Calculus is a tool to help us understand how functional relationships change, such as the position or speed of a moving object as a function of time.

→ changing slope of a curve
→ rotation of a body
→ Study dynamics of a computer algorithm
→ Convergence of the algorithm to the solution
→ Minimize computational time.
→ Maximize the efficiency of an algorithm

## Limit of a function

$$f : \mathbb{R} \to \mathbb{R}.$$

Suppose $f(x)$ is defined on an open interval about $x_0$, except possibly at $x_0$ itself.

$$f : (a, b) \to \mathbb{R}$$

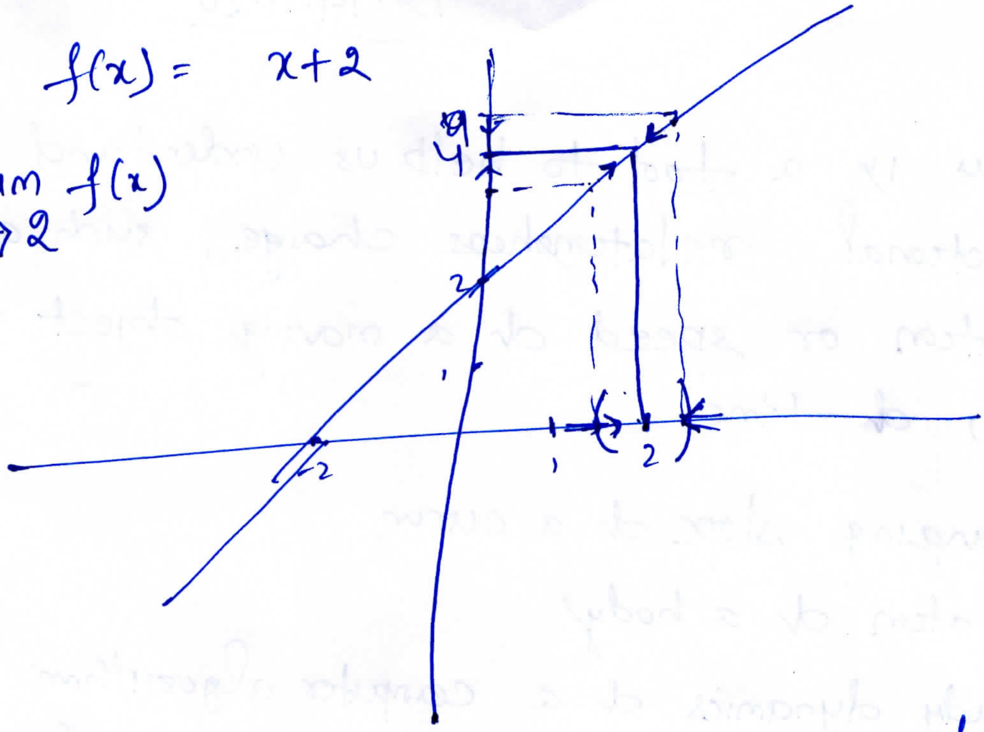
If $f(x)$ can be brought arbitrarily close to $L$ (as close as we like) for all $x$ sufficiently close to $x_0$, we say that $f(x)$ approaches the limit $L$ as $x$ approaches $x_0$.

We write $\lim_{x \to x_0} f(x) = L$.

Meaning: The values of $f(x)$ are close to the number $L$ whenever $x$ is close to $x_0$.

Ex: $f(x) = x+2$

$\lim_{x\to 2} f(x)$




Note-1
The limit value of a function does not depend on how the function is defined at the point being approached.

Ex: $f(x) = \frac{x^2-1}{x-1}$

$f$ is not defined at $x=1$

But $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} (x+1) = 2$ exists.

So for existence of limit, function may not be defined at the point being approached.

