① L-1 After 2nd Th Sem Dt. 22.10.18

Complex Variables

Advanced Engineering Mathematics
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Complex numbers, Complex Plane

Dealing with solving $x^2+1=0$
we donot have real solution, but it is a quadratic equation, we solve it

$$x = \pm\sqrt{-1}$$

We denote $\sqrt{-1} = i$ $\Rightarrow$ $\boxed{i^2=-1}$

$$x = \pm i$$

A complex number z is an ordered pair (x,y) of real numbers x & y, written as $z = x+iy$ in practice we write.

$z = (x,y)$ or

x is real part
y is imaginary part.

Two complex numbers are equal if their real parts are equal & their imaginary parts are equal.

Addition of two complex numbers

$z_1 = x_1+iy_1$ $\quad$ $z_2 = x_2+iy_2$

$$z_1+z_2 = (x_1+x_2)+i(y_1+y_2).$$

$$z_1-z_2 = (x_1-x_2)+i(y_1-y_2)$$

$$z_1z_2 = (x_1+iy_1)(x_2+iy_2) = (x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$$

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} = \frac{x_1x_2+y_1y_2+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$$

$$= \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i\,\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}$$

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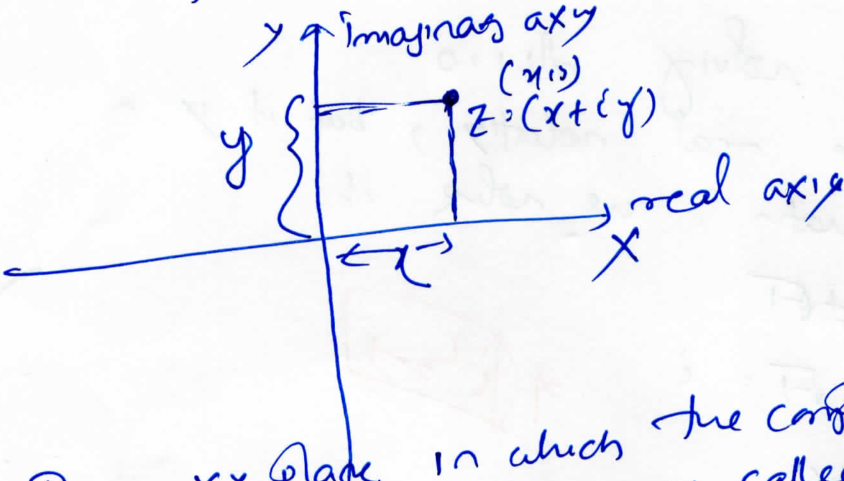
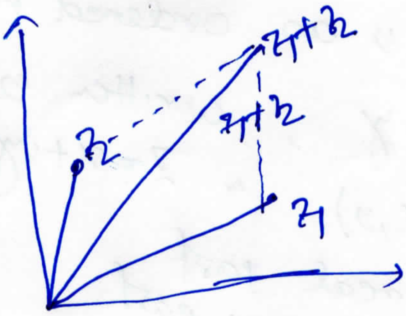
Complex Plane

The geometrical representation of complex numbers as points in a plane.

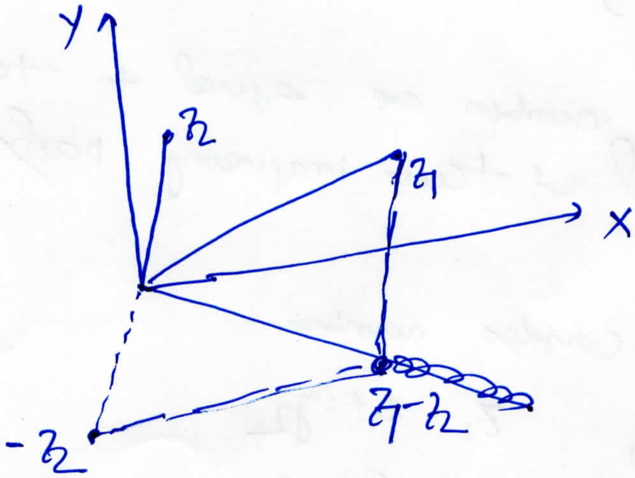


The xy plane in which the complex numbers are represented in this way is called complex plane.

$z_1 + z_2$



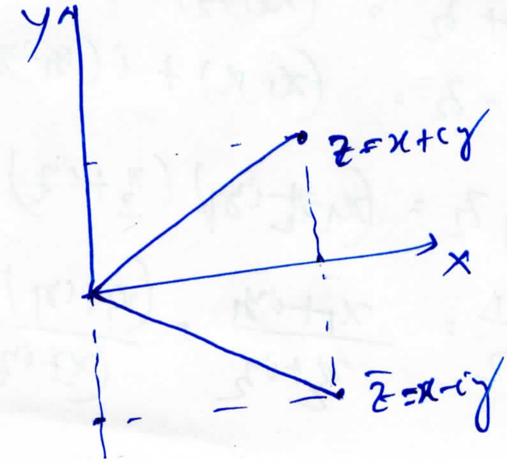
$z_1 - z_2$



Complex conjugate

$z = x + iy$

$\bar{z} = x - iy$



3) $\operatorname{Re} z$ = Real part of $z = x = \frac{z+\bar{z}}{2}$

$\operatorname{Im} z$ = Imaginary part of $z = y = \frac{z-\bar{z}}{2i}$

$$\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$$

$$\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

Polar form of Complex numbers, powers and roots

$z = x+iy$

$x = r\cos\theta$, $y = r\sin\theta$

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Absolute value of z $\quad |e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta+\sin^2\theta} = 1$

$|z| = |re^{i\theta}| = r = \sqrt{x^2+y^2}$

$= \sqrt{z\bar{z}}$

$|z| \to$ distance of the point z from origin.

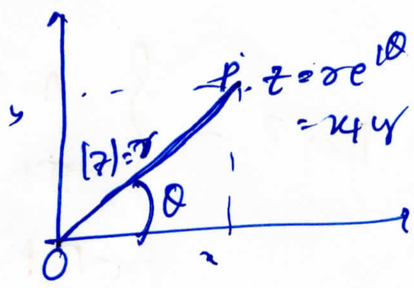
$|z_1 - z_2| \to$ distance between z_1 & z_2.

Argument of z

θ is called the argument of z, denoted by $\arg(z)$.

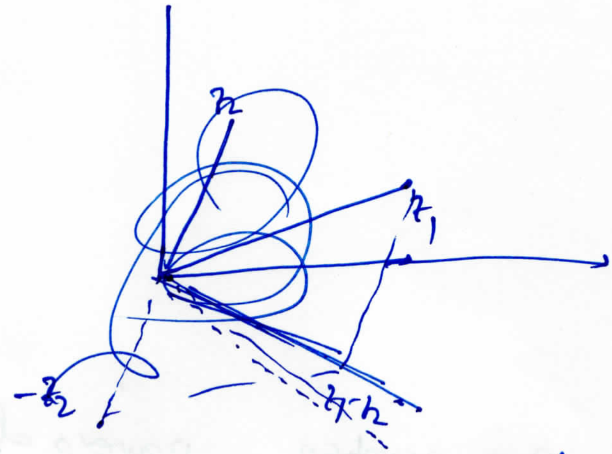
$\theta = \arg z = \arctan \frac{y}{x}$

θ is the directed angle from positive x-axis to OP



All angles are measured in radians, and positive in the counter clockwise.

$|z_1 - z_2|$



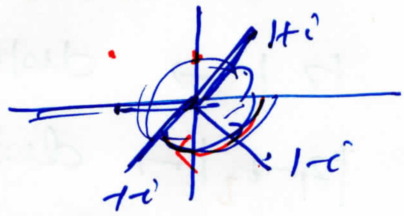
Principal value of the argument of z

The value of θ that lies in the interval $-\pi < \theta \le \pi$ is called the principal value of the argument of ($z \ne 0$).

Denoted by Arg z.

$$-\pi < \text{Arg } z \le \pi$$

Note We must be careful to the quadrant in which z lies.



$z \to (1+i)$

$\arg(1+i) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$\arg(-1-i) = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$ $\quad -\pi + \frac{\pi}{4}$ $\quad -\frac{3\pi}{4}$

$$\text{Arg } z = \begin{cases} \text{Arctan}(y/x) & \text{if } x > 0 \\ \pi + \text{Arctan}(y/x) & \text{if } x < 0 \text{ and } y \ge 0 \\ -\pi + \text{Arctan}(y/x) & \text{if } x < 0,\ y < 0 \\ \frac{\pi}{2} & \text{if } x = 0,\ y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0,\ y < 0 \end{cases}$$

$\text{Arg } i = \frac{\pi}{2}$

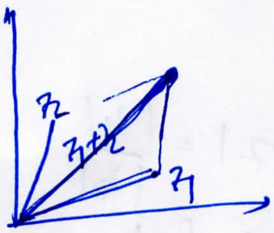
$\text{Arg}(1-i) = -\frac{\pi}{4}$

$\text{Arg}(-1) = \pi$

$\text{Arg}(-1-i) = -\frac{3\pi}{4}$

(5) Triangle inequality

$$|z_1+z_2| \le |z_1|+|z_2|$$



$$|z_1+z_2+z_3| \le |z_1|+|z_2|+|z_3|$$

$$|z_1+\cdots+z_n| \le |z_1|+|z_2|+\cdots+|z_n|.$$

Multiplication and division in polar form

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

multiplication

$z_1 z_2 = r_1 r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right]$

$= r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right] = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$ — (1)

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ — (2)

division The quotient $z = \frac{z_1}{z_2}$ is the number z satisfying $z z_2 = z_1$.

