DATE

3) Vogel's approximation

| | | | | | |
|---|---|---|---|---|---|
| 19 [5] | 30 | 50 | 10 [2] | 7 (2) (0) | 9 / 9 / 40 / 40 |
| 70 | 30 | 40 [7] | 60 [2] | 9 (2) (0) | 10 / 20 / 20 / 20 |
| 40 | 8 [8] | 70 | 20 [10] | 18 (10) (0) | 12 / 20 / 50 |
| 5 (0) | 8 (0) | 7 (0) | 14 (4) (2) (0) | | |
| 21 | 22 | 10 / 10 | 10 / 50 | | |

We will calculate the different between the minimum & next minimum for each row & each column.
Pick row/column that has maximum difference.
second column in this case.
Minimum value is 8 & $x_{32} = 8$
This column is exhausted.
Repeat the process again by finding differences. (since column was deleted, compute only row difference again)

$x_{11} = 5$, $x_{14} = 2$, $x_{23} = 7$, $x_{24} = 2$, $x_{32} = 8$, $x_{34} = 10$

cost (z) = $5 \times 19 + 2 \times 10 + 7 \times 40 + 2 \times 60 + 8 \times 8 + 10 \times 20$

→ Optimal solution for transportation problem: UV Method.
Basic cells are cells where positive value is allocated. We calculate $u_i$ and $v_j$

| | | | | $a_i$ ; $u_i$ |
|---|---|---|---|---|
| [x] 6 | [x] 9 | 8 | 7 | 0 ; 3 |
| 4 | [x] 6 | [x] 4 | [x] 7 | 0 ' 0 |
| 9 | 5 | [x] 4 | 2 | 0 ' 0 |
| 7 | 1 | 6 | [x] 8 | 0 , 1 |
| $b_j$: 0 | 0 | 0 | 0 | |
| $v_j$: 3 | 6 | 4 | 7 | |

- Look at row or column that contains maximum number of basic cells
- We select second row since it contains max no. of assigned cells, so it has a value 0
- Assign values of $u_i$ and $v_j$ such that $u_i + v_j = c_{ij}$
- If there are ties, we can choose any of the rows/columns

For all non-basic cells, compute $z_{ij} = u_i + v_j - c_{ij}$, $z_{ij} \le 0$ $\forall$ non-basic cells

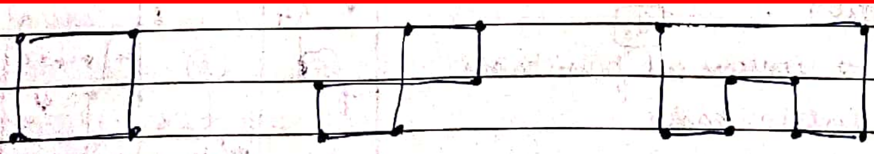
If $z_{ij} \le 0$, for all non-basic cells, solution is optimal, else solution is not optimal or degenerate

values of $z_{ij}$ are circled →

| | | | |
|---|---|---|---|
| [x] 6 | [x] 9 | -1 8 | 3 7 |
| -1 4 | [x] 6 | [x] 4 | [x] 7 |
| -6 9 | 1 5 | [x] 4 | 5 2 |
| 1 7 | -6 1 | -1 6 | [x] 8 |

Here, since there is atleast one $z_{ij} > 0$, the solution is not optimal.

A loop contains a non-basic cell and any odd number of basic cells.
- Find a non-basic cell and odd number of basic cells, find a loop around the cells.



We choose the non basic cell that has the maximum positive value

Typo

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| | | | |
|---|---|---|---|
| [x] 6 | [x] 9 | -1 8 | 3 7 |
| -1 4 | [x] 6 | [x] 4 | [x] 7 |
| -6 9 | 1 5 | [x] 4 | 5 2 |
| 1 7 | 6 1 | -1 6 | [x] 8 |

Non basic cells are the ones with values of $z_{ij}$ noted down, the basic one are marked with [x]

· We have to increase / decrease the allocation :
· after performing allocation, we will again find $z_{ij}$

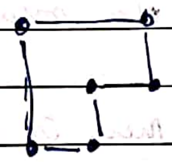
| | | | | $a_i$ |
|---|---|---|---|---|
| | 4 | 6 | 9 | 5 | 16 |
| | 2 | 6 | 4 | 1 | 12 |
| | 5 | 7 | 2 | 9 | 15 |
| $b_j$ | 12 | 14 | 9 | 8 | |

· Use any of the 3 methods to obtain the initial basic feasible solution.

