Example:

We apply this interpolation method to the example of $y = 1/x$ at $x = 3.44$ (used to find the cubic-order Lagrange polynomial).

1/. The Linear Interpolation:

$x_0 = 3.35$
$x_1 = 3.40$
$f(x_0) = 0.298507$
$f(x_1) = 0.294118$

$$P_1(x) = f(x_0) + (x - x_0) f[x_0, x_1].$$

$$\Rightarrow P_1(x) = 0.298507 + (3.44 - 3.35) \times \left(\frac{0.294118 - 0.298507}{3.40 - 3.35} \right)$$

$$\Rightarrow P_1(x) = 0.298507 + (3.44 - 3.35) \times -0.08778$$

$$\Rightarrow P_1(x) = 0.298507 - 0.007900 = 0.290607$$

2/. The Quadratic Interpolation:

$$P_2(x) = P_1(x) + (x - x_0)(x - x_1) f[x_0, x_1, x_2].$$

$$f[x_0, x_1, x_2] = \frac{1}{x_2 - x_0} \left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right]$$

$x_2 = 3.50, \ f(x_2) = 0.285714$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f[x_1, x_2] = \frac{0.285714 - 0.294118}{3.50 - 3.40}$$

$$\Rightarrow f[x_1, x_2] = -0.084040, \quad f[x_0, x_1] = -0.08778.$$

$$\Rightarrow P_2(x) = 0.290607 + \frac{(3.44 - 3.35)(3.44 - 3.40)}{3.50 - 3.35} \times [-0.084040 - (-0.087780)]$$

$$f[x_0, x_1, x_2] = \frac{-0.084040 - (-0.087780)}{3.50 - 3.35} = 0.02493$$

$$P_2(x) = 0.290607 + (3.44 - 3.35)(3.44 - 3.40) \times 0.024933 = 0.290697$$

$x_3 = 3.60$
$f(x_3) = 0.277778$

3/. <u>The Cubic Interpolation</u> :

$$P_3(x) = P_2(x) + (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x_3]$$

$$f[x_0, x_1, x_2, x_3] = \frac{1}{x_3 - x_0}\Big[f[x_1, x_2, x_3] - f[x_0, x_1, x_2] \Big]$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_2}$$

$f[x_1, x_2] = -0.084040$

$$f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{0.277778 - 0.285714}{3.60 - 3.50} = -0.07936$$

$$\therefore f[x_1, x_2, x_3] = \frac{-0.07936 - (-0.08404)}{3.6 - 3.4} = 0.02340$$

Also, $f[x_0, x_1, x_2] = 0.024933$ (obtained for quadratic interpolation.

$$\Rightarrow f[x_0, x_1, x_2, x_3] = \frac{0.02340 - 0.024933}{3.6 - 3.35} = -6.132 \times 10^{-3}$$

$$\Rightarrow P_3(x) = 0.290697 + (3.44 - 3.35)(3.44 - 3.40)(3.44 - 3.5) \times -6.132 \times 10^{-3}$$

$$\Rightarrow P_3(x) = 0.290697 + 0.0000013 = 0.290698$$

<u>The function in $y = 1/x$</u>. At $x = 3.44$, $y = \frac{1}{3.44}$
$= 0.290698$ (up to 6 decimal places.)

$P_1(x) = 0.290607$, $P_2(x) = 0.290697$, $P_3(x) = 0.290698$, Hence, $P_3(x)$ matches the value of $y = 1/x$ correctly up to 6 decimal places. Also, the Newton interpolation at every higher order simply <u>adds</u> an <u>extra correction</u> to the previous order.

Forward Difference

The nodes $\{x_j\}$ being evenly spaced, we define $\boxed{x_j = x_0 + jh}$ $(j = 0, 1, \ldots, n)$. We also define the first-order forward difference as

$$\boxed{\Delta f(x_j) = \Delta f_j = f_{j+1} - f_j}$$

and likewise, the second-order forward difference as

$$\boxed{\begin{aligned}\Delta^2 f(x_j) = \Delta f_{j+1} - \Delta f_j &= (f_{j+2} - f_{j+1}) - (f_{j+1} - f_j) \\ &= f_{j+2} - 2f_{j+1} + f_j\end{aligned}}$$

Generalising to any higher order k, $(k \geq 2)$, we have $\boxed{\Delta^k f_j = \Delta^{k-1} f_{j+1} - \Delta^{k-1} f_j}$.

