 4) $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = L/M$ Provided $M \neq 0$

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5) $\lim_{(x,y)\to(x_0,y_0)} (f(x,y))^n = L^n$

where n is positive

6) $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$.

where n is +ve integer and

when n is even we assume $L > 0$

<u>example</u>: $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

$\lim_{(x,y)\to(0,0)} \frac{\cancel{\sqrt{x}}(x^{3/2} - \sqrt{xy})}{\cancel{\sqrt{x}}(1-} \quad \frac{x(x-y)(\sqrt{x}+\sqrt{y})}{x-y}$

$= 0.$

<u>example</u> Does $\lim_{(x,y)\to(0,0)} y/x$ exist.

suppose we approach $(0,0)$ along x axis

so $y=0$ and $x \neq 0$

so $\lim_{(x,y)\to(0,0)} 0/x = 0$

But if we approach the origin $y=x$ line

then $\lim_{(x,y)\to(0,0)} y/x = 1.$

For different path of approach we are getting a different limit. this implies limit does not exist.

in other way if we approach origin along $y = mx + \cancel{c}$ line

so $\lim_{(x,y)\to(x_0,y_0)} y/x = mx/x = m$

so for different values of m we get different limit

so limit does not exist.

changing to polar co-ordinate:

IF you can not make any head way with $\lim_{(x,y)\to(0,0)} f(x,y)$ in rectangular co-ordinates, try changing to polar co-ordinate

Substitute $x = r\cos\theta$ and $y = r\sin\theta$ and investigate the limit of the resulting expression $r \to 0$.

Given $\varepsilon > 0$, there exist $\delta > 0$ such that for all r and θ

$$0 < |r| < \delta \Rightarrow |f(r,\theta) - L| < \varepsilon.$$

means

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} F(r,\theta) = L$$

example: $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$

$x = r\cos\theta \quad y = r\sin\theta$

so $F(r,\theta) = \frac{4r\cos\theta \cdot r^2\sin^2\theta}{r^2}$

$= 4r\cos\theta \cdot \sin^2\theta$

now $\lim_{r\to 0} F(r,\theta) = 4r\cos\theta \cdot \sin^2\theta$

$= 4(0)$

$= 0.$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0.$

By using precise defination

$$|f(x) - L| < \varepsilon$$

$$|4r\cos\theta \cdot \sin^2\theta| < 4r < \varepsilon$$

$$4r < \varepsilon$$

$$r < \varepsilon/4$$

$$\text{so } \boxed{\delta = \varepsilon/4}$$

example: $\lim_{(x,y)\to(0,0)} \dfrac{x^3}{x^2+y^2}$

taking $x = r\cos\theta$ and $y = r\sin\theta$

then $\lim_{(r,\theta)\to 0} \dfrac{r^3\cos^3\theta}{r^2}$

$= 0$

so $\lim_{(x,y)\to(0,0)} \dfrac{x^3}{x^2+y^2} = 0$

By using Pricize defination

$$|f(x) - L| < \varepsilon$$

$$|r\cos^3\theta| < r < \varepsilon$$

$$\text{so } \boxed{\varepsilon = \delta}$$

example: $\lim_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$

$x = r\cos\theta \quad y = r\sin\theta$

$\lim_{r\to 0} \cos^2\theta$

so limit does not exist. because for different value of θ value of limit will be diffrent

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example: $f(x,y) = \frac{2x^2y}{x^4+y^2}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$

$$\lim_{r\to 0} \frac{2\cdot r^2\cos^2\theta \cdot r\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta}$$

$$\lim_{r\to 0} \frac{2r^3\cos^2\theta\cdot\sin\theta}{r^2(r^2\cos^4\theta+\sin^2\theta)}$$

$$\lim_{r\to 0} \frac{2r\cos^2\theta\cdot\sin\theta}{r^2\cos^4\theta+\sin^2\theta}$$

$= \cancel{0}.$ at $\theta = 0, 2\pi, \pi$

limit does not exist.

so $f(r,\theta)$ does not exist when $\theta = 0, \pi, 2\pi, \dots$

If we approach $(0,0)$

By $y = mx^2$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2\cdot mx^2}{x^4+m^2x^4} = \frac{2m}{1+m^2}$$

so for different values of m we will get diffrent ans

example: $\lim_{(x,y)\to(0,0)} \frac{2x}{x^2+x+y^2}$

taking $x = r\cos\theta$ and $y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{2r\cos\theta}{r^2+r\cos\theta} = \lim_{\cancel{(x,y)}\, r\to 0} \frac{2r\cos\theta}{r(\cancel{r}+\cos\theta)}$$

when $\theta = \pi/2, 3\pi/2,$

so we can not cancel out.

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Continuity :

A function $f(x,y)$ is continuous at the point (x_0, y_0) if

(i) f is defined at (x_0, y_0)

(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exist.

and both should be same

example: $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & ; (x,y) \neq 0 \\ 0 & ; x = 0 \end{cases}$

At $(0,0)$

$$f(0,0) = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim_{r\to 0} \frac{2r^2 \sin\theta \cdot \cos\theta}{r^2} = \sin 2\theta$$

For diffrent value of θ value of $\sin 2\theta$ will diffrent.

so limit does not exist.

$\Rightarrow$ $\boxed{\text{If } f \text{ is continuous at } (x_0, y_0) \text{ and } g \text{ is a single variable function continuous at } f(x_0, y_0), \text{ then the composite function is continuous at } (x_0, y_0)}$

example: $\cos\left(\dfrac{xy}{x^2+1}\right)$

Similarlly $\ln(x^2y^2+1)$ is continuous.

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Function with 3 variables:

In 3 dimention case a δ radii disc will be replace by δ radius sphere.

so defination will be

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta \Rightarrow |f(x,y,z) - L| < \varepsilon$$

Partial derivative:

The partial derivative of $z = f(x,y)$ with respect to x at point (x_0, y_0) is $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$

we can denote it by f_x or $\frac{\partial f}{\partial x}$, Z_x, $\frac{\partial f}{\partial x}$

means this is equal to $\frac{d}{dx} f(x, y_0)$

we treat y_0 as a constant.

[Rate of change of surface in x direction]

→ The partial derivative of $f(x,y)$ with respect to y at the point (x_0,y_0) is $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0}$.

$$= \lim_{k \to 0} \frac{f(x_0, y+k) - f(x_0, y_0)}{k}.$$

example: $f(x,y) = \frac{2y}{y+\cos x}$

$$\frac{\partial f}{\partial x} = 2y \cdot \frac{d}{dx} \frac{1}{y+\cos x} = 2y \cdot \left(\frac{-1}{(y+\cos x)^2}\right) \cdot (-\sin x)$$

$$= \frac{2y\sin x}{(y+\cos x)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(y+\cos x)\cdot 2 - 2y(1+0)}{(y+\cos x)^2} = \frac{2\cos x}{(y+\cos x)^2}$$

Implicict partial derivative:

$$yz - \ln z = x+y$$

z is a function of two independent variable x and y.

$$\frac{\partial z}{\partial x} = ? \qquad \frac{\partial z}{\partial y} = ?$$

taking the partial derivative with respect to x.

$$\frac{\partial}{\partial x} yz - \frac{\partial}{\partial x}(\ln z) = \frac{\partial}{\partial x}(y+x)$$

$$y\frac{\partial z}{\partial x} - \frac{\partial}{\partial x}(\ln z) = 1$$

$$\therefore y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{dz}{dx} = 1$$

$$\frac{\partial z}{\partial x} = \frac{1}{(y-1/z)}$$

⇒ The plane $x=1$ intersects the paraboloid $z=x^2+y^2$ in a parabola find the slop of a tangent to the parabola at point $(1,2,5)$ in y direction.

$$z = x^2+y^2$$

$$\left.\frac{\partial z}{\partial y}\right|_{y=2} = 2y = 2(2) = 4.$$

functions having more than two variables, the concepts of partial derivadive are similar

$\Rightarrow$ IF there exists partial derivative of function, It is not compulsary that function is continuous.

example: $f(x,y) = \begin{cases} 0 & , xy \neq 0 \\ 1 & , xy = 0 \end{cases}$ At $(0,0)$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1-1}{h}$$

$$= 0$$

Similarly $\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{K\to 0} \frac{f(0,0+K) - f(0,0)}{h}$

$$= \lim_{K\to 0} \frac{\cancel{0}\ 1-1}{h} = 0$$

if we approch the origion along the line $y = x$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

now we will approch along the x axis

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1$$

So we can say thet function is not continuous.

Second order partial derivative:

$$f_{xy} = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \cdot \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{xx} = \frac{\partial^2 f}{\partial x \cdot \partial x} = \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y \cdot \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

example: $f(x,y) = x\cos y + ye^x$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(x\cos y + ye^x)\right) = \frac{\partial}{\partial x}(-x\sin y + e^x)$$

$$= -\sin y + e^x$$

$$= e^x - \sin y$$

both are same (it is not true for all the fyn^n)

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(x\cos y + ye^x)\right) = \frac{\partial}{\partial y}(\cos y + ye^x)$$

$$= -\sin y + e^x$$

$$\partial_{xx} = \frac{\partial}{\partial x}(\cos y + ye^x) = 0 + ye^x$$

$$\partial_{yy} = \frac{d}{\partial y}(x(-\sin y) + e^x))$$

$$= -\cos y\, x + 0.$$

Mixed derivative theorem:

IF $f(x,y)$ and it's partial deriatives f_x, f_y, f_{xy}, and f_{yx} exist throughout an open region contuing the point (a,b) and are all continuous at (a,b) then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

<u>example</u>: $f(x,y) = \dfrac{x^3}{x^2+y^2}$. Find derivative at $(0,0)$

$$\partial f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{x^3}{x^2+y^2}\right)\right) = \frac{\partial}{\partial x}\, x^3\left(\frac{-1}{(x^2+y^2)^2}\right)(2y)$$

Differentiablity :

A function $z = f(x, y)$ is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist at Δz satisfying the eq^n of form

$$\Delta z = f_x(x_0, y_0) \cdot \Delta x + f_y(x_0, y_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of ε_1 and ε_2 goes to zero as both Δx and Δy goes to zero.

$z = f(x, y)$ so $\left.\partial z\right|_{(x_0, y_0)} = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} + \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}$

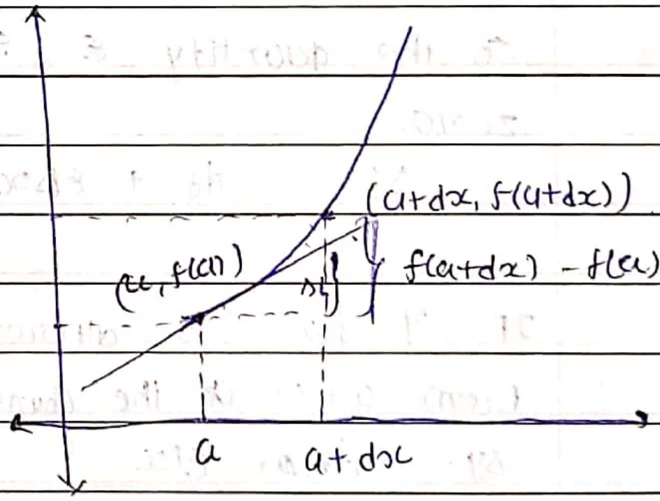
→ suppose $w = f(x, y, z)$

$$dw = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

→ $y = f(x)$

$$\frac{dy}{dx} = f'(x)$$

$\therefore$ $dy = f'(x)\, dx$



• Let $x = a$ and set $dx = \Delta x$.

the corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a)$$

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$\Delta L = L(a + dx) - L(a)$

$= f(a) + f'(a)(a + dx - a) - f(a)$

$\Delta L = f'(a)\,dx$

⇒ Error in the differential approximation

True change in the function

$\Delta f = f(a + \Delta x) - f(a)$

and differential estimate is

$df = f'(a)\,\Delta x$

So Approximate error $= \Delta f - df$

$= f(a + \Delta x) - f(a) - f'(a)\,\Delta x$

$= \left(\frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a)\right)\Delta x$

as $\Delta(x)$ tends to zero

$= \left(f'(a) - f'(a)\right)dx$

$= 0$

So the quantity ϵ goes to zero when Δx goes to zero.

$\Delta f = df + \epsilon\Delta x$

If $y = f(x)$ is differentiable at $x = a$ and x changes from $a + \Delta L$ on the change Δy in f is given by

$\Delta y = f'(a)\Delta x + \epsilon\Delta x$.

In which $\epsilon \to 0$ then $\Delta x \to 0$.

two variable case :

$z = f(x, y)$

suppose that the 1st Partial derivative of $f(x,y)$ are defined throughout the open region R containing the point (x_0, y_0) and f_x at f_y are continuous at (x_0, y_0) then

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

In which ε_1 and $\varepsilon_2 \to 0$ then Δx and Δy goes to zero.

Defination:

A function $z = f(x, y)$ is differentiate if (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exists and continuous also and Δz satisfy this eq^n

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta x$$

when ε_1 and ε_2 goes to zero then Δx and Δy goes to zero.

$$dz = \frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz$$

If $w = f(x, y, z)$

then

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

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If the partial derivative f_x and f_y of a function $f(x,y)$ exist and are continuous then f is differentiable

If a function is continuous at (x_0, y_0), then it is continuous at (x_0, y_0).

Proof: suppose f is differentiable at (x_0, y_0), then

$$\Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$$

$$= f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

this goes to zero as $\Delta x \to 0$ and Δy goes to zero.

$$\lim_{(x,y)\to(x_0,y_0)} f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) = 0$$

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x_0+\Delta x, y+\Delta y) = f(x_0, y_0)$$

so f is continuous.

chain rule:

If $\omega = f(x,y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable at t, then $\omega = f(x(t), y(t))$ is differentiable with respect to t.

