Q.1 Find the first order partial derivatives of the following functions at the point (x, y) from the first principles / definitions.

(i) $f(x,y) = x^2 + y^2 + x$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + y^2 + x + h - (x^2 + y^2 + x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + h^2 + 2xh + \cancel{y^2} + \cancel{x} + h - \cancel{x^2} - \cancel{y^2} - \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{h(2x+1) + h^2}{h}$$

$$= \lim_{h \to 0} 2x + 1 + h$$

$$\therefore \boxed{\frac{df}{dx} = 2x + 1}$$

$$\frac{df}{dy} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + (y+h)^2 + x - (x^2 + y^2 + x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + \cancel{y^2} + h^2 + 2yh + \cancel{x} - \cancel{x^2} - \cancel{y^2} - \cancel{x}}{h}$$

$$\therefore \frac{df}{dy} = \lim_{h \to 0} \frac{h(2y+h)}{h}$$

$$= \lim_{h \to 0} 2y + h$$

$$\therefore \boxed{\frac{df}{dy} = 2y}$$

(ii) $f(x,y) = \sin(2x+3y)$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{\sin[2(x+h)+3y] - \sin(2x+3y)}{h}$$

$$= \lim_{h \to 0} \frac{2\cos(2x+3y+h) \cdot \sin h}{h}$$

$$\left(\because \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) \right)$$

$$\therefore \frac{df}{dx} = \lim_{h \to 0} 2\cos(2x+3y+h) \lim_{h \to 0} \frac{\sin h}{h}$$

($\because$ Provided the limit exists)

$$\therefore \boxed{\frac{df}{dx} = 2\cos(2x+3y)}$$

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\sin[2x + 3(y+h)] - \sin(2x+3y)}{h}$$

$$= \lim_{h\to 0} \frac{2\cos(2x+3y+3h/2)\sin 3h/2}{h}$$

$$= \lim_{h\to 0} 3\cos(2x+3y+3h/2) \cdot \lim_{h\to 0} \frac{\sin 3h/2}{3h/2}$$

$$\therefore \boxed{\frac{\partial f}{\partial y} = 3\cos(2x+3y)}$$

Q.2 Show the function

$$f(x,y) = \begin{cases} (x+y)\sin\frac{1}{x+y} & ; x+y \neq 0 \\ 0 & ; x+y = 0 \end{cases}$$

is continuous at (0,0) but its partial derivatives f_x and f_y do not exist at (0,0).

we have, $\forall \varepsilon > 0 \; \exists \delta > 0$.

$$|f(x,y) - f(0,0)| = \left|(x+y)\sin\left(\frac{1}{x+y}\right)\right| \leq |x+y|$$

$$\leq |x| + |y|$$

$$\leq \sqrt{x^2} + \sqrt{y^2}$$

$$\leq \sqrt{x^2+y^2} + \sqrt{x^2+y^2}$$

$$\leq 2\sqrt{x^2+y^2} < 2\delta < \varepsilon$$

for $\delta < \frac{\varepsilon}{2}$

So, $|f(x,y) - f(0,0)| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$.

Therefore

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

Hence the given function is continuous at $(0,0)$.

Now,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h\sin(1/h) - 0}{h}$$

$$= \lim_{h\to 0} \sin\left(\frac{1}{h}\right) \text{ does not exist.}$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{k\sin(1/k) - 0}{k}$$

$$= \lim_{k\to 0} \sin\frac{1}{k} \text{ does not exist.}$$

Therefore, the partial derivatives f_x and f_y do not exist at $(0,0)$.

Q.3 Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2} & ; \ (x,y) \neq (0,0) \\ 0 & ; \ (x,y) = (0,0) \end{cases}$$

is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exist at $(0,0)$.

· choose the path $y = mx$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{mx^2}{(1+2m^2)x^2} = \frac{m}{1+2m^2}$$

depends on m, so the function is not continuous at $(0,0)$.

Now,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

Therefore, the partial derivatives f_x and f_y exist at $(0,0)$.

Q.4 Show that the function,

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & ; \ (x,y) \neq (0,0) \\ 0 & ; \ (x,y) = (0,0) \end{cases}$$

has partial derivatives $f_x(0,0)$, $f_y(0,0)$, but the partial derivatives are not continuous at $(0,0)$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

$$f_x(x,y) = \frac{\partial}{\partial x}\left[\frac{xy}{\sqrt{x^2+y^2}}\right]$$

$$= y\left[\frac{\sqrt{x^2+y^2}\cdot 1 - x\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x}{x^2+y^2}\right]$$

$$= y\left[\frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2}\right]$$

$$= y\left[\frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}}\right]$$

$$\therefore f_x(x,y) = \frac{y^3}{(x^2+y^2)^{3/2}}$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left[\frac{x\cdot y}{\sqrt{x^2+y^2}}\right]$$

