Lecture - 21

Recap: Exponential random variables

$E[X] = \frac{1}{\lambda}$

$Var(X) = \frac{1}{\lambda^2}$

$\Bigg\}$ Homework.

Q: Suppose that the length of a phone call in minutes is an e.r.v with $\lambda = \frac{1}{10}$.

What is the probability that you need to wait for > 10 minutes?

X: no. of minutes that you wait

$P(X > 10) = 1 - P(X \leq 10) = 1 - F(10)$

$$F(a) = 1 - e^{-\lambda a}$$

$$F(10) = 1 - e^{-\frac{1}{10}\cdot 10} = 1 - e^{-1}$$

$$\underline{P(X > 10)} = e^{-1} = \frac{1}{e} = 1 - F(a) = P(X > a)$$

Memoryless random variable.

$$P(\overbrace{X > s+t}^{A} \mid \overbrace{X > t}^{B}) = P(X > s) = e^{-\lambda s}$$

X = battery life

t = 100 hours

s = 50 hours.

$$\frac{P(X > s+t,\ X > t)}{P(X > t)} = \frac{P(X > s+t)}{P(X > t)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = \text{R.H.S.}$$

For an exponential r.v;

$$P(X>a) = 1 - F(a)$$
$$= 1 - (1 - e^{-\lambda a})$$

$$\underline{P(X>a) = e^{-\lambda a}}$$

Functions of continuous random variables

e.g.: X is Uniformly distributed over $(0,1)$.

$$Y = X^3$$

$f_Y(y)$ = density function for Y.

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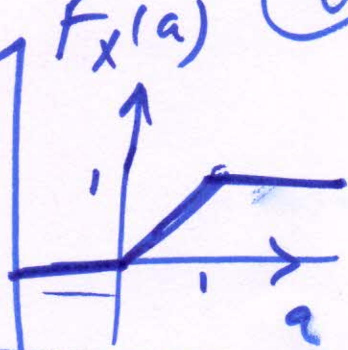
$$F_X(a) = \int_0^a 1 \cdot dx = a \qquad 0 \le a \le 1$$

for $a \geq 1$, $F_X(a) = 1$



$$F_Y(y) = P(Y \le y)$$

$$= P(X^3 \le y)$$

$$= P(X \le y^{1/3})$$

$$= F_X(y^{1/3})$$

$$F_Y(y) = y^{1/3}$$

$$f_Y(y) = \frac{1}{3\,y^{2/3}}$$

$f_X(x) = 1$ if $0 \le x \le 1$

0 otherwise.

e.g.:

X is continuous random variable,

f_X .

$Y = X^2$

What is f_Y in terms of f_X?

$$F_Y(y) = P(Y \leq y)$$

$$= P(X^2 \leq y)$$

$$= P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$$

e.g.

$X : f_X$

$Y = |X|$

$f_Y(y)$ in terms of f_X

$$F_Y(y) = P(Y \leq y)$$

$$= P(|X| \leq y)$$

$$= P(-y \leq X \leq y)$$

$$= F_X(y) - F_X(-y)$$

$$f_Y(y) = f_X(y) + f_X(-y)$$

Theorem:

$X, \quad f_X(x)$

$g(x)$ is strictly monotonic (increasing or decreasing), differentiable function of x.

Then $Y = g(X)$ has a probability density function.

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right| & \text{if } y = g(x) \\ 0 & \text{if } y \neq g(x) \ \forall x \end{cases}$$