Supplementary Topic

Concept of Rescaling Equations

A general form of the Gaussian equation is $\boxed{y = y_0 e^{-a(x-\mu)^2}}$.

Define $\boxed{Y = \frac{y}{y_0}}$ and $\boxed{X = \frac{x-\mu}{\sqrt{1/a}}}$ to get $\boxed{Y = e^{-X^2}}$

5/. $\boxed{y = x \ln x}$ i.) x is always > 0. When $x > 1$, $y > 0$ and when $0 < x < 1$, $y < 0$. Hence the function is in the first and fourth quadrants.

ii.) when $x = 1$, $y = 0$. $\therefore$ The function passes through the point $(1, 0)$.

iii.) $\boxed{\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x}$ when $\ln x + 1 = 0$, $\Rightarrow x = e^{-1} = 1/e$. There is a turning point of the function at $x = 1/e$.

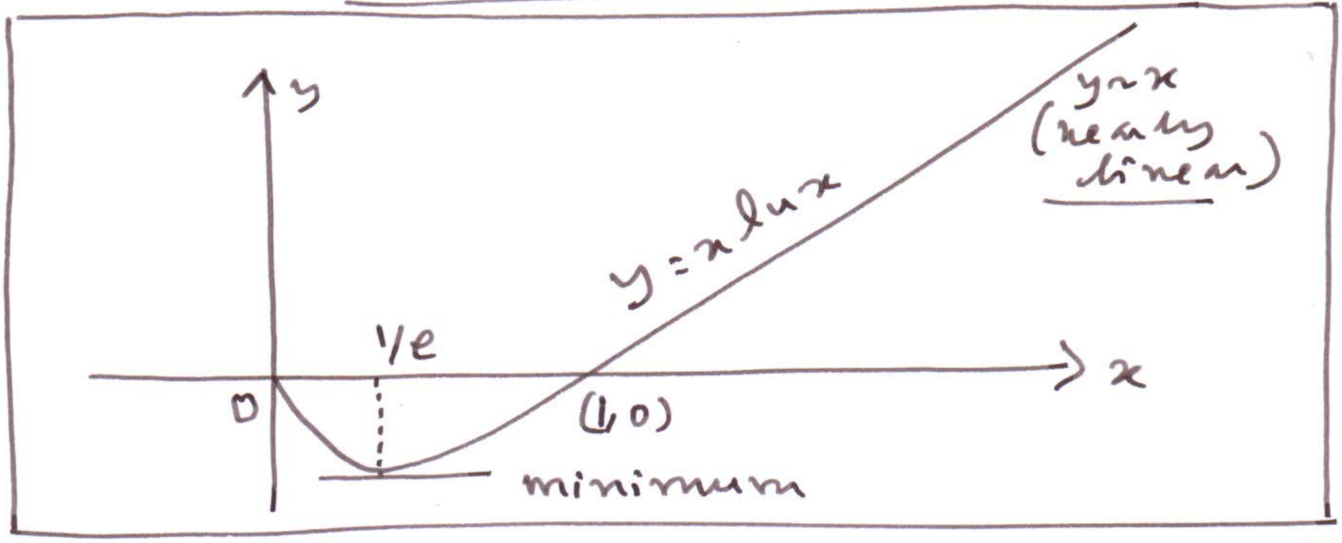
$\boxed{\frac{d^2y}{dx^2} = \frac{1}{x}}$. At $x = 1/e$ $\frac{d^2y}{dx^2} = e > 0$.
$\Rightarrow$ The turning point is a minimum

iv.) When $x \to \infty$, $\ln x$ changes very slowly and is ~~pr~~ practically a constant.
$\Rightarrow y \sim x$ for large values of x.

v.) When $x \to 0$, $y \to 0 \times -\infty$.

