— Manish K. Gupta www.mankg.com

Derivatives Let f a real-valued fn. on an interval $J \subset \mathbb{R}$

$c \in J$ we say that f has a derivative at c

if $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists.

— If this limit exists we denote it by $f'(c)$

Note

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$$

THEOREM If $f'(c)$ exist then fn. f is conts at c.

REMARK It is possible that $f'(c)$ does not exist even though f is conts at c

Example $f(x) = |x| \quad (-\infty < x < \infty)$

then $\frac{f(x)-f(0)}{x-0} = 1$ if $x > 0$

while $\frac{f(x)-f(0)}{x-0} = -1$ if $x < 0$

Hence $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$ does not ~~exist~~.

Thus $f(x)$ does not have a derivative at 0 but it is conts at 0 (Verify that!)

~~**Example** The function $g(x) = x^2$ is cont.s at each point in $[0,1]$ but does not have a derivative at any pt. in $[0,1]$~~

— Give an example of a fn. f which is cont.s every where on $\mathbb{R}$ but $f'(a)$ does not exist at any $a \in \mathbb{R}$.

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos 3^n x}{2^n} \qquad (x \in \mathbb{R})$$

(Weierstrass example)

THEOREM Every differentiable function is cont.s.

$\square$ For $x \neq c$ we have

$$f(x) - f(c) = \left\{ \frac{f(x) - f(c)}{(x-c)} \right\} (x-c)$$

Now since $\lim_{x \to c} \left\{ \frac{f(x) - f(c)}{x - c} \right\} = f'(c)$ & $\lim_{x \to c} (x - c) = 0$

we have

$$\lim_{x \to c} \{ f(x) - f(c) \} = 0$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c) \qquad ▨$$

POINTS

① If $f'(c) > 0$ the curve $y = f(x)$ is <u>'ascending'</u> at c

② If $f'(c) < 0$ the curve $y = f(x)$ is <u>'descending'</u> at c

③ If $f'(c) = 0$ the curve has a horizontal tangent at c.

— If $f'(c)$ exist the curve $y = f(x)$ is smooth at c.

Def: n^{th} derivative

$$f^{(n)}(c) = \lim_{x \to c} \frac{f^{(n-1)}(x) - f^{(n-1)}(c)}{x - c}$$

provided $f^{(n-1)}(x)$ exists $\forall x$ in an interval containing c & provided the limit exists.

$\Rightarrow$ If $f^{(n)}(c)$ exists then $f^{(n-1)}$ is conts at c.

PRE-ROLLE'S THEOREM Let f be a conts real valued fn. on the closed bounded interval $[a,b]$. If the maximum value for f is attained at c, where $a < c < b$ and if $f'(c)$ exists then $f'(c) = 0$

$\square$ Suppose $f'(c) \neq 0$.

- If $f'(c) > 0$ then $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$ & so

$$\frac{f(x) - f(c)}{x - c} > 0 \quad \text{for } 0 < |x - c| < \delta_1$$

δ_1 is a suitable +ve no.

If $x \in (c, c + \delta_1)$ then $x - c > 0$ and hence we have $f(x) - f(c) > 0$ This is a contradiction to the fact that f attains a max. at c.

- If $f'(c) < 0$ then $\frac{f(x) - f(c)}{x - c} < 0$ for $0 < |x - c| < \delta_2$

If $x \in (c - \delta_2, c)$ then $x - c < 0$ and hence $f(x) - f(c) > 0$ again a contradiction.

$\Rightarrow f'(c) = 0$ ■

ROLLE'S THEOREM

Let f be a cont.s real-valued fn. on the closed bounded interval $[a, b]$ with $f(a) = f(b) = 0$. If $f'(x)$ exists for all x in (a, b), then there is some point $c \in (a, b)$ where $f'(c) = 0$

$\square$ If $f \equiv 0$ on $[a, b]$ nothing to prove.

If $f(x) > 0$ for some $x \in (a, b)$ then the max. value of f on $[a, b]$ will not be attained at a or b (end points since $f(a) = f(b) = 0$)

Note: If the real valued fn. f is cont.s on the closed bounded interval $[a, b]$ then f attains a max. or min. value at points of $[a, b]$

Hence f will attain a max value at some $c \in (a, b)$. Now the result follows from PRE-ROLLE'S THEOREM.

If $f(x) < 0$ similar arguments holds both for PRE-ROLLE'S THEOREM & above proof.

$\blacksquare$

$\boxed{\text{Example}}$

$$f(x) = \sqrt{1-x^2} \qquad (-1 \le x \le 1).$$

f obeys hypothesis of Rolles Thm. with $a=-1$, $b=1$

Here $f'(x) = -\dfrac{x}{\sqrt{1-x^2}}$ for $-1<x<1$

& f does not have a derivative at -1 or 1

and for this f, $c=0$

$\Rightarrow f'(c) = f'(0) = 0.$

$\boxed{\text{Example}}$ $g(x) = 1-|x| \quad (-1 \le x \le 1)$

then $g(-1) = g(1) = 0$ & g is conts on $[-1,1]$.

Also $g'(x)$ exists for all x in $(-1,1)$ except at $x=0$

$\therefore$ g obeys everything except fails to have a derivative at 0.

