01/03/2019

⇒ Minimum Mean Squared Estimation (MMSE) of a R.V.

$$\underset{\text{R.V.}}{Y} \to \boxed{\text{channel}} \to \underset{\text{R.V.}}{X}$$

X is accessible, how will you estimate Y?
$\hat{Y}$ has to be as close to Y as possible

- One of the ways in which this can be achieved is by using MMSE estimate of Y, i.e. $\min (Y-\hat{Y})^2 = \min (Y-g(X))^2$
- Let us say we want to estimate Y as a constant, $\hat{Y} = b$
then we have to minimize $E((Y-b)^2)$ wrt b → differentiate

$$E((Y-b)^2) = \int (y-b)^2 f_Y(y)\, dy$$

diff. wrt b & equating to 0

$$-2\int (y-b) f_Y(y)\, dy = 0$$

$$E((y-b)) = 0$$

$$E(y) = b \to \hat{Y}$$

$$\min E((Y-\hat{Y})^2) = E((Y-E(Y))^2) = \text{Var}(Y)$$

- Let us try to estimate Y (i.e. $\hat{Y}$) as a linear function of X (observed)
Now g is a linear function of X

$\hat{Y} = aX + b$ where $\underbrace{a, b}_{\text{constants}}$ are to be determined using the MMSE

$$\min_{a,b} E(Y-\hat{Y})^2$$

$$E[(Y-\hat{Y})^2] = E[(Y-aX-b)^2]$$

Let $(Y-aX) = Z$ another R.V.

then $E[(Y-\hat{Y})^2] = E[(Z-b)^2]$

Now $\min_b E[(Z-b)^2]$, from this, $b^* = \hat{b} = E(Z)$
$= E(Y-aX)$

To get a^*, minimize $E[(Y-\hat{Y})^2]$ w.r.t a

$$\min_a [E(Y-aX-E(Y-aX))^2]$$

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$$= \min_a \left\{ E\left[\{ (Y - E(Y)) - a(X - E(X)) \}^2 \right] \right\}$$

Differentiate w.r.t a & equate to zero

$$-2E\left[\left((Y - E(Y)) - a(X - E(X)) \right)(X - E(X)) \right] = 0$$

The error is orthogonal to the observation X

- orthogonality condition in MMSE estimation

$$-2(\mathrm{cov}(X,Y) - a\,\mathrm{Var}(X)) = 0$$

$$a^* = \hat{a} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X^2} \qquad \sigma_X^2 = \mathrm{Var}(X)$$

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}, \quad a^* = \frac{\rho_{XY}\sigma_X\sigma_Y}{\sigma_X^2} \qquad \rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}$$

$(-1 \text{ to } 1)$

$$= \frac{\rho_{XY}\sigma_Y}{\sigma_X}$$

$$\hat{Y} = a^* X + b^*$$

$$= \frac{\rho_{XY}\sigma_Y}{\sigma_X} X + E\left(Y - \frac{\rho_{XY}\sigma_Y}{\sigma_X} X \right)$$

$$= \frac{\rho_{XY}\sigma_Y}{\sigma_X}(X - E(X)) + E(Y)$$

→ when $\hat{Y} = b$ then the MMSE $= E(Y - E(Y))^2 = \sigma_Y^2$

Observed image X → $\boxed{h_w(x,y)}$ →

Weiner Filter.

True image → Y

Estimate of Y is $E[(Y - \hat{Y})^2] = E[Y - (\ \ )]$

convolution of X & $h_w(x,y)$

$x_n * h_w(x,y)$

→ When $\hat{Y} = a^* X + b^*$ then MMSE ~~E(... (Y-E(Y~~ $E[(Y-\hat{Y})^2]$

We have

$$E[((Y-aX) - E(Y-aX))^2]$$

$$\text{MMSE} = E[\{(Y-E(Y)) - a^*(X-E(X))\}^2]$$

$$= E\Big\{ (Y-E(Y))[(Y-E(Y)) - a^*(X-E(X))] - a^*(X-E(X)) \underbrace{[(Y-E(Y)) - a^*(X-E(X))]}_{0} \Big\}$$

error is orthogonal to the data.

$$= E[(Y-E(Y))(Y-E(Y))] - a^* E[(X-E(X))(Y-E(Y))]$$

$$= \text{Var}(Y) - a^* \text{Cov}(X,Y)$$

$$= \sigma_Y^2 - \frac{\text{Cov}(X,Y)}{\sigma_X^2} \cdot \text{Cov}(X,Y)$$

$$= \sigma_Y^2 - \frac{\rho_{XY}^2 \cancel{\sigma_X^2} \sigma_Y^2}{\cancel{\sigma_X^2}}$$

$$= \sigma_Y^2 - \rho_{XY}^2 \sigma_Y^2 = \sigma_Y^2(1-\rho_{XY}^2)$$

If $|\rho_{XY}| = 1$ then MMSE $= 0$ → highly correlated

If $\rho_{XY} = 0$ then MMSE $= \sigma_Y^2$ →

* If X & Y are jointly gaussian, then the linear estimate is same as the non-linear estimate → application: Wiener filter.

→ Let us consider $\hat{Y} = g(X)$ where g maybe linear/non-linear

To obtain best/optimum estimate (MMSE) of Y (true RV.)

$$\min E(Y-\hat{Y})^2 = E[(Y-g(X))^2]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-g(x))^2 f_{XY}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{\infty} (y-g(x))^2 f_{Y|X}(y|x)\, dy$$

Discrete form $\sum p(x_i) \cdot \sum (y_i - g(x_i))^2 p(y_i|x_i)$

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$= p(x_1)(y_1 - g(x_1))^2 p(y_1|x_1) + p(x_1)(y_2 - g(x_1))^2 p(y_2|x_1) + \dots$
$+ p(x_2)(y_1 - g(x_2))^2 p(y_2|x_2) + p(x_2)(y_2 - g(x_2))^2 p(y_2|x_2) + \dots$

- $g(x)$ has to be chosen such that it is minimized $\forall x$
- Consider the case of what we did when we estimated Y as some constant $\min_b E((Y-b)^2)$, $\hat{Y} = E(Y)$

$$\int (y-b)^2 f(y)\,dy \text{ is minimized w.r.t } b$$

- For the problem where we choose $\hat{Y} = aX+b$ we need

$$\int_{-\infty}^{\infty} (y - g(x))^2 f_Y(y|x)\,dy \text{ to be minimum for every } x$$

- So, in the case of choosing $\hat{Y} = b$, it is like choosing only one value of x. So the solution is,

$$\underline{\hat{Y} = E(Y|X)} \rightarrow \text{Best MMSE estimate of } Y \text{ for any } g(x)$$

→ Let X be uniformly distributed R.V. in the interval $(-1,1)$ & $Y = X^2$. Find the best linear estimator in terms of X. Compare the performance with the best estimator.

- Accessible is X - uniform $(-1,1)$ True $Y = E\{X^2\}$
- Best linear estimator $\hat{Y} = a^* X + b^*$

$E(X) = 0$ (uniform)

$$a^* = \frac{Cov(X,Y)}{Var(X)} = \frac{E(XY) - E(X)E(Y)}{E(X^2) - E(X)^2} \quad 0.$$

$$= \frac{E(X^3)}{E(X^2)} = \frac{\int_{-1}^{1} x^3 f_X(x)\,dx}{\int_{-1}^{1} x^2 f_X(x)\,dx} = \frac{1}{2}\left[\frac{x^4}{4}\right]_{-1}^{1} = \underline{\underline{0}}$$

$$b^* = E(Y) - E(X)\frac{cov(X,Y)}{Var(X)}$$

$$= E(Y) = E(X^2) = \int_{-1}^{1} x^2 f_X(x)\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{6} + \frac{1}{6} = \underline{\underline{\frac{1}{3}}}$$

$\frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}\right)$

$$\text{Error MMSE} = E((Y - E(Y))^2) = E((Y-\hat{Y})^2) = Var(Y)$$
$$= E(Y^2) - E(Y)^2 = \frac{1}{5} - \frac{1}{9} = \underline{\underline{\frac{4}{45}}}.$$

- Best estimator $\hat{Y} = E(Y|X) = E(X^2|X)$

$\hat{y}_1 = E(X^2|x_1) = x_1^2$ ||ly $\hat{y}_2 = E(X^2|x_2) = x_2^2$ and so on..

$$\hat{Y} = E(Y|X) = E(X^2|X) = X^2$$

$$E((Y-\hat{Y})^2) = E(X^2 - X^2) = \underline{0}$$

- Jointly distributed gaussian Random Variables

1. How to represent the PDF
2. Propagation of Jointly distributed gaussian RV's.

X , $F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2\sigma^2}(x-m)^2}$

$\downarrow$

gaussian R.V

m σ^2

Now consider two R.Vs X_1 and X_2

$\underbrace{m_1 \ \sigma_1^2}_{X_1}$ $\underbrace{m_2 \ \sigma_2^2}_{X_2}$

If these two are considered statistically independent, their joint gaussian distribution can be expressed as (i.e. if X_1 and X_2 are jointly gaussian and are independent)

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2}\frac{(x_1-m_1)^2}{\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2}\frac{(x_2-m_2)^2}{\sigma_2^2}}$$

$$f_{X_1}(x_1) \cdot f_{X_2}(x_2)$$

then $C_X = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$ $\quad C_X^{-1} = \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix} \frac{1}{\sigma_1^2\sigma_2^2}$

$$= \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{bmatrix}$$

$\det\{C_X\} = |C_X| = \sigma_1^2\sigma_2^2$

$|C_X|^{1/2} = \sigma_1\sigma_2$

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{(2\pi)^{2/2} C_X^{1/2}} e^{\underbrace{-\frac{1}{2}\underset{1\times 2}{[(x_1-m_1)\ (x_2-m_2)]} C_X^{-1} \underset{2\times 1}{\begin{bmatrix} x_1-m_1 \\ x_2-m_2 \end{bmatrix}}}_{}}$$

$$\frac{(x_1-m_1)^2}{\sigma_1^2} + \frac{(x_2-m_2)^2}{\sigma_2^2}$$

Even if RVs are not independent, then C_X can be accordingly substituted in this equation.

$$C_X = \begin{bmatrix} \sigma_1^2 & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_2 X_1) & \sigma_2^2 \end{bmatrix}$$

If we consider $X_1, X_2, \ldots X_N$ and if they are jointly gaussian, the joint PDF can be written as.

$$f_{X_1 X_2 X_3 \ldots X_N}(x_1, x_2 \ldots x_N) = \frac{1}{(2\pi)^{N/2} |C_x|^{1/2}} e^{-\frac{1}{2} \underbrace{(\underline{x} - \underline{m}_x)^T}_{1\times N} \underbrace{C_x^{-1}}_{N\times N} \underbrace{(\underline{x} - \underline{m}_x)}_{N\times 1}}$$

$x \rightarrow$ vector of RVs.
$m \rightarrow$ mean vector of RVs.