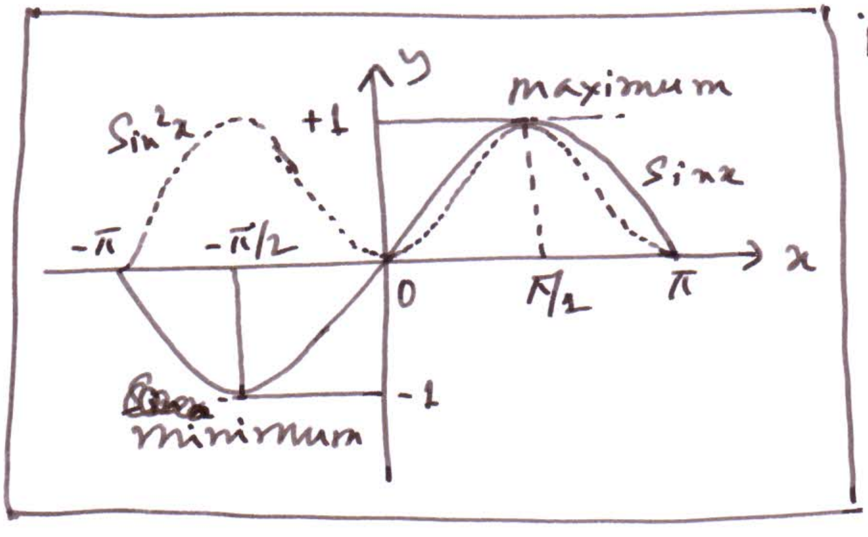
Apply L'Hospital's Rule on $\boxed{y = x \ln x = \frac{\ln x}{1/x}}$ when $x \to 0$. $\Rightarrow y = \frac{1/x}{-1/x^2} = -x \to 0$ when $x \to 0$.

Hence at $\boxed{x = 0, \quad y = 0}$ (will not work on a computer)



6/. $\boxed{y = \sin^2 x}$ i.) For any x, $\sin x$ varies between $\boxed{-1 < \sin x < 1}$. Hence, $\boxed{0 < \sin^2 x < 1}$. This is an even function, i.e. $y(x) = y(-x)$.

ii.) Since $|\sin x| \leq 1$, $\boxed{|\sin^2 x| < |\sin x|}$, except at $\underline{\sin x = 0}$ and $\underline{\sin x = 1}$.



iii.) Average of $\sin x$ over one full cycle $= 0 \Rightarrow \boxed{\langle \sin \rangle = 0}$. This is due to the symmetry above and below the line $y = 0$.

iv.) Over a full cycle, $\boxed{\langle \sin^2 x \rangle = 1/2}$.

Monomials : A Single Power

1/. $\boxed{y \sim x^{\alpha}} \Rightarrow \boxed{\dfrac{dy}{dx} \sim \alpha\, x^{\alpha - 1}}$

Case I: $\boxed{\alpha = 0} \Rightarrow \boxed{y \sim \text{constant}}$

The plot is a simple horizontal line.

Case II: $\boxed{\alpha = 1} \Rightarrow \boxed{y \sim x}$.

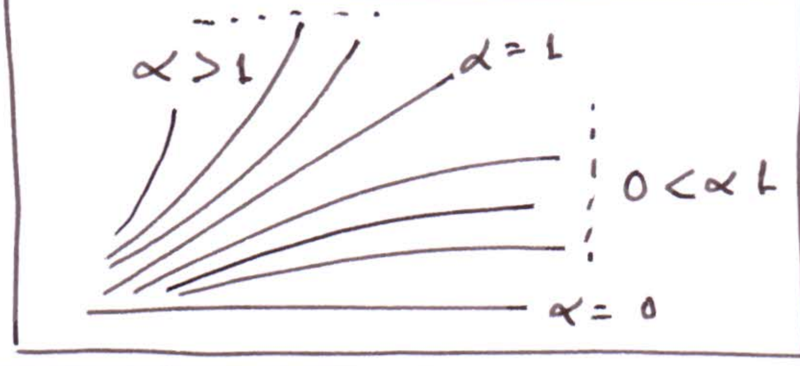
The plot is a straight line, growing at a constant rate.

