CCV–1

Feb 28, 2008

SC216 Calculus with Complex Variables

Analytic fn. & Singular points

— A pt. at which $f(z)$ fails to be analytic is called a singular pt. of $f(z)$.

Types of Singularities

(1) Isolated Singularities The pt. $z=z_0$ is called an isolated singularity or isolated singular pt. if we can find $\delta>0$ s.t. the circle $|z-z_0|=\delta$ encloses no singular pt. other than z_0.

(2) Poles If we can find a +ve integer n s.t.

$\lim_{z\to z_0}(z-z_0)^n f(z) = A \neq 0$ than $z=z_0$ is called a pole of order n. If $n=1$ z_0 is called a simple pole.

Example

$$f(z) = \frac{1}{(z-2)^4}$$ has a pole of order 4 at $z=2$

$$f(z) = \frac{3z-2}{(z-1)^3(z+1)(z-4)}$$ has a pole of order 3 at $z=1$ & simple pole at $z=-1$ & $z=4$

If $g(z) = (z-z_0)^n f(z)$ where $f(z_0)\neq 0$ & n a +ve integer then $z=z_0$ is called a zero of order n of $g(z)$. If $n=1$ z_0 is called a simple zero.

(3) **Branch points** of multiple valued fns. are singular pts.

Example $f(z) = (z-3)^{1/2}$ has a branch pt. at $z=3$

Example $f(z) = \log(z^2+z-2)$ has branch pts at $z^2+z-2=0$ i.e; at $z=1$ & $z=-2$

(4) **Removable Singularities**

The singular pt. z_0 is called a removable singularity of $f(z)$ if $\lim_{z \to z_0} f(z)$ exists.

Example The singular pt. $z=0$ is a removable singularity of $f(z) = \frac{\sin z}{z}$

(5) **Essential Singularities**

A singularity which is not a pole, branch pt. or removable singularity is called an <u>essential singularity</u>.

Example $f(z) = e^{1/(z-3)}$ has an essential singularity at $z=3$

(6) **Singularities at ∞**

The type of singularity of $f(z)$ at $z=\infty$ is same as that of $f(1/w)$ at $w=0$

$f(z)=z^3$ has a pole of order 3 at $z=\infty$

Integrals

Consider fns. $w : \mathbb{R} \to \mathbb{C}$

$w(t) = u(t) + i\,v(t)$ complex-valued fn. of a real variable $t \in \mathbb{R}$

$\Rightarrow \frac{d}{dt}\{w(t)\} = w'(t) = u'(t) + i\,v'(t)$

- For such kind of function not all properties holds true. for example, the mean value theorem (MVT) no longer applies.

□ Suppose $w(t)$ is conts on $a \le t \le b \Rightarrow$ its component fns. $u(t)$ & $v(t)$ are conts there. Now even if $w'(t)$ exist for $a < t < b$ MVT does not hold. i.e, there may not exist c in $a < t < b$ s.t.

$$w'(c) = \frac{w(b) - w(a)}{b - a}$$

If $w(t) = e^{it}$ on $0 \le t \le 2\pi$

then $|w'(t)| = 1 \Rightarrow w'(t) \neq 0$ while

$w(2\pi) - w(0) = 0$ ▨

The Definite Integrals of fns. $w(t) : \mathbb{R} \to \mathbb{C}$

- $$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

when the R.H.S. integrals exist.

$\Rightarrow$

$$\mathrm{Re}\int_a^b w(t)\,dt = \int_a^b \mathrm{Re}[w(t)]\,dt \quad \&$$

$$\mathrm{Im}\int_a^b w(t)\,dt = \int_a^b \mathrm{Im}[w(t)]\,dt$$

Example

$$\int_0^1 (1+it)^2 = \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt = \frac{2}{3} + i$$

$$- \int_a^b w(t)\,dt = W(b) - W(a) = W(t)\Big|_a^b$$

where $w(t) = u(t) + i\,v(t)$ &

$W(t) = U(t) + i\,V(t)$ are cont's on $a \le t \le b$

$W'(t) = w(t)$ when $a \le t \le b \Rightarrow U'(t) = u(t)$ & $V'(t) = v(t)$.

$\boxed{\text{Example}}$: $(e^{it})' = i e^{it}$

$$\Rightarrow \int_0^{\pi/4} e^{it}\,dt = \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$- \left| \int_a^b w(t)\,dt \right| \le \int_a^b |w(t)|\,dt \quad (a \le b)$$

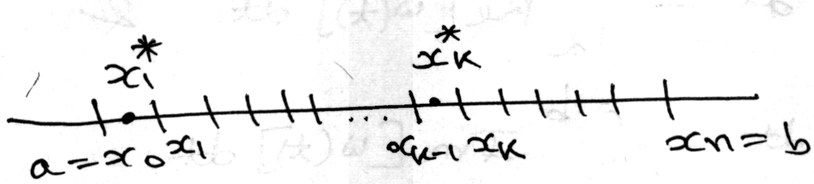
$$- \left| \int_a^\infty w(t)\,dt \right| \le \int_a |w(t)|\,dt$$

REVIEW ON REAL INTEGRALS

① Definite Integral for $f(x) : [a,b] \to \mathbb{R}$

- 1° Let f be a fn. of a single variable x defined $\forall\, x \in [a,b]$
- 2° Let P be a partition

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$



Partition of $[a,b]$ with x_k^* in each subinterval $[x_{k-1}, x_k]$

– 3° Let $\|P\|$ is the length of the longest subinterval

– 4° Choose a no. x_k^* in each subinterval $[x_{k-1}, x_k]$ of $[a,b]$

– 5° Form the products $f(x_k^*)\Delta x_k$ $(k=1,2,\ldots,n)$

& then sum $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$.

