Solutions Tutorial-10

Q.1 (i) $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$\Rightarrow \dfrac{dx}{dy} = \dfrac{\tan^{-1}y - x}{1+y^2} = \dfrac{\tan^{-1}y}{1+y^2} - \dfrac{x}{1+y^2}$

$\Rightarrow \dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{\tan^{-1}y}{1+y^2}$

I.F. $= e^{\int P(y)\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

Solution.

$x \times \text{I.F} = \int \text{I.F.}\; Q(y)\,dy$

$\Rightarrow x\, e^{\tan^{-1}y} = \int e^{\tan^{-1}y} \cdot \dfrac{\tan^{-1}y}{1+y^2}\,dy + C$

$= \int z e^{z}\,dz + C$ $\qquad z = \tan^{-1}y$, $dz = \dfrac{1}{1+y^2}dy$

$= z e^{z} - e^{z} + C$

$\Rightarrow x\, e^{\tan^{-1}y} = \tan^{-1}y\, e^{\tan^{-1}y} - e^{\tan^{-1}y} + C$

$\Rightarrow x = \tan^{-1}y - 1 + C e^{-\tan^{-1}y}$

(ii) $x\,dy + (y+4)\,dx = 0 \Rightarrow x\,dy = -(y+4)\,dx$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x} - \frac{4}{x}$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = -\frac{4}{x}$ $\quad P(x) = \frac{1}{x}, \quad Q(x) = -\frac{4}{x}$

I.F. $= e^{\int \frac{1}{x}dx} = e^{\log x} = x$

$y \times I.F. = \int Q(x) \cdot I.F.\, dx$

$\Rightarrow yx = \int -\frac{4}{\cancel{x}} \cdot \cancel{x}\, dx = -4x + C$

$\Rightarrow y = -4 + \frac{C}{x}, \quad x > 0$

(iii) $(2y - x^3)\,dx = x\,dy$

$\frac{dy}{dx} = \frac{2y}{x} - x^2$

$\Rightarrow \frac{dy}{dx} - \frac{2y}{x} = -x^2$

I.F. $e^{\int -\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$

$y \times \frac{1}{x^2} = \int \frac{1}{x^2} \cdot (-x^2)\,dx = \int -dx = -x + C$

$\Rightarrow y = -x^3 + Cx^2$

(v) $3y^2 \frac{dy}{dx} + xy^3 = x$

put $y^3 = z \Rightarrow 3y^2 \frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{dz}{dx} + zx = x$

Linear equation in z dependent variable
x independent variable.

I.F. $= e^{\int x\,dx} = e^{\frac{x^2}{2}}$

$\frac{x^2}{2} = p \Rightarrow x\,dx = dp$

$z \cdot e^{x^2/2} = \int x e^{x^2/2} dx = \int e^p dp$
$= e^p = e^{x^2/2} + C$

$\Rightarrow y^3 = 1 + Ce^{-x^2/2}$

(v) ~~$2x^2 \frac{dy}{dx} = (x-1)(y^2-x^2) + 2xy$~~

(v) $\sin y \frac{dy}{dx} = \cos x (2\cos y - \sin^2 x)$

~~put~~

$\Rightarrow \sin y \frac{dy}{dx} = 2\cos x \cos y - \sin^2 x \cos x$

putting $\cos y = z$
$-\sin y \frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow -\frac{dz}{dx} - 2\cos x\, z = -\sin^2 x \cos x$

$\Rightarrow \frac{dz}{dx} + 2z\cos x = \sin^2 x \cos x$
Linear equation

I.F. $e^{\int 2\cos x\,dx} = e^{2\sin x}$

$\sin x = t$
$\cos x\,dx = dt$

so $z e^{2\sin x} = \int e^{2\sin x} \sin^2 x \cos x\,dx$
$= \int t^2 e^{2t} dt + C = \frac{t^2}{2} e^{2t} - \frac{t}{2} e^{2t} + \frac{e^{2t}}{4} + C$

$\Rightarrow \cos y\, e^{2\sin x} = \frac{1}{2}\sin^2 x\, e^{2\sin x} - \frac{1}{2}\sin x\, e^{2\sin x} + \frac{1}{4} e^{2\sin x} + C$