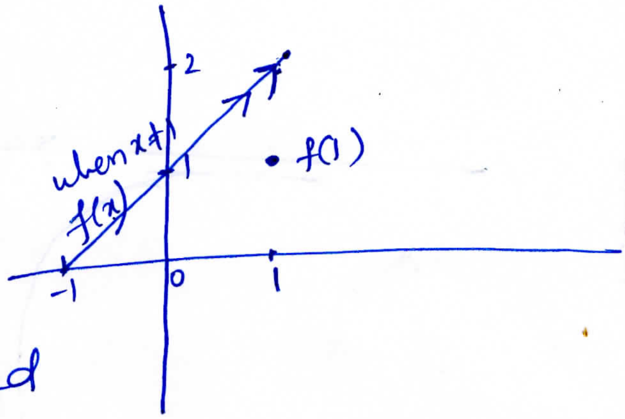
Note-2
function may be defined at a point $x_0$ but may not be equal to limiting value at $x_0$.

That is $f(x_0)$ is defined but

$$\lim_{x \to x_0} f(x) \neq f(x_0)$$

EXP

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{for } x \neq 1 \\ 1 & \text{for } x = 1 \end{cases}$$




Here $f(1) = 1$ is defined

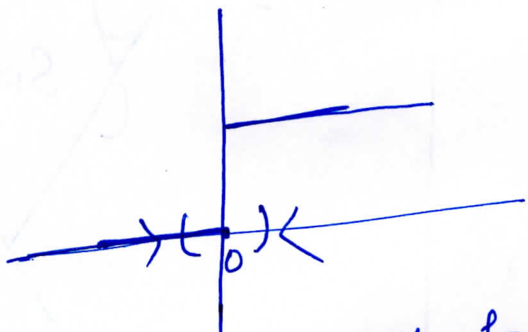
But $\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-1}{x-1} = 2 \neq f(1)$.

Note-3 Sometimes limit can fail to exist.

EXP

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

$\lim_{x \to 0} f(x)$  $x_0 = 0$



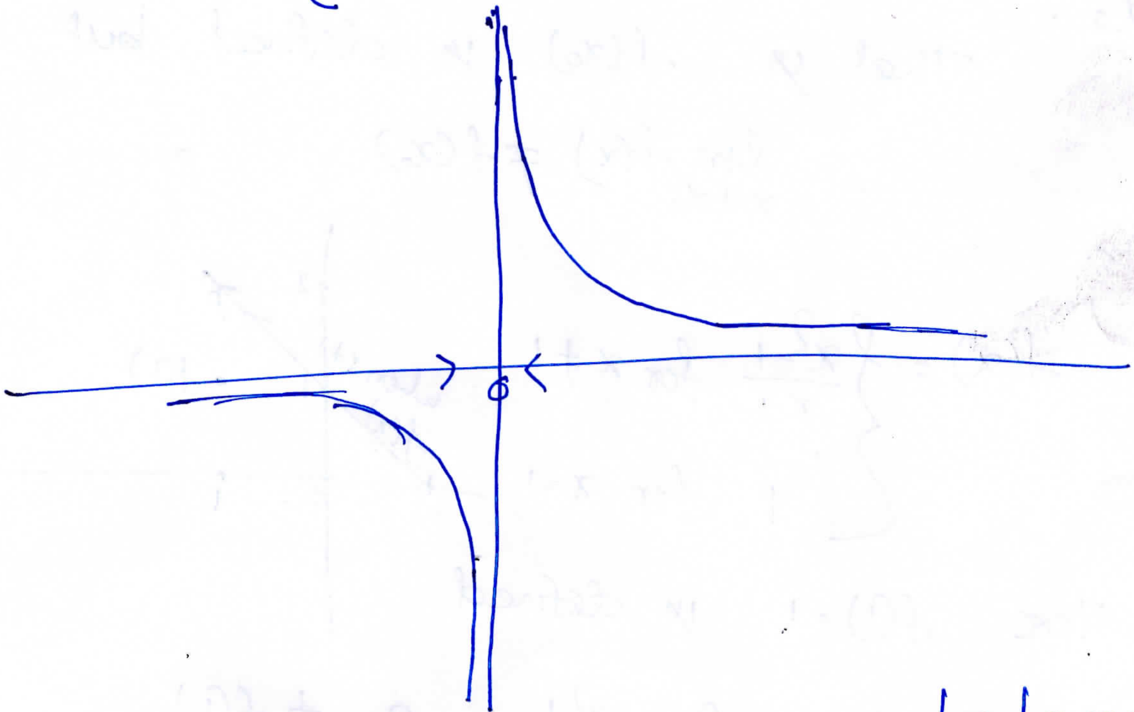

For -ve values of $x$, we approach from left side to 0, $f(x)$ value is arbitrarily close to 0. $\lim_{x \to 0^-} f(x) = 0$

If we approach 0 from right side of 0, $f(x)$ value is arbitrarily close to 1. $\lim_{x \to 0^+} f(x) = 1$

But we can not $f(x)$ arbitrarily close to one single point when we approach 0 from both side.

