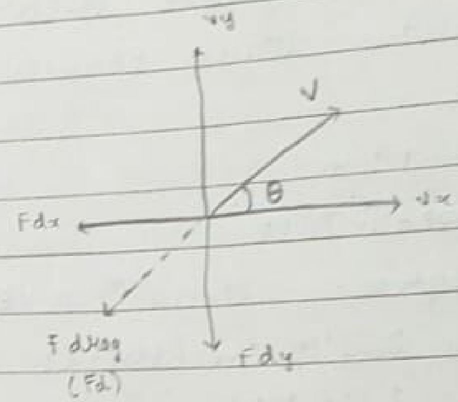
Here, to consider the effect of drag fo[rce]
take its components.



$$m\frac{dv_x}{dt} = 0 - Fd\cos\theta$$

$$m\frac{dv_y}{dt} = mg - Fd\sin\theta$$

But the θ here creates a problem while updating so we want to get rid of it. so here $\sin\theta = \frac{v_y}{v}$; $\cos\theta = \frac{v_x}{v}$

where $v = \sqrt{v_x^2 + v_y^2}$

Now typically $Fd = -Bv^2$ (High speed in projectile so linear proportionality of F on v is neglected).

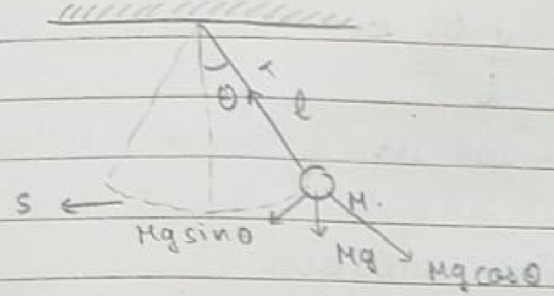
$$Fdx = Fd\cos\theta = Fd\frac{v_x}{v} = -B_2 v v_x$$

$$Fdy = Fd\sin\theta = Fd\frac{v_y}{v} = -B_1 v v_y$$

Finally; $m\frac{dv_x}{dt} = -B_2 v v_x$ $m\frac{dv_y}{dt} = mg - B_1 v v_y$

Oscillation

→ To and fro change of state around the equilibrium position



Arc of motion = s.
since θ very small
$s = l\theta$

Approximation -

→ Point mass M

→ string massless and inextensible (tight)

→ No air drag.

→ The forces acting here are tension in the string and mg downwards.

→ Here, since string is tight, T and Mg cos balance each other

→ Here mg sin θ is the restoring force. Hence

$F_\theta = -mg\sin\theta$

→ we here assume that θ is very small

so $\sin\theta \approx \theta$ so $F_\theta = -mg\theta$

So. $m\dfrac{d^2s}{dt^2} = -mg\theta$

$s = l\theta \Rightarrow \dfrac{ds}{dt} = l\dfrac{d\theta}{dt} \Rightarrow \dfrac{d^2s}{dt^2} = l\dfrac{d^2\theta}{dt^2}$

→ Hence $\dfrac{d^2\theta}{dt^2} = -\dfrac{g}{l}\theta$

Solution : $\theta = \theta_0 \sin \omega t$

→ Here $\frac{d\omega}{dt} = -\frac{g}{l}\theta$ and $\frac{d\theta}{dt} = \omega$

→ In code

$\omega_{i+1} = \omega_i - \frac{g}{l}\theta_i \Delta t$

$\theta_{i+1} = \theta_i + \underline{\omega_i} \Delta t$ should be ω_{i+1}

→ Important observations

① Amplitude remains constant

② Motion is COMPLETELY INDEPENDENT of mass

→ characteristic time scale of system is of order of time period $T = 2\pi\sqrt{\frac{l}{g}}$

timeperiod may not be integer

```
clear all;
length = 1;
g = 9.8;
timeperiod = 2*pi*sqrt(length/g);
dt = timeperiod/100
simulations = 10*timeperiod;
npoints = round(simulation/dt);
omega = zeros(npoints,1);
theta = zeros(npoints,1);
time = zeros(npoints,1);

theta(1) = 0.2;    (in radians)
omega(1) = 0;
time(1) = 0;
```

★ Here energy conservation is violated as v is increasing (so we need correction)

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for step = 1 : npoints - 1
omega (step+1) = omega (step) - (g/ length)* dt* theta (step) ;

theta (step+1) = theta (step) + omega (step)* dt;
time (step+1) = time (step) + dt ;
end

need omega (step+1) here

theta = 57.5 * theta ;
plot (time, theta) ; ⟶ Not a stable graph (gives increasing sinusoid — solution : use ode). ★ correction...

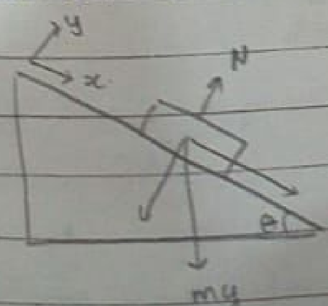
Note : Sources of error ...

→ wrong equation.
→ wrong initial condition
→ dt not sufficiently small
→ simulation time not sufficiently large.
→ Truncation / Round off error.

Note — If there's an error in the system, it will accumulate with time. So to make sure if trend worsens with time or not, increase simulation time to make sure if there's an error or not

Types of forces

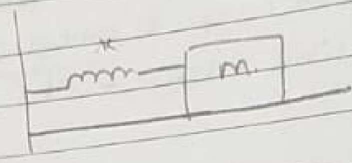
→ Contact / Non contact (field) forces



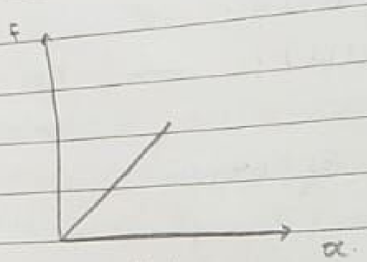
$$mg\sin\theta = m\frac{dv_x}{dt}$$

$$mg\sin\theta = m\frac{d^2x}{dt^2}$$

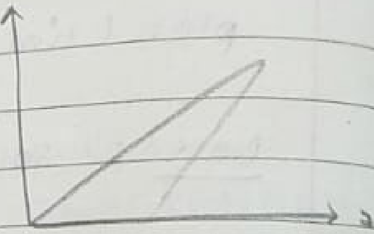
→ If we include the frictional force
$ma = mg\sin\theta - f_s$

→ 

Restoring force $\propto$ displacement
$(F = -kx)$



Hooke' law



Beyond Hooke's law. (does not come back to equilibrium - there is a permanent distortion).

Here $ma = -kx$

$\frac{d^2x}{dt^2} = -\frac{kx}{m}$. Here $\omega = \sqrt{\frac{k}{m}}$

(Pendulum $\omega \to$ independent of mass
spring system $\omega \to$ dependent on mass)

→ A system with a body falling and drag force. → $\frac{dy}{dx} + P(x)y = Q(x)$ ....

$ma = mg - kv$

$\frac{dv}{dt} = g - \frac{kv}{m}$

$\frac{dv}{dt} + \frac{kv}{m} = g$

$y = \frac{1}{\mu(x)}\int \mu(x)\,Q(x)\,dx$

where $\mu(x) = e^{\int P(x)\,dx}$

→ Using $t=0 \Rightarrow v = v_0$ gives

$v = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}$

→ Here as $t \to \infty$, v becomes constant
$(v = mg/k)$

→ given v small - ignore v^2 (in drag force)
v large - ignore v (in drag force)

→ when Euler method gives unstable results, use Euler Cromer method

→ Projectile Motion - 4 function changing with time (some function u)
Here $v_1 = \frac{dx}{dt}$ $v_2 = \frac{dy}{dt}$ $v_3 = \frac{dv_x}{dt}$ $v_4 = \frac{dv_y}{dt}$

ODE solver

[t, v] = ode45(@myfunc, [tstart, tfinal], v0, options)

v → coloumn vector → $\begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix}$

v_0 → coloumn vector → $\begin{matrix} v_{01} \\ v_{02} \\ v_{03} \\ v_{04} \end{matrix}$

Basically this equation is doing integration from tstart to tfinal.

