Boolean Algebra

Elements: '0' and '1'

Operators: plus, times and overbar
+ · —

+	0	1
0	0	1
1	1	1

.	0	1
0	0	0
1	0	1

$\overline{1} = 0$ and $\overline{0} = 1$

Property of duality of Boolean algebra

The dual of an algebraic expression is obtained by

1. Interchanging OR and AND operations,
2. Replacing 1's by 0's and 0's by 1's.

Ex.

$$X.(Y+Z) = X.Y + X.Z$$

Dual form:

$$X + Y.Z = (X+Y).(X+Z)$$

Ex. $$\overline{A+B} = \overline{A} \cdot \overline{B}$$

Dual $$\overline{A.B} = \overline{A} + \overline{B}$$

Ex. $$0.X = 0$$

$$\Rightarrow 1 + X = 1$$

$$X.(Y+Z) = X.Y + X.Z$$

Proof: (Using truth table)

X	Y	Z	Y+Z	X.(Y+Z)	X.Y	X.Z	X.Y+X.Z
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

Venn Diagram :

Theorems of Boolean Algebra:

- **Theorem 1:**

  (a) $0.X = 0$ (b) $1+X = 1$

- **Theorem 2:**

  (a) $1.X = X$ (b) $0+X = X$

- **Theorem 3 (Idempotent or Identity Laws):**

  (a) $X.X. . .X = X$ (b) $X+X+ \quad +X = X$

- **Theorem 4 (Complementation Law):**

  (a) $X.\overline{X} = 0$ (b) $X + \overline{X} = 1$

- **Theorem 5 (Commutative Laws):**

  (a) $X + Y = Y+X$ (b) $X.Y = Y.X$

- **Theorem 6 (Associative Laws):**

  (a) X + (Y + Z) = (X+Y) + Z
  (b) X.(Y.Z) = (X.Y).Z

- **Theorem 7 (Distributive Laws):**

  (a) X.(Y+Z) = X.Y + X.Z
  (b) X + Y.Z = (X+Y). (X+Z)

- **Theorem 8 :**

  (a) $X.Y + X.\overline{Y} = X$
  (b) $(X + Y).(X + \overline{Y}) = X$

- **Theorem 9 :**

  (a) $(X + \cancel{\overline{Y}}).Y = XY$
  (b) $X.\overline{Y} + Y = X + Y$ **(Using dual of Theorem 9(a)**

- **Theorem 10 (Absorption Law or Redundancy Law)**

  (a) X + X.Y = X
  (b) X.(X+Y) = X

- **Theorem 11**

(a) $Z.X + Z.\overline{X}.Y = Z.X + Z.Y$
(b) $(Z + X).(Z + \overline{X} + Y) = (Z + X).(Z + Y)$

- **Theorem 12 (Consensus Theorem)**

(a) $X.Y + \overline{X}.Z + \cancel{Y}.Z = X.Y + \overline{X}.Z$
(b) $(X + Y).(\overline{X} + Z)(Y + Z) = (X + Y).(\overline{X} + Z)$

- **Theorem 13 (DeMorgan's Theorem)**

(a) $\overline{X_1 + X_2 + \cdots + X_n} = \overline{X_1}.\overline{X_2}\,\ldots.\overline{X_n}$
(b) $\overline{X_1.X_2\,\ldots.X_n} = \overline{X_1} + \overline{X_2} + \cdots + \overline{X_n}$

- **Theorem 14 (Transposition Theorem)**

(a) $X.Y + \overline{X}.Z = (X + Z)(\overline{X} + Y)$
(b) $(X + Y)(\overline{X} + Z) = X.Z + \overline{X}.Y$

- **Theorem 15 (Involution Law)**

$$\overline{\overline{X}} = X$$

Simplification of Boolean Expressions:

(Using Theorems)