– 6° The definite integral of f on $[a,b]$ is

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k \qquad \text{— (1)}$$

When this limit exist we say f is integrable on $[a,b]$.

TERMINOLOGY	Types of curves

– Suppose a curve C in the plane is parametrized by $x = x(t)$ & $y = y(t)$ $a \le t \le b$
$\underbrace{}_{\text{conts real fns}}$

– $(x(a), y(a))$ – initial pt. A (say) of curve C
$(x(b), y(b))$ – terminal pt. B (say)

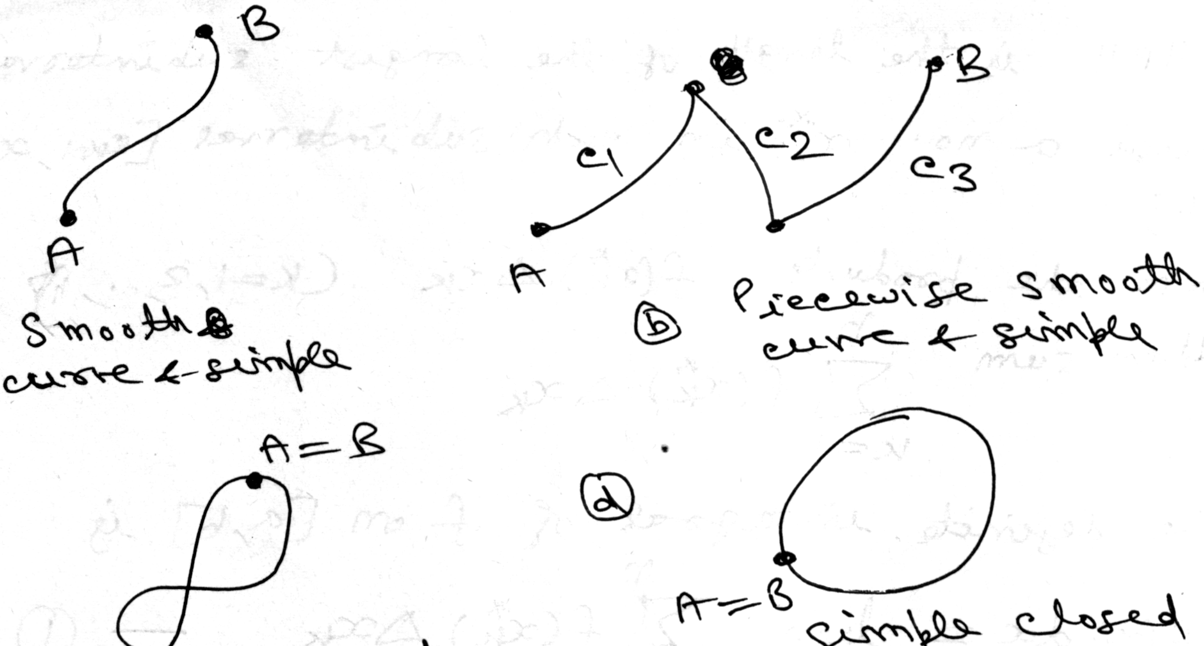
– (i) C is <u>smooth curve</u> if x' & y' are conts on $[a,b]$ & not simultaneously zero on (a,b).

(ii) C is a <u>piecewise smooth curve</u> if it consists of a finite no. of smooth curves $C_1, C_2, \ldots, C_n$ joined end to end.

(iii) C is a <u>simple curve</u> if the curve C does not cross itself except possibly at $t=a$ & $t=b$

(iv) C is a <u>closed curve</u> if $A = B$

(v) C is a <u>simple closed curve</u> if the curve C does not cross itself and $A = B$ i.e. C is simple & closed.



(a) Smooth curve & simple

(b) Piecewise smooth curve & simple

(c) Closed but not simple

(d) simple closed curve

Types of curves in the plane

② Line Integrals in the plane (curve integrals)

1° Let G be a fn. of 2 variables x & y defined at all points on a smooth curve c that lies in some region of the xy-plane. Let c has parametrization $x = x(t)$ & $y = y(t)$, $a \le t \le b$.