Eq. $\frac{d\theta}{dt} = \omega$ $\frac{d\omega}{dt} = -\frac{g}{l}\theta$

function F = rhs(t, v)
F(1) = v(2)
F(2) = - cnst * v(1)

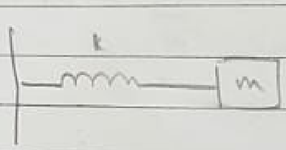
→ zero vel, constant acc — ball thrown up (at highest point)

→ vel, acc in opp direction — ball thrown up (while upward motion)

→ zero acc, finite vel — motion with constant velocity

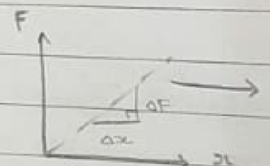
Phase 2

→ Difficult to draw all forces as vectors so we use Hamiltonian dynamics (and Lagrangian) — here only scalars used



$F \propto x$

$F = -kx$



$$\text{slope} = k = \frac{\Delta F}{\Delta x}$$

Writing F as Taylor series

$$F(x) = F_0 + x\left(\frac{dF}{dx}\right)_{x=0} + \frac{1}{2}x^2\left(\frac{d^2F}{dx^2}\right)_{x=0}$$

Has continuous derivatives.

→ x very small — neglect higher order terms

→ F_0 is force at $x = 0$ (Take it $F_0 = 0$):

$$F(x) \approx \left(\frac{dF}{dx}\right)_{x=0} x \;\rightarrow\; \text{Resembles } F = -kx$$

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→ Within Hookes elastic limit, no problem
Else permanent distortion.

Very Imp! $\boxed{\text{SHM Equation}}$: $\ddot{x} = -\frac{kx}{m} = -\omega_0^2 x$.

solution — $x = A\cos(\omega_0 t + \phi)$.

Eg. suppose at $t=0$, $x=0$, $v=-3$, $k=10$, $m=0.1$

$\omega_0 = \sqrt{\frac{10}{0.1}} = 10$ rad/s.

$T = \frac{2\pi}{\omega} = 6.28$ sec

→ At $t=0$, $x=0 \Rightarrow \phi = \pi/2, 3\pi/2$
→ $\dot{x} = -A\omega_0 \sin(\omega_0 t + \phi)$
At $t=0$, $v=-3 \Rightarrow A = -3$

So $x = -3\cos(10t + \pi/2)$.

→ Imp Conclusions — Initial conditions decide amplitude and phase angle. Angular frequency does not depend on amplitude. $\omega_0 = \sqrt{k/m}$

→ Work done on a particle by force F
$W = \int_1^2 F \cdot dr$

→ But this is difficult to analyse so we want to rewrite in terms of power/energy.

$$F \cdot dr = m \frac{dv}{dt} \frac{dr}{dt} dt$$

$$= m \frac{dv}{dt} v \, dt$$

$$= \frac{m}{2} \frac{d}{dt} (v \cdot v) \, dt$$

$$F \cdot dr = d(\tfrac{1}{2} m v^2)$$

$\downarrow$

Imp conclusion – small change in kinetic energy is work done.

→ $W_{12} = \frac{1}{2} m (v_2^2 - v_1^2) = T_2 - T_1$

→ $T_2 > T_1$: work done on particle (KE ↑)

$T_1 > T_2$: work done by particle (KE ↓).

very Imp. $\boxed{\rightarrow}$ Also, change in potential energy can lead to work done.

→ Absolute P.E no meaning. The change in P.E is important. This change gives force which is required for motion

→ Here we are taking conservative forces - does not depend on path, depends only on start-end point.

→ capacity to do work = P.E.

$$U_1 - U_2 = \int_1^2 F \cdot dr$$

→ Potential — very important — scalar quantity easy to analyse. (Shape of potential wrt distance can give us info about force need for motion)

$$\int_1^2 F \cdot dr = -\int_1^2 dU$$

→ $F = -\nabla U$ — only when force is CONSERVATIVE

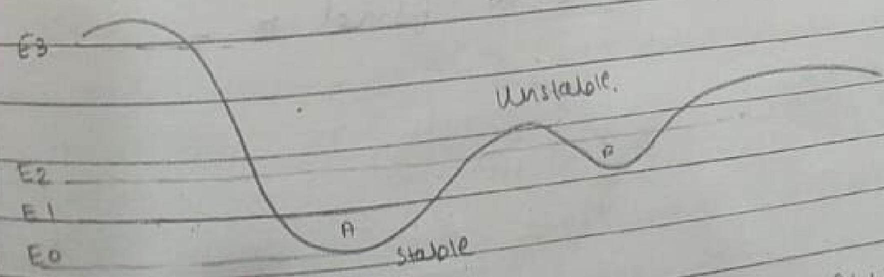
→ At times $F = -\nabla U(x,t)$
goal is to understand potential as a function of space and time.

→ Here we take just $F = -\nabla U(r)$

→ Potential of S.H. oscillator $= \frac{1}{2} k x^2$
(parabolic)

→ SHM

So if we have $U = \frac{1}{2} k x^2$ we can surely say there will be SHM.



Consider particles with different energy at any point
E0 — can be present only at A (bounded)
E1 — oscillatory motion at A (bounded)
E2 — oscillatory at A / B

We also looked at the oscillatory motion.

E 3 — unbounded (comes from ∞, strikes and goes back to ∞)

→ $\frac{dU}{dx} = 0 \rightarrow$ Equilibrium

$\frac{d^2U}{dx^2} > 0 \rightarrow$ Stable $\qquad \frac{d^2U}{dx^2} < 0 \rightarrow$ Unstable

(Minima) $\qquad$ (Maxima)

↓ $\qquad$ ↓

Less Energy More Stable. $\qquad$ More Energy Less Stable

Goal — understand $x(t)$ from $U(x)$

$F = -\nabla U(x)$.

Now $E = K + U$

$E = \frac{1}{2}mv^2 + U(x)$

$\frac{dx}{dt} = v = \sqrt{2/m\,(E - U(x))}$

Integrate this to find x....

$$\int_{x_0}^{x} \frac{dx}{\sqrt{E - U(x)}} = \int_{t_0}^{t} \sqrt{\frac{2}{m}}\, dt.$$

↓

Here; $U(x) = \frac{1}{2}kx^2$ for SHO

$U(x) = \frac{-k}{x}$ for gravitational force so given potential and total energy, we can know about the type of motion that body does

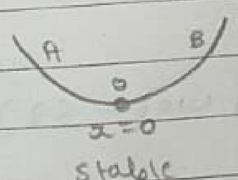
→ $v(x) = V_0 + x\left(\frac{dv}{dx}\right)_{x=0} + \frac{x^2}{2!}\left(\frac{d^2v}{dx^2}\right)_{x=0} + \ldots$

considering small oscillations about the equilibrium point, x is very small so higher terms neglected

At equilibrium : $V(x) = \frac{x^2}{2!}\left(\frac{d^2v}{dx^2}\right)$

since $x=0$ is stable equilibrium, on both sides of $x=0$, we definitely need $V(x) > 0$. so $\frac{d^2v}{dx^2}$ must be > 0

Hence this is the condition of stability



Particle prefers to stay in min potential energy. since $x=0$ is equilibrium, $V(x)$ at A and B must be $> V(x)$ at 0 since particle wants to stay at 0.

→ Note : Newtonian mechanics is deterministic we need initial conditions to evolve the motion of system

→ Autonomous system – F does not change with t. (does not depend on t)

→ Non autonomous – F depends on t

→ Difference between $F(x, v)$ and $F(x, v, t)$
unique solution multiple solution

$x, v, t \rightarrow$ independent dynamic variable

→ Explanation of total energy proportional to square of amplitude

$T = K + U$. Here $x = A \sin(\omega_0 t - \delta)$

$v = A\omega_0 \cos(\omega_0 t - \delta)$

Here $ma = -kx$

$\frac{d^2x}{dt^2} = -\frac{k}{m}x$ so $\omega^2 = \frac{k}{m}$.

Now $dW = -F\,dx = kx\,dx$.

