Planetary Motion/ Gravitation
Central Force Motion

Objective: The motion of a system consisting of two bodies in the presence of a Central Force.

Central force→ force directed along the line connecting the centers of the two bodies.

Lagrangian for such a system

Center of Mass

System of N particles $\alpha = 1, \ldots, N$, with masses m_α and positions $\mathbf{r}_\alpha$ →The center of mass (or CM) is defined to be the position

$$R = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha = \frac{m_1\mathbf{r}_1 + \cdots + m_N\mathbf{r}_N}{M}$$

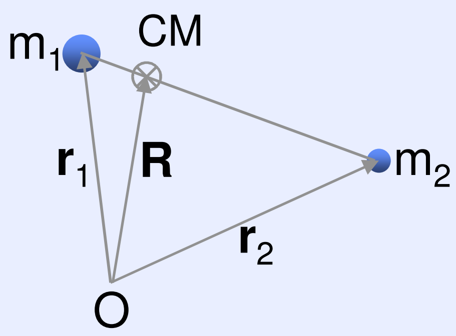
Vector equation → separate equations for each of the components (X, Y, Z):

$$X = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha x_\alpha, \quad Y = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha y_\alpha, \quad Z = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha z_\alpha.$$

weighted average of the positions of each mass element.

center of mass for a two particle system

$$R = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$



The distance of the CM from m_1 and m_2 is in the ratio $\mathrm{m}_2/\mathrm{m}_1$.

if $\mathrm{m}_1 >> \mathrm{m}_2$, then the CM will be very close to m_1.

Momentum of N-particle system?

Velocity of CM ?

Center of Mass and Equation of Motion

time derivative of the center of mass for N particles → the CM velocity

$$\dot{\mathbf{R}} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \dot{\mathbf{r}}_\alpha = \frac{1}{M}\sum_{\alpha=1}^{N} \mathbf{p}_\alpha$$

momentum of an N-particle system is

$$\mathbf{P} = M\dot{\mathbf{R}}$$

<u>The equation of motion:</u>

$$\mathbf{F}_{\text{ext}} = M\ddot{\mathbf{R}}$$

Motion of the CM of a collection of particles → the external forces on all the individual particles is concentrated at the CM.

Central Force Problem

- Electrostatics and gravitation. Angular momentum is conserved.
- True for two-body problems, and for spherically symmetric masses or charges.
- Let us look at the gravitation problem, ideas apply directly to any central force problem.

The Gravitation 2-Body Problem

We have two gravitating bodies of mass m_1 and m_2, at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. The potential energy is

$$U(\mathbf{r}_1, \mathbf{r}_2) = -\frac{G m_1 m_2}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|}.$$

This depends only on the separation between the masses, not on $\mathbf{r}_1$ and $\mathbf{r}_2$ separately. It depends only on the magnitude $\left|\mathbf{r}_1 - \mathbf{r}_2\right|$

$$U(\mathbf{r}_1, \mathbf{r}_2) = U(\left|\mathbf{r}_1 - \mathbf{r}_2\right|).$$

a new variable, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, which is the position of body 1 relative to body 2.

$$U = U(r).$$

What is the Lagrangian?

The Gravitation 2-Body Problem

In terms of Lagrangian mechanics, we have for the two-body problem:

$$\mathcal{L} = T - U = \tfrac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\mathbf{r}}_2^2 - U(r).$$

Write r_1 and r_2 in terms of the center of mass R.

CM, Relative Coordinates

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}.$$

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r} \quad \text{and} \quad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}.$$

Putting these into the kinetic energy

$$T = \tfrac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\mathbf{r}}_2^2 = \tfrac{1}{2} m_1 \left[\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right]^2 + \tfrac{1}{2} m_2 \left[\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right]^2.$$

Solve this …..

Reduced Mass

$$T = \tfrac{1}{2} M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 .$$

μ for the **reduced mass**: $\boxed{\mu = \dfrac{m_1 m_2}{m_1 + m_2}.}$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

μ is less than the smaller of m_1 and m_2.

The reduced mass of the Sun-Earth system is almost exactly the mass of the Earth.

Lagrangian of the 2 body system

Lagrangian

$$\mathcal{L} = T - U = \tfrac{1}{2} M \dot{\mathbf{R}}^2 + \left(\tfrac{1}{2} \mu \dot{\mathbf{r}}^2 - U(r) \right)$$

$$= \mathcal{L}_{\text{CM}} \qquad + \qquad \mathcal{L}_{\text{rel}}.$$

CM and relative cords → generalized cords which split the problem into two parts.
What if CM is origin !!

The Equations of Motion

Lagrangian $$\mathcal{L} = T - U = \tfrac{1}{2} M\dot{\mathbf{R}}^2 + \left(,\tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)\right)$$

What are The equations of motion ?

The CM equation is : $$M\ddot{\mathbf{R}} = 0 \quad \text{or} \quad \dot{\mathbf{R}} = \text{const.}$$

Our 2-body problem is an isolated system, hence no outside forces are acting (Newton's 1st Law).

The CM coordinate is ignorable.

The Lagrangian does not depend on $\mathbf{R}$ → a conservation law (conservation of momentum).

The Equations of Motion

The Lagrange equation for the other coordinate, the relative position $\mathbf{r}$

$$\mu\ddot{\mathbf{r}} = -\nabla U(r),$$

This is the equation of motion for a single free particle of mass μ (reduced mass) subject to potential energy $U(r)$.

The CM Reference Frame

Since the velocity of the CM is constant, we can change to a frame moving with this constant velocity → alternate inertial frame, $\dot{\mathbf{R}} = 0.$

In the CM frame, the Lagrangian is just

$$\mathcal{L} = \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)$$

and the problem is reduced to a one-body problem.

Kind of pseudo-single body system.

- **The "single body" is of reduced mass μ, and the center of its orbit is the other body NOT the CM.**
- The choice of origin that led to the above Lagrangian is the CM.

Origin at CM, Path relative to CM

Equivalent one-dimensional problem