Tutorial- 8

Q.1 Solve the following linear differential equations.

(i) $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

(ii) $x\,dy + (y+y)\,dx = 0$

(iii) $(2y - x^3)\,dx = x\,dy$

(iv) $3y^2 \frac{dy}{dx} + xy^3 = x$

~~(v) $2x^2 \frac{dy}{dx} = (x-1)(y^2-x^2) + 2xy$~~

(vi) $\sin y \frac{dy}{dx} = \cos x\,(2\cos y - \sin^2 x)$

(vii) $(1-x^2)\frac{dy}{dx} - xy - axy^2 = 0$

Q.2 Find the general solution of the following differential equations

(i) $4y'' + 16y' + 52y = 0$

(ii) $4y'' + 4y' + y = 0$

Q.3 (i) Show that there is no solution to the boundary value problem

$$y'' + 4y = 0, \quad y(0) = 0, \quad y(\pi) = 1$$

(ii) Show that there are infinitely many solutions to the boundary value problem

$$y'' + 4y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

Q.4 Solve the nonhomogeneous equation by the method of undetermined coefficients

(1) $y'' + y = 2x + 3e^{x}$

(2) $y'' - y' - 6y = e^{-x} - 7\cos x$

Solutions

Tutorial. 8

Q.1 (i) $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$I.F. = e^{\int P(y)\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Solution.

$$x \times I.F = \int I.F.\ Q(y)\,dy$$

$$\Rightarrow x\,e^{\tan^{-1}y} = \int e^{\tan^{-1}y} \cdot \frac{\tan^{-1}y}{1+y^2}\,dy + C$$

$$= \int z e^{z}\,dz + C \qquad z = \tan^{-1}y, \quad dz = \frac{1}{1+y^2}dy$$

$$= z e^{z} - e^{z} + C$$

$$\Rightarrow x\,e^{\tan^{-1}y} = \tan^{-1}y\, e^{\tan^{-1}y} - e^{\tan^{-1}y} + C$$

$$\Rightarrow x = \tan^{-1}y - 1 + C e^{-\tan^{-1}y}$$

(ii) $x\,dy + (y+4)\,dx = 0 \Rightarrow x\,dy = -(y+4)\,dx$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x} - \frac{4}{x}$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = -\frac{4}{x}$ $\qquad P(x) = \frac{1}{x} \qquad Q(x) = -\frac{4}{x}$

I.F. $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$y \times \text{I.F.} = \int Q(x) \cdot \text{I.F.}\, dx$

$\Rightarrow yx = \int -\frac{4}{\cancel{x}} \cdot \cancel{x}\, dx = -4x + C$

$\Rightarrow y = -4 + \frac{C}{x}, \quad x > 0$

(iii) $(2y - x^3)\,dx = x\,dy$

$\frac{dy}{dx} = \frac{2y}{x} - x^2$

$\Rightarrow \frac{dy}{dx} - \frac{2y}{x} = -x^2$

I.F. $= e^{\int -\frac{2}{x} dx} = e^{-2\log x} = \frac{1}{x^2}$

$y \times \frac{1}{x^2} = \int \frac{1}{x^2} \cdot (-x^2)\, dx = \int -dx = -x + C$

$\Rightarrow y = -x^3 + Cx^2$

(v) $3y^2 \frac{dy}{dx} + xy^3 = x$

put $y^3 = z \Rightarrow 3y^2 \frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{dz}{dx} + zx = x$

Linear equation in z dependent variable
x independent variable.

I.F. $e^{\int x\,dx} = e^{\frac{x^2}{2}}$

$\frac{x^2}{2} = t$
$\Rightarrow x\,dx = dt$

$z \cdot e^{x^2/2} = \int x e^{x^2/2}\,dx = \int e^t\,dt$
$= e^t = e^{x^2/2} + C$

$\Rightarrow y^3 = 1 + c e^{-x^2/2}$

(v) ~~$2x^2 \frac{dy}{dx} = (x-1)(y^2-x^2) + 2xy$~~

(v) $\sin y \frac{dy}{dx} = \cos x (2\cos y - \sin^2 x)$

$\Rightarrow \sin y \frac{dy}{dx} = 2\cos x \cos y - \sin^2 x \cos x$

putting $\cos y = z$
$-\sin y \frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow -\frac{dz}{dx} - 2\cos x\, z = -\sin^2 x \cos x$