| | | | | $a_i$ | $u_i$ |
|---|---|---|---|---|---|
| [12] 4 | 6 | 9 | [4] 5 | 16 | 4 |
| 2 | 6 | [8] 4 | [4] 1 | 12 | 0 |
| 5 | [14] 7 | [1] 2 | 9 | 15 | -2 |
| $b_i$ 12 | 14 | 9 | 8 | | |
| $v_j$ 0 | 9 | 4 | 1 | | |

| | | | | $a_i$ | $u_i$ |
|---|---|---|---|---|---|
| [12] 4 | 7 (+4) 6 | -1 9 | [4] (-4) 5 | 16 | 4 |
| -2 2 | -3 6 | [8] (-4) 4 | [4] (+4) 1 | 12 | 0 |
| -7 5 | [14] (-4) 7 | [1] (+4) 2 | -10 9 | 15 | -2 |
| $b_i$ 12 | 14 | 9 | 8 | | |
| $v_j$ 0 | 9 | 4 | 1 | | |

Loop is



3 remaining points [14] 7, [8] 4, [4] 5

New allocation :

[4] 5 4-4=0 → remove cell from basis

[7] 6 enters basis

| | | | | $u_i$ |
|---|---|---|---|---|
| [12] 4 | [4] 0+4=4 6 | -8 9 | -7 4-4=0 5 | 6 |
| 5 2 | 3 6 | [4] 8-4=4 4 | [8] 4+4=8 1 | -9 |
| 0 5 | [10] 14-4=10 7 | [5] 1+4=5 2 | -10 9 | 7 |
| $v_i$ -2 | 0 | -5 | -8 | |

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- Maximum the value with $\boxed{5}\ 2$, $\cancel{\boxed{\ }\ 2}$

We have 3 cells $\boxed{10}\ 7$, $\boxed{4}\ 4$ and $\boxed{12}\ 4$

| | | | |
|---|---|---|---|
| 8<br>12-4=8 | 8<br>4+4=8 | | |
| 4<br>0+4=4 | | 4-4=0 | 8 |
| | 6<br>10-4=6 | 9<br>5+4=9 | |

- Continue till we don't get all $z_{ij} \leq 0$
- If solution is optimal, we won't get a loop.

## GAME THEORY

In this world of reality, two or more opponents engaging themselves in the same field, have conflict of interest.

Exm Two businessmen A & B in the same field are fighting for the same business. A has executives $A_1, A_2, A_3, A_4$ and B has executives $B_1, B_2, B_3, B_4$ (or strategies). At any point of time A can employ the service of executive at his hand, without knowing which executive B is employing. Same is the case for B. What would be the outcome of the process at the end?

Defns:

i) A and B are decision makers.

ii) $A_1 A_2 .. A_n$ are strategies of A   iii) $B_1 B_2 ... B_m$ are strategies of B.

iv) Profit/loss is attached with each strategy. Profit for A is loss for B and vice versa.

v) A is a maximizing player then B is the minimizing player and vice versa.

vi) The profit/loss of every player is shown by a matrix called pay-off matrix

Exm

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $A_1$ | 4 | 2 | -3 | 1 |
| $A_2$ | 0 | -3 | 4 | 6 |
| $A_3$ | 3 | 2 | 2 | -2 |

→ pay-off matrix for player A.

-ve values correspond to profit for B.

If A adopts strategy $A_1$ and B adopts $B_1$, then gain/profit of player A is 4 units (≡ loss of player B is 4 units)

→ Game: A game is a solution in which two or more decision makers (players) choose course of actions/strategies available to them and the outcome is affected by the strategies adopted by the players collectively.

A game involves a set of rules -

i) $n \geq 2$ decision makers with ~~oppos~~ mutually opposed interests.

ii) course of actions available to each player are known to him.

iii) A clearly defined set of end states (win/loss/draw) terminates the competition.

iv) Pay-off to each player at the end of each play is known.

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→ Problem of the game:
If A is a maximizing player and B is a minimizing player (assuming for a game of 2 players), then A's problem is to maximize his minimum profit
B's problem is to minimize his maximum ~~profit~~ loss.

→ Zero-sum game
If $P_i$ is the pay-off to player $i$ at the end, then if $\sum_{i=1}^{n} P_i = 0$ then the game is called a zero sum game. Otherwise it is a non-zero-sum game
($n$ = # of players)
For a two person zero sum game, $P_1 + P_2 = 0$ (profit one one is loss of the other).