Case III: $\boxed{0 < \alpha < 1} \Rightarrow \boxed{\dfrac{dy}{dx} \sim \dfrac{\alpha}{x^{1-\alpha}}}$.

The plot is a function that increases at a decreasing rate. When $x \to \infty$, $\dfrac{dy}{dx} \to 0$.

Case IV: $\boxed{\alpha > 1} \Rightarrow \boxed{\dfrac{dy}{dx} \sim \alpha\, x^{\alpha - 1}}$, $\boxed{\alpha - 1 > 0}$.

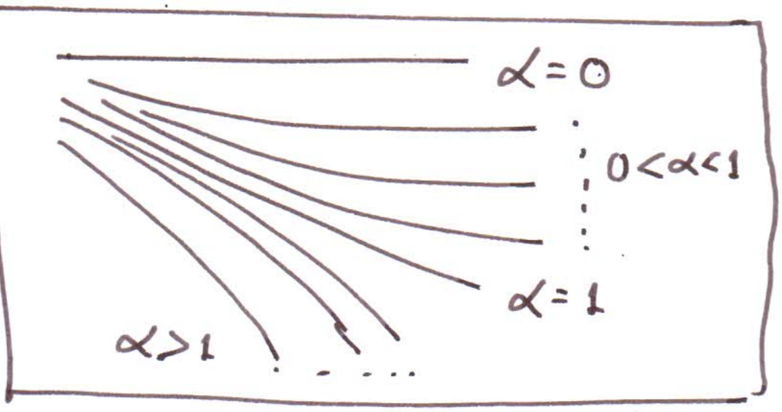
The plot is a function that increases at an increasing rate. When $x \to \infty$, $\dfrac{dy}{dx} \to \infty$.



These solutions show growth because $\boxed{y \sim + x^{\alpha}}$. With a negative sign y will decay.

2/. $\boxed{y \sim -x^{\alpha}}$

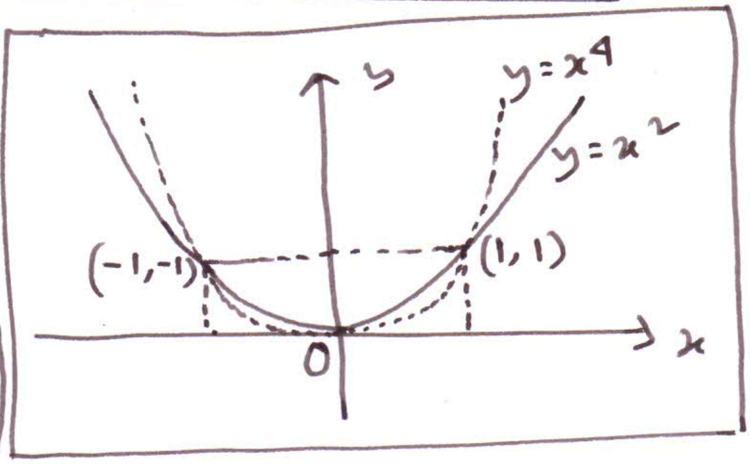
$\boxed{\frac{dy}{dx} \sim -\alpha x^{\alpha-1}}$



Plots of functions $\boxed{y \sim x^{\alpha}}$ with $\boxed{\alpha > 1}$ (α is an integer).

