GRAPH THEORY

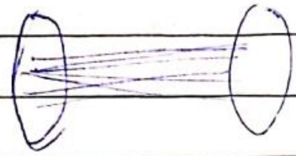
Books:

1. Introduction to Graph Theory by Douglas B. West, Prentice Hall
2. 'Graph Theory' by Frank Harary.

Puzzle

Q. A city contains good wizards & bad wizards. A good wizard will always tell the truth, a bad wizard is unreliable. The king wants to get rid of all the bad wizards. Once he gets rid of a wizard he will come to know if he was good or bad. How can he do so ~~without~~ by getting rid of atmost 1 good wizard?

A: Pick one wizard & ask him if all the others are good or bad & get rid of him. If he was good, we can now get rid of all the bad wizards he pointed to. If he was bad, repeat the process.



one set of people are pointing to the other set as bad.

colluding

Either all are good or all are bad

DEFINITION OF GRAPH

- Hierarchy of Rules | breaking ties.

  eg. finding from the first letter in a dictionary.

  $A_1, A_2, A_3, A_4 \ldots$ - a set of attributes to be compared.

  Starting with the first attribute, loop until you get unequal values.

- SET: collection of distinct, well-defined objects.

  Truth vs Compatibility / well defined but unknown

  Well defined: Membership cannot be ambiguous

- MULTI SET: collection of well-defined
- Cartesian product: All possible ~~values~~ of combinations of x & y s.t. $x \in S1$ & $y \in S2$
- Relation ~~of~~: a subset of the Cartesian Product.
- subset ?

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→ Functions → Domain & Co-domain $\in D$ → Domain

n^m ↳ Special class of Relations $- 2^{|A|\times|B|}$

↳ Subset of Cartesian product → $|A|\times|B|$

all pairs such that first is from set A & second from set B.

· or dimensional vectors.

functions → no two vectors have the same first co-ordinate

Co-domain vs Range — Range is the subset of co-domain in which all elements have a pre-image.

$\frac{n!}{(m-n)!}$ • Injective function (one-one) has $\forall x, y \in D$

$|D| \leq |C|$ $x \neq y \Rightarrow F(x) \neq F(y)$

• Surjective function Range = Co-domain (onto)

$n!$ • Bijective (injective & surjective or one-one & onto)

$|D| \geq |C|$ $|D| = |C|$

eg. Using bijective function to count a set.

• Function composition $g(f(x))$

f must be defined at x & g must be defined at $f(x)$

i.e. $x \in D_f$ & $f(x) \in D_g$

$R_f \subseteq D_g$ | $R_f \leq D_g$.

$\in$ or C_f.

• Iteration → composing a function with itself.

Fixed point. Order of group

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 4 & 5 & 6 & 7 & 3 \end{pmatrix}$ LCM of 2 & 5 → 10 — Order?

• Cartesian product → Binary operator.

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⇒ $G = (V, E)$

Simple graphs — No self loops (Reflexivity ✗), no more than 1 edge between a pair of vertices.

A finite $\boxed{\text{graph}}$ is a combinations of two sets — a finite set V of vertices and a binary, irreflexive, symmetric (undirected graph) relation on V called E.

↓ Subset of cartesian product which is a set, hence duplicates are not allowed.

✱ Look up: Automorphism, Isomorphism, Partial order, hasse diagram, etc.

⇒ A $\boxed{\text{Sub graph}}$ $H = (V', E')$ is a graph where $V' \subseteq V$ and $E' \subseteq E$ (E' cannot have an edge whose vertices $\notin V'$)

$V = \{A, B, C, D, E, F\}$

$V' = \{C, D, E, F\}$ ✗ $E' = \{(A,B), (A,F), (C,E)\}$

⇒ Spanning sub-graph & Induced sub-graph

Spanning sub-graph: ↓ $V' = V$

No. of spanning subgraphs $= 2^{|E|}$ ↓ may not be connected.

Induced sub-graph: Collusion

Edge Maximal sub graph for specified vertex set.

- all edges from the graph whose end vertices $\in$ vertex set.
- only 1 possible for given vertex set.