$\int_0^x dW = U = \frac{1}{2}kx^2$

Also $K = \frac{1}{2}mv^2$

so $T = \frac{1}{2}m(A^2\omega_0^2 \cos^2(\omega_0 t - \delta)) +$

$\frac{1}{2}k(A^2 \sin^2(\omega_0 t - \delta))$

$T = \frac{1}{2}kA^2$ (Using $\omega^2 m = k$)

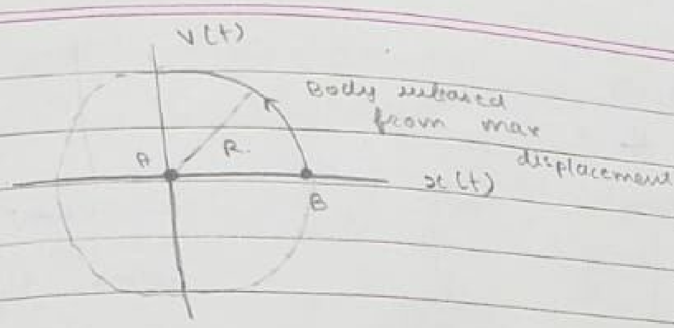
Hence, the general result for a linear system: $T = \frac{1}{2}kA^2$

→ $m\ddot{x} = F(x) = -\frac{dU(x)}{dx} = -\dot{U}(x)$.

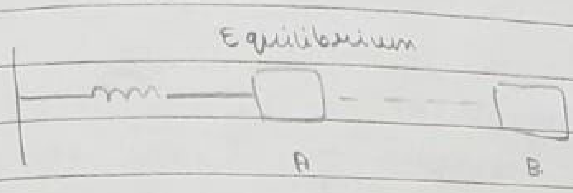
Now, $\dot{x} = v$

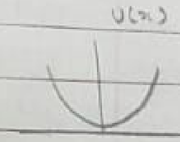
so $\dot{v} = -\frac{\dot{U}(x)}{m}$ → so both x, v independent dynamic variable

Radius = R = Total Energy
(constant)



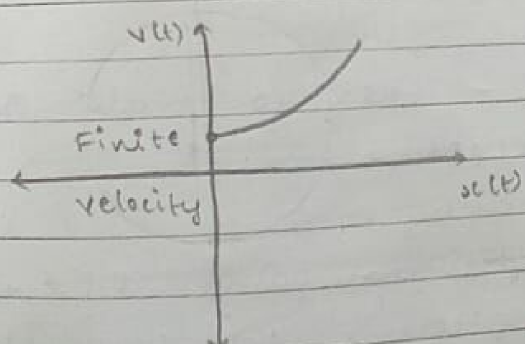
Note:
State of system is completely determined by x and v





Here potential looks like $\frac{1}{2}kx^2$

→ If Potential = $-\frac{1}{2}kx^2$, given finite velocity, draw trajectory of particle

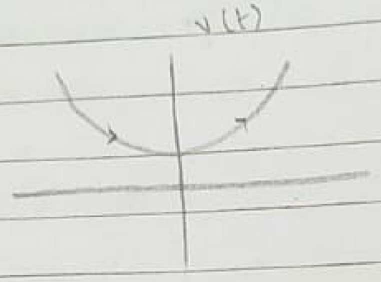


→ Phase space trajectories

Time automatically comes in picture when we plot $v(t)$ vs $x(t)$ because space itself is a function of time.

Real Life Problem — 6 dimensional space
3 dimensions for x and 3 dimensions for v

If particle is coming with some finite energy $E > V_0$ from $-\infty$ then



Here also
$U = -\frac{1}{2}kx^2$

Autonomous system - Phase trajectories cannot intersect as we always need a unique solution

→ x, v, t → independent dynamic variable

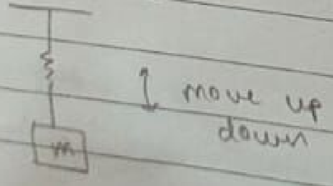
→ Newtonian mechanics is deterministic
We need initial conditions

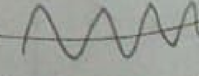
→ Autonomous system - F does not depend on t.

Isolated system - easy to make assumption that F is conservative

→ Phase space trajectories point $P(x, v)$ on phase space represent the complete state of system

→ Closed phase space trajectory means the motion is oscillatory.



time flow → Hence 

(sinusoidal)

→ $\dot{x} = v \quad (dx/dt)$

→ $\dot{v} = -\omega^2 x \quad (dv/dt)$

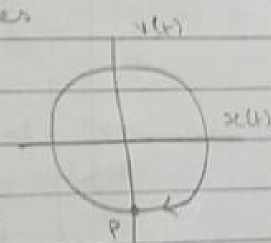
→ $\dfrac{dv}{dx} = \dfrac{-\omega^2 x}{v}$ → slope in phase space.

↓

Integrate this to get phase trajectories in phase space. Then we can analyse complete motion.

Body released from max displacement.

Then displacement decreases and velocity in −ve direction. At some point P, $x = 0$ still v −ve and mass continues to move until it reaches max displacement on other side where $v = 0$. Then direction of motion changes, v becomes +ve and body again moves towards equilibrium $x = 0$ and v max and then to the max displacement from where it started



→ Radius = Total energy ...

$$\frac{dv}{dx} = -\frac{\omega^2 x}{v}$$

$$v\,dv + \omega^2 x\,dx = 0$$

$$\frac{1}{2}v^2 + \frac{1}{2}\omega^2 x^2 = c$$

$$\underbrace{\frac{1}{2}mv^2}_{KE} + \underbrace{\frac{1}{2}m\omega^2 x^2}_{PE} = c$$

→ Total Mechanical energy for conservative force.

→ Let the constant $C = E$.

So $\frac{x^2}{2E/m} + \frac{v^2}{2E/m\omega^2} = 1$

↓

Different E, different graphs represent systems with different total energy

<u>Simple Harmonic Oscillation</u>

$A = \sqrt{a^2+b^2}$

$\phi = \tan^{-1}(b/a)$

→ $\tilde{x}(t) = \tilde{A}e^{i\omega t}$ where $\tilde{A} = Ae^{i\phi}$

$\tilde{A} = a + ib$

→ $\tilde{x}(t) = (a+ib)e^{i\omega t}$

→ $\tilde{v}(t) = \dot{\tilde{x}}(t) = \tilde{A}i\omega e^{i\omega t} = i\omega\tilde{x}(t)$

$= e^{i\pi/2}\omega\tilde{x}(t)$

→ Imp conclusion - For ideal system, v and x are phase shifted by $\pi/2$ (because of multiplication by i)

→ 2^{nd} order linear constant coefficient

$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$.

→ If $f(x) = 0$ then it becomes homogenous

$y'' + ay' + by = 0$.

↓

Make substitution $y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2e^{rx}$

This gives $r^2 + ar + b = 0$

So $r = -\frac{a}{2} \pm \frac{\sqrt{a^2-4b}}{2}$.

general solution : $y = \underline{t_1 e^{r_1 x}} + \underline{t_2 e^{r_2 x}}$

2 linearly independent functions

→ Now, we will take drag force into consideration. $ma = -kx - bv$

restoring force, damping force

(Both -ve because both oppose motion)

$m\ddot{x} + kx + bv = 0$

$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$ → Linear DE as power of higher order of x is 1

b/m = damping parameter

k/m = characteristic angular frequency (time scale of system).

solution : $x(t) = Ae^{\alpha t}$

$\dot{x}(t) = A\alpha e^{\alpha t}$

$\ddot{x}(t) = A\alpha^2 e^{\alpha t}$

$\alpha^2 + \beta\alpha + w_0^2 = 0$

$\alpha = -\beta \pm \sqrt{\beta^2 - w_0^2}$

$w_0 > \beta$ → Complex

$w_0 < \beta$ → Real

$w_0 = \beta$ → Natural frequency

$\beta = b/m$ $\quad w_0 = k/m$

General solution —

$x(t) = e^{-\beta t} [A_1 \exp(\sqrt{\beta^2 - \omega_0^2})t + A_2 \exp(-\sqrt{\beta^2 - \omega_0^2})t]$

$\downarrow$

System damps

$\omega = \sqrt{k/m}$

$\beta = b/2m$

→ Undamped ossillation —

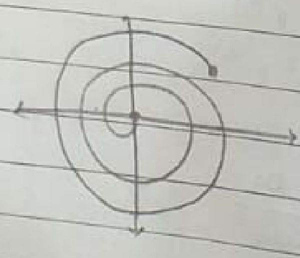
1. Total energy remains constant.
2. Velocity and displacement out of phase by
3. From phase space trajectory, we can say it's a bounded motion and also periodic (circle)
4. As amplitude changes, energy changes but time period constant

Cases

1. $\omega_0^2 > \beta^2$ so $\sqrt{\beta^2 - \omega_0^2}$ is imaginary.

decrement of mo (amplit

Damping alters frequency of system. New frequency $\omega_1^2 = \omega_0^2 - \beta^2$ (Underdamped)



→ Energy decreases.