Hence $|z z_2| = |z||z_2| = |z_1|$

$\Rightarrow \arg(z z_2) = \arg(z) + \arg z_2 = \arg z_1$

$\Rightarrow \arg(z) = \arg z_1 - \arg z_2$

$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg z_2$. — (4)

upto multiple of 2π

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$\left|\frac{z_1}{z_2}\right|$ we have

$\arg(z z_2) = \arg(z$

$z = \frac{z_1}{z_2} \Rightarrow \quad z z_2 = z_1$

$|z z_2| = z_1$

$\Rightarrow |z||z_2| = |z_1| \quad \Rightarrow |z| = \frac{|z_1|}{|z_2|}$

$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right].$

De Moivre's formula

when $z_1 = z_2 = \dots = z_n = z$

From (1) & (2)

$z = r(\cos\theta + i\sin\theta)$

$z^n = r^n(\cos n\theta + i\sin n\theta)$

when $|z| = 1$ $\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

Roots

If $z = \omega^n \qquad z \neq 0.$

$\omega = z^{1/n}$ nth root of z.

This is multivalued.

If we write

$z = r(\cos\theta + i\sin\theta)$

and $\omega = R(\cos\phi + i\sin\phi)$

Then $\omega^n = R^n(\cos n\phi + i\sin n\phi) = z = r(\cos\theta + i\sin\theta)$

Equating $R^n = r \Rightarrow R = \sqrt[n]{r}$

where the root is real positive

thus uniquely determined

(7)

Again $n\phi = \theta + 2k\pi$

$\Rightarrow \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$ $\qquad k = 0, 1, 2, \ldots, n-1$.

For $k = 0, 1, 2, \ldots, n-1$, we get n distinct values of ω.

So if $z = r(\cos\theta + i\sin\theta)$

$$z^{1/n} = r^{1/n}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$$

$k = 0, 1, 2, \ldots n-1$.

These n values lie on a circle of radius $r^{1/n}$ with center at the origin and constitute the vertices of a regular polygon of n sides.

If we take $z = 1$, we have $|z| = r = 1$.

and $\operatorname{Arg} z = 0$.

Then $\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, $k = 0, 1, \ldots, n-1$

These n values are called the nth roots of unity.

They lie in a circle of radius 1 and centre 0.

If ω denote the value corresponding to $k = 1$, then the n values of $\sqrt[n]{1}$ can be written as

$1, \omega, \omega^2, \ldots, \omega^{n-1}$.

$\boxed{\omega^n = 1}$, $\boxed{1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0}$

(8) Complex Derivative, Analytic functions

The unit circle $|z|=1$, $\sqrt{x^2+y^2}=1$

$x^2+y^2=1$

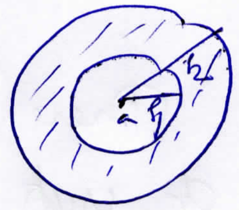
$|z-a| = \rho$. circle with center a, radius ρ.

$|z-a| \le \rho$ closed disk with center a radius ρ.

$|z-a| < \rho$ open disk with center a, radius ρ.

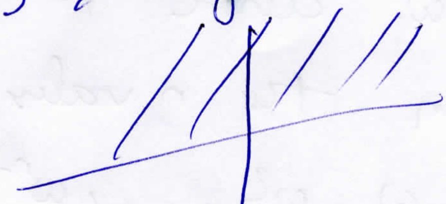
An open circular disk $|z-a|<\rho$ is called a neighborhood of a.

Open annulus $\rho_1 < |z-a| < \rho_2$



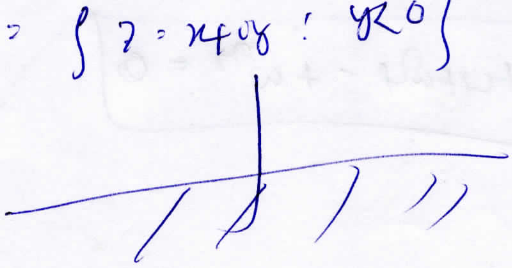
Half planes Upper half plane

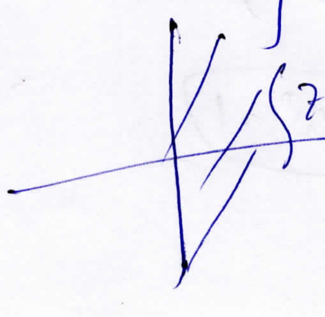
= set of all points $z = x+iy$ such that $y>0$



Lower half plane

$= \{z = x+iy : y<0\}$





$\{z = x+iy : x>0\}$ Right half plane

Left half plane

9 Complex function

Let $D \subset \mathbb{C}$ D subset of complex plane

$$f : D \to \mathbb{C}$$

is called a complex funct

$$w = f(z).$$

z varies in D.

z is complex variable

D is called domain of f.

The set of all values of f is called range of f.

We write

$$w = f(z) = \underbrace{u(x,y)}_{\text{real part}} + i\underbrace{v(x,y)}_{\text{imaginary part}}$$

Exn

$$w = f(z) = z^2 + 3z$$

$$u(x,y) = Re(f(z)) = x^2 - y^2 + 3x$$

$$v(x,y) = Im(f(z)) = 2xy + 3y.$$

Limit, Continuity

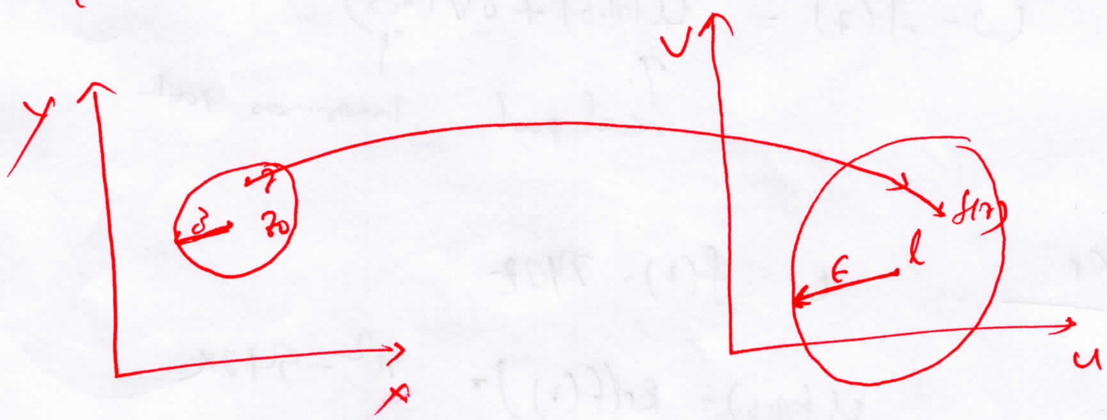
A function $f(z)$ is said to have the limit l as z approaches a point z_0, written as

$$\lim_{z \to z_0} f(z) = l$$

If f is defined in a nbd of z_0 (except perhaps at z_0 itself) and if the values of f are close to l for all z close to z_0.

That is
For every $\epsilon > 0$, $\exists$ $\delta > 0$ such that for all $z \neq z_0$ in the disk $|z - z_0| < \delta$, we have

$$|f(z) - l| < \epsilon.$$



If a limit exists it is unique.

Continuous function

A function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0).$$

$f(z)$ is continuous in a domain if it is continuous at each point of the domain.

L-3 Complex
Dt. 28.10.'18 (1)

Derivative

The derivative of a complex funct f at a point z_0 is written as $f'(z_0)$ and is defined by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

provided this limit exists.

Then f is said to be differentiable at z_0.

If we write $z - z_0 = \Delta z$

Then $$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Properties

$(cf)' = cf'$ $\quad (fg)' = f'g + fg'$

$(f+g)' = f'+g'$ $\quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ $\quad g \neq 0$.

$(f-g)' = f'-g'$

Chain rule

$(z^n)' = nz^{n-1}$

If $f(z)$ is differentiable at z_0, it is continuous at z.