(vi) $(1-x^2)\frac{dy}{dx} - xy - axy^2 = 0$

$$\Rightarrow \frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{axy^2}{1-x^2}$$

$$\Rightarrow y^{-2}\frac{dy}{dx} - \frac{x}{1-x^2}y^{-1} = \frac{ax}{1-x^2}$$

$$y^{-1} = z \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow -\frac{dz}{dx} - \frac{x}{1-x^2}z = \frac{ax}{1-x^2}$$

$$\Rightarrow \frac{dz}{dx} + \frac{x}{1-x^2}z = \frac{-ax}{1-x^2}$$

I.F. $e^{\int \frac{x}{1-x^2}dx} = e^{-\frac{1}{2}\int \frac{1}{t}dt}$ $\quad 1-x^2 = t$, $-2x\,dx = dt$

$$= e^{-\frac{1}{2}\log t} = t^{-\frac{1}{2}} = (1-x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}}$$

$$z\frac{1}{\sqrt{1-x^2}} = \int \frac{-ax}{(1-x^2)}\cdot\frac{1}{\sqrt{1-x^2}}dx + C$$ $\quad 1-x^2 = p$, $-2x\,dx = dp$

$$= -a\int \frac{x}{(1-x^2)^{3/2}}dx + C$$

$$= \frac{-a}{-2}\int \frac{1}{p^{3/2}}dp + C = \frac{a}{2}\frac{p^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C$$

$$= \frac{a}{2}\frac{p^{-\frac{1}{2}}}{-\frac{1}{2}} + C = -a(1-x^2)^{-1/2} + C$$

$$= -a\frac{1}{\sqrt{1-x^2}} + C$$

$$\Rightarrow z = -a + C\sqrt{1-x^2}$$

$$\Rightarrow \frac{1}{y} = -a + C\sqrt{1-x^2} \Rightarrow \left(C\sqrt{1-x^2} - a\right)y = 1$$

(Ans)

(i) $4y'' + 16y' + 52y = 0$

Putting $y = e^{mx}$ we get the auxiliary equation

$$4m^2 + 16m + 52 = 0$$

$\Rightarrow m^2 + 4m + 13 = 0$ $\quad \Rightarrow m = \frac{-4 \pm \sqrt{16 - 4 \cdot 1 \cdot 13}}{2}$

$= \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$

$\Rightarrow$

$m = -2 \pm 3i$

$\alpha = -2, \quad \beta = 3$

$\therefore$ General solution is

$y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x$

$= e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$.

(ii) $4y'' + 4y' + y = 0$

Putting $y = e^{mx}$ we get the auxiliary equation

$4m^2 + 4m + 1 = 0 \quad \Rightarrow m = \frac{-4 \pm \sqrt{16 - 4 \cdot 4 \cdot 1}}{2 \cdot 4} = \frac{-4}{8} = -\frac{1}{2}$

$m = -\frac{1}{2}$ is a repeated root

So the general solution is

$y(x) = c_1 e^{mx} + c_2 x e^{mx}$

$= (c_1 + c_2 x) e^{-\frac{1}{2}x}$

Q.3

(I) $y'' + 4y = 0$, $y(0) = 0$, $y(\pi) = 1$

Put $y = e^{mx}$ to get the auxiliary equation

$m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m = \pm\sqrt{-4} = \pm 2i$

$\alpha = 0$, $\beta = 2$

So general solution is

$y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) = c_1 \cos 2x + c_2 \sin 2x$

$y(0) = 0 \Rightarrow c_1 \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0$

$y(\pi) = 1 \Rightarrow c_1 \cos 2\pi + c_2 \sin 2\pi = 1$

But $c_1 = 0$

$\Rightarrow \underline{\underline{c_2 \sin 2\pi = 1}}$ but $\sin 2\pi = 0$

$\Rightarrow c_2 \sin 2\pi = 0$

which is not 1

So there is no solution to the boundary value problem.

(II) General solution is

$y(x) = c_1 \cos 2x + c_2 \sin 2x$

$y(0) = 0 \Rightarrow c_1 = 0$

$y(\pi) = 0 \Rightarrow c_1 \cos 2\pi + c_2 \sin 2\pi = 0$ as $c_1 = 0$

$\Rightarrow c_2 \sin 2\pi = 0$

This is true for any value of c_2 as $\sin 2\pi = 0$

So there are infinite number of solutions..

Q.4 (1) $y''+y = 2x+3e^x$ — (1)

First solve the associated homogeneous equation

$y''+y=0$

A. eqn $m^2+1=0 \Rightarrow m=\pm i$

$y_c(x) = C_1\cos x + C_2\sin x$

To find particular integral

$G(x)$ contains terms $2x$ and $3e^x$

Also 1 is not a root of auxiliary equation

0 is not a " " " "