For $\underline{k=2}$, $\boxed{\Delta^2 f(x_j) = f_{j+2} - 2f_{j+1} + f_j} \rightarrow$ Centred-difference polynomial.

Now $\boxed{f[x_j, x_{j+1}] = \dfrac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j} = \dfrac{f_{j+1} - f_j}{\cancel{x_0} + (j+1)h - \cancel{x_0} - jh} = \dfrac{\Delta f_j}{h}}$

Similarly, $f[x_j, x_{j+1}, x_{j+2}] = \dfrac{f[x_{j+1}, x_{j+2}] - f[x_j, x_{j+1}]}{x_{j+2} - x_j}$

$\Rightarrow f[x_j, x_{j+1}, x_{j+2}] = \dfrac{(1/h)(f_{j+2} - f_{j+1}) - (1/h)(f_{j+1} - f_j)}{\cancel{x_0} + (j+2)h - \cancel{x_0} - \cancel{jh}}$ $\boxed{\dfrac{1}{h}\Delta^2 f_j}$

$\Rightarrow \boxed{f[x_j, x_{j+1}, x_{j+2}] = \dfrac{\Delta^2 f_j}{2h^2}}$.

Again $f[x_j, x_{j+1}, x_{j+2}, x_{j+3}] = \dfrac{f[x_{j+1}, x_{j+2}, x_{j+3}] - f[x_j, x_{j+1}, x_{j+2}]}{(x_{j+3}) - x_j}$

Now, $f[x_j, x_{j+1}, x_{j+2}] = \frac{1}{2h^2}\Delta^2 f_j$. By induction we can say $\boxed{f[x_{j+1}, x_{j+2}, x_{j+3}] = \frac{\Delta^2 f_{j+1}}{2h^2}}$.

Also $x_{j+3} - x_j = \cancel{x_0} + (\cancel{j}+3)h - \cancel{x_0} - \cancel{jh} = 3h$.

$$\therefore \boxed{f[x_j, x_{j+1}, x_{j+2}, x_{j+3}] = \frac{\frac{\Delta^2 f_{j+1}}{2h^2} - \frac{\Delta^2 f_j}{2h^2}}{3h}}.$$

$$\Rightarrow \boxed{f[x_j, x_{j+1}, x_{j+2}, x_{j+3}] = \frac{\Delta^3 f_j}{3\cdot 2\cdot h^3}}$$

$$f[x_j, x_{j+1}, x_{j+2}, x_{j+3}, x_{j+4}] = \frac{f[x_{j+1}, x_{j+2}, x_{j+3}, x_{j+4}] - f[x_j, x_{j+1}, x_{j+2}, x_{j+3}]}{x_{j+4} - x_j}$$

Since, $f[x_j, x_{j+1}, x_{j+2}, x_{j+3}] = \frac{\Delta^3 f_j}{3\cdot 2\cdot h^3}$, by induction we see that $\boxed{f[x_{j+1}, x_{j+2}, x_{j+3}, x_{j+4}] = \frac{\Delta^3 f_{j+1}}{3\cdot 2\cdot h^3}}$.

And $x_{j+4} - x_j = \cancel{x_0} + (\cancel{j}+4)h - \cancel{x_0} - \cancel{jh} = 4h$.

$$\therefore \boxed{\begin{aligned} f[x_j, x_{j+1}, x_{j+2}, x_{j+3}, x_{j+4}] &= \frac{1}{4h}\cdot\left[\frac{\Delta^3 f_{j+1} - \Delta^3 f_j}{3\cdot 2\cdot h^3}\right] \\ &= \frac{\Delta^4 f_j}{4\cdot 3\cdot 2\, h^4} \end{aligned}}$$

Hence $\boxed{f[x_j, x_{j+1}, \ldots, x_{j+k}] = \frac{\Delta^k f_j}{k!\, h^k}}$, Generalising to any order k, by induction.

Newton's Forward Difference Interpolation Polynomial

$$\boxed{P_n(x) = f(x_0) + (x-x_0) f[x_0, x_1] + \cdots + (x-x_0)(x-x_1)(x-x_{n-1}) \times f[x_0, x_1, \ldots, x_n]}$$

Now define $\boxed{x_j = x_0 + jh}$ for evenly-spaced nodes $\{x_j\}$,

and $\boxed{x = x_0 + \mu h}$. $\Rightarrow$ $\boxed{x - x_j = (\mu - j)h}$, $\boxed{f(x_0) = f_0}$..

