SC 216 Calculus with Complex Variables Jan 9, 2008

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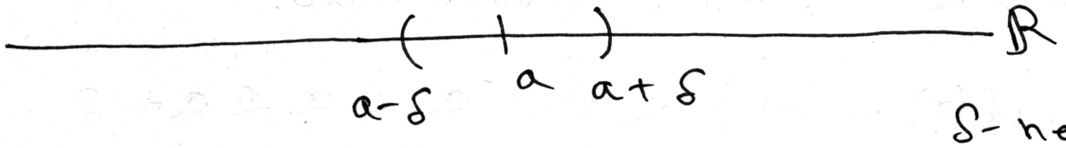
Limit of a fn. on the real line

$a \in \mathbb{R}$ $f(x)$ is a real valued fn.

$\delta > 0$



δ- neighborhood of a

$0 < |x-a| < \delta$ is called deleted nbd. of a

Distance between two pt.s on $\mathbb{R}$

$$d(x_1, x_2) = |x_2 - x_1| \quad , x_1, x_2 \in \mathbb{R}$$

$$f(x) \longrightarrow L \quad (L \in \mathbb{R}) \quad \text{as} \quad x \longrightarrow a$$

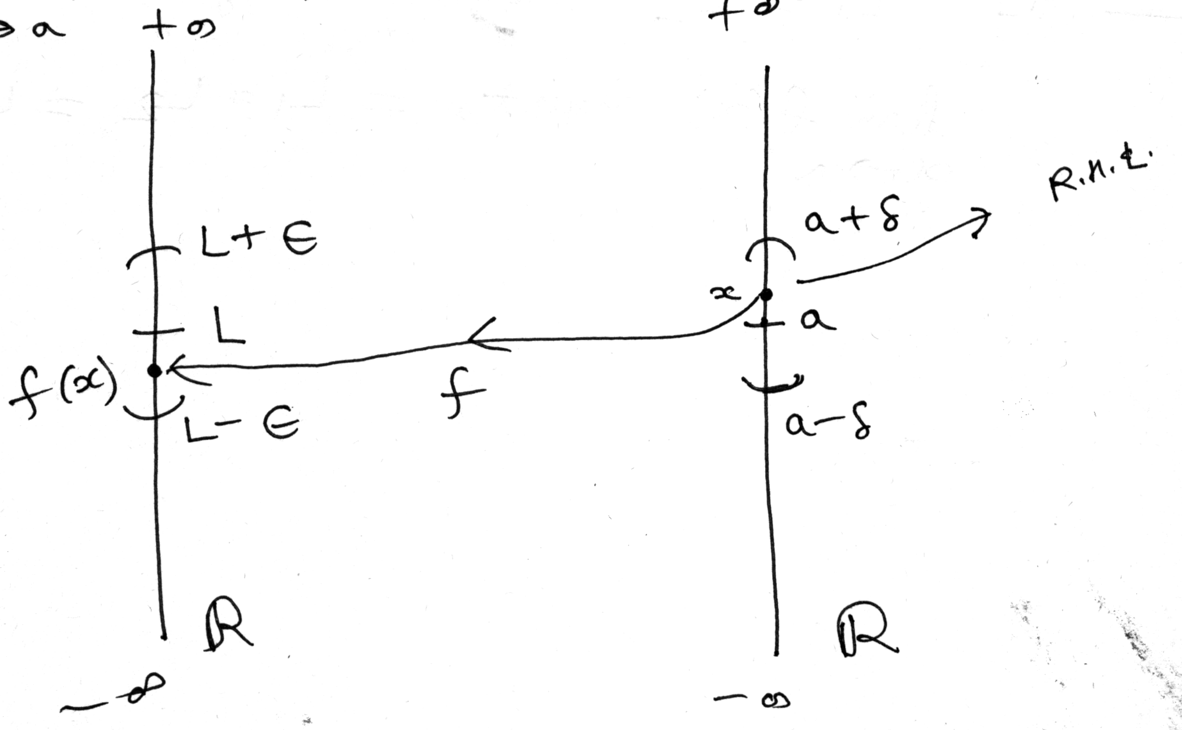
if given $\epsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|f(x) - L| < \epsilon \quad (0 < |x-a| < \delta)$$

$\downarrow$

$a-\delta < x < a+\delta$

$$\lim_{x \to a} f(x) = L$$



— The pt. a need not be in the domain of f.

$$\lim_{x \to 0} \left(\frac{\sin x}{x}\right) = 1$$

$\frac{\sin x}{x}$ is not defined at 0

$\boxed{\text{R.H.L.}}$

$x > a$ $\quad x \to a$ from R.H.S.

