Vertex Cover
Edge Cover

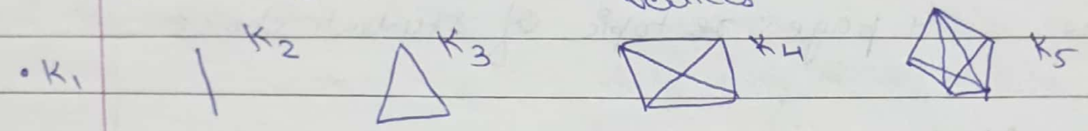
Graph Theory

Complete graph

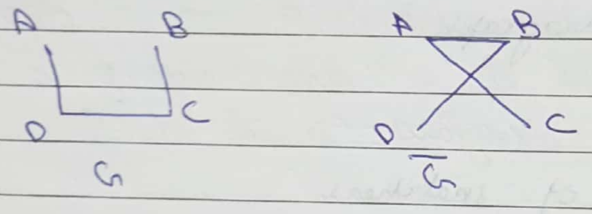
$E = \{\text{All vertex pairs}\}$ $\quad {}^nC_2$ edges

Graph complement

- Applied to potential vertex set and not applied vertex set.
- $G \longrightarrow \bar{G}$ (complement)
- $E(G) \cap E(\bar{G}) = \phi$
- $E(G) \cup E(\bar{G}) = K_V \longrightarrow$ Symbol for complete graph with V vertices

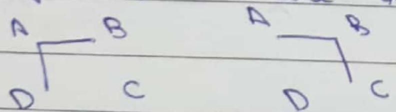


Every graph has a unique complement



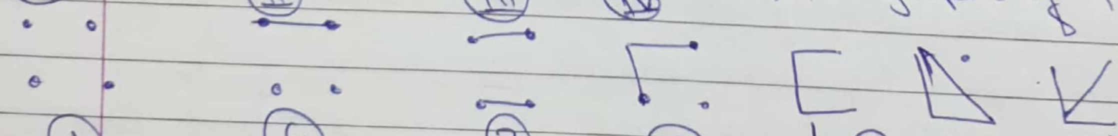
No. of graphs with 4 labelled vertices
- 6 edges in complete graph
- Either include or exclude edge
- 2^6 graphs possible

11 graphs which are distinct.



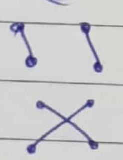
$\Rightarrow$ Identical

I   II   III   IV $\longrightarrow$ 12 graphs of this type



1   6   3   12   12   4   4 $\rightarrow$ choosing degree 3 4 choice

6 ways to decide 2 end points



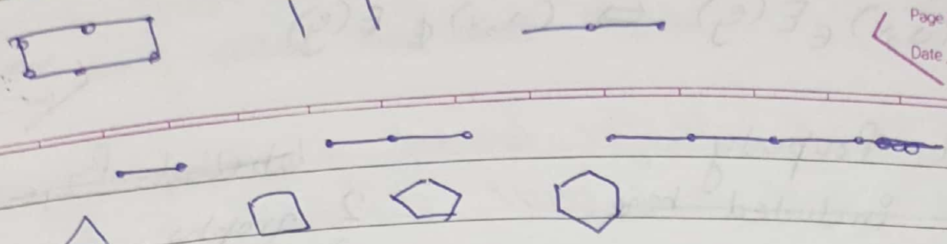
6×2 — 6 ways to select end points — 2 ways to select middle vertex

6×2 — 6 ways to select 2 end points

4 ways to decide isolated vertices — 2! (ways of going through 2 middle points)

P_n

C_n



Edgeless graph $= \overline{K_n}$ (graph with 0 edges and n vertices.

Complement of graphs (I), (II), (III), (IV)
↳ 6 − 2, 6 − 1, 6 − 0
(4) (5) (6) edges

Bijective mapping ∴ $\{1 + 6 + 3 + 12 + 12 + 4 + 4$
$\{ + 1 + 6 + 3 + 12 = 64$ total graphs.
↓
Every graph has a complement

$G_{n,m}$ $\quad \overline{G}_{n, {}^nC_2 - m}$
→ Partition of complement graph

11 graphs are distinct
4 + 4 + 3

Complete graphs : Have high degree of symmetry

~~Self~~ Self complementary
− If graph is structurally isomorphic to its complement graph. $G \cong \overline{G}$

$C_5 \cong \overline{C_5}$ $\quad \boxed{P_4 \cong \overline{P_4}} \rightarrow ?$

Isomorphism

Closely related to ~~isomorphic~~ similar structure.

