SC216 Calculus with complex Variables — March 17, 2008

Power Series

$$\sum_{n=0}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$

— Series converges for $z=a$.

— If $\exists$ a +ve no. R s.t. series converges for $|z-a| < R$

$R \to$ Radius of convergence.

series diverges for $|z-a| > R$ & for $|z-a| = R$ it may or may not convergent.

Example

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots (-1)^{n-1}\frac{z^n}{n} + \cdots \qquad |z|<1$$

$\therefore R=1$ & $a=0$

$$(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}z^n + \cdots$$

$R=1$ $\qquad |z|<1$

Taylor's Theorem

Let $f(z)$ be analytic inside & on a simple closed curve c. Let a & $a+h$ be two points inside c, then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2 f''(a)}{2!} + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots$$

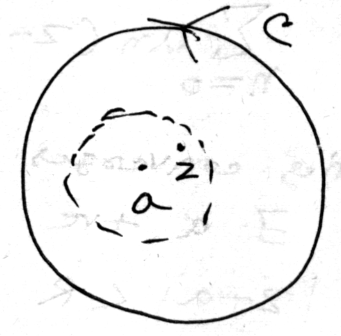
or writing $z = a+h$ & $h = z-a$

$$\Rightarrow f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

Taylor series

- Region of convergence of this is $|z-a|<R$.
- For $a=0$ it is called as Maclaurin series.

Proof:



Let z be any pt. inside C.
Construct a circle C_1 with center at a & enclosing z. then by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\,dw \qquad ---(1)$$

Now
$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)}\left\{\frac{1}{1-\frac{(z-a)}{(w-a)}}\right\}$$

$$= \frac{1}{w-a}\left\{1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\dots+\left(\frac{z-a}{w-a}\right)^{n-1} + \left(\frac{z-a}{w-a}\right)^n\frac{1}{1-\frac{z-a}{w-a}}\right\}$$

$\Rightarrow$

$$\frac{1}{w-z} = \frac{1}{w-a}+\frac{z-a}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+\dots+\frac{(z-a)^{n-1}}{(w-a)^n} + \left(\frac{z-a}{w-a}\right)^n\frac{1}{w-z} \qquad ---(2)$$

Multiplying (2) by $f(w)$ & using (1) we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw + \dots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw + U_n \qquad ---(3)$$

where $U_n = \frac{1}{2\pi i} \oint_{C_1} \left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z} dw$

Now using cauchy's integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw \quad n = 0, 1, 2, 3 \ldots$$

(3) becomes

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

$$+ \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + U_n$$

$\therefore$ Now we only want to show that as $n \to \infty$ $U_n \to 0$.

To do this we note that $\because$ w is on C_1

we have $\left|\frac{z-a}{w-a}\right| = r < 1$ $\quad$ r is a const.

Also, we have $|f(w)| < M$ $\quad$ M is a const.

& $|w-z| = |(w-a) - (z-a)| \geq r_1 - |z-a|$

where r_1 is radius of C_1

$\Rightarrow$ $$|U_n| = \frac{1}{2\pi} \left| \oint_{C_1} \left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{(w-z)} dw \right|$$

$$\leq \frac{1}{2\pi} \frac{r^n M}{r_1 - |z-a|} \cdot 2\pi r_1 = \frac{r^n M . r_1}{r_1 - |z-a|}$$

$$\longrightarrow 0 \text{ as } n \to \infty$$

$(\because r < 1)$

Laurent's theorem

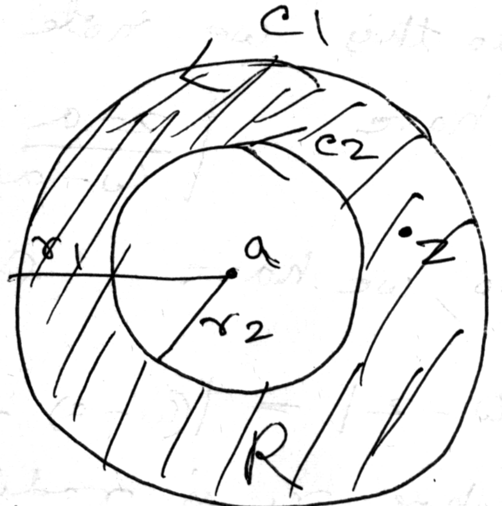
If $f(z)$ is analytic inside and on the boundary of the ring-shaped region R bounded by two concentric circles C_1 & C_2 with center at a & respective radii r_1 & r_2 $(r_1 > r_2)$ then $\forall z \in R$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