**Exp**

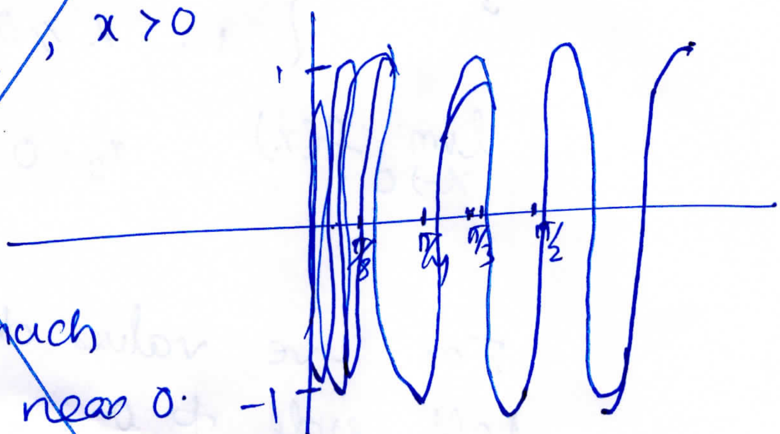
$$g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$




As $x \to 0$, $g(x)$ grows too large to have a limit.

**Exp**

$$f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$$




It oscillates too much to have a limit near 0.

It oscillates between $-1$ and $+1$ in every open interval containing 0.

At $x$ = , $\frac{1}{2\pi}$ $-\frac{1}{\pi}$ 0 $\frac{1}{\pi}$, $\frac{1}{2\pi}$, $\frac{1}{3\pi}$, ... ~~$\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{5}$, $\frac{\pi}{6}$,~~ $f(x) \to 0$

At $x = \frac{1}{(2n+1)\frac{\pi}{2}}$, $f(x) = 1$ $\frac{\pi}{2}$,

## Precise definition of limit

Let $f(x)$ be defined on an open interval about $x_0$; except possibly at $x_0$.

Then $\lim_{x \to x_0} f(x) = L$

If for every $\epsilon > 0$, there exists ($\exists$) a $\delta > 0$ such that for all $x$ in $0 < |x - x_0| < \delta$, $|f(x) - L| < \epsilon$.

$|f(x) - L| < \epsilon \Rightarrow -\epsilon < f(x) - L < \epsilon$

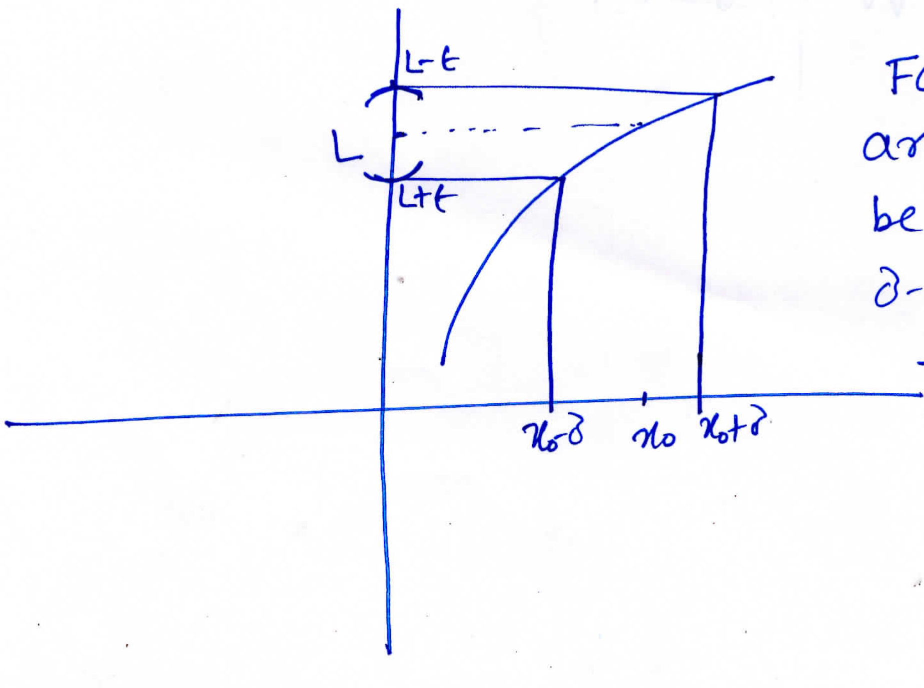
$\Rightarrow L - \epsilon < f(x) < L + \epsilon$

