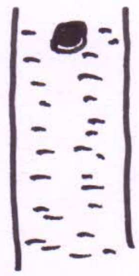
Object Falling through a Liquid Column



For an object falling through a liquid column, there are three forces acting on it. They are:

i.) Gravity (Weight) $\boxed{mg}$, ii.) Buoyancy (Upthrust = Weight of liquid displaced $\boxed{\rho_\ell V g}$, where ρ_ℓ is the liquid density and V is the volume of the object. iii.) Viscous drag $\boxed{kv}$, in which $v \equiv v(t)$ is the velocity of the falling object and k is the drag coefficient.

Friction (or viscosity in a liquid) opposes motion. It acts only when there is motion. The drag force due to friction is then $\boxed{D \equiv D(v)}$. The simplest form of D is a linear function $\boxed{D = kv}$. For a sphere the drag coefficient is $\boxed{k = 6\pi\eta r}$, with $r \rightarrow$ radius of the sphere and $\eta \rightarrow$ the dynamic viscosity of the liquid. The linear function of $\boxed{D = kv}$ applies well for a small velocity of fall in a liquid.

The force balance equation is then

$$\boxed{m\frac{dv}{dt} = \underset{(\text{weight})\downarrow}{mg} - \underset{(\text{buoyancy})\uparrow}{\rho_\ell V g} - \underset{(\text{drag})\uparrow}{Kv}}$$

. We write $\boxed{m = \rho V}$

in which ρ is the density of the falling object.

$\Rightarrow \boxed{\frac{dv}{dt} = g - \frac{\rho_\ell}{\rho} g - \frac{K}{m} v = \bar{g} - \frac{K}{m} v}$, where

$\boxed{\bar{g} = g\left(1 - \frac{\rho_\ell}{\rho}\right)}$. The above equation is in the form of $\boxed{\frac{dx}{dt} = a - bx}$

where $\boxed{a \to \bar{g}}$ and $\boxed{b \to K/m}$. The solution of $v \equiv v(t)$ is $\boxed{v = v_T\left(1 - e^{-mt/K}\right)}$ in which the terminal velocity $\boxed{v_T = m\bar{g}/K}$ (Stokes's law of terminal velocity). The early growth of v is linear $\boxed{v \simeq \bar{g}t}$, and after a time scale of $\boxed{t_0 = m/K}$, $\boxed{v \to v_T \text{ for } t \to \infty}$.

The terminal (saturation) velocity is also $\boxed{v_T = \bar{g}\, t_0}$.

The initial condition of the fall is $\boxed{t = 0, v = 0}$.

If z is the depth from the starting point, we can write $\boxed{v = \frac{dz}{dt} = v_T\left(1 - e^{-mt/K}\right)}$.

The integral is $z = v_T t - v_T \int e^{-t/t_0}\, dt$ (P.T.O.)

$$\Rightarrow \boxed{z = V_T t - V_T \frac{e^{-t/t_0}}{-1/t_0} + c = V_T t + t_0 V_T e^{-t/t_0} + c}$$

When $t=0$, $z=0 \Rightarrow 0 = 0 + t_0 V_T + c \Rightarrow \boxed{c = -t_0 V_T}$.

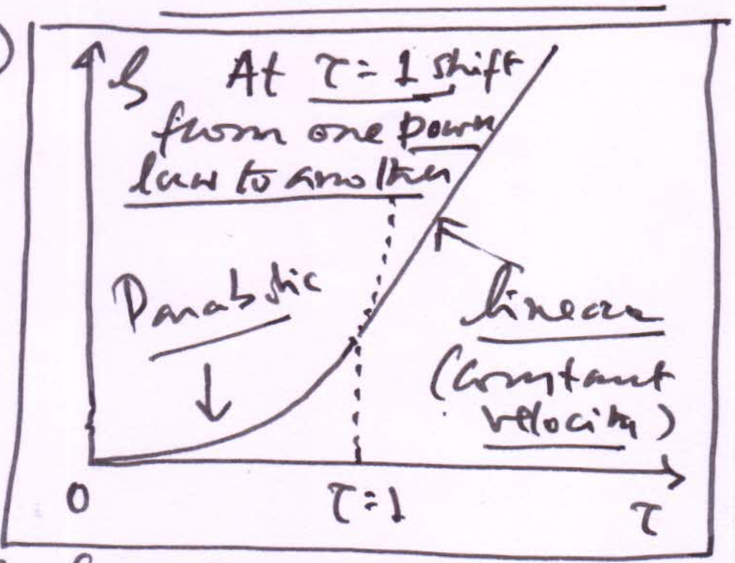
$$\Rightarrow \boxed{z = V_T t + V_T t_0 \left(e^{-t/t_0} - 1\right)}$$ Define $\boxed{\zeta = \frac{z}{V_T t_0}}$ and $\boxed{\tau = t/t_0}$.

$$\therefore \frac{z}{V_T t_0} = \frac{t}{t_0} + e^{-t/t_0} - 1 \Rightarrow \boxed{\zeta = (\tau - 1) + e^{-\tau}}.$$