where
$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)}{(\omega-a)^{n+1}} d\omega \ ; \ n = 0,1,2,\ldots$$

$$a_{-n} = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\omega)}{(\omega-a)^{-n+1}} d\omega \ ; \ n = 1,2,3,\ldots$$

□ By Cauchy integral formula



$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)}{\omega - z} d\omega - \frac{1}{2\pi i} \oint_{C_2} \frac{f(\omega)}{\omega - z} d\omega$$

Now the proof is similar to Taylor's series proof by expanding $\frac{1}{\omega - z}$ etc. ▨

— The part $a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$ is called the analytic part & the part with inverse power of $(z-a)$ is called principal part.

$\boxed{\text{Example}}$ $f(z) = \ln(1+z)$

Expand $f(z)$ in Taylor series about $z=0$

□ $f(z) = \ln(1+z) = f(0) + f'(0)z + \dfrac{f''(0)z^2}{2!} + \dots$

$= z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + \dots$

<u>Alternative method:</u>

$\because$ for $|z|<1$, $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 + \dots$

integrating from 0 to z yields

$\ln(1+z) = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + \dots$

$\therefore R = 1$ ($\because |z|<1$)

Classification of Singularities

- **Poles** If Laurent's expansion of $f(z)$ has finite no. of terms having negative powers of $(z-a)$. Eg.

if $f(z)$ has

$$\frac{a_1}{(z-a)} + \frac{a_2}{(z-a)^2} + \dots + \frac{a_n}{(z-a)^n}$$

$a_n \neq 0$

$\Rightarrow$ $z=a$ is a pole of order n.

- **Removable Singularity**

If $f(z)$ is not defined at $z=a$ but $\lim_{z \to a} f(z)$ exists $z=a$ called removable singularity

$\boxed{\text{Example}}$ Find Laurent series about $z=1$ and name the singularity

$$\frac{e^{2z}}{(z-1)^3}$$

~~let z=1+5~~

□ Let $z-1=u \Rightarrow z=1+u$

$$\Rightarrow \frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2}{u^3}\cdot e^{2u}$$

$$= \frac{e^2}{u^3}\left\{1+2u+\frac{(2u)^2}{2!}+\frac{(2u)^3}{3!}+\cdots\right\}$$

$$= \frac{e^2}{(z-1)^3}+\frac{2e^2}{(z-1)^2}+\frac{2e^2}{(z-1)}+\frac{4e^2}{3}+\frac{2e^2}{3}(z-1)+\cdots$$

$\Rightarrow z=1$ is a pole of order 3.

$\boxed{\text{Residues}}$ $f(z)$ single valued & analytic inside & on C except at $z=a$

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-a)^n$$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \quad , n = 0, \pm 1, \pm 2, \ldots$$

For $n=-1$

$$\oint_C f(z)\,dz = 2\pi i\, a_{-1}$$

$$\boxed{a_{-1} = \text{Residue of } f(z) \text{ at } z=a}$$

$\boxed{\text{Calculation of Residues}}$

$$a_{-1} = \lim_{z\to a} \frac{1}{(k-1)!}\cdot \frac{d^{k-1}}{dz^{k-1}}\left\{(z-a)^k f(z)\right\}$$

if $z=a$ is a pole of order k.

