# Tute-6 Sol$^n$

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**Sol$^n$-1** X = Hypergeometric random variable with parameters $n, N$ & $m$.

$E[X] = ?$

$Var[X] = ?$

here, $P(X=i) = \dfrac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$.

to find in general, compute $E[X^k]$

$$E[X^k] = \sum_{i=0}^{n} i^k P[X=i]$$

$$= \sum_{i=1}^{n} i^k \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$$

Now, $i\binom{m}{i} = m\binom{m-1}{i-1}$

and $n\binom{N}{n} = N\binom{N-1}{n-1}$

$$\therefore E[X^k] = \sum_{i=0}^{n} \frac{i^{k-1}\, i\binom{m}{i}\binom{N-m}{n-i} \times n}{n \times \binom{N}{n}}$$

$$= \sum_{i=0}^{n} \frac{i^{k-1} nm \binom{m-1}{i-1}\binom{N-m}{n-i}}{N\binom{N-1}{n-1}}$$

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$$= \sum_{i=0}^{n} \frac{nm}{N} \, i^{k-1} \frac{\binom{m-1}{i-1}\binom{N-m}{n-i}}{\binom{N-1}{n-1}}$$

the above term is not defined for $i = 0$

so, taking $j = i-1$

$$\therefore E[X^k] = \frac{nm}{N} \sum_{j=0}^{n} (j+1)^{k-1} \frac{\binom{m-1}{j}\binom{N-m}{n-1-j}}{\binom{N-1}{n-1}}$$

$$= \frac{nm}{N} E\left[(Y+1)^{k-1}\right]$$

Here $Y$ is a hypergeometric random variable with parameters $n-1, N-1$ & $m-1$.

Now putting $k=1$

$$\Rightarrow E[X] = \frac{nm}{N} E[1]$$

$$\Rightarrow \boxed{E[X] = \frac{nm}{N}}$$

• $k=2 \Rightarrow E[X^2] = \frac{nm}{N} E[Y+1]$ $\quad \because (E[Y+1] = E[Y]+1)$

$$\Rightarrow \boxed{E[X^2] = \frac{nm}{N}\left[\frac{(n-1)(m-1)}{(N-1)} + 1\right]}$$

now, $Var[X] = E[X^2] - [E[X]]^2$

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$$\therefore Var[X] = \frac{nm}{N}\left[\frac{(n-1)(m-1)}{N-1}+1\right] - \frac{n^2m^2}{N^2}$$

$$\boxed{\therefore Var[X] = \frac{nm}{N}\left[\frac{(n-1)(m-1)}{N-1}+1-\frac{nm}{N}\right]}$$

now, Let $p = \frac{m}{N} \Rightarrow \boxed{m = pN}$

$$\therefore Var[X] = np\left[\frac{(n-1)(Np-1)}{N-1}+1-np\right]$$

$$= np\left[(n-1)\left(p - \frac{(1-p)}{N-1}\right)+1-np\right]$$

$$= np\left[(n-1)p - \frac{(n-1)(1-p)}{N-1}+1-np\right]$$

$$= np\left[(1-p) - \frac{(n-1)(1-p)}{N-1}\right]$$

$$\boxed{\therefore Var[X] = np(1-p)\left[1-\frac{n-1}{N-1}\right]}$$

Soln-2 n fair dice are rolled.

$E[X] = ?$, ~~Var[X]~~ ?

here X : Random variable that shows the sum of obtained when n fair dice are rolled.

let's define Random variables $X_1, X_2, \ldots, X_n$

$X_i$ = The number obtained in ith dice.

So, $X = X_1 + X_2 + \ldots + X_i$

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$$\therefore \boxed{X = \sum_{i=1}^{n} X_i} \longrightarrow (1)$$

and, $E[X_i] = \sum_{j=1}^{6} j \left(\frac{1}{6}\right)$

$$= \frac{21}{6} = \boxed{\frac{7}{2}}$$

$\therefore$ From eq$^n$ (1)

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right]$$

$$= \sum_{i=1}^{n} E[X_i]$$

$$= \sum_{i=1}^{n} \frac{7}{2}$$

$$= n \cdot \frac{7}{2}$$

$$\therefore \boxed{E[X] = 3.5n}$$

Sol$^n$.3 The distribution function of the Random variable X -

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \leq x < 1 \\ \frac{2}{3} & 1 \leq x < 2 \\ \frac{11}{12} & 2 \leq x < 3 \\ 1 & 3 \leq x. \end{cases}$$

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Sol$^n$ → First of all let's get clear about the cumulavive distribution function.