→ Oscillatory, but not periodic.

?? ← $\tilde{x}(t) = \tilde{A}\, e^{-\beta t} e^{i\omega_1 t}$

$\tilde{v}(t) = (i\omega_1 - \beta)\, \tilde{x}(t)$ $(\tilde{A} = a + ib)$

↳ If $\beta = 0$, we go back to original system.

★ $\frac{dE}{dt} \propto v^2$ → Rate of energy loss max when v max (near equilibrium)

Rate of energy loss = 0 when $v = 0$ and x max.

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→ $\beta = 0$ $\quad \tilde{v}(t) = e^{i\pi/2}\, \omega\, \tilde{x}(t)$.

→ $\beta \neq 0$ $\quad$ Amplitude $(i\omega_1 - \beta) = \sqrt{\omega_1^2 + \beta^2}$

$= \sqrt{\omega_0^2 - \beta^2 + \beta^2}$

$= \omega_0$

Phase $(i\omega_1 - \beta) = \tan^{-1}\left(-\frac{\omega_1}{\beta}\right)$

$= \phi$.

$\tilde{v}(t) = \omega_0 e^{i\phi}\, \tilde{x}(t)$. with $\phi = \tan^{-1}\left(-\frac{\sqrt{\omega_0^2 - \beta^2}}{\beta}\right)$

when $\beta = 0$, $\quad \phi = \pi/2$

$\beta \to \omega_0$, $\quad \phi = 0$.

→ when system damps, its frequency decreases and phase between velocity and displacement decreases as $\beta \to \omega_0$

→ Also, amplitude is decreasing

→ To find the rate of energy loss ★

decrement of motion (amplitude) → We have $x(t)$ and $x(t+T)$ Take $t = 0$

$$\frac{x(0)}{x(T)} = \frac{1}{e^{-\beta 2\pi/\omega}} = e^{\beta 2\pi/\omega}.$$

$$\ln\left(\frac{x(0)}{x(T)}\right) = \frac{\beta 2\pi}{\omega}$$ → logarithmic decrement

→ We have $\tilde{x}(t) = \tilde{A} e^{(i\omega - \beta)t}$ We want $\tilde{x}(t)$ in term of initial conditions x_0, v_0

$v_0 = Re(\tilde{v}(t))$??

$= Re((i\omega - \beta)(a + ib))$

Note : $\tilde{A} = a + ib$. Real part of $\hat{x}(t) = Re(\tilde{A})$

so $x_0 = a$. ??

$v_0 = -\beta a - \omega b = -\beta x_0 - \omega b$

This gives $b = (-v_0 - \beta x_0)/\omega$.

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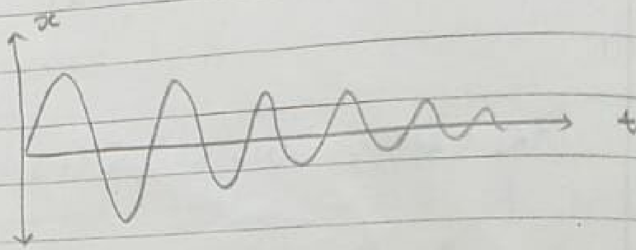
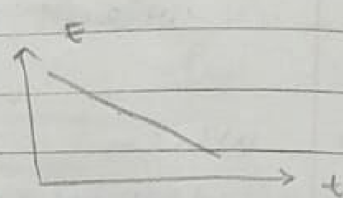
So $\tilde{x}(t) = \underbrace{x_0}_{a} - i\underbrace{\left[\frac{v_0 + \beta x_0}{\omega}\right]}_{-b} e^{-\beta t} e^{i\omega t}$

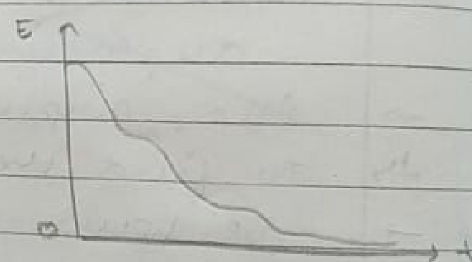
System :



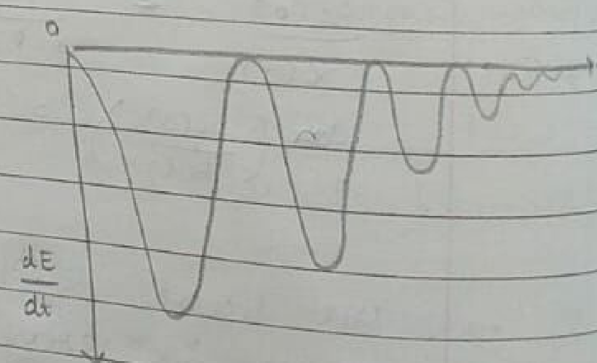
Energy : Trend (approx) $\rightarrow$



Actual $\rightarrow$



$\frac{dE}{dt}$: proportional to v^2



$\Downarrow$

Explanation ??

(2) $\beta^2 > \omega_0^2$ so $\sqrt{\beta^2 - \omega_0^2}$ is real.

Overdamped ...

→ Both roots real and negative

$$\alpha_1 = -\beta + \sqrt{\beta^2 - \omega_0^2} = -\gamma_1$$

$$\alpha_2 = -\beta + \sqrt{\beta^2 - \omega_0^2} = -\gamma_2$$

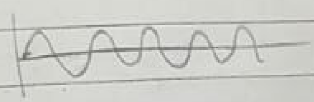
$\gamma_2 > \gamma_1$ so $x(t) = A_1 e^{-\gamma_2 t} + A_2 e^{-\gamma_2 t}$

Clearly,

→ Here there are no oscillations.

→ One will decay faster than other and faster will control the system.

→ Damping time scale > Characteristic time scale so no oscillations.



Damping

characteristic

Before even if observe one complete oscillation, system damps. so no oscillations

→ $T_0 \propto 1/\omega_0$ (Time period of system)

$T_D \propto 1/\beta$ (Time period of damping)

so here $T_0 > T_D$ hence no oscillation

→ In $T_D > T_0$, we see oscillations but they gradually decrease.

→ When $\beta >> \omega_0$, $\sqrt{\beta^2 - \omega_0^2} = \beta\sqrt{1 - \omega_0^2/\beta^2}$

$$\simeq \beta\left[1 - \frac{\omega_0^2}{2\beta^2}\right]$$

(Approximating by binomial expansion.

Now observing the roots ...

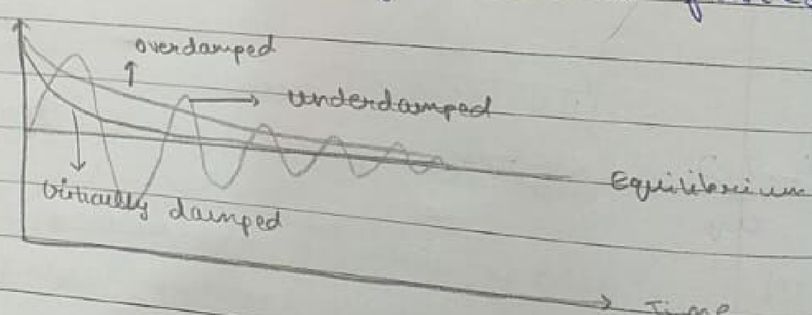
$\rightarrow$ $-\gamma_1 = -\beta + \beta - \omega_0^2 / 2\beta$

$\gamma_1 = \frac{\omega_0^2}{2\beta} \longrightarrow$ can ignore this when $\beta >> \omega_0$

$\rightarrow$ $\gamma_2 = 2\beta \longrightarrow$ decays very fast than first case

Note

Take a real life example where we want a door spring in order for the door to come to rest. Then using a critically damped system is the best option because it comes to equilibrium fastest