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \quad \begin{bmatrix} m_{X_1} \\ m_{X_2} \\ \vdots \\ m_{X_N} \end{bmatrix}$$

Q. Does uncorrelatedness imply independence? Yes, for jointly gaussian

Properties

* If $X_1, X_2, \ldots X_N$ are jointly gaussian and are uncorrelated, they are independent.
* If $X_1, X_2 \ldots X_N$ are jointly gaussian, the marginals are also gaussian, i.e. $f_{X_1}(x_1), f_{X_2}(x_2) \ldots$ are gaussian. But if $X_1, X_2 \ldots X_N$ are marginally gaussian, they need not be jointly gaussian.
* $\underline{Y} = A\underline{X}$ – linear transformation of jointly gaussian R.V.s leads to gaussian distribution of Y.
* Other properties from book
* Conditional PDFs of $X_1 X_2 \ldots X_N$ jointly gaussian are also gaussian
* Best estimated $\hat{Y}$ of X, Y jointly gaussian R.Vs is same as the linear estimate

  *: from exam P.O.V.

Central Limit Theorem

$X_1, X_2 \ldots X_N$ iid r.v.s, (m, σ^2) — iid = independent identically distributed, same (m, σ).

$S_N = X_1 + X_2 + X_3 + \cdots X_N$ → another R.V.

mean of $S_N = Nm$

Variance of $S_N = N\sigma^2$ ($\because$ covariance $= 0$)

Now, form a R.V. Z_N which zeromean & variance $= 1$.

$$Z_N = \frac{S_N - Nm}{\sqrt{N}\,\sigma}$$

As per central limit theorem

$$\lim_{N\to\infty} P(Z_N \le z) = \frac{1}{\sqrt{2\pi}\cdot 1}\int_{-\infty}^{z} e^{-x^2/2}\,dx$$

As $N \to \infty$ $Z_N \xrightarrow{\text{has}}$ gaussian distribution

- Must be normalised (centralised) else mean of Z_N will $\to \infty$

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1. $$f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\left[\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right]}$$

TP: $f_{Y|X}(x|x)$ is also gaussian.

$$f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\left[\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right]}}{\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-x^2/2\sigma_1^2}}$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}\,\sigma_2} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right) + \frac{x^2}{2\sigma_1^2}\right] \quad 1)$$

$$= \frac{1}{\sigma_2\sqrt{2\pi(1-\rho^2)}} \exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{-2\rho xy}{\sigma_1\sigma_2}\right) + \underbrace{\frac{x^2}{2\sigma_1^2} - \frac{x^2}{2\sigma_1^2(1-\rho^2)}}_{\frac{\cancel{x^2} - \cancel{x^2} + \rho^2 x^2}{2\sigma_1^2(1-\rho^2)}}\right] \quad 2$$

$$= \frac{1}{\sigma_2\sqrt{2\pi(1-\rho^2)}} \exp\left[\frac{-1}{2(1-\rho^2)}\left[\frac{y^2}{\sigma_2^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{x^2\rho^2}{\sigma_1^2}\right]\right]$$

$$= \frac{1}{\sigma_2\sqrt{2\pi(1-\rho^2)}} \exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{y}{\sigma_2} - \frac{\rho x}{\sigma_1}\right)^2\right]$$

Q: Show that marginally gaussian does not imply jointly gaussian.

Laws of large numbers

r.v. X ; true mean $E(X) = \mu$ (unknown), true variance is say σ^2
Let $X_1, \ldots, X_n$ be the iid sample having the same mean & variance

$E(X_1) = E(X_2) = E(X_3) = \ldots = E(X_n) = \mu$

$Var(X_1) = Var(X_2) = \ldots = Var(X_n) = \sigma^2$

Let us compute $M_n = \frac{1}{n}[X_1 + X_2 + X_3 + \ldots + X_n]$

$= \frac{1}{n}\sum_{i=1}^{n} X_i$, call it a sample mean of X

M_n itself is a r.v.

We want to have M_n such properties on the avg. M_n has to be close to μ & variance of M_n (its deviation from mean μ has to be close to 0)

i.e., $E(M_n) = \mu$ & $E(M_n - E(M_n))^2 \longrightarrow 0$

Look at $E(M_n) = E(\frac{1}{n}(X_1 + \ldots + X_n))$

$= E(\frac{1}{n}[\mu + \mu + \ldots + \mu] = \mu$ ← ~~unbiased estimator~~.

So sample mean = true mean, i.e., E(estimate) = true

Now $E(M_n - \mu)^2 = Var(M_n)$

$= Var(\frac{X_1}{n} + \frac{X_2}{n} + \ldots + \frac{X_N}{n}) = n \cdot \frac{\sigma^2}{n^2} = \frac{\sigma^2}{n}$

$Var(X_1) = \sigma^2$

$Var(\frac{X_1}{n}) = E\left[\left(\frac{X_n}{n}\right)^2\right] - \left(E\left[\frac{X_n}{n}\right]\right)^2$

$= \frac{\sigma^2}{n^2}$

Let us Chebyshev's inequality of M_n ↳ $\mu, \frac{\sigma^2}{n}$

$P(|X-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$, $\varepsilon > 0$

$\therefore P(|M_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n \cdot \varepsilon^2}$

$\therefore P(|M_n - \mu| < \varepsilon) \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$

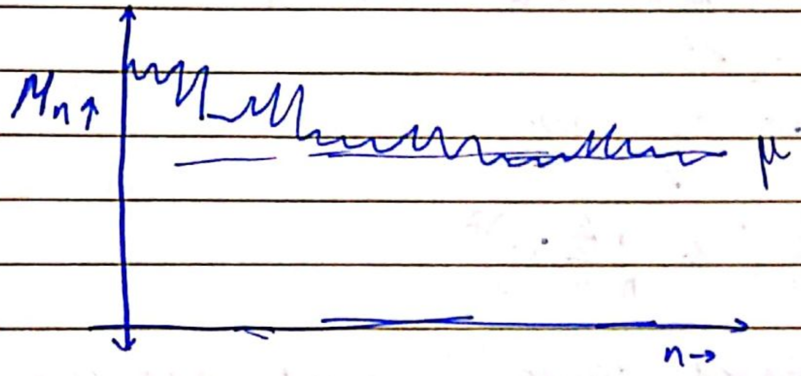
(M_n → sample mean, μ ↳ true mean)

$\lim_{n \to \infty} P(|M_n - \mu| < \varepsilon) = 1$

Weak Law of Large Number

$\lim_{n \to \infty} P(|M_n - \mu| < \varepsilon) = 1$ $\varepsilon > 0$

For any $\varepsilon > 0$, sufficiently large n $(n \to \infty)$, the chance that the sample mean deviating from true mean by an amount ε approaches to 1 (ε is chosen). This is the case for some fixed n.



$M_1 = X_1$

$M_2 = \frac{X_1 + X_2}{2}$

$\vdots$

$M_N = \frac{X_1 + X_2 + \cdots X_n}{n}$

Strong Law of Large Numbers

Every sequence sample mean approaches to true mean as $n \to \infty$

$P[\lim_{n \to \infty} M_n = \mu] = 1$

Estimation of mean by using samples drawn from the distribution of X

true mean - μ sample mean - estimate of μ : $M_n = \hat{\mu} = \sum_{i=1}^{n} X_i$

If $E(\hat{\mu}) = \mu \to$ then the estimator is called as unbiased estimator

Estimating μ by using the estimator as

$\frac{\text{sum of samples}}{n} = \sum_{i=1}^{n} X_i$ is an unbiased estimator

---

two

→ θ - true parameter (unknown)
$\hat{\theta}$ - estimate of θ (R.V.)
$E(\hat{\theta}) = \theta$ means estimator is unbiased
] general case.