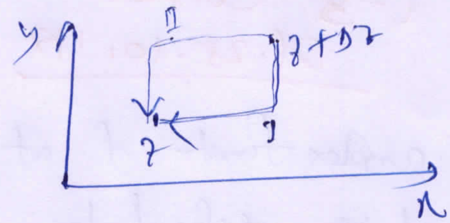
$\bar{z}$ is not differentiable

$f(z) = \bar{z} = x - iy$

If we write
$\Delta z = \Delta x + i\Delta y$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$, this is $+1$
if $\Delta x = 0$, this is -1



Thus $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \to +1$ along path when $\Delta y = 0$

$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \to -1$ along the path $\Delta x = 0$.

Hence the limit does not exist.

$\Rightarrow$ funct is not differentiable at any z.

Analytic function

D open connected set

A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D.

The function $f(z)$ is said to be analytic at a point $z_0 \in D$ if $f(z)$ is ~~analytic~~ differentiable in a neighbour of z_0.

By analytic function we mean a function that is analytic in some domain.

(Connected no separation)

Note **Domain**
An open connected set is called a domain.

Region A region is a set consisting of a domain plus, perhaps some or all of its boundary points.

(3)

Analyticity of $f(z)$ at z_0 means $f(z)$ has a derivative at every point in some nbd of z_0 (including z_0 itself).

This concept is motivated by the fact that it is of no practical interest if a function is differentiable merely at a single point z_0 but not throughout some nbd of z_0.

Ex: All polynomial in z are analytic functions

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + z + c$$

$\sin z$, $\cos z$, e^z are analytic functions in the whole complex plane.

$$f(z) = \frac{g(z)}{h(z)} \quad \text{where } g(z) \text{ \& } h(z) \text{ are polynomials}$$

then $f(z)$ is analytic except at the points where $h(z) = 0$.

Ex:

$f(z) = \operatorname{Re} z$ $\quad z = x+iy \quad \operatorname{Re} z = x$

$\Delta z = \Delta x + i\Delta y$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\operatorname{Re}(z+\Delta z) - \operatorname{Re} z}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\cancel{x} + \Delta x - \cancel{x}}{\Delta x + i\Delta y}$$

On the direction If $\Delta y = 0$, then $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = 1$

On the direction $\Delta x \to 0$, then $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = 0$

Two different limit

So function is not differentiable at any $z \in \mathbb{C}$.

$\Rightarrow f(z)$ is not analytic at any $z \in \mathbb{C}$

(4)

Tutorial

Ex7 Show that $f(z) = |z|^2$ is differentiable only at $z=0$; hence it is nowhere analytic.

Cauchy-Riemann Equations Laplace's equation

Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some nbd of a point $z = x+iy$ and ~~differentiable~~ analytic at z itself. Then at that point the 1st order partial derivatives of u and v exist and satisfy the Cauchy Riemann equations,

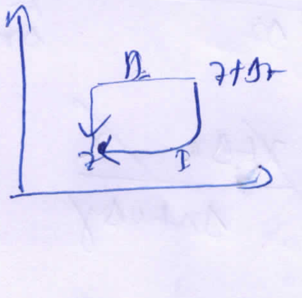
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proof

By assumption the derivative $f'(z)$ of z exists,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

By definition of a limit in complex plane, we can let $\Delta z \to 0$ along any path in a nbd of z.



Write $\Delta z = \Delta x + i\Delta y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

If we let $\Delta y \to 0$ first and then $\Delta x \to 0$

After Δy is zero, $\Delta z = \Delta x$

Then
$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta y}$$

Since $f'(z)$ exists, both limits exist

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} \quad \text{— (4)}$$

Similarly if we choose path II, we let $\Delta x \to 0$ first and then $\Delta y \to 0$

After $\Delta x \to 0$, $\Delta z = i\Delta y$

$$\text{Then } f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{(5)}$$

Comparing (4) & (5)

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Exp $f(z) = z^2$ is analytic ~~for~~ all z

$$f(z) = (x+iy)^2 = x^2 - y^2 + i\,2xy$$

$$u(x,y) = x^2 - y^2, \quad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$p \Rightarrow q$
$\sim q \Rightarrow \sim p$.

Exp $f(z) = \bar{z} = x - iy$

$u = x, \quad v = -y$

$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1$

$\Rightarrow$ Cauchy Riemann equation is not satisfy

$\Rightarrow$ The function is not differentiable.

Theorem
If two real-valued continuous function $u(x,y)$ & $v(x,y)$ have continuous first partial derivatives that satisfy C-R. equation in some domain D. then the function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in D.

Q Is z^3 analytic?

$u = x^3 - 3xy^2, \quad v = 3x^2y - y^3$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial v}{\partial x} = 6xy$

C.R. equation are satisfied & all partial derivative are continuous. Hence $f(z) = z^3$ is analytic in D.

Ex. An analytic function of constant absolute value is constant.

Proof Let $f(z)$ be an analytic function whose modulus is constant.

Let $f(z) = u + iv$

$\Rightarrow |f(z)| = \sqrt{u^2+v^2} = k \Rightarrow u^2+v^2 = k^2.$

Differentiating w.r.t. x

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \text{—(1)}$$

Differentiating w.r.t. y we get

~~Since $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$~~

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad \text{—(2)}$$

Since $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ put in (1)

& $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ put in (2)

(1) becomes $\quad u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \quad$ —(3)

(2) becomes $\quad u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0 \quad$ —(4)

Multiplying (3) with u & (4) by (v)

$$\Rightarrow u^2\frac{\partial u}{\partial x} - uv\cancel{\frac{\partial u}{\partial y}} = 0$$

$$\& \quad uv\cancel{\frac{\partial u}{\partial y}} + v^2\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow (u^2+v^2)\frac{\partial u}{\partial x} = 0 \quad \text{—(5)}$$

Similarly eliminating $\frac{\partial u}{\partial x}$ from (3) & (4) we get

$$(u^2+v^2)\frac{\partial u}{\partial y} = 0 \quad \text{—(6)}$$

~~Since~~ $\kappa^2 = u^2+v^2 = 0$ then $u = v = 0 \Rightarrow f = 0$

$\Rightarrow f$ is const.

If $\kappa^2 = u^2+v^2 \neq 0 \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$

Hence by Cauchy Riemann equation

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} = 0 ,$$

To getere $u = \text{Const}$ and $v = \text{Const}$.

Hence $f(z) = u + iv$ is constant.

Laplace's Equation

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

( ) Laplace equation

U is said to be Harmonic function

if it satisfies Laplace equation.

Note

If given u, that is real part of $f(z) = u + iv$, then we can find v, by using C.R equation

10-4 Dt. 29.10.18 (1)

Exponential function

$$e^z = e^x(\cos y + i\sin y) = e^{x+iy} = e^x \cdot e^{iy}$$

e^z is an entire function.

Entire function: ~~a function~~ which is analytic in the whole complex plane.

Trigonometric function

$$e^{iz} = \cos z + i\sin z$$

$\cos z = \frac{1}{2}(e^{iz}+e^{-iz})$ $\sin z = \frac{1}{2i}(e^{iz}-e^{-iz})$

$\sinh z = \frac{1}{2}(e^{z}-e^{-z})$

$\cosh z = \frac{1}{2}(e^{z}+e^{-z})$

Logarithm

natural logarithm $\ln z$

$\cancel{e^w = z}$ ~~then~~

$\cancel{e^w = e^{u+iv} = re^{i\theta}}$

$\ln z = \ln re^{i\theta}$
$= \ln r + i\theta$

$|z|>0,$
$\theta = \arg z$

$\ln z$ is many valued function

The value of $\ln z$ corresponding to principal value $\operatorname{Arg} z$.

$$\boxed{\operatorname{Ln} z = \ln|z| + i\operatorname{Arg} z}$$

$$\ln z = \operatorname{Ln} z \pm 2n\pi i$$

Complex integration

Line integral in the complex plane

$$\int_C f(z)\,dz$$

The integrand $f(z)$ is integrated over a given curve C in the complex plane. C is called the path of integration.

We may represent a curve C by a parametric representation $z(t) = x(t) + iy(t) \quad (a \le t \le b) \quad -(1)$

The sense of increasing t is called positive sense in C. and we say that in this way orients C.

We subdivide the interval $a \le t \le b$ by pts

$t_0 = a, t_1, \ldots, t_{n-1}, t_n = b$

$t_0 < t_1 < \ldots < t_n$

To this subdivision we have made subdivision to C by points

$z_0, z_1, \ldots, z_{n-1}, z_n = Z.$

$z_j = z(t_j).$

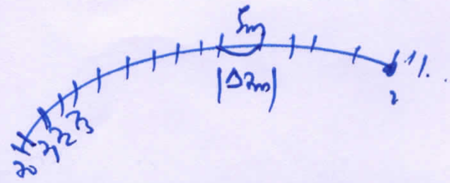
On each portion of subdivision of C we choose arbitrary pt say ξ_1 in (z_0, z_1) $\quad \xi_1(t) = z(t)$ where $t_0 \le t \le t_1$

ξ_2 in (z_1, z_2) and so on

$S_n = \sum_{m=1}^{n} f(\zeta_m)\,\Delta z_m$ $\qquad \Delta z_m = z_m - z_{m-1}$

When $n \to \infty$, the partition is infinitely small

$I = \lim_{n\to\infty} S_n$



If C is a closed path then it is denoted by

$$\oint_C f(z)\,dz$$

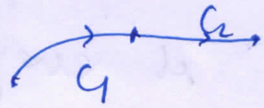
Three basic properties

① $\int_C \left(k_1 f_1(z) + k_2 f_2(z)\right)dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz$

1 ~~two path~~

② $\int_{z_0}^{z} f(z)\,dz = -\int_{z}^{z_0} f(z)\,dz$

③ $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$ if $C = C_1 \cup C_2$



Existence of integral

If $f(z)$ is continuous at C is piecewise smooth, then the integral $\int_C f(z)dz$ exists.

