Tutorial - 5

Find the limit of f as $(x,y) \to (0,0)$ or show that the limit does not exist. (By using polar co-ordinates)

(1) $f(x,y) = \dfrac{x^3 - xy^2}{x^2+y^2}$

(2) $f(x,y) = \cos\left(\dfrac{x^3-y^3}{x^2+y^2}\right)$

(3) $f(x,y) = \dfrac{y^2}{x^2+y^2}$

(4) $f(x,y) = \dfrac{2x}{x^2+x+y^2}$

(5) $f(x,y) = \tan^{-1}\left(\dfrac{|x|+|y|}{x^2+y^2}\right)$

(6) $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

Show that the following functions have no limit as $(x,y) \to (0,0)$

(7) $f(x,y) = -\dfrac{x}{\sqrt{x^2+y^2}}$

(8) $f(x,y) = \dfrac{x^4}{x^4+y^2}$

(9) $f(x,y) = \dfrac{x^4-y^2}{x^4+y^2}$

(10) $f(x,y) = \dfrac{xy}{|xy|}$

(11) $f(x,y) = \dfrac{x-y}{x+y}$

(12) $f(x,y) = \dfrac{x+y}{x-y}$

(13) $f(x,y) = \dfrac{x^2+y}{y}$

(14) $f(x,y) = \dfrac{x^2}{x^2-y}$

(15)

~~Whether the following limit does not exist? If yes~~

Show that the following limit does not exist

(15) $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy^2-1}{y-1}$

(16) $\lim\limits_{(x,y)\to(1,-1)} \dfrac{xy+1}{x^2-y^2}$

Use (ϵ,δ) definition for the following problems.

Given $f(x,y)$ of a positive ϵ. In each of the following problem show that there exist $\delta>0$ such that for all (x,y)

$\sqrt{x^2+y^2}<\delta \Rightarrow |f(x,y)-f(0,0)|<\epsilon$.

(17) $f(x,y)=x^2+y^2$, $\epsilon=0.01$

(17) $f(x,y)=\dfrac{y}{x^2+1}$, $\epsilon=0.05$

(19) $f(x,y)=\dfrac{x+y}{x^2+1}$, $\epsilon=0.01$

(20) $f(x,y)=\dfrac{x+y}{2+\cos x}$, $\epsilon=0.02$

① $\lim_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2}$

$x = r\cos\theta \quad , \quad y = r\sin\theta$

$= \lim_{r\to 0} \frac{r^3\cos^3\theta - r\cos\theta\, r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$

$= \lim_{r\to 0} \frac{r^3(\cos^3\theta - \cos\theta\sin^2\theta)}{r^2}$

$= \lim_{r\to 0} \frac{r(\cos^3\theta - \cos\theta\sin^2\theta)}{1} = 0$

②

Ans 1

③

$\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$

$= \lim_{r\to 0} \sin^2\theta = \sin^2\theta$

The limit does not exist since $\sin^2\theta$ depends on θ.

It varies in between 0 and 1

④

$\lim_{(x,y)\to(0,0)} \frac{2x}{x^2+x+y^2}$

$= \lim_{r\to 0} \frac{2r\cos\theta}{r^2\cos^2\theta + r\cos\theta + r^2\sin^2\theta} = \lim_{r\to 0} \frac{2r\cos\theta}{r^2 + r\cos\theta}$

$= \lim_{r\to 0} \frac{2\cos\theta}{r+\cos\theta} = \frac{2\cos\theta}{\cos\theta}$

The limit does not exist for $\cos\theta = 0$.

(5)

$$\lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{|x|+|y|}{x^2+y^2}\right)$$

$$= \lim_{r\to 0} \tan^{-1}\left(\frac{|r\cos\theta| + |r\sin\theta|}{r^2}\right)$$

$$= \lim_{r\to 0} \tan^{-1}\left[\frac{|r|\left(|\cos\theta| + |\sin\theta|\right)}{r^2}\right]$$

If $r\to 0^+$

$$\lim_{r\to 0^+} \tan^{-1}\left[\frac{|r|\left(|\cos\theta| + |\sin\theta|\right)}{r^2}\right]$$

$$= \lim_{r\to 0^+} \tan^{-1}\left[\frac{|\cos\theta| + |\sin\theta|}{r}\right] = \frac{\pi}{2}$$

If $r\to 0^-$

$$\lim_{r\to 0^-} \tan^{-1}\left[\frac{|r|\left(|\cos\theta| + |\sin\theta|\right)}{r^2}\right]$$

$$= \lim_{r\to 0^-} \tan^{-1}\left[\frac{|\cos\theta| + |\sin\theta|}{-r}\right] = \frac{\pi}{2}$$

So the limit ~~is~~ $\frac{\pi}{2}$

(6)

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

$$= \lim_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2} = \lim_{r\to 0} \cos^2\theta - \sin^2\theta$$

$$= \lim_{r\to 0} \cos 2\theta$$ ~~whose~~ value ranges from ~~0~~ -1 to 1 depends on θ.