$[E(\hat{\theta}) - \theta]$ is called bias of the estimator (0 for good estimator)
Variance of estimator $E([(\hat{\theta} - E(\hat{\theta}))^2])$

H.W. → Mean squared Error = $bias^2$ + Variance = $E(\theta - \hat{\theta})^2$
b/n estimated & true parameter

→ I want to estimate variance of a R.V. X
true variance = σ^2
estimated variance = $\hat{\sigma}^2$ R.V.
$\downarrow$
Z

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \frac{1}{n} \sum_{i=1}^{n} X_i \right)^2$$

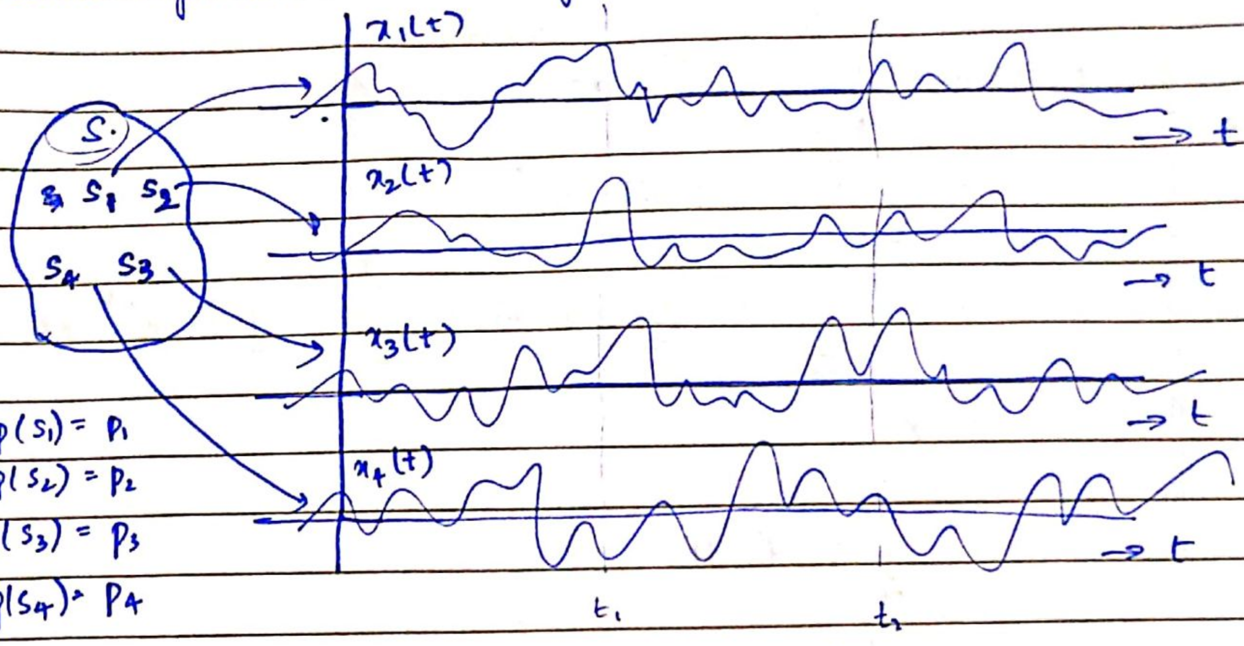
Is this an unbiased estimator? If not, what change has to be done in the formula to make it an unbiased estimator?
See whether $E[\hat{\sigma}^2] = \sigma^2$

→ Random / Stochastic Process
Random variable - probabilistic description of a random quantity. without considering the time (space)
In practice we encounter random signals. How can these random signals be analyzed by probabilistic approach leads to the concept of random process (R.P.).
Random signals are functions of time.

→ Examples (of R.P.)
- temperature monitoring in a room from 9am - 12am for 300 days.
- 100 peoples saying the same sentence - waveforms
- noise in resistor

* Random Process: A probability system consisting of sample space S, ensemble (collection) of waveforms and probability

measure for all these waveforms, is called R.P.



$p(s_1) = p_1$
$p(s_2) = p_2$
$p(s_3) = p_3$
$p(s_4) = p_4$

Every sample point is mapped to a waveform

$\sum_{i=1}^{4} p_i = 1$

Theoretically waveforms exists from $-\infty$ to ∞ time (not practically)

R.P is denoted $X(t)$

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$R_X(t_1, t_2) = E(X(t_1)\, X(t_2))$

$R_X(\tau) = E(X(t)\, X(t+\tau))$

Properties : $R_X(0) > R_X(\tau)$ — $R_X(0) \to$ Mean square Value

$R_X(\tau) = R_X(-\tau)$

$\mathcal{F}(R_X(\tau)) = P.S.D. = S_X(f) \to$ Real, +ve & even f^n.

Q.1. a) $e^{-\tau^2}$ Valid

b) $|\tau| e^{-|\tau|}$ Invalid

c) $10\, e^{-|\tau+2|}$ Invalid

d) $\left(\frac{\sin \pi t}{\pi t}\right)^2$ Valid

e) Invalid.

Q.2. a) Valid

b) Invalid not even.

c) Invalid -ve at zero

d) Invalid complex

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Q.3. $S_x(f) = \begin{cases} 1 + \frac{1}{4}|f| & |f| \le 4 \\ 0 & \text{elsewhere} \end{cases}$

$$R_x(\tau) = \int_{-f}^{f} S_x(f)\, e^{\tau t}\, dt$$

$$= \int_{-4}^{4} \left(1 + \frac{1}{4}|f|\right) e^{\tau t}\, dt$$

$$= \int_{-4}^{0} \left(1 - \frac{1}{4}f\right) e^{\tau t}\, dt + \int_{0}^{4} \left(1 + \frac{1}{4}f\right) e^{\tau t}\, dt$$

$= \cancel{\phi}$

$$\int_{-f}^{f} S_x(f)\, dt$$

$$= \int_{-4}^{0} \left(1 - \frac{1}{4}f\right) df + \int_{0}^{4} \left(1 + \frac{1}{4}f\right) df$$

$$= \left(f - \frac{f^2}{8}\right)_{-4}^{0} + \left(f + \frac{f^2}{8}\right)_{0}^{4}$$

$$= -\left(-4 - \frac{16}{8}\right) + \left(4 + \frac{16}{8}\right)$$

$$= 8 + 4 = \underline{12}.$$

Q.4. $Y(t) = X(t)\, A\cos(\omega_c t + \phi)$

$\int x f_x(x)\, dx$

$$R_Y(\tau) = E[Y(t)\,Y(t+\tau)]$$

$$= \int_{-\infty}^{\infty} E\left[X(t)\, A\cos(\omega_c t + \phi) \cdot X(t+\tau)\, A\cos(\omega_c(t+\tau) + \phi)\right]$$

$$= A^2\, E\left[X(t)\, X(t+\tau) \cdot \cos(\omega_c t + \phi) \cos(\omega_c t + \omega_c\tau + \phi)\right]$$

$\frac{A^2}{2} R_x(\tau) \cos\theta$

$$= A^2\, E[X(t)\, X(t+\tau)]\, E[\cos(\omega_c t + \phi) \cos(\omega_c t + \omega_c \tau + \phi)]$$

$$= A^2\, E[X(t)\, X(t+\tau)] \cdot \frac{1}{2} \left\{ E\left[\underbrace{\cos(2\omega_c t + 2\phi + \omega_c \tau)}_{\to 0} + \cos(\omega_c \tau) \right]\right.$$

$$= \frac{A^2}{2} R_x(\tau)$$

→ Given a random process $X(t)$, we can identify the following quantities

$X(t)$: random process (ensemble of waveforms)

$x_j(t)$: The sample function corresponding to sample point s_j

$X(t_i)$: Random variable at $t = t_1$

$x_j(t_i)$: Real number giving the value of $x_j(t)$ at $t = t_i$

→ Example to illustrate the concept of R.P.

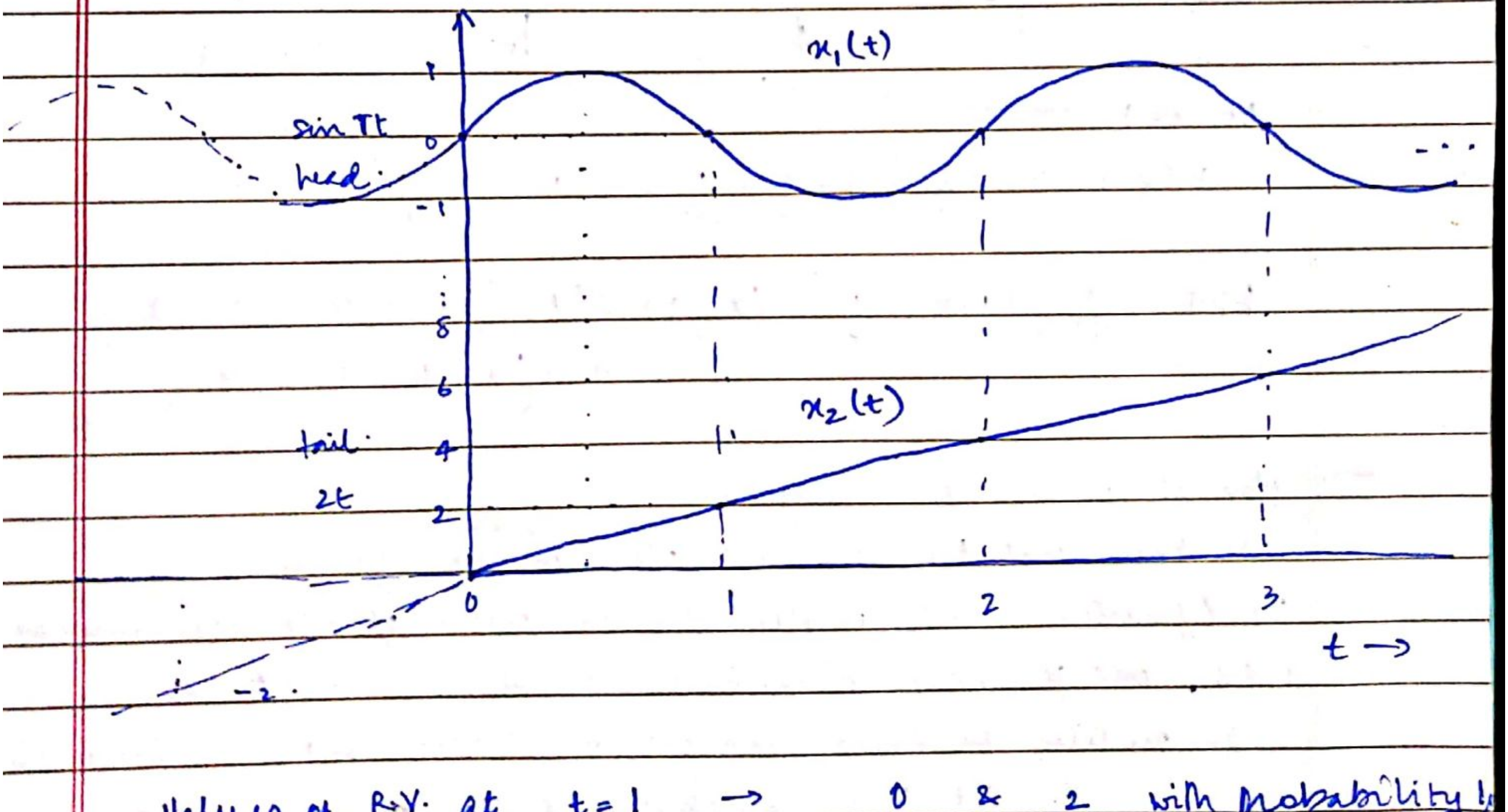
Consider the experiment of tossing a fair coin. Let us say the R.P. is defined as follows -

$X(t) = \sin \pi t$ if head occurs and

$X(t) = 2t$ if it results in a tail.