Result

If $f(z)$ is analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0) \quad \{F'(z) = f(z)\}$$

Simply connected domain

A domain D is simply connected if every simple closed curve in D encloses only points of D

Exp

$$\int_0^{1+i} z^2 dz = \frac{z^3}{3}\Big]_0^{1+i} = -\frac{2}{3} + \frac{2}{3}i$$

$$\int_{-i}^{i} \frac{dz}{z} = (\ln z)_{-i}^{i} = \ln i - \ln(-i)$$
$$= i\frac{\pi}{2} - \left(-i\frac{\pi}{2}\right) = i\pi$$

Integration of function over a path

Let C be a piecewise smooth path, represented by $z = r(t)$, $a \le t \le b$. Let $f(z)$ be continuous function on C. Then

$$\int_C f(z)\,dz = \int_a^b f(r(t))\, r'(t)\,dt.$$

Proof

$$\int_C f(z)\,dz \qquad C: z = r(t), \quad a \le t \le b$$

$$= \int_{t=a}^{t=b} f(r(t))\, r'(t)\,dt$$

$z = r(t)$

$\frac{dz}{dt} = r'(t)$

$\Rightarrow dz = r'(t)\,dt$

Ex 1

$$\oint_C \frac{dz}{z}$$

C is the unit circle counterclockwise direction.

Sol

C is $r(t) = e^{it}$, $0 \le t \le 2\pi$

Counterclockwise integration corresponds to an increase of t from 0 to 2π

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{e^{it}}\, i e^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i$$

$z = e^{it}$

$dz = i e^{it}\,dt$

Simply connectedness is essential ?

$$\oint_C \frac{1}{z}\,dz$$

$$\oint_C \frac{1}{z}\,dz = 0 \quad \text{by} \quad \int_{z_0}^{z_0} \frac{1}{z}\,dz$$

simple closed path

but $\frac{1}{z}$ is not analytic at $z=0$.

Any simply connected domain containing the unit circle must contain $z=0$, so that thm 1 does not apply.

So it is not enough that $\frac{1}{z}$ is analytic in an annulus, because an annulus is not simply connected.

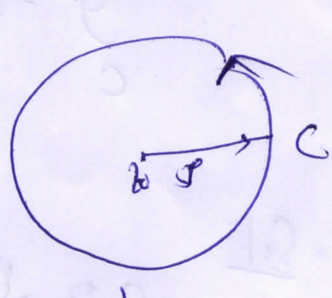
Exp →

$$\int_C \frac{1}{(z-z_0)^m}\,dz$$

m is an integer, z_0 is a constant. $m > 1$

C is the circle with radius ρ with center at z_0.

Direction counterclockwise



Sol^n

$z(t) = z_0 + \rho e^{it} = z_0 + \rho(\cos t + i\sin t)$

Then $(z-z_0)^m$ over C is

$$\int_C \frac{1}{(z-z_0)^m}\,dz = \int_{t=0}^{2\pi} \frac{1}{(\cancel{z_0}+\rho e^{it}-\cancel{z_0})^m}\, i\rho e^{it}\,dt$$

$$= \frac{i}{\rho^{m-1}} \int_0^{2\pi} e^{i(1-m)t}\,dt$$

$$= \frac{i}{\rho^{m-1}} \left[\frac{e^{i(1-m)t}}{i(1-m)} \right]_0^{2\pi} \qquad m > 1$$

$$= \frac{i}{\rho^{m-1}} \left(\frac{e^{-i(m-1)t}}{i(1-m)} \right)_0^{2\pi}$$

$$= \frac{1}{\rho^{m-1}(1-m)} \left[1 - 1 \right) = 0$$

$\int \sin x \, dx$

$\int \frac{e^{ix} - e^{-ix}}{2i} \, dx$

e^{it}, re^{it}

$\cos(m-1)t - i\sin(m-1)t$

$\cos(m-1)2\pi -$

$1 - 0$

$\int_0^{2\pi} i \sin(e^{it}) e^{it} \, dt$

e^{it}

D.T. : 10.16

EXP - 2

$$\oint_C (z - z_0)^m \, dz$$

z_0 const

C : circle with center z_0 radius ρ.

$$r(t) = z_0 + \rho e^{it} = z_0 + \rho(\cos t + i \sin t).$$

$$\oint_C (z - z_0)^m \, dz = \int_0^{2\pi} (z_0 + \rho e^{it} - z_0)^m \, i\rho e^{it} \, dt$$

$$= \int_0^{2\pi} \rho^m e^{mit} \cdot i\rho e^{it} \, dt$$

$$= i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t} \, dt$$

$$= i\rho^{m+1} \left[\int_0^{2\pi} \cos(m+1)t \, dt + i \int_0^{2\pi} \sin(m+1)t \, dt \right]$$

$\cos 0 = 1$, $\sin 0 = 0$

If $m = -1$, we have $\rho^{m+1} = 1$, (the integral value

we thus obtain $2\pi i$

For integer $m \neq -1$, each of the two integrals is zero, because we integrate over an interval of length 2π, equal to period of sine & cosine.

Hence
$$\oint_C (z-z_0)^m\, dz = \begin{cases} 2\pi i & \text{if } m=-1 \\ 0 & (m \neq -1 \text{ and integer}) \end{cases}$$

Note

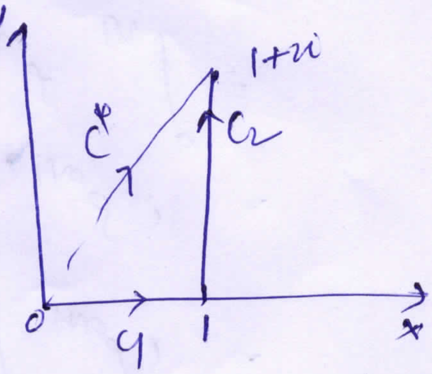
A complex line integral depends on the path also although the end points are same.

Integral of nonanalytic function. Dependence on path

$$\boxed{\begin{array}{c} z_0 \to z_1 \\ r(t) = (1-t)z_0 + t z_1 \end{array}} \quad \boxed{0 \le t \le 1}$$

$\int f(z)\, dz$ $\quad f(z) = \operatorname{Re} z = x$ for 0 to $1+2i$

along path C^* & path $C = C_1 \cup C_2$.



C^* can be represented by
$r(t) = t + 2it \quad 0 \le t \le 1$
$r'(t) = 1+2i$

$$\int_{C^*} \operatorname{Re} z\, dz = \int_0^1 t(1+2i)\, dt = \frac{1}{2}(1+2i) = \frac{1}{2} + i$$

Now $C_1: r(t) = t \qquad r'(t) = 1 \qquad 0 \le t \le 1$

$C_2: r(t) = 1 + it \qquad r'(t) = i \qquad 0 \le t \le 2$

$$\int_{C = C_1 \cup C_2} \mathrm{Re}\, z \, dz = \int_{C_1} \mathrm{Re}\, z \, dz + \int_{C_2} \mathrm{Re}\, z \, dz$$

$$= \int_0^1 t \cdot dt + \int_0^2 1 \cdot i \, dt$$

$$= \frac{t^2}{2}\Big]_0^1 + it\Big]_0^2$$

$$= \frac{1}{2} + 2i$$

So for different paths the values of the integral will be different.

ML- Inequality

$$\left| \int_C f(z)\,dz \right| \le ML$$

L length of C

M is a constant such that $|f(z)| \le M$ $\forall z \in C$.