X → Random variable

$\cancel{F(a) = \sum_{i=-\infty}^{a} P(a)}$ $\boxed{F(a) = \sum_{\text{all } x < a} p(x)}$

$\boxed{P(a < X \leq b) = F(b) - F(a)}$ $[a < b]$

now, $\{X \leq b\} = \{X \leq a\} \cup \{a < X \leq b\}$

$\Rightarrow P\{X \leq b\} = P\{X \leq a\} + P\{a < X \leq b\}$

now, $P\{X < b\} = P\left(\lim_{n \to \infty} \left\{X \leq b - \frac{1}{n}\right\}\right)$

$= \lim_{n \to \infty} P\left\{X \leq b - \frac{1}{n}\right\}$

$\therefore \boxed{P\{X < b\} = \lim_{n \to \infty} F\left(b - \frac{1}{n}\right)}$

$\Rightarrow \boxed{P(b) = P(X = b) = F(b) - \lim_{n \to \infty} F\left(b - \frac{1}{n}\right)}$

→ because at this point we need to find the value of the cumulative distribution function at a value 'x'. x is the greatest number less than b.

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a] $P\{X<3\} = \lim_{n\to\infty} F\left(3-\frac{1}{n}\right)$ $\longrightarrow <3$

$$\boxed{=\frac{11}{12}}$$

b] $P\{X=1\} = F(1) - \lim_{n\to\infty} F\left(1-\frac{1}{n}\right)$

$$= \frac{2}{3} - \frac{1}{2}$$

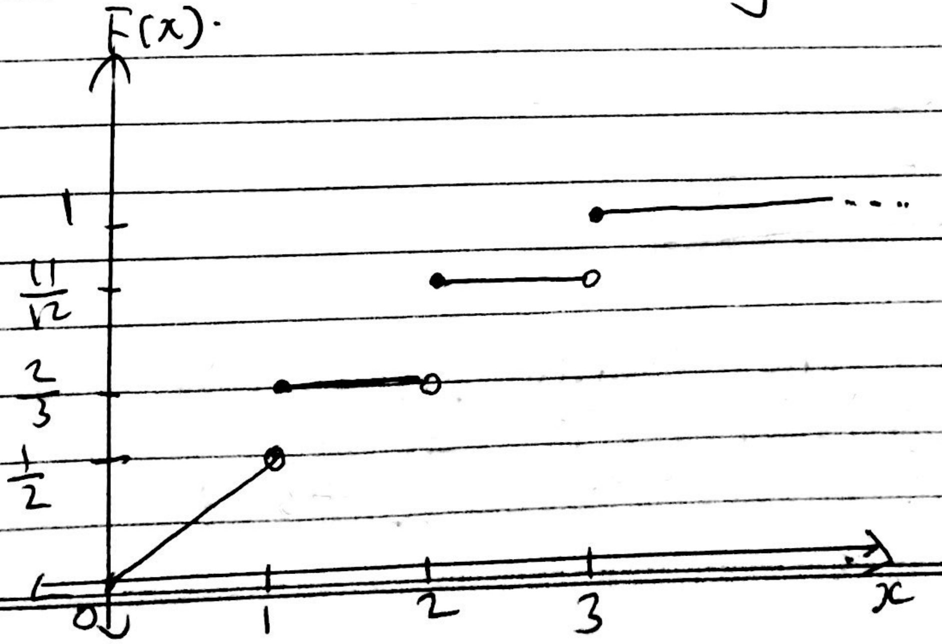
$$\boxed{=\frac{1}{6}}$$

c] $P(2<X\leq 4) = F(4) - F(2)$

$$= 1 - \frac{11}{12}$$

$$\boxed{=\frac{1}{12}}$$

- Just to find limits easily follow the graph.