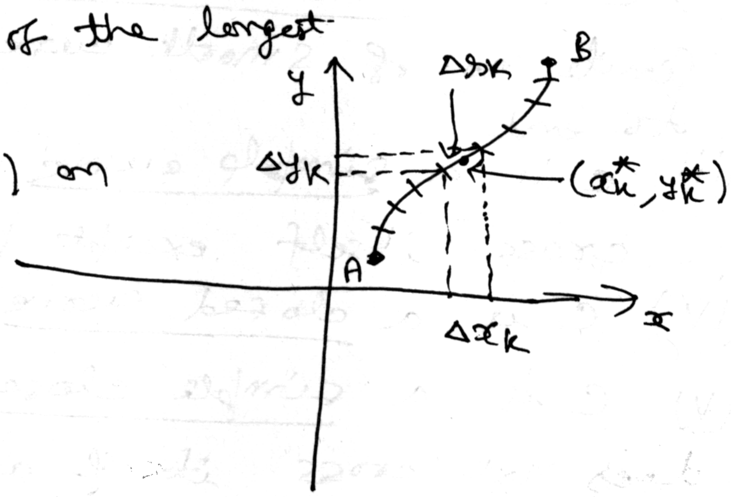
2° Let P be a partition of $[a,b]$ into sub intervals $[t_{k-1}, t_k]$ of length $\Delta t_k = t_k - t_{k-1}$:

$$a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b$$

This P induces a partition of the curve c into n subarcs of length Δs_k. Let the projection of each subarc onto x & y axes have lengths Δx_k & Δy_k

3° If $\|P\|$ denotes the length of the longest subinterval and

4° if we choose a pt. (x_k^*, y_k^*) on each subarc of C



5° Form n products

$$G(x_k^*, y_k^*)\,\Delta x_k$$
$$G(x_k^*, y_k^*)\,\Delta y_k \quad \&$$
$$G(x_k^*, y_k^*)\,\Delta s_k$$

sum

$$\sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta x_k \ , \ \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta y_k \ \text{and}$$

$$\sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta s_k \quad \text{then}$$

(i) The line integral of G along c w.r.to x is

$$\int_c G(x,y)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta x_k$$

(ii) The line integral of G along c w.r.to y is

$$\int_c G(x,y)\,dy = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta y_k$$

(iii) The line integral of G along c w.r.to arc length s is

$$\int_c G(x,y)\,ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta s_k .$$

— To find line integrals < c is a parametric curve $\boxed{\text{Case A}}$; c is defined by explicit fn. $\boxed{\text{Case B}}$

$\boxed{\text{Case A}}$

$$\int_c G(x,y)\,dx = \int_a^b G(x(t), y(t))\,x'(t)\,dt$$

$$\int_c G(x,y)\,dy = \int_a^b G(x(t), y(t))\,y'(t)\,dt$$

$$\int_c G(x,y)\,ds = \int_a^b G(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$c: \begin{array}{l} x = x(t) \\ y = y(t) \end{array}$$

$\boxed{\text{Example}}$

Evaluate

(a) $\int_c xy^2\,dx$ (b) $\int_c xy^2\,dy$ (c) $\int_c xy^2\,ds$

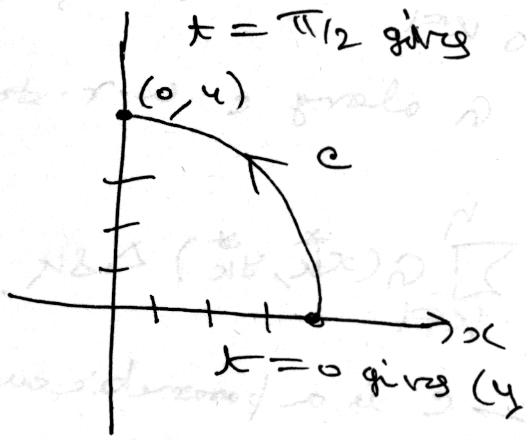
where path of integration c is the quarter circle defined by $x = 4\cos t$ & $y = 4\sin t\ (0 \le t \le \frac{\pi}{2})$.

□ (a) Since $dx = -4\sin t\,dt$

$$\Rightarrow \int_c xy^2\,dx = \int_0^{\pi/2} (4\cos t)(4\sin t)^2(-4\sin t)\,dt$$

$$= -256\int_0^{\pi/2} \sin^3 t\cos t\,dt = -256\left[\frac{1}{4}\sin^4 t\right]_0^{\pi/2}$$

$$= -64$$



Path c of integration

(b) Since $dy = 4\cos t\,dt$

$$\Rightarrow \int_c xy^2\,dy = \int_0^{\pi/2} (4\cos t)(4\sin t)^2(4\cos t\,dt)$$

$$= 256\int_0^{\pi/2} \sin^2 t\cos^2 t\,dt$$

$$= 64\int_0^{\pi/2} \frac{1}{2}(1-\cos 4t)\,dt$$

$$= 32\left[t - \frac{1}{4}\sin 4t\right]_0^{\pi/2} = 16\pi$$

(c) Since $ds = \sqrt{16(\sin^2 t + \cos^2 t)}\, dt = 4\,dt$

$$\Rightarrow \int_C xy^2\, ds = \int_0^{\pi/2} (4\cos t)(4\sin t)^2 (4dt)$$

$$= 256 \int_0^{\pi/2} \sin^2 t \cos t\, dt$$

$$= 256\left[\tfrac{1}{3}\sin t\right]_0^{\pi/2} = \frac{256}{3}$$

Case B

— If path of integration C is $y = f(x)$, $a \le x \le b$

$\Rightarrow dy = f'(x)\, dx$

& $ds = \sqrt{1+(f'(x))^2}\, dx$

$$\Rightarrow \boxed{\begin{aligned} \int_C G(x,y)\, dx &= \int_a^b G(x, f(x))\, dx \\ \int_C G(x,y)\, dy &= \int_a^b G(x, f(x))\, f'(x)\, dx \\ \int_C G(x,y)\, ds &= \int_a^b G(x, f(x))\, \sqrt{1+(f'(x))^2}\, dx \end{aligned}}$$