$$\Rightarrow P_n(x) = f(x_0) + \mu \cancel{h} \frac{\Delta f_0}{\cancel{h}} + \mu \cancel{h} (\mu - 1) \cancel{h} \frac{\Delta^2 f_0}{2! \cancel{h^2}}$$

$$+ \mu \cancel{h} (\mu - 1) \cancel{h} (\mu - 2) \cancel{h} \frac{\Delta^3 f_0}{3! \cancel{h^3}} + \cdots + \mu(\mu-1)\cdots(\mu-n+1) \cancel{h^n} \times \frac{\Delta^n f_0}{n! \cancel{h^n}}$$

$$\Rightarrow P_n(x) = f(x_0) + \mu(\mu-1) \frac{\Delta^2 f_0}{2!} + \mu(\mu-1)(\mu-2) \frac{\Delta^3 f_0}{3!} + \cdots + \mu(\mu-1)\cdots(\mu-n+1) \frac{\Delta^n f_0}{n!}$$

The coefficients can be recast as $\boxed{\dfrac{\mu(\mu-1)\cdots(\mu-k+1)}{k!} = \dfrac{\mu!}{k!(\mu-k)!}}$.

$\Rightarrow$ $\boxed{\dfrac{\mu!}{k!(\mu-k)!} = {}^{\mu}C_k}$ as in the binomial expansion.

$$\Rightarrow \boxed{P_n(x) = \sum_{k=0}^{n} \frac{\mu!}{k!(\mu-k)!} \Delta^k f_0 = \sum_{k=0}^{n} {}^{\mu}C_k \, \Delta^k f_0}$$

Newton's Forward Difference Interpolation Polynomial.

Backward Difference

Define $\boxed{\nabla f_j = f_j - f_{j-1}}$, $\boxed{\nabla^2 f_j = \nabla f_j - \nabla f_{j-1}}$

Generalising to $\boxed{\nabla^k f_j = \nabla^{k-1} f_j - \nabla^{k-1} f_{j-1}}$.

Further, $\boxed{x_{-j} = x_0 - jh}$ $\Rightarrow$ $\boxed{x_{-j} - x_0 = -jh}$ $(j>0)$.

$$\therefore \boxed{f[x_0, x_{-1}] = \frac{f_{-1} - f_0}{x_{-1} - x_0} = \frac{f_{-1} - f_0}{-h} = \frac{f_0 - f_{-1}}{h} = \frac{\nabla f_0}{h}}.$$

$$f[x_0, x_{-1}, x_{-2}] = \frac{f[x_{-1}, x_{-2}] - f[x_0, x_{-1}]}{x_{-2} - x_0}.$$

$f[x_0, x_{-1}] = \frac{\nabla f_0}{h}$

$x_{-2} - x_0 = -2h$

By induction we see $\boxed{f[x_{-1}, x_{-2}] = \frac{\nabla f_{-1}}{h}}$.

$$\therefore \boxed{f[x_0, x_{-1}, x_{-2}] = \frac{\nabla f_{-1}/h - \nabla f_0/h}{-2h} = \frac{\nabla f_0 - \nabla f_{-1}}{h\,(2h)} = \frac{\nabla^2 f_0}{2h^2}}$$

$$f[x_0, x_{-1}, x_{-2}, x_{-3}] = \frac{f[x_{-1}, x_{-2}, x_{-3}] - f[x_0, x_{-1}, x_{-2}]}{x_{-3} - x_0}$$

$x_{-3} - x_0 = -3h$.

$\because f[x_0, x_{-1}, x_{-2}] = \frac{\nabla^2 f_0}{2h^2}$, by induction $\boxed{f[x_{-1}, x_{-2}, x_{-3}] = \frac{\nabla^2 f_{-1}}{2h^2}}$.

$$\Rightarrow \boxed{f[x_0, x_{-1}, x_{-2}, x_{-3}] = \frac{1}{2h^2}\,\frac{\nabla^2 f_{-1} - \nabla^2 f_0}{(-3h)} = \frac{\nabla^2 f_0 - \nabla^2 f_{-1}}{3\cdot 2\cdot h^3} = \frac{\nabla^3 f_0}{3\cdot 2\cdot h^3}}$$

Similarly, it can be shown $\boxed{f[x_0, x_{-1}, x_{-2}, x_{-3}, x_{-4}] = \frac{\nabla^4 f_0}{4!\,h^4}}$.

Generalising this argument to order k $\boxed{f[x_0, x_{-1}, \ldots, x_{-k}] = \frac{\nabla^k f_0}{k!\,h^k}}$.