$\rightarrow$ $F = \cos(\omega t + \phi_0) \rightarrow$ Any arbitrary force is a function of time

$F(t) = \sum_{\omega} F_\omega \cos(\omega t + \psi_0)$

$\rightarrow$ $m\ddot{x} + kx = F\cos(\omega t + \psi)$

$\ddot{x} + \omega_0^2 x = \frac{F}{m}\cos(\omega t + \psi)$

$\ddot{\tilde{x}} + \omega_0^2 \tilde{x} = \tilde{f} e^{i\omega t}$

Writing in complex notation

where $\tilde{f} = \frac{F}{m} e^{i\psi}$

→ Note — This is a non homogenous D.E. so we find a particular solution and then the solution of complimentary function (i.e the corresponding homogenous part)

→ Imp conclusion — Here system has its own frequency w_0 (without damp) and therer a frequency in force w Hence there's an interplay of 2 systems

→ solution of complimentary function

$$C.F = A_1 e^{iw_0 t} + A_2 e^{-iw_0 t} \quad — (1)$$

→ For P.S — trial solution —

$$\tilde{x}(t) = \tilde{B} e^{iwt}$$

$$-w^2 \tilde{B} e^{iwt} + w_0^2 \tilde{B} e^{iwt} = \tilde{f} e^{iwt}$$

$$\tilde{\beta}(w_0^2 - w^2) = \tilde{f}$$

$$\tilde{\beta} = \frac{\tilde{f}}{w_0^2 - w^2}$$

$$\text{Here } \tilde{x}(t) = \frac{\tilde{f}}{w_0^2 - w^2} e^{iwt} \quad — (2)$$

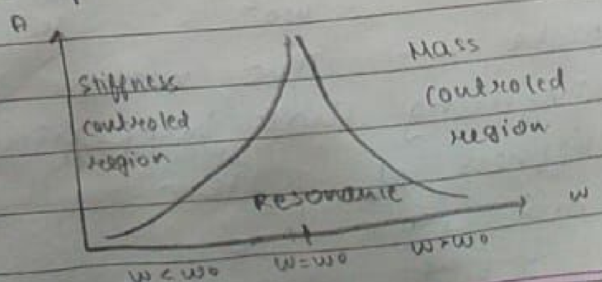
natural frequency ← w_0^2 - w^2 → frequency of external force

Final solution is (1) + (2) ...

→ Question we are interested in...

Oscillation and

(1) Amplitude and phase of oscillation and its relation to amplitude and phase of the force.

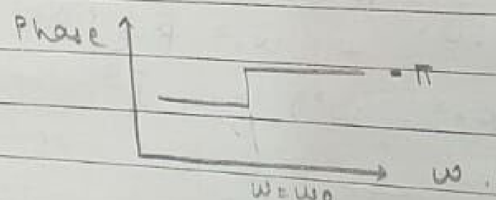


$\rightarrow |\hat{x}| = \dfrac{f}{|\omega_0^2 - \omega^2|}$ $\rightarrow$ amplitude

$\rightarrow \tilde{f} = F \dfrac{e^{i\psi}}{m}$

① $\omega < \omega_0$ — +ve number multiplied to complex number, so no phase change (system and force in same phase).

② $\omega > \omega_0$ — -ve multiplied with complex so phase shift by π
$\tilde{f} \times (-1) = \tilde{f} e^{-i\pi}$ (Phase lag between system and force



③ $\omega = \omega_0$ — Resonance

$\rightarrow$ $\omega << \omega_0$ — system oscillating very fast. $\hat{x}(t) = \dfrac{\tilde{f}}{\omega_0^2} e^{i\omega t}$

when $\omega \rightarrow 0$, we are not introducing new frequency.
$\hat{x}(t) = \dfrac{\tilde{f}}{\omega_0^2} = \dfrac{F}{K} e^{i(\omega t + \phi)}$.

$\rightarrow$ Note — The mass lost its important $(K = \omega_0^2 m)$
Anything that happening is because of

$\rightarrow$ stiffness of system.
$m\ddot{x} + Kx = F$
$x = F/K \rightarrow$ independent of mass.

→ Here, we are just shifting systems equilibrium, not tampering its frequency

→ This is called stiffness controlled region

→ $\omega >> \omega_0$

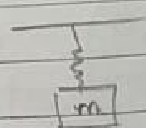
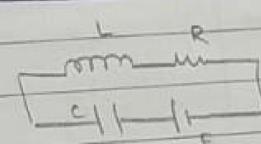
$$\tilde{x}(t) = \frac{\tilde{f}}{\omega^2} e^{i\omega t}$$

$$= \frac{-F}{m\omega^2} e^{i(\omega t + \phi)}$$

Note – No k here so spring loses its effect

This is mass controlled region.

→ RC circuit – Non mechanical oscillation

Mechanical	Non-mechanical
	
x = displacement	q = charge
v = velocity	I = current
m = mass (inertia)	L = inductance
b = damping	R = resistance
$1/k$ = spring factor	C = capacitance
F = amplitude of force	E = EMF voltage

→ $m\ddot{x} + b\dot{x} + kx = F\cos\omega t$ (or 0)
(for mechanical)

→ Now $V_I = L\, dI/dt$

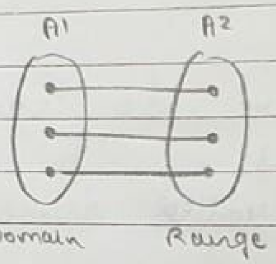
$V_R = RI$

$V_C = q/c$

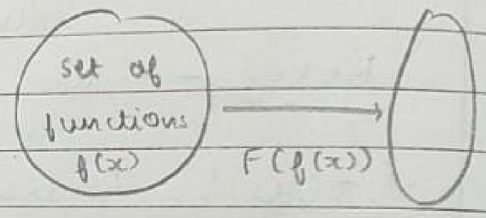
→ So $L\ddot{q} + R\dot{q} + \frac{q}{c} = F_0 \sin \omega_0 t$

→ For mechanical — $\omega_0 = \sqrt{k/m}$
for non mechanical — $\omega_0 = 1/\sqrt{LC}$

<u>Calculus of variation</u>



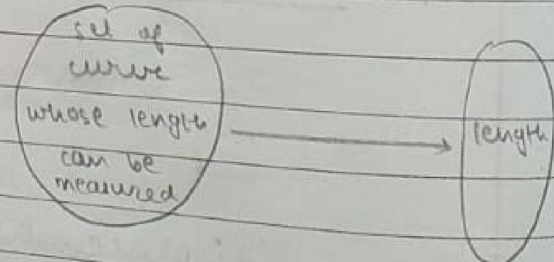
$f(x) = x^2$



Functionals
(function of functions)

↓

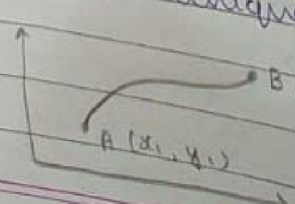
Example...

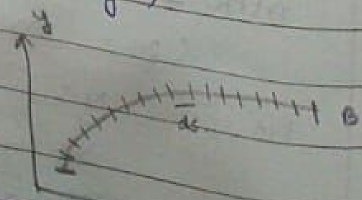


$F(y(x))$ = length of curve
$F(y(x)) = 2\pi r$

→ A question — Whats the minimum distance between 2 point on a sphere ? (The answer is not obvious — This will help us understand why an object thrown takes a unique path only).

First we see min distance b/w 2 pts on plane





→ $ds^2 = dx^2 + dy^2$

→ Length of curve $= \lim_{n \to \infty} \sum_{n \text{ points}} ds$

$= \int_A^B ds.$

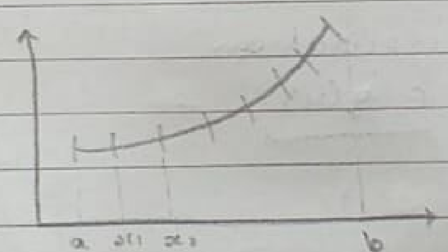
→ $ds = \sqrt{1 + y'^2}\, dx$ $(y' = dy/dx).$

→ $F(y) = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$

→ $y = f(x)$ function

→ $F(y)$ Functional

Find $\underbrace{y(x)}_{\text{curve}}$ so that $\underbrace{F(y)}_{\text{length of curve}}$ is minimum.



$$J(y) = \int_a^{x_1} F(x, y, y')\, dy + \int_{x_1}^{x_2} F(x, y, y')\, dy + \cdots + \int_{x_{n-1}}^{b} F(x, y, y')\, dy$$

→ If we calculate value of functionals in small parts and add them all to get the global functional, then such functional is called local functional

→ Note : Center of Mass is not a local functional.