$$\int_C G(x,y)\, ds = \int_{C_1} G(x,y)\, ds + \int_{C_2} G(x,y)\, ds$$

(piecewise smooth curve)

— Notation: $\int_C P(x,y)\, dx + \int_C Q(x,y)\, dy$

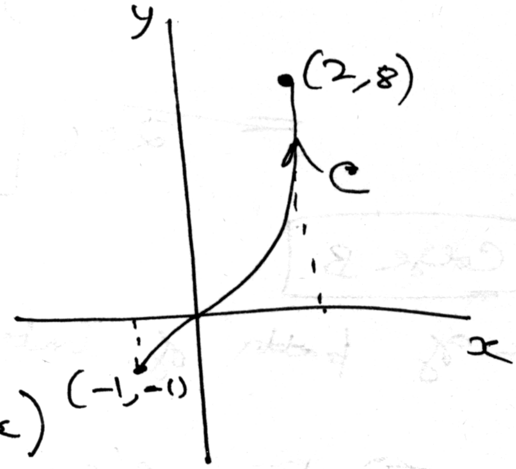
$$\equiv \int_C P(x,y)\, dx + Q(x,y)\, dy$$

or simply $\int_C P\, dx + Q\, dy$

- A line integral along a closed curve c is denoted by

$$\oint_c P\,dx + Q\,dy$$

$\boxed{\text{Example}}$ $\int_c xy\,dx + x^2\,dy$ where c is the graph of $y = x^3$, $-1 \le x \le 2$



$\square$ $\because y = x^3$

$\Rightarrow dy = 3x^2\,dx$

$\Rightarrow \int_c xy\,dx + x^2\,dy$

$= \int_{-1}^{2} x(x^3)\,dx + x^2(3x^2\,dx)$

$= \int_{-1}^{2} 4x^4\,dx = \frac{4}{5} x^5 \Big|_{-1}^{2} = \frac{132}{5}$ $\blacksquare$

$\boxed{\text{Example}}$ $\oint_c x\,dx$ c is the circle defined by

$x = \cos t$, $y = \sin t$ $0 \le t \le 2\pi$

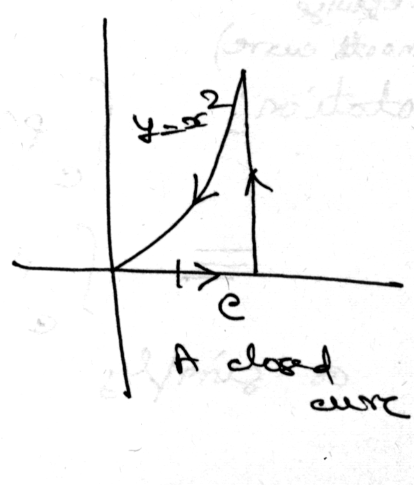
$\square$ $\because x = \cos t \Rightarrow dx = -\sin t\,dt$

$\Rightarrow \oint_c x\,dx = \int_0^{2\pi} \cos t\,(-\sin t\,dt)$

$= \frac{1}{2}\cos^2 t \Big|_0^{2\pi} = \frac{1}{2}(1-1) = 0$

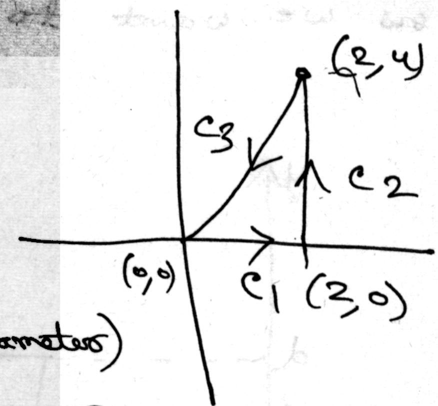
$\boxed{\text{Example}}$ $\oint_c y^2\,dx - x^2\,dy$

c is closed curve shown in fig



$\square$

$C = C_1 \cup C_2 \cup C_3$



$$\Rightarrow \oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

On C_1 $y = 0 \Rightarrow dy = 0$ (x is a parameter)

$$\Rightarrow \int_{C_1} y^2 dx - x^2 dy = \int_0^2 0\, dx - x^2(0) = 0$$

On C_2 (y is a parameter & $x = 2 \Rightarrow dx = 0$

$$\Rightarrow \int_{C_2} y^2 dx - x^2 dy = \int_0^4 y^2(0) - 4\, dy = -\int_0^4 4\, dy = -16$$

