CCV-7   SC216 Calculus with complex variables   Jan 31, 2008

Numerical Integration

Please look at visual calculus web site for it.

Directional Derivatives

Note that for a fn. $z=f(x,y)$ the partial derivatives ~~represent the ro~~ f_x & f_y represent the rate of change of f in the directions of the unit vectors $\hat{i}$ & $\hat{j}$

— How to find derivative of $f(x,y)$ in any direction?

Note $$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

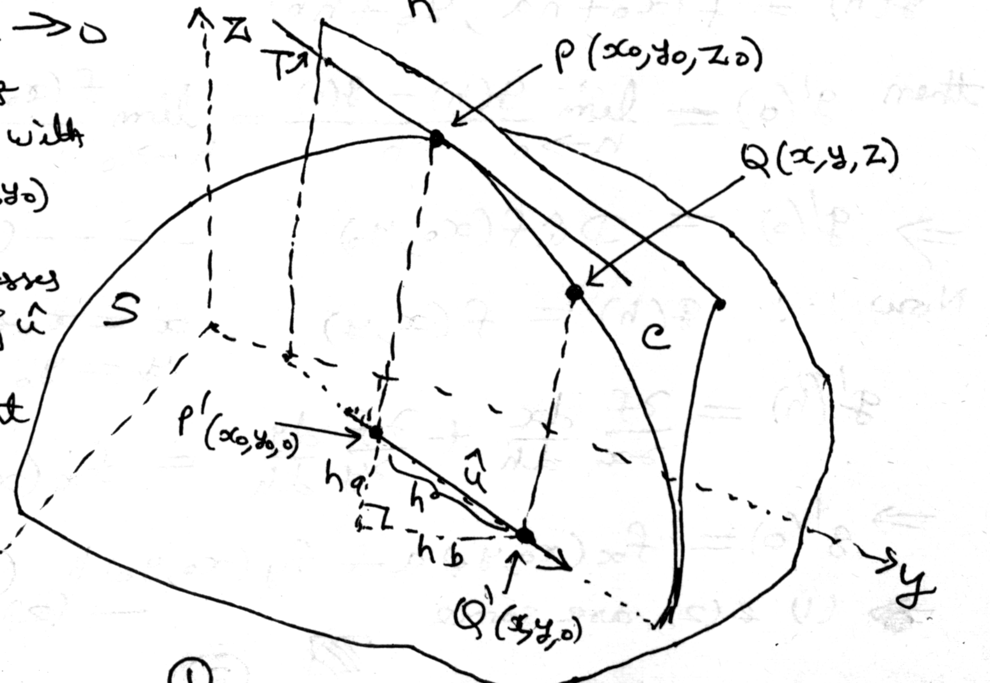
& $$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

So ~~if~~ if $\hat{u} = \langle a, b\rangle$ is an arbitary unit vector we can define what we call a directional derivative of f at (x_0,y_0) in the direction ~~of~~ of a unit vector $\hat{u} = \langle a,b\rangle$ as

$$D_uf(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

if this limit exists

- The graph of f = surface S with eqn $z=f(x,y)$ and $z_0 = f(x_0,y_0)$
- Pt. $P(x_0,y_0,z_0)$ lies on S
- The vertical plane that passes through P in the direction of $\hat{u}$ intersects S in a curve C
- The slope of the tangent line T to C at pt P is the rate of change of z in the dirn of $\hat{u}$



— If $Q(x, y, z)$ is another pt. on C and P' & Q' are projections of P, Q on the xy-plane then the vector $\overrightarrow{P'Q'}$ is $\|^{al}$ to $\hat{u}$ and so

$$\overrightarrow{P'Q'} = h\hat{u} = \langle ha, hb \rangle \quad \text{for some scalar } h$$

$$\therefore \left.\begin{array}{l} x - x_0 = ha \\ \& \ y - y_0 = hb \end{array}\right\} \Rightarrow \frac{\Delta z}{h} = \frac{z - z_0}{h}$$

$$= \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

— If we take limit $h \to 0$ this is rate of change of z (w.r.t. to distance) in the direction of $\hat{u}$ is called directional derivative

— Note if $\hat{u} = \hat{i} = \langle 1, 0 \rangle$ then $D_{\hat{i}} f = f_x$ & simly for $\hat{u} = \hat{j} = \langle 0, 1 \rangle$