Hence, $\boxed{\frac{d\zeta}{d\tau} = 1 - e^{-\tau}}$. ~~Hence~~ When $\frac{d\zeta}{d\tau} = 0$
$\Rightarrow \boxed{\tau = 0}$ or $e^{-\tau} = 1$

The second derivative, $\boxed{\frac{d^2\zeta}{d\tau^2} = e^{-\tau}}$. When $\tau = 0$,
$\boxed{\frac{d^2\zeta}{d\tau^2} = 1 > 0}$ $\therefore$ $\tau = 0$ is a minimum point.
and ζ (or z) increases monotonically for ~~t(or t)~~ τ (or t) > 0. i.) When $\tau \to 0$, a series expansion gives $\zeta \simeq \cancel{\tau} - \cancel{1} + \left(\cancel{1} - \cancel{\tau} + \frac{\tau^2}{2} + \dots\right)$ [Ignore the higher orders for $\tau \ll 1$]
$\Rightarrow \boxed{\zeta \simeq \tau^2/2}$ (Parabola)

Alternatively for small t,
$\boxed{v \simeq \bar{g} t} \Rightarrow \frac{dz}{dt} \simeq \bar{g} t$
or $\boxed{z \simeq \bar{g}\frac{t^2}{2}}$ (Parabola).



ii.) When ~~t~~ $\to \infty$ $(\tau \gg 1)$
$\boxed{\zeta \simeq \tau}$ (Linear) Alternatively, for large t, $\boxed{v = \frac{dz}{dt} \simeq V_T} \Rightarrow \boxed{z \simeq V_T t}$ (Linear)

Long Fall through Air

Eg. Free fall of a parachutist

Turbulence is a common phenomenon in the atmosphere. If the dynamic viscosity of air is η, its density is ρ, then the kinematic viscosity is $\boxed{\nu = \eta/\rho}$. If the atmospheric turbulent eddies have a characteristic length scale of l and a characteristic eddy velocity v, then

Reynold's number: $\boxed{R_e = \frac{lv}{\nu} = \frac{\rho l v}{\eta}}$.

Dimensionally $\boxed{[\nu] = L^2 T^{-1}}$. Hence R_e is a dimensionless number ($[l] = L, [v] = LT^{-1}$)

Typically $\boxed{\nu_{water} \sim 10^{-2}\, m^2 s^{-1}}$, $\boxed{\nu_{air} \sim 10^{-4}\, m^2 s^{-1}}$

For any object travelling through a fluid (either air or water), the fluid drag is $\boxed{D \propto v^r}$, $v \to$ velocity, $r \to$ an exponent.

i.) When $\boxed{R_e \sim 10}$ (low v, high ν), $\boxed{r = 1}$ (liquid)
ii.) When $\boxed{10 < R < 10^3}$, r is uncertain.
iii.) When $\boxed{R_e \sim 10^3}$ (high v, low ν), $\boxed{r = 2}$ (air)
This is due to the turbulence in air

For a parachutist falling from a height of 32,000 ft to 2,000 ft (till the parachute is opened), we take $\boxed{D = kv^2}$, in which the drag coefficient $k > 0$. The equation of the fall is $\boxed{m\frac{dv}{dt} = mg - kv^2}$, neglecting buoyancy in air (a very small effect)

$\therefore \boxed{\frac{dv}{dt} = g - \frac{k}{m}v^2}$, in the form $\boxed{\frac{dx}{dt} = a - bx^2}$,

with $a \rightarrow g$ and $b \rightarrow k/m$. The integral solution for $t = 0$, $v = 0$ (the initial condition) is $\boxed{v = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}\, t\right)}$.

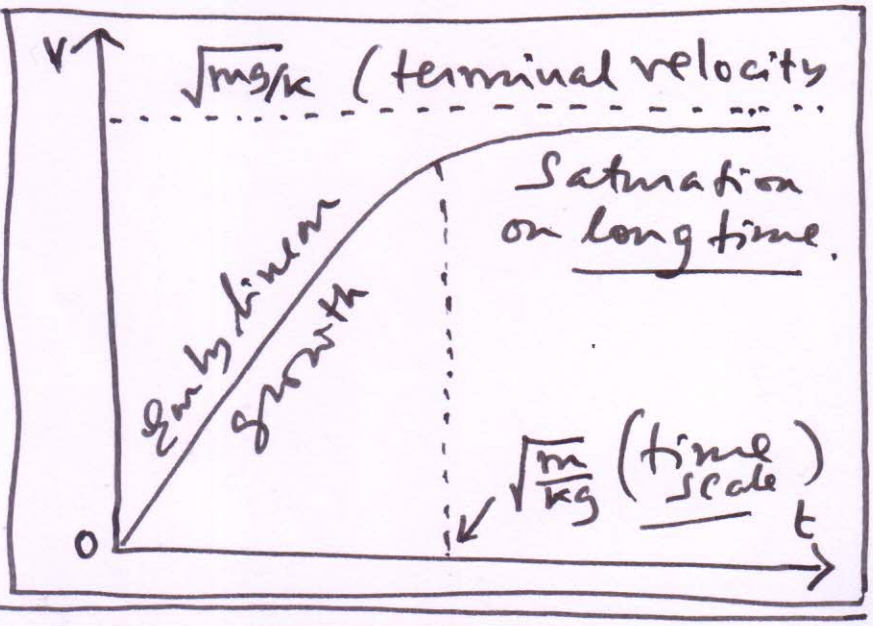
i.) When $t \rightarrow \infty$, $\boxed{v \rightarrow \sqrt{\frac{mg}{k}}}$ (terminal velocity)

ii.) When $t \rightarrow 0$, $\boxed{\tanh\left(\sqrt{\frac{kg}{m}}\, t\right) \simeq \sqrt{\frac{kg}{m}}\, t}$