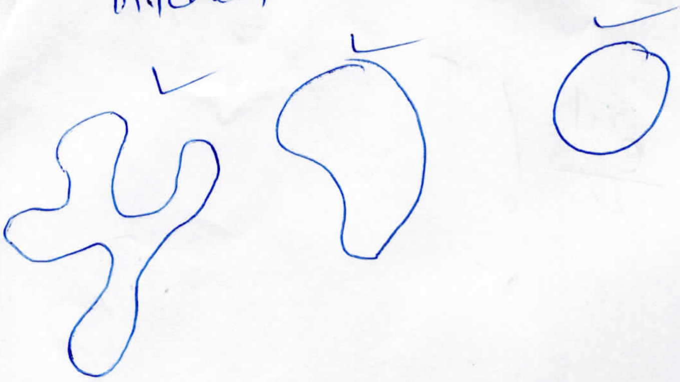
$$|S_n| = \left| \sum_{i=1}^{n} f(\xi_i)\Delta z_i \right| \le \sum_{i=1}^{n} |f(\xi_i)||\Delta z_i|$$

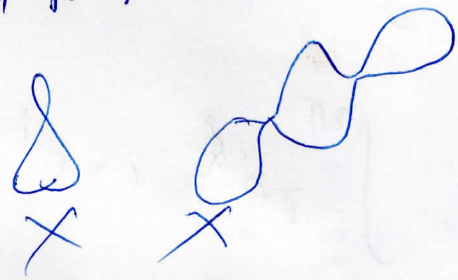
$$\le M \sum_{i=1}^{n} |\Delta z_i| = ML$$

Cauchy Integral Theorem

Dt. 12-11-18 (1)

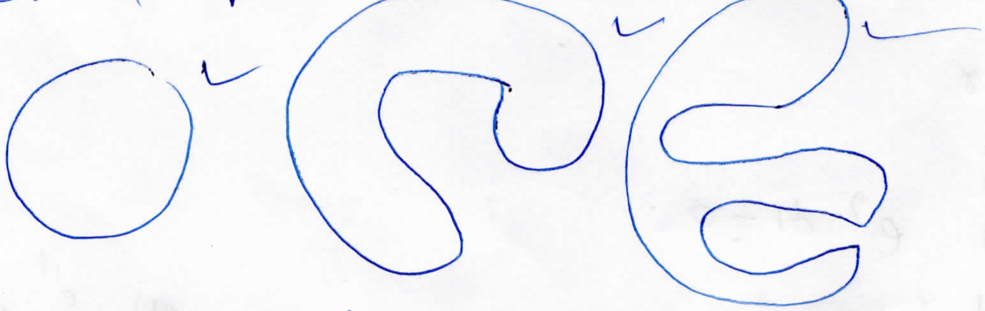
A simple closed path is a closed path that does not intersect or touch itself

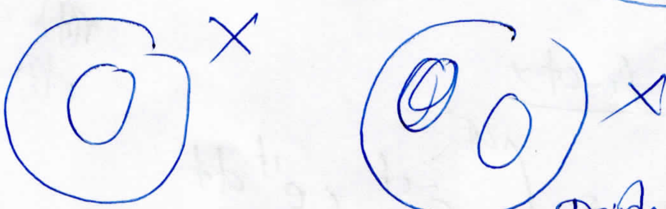




Simply connected domain:

It is a domain such that every simple closed path in D encloses only points of D.





doubly connected

Triply connected.

Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D

$$\oint_C f(z)\,dz = 0.$$

$\sin z \cdot \cos \pi/2 - \cos z \cdot \cos \pi/2$

$\sin(z+\pi/2) = \sin z \cdot \cos \pi/2 + \cos z \cdot \sin \pi/2$

②

Exm) $\oint_C z\,dz \qquad C: |z| = r$

$\gamma(t) = re^{it}$

$\gamma'(t) = rie^{it}$

$= \int_0^{2\pi} re^{it} \cdot ire^{it}\,dt$

$= ir^2 \int_0^{2\pi} e^{2it}\,dt = ir^2 \left[\frac{e^{2it}}{2i}\right]_0^{2\pi}$

$= \frac{\cancel{i}r^2}{2\cancel{i}}\left[e^{4\pi i} - e^0\right] = 0$

Exm-2 $\oint_{|z|=r} \cos z\,dz = 0$

Exm) $\oint_{|z|=r} e^z\,dz = 0.$

For nonanalytic function

$\gamma(t) = re^{it}$

$\gamma'(t) = ire^{it}$

Exm But $\oint_{C,\ |z|=1} \bar{z}\,dz = \int_0^{2\pi} e^{-it} \cdot ie^{it}\,dt$

$= 2\pi i$ which is not zero.

Ex) <u>Analyticity is sufficient, Not necessary</u>

$$\oint_{|z|=1} \frac{dz}{z^2} = 0$$

$f(z) = \frac{1}{z^2}$ is not analytic at ~~z~~ 0. So inside the ~~domain of integration~~ circle.

But $I = \int_0^{2\pi} e^{-2it} \cdot ie^{it}\, dt$ $\qquad r(t) = e^{it}$, $z^2 = e^{2it}$, $r'(t) = ie^{it}$

$= i \int_0^{2\pi} e^{-it}\, dt$

$= i \int_0^{2\pi} (\cos t - i\sin t)\, dt$

$= i \left([\sin t]_0^{2\pi} + i\,[-\cos t]_0^{2\pi} \right)$

$= 0 + i\,[\cos 2\pi - \cos 0] = i\,[1-1] = 0$

$f(z) = \frac{1}{z^2}$ is not analytic still $\oint_{|z|=1} f(z)\,dz = 0$.

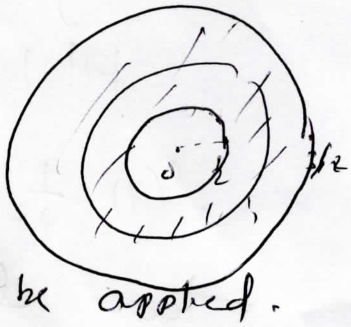
So the condition analyticity is not necessary condition ~~for~~ for the integral to be 0.

(4)

Exp
Simply connected ness is essential

We have seen $\oint_C \frac{dz}{z} = 2\pi i$ $\quad C = |z| = 1.$

If C lies in the annulus $\frac{1}{2} < |z| < \frac{3}{2}$



In the annulus $f(z) = \frac{1}{z}$ analytic

But this domain is not simply connected.
So that Cauchy's theorem cannot be applied.
Hence the condition that the domain D be simply connected is essential.

Independent of path *imp*

If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

Cauchy's Integral formula

Th^n If $f(z)$ is analytic in a simply connected domain D. Then for any point $z_0 \in D$ and any simple closed path C in D that encloses z_0

$$\oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0) \quad \text{--- (1)}$$

the integration being taken counterclockwise.

or $$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\, dz = f(z_0)$$

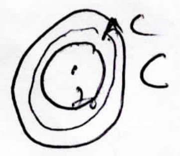
Proof

$$f(z) = f(z_0) + [f(z) - f(z_0)]$$

$$\oint_C \frac{f(z)}{z-z_0}\, dz = \oint_C \frac{f(z_0)}{z-z_0}\, dz + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\, dz$$

$$= f(z_0) \oint_C \frac{1}{z-z_0}\, dz + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\, dz$$

$$\oint_C \frac{1}{z-z_0}\, dz =$$



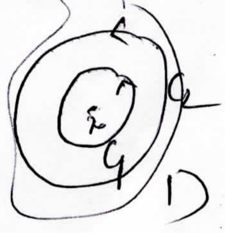
$C: z_0 + \rho e^{it}$

$$\int_0^{2\pi} \frac{1}{z_0 + \rho e^{it} - z_0}\, i\rho e^{it}\, dt$$

$$= 2\pi i$$

Result

$$\int_{C_1} f(z)\, dz = \int_C f(z)\, dz$$



$$\oint_{C_2} \frac{f(z)-f(z_0)}{z-z_0}\, dz$$

The integrand is analytic except at z_0.

Since $f(z)$ is analytic it is continuous

$\Rightarrow |f(z)-f(z_0)| < \epsilon$ when $|z-z_0| < \delta$.

(choose $\rho < \delta$)

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \leq \frac{|f(z)-f(z_0)|}{|z-z_0|} < \frac{\epsilon}{\rho}$$

The length of the C_2 is $2\pi\rho$.

By ML inequality

$$\Rightarrow \left| \oint_{C} \frac{f(z)-f(z_0)}{z-z_0}\, dz \right| < \frac{\epsilon}{\rho}\, 2\pi\rho = 2\pi\epsilon \to 0.$$

$$\Rightarrow \oint \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0).$$

DT- 13-11-18

Exp 1 $\oint_C \frac{e^z}{z-2} dz$

C is any contour enclosing $z_0 = 2$

By Cauchy integral formula

$$\oint_C \frac{e^z}{z-2} dz = 2\pi i f(z_0) = 2\pi i f(2) = 2\pi i e^2$$

Exp 2 $\oint_C$ If C is any contour for which z_0 lies outside

Then inside C $\frac{e^z}{z-2}$ is analytic. So by Cauchy's theorem $\oint_C \frac{e^z}{z-2} dz = 0$.