Ex. Simplify the Boolean function

$$F = AB + \underline{BC + \overline{B}C}.$$

$$= AB + (B+\overline{B})C$$

$$= AB + 1.C$$

$$= AB + C$$

Ex. Simplify the Boolean function

$$F = A + \overline{A}\overline{B}.$$

$$= (A+\overline{A}).(A+\overline{B})$$

$$= 1.(A+\overline{B})$$

$$= A+\overline{B}$$

Ex. Simplify the Boolean function

$$F = AB + \overline{AC} + A\overline{B}C(AB + C).$$

$$= AB + \overline{AC} + A\overline{B}CAB + A\overline{B}CC$$
$$(=0)$$

$AB(C+\overline{C})$
$= ABC + AB\overline{C}$

$$= AB + \overline{AC} + A\overline{B}C$$

$$= AB + ABC + A\overline{B}C + \overline{AC}$$

$$= AB + AC(B+\overline{B}) + \overline{AC}$$

$$= AB + AC + \overline{AC} = 1$$

Ex. Simplify the Boolean function

$$F = ABC + AB\overline{C} + A\overline{B}C.$$

$$= ABC + AB\overline{C} + ABC + A\overline{B}C$$

$$= AB(C+\overline{C}) + AC(B+\overline{B})$$

$$= AB + AC$$

$$= A(B+C)$$

$$(x+y)(x+\bar{y})$$

$$= x.x + \underline{x\bar{y} + yx} + y\bar{y}$$

$$= x + x(\underline{\bar{y}+y}) + 0$$

$$= x + x.1$$

$$= x + x$$

$$= x$$

$$x\bar{y} + y$$

$$= x\bar{y} + \underline{y + yx}$$

$$= x(\bar{y}+y) + y$$

$$= x + y$$

$$y = y(x+\bar{x})$$

$$= yx + y\bar{x}$$

$$y + yx$$

$$= y(1+x)$$

$$= y.1 = y$$

An arbitrary logic function can be expressed in the following standard forms:

1. Sum of Products (SOP)
2. Product of Sums (POS)

Sum of Products (SOP)

It is an OR operation on AND operated variables.

Ex. $F(x,y,z) = x\bar{y} + \bar{x}\bar{y}z$

Product terms:

$x\bar{y}, \bar{x}\bar{y}z$

Ex. $F(x,y,z) = \bar{x}y\bar{z} + x\bar{y}z + xyz$

Product terms:

$\bar{x}y\bar{z}, x\bar{y}z, xyz$

Minterm:

A product term containing all n variables of the function in either true or complemented form is called the minterm.

X	Y	Z	Minterm	
0	0	0	$\bar{x}\bar{y}\bar{z}$	m_0
0	0	1	$\bar{x}\bar{y}z$	m_1
0	1	0		m_2
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1	xyz	m_7

Canonical Sum of Products:

If SOP contains only minterms, then it is referred to as canonical sum of products expression.

Ex. $F(x,y,z) = x\bar{y}z + x\bar{y}\bar{z} + xyz$

$= m_5 + m_4 + m_7$

$= \sum m(4,5,7)$

$= \sum (4,5,7)$

Ex. Obtain the canonical sum of products form of the following function.

$$F(A,B,C) = A + BC$$

$$F(A,B,C) = A + BC$$

$$= A(B+\bar{B})(C+\bar{C}) + (A+\bar{A})BC$$

$$= ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C} + ABC + \bar{A}BC$$

$$= ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C} + \bar{A}BC$$

$$= m_7 + m_6 + m_5 + m_4 + m_3$$

$$= \sum m(3,4,5,6,7)$$

$$= \sum (3,4,5,6,7)$$

Implementation of SOP equations:

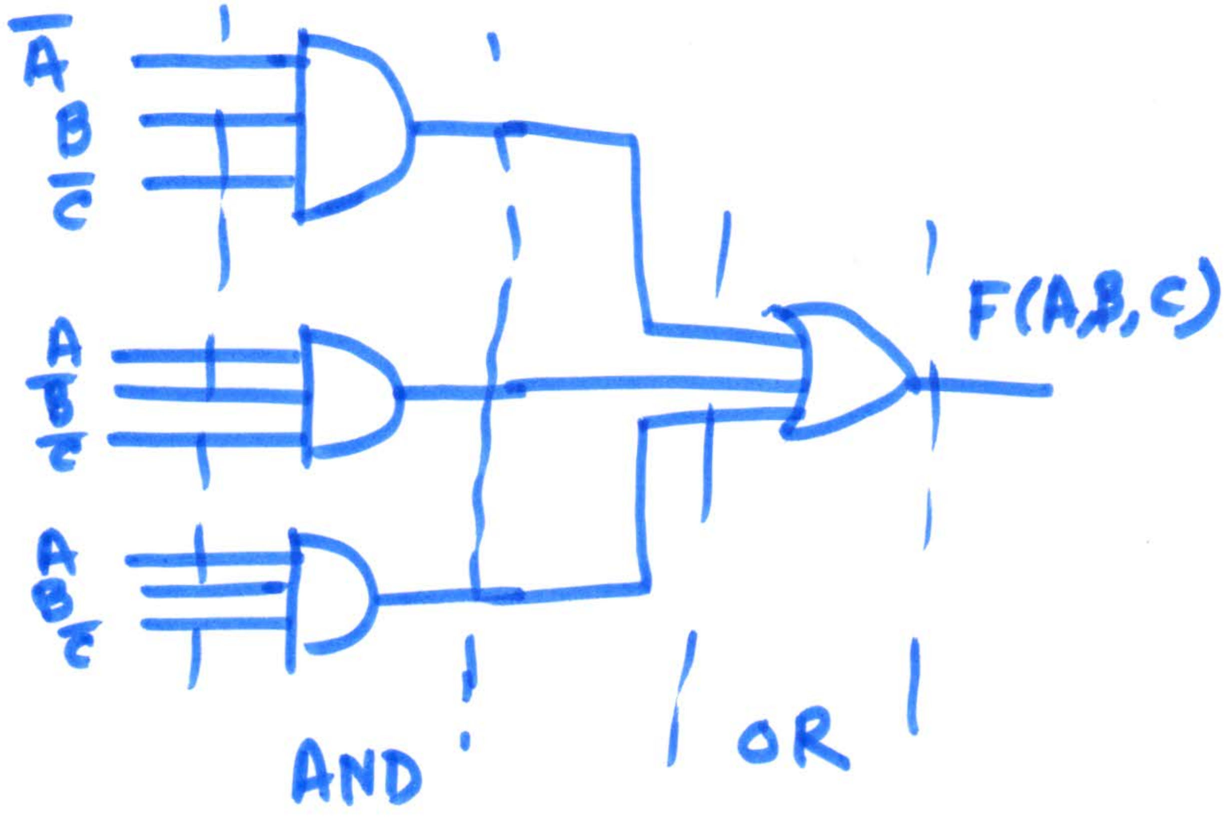
We may implement SOP equations by an AND-OR network, or a NAND-NAND network.

Ex.