→ Course of Action
· set of actions (strategies)
· set of probabilities associated with each action.

| | | $y_1$ | $y_2$ | $\cdots$ | $y_n$ |
|---|---|---|---|---|---|
| | | $B_1$ | $B_2$ | $\cdots$ | $B_n$ |
| $x_1$ | $A_1$ | | | | |
| $x_2$ | $A_2$ | | $a_{ij}$ | | |
| $\vdots$ | $\vdots$ | | | | |
| $x_n$ | $A_n$ | | | | |

$a_{ij}$ = pay-off of player-1
$x_i$ = probability with which player 1 chooses strategy $A_i$
$y_j$ = probability with which player 2 chooses strategy $B_j$

$(A_1, A_2 \cdots A_m)$ Actions of A } strategies of A
$X = (x_1, x_2 \ldots x_i \cdots x_m)$ probabilities associated with each action
$(B_1, B_2 \cdots B_n)$ Actions of B } strategies of B
$Y = (y_1, y_2 \cdots y_n)$ probabilities associated with each action → $i^{th}$ position.

In particular if $X = \xi_i = (0, 0, \ldots 1, \ldots 0)$ → Deterministic
$Y = \eta_j = (0, 0 \ldots 1, \ldots 0)$
↳ $j^{th}$ position.
then the game is called pure.
Otherwise the game has mixed strategy

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→

| | $B_1$ | $B_2$ | $B_2$ |
|---|---|---|---|
| $A_1$ | -2 | 0 | $\boxed{2}$ |
| $A_2$ | -1 | 1 | 0 |
| $A_3$ | $\boxed{1}$ | $\boxed{2}$ | 1 |

For a pure game, if A chooses $A_1$, B chooses $B_1$
→ For A - max. profit
B - min. loss

· For each row we find the minimum value → B's choices corresponding to A's strategies.
$\boxed{\cdot}$ For each column we find the maximum value → A's choices corresponding to B's strategies
$\boxed{\ }$ Value of the game & the cell shows the optimal strategy.

We are basically finding $\max_i \min_j a_{ij} = \underline{v}$

and $\min_j \max_i a_{ij} = \bar{v}$

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→ When $\underline{v} = \bar{v} = v$ ← value of the game

☐ ← This cell is called the saddle point.

For a two person if there exists a saddle point, it is the value of the game and the optimal strategy.

→ Else if $\underline{v} \neq \bar{v}$ then the value of the game $v$ is $\underline{v} \leq v \leq \bar{v}$

eg.

(1)

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $A_1$ | 4 | 6 | -2 | 1 |
| $A_2$ | 3 | 3 | 4 | 2 |
| $A_3$ | 4 | 5 | 5 | 1 |

→ value of the game & optimal strategy is $A_2 B_4$

(2)

| | $B_1$ | $B_2$ |
|---|---|---|
| $A_1$ | 4 | -1 |
| $A_2$ | 2 | 3 |

$\underline{v} = 2$

$\bar{v} = 3$

$2 \leq v \leq 3$

↑

NO saddle point → pure strategy will not work

↓

(3)

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $A_1$ | 0 | 2 | -3 | 0 |
| $A_2$ | -2 | 0 | 0 | 3 |
| $A_3$ | 3 | 0 | 0 | -4 |
| $A_4$ | 0 | -3 | 4 | 0 |

$\underline{v} = -2$

$\bar{v} = 2$

$-2 \leq v \leq 2$

$A_1, A_2, \ldots A_m$ : Actions of A

$x_1, x_2 \ldots x_m$ : probabilities $\sum x_i = 1$

$B_1, B_2 \ldots B_n$ : Actions of B

$y_1, y_2 \ldots y_n$ : probabilities $\sum y_j = 1$

} For Mixed strategy game

$A = (a_{ij})_{m \times n}$ → pay-off matrix

Expected pay-off to player A, where $A_\gamma$ is chosen.

$$E(\xi_r, Y) = \sum_{j=1}^{n} a_{\gamma j} y_j$$

Expected pay-off for player A is,

$$E(X, Y) = \sum_{\gamma=1}^{m} \left( \sum_{j=1}^{n} a_{\gamma j} y_j \right)$$

Similarly, Expected pay-off for player A when B chooses $B_s$,

$$E(X, \eta_s) = \sum_{i=1}^{m} a_{is} x_i$$

Expected pay-off for player A is,

$$E(X, Y) = \sum_{s=1}^{n} \left( \sum_{i=1}^{m} a_{is} x_i \right)$$

Typo

Expected pay off for player A is,

$$E(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$$

→ Finding the strategy.