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{dt}$$

$\omega = f(x,y,z)$ $\quad x(t),\ y(t),\ z(t)$

$$\frac{\partial w}{\partial t} = \frac{dw}{dx}\cdot\frac{dx}{dt} + \frac{dw}{dy}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

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example: $\omega = xy + z$ $x = \cos t$

$y = \sin t$

$z = t$

Find $\frac{dw}{dt} = ?$

$\omega = xy + z$

$$\frac{dw}{dt} = x\frac{dy}{dt} + y \cdot \frac{dx}{dt} + \frac{dz}{dt}$$

$$= x(\cos t) + y(-\sin t) + 1$$

$$= x\cos t - y\sin t + 1$$

$$= \cos^2 t - \sin^2 t + 1$$

$$= 2\cos^2 t$$

Chain rule for two indtipendent variable and three intermediate variable

$\omega = f(x, y, z)$

$x = g(r, s)$

$y = h(r, s)$

$z = k(r, s)$

If all four function are differentiable then ω has partial derivative with respect to r and s. and

$$\frac{\partial w}{\partial r} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$\frac{dw}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}.$$

example. $\omega = x + 2y + z^2$ $x = \frac{r}{2}$ $y = r^2 + \ln s$ $z = 2r$

Find $\frac{\partial \omega}{\partial r}$ and $\frac{\partial \omega}{\partial s}$

$$\frac{\partial \omega}{\partial r} = \frac{\partial x}{\partial r} + \frac{2\partial y}{\partial r} + \frac{2z \cdot \partial z}{\partial r}$$

$$= \frac{1}{2} + 2 \cdot \left(2r + \frac{1}{s} \cdot \frac{\partial s}{\partial r}\right) + 2z \cdot 2$$

$$= \frac{1}{2} + 4r + \frac{1}{s}\frac{\partial s}{\partial R} + 4r$$

$$\frac{\partial \omega}{\partial s} =$$

Implict defferentiation:

(i) suppose $F(x,y)$ is diffrentiable

(ii) the function $F(x,y)=0$ defines y implicity as a diffrentiable function of x.

Since $w = F(x,y) = 0$

$$\frac{dw}{dx} = 0$$

$$\therefore 0 = \frac{dw}{dx} = \frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{dx}$$

$$= \cancel{f_x}$$

$$= F_x + F_y \cdot \frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-F_x}{F_y}}$$ provided $F_y \neq 0$.

dependent

IF $f(x,y,z)=0$ is diffrentiable

then

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= \frac{-F_x}{F_z} \\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} \end{aligned}}$$

example: $x^3 + z^2 + ye^{xz} + z\cos y = 0$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(means z is dependent and x and y is independent)

Directional Derivative :

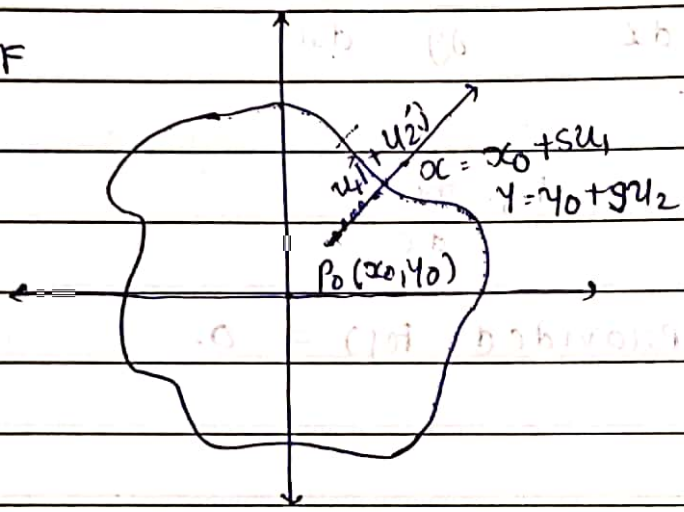
suppose that that function $f(x,y)$ is defined througout the rigion R in the xy plane.

$P_0(x_0, y_0)$ is a point in R

and $u = u_1\hat{i} + u_2\hat{j}$ is a unit vector.

Then the eq^n $x = x_0 + su_1$, $y = y_0 + su_2$ parameter eq^n of the line through P_0 $\parallel^r$ to the given u.

If



The rate of change of f at P_0 in the directio of u is

$$\left(\frac{df}{ds}\right)_{u, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + sy_2) - f(x_0, y_0)}{s}$$

provided that limit exist

If $u = \hat{i}$ $u_1 = 1$, $y_2 = 0$

so $$\left(\frac{df}{ds}\right)_{u, P_0} = \lim_{s \to 0} \frac{f(x_0 + s, y_0 + 0) - f(x_0, y_0)}{s}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0) = P_0.}$$

Q: $u = \hat{j}$ $\quad u_1 = 0, \; u_2 = 1$

$$\frac{df}{ds} = \lim_{s \to 0} \frac{f(x_0 + 0, y_0 + s) - f(x_0, y_0)}{s}$$

$$= \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}$$

example: Find the derivative of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction of the unit vector

$$u = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$\left.\frac{\partial f}{\partial s}\right|_{u, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

$$= \lim_{s \to 0} \frac{f(x_0 + s/\sqrt{2}, y_0 + s/\sqrt{2}) - f(x_0, y_0)}{s}$$

$$= \lim_{s \to 0} \frac{(x_0 + s/\sqrt{2})^2 + (x_0 + s/\sqrt{2})(y_0 + s/\sqrt{2}) - 3}{s}$$

$$= \lim_{s \to 0} \frac{(1 + s/\sqrt{2})^2 + (1 + s/\sqrt{2})(2 + s/\sqrt{2}) - 3}{s}$$

$$= \lim_{s \to 0} \frac{\cancel{1} + \sqrt{2}s + s^2/2 + \cancel{2} + s/\sqrt{2} + \sqrt{2}s + s^2/2 - \cancel{3}}{s}$$

$$= \lim_{s \to 0} \sqrt{2} + \frac{s}{2} + \frac{1}{\sqrt{2}} + \sqrt{2} + \frac{s}{2}$$

$$= \sqrt{2} + 1/\sqrt{2} + \sqrt{2}$$

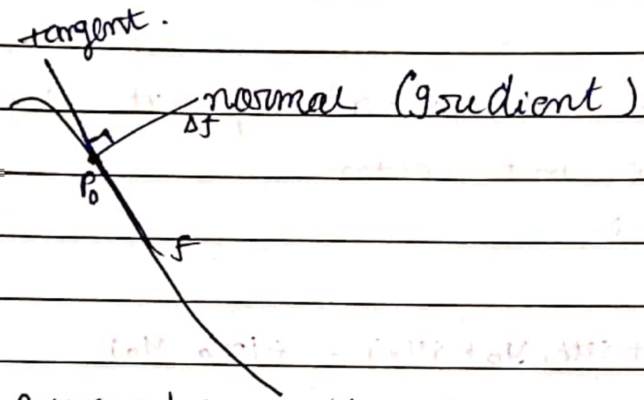
$$= 2\sqrt{2} + 1/\sqrt{2}$$

$$5/\sqrt{2}$$

Gradient vector :

The gradiand vector or gradient of $f(x,y)$ at a point $P_0(x_0, y_0)$ is

$$\nabla f = \left.\frac{\partial f}{\partial x}\right|_{P_0} \hat{i} + \left.\frac{\partial f}{\partial y}\right|_{P_0} \hat{j}$$



Relationship with directional Derivative

$x = x_0 + su_1$

$y = x_0 + su_2$

$$\left.\frac{df}{ds}\right|_{P_0(x_0,y_0)} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \cdot \frac{\partial x}{\partial s} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \frac{dy}{\partial s}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{df}{\partial y}\right)_{P_0} u_2$$

$$= \left(\left(\frac{\partial f}{\partial x}\right)\hat{i} + \left(\frac{df}{dy}\right)\hat{j}\right) \cdot (u_1\hat{i} + u_2\hat{j})$$

$$= \nabla f|_{P_0} \cdot u$$

$$\frac{df}{ds} = D_u f = \vec{\nabla F} \cdot \vec{u}$$

$$= |\vec{\nabla F}||\vec{u}| \cos\theta$$

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1) when $\theta = 0$ then $\cos\theta = 1$

$\frac{\partial f}{\partial s}$ the function f increases most rapidly when u is in the direction of the f.

2) similarly f decreses most rapidly in the direction of $-\nabla f$

3) when $\theta = \pi/2$ then $d_u f$

$d_u f = 0$ or $f \neq 0$

$$D_u f = \nabla f \cdot \vec{u} = |0| = 0$$

any direction u orthogonal to the gradient ∇f ($\neq 0$) is a direction.

example: Find the direction in which $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

(i) incresing most rapiduly at the point (1,1)

(ii) decreses most rapidly at (1,1)

(iii) what are the direction of no change in f at (1,1)?

Soln:

$$\nabla f|_{(1,1)} = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j}$$

$$= x\hat{i} + y\hat{j}\,|_{(1,1)}$$

$$= \hat{i} + \hat{j}$$

it's direction $= \frac{\hat{i} + \hat{j}}{\sqrt{2}}$

(ii) $= \frac{-1}{\sqrt{2}}\hat{i} + \frac{-1}{\sqrt{2}}\hat{j}$

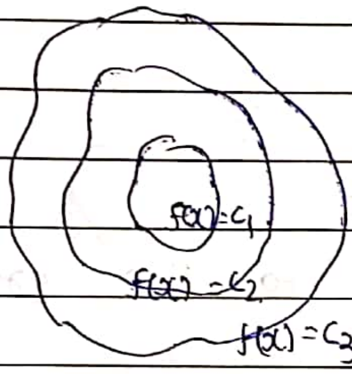
(iii) the directions of no change

$\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$ and $\frac{-1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

Gradient AND TANGENT to LEAVEL CURVE

If a differentiable function $f(x, y)$ has a constant value c along a smooth curve

$$r = g(t)\,\hat{i} + h(t)\,\hat{j}$$



$$\frac{d}{dt}\Big(f\big(g(t), h(t)\big)\Big) \qquad x = g(t), \quad y = h(t)$$

$$= \frac{d}{dt}(c) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x}\cdot\frac{dg}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dg}{dt} = 0$$

$$\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\cdot\left(\frac{dg}{dt}\hat{i} + \frac{dg}{dt}\hat{j}\right) = 0$$

$$\nabla f \cdot \frac{dr}{dt} = 0$$

normal to curve

tangent of curve

The eq^n of tangent line at (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} (y-y_0) = 0$$

eq^n: Find the eq^n of the tangent line to the elips $\frac{x^2}{4} + y^2 = 2$ at Point $(-2, 1)$

sol^n:

$$\frac{x}{2} + 2y\frac{dy}{dx} = 0$$

$$-1 + 2\frac{dy}{dx} = 0$$

$$\left(\frac{dy}{dx}\right)_{(-2,1)} = \frac{1}{2}$$

eq^n of line

$$(y-1) = \frac{1}{2}(x+2)$$

$$2y - 2 = x + 2$$

$$x - 2y + 4 = 0$$

$$\frac{x_0}{2}(x-x_0) + 2y_0(y-y_0) = 0$$

$$-(x+2) + 2(y-1) = 0$$

$$-x - 2 + 2y - 2 = 0$$

$$2y - x - 4 = 0$$

$$x - 2y + 4 = 0$$

property.

① $\nabla(f+g)$

② ∇

③

tangent plane:

Tangent plane to $f(x,y,z) = C$ at $P_0(x_0, y_0, z_0)$ is

$$\left.\frac{\partial f}{\partial x}\right|_{P_0}(x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{P_0}(y-y_0) + \left.\frac{\partial f}{\partial z}\right|_{P_0}(z-z_0) = 0$$

Normal line to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$ is

$x = x_0 + f_x(P_0)t$ $\quad f_x = \partial f/\partial x$

$y = y_0 + f_y(P_0)t$ $\quad f_y = \partial f/\partial y$

$z = z_0 + f_z(P_0)t.$ $\quad f_z = \partial f/\partial z$

Example: Find the tangent plane and normal line to the surface $f(x,y,z)$ $x^2+y^2+z-9=0$ at $P_0 : (1,2,4)$

$\left.\frac{\partial f}{\partial x}\right|_{P_0} = 2x = 2$ $\quad \left.\frac{\partial f}{\partial y}\right|_{P_0} = 2y = 4$ $\quad \left.\frac{\partial f}{\partial z}\right|_{P_0} = 1$

→ tangent plane at point P_0

$$2(x-1) + 4(y-2) + \overset{1}{\cancel{4}}(z-4) = 0$$

$$2x - 2 + 4y - 8 + \cancel{4z} - \overset{4}{\cancel{16}} = 0$$

$$\cancel{2x} + \cancel{4y} + \cancel{4z} - \cancel{26} = 0$$

$$x + 2y + 2z - 13 = 0$$

$$2x + 4y + z = 14$$

eqn of the normal line

$x = 1 + 2t$

$y = 2 + 4t$

$z = 4 + t$

extreme value of the surface :

Def^n : Let $f(x,y)$ be defined on a region R contuing the point (a,b).

(i) Then the point (a,b) is sat to be a local max^m and f is $f(a,b) \geq f(x,y)$ for all (x,y) in open disc centered at (a,b)

(ii) (a,b) is a local minimum and if for (a,b) $f(a,b) \leq f(x,y)$ for all (x,y) in open disc centered at (a,b).

first derivative test :

If $f(x,y)$ has a local maximum or minimum at an interior point (a,b) of its domain and if the 1st partial derivative exist there, then

$$f_x(a,b) = 0 \quad \text{and} \quad f_y(a,b) = 0$$

Critical Point :

An interior point of the domain of a function $f(x,y)$ where both f_x and f_y are zero or does not exist is called critical point of f.

point of inflection :

Def^n : saddle point

A differentiable function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b) there are domain points (x,y) where $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$ then $((a,b), f(a,b))$ is a saddle point of the surface

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ $f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

Second derivative test :

suppose that $f(x,y)$ and it's First and second Partial derivative are continuous through out all disk centered at (a,b) and $f_x(a,b) = f_y(a,b) = 0$ Then

(I) f has a local maximum at (a,b)
$f_{xx} < 0$ and $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$ at the the point ab.

(II) f has a local minimum at (a,b) if
$f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b)

(III) f has a saddle point at (a,b) if
$f_{xx} \cdot f_{yy} - f_{xy}^2 < 0$ at (a,b).

(IV) the test is inconclussive if
$f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a,b).

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$ Hessian matrix.

$$\det(H) = f_{xx} f_{yy} - f_{xy}^2.$$

example: find the local extreme values of
$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$$

$\frac{\partial f}{\partial x} = y - 2x - 2 = 0$

$\frac{\partial f}{\partial y} = x - 2y - 2 = 0$

the function is defined and differentiable for all x and y and it's domain has no boundry.

the function has extreme value only at the Point where f_x any f_y symuntesly zero.

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by sloving both eq^n

we get $x = -2$ and $y = -2$

$f_{xx} = -2 < 0$

so $(-2,-2)$ is the critical point.

