Central Force Motion …

Summary

- Started with **6d, 2 body problem.**
- Reduced it to **2, 3d 1 body problems,** one (CM motion) of which is trivial.
- **Angular momentum conservation** reduces 2nd 3d problem (relative motion) **from 3d to 2d** (motion in a plane)!
- **Lagrangian** ($\mu \rightarrow$ **m** , conservative, central forces):

$$L = (½)m|\dot{r}|^2 - V(r)$$

- **Motion in a plane**

plane polar coordinates to do the problem:

$$L = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

Energy

Total mechanical energy

$$E = T + V = \text{constant}$$

$$E = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

angular momentum :

$$\ell \equiv mr^2\dot{\theta} = \text{const}$$

$$\dot{\theta} = [\ell/(mr^2)]$$

$$\Rightarrow \quad E = (½)m\dot{r}^2 + (½)[\ell^2/(mr^2)] + V(r) = \text{const}$$

If V(r) is specified the above eqn. completely describes the system. General soln. in terms of E and ℓ.

Equations of motion

$$E = (\tfrac{1}{2})m\dot{r}^2 + [\ell^2/(2mr^2)] + V(r) = \text{const}$$

Energy Conservation allows us to get solutions to the eqns of motion in terms of **r(t)** & **θ(t)** and **r(θ)** or **θ(r)** ≡ **The orbit of the particle!**

Eqn of motion to get **r(t)**: One degree of freedom

$\Rightarrow$ *Very similar to a 1 d problem!*

- $\dot{r} = (dr/dt)$:

$$\dot{r} = \pm(\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{1/2}$$

Equivalent "1d" Problem

The 2 body Central Force problem has been **reduced to evaluation of 2 integrals**, which will give **r(t)** & **θ(t)** : (Given **V(r)**)

$$t(r) = \pm\int dr(\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{-1/2} \quad (1)$$

- *Limits determined by initial conditions*
- *Invert this to get* ***r(t)*** *& use that in* ***θ(t)***

$$\theta(t) = (\ell/m)\int(dt/[r^2(t)]) + \theta_0 \qquad (2)$$

- *Limits* $0 \rightarrow t$, θ_0 *determined by initial condition*

➢ Need **4 integration constants:** E, ℓ, r_0, θ_0

<u>Most cases: **(1), (2)** can't be done except numerically</u>

- Once the central force is specified, we know **V(r)** & can, in principle, do the integral & get the orbit **θ(r),** or **r(θ)**.

$\Rightarrow$ Assuming only a central force law & nothing else:

Reduced the original 6-d problem of 2 particles to a 2-d problem with only 1 degree of freedom.

The solution for the orbit ?

Orbits

Path in the **r, θ** plane: **r(θ)** or **θ(r)** ≡ ***<u>The orbit.</u>***

Orbits

chain rule:

$$(d\theta/dr) = (d\theta/dt)(dt/dr) = (d\theta/dt)/(dr/dt)$$

Or: $$(d\theta/dr) = (\dot{\theta}/\dot{r})$$

Orbits

$$(d\theta/dr) = (\dot{\theta}/\dot{r})$$

$$\ell \equiv mr^2\dot{\theta} = \text{const} \Rightarrow \dot{\theta} = [\ell/(mr^2)]$$

$$\dot{r} = \pm (\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{1/2}$$

$$\Rightarrow (d\theta/dr) = \pm [\ell/(mr^2)](\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{-1/2}$$

Or:

$$(d\theta/dr) = \pm (\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}$$

Integrating this will give $\theta(r)$.

Integrating this gives a formal eqn for the orbit:

$$\boldsymbol{\theta(r) = \pm \int (\ell/r^2)(2m)^{-½}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-½}\, dr}$$

IF the central force is specified, we can calculate **V(r)** & can, do the integral & get the orbit **θ(r),** or, **r(θ)**.

Form of the force law!!

Assignment

Finding the force law for a central force field that gives a particular known orbit.

Example :
Simple harmonic oscillator.

Radial velocity of a particle (central field)

$\dot{r} = \pm \left(\{2/m\}[E - V(r)] - [\ell^2/(m^2 r^2)]\right)^{1/2}$

If radial velocity =0 → turning point in the motion has reached.

$E - V(r) - \ell^2/(2mr^2) = 0$ Generally two roots.

r=constant, what about the orbit?

Lagranges eqn of motion

In polar coordinates. $\dot{\mathbf{r}} = \frac{d}{dt} r\hat{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}.$

We have $$\mathcal{L} = \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r),$$

Thus, the Lagrangian becomes $$\mathcal{L} = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r),$$

which is independent of ϕ, from which we also see that angular momentum is conserved.

Write down the Lagranges eqns. Of motion?

The Two Equations of Motion

ϕ equation for the Lagrangian $\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r),$

is $\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \text{const} = \ell.$ (angular momentum)

- The radial equation, is $\frac{\partial \mathcal{L}}{\partial r} = \mu r \dot{\phi}^2 - \frac{dU}{dr} = \mu \ddot{r}.$
- Equation depends on $\dot{\phi}$, so the two equations are coupled, simply replace $\dot{\phi} = \frac{\ell}{\mu r^2}$

  and find

$$\mu \ddot{r} = \mu r \left(\frac{\ell}{\mu r^2}\right)^2 - \frac{dU}{dr} = \frac{\ell^2}{\mu r^3} - \frac{dU}{dr}.$$

- This depends only on r, one-dimensional problem.
- The above equation is a force equation, so each term is a force.
- The first term on the right can be identified as the “centrifugal force” F_{cf}.

- We can write this as the (1-d) gradient of a potential energy ?

Centrifugal potential energy

$$F_{\mathrm{cf}} = -\nabla U_{\mathrm{cf}} = \frac{\ell^2}{\mu r^3} = -\frac{d}{dr} U_{\mathrm{cf}} \quad \Rightarrow \quad U_{\mathrm{cf}} = \frac{\ell^2}{2\mu r^2}.$$

Write down the eqn. of motion using the gradient of potential ….

The Equiv. 1-D Problem

- Using the gradient of a potential, the equation of motion becomes

$$\mu \ddot{r} = -\frac{dU}{dr} - \frac{dU_{\mathrm{cf}}}{dr} = -\frac{d}{dr}\left[U(r) + U_{\mathrm{cf}}(r)\right] = -\frac{d}{dr} U_{\mathrm{eff}}(r),$$

where U_{eff} is the effective potential energy, i.e. the sum of the actual potential energy $U(r)$ and the centrifugal potential energy $U_{\mathrm{cf}}(r)$:

$$U_{\mathrm{eff}} = U(r) + \frac{\ell^2}{2\mu r^2}.$$

This is actually a correct equation for any two-body central force problem.

For the specific case of a gravitational potential energy, we have

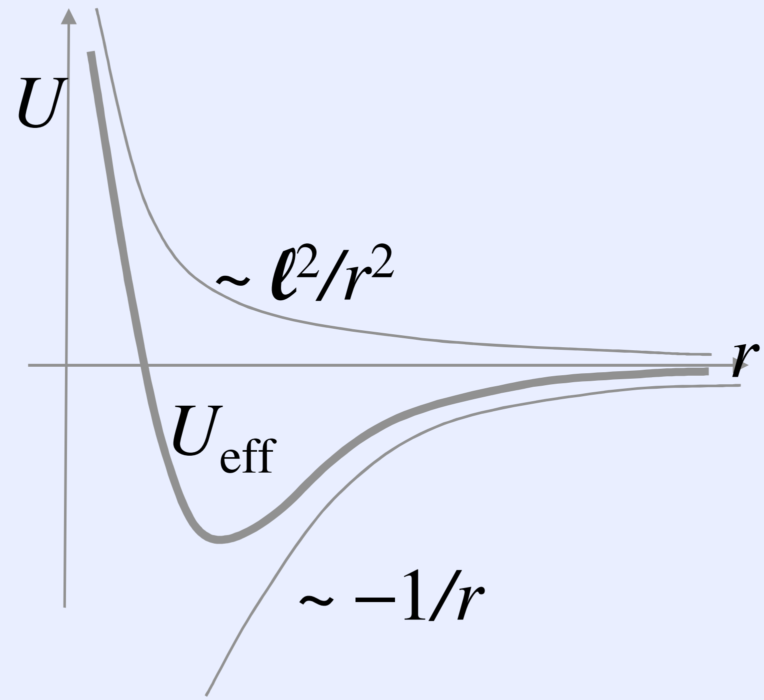
$$U_{\mathrm{eff}} = -\frac{G m_1 m_2}{r} + \frac{\ell^2}{2\mu r^2}.$$

one term of this is negative, while the other is positive. Plot this.