K_n ⇒ complete Graph · unique for a selected vertex set

- $E = \{\text{All vertex pairs}\}$ $^nC_2$ edges

⇒ graph complement G'

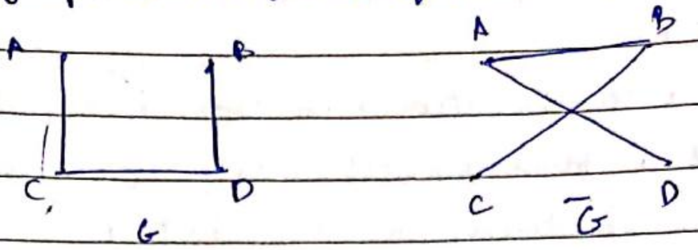
- Retaining the same vertex ~~edge~~ set, Removing all edges from E and adding all edges that were not present in E

$E(G) \cup E(\bar{G}) = K_V$ $(u,v) \in E(G) \Leftrightarrow (u,v) \notin E(\bar{G})$

$E(G) \cap E(\bar{G}) = \phi$

K_V = Complete graph on vertex set V.

· Any graph G has a unique complement $\bar{G}$.



eg. No. of edges in complete graph of 4 vertices

$= {}^4C_2 = 6$

(Labelled Graphs)

No. of graphs possible with 4 vertices $= 2^6 = 64$

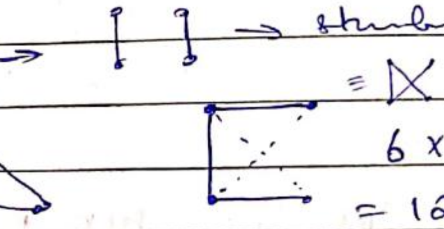
How many distinct graphs are there such that no two are identical? (Isomorphic) - Undirected -

6 edges ≡ 0 edges - 1 graph

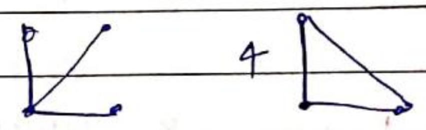
5 edges ≡ 1 edge - 6 graphs → structure, 1 isolated vertex.

4 edges ≡ 2 edges - 12 + 3 graphs → structure

(Complement graph) (15)



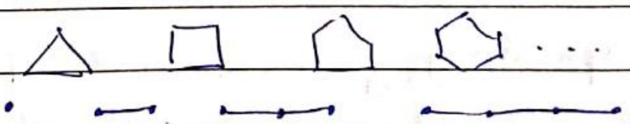
≡ $6 \times 2 = 12$

3 edges - 4  4

Bijective function (20)

+

$20 + 2\times15 + 2\times6 + 2\times1 = 64$.

* Cycles (C_n) 
* Paths (P_n)
* Edgeless graph ($\overline{K_n}$) · No edges.

→ No need to list more because unique complements of 2 edges will have 4 edges & cardinality of sets must be equal

And (11) Isomorphic graphs.

$2\times1 + 2\times1 + 2\times2 + 3 = 11$

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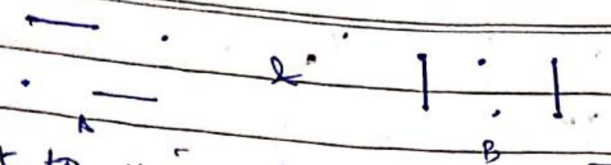
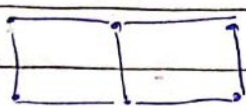
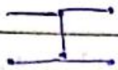
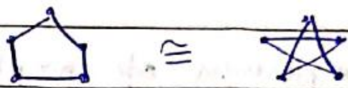
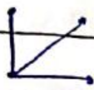


Isomorphic

w.r.t to universal graph 

$\bar{A}$  & $\bar{B}$  Not Isomorphic.

* Self-complementary graphs
If the graph is structurally identical (isomorphic) to its complement graph.
eg. $C_5 \cong \overline{C_5}$ 
$P_4 \cong \overline{P_4}$

 $\bar{G}$  G

Structural Properties

→ Two graphs are isomorphic if they have the same :
- No of vertices
- No of Edges
- No. of Spanning graphs.
- Degree sequence
- No. of cut vertices etc.

