Lecture - 27

P 1

Recap

$$E[g(X,Y)] = \iint g(x,y) f(x,y)\, dx\, dy$$

$$E[X+Y] = E[X] + E[Y]$$

Binomial, hypergeometric, Boole's inequality

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e.g. Expected number of runs of 1's in a bit string having m 0's and n 1's.

$$\frac{(m+n)!}{m!\,n!}$$

$m=2, \; n=3$

$00\underline{111} \to 1$

$\underline{1}0\underline{1}0\underline{1} \to 3$

$0\underline{1}0\underline{11} \to 2$

$\underline{11}00\underline{1} \to 2$

X = no. of runs of 1's. ②

$X_1 \quad . \quad . \quad . \qquad X_{m+n}$

$$
\begin{array}{ccccccccccc}
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
x_1 & x_2 & x_3 & x_4 & & & & & & & x_{11}
\end{array}
\qquad
\begin{array}{l}
m=5 \\
n=6 \\
X=3
\end{array}
$$

$X_i = 1$ if a new run of 1's is starting from the i^{th} position.

$ 0$ otherwise.

$$E[X] = \sum_{i=1}^{m+n} E[X_i]$$

$$E[X_i] = ?$$

$$E[X_1] = \frac{n}{m+n}$$

$$E[X_2] =$$

$$X_1 \rightarrow 1 \qquad X_1 \rightarrow 0 \rightarrow \frac{m}{m+n}$$

if the 1^{st} bit $= 1$

$$\frac{n}{m+n}$$

$\underset{1}{\underline{0}} \quad \underset{2}{\underline{1}} \quad \_ \quad \_ \quad \ldots$

$X_2 = 1$

$P(X_2 = 1) = P(\text{previous bit} = 0)$
$P(\text{current bit} = 1)$

$$= \left(\frac{m}{m+n}\right) \frac{n}{(m+n-1)} = E[X_2]$$

$$E[X_3] = E[X_2] = \underset{i \geq 2}{E[X_i]}$$

$\underset{1}{\underline{\quad}} \quad \underset{2}{\underline{0}} \quad \underset{3}{\underline{1}} \quad \underline{\quad} \quad \ldots$

$$E[X] = E[X_1] + (m+n-1) E[X_2]$$

$$= \frac{n}{m+n} + \frac{\cancel{(m+n-1)}(mn)}{(m+n)\cancel{(m+n-1)}} = \frac{mn+n}{m+n}$$

e.g.: Coupon collecting Problem (4)

N types of coupons.

A B C D E

$\underbrace{C}_{}$ $\overbrace{C \quad A}$ $\overbrace{C \quad A \quad B}$ $\overbrace{C \; C \quad A \; B \; D}$ $\overbrace{E}$

1 2 3 4 5 6 7 8 9 10 11 12

$x_0 = 1, \; x_1 = 2, \; x_2 = 3, \; x_3 = 5, \; x_4 = 1$

X = no. of items you need to purchase in order to collect all types of coupons.

E[X]

X_i = Assume that i type of coupons have already been collected. X_i is the ADDITIONAL no. of coupons that you need to collect in order to get one more type.

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$X = X_0 + \cdots + X_4$

$$E[X] = \sum_{i=0}^{N-1} E[X_i]$$

$E[X_i] = ?$ $\quad X_0 \in \{1\}$

$\cancel{X_1 =}$ $\quad E[X_0] = 1$

$E[X_i]$ $\quad i \geq 1$

$X_1 \in \{1, 2, 3 \quad \cdots \quad \}$

C C C C C C C C . . . A

C . C . B

1 2 3

X_1	p	
1	$\frac{N-1}{N}$	
2	$\frac{N-1}{N} \cdot \frac{1}{N}$	C . C . C . D
3	$\frac{N-1}{N} \cdot \frac{1}{N^2}$	C A $\quad \frac{1}{N}$

X_i	p
1.	$\frac{N-i}{N}$
2	$\frac{N-i}{N} \cdot \frac{i}{N}$
3	$\frac{N-i}{N} \cdot \left(\frac{i}{N}\right)^2$
4	$\frac{N-i}{N} \cdot \left(\frac{i}{N}\right)^3$
$\vdots$	
∞	

Geometric R.V.

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$$C\,C\,C\,B\,.C\,C\,A\,\underbrace{ABCBCCCCCCABD}_{X_3}$$

$i = 3$

$$A\,\underbrace{B\,D}_{X_3 = 2}$$

$$E[X] = E[X_0] + E[X_1] + \dots + E[X_{N-1}]$$