The Equiv. 1-D Problem

we have $$U_{\text{eff}} = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}.$$

V(r) is kind of fictitious potential that has the real potential U(r) and the energy term associated with the angular motion about the center of force.



$$m\ddot{r} - [\ell^2/(mr^3)] = -(\partial V/\partial r) \equiv f(r)$$

(**f(r)** ≡ force along **r**)

$$m\ddot{r} = f(r) + [\ell^2/(mr^3)] \qquad (1)$$

(1) involves only **r**

Same Eqtn of motion (Newton's 2nd Law) as for a **fictitious** (effective) 1d **(r)** problem of mass **m** subject to a force:

$$f'(r) = f(r) + [\ell^2/(mr^3)]$$

Centrifugal "Force" & Potential

- Effective 1d **(r)** problem: **m** subject to a force:

$$f'(r) = f(r) + [\ell^2/(mr^3)]$$

Using $\ell \equiv mr^2\dot{\theta}$:

$$[\ell^2/(mr^3)] \equiv mr\dot{\theta}^2 \equiv m(v_\theta)^2/r \equiv \text{"Centrifugal Force"}$$

- Equivalently, energy:

$$E = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = (½)m\dot{r}^2 + (½)[\ell^2/(mr^2)] + V(r) = \text{const}$$

- **Same energy Eqn as** for a **fictitious** (or effective) 1d **(r)** problem of mass **m** subject to a potential:

$$V'(r) = V(r) + (½)[\ell^2/(mr^2)]$$

  - *Easy to show that* $f'(r) = -(\partial V'/\partial r)$
  - *Can clearly write* $E = (½)m\dot{r}^2 + V'(r) = const$

Centrifugal "Force" & Potential

- Consider: $\mathbf{E = (1/2)m\dot{r}^2 + (1/2)[\ell^2/(mr^2)] + V(r)}$
- ***Term*** → $\mathbf{[\ell^2/(2mr^2)]}$.

Conservation of angular momentum: put ℓ **in this eqn.**

Centrifugal "Force" & Potential

- Consider: $\mathbf{E = (\tfrac{1}{2})m\dot{r}^2 + (\tfrac{1}{2})[\ell^2/(mr^2)] + V(r)}$
- ***Term*** → $[\ell^2/(2mr^2)]$. Conservation of angular momentum: $\ell = mr^2\dot{\theta} \Rightarrow [\ell^2/(2mr^2)] \equiv (\tfrac{1}{2})mr^2\dot{\theta}^2$

  ≡ ***Angular part of kinetic energy*** of mass **m**.
- Because of the form $[\ell^2/(2mr^2)]$, this contribution to the energy depends only on **r**:
- ***When analyzing the* r *part of the motion***, can treat this as **an additional part of the *potential energy.***

⇒ Call it another potential energy term ≡ ***"Centrifugal" Potential Energy***

- $[\ell^2/(2mr^2)] \equiv$ ***"Centrifugal" PE*** $\equiv V_c(r)$

$\Rightarrow$ "Force" associated with $V_c(r)$:

$$f_c(r) \equiv -(\partial V_c/\partial r) = [\ell^2/(mr^3)]$$

Or, using $\ell = mr^2\dot{\theta}$:

$$f_c(r) = [\ell^2/(mr^3)] = mr\dot{\theta}^2 \equiv m(v_\theta)^2/r$$

$\equiv$ ***"Centrifugal Force"***

Effective Potential

- For both qualitative & quantitative analysis of the ***RADIAL*** motion for "particle" of mass **m** in a central potential **V(r)**, $V_c(r) = [\ell^2/(2mr^2)]$ acts as an additional potential
  - *But **physically,** it comes from the **Kinetic Energy** of the particle!*

$\Rightarrow$ Combine **V(r)** & $V_c(r)$ together into an ***Effective Potential*** $\equiv$

$$V'(r) \equiv V(r) + V_c(r)$$
$$\equiv V(r) + [\ell^2/(2mr^2)]$$

- ***Effective Potential*** $\equiv$

$$V'(r) \equiv V(r) + V_c(r) \equiv V(r) + [\ell^2/(2mr^2)]$$

- Consider now:

$$E = (½)m\dot{r}^2 + (½)[\ell^2/(mr^2)] + V(r) = (½)m\dot{r}^2 + V'(r) = \text{const}$$

$$\Rightarrow \qquad \dot{r} = \pm (\{2/m\}[E - V'(r)])^{½}$$

- Given **U(r)**, can use **the above eqn.** to **qualitatively analyze the *RADIAL* motion** for the "particle".

Get turning points, oscillations, etc.

Gives **r** *vs.* $\dot{\mathbf{r}}$ phase diagram.

  - *Similar to analysis of 1 d motion where one analyzes particle motion for various **E** .*

Inverse square law central force: $f(r) = -(k/r^2) \Rightarrow V(r) = -(k/r)$

- *Taking* $V(r \to \infty) \to 0$
- $k > 0$: *Attractive force.* $k < 0$: *Repulsive force.*
- *Gravity:* $k = GmM$. *Always attractive!*
- *Coulomb (SI Units):* $k = (q_1q_2)/(4\pi\varepsilon_0)$. *Could be attractive or repulsive!*

For **inverse square law force,** effective potential is:

$$V'(r) \equiv V(r) + [\ell^2/(2mr^2)] = -(k/r) + [\ell^2/(2mr^2)]$$

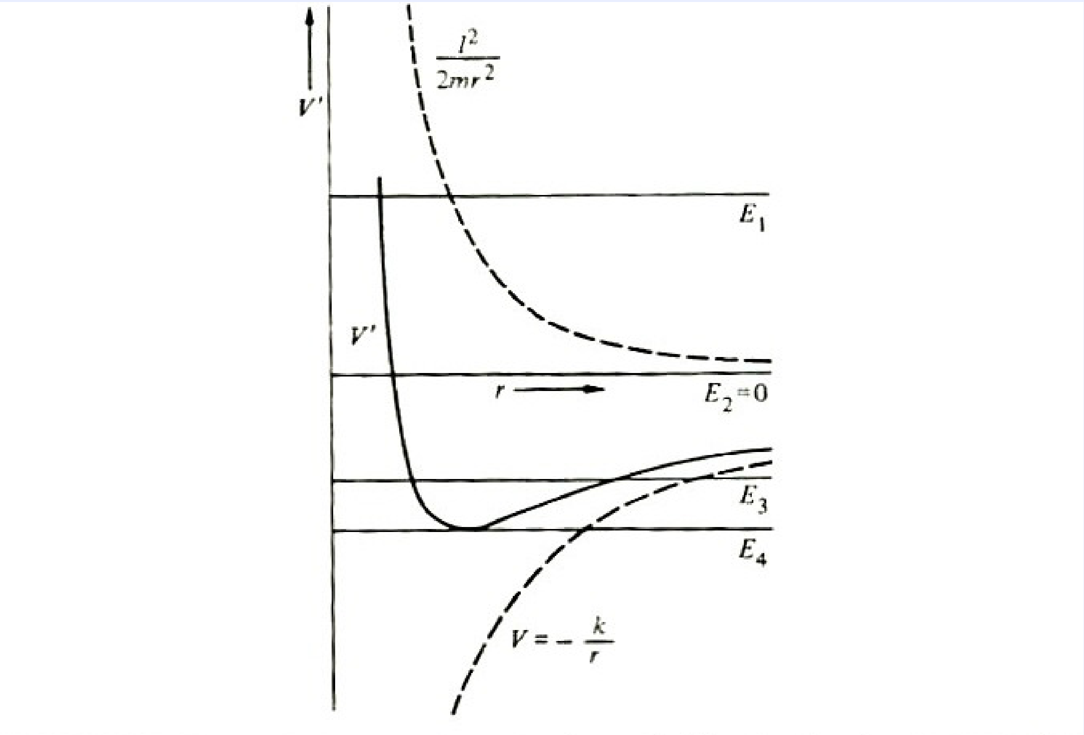
$V'(r)$ for Attractive r^{-2} Forces

- **Qualitatively analyze motion** for different energies **E** in effective potential for inverse square law force.

$$V'(r) = -(k/r) + [\ell^2/(2mr^2)]$$

$$E = (\tfrac{1}{2})m\dot{r}^2 + V'(r) \Rightarrow E - V'(r) = (\tfrac{1}{2})m\dot{r}^2 \geq 0$$

$\Rightarrow \dot{r} = 0$ at turning points ($E = V'(r)$)



NOTE: This analysis is for the **r** ***part of the motion only***. To get the particle orbit **r(θ),** must superimpose **θ** motion on this!