$f_{yy} = -2 < 0$

$f_{xy}\big|_{(-2,-2)} = 1$

so $f_{xx} f_{yy} - (f_{xy})^2 = 3 > 0$

so f has a local $maxi^m$ at (a,b).

and value is $f(-2,-2) = 4 - 4 - 4 + 4 + 4 + 4$

$= 8$

since there are no other local maximum and no boundry $(-2,-2)$ is global maximum $f = f(-2,2) = 8$

find the local extreme values of.

$f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

the function is defined and differentiable for all x and y and it's domain has no boundry.

sol^n: $f_x = -6x + 6y = 0$

$x = y$

$f_y = 6y - 6y^2 + 6x = 0$

$\left| \begin{array}{c} x = 0 \\ y = 0 \end{array} \right|$ and $(2,2)$

$f_{xx} = -6 < 0$

$f_{yy} = 0$ so at $(0,0)$ f has local max^m

$f_{xy}\big| = \frac{\partial}{\partial x}\left(6x\right) = 6 > 0$ saddle point $(0,0)$

$f_{xx} f_{yy} - (f_{xy})^2 = -36 < 0$

DOMS

$f_{xx}\Big|_{(2,2)} = -6 < 0$

$f_{xy} = \dfrac{\partial}{\partial x}\left(6y - 6y^2 + 6x\right) = 6 > 0$

$\therefore$ f has max^m value at $(2,2)$

So at point $(0,0)$ and $(2,2)$ f has maximum value.

Global maximum and minimum on closed bounded region

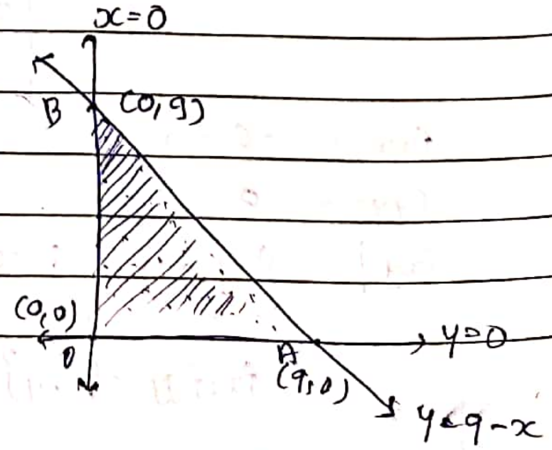
1) List the interior point of R where f may have local maximum and minimum and evolute f at there points. there are crical Points of f.

2) List the boundary Points of R where f has local maximum and minimum and evaluate f at these point.

3) Look through the list for the maximum and minimum values of f. these will be the global max^m and min^m values of f on R.

example: Find the absolute maximum and minimum values of f $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on a triangular region in the first quadrent bounded by the lines $x=0$, $y=0$, $y = 9 - x$

Sol^n



DOMS

Since f is differentiable, the only places where f can assume the values are points inside the triangle where $f_x = f_y = 0$, and points on the boundary.

1) Interior point

$f_x = 0$, and $f_y = 0$

$2 - 2x = 0$ $\quad$ $2 - 2y = 0$

$x = 1$ $\quad$ $y = 1$

at the point $(1,1)$

$f(1,1) = 4$

2) Boundary points.

on the line sagment OA

$(0,0) \rightarrow f(0,0) = 2$

and $f(9,0) = -61$

It extreme value may occure at interior points of OA

$f(x,y) = f(x,0) = 2 + 2x - x^2$

$f'(x) = 2 - 2x \Rightarrow x = 1, y = 0$

$f(1,0) = 3$

on the line sagment Ob

$(0,9) \Rightarrow f(0,9) = -61$

$(0,0) \Rightarrow 2$

$f(x,y)$ for interior point on OB

$f(x,y) = f(0,y) = 2 + 2y - y^2$

$f'(y) = 2 - 2y$

$y = 1$

$f(0,1) = 3.$

for AB sagment $y = 9 - x$

$f(x, 9-x) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$

$= 2 + \cancel{2x} + 18 - \cancel{2x} - \cancel{x^2} - 81 + 18x - \cancel{x^2}$

$= -61 + 18x - x^2$

81
20
61

$f'(x) = 18 - 2x$ $\quad$ $x = 9$ $\quad$ $f(9/2, 9/2)$

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$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} \quad z$$

$f(9/2, 9/2) = -41/2$

Global maximu

$f(1,1) = 4$ $\quad f(1,0) = 3$ $\quad f(9/2, 9/2) = -41/2$

$f(0,0) = 2$ $\quad f(0,9) = -61$

$f(9,0) = -61$ $\quad f(0,1) = 3$

Global minimum

Hassian matrix of 3 variabe function

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

$A_1 = |f_{xx}|$ $\quad A_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yz} & f_{yy} \end{vmatrix}$ $\quad A_3 = \text{Det } |u|$

$A_1 \to < 0$, $\quad A_2 \to > 0$, $\quad A_3 \to < 0$

then it is local max^m

Defination:

A function $z = f(x, y)$ is differentiable at (x_0, y_0) if

(i) $f_x(x_0, y_0)$ exist.

(ii) $f_y(x_0, y_0)$ exist.

and the change $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) \cancel{z_0 + \Delta z} - f(x_0, y_0)$

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

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example: show that

$$f(x,y) = \begin{cases} \dfrac{xy\,(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$

$$f_x|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h}$$

$$= 0$$

$$f_y|_{(0,0)} = \lim_{h\to 0} \frac{f(0, 0+h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} 0-0/h$$

$$= 0.$$

$$\Delta z = f_x(0,0)\Delta x + f_y(0,0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$= 0 + 0 + \epsilon_1 \Delta x + \epsilon_2 \Delta y \quad \text{--- (1)}$$

$$\Delta z = f(0+\Delta x, 0+\Delta y) - f(0,0)$$

$$= f(\Delta x, \Delta y) - 0$$

$$= \frac{\Delta x \cdot \Delta y\,(\Delta x^2 - \Delta y^2)}{\Delta x^2 + \Delta y^2} \quad \text{--- (2)} \quad \cdot \quad \frac{\Delta x^3 \Delta y}{\Delta x^2 + \Delta y^2} - \frac{\Delta y^3 \Delta x}{\Delta x^2 + \Delta y^2}$$

By computing eq^n (1) and (2)

$$\epsilon_1 = \frac{-\Delta y^3}{\Delta x^2 + \Delta y^2} \qquad \epsilon_2 = \frac{\Delta x^3}{\Delta x^2 + \Delta y^2}$$

But now $\Delta x \to 0$ and $\Delta y \to 0$

then ϵ_1 and ϵ_2 tends to zero.

f is differential at $(0,0)$

Taylor series :

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

$$= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots$$

$$f''(x) = 2a_2 + 6a_3(x-a) + \cdots + n(n-1)a_n(x-a)^{n-2} + \cdots$$

$$f'''(x) = 1\cdot2\cdot3\, a_3 + 2\cdot3\cdot4\, a_4(x-a) + \cdots + n(n-1)(n-2)a_n(x-a)^{n-3} + \cdots$$

$f^n(x) = n!\, a_n$ + sum of terms with $(x-a)$ as a factor

$f(a) = a_0$

$f'(a) = a_1$

$f''(a) = 2a_2 \Rightarrow a_2 = f''(a)/2!$

$f^n(a) = n!\, a_n$

$$a_n = \frac{f^n(a)}{n!}$$

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k$$

Taylor's Series of $f(x)$ about point a

Taylor's theorem:

If f and its 1^{st} n derivative $f', f'', \ldots f^n$ all are continuous on the close interval $[a, x]$, there exist a real number $c \in (a, x)$ such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \underbrace{R_n(x)}_{\text{Reminder}}$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$$|R_n(x)| \le \frac{M\,|x-a|^{n+1}}{(n+1)!} \quad \text{where} \quad |f^{(n+1)}(c)| \le M, \; c \in (a, x)$$

example: Find the taylor's series of $f(x) = e^x$ about $x = 0$ and find the error term

$f(x) = e^x$

$f(0) = 1$

$f'(x) = e^x \quad f'(0) = 1$

$f''(0) = 1 \ldots \quad f^n(0) = 1$

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-a) + \frac{f''(0)}{2!}(x-a)^2 + \cdots + \frac{f^n(0)}{n!}(x-a)^2$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \cancel{f^n t}\,\frac{x^n}{n!}$$

$$R_n(x) = \frac{f^{(n+1)}(x)}{(n+1)}\,\cancel{f}\,(x-a)^{n+1}$$

$$\frac{e^c}{(n+1)!}\,x^{n+1}$$

$$|R_n(x)| \le \frac{M\,|x|^{n+1}}{(n+1)!}$$

Taylor series for a function of two veriable:

Let $f(x,y)$ has continuous partial derivative in an open region R containg a point (a,b)

Let (h,k) be incrementaly small enough to put the point $s(a+h, b+k)$ and the line sagment Joining it to (a,b) inside R.

$F(t) = f(a+th, b+tk)$ (any point on this line can written by this)

$$F'(t) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= \cancel{h f(x)}\ h f_x + k f_y$$

$$F''(\cancel{x}t) = \frac{\partial}{\partial x} F' \cdot \frac{dx}{dt} + \frac{\partial F'}{\partial x} \cdot \frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}(h f_x + k f_y) \cdot h + \frac{\partial}{\partial y}(h f_x + k f_y)\, k$$

$$= h^2 f_{xx} + k^2 f_{xx} + hk f_{xy} + hk f_{xy}$$

$$F''(t) = h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}.$$

$\cancel{F''(t)}$

$$F(1) = F(0) + F'(0)(1-0) + \frac{F''(0)}{2!}(1-0)^2 + \dots + \frac{f'''(c)}{3!}(1)^3$$

$c \in (0,1)$

$$f(a+h, b+k) = f(a,b) + h f_x(a,b) + k f_y(a,b)$$
$$+ \frac{1}{2!}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)(a,b) + \dots$$

Exe: Find a quadratic approximation to $f(x,y) = \sin x \cdot \sin y$ near $(0,0)$

How accurate is the approximation at $|x| \leq 0.1$ and $|y| \leq 0.1$

$$f(x,y) = f(0,0) + 1(x f_x + y f_y)_{(0,0)} + \frac{1}{2!}(x^2 f_{xx} + 2xy f_{xx} + y^2 f_{yy})_{(0,0)}$$

$f(x,y) = \sin x \cdot \sin y$

$f(0,0) = 0$

$f_x = \cos x \cdot \sin y \qquad f_x|_{(0,0)} = 0$

$f_y = \sin x \cdot \cos y \qquad f_y|_{(0,0)} = 0$

$f_{xx} = -\sin x \cdot \sin y \qquad f_{xx}|_{(0,0)} = 0$

$f_{xy} = \cos x \cdot \cos y \qquad f_{xy}|_{(0,0)} = 1$

$f_{yy} = -\sin x \cdot \sin y \qquad f_{yy}|_{(0,0)} = 0$

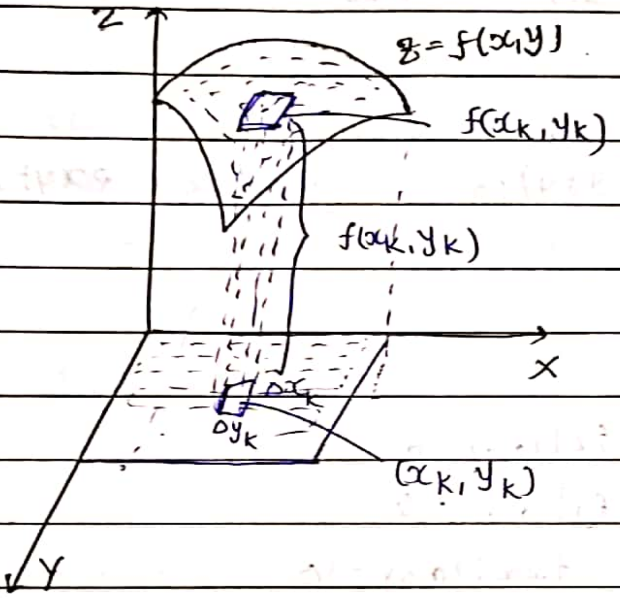
$$f(x,y) = 0 + 0 + \frac{1}{2}(x^2 0 + 2xy \cdot 1 + y^2 \cdot 0)$$

$$= xy$$

$$E(x,y) = \frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3yx^2 f_{yyx} + y^3 f_{yyy})$$

multiple Integral.

Double Integral over rectangle



Partition the plane rectangle $a \le x \le b$, $c \le y \le d$.

Area of the small rectangle $\Delta A_k = \Delta x_k \cdot \Delta y_k$

Volume of the beam = $h \times$ Base

$= f(x_k, y_k)\ \Delta x_k \cdot \Delta y_k$

sum of the all those Beam = $f(x_k, y_k)\ \Delta A_k$

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\ \Delta A_k$$

As $n \to \infty$ or the partition is very small $\|P\| \to 0$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\ \Delta A_x$$

$$= \sum_{k=1}^{\infty} f(x_k, y_k)\ \Delta x_k$$

$$= \iint_R f(x,y)\ dx\ dy$$

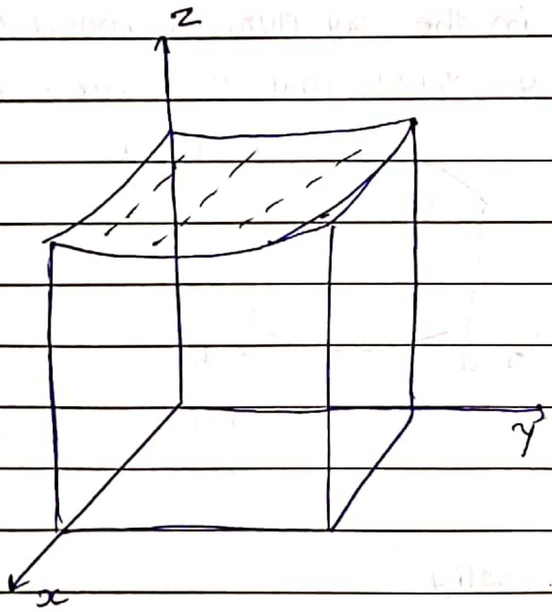
This gives volume of the solid determine by the surface over the rectangulyr rigion R.