Graph complement
$(u,v) \in E(G) \Leftrightarrow (u,v) \notin E(\bar{G})$

Structural Property
~~Vertices are included here~~
- No. of vertices, No. of edges,
  Spanning trees, Cycles,
  Blocks, Cut-vertices,
  Degree sequence

~~Labelled~~ Property
2 graphs
Identical in structural properties are isomorphic to each other.

Two graphs G and H are isomorphic if there is bijective mapping for vertices in G and H such that $f(u)=v$ where $u \in G$ and $v \in H$ and for every u,v adjacent in G, $f(u), f(v)$ are adjacent in H and u,v not adjacent in G ; $f(u), f(v)$ are not adjacent in H.

G is isomorphic to H if f is a bijective function
$f: V(G) \to V(H)$
Such that $(u,v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(H)$

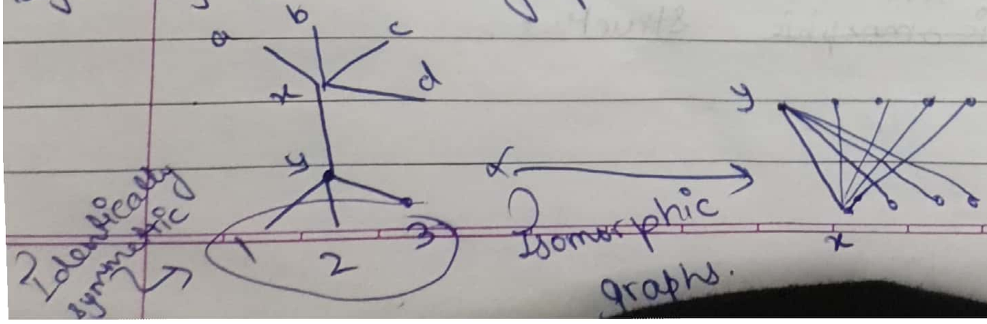
Two graphs are isomorphic and hence mapping between them is isomorphism

Super Polynomial Non exponential time problem
Quasi polynomial time problem

f is called an isomorphism

$G \cong H \Leftrightarrow \bar{G} \cong \bar{H}$

Automorphism
↓
Symmetry within a graph



Introduction to Graph theory by Agost Doughlus

Regular graph: Every vertex has same degree

An automorphism is a ~~permut~~ bijective function f from the vertex set of a graph G to itself such that $(u,v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(G)$

Isomorphism of graphs is an equivalence relation.

Equivalence relation requires 3 relations

- Reflexive, Symmetric, Transitive

$\downarrow$ Identity

$\downarrow$ Bijective functions are invertible

$\downarrow$ composition

$f \quad g$

G_1, G_2, G_3

$(u,v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2 \Leftrightarrow (g(f(u)), g(f(v))) \in E_3$

Isomorphism hence is an equivalence relation

Two graphs are isomorphic $\Leftrightarrow$ the complement of two graphs are isomorphic

Every 2 regular graph is a collection of disjoint cycles.

2 distinct 2 regular graphs on 7 vertices.

Every graph has identity automorphism

Maximum automorphisms: $n!$

G, f_1, f_2

If f is an automorphism then f inverse is also automorphism

$(u,v) \in E \Leftrightarrow (f(u), f(v)) \in E$

$(f^{-1}(x), f^{-1}(y)) \in E \Leftrightarrow (x,y) \in E$

$a \rightarrow b \quad f$

$b \rightarrow a \quad f$ inverse

Automorphism : Preserving adjacency and non adjacency

Every vertex is identical to itself

$f(x) = y$ $\quad\quad g(f(x)) = z$

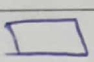
$g(y) = z$

x is identical to y, y is identical to z,
x is identical to z

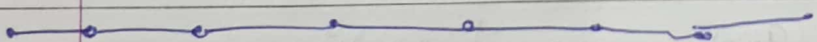
Binary
Relation between vertices based on existence of an automorphism between any pair is an equivalence relation.

Every automorphism that works for a graph also work for complement graph.

Edgeless Graph
Complete graph } $n!$ automorphism

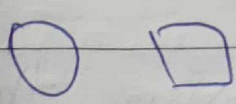


$f(A) = C$
$f(B) = A$ } → Not automorphism.



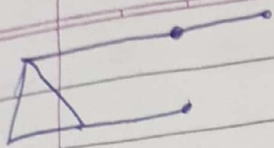
Map vertex to itself or to a vertex equidistant from other end.

