Two-Point Boundary Value Problem

Consider $\boxed{y''(x) = f(x, y, y')}$ with the initial values provided at two distinct values of x (two-point boundary value)

For instance, $\boxed{y''(x) = p(x)\, y'(x) + q(x)\, y(x) + r(x)}$ is a linear equation, with two boundaries $\boxed{y(a) = g_1}$, $\boxed{y(b) = g_2}$, $\boxed{a \le x \le b}$. The above are boundary conditions, and $p(x)$, $q(x)$, $r(x)$ are continuous on $[a, b]$.

If $q(x) > 0$, there is a unique smooth solution.

Finite Differencing

Divide $[a, b]$ into N equal parts.

$\boxed{a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b}$ $\quad \boxed{h = \frac{b-a}{N}}$

$\boxed{x_i = a + ih}$ $(0 \le i \le N) \Rightarrow x_N = a + Nh = b$

Non-uniform intervals are allowed for functions that are flat in some region and rapidly varying in other regions.

$\boxed{p_i = p(x_i)}$, $\boxed{q_i = q(x_i)}$, $\boxed{r_i = r(x_i)}$.

$\boxed{Y_i = Y(x_i)}$. Its numerical approximation is y_i.

Now, by the central difference formula

$$\boxed{Y'(x_i) = \frac{Y(x_{i+1}) - Y(x_{i-1})}{2h}}$$

and by the method of undetermined coefficients.

$$\boxed{Y''(x_i) \simeq \frac{Y(x_{i+1}) - 2Y(x_i) + Y(x_{i-1})}{h^2} = D_h^2(Y).}$$

Using these we get, (for $1 \le i \le N-1$)

$$\boxed{\frac{Y_{i+1} - 2Y_i - Y_{i-1}}{h^2} = p_i \frac{Y_{i+1} - Y_{i-1}}{2h} + q_i Y_i + r_i + O(h^2)}$$

$$\Rightarrow \boxed{\frac{y_{i+1} - 2y_i - y_{i-1}}{h^2} = p_i \frac{y_{i+1} - y_{i-1}}{2h} + q_i y_i + r_i}$$

Gathering all common terms of y_{i-1}, y_i, y_{i+1},

$$\boxed{-\left(1 + \frac{h}{2} p_i\right) y_{i-1} + \left(2 + h^2 q_i\right) y_i + \left(\frac{h}{2} p_i - 1\right) y_{i+1} = -h^2 r_i}$$

$(1 \le i \le N-1)$

There are $(N-1)$ equations for $N+1$ unknown. $y_0 = g_1$, $y_N = g_2$. We find $y_0, y_1, \dots, y_N$ from the foregoing formula.

~~Special~~ Value at the boundary $(i=1)$:

$$\boxed{(2+h^2 q_1)\, y_1 + \left(\frac{h p_1}{2} - 1\right) y_2 = -h^2 r_1 + \left(1+\frac{h}{2} p_1\right) g_1}$$

Similarly, value at the boundary $(i = N-1)$:

$$\boxed{-\left(1+\frac{h p_{N-1}}{2}\right) y_{N-2} + \left(2+h^2 q_{N-1}\right) y_{N-1} = -h^2 r_{N-1} - \left(\frac{h}{2} p_{N-1} - 1\right) g_2}$$

Boundary Conditions with a Derivative

Consider the difference equation,

$$\boxed{\frac{y_{i+1} - 2y_i - y_{i-1}}{h^2} = f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)}$$

whose one boundary condition is given as a derivative $\boxed{y'(b) + k\, y(b) = g_2}$.

Now $\boxed{y'(b) \simeq \frac{y_N - y_{N-1}}{h}}$. Hence, the derivative equation is $\boxed{\frac{y_N - y_{N-1}}{h} + k y_N = g_2}$

The above formula has an accuracy of $O(h)$ only.