(a) Draw the sample functions

(b) Write the pdfs of R.V.s at $t = 0$ & $t = 1$



Values of R.V. at $t = 1$ → 0 & 2 with probability $\frac{1}{2}$

at $t = 1$ $\quad f_{X(1)}(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-2)$

at $t = 0$ $\quad f_{X(0)}(x) = \delta(x)$ $\quad$ (at $t=0$ $x_1(t)$ $x_2(t)$ are both zero)

→ Consider the experiment of throwing a fair die

· sample space has 6 sample points $(s_1 \cdots s_6)$

Let us say the sample functions are $x_i(t) = \frac{1}{2}t + (i-1)$

$s = s_i, \quad i = 1 \ldots 6$

Q. Find the mean value of the R.V. at $X(t)\big|_{t=1}$ i.e. at $X(1)$.

$$= \frac{1}{6}\sum_{i=1}^{6}\left(\frac{1}{2} + (i-1)\right) = \underline{\underline{3}}$$

$x_1(t) = \frac{1}{2}t$ $1/2$

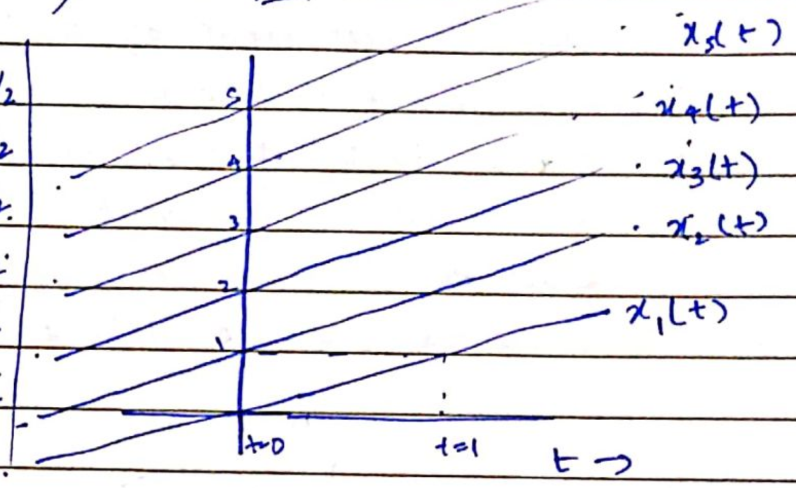
$x_2(t) = \frac{1}{2}t + 1$ $3/2$

$x_3(t) = \frac{1}{2}t + 2$ $5/2$

$x_4(t) = \frac{1}{2}t + 3$ $7/2$

$x_5(t) = \frac{1}{2}t + 4$ $9/2$

$x_6(t) = \frac{1}{2}t + 5$ $11/2$



At $t=1$

$$E[X(1)] = \frac{1}{6}\left(\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \frac{7}{2} + \frac{9}{2} + \frac{11}{2}\right) = \underline{3}$$

P.D.F $$f_{X(1)}(x) = \frac{1}{6}\left[\delta\left(x-\frac{1}{2}\right) + \delta\left(x-\frac{3}{2}\right) + \delta\left(x-\frac{5}{2}\right) + \delta\left(x-\frac{7}{2}\right) + \delta\left(x-\frac{9}{2}\right) + \delta\left(x-\frac{11}{2}\right)\right]$$

→ How to specify R.P. say $X(t)$.

We know that R.P. is a collection of waveforms

By definition $X(t)$ implies the existence of infinite number of R.V.s, one for each time instant where $-\infty < t < \infty$

In practice, we have sample functions with limited time range. So, let $X(t)$ be observed at n time instances (for some range) $t_1, t_2 \ldots t_n$. Corresponding to this we have n R.V.s $X(t_1), X(t_2), \ldots X(t_n)$. With these R.V.s multivariate (joint) PDF can be defined.

$$F_{X(t)}(x) = F_{X(t_1), X(t_2), \ldots X(t_n)}(x_1, x_2, x_3 \ldots x_n) = P(X(t_1) \le x_1, X(t_2) \le x_2 \ldots X(t_n) \le x_n)$$

The joint P.D.F of $X(t)$ can now be written as -

$$f_{X(t)}(x) = \frac{\partial^n}{\partial x_1 \partial x_2 \partial \cdots \partial x_n} F_{X(t)}(x)$$

In this case a R.P. $X(t)$ can be specified if and only if a rule is given or implied for determining $F_{X(t)}(x)$ or $f_{X(t)}(x)$ for any finite set of observations.

→ In a number of applications that we encounter, the R.P. can be specified as follows.

- One of the ways to specify the R.P. is that use rule the joint PDF must depend in a known way on time.

  Based on this we define the Strictly-sense Stationary or wide-sense stationary processes.

$$f_{X(t_1), X(t_2)}(x_1, x_2) = f_{X(t_1+\tau), X(t_2+\tau)}(x_1, x_2)$$

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- Another way to specify a R.P. $X(t)$ is to use a time function involving one or more parameters

$$X(t) = A\cos(\omega_0 t + \theta)$$

where one can consider A & ω_0 as constants and θ as a random variable R.V - θ uniform $(0, 2\pi)$

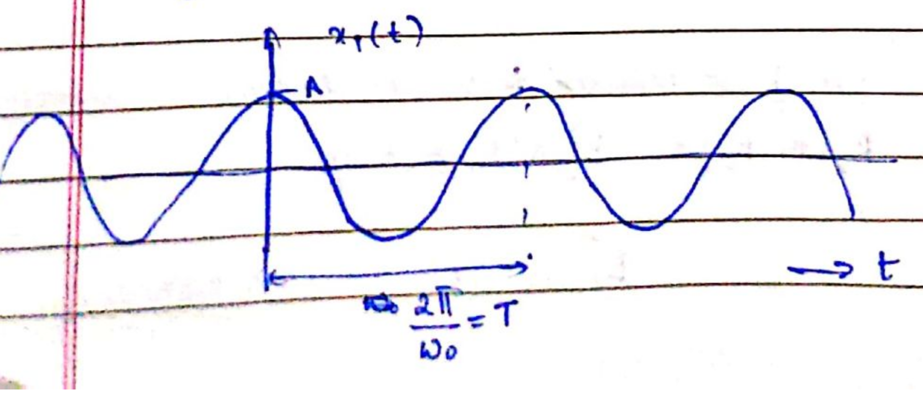
$\theta = 0$, $x_1(t) = A\cos(\omega_0 t)$ $\quad -\infty < t < \infty$

$\theta = \pi/100$, $x_2(t) = A\cos(\omega_0 t + \pi/100)$

$\theta = \pi/2$ $\quad x_3(t) = A\cos(\omega_0 t + \pi/2)$

$\vdots$

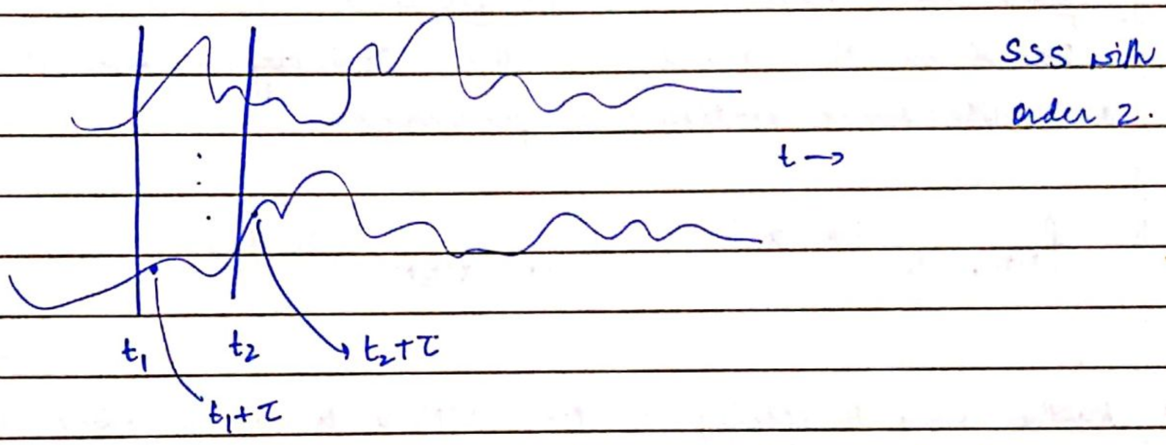
any number of sample functions possible. (up to ∞)



Other samples functions will just be shifted versions of this.

⇒ Strict Sense Stationary Process (SSS)
Let us say, we would like to specify $X(t)$ in terms of density function by considering n.
Let $n=1$, if we make assumption that
$$f_{X(t)}(x) = f_{X(t+\tau)}(x)$$
then it is called SSS process of order 1 (or 1^{st} order SSS process).
eg. White gaussian noise is an SSS process of any order n.

if $n=2$, $f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(t_1+\tau)X(t_2+\tau)}(x_1, x_2)$



|||^{ly}, general expression is, $f_{X(t_1)\dots X(t_n)}(x_1, x_2 \dots x_n) = f_{X(t_1+\tau)\dots X(t_n+\tau)}(x_1, \dots x_n)$

eg. Images are modelled by Random Fields (matrix)
2x2 Binary Image - 2^{16} samples - not possible to find in speech

⇒ Wide Sense Stationary Process
most communication systems can be modelled as WSS.