$$F(A,B,C) = \sum(4, 2, 6)$$

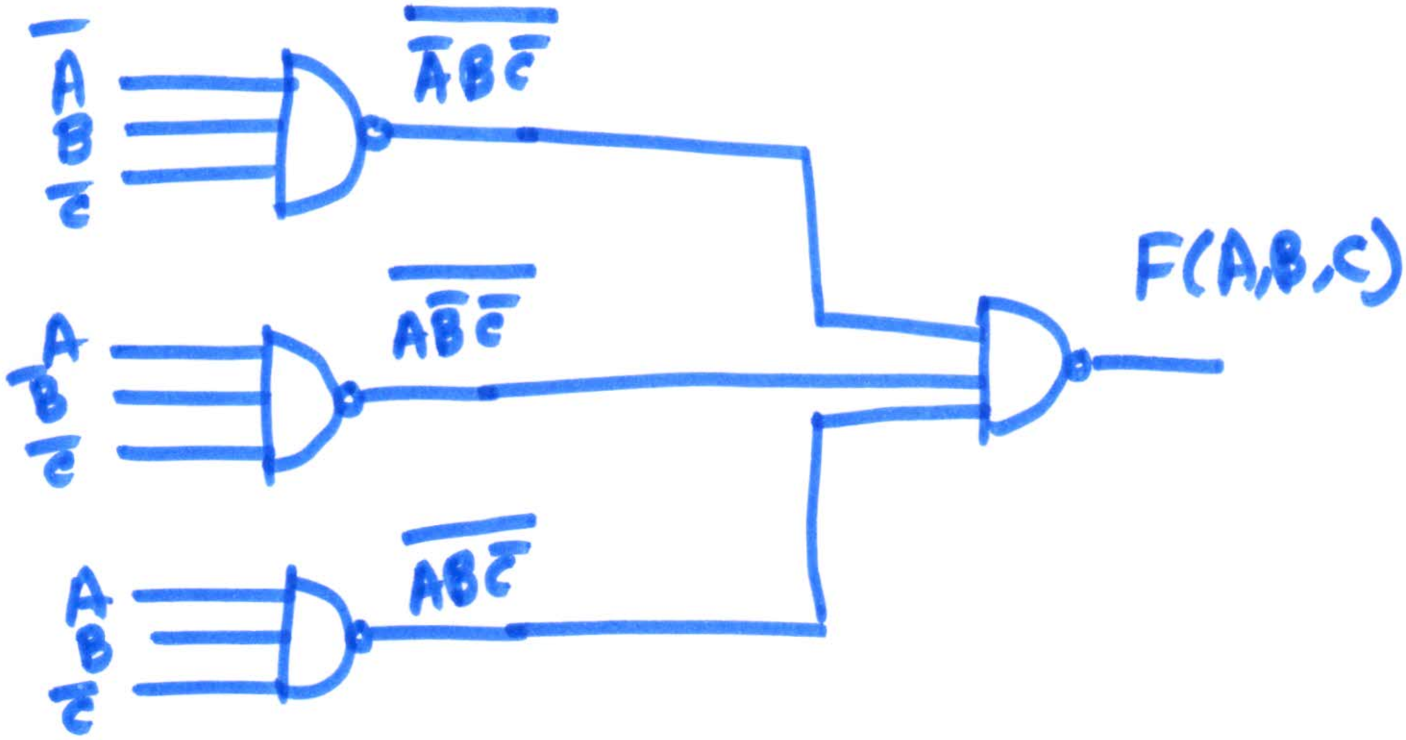
$$= m_2 + m_4 + m_6$$

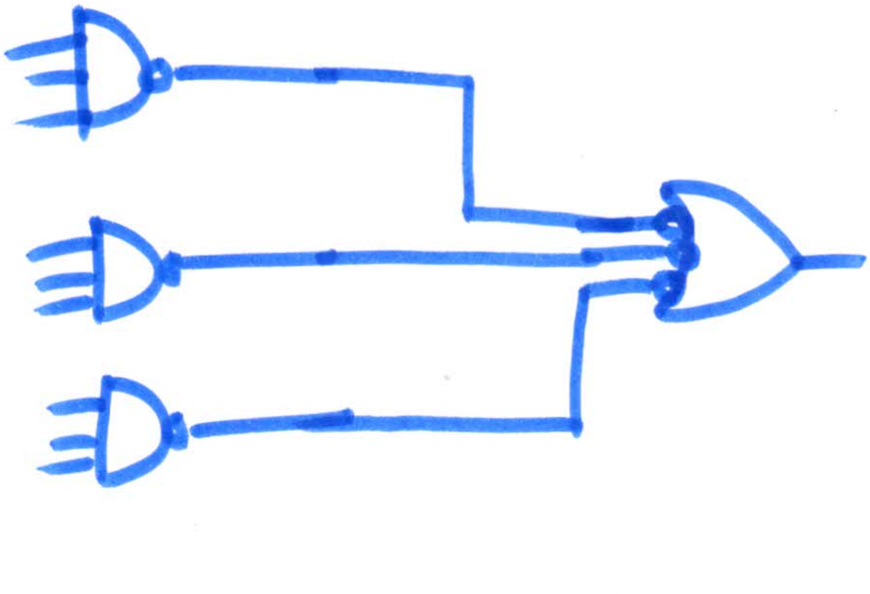
$$= \overline{A}B\overline{C} + A\overline{B}\overline{C} + AB\overline{C}$$



$$F(A,B,C) = \overline{\overline{\bar{A} B \bar{C} + A \bar{B} \bar{C} + A B \bar{C}}}$$

$$= \overline{\overline{\bar{A} B \bar{C}} \cdot \overline{A \bar{B} \bar{C}} \cdot \overline{A B \bar{C}}}$$





Product of Sums (POS)

It is an AND operation on OR operated variables.