Exp $\oint_C \frac{z^3-6}{2z-i}$: z^3-6 is analytic

C is a circle at $z_0 = \frac{i}{2}$ lies inside C.

$$= \oint_C \frac{\frac{1}{2}z^3-3}{z-\frac{1}{2}i} dz = 2\pi i \left[f\left(\frac{i}{2}\right)\right]$$

$$= 2\pi i \left[\frac{1}{2}\left(\frac{i}{2}\right)^3 - 3\right]$$

$$= 2\pi i \left(\frac{1}{28}(-i) - 3\right)$$

$$= \frac{\pi}{8} - 6\pi i$$

If $z_0 = \frac{i}{2}$ lies outside C, then

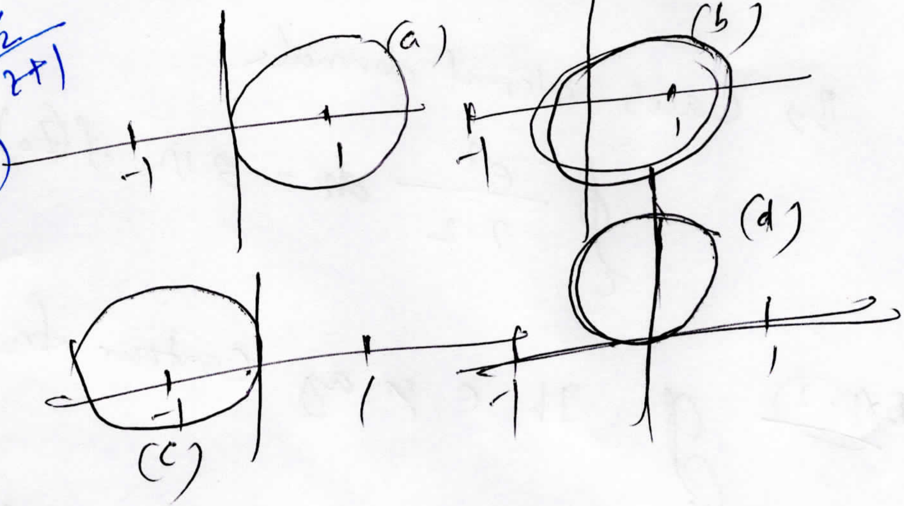
$$\oint_C \frac{z^3-6}{2z-i} dz = 0$$

Ex Integrate (7)

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z+1)(z-1)}$$

Counterclockwise around each of the circles

$$\frac{1}{(z+1)(z-1)} = \frac{\frac{1}{2}}{z-1} - \frac{\frac{1}{2}}{z+1} = \frac{\frac{1}{2}(z+1) - \frac{1}{2}(z-1)}{(z-1)(z+1)}$$



Sol^n (a) The circle $C: |z-1| = 1$

$g(z)$ is not analytic except $z=-1$ & $z=1$.

These are the points we have to take care of.

$C: |z-1| = 1$ encloses $z=1$ where $g(z)$ is not analytic.

$$g(z) = \frac{z^2+1}{(z+1)(z-1)} = \frac{\frac{z^2+1}{z+1}}{(z-1)} = \frac{f(z)}{z-1}$$

$f(z)$ is analytic on and inside C.

$$\int g(z)\,dz = \oint_{|z-1|=1} \frac{\frac{z^2+1}{z+1}}{z-1}\,dz = 2\pi i\, f(1)$$
$$= 2\pi i \left(\frac{1+1}{1+1}\right) = 2\pi i$$

(b) Same result as (a)

(19) (9)

(c) $g(z)$: C: $|z+1|=1$ centre -1, radius 1

C encloses the point -1 at which $g(z)$ is not analytic.

$$g(z) = \frac{z^2+1}{(z-1)(z+1)} = \frac{\frac{z^2+1}{z-1}}{z+1} = \frac{f(z)}{z+1}$$

$f(z)$ is analytic on and inside C.

By Cauchy integral formula

$$\oint_{|z+1|=1} g(z)\,dz = \oint_{|z+1|=1} \frac{\frac{z^2+1}{z-1}}{z+1}\,dz$$

$$= 2\pi i\, f(-1) = 2\pi i\left(\frac{(-1)^2+1}{-1-1}\right)$$

$$= 2\pi i\,(-1) = -2\pi i$$

(d) Both -1 and $+1$ lies outside the circle C.

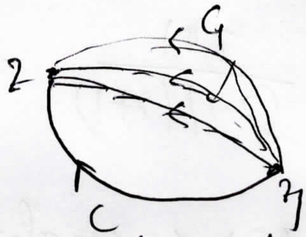
$\Rightarrow g(z)$ is analytic inside and on the circle C.

So by Cauchy Theorem $\oint_C g(z)\,dz = 0$.

Principle of deformation of path

This idea is related to path independence.

We may imagine that the path C_2 was obtained from C_1 by continuously moving C_1 (with ends fixed) until it coincides with C_2.



As long as our deforming path always contains only points at which $f(z)$ is analytic, the integral retains the same value.

This is called the principle of deformation of path.

~~Cauchy's Theorem of multiply connected domain~~

~~Def~~ Derivatives of analytic functions

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

and in general

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz} \quad n = 1, 2, \ldots$$

Here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D and we integrate counterclockwise.

Cauchy's inequality

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$|z-z_0| \le r$

$$\Rightarrow \left|f^{n}(z_0)\right| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right|$$

$\frac{f(z)}{(z-z_0)^{n+1}} \le \frac{M}{r^{n+1}}$

$$\le \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r$$

ML inequality.

$$\Rightarrow \boxed{\left|f^{n}(z_0)\right| \le \frac{n!\, M}{r^{n}}}$$

Liouville's Theorem

Entire function: A function which is analytic in the whole complex plane.

Thm If an entire function $f(z)$ is bounded in absolute value for all z, then $f(z)$ must be a constant.

Morera's Theorem (Converse of Cauchy's Theorem)

If $f(z)$ is continuous in a simply connected domain D and if $\oint_C f(z)\,dz = 0$ for every closed path in D, then $f(z)$ is analytic in D.

15.11.18

Power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + \cdots$$

This power series in powers of $(z-z_0)$

z complex variable

$a_0, a_1, \ldots a_n \ldots$ are complex coefficients.

The power series converges for $|z-z_0| < R$, diverges for $|z-z_0| > R$. circle $|z-z_0| = R$ of convergence

The geometric series:

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

Converges with sum $\dfrac{1}{1-z}$ at $|z| < 1$. & diverges for $|z| \geq 1$.

Ratio test

Let $\sum_{n=0}^{\infty} z_n = z_1 + z_2 + \cdots$

such that $\lim_{n\to\infty} \left| \dfrac{z_{n+1}}{z_n} \right| = L$

(i) The series converges absolutely if $L < 1$

(ii) The series diverges if $L > 1$.

(iii) If $L = 1$ test fails.

Exn $\sum_{n=0}^{\infty} \dfrac{(100+75i)^n}{n!}$.

$$\left| \frac{z_{n+1}}{z_n} \right| = \left| \frac{(100+75i)^{n+1}}{(n+1)!} \times \frac{n!}{(100+75i)^n} \right| = \frac{|100+75i|}{n+1} = \frac{125}{n+1} \to 0 \text{ as } n\to\infty$$

Here $L = 0$

$L < 1$. So the series is ~~absolutely~~ convergent.

Root test

$\sum z_n = z_1 + z_2 + \cdots$

If $\lim_{n\to\infty} |z_n|^{1/n} = L$

if $L<1$, then the series converges absolutely

$L>1$, the series diverges.

If $L=1$, test fails.

Exm $\sum_{n=0}^{\infty} \frac{i^n}{n+i}$

$\lim_{n\to\infty} |z_n|^{1/n} = \lim_{n\to\infty} \left|\frac{i^n}{n+i}\right|^{1/n}$

$= \lim_{n\to\infty} \frac{|i|}{|n+i|^{1/n}}$

$\sum_{n=1}^{\infty} n^2\left(\frac{i}{2}\right)^n$

$\lim_{n\to\infty} \left|n^2\left(\frac{i}{2}\right)^n\right|^{1/n}$

$= \lim_{n\to\infty} \left|n^{2/n} \cdot \frac{i}{2}\right|$

$= \lim_{n\to\infty} (n^{1/n})^2 \left|\frac{i}{2}\right|$

$= 1 \times \frac{1}{2} = \frac{1}{2} < 1$

$\lim_{n\to\infty} n^{1/n} = 1$

$\lim_{n\to\infty} \frac{1}{n}\ln n = \lim_{n\to\infty} \frac{1/n}{1}$

So the series is convergent by root test.

$$\boxed{R = \lim_{n\to\infty} |a_n|^{1/n}}$$

r:x0

Find the radius of convergence of the power

$\frac{n!}{n^n}(z+\pi)^n$ centre $= -\pi$

radius

$\sum_{n=1}^{\infty} \frac{(z-2i)^n}{n^n}$

$a_n = \frac{n!}{n^n}$ $\cancel{a_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}}$