On C_3 x is a parameter & $\because y = x^2 \Rightarrow dy = 2x\,dx$

$$\int_{C_3} y^2 dx - x^2 dy = \int_2^0 (x^2)^2 dx - x^2 (2x\,dx)$$

$$= \int_2^0 (x^4 - 2x^3)\, dx = \left(\frac{1}{5}x^5 - \frac{1}{2}x^4\right)\Big|_2^0 = \frac{8}{5}$$

$$\Rightarrow \oint_C y^2 dx - x^2 dy = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + (-16) + \frac{8}{5} = -\frac{72}{5}$$ ■

Double Integrals

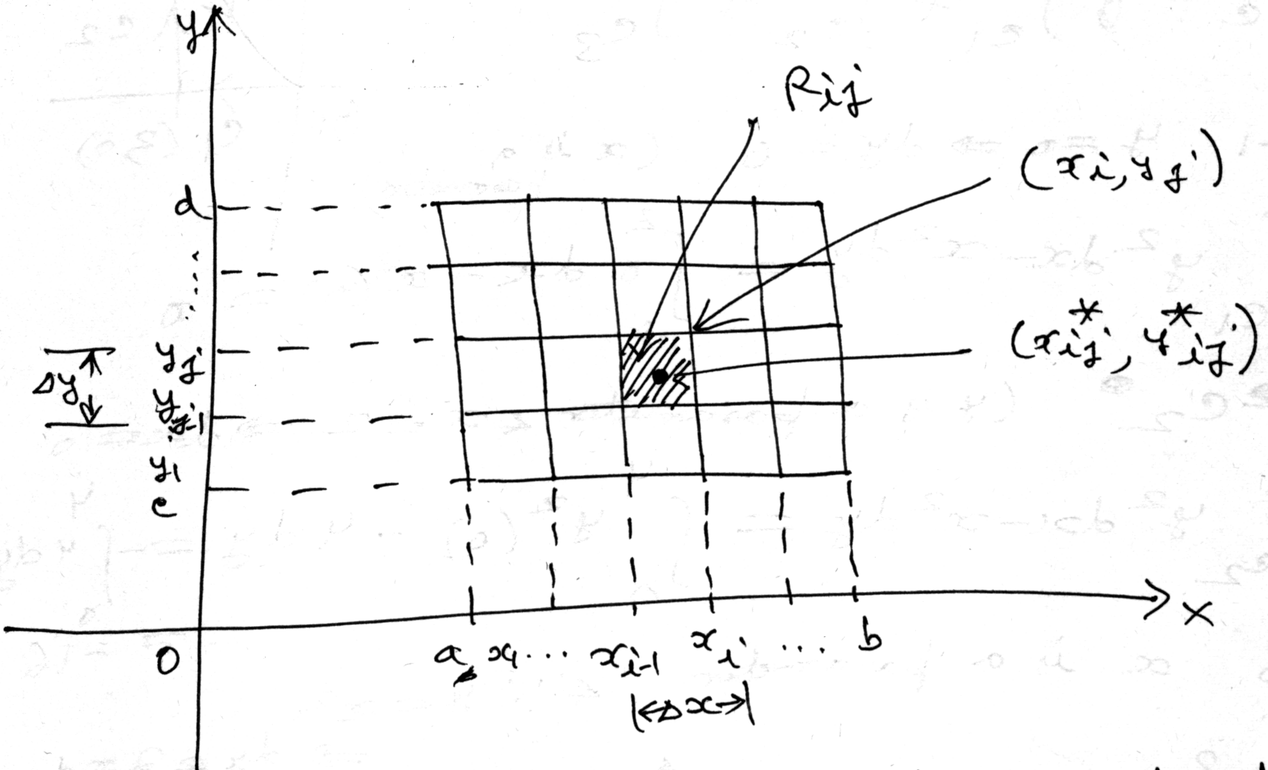
$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$$

A rectangle

Let f be a fn. define on rectangle R. & first suppose that $f(x,y) \geq 0$. The graph of f is a surface $z = f(x,y)$. Let S be the solid that lies above R and under the graph of f i.e.

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z \le f(x,y),\ (x,y) \in R\}$$

Now we want to find the volume of S.



— First we divide the rectangle R into subrectangles. For this we divide $[a,b]$ into m sub intervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{(b-a)}{m}$ & dividing $[c,d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = \frac{d-c}{n}$. This gives small rectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

$$= \left\{ (x,y) \;\middle|\; \begin{array}{l} x_{i-1} \le x \le x_i, \\ y_{j-1} \le y \le y_j \end{array} \right\}$$ each with

area $\Delta A = \Delta x \, \Delta y$ (see fig)

— Choose a sample pt. (x^*_{ij}, y^*_{ij}) in each R_{ij} then we can approximate the part of S that lies above each R_{ij} by a thin rectangle box (or column) with base R_{ij} & height $f(x^*_{ij}, y^*_{ij})$ as shown

— The volume of this box is the height of the box times the area of base rectangle:

$$f(x^*_{ij}, y^*_{ij}) \, \Delta A$$

After summation of all boxes we get an approximation to the total volume of S

$$V \simeq \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*})\,\Delta A$$ (Double Riemann sum)

$$\Rightarrow V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*})\,\Delta A$$

Def: The double integral of f over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*})\,\Delta A$$

if this limit exists.