Motion of particle with energy $E_1 > 0$

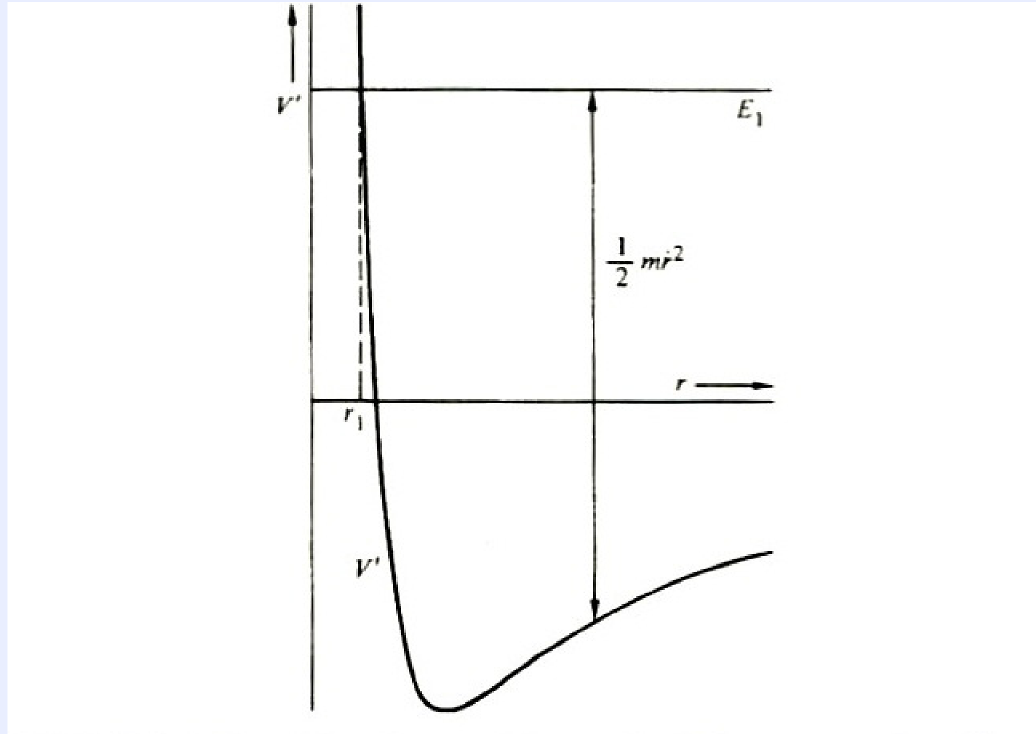
$E_1 - V'(r) = (\frac{1}{2}) m\dot{r}^2 \geq 0$

turning point **?**

min distance of approach ?

max **?**

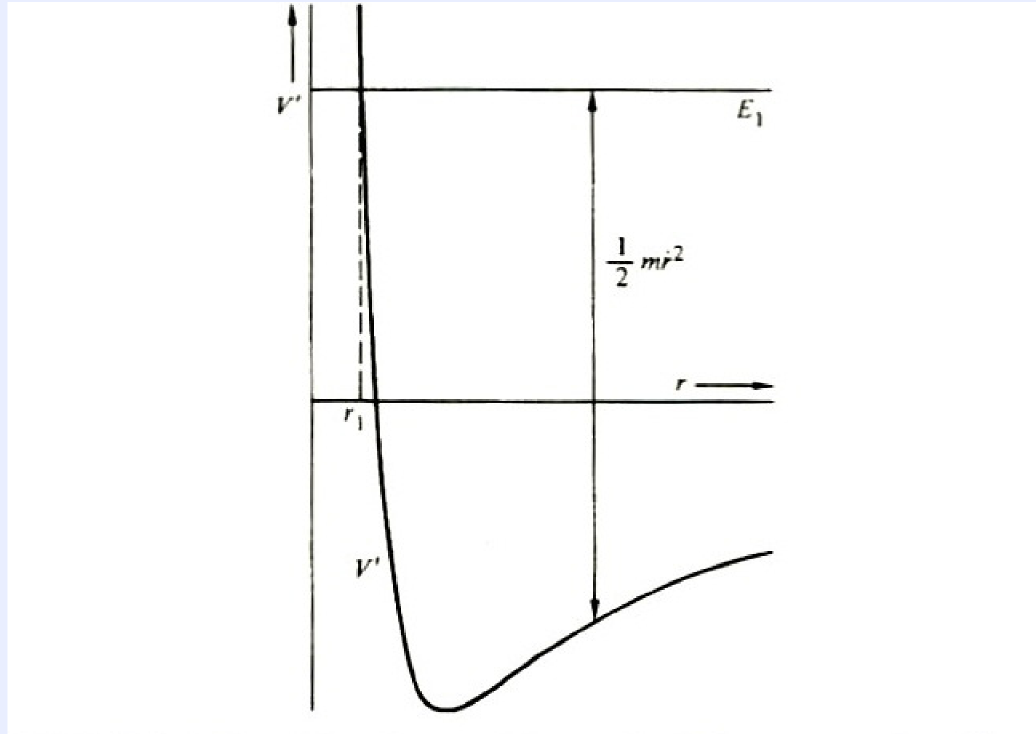
Bounded or **Unbounded orbit?**



Particle from **r** $\rightarrow \infty$ comes in towards **r=0**.
What happens at $\mathbf{r = r_1}$.
It speeds up until **which point.**

Motion of particle with energy $E_1 > 0$

$E_1 - V'(r) = (\frac{1}{2})m\dot{r}^2 \geq 0$



turning point **r1.**

min distance of approach = r1

No. max.

Unbounded orbit.