$a_n = n^{-n}$

$\lim_{n} (a_n)^{1/n} = \lim_{n} (n^{-n})^{1/n} = \lim_{n\to\infty} n^{-1} = 0$

$R = 0$ r:x

$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$

$= \lim_{n\to\infty} \frac{(n!)^{1/n}}{(n^n)^{1/n}} = e$

$= \cancel{(n-1)!}$

$\left(\frac{n!}{n^n}\right)^{1/n}$ $2 \cancel{\log} n$

$2^{2^{1/2}}$

$\cancel{\ln} n$

Find the centre of radius of convergence of

① $\sum_{n=1}^{\infty} n(z+i\sqrt{2})^n$

② $\sum_{n=0}^{\infty} \frac{2^{20n}}{n!}(z-3)^n$

③ $\sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n (z-\pi i)^n$

④ $\sum_{n=0}^{\infty} \frac{i^n n^3}{2^n} z^{2n}$

⑤ $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$

Radius of convergence of a power series

$$\boxed{R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|}$$

Ex: Find the radius of convergence of the power series $\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}(z-3i)^n$

Centre $3i$

Sol: $a_n = \frac{(2n)!}{(n!)^2}$

$a_{n+1} = \frac{(2(n+1))!}{((n+1)!)^2}$

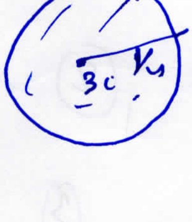
$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{\frac{(2n)!}{(n!)^2}}{\frac{(2n+2)!}{((n+1)!)^2}}$$

$$= \lim_{n\to\infty}\frac{(2n)!}{(n!)^2}\times\frac{((n+1)!)^2}{(2n+2)!}$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)}$$

$$= \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n+2} = \lim_{n\to\infty}\frac{\cancel{n^2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{\cancel{n^2}\left(4+\frac{6}{n}+\frac{2}{n^2}\right)}$$

$$= \frac{1}{4}$$



Power series represent analytic function.

A power series with a nonzero radius of convergence R represents an analytic function at every point interior to its circle of convergence.

Reverse
Every analytic function can be represented by a power series about a point.

Taylor's series

It is an example of a power series about a point z_0 with nonzero radius of convergence.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

or $a_n = \dfrac{1}{2\pi i} \oint_C \dfrac{f(z)}{(z-z_0)^{n+1}} dz$

with counterclockwise integration around a simple closed path C that contains z_0 such that $f(z)$ is analytic on and everywhere inside C.

$$f(z) = f(z_0) + \frac{z-z_0}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z-z_0)^n}{n!} f^{(n)}(z_0) + R_n(z).$$

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)} dz^*$$

T.I.30

geometric series $f(z) = \frac{1}{1-z}$

$$\frac{1}{1-z} = 1+z+z^2+\cdots = \sum_{n=0}^{\infty} z^n \quad \text{when } |z|<1$$

$$\frac{1}{1+z} = 1-z+z^2-z^3+\cdots = \sum_{n=0}^{\infty} (-1)^n z^n \quad \text{when } |z|<1$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots$$

Find the Maclaurin series of $f(z) = \tan^{-1} z$

$$f'(z) = \frac{1}{1+z^2} = (1+z^2)^{-1} = 1 - z^2 + z^4 - z^6 + \cdots$$

Integrate term by term

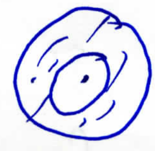
$$f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}$$

②

Exp 1 Find the Laurent series of

(a) $z^2 e^{1/z}$ with center 0.



$$z^2 e^{1/z} = z^2\left(1 + \frac{\frac{1}{z}}{1!} + \frac{\frac{1}{z^2}}{2!} + \frac{\frac{1}{z^3}}{3!} + \cdots\right)$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{3!\,z} + \frac{1}{4!\,z^2} + \frac{1}{5!\,z^3} + \cdots$$

Exp-2

$$f(z) = \frac{1}{z^3 - z^4} = \frac{①}{}$$

$$= \frac{1}{z^3(1-z)} = \frac{1}{z^3}(1-z)^{-1}$$

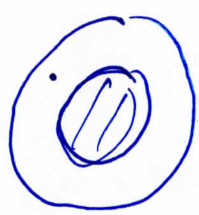
$$= \frac{1}{z^3}\left(1 + z + z^2 + z^3 + \cdots\right)$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$$

Exp-3

$$f(z) = \frac{-2z+3}{z^2 - 3z + 2} \quad \text{with center 0.}$$

$$\frac{1}{z^2 - 3z + 2} = \frac{-1}{z-1} + \frac{1}{z-2} \qquad \cancel{\frac{(z-2) - (z-1)}{(z-1)(z-2)}} =$$



$$f(z) = (-2z+3)\left(-\frac{1}{z-1} + \frac{1}{z-2}\right)$$

$$= -(-2z+3)\frac{1}{z-1} + (-2z+3)\frac{1}{z-2}$$

$$= (-2z+3)\frac{1}{1-z} + (-2z+3)\frac{1}{2-z}$$

$$= (-2z+3)\frac{1}{1-z} + (-2z+3)\frac{1}{2(1-\frac{z}{2})}$$

for $|z| < 1$

& $\left|\frac{z}{2}\right| < 1$ $\Rightarrow |z| < 2$

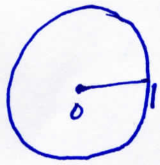
$$= (-2z+3)(1 + z + z^2 + \cdots) + (-2z+3)\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right)$$

Q Find Laurent expansion of $\frac{-2z+3}{z^2-3z+2}$ with centre 0. Dt - 19.11.18 (3)

Sol By Partial fraction

$$f(z) = -\frac{1}{z-1} - \frac{1}{z-2}$$

$$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{for } |z|<1 \quad \text{--- (1)}$$



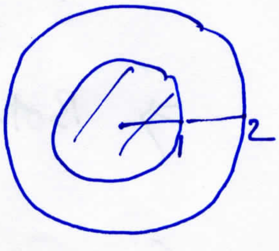
$$-\frac{1}{z-2} = \frac{1}{2-z} = \frac{1}{2(1-\frac{z}{2})}$$

$$= \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} = \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right) = \frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\cdots$$

For $\left|\frac{z}{2}\right|<1$

$\Rightarrow |z|<2$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \quad \text{--- (2)}$$



$$\text{So } f(z) = \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \quad \text{for } |z|<1$$

$$= \sum_{n=0}^{\infty}\left(1+\frac{1}{2^{n+1}}\right) z^n$$

Now in $1<|z|<2$

$$-\frac{1}{z-1} = \frac{-1}{z(1-\frac{1}{z})} = \frac{-1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

for $\left|\frac{1}{z}\right|<1$

$\Rightarrow |z|>1$

$$= \frac{-1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

$$= -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}$$

$$= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \quad \text{--- (3)}$$

From (2) & (3), both the series in (2) & (3) valid in $1<|z|<2$

$$\therefore f(z) = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n$$

(4)

$\underline{|z|>2}$ $\quad \frac{-1}{z-1} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$ for $|z|>1$ — (4)

& $-\frac{1}{z-2} = \frac{-1}{z(1-\frac{2}{z})} = -\frac{1}{z}(1-\frac{2}{z})^{-1}$

$= -\frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots\right)$ for $|\frac{2}{z}|<1$ $\Rightarrow |z|>2$

$= -\left(\frac{1}{z}+\frac{2}{z^2}+\frac{2^2}{z^3}+\cdots\right)$

$= -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$ for $|z|>2$

— (5)

(4) is valid for $|z|>2$

Also (4) is valid for $|z|>1$ then it is also valid for $|z|>2$

$\Rightarrow$ Both (4) & (5) are valid in $|z|>2$.

So $f(z) = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$

$= -\sum_{n=0}^{\infty} (1+2^n) \frac{1}{z^{n+1}}$.

Singularities

A function $f(z)$ has a singular point at $z = z_0$ if it is not analytic (perhaps not even defined) at $z = z_0$, but every nbd of z_0 contains points at which $f(z)$ is analytic.

We say z_0 is a singular point.

$f(z) = \frac{1}{z}$ $\quad z=0$ is a singular point of $\frac{1}{z}$

$f(z) = \frac{1}{z-\pi}$ $\quad z=\pi$ is a " " " $\frac{1}{z-\pi}$

$f(z) = \frac{1}{\sin z}$ $\quad z = \pm n\pi$, $n=0,1,2$ — are all singular points of cosec z

$f(z) = e^{1/z}$ $\quad z=0$ is a singular point.

Isolated singularities

z_0 is said to be an isolated singularity of $f(z)$ if z_0 has a nbd in which there is no other singular points of $f(z)$

Example $f(z) = \frac{1}{z}$, $z=0$ is a isolated singular point

In All the above example the singular points are isolated.

Non isolated

$\frac{1}{\sin 1/z}$ $\quad \sin 1/z = 0 \Rightarrow \frac{1}{z} = n\pi$ $\quad n \in \mathbb{Z}$

$\frac{1}{z} = n\pi$ $\quad n$ is an integer

$z = \frac{1}{n\pi}$ $\quad n = 0, \pm 1, \pm 2, —$

$z=0$ is not isolated singularity because take any nbd of z, we get other singular points.

Classification of isolated singularities

By using Laurent series:

Let z_0 be an isolated singularity of $f(z)$

Let the Laurent series of $f(z)$ is

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{principal part}}$$

Valid in the immediate nbd of z_0 except at z_0 itself that is in the region $0<|z-z_0|<R$.