Newton's Backward Difference Interpolation Polynomial

$$\boxed{\begin{aligned} P_n(x) = {} & f(x_0) + (x-x_0)\, f[x_0, x_{-1}] + (x-x_0)(x-x_{-1})\, f[x_0, x_{-1}, x_{-2}] \\ & + \cdots + (x-x_0)(x-x_{-1})\cdots(x-x_{-n+1})\, f[x_0, x_{-1}, \ldots, x_{-n}] \end{aligned}}$$

Now define $\boxed{x_{-j} = x_0 - jh}$ for evenly-spaced nodes $\{x_j\}$. $\boxed{f(x_0) = f_0}$

and $\boxed{x = x_0 - \nu h}$ $\Rightarrow$ $\boxed{x - x_{-j} = \cancel{x_0} - \nu h - \cancel{x_0} + jh = (-\nu + j)h}$.

$$\Rightarrow P_n(x) = f_0 + (-\nu)\cancel{h}\, \frac{\nabla f_0}{\cancel{h}} + (-\nu)\cancel{h}(-\nu+1)\cancel{h}\, \frac{\nabla^2 f_0}{2!\,\cancel{h^2}}$$

$$+ (-\nu)\cancel{h}(-\nu+1)\cancel{h}(-\nu+2)\cancel{h}\, \frac{\nabla^3 f_0}{3!\,\cancel{h^3}} + \cdots + (-\nu)(-\nu+1)\cdots(-\nu+n-1)\cancel{h^n}\, \frac{\nabla^n f_0}{n!\,\cancel{h^n}}$$

$$\Rightarrow \boxed{\begin{aligned} P_n(x) = {} & f_0 + (-1)^1 \nu \nabla f_0 + (-1)^2 \nu(\nu-1) \frac{\nabla^2 f_0}{2!} \\ & + (-1)^3 \nu(\nu-1)(\nu-2) \frac{\nabla^3 f_0}{3!} + \cdots + (-1)^k \frac{\nu(\nu-1)\cdots(\nu-n+1)\, \nabla^n f_0}{n!} \end{aligned}}$$

Again we write $\boxed{\dfrac{\nu(\nu-1)\cdots(\nu-k+1)}{k!} = \dfrac{\nu!}{k!(\nu-k)!}}$

But $\boxed{\dfrac{\nu!}{k!(\nu-k)!} = {}^{\nu}C_k}$ again in the form of a binomial expansion.

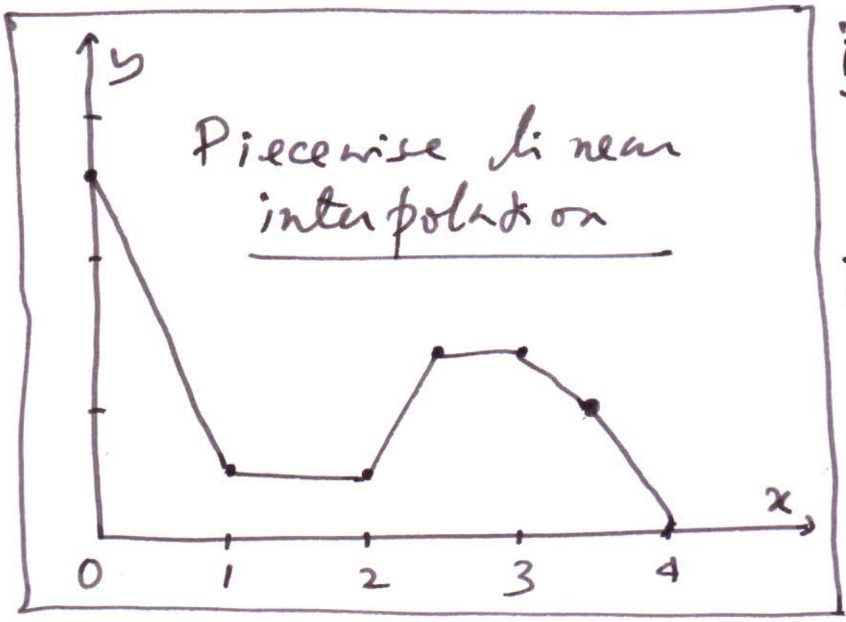
$$\therefore \boxed{P_n(x) = \sum_{k=0}^{n} (-1)^k \frac{\nu!}{k!(\nu-k)!} \nabla^k f_0 = \sum_{k=0}^{n} (-1)^k\, {}^{\nu}C_k\, \nabla^k f_0}.$$

Newton's Backward Difference Interpolation Polynomial.