Fubini's theorem:

If $f(x,y)$ is continous through out the rectangle $R: a \le x \le b,\ c \le y \le d$ then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_a^b \int_c^d f(x,y)\, dx\, dy$$

example: Find the volume of the regin bounded above by the elliptical parabolic $z = 10 + x^2 + 3y^2$ and below by the rectangle $R: 0 \le x \le 1,\ 0 \le y \le 2$



$$V = \iint_R f(x,y)\, dx\, dy$$

$$= \int_0^2 \int_0^1 (10 + x^2 + 3y^2)\, dx\, dy$$

$$= \int_0^2 \left[10x + x^3/3 + 3y^2 x \right]_0^1 dy$$

$$= \int_0^2 \left[10 + 1/3 + 3y^2 \right] dy$$

$$= \left[10y + \frac{y}{3} + y^3 \right]_0^2$$

$$= 20 + \frac{2}{3} + 8$$

$$= 28 + 2/3$$

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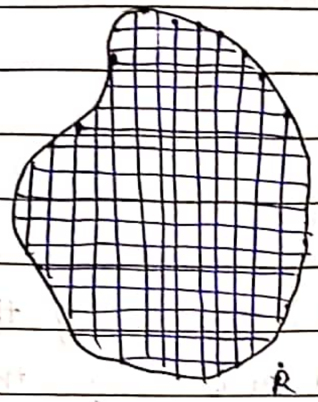
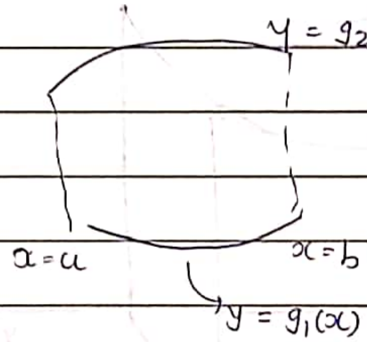
Double integral over bounded non rectangular region.


R

$$V = \iint f(x,y)\, dA$$

volume by using cross-section:

If R is the rigion in the xy plane bounded above and below by $y = g_2(x)$ and $y = g_1(x)$ and the sides by $x=a$ and $x=b$



crossectional area

$$A(x) = \int_{y=g_1(x)}^{g_2(x)} f(x,y)\, dy.$$

Then the volume of the solid

$$V = \int_a^b A(x)\, dx$$

$$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx$$

fubini's theorem:

Let $f(x,y)$ be continous on R

(1) If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$ and g_1 and g_2 continuous on $[a,b]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

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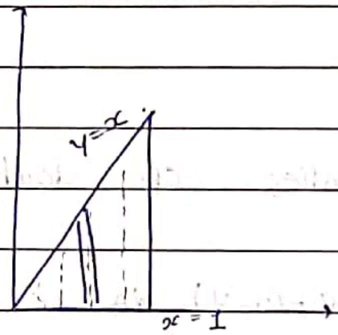
2) If R is defined by the region $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$ when h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

example: Find the volume of the Prism whose Base is the triangle in the x-y plane bounded by the x axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x, y) = 3 - x - y$.



$$V = \int_0^1 \int_0^x f(x,y)\, dy\, dx$$

$$= \int_0^1 \left(\int_0^x (3 - x - y)\, dy \right) dx$$

$$= \int_0^1 \left[3y - xy - y^2/2 \right]_0^x dx$$

$$= \int_0^1 \left[3x - x^2 - x^2/2 \right] dx$$

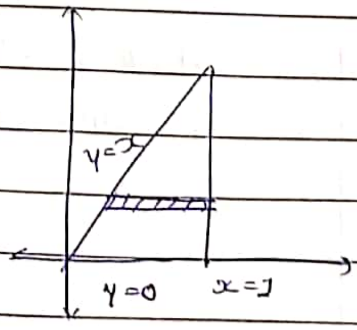
$$= \int_0^1 \left[3x - 3x^2/2 \right] dx.$$

$$= \left[3x^2/2 - 3x^3/6 \right]_0^1$$

$$= 3/2 - 3/4$$

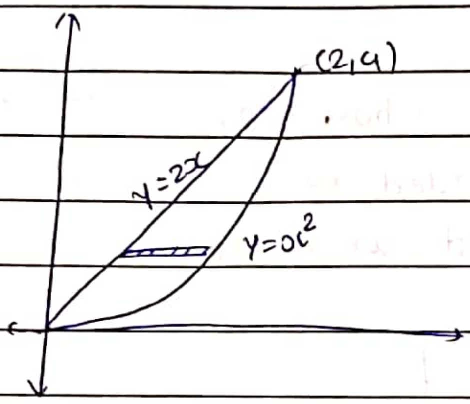
or

$$\int_{y=0}^{1} \left(\int_{x=y}^{x=1} (3 - x - y)\, dx \right) dy$$



Q: change the order of Integration

$$\int_0^2 \int_{y=x^2}^{2x} (4x+2)\, dy\, dx$$



for that we have to take horizontal limit

$$V = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\, dx\, dy$$

Properties of double Integrals.

1) $\iint_R c f(x,y)\, dA = c \iint_R f(x,y)\, dA$

2) $\iint_R (f(x,y) \pm g(x,y))\, dA = \iint_R f(x,y)\, dA \pm \iint_R g(x,y)\, dA.$

3) (a) $\iint_R f(x,y)\, dA \geq 0$ if $f(x,y) \geq 0$; $x \in R$

(b) $\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA$ if $f(x,y) \geq g(x,y)$ $\forall (x,y) \in R$

4) $\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$

If R is the union of two non overlaping region R_1 and R_2

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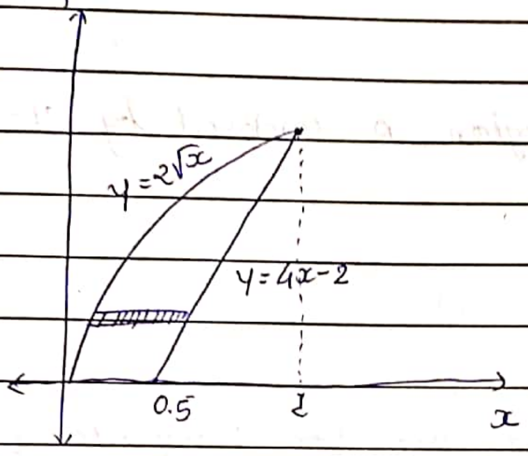
example: Find the volume of the solid that lies between the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$ the line $y = 4x - 2$ and the x-axis

Soln:

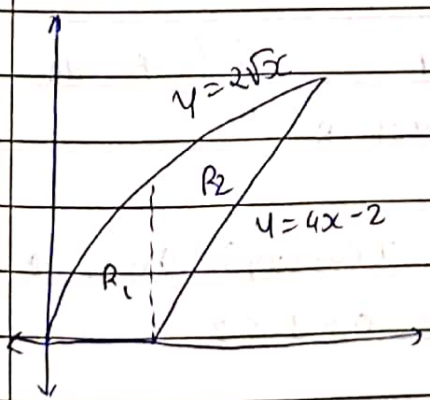


$$V = \iint_R f(x, y)\, dA$$

$$= \int_0^1 \int_{y^2/4}^{\frac{y+2}{4}} (16 - x^2 - y^2)\, dx\, dy$$

$$= 12.4$$

If we want to take fi Integrate First by dy

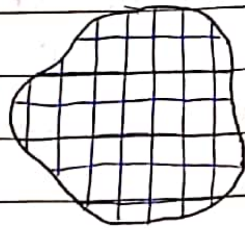


$$V = \int_0^{0.5} \int_{y=0}^{y=2\sqrt{x}} (16 - x^2 - y^2)\, dy\, dx$$

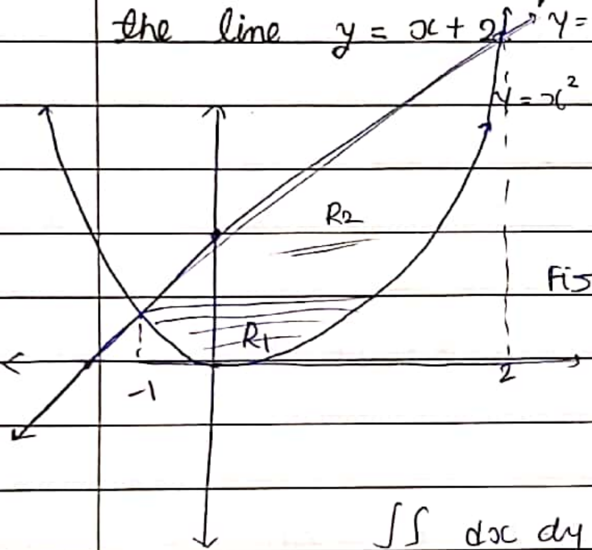
$$+ \int_{x=0.5}^{1} \int_{y=0}^{y=2\sqrt{x}} (16 - x^2 - y^2)\, dy\, dx$$

Area by double integrals:

$$A = \iint_R dx\,dy$$



ex: Find the area of the region R enclosed by $y = x^2$ and the line $y = x + 2$



First we count to integrate w.r.t. x

$$\iint_{R_1} dx\,dy + \iint_{R_2} dx\,dy$$

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^4 \int_{y-2}^{\sqrt{x}} dx\,dy$$

Average value of f over R $= \dfrac{1}{\text{area of } R} \iint_R f\,dA$

example: Find the average value of $f(x, y) = x\cos(xy)$ over the rectangle $R : 0 \le x \le \pi \; ; \; 0 \le y \le 1$

average value of f is $= \dfrac{1}{\text{area of } R} \iint_R f\,dx\,dy$

area of R = $\pi \times 1 = \pi$

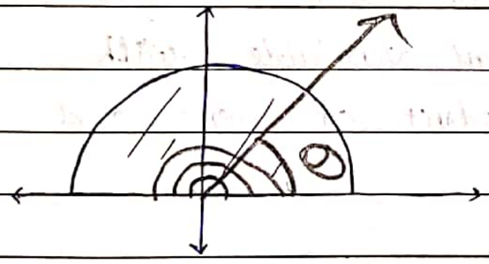
$$\int_0^{\pi}\int_0^{1} x.\cos(xy)\, dy\, dx$$

$$\int_0^{\pi} \left[\frac{x \cancel{\cos} \sin(xy)}{x} \right]_0^1 dx$$

$= 2$

so Average value of $x\cos xy$ over $0 \le x \le \pi$, $0 \le y \le 1$.

example: Evaluate $\iint_R e^{x^2+y^2}\, dy\, dx$. where R is the semi-circular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$



$x = r\cos\theta$

$y = r\sin\theta$

$f(r,\theta) = e^{r^2}$

$$\iint_R f(x,y)\, dx\, dy = \int_\theta \int_r F(r,\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{1} e^{r^2} . r\, dr\, d\theta.$$

Volume using triple integral.

$V = \iiint dx\, dy\, dz$

Differential eq^n :

First order differential eq^n :

Order of a differential eq^n :

the order of the highest derivative involved in the differential eq^n.

$$\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^2 + 3y^2 = 5\sin x$$

Order of eq^n : 3

Degree of diff eq^n : 1 (Power of the highest order derivative)

→ a differential eq^n ~~not~~ said to be linial if it does not contain product of dependent variable with product of derivatives and product of dependend variable with its derivative

$$x^2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = \sin x \qquad \text{liniar}$$

$$x^2\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} = x\sin x \qquad \text{not liniar}$$

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = x\sin x \qquad \text{non liniar}$$

$$\frac{dy}{dx} = y^2\sin x$$

non liniar

A differentiable function $y(x)$ is ~~said to~~ which satisfyes the given differential equation is called a solution of differential eq^n.

First order diff. eq^n:

$$\frac{dy}{dx} = f(x,y)$$

The general solution to a 1^{st} order diff. eq^n is a solution that contains all possible solutions.

exam: show that the function $y = x+1 - \frac{e^x}{3}$ is a solution to the first initial value problem

$$\frac{dy}{dx} = y - x \quad , y(0) = 2/3$$

$$y = x+1 - \frac{e^x}{3}$$

$$\frac{dy}{dx} = 1 - \frac{e^x}{3}$$

and $y - x = \cancel{x}\ x+1 - \frac{e^x}{3} - x = 1 - \frac{e^x}{3}$

so $\frac{dy}{dx} = y - x$

$y(0) = 1 - 1/3 = 2/3$

so the function $y = x+1 - e^x/3$ is a solution to the first intial value problem

Separable differential eqⁿ:

$\because \frac{dy}{dx} = f(x,y)$

If $f(x,y)$ can be written as

$$f(x,y) = g(x)\cdot h(x)$$

$$\frac{dy}{dx} = g(x)\cdot h(y)$$

$$\Rightarrow \frac{dy}{h(y)} = g(x)\, dx.$$

integrating directly we get solution.

example: $\frac{dy}{dx} = \frac{1}{y}\cdot \log x$

$$\int y\, dy = \int \log x\, dx$$

$$\frac{y^2}{2} = x\log x - x + c$$

homogenous diff eqⁿ:

$$\frac{dy}{dx} = f(x,y) \quad \text{---} (1)$$

$f(x,y)$ is said to be homogenous of degree n if $f(tx, ty) = t^n f(x,y)$

where t is independent of x and y.