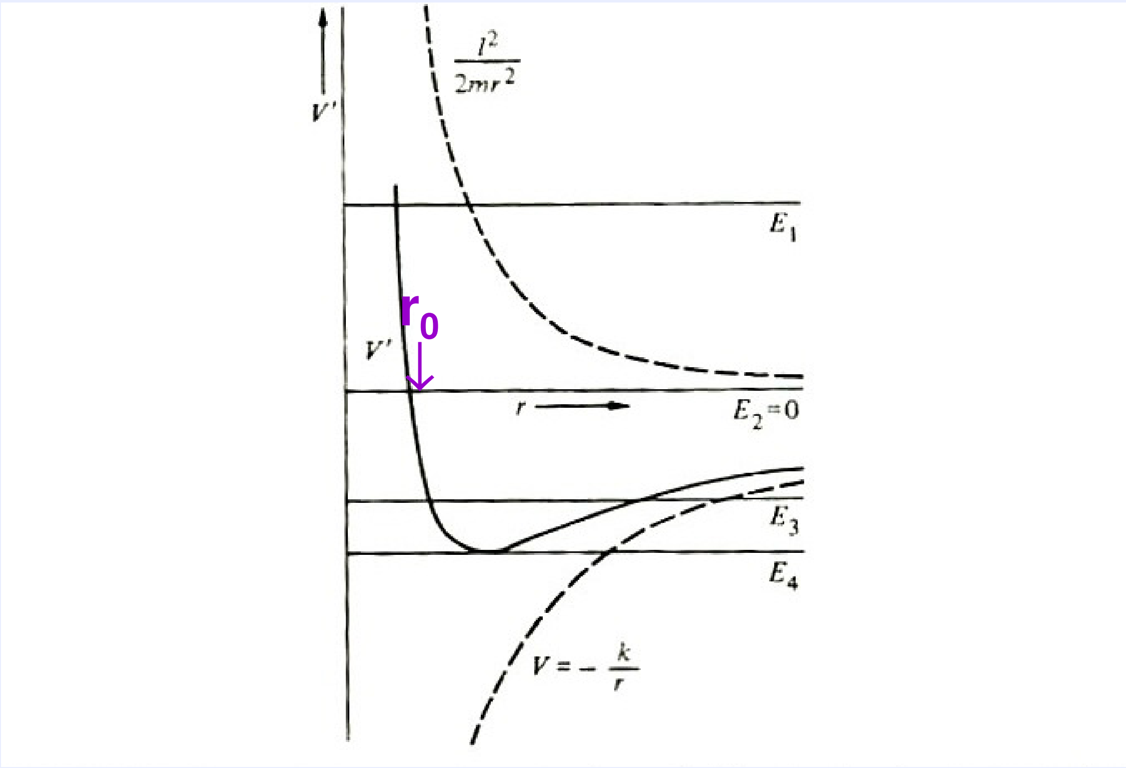
Particle from $r \to \infty$ comes in towards $r=0$. At $r = r_1$, it "strikes" the "repulsive centrifugal barrier", is repelled (turns around) & travels back out towards $r \to \infty$. It speeds up until $r = r_0$ = min of $V'(r)$. Then, slows down as it approaches r_1. After it turns around, it speeds up to r_0 & then slows down to $r \to \infty$.

Motion of particle with **energy $E_2 = 0$:**

$$E_2 - V'(r) = (½)m\dot{r}^2 \geq 0. \quad \Rightarrow -V'(r) = (½)m\dot{r}^2 \geq 0$$

Qualitative motion ?

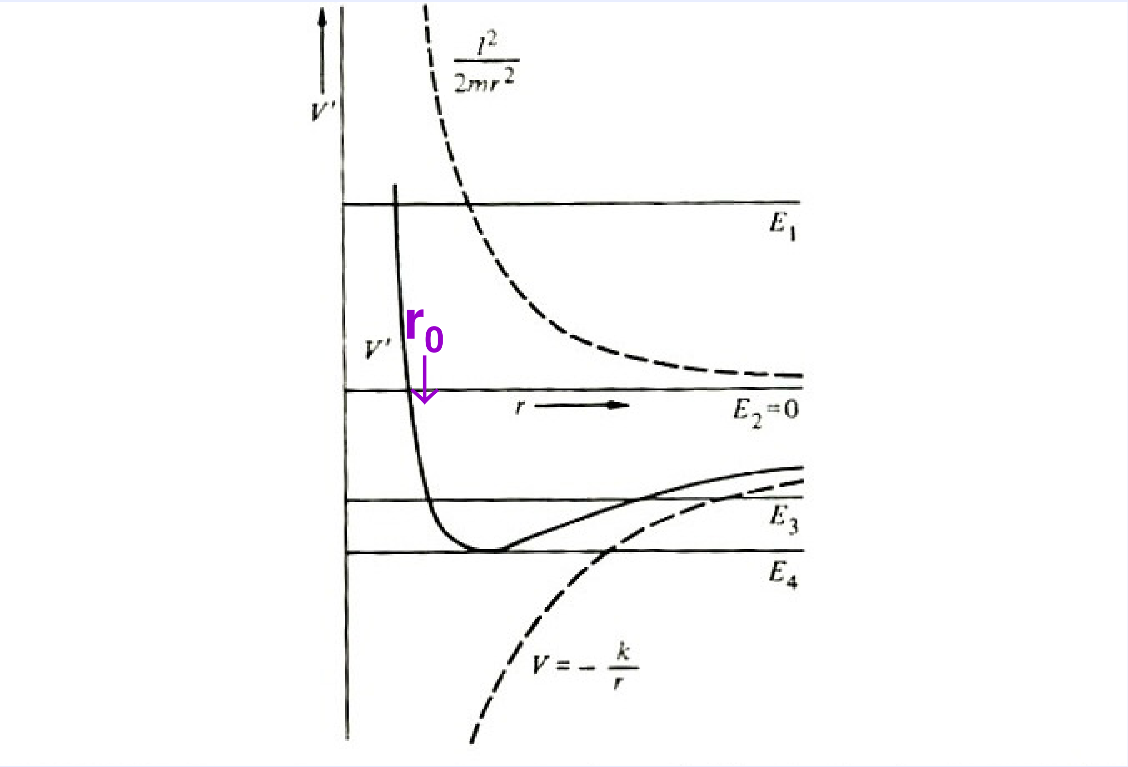
the turning point ?



Motion of particle with **energy $E_2 = 0$:**

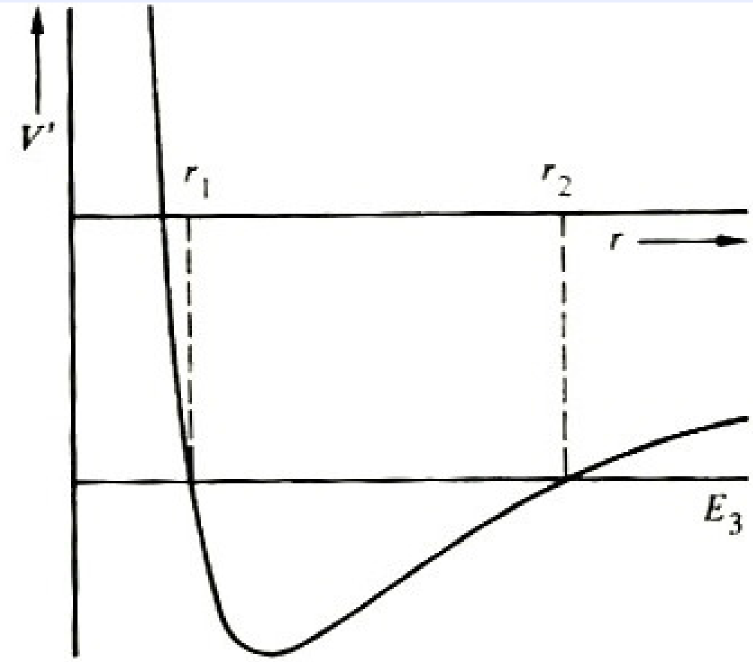
$$E_2 - V'(r) = (\tfrac{1}{2})m\dot{r}^2 \geq 0. \quad \Rightarrow -V'(r) = (\tfrac{1}{2})m\dot{r}^2 \geq 0$$

Qualitative motion is ~ the same as for $\mathbf{E_1}$, except **the turning point is at r_0** (figure):



Motion of particle with energy $E_3 < 0$:

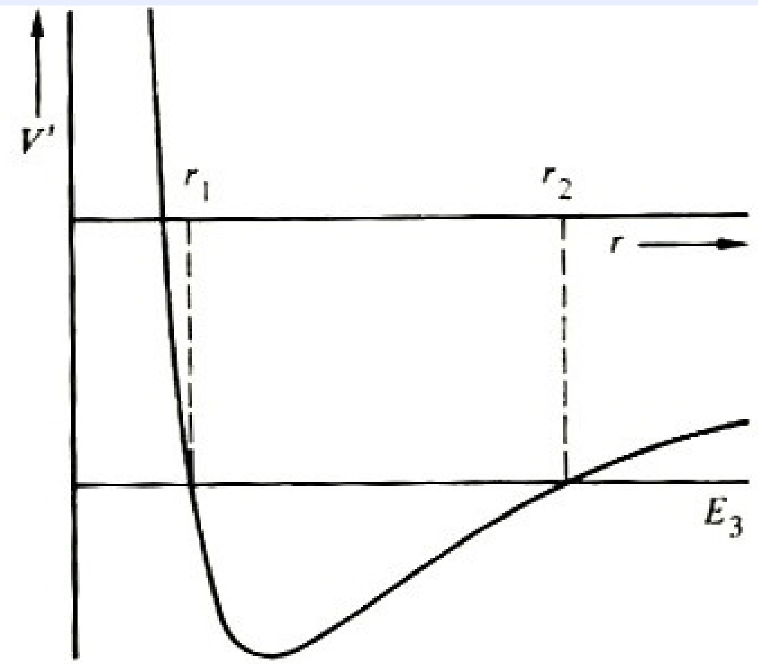
$E_3 - V'(r) = (½)m\dot{r}^2 \geq 0$. **Qualitative motion:**