$|f(x) - L_1| < \epsilon \quad a < x < a + \delta$

$\lim_{x \to a+} f(x) = L_1$ (Right hand limit)

$\boxed{\text{L.H.L.}}$

$x < a$ $\quad x \to a$ from L.H.S.

$|f(x) - L_2| < \epsilon \quad a - \delta < x < a$

$\lim_{x \to a-} f(x) = L_2$ (Left hand limit)

— If $L_1 = L_2$ then we say limit exist

$\lim_{x \to a} f(x)$ exist $= L_1 = L_2 = L$ (say)

Example $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

$\square$ Suppose $\exists\, L \in \mathbb{R}$ s.t. $\lim_{x\to 0} \sin\frac{1}{x} = L$

Then for $\epsilon = 1$ $\exists\, \delta > 0$ s.t.

$$\left|\sin\frac{1}{x} - L\right| < 1 \quad (0 < |x| < \delta) \quad \text{---(1)}$$

Now $\sin\left(2n\pi + \frac{\pi}{2}\right) = \sin\frac{(4n+1)\pi}{2} = 1$

for any $n \in \mathbb{N}$.

$\Rightarrow \sin\frac{1}{x} = 1$ for $x = \frac{2}{\pi(4n+1)}$

hence for some $x \in (0, \delta)$

$\because \lim_{n\to\infty} \frac{2}{\pi(4n+1)} = 0$ for this x (1)

will give $\boxed{|1 - L| < 1}$ —(2)

Similarly, $\sin\left(2n\pi + \frac{3\pi}{2}\right) = -1$ for $n \in \mathbb{N}$

$\Rightarrow$ there will be $x \in (0, \delta)$ for which

$\sin\frac{1}{x} = -1$ again by (1)

$$\boxed{|-1 - L| < 1} \quad \text{—(3)}$$

(2) & (3) gives contradiction (Why?)

$\Rightarrow \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Plot $\sin\frac{1}{x}$ (Exercise)

~~Example:~~

$f(x) \to L$ as $x \to \infty$ if given $\epsilon > 0$, $\exists\ M \in \mathbb{R}$ s.t.

$$|f(x) - L| < \epsilon \quad (x > M)$$

– $\lim\limits_{x \to \infty} f(x) = L$

Example: $\lim\limits_{x \to \infty} \frac{1}{x^2} = 0$

Given $\epsilon > 0$ we must find $M \in \mathbb{R}$ s.t.

$$\left|\frac{1}{x^2} - 0\right| < \epsilon \quad (x > M)$$

$$\Rightarrow \quad \frac{1}{x} < \sqrt{\epsilon} \quad (x > M)$$

Thus if we choose $M = \frac{1}{\sqrt{\epsilon}}$ we are done.

Example: $f(x) = 3 \quad (0 \le x < 1)$

$\qquad\qquad 3 - x \quad (1 \le x \le 2)$

verify that $\lim\limits_{x \to 1-} f(x) = 1$

$\lim\limits_{x \to 1+1} f(x) = 2$

Example: Show that

$$\lim_{x \to 3} (x^2+2x) = 15$$

$\square$ $f(x) = x^2+2x$ $L = 15$

$a = 3$

Given $\epsilon > 0$ we must find $\delta > 0$ s.t.

$$|(x^2+2x) - 15| < \epsilon \quad (0 < |x-3| < \delta) \quad \text{—— (1)}$$

Now $|(x^2+2x) - 15| = |x-3| \cdot |x+5|$

Choose $\delta < 1 \Rightarrow |x-3| < 1$

$\Rightarrow x \in (2,4)$

& so $x+5 \in (7,9)$

$\Rightarrow |x+5| < 9$ if $|x-3| < \delta < 1$

$\Rightarrow |x-3| \cdot |x+5| < \delta \cdot 9$

if $|x-3| < \delta$ & $\delta < 1$

Let $\delta = \min(1, \frac{\epsilon}{9})$ ✓

Then $|x-3| \cdot |x+5| < 9\delta \le \epsilon$

$\therefore$ This gives (1) $|x-3| < \delta$

$\therefore$ we have found a δ for which (1) holds

$\Rightarrow \lim_{x \to 3} (x^2+2x) = 15$

$\blacksquare$

Example: Show that

$$\lim_{x \to 1} \sqrt{x+3} = 2$$