Interpolation Using Spline Functions

Consider the following data points:

x	0	1	2	2.5	3	3.5	4
y	2.5	0.5	0.5	1.5	1.5	1.125	0



i) The data points (nodes) are not monotonic.

ii) Piecewise linear interpolation is discontinuous at the nodes.

iii) Spline interpolation makes it continuous

For n data points (x_i, y_i), $i = 1, 2, \ldots, n$, and $x_1 < x_2 < \cdots < x_n$, seek a function $S(x)$ defined on $[a, b]$ ($a = x_1$ and $b = x_n$), such that $\boxed{S(x_i) = y_i}$, $i = 1, 2, \ldots, n$.

1/ For smooth interpolation, $S'(x)$ and $S''(x)$ are continuous.

2/ The linear interpolation is to be followed closely.

3/ Hence, $S'(x)$ must not change rapidly, and $S''(x)$ must be very small. (For linear interpolation, $S'(x)$ = constant and $S''(x) = 0$).

Natural Cubic Spline Interpolation

1/. $S(x)$ is a polynomial of degree ≤ 3 on subintervals $[x_{j-1}, x_j]$, $j = 2, 3, \ldots, n$.

2/. $S(x)$, $S'(x)$, $S''(x)$ are continuous for $a \leq x \leq b$.

3/. $S''(x_1) = S''(x_n) = 0$.

$S(x)$ is a natural cubic spline function.

If $S(x)$ is cubic, $\Rightarrow$ $S''(x)$ is linear.

Now introduce values M_i $(i = 1, 2, 3, \ldots, n)$, such that $\boxed{M_i = S''(x_i)}$. On an interval $[x_{j-1}, x_j]$, $\boxed{S''(x_{j-1}) = M_{j-1}}$ and $\boxed{S''(x_j) = M_j}$.

With these two values we interpolate a linear function, $S''(x)$, between x_{j-1} and x_j.

Now, given two coordinates (x_1, y_1) and (x_2, y_2), the line joining them is given by.

$$\boxed{y = \frac{y_1(x - x_2)}{(x_1 - x_2)} + \frac{y_2(x - x_1)}{x_2 - x_1} = \frac{(x_2 - x)y_1 + (x - x_1)y_2}{x_2 - x_1}}$$

We set the equivalence $\boxed{y \to S''(x)}$, $\boxed{x_1 \to x_{j-1}}$ and $\boxed{x_2 \to x_j}$ to get

(P. T. O.)

$$S''(x) = \frac{(x_j - x)\, S''(x_{j-1}) + (x - x_{j-1})\, S''(x_j)}{x_j - x_{j-1}}$$

Since, $S''(x_i) = M_i$ we finally write

$$\boxed{S''(x) = \frac{(x_j - x) M_{j-1} + (x - x_{j-1}) M_j}{x_j - x_{j-1}}}$$

, which is the linear interpolation function of $S''(x)$.

On integrating it we get,

$$S'(x) = \frac{M_{j-1}}{x_j - x_{j-1}} \int (x_j - x)\, dx + \frac{M_j}{x_j - x_{j-1}} \int (x - x_{j-1})\, dx$$

$$\Rightarrow \boxed{S'(x) = \frac{M_{j-1}}{x_j - x_{j-1}} \times -\frac{(x_j - x)^2}{2} + \frac{M_j}{x_j - x_{j-1}} \frac{(x - x_{j-1})^2}{2} + A}$$

with A being an arbitrary integration constant.

~~On other~~ $S'(x)$ gives a quadratic function.

On integrating $S'(x)$ we get a cubic function as,

$$S(x) = \frac{M_{j-1}}{x_j - x_{j-1}} \times -\frac{1}{2} \int (x_j - x)^2 dx + \frac{M_j}{x_j - x_{j-1}} \frac{1}{2} \int (x - x_{j-1})^2 dx + \int A\, dx$$

$$\Rightarrow \boxed{S(x) = \frac{M_{j-1}}{x_j - x_{j-1}} \times \frac{(x_j - x)^3}{6} + \frac{M_j}{x_j - x_{j-1}} \times \frac{(x - x_{j-1})^3}{6} + Ax + B}$$

with B being another integration constant.

We recast $\boxed{Ax + B = C(x_j - x) + D(x - x_{j-1})}$.