$\Rightarrow \frac{dz}{dx} + 2z\cos x = \sin^2 x \cos x$ Linear equation

I.F. $e^{\int 2\cos x\,dx} = e^{2\sin x}$

$\sin x = t$
$\cos x\,dx = dt$

So $z e^{2\sin x} = \int e^{2\sin x} \sin^2 x \cos x\,dx$
$= \int t^2 e^{2t}\,dt + C = \frac{t^2}{2} e^{2t} - \frac{t}{2} e^{2t} + \frac{e^{2t}}{4} + C$

$\Rightarrow \cos y\, e^{2\sin x} = \frac{1}{2}\sin^2 x\, e^{2\sin x} - \frac{1}{2}\sin x\, e^{2\sin x} + \frac{1}{4} e^{2\sin x} + C$

(vi) $(1-x^2)\frac{dy}{dx} - xy - axy^2 = 0$

$$\Rightarrow \frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{axy^2}{1-x^2}$$

$$\Rightarrow y^{-2}\frac{dy}{dx} - \frac{x}{1-x^2}y^{-1} = \frac{ax}{1-x^2}$$

$$y^{-1} = z \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow -\frac{dz}{dx} - \frac{x}{1-x^2}z = \frac{ax}{1-x^2}$$

$$\Rightarrow \frac{dz}{dx} + \frac{x}{1-x^2}z = \frac{-ax}{1-x^2}$$

I.F. $e^{\int \frac{x}{1-x^2}dx} = e^{-\frac{1}{2}\int \frac{1}{t}dt}$ $\quad 1-x^2 = t$, $-2x\,dx = dt$

$$= e^{-\frac{1}{2}\log t} = t^{-\frac{1}{2}} = (1-x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}}$$

$$z\frac{1}{\sqrt{1-x^2}} = \int \frac{-ax}{(1-x^2)}\cdot\frac{1}{\sqrt{1-x^2}}dx + C$$

$1-x^2 = p$, $-2x\,dx = dp$

$$= -a\int \frac{x}{(1-x^2)^{\frac{3}{2}}}dx + C$$

$$= \frac{-a}{-2}\int \frac{1}{p^{\frac{3}{2}}}dp + C = \frac{a}{2}\frac{p^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C$$

$$= \frac{a}{2}\frac{p^{-\frac{1}{2}}}{-\frac{1}{2}} + C = -a(1-x^2)^{-\frac{1}{2}} + C$$

$$= -a\frac{1}{\sqrt{1-x^2}} + C$$

$$\Rightarrow z = -a + C\sqrt{1-x^2}$$

$$\Rightarrow \frac{1}{y} = -a + C\sqrt{1-x^2} \Rightarrow \left(C\sqrt{1-x^2} - a\right)y = 1$$

(Ans)

2 (i) $4y'' + 16y' + 52y = 0$

Putting $y = e^{mx}$ we get the auxiliary equation

$$4m^2 + 16m + 52 = 0$$

$\Rightarrow m^2 + 4m + 13 = 0 \Rightarrow m = \dfrac{-4 \pm \sqrt{16 - 4 \cdot 1 \cdot 13}}{2}$

$= \dfrac{-4 \pm \sqrt{-36}}{2} = \dfrac{-4 \pm 6i}{2} = -2 \pm 3i$

$\Rightarrow$

$m = -2 \pm 3i$

$\alpha = -2, \quad \beta = 3$

$\therefore$ General solution is

$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x$

$= e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$.

(ii) $4y'' + 4y' + y = 0$

Putting $y = e^{mx}$ we get the auxiliary equation

$4m^2 + 4m + 1 = 0 \Rightarrow m = \dfrac{-4 \pm \sqrt{16 - 4 \cdot 4 \cdot 1}}{2 \cdot 4} = \dfrac{-4}{8} = -\dfrac{1}{2}$

$m = -\frac{1}{2}$ is a repeated root

So the general solution is

$y(x) = c_1 e^{mx} + c_2 x e^{mx}$

$= (c_1 + c_2 x) e^{-\frac{1}{2}x}$

Q.3

(I) $y'' + 4y = 0$, $y(0) = 0$, $y(\pi) = 1$

Put $y = e^{mx}$ to get the auxiliary equation

$m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m = \pm\sqrt{-4} = \pm 2i$

$\alpha = 0$, $\beta = 2$

So general solution y

$y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) = c_1 \cos 2x + c_2 \sin 2x$

$y(0) = 0 \Rightarrow c_1 \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0$

$y(\pi) = 1 \Rightarrow c_1 \cos 2\pi + c_2 \sin 2\pi = 1$

But $c_1 = 0$ but $\sin 2\pi = 0$

$\Rightarrow c_2 \sin 2\pi = 1$ $\Rightarrow c_2 \sin 2\pi = 0$

which is not 1

So there is no solution for the boundary value problem.