— One can prove that this limit exists if f is a cont's fn.

— We may also choose that sample pt. (x_{ij}^{*}, y_{ij}^{*}) to be (x_i, y_j) & then integral looks simpler

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_i)\,\Delta A$$

$\Rightarrow$ If $f(x,y) \geq 0$ then the volume V of the solid that lies above the rectangle R & below the surface $z = f(x,y)$ is $V = \iint_R f(x,y)\,dA$

$\boxed{\text{Example}}$ If $R = \{(x,y) \mid -1 \leq x \leq 1, -2 \leq y \leq 2\}$

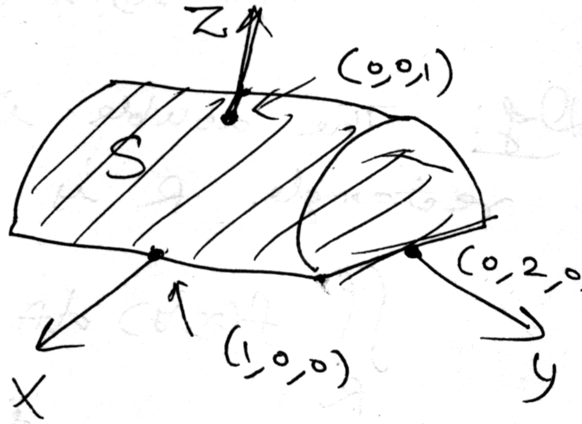
find

$$\iint_R \sqrt{1-x^2}\,dA$$

$\square$ $\because \sqrt{1-x^2} \geq 0$ put $z = \sqrt{1-x^2}$

$\Rightarrow x^2 + z^2 = 1$ & $z \geq 0$

$\Rightarrow$ double integral represents the volume of the solid that lies below the circular cylinder $x^2+z^2=1$ & above the rectangle R.



$\Rightarrow \iint_R \sqrt{1-x^2}\, dA$

$= \frac{1}{2}\pi(1)^2 \times 4$

$= 2\pi$

$\because$ The volume of S is the area of a semicircle with radius 1 times the length of the cylinder. $\blacksquare$

Properties

① $\iint_R [f(x,y) + g(x,y)]\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$

② $\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA$, c is a const.

③ If $f(x,y) \geq g(x,y)$ $\forall (x,y) \in R$ then

$$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA$$

Iterated Integrals

Double integrals can be expressed as a iterated integral.

- Suppose $f(x,y)$ is cont's on $R = [a,b] \times [c,d]$
- $\int_c^d f(x,y)\,dy \triangleq$ f is integrated w.r. to y from $y=c$ to $y=d$ keeping x fixed.
- If $A(x) = \int_c^d f(x,y)\,dy$

$$\Rightarrow \int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

(Iterated integral).

$$\Rightarrow \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

and simly

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

Example Evaluate

(a) $\int_0^3 \int_1^2 x^2 y\,dy\,dx$ (b) $\int_1^2 \int_0^3 x^2 y\,dx\,dy$

$\square$ (a) $\int_1^2 x^2 y\,dx = \left[\frac{x^2 y^2}{2}\right]_{y=1}^{2} = \frac{3}{2}x^2$

$$\Rightarrow \int_0^3 \int_1^2 x^2 y\,dx = \int_0^3 \frac{3}{2}x^2\,dx = \frac{27}{2}$$

we will get here same answer even if we change the order of integration.

$\boxed{\text{Fubini's Theorem}}$ If f is cont's on the rectangle $R = \{(x,y) \mid a \le x \le b, \; c \le y \le d\}$

then $$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx$$
$$= \int_c^d \int_a^b f(x,y)\, dx\, dy$$

— In general this is true if f is bounded on R, f is discont's only on a finite # of smooth curves & the iterated integrals exist.

$\boxed{\text{Example}}$ $$\iint_R (x-3y^2)\, dA$$
$$R = \{(x,y) \mid 0 \le x \le 2, 1 \le y \le 2\}$$

☐ Using Fubini's theorem

$$\iint_R (x-3y^2)\, dA = \int_0^2 \int_1^2 (x-3y^2)\, dy\, dx$$
$$= \int_0^2 \Big[xy - y^3 \Big]_{y=1}^{y=2} dx$$
$$= \int_0^2 (x-7)\, dx = \frac{x^2}{2} - 7x \Big|_0^2 = -12$$

$\boxed{\text{Example}}$ $$\iint_R y \sin(xy)\, dA, \quad R = [1,2] \times [0,\pi]$$

$\boxed{\text{Answer} = 0}$

– In the special case, when $f(x,y) = g(x)\,h(y)$

$R = [a,b] \times [c,d]$

then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d g(x)\,h(y)\,dx\,dy$$

$$= \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

$\boxed{\text{Example}}$ $R = [0, \pi/2] \times [0, \frac{\pi}{2}]$

□ $$\iint_R \sin x \cos y \, dA = \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy$$

