Tutorial-2

Q.1 Show that $f(z) = |z|^2$ is differentiable only at $z = 0$; no where else. So it is nowhere analytic.

Sol^n

$$f(z) = |z|^2 = x^2 + y^2 \qquad , z = x + iy$$
$$\Delta z = \Delta x + i\Delta y$$

$$f(z+\Delta z) = |z+\Delta z|^2 = (x+\Delta x)^2 + (y+\Delta y)^2$$

At $z = 0$

$$\lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|\Delta z|^2 - 0}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (\Delta x - i\Delta y) = 0$$

$\Rightarrow$ function $f(z) = |z|^2$ is differentiable at $z = 0$.

When $z \neq 0$ ($x \neq 0$ or $y \neq 0$ or both $\neq 0$)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(x+\Delta x)^2 + (y+\Delta y)^2 - (x^2+y^2)}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(\Delta x)^2 + 2x\Delta x + (\Delta y)^2 + 2y\Delta y}{\Delta x + i\Delta y}$$

When ($\Delta x \to 0$ first, then $\Delta y \to 0$,

$$= \lim_{\Delta y \to 0} \frac{(\Delta y)^2 + 2y\Delta y}{i\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y + 2y}{i} = \frac{2y}{i}$$

$$= -2iy$$

when $\Delta y \to 0$ first, then $\Delta x \to 0$.

$$f'(z) = \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 2x\Delta x}{\Delta x} = \lim_{\Delta x \to 0} (\Delta x + 2x) = 2x$$

$f'(z) = -2iy = 2x$ this is true only when both $x = y = 0$ otherwise it is not.

Hence function is not differentiable for $z \neq 0$.

$\Rightarrow$ function $f(z) = |z|^2$ is not analytic ~~at any point~~ at any point z. That is function is nowhere analytic.

<u>Q.2</u> Prove that an analytic function whose real part is constant is a constant function.

<u>Sol^n</u> Let $f(z)$ is an analytic function whose real part is constant.

If $f(z) = u + iv$
u is constant.

$$\Rightarrow \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 0$$

By C.R. equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = 0$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \Rightarrow v$ is constant.

$\Rightarrow u + iv$ is constant.

$\Rightarrow f(z)$ is constant function.

Q.1 Prove that an analytic function whose modulus is constant is a constant function.

Sol^n Let $f(z)$ be an analytic function whose modulus is constant.

$f(z) = u + iv$

$|f(z)| = u^2 + v^2 = k^2$ constant.

Differentiating w.r.t. x

$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$ — (1)

Differentiating w.r.t y

$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$ — (2)

By C.R. equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Putting $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ in (1)

$u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial x} = 0$ — (3)

Putting $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ in (2)

$-u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0$ — (4)

(3) $\times u$ + (4) $\times v$

$u^2\frac{\partial v}{\partial y} + \cancel{uv\frac{\partial v}{\partial x}}$

$+ \cancel{-uv\frac{\partial v}{\partial x}} + v^2\frac{\partial v}{\partial y} = 0$

$\Rightarrow (u^2+v^2)\frac{\partial v}{\partial y} = 0$

If $u^2+v^2=0$ then it must be that $u=0, v=0$

$\Rightarrow f = 0 \Rightarrow f$ is constant.

⇒ Q-3 ~~Contours~~ Continued

If $u^2+v^2 \neq 0$, then $\frac{\partial v}{\partial y} = 0$

By C.R. equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$

Similarly $\cancel{(3) \times v \pm (4) \times u}$

$\cancel{uv\frac{\partial v}{\partial y} + v^2\frac{\partial v}{\partial x} = 0}$

$\cancel{-u^2\frac{\partial v}{\partial x} + v^2}$

Putting $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ in equation (1) and (2)

we get $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$

⇒ u and v are constant.

⇒ $f = u+iv$ is constant.

