SYSTEMS OF EQUATIONS

Solution of Systems of Linear Equations

1/. Direct method – Gaussian elimination
2/. Iterative methods – Jacobi/Gauss-Seidel methods.

Systems of Linear Equations

Example: $\boxed{ax + by = c}$ and $\boxed{dx + ey = f}$, give a simple system of order 2.

A general system of order n can be written as

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{array}}$$

In the coefficients a_{ij}, $i \rightarrow$ equation number and $j \rightarrow$ variable number.

Defining in terms of matrix and vectors

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \vec{b} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

We can write the system as $\boxed{\tilde{A}\vec{x} = \vec{b}}$.

$\tilde{A} \rightarrow$ matrix $\quad \vec{x}, \vec{b} \rightarrow$ vectors.

Matrix Arithmetic

Given a matrix

$$\tilde{A} \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Order is $m \times n$

$m \to$ No. of rows

$n \to$ No. of Columns

its transpose is

$$\tilde{A}^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & a_{22} & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}$$

obtained by exchanging the row and column elements, rotating about the diagonal element

Example: $\tilde{A} \equiv \begin{bmatrix} a & b & 1 \\ 2 & c & d \end{bmatrix}$ $\tilde{A}^T = \begin{bmatrix} a & 2 \\ b & c \\ 1 & d \end{bmatrix}$

Equality of Matrices: Two matrices are equal if they have the same order and if their corresponding elements are equal.

Arithmetic Operations

1/. Multiplication by a scalar:

$$\tilde{A} \equiv \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} \Rightarrow b\tilde{A} = \begin{bmatrix} b a_{11} & \cdots & b a_{1n} \\ \vdots & & \vdots \\ b a_{m1} & \cdots & b a_{mn} \end{bmatrix}$$

b is a scalar number. When $b = -1$,

$b\tilde{A} = -\tilde{A} \to$ negative of $\tilde{A}$. Note: The Scalar b is an arbitrary real number.

2/. Summation of two matrices :

If $\tilde{A}$ and $\tilde{B}$ are two matrices of the same ~~odd~~ order $(m \times n)$, then,

$$\tilde{A} + \tilde{B} = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1m} + b_{1m} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

Each element of the new matrix = $\boxed{a_{ij} + b_{ij}}$

The zero matrix $\tilde{O} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ for any order $(m \times n)$.

↳ (all elements are zero)

$$\Rightarrow \boxed{\tilde{O} + \tilde{A} = \tilde{A} + \tilde{O} = \tilde{A}}$$

The difference of two matrices is defined as

$$\boxed{\tilde{A} - \tilde{B} = \tilde{A} + (-1)\tilde{B}} \Rightarrow \boxed{\tilde{A} + (-1)\tilde{A} = \tilde{O}}.$$

3/. Multiplication of two matrices :

If $\tilde{A}$ is a matrix of order $m \times n$ and $\tilde{B}$ is a matrix of order $n \times p$, then $\boxed{\tilde{C} = \tilde{A}\tilde{B}}$ has the order $m \times p$, in which,

$$\boxed{C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}} \quad \text{with} \quad \boxed{1 \le i \le m} \text{ and } \boxed{1 \le j \le p}.$$

If $\tilde{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ then $\tilde{A}\vec{x}$

(order $m \times n$) (Order $n \times 1$)

$$= \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$$

[order $m \times 1$]

Hence, a <u>system of linear equations</u>

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n \end{array}}$$

- Can be written in terms of <u>matrices and vectors</u>

$$\tilde{A} \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \vec{x} \equiv \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{b} \equiv \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

as $\boxed{\vec{A}\vec{x} = \vec{b}}$ (in a <u>concise form</u>)

<u>Row Operations</u>

1/. Interchange two rows.
2/. Multiply a row by a non-zero scalar.
3/. Add multiple of one row to another.