Motion of particle with energy $E_3 < 0$:

$$E_3 - V'(r) = (\tfrac{1}{2})m\dot{r}^2 \geq 0.$$

"oscillatory" in r



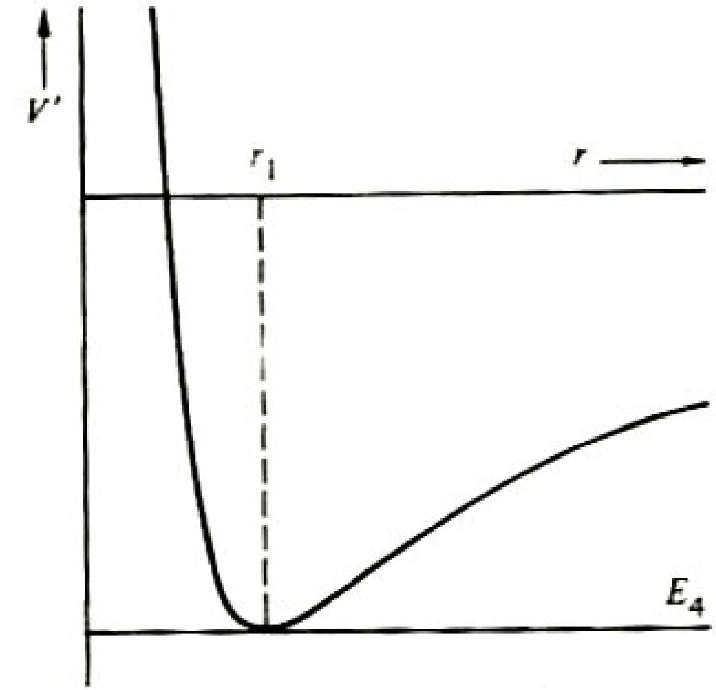
2 turning points, min & max **r**: (r_1 & r_2).

Turning points given by solutions to $E_3 = V'(r)$.

Orbit is bounded. r_1 & r_2 ≡ "apsidal' distances.

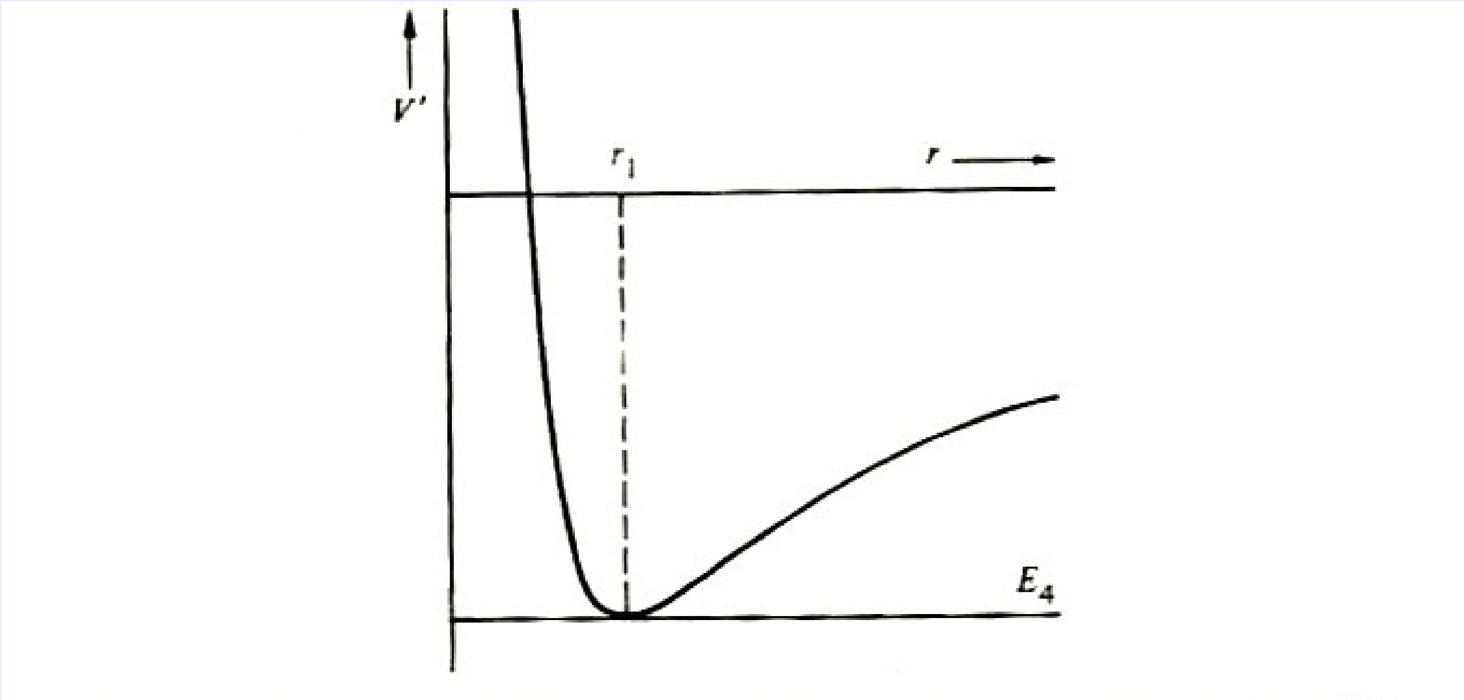
Motion of particle with **energy $E_4 < 0$:**

- **$E_4 - V'(r) = 0$**



Motion of particle with **energy $E_4 < 0$:**

- **E_4** -**V´(r)** = **0 ($\dot{r}$ = 0)** ⇒ **r = r_1** (min **r** of **V´(r)**) **=** constant ⇒ **Circular orbit** (& bounded) **r(θ) = r_1**!



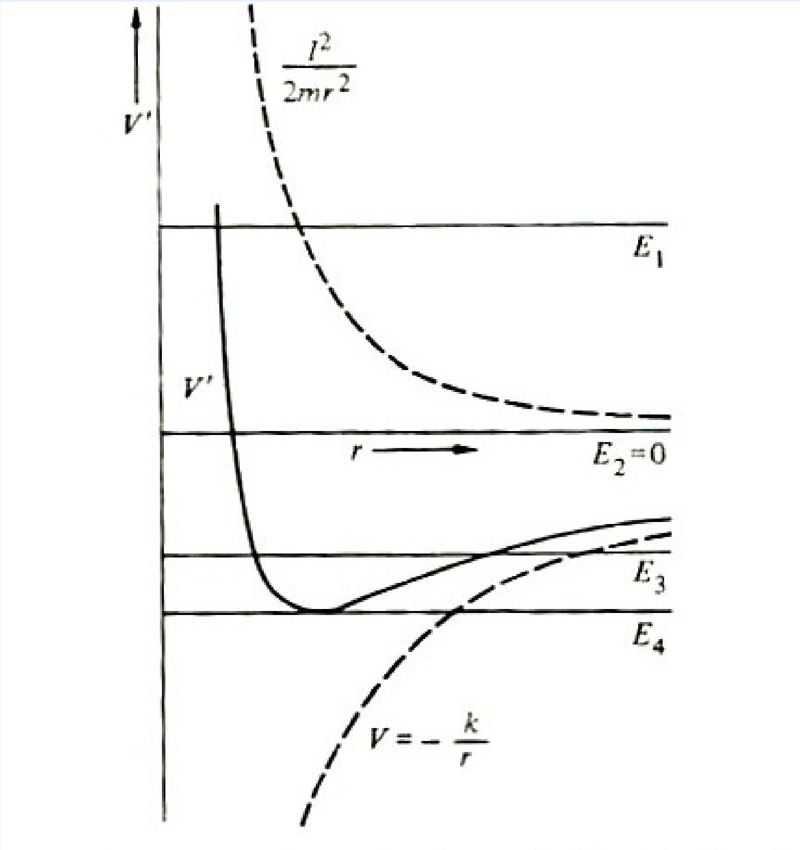
Energy E < E_4? $\Rightarrow$ E $-V'(r) = (½)m\dot{r}^2 < 0$

Unphysical! Requires $\dot{\mathbf{r}}$ = imaginary.