Q.4 Find the principal value of the argument for the following

(i) $1-i$ (iv) $-5+5i$

(ii) $3+4i$

(iii) $-\pi-\pi i$

Sol (i) $z = 1-i$ $x=1,\ y=-1$, $x>0$.

$$\operatorname{Arg} z = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1) = -\pi/4$$

(ii) $z = 3+4i$, $x=3,\ y=4$, $x>0$

$$\operatorname{Arg} z = \tan^{-1}(y/x) = \tan^{-1}(4/3) = 0.9273$$

(iii) $z = -\pi-\pi i$, $x=-\pi,\ y=-\pi$, $x<0,\ y<0$

$$\operatorname{Arg}(z) = -\pi + \tan^{-1}(y/x) = -\pi + \tan^{-1}\left(\frac{-\pi}{-\pi}\right)$$
$$= -\pi + \tan^{-1}(1) = -\pi + \pi/4 = -3\pi/4$$

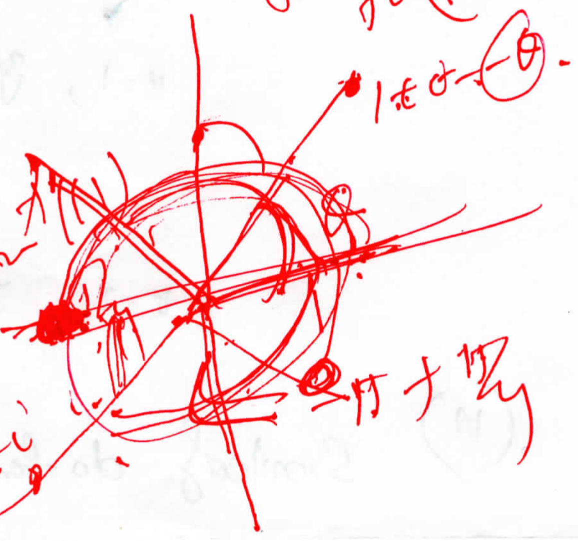
(iv) $z = -5+5i$, $x=-5<0,\ y=5>0$

$$\operatorname{Arg}(z) = \pi + \tan^{-1}(y/x) = \pi + \tan^{-1}\left(\frac{5}{-5}\right)$$
$$= \pi + \tan^{-1}(-1) = \pi + (-\pi/4) = \frac{3\pi}{4}$$

$\tan^{-1}(1)$

$\tan^{-1}(y/x) = \tan^{-1}(1)$

$= \tan^{-1}\frac{-1}{-1}$



Q.5 Find (i) ~~$\sqrt[4]{-7+24i}$~~ $\sqrt[3]{1+i}$

(ii) $\sqrt[4]{-4}$

Sol^n $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$

$k = 0, 2, \dots, n-1$

r is modulus of z

θ is principal value of ~~the~~ argument of z.

~~(i) $\sqrt[4]{-7+24i}$~~

~~Here $z = -7+24i$~~

~~$r = |z| = \sqrt{(-7)^2+(24)^2} = \sqrt{49+576} = \sqrt{625} = 25$~~

~~$\theta = Arg(z) = \pi + \tan^{-1}(y/x) = \pi + \tan^{-1}\left(\frac{24}{-7}\right)$~~

(i) $\sqrt[3]{1+i}$ Here $z = 1+i$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = Arg(z) = \tan^{-1}(y/x) = \tan^{-1}(1) = \pi/4$

$$\sqrt[3]{1+i} = (\sqrt{2})^{1/3}\left(\cos\frac{\pi/4+2k\pi}{3} + i\sin\frac{\pi/4+2k\pi}{3}\right)$$

$k = 0, 1, 2$

For $k=0$, $\sqrt[3]{1+i} = (\sqrt{2})^{1/3}\left(\cos \pi/12 + i\sin \pi/12\right)$

$k=1$, $\sqrt[3]{1+i} = (\sqrt{2})^{1/3}\left(\cos\frac{\pi/4+2\pi}{3} + i\sin\frac{\pi/4+2\pi}{3}\right)$