So y_p is of the form

$$y_p = (Ax+B)+Ce^x$$

$$y_p' = A+Ce^x$$

$$y_p'' = Ce^x$$

Putting in (1)

$$Ce^x+(Ax+B)+Ce^x = 2x+3e^x$$

$$\Rightarrow 2Ce^x + Ax+B = 2x+3e^x$$

Comparing the coefficients

$$2C=3 \Rightarrow C=3/2, \quad A=2, \quad B=0$$

So particular solution is

$$y_p = 2x+\frac{3}{2}e^x$$

General solution is $y_c+y_p = y(x)$

$$y(x) = C_1\cos x + C_2\sin x + 2x + \frac{3}{2}e^x.$$

(11) Q.4 $y'' - y' - 6y = e^{-x} - 7\cos x$ — (1)

Associated homogeneous equation

$y'' - y' - 6y = 0$

Auxiliary equation $m^2 - m - 6 = 0$

$\Rightarrow (m-3)(m+2) = 0$

$\Rightarrow m_1 = 3, \quad m_2 = -2$

$y_c = c_1 e^{3x} + c_2 e^{-2x}$

$G(x) = e^{-x} - 7\cos x$

It contains a term e^{-x} ~~of~~ -1 is not a root of A. eq^n.

So y_p contains a term Ae^{-x}

$G(x)$ contains $\cos x$ ~~of~~ i is not a root of auxiliary eq^n.

So y_p contains $B\cos x + C\sin x$

We expect $y_p = Ae^{-x} + B\cos x + C\sin x$

$y_p' = -Ae^{-x} - B\sin x + C\cos x$

$y_p'' = Ae^{-x} - B\cos x - C\sin x$

putting in (1)

$y'' - y' - 6y = e^{-x} - 7\cos x$

$\Rightarrow Ae^{-x} - B\cos x - C\sin x + Ae^{-x} + B\sin x - C\cos x - 6Ae^{-x}$
$- 6B\cos x - 6C\sin x = e^{-x} - 7\cos x$

$\Rightarrow$ $-4Ae^{-x} + (B - C - 6B)\cos x + (-C + B - 6C)\sin x = e^{-x} - 7\cos x$

$\Rightarrow -4A = 1 \Rightarrow A = -\frac{1}{4}$

$-7B - C = -7$ and $B - 7C = 0 \Rightarrow B = 7C$

$-7B - C = -49C - C = -7 \Rightarrow -50C = -7$

$\Rightarrow C = \frac{7}{50}$

$B = 7C = \frac{49}{50}$

(ii) continue

$$y_p = -\frac{1}{4}e^{-x} + \frac{49}{50}\cos x + \frac{7}{50}\sin x$$

$$y(x) = y_c + y_p = c_1 e^{3x} + c_2 e^{-2x} - \frac{1}{4}e^{-x} + \frac{49}{50}\cos x + \frac{7}{50}\sin x$$

(Ans)

Q.5 Solve the following differential equation by the method of variation of parameters.

(i) $\frac{d^2y}{dx^2} + y = \operatorname{Cosec} x$

(ii) $(D^2+4)y = 2\tan 2x$

Sol^n

(i) The complementary function is the solution of associated homogeneous equation

$$\frac{d^2y}{dx^2} + y = 0$$

Auxiliary equation $m^2+1=0 \Rightarrow m^2=-1$

$\Rightarrow m = \pm i$ complex roots.

Complementary function $y_c = c_1 \cos x + c_2 \sin x$

Let the particular integral

$$y_p = c_1(x)\cos x + c_2(x)\sin x$$

Here $y_1 = \cos x \qquad y_2 = \sin x$

$y_1' = -\sin x \qquad y_2' = \cos x$

Wronskian $W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$c_1(x) = \int -\frac{y_2 f(x)}{W} dx = \int -\sin x \cdot \operatorname{Cosec} x \, dx = \int -dx = -x$

$c_2(x) = \int \frac{y_1 f(x)}{W} dx = \int \cos x \cdot \operatorname{Cosec} x \, dx = \int \cot x \, dx$

$= \log(\sin x)$

So $y_p = c_1(x)\cos x + c_2(x)\sin x$

$= -x\cos x + \sin x \log(\sin x)$

General solution is $y(x) = y_c + y_p = c_1 \cos x + c_2 \sin x - x\cos x + \sin x(\log \sin x)$