<u>Example</u>: $\tilde{A} \equiv \begin{bmatrix} 3 & 3 & 3 & 1 \\ 2 & 2 & 3 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$ a <u>3 x 4 matrix</u>

i.) <u>(row 2) x (-1)</u> + <u>(row 1)</u> and <u>(row 3) x (-1) + (row 2)</u> give

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix} \longrightarrow$$

ii.) <u>(row 2) x (-1)</u> <u>+ row 1</u> and also <u>+ row 3</u>

$$\longrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

(P.T.O.)

iii.) (Row 1) × (−2) + (Row 2) → $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

iv.) Interchange (Row 1) and (Row 2) → $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

v.) Multiply (Row 3) by (1/3) → $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$

The Matrix Inverse

For any real number x, $\boxed{1 \cdot x = x \cdot 1 = x}$.

With matrix multiplication, the scalar 1 is replaced by the identity matrix. $\tilde{I}$.

If $n \geq 1$, then $\tilde{I}_n \equiv \begin{bmatrix} 1 & 0 & \cdots \\ \vdots & 1 & \cdots \\ 0 & \vdots & \\ 0 & 0 & \cdots 1 \end{bmatrix}$, which is a square matrix of order n, with elements given by the formula $\boxed{\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}}$ → The Kronecker Delta function

For any matrix $\tilde{A}$ of order $m \times n$, we set $\boxed{\tilde{A} = \tilde{I}_m \tilde{A}}$ or $\boxed{\tilde{A} = \tilde{A}\tilde{I}_n}$.

Now if $\tilde{A}$ and $\tilde{B}$ are square matrices, and $\boxed{\tilde{A}\tilde{B} = \tilde{I}}$, $\boxed{\tilde{B}\tilde{A} = \tilde{I}}$, then $\tilde{B}$ is the inverse of $\tilde{A}$.

Also $\tilde{A}$ can have exactly one inverse.

The inverse of $\tilde{A}$ is $\tilde{A}^{-1}$. The inverse of a matrix exits if $\boxed{\det(\tilde{A}) \neq 0}$ (a non-singular matrix). If $\boxed{\det(\tilde{A}) = 0}$ then $\tilde{A}$ is a singular matrix.

$$\boxed{\tilde{A}^{-1} = \frac{1}{|\tilde{A}|} \tilde{A}^{CT}}$$

$|\tilde{A}| \to$ determinant of $\tilde{A}$.

$\tilde{A}^{C} \to$ Cofactor of $\tilde{A}$

$\tilde{A}^{CT} \to$ Transpose of Cofactor.

Example: $\tilde{A} \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\tilde{A}^{C} \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$A_{11} = (-1)^{1+1} d = d$, $A_{12} = (-1)^{1+2} c = -c$.

$A_{21} = (-1)^{2+1} b = -b$, $A_{22} = (-1)^{2+2} a = a$

Hence, $\tilde{A}^{C} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ $\therefore$ $\boxed{\tilde{A}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$

Example: $\tilde{A} \equiv \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \boxed{\begin{aligned} |\tilde{A}| &= 2(4-1) \\ &- 1(2-0) + 0 \\ &= 6 - 2 = 4 \end{aligned}}$

The cofactor matrix is

$\tilde{A}^{C} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ Hence, $A_{11} = (-1)^{1+1}(4-1) = 3$,

$A_{12} = (-1)^{1+2}(2-0) = -2$,

$A_{13} = (-1)^{1+3}(1-0) = 1$,

$A_{21} = (-1)^{2+1}(2-0) = -2$, $A_{22} = (-1)^{2+2}(4-0) = 4$

$A_{23} = (-1)^{2+3}(2-0) = -2$, $A_{31} = (-1)^{3+1}(1-0) = 1$

$A_{32} = (-1)^{3+2}(2-0) = -2$, $A_{33} = (-1)^{3+3}(4-1) = 3$ (P.T.O.)

$$\Rightarrow \tilde{A}^C = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{\tilde{A}^{CT}}{|\tilde{A}|} = \frac{1}{4}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

Rules of Matrix Algebra

Associative Laws:

1/. $(\tilde{A}+\tilde{B})+c = \tilde{A}+(\tilde{B}+\tilde{C})$, 2/. $(\tilde{A}\tilde{B})\tilde{C} = \tilde{A}(\tilde{B}\tilde{C})$

The Commutative Law: 3/. $\tilde{A}+\tilde{B} = \tilde{B}+\tilde{A}$

The Distributive Laws:

4/. $\tilde{A}(\tilde{B}+\tilde{C}) = \tilde{A}\tilde{B}+\tilde{A}\tilde{C}$, 5/. $(\tilde{A}+\tilde{B})\tilde{C} = \tilde{A}\tilde{C}+\tilde{B}\tilde{C}$

6/. $(\tilde{A}\tilde{B})^T = \tilde{B}^T\tilde{A}^T$, 7/. $(\tilde{A}+\tilde{B})^T = \tilde{A}^T+\tilde{B}^T$

8/. $(c\tilde{A})^{-1} = \frac{1}{c}\tilde{A}^{-1}$, 9/. $(\tilde{A}\tilde{B})^{-1} = \tilde{B}^{-1}\tilde{A}^{-1}$

[$c \rightarrow$ non-zero constant]

10/. $\det(\tilde{A}\tilde{B}) = \det(\tilde{A})\det(\tilde{B})$, 11/. $\det(\tilde{A}^T) = \det(\tilde{A})$

12/. $|c\tilde{A}| = c^n|\tilde{A}|$, $n = \text{order}(\tilde{A})$.

Solvability of Linear Systems

Given $\boxed{\tilde{A}\vec{x} = \vec{b}}$, $\boxed{\vec{x} = \tilde{A}^{-1}\vec{b}}$. A solution exists only if $|\tilde{A}| \neq 0$. Further, $|\tilde{A}|$ is a square matrix, and $\boxed{\tilde{A}\vec{x} = \vec{b}}$ has a unique solution. This is implicitly assumed when finding determinants/inverses.

Gaussian Elimination

Example:

$$\boxed{\begin{array}{lr} x_1 + 2x_2 + x_3 = 0 & \text{——— } (E1) \\ 2x_1 + 2x_2 + 3x_3 = 3 & \text{——— } (E2) \\ -x_1 - 3x_2 + 0x_3 = 2 & \text{——— } (E3) \end{array}}$$

i.) $(E2) - 2(E1)$ and $(E3) - (-1)(E1)$ will eliminate x_1 from $(E2)$ and $(E3)$.

Hence. (Elimination step starts)

$$\boxed{\begin{array}{lr} x_1 + 2x_2 + x_3 = 0 & \text{——— } (E1) \\ \quad -2x_2 + x_3 = 3 & \text{——— } (E2) \\ \quad -x_2 + x_3 = 2 & \text{——— } (E3) \end{array}}$$

ii.) $(E3) - \left(\frac{1}{2}\right)(E2)$ will eliminate x_2 from $(E3)$.

Hence, (Elimination ends)

$$\boxed{\begin{array}{lr} x_1 + 2x_2 + x_3 = 0 & \text{——— } (E1) \\ \quad -2x_2 + x_3 = 3 & \text{——— } (E2) \\ \quad\quad +\frac{x_3}{2} = \frac{1}{2} & \text{——— } (E3) \end{array}}$$

Upper triangular system of linear equations

$\Rightarrow \boxed{x_3 = 1}$, $\boxed{-2x_2 + 1 = 3}$ $\Rightarrow \boxed{x_2 = -1}$ and

$\boxed{x_1 - 2 + 1 = 0}$ $\Rightarrow \boxed{x_1 = 1}$ (x_1 and x_2 are obtained by back substitution).

The principle of Gaussian elimination:

i.) Elimination steps.

ii.) Obtain upper triangular system of linear equations.

iii.) Back substitution.

(Works for several hundred equations).