Partial derivate ① wrt ϵ

Derivate ① wrt x

Derivate above eq. wrt ϵ

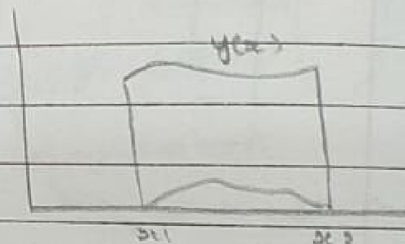
→ $ds = \sqrt{1+(y')^2}\, dx$

→ $s = \int_{x_1}^{x_2} ds$

$= \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$ → Goal is to find shortest path

$I = \int_{IP}^{FP} F(x, y, y')\, dx$

suppose we know the solution and its $y(x)$.



→ Parametric representation

① $\underset{\downarrow}{Y(x)} = y(x) + \underline{\epsilon\, \eta(x)}$ → neighbouring solutions.

Family of curves

Not shortest but a possible path. ↓ $y = 0$ at both ends

→ Note: both ϵ and η are parameters

$y(\epsilon, x) = y(0, x) + \epsilon\, \eta(x)$.

$\epsilon = 0$ gives THE SOLUTION. All other solutions give a little larger path

→ $\eta(x)$:- some function of x; that has continuous first derivatives; that vanishes at x_1 and x_2.

$\eta(x_1) = \eta(x_2) = 0$.

→

→

→

→

→

→

Note : $\frac{\partial x}{\partial \epsilon} = 0$ as x is an independent variable.

y is function of x.

Partial derivative ① wrt ϵ

$$\frac{\partial Y(\epsilon, x)}{\partial \epsilon} = 0 + \frac{\partial \epsilon}{\partial \epsilon}\eta(x) + \epsilon \underline{\frac{\partial \eta}{\partial \epsilon}} = \eta(x)$$

$= 0$ because ϵ is function of x.

Derivative ① wrt x

$$Y'(x) = y'(x) + \epsilon \eta'(x).$$

Derivative above eq. wrt ϵ

$$\frac{\partial Y'}{\partial \epsilon} = \eta'(x).$$

Y is a representative member of family which gives the desired solution

→ $$I(\epsilon) = \int_{x_1}^{x_2} \underline{F(x, Y, Y')}\, dx.$$ THE FUNCTIONAL giving shortest path

I has to be independent of ϵ in first order along the <u>path giving minimum</u>

→ $$\left.\frac{\partial I}{\partial \epsilon}\right|_{\text{at } \epsilon = 0} = 0.$$ Here ϵ must be 0

Derivative must be $= 0$ for minima

→ $$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial Y}\frac{\partial Y}{\partial \epsilon} + \frac{\partial F}{\partial Y'}\frac{\partial Y'}{\partial \epsilon}\right]dx \;\star$$

$$= \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial Y}\eta(x) + \underbrace{\frac{\partial F}{\partial Y'}\eta'(x)}\right]dx.$$

$$= 0$$

→ using <u>integration by parts</u> on second term

$v = \eta'(x)$ $\quad \int v = \eta(x) \longrightarrow [u\int v - \int u'\int v]$

$u = \frac{\partial F}{\partial Y'}$ $\quad u' = \frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right)$

→ $$\left.\frac{\partial I}{\partial \epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2}\frac{\partial F}{\partial Y}\eta(x)\,dx + \left[\underbrace{\frac{\partial F}{\partial Y'}[\eta(x)]_{x_1}^{x_2}}_{=0 \text{ as } \eta(x_1) = \eta(x_2) = 0} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial F}{\partial Y'}\right)\eta(x)\,dx\right] = 0.$$

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At $\epsilon = 0$; $Y = y$ ...

$$\int_{x_1}^{x_2} \underbrace{\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right]}_{\text{must be } = 0 \text{ as } \eta(x) \text{ is arbitrary}} \eta(x) \, dx = 0$$

This is the necessary condition to get min. This is Euler's condition (Not a sufficient condition. Also, it tells about extremum in general, not particular max or min)

→ In our case, $F = \sqrt{1+(y')^2}$

$\frac{\partial F}{\partial y} = 0$ → No terms of y

$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0.$$

$$\frac{y'}{\sqrt{1+(y')^2}} = c'$$

$$(y')^2 = c\,(1+(y')^2)$$

$$(y')^2\,(1-c) = c$$

$y' =$ constant (suppose a)

$y = ax + b$ → Integrating and taking constant of integration as b

Note: If $y' = a$, then the only possible solution of this is a straight line.

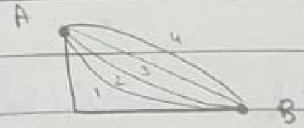
We can find constants a and b from given initial conditions. Suppose 2 boundary points are (x_1, y_1) and (x_2, y_2). By making 2 equations and solving, we get ...

$$(y - y_1) = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1).$$

→ Minimisation principle – More fundamental than Newton's law

→ Minimum distance does not necessarily mean minimum time.

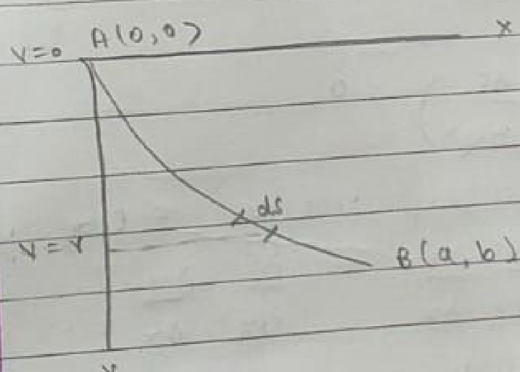
Brachistochrone Problem



→ Need to go from A to B in min time (consider something falling under gravity)

$$\text{Time} = \frac{\text{Distance}}{\text{speed}}$$ → we need to optimize two quantities.

Brachistos – Shortest } Problem of quickest descent (no friction)
Kronos – Time



v is constant During ds..

$$dt = \frac{ds}{v}$$

$$T = \int_A^B dt = \int \frac{ds}{v(x,y)}$$

$$= \int \frac{\sqrt{1+(y')^2}}{v}\, dx.$$ → Eliminate by using energy.

This integration is a problem. We need to get rid of v.

* y and y' are functions of only x. So
$\frac{\partial y}{\partial x} = \frac{dy}{dx} = y'$ and $\frac{\partial y'}{\partial x} = \frac{dy'}{dx} = y''$

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Now Loss of P.E. = gain in K.E

$$mgy = \frac{1}{2} mv^2$$

$$v = \sqrt{2gy}$$

$$T(y) = \int_0^a \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}} \, dx.$$

Our goal is to find y for which $T(y)$ is minimum. We have written the function. so now we use Euler lagrange equation.

→ $\frac{\partial F}{\partial y} = \frac{d}{dx} \frac{\partial F}{\partial y'}$ Here $F = \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}$

so $\frac{\partial F}{\partial x} = 0$.

→ Multiply by y'

$$y' \frac{\partial F}{\partial y} - y' \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

→ Now we actually have $F(x, y, y')$

so
$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial x}$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y'' \longrightarrow *$$

$$\underbrace{y' \frac{\partial F}{\partial y}} = \frac{dF}{dx} - \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y'} y''$$

$$\frac{dF}{dx} - \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y'} y'' - \underbrace{y' \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)} = 0$$

$$\frac{dF}{dx} - \frac{\partial F}{\partial x} - \underbrace{\left[y'' \frac{\partial F}{\partial y'} + y' \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right]} = 0.$$

→

$$\frac{d}{dx}\left(y' \frac{\partial F}{\partial y'} \right) = y' \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'} y''$$

so $$\frac{dF}{dx} - \frac{\partial F}{\partial x} - \frac{d}{dx}\left(y' \frac{\partial F}{\partial y'} \right) = 0.$$

$$\frac{dF}{dx} - \frac{d}{dx}\left(y' \frac{\partial F}{\partial y'} \right) = \frac{\partial F}{\partial x}$$

if $\frac{\partial F}{\partial x} = 0$; $$\underbrace{F - y' \frac{\partial F}{\partial y'} = c}_{\text{Beltrami's identity.}}$$

Use this when functional not dependent on x (the variable of integration) ← Beltrami's identity.