Ex. $F(A,B,C) = (A+\overline{B}).(\overline{A}+B+\overline{C})$

Sum terms:

$A+\overline{B}$, $\overline{A}+B+\overline{C}$

Ex. $F(A,B,C) = (\overline{A}+\overline{B}+\overline{C})(A+B+C)$

Sum terms:

$(\overline{A}+\overline{B}+\overline{C})$, $(A+B+C)$

Maxterm:

A sum term containing all n variables of the function in either true or complemented form is called the maxterm.

X	Y	Z	Maxterm	
0	0	0	$X+Y+Z$	M_0
0	0	1	$X+Y+\bar{Z}$	M_1
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1	$\bar{X}+\bar{Y}+\bar{Z}$	M_7

Canonical Product of Sums:

If POS contains only maxterms, then it is referred to as canonical product of sums expression.

Ex. $F(X,Y,Z) = (X+\bar{Y}+Z).(X+Y+Z)$

$= M_2 . M_0$

$= \prod M(0,2)$

$= \prod (0,2)$

Ex. Obtain the canonical ~~sum of~~ product~~s~~ of sums form of the following function.

$$F(A,B,C) = (A + B)(A+C)$$

$$= (A+B+C\bar{C})(A+C+B\bar{B})$$

$$= (A+B+C)(A+B+\bar{C})(A+C+B)(A+C+\bar{B})$$

$$= (A+B+C)(A+B+\bar{C})(A+C+\bar{B}) = M_0 M_1 M_2$$

Ex. Obtain the canonical ~~sum of~~ product~~s~~ of sums form of the following function.

$$F(A,B,C) = A + \bar{B}C$$

$$= (A+\bar{B})(A+C)$$

$$= (A+\bar{B}+C\bar{C})(A+C+B\bar{B})$$

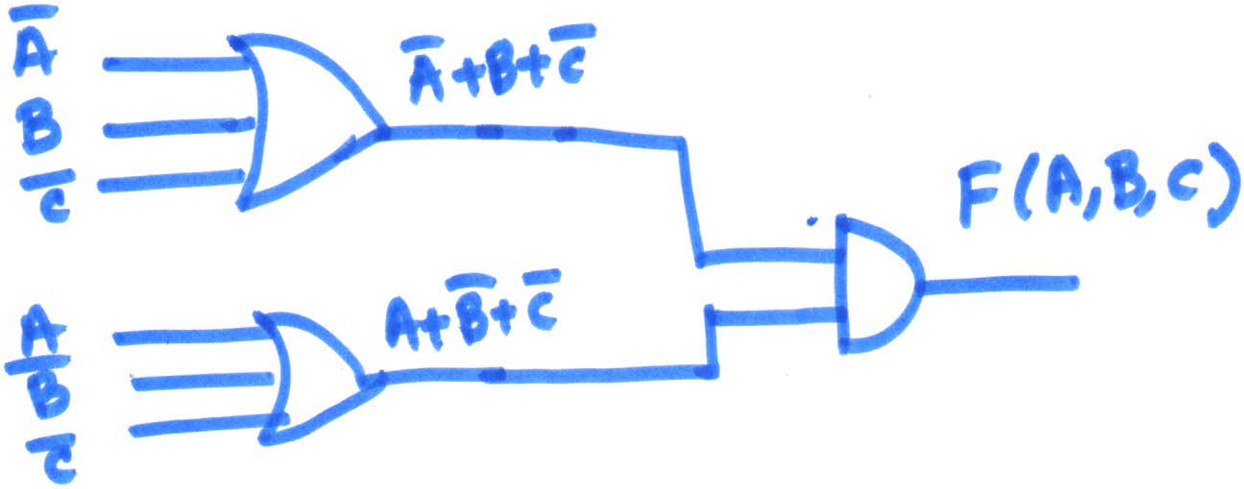
$$= (A+\bar{B}+C)(A+\bar{B}+\bar{C})(A+C+B)(A+C+\bar{B})$$

$$= M_2 \cdot M_3 \cdot M_0 \cdot \cancel{M} = \prod(0,2,3)$$

Implementation of POS equations:

We may implement POS equations by an OR-AND network, or a NOR-NOR network.