Case I: $\boxed{\alpha \text{ is even}}$

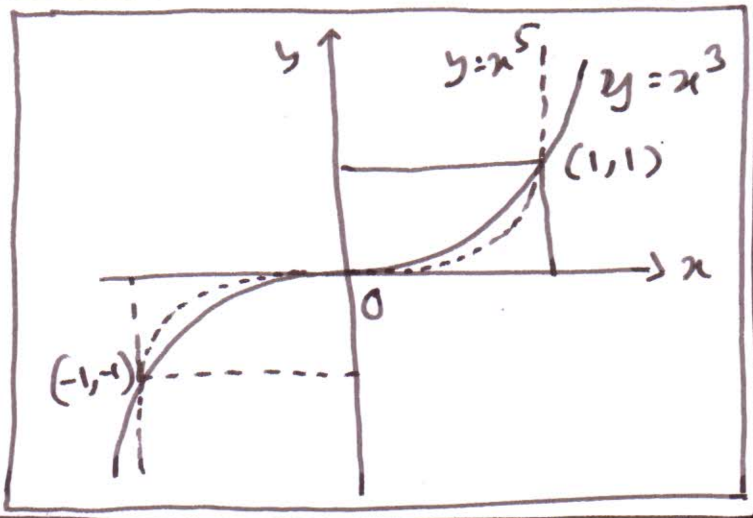
Compare $\boxed{y = x^2}$ and $\boxed{y = x^4}$ $\boxed{\text{At } x=0 \;\; \frac{dy}{dx}=0}$



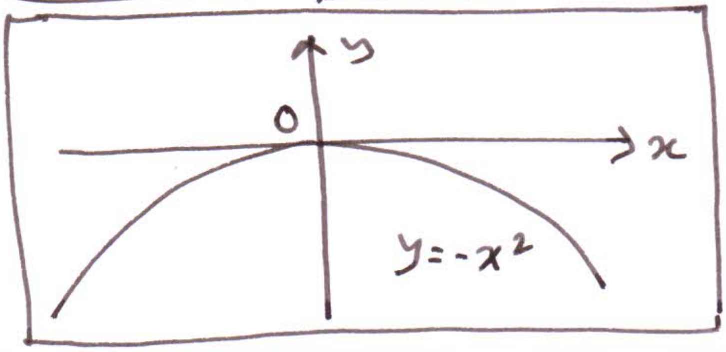
Case II: $\boxed{\alpha \text{ is odd}}$

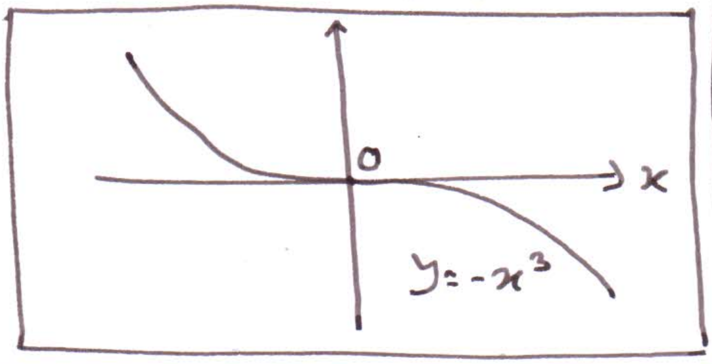
Compare $\boxed{y = x^3}$ and $\boxed{y = x^5}$. At $x=0$

$\boxed{\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0}$.



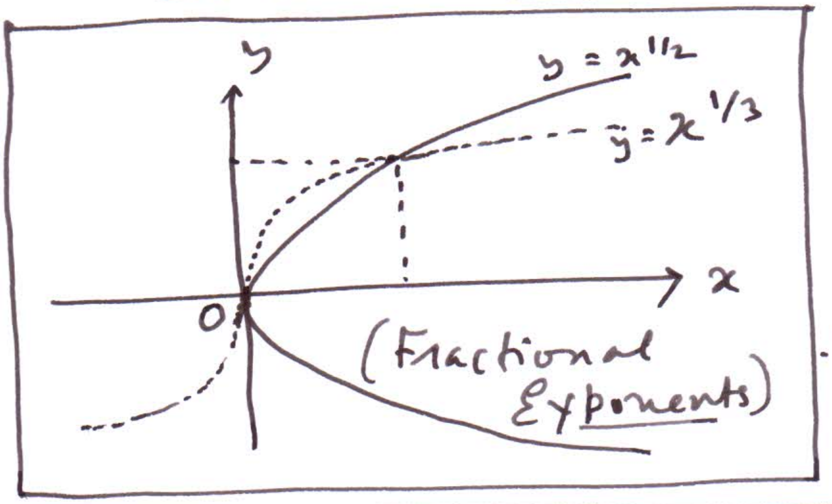
Case III: $\boxed{y \sim -x^{\alpha}}$ with $\boxed{\alpha = 2, 3}$ one odd and one even.