→ using this in our problem

$$F - y' \frac{\partial F}{\partial y'} = c.$$

Here $$\frac{\partial F}{\partial y'} = \frac{1}{2} \frac{(1+(y')^2)^{-1/2}}{\sqrt{2gy}} 2y'$$

$$= \frac{y'}{\sqrt{2gy(1+(y')^2)}}$$

so $$\frac{\sqrt{1+(y')^2}}{\sqrt{2gy}} - \frac{(y')^2}{\sqrt{2gy(1+(y')^2)}} = c$$

$$\sqrt{1+(y')^2} - \frac{(y')^2}{\sqrt{1+(y')^2}} = c\sqrt{2gy}$$

(LCM and squaring)

$$1 = c^2 (2gy)(1+(y')^2).$$

$$(1+(y')^2)y = \frac{1}{c^2 2g} = c_1$$

$$y + y(y')^2 = c_1$$ → computer can solve this

$$y'^2 = \frac{c_1 - y}{y}.$$

$$\rightarrow \frac{dy}{dx} = \sqrt{\frac{c_1 - y}{y}}$$

$$dx = \sqrt{\frac{y}{c_1 - y}}\, dy$$

$$x = \int \sqrt{\frac{y}{c_1 - y}}\, dy$$

Solve hint — $y = c_1 \sin^2\theta$

Alt.. $y = \frac{c_1}{2}(1 - \cos\theta)$

$$dy = \frac{c_1}{2} \sin\theta \, d\theta$$

$$x = \int \sqrt{\frac{c_1(1-\cos\theta)}{2(c_1 - c_1/2 + c_1/2 \cos\theta)}}\ \frac{c_1}{2} \sin\theta \, d\theta$$

$$= \int \sqrt{\frac{c_1(1-\cos\theta)}{c_1 + c_1\cos\theta}}\ \frac{c_1}{2} \sin\theta \, d\theta$$

$$= \int \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}}\ \frac{c_1}{2} \sin\theta \, d\theta$$

$$= c_2 \int (1 - \cos\theta)\, d\theta$$

$x = c_2(\theta - \sin\theta) + \text{constant}$
$y = c_2(1 - \cos\theta)$] cycloid

To find constants, use boundary conditions
(c_2 and constant)

$y = 0,\ x = 0,\ \theta = 0 \rightarrow$ constant $= 0$.

$x = L,\ y = h$

Given ?? $L = c_2(\theta_2 - \sin\theta_2)$

$h = c_2(1 - \cos\theta)$

How ??

Note — Light follows the path with minimum time (not necessarily minimum distance).

Solution?

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$\rightarrow \quad T = \int_1^2 \frac{ds}{v} \qquad\qquad ds = \sqrt{dx^2 + dy^2} = dy\sqrt{(x')^2+1}$

$= \int_1^2 \frac{\sqrt{(x')^2+1}}{\sqrt{2gy}}\, dy$

$= \frac{1}{\sqrt{2g}} \int_0^y \frac{\sqrt{(x')^2+1}}{\sqrt{y}}\, dy.$

$\downarrow$

Once we get the functional, apply Euler Lagrange to get the desired minimum quantity

$\frac{\partial f}{\partial x} = \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) \quad \rightarrow \quad f = \frac{\sqrt{(x')^2+1}}{y}$

Here $\frac{\partial f}{\partial x} = 0 \overset{?}{\Longrightarrow}$ so $\frac{\partial f}{\partial x'} = \underline{\text{constant}}$

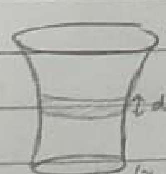
can be anything just independent of y

so $x = \int \sqrt{\frac{y}{2a-y}}\, dy$??

Soap Film problem

Solution?!

Note : ds is the width of strip



$dA = 2\pi x \, ds$

$= 2\pi x \sqrt{1+(y')^2}\, dx$

$A = \int_{x_1}^{x_2} 2\pi x \sqrt{1+(y')^2}\, dx$

$\downarrow$

If this functional has to be minimum, it must satisfy Euler Lagrange

Alt. $\quad A = \int_{y_1}^{y_2} 2\pi x \sqrt{1+(x')^2}\, dy$

$\downarrow$ independent of y — so can use Beltrami

object thrown — 3 degree of freedom
ship floating — 6 degree of freedom (independent)

→ Action $s = \int KE - PE \, dt$. Body prefers a path where action is minimum

→ Lagrangian concept is definitely better as no more dealing with vector

EQ. Consider object moving vertically in gravitational field. Write Lagrangian

$$L = \frac{1}{2} m (\dot{y})^2 - mgy$$

Mapping
$y \to y$
$y' \to v$
$x \to t$

→ Using Euler Lagrange ...

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = 0$$

$$-mg - \frac{d}{dt}\left(m\dot{y}\right) = 0$$

$$-mg = my''$$

$$y'' = -g$$

$$\frac{dv}{dt} = -g$$

↓

same formula that we obtain from Newtonian Mechanics i.e. $ma = -mg$ for a body thrown verticall upwards

Note: For using Euler Lagrange, map the independent variable to x and dependent variable and its derivative to y and y'

Q. Write Lagrangian for 1D oscillator.

$$L = \frac{1}{2} m (\dot{x})^2 - \frac{1}{2} k x^2$$

(1) $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ (2) $\frac{\partial L}{\partial x} = -kx$.

(3) $\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m\ddot{x}$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} + kx = 0$$ [same as $ma = -kx$]

Note — simple pendulum has one degree of freedom and that is along θ

→ How can we accomodate constraints in equation of motion? (we want to do this so that we can simplify the motion by reducing number of equations)

→ There are 2 types of constraints:

(1) Holonomic (removes DOF)

(2) Non-holonomic (does not remove DOF)

→ State of system with n particles - we need n radius vectors and $3n$ quantities
↓
degree of freedom of the system.

→ Now, if there are m constants (i.e 2 particles connected then 1 constraint, 3 particles connected then 2 constraints ___) (reduced DOF, so then DOF = $3n - m$. less equations

→ S can be anything of any dimension - Rectangular, spherical, cylindrical (If we want 3 coordinates, we can select each from different coordinate system)

→ generalised coordinates (q_1, q_2, q_3)
generalised velocity $(\dot{q}_1, \dot{q}_2, \dot{q}_3)$

suppose we have n particles and $\alpha = 1, 2, \ldots$

$x_{\alpha,i} = x_{\alpha i}(q_1, q_2, \ldots, q_s\ t)$

$= x_{\alpha i}(q_j, t)$ where $j = 1 \ldots s$

what is x??

(DOF) ← generalised coordinates

i is from 1 to 3 (each particle has 3 direction of motion)

$f_k(x_{\alpha,i}, t) = 0$ → equation of constraint

$(k = 1 \text{ to } m, \quad s = 3n - m)$

what is f??

Moving on sphere — equation of constraint is $r = c$ (constant radius) Now we just need θ and ϕ to find any point on sphere. So here DOF reduced from 3 to 2

IMP — Identify constraints and write its equation

→ Concept of constraints — classical. Not applicable in quantum as there we have uncertainity (But we need to define absolutely certain constraints)

Steps of problem solving ...

① Identify DOF (check for hidden constraints)
② How to construct a Lagrangian
③ Determine action.
④ Find the true path that nature takes

↓

Follow these steps in general

Charge Particle Dynamics in EM fields

→ Charge Particles oscillate to generate wave, which is the backbone of communication today.

→ To move very small mass (like e^-) we need Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Observations ... when $\vec{B} = 0$

→ Force parallel to $\vec{E}$

→ Force depends directly on charge

Observations ... when $\vec{E} = 0$

→ No effect of magnetic field if no intial velocity is given

Applications ...