Ex. $F(A,B,C) = (\bar{A}+B+\bar{C})(A+\bar{B}+\bar{C})$



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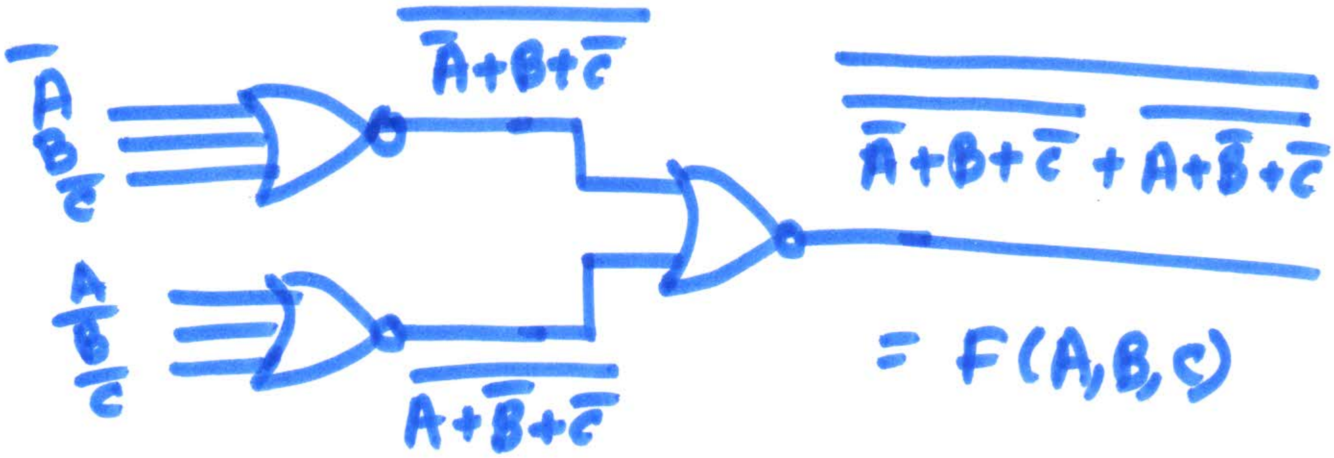

$$F(A,B,C) = \overline{\overline{(\overline{A}+B+\overline{C})(A+\overline{B}+\overline{C})}}$$

$$= \overline{\overline{\overline{A}+B+\overline{C}} + \overline{A+\overline{B}+\overline{C}}}$$

Conversion between Canonical Forms:

Ex.

X	Y	Z	F	Minterm	Maxterm
0	0	0	0		$X+Y+Z$
0	0	1	0		$X+Y+\bar{Z}$
0	1	0	1	$\bar{X}Y\bar{Z}$	
0	1	1	0		$X+\bar{Y}+\bar{Z}$
1	0	0	1	$X\bar{Y}\bar{Z}$	
1	0	1	0		$\bar{X}+Y+\bar{Z}$
1	1	0	1	$XY\bar{Z}$	
1	1	1	1	XYZ	

$$F(X,Y,Z) = \sum m(2,4,6,7)$$

$$F(X,Y,Z) = \prod M(0,1,3,5)$$

Karnaugh Map (K-map)

- A Karnaugh map is basically a visual display of minterms used in a canonical sum of products form.
- K-map is also used to display the maxterms of a canonical product of sums form.
- We use K-map to simplify Boolean functions.