Very difficult to prove 2 graphs Isomorphic; but proving that they are Not isomorphic ?

→ Two graphs are isomorphic if there is a bijective function F such that, $F: V(G) \to V(H)$

$(u,v) \in E(G) \iff (F(u), F(v)) \in E(H)$ (if & only if)

The function F is called an isomorphism.
Two graphs are isomorphic if there is an isomorphism.

* Super-polynomial, sub-exponential time algorithm.

$p \leftrightarrow q$

$\sim p \leftrightarrow \sim q$

$(u,v) \notin E(G) \iff (F(u), F(v)) \notin E(H)$

$(u,v) \in E(\bar{G}) \iff (F(u), F(v)) \in E(\bar{H})$

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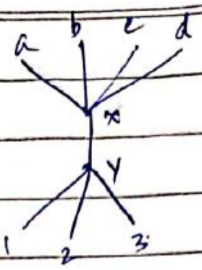
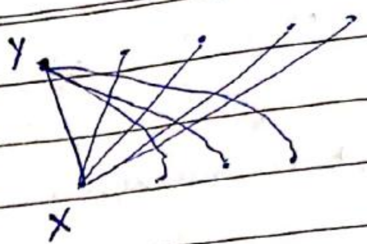
* Automorphism





Structurally identical [

No choice for mapping X & Y]

→ An automorphism is a bijective ~~graph~~ function F from the vertex set of a graph to itself, such that $(u, v) \in E(G) \leftrightarrow (F(u), F(v)) \in E(G)$

(Adjacency is preserved)

- Isomorphism of graphs is an Equivalence relation
  (Reflexive, symmetric, Transitive)
  graph is isomorphic to itself ↳ Identity | inverse | ↳ composition.

→ G_1, G_2, G_3 are 3 graphs, we want to argue that isomorphism is a transitive relation

$(u,v) \in E_1 \leftrightarrow (F(u), F(v)) \in E_2 \leftrightarrow (G(F(u)), G(F(v))) \in E_3$

where F is an isomorphism b/n G_1 & G_2 & G is an isomorphism b/n G_2 & G_3 $F(G(x))$ → the composition is an isomorphism b/n G1 & G3.

→ Regular graph → every vertex has the same degree

4 regular graph $\xrightarrow{\text{complement}}$ 2 regular graph

collection of disjoint cycles. — very distinct structures

→ Sometimes it is easier to establish isomorphism b/n complements of graph as in the above case.

→ No. of automorphisms possible |V| !

→ Inverse of an automorphism is also an automorphism.

→ Binary relation between vertices based on the existance of an automorphism between any pair is an equivalence relation

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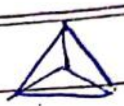
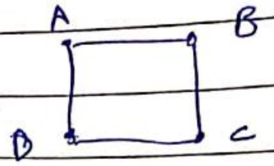


→ Automorphisms of a complete graph $n!$

→ If graph is symmetric

↪ No. of equivalence classes - 1



→ graph is a path - Exactly 2 automorphisms

Identity + Reverse

No. of equivalence classes - $n/2$.

Size of equivalence classes - 2

→ For a cycle - no. of automorphisms = $2 * n$

no. of Equivalence classes - 1

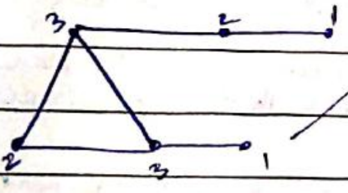
⇒ For graphs with only 1 equivalence class of vertices, they are called Vertex Transitive graphs

completely symmetrical

size N.

(High Flexibility) K_n, $\overline{K_n}$, C_n, $\overline{C_n}$, Q_n, $\overline{Q_n}$ . . .

⇒ Rigid graphs - Every vertex is unique, N equivalence classes.



→ Distance to cycle is different.

Repetitive Degrees.

⇒ Regular Rigid Graph - 12 vertices

* MISSED CLASS 19/8

Vertex Transitive graphs.

Edge Transitive graphs.

Different / Extended notions of symmetry.