$\Rightarrow v \simeq \sqrt{\frac{\cancel{m}g}{\cancel{k}}}\sqrt{\frac{\cancel{k}g}{\cancel{m}}}\, t$

$\Rightarrow \boxed{v \simeq gt}$, i.e. early growth is linear.

iii.) Natural scale of time is $\sqrt{\frac{m}{kg}}$



Bicycle Motion Against Air Resistance

The equation of motion $\boxed{m\frac{dv}{dt} = F}$,
with $m \to$ mass of the bicycle and rider, $v \to$ velocity and $F \to$ force exerted by the rider on the bicycle. A-priori F is not known. The kinetic energy of the bicycle-rider combination is $\boxed{\mathcal{E} = \frac{1}{2}mv^2}$.

$$\Rightarrow \frac{d\mathcal{E}}{dt} = mv\frac{dv}{dt} \Rightarrow \boxed{m\frac{dv}{dt} = \frac{1}{v}\frac{d\mathcal{E}}{dt} = F}$$ (without air resistance)

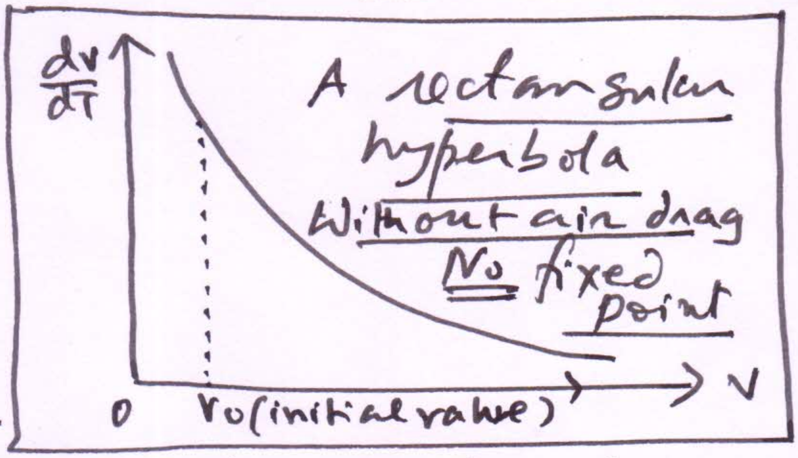
Now the power output of the rider is $\boxed{P = \frac{d\mathcal{E}}{dt}}$.
We approximate it as a constant quantity.

$\Rightarrow \boxed{\frac{dv}{dt} = \frac{P}{mv}}$ → an autonomous equation in the form $\boxed{\frac{dv}{dt} = f(v)}$.

Writing (rescaling) $\boxed{T = \frac{Pt}{m}}$

we get $\boxed{\frac{dv}{dT} = \frac{1}{v}}$.

$\boxed{\frac{dv}{dT} = 0}$ when $v \to \infty$



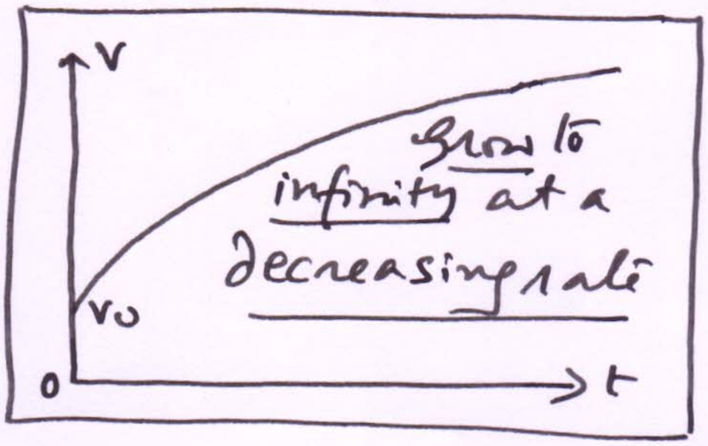
$\Rightarrow$ No fixed point for any finite value of v, with no air drag.