$\square$ $\because$

$$f(z) = \frac{a_{-k}}{(z-a)^k} + \frac{a_{-k+1}}{(z-a)^{k-1}} + \cdots + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + \cdots \quad \text{--- (1)}$$

as $z=a$ is a pole of order k

Multiply both sides by $(z-a)^k$ & then differentiating both sides of (1) $(k-1)$ times we get

$$\frac{d^{k-1}}{dz^{k-1}}\left\{(z-a)^k f(z)\right\} = (k-1)!\, a_{-1} + k(k-1)\ldots 2\, a_0 (z-a) + \ldots$$

let $z\to a$

$$\lim_{z\to a} \frac{d^{k-1}}{dz^{k-1}}\left\{(z-a)^k f(z)\right\} = (k-1)!\, a_{-1}$$

Example

$$f(z) = \frac{z}{(z-1)(z+1)^2}$$

$z=1$ is a simple pole

$\therefore$ Residue at $z=1$ is

$$\lim_{z\to 1} (z-1)\left\{\frac{z}{(z-1)(z+1)^2}\right\} = \frac{1}{4}$$

$z=-1$ is a pole of order 2

$\Rightarrow$ Residue at $z=-1$ is

$$\lim_{z\to -1} \frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2\left(\frac{z}{(z-1)(z+1)^2}\right)\right\} = -\frac{1}{4}$$

If $z=a$ is an essential singularity the residue can be found by using known series expansions.

Example $f(z) = e^{-1/z}$.

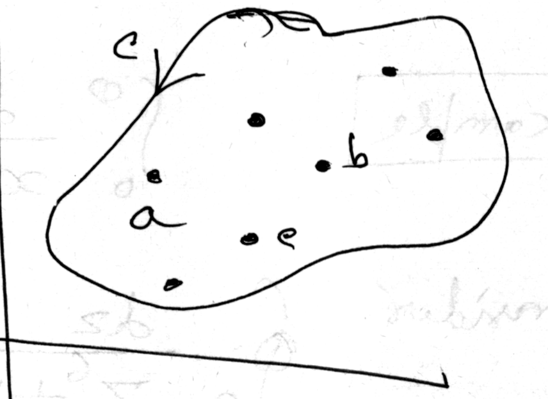
$z=0$ is essential singularity

& $\because$ $e^{-1/z} = 1 - \frac{1}{z} + \frac{1}{2!\,z^2} - \frac{1}{3!\,z^3} + \cdots$

$\therefore$ coeff of $\left(\frac{1}{z}\right) = -1 =$ Residue at $z=0$

THE RESIDUE THEOREM

Let $f(z)$ be single-valued and analytic inside & on a simple closed curve c except at the singularities $a, b, c, \ldots$ inside c which have residues given by $a_{-1}, b_{-1}, c_{-1}, \ldots$ then the residue theorem states that

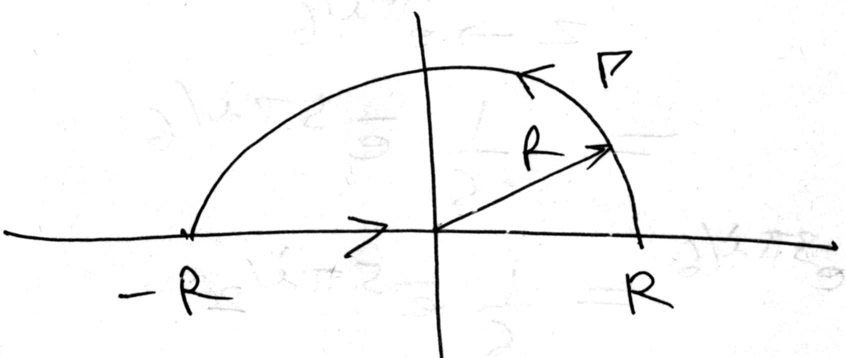


$$\oint_c f(z)\, dz = 2\pi i \left(a_{-1} + b_{-1} + c_{-1} + \cdots\right)$$

Evaluation of Definite Integrals

① $\int_{-\infty}^{+\infty} F(x)\, dx$, $F(x)$ is a rational fn.