H.W. Q. If $X(t)$ is SSS with order n, then it is SSS with order $< n$. Prove this.

To understand WSS, we first defined Auto-Correlation function of a R.P. $X(t)$. $R_X(t_1, t_2) = E[X(t_1)X(t_2)]$

* Auto-Covariance $E[X(t_1)X(t_2) - E[X(t_1)X(t_2)]$

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$R_X(1,1) = E[X^2(1)]$
$R_X(t_1,t_1) = E[X^2(t_1)]$ → Mean Square Value of the R.P. at $t=t_1$ ≈ Total Power Content at $t=t_1$

$R_X(t_1,t_2)$ is a function of time instances t_1 and t_2 for a R.P. $X(t)$.
Let us consider $t_1 = t$ and $t_2 = t+\tau$

$R_X(t, t+\tau) = E[X(t)X(t+\tau)]$ ~~$= R_X(\tau)$~~

If $R_X(t,t+\tau)$ does not depend on the time instance, but only on the time difference then $E[X(t)X(t+\tau)] = R_X(\tau)$

i.e. $R_X(t_1, t_1+\tau) = R_X(t_2,t_2+\tau) = \cdots R_X(t_n, t_n+\tau)$ i.e. any time Instance

This assumption is made for speech signals.

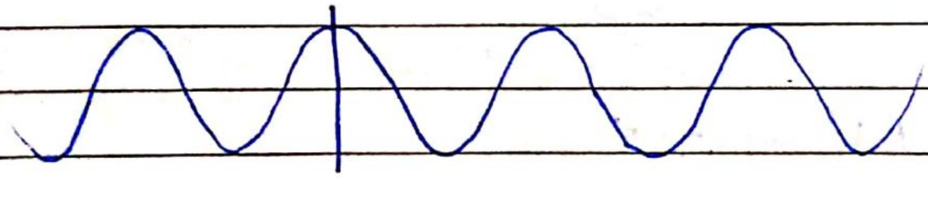
$R_X(t_1,t_2) = R_X(t_2, t_3)$

- Definition of WSS : A R.P. $X(t)$ is WSS if $E[X(t)]$ = constant & $R_X(\tau) = E[X(t_1)X(t_1+\tau)] = R_X(\tau)$ i.e. R_X depends only on the difference. $E[X(t_2)X(t_2+\tau)]$

H.W. Q. If a process is SSS, it is WSS. Prove this. The converse is not true

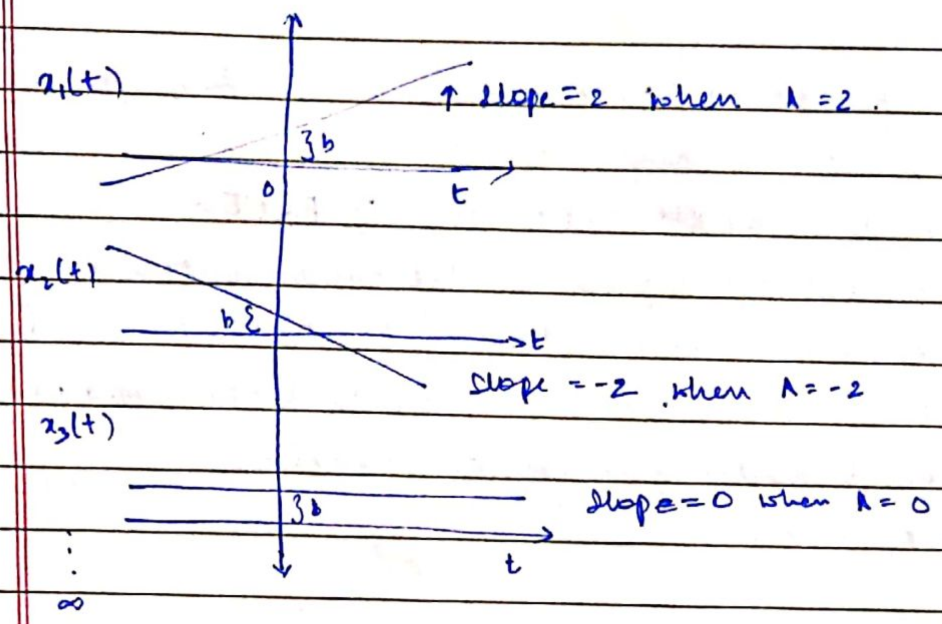
$R_X(0) = E[X(t_1)X(t_1+0)] = E[X^2(t_1)]$ MS value of R.P. $x(t)$
$R_X(0) = E[X(t_2)X(t_2+0)] = E[X^2(t_2)]$ Total power of R.P. $x(t)$
$E[X^2(t)] = E[X^2(t+\tau)] = R_X(0)$ does not matter.

→ Sketch the ensemble of R.P. $x(t)$ given $x(t) = K\cos(\omega_0 t+\theta)$ $-\infty < t < \infty$ where ω & θ are constants and K is a uniform R.V. in range $(-A, A)$. Looking at the waveform, what can you say about the stationarity of the process.



→ Sketch the ensemble of waveforms of the R.P $x(t)$

$X(t) = At + b$ — b is a constant & A is uniform in $(-2,2)$



Decide whether the process is WSS or not.

$E[X(t_1)] = E[At_1 + b] = t_1 E[A] + b = \underline{b}$ → constant ∀ t

$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\} = E[(At_1+b)(At_2+b)]$

$= E[A^2 t_1 t_2 + b^2 + \underbrace{Ab(t_1+t_2)}_{\to 0}] = \frac{4}{3} t_1 t_2 + b^2$

∴ Autocorrelation is not dependent on τ. ∴ it is dependent on t_1 & t_2. It is not WSS, even though mean is constant.

→ Test for WSS of $X(t) = A\cos(\omega_0 t + \theta)$ when θ is a R.V. uniformly distributed in the range of $(0, 2\pi)$

$E[X(t)] = E[A\cos(\omega_0 t + \theta)] = A\, E[\cos(\omega t + \theta)]$

~~$= AE[\cos\omega_0 t\cos\theta - \sin\omega t\sin\theta]$~~

$= A \int \underbrace{\cos(\omega t + \theta)}_{g(\theta)} . \underbrace{\frac{1}{2\pi}}_{f(\theta)} d\theta = 0$

$R_X(t_1, t_2) = A^2 E[\cos(\omega t_1 + \theta)$

$= \frac{A^2}{2}\cos(\omega_0(t_1 - t_2)) = \frac{A^2}{2}\cos(\omega\tau) = R_X(\tau)$

∴ WSS.

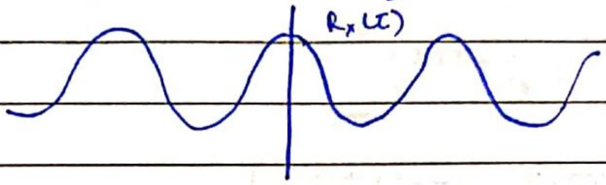
Consider a WSS process $X(t)$. For this process

$R_X(\tau) = E[X(t)X(t+\tau)]$

$R_X(-\tau) = E[X(t)X(t-\tau)]$

$R_X(\tau) = R_X(-\tau)$ $\therefore R_X(\tau)$ is symmetric.

For this example $R_X(\tau) = \frac{A^2}{2}\cos(\omega\tau)$



The R.P. is periodic & hence $R_X(\tau)$ is also periodic

Tutorial - 10

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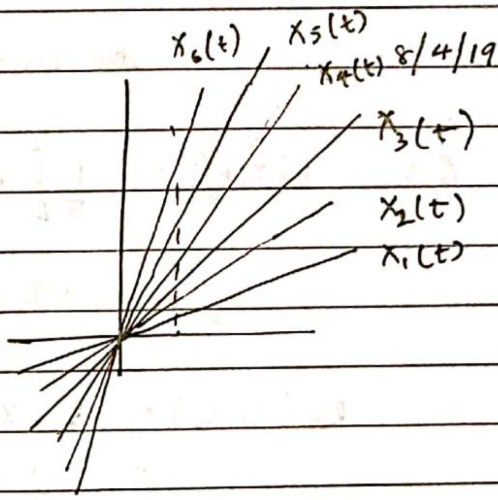


Q.1. $X_i(t) = it$, $i = 1, 2, \cdots 6$

a) $E[X] = \frac{1}{6}(1+2+3\cdots 6) = \frac{21}{6} = 3.5$

$E[Y] = \frac{2}{6}(1+2+\cdots 6) = \frac{42}{6} = \underline{\underline{7}}$

b) $f_{X,Y}(x,y) = \frac{1}{6}\sum_{i=1}^{6}\delta(x-i, y-2i)$

c) $R_X(1,2) = E[X_i(1)X_i(2)] = \frac{1}{6}\times 1\times 2 + \frac{1}{6}\times 2\times 4 + \frac{1}{6}\times 3\times 6 \cdots \frac{1}{6}\times 6\times 12$

$= \frac{2}{6}[1+4+9+16+25+36] = \frac{182}{6}$

Q.2. $X(t) = A\cos(\omega_c t + \alpha)$ $\alpha \to$ uniform $(0, 2\pi)$

a) 0 b) $\frac{A^2}{2}\cos\omega_c t$

Q.3. $X(t) = At + b$

$R_X(t_1, t_2) = E[(At_1+b)(At_2+b)] = E[A^2 t_1 t_2 + At_1 b + At_2 b + b^2]$

$E[X(t)] = E[At+b] = tE[A] + b = \cancel{0} + b$

$= \underline{\underline{\frac{4}{3}t_1 t_2}} + \cdots$

Not WSS.