2 equivalence classes ; 2 automorphisms.

$2n$ for cycle Automorphism]
$n = 4 \Rightarrow 8$ →  No. of Equivalence class = 1 (Pick 2 and choose neighbours)

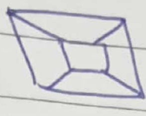
Vertex transitive Graph → only one equivalence class of vertex
→ Degree of all vertices are equal.

Rigid Graph : n equivalence classes each of size 1



No. two vertices that are identical

Vertex Transitive Graph

- All cycle graphs 
- Hypercubes of any dimension 

Knesser Graphs

- They are two parameter graphs. n: n is size of Universal set, K: K element subset
- KN (n, K) set theoretic definition of graph
- Every vertex is labelled with set.
- Disjointness Graph

Application → Interval Graphs (OS Scheduling)

Dual of disjointness graphs is intersection graphs.
↓
Complement of disjointness graph.

Two sets intersect if and only if they are disjoint.

Interval Graph : One specific kind of intersection graphs.

⊙ Vertices represent geometric sets called interval

KN(n, K) → Set cardinality, ↳ Uniform subset sizes → No. of vertices = ^nC_K

One vertex for each subset of size K of a set of size n.

Adjacency ≡ Disjointness

Matching graph (KN(n/2, n/2) Every Knesser graph of this type is matching graph.)

Degree of vertex in Knesser Graph

KN (n, K) $^{n-K}C_K$ { K vertices subset, n−K vertices left } (Always take K vertices)

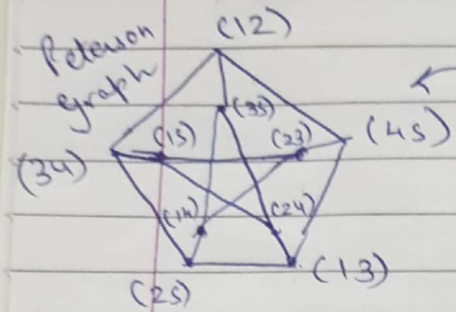
Complete graphs are also Knesser graph

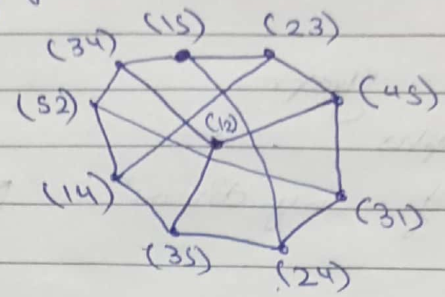
∴ $^{n-K}C_K$ ← degree of every vegree

Peterson graph

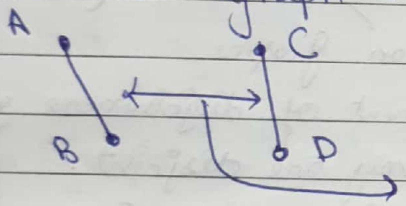
n = Odd such that 2K+1

KN(2K+1, K) { Odd graphs
Special graph : K = 2





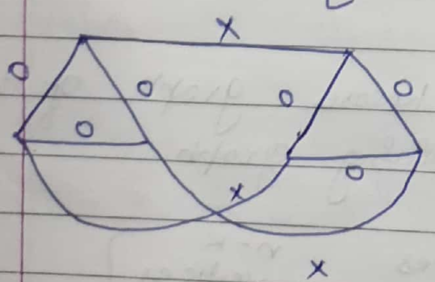
Edge Transitive graph | All edge equivalence
Every edge can be mapped to other edge
(In Peterson graph : P_4)



Edge mapping ⇒ f(A) = C and f(B) = D
or
f(A) = D and f(B) = C

A graph can neither be edge transitive nor edge ~~ridge~~ rigid

P_4 a b c d
P_4 b c d e
{ Peterson is edge transitive, vertex transitive }.



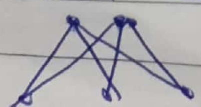
{ All vertices are identical.
Example of vertex transitive and not edge transitive

DOUBT

2 equivalence classes of edges (One with O's are identicals and with X's are identical)

Complete Bipartite graph ~~with assymetric sizes~~

$K_{2,3}$



{ Not vertex transitive
All edges are symmetric
Edge transitive graph

No. of mappings = 2! 3! = No. of automorphisms.