So The limit does not exist.

Q.7

$$\lim_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2+y^2}}$$

Along the line $y = mx$

$$f(x,y) = \frac{-x}{\sqrt{x^2+m^2x^2}} = \frac{-\cancel{x}}{\cancel{x}\sqrt{1+m^2}} = \frac{-1}{\sqrt{1+m^2}}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{-1}{1+m^2}$$ which is different for different values of m

So limit does not exist.

Q.8

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$$

Along the curve $y = mx^2$

$$f(x,y) = \frac{x^4}{x^4+m^2x^4} = \frac{1}{1+m^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{1+m^2}$$ which varies with different values of m.

So limit does not exist.

Q.9

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2}$$ $\boxed{\text{Hint } y = mx^2}$

Q.10

$$\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|}$$

$y = mx$

$$\frac{xy}{|xy|} = \frac{x\cdot mx}{|x\cdot mx|} = \frac{m\cancel{x^2}}{|m|\,\cancel{x^2}} = \frac{m}{|m|}$$

value are +1 and −1

So limit does not exist.

Q.11

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$$ $\boxed{\text{Hint } y = mx}$

Q.12

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$$ $\boxed{\text{Hint } y = mx}$

(13) $\lim_{(x,y)\to(0,0)} \frac{x^2+y}{y}$ $\boxed{\text{Hint } y = mx^2}$

(14) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2-y}$ $\boxed{\text{Hint } y = mx^2}$

(15) $\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$ ~~Hint Along x=1~~

<u>Along $x=1$</u> $\lim_{\substack{(x,y)\to(1,1)\\ \text{along } x=1}} \frac{xy^2-1}{y-1} = \lim_{y\to 1} \frac{y^2-1}{y-1} = \lim_{y\to 1} y+1 = 2$

<u>Along $y=x$</u> $\lim_{\substack{(x,y)\to(1,1)\\ \text{along } y=x}} \frac{xy^2-1}{y-1} = \lim_{(x,y)\to(1,1)} \frac{y^3-1}{y-1} = \lim_{y\to 1} (y^2+y+1) = 3$

So limit does not exist.

(16) <u>Hint</u> Along $y=1$, limit is $-\frac{1}{2}$

Along $y=-x^2$, limit is $\frac{3}{2}$

So limit does not exist.

(17)

(7)

$f(x,y) = x^2+y^2$, $f(0,0)=0$

$\epsilon = 0.01$

$\Rightarrow |f(x,y)-f(0,0)| < 0.01$

$\Rightarrow |x^2+y^2-0| < 0.01$

$\Rightarrow$ ~~$\sqrt{x^2+y^2} < 0.1$~~

This above will be true if we take $\sqrt{x^2+y^2} < \boxed{0.1 = \delta}$

(18)

$f(x,y) = \dfrac{y}{x^2+1}$, $f(0,0)=0$, $\epsilon = 0.05$

$|f(x,y)-f(0,0)|$

$= \left|\dfrac{y}{x^2+1}\right| \le |y| \le \sqrt{x^2+y^2}$

so $|f(x,y)-f(0,0)| < 0.05$ when $\sqrt{x^2+y^2} < 0.05$

so $\boxed{\delta = 0.05}$

(19)

$f(x,y) = \dfrac{x+y}{x^2+1}$, $f(0,0)=0$, $\epsilon = 0.01$

$|f(x,y)-f(0,0)| = \left|\dfrac{x+y}{x^2+1}\right| \le |x+y| \le |x|+|y|$

$\le \sqrt{x^2+y^2} + \sqrt{x^2+y^2}$

$= 2\sqrt{x^2+y^2} < 0.01$

when $\sqrt{x^2+y^2} < \dfrac{0.01}{2} = 0.005$

$\boxed{\delta = 0.005}$

(20)

$f(x,y) = \dfrac{x+y}{2+\cos x}$, $f(0,0)=0$, $\epsilon=0.02$

$|f(x,y)-f(0,0)| = \left|\dfrac{x+y}{2+\cos x}\right|$

$-1 \le \cos x \le 1$

$1 \le 2+\cos x \le 3 \Rightarrow \dfrac{1}{3} \le \dfrac{1}{2+\cos x} \le 1$

$\Rightarrow \dfrac{|x+y|}{3} \le \left|\dfrac{x+y}{2+\cos x}\right| \le |x+y| \le |x|+|y| \le \sqrt{x^2+y^2}+\sqrt{x^2+y^2} < 0.02$

when $\sqrt{x^2+y^2} < 0.01$

$\boxed{\delta = 0.01}$