$\boxed{\text{Theorem}}$ If f is a differentiable fn. of x & y then f has a directional derivative in the direction of any unit vector $\bar{u} = \langle a, b \rangle$ and

$$D_{\hat{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

□ If we define a fn. g of single variable

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then $g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$

$\Rightarrow g'(0) = D_{\hat{u}} f(x_0, y_0)$ ----(1)

Now $\because g(h) = f(x, y)$, $\left.\begin{array}{l} x = x_0 + ha \\ y = y_0 + hb \end{array}\right\} \Rightarrow$ By chain rule

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)\, a + f_y(x, y)\, b$$

$\Rightarrow g'(0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$ —(2) $\left(\because \text{for } h = 0 \ \ x = x_0, y = y_0\right)$

$\Rightarrow$ (1) & (2) are same

– If the unit vector $\hat{u}$ makes an angle θ with the x-axis then $\hat{u} = \langle \cos\theta, \sin\theta \rangle$ ~~& then~~

$$\Rightarrow \boxed{D_{\hat{u}} f(x,y) = f_x(x,y) \cos\theta + f_y(x,y) \sin\theta}$$

Example Find the directional derivative $D_{\hat{u}} f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ and $\hat{u}$ is the unit vector given by angle $\theta = \pi/6$

What is $D_{\hat{u}} f(1,2) = ?$

□ $\because \theta = \pi/6$

$$\Rightarrow D_{\hat{u}} f(x,y) = f_x(x,y) \cos\frac{\pi}{6} + f_y(x,y) \sin\pi/6$$

$$= (3x^2 - 3y) \frac{\sqrt{3}}{2} + (-3x + 8y) \cdot \frac{1}{2}$$

$$\therefore D_{\hat{u}} f(1,2) = \frac{13 - 3\sqrt{3}}{2}$$

Gradient vector

$$D_{\hat{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b$$ ← dot product

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle$$

$$\boxed{D_{\hat{u}} f(x,y) = \underbrace{\langle f_x, f_y \rangle}_{\text{Gradient vector}} \cdot \hat{u}}$$

$$\nabla f = \nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$= \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j}$$

Example If $f(x,y) = \sin x + e^{xy}$

then $\nabla f = \langle f_x, f_y \rangle = \langle \cos x + y e^{xy}, x e^{xy} \rangle$

& $\nabla f(0,0) = \langle 2, 0 \rangle$

- Thus the directional derivative can be written as

$$\boxed{D_u f(x,y) = \nabla f(x,y) \cdot \hat{u}}$$

(scalar projection of the gradient vector onto $\hat{u}$)

Example Find the dir.al derivative of the fn. $f(x,y) = x^2y^3 - 4y$ at $(2,-1)$ in the direction of the vector $\bar{v} = 2\hat{i} + 5\hat{j}$.

□ The gradient vector $\nabla f = 2xy^3\hat{i} + (3x^2y^2 - 4)\hat{j}$ at $(2,-1)$

$$\Rightarrow \nabla f(2,-1) = -4\hat{i} + 8\hat{j}$$

Now $\bar{v}$ is not a unit vector but $|\bar{v}| = \sqrt{29}$

$$\Rightarrow \hat{u} = \frac{\bar{v}}{|\bar{v}|} = \frac{2}{\sqrt{29}}\hat{i} + \frac{5}{\sqrt{29}}\hat{j}$$

Thus
$$D_u f(2,-1) = \nabla f(2,-1) \cdot \hat{u}$$
$$= (-4\hat{i} + 8\hat{j}) \cdot \left(\frac{2}{\sqrt{29}}\hat{i} + \frac{5}{\sqrt{29}}\hat{j}\right)$$
$$= \frac{32}{\sqrt{29}}$$

FUNCTIONS OF 3 VARIABLES

- The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\bar{u} = \langle a, b, c \rangle$ is

$$D_{\hat{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

— Using vector notation we can write

$$D_{\hat{u}} f(\bar{x}_0) = \lim_{h \to 0} \frac{f(\bar{x}_0 + h\hat{u}) - f(\bar{x}_0)}{h}$$

where $\bar{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$

$\bar{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$

— Note this is ~~a~~ reasonable since the vector eq^n of the line through $\bar{x}_0$ in the direction of the vector $\hat{u}$ is given by $\bar{x} = \bar{x}_0 + t\hat{u}$

& so $f(\bar{x}_0 + h\hat{u})$ represents the value of f at a pt. on this line.

