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TUTORIAL 2 SOLUTIONS:

Q:1 Find the limits:

(1) $\lim_{x\to\infty} (\sqrt{x^2+3x} - \sqrt{x^2-2x}) = \lim_{x\to\infty} (\sqrt{x^2+3x} - \sqrt{x^2-2x}) \times \frac{(\sqrt{x^2+3x}+\sqrt{x^2-2x})}{(\sqrt{x^2+3x}+\sqrt{x^2-2x})}$

$= \lim_{x\to\infty} \frac{(x^2+3x)-(x^2-2x)}{\sqrt{x^2+3x}+\sqrt{x^2-2x}} = \lim_{x\to\infty} \frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$

$= \lim_{x\to\infty} \frac{5\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}\right)} = \frac{5}{1+1} = \frac{5}{2}.$

(2) $\lim_{x\to\infty} (\sqrt{x^2+x} - \sqrt{x^2-x}) = \lim_{x\to\infty} (\sqrt{x^2+x} - \sqrt{x^2-x}) \times \frac{(\sqrt{x^2+x}+\sqrt{x^2-x})}{(\sqrt{x^2+x}+\sqrt{x^2-x})}$

$= \lim_{x\to\infty} \frac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}} = \lim_{x\to\infty} \frac{2\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}\right)} = \frac{2}{2} = 1.$

Q.2 Use the formal definition to prove that

(a) $\lim_{x\to\infty} \frac{1}{|x|} = \infty.$

Definition: for every real number $B>0$, we must find a $\delta>0$ such that for all x, $0<|x-0|<\delta \Rightarrow \frac{1}{|x|} > B.$

$0<|x-0|<\delta.$

Choose $\delta = \frac{1}{B}.$

$|x| < \frac{1}{B}.$

$\frac{1}{|x|} > B.$

So for every B, we can find a δ, such that $f(x) > B.$

$\therefore \lim_{x\to 0} \frac{1}{|x|} = \infty$

(b) $\lim_{x\to 1^-} \frac{1}{1-x^2} = \infty$.

$B>0$, $0<x<1$, $\frac{1}{1-x^2} > B$.

Since $x<1$, $\frac{1+x}{2} < 1$.

$1-\delta < x < 1 \Rightarrow 1-x < \delta$.

Choose $\delta = \frac{1}{2B}$.

$1-x < \frac{1}{2B}$.

$(1-x)(1+x) < \frac{1}{B}\frac{(1+x)}{2}$ (Multiply $1+x$ both sides)

$(1-x)(1+x) < \frac{1}{B}$ (since $\frac{1+x}{2} < 1$).

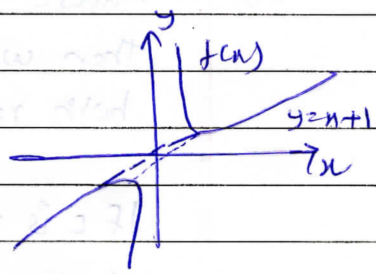
$1-x^2 < \frac{1}{B}$.

$\frac{1}{1-x^2} > B$ for $0<x<1$

$\Rightarrow \lim_{x\to 1^-} \frac{1}{1-x^2} = \infty$.

Q.3 Find the oblique asymptotes of.

(a) $f(x) = \frac{x^2+1}{x-1}$



$f(x) = y = \frac{x^2+1}{x-1} = \underbrace{(x+1)}_{g(x)} + \frac{2}{x-1}$

$\lim_{x\to\infty} \frac{2}{x-1} = 0$.

So $f(x)$ and $g(x)$ are very close when $x\to\infty$.

So $g(x) = 1+x$ is the oblique asymptote.

(b) $f(x) = \frac{x^3+1}{x^2} = x + \frac{1}{x^2}$

As $x \to \infty$, $\frac{1}{x^2} \to 0$.

$f(x)$ and x ~~[illegible]~~ are very close as $x \to \infty$

$y = x$ is the oblique asymptote.

Q.4 $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

(a) Show $f(x)$ is continuos at $x=0$.