$(L-\epsilon, L+\epsilon)$ open interval around L.

$0 < |x - x_0| < \delta$, $(x_0 - \delta, x_0 + \delta) - \{x_0\}$

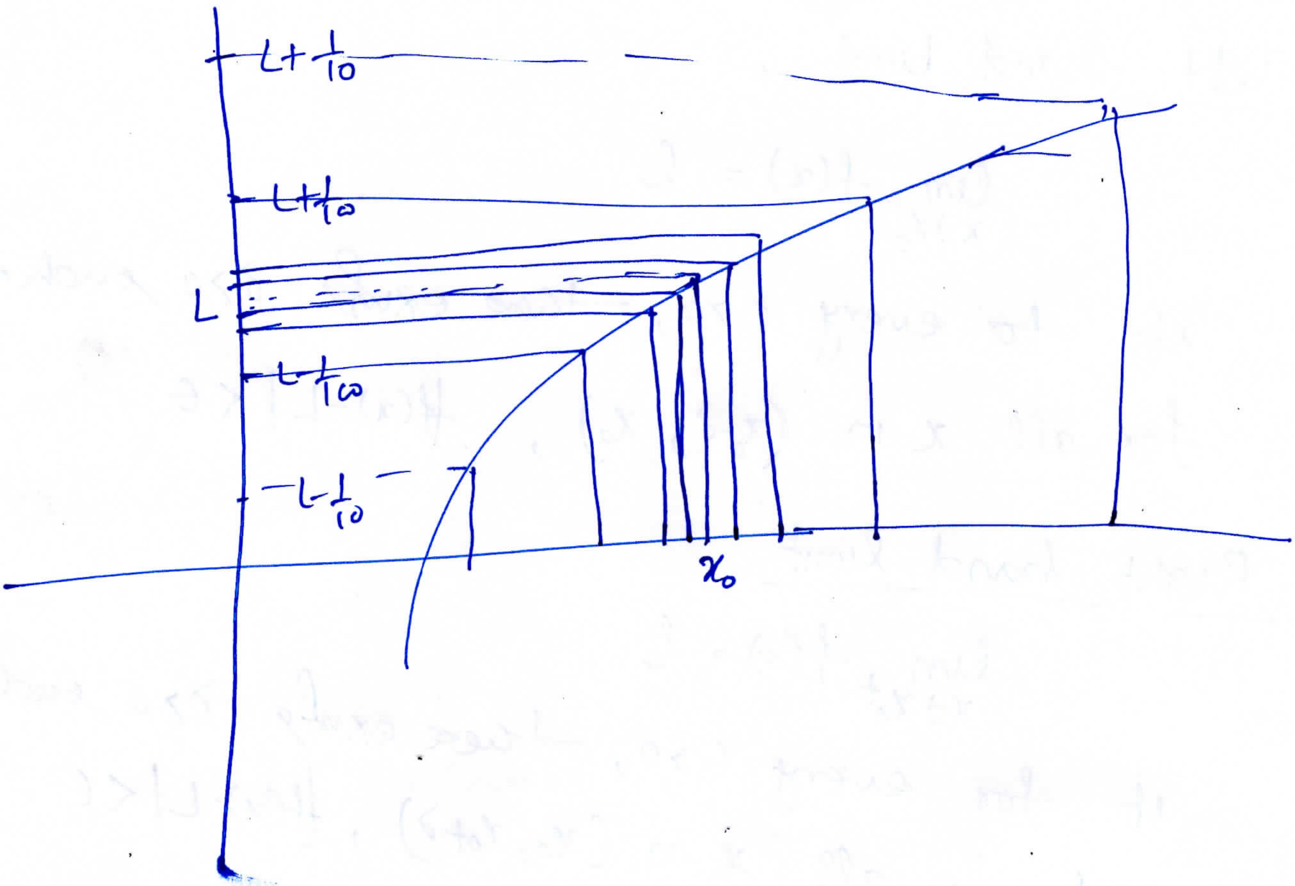
deleted nbd of $x_0$.