Consider $\oint_c F(z)\, dz$ along a contour c



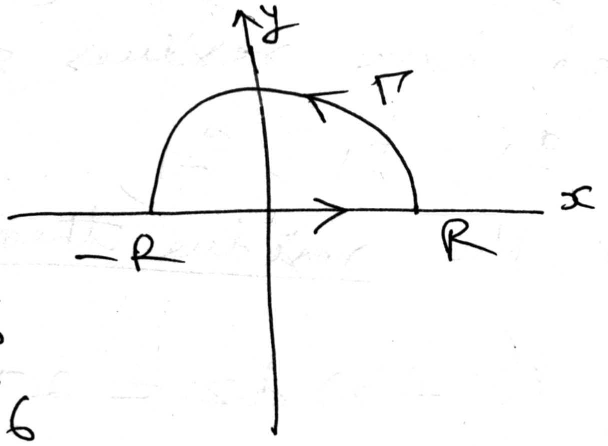
consisting of the line along the x-axis from $-R$ to R & the semicircle Γ above the x-axis having the line as diameter. then let $R \to \infty$

- If $F(x)$ is an even fn. this can be ~~used~~ used to evaluate $\int_0^\infty F(x)\,dx$.

$\boxed{\text{Example}}$ $\int_0^\infty \frac{dx}{x^6+1}$

□

Consider $\oint_C \frac{dz}{z^6+1}$ where C as shown



$\because z^6+1=0$ gives

$z = e^{\pi i/6}$ $\quad z=e^{7\pi i/6}$

$z = e^{3\pi i/6}$ $\quad z=e^{9\pi i/6}$

$z = e^{5\pi i/6}$ $\quad z=e^{11\pi i/6}$

as simple poles but only the poles $e^{\pi i/6}$, $e^{3\pi i/6}$ & $e^{5\pi i/6}$ lie within C

$\Rightarrow$ Residue at $e^{\pi i/6} = \lim_{z \to e^{\pi i/6}} \left\{ (z - e^{\pi i/6}) \frac{1}{z^6+1} \right\}$

$= \frac{1}{6} e^{-5\pi i/6}$

Residue at $e^{3\pi i/6} = \frac{1}{6} e^{-5\pi i/2}$

Residue at $e^{5\pi i/6} = \frac{1}{6} e^{-25\pi i/6}$

$$\Rightarrow \oint_C \frac{dz}{z^6+1} = 2\pi i \left\{ \frac{1}{6} e^{-5\pi i/6} + \frac{1}{6} e^{-5\pi i/2} + \frac{1}{6} e^{-25\pi i/6} \right\}$$

(By Residue theorem) $= \frac{2\pi}{3}$

i.e.,

$$\int_{-R}^{R} \frac{dx}{x^6+1} + \int_{\Gamma} \frac{dz}{z^6+1} = \frac{2\pi}{3}$$

$\longrightarrow 0$

as $R \to \infty$ it can be shown that $\int_{\Gamma} \frac{dz}{z^6+1} \to 0$

& we get $\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}$

$$\Rightarrow 2 \cdot \int_0^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$$

(2) $\int_0^{2\pi} G(\sin\theta, \cos\theta)$, where $G(\sin\theta, \cos\theta)$ is a rational fn. of $\sin\theta$ & $\cos\theta$

Let $z = e^{i\theta}$ & $\sin\theta = \frac{z - z^{-1}}{2i}$

$\cos\theta = \frac{z + z^{-1}}{2}$

& $dz = i e^{i\theta} d\theta$

$$\oint_c F(z)\,dz \quad \text{where } c \text{ is given as}$$

Example

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$$

Let $z = e^{i\theta}$, $\sin\theta = \dfrac{z-z^{-1}}{2i}$

$$\cos\theta = \frac{z+z^{-1}}{2}$$

$$dz = iz\,d\theta$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_c \frac{2\,dz}{(1-2i)z^2+6iz-1-2i}$$