$(X_0, Y_0)$ be the strategy if $E(X_0,Y_0) \geq E(X,Y_0) \; \forall \; x \neq x_0$

$\leq E(X_0, Y) \; \forall \; y \neq y_0$

$$E(X_0,Y_0) = \max_X \min_Y E(X,Y) = \min_Y \max_X E(X,Y)$$

$(X_0 \, Y_0)$ is the strategic saddle point.

Ex$^m$

| | | $y$ $B_1$ | $1-y$ $B_2$ |
|---|---|---|---|
| $x$ | $A_1$ | 5 | 1 |
| $1-x$ | $A_2$ | 3 | 4 |

$$E(X,Y) = 5xy + x(1-y) + 3y(1-x) + 4(1-x)(1-y) \quad \text{—①}$$

$$= C + D(x-K)(y-l)$$

$$= C + D(xy - Ky - lx + Kl) \quad \text{—②}$$

Writing ① in the form of ②

$$5xy + x - xy + 3y - 3xy + 4 - 4x - 4y + 4xy$$

$$= 5xy - 3x - y + 4 \quad \text{—③}$$

Equating ② and ③

$D = 5$, $l = 3/5$, $k = 1/5$ and $C = 17/5$

$$E(X,Y) = \frac{17}{5} + 5\left(x - \frac{1}{5}\right)\left(y - \frac{3}{5}\right)$$

The value of the game is $\frac{17}{5}$

optimal strategy for player A $\left(\frac{1}{5}, \frac{4}{5}\right)$

for player B $\left(\frac{3}{5}, \frac{2}{5}\right)$

⇒ Solving $2 \times n$ (or $m \times 2$) game (Graphically) 13·11·19

Let the pay-off matrix be –

| | | $y_1$ $B_1$ | $y_2$ $B_2$ | $y_3$ $B_3$ | $y_4$ $B_4$ | $y_5$ $B_5$ |
|---|---|---|---|---|---|---|
| $x$ | $A_1$ | 1 | 4 | -1 | -5 | 6 |
| $1-x$ | $A_2$ | 3 | 2 | 6 | 4 | -3 |

There is no saddle point.

$0 \leq y_i \leq 1$ and $\sum y_i = 1$

Expected pay-off to player A, when B chooses $B_1$ is

$$x + (1-x)3 = 3 - 2x$$

When B chooses $B_2$, it will be $4x + 2(1-x) = 2x + 2$

When B chooses $B_3$, it will be $-x + 6(1-x) = 6 - 7x$

When B chooses $B_4$, it will be $-5x + 4(1-x) = 4 - 9x$

When B chooses $B_5$, it will be $6x - 3(1-x) = 9x - 3$

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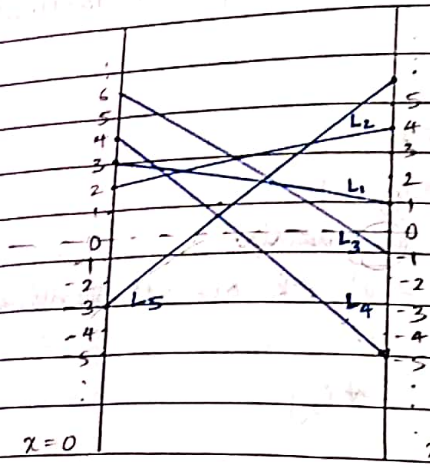


$B_1$ $(3-2x)$ is the line joining points $(0,3)$ and $(1,1)$

$B_2$ $(2x+2)$ joins $(0,2)$ and $(1,4)$

$B_3$ $(6-2x)$ joins $(0,6)$ and $(1,-1)$

$B_4$ $(4-9x)$ joins $(0,4)$ and $(1,-5)$

$B_5$ $(9x-3)$ joins $(0,-3)$ and $(1,6)$

Since A's goal is to maximize his minimum profit, it will lie in the lower envelope. To find $\max_y \min_y E(X,Y)$, we need to show $L_5 = L_4$

$9x_0 - 3 = 4 - 9x_0 \implies 18x_0 = 7 \implies x_0 = 7/18$

Strategy for A is $\left(\frac{7}{18}, \frac{11}{18}\right)$ and $Y = (0, 0, 0, y_4, y_5)$

Since $y_1, y_2, y_3$ are 0, matrix can be reduced.

| | | $y$ | $1-y$ |
|---|---|---|---|
| | | $B_1$ | $B_2$ |
| $7/18$ $x$ | $A_1$ | $-5$ | $6$ |
| $11/18$ $1-x$ | $A_2$ | $4$ | $-3$ |