If $f(x,y)$ is homogenous in x and y then eqⁿ (1) is called a 1st order homogeneous differential eqⁿ.

solution method :

Put $y = vx$ in eq^n(1) that reduce it to a separable eq^n.

ex: Solve $x^2 y\, dx - (x^3 + y^3)\, dy = 0$

$$x^2 y\, dx = (x^3 + y^3)\, dy$$

$$\therefore \frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$$

homogeneous differential eq^n.

taking $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^2 \cdot vx}{x^3 + (vx)^3}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{x^3} v}{\cancel{x^3}(1+v^3)}$$

$$\therefore x\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$\therefore x\frac{dv}{dx} = \frac{\cancel{v} - \cancel{v} - v^4}{1+v^3}$$

$$\therefore x\frac{dv}{dx} = \frac{-v^4}{1+v^3}$$

$$\therefore \int \frac{1+v^3}{v^4}\, dv = \int \frac{-1}{x}\, dx$$

$$\therefore \int v^{-4} + v^{-1}\, dv = -\log x + c$$

$$\therefore \frac{v^{-3}}{-3} + \log v = -\log x + c$$

$$\therefore \frac{-1}{3v^3} + \log v = -\log x + c$$

$$\therefore \frac{-x^3}{3y^3} + \log\left(\frac{y}{x} \cdot x\right) = c$$

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$$\therefore \frac{-x^3}{3y^3} + \log y = c$$

Q: $\frac{dy}{dx} = \frac{x+y}{x-y}$ (this eq^n is homogeneous eq^n)

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} = \frac{x+vx}{x-vx} = \frac{1+v}{1-v}$

$x\frac{dv}{dx} = \frac{1+v}{1-v} - v$

$x\frac{dv}{dx} = \frac{1+\cancel{v}-\cancel{v}+v^2}{1-v}$

$x\frac{dv}{dx} = \frac{1+v^2}{1-v^2}$

$\frac{1-v}{1+v^2}\,dv = \frac{1}{x}\,dx$

$\int\frac{1}{1+v^2}\,dv - \frac{1}{2}\int\frac{2v}{1+v^2}\,dv = \int\frac{1}{x}\,dx$

$c + \tan^{-1}v - \frac{1}{2}\log(1+v^2) = \log x$

$\tan^{-1}(y/x) = \log(x\sqrt{1+v^2}) + c$

$\tan^{-1}(y/x) = \log\left(x\sqrt{\frac{1+y^2}{x^2}}\right) + c$

$\tan^{-1}(y/x) = \log(\sqrt{1+y^2}) + c.$

Q: $\frac{dy}{dx} = \frac{x+y+3}{x-y+1}$

we can make this eq^n homogeneous

Let's take $x = X+h$

$y = Y+k$

$\frac{dy}{dx} = \frac{X+h+Y+k+3}{X+h-Y-k+1}$

$h+k = -3$

$h-k = -1$

$2h = -4$

$h = -2$ $\quad k = -1$

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take $x = X - 2$

$y = Y - 1$

$$\therefore \frac{dY}{dX} = \frac{X+Y}{X-Y}$$

second order liniar differential eq^n:

$$P(x)\, y''(x) + Q(x)\, y'(x) + R(x) \cdot y(x) = G(x) \quad (1)$$

P, Q, R and G are continuous function of x.

If $G(x) = 0$ $\forall(x)$. then it eq^n (1) becomes

$$P(x) \cdot y''(x) + Q(x) \cdot y'(x) + R(x) \cdot y(x) = 0$$

is called liniar homogeneous eq^n.

If $G(x) \neq 0$ then (1) is said to be non-homogeneous.

we assume that $P(x) \neq 0$.

second order liniar homogeneous eq^n

$$P(x) \cdot y'' + Q(x) \cdot y' + R(x) \cdot y(x) = 0$$

Principle of superposition:

if $y_1(x)$ and $y_2(x)$ are two solution of liniar homogeneous eq^n (2), then for any constants c_1 and c_2 the function $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is also solution of (2)

$$P(x)\, y_1'' + Q(x)\, y_1' + R(x)\, y_1 = 0$$

$$P(x)\, y_2'' + Q(x)\, y_2' + R(x)\, y_2 = 0.$$

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$P(x)y'' + Q(x)y' + R(x)y$

$= P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2)$

$= c_1\underbrace{(P(x)\cdot y_1'' + Q(x)y_1' + R(x)\cdot y_1)}_{=0} + c_2\underbrace{(P(x)y_2'' + Q(x)y_2' + R(x)y_2)}_{=0}$

$= 0.$

→ If P, Q and R are continuous function over the interval I at $P(x) \neq 0$ on I. then the linier homogeneous eqn (2) has two linearly independent solution y_1 and y_2.

then $c_1y_1 + c_2y_2$ is also a solution.

and it is called general solution.

$y_1 = x$ and $y_2 = 2x$ x and $2x$ are liniarly dependent

$y_1 = x$ and $y_2 = x^2$ then x and x^2 are liniarly independent

$y = e^x$ $y_2 = x$ liniarly indepent.

$y_1 = e^x$ $y_2 = xe^x$ liniarly independent

wronskien method

if for two function y_1 and y_2

$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2^2 \end{vmatrix} = 0$ then they both are liniarly dependend

$y_1 = x \quad y_2 = 2x \quad y_3 = x^2$

for three function

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & 2x & x^2 \\ 1 & 2 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 0$$

2^{nd} order homogeneous eq^n with constant co-efficient.

$$ay'' + by' + cy = 0 \quad \dots (3)$$

a, b and c are constant.

one choice of the solution is the form of $y = e^{mx}$.

$y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2 e^{mx}$

putting this in eq^n

$$a \cdot m^2 e^{mx} + bme^{mx} + ce^{mx} = 0 \quad (\because e^{mx} \neq 0)$$

$[am^2 + bm + c = 0]$ Auxiliary eq^n (characteristic eq^n) .. (4)

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (m_1 \text{ and } m_2)$$

case: 1 when $b^2 - 4ac > 0$ (real roots)

two distinct real roots. m_1 and m_2.

so $y_1 = e^{m_1 x}$ is solution

and $y_2 = e^{m_2 x}$ is a solution

y_1 and y_2 are liniarly indipendent because m_1 and m_2 are distinc

so By superposition principle

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

example: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ find general sol^n of this eq^n

for $y = e^{mx}$ to be a solution

$$m^2 e^{mx} - me^{mx} - 6 \cdot e^{mx} = 0$$

$$m^2 - m - 6 = 0$$

$(m-3)(m+2) = 0$ so general solution is

$\boxed{m_1 = 3}$ $\boxed{m_2 = -2}$ $= c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{3x} + c_2 e^{-2x}$

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case: 2 when both roots are equal

$m_1 = m_2 = \frac{-b}{2a} = m$ $\therefore \boxed{2am + b = 0}$

when $b^2 - 4ac = 0.$

$y_1 = e^{m_1 x}$ is a solution

and other solution

will $y_2 = x e^{mx}$

$y_2' = x \cdot m e^{mx} + e^{mx}$

$y_2'' = m[x m e^{mx} + e^{mx}]$

$= x m^2 e^{mx} + m e^{mx} + m e^{mx}$

$a y_2'' + b y_2' + c y_2 =$

$= a\left(x m^2 e^{mx} + 2m e^{mx}\right) + b\left(e^{mx} + m x e^{mx}\right) + c x e^{mx}$

$= \underbrace{(am^2 + bm + c)}_{=0} x e^{mx} + e^{mx} \underbrace{(2am + b)}_{=0}$

$= 0$

that means $y_2 = x e^{mx}$ is also solution of eq^n.

so the general solution will be $= c_1 e^{mx} + c_2 x e^{mx}$

$= (c_1 + c_2 x) e^{mx}$

example: $y'' + 4y' + 4y = 0$ find general sol^n.

$y = e^{mx}$

$\therefore m^2 e^{mx} + 4m e^{mx} + 4 e^{mx} = 0$

$\therefore m^2 + 4m + 4 = 0$

$\therefore (m+2)^2 = 0$

$\therefore m = -2$

and other $y_2 = x e^{-2x}$

so general sol^n

$= c_1 e^{-2x} + x c_2 e^{-2x}$

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Case: 3 when $b^2-4ac < 0$.

m_1 and m_2 are complex roots

$m_1 = \alpha + ib$ and $m_2 = \alpha - ib$

$y_1 = e^{(\alpha+ib)x} \cdot e^{\alpha x}$.

$y_2 = e^{m_2 x} = e^{(\alpha - ib)x}$.

$e^{(\alpha+i\beta)x} = e^{\alpha x} \cdot e^{ibx}$ $\qquad$ $e^{(\alpha-ib)x} = e^{\alpha x} \cdot e^{-ibx}$

$y_1 = e^{\alpha x} \cdot (\cos\beta x + i\sin\beta x)$ $\qquad$ $y_2 = e^{\alpha x} \cdot (\cos\beta x - i\sin\beta x)$

By principle of superposition

$\frac{1}{2} y_1 + \frac{1}{2} y_2 = e^{\alpha x} \cdot \cos\beta x = y_3$

$\frac{1}{2i} y_1 - \frac{1}{2i} y_2 = \frac{1}{2i} (\cancel{e^{\alpha x} \cdot \cos\beta x}) + \frac{i}{2i} e^{\alpha x} \cdot \sin\beta x$

$- \frac{1}{2i} (\cancel{e^{\alpha x} \cdot \cos\beta x}) + \frac{i}{2i} e^{\alpha x} \cdot \sin\beta x$

$y_4 = e^{\alpha x} \sin\beta x$

the general solution is $c_1 e^{\alpha x} \cos\beta x + c_2 e^{\alpha x} \sin\beta x$

$$\boxed{y(x) = e^{\alpha x} (c_1 \cos\beta x + c_2 \sin\beta x)}$$

example: Find the general Solution of

$$y'' - 4y' + 5y = 0$$

$\Rightarrow$ put $y = e^{mx}$

$$m^2 - 4m + 5 = 0 \quad \text{Auxiliary eq}^n$$

$$D = b^2 - 4ac$$
$$= 16 - 4(5)$$
$$= -4$$

$$m_1 = \frac{4 + 2i}{2} \qquad m_2 = \frac{4 - 2i}{2}$$

$$m_1 = \underset{\alpha}{2} + \underset{\beta}{i} \qquad m_2 = 2 - i$$

So general Solution will be = $e^{2x}(c_1 \cos x + c_2 \sin x)$

example: find the particle soln

to $y'' - 2y' + y = 0$ $\quad y(0) = 1$

$y'(0) = -1$

Soln: Find the general solution

$y = e^{mx}$

$$m^2 - 2m + 1 = 0$$

$$(m + \cancel{1})(m - 1) = 0$$

$$\cancel{m = 2} \quad m = 1$$

general soln = e^{x}, e^{x}.

So roots are real and eq.

general soln = $c_1 e^x + c_2 x e^x$

$y(0) = 1$

$y(x) = c_1 e^{x} + c_2 x e^{x}$

$y(0) = 1 - c_1 + 0$

$c_1 = 1$

$y'(x) = c_1 e^{x} + c_2 (x e^{x} + e^{x})$

$-1 = e^{0} + c_2 (0 + 1)$

$c_2 = -2$

example:: Slove $y'' + 4y' = 0$

$y(0) = 0 \quad y(\pi/12) = 1$

eq^n will be $m^2 + 4m = 0$

$m(m + 4) = 0$

$m = 0$ and $m - 4$

solution of eq^n will be $y = e^{m_1 x}$ and $y_2 = e^{m_2 x}$

$sol^n: 1, e^{4x}$

General Solution will $= c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= 1$

Nonhomogeneous linear equation with constant coefficient.

$$ay'' + by' + cy = G(x) \quad --(i)$$

$G(x) \neq 0$

$G(x)$ is continuous function over some open interval I.

The associated homogeneous eqⁿ y

$$ay'' + by' + cy = 0 \quad --(ii)$$

The associated homogeneous eqⁿ

the solution of eqⁿ $y_c = c_1y_1 + c_2y_2$

complimentry solution

Suppose we some how how get one particular solution $y = y_c + y_p$ is also solⁿ of y

$$y' = y_c' + y_p'$$

$$y'' = y_c'' + y_p''$$

putting in eqⁿ 1

$$a(y_c'' + y_p'') + b(y_c' + y_p') + c(y_c + y_p) = 0.$$

$$\therefore \underbrace{(ay_c'' + by_c' + cy_c)}_{=0} + \underbrace{(ay_p'' + by_p' + cy_p)}_{=0}$$

(y_c is solution of (ii))

the general solution of eqⁿ of (i) is

$$y = y_c + y_p$$

y_c = complimentry solution

y_p = particular solⁿ.

To Find the particular integral y_P.

$$ay'' + by' + cy = G(x) \quad \text{--}(1)$$

By the method of ~~eqn~~ undetermined coefficient we can find particular integral of (1) if $G(x)$ have some particular task.

Note: If $G(x)$ has a term that is a constant multiple of

If $G(x)$ has a term that is a constant multiple.	and if	then include the expression into trial function of y_p
e^{rx}	(i) r is not a root of Auxiliary eqn	$A \cdot e^{rx}$
	(ii) r is a single root of Auxiliary eqn	Axe^{rx}
	(iii) r is a double root of Auxiliary eqn	Ax^2e^{rx}
$\sin kx, \cos kx$	ki is not a root of Auxiliary eqn	$B\cos kx + C\sin kx$
$ax^2 + bx + c$	0 is not a root of Auxiliary eqn	$Ax^2 + Bx + C$
	0 is ~~simple~~ single root of A eqn	$Ax^3 + Bx^2 + Cx$
	0 is a double root of eqn	$Ax^4 + Bx^3 + Cx^2$

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example: $y'' - 2y' - 3y = 1 - x^2$

sol^n: The associated homogeneous eq^n $y'' - 2y' - 3y = 0$

Auxiliary eq^n will $m^2 - 2m - 3 = 0$

$m_1 = -1, \quad m_2 = 3$

$y_c = c_1 e^{-x} + c_2 e^{3x}$

$G(x) = 1 - x^2$ (Polynomial)

let $y_p = Ax^2 + Bx + c$

$y_p' = 2Ax + B$

$y_p'' = 2A$

since y_p is to be solution it should satisfy eq^n.

$y_p'' - 2y_p' - 3y_p = 1 - x^2$

$2A - 2(2Ax + B) - 3(Ax^2 + Bx + c) = 1 - x^2$

$-3Ax^2 + x(-4A - 3B) + 2A - 2B - 3C = 1 - x^2$

coffi of x^2

$-3A = -1$

$A = 1/3$

$-4A - 3B = 0$

$\frac{-4}{3} = 3B$

$B = -4/9$

$2A - 2B - 3C = 1$

$\frac{3 \times 2}{3 \times 3} - \frac{8}{9} - 3C = 1$

$-2/9 - 3C = 1$

$-3C = 1 + 2/9$

$-3C = 11/9$

$C = -\cancel{11}\ {+5}/27$

$Y = Y_c + Y_p$

$= c_1 e^{-x} + c_2 e^{3x} + \frac{x^3}{3} - \frac{4x}{9} + \frac{5}{27}$

Q: Find a particular solution of

$$y'' - y' = 2\sin x$$

Associated homogeneous eqn

$$y'' - y' = 0$$

Auxiliary eqn is $m^2 - m = 0$

$m_1 = 0,\ m_2 = 1$

$$y_c = c_1 e^0 + c_2 e^x$$

$$= c_1 + c_2 e^x.$$

$G(x) = 2\sin x$

Let $y_p = A\sin x$

$y_p' = A\cos x$

$y_p'' = -A\sin x$

$y_p'' - y_p' = 2\sin x$

$-A\sin x - A\cos x = 2\sin x$

$-A = 2$

$A = -2.$ and $A = 0$

So this is not possible

Not necessary.