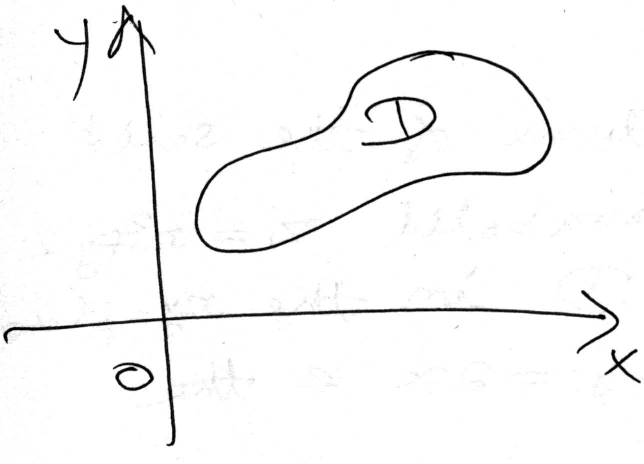
$$= 1 \cdot 1 = 1$$ ▨

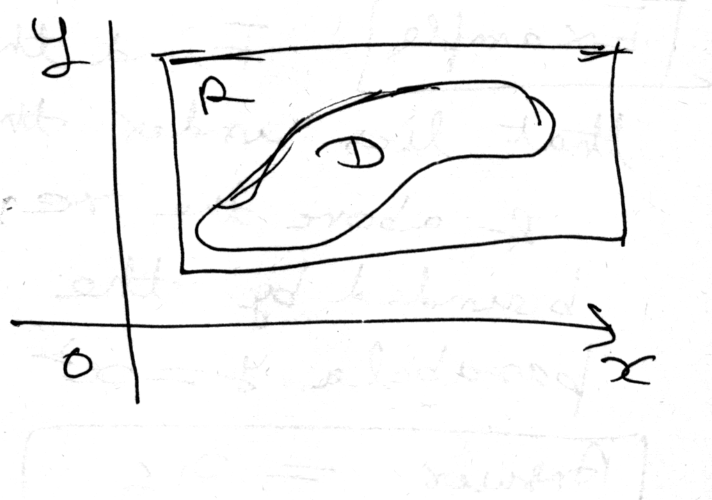
Double Integrals over a region D

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

D – bounded region

where $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R \text{ but } \notin D \end{cases}$





- If f is cont.s on type I region D s.t.

$$D = \{(x,y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

then
$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

- ~~If f is cont~~ Similarly

If $D = \{(x,y) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$

type II region D then
$$\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Example $\iint_D (x+2y)\, dA$, D is region bounded by the parabolas $y = 2x^2$ & $y = 1+x^2$

$$\boxed{\text{Answer} = \frac{32}{15}}$$

Example Find the volume of the solid that lies under the paraboloid $z = x^2+y^2$ & above the region D in the xy-plane bounded by the line $y = 2x$ & the parabola $y = x^2$

$$\boxed{\text{Answer} = \frac{216}{35}}$$

Example Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$$

$\boxed{\text{Answer} = \frac{1}{2}(1-\cos 1)}$

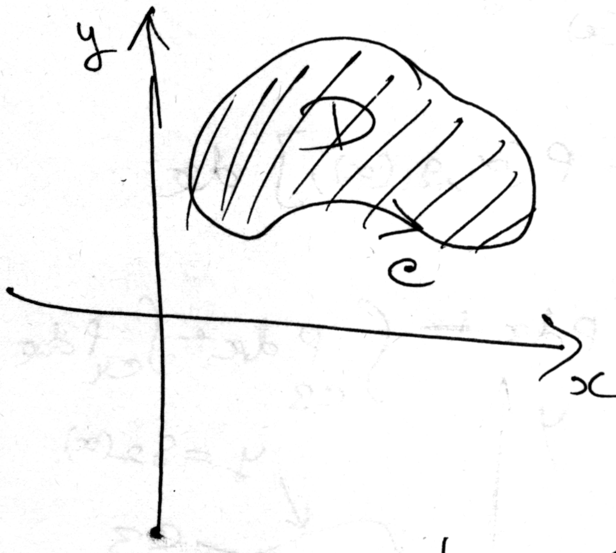
Similarly one can work on triple integrals.

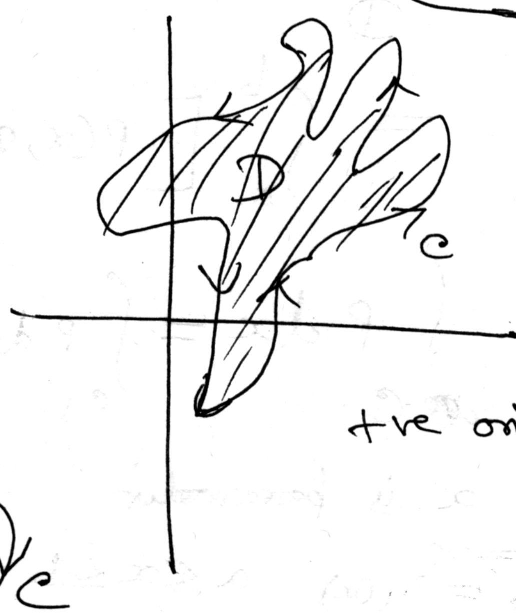
GREEN'S THEOREM

Let C be a +vely oriented, piecewise-smooth, simple closed curve in the plane & let D be the region bounded by C. If P & Q have cont.s partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Some denote $C = \partial D$





+ve orientiation

-ve orientiation

☐ We will prove it for simple case when

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

g_1 & g_2 are cont's fn.