$$= x\left[\frac{\sqrt{x^2+y^2}\cdot 1 - y\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y}{x^2+y^2}\right]$$

$$= x\left[\frac{\sqrt{x^2+y^2} - \frac{y^2}{\sqrt{x^2+y^2}}}{x^2+y^2}\right]$$

$$\therefore f_y(x,y) = x\left[\frac{x^2+y^2-y^2}{(x^2+y^2)^{3/2}}\right]$$

$$= \frac{x^3}{(x^2+y^2)^{3/2}}$$

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{3/2}}$$

(along the path $y = mx$)

$$= \lim_{x\to 0} \frac{m^3x^3}{(x^2+m^2x^2)^{3/2}} = \cancel{\lim_{x\to 0}} \frac{m^3}{(1+m^2)^{3/2}}$$

which is different for different values of m. So, the limit does not exist. So f_x is not continuous.

Similarly,

$$\lim_{(x,y)\to(0,0)} f_y(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3}{(x^2+y^2)^{3/2}}$$

(along the path $y = mx$)

$$= \lim_{x\to 0} \frac{x^3}{(x^2+m^2x^2)^{3/2}} = \frac{1}{(1+m^2)^{3/2}}$$

which is different for different values of m. So, limit does not exist. So, f_y is not continuous

Q.5 let $f(x,y) = x^2 - xy + y^2 - y$. Find the direction u and the value of $D_uf(1,-1)$ for which,

(i) $D_u f(1,-1)$ is largest

$f(x,y) = x^2 - xy + y^2 - y$

$\vec{\nabla} f = (2x-y)\hat{i} + (-x+2y-1)\hat{j}$

$\vec{\nabla} f(1,-1) = 3\hat{i} - 4\hat{j}$

$|\vec{\nabla} f(1,-1)| = \sqrt{3^2+4^2} = 5$

(i) $D_u f(1,-1)$ will be largest in the direction of $\vec{\nabla} f$.

So, $\hat{u} = \dfrac{\vec{\nabla} f(1,-1)}{|\vec{\nabla} f(1,-1)|} = \dfrac{3}{5}\hat{i} - \dfrac{4}{5}\hat{j}$

largest $D_u f = \vec{\nabla} f \cdot \hat{u}\Big|_{(1,-1)} = (3\hat{i} - 4\hat{j}) \cdot \left(\dfrac{3}{5}\hat{i} - \dfrac{4}{5}\hat{j}\right)$

$= \dfrac{9}{5} + \dfrac{16}{5} = \dfrac{25}{5} = \boxed{5}$

(ii) $D_u f(1,-1)$ is smallest.
$D_u f(1,-1)$ is smallest in the negative gradient direction.

So, the direction $\hat{u} = \dfrac{-3}{5}\hat{i} + \dfrac{4}{5}\hat{j}$

Smallest $D_u f(1,-1) = \vec{\nabla} f \cdot \hat{u} = (3\hat{i} - 4\hat{j}) \cdot \left(\dfrac{-3}{5}\hat{i} + \dfrac{4}{5}\hat{j}\right)$

$= -5$

(iii) $D_u f(1,-1) = 0$.

$D_u f(1,-1) = 0$ in the perpendicular direction to the direction of $\vec{\nabla} f$.

So, $\hat{u} = \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$

or $\hat{u} = \frac{-4}{5}\hat{i} - \frac{3}{5}\hat{j}$

(iv) $D_u f(1,-1) = 4$

$\vec{u} = u_1\hat{i} + u_2\hat{j}$ unit vector

$|\vec{u}| = \sqrt{u_1^2 + u_2^2} = 1$

$D_u f(1,-1) = 4$

$\Rightarrow \nabla f(1,-1) \cdot \vec{u} = 4$

$\Rightarrow (3\hat{i} - 4\hat{j}) \cdot (u_1\hat{i} + u_2\hat{j}) = 4$

$\Rightarrow 3u_1 - 4u_2 = 4 \Rightarrow u_2 = \frac{-4 + 3u_1}{4} = \frac{3}{4}u_1 - 1$

but $u_1^2 + u_2^2 = 1$

$\Rightarrow u_1^2 + \frac{9}{16}u_1^2 + 1 - \frac{6}{4}u_1 = 1$

$\Rightarrow \frac{25}{16}u_1^2 - \frac{6}{4}u_1 = 0$

$\Rightarrow u_1\left(\frac{25}{16}u_1 - \frac{6}{4}\right) = 0$

$\Rightarrow u_1 = 0$ or $u_1 = \frac{6}{4} \times \frac{16}{25} = \frac{24}{25}$

if $u_1 = 0$ then $u_2 = -1$

so, one vector is $\vec{u} = -\hat{j}$ — (1)

if $u_1 = \frac{24}{25}$, then, $u_2 = \frac{3}{4} \times \frac{24}{25} - 1 = \frac{18}{25} - 1$
$= -7/25$

So, another vector is,

$\vec{u} = \frac{24}{25}\hat{i} - \frac{7}{25}\hat{j}$ — (2)

for both the vector in (1) and (2) $Duf\big|_{(1,-1)} = 4$.

