Lecture-11

Recap:

Bernoulli random variable

Binomial random variable.

$$E(X) = np$$
$$Var(X) = np(1-p)$$

H.W.

H.W.

	Fast bowler	Spin bowler	
Kohli	180 runs / 202 balls	282 / 250	462 / 452
Dhoni	232 runs / 250 balls	128 / 108	360 / 358

Simpson's paradox

Dept.	Women	Men
Maths	$\frac{20 \text{ admitted}}{50 \text{ applied}}$	$\frac{10}{30}$
C.S.	$\frac{30}{40}$	$\frac{50}{70}$
University	$\frac{50}{90}$	$\frac{60}{100}$ higher

Poisson Random Variable

Let $X = 0, 1, 2, 3, \ldots$

$$P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

$$\sum_{i=0}^{\infty} P(X=i) = 1$$

$$= \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!}$$

$$\sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!}$$

$$e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$$

$$E(X) = ? = \sum x_i p_i$$

$$\sum_{i=0}^{\infty} x_i p_i = \sum_{i=0}^{\infty} i \cdot e^{-\lambda} \frac{\lambda^i}{i!}$$

$$\cancel{e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{(i-1)!}} \qquad j = i-1$$

$$e^{-\lambda} \sum_{j=1}^{\infty}$$

$$\sum_{\substack{i=0 \\ }}^{\infty} \underset{0}{\overset{i}{\shortparallel}} e^{-\lambda} \frac{\lambda^i}{i!} = E(X) \qquad (4)$$

$$= \sum_{i=1}^{\infty} i \, e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!}$$

$\underline{j = i-1} \Rightarrow i = j+1$

$$= e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{j+1}}{j!}$$

$$= e^{-\lambda} \lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

$$= \lambda = E(X)$$

$\text{Var}(X) = \lambda$ (H.W.)

Poisson is a good approximation to binomial for n large, p small with np of moderate size

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$$E(X) = np = \lambda$$

$$\text{Binomial} \qquad \Downarrow \qquad \text{Poisson}$$

$$p = \frac{\lambda}{n}$$

$$P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i} \xrightarrow[\text{Poisson}]{\text{tends to}}$$

$$= \frac{n!}{i!\,(n-i)!} \frac{\lambda^i}{n^i} \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{\lambda^i}{i!} \frac{n!}{(n-i)!\, n^i} \frac{(1-\lambda/n)^{n-i}\,(1-\lambda/n)^i}{(1-\lambda/n)^i}$$

$$= \frac{\lambda^i}{i!} \underbrace{\frac{n}{n}}_{1} \underbrace{\frac{n-1}{n}}_{1} \underbrace{\frac{n-2}{n}}_{1} \cdots \underbrace{\frac{n-i+1}{n}}_{1} \frac{\overbrace{(1-\lambda/n)^n}^{\to e^{-\lambda}}}{\underbrace{(1-\lambda/n)^i}_{\to 0}}$$

$$= \frac{\lambda^i}{i!} e^{-\lambda}$$

Poisson distribution.

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e.g. probability of a certain item being defective 0.1

Sample of 10 items.

P(atmost 1 item out of 10 is defective, i.e., atleast 9 are not defective.)

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X = no. of defective items.

$P(X=0) + P(X=1)$
$\quad k=0 \qquad\qquad k=1$

Binomial: $n = 10, p = 0.1$

$$\binom{10}{0} p^0 (1-p)^{10} + \binom{10}{1} p^1 (1-p)^9$$

$$= (0.9)^{10} + \underbrace{10 \cdot (0.1)}_{1} (0.9)^9$$

$$= (0.9)^9 (0.9 + 1)$$

$$= (1.9) * (0.9)^9 = \underline{0.736}$$

Poisson: $\lambda = np = 1$

$$e^{-\lambda}\left(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!}\right) = e^{-1}(1+\lambda) = \frac{1+1}{e}$$

$$0.7357 = \frac{2}{e}$$

Geometric Random Variable.

Repeating an experiment until the 1st success.

P(success) = p

P(failure) = $1-p$

X = no. of trials until you get the 1st success.

$X \in \{1, 2, 3, \ldots\}$

~~X~~ $X=i$	$P(X=i)$
1	p
2	$(1-p)\,p$
3	$(1-p)^2\,p$
k	$(1-p)^{k-1} \cdot p$

Sum = 1

$$\sum p_i = 1$$

$$\sum_{i=1}^{\infty} (1-p)^{i-1} \cdot p = S$$

$$S = p + (1-p)p + (1-p)^2 p + \dots$$

$$= p(1 + (1-p) + (1-p)^2 + \dots)$$

$$= \frac{p \cdot 1}{1-(1-p)} = 1$$

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e.g.: 20 white balls
30 black balls.

taking out a ball, noting its color, and putting it back in the bag. Repeat until you get a black ball.

$P(X = k)$

$$= \left(\frac{2}{5}\right)^{k-1} \cdot \frac{3}{5}$$

$$= \frac{2^{k-1} \cdot 3}{5^k}$$

What is the probability that you take $\geq k$ attempts?

$$\sum_{i=k}^{\infty} P(X = k) = \sum_{i=k}^{\infty} \frac{2^{i-1} \cdot 3}{5^i}$$

g.p.

H.W.

H.W.