$E(X,Y) = -5 \times \frac{7}{18} y + 6 \times \frac{7}{18} \times (1-y) + 4 \times \frac{11}{18} y - 3 \times \frac{11}{18} \times (1-y)$

$E(X,Y) = -5xy + 6x(1-y) + 4y(1-x) - 3(1-x)(1-y)$

$\equiv C + D(x-k)(y-l)$

Solve and we will get $x = \frac{7}{18}$, $y = \frac{1}{2}$ and $V = 1/2$

→ **Dominance Property**

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $A_1$ | -3 | 3 | 1 | 20 |
| $A_2$ | 5 | 5 | 4 | 6 |
| $A_3$ | -4 | -2 | 0 | 6 |

Ignoring $A_2$ for the time being, A will always choose $A_1$ over $A_3$ for whichever strategy B chooses. So $A_3$ is dominated by $A_1$ (Every value of $A_1$ is $\geq$ value of $A_3$) so $A_3$ is redundant and can be removed.

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $A_1$ | -3 | 3 | 1 | 20 |
| $A_2$ | 5 | 5 | 4 | 6 |

Similarly $B_2$ is dominated by $B_1$ and can be removed. (every value of $B_1 \leq B_2$)

and $B_4$ is dominated by $B_1$

| | $B_1$ | $B_3$ |
|---|---|---|
| $A_1$ | -3 | 1 |
| $A_2$ | 5 | 4 |

Again, $A_1$ is dominated by $A_2$ and again $B_1$ is dominated by $B_3$

| | $B_1$ | $B_3$ |
|---|---|---|
| $A_2$ | 5 | 4 |

$\therefore$ Optimal strategy is $A_2 B_3$ and $V = 4$

(also the saddle point of above 2×2 matrix)

Typo

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→ solving an $m \times n$ game using LPP

Let $A_{m \times n}$ be a pay-off matrix

$X = (x_1, x_2, \dots x_m)$ , $0 \le x_i \le 1$ , $\Sigma x_i = 1$

$Y = (y_1, y_2, \dots y_n)$ , $0 \le y_j \le 1$ , $\Sigma y_j = 1$

Consider a new ~~matrix~~ game with pay-off matrix

$A_{m \times n} = ((a'_{ij} + k))$ such that $a_{ij} = a'_{ij} + k$ are all positive $\forall i, j$

$$\begin{array}{c|cccc} & B_1 & B_2 & \dots & B_n \\ \hline A_1 & a_{11} & a_{12} & \dots & a_{1n} \\ A_2 & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ A_m & a_{m1} & a_{m2} & \dots & a_{mn} \end{array}$$

when $a_{ij} \ge 0 \quad \forall i, j$

$E_j(x) = a_{1j} x_1 + a_{2j} x_2 + \dots + a_{mj} x_m$ (Expected pay-off when B chooses $B_j$)

assume $\min_j E_j(x) = U$

A's problem is to find $x$ such that $V$ attains its maximum.

so, $E_j(x) \ge u \quad \forall j$

A's problem is to maximize $V$ subject to

$E_j(x) \ge u \quad \forall \quad j = 1, 2, \dots n$

and maximize $u \equiv$ minimize $\frac{1}{u}$ (true since all $a_{ij}$ & $x_i$ are +ve)

so, the LPP problem is:

- minimize $Z = \frac{1}{u}$

  subject to $E_j(x) \ge u$ , $x \ge 0$

- minimize $Z = \frac{1}{u} = \frac{x_1 + x_2 + \dots + x_m}{u}$

  subject to $a_{1j} x_1 + a_{2j} x_2 + \dots + a_{mj} x_j \ge u \qquad \forall j = 1, 2, \dots n$

  $x_i \ge 0 \qquad \forall i = 1, 2 \dots m$

→ Let us assume that $\frac{x_i}{u} = x_i'$

- Minimize $Z = x_1' + x_2' + \dots + x_2'$

  subject to $a_{1j} x_1' + a_{2j} x_2' + \dots + a_{mj} x_m' \ge 1$

  $x_i' \ge 0$

Now we solve this LPP and then some operations will have to be carried out to get the solution to the original problem.