(v) $Duf(1,-1) = -3$.

let $\vec{u} = u_1\hat{i} + u_2\hat{j}$ $\quad |\vec{u}| = \sqrt{u_1^2 + u_2^2} = 1$

$Duf(1,-1) = -3$

$\Rightarrow \nabla f(1,-1)\,\vec{u} = -3$

$\Rightarrow (3\hat{i} - 4\hat{j}).(u_1\hat{i} + u_2\hat{j}) = -3$

$\Rightarrow 3u_1 - 4u_2 = -3 \Rightarrow u_1 = \frac{-3 + 4u_2}{3} = -1 + \frac{4}{3}u_2$

but $u_1^2 + u_2^2 = 1$

$\Rightarrow \left(\frac{4}{3}u_2 - 1\right)^2 + u_2^2 = 1$

$\Rightarrow \frac{16}{9}u_2^2 - \frac{8}{3}u_2 + 1 + u_2^2 = 1$

$$\Rightarrow \frac{25}{9}u_2^2 - \frac{8}{3}u_2 = 0$$

$$\Rightarrow u_2\left(\frac{25}{9}u_2 - \frac{8}{3}\right) = 0$$

$$\Rightarrow u_2 = 0 \quad \text{or} \quad u_2 = \frac{8}{\cancel{3}} \times \frac{\cancel{9}^{3}}{25} = \frac{24}{25}$$

if $u_2 = 0$ then $u_1 = -1$

so, one vector is $\vec{u} = -\hat{i}$ — (1)

if $u_2 = \frac{24}{25}$ then $u_1 = \left(\frac{4}{3}\right)\left(\frac{24}{25}\right) - 1 = \frac{7}{25}$

so, another vector is $\vec{u} = \frac{7}{25}\hat{i} + \frac{24}{25}\hat{j}$

for both the vector in (1) and (2) $Duf|_{(1,-1)} = -3$

Q.6 Is there a direction u in which the rate of change of $f(x,y) = x^2 - 3xy + 4y^2$ at $P(1,2)$ equals 14? Give reason for your answer.

$f(x,y) = x^2 - 3xy + 4y^2$

$P(1,2)$

$\vec{\nabla} f = (2x - 3y)\hat{i} + (-3x + 8y)\hat{j}$

$\vec{\nabla} f \Big|_{(1,2)} = -4\hat{i} + 13\hat{j}$

$\left| \vec{\nabla} f \big|_{(1,2)} \right| = \sqrt{(-4)^2 + (13)^2} = \sqrt{185}$

So, maximum rate of change ~~of~~ is $|\vec{\nabla} f|$.

So, $\sqrt{185}$ is the maximum rate of change of f at (1,2).

So, there is no direction u in which the rate of change of f(x,y) at P(1,2) equals 14.

Q7 The derivative of f(x,y,z) at a point P is greatest in the direction of $\vec{v} = \hat{i} + \hat{j} - \hat{k}$. In this direction the value of the derivative is $2\sqrt{3}$.

(i) What is ∇f at P? Give reason for your answer.

We are given that at $\hat{u} = \frac{\vec{v}}{|\vec{v}|}$, then

$$D_u f = 2\sqrt{3} \text{ at } P. \quad \text{——(1)}$$

$$\hat{u} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} - \hat{k})$$

We also know that,

$$D_u f = \vec{\nabla} f \cdot \hat{u}$$

Again it is given that $D_u f$ is maximum in the direction of $\vec{v}$, and it is fact that maximum rate of change occurs in the gradient direction.

So, the angle between ∇f and $\hat{u}$ is zero.

$$D_u f = |\vec{\nabla} f| \cdot |\hat{u}| \cdot \cos 0$$

$$= |\vec{\nabla} f| \quad \text{——(2)}$$

from (1) and (2)

$$|\vec{\nabla} f| = 2\sqrt{3}$$

$$\Rightarrow \quad \vec{\nabla} f = |\vec{\nabla} f| \cdot \hat{u}$$

$$= 2\sqrt{3}\,(\hat{i} + \hat{j} - \hat{k})\,\frac{1}{\sqrt{3}}$$

$$= 2\hat{i} + 2\hat{j} - 2\hat{k}$$

(ii) What is the derivative of f at P in the direction of $\hat{i} + \hat{j}$?

if $\quad u = \hat{i} + \hat{j}$

$$\hat{u} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$D_u f = \vec{\nabla} f \cdot \hat{u}$$

$$= (2\hat{i} + 2\hat{j} - 2\hat{k}) \cdot \left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$$

$$= \boxed{2\sqrt{2}}$$