$\Rightarrow \boxed{A = D - C}$ and $\boxed{B = C x_j - D x_{j-1}}$.

$$\therefore \boxed{S(x) = \frac{M_{j-1}}{x_j - x_{j-1}} \cdot \frac{(x_j - x)^3}{6} + \frac{M_j}{x_j - x_{j-1}} \cdot \frac{(x - x_{j-1})^3}{6} + C(x_j - x) + D(x - x_{j-1})}.$$

Noting $\boxed{S(x_{j-1}) = y_{j-1}}$ and $\boxed{S(x_j) = y_j}$,

when $\underline{x = x_j}$, $\boxed{S(x_j) = y_j = \frac{M_j}{6}(x_j - x_{j-1})^2 + D(x_j - x_{j-1})}$

and

when $\underline{x = x_{j-1}}$, $\boxed{S(x_{j-1}) = y_{j-1} = \frac{M_{j-1}}{6}(x_j - x_{j-1})^2 + C(x_j - x_{j-1})}$

Hence $\boxed{D = \frac{y_j}{x_j - x_{j-1}} - \frac{M_j}{6}(x_j - x_{j-1})}$ and

likewise $\boxed{C = \frac{y_{j-1}}{x_j - x_{j-1}} - \frac{M_{j-1}}{6}(x_j - x_{j-1})}$.

$$\therefore \boxed{D - C = \frac{y_j - y_{j-1}}{x_j - x_{j-1}} - \frac{(x_j - x_{j-1})}{6}(M_j - M_{j-1})}.$$

$(A = D - C)$

$$\boxed{S'(x) = \frac{-M_{j-1}}{x_j - x_{j-1}} \frac{(x_j - x)^2}{2} + \frac{M_j}{x_j - x_{j-1}} \cdot \frac{(x - x_{j-1})^2}{2} + D - C}.$$

$$\Rightarrow \boxed{S'(x) = \frac{-(x_j - x)^2 M_{j-1} + (x - x_{j-1})^2 M_j}{2(x_j - x_{j-1})} + \frac{y_j - y_{j-1}}{x_j - x_{j-1}} - \frac{(x_j - x_{j-1})}{6}(M_j - M_{j-1})}$$

For the interval $[x_{j-1}, x_j]$ $S(x_j)$ <u>must be equal to</u> the value of $S(x_j)$ for the interval $[x_j, x_{j+1}]$ for ~~[illegible]~~ smooth matching.

The function $S'(x)$ has been obtained for the interval $[x_{j-1}, x_j]$. For the interval $[x_j, x_{j+1}]$, we simply transform $\boxed{j-1 \rightarrow j}$ and $\boxed{j \rightarrow j+1}$.

$$\therefore \boxed{S'(x) = \frac{-(x_{j+1} - x)^2 M_j + (x - x_j)^2 M_{j+1}}{2(x_{j+1} - x_j)} + \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{(x_{j+1} - x_j)}{6}(M_{j+1} - M_j)}$$

The common boundary point of the intervals $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$ is x_j. When $x = x_j$ the former interval gives,

$$\boxed{S'(x_j) = \frac{1}{2} M_j (x_j - x_{j-1}) + \left(\frac{y_j - y_{j-1}}{x_j - x_{j-1}}\right) - \frac{(x_j - x_{j-1})}{6}(M_j - M_{j-1})}$$

while the latter interval gives,

$$\boxed{S'(x_j) = -\frac{1}{2} M_j (x_{j+1} - x_j) + \left(\frac{y_{j+1} - y_j}{x_{j+1} - x_j}\right) - \frac{(x_{j+1} - x_j)}{6}(M_{j+1} - M_j)}$$

For a smooth interpolation ~~across~~ between the two intervals, the two values of $S'(x_j)$ must match.

Hence, $$\frac{1}{2} M_j (x_j - x_{j-1}) + \left(\frac{y_j - y_{j-1}}{x_j - x_{j-1}}\right) - \frac{x_j - x_{j-1}}{6}(M_j - M_{j-1})$$

$$= -\frac{1}{2} M_j (x_{j+1} - x_j) + \left(\frac{y_{j+1} - y_j}{x_{j+1} - x_j}\right) - \frac{x_{j+1} - x_j}{6}(M_{j+1} - M_j)$$

The function $S'(x)$ has been obtained for the interval $[x_{j-1}, x_j]$. For the interval $[x_j, x_{j+1}]$, we simply transform $\boxed{j-1 \to j}$ and $\boxed{j \to j+1}$.

$$\therefore \boxed{S'(x) = \frac{-(x_{j+1} - x)^2 M_j + (x - x_j)^2 M_{j+1}}{2(x_{j+1} - x_j)} + \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{(x_{j+1} - x_j)}{6}(M_{j+1} - M_j)}$$