PREferent graph.

unique common neighbour

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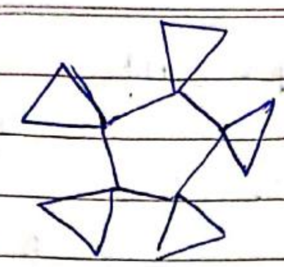


Not vertex transitive
not edge transitive
'Triangle' transitive ?
P_3 / K_3

2 equivalence classes of automorphism.

→ DECOMPOSITIONS

Partitioning of an edge set of graph ~~into complete graphs~~ into arbitrary no. of graphs.

Identical ; → Reusing edges X
Similar ; Leaving out edges X
Arbitrary ;

eg. Self-complementary graphs

→ K_3
→ K_2

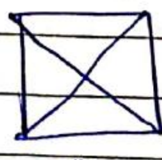
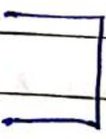
or decompose into cycles (centre cycle + triangles)

$$d_G(v) = \sum_{i=1}^{k} d_{G_i}(v)$$

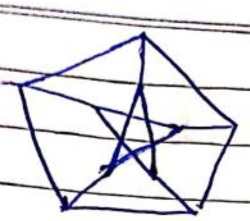
Degree of vertex = sum of its degree in all the decompositions.

eg. Partitioning K_6 into K_3's. → every vertex has degree 2.
Not possible → Degree 5
5 cannot be expressed in multiple of 2

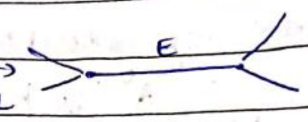
→ Self complementary graphs.
- Decomposing into 2 graphs isomorphic to each other
- Even no. of edges. (~~to~~ partition into 2 isomorphic)
- If graph has to be complete, $\frac{n(n-1)}{2}$ should be a multip of 2 ∴ n or (n-1) should be a multiple of 4.

eg. K_4   & 

K_5   & 

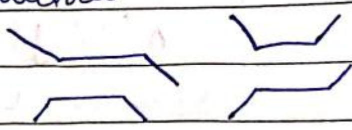


every edge has this structure



↳ No. of edges = 15

No. of 5 cycles. each edge has structure



using 3 cycles of this structure

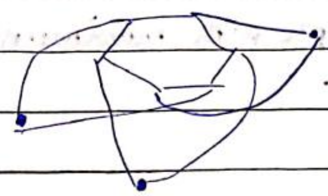
4 ways to consider edges in ~~graph~~ cycle

so 15 × 4 → 60 cycles → Over counting.

5 length cycle - each cycle counted 5 times

∴ 60/5 → 12 cycles in above cycle.

? K_{13}

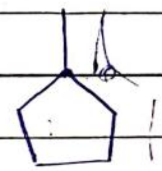


5 cycles.

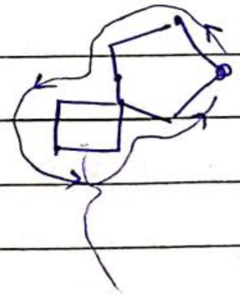
* MISSED CLASSES   Fixdt theorem of graph theory, → Degree = 2× no. of edges, Instance matrix

Path, trail, walk, cycle   odd/even   closed walk.

→ every closed odd walk contains an odd cycle.



We may not traverse the edges of the odd cycle in order (continuously).



* Every u → v walk contains a u → v path.

→ Adjacency Matrix   A[i, j]

$A^2[i,j]$ = ?   Sychronized edges?

. common neighbors of i & j

. & diagonal gives degree of each vertex ↷ Principle diagonal gives the degree sequence of the graph.

. gives no. of walks of length 2 (because A^2) in general A^n will give no. of walks of length n.

Prove my induction.

$$\sum_{1 \le k \le n} \underbrace{A^{t-1}[i,k]}_{\text{no. of walks of length } t-1} \cdot A[k,j] = A^t[i,j]$$

$\therefore A^t[i,j]$ represents the no. of walks of length t between i and j (from i to j in case of directed graph)

- vertex transitive graph, cut vertex, maximal path etc.

Path length $\geq$ minimum degree

EXTREMAL GRAPH THEORY

$\Rightarrow$ A graph with minimum degree k has a cycle of length atleast $(k+1)$.