Augmented Matrix

The previous example can be solved by an augmented matrix $[\tilde{A}|b]$.

$$\therefore [\tilde{A}|b] \equiv \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 2 & 3 & 3 \\ -1 & -3 & 0 & 2 \end{array}\right]$$

Following the elimination steps we get

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right] \text{ and then } \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 1/2 & 1/2 \end{array}\right]$$

This is the equivalent of

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1/2 \end{bmatrix}$$

whose solution is $x_1 = 1$, $x_2 = -1$, $x_3 = 1$.

Gaussian Elimination with General Coefficients

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \quad \text{——} (E1) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \quad \text{——} (E2) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \quad \text{——} (E3) \end{aligned}}$$

Order $\boxed{n = 3}$

$\boxed{a_{11} \neq 0}$. Define $\boxed{m_{21} = \frac{a_{21}}{a_{11}}}$ and $\boxed{m_{31} = \frac{a_{31}}{a_{11}}}$.

The first subscript is the equation number and the second subscript is the variable number.

Apply $\boxed{(\mathcal{E}2) - m_{21} \times (\mathcal{E}_1)}$ and $\boxed{(\mathcal{E}3) - m_{31} \times (\mathcal{E}_1)}$ to eliminate x_1 from $(\mathcal{E}2)$ ~~&~~ $(\mathcal{E}3)$

$$\Rightarrow \boxed{\begin{array}{rl} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1 & \text{———} (\mathcal{E}1) \\ a_{22}^{(2)} x_2 + a_{23}^{(2)} x_3 = b_2^{(2)} & \text{———} (\mathcal{E}2) \\ a_{32}^{(2)} x_2 + a_{33}^{(2)} x_3 = b_3^{(2)} & \text{———} (\mathcal{E}3) \end{array}}$$

$$\boxed{\begin{array}{ll} a_{22}^{(2)} = a_{22} - m_{21} a_{12}, & a_{23}^{(2)} = a_{23} - m_{21} a_{13} \\ a_{32}^{(2)} = a_{32} - m_{31} a_{12}, & a_{33}^{(2)} = a_{33} - m_{31} a_{13} \\ b_2^{(2)} = b_2 - m_{21} b_1, & b_3^{(2)} = b_3 - m_{31} b_1 \end{array}}$$

$\boxed{a_{22}^{(2)} \neq 0}$. Define $\boxed{m_{32} = \dfrac{a_{32}^{(2)}}{a_{22}^{(2)}}}$. To eliminate x_2 from $(\mathcal{E}3)$ apply $\boxed{(\mathcal{E}3) - m_{32} \times (\mathcal{E}2)}$, to get,

$$\boxed{\begin{array}{rl} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b & \text{———} (\mathcal{E}1) \\ a_{22}^{(2)} x_2 + a_{23}^{(2)} x_3 = b_2^{(2)} & \text{———} (\mathcal{E}2) \\ a_{33}^{(3)} x_3 = b_3^{(3)} & \text{———} (\mathcal{E}3) \end{array}}$$

$$\boxed{a_{33}^{(3)} = a_{33}^{(2)} - m_{32} a_{23}^{(2)}, \quad b_3^{(3)} = b_3^{(2)} - m_{32} b_2^{(2)}}$$

By back substitution we successively get

$\boxed{x_3 = \dfrac{b_3^{(3)}}{a_{33}^{(3)}}}$, $\boxed{x_2 = \dfrac{b_2^{(2)} - a_{23}^{(2)} x_3}{a_{22}^{(2)}}}$ and

$\boxed{x_1 = \dfrac{b_1 - a_{12} x_2 - a_{13} x_3}{a_{11}}}$

This method of solving $n = 3$ system can be extended for any n.