Let $\epsilon > 0$ be given

If x is rational, then $f(x) = x$

$\Rightarrow |f(x) - 0| = |x - 0| < \epsilon$.

$|x - x_0| < \delta \Rightarrow |x - 0| < \delta \Rightarrow |x| < \delta$

We can chose a $\delta = \epsilon$, such that above is satisfied.

If x is irrational $f(x) = 0$.

$\Rightarrow |f(x) - 0| < \epsilon \Rightarrow 0 < \epsilon$ which is always true.

no matter how close irrational x is to 0.

We can chose δ such that $|f(x) - 0| < \epsilon$.

$\Rightarrow f$ is continuos at $x = 0$.

(b) Choose $x = c > 0$.

Then within the interval $(c-\delta, c+\delta)$ there are both rational and irrational numbers.

If c is rational, pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$, there is an irrational number x in $(c-\delta, c+\delta)$

$|f(x) - c| = |0 - c| = c > \frac{c}{2} = \epsilon$.

$\Rightarrow f$ is not continuos at any rational $c > 0$.

If c is irrational $\Rightarrow f(c)=0$.

pick $\epsilon = \frac{c}{2}$.

No matter how small δ we choose, there is an irrational and rational number x in $(c-\delta, c+\delta)$

$$|x-c|<\frac{c}{2} \Rightarrow \frac{c}{2}<x<\frac{3c}{2}$$

$$|f(x)-f(c)| = |x-0| = |x| > \frac{c}{2} = \epsilon$$

f is not continuous at any irrational $c>0$

If $x=c<0$, repeat the argument taking $\epsilon = \frac{|c|}{2} = \frac{-c}{2}$

Therefore f fails to be continuous at any arbitrary point c

Q.5

(a) True

If $\lim_{x\to a} (f(x)+g(x))$ exists then

$$\lim_{x\to a} (f(x)+g(x)) - \lim_{x\to a} f(x) = \lim_{x\to a} (f(x)+g(x)-f(x))$$

$$= \lim_{x\to a} g(x) \text{ exists.}$$

(b) False.

$$f(x) = \frac{1}{x} \qquad g(x) = \frac{-1}{x}.$$

$\lim_{x\to 0} f(x)$ does not exist

$\lim_{x\to 0} g(x)$ does not exist

but $\lim_{x\to 0} f(x)+g(x) = \lim_{x\to 0} \frac{1}{x} - \frac{1}{x} = \lim_{x\to 0} 0 = 0$ exists.

(c) True.

$g(x) = |x|$ is continuous

$\Rightarrow g(f(x)) = |f(x)|$ is continuous.

Composition of two functions.

(d) False.

$$f(x) = \begin{cases} -1 & x \le 0 \\ 1 & x > 0 \end{cases}$$

$f(x)$ is discontinuous at $x = 0$

But $|f(x)| = 1$ is continuous at $x = 0$.

6. Identify the horizontal, vertical and oblique asymptotes for the following.

(a) $f(x) = \dfrac{\sqrt{5x^2+7}}{2x+3}$

$$\lim_{x\to\infty} \frac{\sqrt{5x^2+7}}{2x+3} = \lim_{x\to\infty} \frac{|x|\sqrt{5+7/x^2}}{x\left(2+\frac{3}{x}\right)} = \lim_{x\to\infty} \frac{\cancel{x}\cdot\sqrt{5+7/x^2}}{\cancel{x}(2+3/x)}$$

$$= \frac{\sqrt{5}}{2}$$

$$\lim_{x\to-\infty} \frac{\sqrt{5x^2+7}}{2x+3} = \lim_{x\to-\infty} \frac{|x|\sqrt{5+7/x^2}}{x(2+3/x)} = \lim_{x\to-\infty} \frac{-\cancel{x}\sqrt{5+7/x^2}}{\cancel{x}(2+3/x)}$$

$$= -\sqrt{5}/2$$

$y = \sqrt{5}/2$, $y = -\sqrt{5}/2$ are horizontal asymptotes.