$= (\sqrt{2})^{1/3}\left(\cos\frac{9\pi}{12} + i\sin 9\pi/12\right)$

$k=2$, $\sqrt[3]{1+i} = (\sqrt{2})^{1/3}\left(\cos\frac{\pi/4+4\pi}{3} + i\sin\frac{\pi/4+4\pi}{3}\right)$

$= (\sqrt{2})^{1/3}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$

(ii) Similarly do for $\sqrt[4]{-4}$ | ~~Ans~~

Q-6 Show that an analytic function is independent of $\bar{z}$.

Sol Let $f(z) = u + iv$ is an analytic function

$z = x + iy$

We have $x = \frac{z+\bar{z}}{2}$ $y = \frac{z-\bar{z}}{2i}$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$= \frac{\partial f}{\partial x}\cdot\frac{1}{2} + \frac{\partial f}{\partial y}\cdot\frac{1}{2i} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y}\right]$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial f}{\partial x}\cdot\frac{1}{2} + \frac{\partial f}{\partial y}\cdot\left(-\frac{1}{2i}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right]$$

As $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$= \frac{1}{2}\left[\cancel{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}}^{\,0} + i\left(\cancel{\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}}^{\,0}\right)\right]$$

$$= 0$$

$\therefore \frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow f$ is independent of $\bar{z}$.

Q.2 Show that
$f(z) = |\operatorname{Re} z \operatorname{Im} z|^{\frac{1}{2}}$ satisfies the C.R. equation at the origin, but is not differentiable at origin.

Kajib
Friday Lab
GR-5

Sol^n $\quad z = x+iy$

$f(z) = u+iv \qquad \operatorname{Re} z = x$

$= |xy|^{\frac{1}{2}} \qquad \operatorname{Im} z = y$

$u(x,y) = |xy|^{\frac{1}{2}}, \quad v(x,y) = 0.$

$$\frac{\partial u}{\partial x}(0,0) = \lim_{h\to 0} \frac{u(0+h,0)-u(0,0)}{h} = \lim_{h\to 0}\frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

Similarly $\frac{\partial v}{\partial x}(0,0) = 0$

$\frac{\partial u}{\partial y}(0,0) = 0 \qquad \frac{\partial v}{\partial y}(0,0) = 0.$

So $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ at the origin.

$\Rightarrow$ C.R. equation are satisfied at the origin.

But $f(z)$ is not differentiable at $z=0$.

Reason

Putting into polar form $x = r\cos\theta, \; y = r\sin\theta$.

$$f(z) = |xy|^{\frac{1}{2}} = |r\cos\theta \cdot r\sin\theta|^{\frac{1}{2}}$$

$$= (r^2)^{\frac{1}{2}} |\cos\theta\sin\theta|^{\frac{1}{2}}$$

$$= r\frac{|\sin 2\theta|^{\frac{1}{2}}}{\sqrt{2}}$$

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = \lim_{z\to 0}\frac{\frac{\cancel{r}|\sin 2\theta|^{\frac{1}{2}}}{\sqrt{2}} - 0}{\cancel{r}(\cos\theta + i\sin\theta)} = \frac{e^{-i\theta}|\sin 2\theta|^{\frac{1}{2}}}{\sqrt{2}}$$

which is different for different values of θ.

So $\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}$ does not exist $\Rightarrow f$ is not differentiable at ~~z=0~~.

(8) Sketch the following sets in the complex plane and decide whether they are open, closed or a domain.

(i) $S = \{ z \mid |z-1| < 1 \text{ or } |z+1| < 1 \}$

(ii) $|\text{Arg } z| < \frac{\pi}{4}$

Sol^n (i) $|z-1| < 1$ $\quad (x-1)^2 + y^2 < 1$ centre at $(1,0)$ radius 1

$|z+1| < 1 \Rightarrow (x+1)^2 + y^2 < 1$ center $(-1,0)$ radius 1.