A General n-Order Linear System

$$\boxed{\begin{array}{l} a_{11}^{(1)} x_1 + \cdots + a_{1n}^{(1)} x_n = b_1^{(1)} \text{ ——— } (\mathcal{E}1) \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{n1}^{(1)} x_1 + \cdots + a_{nn}^{(1)} x_n = b_n^{(1)} \text{ ——— } (\mathcal{E}n) \end{array}}$$

For $\boxed{k = 1, 2, \ldots, n-1}$, carry out elimination.
Eliminate x_k from $(\mathcal{E}(k+1))$ through $(\mathcal{E}n)$.
The preceding steps from $1, \ldots, k-1$ will give a system of the form

$$\boxed{\begin{array}{l} a_{11}^{(1)} x_1 + a_{12}^{(1)} x_2 + \cdots \qquad + a_{1n}^{(1)} x_n = b_1^{(1)} \text{ — } (\mathcal{E}1) \\ \qquad a_{22}^{(2)} x_2 + \cdots \qquad + a_{2n}^{(2)} x_n = b_2^{(2)} \text{ — } (\mathcal{E}2) \\ \qquad\qquad \ddots \qquad\qquad \vdots \\ \qquad\qquad a_{kk}^{(k)} x_k + \cdots + a_{kn}^{(k)} x_n = b_k^{(k)} \text{ — } (\mathcal{E}3) \\ \qquad\qquad \vdots \qquad\qquad \vdots \\ \qquad\qquad a_{nk}^{(k)} x_k + \cdots + a_{nn}^{(k)} x_n = b_n^{(k)} \text{ — } (\mathcal{E}n) \end{array}}$$

For $\boxed{a_{kk}^{(k)} \neq 0}$ define $\boxed{m_{ik} = \dfrac{a_{ik}^{(k)}}{a_{kk}^{(k)}}}$, $i = k+1, \ldots, n$

For equations $i = k+1, \ldots, n$ subtract $m_{ik} \times (\mathcal{E}k)$ from $(\mathcal{E}i)$ to eliminate x_k from $(\mathcal{E}i)$

(P.T.O.)

$\Rightarrow$ $\boxed{\mathcal{E}i - m_{ik}(\mathcal{E}k)}$ to eliminate x_k from $(\mathcal{E}i)$.

Hence, $$\boxed{\begin{aligned} a_{ij}^{(k+1)} &= a_{ij}^{(k)} - m_{ik}\, a_{kj}^{(k)} \\ b_i^{(k+1)} &= b_i^{(k)} - m_{ik}\, b_k^{(k)} \end{aligned}}$$

$\rightarrow$ $i, j = k+1, \ldots, n$

$\rightarrow$ $i = k+1, \ldots, n$

For all $n-1$ steps, the upper triangular linear system will be formed.

$$\boxed{\begin{aligned} u_{11}x_1 + \cdots + u_{1n}x_n &= g_1 \\ + u_{22}x_2 + \cdots + u_{2n}x_n &= g_2 \\ &\vdots \\ u_{nn}x_n &= g_n \end{aligned}}$$

$\rightarrow$ The upper triangular system of linear equations after all elimination steps.

Here, $\boxed{u_{ij} = a_{ij}^{(i)}}$, $\boxed{g_i = b_i^{(i)}}$.

Solving successively and back substituting for $x_n, x_{n-1}, \ldots, x_1$ will give

$$\boxed{\begin{aligned} x_n &= g_n / u_{nn} \\ &\vdots \\ x_i &= \frac{g_i - \sum_{j=i+1}^{n} u_{ij}\, x_j}{u_{ii}} \end{aligned}}$$

$i = n-1, \ldots, 1$

This provides the full solution of the n-order system by Gaussian elimination.

Example:

A system with $n=4$

$$\boxed{\begin{aligned} 4x_1 + 3x_2 + 2x_3 + x_4 &= 1 \\ 3x_1 + 4x_2 + 3x_3 + 2x_4 &= 1 \\ 2x_1 + 3x_2 + 4x_3 + 3x_4 &= -1 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= -1 \end{aligned}}$$