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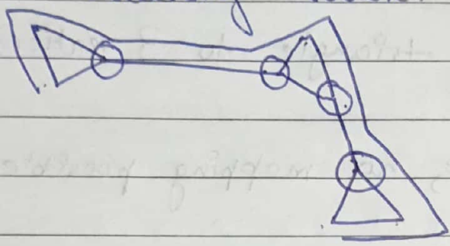
$K_{n,m} \Rightarrow n!\,m!$ no. of automorphisms.

All possible combinations are possible

Edge ~~transitive~~	Vertex transition
0	0
0	1
1	0
1	1

Cut Vertex

- Removing a ~~vero~~ cut vertex increases no. of connected components in a graph.
- Communication breaking vertex.



{ 4 vertices marked are cut vertices.

Within a graph; there is a concept of path.

Concept of path within a graph.

Maximal path. (Every graph has a maximal path)

↳ A path is maximal when we cannot extend its ~~end~~ end point

why further covering not possible?

1. No more edges
2. Including edges violates definition.

$\boxed{\text{Maximal path end points are not cut vertices. (Every non trivial Graph has atleast 2 non cut vertices)}}$

⇓

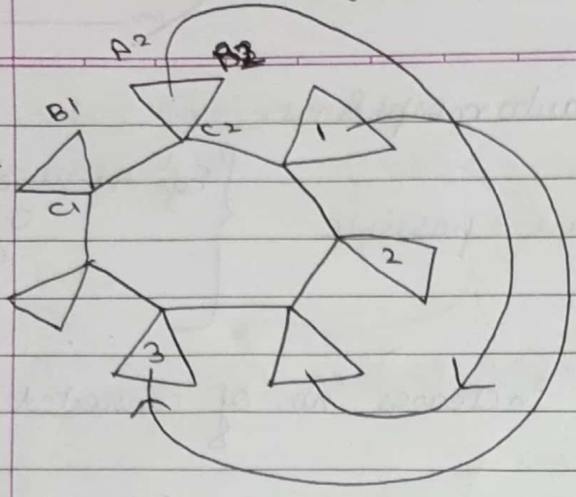
Vertex Transitive Graph cannot have cut vertices.

- No cut vertex present
- If x is cut vertex, y is also cut vertex.
- No mapping of cut vertex and non cut Vertex.

All Knesser Graphs are vertex transitive.

Labelling of Peterson Graph.

A and B are flip symmetric.



2 equivalence classes of vertices.

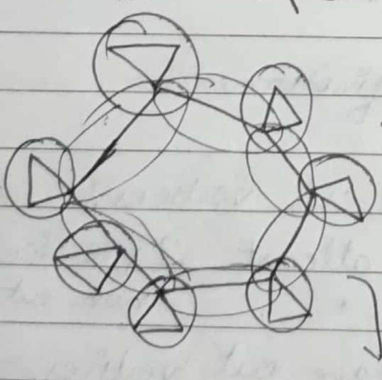
Now the mapping freezes
No choice to whom to map
When only one map was there
1 could be mapped to 2 or 3.

Triangle transitive
Mapping 3 vertices of any triangle to 3 vertices of any triangle.

If vertices are not similar ; no mapping possible.

Decompositions

Partitioning edge set of a graph
Identical ; similar ; Arbitary
Self complementary Graphs
① Decomposition into complete graphs



7 triangles
+ 7 links
14 complete graphs

Decompose into cycles

8 cycles → 7 triangle
1 heptagon

$$d_G(v) = \sum_{i=1}^{k} d_{G_i}(v)$$

Degree of vertex in a graph is sum of its degree in all decompositions.

Not possible to decompose K_6 into K_3's
Reason : Degree of every vertex in $K_6 = 5$
Degree of every vertex in $K_3 = 2$

$2 \times x = 5$
(Not possible)

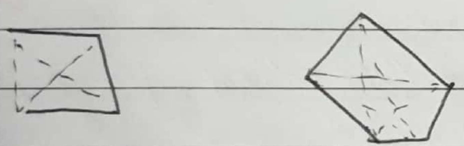
Self Complementary : Decompose complete graph into ~~two parts~~ such that both are isomorphic to each other.