Considering above matrix again,

$E_i(Y) = a_{i1} y_1 + a_{i2} y_2 + \dots + a_{in} y_n \qquad i = 1, 2, \dots n$

and assume $\max_i E_i(Y) = w$

B's problem is to find $Y$ such that $w$ attains its minimum

$E_i(Y) \le w \quad \forall i$

and minimize $w \equiv$ maximize $\frac{1}{w}$

Maximize $T = \frac{1}{w}$ subject to $E_i(Y) \le w \quad \forall i, \; y_i \ge 0$

∴ Maximize $T = y_1' + y_2' + \dots + y_n'$

subject to $a_{i1} y_1' + a_{i2} y_2' + \dots + a_{in} y_n' \le 1 \qquad i = 1, 2, \dots m$

$y_i' \ge 0$

By solving these two LPP's we have,

$\min Z = \max T = v^*$

and original value of the game is $\left(\frac{1}{v^*} - K\right)$ and $x_i$ & $y_j$ are the strategies

Exm $B =$

| | | |
|---|---|---|
| -1 | -2 | 8 |
| 7 | 5 | -1 |
| 6 | 0 | 12 |

to make all values +ve choose $k = 2$

$A = B + 2$

$A =$

| | | |
|---|---|---|
| 1 | 0 | 10 |
| 9 | 7 | 1 |
| 8 | 2 | 14 |

Let the strategies be $X = (x_1\, x_2\, x_3)$ and $Y = (y_1\, y_2\, y_3)$

$\min Z = x_1' + x_2' + x_3'$

subject to $x_1 + 9x_2 + 8x_3 \ge 1$

$7x_2 + 2x_3 \ge 1$

$10x_1 + x_2 + 14x_3 \ge 1 \quad , \quad x_1, x_2, x_3 \ge 0$

Above is the primal, the corresponding dual is

$\max T = y_1 + y_2 + y_3$

subject to $y_1 + 10y_3 \ge 1$

$9y_1 + 7y_2 + y_3 \ge 1$

$8y_1 + 2y_2 + 14y_3 \ge 1 \quad , \quad y_1, y_2, y_3 \ge 0$

Solving this, we will get $Y = \left(0, \frac{13}{96}, \frac{5}{96}\right)$ (dual)

(optimal solⁿ)

$X = \left(0, \frac{1}{8}, \frac{1}{16}\right)$ (primal)

and $u_{min} = w_{max} = \text{value} = \frac{3}{16}$

→ For the original problem

$u = w = \frac{3}{16}$

$x_i^* = \frac{x_i}{u}$ (we took $x_i$ as $x_i'$ here) $\qquad y_i^* = \frac{y_i}{w}$

$X^* = \left(0, \frac{2}{3}, \frac{1}{3}\right)$ ~~and~~ and $Y^* = \left(0, \frac{13}{18}, \frac{5}{18}\right)$

Typo

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Value of the game is $\left(\frac{1}{r^*} - k\right) = \frac{1}{3/16} - 2 = \frac{10}{3}$

⇒ THE QUEUEING MODEL

arrival → entity enters system

service → entity requires some of the system's services

queue → associated with the system

→ follow some distribution

eg. Railway ticket booking counter.

Types of Queueing Model :

Doctor

Queue

1) Single server

2) Multiple server

3) Finite Queue length system - system has certain service limits so length of queue is finite.

4) Infinite Queue length

5) Finite Population → anybody can join the system

6) Infinite Population → only a certain group of entities are permitted.

Railway ticket counter - 2, 4, 6

Doctor clinic - 1, 4, 5/6 → if doctor will check only 10 ppl in a day. institute doctor will only check students.

garage (car service) - 1/2, 3, 6

- We mostly focus on Infinite Population systems.

The arrivals follow a Poisson distribution & services follow an exponential distributions.

Arrival rate → $\lambda$/hr   Service rate → $\mu$/hr

for infinite population, the relation b/n $\lambda$ and $\mu$ →

$$\frac{\lambda}{\mu} < 1$$

Three terms :

1. Balking - in a long queue, you estimate the time & end up leaving the system without joining
2. Reneging - Initially you estimate that you will be serviced, but later realise otherwise & leave the system
3. Jockeying - when multiple queues are present, you select the shortest 'zigzagging' queue. on observing a faster moving queue you move to it. Happens only in the beginning, while entering.

1) Single server, infinite queue length

Kendal's Notation.

$$M / M / 1 / \infty / \infty$$

arrivals Poisson $\lambda$/hr

service exponential $\mu$/hr

single service

infinite queue length

infinite population.