Two-variable Map:

X	Y	Minterm
0	0	$\bar{X}\bar{Y}$ (m_0)
0	1	$\bar{X}Y$ (m_1)
1	0	$X\bar{Y}$ (m_2)
1	1	XY (m_3)

X\Y	0	1
0	m_0	m_1
1	m_2	m_3

Three-variable Map:

X	Y	Z	Minterm
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

X \ YZ	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

XY \ Z	0	1
00	m_0	m_1
01	m_2	m_3
11	m_6	m_7
10	m_4	m_5

Four-variable Map:

A	B	C	D	Minterm
0	0	0	0	m_0
0	0	0	1	m_1
0	0	1	0	m_2
0	0	1	1	m_3
0	1	0	0	m_4
0	1	0	1	m_5
0	1	1	0	m_6
0	1	1	1	m_7
1	0	0	0	m_8
1	0	0	1	m_9
1	0	1	0	m_{10}
1	0	1	1	m_{11}
1	1	0	0	m_{12}
1	1	0	1	m_{13}
1	1	1	0	m_{14}
1	1	1	1	m_{15}

AB \ CD	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

Simplifying a Boolean Function Using K-map:

For Pairs:

Ex. $F(X,Y,Z) = \sum m(5,7) = X\bar{Y}Z + XYZ$

$= XZ(\bar{Y}+Y)$

$= XZ$

X \ YZ	00	01	11	10
0				
1		1	1	

XY \ Z	0	1
00		
01		
11		1
10		1

$F = XZ$

Ex. $F(A,B,C,D) = \sum m(11,15) = A\bar{B}CD + ABCD$

$= ACD(\bar{B}+B)$

$= ACD$

AB \ CD	00	01	11	10
00				
01				
11			1	
10			1	

$F = ACD$

$$F(A,B,C,D) = \sum m(8,10)$$

Ex.

AB \ CD	00	01	11	10
00				
01				
11				
10	1			1

$$F = A\bar{B}\bar{D}$$

$F(A,B,C,D)$

Ex.

AB \ CD	00	01	11	10
00		1		
01	1	1	1	
11		1		
10				

$$F = \bar{A}B\bar{C} + B\bar{C}D + \bar{A}BD + \bar{A}\bar{C}D$$

Ex. $F(X,Y,Z)$

X \ YZ	00	01	11	10
0		1		
1	1	1		1

$$F = X\bar{Z} + \bar{Y}Z$$

For Quad:

Ex. $F(A,B) = \sum m(0,1,2,3)$ $= 1$

A\B	0	1
0	1	1
1	1	1

Ex. $F(X,Y,Z) = \sum m(4,5,6,7)$ $= x\bar{y}\bar{z} + x\bar{y}z + xy\bar{z} + xyz$

$= x(\bar{y}\bar{z} + \bar{y}z + y\bar{z} + yz)$

$= x$

x\yz	00	01	11	10
0				
1	1	1	1	1

$F = x$

xy\z	0	1
00		
01		
11	1	1
10	1	1

$F = x$

Ex. $F(A,B,C,D) = \sum m(5,7,13,15)$

AB \ CD	00	01	11	10
00				
01		1	1	
11		1	1	
10				

$F = BD$

Ex. $F(A,B,C,D)$

AB \ CD	00	01	11	10
00				
01				
11	1			1
10	1			1

$F = A\overline{D}$

Ex. F(A,B,C,D)

AB\CD	00	01	11	10
00	1			1
01				
11				
10	1			1

$F = \overline{B}\,\overline{D}$

Ex. F(A,B,C)

A\BC	00	01	11	10
0	1	1		1
1	1	1		1

$F = \overline{B} + \overline{C}$

For Octet:

Ex. F(A,B,C)

A\BC				
	1	1	1	1
	1	1	1	1

$F = 1$

F(A,B,C,D)

Ex.

AB\CD	00	01	11	10
00				
01	1	1	1	1
11	1	1	1	1
10				

$F = B$

F(A,B,CD

Ex.

AB\CD	00	01	11	10
00	1			1
01	1			1
11	1			1
10	1			1

$F = \overline{D}$

More Examples:

$F(A,B,C,D)$

Ex.

AB\CD	00	01	11	10
00	1	1	1	1
01		1	1	
11		1		
10	1			1

not required

$$F = \cancel{\bar{A}\bar{B}} + \bar{B}\bar{D} + \bar{A}D + B\bar{C}D$$

Ex.

			1
1	1	1	
1	1	1	
1	1		1