So we will take $y_p = A\sin x + B\cos x$

$y_p' = A\cos x - B\sin x$

$y_p'' = -A\sin x - B\cos x$

$y_p'' - y_p' = 2\sin x$

$-A\sin x - B\cos x - A\cos x + B\sin x = 2\sin x$

$(B-A)\sin x - (A+B)\cos x = 2\sin x$

$B - A = 2$ $\quad$ $A + B = 0$

$2B = 2$

$B = 1$ And $A = -1.$

So particular solution of diffn eqn is

$$\boxed{y_p = -\sin x + \cos x}$$

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example: $y'' - 3y' + 2y = 5e^{x}$

sol^n:

associated homogeneous eq^n

$$y'' - 3y' + 2y = 0$$

Auxiliary eq^n $m^2 - 3m + 2 = 0$

$$(m-2)(m-1) = 0$$

$$m_1 = 2 \quad m_2 = 1$$

$$y = c_1 e^{+2x} + c_2 e^{x}$$

For particular ~~soln~~ integral

$$g(x) = 5e^{x}$$

here $r=1$ is a one root of Auxiliary eq^n

so let us take $y_p = Ax e^{x}$

$$y_p = Axe^{x}$$

$$y_p' = A[xe^{x} + e^{x}]$$

$$y_p'' = A\{[xe^{x} + e^{x} + e^{x}]$$

$$A[xe^{x} + 2e^{x}] - 3A[xe^{x} + e^{x}] + 2Axe^{x} = 5e^{x}$$

$$xe^{x}[A - 3A + 2A] + e^{x}[2A - 3A] = 5e^{x}$$

$$2A - 3A = 5$$

$$-A = 5 \Rightarrow \boxed{A = -5}$$

So $y_p = -5xe^{x}$

So solution of the eq^n

$$y = y_c + y_p$$

$$= c_1 e^{2x} + c_2 e^{x} - 5xe^{x}$$

Find the particular sol^n of

$$y'' - 6y' + 9y = e^{3x}$$

sol^n: associated homogenous eq^n

$$y'' - 6y' + 9y = 0$$

Auxiliary eq^n $m^2 - 6m + 9 = 0$

$$(m-3)^2 = 0$$

$$m_1 = 3 \quad m_2 = 3$$

here r is a double root of Auxilary eq^n

$$y_p = Ax^2 e^{3x}$$

$$y_p' = A[3x^2 e^{3x} + 2x e^{3x}]$$

$$y_p'' = A[3\{e^{3x} \cdot 2x + x^2 \cdot 3e^{3x}\} + 2\{3e^{3x} \cdot x + e^{3x}\}]$$

$$6Ae^{3x} \cdot x + 3Ax^2 e^{3x} + 6xe^{3x} + 2e^{3x}$$

$$A = 1/2$$

example: $y'' - y' = 5e^x - \sin 2x$

associated Auxilary eq^n $m^2 - m = 0$

$$m(m-1) = 0$$

$$m = 0 \quad m = 1$$

Let us take $y_p = Axe^x - B\sin kx - C\cos kx$

$$y_p' = Axe^x + Ae^x - Bk\cos kx + Ck\sin kx$$

$$y_p' = Axe^x + Ae^x - 2B\cos 2x + 2C\sin 2x$$

$$y_p'' = A\{xe^x + e^x\} + Ae^x + 4B\sin 2x + 4C\cos 2x$$

$$\cancel{Axe^x} + 2Ae^x + 4B\sin 2x + 4C\cos 2x - Axe^x - Ae^x + 2B\cos 2x - 2C\sin 2x = 5e^x - \sin 2x$$

$$Ae^x + \sin 2x(4B - 2C) + (4C + 2B)\cos 2x = 5e^x - \sin 2x$$

$\boxed{A = 5}$ $\quad 4B - 2C = -1 \quad 4C + 2B = 0$

$-8C - 2C = -1 \quad \boxed{C = 1/10} \quad B = -2C \quad \boxed{B = -1/5}$

Method of variation of parameter :

$ay'' + by' + cy = G(x)$ -- (1)

the associated homogeneous eq^n .

$ay'' + by' + cy = 0$

$y = e^{mx}$

Auxiliary eq^n

$am^2 + bm + c = 0$

Let $y_c = c_1 y_1(x) + c_2 y_2(x)$

we assume that particular integral of the form

$\left[y_p = c_1(x)\, y_1(x) + c_2(x)\, y_2(x) \right]$

where $c_1(x)$ and $c_2(x)$ are unknown function of x, that are to be determined

y_p should satisfy eq^n(1).

To find out two unknows $c_1(x)$ and $c_2(x)$ we ~~for~~ need another eq^n.

for simplicity let us take the other eq^n as

$\left[c_1'(x) \cdot y_1(x) + c_2'(x)\, y_2(x) = 0 \right]$ -- (3)

$y_p = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$

$y_p' = \underbrace{c_1'(x)\, y_1(x) + c_2'(x)\, y_2(x)}_{=0} + y_1'(x) \cdot c_1(x) + y_2'(x) \cdot c_2(x)$

$y_p' = c_1(x) \cdot y_1'(x) + y_2'(x) \cdot c_2(x)$

$y_p'' = c_1(x)\, y_1''(x) + c_1'(x) \cdot y_1'(x) + c_2(x) \cdot y_2''(x) + c_2'(x) \cdot y_2'(x)$

putting this values in eq^n(1).

$ay_p'' + by_p' + cy_p = \cancel{0}\ G(x)$

$a(c_1'y_1' + c_1y_1'' + c_2y_2'' + c_2'y_2') + b(c_1y_1' + y_2'c_2) + c(c_1y_1 + c_2y_2) = G(x)$

$$\Rightarrow c_1(\cancel{ay_1'' + by_1' + cy_1}) + c_2(\cancel{ay_2'' + by_2' + cy_2}) + a(c_1'y_1' + c_2'y_2') = G(x)$$

$$\Rightarrow a(c_1'y_1' + c_2'y_2') = G(x)$$

$$\Rightarrow c_1'y_1' + c_2'y_2' = \frac{G(x)}{a} \quad \dots (4)$$

and we have assume

$$c_1'y_1 + c_2'y_2 = 0.$$

$$c_1' = \frac{w_1}{w} = \qquad c_2' = \frac{w_2}{w} =$$

where $w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ $\quad w_1 = \begin{vmatrix} 0 & G(x)/a \\ G(x)/a & \end{vmatrix}$

$$w_1 = \begin{vmatrix} 0 & y_2 \\ G(x)/a & y_2' \end{vmatrix} \qquad w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & G(x)/a \end{vmatrix}$$

$$\int c_1'\,dx = c_1$$
$$\int c_2'\,dx = c_2.$$

example: $y'' + y = \tan x$

$\rightsquigarrow$ associated homogeneous eqn

$$m^2 + 1 = y'' + y = 0$$

taking one solution $y = e^{mx}$

$$y'' = m^2 e^{mx}$$

Auxiliary homogeneous eqn

$$m^2 + 1 = 0$$

$$m_1 = +i \qquad m_2 = -i$$

$$y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

$$= e^0(c_1 \cos x + c_2 \sin x) = \cancel{c_1 \sin x}\ c_1\cos x + c_2 \sin x$$

two linearly independent solution are

$y_1(x) = \cos x \qquad y_2(x) = \sin x$

let $y_p = c_1(x) \cdot y_1(x) + c_2(x) \cdot y_2(x)$

$= c_1(x) \cdot \cos x + c_2(x) \cdot \sin x$

$c_1'(x) = \dfrac{\omega_1}{\omega}$

$\omega_1 = \begin{vmatrix} 0 & y_2 \\ G(x)/a & y_2' \end{vmatrix} \qquad \omega_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & G(x)/a \end{vmatrix}$

$= \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} \qquad = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}$

$= -\sin x \cdot \tan x \qquad = \tan x \cdot \cos x$

$= \sin x$

$\omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x$

$= 1.$

$c_1'(x) = -\dfrac{\sin^2 x}{\cos x} \qquad c_2'(x) = \sin x$

$c_1(x) = \int \dfrac{-\sin^2 x}{\cos x}\, dx \qquad c_2(x) = \int \sin x \, dx$

$= \int \dfrac{\cos^2 x - 1}{\cos x}\, dx \qquad = -\cos x$

$= \int (\cos x - \sec x)\, dx$

$= \sin x - \log|\sec x + \tan x|$

$y_p = (\sin x - \log|\sec x + \tan x|) \cdot \cos x + -\cos x \cdot \sin x$

$= \cos x \cdot \sin x - \cos x \cdot \log|\sec x + \tan x| - \cos x \cdot \sin x$

$= -\cos x (\log|\sec x + \tan x|)$

Solution of the eq^n

$$y = y_c + y_p$$

$$= c_1 \cos x + c_2 \sin x - \cos x \cdot \log(\sec x + \tan x)$$

Ex: Slove $y'' + y' - 2y = xe^x$

→ associated homogeneous eq^n

$$y'' + y' - 2y = 0$$

Auxiliary eq^n $\qquad y = e^{mx}$

$$m^2 + m - 2 = 0$$

$$(m+2)(m-1) = 0$$

$$m = -2 \qquad m = 1$$

$$y_1 = e^{-2x} \text{ and } y_2 = e^x$$

→ $y_c = c_1 e^{-2x} + c_2 e^x$

→ let $y_p = c_1(x) \cdot e^{-2x} + c_2(x) \cdot e^x$

$$w_1 = \begin{vmatrix} 0 & y_2 \\ g(x)/a & y_2' \end{vmatrix} \qquad w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x)/a \end{vmatrix} \qquad w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} 0 & e^x \\ xe^x & e^x \end{vmatrix} \qquad = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & xe^x \end{vmatrix} \qquad = \begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix}$$

$$= -xe^{2x} \qquad = x \cdot e^{-x} \qquad = e^{-x} + 2e^{-x}$$

$$= 3e^{-x}$$

$$c_1'(x) = \frac{w_1}{w} = \frac{-xe^{2x}}{3e^{-x}} \qquad c_2'(x) = \frac{w_2}{w}$$

$$= \frac{-xe^{3x}}{3} \qquad = \frac{x}{3}$$

$$c_1(x) = \int c_1'(x)\,dx \qquad c_2(x) = \int x/3 \, dx$$

$$= \frac{-1}{3} \int x \cdot e^{3x}\,dx \qquad = \frac{x^2}{6}$$

$$= \frac{-1}{3}\left\{ x \cdot \frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}\,dx \right\}$$

$$= -\frac{1}{3}\left\{ \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} \right\} = \frac{e^{3x}}{27} - \frac{xe^{3x}}{9}$$

DOMS

$y_p = \left(\frac{3e^{3x}}{27} - \frac{1x e^{3x}}{9}\right) e^{-2x} + \frac{x^2}{6}(e^x)$

→ solution of differential eqⁿ

$= c_1 e^{-2x} + c_2 e^{x} + \left(\frac{3e^{3x}}{27} - \frac{1}{9}xe^{3x}\right) \cdot e^{-2x} + \frac{x^2}{6}(e^x)$

example: $y''' + 3y'' + y' - y = 0$

put $y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2 e^{mx}$

$y''' = m^3 e^{mx}$

$m^3 + 3m^2 + m - 1 = 0$

$(m+1)(m^2+2m-1) = 0$

$$
\begin{array}{r|l}
 & m^2 + 2m - 1 \\
m+1 & m^3 + 3m^2 + m - 1 \\
 & m^3 + m^2 \\
 & \quad 2m^2 + m - 1 \\
 & \quad 2m^2 + 2m \\
 & \qquad -m - 1
\end{array}
$$

m_1, m_2, m_3 are real and distinct.

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_2 x}$

If $m_1 = m_2 =$ real, m_3

$y_c = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x}$.

If $m_1 = m_2 = \alpha \pm i\beta$, m_3 real

$y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x}$.

suppose we have non-homogeneous eq^n

$$y''' + 3y'' + y' - y = \sec x.$$

y_1, y_2, y_3 are three linearly independent solu^n of associated homogeneous eq^n.

$$y_p = c_1(x)\cdot y_1 + c_2(x)\cdot y_2 + c_3(x)\cdot y_3$$

P.I. $= y_p$

$$c_1'(x) = \frac{\omega_1}{\omega}$$

$$c_2'(x) = \frac{\omega_2}{\omega}$$

$$c_3'(x) = \omega_3/\omega$$

$$\omega = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$\omega_1 = \begin{vmatrix} \cancel{0} & \cancel{y_2} \\ \cancel{0} & \cancel{y_2'} \\ \cancel{g(x)/a} & \cancel{y_2''} \end{vmatrix}$$

$$\omega_1 = \begin{vmatrix} g(x)/a & y_2 & y_3 \\ 0 & y_2' & y_3' \\ G(x)/a & y_2'' & y_3'' \end{vmatrix} \qquad \omega_2 = \begin{vmatrix} y_1 & g(x)/a\ 0 & y_3 \\ y_1' & 0\ g(x)/a & y_3' \\ y_1'' & G(x)/a & y_3'' \end{vmatrix}$$

$$\omega_3 = \begin{vmatrix} y_1 & y_2 & g(x)/a \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & G(x)/a \end{vmatrix}$$

$64 - 48 + 24 - 6$

__/__/__

example: $y''' - 3y'' + 6y' - 6y = x^4 \ln x$

→ associated homogeneous eqⁿ

$y''' - 3y'' + 6y' - \cancel{6y} = \cancel{x^4 \ln x}\ 0$

$9 - 4(6)$

$9 - 24$

15

auxiliary eqⁿ

$m^3 - 3m^2 + 6m - \cancel{6} = 0$

$8 - 12 + 12$

$\cancel{27 - 27 +}$

$m(m^2 - 3m + 6) = 0$

$m_1 = 0 \qquad m_2, m_3 = \dfrac{3 \pm i\sqrt{15}}{2}$

$\alpha = 3/2 \qquad \beta = \sqrt{15}/2$

$y_c = c_1 e^{m_1 x} + e^{\alpha x}(c_2 \cos \beta x + c_3 \sin \beta x)$

$= c_1 + e^{\frac{3}{2}x}\left(c_2 \cos \frac{\sqrt{15}}{2} x + c_3 \sin \frac{\sqrt{15}}{2} x\right)$

$y_1 = 1 \qquad y_2 = e^{\frac{3}{2}x} \cdot \cos \frac{\sqrt{15}}{2} x \qquad y_3 = e^{\frac{3}{2}x} \sin \frac{\sqrt{15}}{2} x$

example: $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0 \quad (x \neq 0)$

put $y = x^m$ [euler - cauchy eqⁿ]