The common boundary point of the intervals $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$ is x_j. When $x = x_j$ the former interval gives,

$$\boxed{S'(x_j) = \frac{1}{2} M_j (x_j - x_{j-1}) + \left(\frac{y_j - y_{j-1}}{x_j - x_{j-1}}\right) - \frac{(x_j - x_{j-1})}{6}(M_j - M_{j-1})}$$

while the latter interval gives,

$$\boxed{S'(x_j) = -\frac{1}{2} M_j (x_{j+1} - x_j) + \left(\frac{y_{j+1} - y_j}{x_{j+1} - x_j}\right) - \frac{(x_{j+1} - x_j)}{6}(M_{j+1} - M_j)}.$$

For a smooth interpolation, between the two intervals, the two values of $S'(x_j)$ must match.

Hence, $$\frac{1}{2} M_j (x_j - x_{j-1}) + \left(\frac{y_j - y_{j-1}}{x_j - x_{j-1}}\right) - \frac{x_j - x_{j-1}}{6}(M_j - M_{j-1})$$

$$= -\frac{1}{2} M_j (x_{j+1} - x_j) + \left(\frac{y_{j+1} - y_j}{x_{j+1} - x_j}\right) - \frac{x_{j+1} - x_j}{6}(M_{j+1} - M_j)$$

$$\Rightarrow \frac{1}{2} M_j \left(\cancel{x_j} - x_{j-1} + x_{j+1} - \cancel{x_j} \right) - \frac{1}{6} M_j \left(\cancel{x_j} - x_{j-1} + x_{j+1} - \cancel{x_j} \right) + \frac{M_{j-1}}{6} \left(x_j - x_{j-1} \right) + \frac{M_{j+1}}{6} \left(x_{j+1} - x_j \right)$$

$$= \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

$$\Rightarrow \boxed{\frac{M_{j-1}}{6} \left(x_j - x_{j-1} \right) + \frac{1}{3} M_j \left(x_{j+1} - x_{j-1} \right) + \frac{M_{j+1}}{6} \left(x_{j+1} - x_j \right) = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}}$$

For n data points, $x_1, x_2, \ldots, x_n$, the above equation matches derivatives at $x_2, x_3, \ldots, x_{n-1}$, i.e. at $n-2$ data points. $(M_1 = M_n = 0)$.

Example: Find the natural cubic spline to interpolate $(1,1)$, $(2, 1/2)$, $(3, 1/3)$, $(4, 1/4)$.

$\boxed{S''(x_1) = M_1 = S''(x_4) = M_4 = 0}$. The derivatives are to be matched for $j = 2, 3$ at x_2 and x_3.

Also, all $\boxed{x_j - x_{j-1} = 1}$ and $\boxed{x_{j+1} - x_j = 1}$.

Further, $\boxed{x_{j+1} - x_{j-1} = 2}$ $\left(x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4 \right)$

for $j = 2$ $\qquad \frac{M_1}{6} + \frac{2M_2}{3} + \frac{M_3}{6} = \left(\frac{1}{3} - \cancel{\frac{1}{2}} \right) - \left(\cancel{\frac{1}{2}} - 1 \right)$

$$\Rightarrow \boxed{\frac{M_1}{6} + \frac{2M_2}{3} + \frac{M_3}{6} = \frac{1}{3}}$$

For $j = 3$, $\quad \frac{M_2}{6} + \frac{2M_3}{3} + \frac{1}{6}M_4 = \left(\frac{1}{4} - \frac{1}{3}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$

$$= \frac{1}{4} - \frac{2}{3} + \frac{2}{4} = \frac{1}{12}$$

$\Rightarrow \boxed{\frac{M_2}{6} + \frac{2M_3}{3} + \frac{M_4}{6} = \frac{1}{12}}$ But $\boxed{M_1 = M_4 = 0}$.

$\Rightarrow \boxed{\frac{2}{3}M_2 + \frac{M_3}{6} = \frac{1}{3}}$ and $\boxed{\frac{M_2}{6} + \frac{2M_3}{3} = \frac{1}{12}}$.

$\Rightarrow \boxed{4M_2 + M_3 = 2}$ and $\boxed{2M_2 + 8M_3 = 1}$

$\Rightarrow 4M_2 + M_3 = 2$ and $4M_2 + 16M_3 = 2$.

Subtracting the first equation from the second, we get, $\underline{15M_3 = 0} \Rightarrow \boxed{M_3 = 0}$.