(11) $(D^2+4)y = 2\tan 2x$

Sol^n Associated homogeneous equation

$$(D^2+4)y = 0 \quad \text{ie} \quad \frac{d^2y}{dx^2} + 4y = 0$$

Auxiliary equation $m^2+4=0 \Rightarrow m^2 = -4$
$\Rightarrow m = \pm 2i$

Complementary function $y_c = C_1\cos 2x + C_2\sin 2x$

Let $y_p = C_1(x)\cos 2x + C_2(x)\sin 2x$

Here $y_1(x) = \cos 2x$, $y_2(x) = \sin 2x$

$$W(y_1,y_2) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x$$
$$= 2(\cos^2 2x + \sin^2 2x) = 2$$

$$C_1(x) = \int -\frac{y_2(x) f(x)}{W}\,dx = -\frac{1}{\cancel{2}}\int \sin 2x \cdot \cancel{2}\tan 2x\,dx$$

$$= -\int \frac{\sin^2 2x}{\cos 2x}\,dx =$$

$$= -\frac{1}{2}\left(\ln|\sec 2x + \tan 2x| - \sin 2x\right)$$

$$C_2(x) = \int \frac{y_1(x) f(x)}{W}\,dx = \int \frac{\cos 2x \cdot \cancel{2}\tan 2x\,dx}{\cancel{2}}$$

$$= \int \sin 2x\,dx = -\frac{1}{2}\cos 2x$$

$$y_p = -\frac{1}{2}\left(\ln|\sec 2x + \tan 2x| - \sin 2x\right)\cdot\cos 2x$$
$$-\frac{1}{2}\cos 2x \cdot \sin 2x$$

General solution $y(x) = y(c) + y_p$

Q.6

(a) Solve $\frac{dy}{dx} = \frac{x-y+1}{x+y-3}$

Sol^n

put $x = X+h$
$y = Y+K$

$$\frac{dY}{dX} = \frac{X-Y+h-K+1}{X+Y+h+K-3} = \frac{X-Y}{X+Y} \quad \text{for } h=1, \; K=2$$

(Solve $h-K+1=0$, $h+K-3=0$)

It is homogeneous in Y & X.

put $Y = vX$

$$v + X\frac{dv}{dX} = \frac{X-vX}{X+vX} = \frac{1-v}{1+v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1-v}{1+v} - v = \frac{1-v-v-v^2}{1+v} = \frac{1-2v-v^2}{1+v}$$

$$\Rightarrow \frac{1+v}{1-2v-v^2}\,dv = \frac{dX}{X}$$

Integrating $-\frac{1}{2}\log(1-2v-v^2) = \log X + \log C = \log CX$

$$\Rightarrow \frac{1}{\sqrt{1-2v-v^2}} = CX$$

$$\Rightarrow \frac{1}{\sqrt{1-2\frac{Y}{X}-\frac{Y^2}{X^2}}} = CX \Rightarrow \frac{X}{\sqrt{X^2-2XY-Y^2}} = CX$$

$$\Rightarrow \frac{1}{\sqrt{X^2-2XY-Y^2}} = C$$

$$\Rightarrow X^2-2XY-Y^2 = K \quad \text{Constant}$$

$$\Rightarrow (x-1)^2 - 2(x-1)(y-2) - (y-2)^2 = K$$

$$\Rightarrow x^2-2xy-y^2+2x+6y-7 = K$$

$$\Rightarrow x^2-2xy-y^2+2x+6y = K_1 \quad \text{Constant.}$$

Q.6(b) Solve $(x+y)(dx-dy) = dx+dy$

Sol^n $\quad dx-dy = \frac{dx+dy}{x+y}$

Integrating $\quad \log(x+y) = x-y+C$

Q.6(c) Solve $(2x+3y-5)\frac{dy}{dx} + 2x+3y+1 = 0$

Sol^n $\quad \frac{dy}{dx} = \frac{-2x-3y-1}{2x+3y-5}$

put $2x+3y = z$

$\Rightarrow 2 + 3\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow 3\frac{dy}{dx} = \frac{dz}{dx} - 2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{3}\frac{dz}{dx} - \frac{2}{3}$

$\Rightarrow \frac{1}{3}\frac{dz}{dx} - \frac{2}{3} = \frac{-z-1}{z-5}$

$\Rightarrow \frac{1}{3}\frac{dz}{dx} = \frac{-z-1}{z-5} + \frac{2}{3}$

$= \frac{-3z-3+2z-10}{3(z-5)} = \frac{-z-13}{3(z-5)}$

$\Rightarrow \frac{dz}{dx} = \frac{-z-13}{z-5} \Rightarrow -\frac{z-5}{z+13}dz = dx$

$\Rightarrow -\frac{z+13-18}{z+13}dz = dx$

$\Rightarrow -\left(1-\frac{18}{z+13}\right)dz = dx$

Integrating $\quad -z + 18\log(z+13) = x+C$

$\Rightarrow -2x-3y+18\log(2x+3y+13) = x+C$

$\Rightarrow -3(x+y) + 18\log(2x+3y+13) = C$

$\Rightarrow x+y-6\log(2x+3y+13) = \frac{C}{-3} = K$

(Ans).