The integral solution is found with an initial value of $v = v_0$ at $t = 0$.

$$\boxed{\int_{v_0}^{v} v\,dv = \frac{P}{m}\int_0^t dt}$$

(P. T. O.)

$$\Rightarrow \frac{v^2}{2}\Big|_{v_0}^{v} = \frac{Pt}{m}\Big|_0^t \Rightarrow \boxed{v^2 = v_0^2 + \frac{2Pt}{m}}.$$



When $t \to \infty$, $v^2 \to \infty$

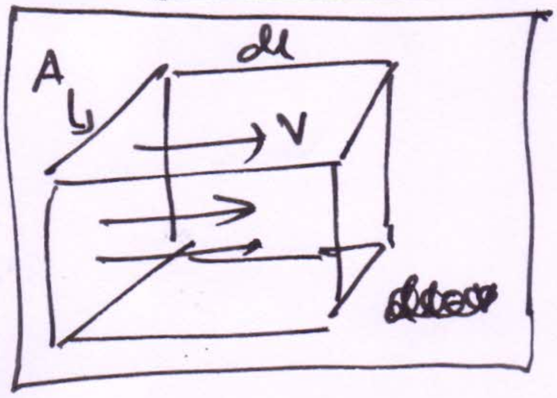
$$\Rightarrow \boxed{v = \sqrt{v_0^2 + \frac{2Pt}{m}}}$$

$\boxed{v \sim t^{1/2}}$ There is no limit on v as $t \to \infty$.

Integrating $\boxed{\frac{dv}{dt} = \frac{P}{mv}}$ by Euler's method,

$$\boxed{\frac{v_{i+1} - v_i}{\Delta t} \simeq \frac{P}{m v_i}} \Rightarrow \boxed{v_{i+1} = v_i + \frac{P}{m v_i}\Delta t}$$

So far the air resistance has not been considered. The air resistance offers a drag force, $\boxed{D \simeq -b_1 v - b_2 v^2}$ opposing the motion. The two terms on the right hand side of D resemble a Taylor expansion in v. For high velocities we usually ignore the first-order term in v and keep only the v^2 term.



Consider an object with a frontal surface area A moving forward through a distance dl in time dt.

The mass of air displaced (pushed out of the way) by the frontal surface in moving a distance dl is $\boxed{\rho A\,dl}$, in which ρ is the density of air. The frontal surface travels with a velocity $\boxed{v = \frac{dl}{dt}}$.

The displaced air acquires a velocity of the same order. The kinetic energy of the displaced air is $\boxed{\mathcal{E}_{kin} \sim (\rho A\,dl)\frac{v^2}{2}}$.

This energy is provided by the work done by the cyclist. The work is being done against the air drag. Hence, the work done is $\boxed{-\mathcal{D}\,dl}$, the negative sign indicating that the force applied is opposite to the drag. $\therefore$ $\boxed{-\mathcal{D}\,\cancel{dl} \simeq (\rho A\,\cancel{dl})\frac{v^2}{2}}$.

$\Rightarrow \boxed{\mathcal{D} \simeq -\frac{1}{2}\rho A v^2}$ or more generally,

$\boxed{\mathcal{D} = -b\rho A v^2}$, in which $b \simeq 1/2$ is the drag coefficient. It can have a slightly varying value. The drag force can be reduced by reducing the frontal area A (bicyclist crouching). It is also reduced for lower air density ρ.

Accounting for air drag, the equation for a bicycle motion becomes modified as

$$\boxed{m\frac{dv}{dt} = \frac{P}{v} - b\rho A v^2} \Rightarrow \boxed{\frac{dv}{dt} = \frac{P}{mv} - \frac{b\rho A v^2}{m}}.$$

Multiplying throughout by $\frac{mv}{b\rho A}$ we get

$$\boxed{\frac{mv}{b\rho A}\frac{dv}{dt} = \frac{P}{b\rho A} - v^3}.$$ Now define $\boxed{a^3 = \frac{P}{b\rho A}}$.

$$\Rightarrow \boxed{\frac{m}{b\rho A}\, v\frac{dv}{dt} = a^3 - v^3}.$$ Define $\boxed{u = \frac{v}{a}}$

and divide throughout by a^3 to get,

$$\boxed{\frac{m}{a b\rho A}\, u\frac{du}{dt} = 1 - u^3}$$ Define $\boxed{T = \frac{a b\rho A}{m}t}$,

to get, $\boxed{u\frac{du}{dT} = 1 - u^3} \Rightarrow \boxed{\frac{du}{dT} = \frac{1-u^3}{u} = \frac{1}{u} - u^2}$

in an autonomous form $\boxed{\frac{du}{dT} = f(u)}$.

The fixed points are found from $\boxed{\frac{du}{dT} = 0}$.

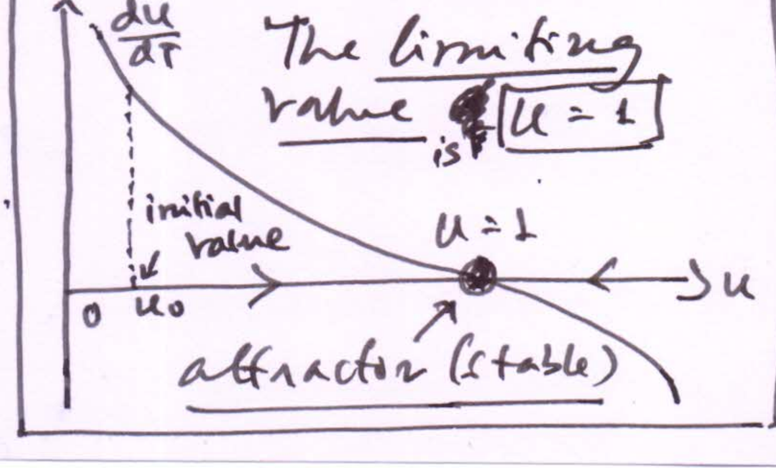
$$\Rightarrow \boxed{\frac{du}{dT} = \frac{(1-u)(1+u+u^2)}{u} = 0}.$$ This will be satisfied when

$\boxed{u \to \infty}$ or $\boxed{u = 1}$ or

$\boxed{1+u+u^2 = 0} \to$ both roots are complex.