① 99% of universe is charge so we need to understand the dynamics of charge.
② Chips - etching (we can give high velocity to charges)
③ Tubelights
④ Biomedical

$$v \times B = \begin{vmatrix} i & j & k \\ v_x & v_y & v_z \\ 0 & 0 & B_z \end{vmatrix} = v_y B_z i - v_x B_z j$$

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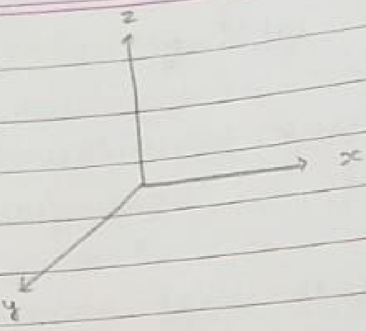

① Take $B = 0$

$E = E_x$

② Take $E = 0$

$B = B_z$

(effective only if we ha[ve] some initial velocity)

① Equation of motion ... $(a_y = a_z = 0)$

$ma = qE$

$m \frac{dv_x}{dt} = qE_x$

$v(x) = \frac{qEt}{m} + \underline{v_x(0)}$ initial at $t = 0$. So even if we give no initial velocity we will get some $v(x)$

② Equation of motion ... First step - calculate Forces in all directions

$ma = q(v \times B)$

This is equivalent to ...

$m\dot{v} = q(i v_y B_z - j v_x B_z)$

→ $m\dot{v}_x = q v_y B_z$

$m\dot{v}_y = q v_x B_z$

$m\dot{v}_z =$ constant

$\dot{v}_x = \frac{qB_z}{m} v_y$

$\dot{v}_y = -\frac{qB_z}{m} v_x$

Observations ...

① Equation are symetric

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(2) $\frac{qB_z}{m}$ is common in both and has dimension $M^0L^0T^{-1}$ i.e unit is sec^{-1} (d/dt). This is the cyclotron frequency $\omega_c = \frac{qB}{m}$

→ $\ddot{V_x} = \omega_c \dot{V_y} = -\omega_c^2 V_x$
→ $\ddot{V_y} = -\omega_c \dot{V_x} = -\omega_c^2 V_y$

These equations represent SHM

→ Looking at the unit of force ...

$F = qVB$

$F \rightarrow C \times \frac{m}{s} \times \frac{N}{Am} = N$ $\quad (F = IlB)$

$(A = C/s)$

→ Particle moving with some velocity in B. so it experience a force $\perp^r$ to both v and B and hence circular motion. So centripetal acceleration comes into picture

→ $F_c = \frac{mv^2}{R}$

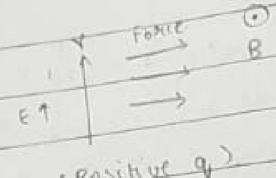
→ $\frac{mv^2}{R} = qvB \Rightarrow R = \frac{mv}{qB}$

→ When m, v large, R is large so charge held loosely

→ When q, B large, R is small, circle tight so charge held tightly.

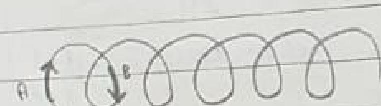
★ Note: Here E is upwards. so if +ve q then v
if -ve q then v downwards
So ultimately $F = q(v \times B)$
is in same direction in
both case

Now consider effects of both E and B



★ Using right hand thumb rule, we get direction of F due to B (towards the right).

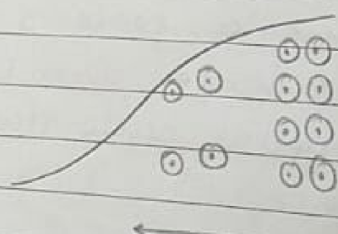
+ve charge – Right Hand Thumb rule
-ve charge – Left Hand Thumb rule



Along A, E support motion so radius increases. Along B, E opposes motion so radius decreases

Hence, helical motion instead of circular motion.

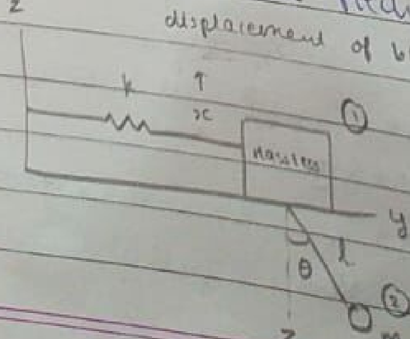
→ Note – whether we take +ve or -ve charge, the direction of motion will be same. It changes with changing direction of B.

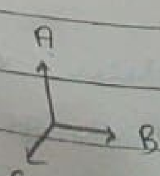


$\vec{B}$ coming out of the paper is constant (i.e magnitude of $\vec{B}$ is constant in that direction)

ΔB in this direction

→ Back to Classical Mechanics ...





step 1: Degrees of freedom = 6 - 1 - 2 - 1 = 2

6 → 2 objects in system so DOF = 6

Now removing the constraints

1 → l is fixed

2 → (1) cannot move in A or C

1 → (2) cannot move in C

Now the 2 DOF that we have are x and θ (Note - They need not be from same coordinate system)

step 2: Find energies to write Lagrangian

(1) Massless so KE = 0

PE = $\frac{1}{2}kx^2$

(2) Now here we have v_x and v_y only. To observe, freeze system (1) so now like simple pendulum

$$y = x + l\sin\theta$$

$$z = -l\cos\theta$$

$$\dot{y} = v_y = \dot{x} + l\dot{\theta}\cos\theta$$

$$\dot{z} = v_z = l\dot{\theta}\sin\theta$$

→ Now, writing the total energies

$$KE = \frac{1}{2}m(v_y^2 + v_z^2)$$

$$= \frac{m}{2}(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta)$$

$$PE = \frac{1}{2}kx^2 + mgz$$

$$= \frac{1}{2}kx^2 - mgl\cos\theta$$

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Step 3: Write Euler Lagrange equation

$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{m}{2}\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right) - \frac{kx^2}{2} + mgl\cos\theta$

(we'll get 2 equations — one for x and one for θ) → DOF = 2 so 2 sets of equat

set 1 → $\frac{\partial L}{\partial \dot{x}} = m(\dot{x} + l\dot{\theta}\cos\theta)$

→ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} + ml\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)$

→ $\frac{\partial L}{\partial x} = -kx$

set 2 → $\frac{\partial L}{\partial \dot{\theta}} = ml(l\dot{\theta} + \dot{x}\cos\theta)$

→ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + ml(\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta)$

→ $\frac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta$

→ From these equations ...

set 1 $m\ddot{x} + kx = ml(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta)$

set 2. $\ddot{\theta} + \frac{g}{l}\sin\theta = -\frac{\ddot{x}}{l}\cos\theta$

Freeze ② ... set 1 gives $m\ddot{x} + kx = 0$

Freeze ① ... set 2 gives $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

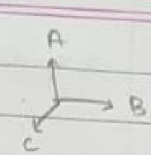
which match the individual analysis of these systems done earlier.

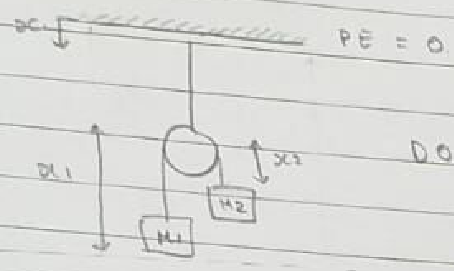
Step 1:

Step 2:

Step

Pulley system





DOF = 6 - 4 - 1 = 1

Step 1: 6 → 2 particles so 6 DOF

4 → No motion in B or C for both masses

1 → Length of string constant

Step 2: $v_1 = \dot{x}$

$v_2 = -\dot{x}$

$x_1 + x_2 = l$

Here we need only 1 generalised coordinate $x_1 = x$

Total energy = KE + PE

$$KE = \frac{1}{2} M_1 \dot{x}^2 + \frac{1}{2} M_2 (-\dot{x})^2$$

$$PE = -M_1 g x - M_2 g (l - x)$$ (because PE = 0 at surface)

Step 3: $$L = \frac{1}{2} M_1 \dot{x}^2 + \frac{1}{2} M_2 \dot{x}^2 + M_1 g x + M_2 g l - M_2 g x$$

$$\frac{\partial L}{\partial \dot{x}} = M_1 \dot{x} + M_2 \dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M_1 \ddot{x} + M_2 \ddot{x}$$

$$\frac{\partial L}{\partial x} = M_1 g - M_2 g$$