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

$y''' = m(m-1)(m-2) x^{m-3}$

$m(m-1)(m-2)x^m - 3m(m-1)x^m + 6m x^m - 6x^m = 0$

$m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0$

$(m^2 - m)(m-2) - 3m^2 + 3m + 6m - 6 = 0$

$m^3 - 2m^2 - m^2 + 2m - 3m^2 + 3m + 6m - 6 = 0$

$m^3 - 6m^2 + 11m - 6 = 0$

$(m-1)(m-2)(m)$

$(m-1)(m-2)(m-3) = 0$

DOMS $\quad m_1 = 1, \ m_2 = 2, \ m_3 = 3$

Solutions are

$$y_1 = x \quad , y_2 = x^2 \quad , \quad y_3 = x^3$$

General Solution will be

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

if eqn is

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 6\, x^4 \ln x$$

$$\omega = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

$$\omega_1 = \begin{vmatrix} \cancel{\frac{x^4 \ln x}{x^3}}\ 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ x \ln x & 2 & 6x \end{vmatrix}$$

$$= \cancel{x \ln x \,[12x^2 - 6x^2]}$$

$$= \cancel{6x^3 \ln x}$$

$$= x \ln x \,[3x^4 - 2x^4]$$

$$= x^5 \ln x$$

$$\omega_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & x \ln x & 6x \end{vmatrix}$$

$$= -x \ln(x) \,[3x^3 - x^3]$$

$$= -2x^3 \cdot x \cdot \ln x$$

$$= -2x^4 \ln x$$

$$\omega_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & x \ln x \end{vmatrix} = x \ln x \,[2x^2 - x^2]$$

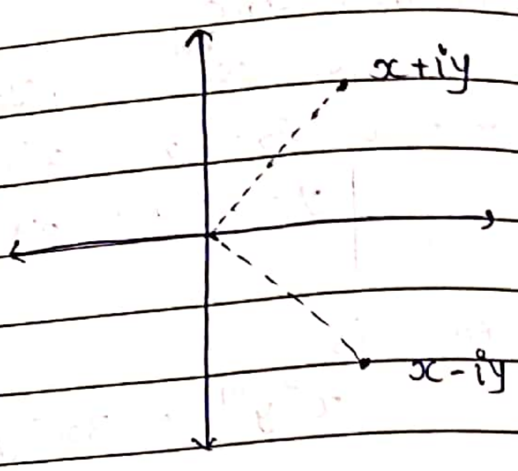
$$= x^3 \ln x$$

$$y_p = c_1(x) \cdot y_1 + c_2(x)\, y_2 + c_3(x)\, y_3$$

$$c_1(x) = \int \frac{\omega_1}{\omega}\, dx \qquad c_3(x) = \int \frac{\omega_3}{\omega}\, dx$$

$$c_2(x) = \int \frac{\omega_2}{\omega}\, dx$$

complex variables.



Complex conjugate

$z = x + iy$

$\bar{z} = x - iy$

reflection about x axis

$x = \frac{z + \bar{z}}{2} = Re(z)$

$y = \frac{z - \bar{z}}{2i} = Im(z)$

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Polar form of a complex number.

$z = x + iy$

$x = r\cos\theta \qquad y = r\sin\theta$

$z = r\cos\theta + i r\sin\theta$

$= r(\cos\theta + i\sin\theta) = re^{i\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta)$

⇒ Absolute value of z (modulus of z)

$z = x + iy$

$|z| = \sqrt{x^2 + y^2}$

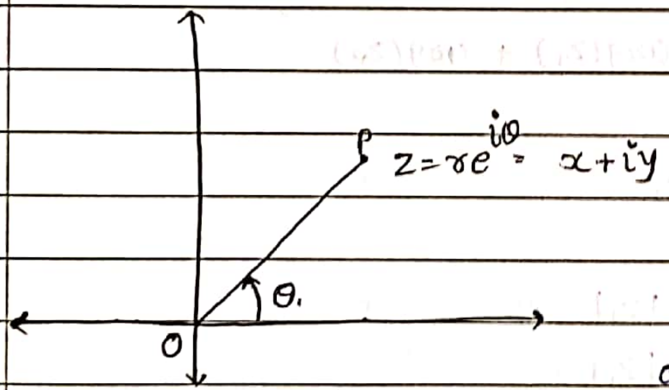
$|z| = |re^{i\theta}|$

$= |r|\,|e^{i\theta}|$

$= r$

⇒ Argument of z.

θ = argument of z = $\arg(z)$ = $\arctan\left(\frac{y}{x}\right)$ = $\tan^{-1}\left(\frac{y}{x}\right)$



θ is the directed angle from positive x-axis to the line OP.

angle is messured is to counter clockwise direction from +ve x axis.

Principle value of the argument of z.

The value of θ that lies in the interval $-\pi < \theta < \pi$ is called the principle value of the argument of z.

~~Demo~~

Denoted by $Arg(z)$

$-\pi < Arg(z) < \pi$.

ex: $Arg(i) = \pi/2$

$Arg(1-i) = -\pi/4$

$Arg(-1-i) = -3\pi/4$

DOMS

⇒ Triangle inequality :

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$$

⇒ multiplication and division in polar form

$$z_1 = r_1 e^{i\theta_1} = r_1 (\cos\theta_1 + i\sin\theta_2)$$

$$z_2 = r_2 \cdot e^{i\theta_2} = r_2 (\cos\theta_2 + i\sin\theta_2)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2 \; e^{i(\theta_1 + \theta_2)}$$

$$|z_1 + z_2| = r_1 r_2$$

$$\arg(z_1 \cdot z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

Let $z = \dfrac{z_1}{z_2} \Rightarrow z z_2 = z_1$

$$|z z_2| = |z_1|$$

$$\therefore |z| = \frac{|z_1|}{|z_2|}$$

$$\arg(z) = \arg(z_1) - \arg(z_2)$$

⇒ $z = r e^{i\theta} = r(\cos\theta + i\sin\theta)$

$$z^n = r^n (e^{i\theta})^n = r^n (\cos\theta + i\sin\theta)^n$$

$$= r^n (e^{ni\theta})$$

$$= r^n (\cos n\theta + i\sin(n\theta))$$

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if $|z| = 1 \Rightarrow r = 1$

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

De Moivre's theorem

Roots :

IF $z = \omega^n$

then $\omega = z^{1/n}$ nth root of z.

$z = r(\cos\theta + i\sin\theta)$ $\qquad \omega = R(\cos\phi + i\sin\phi)$

$$\omega^n = \cancel{z}\ R^n(\cos n\phi + i\sin n\phi)$$

$$\boxed{\begin{aligned} R^n = r &\Rightarrow R = r^{1/n} \\ n\phi = \theta &\Rightarrow \phi = \theta/n \end{aligned}}$$

$$n\phi = \theta + 2k\pi$$

$$\phi = \frac{1}{n}(\theta + 2k\pi) \qquad k = 0, 1, \ldots, n-1$$

$$\omega = z^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right)$$

if $z = 1$ $\qquad \operatorname{Arg}(z) = 0$

$|z| = 1$

$$z^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right)$$

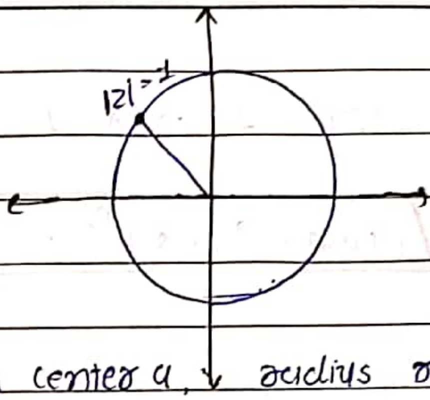
$$= \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \quad k = 0, 1, 2, \ldots, (n-1)$$

n^{th} roots of unity.

if ω denotes value for $k = 1$ $\qquad 1, \omega, \omega^2, \ldots, \omega^{n-1}$

$\omega^n = 1$, $\quad 1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$

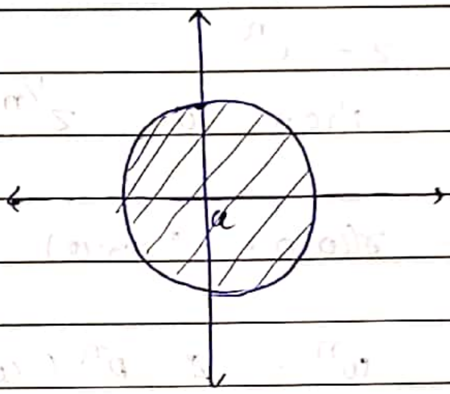
Unit circle in complex plane.



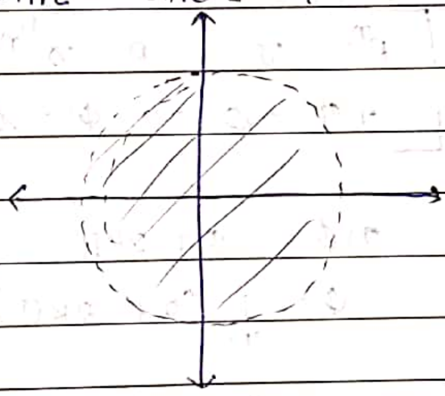
$|z| = 1.$

circle when center a and radious r $|z-a| = r$

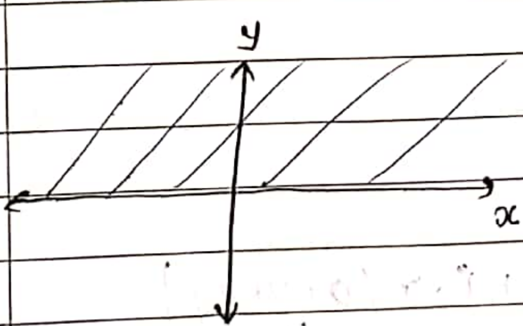
$|z-a| \leq r$. closed disc with center a, radius r.



$|z-a| < r$ open disc with center a, radious r.



upper half plane



$z = x+iy$ such that $y > 0$

lower half plane.

$\{z = x+iy$ such that $y<0\}$

Right half plane.

$\{z = x+iy$ such that $x>0\}$

Left half plane

$\{z = x+iy$ such that $x<0\}$

Complex Function:

Let $D \subset \mathbb{C}$ (complex plane)

$f : D \to \mathbb{C}$ is called a complex function.

$\omega = f(z) = u(x,y) + i\,v(x,y)$

so $u(x,y)$ is the real part and $v(x,y)$ is the imaginary part.

example: Real part of $z^2 + 37$

$f(z) = z^2 + 37$

$= (x+iy)^2 + 37$

$= x^2 - y^2 + 37 + 2ixy$

Real part of $f(z) = x^2 - y^2 + 37$

and imaginary part $= 2xy$

Limits:

$\lim_{z \to z_0} f(z) = l$

if f is defined in nbd of z_0 (except perhaps at z_0 it self) and if the value of f are close to l for all z close to z_0.

For every $\varepsilon > 0$, there exist $\delta > 0$ such that for all $z \neq z_0$ in the disc $|z - z_0| < \delta$, we have $|f(z) - l| < \varepsilon$

$0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \varepsilon$

$\Rightarrow$ continuous function :

A function $f(z_0)$ is said to be continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$

$\Rightarrow$ Derivative :

The Derivative of a complex function f at a point z_0 is defined by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Provided this limit exist.

$z = x + iy \quad \Delta z = \Delta x + i\Delta y.$

Properties:

(i) $(f \pm g)' = f' \pm g'$

(ii) $(kf)' = k(f')$ $\quad k \in R$

(iii) $(fg)' = f'g + g'f$

(iv) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad g \neq 0$

if $f'(z)$ is differentiable at z_0 then $f(z)$ is continuous at z_0

$f(z) = z^2$

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2z\Delta z + \Delta z^2}{\Delta z}$$

$$= 2z$$

$f(z) = \sin z$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\sin(z+\Delta z) - \sin z}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\cancel{2}\cos\left(\frac{2z+\Delta z}{2}\right) \cdot \sin\left(\Delta z/2\right)}{\Delta z/2}$$

$$= \cos z$$

exam: $f(z) = \bar{z}$ is not differentiable

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{(\overline{z+\Delta z}) - \bar{z}}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

case: 1 Δz is purely real number

$$\lim_{\Delta z \to z} \frac{\overline{\Delta z}}{\Delta z} = 1$$

Δz is purly imaginary

$$\lim_{\Delta z \to z} = -1$$

so limit does not exist so $f(z) = \bar{z}$ is not differentiable

for different path of approach we have different limit so limit does not exist.

ex: $f(z) = Re(z)$

$z = x + iy$

$$\lim_{\Delta z \to 0} \frac{Re(z+\Delta z) - Re(z)}{\Delta z}$$

$\Delta z = \Delta x + \Delta y\, i$

$Re(z+\Delta z) = x + \Delta x$

$$\lim_{\Delta z \to 0} \frac{x + \Delta x - x}{\Delta z}$$

when $\Delta x \to 0$ $\lim_{\Delta y \to 0} \frac{\Delta x}{\Delta z} = 0$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x}{\Delta x + i\Delta y}$$

when $\Delta y \to 0$ $\lim_{\Delta x \to 0} \neq 1$

So limit does not exist

DOMS

ex: Show that $f(z) = |z|^2$ is differentiable only at $z=0$ and no where else

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

Analytic function:

A function is said to be Analytic in a domain D if $f(z)$ is ~~determined~~ define and differentiable at all points of D

D → open connected set.

Domain: open connected set.

A function $f(z)$ is said to be analytic at a point $z_0 \in D$ if $f(z)$ is differentiable in a nbd of z_0.

ex: $f(z) = |z|^2$ is differentiable only at $z=0$.
So function is not Analytic

All polynomials are Analytic function

$\sin z$, $\cos z$, e^z are Analytic function

$$\left.\begin{array}{l} f(z) = \bar{z} \\ f(z) = Re(z) \\ f(z) = Im(z) \end{array}\right\}$$ is not analytic function.