$\Rightarrow \boxed{u = 1}$ or $\boxed{v = a}$ or

$\boxed{v = \frac{P}{b\rho A}}$ in the limit of the velocity



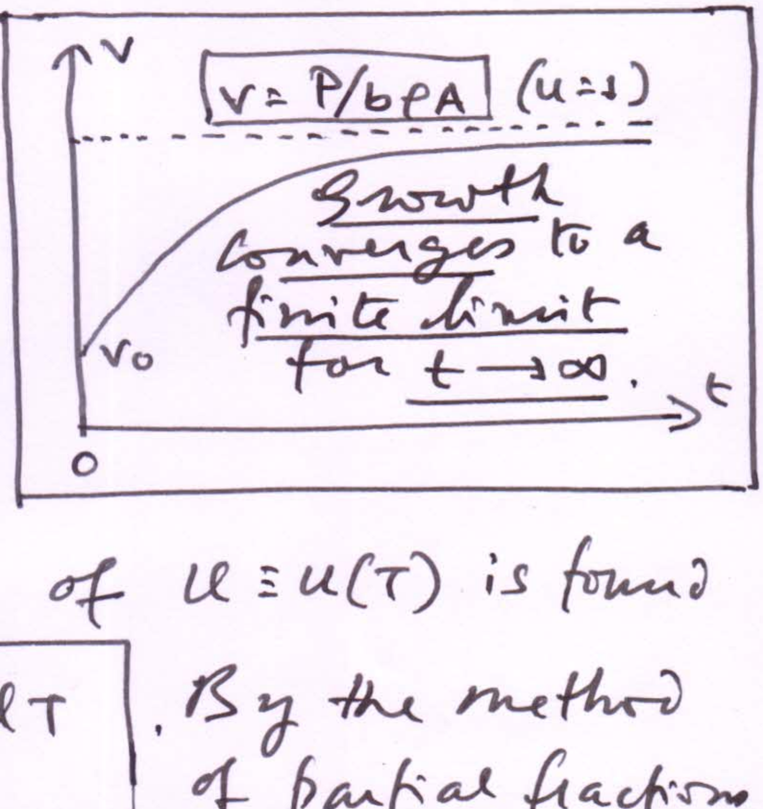
i.) For $u \ll 1$, $\boxed{\frac{du}{dT} \simeq \frac{1}{u}}$ because $\boxed{\frac{1}{u} \gg u^2}$

ii.) For $u \gg 1$, $\boxed{\frac{du}{dT} \simeq -u^2}$ because $\boxed{\frac{1}{u} \ll u^2}$

Around $u = 1$ both terms balance out and $\boxed{\frac{du}{dT} = 0}$ to give a fixed point at $\boxed{u = 1}$.

$\therefore$ For $\boxed{u \ll 1}$, the phase plot has the form of a rectangular hyperbola and for $\boxed{u \gg 1}$ the phase plot has the form of a parabola.

Since the stable fixed point at $u = 1$ is approached for $t \to \infty$, there is a finite limit to the velocity $\boxed{V = \frac{P}{b\rho A}}$



The integral solution of $u \equiv u(T)$ is found from $\boxed{\int \frac{u\,du}{1-u^3} = \int dT}$. By the method of partial fractions we write

$$\boxed{\frac{u}{1-u^3} \equiv \frac{u}{(1-u)(1+u+u^2)} \equiv \frac{A}{1-u} + \frac{Bu+C}{1+u+u^2}}$$

$\Rightarrow \boxed{u = A(1+u+u^2) + (Bu+C)(1-u)}$ i) When $\boxed{u = 1}$ $A \cdot 3 = 1 \Rightarrow \boxed{A = \frac{1}{3}}$

ii.) When $\boxed{u=0}$ $\Rightarrow$ $0 = A + C$ $\Rightarrow$ $\boxed{C = -A = -1/3}$

iii.) When $\boxed{u=-1}$ $\Rightarrow$ $-1 = A + (-B+C)\cdot 2$ $\Rightarrow$ $\boxed{B = 1/3}$.

$$\Rightarrow \boxed{\frac{1}{3}\int\frac{du}{1-u} + \frac{1}{3}\int\frac{u-1}{u^2+u+1}\,du = \int dT}$$ The $u-T$ integral

$$\Rightarrow -\frac{1}{3}\int\frac{d(-u)}{1-u} + \frac{1}{6}\int\frac{(2u+1-3)\,du}{u^2+u+1} = \int dT$$

$$\Rightarrow \boxed{-\frac{1}{3}\ln(1-u) + \frac{1}{6}\ln\left(u^2+u+1\right) - \frac{1}{2}\int\frac{du}{u^2+u+1} = T}$$