$200 \text{ sinc}(2\pi 1000 \tau)$

Q.4. $S_x(f) = 0.1, \quad |f| < 1000 \text{ Hz}$ - Stationary.

0 elsewhere.

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f\tau} df \qquad MSE = \int_{-1000}^{1000} 0.1 \, df.$$

$$= 0.1 \int_{-1000}^{1000} e^{j2\pi f\tau} df$$

$$= \frac{0.1}{j2\pi\tau} \left(e^{j2\pi f t} \right)_{-1000}^{1000}$$

$$= 0.1 \left[\frac{e^{2\pi j 1000\tau} - e^{-2\pi j 1000\tau}}{j2\pi\tau} \right] = \frac{0.2 \sin(2\pi 1000 \tau)}{2\pi\tau}$$

$$= 200 \text{ sinc}(2\pi \cdot 1000 \tau)$$

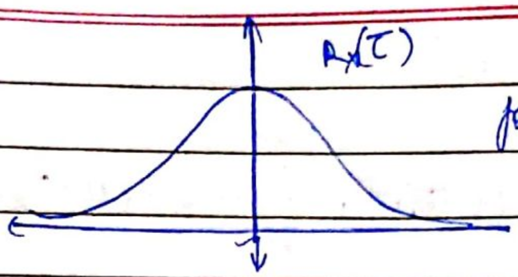
Q.5 $E[X(t)] = \frac{1}{5}\{ \cos 2\pi t - \sin 2\pi t + \sin 2\pi t - \cos 2\pi t - \sqrt{2}\cos t - \sqrt{2}\sin t + x_5(t) \} = 0$

$$x_5(t) = \sqrt{2}\cos t + \sqrt{2}\sin t$$

$$R_x(t_1, t_2) = E[X(t_1) X(t_2)]$$

$$= \frac{1}{5}\Big[(\cos 2\pi t_1 - \sin 2\pi t_1)(\cos 2\pi t_2 - \sin 2\pi t_2) + (\sin 2\pi t_1 - \cos 2\pi t_1)(\sin 2\pi t_2 - \cos 2\pi t_2) + 2\cos t_1 \cos t_2 + 2\sin t_1 \sin t_2 + (\sqrt{2}\cos t_1 + \sqrt{2}\sin t_1)(\sqrt{2}\cos t_2 + \sqrt{2}\sin t_2) \Big]$$

$$= \frac{1}{5}\Big[(\cos 2\pi t_1 \cos 2\pi t_2 + \sin 2\pi t_1 \sin 2\pi t_2 - \sin 2\pi t_1 \cos 2\pi t_2 - \cos 2\pi t_1 \sin 2\pi t_2) + (\cos 2\pi t_1 \cos 2\pi t_2 + \sin 2\pi t_1 \sin 2\pi t_2 - \sin 2\pi t_2 \cos 2\pi t_1 - \cos 2\pi t_2 \sin 2\pi t_1) + 2\cos t_1 \cos t_2 + 2\sin t_1 \sin t_2 + (2\cos t_1 \cos t_2 + 2\sin t_1 \sin t_2 + 2\cos t_1 \sin t_2 + 2\cos t_2 \sin t_1) \Big]$$



for speech

→ $x(t) = A\cos(w_c t + \theta)$ $\theta \to$ uniform in $(0, 2\pi)$

$E[X(t)] = 0$ $R_X(\tau) = \frac{A^2}{2}\cos w_c \tau$

$R_X(\tau) = E[X(t)X(t+\tau)]$

$R_X(0) = E[X^2(t)] = \frac{A^2}{2}$ → Total power in R.P. $X(t)$

• Random signal applied as input to an LTI system characterised by impulse response ← deterministic

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

If $X(t)$ is a R.P., $Y(t)$ is also a R.P.

Q. Let us say that $X(t)$ is WSS, what can you say about $Y(t)$? Is $Y(t)$ WSS?

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau$$

→ constant, linear operation.

$$E[Y(t)] = \int_{-\infty}^{\infty} h(\tau)\, E[X(t-\tau)]\, d\tau$$

If $X(t)$ is WSS, $E[X(t-\tau)] = \mu_X$ · some constant

$$E[Y(t)] = \mu_X \underbrace{\int_{-\infty}^{\infty} h(\tau)\, d\tau}_{\text{constant}}$$

$\therefore$ $Y(t)$ ~~is also WSS~~. has constant Mean.

$$H(f) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

$$H(0) = \int_{-\infty}^{\infty} h(\tau)\, d\tau$$

$$\therefore E[Y(t)] = \mu_X H(0)$$

$$R_Y(t, u) = E[Y(t)\, Y(u)]$$
$t_1 \quad t_2$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, x(u-\tau_2)\, d\tau_2 \right]$$

$$= \int h(\tau_1)\, d\tau_1 \int h(\tau_2)\, d\tau_2 \; E\left[x(t-\tau_1)\, x(\mu-\tau_2) \right]$$

Now, $R_X(t, u) = R_X(t-\mu)$ because $x(t)$ is WSS.

$$\therefore R_Y(t, u) = \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, d\tau_2 \; R_X(\underbrace{t-u}_{}-\tau_1+\tau_2)$$

$~~R_X(t)~~$ t, u $t-u$ τ

$R_Y(t, u)$ is dependent on $t-\mu$ $\therefore$ It is WSS.

⇒ Power Spectral Density

$R_X(\tau)$ of $X(t)$ → autocorrelation] Fourier transform
$S_X(f)$ of $X(t)$ → Power Spectral Density] Pairs.

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f\tau}\, df = \mathcal{F}^{-1}[S_X(f)]$$

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-2\pi f\tau}\, d\tau = \mathcal{F}[R_X(\tau)]$$

As computed from previous example,

$$R_X(\tau) = \frac{A^2}{2} \cos w_c\tau \qquad -\infty < \tau < \infty$$

$$S_X(f) = \frac{A^2}{2} \int_{-\infty}^{\infty} \cos w_c\tau \; e^{-j2\pi f\tau}\, d\tau$$ → This is actually Power/Hz → integrate to get total power.

$$= \frac{A^2}{4} \left[\delta(f-f_0) + \delta(f+f_0) \right]$$

Total Power $= \dfrac{A^2}{2}$

$$\left(\frac{e^{jw_c\tau} + e^{-jw_c\tau}}{2} \right) e^{-j2\pi f\tau}\, d\tau$$

$$\left(\frac{e^{j2\pi f_c\tau} + e^{-j2\pi f_c\tau}}{2} \right) e^{-2\pi f\tau}$$

$$\boxed{\mathcal{F}(\delta(t-a)) = e^{-i2\pi fa}}$$

$$\therefore \mathcal{F}^{-1}\left[e^{-i2\pi fa} \right] = \delta(t-a)$$

$$\frac{A^2}{4} \left[\delta(f+f_0) + \delta(f-f_0) \right]$$

→ Power Spectral Density of WSS process $(S_X(f))$

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

$X(t)$ if WSS, gives WSS $Y(t)$

$$R_Y(\tau) = E[Y(t)\,Y(t+\tau)]$$
$$= \int_{-\infty}^{\infty} h(\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\,d\tau_2\, R_X(\tau - \tau_1 + \tau_2) \quad \text{where } \tau = t_1 - u$$

Substitute $\tau = 0$, then

$$R_Y(0) = E[Y^2(t)] \Rightarrow \text{total power in the random process } Y(t)$$
$$= \int_{-\infty}^{\infty} h(\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\,d\tau_2\, R_X(\tau_2 - \tau_1)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\,h(\tau_2)\, R_X(\tau_2 - \tau_1)\, d\tau_1\, d\tau_2$$

$$h(\tau_1) = \int_{-\infty}^{\infty} H(f)\, e^{j2\pi f \tau_1}\, df$$

$$\therefore\ R_Y(0) = \int_{-\infty}^{\infty} H(f)\,df \int_{-\infty}^{\infty} h(\tau_2)\,d\tau_2 \int_{-\infty}^{\infty} \underbrace{R_X(\tau_2 - \tau_1)\, e^{j2\pi f \tau_1}}\, d\tau_1$$

Let $\tau_1 - \tau_2 = \lambda \rightarrow \tau_1 = \tau_2 + \lambda \rightarrow d\tau_1 = d\lambda$

3^{rd} term ↳ $$\int_{-\infty}^{\infty} R_X(-\lambda)\, e^{j2\pi f(\tau_2 + \lambda)}\, d\lambda$$

×

$$= \int_{-\infty}^{\infty} e^{j2\pi f \tau_2}\, R_X(\lambda)\, e^{j2\pi f \lambda}\, d\lambda \qquad \left[\because R_X(-\lambda) = R_X(\lambda) \text{ symmetric}\right]$$

can be + or −

constant taken outside

$$\underbrace{\qquad}\ \mathcal{F}\{R_X(\tau)\} = S_X(f)$$

take $\tau_2 - \tau_1 = \lambda \rightarrow \tau_1 = \tau_2 - \lambda \rightarrow d\tau_1 = -d\lambda$

$-\lambda = \tau$
$-d\lambda = d\tau$

$$\int_{-\infty}^{\infty} H(f)\,df \int_{-\infty}^{\infty} \overbrace{h(\tau_2)\, e^{-j2\pi f \tau_2}}^{H^*(f)} \int_{-\infty}^{\infty} R_X(\tau)\, e^{\pm j2\pi f \tau}\, d\tau$$

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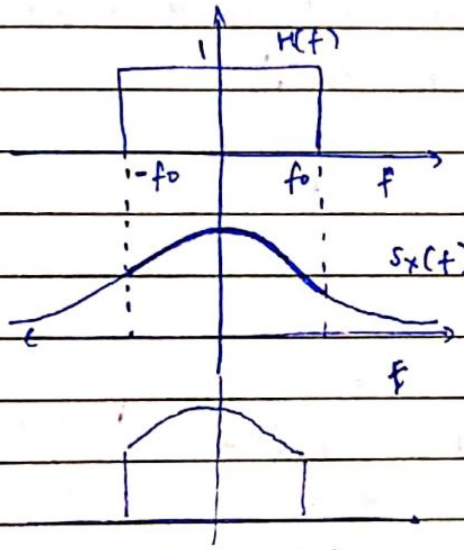
$$E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 \, S_X(f) \, df$$

which is always +ve

represents total power in $Y(t)$

$$x(t) \longrightarrow \boxed{\begin{matrix} h(t) \\ H(f) \end{matrix}} \longrightarrow Y(t)$$

Total power can be found by taking the square of magnitude of the frequency response $(|H(f)|^2)$

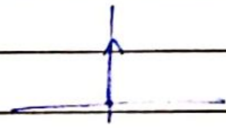


} acts as LPF

→ power content in each frequency component.
Total given by integrating $S_X(f)$

O/p power content < I/p power content.