So as want to make $f(x)$ as ~~close as to~~ close to L as much as we want in the nbd of $x_0$.

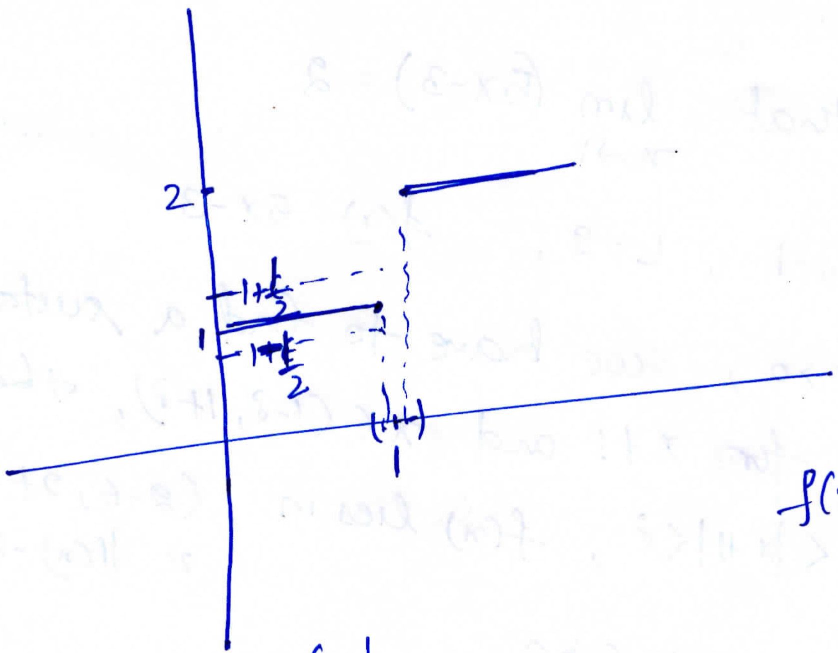



For every $\epsilon$-nbd around L, we should be able to get a $\delta$-nbd around $x_0$.

Then we say $\lim_{x \to x_0} f(x) = L$.




Exp




$$f(x)=\begin{cases}1 & \text{if } x\le 1\\ 2 & \text{if } x>1\end{cases}$$

$\epsilon=\frac{1}{2}$

For every $\delta$-nbd around 1, there are some $x$, such that $|f(x)-1|>\frac{1}{2}$

So $\lim\limits_{x\to 1} f(x)\neq 1$.

Similarly $\lim\limits_{x\to 1} f(x)\neq 2$

Also $\lim\limits_{x\to 1} f(x)\neq L$.

## Left hand Limit

$$\lim_{x \to x_0^-} f(x) = l$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x$ in $(x_0 - \delta, x_0)$, $|f(x) - L| < \epsilon$

## Right hand limit

$$\lim_{x \to x_0^+} f(x) = l$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x$ in $(x_0, x_0 + \delta)$, $|f(x) - L| < \epsilon$.

**Exp** Show that $\lim_{x \to 1} (5x - 3) = 2$.

**Sol$^n$** Here $x_0 = 1$, $L = 2$, $f(x) = 5x - 3$

For any $\epsilon > 0$, we have to find a suitable $\delta > 0$ such that for $x \neq 1$ and $x \in (1-\delta, 1+\delta)$, that is when $0 < |x - 1| < \delta$, $f(x)$ lies in $(2 - \epsilon, 2 + \epsilon)$ i.e. $|f(x) - 2| < \epsilon$.