Local degree divisibility
Total edge divisibility } Divisibility

Peterson graph is P_4 transitive, edge transitive, vertex transitive.
P_4 → 4 vertex path ⟹ P_3 transitive

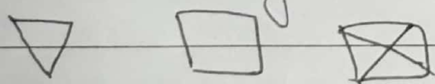
For Self complementary graph
No. of edges should be divisible into 2 parts ⇒ $(n-1)n/2$
multiple of 2

$\therefore n(n-1) =$ multiple of 4



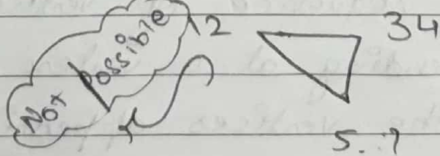
$n = 4, 5, 8, 9$
$n = (4K, 4K+1)$

Why there are no triangles in Peterson Graph?



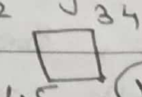
$C_3 = K_3$ C_4 K_4

Not possible
Adjacent vertices belong to disjoint sets of cardinality 2.



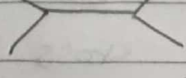
6 element set
But available only 5 vertices

Diameter of peterson graph = 2
Peterson graph does not have 4 length cycle C_4 ✗ C_3 ✗



(12) ← Not possible.

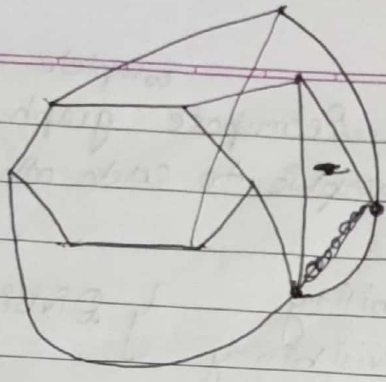
Every edge involves 4 cycles



15 edges

$\frac{15 \times 4}{5} = 12$ cycles of 5 edges (K_5)
overcounting of cycles

Induced 6 cycle



Adjacency matrix
Adjacency List
Incidence matrix

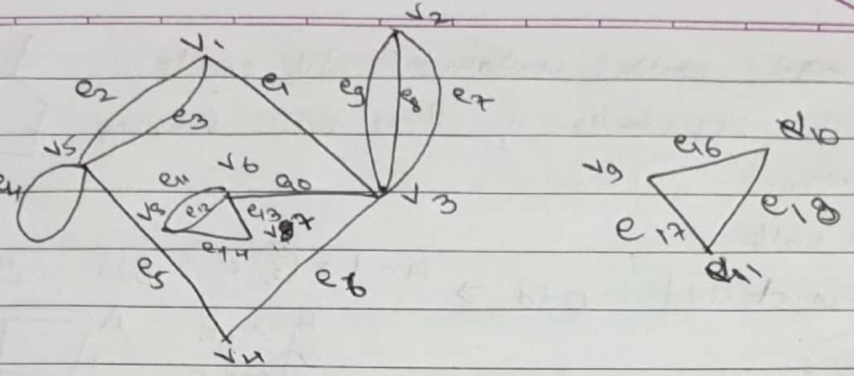
Degree of vertex : Add number of ones in a row.

	e_1	e_2	e_3
u_1	.	.	.
u_2	.	.	
u_2	.	.	.

$$\sum_{v \in V'} \text{degree}(v) = 2|E|$$ No. of edges

Handshaking lemma.

Degree of a vertex is number of adjacent edges.
Walk, Trail, Path, Cycle
Walk : An alternating sequence of vertices and edges beginning at a vertex and ending at a vertex such that for each edge is a sequence the vertices appearing immediately before and after are its two end points.
Length : No. of edges in a sequence (No. of occurrences of edges in a sequence including duplicates).
Odd walk, Even walk
Closed walk, Open walk
Closed walk : First vertex same as last vertex
Walk cannot help you cross graphs (components)



$v_1 e_{16} v_{10} e_{16} v_9 e_{16} v_{10} e_{18} v_{11}$

$v_1 e_2 v_5 e_4 v_5 e_5 v_4 e_5 v_5$

Trail : Trail is a walk without repeating edges. Allows reuse of ~~edge~~ vertices.

Forbidding repeat of vertices implies forbidding of repeating of edges.

Path : Vertices cannot be repeated in a trail.

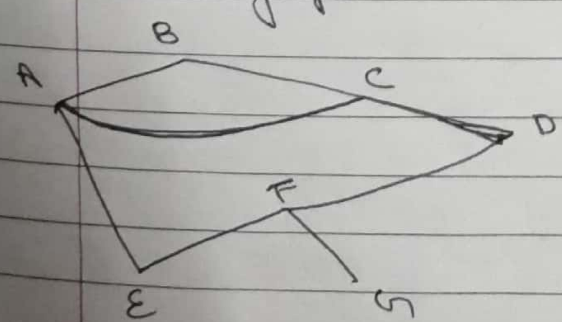
Length of cycle : No. of edges.