Now poles of integrand are simple poles $z = 2-i$

$$\frac{2-i}{5}$$

Only $\dfrac{2-i}{5}$ lies inside C

& Residue at $\dfrac{2-i}{5} = \dfrac{1}{2i}$

$$\Rightarrow \oint_c \frac{2\,dz}{(1-2i)z^2+6iz-1-2i} = 2\pi i\left(\frac{1}{2i}\right)$$

$$= \pi$$

(3) $\displaystyle\int_{-\infty}^{+\infty} F(x) \left\{ \begin{matrix} \cos mx \\ \sin mx \end{matrix} \right\}$ $F(x)$ is a rational fn.

Here we consider $\displaystyle\oint_C F(z)\, e^{imz}\, dz$

$\boxed{\text{Example}}$ $\displaystyle\int_0^{\infty} \frac{\cos mx}{x^2+1}\, dx = \frac{\pi}{2} e^{-m}$ $(m>0)$

$\square$ $\displaystyle\oint_C \frac{e^{imz}}{z^2+1}\, dz$

The integrand has poles at $z = \pm i$
& only $z = i$ lies inside C

Residue at $z = i$ = $\dfrac{e^{-m}}{2i}$

$$\oint_C \frac{e^{imz}}{z^2+1} = 2\pi i \left(\frac{e^{-m}}{2i} \right) = \pi e^{-m}$$

$$\int_{-R}^{R} \frac{e^{imx}}{x^2+1}\, dx + \int_{\Gamma} \frac{e^{imz}}{z^2+1}\, dz = \pi e^{-m}$$

$\to 0$

$$2\int_0^{R} \frac{\cos mx}{x^2+1}\, dx = \pi e^{-m}$$

as $R \to \infty$

SUMMATION OF SERIES

1. $\sum_{-\infty}^{+\infty} f(n) = -\{$ Sum of Residues of $\pi \cot \pi z\, f(z)$ at all the poles of $f(z)\}$

2. $\sum_{-\infty}^{+\infty} (-1)^n f(n) = -\{$ sum of residues of $\pi\ \cancel{\cot}\ \csc \pi z\, f(z)$ at all the poles of $f(z)\}$

3. $\sum_{-\infty}^{+\infty} f\left(\frac{2n+1}{2}\right) = \{$ Sum of residues of $\pi \tan \pi z\, f(z)$ at all the poles of $f(z)\}$

4. $\sum_{-\infty}^{+\infty} (-1)^n f\left(\frac{2n+1}{2}\right) = \{$ Sum of residues of $\pi \sec \pi z\, f(z)$ at all the poles of $f(z)\}$

Example

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

□ $$F(z) = \frac{\pi \cot \pi z}{z^2} = \frac{\pi \cos \pi z}{z^2 \sin \pi z}$$

$$\Rightarrow F(z) = \frac{\left(1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - \cdots\right)}{z^3\left(1 - \frac{\pi^2 z^2}{3!} + \frac{\pi^4 z^4}{5!} - \cdots\right)}$$

$$= \frac{1}{z^3}\left(1 - \frac{\pi^2 z^2}{2!} + \cdots\right)\left(1 + \frac{\pi^2 z^2}{3!} + \cdots\right)$$

$$= \frac{1}{z^3}\left(1 - \frac{\pi^2 z^2}{3} + \cdots\right)$$

$\Rightarrow$ Residue at $z = 0$ is $-\frac{\pi^2}{3}$.

$\Rightarrow$

$$\oint_{C_N} \frac{\pi \cot \pi z}{z^2}\, dz = \sum_{n=-N}^{-1} \frac{1}{n^2} + \sum_{n=1}^{N} \frac{1}{n^2} - \frac{\pi^2}{3}$$

$$= 2\sum_{n=1}^{N} \frac{1}{n^2} - \frac{\pi^2}{3}$$

as $N \to \infty$ $\quad$ L.H.S. $\to 0$

$$\Rightarrow \quad 2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$\blacksquare$