Reduce fo such that PSD of $Y(t)$ is an impulse



& gives Power Content / Hertz.

- For a deterministic signal, we can take the fourier transform of the waveform. But in R.P., fourier transform will have to be taken for every sample function - collective characterization not possible. Hence, we do it for the autocorrelation.

- On passing fewer frequencies through $H(t)$, R.P. will become less Random.

* Weiner Khinchine Theorem: autocorrelation function & power spectral density function form a Fourier Transform pair

⇒ PSD of a WSS process
ACF & PSD are FT pair, which is referred to as Wiener Khinchine theorem.

$$\underbrace{R_x(\tau)}_{\substack{\text{ACF of } x(t)\\ x(t) \text{ in TD}}} = \int_{-\infty}^{\infty} \underbrace{S_x(f)}_{\substack{\text{PSD of } x(t)\\ x(t) \text{ in FD}}} e^{j2\pi f\tau}\, df \qquad \forall \tau$$

$$S_X(f) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad \forall f$$

$$= \int_{-\infty}^{\infty} R_x(\tau) \cos 2\pi f\tau \, d\tau$$

- $R_x(\tau) = R_x(-\tau)$ → it is real & symmetric (even) function.

$S_x(f) = S_x(-f)$

$$S_x(f) = \int_{-\infty}^{\infty} S_x(f) \cos 2\pi f\tau \, df$$

$S_x(f)$ cannot be -ve values because $S_x(f)$ is Power/Hz. Since Power cannot be -ve, hence $S_x(f)$ is always +ve.

$$S_x(0) = \int_{-\infty}^{\infty} R_x(\tau)\, d\tau$$, represents dc power.

$$E[x^2(t)] = R_x(0) = \int_{-\infty}^{\infty} S_x(f)\, df$$

$E[x(t)x(t+0)]$

→ Let us say,

$$R_x(\tau) = e^{-2\alpha|\tau|}, \quad \alpha>0$$

$S_x(f) = ?$

$$\Rightarrow S_x(f) = \int_{-\infty}^{0} e^{-2\alpha\tau} \cdot e^{-j2\pi f\tau}\, df + \int_{0}^{\infty} e^{2\alpha\tau} \cdot e^{-j2\pi f\tau}\, df$$

$$= \int_{-\infty}^{0} e^{-(2\alpha\tau + j2\pi f\tau)}\, df + \int_{0}^{\infty} e^{2\alpha\tau + (-j2\pi f\tau)}\, df$$

$$= e^{-2\alpha\tau} \cdot \int_{-\infty}^{0} e^{-j2\pi f\tau}\, df + e^{2\alpha\tau} \int_{0}^{\infty} e^{-j2\pi f\tau}\, df$$

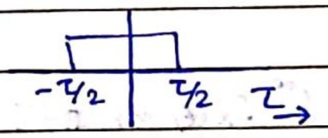
$$= -e^{-2\alpha\tau} \left[\frac{e^{-j2\pi f\tau}}{j2\pi\tau} \right]_{-\infty}^{0} + -e^{2\alpha\tau} \left[\frac{e^{-j2\pi f\tau}}{j2\pi\tau} \right]_{0}^{\infty}$$

$$= -e^{-2\alpha\tau}\frac{1}{j2\pi\tau} + e^{+2\alpha\tau}\left[\frac{1}{j2\pi\tau}\right]$$

$$= \frac{-e^{-2\alpha\tau}}{j2\pi\tau} + \frac{e^{2\alpha\tau}}{j2\pi\tau}$$

This is a valid auto-correlation function.

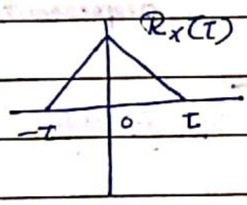
→ Consider $R_X(\tau)$ as a gate function.



FT of this is sinc function which has -ve values of in PSD which cannot be -ve.

* If there is a discontinuity in $R_X(\tau)$, Autocorrelation function will not be valid.

→ Consider



It is valid auto-correlation as it is sinc^2 function

→ $E[y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\, df$

$\int_{-\infty}^{\infty} S_y(f)\, df$

$$S_y(f) = S_X(f)\,|H(f)|^2$$

FT is not defined for r.p. but PSD exists.

$$Y(f) = H(f)\,X(f)$$

Ex→ Find PSD for r.p. $X(t) = A\cos(\omega_0 t + \theta)$, θ is uniform $(0, 2\pi)$

$R_X(\tau) = A^2/2 \times \cos\omega_0\tau$, $\forall \tau$

$S_X(f) = FT[R_X(\tau)] = \frac{A^2}{4}\left[\delta(f-f_0) + \delta(f+f_0)\right]$

Total power in $X(t) = R_X(0) = \frac{A^2}{2}$ OR $\int_{-\infty}^{\infty} S_X(f)\, df = \frac{A^2}{2}$

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Ex → Let $Y(t) = X(t)\cos(\omega_c t + \theta)$, $X(t)$ and θ are independent, θ is uniform $(0, 2\pi)$, $X(t)$ is a WSS process with known $R_X(\tau)$ and $S_X(f)$. Find ACF & PSD of $Y(t)$.

It is like ~~DSBC~~ DSBSC

$$R_Y(\tau) = E[Y(t)Y(t+\tau)]$$

$$= E[X(t)\cos(\omega_c t + \theta)\, X(t+\tau)\cos(\omega_c(t+\tau) + \theta)]$$

$$= E[X(t)X(t+\tau)] \times E[\cos(\omega_c t + \theta)\cos(\omega_c(t+\tau) + \theta)]$$

$$= \frac{1}{2} R_X(\tau)\cos\omega_c \tau \quad , \forall \tau$$

$$S_Y(f) = FT[R_Y(\tau)] = \frac{1}{4}[S_X(f - f_c) + S_X(f + f_c)]$$

* ACF & PSD of a WSS R.P. $X(t)$

$$R_X(\tau) \xleftrightarrow{\mathcal{F}} S_X(f)$$

time domain representation of a R.P. — frequency domain representation of a R.P.

Total power in $X(t)$ = $E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f)\,df = R_X(0)$

→ One can show that $R_X(0) \geq R_X(\tau)$.

To show this, consider $E[X(t) - X(t+\tau)]^2 \geq 0$

$$\underbrace{E[X^2(t)]}_{R_X(0)} - 2\underbrace{R_X(\tau)}_{R_X(\tau)} + E[X^2(t+\tau)]$$

$$R_X(0) \geq 2R_X(\tau) - E$$

For speech as a random process

$$\lim_{\tau\to\infty} R_X(\tau) = \lim_{\tau\to\infty} E[X(t)X(t+\tau)] = \lim_{\tau\to\infty} E[X(t)]\cdot E[X(t+\tau)]$$

i.e. No dependency (independent) / No correlation when time difference becomes very large.

for $R_X(\tau) = \underbrace{\cos\omega_c\tau}_{\text{periodic}}$ — this is not applicable

classmate
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⇒ GUASSIAN RANDOM PROCESS.

Consider a R.P. $X(t)$ sampled at $t_1, t_2 \dots t_n$ $n \geq 1$

at every t, we have a random variable taking values as

$x_1(t_1), x_2(t_1), x_3(t_1) \rightarrow X(t_1)$

$X(t_1), X(t_2) \dots X(t_n)$ are Random Variable.

If the Joint P.D.F of these R.V.s is guassian, then the random process is guassian.

$\hookrightarrow f_{X(t_1) X(t_2) \dots X(t_n)}(x_1, x_2 \dots x_n)$

$$f_{X(t_1) X(t_2) \dots X(t_n)}(x_1, x_2 \dots x_n) = \frac{1}{(2\pi)^{n/2} |C_{X(t)}|^{1/2}} \exp\left[-\frac{1}{2}(\underline{x}-\underline{m}_x)^T C_{X(t)}^{-1} (\underline{x}-\underline{m}_x)\right]$$

$\underset{n \times n}{C_x} \rightarrow$ covariance matrix of R.V.s $X(t_1) \dots X(t_n)$

$X(t)$ as guassian Process depends only on means & covariances.

If this process is WSS, then it is a WSS Guassian Process.