Now $u^2+u+1 = u^2+2\cdot\frac{1}{2}u+1 = \left(u+\frac{1}{2}\right)^2+\frac{3}{4}$

$$\therefore -\frac{1}{2}\int\frac{du}{u^2+u+1} = -\frac{1}{2}\int\frac{du}{\frac{3}{4}+\left(u+\frac{1}{2}\right)^2} = -\frac{1}{2}\cdot\frac{4}{3}\int\frac{du}{1+\left(\frac{2}{\sqrt{3}}u+\frac{1}{\sqrt{3}}\right)^2}$$

Define $\boxed{\tan\theta = \frac{2u+1}{\sqrt{3}}}$, $\boxed{\sec^2\theta = 1+\tan^2\theta}$ $\Big|$ $\frac{4}{3}\left(u+\frac{1}{2}\right)^2 = \left(\frac{2}{\sqrt{3}}u+\frac{1}{\sqrt{3}}\right)^2$

$\Rightarrow \sec^2\theta\, d\theta = \frac{2}{\sqrt{3}}\,du \Rightarrow \boxed{du = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta}$.

$$\boxed{\begin{aligned}&\Rightarrow -\frac{1}{2}\cdot\frac{4}{3}\int\frac{du}{1+\left(\frac{2u+1}{\sqrt{3}}\right)^2} = -\frac{1}{2}\cdot\frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int\frac{\cancel{\sec^2\theta}\,d\theta}{\cancel{\sec^2\theta}}\\ &\text{The result is} \longrightarrow = -\frac{1}{\sqrt{3}}\int d\theta = \frac{-1}{\sqrt{3}}\arctan\left(\frac{2u+1}{\sqrt{3}}\right)\end{aligned}}$$

The full integral solution is (not in closed form)

$$\boxed{-\frac{1}{3}\ln(1-u) + \frac{1}{6}\ln\left(u^2+u+1\right) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2u+1}{\sqrt{3}}\right) = T + \text{constant}}$$

[As $u \to 1$, $T \to \infty$]

Integrating numerically by Euler's method, $\boxed{\frac{v_{i+1}-v_i}{\Delta t} \simeq \frac{P}{mv_i} - \frac{b\rho A}{m}v_i^2}$,

$$\boxed{v_{i+1} \simeq v_i + \frac{P}{mv_i}\Delta t - \frac{b\rho A}{m}v_i^2\Delta t}$$ (upto the first order)

Projectile Motion

Example: Trajectory of an artillery shell.

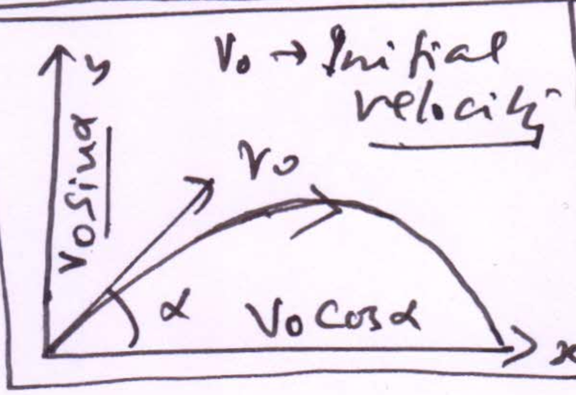
Projectile motion has two components, one along the x-axis (horizontal motion) and the other along the y-axis (vertical motion).

The equations of motion are (without air drag)

i.) $\boxed{\dfrac{d^2x}{dt^2} = 0}$ in the horizontal direction.

ii.) $\boxed{\dfrac{d^2y}{dt^2} = -g}$ in the vertical direction (against gravity) (No air drag)

The projectile has an initial velocity of v_0, with a projection angle α.



$\therefore$ Along the horizontal direction we write $\boxed{\dfrac{dx}{dt} = V_x \text{ (constant)}}$.

and along the vertical direction, we write, $\boxed{\dfrac{dy}{dt} = V_y}$. Initial value of V_x at $t=0$ is $V_0 \cos\alpha$ (stays the same all time).

The initial value of V_y at $t=0$ is $V_0 \sin\alpha$.

Since V_x is constant $\boxed{\dfrac{dV_x}{dt} = 0}$. The x-t integral is $\boxed{x = V_x t + A_1}$, where A_1 is an integral constant.

When $t=0$, $x=0$ $\Rightarrow 0 = 0 + A_1 \Rightarrow \boxed{A_1 = 0}$

$\therefore$ $\boxed{x = V_x t = V_0(\cos\alpha)\, t}$ (V_x is constant)

Since $\boxed{\frac{dy}{dt} = V_y}$, we write $\boxed{\frac{d^2y}{dt^2} = \frac{dV_y}{dt} = -g}$

($A_2 \rightarrow$ Integral constant)

$\Rightarrow \boxed{V_y = -gt + A_2}$ When $t=0$, $V_y = V_0 \sin\alpha$.