You can see that for such a fn. g there is no c in $(-1,1)$ for which $g'(c)=0$

LAW OF THE MEAN

If f is a cont:s fn. on the closed bounded interval $[a,b]$, and if $f'(x)$ exists for all x in (a,b), then $\exists$ c in (a,b) s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Let

$\square$ $$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

$$(a \leq x \leq b)$$

Then $h(a) = 0 = h(b)$ and this fn. obeys the other hypotheses of Rolle's theorem as well.

$\Rightarrow$ $\exists\ c \in (a,b)$ s.t. $h'(c) = 0$

but $$h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

Q.E.D.

— Fundamental theorem of calculus ?

TAYLOR'S THEOREM

Suppose $\forall x$ in some interval J the function f may be expressed as

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \cdots + A_n(x-a)^n + \cdots \quad \text{—— (1)}$$

$a \in J$

We say that (1) is an expansion of f in powers of $(x-a)$.

How do you compute coff. A_i?

METHOD

Put $x=a$ in (1)

$$f(a) = A_0$$

differentiate both sides of (1)

$$f'(x) = A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + \cdots$$

& put $x=a$

$$f'(a) = A_1 \quad \text{\& so on.}$$

$$\cdots \quad f^{(n)}(a) = n!\,A_n$$

$\therefore$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \quad \text{-- (2)}$$

We have assumed

① $f'(x)$ exists

② ~~to~~ the ~~deri~~ derivative of R.H.S. of (1) can be computed by taking the derivative separately. (even though there are ∞ terms)

For $a=0$ it is called Maclaurin series

$$\boxed{\text{What is the Maclaurin series for } f(x) = e^{-1/x^2}\,?}$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x)$$

$$\lim_{n \to \infty} R_{n+1}(x) = 0$$

$\boxed{\text{Lemma:}}$ Let f be a real-valued fn. on the interval $[a, a+h]$ s.t. $f^{(n+1)}(x)$ exists $\forall$ $x \in [a, a+h]$ & $f^{(n+1)}$ is cont.s on $[a, a+h]$.

Let $$R_{k+1}(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\,dt$$

$x \in [a, a+h]$

then $k = 0, 1, 2, \dots, n$

$$R_k(x) - R_{k+1}(x) = \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$x \in [a, a+h]$

The same results holds if $h<0$ $k = 1, 2, \dots, n$.
& $[a, a+h]$ is replaced by $[a+h, a]$.

☐ Lemma can be proved ~~b~~ using integration by parts.

$\boxed{\text{Taylor's Formula}}$

Let f be a real-valued fn. on $[a, a+h]$ s.t. $f^{(n+1)}(x)$ exists $\forall x \in [a, a+h]$ & $f^{(n+1)}$ is conts on $[a, a+h]$ then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_{n+1}(x)$$

$$x \in [a, a+h]$$

where

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\, dt$$

$\boxed{\text{Example}}$

$$g(x) = e^{-1/x} \quad (x > 0)$$

$$g(0) = 0$$

$$g'(0) = \lim_{x \to 0+} \frac{g(x) - g(0)}{x} = \lim_{x \to 0+} \frac{e^{-1/x}}{x} = 0$$

(Why?)

Note $\lim_{x \to \infty} \frac{x^n}{e^x} = 0 \quad (n \in \mathbb{N})$

& $\lim_{x \to 0+} \frac{1}{x^n e^{1/x}} = 0 \quad (n \in \mathbb{N})$

$\Rightarrow g^{(n)}(x) = 0$

i.e. $g(0) = g'(0) = \cdots = g^{(n)}(0) = \cdots = 0$

$\therefore$ Maclaurin series for g is $\equiv 0$ &

$\therefore$ does not $\longrightarrow$ to $g(x)$ for any $x > 0$.

This shows that existence of all derivatives of a fn. f at a pt. a does not imply

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!} (x-a) \quad \text{for any } x \neq a$$

$\mathscr{C}[a,b] = \{$ set of cont.s real-valued fns. f on a closed interval $[a,b]$ $\}$

<u>Define</u> : (Norm)

Let $f \in \mathscr{C}[a,b]$

$\|f\| = \max\limits_{a \le x \le b} |f(x)|$

(norm of f)

Verify that ① $\|f\| \geq 0$

② $\|f\| = 0 \Leftrightarrow f(x) = 0 \quad (a \le x \le b)$

③ $\|\lambda f\| = |\lambda| \, \|f\| \quad (\lambda \in \mathbb{R})$

④ $\|f+g\| \le \|f\| + \|g\|$

$\boxed{\text{Metric (distance)}}$ $f, g \in \mathscr{C}[a,b]$

$d(f,g) = \|f-g\|$

distance

$\boxed{\text{The Weierstrass approximation theorem}}$

Let f be any fn. in $\mathscr{C}[a,b]$. Then given $\epsilon > 0$, $\exists$ a polynomial P

defined as $P(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$

$x \in \mathbb{R}^1$

such that

$|P(x) - f(x)| < \epsilon \quad (a \le x \le b)$

-Note You can also write it as $\|P - f\| < \epsilon$