→ Consider an $n=2$, SSS guassian process. Then we have

$$\therefore f_{X(t_1) X(t_2)}(x_1, x_2) = f_{X(t_1+\tau) X(t_2+\tau)}(x_1, x_2)$$

→ Covariance matrix of WSS guassian process with $n = n$.

$$\cancel{\begin{bmatrix} \sigma^2_{X(t_1)} & \mathrm{Cov}(X(t_1), X(t_2)) \\ \mathrm{Cov}(X(t_2), X(t_1)) & \sigma^2_{X(t_2)} \end{bmatrix}} \quad \begin{bmatrix} \sigma^2_{X(t_1)} & \mathrm{Cov}(X(t_1)\, X(t_1+\tau)) \\ & \sigma^2_{X(t_2)} \end{bmatrix}$$

$\Downarrow$ $\sigma^2_{X(t_1+\tau)}$

$$\mathrm{Cov}[X(t_1)\, X(t_1+\tau)] = E[X(t_1)\, X(t_1+\tau)] - \mu^2_{X(t_1)}$$

same for ~~all~~ $x(t)$.

$$\downarrow$$

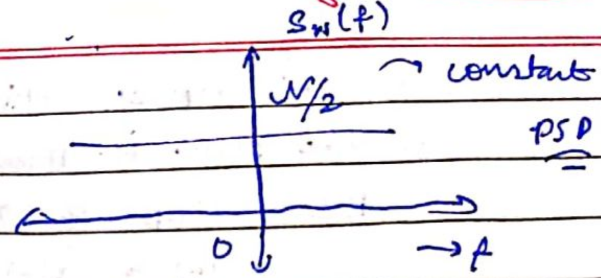
$$\mathrm{Cov}(t_1 - t_2) = R_X(t_1 - t_2) - \mu^2_{X(t_1)}$$

* A WSS guassian process is also an SSS guassian process.

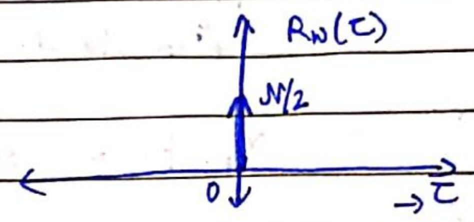
Scanned by CamScanner

* White Process $W(t)$

$W(t)$ has constant PSD given by $\frac{N}{2}$



Autocorrelation is an impulse at $\tau = 0$ of strength $\frac{N}{2}$



zero for $\forall$ τ other than $\tau = 0$

Indicates High uncorrelation

Total power content $= \infty \leftarrow \int_{-\infty}^{\infty} S_W(f)\, df$

White noise is fictitious

→ Band-Pass Signal Representation. (Analytic representation of signals)

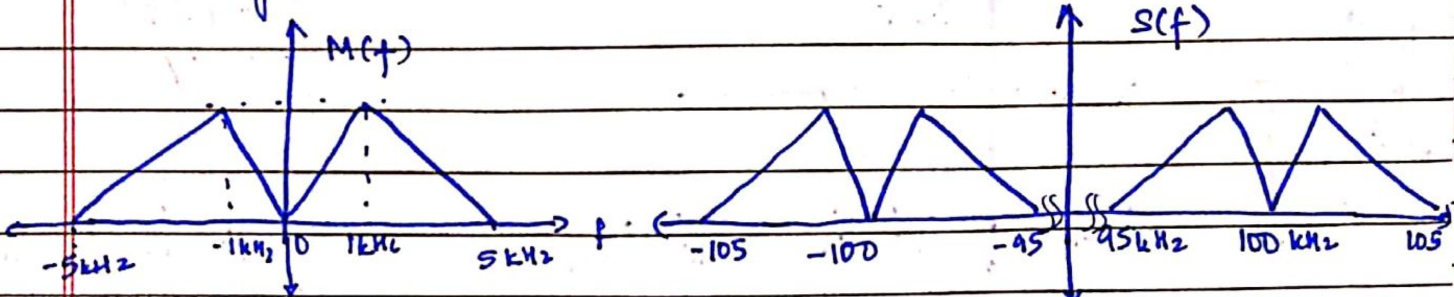
Consider a communication system that transmits

$$s(t) = m(t) \cos 2\pi f_c t = m(t) \cos \omega_c t$$

$m(t)$ → modulating / baseband signal $\qquad f_c \to$ carrier frequency.

$$S(f) = \mathcal{F}\{s(t)\} = \frac{1}{2}\left[M(f-f_c) + M(f+f_c) \right]$$

If $f_c = 100$ kHz



S(f) is called the Band-Pass signal

It can also be referred to as narrow band bandpass signal (NBBP)

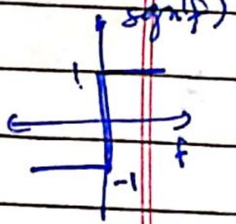
$f_c = 100$ kHz $\gg$ B.W. of modulated signal $= 10$ kHz

- Carrier does not carry information about the message signal.
- Studying NBBP signals & systems using analytic representation of signal.

In analytic representation of bandpass signals, allows us to represent them in terms of their 'low pass equivalents'

Pre-envelope of $x(t)$

$$\underbrace{x_{pe}(t)}_{\text{complex signal}} = x(t) + j\,\underbrace{\hat{x}(t)}_{\text{Hilbert transform of } x(t)}$$

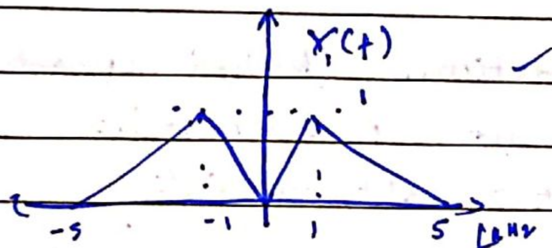


$$X_{pe}(f) = X(f) + j\left[\underbrace{-j\,\text{sgn}(f)}_{H(f)}\; X(f)\right]$$

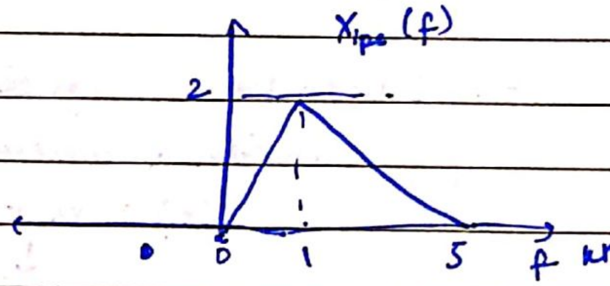
$$= X(f) + \text{sgn}(f)\, X(f)$$

$$= \begin{cases} 2X(f) & \text{for } f>0 \\ X(0) & \text{for } f=0 \\ 0 & \text{for } f<0 \end{cases}$$

Low pass (band limited)



$\longrightarrow$



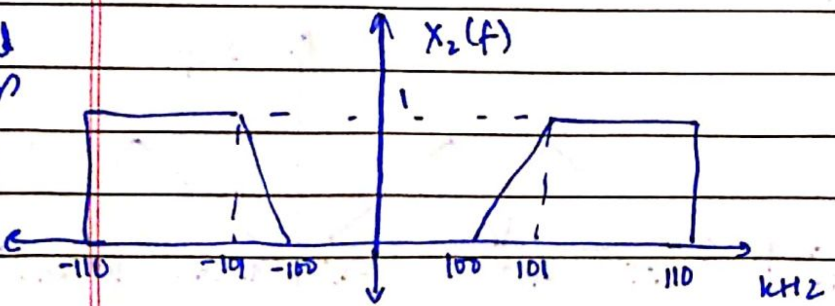
real signal

$\mathcal{F}\{x_1(t)\} = X_1(f)$

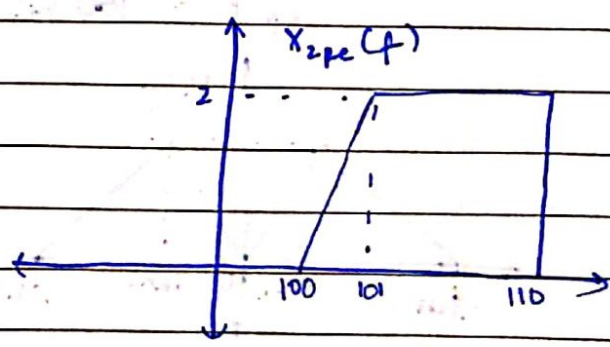
complex signal

$\mathcal{F}\{x_1(t) + j\hat{x}_1(t)\} = X_{1pe}(f)$

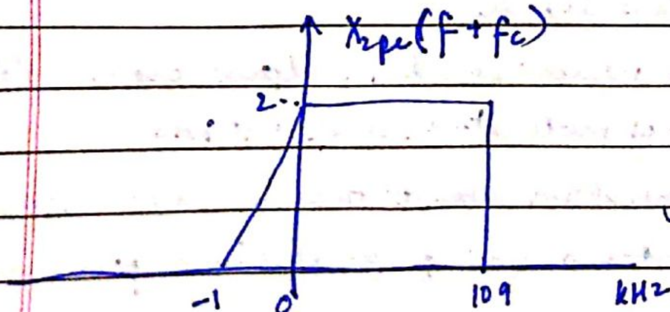
Band pass NBBP



Real signal



complex signal



$f_c = 100$ kHz

complex signal.

$$\mathcal{F}^{-1}\left[X_{pe}(f+f_c)\right] = x_{pe}(t)\, e^{-j2\pi f_c t} = x_{ce}(t)$$

$$x_{ce}(t) = x_{pe}(t)\, e^{-j2\pi f_c t}$$

$\mathcal{F} \downarrow$

$$X_{ce}(f) = X_{pe}(f+f_c)$$

$$x_{pe}(t) = e^{j2\pi f_c t}\, x_{ce}(t)$$

$\hookrightarrow$ in general, complex $\rightarrow$ but could be real also

$$= x_{ce}(t)\, e^{j2\pi f_c t}$$

$$= \underbrace{[\,x_R(t) + j\,x_I(t)\,]}_{\text{low pass signal}}\, e^{j2\pi f_c t}$$

real part — img. part.

also since $x_{pe}(t) = x(t) + j\,\hat{x}(t)$

$$x(t) = \mathrm{Re}[x_{pe}(t)]$$

$$x_{pe}(t) = [x_R(t) + j\,x_I(t)]\,[\cos 2\pi f_c t + j \sin 2\pi f_c t]$$

$$x(t) = \mathrm{Re}[x_{pe}(t)] = \underbrace{(x_R(t)\cos 2\pi f_c t - x_I(t)\sin 2\pi f_c t)}_{\text{canonical representation of } x(t)}$$

$x_R(t)$: low pass, in-phase component

$x_I(t)$: quadrature component (cos & sin 90° out of phase)