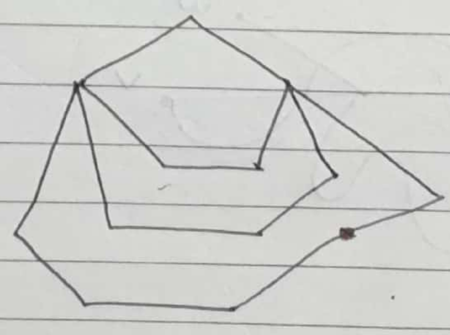
Subgraph : Subset of vertices and edges.

Peterson graph is not a cycle. But it contains cycle.

Unweighted graph : BFS
Weighted graph : Dijkstra's Algorithm
} Shortest path finding in a graph.

Induced Subgraph





Every shortest path has to be induced

A shortest cycle in whole graph is induced.

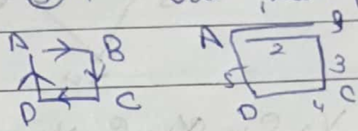
A shortest cycle must be induced.

Every closed odd walk contains odd cycle

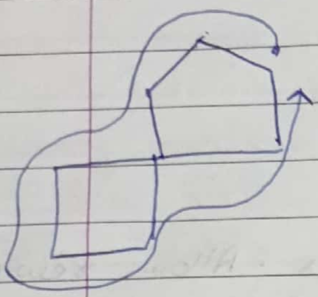
↳ In a string with repetitions; there exist substring with no repetitions.

If even closed walk

↳ Event odd = odd ⇒ $4-1=3$ ← odd walk | Odd cycle



skip A and go farther

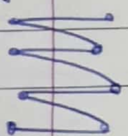


Traversal does not include odd cycle necessarily.

↓

But set of edges contain odd cycle.

Every closed even walk may not necessarily contain cycle.



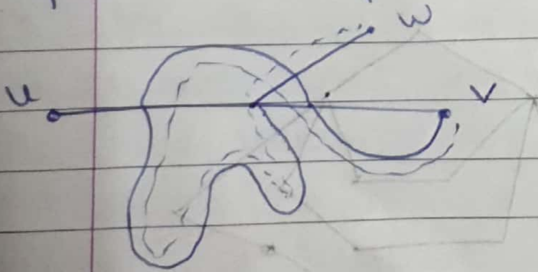
Connected Graph

- For every pair of vertices; there is a path

For every $u \to v$ walk; it contains $u \to v$ path.

Path from $u \to v$, Path from $v \to w$ $\not\Rightarrow$ Path from $u \to w$.

As there might be some intersection of $u \to v$ and $v \to w$ paths (except for v)



Adjacency Matrix

$A^2[i,j]$: Gives common neighbour of i,j

$A^2[i,i]$: Degree of i

Degree sequence = Principal diagonal entries of A^2

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Adjacency matrix gives number of walks of length 1
A^n gives number of walks of length n.
↳ $A^n(i,j)$: No of walks between i & j ~~in A~~ of length n.

$i \overset{t-1}{\sim\sim\sim} \underset{t}{—} j$ ~~If we want to reach j in t steps from i then we need to reach neighbour of j in (t-1) steps.~~

t^{th} power of adjacency matrix (A^t) entry (i,j)
$A^t[i,j]$ represents the no. of walks of length t between i and j.

Bipartite graphs
Euler graphs

Decomposition algorithms

Has to have cycle of atleast K+1 length ~~mind~~

If there is a graph with minimum degree K ; it gaurantees path of some length
A vertex of degree K : K+1 vertices atleast.

Minimum degree : δ $\delta = K$
Extremal graph theory

—x—x—x—x—x—x—

Trails and decomposition
Eulerian Trails
- Closed trails that covers all edges { Closed , Spanning Trail }
Edge cannot be reused.

Degree of endpoint in a closed trail is even.
Degree of ~~integer~~ interior vertex in trail is even.

~~Any trail~~

$d_G(v) = d_{G_1}(v) + d_{G_2}(v) + \ldots$

Decomposition into K graphs.

~~Inter~~ End point = Even Degree (Closed trail)
~~or~~ End point = odd degree (open trail)
→ [You need to be out]

Degree of starting and end vertex in a trail = Odd number

End points have odd degree

Path is special trail : (End points degree = 1)

String concatenation

<u>Question</u> : Decompose graph into open trail.