$\therefore$ $\boxed{A_2 = V_0 \sin\alpha} \Rightarrow \boxed{V_y = V_0(\sin\alpha) - gt}$ Integral constant

$\Rightarrow \boxed{\frac{dy}{dt} = V_0(\sin\alpha) - gt} \Rightarrow \boxed{y = V_0(\sin\alpha)t - \frac{gt^2}{2} + A_3}$

When $t=0$, $y=0$ $\Rightarrow \boxed{A_3 = 0}$ $\therefore \boxed{y = V_0(\sin\alpha)t - \frac{gt^2}{2}}$

Hence, we have a set of parametric equations in the form $\boxed{x \equiv x(t)}$ and $\boxed{y \equiv y(t)}$. Noting that $\boxed{t = \frac{x}{V_0 \cos\alpha}}$, we get an x-y equation,

$$\boxed{y = \cancel{V_0} \sin\alpha \cdot \frac{x}{\cancel{V_0}\cos\alpha} - \frac{g}{2}\frac{x^2}{V_0^2\cos^2\alpha}}$$

, which is finally written as

$$\boxed{y = (\tan\alpha)\,x - \frac{1}{2}\, g \frac{x^2}{V_0^2\cos^2\alpha}}$$

$\rightarrow$ The equation of a parabola.

[The trajectory of a projectile without air drag. When $y=0$, $\boxed{x=0}$ and $\rightarrow$ The range of the projectile.

$$\boxed{x = \frac{2V_0^2}{g}\tan\alpha\cos^2\alpha = \frac{V_0^2}{g}\sin(2\alpha)}$$

Maximum range is $\boxed{x = \frac{V_0^2}{g}}$ when $\boxed{\sin(2\alpha) = 1}$, $\boxed{\alpha = 45°}$

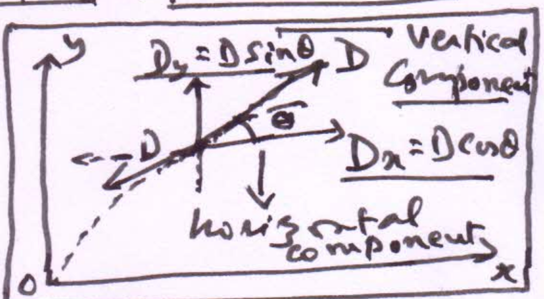
Integrating numerically by Euler's method,

$$\boxed{\frac{dx}{dt} = v_x \Rightarrow x_{i+1} = x_i + v_{x,i}\,\Delta t} \quad \boxed{\frac{dv_x}{dt} = 0 \Rightarrow v_{x,i+1} = v_{x,i}}$$

$$\boxed{\frac{dy}{dt} = v_y \Rightarrow y_{i+1} = y_i + v_{y,i}\,\Delta t} \quad \boxed{\frac{dv_y}{dt} = -g \Rightarrow v_{y,i+1} = v_{y,i} - g\Delta t}$$

Accounting for air drag, the drag force, $\boxed{D = -Bv^2}$.

The negative sign implies opposition to the motion.



i.) The horizontal component of the drag is $\boxed{D_x = D\cos\theta}$, and $\boxed{\cos\theta = v_x/v}$.

ii.) The vertical component of the drag is $\boxed{D_y = D\sin\theta}$, and $\boxed{\sin\theta = v_y/v}$. Since,

$$\boxed{v^2 = v_x^2 + v_y^2}, \quad \boxed{D_x = -Bv^2\frac{v_x}{v} = -B\sqrt{v_x^2 + v_y^2}\cdot v_x}$$

and likewise, $\boxed{D_y = -Bv^2\frac{v_y}{v} = -B\sqrt{v_x^2 + v_y^2}\cdot v_y}$

The equations of motion are modified as

$$\boxed{m\frac{dv_x}{dt} = -B\sqrt{v_x^2 + v_y^2}\; v_x} \quad \text{and} \quad \boxed{\frac{dx}{dt} = v_x}.$$

$$\boxed{m\frac{dv_y}{dt} = -mg - B\sqrt{v_x^2 + v_y^2}\, v_y} \quad \text{and} \quad \boxed{\frac{dy}{dt} = v_y}.$$

In Euler's method only $\boxed{dv_x/dt}$ and $\boxed{dv_y/dt}$ are modified as

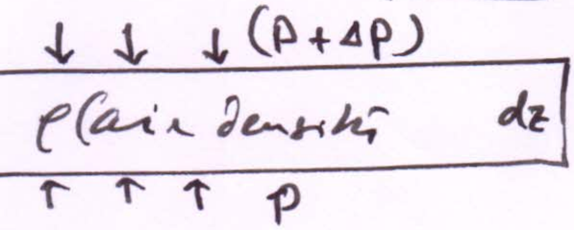
$$\boxed{v_{x,i+1} = v_{x,i} - \left(\frac{B}{m}\sqrt{v_{x,i}^2 + v_{y,i}^2}\right) v_{x,i}\,\boxed{\Delta t}}$$

(multiply by Δt in the last terms)

$$\boxed{v_{y,i+1} = v_{y,i} - g\Delta t - \left(\frac{B}{m}\sqrt{v_{x,i}^2 + v_{y,i}^2}\right) v_{y,i}\,\Delta t}$$

Vertical Variation of Air Density

The density of air affects ~~the~~ projectile motion, because the drag force depends on ~~the~~ air density.