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$\lambda$/hr arrivals → Server ($\mu$/hr)

0 people - $P_0$
1 " - $P_1$
2 " - $P_2$
⋮
n " - $P_n$

$P_i$ — probability that i people are in system

We are interested in -
$L_s$ 1. Length of a system. (Expected no. of people in system)
$L_q$ 2. Length of a Queue (Expected " in queue)
$W_s$ 3. Waiting time in the system (Expected time → total time taken to receive service)
$W_q$ 4. Waiting time in the Queue (Time from joining to getting service)

$(\lambda, \mu, c) \to$ Input  c=1 (single server)

We first derive expression for $P_n$.
Assume that only one event takes place in a small interval of time h.

- $P_n(t+h) = P_{n-1}(t) * (\text{probability of one arrival \& no service})$
  $+ P_{n+1}(t) * (\text{probability of no arrival \& 1 service})$
  $+ P_n(t) * (\text{probability of no arrival \& no service})$

(we do not consider $P_n(t)$ & one arrival & one service because only 1 event occurs in h)

Probability of one arrival is $= \lambda h$  No arrival $= (1-\lambda h)$
Probability of one service is $= \mu h$  No service $= (1-\mu h)$

- $P_n(t+h) = P_{n-1}(t) * \lambda h(1-\mu h) + P_{n+1}(t) * \mu h(1-\lambda h) + P_n(t)(1-\lambda h)(1-\mu h)$

Remove higher order terms (squares)

- $P_n(t+h) = P_{n-1}(t)\lambda h + P_{n+1}(t)\mu h + P_n(t)(1-\lambda h-\mu h)$

- $\dfrac{P_n(t+h)-P_n(t)}{h} = P_{n-1}(t)\lambda + P_{n+1}(t)\mu - P_n(t)(\lambda+\mu)$

↳ tends to 0

- $\lambda P_{n-1} + \mu P_{n+1} = (\lambda+\mu) P_n$ —— (1)

Similarly, $P_0(t+h) = P_1(t) * (\text{no arrival \& one service})$
$+ P_0(t) * (\text{no arrival \& no service})$

- $P_0(t+h) = P_1(t)\mu h(1-\lambda h) + P_0(t)(1-\mu h)(1)$ ~~λh~~

↳ 0 people in queue & no service

- $\dfrac{P_0(t+h)-P_0(t)}{h} = P_1(t)\mu - P_0(t)\lambda$

$P_0'(t)$ = 0

- $\lambda P_0 = \mu P_1 \implies P_1 = \dfrac{\lambda}{\mu} P_0$ —— (2)

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Eqn ① with $n=1$ is -

$$\lambda p_0 + \mu p_2 = (\lambda+\mu) p_1$$

$$\rightarrow \lambda p_0 + \mu p_2 = \lambda p_1 + \cancel{\mu p_1}$$

from ② $\mu p_1 = \lambda p_0$

$$\rightarrow p_2 = \frac{\lambda}{\mu} p_1 = \left(\frac{\lambda}{\mu}\right)^2 p_0 \quad (\text{from } ②)$$

$$\vdots$$

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0 \quad \text{and take } \frac{\lambda}{\mu} = \rho$$

$$\rightarrow p_n = \rho^n p_0$$

- Now we want to find $p_0$.

$$p_0 + p_1 + p_2 + \dots + p_\infty = 1$$

$$p_0 + \rho p_0 + \rho^2 p_0 + \dots + \infty = 1$$

$$p_0 [1 + \rho + \rho^2 + \dots \infty] = 1$$

$$p_0 \frac{1}{1-\rho} = 1$$

$$p_0 = 1-\rho \quad (\lambda < \mu \therefore \rho < 1)$$

then $p_1 = \rho(1-\rho)$

& $p_n = \rho^n (1-\rho)$

- Our target is to calculate $L_s, L_q, W_s, W_q$

→ $L_s$ = expected no. of people in the system

$$= \sum_{j=0}^{\infty} j p_j = \sum_{j=0}^{\infty} j \rho^j p_0$$

$$= \rho p_0 \sum_{j=0}^{\infty} j \rho^{j-1} = \rho p_0 \sum_{j=0}^{\infty} \frac{d}{d\rho}(\rho^j)$$

$$= \rho p_0 \frac{d}{d\rho} \sum_{j=0}^{\infty} \rho^j = \rho p_0 \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)$$

$$= \rho p_0 \left(\frac{1}{1-\rho}\right)^2 = \rho \cancel{(1-\rho)} \cdot \frac{1}{(1-\rho)^{\cancel{2}}}$$

$$\boxed{L_s = \frac{\rho}{1-\rho}}$$

→ $L_q$ = expected no. of people in the queue

$L_s = L_q + \frac{\lambda}{\mu}$ → expected no. of people that are served.