Case IV: $\boxed{y \sim x^{\alpha}}$ $\boxed{0 < \alpha < 1}$ Fractional exponents

Consider $\boxed{\alpha = 1/2,\ 1/3}$ An even root and an odd root.



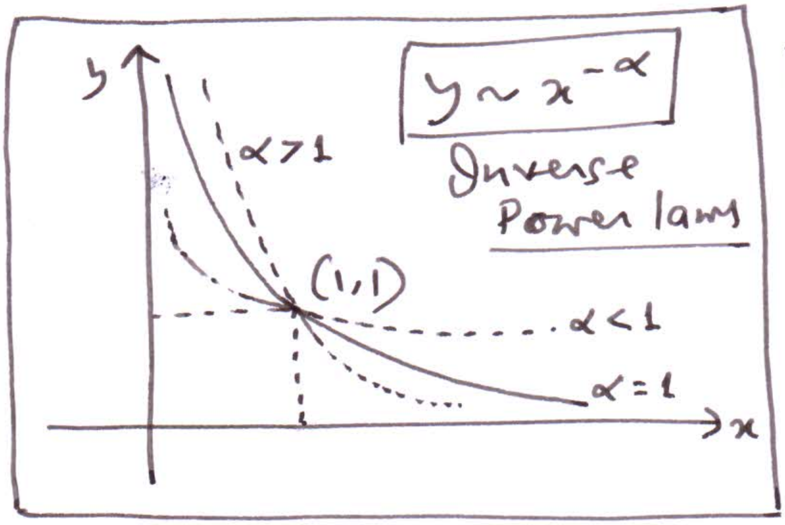
$\boxed{\dfrac{dy}{dx} \sim \dfrac{\alpha}{x^{1-\alpha}}}$ $\Rightarrow$ At the origin $(x = 0)$, $\dfrac{dy}{dx} \to \infty$.

$\therefore$ The function has a vertical tangent at $x = 0$.

Monomials with a Negative Exponent

3/. $\boxed{y \sim x^{-\alpha}}$ and $\boxed{\alpha > 0}$ $\boxed{\dfrac{dy}{dx} \sim \dfrac{-\alpha}{x^{\alpha+1}}}$

$\Rightarrow$ $\boxed{\dfrac{dy}{dx} = \alpha \dfrac{y}{x}}$ $\Rightarrow$ Slope becomes steeper with an increase in α. $(\alpha > 0)$



i.) For $\boxed{y \sim \frac{1}{x}}$ we have the case of a rectangular hyperbola, whose asymptotes are rotated to coincide with the coordinate axes.

ii.) All of the $y \sim x^{-\alpha}$ functions pass through (1,1) as a pivot. For larger values of α, there is a faster convergence when $x \to \infty$.

iii.) Practical examples are the isothermal process ($pV \sim$ constant) and the adiabatic process ($p \propto V^{-\gamma}$).

Taylor Series and Taylor Polynomials

Taylor Series: Given a function $\boxed{y = f(x)}$ about a point $\boxed{x = a}$ we write,

$$\boxed{f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots}$$

in an infinite-order summation series. (The Taylor Series) $\Rightarrow$ $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}}$

Taylor Polynomial: Is extracted from the infinite-order Taylor series, as a finite-order polynomial by applying a truncation up to the desired degree.

Taylor Polynomial of Order 1: $\boxed{p_1(x) = f(a) + f'(a)(x-a)}$
(Equation of a straight line).

Order 2: $\boxed{p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2}$

Order 3: $\boxed{p_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3}$

Order n: $\boxed{p_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$