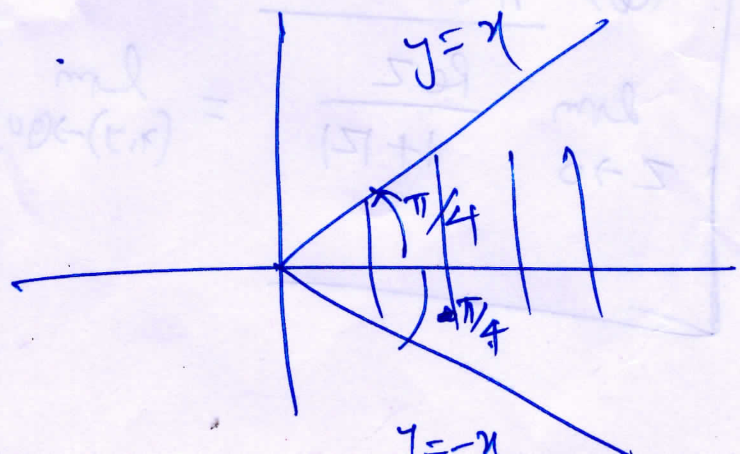
open and disconnected being union of two disjoint open discs.

So not a domain.

(ii) Note the Arg z is the principal argument, i.e. $-\pi < \text{Arg } z \leq \pi$.

except the boundary lines $y = \pm x$.

$\Rightarrow$ open, a domain



9. Show that the function

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

satisfies the Cauchy - Riemann equations at $z=0$ but $f'(0)$ doesn't exist.

Sol^n Writing $f(z) = u(x,y) + iv(x,y)$ we get

$$u(x,y) = \frac{x^3 - y^3}{x^2+y^2} \quad \text{and} \quad v(x,y) = \frac{x^3+y^3}{x^2+y^2}, \; (x,y) \neq (0,0)$$

Since $f(0) = 0$ we have $u(0,0) = v(0,0) = 0$.

Now as $z \to 0$ we obtain at the point $z=0$

$$u_x = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1,$$

$$u_y = \lim_{k\to 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k\to 0} \frac{-y}{y} = -1$$

$$v_x = \lim_{h\to 0} \frac{v(h,0) - v(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$$v_y = \lim_{k\to 0} \frac{v(0,k) - v(0,0)}{k} = \lim_{k\to 0} \frac{k}{k} = 1$$

Therefore at $z=0$, $u_x = v_y$ and $u_y = -v_x$.

Therefore the C-R equations are satisfied

We now have ④

$$f'(0) = \lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{z \to 0} \frac{x^3-y^3+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

$$= \lim_{z \to 0} \frac{(1+i)(x^3+iy^3)}{(x^2+y^2)(x+iy)} = \lim_{z \to 0} \frac{(1+i)(x^3+iy^3)(x-iy)}{(x^2+y^2)(x+iy)(x-iy)}$$

$$= \lim_{z \to 0} \frac{(1+i)(x^3+iy^3)(x-iy)}{(x^2+y^2)^2}$$

Choosing the path $y = mx$ we get

$$\lim_{x \to 0} \frac{(1+i)(1+im^3)(1-im)x^4}{(1+m^2)^2 x^4} = \frac{(1+i)(1+im^3)(1-im)}{(1+m^2)^2}$$

which depends on m. Therefore the limit doesn't exist. Hence $f'(0)$ doesn't exist.

⑩ Show that the derivative of a real valued function $f(z)$ of a complex variable z at any point is either zero or it doesn't exist.

Pf: Consider the quotient

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = F(\Delta z)$$

Since f is real valued the numerator of the quotient is real. Consequently $F(\Delta z) \to$ a real value if $\Delta z \to 0$ along the real axis $(\Delta y \to 0)$

Hence $\lim_{\Delta z \to 0} f(\Delta z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ (5)

doesn't exist or if it exist it must be zero.