Till now considered one value of angular momentum ℓ.

Clearly changing ℓ changes **V´(r)** quantitatively, but not Qualitatively.

$\Rightarrow$ Orbit types will be the same for similar energies.



$V'(r)$ for Attractive r^{-2} Forces

Shape of the orbit ??

- Energy $\mathbf{E_1 > 0}$:
- Energy $\mathbf{E_2 = 0}$:
- Energy $\mathbf{E_3 < 0}$:
- Energy $\mathbf{E_4 = [V'(r)]_{min}}$:

Other Attractive Forces

For **other types of Forces**: Orbits aren't so simple.

For any ***attractive*** **V(r)** still have the same qualitative division into open, bounded, & circular orbits if:

1. V(r) falls off *slower* than r^{-2} as **r** $\to \infty$

Ensures that **V(r)** > **(½)[ℓ^2/(mr^2)]** as **r** $\to \infty$

$\Rightarrow$ **V(r)** dominates the Centrifugal Potential at large **r**.

2. V(r) $\to \infty$ ***slower* than r^{-2}** as **r** $\to$ **0**

Ensures that **V(r)** < **(½)[ℓ^2/(mr^2)]** as **r** $\to$ **0**

$\Rightarrow$ The centrifugal Potential dominates **V(r)** at small **r**.

- If the ***attractive*** potential **V(r)** doesn't satisfy these conditions, the qualitative nature of the orbits will be **different.**

- However, we can still use same method to examine the orbits.

- Example: $\mathbf{V(r) = -(a/r^3)}$ ($\mathbf{a}$ = constant)

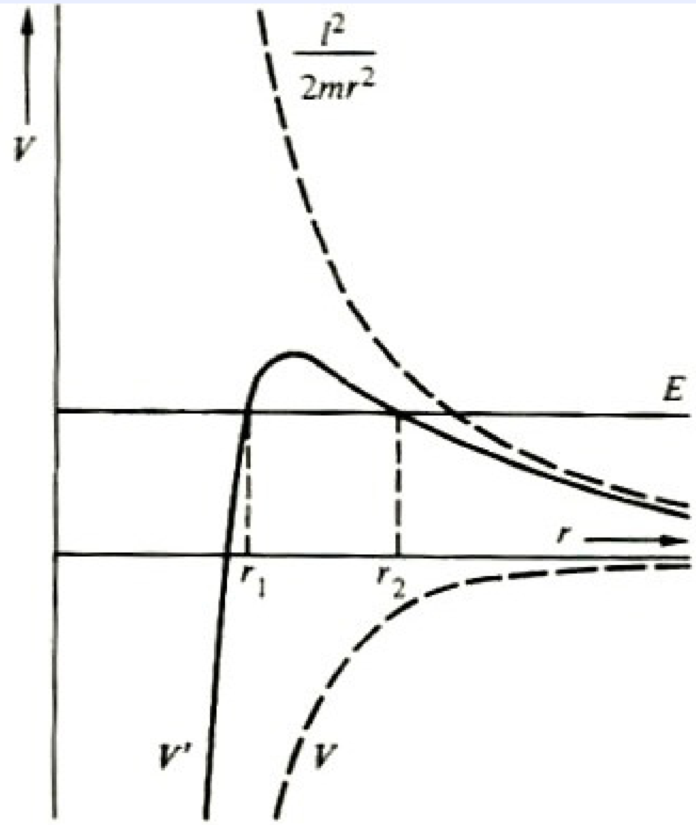
$\Rightarrow$ Force: $\mathbf{f(r) = ?}$

$V'(r)$ for Attractive r^{-4} Forces

- **Example:** $V(r) = -(a/r^3)$; $\Rightarrow$ $f(r) = -(3a/r^4)$.

  Eff. potential: $V'(r) = -(a/r^3) + (½)[\ell^2/(mr^2)]$

Energy E, motion types ?



$V'(r)$ for Attractive r^{-4} Forces

- **Example:** $V(r) = -(a/r^3)$; $\Rightarrow f(r) = -(3a/r^4)$.

  Eff. potential: $V'(r) = -(a/r^3) + (\frac{1}{2})[\ell^2/(mr^2)]$

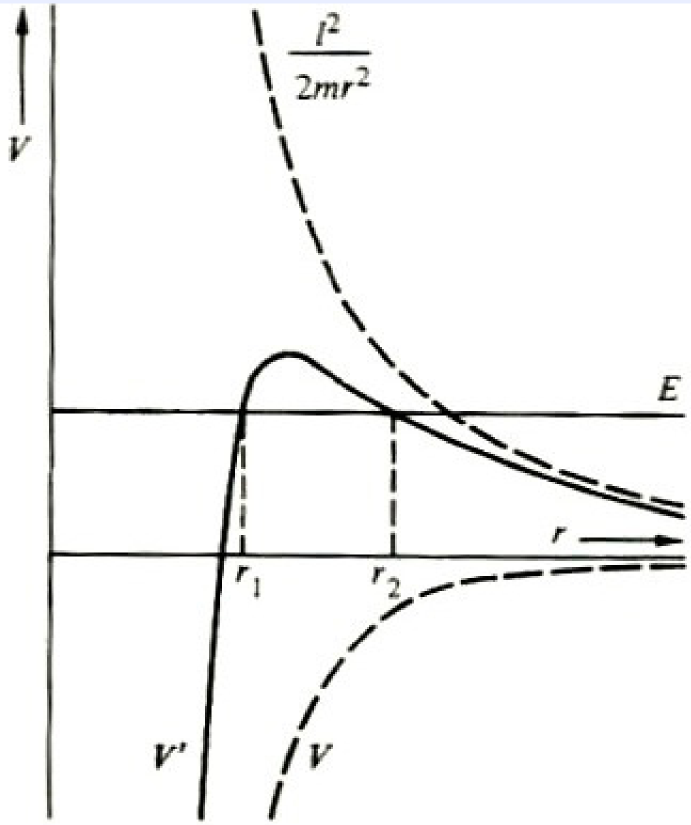
Energy **E** →

motion types, depending on **r**:

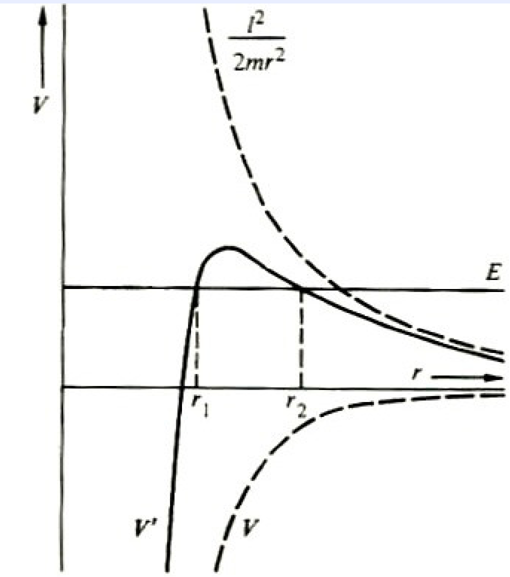
$r < r_1$, ?

$r > r_2$, ?

$r_1 < r < r_2$: ?



$V'(r)$ for Attractive r^{-4} Forces



Energy **E**, 2 motion types, depending on **r**:

- **$r < r_1$**, ***bounded orbit.***

$r < r_1$ always. Particle passes through center of force (**$r = 0$**).

- **$r > r_2$**, ***unbounded orbit***.

$r > r_2$ always. Particle can never get to the center force (**$r = 0$**).

- **$r_1 < r < r_2$**: Not possible physically, since would require **$E - V'(r) = (½)m\dot{r}^2 < 0$** $\Rightarrow$ Unphysical! $\Rightarrow$ **$\dot{r}$** imaginary!