$\therefore \underline{4M_2 = 2} \Rightarrow \boxed{M_2 = 1/2}$.

So we have $\boxed{M_1 = M_3 = M_4 = 0}$ and $\boxed{M_2 = 1/2}$.

Now, $$\boxed{S(x) = \frac{M_{j-1}}{x_j - x_{j-1}} \frac{(x_j - x)^3}{6} + \frac{M_j}{x_j - x_{j-1}} \frac{(x - x_{j-1})^3}{6} + C(x_j - x) + D(x - x_{j-1}).}$$

$\boxed{C = \frac{y_{j-1}}{x_j - x_{j-1}} - \frac{M_{j-1}}{6}(x_j - x_{j-1})}$, $\boxed{D = \frac{y_j}{x_j - x_{j-1}} - \frac{M_j}{6}(x_j - x_{j-1})}$.

1/. $\underline{\text{For } j = 2}$; $\quad S(x) = \underline{\frac{M_2 (x - x_1)^3}{6}} + C(x_2 - x) + D(x - x_1)$

where $\boxed{C = y_1 = 1}$, $\boxed{D = y_2 - \frac{M_2}{6}}$ $\quad (\underline{x_j - x_{j-1} = 1})$

$\underline{M_1 = 0}$ and $\underline{x_1 = 1}$, $\underline{x_2 = 2}$, $\underline{M_2 = 1/2}$, $\underline{y_2 = \frac{1}{2}}$

$$\therefore \boxed{D = \frac{1}{2} - \frac{1}{12} = \frac{6}{12} - \frac{1}{12} = \frac{5}{12}}.$$

$$\Rightarrow S(x) = \frac{1}{12}(x-1)^3 + (2-x) + \frac{5}{12}(x-1)$$

$$\Rightarrow S(x) = \frac{1}{12}(x^3 - 3x^2 + 3x - 1) + 2 - x + \frac{5x}{12} - \frac{5}{12}$$

$$\Rightarrow S(x) = \frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{x}{4} - x + \frac{5x}{12} - \frac{1}{12} + 2 - \frac{5}{12}$$

$$\Rightarrow \boxed{S(x) = \frac{x^3}{12} - \frac{1}{4}x^2 - \frac{x}{3} + \frac{3}{2}} \left(\text{for } \underline{1 \le x \le 2}\right).$$

2/. <u>For $j = 3$</u>: $S(x) = \frac{M_2}{6}(x_3 - x)^3 + C(x_3 - x) + D(x - x_2)$

$\boxed{C = y_2 - \frac{M_2}{6} = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}}$, $\boxed{D = y_3 = \frac{1}{3}}$ | $x_3 = 3$, $x_2 = 2$, $M_2 = 1/2$

$$\therefore S(x) = \frac{1}{12}(3-x)^3 + \frac{5}{12}(3-x) + \frac{1}{3}(x-2)$$

$$\Rightarrow S(x) = \frac{1}{12}\left[27 + 9x^2 - 27x - x^3\right] + \frac{5}{4} - \frac{5x}{12} + \frac{x}{3} - \frac{2}{3}$$

$$\Rightarrow S(x) = -\frac{x^3}{12} + \frac{3}{4}x^2 - \left(\frac{9x}{4} - \frac{5x}{12} + \frac{x}{3}\right) + \left(\frac{27}{12} + \frac{5}{4} - \frac{2}{3}\right)$$

$$\Rightarrow \boxed{S(x) = -\frac{x^3}{12} + \frac{3}{4}x^2 - \frac{7}{3}x + \frac{17}{6}} \left(\text{for } \underline{2 \le x \le 3}\right).$$

3/ <u>For $j = 4$</u>: $S(x) = C(x_j - x) + D(x - x_{j-1})$ | $M_3 = M_4 = 0$, $x_4 = 4$, $x_3 = 3$.

$\boxed{C = y_3 = 1/3}$, $\boxed{D = y_4 = 1/4}$

$$\Rightarrow S(x) = \frac{1}{3}(4-x) + \frac{1}{4}(x-3) = -\frac{x}{3} + \frac{x}{4} + \frac{4}{3} - \frac{3}{4}$$

$$\Rightarrow \boxed{S(x) = -\frac{x}{12} + \frac{7}{12}} \left(\text{for } \underline{3 \le x \le 4}\right).$$

If $x_1, x_2, \ldots, x_n$ are given $S(x)$ is calculated using $n-1$ points, i.e. $x_2, x_3, \ldots, x_n$.