— If $f(x,y,z)$ is differentiable & $\hat{u} = \langle a, b, c \rangle$ then we can show that

$$D_{\hat{u}} f(x,y,z) = f_x(x,y,z)\, a + f_y(x,y,z)\, b + f_z(x,y,z)\, c$$

(Verify!)

— Gradient vector

$$\nabla f \text{ or grad} f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

— $$D_{\hat{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \hat{u}$$

$\boxed{\text{Exercise}}$ If $f(x,y,z) = x \sin yz$ find the gradient of f and find the directional derivative of f at $(1,3,0)$ in the direction of $\bar{v} = \hat{i} + 2\hat{j} - \hat{k}$.

☐ Answer $D_{\hat{u}} f(1,3,0) = -\sqrt{\frac{3}{2}}$ ▨

$\boxed{\text{Maximizing the directional derivative}}$

Given a fn. f of 2 or more variables.
Consider all possible directional derivatives of f at a given pt. ~~This~~ Those give the rates of changes of f in all possible directions.

Question: In which of this direction does f changes fastest and what is the max rate of change?

Theorem Suppose f is differentiable of 2 or 3 variables. The max value of the directional derivative $D_{\hat{u}} f(\bar{x})$ is $|\nabla f(\bar{x})|$ and it occurs when $\hat{u}$ has the same direction as the gradient vector $\nabla f(\bar{x})$.

$\square$ $D_{\hat{u}} f = \nabla f \cdot \hat{u} = |\nabla f| \, |\hat{u}| \cos\theta$

$= |\nabla f| \cos\theta \qquad (\because |\hat{u}| = 1)$

θ = angle between ∇f and $\hat{u}$.

$\cos\theta$ is max when $\theta = 0$ $(\Rightarrow \cos\theta = 1)$

$\therefore$ The max. value of $D_u f$ is $|\nabla f|$ and it occurs at $\theta = 0$ (i.e. when $\hat{u}$ has the same direction as ∇f) $\blacksquare$

$\boxed{\text{Example}}$ If $f(x,y) = xe^{y}$ (a) find the rate of change of f at the pt. $P(2,0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the max rate of change? What is this max. rate of change?

$\square$ (a) $\nabla f = \langle f_x, f_y \rangle = \langle e^{y}, xe^{y} \rangle$

$\Rightarrow \nabla f(2,0) = \langle 1, 2 \rangle$

The unit vector in the direction of $\vec{PQ} = \langle -1.5, 2 \rangle$ is $\hat{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$

$\Rightarrow$ The rate of change of f in the direction from P to Q is $D_{\hat{u}} f(2,0) = \nabla f(2,0) \cdot \hat{u} = \langle 1, 2 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle = 1$

(b) f increases fastest in the direction of the gradient vector $\nabla f(2,0) = \langle 1, 2 \rangle$. The max. rate of change is $|\nabla f(2,0)| = |\langle 1, 2 \rangle| = \sqrt{5}$ ▨

$\boxed{\text{Example}}$ Suppose that the temprature at a pt. (x,y,z) in space is given by $T(x,y,z) = \dfrac{80}{(1+x^2+2y^2+3z^2)}$, where T is measured in deg. Celsius. & x, y, z in meters. In which direction the temp. increases fastest at the pt. $(1,1,-2)$? What is the max rate of increase?

$\square$ $\nabla T = \dfrac{\partial T}{\partial x}\hat{i} + \dfrac{\partial T}{\partial y}\hat{j} + \dfrac{\partial T}{\partial z}\hat{k}$

$$\nabla T = \frac{160}{(1+x^2+2y^2+3z^2)^2}\left(-x\hat{i} - 2y\hat{j} - 3z\hat{k}\right)$$

$\Rightarrow \nabla T(1,1,-2) = \frac{5}{8}\left(-\hat{i} - 2\hat{j} + 6\hat{k}\right)$. Thus the temp. increases faster in the direct of $\nabla T(1,1,-2)$ or $\simeq$ in the direction of $-\hat{i} - 2\hat{j} + 6\hat{k}$ or the unit vector $(-i - 2j + 6k)/\sqrt{41}$. The max. rate of increase is the length of grad. vector $= |\nabla T(1,1,-2)| = \frac{5\sqrt{41}}{8}$

$\frac{5\sqrt{41}}{8} \simeq 4°C/m$. ▨