Decompositions

- Eulerian Trails → no edges are repeated & all edges of graph should be covered & must be a closed trail.

(Konigsberg Bridge problem).

i.e. a closed spanning trail. → open trail degree is odd

- Degree of endpoint of a closed trail is always even, degree of any ~~other~~ interior point of a trial is even.

(In this case degree of vertex in trail may not be equal to its degree in the graph)

But for eulerian trail (since it is spanning), degree must be equal. → even.

$2k$ odd degree vertices → k rounds

$\Rightarrow$ Presence of odd degree vertex forbids eulerian trail.

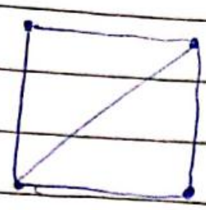
The graph ust be connected ??

Necessary / sufficient conditions

(1) Every vertex is Even degree

(2) Only one non-trivial component.

Q. compute degree sequence of complement of graph.
Only if degree sequence of graph & its complement are same it may be isomorphic
Self complementary graphs $\frac{n-1}{2}$ degree.

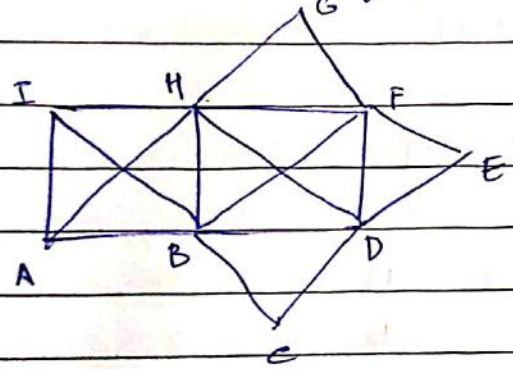


Simple graph with 4 vertices & 3 cycles.
unique?

BIPARTITE GRAPHS

generalised form r-partitite graphs, $r \geq 1$
Vertex set can be partitioned into atmost r sets

+ Clique - induced complete sub-graph
* Independent set - complement of clique.
↳ induced sub-graph has NO edges.



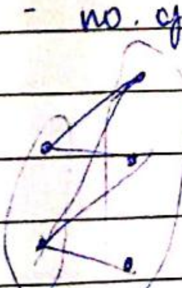
I, G, E, C
A, C, E, G
↳ Independent sets.
B, H, F, D
↳ clique

A & I are automorphic

What could be the intersection b/n a clique & an independent set of a graph? 1 Vertex AT MOST

α - no. of vertices in largest independent set of graph
= 4 for this graph.

Maximal



↳ independent set, but a larger one exists
But it doesn't have a superset, i.e. no more can be included in this set.

ω - size of largest clique $= 4$

$$\boxed{\alpha(G) = \omega(G')}$$

$\alpha \cap \omega \leq 1$

$\alpha + \omega \leq n+1$

eg. complete graph $\alpha = n$, $\omega = 1$

* Edgeless graph of n-vertices is the only 1-partite graph (also n-partite)
* Partitioning into independent sets : graph coloring.
* Bipartite graph - must have a valid bi-partition.
* connected bipartite graphs have unique partitions.
  'If you're not with us, you're against us!'

Easy to characterise because of duality.

Closure Property

Super graph - adding vertices / edges, sub-graph - Removing

	Bipartite graph	Nonbipartite graph.
Super graph	BN	N
sub-graph	B	B

* Bipartite graph is closed under sub-graph operation
* Non-Bipartite graph is closed under super-graph operation
  remains in the set.
* Odd length cycle is Non-bi partite

C_{2k+1}

Proof by contradiction.

cross edges to find odd cycle.

BFS - adjacent nodes cannot differ in distance from a third node by more than 1

generalise to distance x nodes, not differing by more than x

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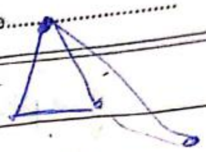

- Horizontal & vertical cross edges.
  same level — differ by 1
- horizontal cross edge will give an odd length cycle

Take edge set, if edge (u,v) has u or v as parent in BFS tree it is a tree edge else it is a cross edge

If it is a cross edge, find if it is horizontal if distance from the root is equal.

→ If graph has an odd cycle it is not bipartite.

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