$L_s = \lambda W_s$
$L_q = \lambda W_q$ ] LITTLE'S EQUATION

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Ex^n $\lambda = 8/hr$, $\mu = 9/hr$ so, $\rho = \frac{\lambda}{\mu} = \frac{8}{9}$

- Probability of no person in the system $= p_0 = 1-\rho = 1-\frac{8}{9} = \frac{1}{9}$
- $(1-p_0) = \frac{8}{9} \rightarrow$ probability that system is busy.

- Probability that there is no person in the queue = Probability that there is no person in the system + probability that there is 1 person in the system (and is served) $= p_0 + p_1 = (1-\rho) + \rho(1-\rho)$

$= \frac{1}{9} + \frac{8}{9}\left(\frac{1}{9}\right) = \frac{1}{9}\left(1+\frac{8}{9}\right) = \frac{17}{81} = \underline{0.2098}$

- Probability of 5 people in the system $= p_5 = \rho^5(1-\rho)$
- Probability of atleast 4 person in the system

$P(n \geq 4) = p_4 + p_5 + \cdots + p_\infty$

$= 1 - (p_0 + p_1 + p_2 + p_3)$

- $L_s = \frac{\rho}{1-\rho} = \frac{8/9}{1/9} = \underline{\underline{8}}$

- $L_q = L_s - \rho = 8 - \frac{8}{9} = \frac{64}{9} = 7.11$

- $W_q = L_q/\lambda = \frac{64/9}{8} = \frac{8}{9} = 0.88$ hr

- $W_s = L_s/\lambda = 8/8 = \underline{\underline{1}}$ hr

$\Rightarrow$ $\underline{M/M/1/N/\infty}$ Model finite queue length, infinite population.

$\rho = \frac{\lambda}{\mu}$ and queue can contain AT MOST N people.

$p_0, p_1, p_2, \cdots P_N$

$p_0 + p_1 + p_2 + \cdots + p_N = 1$

$p_0 + \rho p_0 + \rho^2 p_0 + \cdots + \rho^N p_0 = 1$

$p_0\left(\frac{1-\rho^{N+1}}{1-\rho}\right) = 1 \rightarrow \boxed{p_0 = \frac{1-\rho}{1-\rho^{N+1}}}$

(can't say if $\rho$ is <1 or >1.)

$p_n = \rho^n p_0$ remains same.

and for $L_s = \sum_{j=0}^{N} j p_j = \sum_{j=0}^{N} j \rho^j p_0 = \rho p_0 \sum_{j=0}^{N} j \rho^{j-1}$

$= \rho p_0 \sum_{j=0}^{N} \frac{d}{d\rho}(\rho^j) = \rho p_0 \frac{d}{d\rho} \sum_{j=0}^{N} \rho^j$

$= \rho p_0 \frac{d}{d\rho}\left(\frac{1-\rho^{N+1}}{1-\rho}\right) = \rho p_0 \left[\frac{(1-\rho)(N+1)(-1)(\rho^N) - (1-\rho^{N+1})(-1)}{(1-\rho)^2}\right]$

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$$= \rho p_0 \left[ \frac{(N+1)(\rho^{N+1} - \rho^N) + (1 - \rho^{N+1})}{(1-\rho)^2} \right] = \rho p_0 \left[ \frac{1 - \rho^{N+1} + (N+1)\rho^{N+1} - (N+1)\rho^N}{(1-\rho)^2} \right]$$

$$= \rho p_0 \left[ \frac{1 - \cancel{\rho^{N+1}} + \cancel{\rho^{N+1}} + N\rho^{N+1} - N\rho^N - \rho^N}{(1-\rho)^2} \right] = \frac{\rho p_0}{(1-\rho)^2} \left( 1 + N\rho^{N+1} - (N+1)\rho^N \right)$$

$$= \frac{\rho}{(1-\rho)^{\cancel{2}}} \left( \frac{\cancel{1-\rho}}{1-\rho^{N+1}} \right) \left( 1 + N\rho^{N+1} - (N+1)\rho^N \right)$$

$$\boxed{L_s = \frac{\rho\left(1 + N\rho^{N+1} - (1+N)\rho^N\right)}{(1-\rho)(1-\rho^{N+1})}}$$

'Forced Balking' takes place when queue is full.

For infinite queue length, $L_s = L_q + \frac{\lambda}{\mu}$

$$\lambda_{effective} = \lambda(1 - P_N)$$

and $L_s = L_q + \frac{\lambda_{effective}}{\mu}$

$W_s$ and $W_q$ equations remain same