V´(r): Isotropic Simple Harmonic Oscillator

- **Example:** Isotropic Simple Harmonic Oscillator:

  f(r) = ? , V(r) = ?

  Effective potential: **V´(r) = ?**

V´(r): Isotropic Simple Harmonic Oscillator

Isotropic Simple Harmonic Oscillator:

$$f(r) = -kr, \quad V(r) = (½)kr^2$$

Effective potential: $V'(r) = (½)kr^2 + (½)[\ell^2/(mr^2)]$

For $\ell = 0$

$\Rightarrow V'(r) = V(r) = (½)kr^2$:

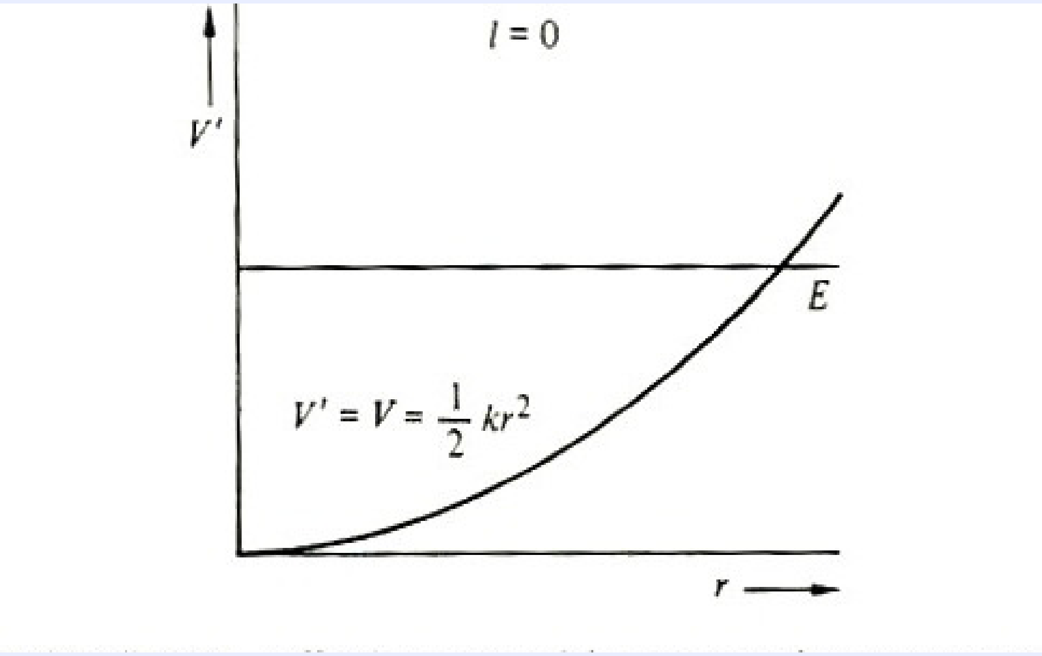
Motion?

Turning point ?

motion amplitude.

$E - V(r) = (½)mr^2 > 0$

Speed ?



$V'(r)$: Isotropic Simple Harmonic Oscillator

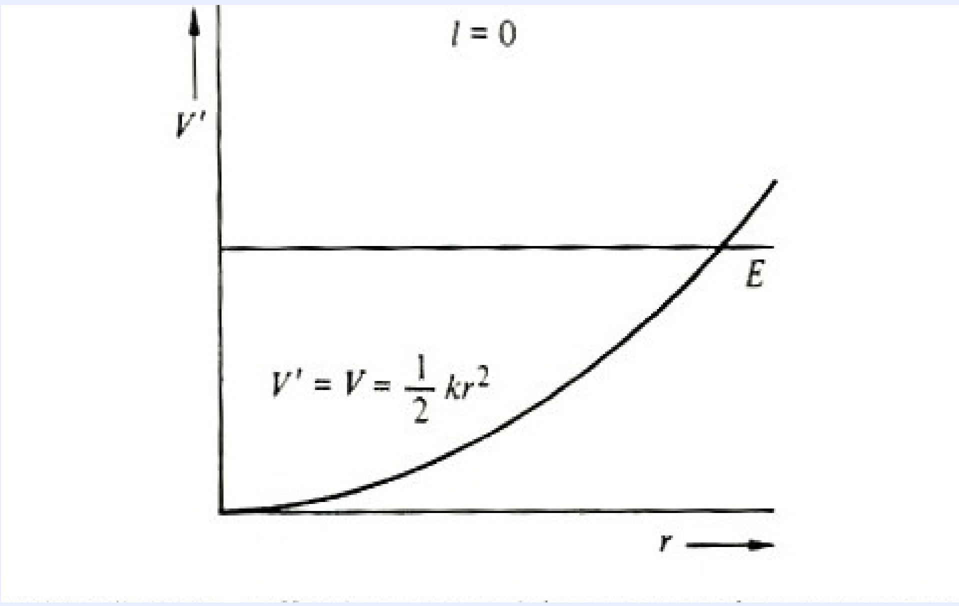
Isotropic Simple Harmonic Oscillator:

$$\mathbf{f(r) = - kr, \; V(r) = (½)kr^2}$$

Effective potential: $\mathbf{V'(r) = (½)kr^2 + (½)[\ell^2/(mr^2)]}$

- ➢ $\ell = 0 \Rightarrow \mathbf{V'(r) = V(r) = (½)kr^2}$:

Any **E >0**: Motion is straight line in "**r**" direction. **Simple harmonic**. Passes through **r = 0**. Turning point at $\mathbf{r_1}$ **=** motion amplitude.



$\mathbf{E - V(r)} = \mathbf{(½)m\dot{r}^2 > 0} \Rightarrow$ Speeds up as heads towards **r = 0**, slows down as heads away from **r = 0**. Stops at $\mathbf{r_1}$, turns around.

Isotropic Simple Harmonic Oscillator:

$f(r) = - kr$, $V'(r) = (½)kr^2 + (½)[\ell^2/(mr^2)]$

Case 2 : $\ell \neq 0$

- Isotropic Simple Harmonic Oscillator:

  $f(r) = - kr$, $V'(r) = (\frac{1}{2})kr^2 + (\frac{1}{2})[\ell^2/(mr^2)]$

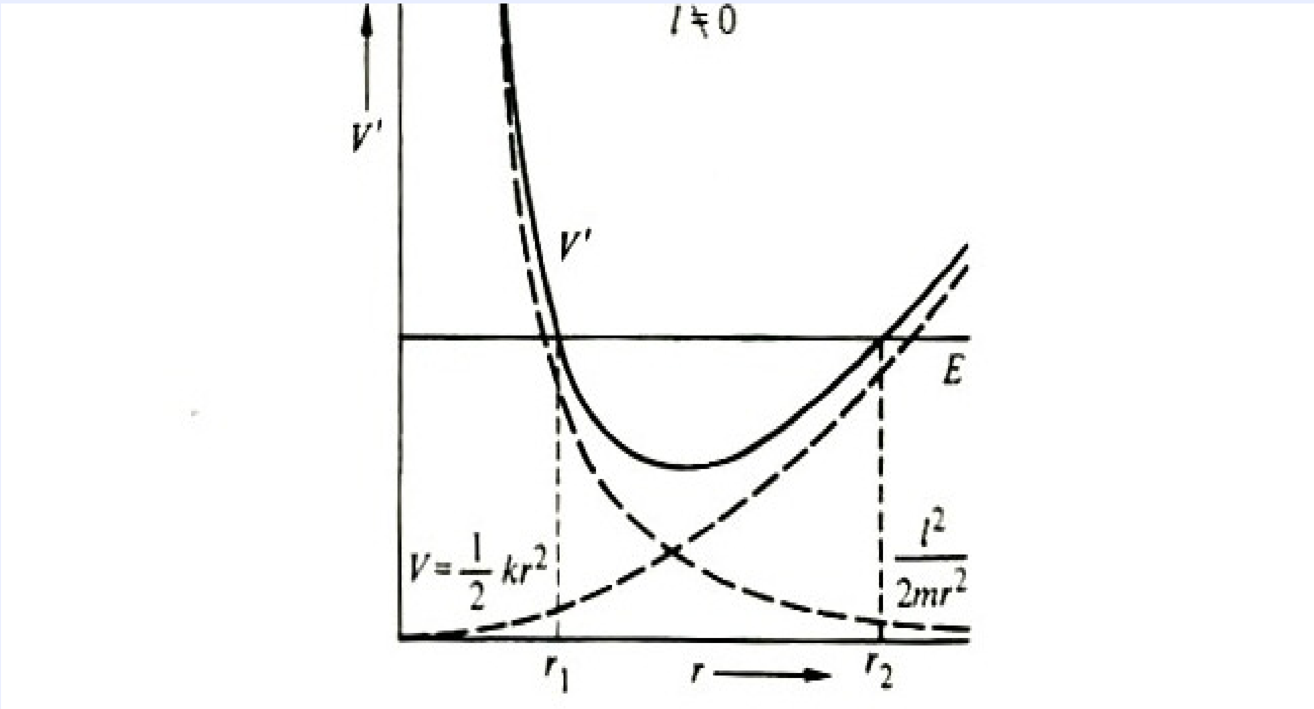
- $\ell \neq 0$

All **E**: **orbit?**

Turning

points **?**

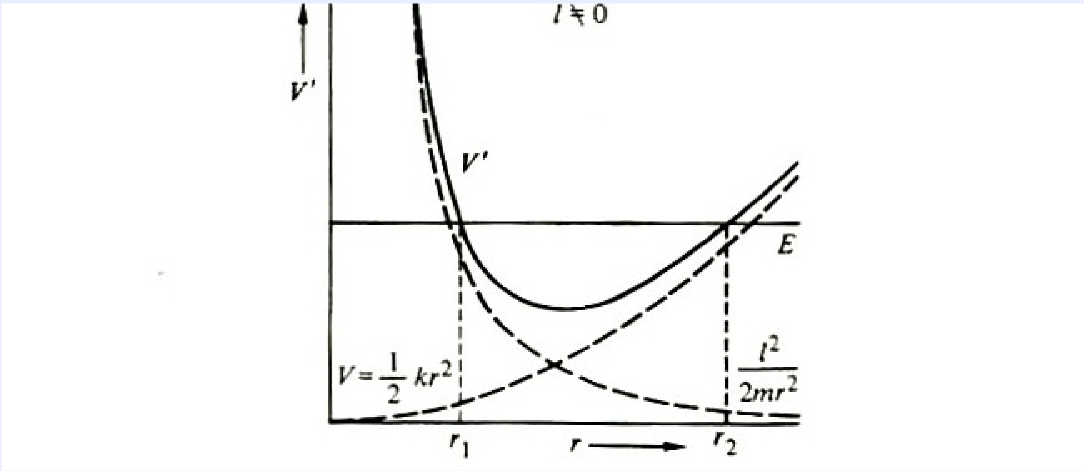
Motion?



Isotropic Simple Harmonic Oscillator:

$$\mathbf{f(r) = - kr, \; V'(r) = (\tfrac{1}{2})kr^2 + (\tfrac{1}{2})[\ell^2/(mr^2)]}$$

- $\ell \neq \mathbf{0} \Rightarrow$ (fig):



All **E**: **Bounded orbit.** Turning points $\mathbf{r_1}$ & $\mathbf{r_2}$.

$\mathbf{E} - \mathbf{V'(r)} = \mathbf{(\tfrac{1}{2})\, m\dot{r}^2 > 0}$

Does not pass through **r = 0**

$\Rightarrow$ Oscillates in **r** between $\mathbf{r_1}$ & $\mathbf{r_2}$. **Motion in plane (r(θ)) is elliptic.**

Question

Find the force law for a central force field that allows a particle to move in a spiral orbit given by $r=k\theta^2$, where k is a constant.

Orbits

$$(d\theta/dr) = (\dot{\theta}/\dot{r})$$

$$\ell \equiv mr^2\dot{\theta} = \text{const} \Rightarrow \dot{\theta} = [\ell/(mr^2)]$$

$$\dot{r} = \pm(\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{1/2}$$

$$\Rightarrow (d\theta/dr) = \pm[\ell/(mr^2)](\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{-1/2}$$

Or:

$$(d\theta/dr) = \pm(\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}$$

Integrating this will give $\theta(r)$.

Solution

$$r = k\theta^2$$

$$\left[\frac{dr}{d\theta}\right]^2 = 4k^2\theta^2 = 4kr$$

Solution

$$r = k\theta^2$$

$$\left[\frac{dr}{d\theta}\right]^2 = 4k^2\theta^2 = 4kr$$

$$4kr = \frac{2\mu r^4}{\ell^2}\left[E - U - \frac{\ell^2}{2\mu r^2}\right]$$

$$U = E - \frac{2k\ell^2}{\mu}\frac{1}{r^3} - \frac{\ell^2}{2\mu}\frac{1}{r^2}$$

$$\boxed{F(r) = -\frac{\ell^2}{\mu}\left[\frac{6k}{r^4} + \frac{1}{r^3}\right]}$$

Orbit Eqn

$$(d\theta/dr) = \pm (\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}$$

- Integrating this gives:

$$\theta(r) = \pm \int (\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}\, dr$$

- **Once the central force is specified, we know V(r) & we can, in principle, do the integral & get the orbit θ(r), or, r(θ).**

We have reduced the original 6d problem of 2 particles to a 2d problem with only 1 degree of freedom. The solution can be obtained simply by doing the above (1d) integral!

(integral can always be done. Usually numerically.)

General Eqn for orbit (***any*** Central Potential **V(r)**) is:

$$(2m)^{1/2}\theta(r) = \pm \int (\ell/r^2)dr/D(r) \qquad (1)$$

$$D(r) \equiv [E - V(r) - \{\ell^2/(2mr^2)\}]^{1/2}$$

- In general, **(1)** must be evaluated numerically.

True for most **power law forces:**

$$f(r) = kr^n \,;\, V(r) = kr^{n+1}$$

- For a few integer & fractional values of **n**, can be done analytically.

Another approach → ***differential eqtn* for the orbit!**

Equation of the Orbit

Start with the equation of motion in terms of forces, and transform it using a couple of tricks. Radial eqn.

$$\mu \ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}.$$

First change variables from r to $u = 1/r$.

Second convert the differential operator d/dt in terms of $d/d\phi$:

Equation of the Orbit

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}.$$

Find $\ddot{r}$

Equation of the Orbit

$$\dot{r} = \frac{d}{dt}(r) = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\frac{1}{u} = -\frac{\ell}{\mu}\frac{du}{d\phi}$$

Equation of the Orbit

$$\dot{r} = \frac{d}{dt}(r) = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\frac{1}{u} = -\frac{\ell}{\mu}\frac{du}{d\phi}$$

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\left(-\frac{\ell}{\mu}\frac{du}{d\phi}\right) = -\frac{\ell^2 u^2}{\mu^2}\frac{d^2 u}{d\phi^2}.$$

Equation of the Orbit

$$\dot{r} = \frac{d}{dt}(r) = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\frac{1}{u} = -\frac{\ell}{\mu}\frac{du}{d\phi}$$

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\left(-\frac{\ell}{\mu}\frac{du}{d\phi}\right) = -\frac{\ell^2 u^2}{\mu^2}\frac{d^2 u}{d\phi^2}.$$

$$\mu\ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}.$$

$$-\mu\frac{\ell^2 u^2}{\mu^2}\frac{\partial^2 u}{\partial\phi^2} = F(r) + \frac{\ell^2 u^3}{\mu} \qquad \text{or}$$

$$\boxed{u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}F(r).}$$

- Usually, given **f(r),** we use the integral formulation.
- However, **differential eqn are most useful for**

The Inverse Problem:

≡ ***Given a known orbit r(θ) or θ(r), determine the force law f(r).***

Problems

1: Find the force law for a central force field that allows a particle to move in a **some orbit** given by **r ~ f(θ)**.

2: Find **r(t)** and **θ(t)** for the same case.

3: What is the total energy of the orbit for the same case?