$$\lim_{x\to-3/2^+} \frac{\sqrt{5x^2+7}}{2x+3} = \lim_{x\to-3/2^+} \frac{\sqrt{5x^2+7}}{2(x+3/2)} \qquad \left(x \to -\tfrac{3}{2}^+ \Rightarrow x+\tfrac{3}{2} \to 0^+ \right.$$

$$= \frac{\sqrt{5\times(9/4)+7}}{2*0^+} = +\infty$$

$$\lim_{x \to -\frac{3}{2}^-} \frac{\sqrt{5x^2+7}}{2x+3} = \frac{\sqrt{5(9/4)+7}}{2 \times 0^-} = -\infty$$

$x = -\frac{3}{2}$ is a vertical asymptote.

(b) $f(x) = \dfrac{1-4x^3}{3+2x-x^2} = \dfrac{1-4x^3}{-(x+1)(x-3)} = \dfrac{4x^3-1}{(x+1)(x-3)}$

$$\lim_{x \to -1^+} \frac{4x^3-1}{(x+1)(x-3)} = \frac{4(-1)-1}{0^+(-4)} = +\infty \quad \left(\begin{array}{c} x \to -1^+ \\ x+1 \to 0^+ \end{array} \right)$$

$$\lim_{x \to -1^-} \frac{4x^3-1}{(x+1)(x-3)} = \frac{4(-1)-1}{0^-(-4)} = -\infty \quad \left(\begin{array}{c} x \to -1^- \\ x+1 \to 0^- \end{array} \right)$$

Similarly

$$\lim_{x \to 3^+} \frac{4x^3-1}{(x+1)(x-3)} = \frac{4 \times 3^3 - 1}{4 \times 0^+} = +\infty$$

$$\lim_{x \to 3^-} \frac{4x^3-1}{(x+1)(x-3)} = \frac{4(3)^3-1}{4 \times 0^-} = -\infty$$

$x = -1$, $x = 3$ are vertical asymptotes.

$$\begin{array}{r|l} 3+2x-x^2 \;) & 1-4x^3 \;\;(\,4x+8 \\ & \underline{12x+8x^2 -4x^3} \\ & 1-12x-8x^2 \\ & \underline{24+16x-8x^2} \\ & -23-28x \end{array}$$

$$f(x) = (4x+8) + \underbrace{\frac{(-23-28x)}{3+2x-x^2}}_{g(x)}$$

As $x \to \pm\infty$ the value of $g(x) \to 0$

and $f(x) \to 4x+8$

$y = 4x+8$ is an oblique asymptote.

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7\. (a) $\dfrac{16-x^2}{x+4}$

$$f(x) = \frac{16-x^2}{x+4} = \frac{(4-x)(4+x)}{4+x} = 4-x \text{ for } x \neq -4$$

$$\lim_{x \to -4} f(x) = 8$$

If we define $f(-4)=8$, then we can remove the discontinuity.

(b) $$f(x) = \frac{x+12}{x^2-9} = \frac{x+12}{(x+3)(x-3)}$$

As $\lim_{x \to \pm 3} f(x) = \pm\infty$

So the function has Infinite discontinuity

(c) $$f(x) = \frac{x^3-27}{|x-3|}$$

As $\lim_{x \to 3^+} f(x)$ and $\lim_{x \to 3^-} f(x)$ exist and different.

So $f(x)$ has a Jump Discontinuity.

8\. $$x+1 \,\big)\, x^2+ax+5 \,\big(\, x+(a-1)$$
$$\underline{x^2+x}$$
$$(a-1)x+5$$
$$\underline{(a-1)x+(a-1)}$$
$$-a+1+5$$

$$\Rightarrow \frac{x^2+ax+5}{x+1} = [x+(a-1)] + \frac{(-a+6)}{x+1}$$

As $\lim_{x \to +\infty} \dfrac{-a+6}{x+1} = 0$, So $x+(a-1) = x+3$

$\Rightarrow a-1=3$

$\Rightarrow a=4$