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**Solⁿ-4** The lifetime in hours of a radio tube is a random variable having a probability density function.

$$f(x) = \begin{cases} 0 & x \leq 100 \\ \dfrac{100}{x^2} & x > 100. \end{cases}$$

- There are 5 tubes.
- What is the probability that exactly 2 radio tubes will die in the first 150 hours

$E_i$ = $i^{th}$ tube will die in the first 150 hours.

$$E_i = \int_0^{150} f(x)\,dx$$

$$= \int_{100}^{150} \frac{100}{x^2}\,dx$$

$$= 100\left[\frac{1}{x}\right]_{150}^{100}$$

$$= 100\left(\frac{1}{100} - \frac{1}{150}\right)$$

$$= \frac{100 \times 50}{100 \times 150}$$

$$\therefore \boxed{E_i = \frac{1}{3}}$$

Hence the desired probability is

$$= \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \boxed{\frac{80}{243}}$$

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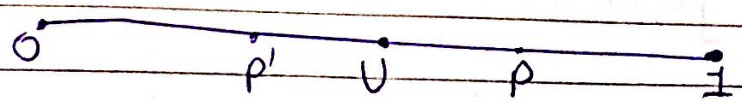
**Sol^n s** Stick of length 1 is split at a point U. that is uniformly distributed over (0,1). Determine the expected length of peices that contains point 'p.', $0 \leq p \leq 1$.




$L_p(U)$: Length of substick that contains the point p.

$$\therefore E[L_p(U)] = \int_0^1 L_p(u)\, du.$$

now, $L_p(U) = \begin{cases} 1-U, & p > U \\ U, & p < U \end{cases}$

$$\therefore E[L_p(U)] = \int_0^1 L_p(u)\, du$$

$$= \int_0^p (1-u)\, du + \int_p^1 u\, du$$

$$= \left[u - \frac{u^2}{2}\right]_0^p + \left[\frac{u^2}{2}\right]_p^1$$

$$= \cancel{1} - \left(\cancel{1} + \cancel{\frac{p^2}{2}}\right) + \cancel{\frac{1}{2}} - \cancel{\frac{p^2}{2}}$$

$$= p - \frac{p^2}{2} + \frac{1}{2} - \frac{p^2}{2}$$

$$= \frac{1}{2} + p - p^2$$

$$\boxed{= \frac{1}{2} + p(1-p)}$$

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<u>Soln 6</u> $X$ is uniformly distributed over $(\alpha, \beta)$.

$$\therefore f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & \alpha \leq x < \beta \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{\alpha}^{\beta} \frac{x}{\beta - \alpha}\, dx$$

$$= \frac{1}{\beta - \alpha} \left[ \frac{x^2}{2} \right]_{\alpha}^{\beta}$$

$$= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)}$$

$$\boxed{= \frac{\beta + \alpha}{2}}$$

$$E[X^2] = \int_{\alpha}^{\beta} \frac{x^2}{\beta - \alpha}\, dx$$

$$= \frac{1}{3(\beta - \alpha)} \left[ x^3 \right]_{\alpha}^{\beta}$$

$$= \frac{\beta^3 - \alpha^3}{3(\beta - \alpha)}$$

$$\begin{array}{r|l} & \beta^2 + \alpha\beta + \alpha^2 \\ \beta - \alpha & \beta^3 - \alpha^3 \\ & \beta^3 - \alpha\beta^2 \\ \hline & \alpha\beta^2 - \alpha^3 \\ & \alpha\beta^2 - \alpha^2\beta \\ \hline & \alpha^2\beta - \alpha^3 \\ & \alpha^2\beta - \alpha^3 \\ \hline & 0 \quad 0 \end{array}$$

$$\therefore \boxed{E[X^2] = \frac{\beta^2 + \alpha\beta + \alpha^2}{3}}$$

$$\therefore Var[X] = E[X^2] - (E[X])^2$$

$$= \frac{\beta^2 + \alpha\beta + \alpha^2}{3} - \frac{(\beta + \alpha)^2}{4}$$

$$= \frac{4\beta^2 + 4\alpha\beta + 4\alpha^2 - 3\beta^2 - 3\alpha^2 - 6\alpha\beta}{12}$$

$$= \frac{\alpha^2 + \beta^2 - 2\alpha\beta}{12}$$

$$\boxed{= \frac{(\alpha - \beta)^2}{12}}$$