Cauchy - Riemann eqⁿ:

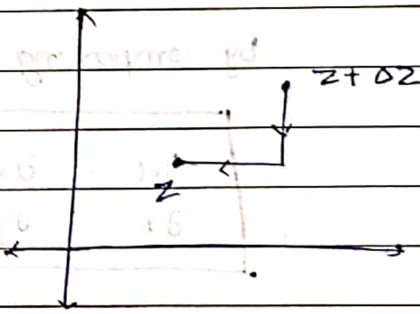
Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some nbd of a point $z = x+iy$ and analytic at z itself then at that point the 1^{st} order partial derivative of u and v exist and satisfy the ~~eqn~~ cauchy - Riemann eqⁿ.

and $$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

PROOF :

By assumption the derivative $f'(z)$ exist

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$



$$\Delta z = \Delta x + i\Delta y$$

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)]}{\Delta x + i\Delta y}$$

If we let $\Delta y \to 0$ first and then $\Delta x \to 0$.

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \lim_{\Delta x \to 0} i\left(\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

DOMS

and If we let $\Delta x \to 0$ first and then $\Delta y \to 0$

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \frac{i\,v(x, y+\Delta y) - i\,v(x,y)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

~ function is differentiable at z so both should be same

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

by comparing real part imaginary part

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$ C.R. eqn.

Theorem: If two real valued continuous function $u(x,y)$ and $v(x,y)$ have continuous first partial derivatives ut that satisfy C.R. eqn in some Domain D. then the function
$f(z) = u + iv$ is analytic in D.

Q: If z^3 is analytic.

$$z^3 = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$$

$u(x,y) = x^3 - 3xy^2$

$v(x,y) = 3x^2y - y^3$

DOMS

$\frac{\partial u(x,y)}{\partial x} = 3x^2 - 3y^2$ $\frac{\partial v(x,y)}{\partial x} = 6xy$

$\frac{\partial u(x,y)}{\partial y} = -6xy$ $\frac{\partial v(x,y)}{\partial y} = 3x^2 - 3y^2$

so $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

all the partial derivatives are continuous.
so Function is Analytic.

exa: $f(z) = \bar{z}$
$= x - iy$

$u(x,y) = x$ $v(x,y) = -y$

$\frac{\partial u}{\partial x} = 1$. $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0$ $\frac{\partial v}{\partial x} = -1$

So Function is not Analytic.
so C.R. eqn are not satisfied so $f(z)$, $\bar{z}$ is not analytic.

Laplace eqn:

IF $f(z) = u + iv$ is analytic in a domain D,
then $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 0}$ Laplace eqn.

IF u satisfy Laplace eqn then it is said to be a harmonic function.

Example: $u(x,y) = x^2 - y^2$ ~~at~~ cheak u is harmonic or not? Find it's harmonic congugate v. such that $u+iv$ is analytic function

Sol^n:
$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial^2 u}{\partial x^2} = 2$$
$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial^2 u}{\partial y^2} = -2$$
$$\text{so } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

so u is a harmonic function

* NOTE every harmonic function in a simply connected domain has it harmonic congugate.

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y$$
$$\Rightarrow v = \int 2x \, dy \qquad \Rightarrow \frac{\partial v}{\partial x} = 2y \quad \text{--- (2)}$$
$$= 2xy + \phi(x) \quad \text{--- (1)}$$

~~$N = 2xy +$~~

From eq^n 1
$$\frac{\partial v}{\partial x} = 2y + \phi'(x) = 2y \quad \text{From eq}^n\ 2$$
$$\text{so } \phi'(x) = 0$$
$$\phi(x) = k \quad (\text{constant})$$
$$\Rightarrow v = 2xy + k.$$

⇒ C.R. eqn in Polar Form:

$u(r, \theta)$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Laplase eqn in Polar form:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Problem: suppose $f(z)$ is an analytic Function whose imaginary part is constant.

$f(z) = u + iv$ and v is constant

Soln: v is a constant $v = K$.

f is analytic, so that real part satisfy CR eqn

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

since $v = 0$

$$\frac{\partial v}{\partial y} = 0 \quad \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0$$

u is constant.

⇒ $f = u + iv$ is also constant.

* Entire Function :

A function f(z) which is analytic in the whole complex plane. is called an entire function

example: e^z, $\sin z$, $\cos z$, any polynomial in z.

$f(z) = \frac{1}{z}$ and $|z| > 1$

function is analytic in $|z| > 1$. but $1/z$ is not entire

$$e^z = e^x \cdot e^{iy} = e^x[\cos y + i\sin y]$$

$$u = e^x \cos y \quad v = e^x \sin y$$

$$e^{iz} = e^{i(x+iy)} = e^{ix} \cdot e^{-y} = e^{-y}(\cos x + i\sin x)$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{1}{2}(e^z - e^{-z})$$

* Logarithmic function :

natural Logarithm $\ln z$

$$\ln z = \ln(re^{i\theta})$$
$$= \ln r + i\theta \qquad |z| > 0$$
$$\theta = \arg z.$$

$\ln z$ is multivalued function.
because it depend on θ.

$Ln(z) = ln|z| + i\,Arg(z)$ (one valued function)

where $Arg(z)$ is principle value of the argument.

$$\ln z - Ln(z) = \pm 2k\pi i$$
$$\ln z = Ln(z) \pm 2k\pi i$$

* ~~Live~~ Line integral in the complex plane.

$\int_C f(z)\,dz$ c is a curve.

we may represent the curve c in a parametric representation

$$z(t) = x(t) + iy(t) \qquad (a \leq t \leq b)$$

the sense of increasing t is called positive sense in c and we say that in this way c orients.

we subdevide the interval $a \leq t \leq b$

$$t_0 = a, t_1, t_2, \ldots, t_n = b$$

the corrosponding values will get

$z_0, z_1, \ldots z_n$ $\qquad z_j = z(t_j)$

$z_j = z(t_j)$

on each partition of subdevition on c. we choose arbitrary point $\xi_1 \in (z_0, z_1)$

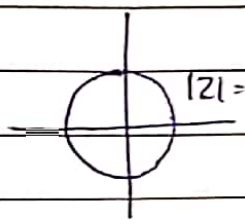
$\xi_2 \in (z_1, z_2)$

$$s_n = \sum_{k=1}^{n} f(\xi_k)\,\Delta z_k$$

As $n \to \infty$ then $s_n \approx \int_C f(z)\,dz$

If C is a close path (curve) then it is denoted by $\oint_C f(z)\,dz$

example



$$\oint_{|z|=1} f(z)\,dz$$

Properties :

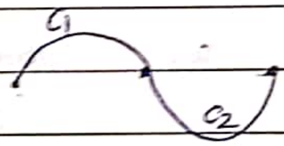
(i) $\int_C \left(k_1 f_1(z) + k_2 f_2(z)\right) dz$

$$= k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz.$$

(ii) $\int_{z_0}^{z_1} f(z)\,dz = -\int_{z_1}^{z_0} f(z)\,dz$

$C = C_1 \cup C_2$

and $C_1 \cap C_2$ is only one point.



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

existance of integrals

IF $f(z)$ is continuous and C is peacewise smooth, then $\int_C f(z)\,dz$ exist.

simply connected domain :

A Domain D is simply connected At every simple closed curve in D only ~~contains~~ encloses only points in D.

integration By using the path :

If c be a piecewise smooth can represented by $z = \gamma(t)$, $a \le t \le b$. Let $f(z)$ be continuous on c. then $\int_c f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$

$$z = \gamma(t)$$

$$\frac{dz}{dt} = \gamma'(t)$$

$$dz = \gamma'(t)\,dt.$$

example: $\oint_c \frac{dz}{z}$ c is unit circle direction is counter clockwise.

$|z| = 1.$

$\gamma(t) = e^{it}$, $0 \le t \le 2\pi$

$$\oint_c \frac{dz}{z} = \int_0^{2\pi} \frac{1}{\cancel{e^{it}}} \cdot i\cancel{e^{it}}\,dt = i\cdot[2\pi - 0] = 2\pi i$$

example:

sol^n: $C: a + re^{it}$; $0 \le t \le 2\pi$

$= \gamma(t)$.

$$\oint_C (z-a)^m \, dz \qquad\qquad f(z) = (z-a)^m$$

$$= \int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t) \, dt$$

$$= \int_0^{2\pi} (a + re^{it} - a)^m \cdot ire^{it} \, dt$$

$$= \int_0^{2\pi} ir^{m+1} e^{i(m+1)t} \, dt$$

when $m = -1$

$$\oint_C \frac{1}{z-a} dz = i \int_0^{2\pi} dt$$

$$= 2\pi i$$

when $m \neq -1$ and m is integer

$$ir^{m+1} \int_0^{2\pi} e^{i(m+1)t} \, dt$$

$$\cancel{i}r^{m+1} \cdot \left[\frac{e^{i(m+1)t}}{i(m+1)} \right]_0^{2\pi}$$

$$\frac{r^{m+1}}{m+1} \left[e^{i(m+1) \cdot 2\pi} - 1 \right]$$

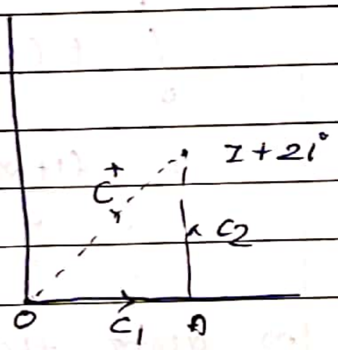
$$= \frac{r^{m+1}}{m+1} \left[\cancel{e^{im2\pi}} \cdot 1 - 1 \right]$$

$$= 0$$

DOMS

$\oint_C (z-a)^m\, dz = 0$ where $m \neq -1$
c is a circle of center a radius r.

ex: $\int_C f(z)\, dz$ where $f(z) = Re(z)$ and c is a path from 0 to 1 + 2i



Along C^*

$\int Re(z)\, dz$

$\int_0^1 Re(\gamma(t)) \cdot \gamma'(t)\, dt$

$= \int_0^1 t \cdot (1+2i)\, dt$

$= \left[\frac{t^2}{2}(1+2i)\right]_0^1$

$= \frac{1}{2}(1+2i)$

para. eq^n of line
$(1-t)(0) + t(1+2i)$
$= t + 2ti = \gamma(t)$
$0 < t < 0$

→ case: 2 $\int_{C_1 \cup C_2} f(z)\, dz$

$\int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$

$C_1 : (1-t)0 + t$
$= t = \gamma(t)$

$C_2 : (1-t)1 + t(1+2i)$
$= 1 - t + t + 2it$
$= 1 + 2it$
$= \gamma_2(t)$

$\int_0^1 Re(\gamma_1(t)) \cdot \gamma_1'(t)\, dt + \int_0^1 Re(\gamma_2(t)) \cdot \gamma_2'(t)\, dt$

$\int_0^1 t \cdot dt + \int_0^1 1 \cdot (2i)\, dt$

$\left[t^2/2\right]_0^1 + \left[2it\right]_0^1$

$\frac{1}{2} + 2i$

Integral of a non-analytic function depends on the path followed.

example:

$$\int_C f(z)\, dt \qquad f(z) = z$$

$C: 0 \text{ to } 1 + 2i$

find integral along both path

$$\int_{c^*} f(\gamma(t)) \cdot \gamma'(t) \cdot dt \qquad c^* = (t-1)0 + t(1+2i)$$

$$= t(1+2i) = \gamma t$$

$$\int_0^1 t(1+2i)(1+2i)\, dt$$

$$= \frac{(1+2i)^2}{2}$$

→ along the other path

$$\int_{C_1 \cup C_2} f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$$

$$C_1 = (t-1)0 + t = t$$

$$C_2 = (t-1)1 + t(1+2i) = 1 + 2it$$

$$= \int_0^1 f(\gamma_1(t)) \cdot \gamma_1'(t)\, dt + \int_0^1 f(\gamma_2(t)) \cdot \gamma_2'(t)\, dt$$

$$= \int_0^1 t \cdot dt + \int_0^1 (1 + 2it)(2i)\, dt$$

$$= \frac{1}{2} + 2i\left[t + it^2\right]_0^1$$

$$= \frac{1}{2} + 2i\left[1 + i\right]$$

$$= \frac{(1+2i)^2}{2}$$

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Cauchy's Integral theorem

IF $f(z)$ is analytic function in a simply connected domain D, then for every single closed path c in D,

$$\int_C f(z)\, dz = 0$$

example $\int_{|z|=1} z \, dz$

$f(z) = z$

$c: |z| = 1$

$0 + e^{it} = \gamma(t)$

$0 \leq t \leq 2\pi$

$$\int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t) dt$$

$$\int_0^{2\pi} e^{it} \, dt \cdot ie^{it}$$

$$= \left[\frac{ie^{2it}}{i}\right]_0^{2\pi}$$

$$= -i\left[e^{i2\pi} - 1\right]$$

$$= 0$$

example: $\int_{|z|=1} \sin z \, dz$

$f(z) = \sin z$

$c: |z| = 1$

$e^{it} = \gamma(t)$

$0 \leq t \leq 2\pi$

$$\int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t) \, dt$$

$$\int_0^{2\pi} \sin(e^{it}) \cdot ie^{it} \, dt$$

$$\int_1^1 \sin m \, dm$$

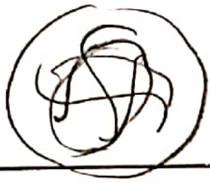
$e^{it} = m$

$ie^{it} dt = dm$

when $t = 0$ $m = 1$

$t = 2\pi$ $m = 1$

$$= 0$$



Anu

Analyticity is sufficient condition but not necessary.

ex. $\displaystyle\int_{|z|=1} \frac{1}{z^2}\, dz$ $\qquad \gamma(t) = e^{it},\quad 0 \le t \le 2\pi$

$$= \int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t)\, dt$$

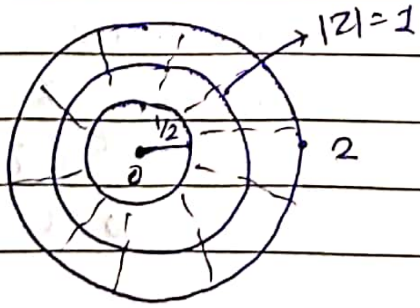
$$= \int_0^{2\pi} \frac{1}{e^{2it}} \cdot i e^{it}\, dt$$

$$= i \int_0^{2\pi} e^{-it}\, dt$$

$$= i \left[\frac{e^{-it}}{-i} \right]_0^{2\pi} = 0$$

Simply connected domain is essential

if domain is not simply connected



$\displaystyle\oint \frac{1}{z}\, dz$ $\qquad \gamma(t) = e^{it}$

$$\int_0^{2\pi} \frac{1}{e^{it}} \cdot i e^{it}\, dt$$

$$= 2\pi i$$