(II) General solution y

$y(x) = c_1 \cos 2x + c_2 \sin 2x$

$y(0) = 0 \Rightarrow c_1 = 0$

$y(\pi) = 0 \Rightarrow c_1 \cos 2\pi + c_2 \sin 2\pi = 0$ as $c_1 = 0$

$\Rightarrow c_2 \sin 2\pi = 0$

This is true for any value of c_2 as $\sin 2\pi = 0$

So there are infinite number of solutions.

Q.4 (1) $y'' + y = 2x + 3e^x$ — (1)

First solve the associated homogeneous equation

$y'' + y = 0$

A. eqn $m^2 + 1 = 0 \Rightarrow m = \pm i$

$y_c(x) = C_1 \cos x + C_2 \sin x$

To find particular integral

$G(x)$ contains terms $2x$ & $3e^x$

Also 1 is not a root of auxiliary equation

0 is not a " " " "

So y_p is of the form

$$y_p = (Ax + B) + Ce^x$$

$$y_p' = A + Ce^x$$

$$y_p'' = Ce^x$$

Putting in (1)

$$Ce^x + (Ax + B) + Ce^x = 2x + 3e^x$$

$$\Rightarrow 2Ce^x + Ax + B = 2x + 3e^x$$

Compare the coefficients

$2C = 3 \Rightarrow C = 3/2$, $A = 2$, $B = 0$

So particular solution is

$$y_p = 2x + \frac{3}{2}e^x$$

General solution is $y_c + y_p = y(x)$

$$y(x) = C_1 \cos x + C_2 \sin x + 2x + \frac{3}{2}e^x.$$

(11) Q.4 $y'' - y' - 6y = e^{-x} - 7\cos x$ — (1)

Associated homogeneous equation

$y'' - y' - 6y = 0$

Auxiliary equation $m^2 - m - 6 = 0$

$\Rightarrow (m-3)(m+2) = 0$

$\Rightarrow m_1 = 3, \quad m_2 = -2$

$y_c = c_1 e^{3x} + c_2 e^{-2x}$

$G(x) = e^{-x} - 7\cos x$

It contains a term e^{-x} of 1 is not a root of A.eqn.

So y_p contains a term Ae^{-x}

$G(x)$ contains $\cos x$ of i is not a root of auxiliary eqn.

So y_p contains $B\cos x + C\sin x$

We expect $y_p = Ae^{-x} + B\cos x + C\sin x$

$y_p' = -Ae^{-x} - B\sin x + C\cos x$

$y_p'' = Ae^{-x} - B\cos x - C\sin x$

putting in (1)

$y'' - y' - 6y = e^{-x} - 7\cos x$

$\Rightarrow Ae^{-x} - B\cos x - C\sin x + Ae^{-x} + B\sin x - C\cos x - 6Ae^{-x}$
$- 6B\cos x - 6C\sin x = e^{-x} - 7\cos x$

$\Rightarrow$ ~~$-4Ae^{-x} = e^{-x}$~~ $-4Ae^{-x} + (-B - C - 6B)\cos x + (-C + B - 6C)\sin x$
$= e^{-x} - 7\cos x$

$\Rightarrow -4A = 1 \Rightarrow A = -\frac{1}{4}$

~~$B =$~~ $-7B - C = -7$ and $B - 7C = 0 \Rightarrow B = 7C$

$-7B - C = -49C - C = -7 \Rightarrow -50C = -7$
$\Rightarrow C = \frac{7}{50}$

$B = 7C = \frac{49}{50}$

(ii) continue

$$y_p = -\frac{1}{4}e^{-x} + \frac{49}{50}\cos x + \frac{7}{50}\sin x$$

$$y(x) = y_c + y_p = c_1 e^{3x} + c_2 e^{-2x} - \frac{1}{4}e^{-x} + \frac{49}{50}\cos x + \frac{7}{50}\sin x$$

(Ans)