Let us take any $\epsilon > 0$,

$|f(x) - L| < \epsilon$ $\Rightarrow$ $|5x - 3 - 2| < \epsilon$

$\Rightarrow |5x - 5| < \epsilon$

$\Rightarrow 5|x - 1| < \epsilon \Rightarrow |x - 1| < \frac{\epsilon}{5}$

So if we choose $\delta = \frac{\epsilon}{5}$

So for $0 < |x - 1| < \frac{\epsilon}{5}$, then $|5x - 3 - 2|$

$= 5|x - 1| < 5 \times \frac{\epsilon}{5} = \epsilon$

i.e. $|5x - 3 - 2| < \epsilon$

$\Rightarrow \lim_{x \to 1} (5x - 3) = 2$

Exp-2 $\lim_{x \to x_0} x = x_0$

Here $f(x) = x_0$, $L = x_0$

Let $\epsilon > 0$, ~~of~~ $|f(x) - L| < \epsilon$

$\Rightarrow |x - x_0| < \epsilon$

So if we choose $\delta = \epsilon$, then

$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| = |x - x_0| < \epsilon$

So for all $x$, in $0 < |x - x_0| < \delta$, we have $|x - x_0| < \epsilon$.

ie $|f(x) - L| < \epsilon$

$\Rightarrow \lim_{x \to x_0} x = x_0$.

Procedure How to find $\delta$ for given $f, L, x_0$ and $\epsilon > 0$.

① Solve the inequality $|f(x) - L| < \epsilon$ to find an open interval $(a, b)$ containing $x_0$ on which the inequality holds for all $x \neq x_0$.

② Find a value of $\delta > 0$ that place the open interval $(x_0 - \delta, x_0 + \delta)$ centered at $x_0$ inside the ~~o~~interval $(a, b)$.

The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this $\delta$-interval.

Exp

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Prove $\lim_{x \to 2} f(x) = 4$

Solⁿ Our task is to show that given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x$, $0 < |x-2| < \delta$ we have $|f(x) - 4| < \epsilon$.

Step-1 Solve the inequality $|f(x) - 4| < \epsilon$ to find an open interval $(a, b)$ containing $x_0 = 2$ on which the inequality holds for all $x \neq 2$.

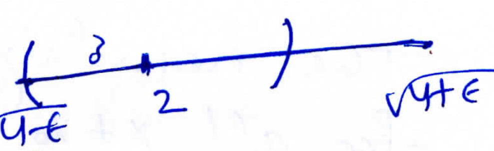
For $x \neq x_0 = 2$, we have $f(x) = x^2$

The inequality to solve is

$$|x^2 - 4| < \epsilon$$
$$\Rightarrow -\epsilon < x^2 - 4 < \epsilon$$
$$\Rightarrow 4 - \epsilon < x^2 < 4 + \epsilon$$
$$\Rightarrow \sqrt{4-\epsilon} < x < \sqrt{4+\epsilon}$$

So we have got an open interval $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$ containing $x_0 = 2$.

Step-2 Find a value of $\delta > 0$ that places the centered interval $(2-\delta, 2+\delta)$ inside $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$




Let $\delta = \min\{2 - \sqrt{4-\epsilon}, \sqrt{4+\epsilon} - 2\}$

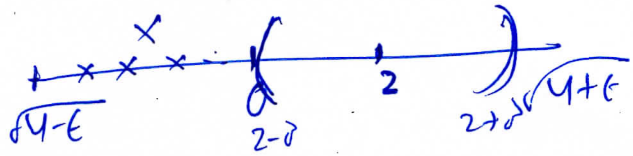
If $\delta$ has this or any smaller positive value the inequality $0 < |x-2| < \delta$ will automatically place $x$ between $\sqrt{4-\epsilon}$ & $\sqrt{4+\epsilon}$ to make $|f(x) - 4| < \epsilon$.

So for all $x$, $0<|x-2|<\delta \Rightarrow |f(x)-4|<\epsilon$

This completes the proof for $\epsilon<4$.

If $\epsilon \geq 4$, then we take $\delta$ to be the distance from $x_0=2$ to the nearest end point of the interval $(0, \sqrt{4+\epsilon})$

In other words $\delta = \min\{2, \sqrt{4+\epsilon}-2\}$.




— $\delta$ —