Now note that Green's theorem will hold if we can show that

$$\boxed{\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA} \quad \text{--- (1)}$$

and

$$\boxed{\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA} \quad \text{--- (2)}$$

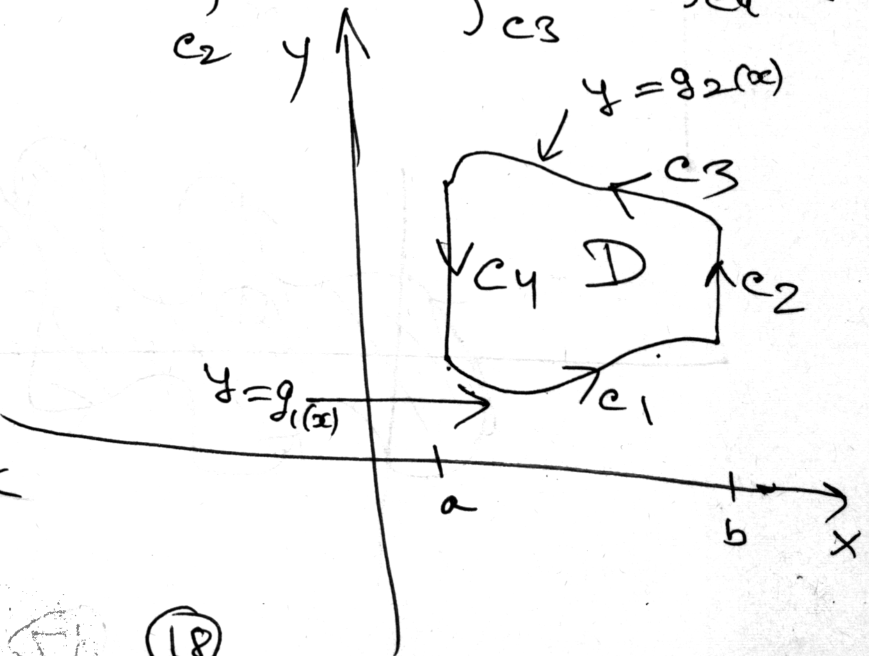
Let us prove (1). (2) can be proved in the same way.

Consider $\displaystyle\iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx$

$$= \int_a^b \left[P(x, g_2(x)) - P(x, g_1(x)) \right] dx$$

Now

$$\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx + \int_{C_4} P\,dx$$



On C_1: x is parameter

$x = x$
$y = g_1(x)$ $\quad a \le x \le b$

$$\Rightarrow \int_{C_1} P(x,y)\,dx = \int_a^b P(x, g_1(x))\,dx$$

Now C_3 goes from right to left but $-C_3$ goes from left to right

$\Rightarrow$ on $-C_3$: $x = x,\ y = g_2(x)$, $a \le x \le b$

$$\Rightarrow \int_{C_3} P(x,y)\,dx = -\int_{-C_3} P(x,y)\,dx = -\int_a^b P(x, g_2(x))\,dx$$

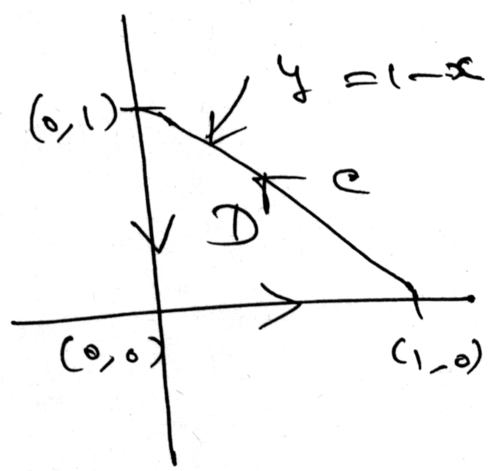
On C_2 or C_4 : x is constant $\Rightarrow dx = 0$

$$\Rightarrow \int_{C_2} P(x,y)\,dx = 0 = \int_{C_4} P(x,y)\,dx$$

$$\Rightarrow \boxed{\int_C P(x,y)\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA}$$

Similarly (2) can be proved. ▨

Example $\int_C x^4\,dx + xy\,dy$ where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$ & from $(0,1)$ to $(0,0)$.



□ $\int_C x^4\,dx + xy\,dy$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_0^1 \int_0^{1-x} (y - 0)\,dy\,dx = \int_0^1 \left[\frac{1}{2}y^2\right]_{y=0}^{y=1-x} dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx = -\frac{1}{6}(1-x^3)\Big|_0^1$$

$$= \frac{1}{6}$$ ▨