The sum of the 1st series is analytic at z_0

The 2nd series is called the principal part.

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

If the principal part has only finite no. of terms in the series that is

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

Then z_0 is called a pole.

m is called the order of the pole.

If the principal part has infinite no. of terms in the series, then the singular point z_0 is called essential singularity.

If z_0 is a singular point of $f(z)$ but the Laurent series of $f(z)$ has no principal part, then z_0 is called a removable singularity.

Ex 1.1 $f(z) = \frac{\sin z}{z}$ $\quad z=0$ is a singular point.

Laurent series of $\frac{\sin z}{z}$ about $z=0$

$$= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

The Laurent series has no principal part.

$\Rightarrow$ $z=0$ is a removable singularity.

Ex 1.2 $f(z) = \frac{1}{(z-1)(z+1)}$ $\quad z=1$, $z=-1$ are singular points

$$\frac{1}{(z-1)(z+1)} = \cancel{\frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)} = \frac{1}{(z-1)(z-1+2)}$$

$$= \cancel{\frac{1}{2}\left(\frac{-1}{1-z} - \frac{1}{z+1+2}\right)}$$

$$= \frac{1}{(z-1)} \frac{1}{2\left(1+\frac{z-1}{2}\right)}$$

$$= \frac{1}{2}\frac{1}{z-1}\left(1+\frac{z-1}{2}\right)^{-1} = \frac{1}{2}\frac{1}{z-1}\left(1 - \frac{z-1}{2} + \frac{(z-1)^2}{2^2} - \frac{(z-1)^3}{2^3} + \cdots\right)$$

$$= \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{2} + \frac{(z-1)}{2^2} - \frac{(z-1)^2}{2^3} + \cdots\right]$$

The series is about 1 as a power of $z-1$.

The principal part of this series is $\frac{1}{2}\frac{1}{z-1}$.

So $z=1$ is a pole of order 1.

Similarly $z=-1$ is also a pole of order 1

Residue integration method

Dt: 20.11.17

We want to evaluate $\oint_C f(z)\,dz$ C is simple closed path.

If $f(z)$ is analytic every inside and on C, then integral is 0.

If $f(z)$ has a singularity at z_0 inside C. Then $f(z)$ has Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\,dz$$

$$\oint_C f(z)\,dz = 2\pi i\, b_1$$

b_1, the coefficient of $\frac{1}{z-z_0}$ is called the residue of $f(z)$ at $z = z_0$.

~~If $f(z)$ has a pole of order~~

If z_0 is a pole of order 1, then

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z-z_0) f(z)$$

If z_0 is a pole of order m, then

$$\underset{\text{at } z=z_0}{\operatorname{Res}} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m f(z)\right)$$

If z_0 is a pole of order 2, then

$$\underset{\text{at } z=z_0}{\operatorname{Res}} f(z) = \frac{1}{(2-1)!} \lim_{z \to z_0} \frac{d}{dz} \left((z-z_0)^2 f(z)\right)$$

Residue Theorem

Let $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \ldots z_k$ inside C. ~~Then the integral of f(z) taken~~ Then

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \underset{z=z_j}{\operatorname{Res}} f(z).$$

Exn

$$\oint_C \frac{4-3z}{z^2-z}\,dz = \oint_C \frac{4-3z}{z(z-1)} \qquad C: |z|=2$$

Sol^n The integrand has single pole at $z=0$, at $z=1$. Both are inside C.

By Residue theorem

$$\oint \frac{4-3z}{z^2-z}\,dz = 2\pi i\left[\underset{at\ z=0}{\operatorname{Res}} f(z) + \underset{at\ z=1}{\operatorname{Res}} f(z)\right]$$

$$\underset{z=0}{\operatorname{Res}} f(z) = \lim_{z\to 0} (z-0) f(z) = \lim_{z\to 0} \cancel{z}\cdot\frac{4-3z}{\cancel{z}(z-1)}$$

$$= \frac{4}{-1} = -4$$

$$\underset{at\ z=1}{\operatorname{Res}} f(z) = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \cancel{(z-1)}\frac{4-3z}{z\cancel{(z-1)}} = 1$$

$$\therefore \oint \frac{4-3z}{z^2-z}\,dz = 2\pi i\left[-4+1\right] = -6\pi i$$

If $C = |z| = \frac{1}{2}$

Then the singular pt $z=1$ is outside the circle.

$$\oint \frac{4-3z}{z^2-z}\,dz = 2\pi i\left[\underset{z=0}{\operatorname{Res}} f(z)\right] = 2\pi i \lim_{z\to 0} \cancel{z}\frac{4-3z}{\cancel{z}(z-1)}$$

$$= 2\pi i \lim_{z\to 0} (z-0)\frac{4-3z}{z^2-z} = 2\pi i \times (-4) = -8\pi i$$

Since $z=1$ is outside, $\underset{at\ z=1}{\operatorname{Res}} f(z) = 0$ as there is no principal part in Laurent series about $z=1$.

~~Ex~~ $\oint_C \frac{e^z}{\cos z}\,dz \qquad C: |z| = 3$

$f(z) = \frac{e^z}{\cos z}$ has single poles at

$z = \pm(2n+1)\frac{\pi}{2} \quad n = 0, 1, 2, \ldots$

$z = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{-3\pi}{2}, \ldots$

Only $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ are inside $C: |z| = 3$

rest of all are outside C.

So $\oint_C \frac{e^z}{\cos z}\,dz = 2\pi i\left[\operatorname*{Res}_{z=\frac{\pi}{2}} f(z) + \operatorname*{Res}_{z=-\frac{\pi}{2}} f(z)\right]$

$\operatorname*{Res}_{z=\frac{\pi}{2}} f(z) = \lim_{z\to\frac{\pi}{2}} \left(z-\frac{\pi}{2}\right)\frac{e^z}{\cos z}$

$= \lim_{z\to\frac{\pi}{2}} -\left(z-\frac{\pi}{2}\right)\frac{e^z}{\sin\left(z-\frac{\pi}{2}\right)} = \lim_{z\to\frac{\pi}{2}} -\frac{e^z}{\frac{\sin\left(z-\frac{\pi}{2}\right)}{z-\frac{\pi}{2}}} = -e^{\frac{\pi}{2}}$

$\operatorname*{Res}_{z=-\frac{\pi}{2}} f(z) = \lim_{z\to-\frac{\pi}{2}} \left(z+\frac{\pi}{2}\right)\frac{e^z}{\cos z} = \lim_{z\to-\frac{\pi}{2}} \left(z+\frac{\pi}{2}\right)\frac{e^z}{\sin\left(z+\frac{\pi}{2}\right)}$

$= \lim_{z\to-\frac{\pi}{2}} \frac{e^z}{\frac{\sin\left(z+\frac{\pi}{2}\right)}{\left(z+\frac{\pi}{2}\right)}} = \frac{e^{-\frac{\pi}{2}}}{1} = e^{-\frac{\pi}{2}}$

$\therefore \oint_C \frac{e^z}{\cos z}\,dz = 2\pi i\left[e^{-\frac{\pi}{2}} - e^{\frac{\pi}{2}}\right]$

$= -2\pi i\left[\frac{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}}{2}\right] \times 2 = -4\pi i \sinh\frac{\pi}{2}$

Ex: $\oint_C \frac{z+1}{z^4-2z^3}\,dz$, $C: |z| = \frac{1}{2}$

Sol: $f(z) = \frac{z+1}{z^4-2z^3} = \frac{z+1}{z^3(z-2)}$

$f(z)$ has simple pole at $z=2$ & a pole of order 3 at $z=0$

$z=2$ is outside C.

$$\oint_C \frac{z+1}{z^4-2z^3}\,dz = 2\pi i\left[\operatorname*{Res}_{z=0} f(z) + \cancel{\operatorname*{Res}_{z=2} f(z)}\right]$$

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left((z-0)^3 f(z)\right) \qquad m=3$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left(\cancel{z^3}\,\frac{z+1}{\cancel{z^3}(z-2)}\right)$$

$$= \frac{1}{2}\lim_{z\to 0}\left(\frac{6}{(z-2)^3}\right)$$

$$= \frac{1}{2}\times\frac{6}{-8} = -\frac{3}{8}$$

$g(z) = \frac{z+1}{z-2}$

$= \frac{z-2+3}{z-2}$

$= 1 + \frac{3}{z-2}$

$g'(z) = \frac{-3}{(z-2)^2}$

$g''(z) = (-3)(-2)(z-2)^{-3}$

$= \frac{6}{(z-2)^3}$

$$\cancel{\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}(z-2)\frac{z+1}{z^3(z-2)} = \frac{2+1}{2^3} = \frac{3}{8}}$$

$$\oint_C \frac{z+1}{z^4-2z^3}\,dz = 2\pi i\left[-\frac{3}{8}\right] = -\frac{3}{4}\pi i$$