Consider a slab of air with a lateral cross sectional area A and thickness dZ (where Z is the height from the surface of the earth). The air pressure above ~~below~~ the slab is $(P+\Delta P)$ and below the slab is P. Weight of air contained in the slab is $(\rho A dZ)g$. The total downward force is $\boxed{(P+\Delta P)A + (\rho A dZ)g}$.

The total upward force is $\boxed{PA}$. On balancing we get $\boxed{PA = (P+\Delta P)A + (\rho A dZ)g}$. Hence

$\cancel{PA} = \cancel{PA} + \cancel{A}\Delta P + (\rho \cancel{A}\, dZ)g \Rightarrow \boxed{\Delta P = -\rho g dZ}$,

i.e. as height increases, pressure decreases.

Now we approximate air as an ideal gas, which follows two requirements, namely
i) Gas molecules do not interact except for the occasional collisions among them.
ii) All gas particles are point-like with a negligible total volume compared to the gas.

The ideal gas equation of state is

$\boxed{PV = nRT}$. We write n (the no. of moles)

as $\boxed{n = \frac{N}{N_A}}$, $N \to$ Total number of particles

$\Rightarrow \boxed{P = \frac{N}{V} \cdot K_B T}$ since $\boxed{K_B = R/N_A} \to$ The Boltzmann constant

If the average mass of air molecules is $\bar{m}$,

then $\boxed{P = \frac{N\bar{m}}{V\bar{m}} K_B T} \Rightarrow \boxed{P = \frac{\rho K_B T}{\bar{m}}}$, $\boxed{\rho = \frac{N\bar{m}}{V}}$.

With an equation state having $\boxed{P, \rho, T}$ we only have intensive thermodynamic variables.

Treating temperature to have a much smaller variation than density, we consider T as almost a constant. Hence

$\boxed{\Delta P = \Delta \rho \left(\frac{K_B T}{\bar{m}}\right)} \Rightarrow \Delta \rho \left(\frac{K_B T}{\bar{m}}\right) = -\rho g \, dz$.

$\Rightarrow \boxed{\frac{\Delta \rho}{\rho} = \frac{-\bar{m} g}{K_B T} dz} \Rightarrow \boxed{\int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = -\frac{\bar{m} g}{K_B T} \int_0^z dz}$

Air density ρ

i.e. when $z = 0$ (Surface of the earth)

$\rho = \rho_0$ (air density at the surface of the earth)

$\Rightarrow \ln \rho - \ln \rho_0 = -\frac{\bar{m} g z}{K_B T} \Rightarrow \boxed{\rho = \rho_0 \exp\left(-\frac{\bar{m} g z}{K_B T}\right)}$

$\boxed{\frac{K_B T}{\bar{m} g} \sim \frac{10^{-23} \times 300}{10^{-27} \times 10} \sim 300 \text{ km}} \to$ Scale for z (in length dimension)

Usually $z \ll 300\,km$. Hence, $\rho(z)$ can be safely approximated by the series expansion of $\boxed{\exp\left(-\frac{\bar{m} g z}{k_B T}\right) \simeq 1 - \frac{\bar{m} g}{k_B T} z}$.

$\Rightarrow \boxed{\rho \simeq \rho_0 \left(1 - \frac{\bar{m} g z}{k_B T}\right)} \rightarrow$ A linear decrease for $\boxed{z \ll k_B T/\bar{m} g}$.

Turbulence in the atmosphere can also affect the motion of a projectile.

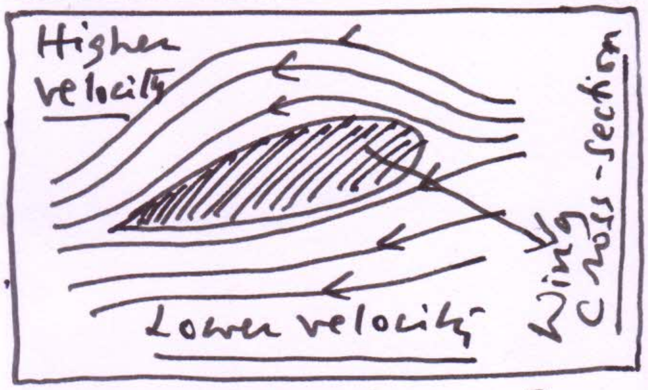
i.) Streamline Motion: Smooth and laminar

ii.) Turbulent Motion: Random and chaotic

Bernoulli Equation: $\boxed{\frac{v^2}{2} + \frac{P}{\rho} + gz = \text{Constant}}$

$v \rightarrow$ velocity, $P \rightarrow$ Pressure, $\rho \rightarrow$ Density, $z \rightarrow$ Height

Lift of an Aircraft



i.) Above the wing closer streamlines have higher velocity. Hence pressure is lower.

ii.) Below the wing the streamlines have lower velocity. Hence at nearly the same height, the pressure is higher.

iii.) Turbulence can also cause velocity and pressure difference in a flying object.