Lecture - 25

Recap:

Distribution of $X+Y$, when X & Y are uniform, independent

triangular distribution

Discrete case $Z = X+Y$

Poisson

$$\left.\begin{array}{ll} X & \lambda_1 \\ Y & \lambda_2 \end{array}\right\} \text{independent}$$

$$P(Z = n) = P(\underset{i}{X} + \underset{n-i}{Y} = n)$$

$$= \sum_{i=0}^{n} P(X=i, Y=n-i)$$

$$\sum_{i=0}^{n} P(X=i, Y=n-i)$$

$$= \sum_{i=0}^{n} P(X=i)\, P(Y=n-i)$$

$$= \sum_{i=0}^{n} e^{-\lambda_1} \frac{\lambda_1^{i}}{i!} \frac{e^{-\lambda_2} \lambda_2^{n-i}}{(n-i)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \lambda_1^{i} \lambda_2^{n-i}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot (\lambda_1+\lambda_2)^{n}$$

$\Rightarrow$ it is Poisson with parameter $\lambda_1 + \lambda_2$.

e.g. Binomial

$$\left.\begin{matrix} X & (n, p) \\ Y & (m, p) \end{matrix}\right\} \text{independent}$$

$$Z = X + Y$$

$$P(Z = k) = P(X + Y = k)$$

$$= \begin{matrix} P(X=0, Y=k) + \\ P(X=1, Y=k-1) + \\ \vdots \\ P(X=k, Y=0) \end{matrix}$$

$$= \sum_{i=0}^{k} P(X=i, Y=k-i)$$

$$= \sum_{i=0}^{k} P(X=i)P(Y=k-i)$$

$$= \sum_{i=0}^{k} \underbrace{\binom{n}{i} p^{i} (1-p)^{n-i}}_{X} \underbrace{\binom{m}{k-i} p^{k-i} (1-p)^{m-k+i}}_{Y}$$

$$= \sum_{i=0}^{k} p^{k} (1-p)^{n+m-k} \binom{n}{i} \binom{m}{k-i}$$

$$= p^{k} (1-p)^{n+m-k} \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}$$

$$= p^{k} (1-p)^{n+m-k} \binom{n+m}{k}$$

$X + Y$ is also binomial with parameters $(n+m, p)$

e.g.:

λ_1 X Poisson } independent

λ_2 Y Poisson }

What is the conditional distribution of X given that $X+Y=n$?

$$P(X=k \mid X+Y=n) =$$

$$\frac{P(X=k,\ X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=k,\ Y=n-k)}{P(X+Y=n)}$$

$$= \frac{\overset{\lambda_1}{P(X=k)}\ \overset{\lambda_2}{P(Y=n-k)}}{\underset{\lambda_1+\lambda_2}{P(X+Y=n)}}$$

⑥

$$\frac{\frac{\cancel{e^{-\lambda_1}}\,\lambda_1^{k}}{k!}\;\frac{\cancel{e^{-\lambda_2}}\,\lambda_2^{n-k}}{(n-k)!}}{\frac{\cancel{e^{-(\lambda_1+\lambda_2)}}\,(\lambda_1+\lambda_2)^{n}}{n!}}$$

$$= \frac{\lambda_1^{k}\,\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^{n}} \cdot \frac{n!}{k!\,(n-k)!}$$

$$= \binom{n}{k} \frac{\lambda_1^{k}\,\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^{k}\,(\lambda_1+\lambda_2)^{n-k}}$$

$$= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{k} \left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-k}$$

$= \text{Binomial with } \left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$

e.g.

$$f(x,y) = \begin{cases} \frac{12}{5} x(2-x-y) & 0 < x < 1 \\ & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Conditional density of X given $Y = y$.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y) = ?}$$

$$f_Y(y) = \int_0^1 f(x,y)\,dx$$

e.g.

$$f(x,y) = \frac{e^{-x/y} \, e^{-y}}{y}$$

$$0 < x < \infty$$
$$0 < y < \infty$$

Compute

$$P(X > 1 \mid Y = y)$$

→ conditional → $f_{X|Y}(x|y) = ?$

→ then integrate $\int_1^\infty \, dx$

H.W.