Writing an augmented matrix

$$\left[\begin{array}{cccc|c} 4 & 3 & 2 & 1 & 1 \\ 3 & 4 & 3 & 2 & 1 \\ 2 & 3 & 4 & 3 & -1 \\ 1 & 2 & 3 & 4 & -1 \end{array}\right] \xrightarrow[\substack{m_{31} = 1/2 \\ m_{41} = \frac{1}{4}}]{m_{21} = 3/4} \left[\begin{array}{cccc|c} 4 & 3 & 2 & 1 & 1 \\ 0 & \frac{7}{4} & \frac{3}{2} & \frac{5}{4} & \frac{1}{4} \\ 0 & \frac{3}{2} & 3 & \frac{5}{2} & -\frac{3}{2} \\ 0 & \frac{5}{4} & \frac{5}{2} & \frac{15}{4} & -\frac{5}{4} \end{array}\right]$$

From the second table we get $\boxed{\begin{aligned} m_{32} &= \tfrac{3}{2} \times \tfrac{4}{7} = 6/7 \\ \text{and } m_{42} &= \tfrac{5}{4} \times \tfrac{4}{7} = 5/7 \end{aligned}}$

This gives,

$$\left[\begin{array}{cccc|c} 4 & 3 & 2 & 1 & 1 \\ 0 & 7/4 & 3/2 & 5/4 & 1/4 \\ 0 & 0 & 12/7 & 10/7 & -12/7 \\ 0 & 0 & 10/7 & 20/7 & -10/7 \end{array}\right] \xrightarrow[\substack{m_{43} \\ = \frac{10}{7} \times \frac{7}{12} \\ = 5/6}]{} \left[\begin{array}{cccc|c} 4 & 3 & 2 & 1 & 1 \\ 0 & 7/4 & 3/2 & 5/4 & 1/4 \\ 0 & 0 & 12/7 & 10/7 & -12/7 \\ 0 & 0 & 0 & 5/3 & 0 \end{array}\right]$$

Hence, $\frac{5}{3} x_4 = 0 \Rightarrow \boxed{x_4 = 0} \Rightarrow \frac{12}{7} x_3 + \frac{10}{7} \cdot 0 = -\frac{12}{7}$

$\Rightarrow \boxed{x_3 = -1}$. Next, $\frac{7}{4} x_2 + \frac{3}{2} \times -1 + \frac{5}{4} \times 0 = \frac{1}{4}$

$\Rightarrow \frac{7}{4} x_2 = \frac{7}{4} \Rightarrow \boxed{x_2 = 1}$. Next, $4x_1 + 3.1 + 2.-1 + 1.0 = 1$

$\Rightarrow 4x_1 + 1 = 1 \Rightarrow \boxed{x_1 = 0}$. $(0, 1, -1, 0)$

Calculation of Matrix Inverses by Gaussian Elimination

Consider two $n=3$ matrices $\tilde{A}$ and $\tilde{X}$.

If $\tilde{A}\,\tilde{X} = \tilde{I}$, then $\boxed{\tilde{X} = \tilde{A}^{-1}}$.

Write $\boxed{\tilde{A}\,\tilde{X} = \tilde{I}}$ in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Taking the product on the <u>left hand side</u> and equating with the <u>right hand side</u>, we get from the first column.

$$\boxed{\begin{aligned} a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} &= 1 \\ a_{21}x_{11} + a_{22}x_{21} + a_{23}x_{31} &= 0 \\ a_{31}x_{11} + a_{32}x_{21} + a_{33}x_{31} &= 0 \end{aligned}}$$

A <u>linear system</u> in x_{11}, x_{21} and x_{31}

Similarly from the second and third columns

$$\boxed{\begin{aligned} a_{11}x_{12} + a_{12}x_{22} + a_{13}x_{32} &= 0 \\ a_{21}x_{12} + a_{22}x_{22} + a_{23}x_{32} &= 1 \\ a_{31}x_{12} + a_{32}x_{22} + a_{33}x_{32} &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} a_{11}x_{13} + a_{12}x_{23} + a_{13}x_{33} &= 0 \\ a_{21}x_{13} + a_{22}x_{23} + a_{23}x_{33} &= 0 \\ a_{31}x_{13} + a_{32}x_{23} + a_{33}x_{33} &= 1 \end{aligned}}$$

In all there are nine x_{ij} variables, which can be solved for by the Gaussian elimination method.