

1. Calculate the laplacian of the following:

(i) $F = x^2 + 2xy + 3z + 4$ (ii) $F = \sin(\hat{k} \cdot \vec{r})$ (iii) $F = \frac{1}{r}$

soln:

(i) $\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 2$

(ii)

$$\begin{aligned}\nabla^2 F &= \frac{\partial^2}{\partial x^2} \sin(\vec{k} \cdot \vec{r}) + \frac{\partial^2}{\partial y^2} \sin(\vec{k} \cdot \vec{r}) + \frac{\partial^2}{\partial z^2} \sin(\vec{k} \cdot \vec{r}) \\ &= -k_x^2 \sin(\vec{k} \cdot \vec{r}) - k_y^2 \sin(\vec{k} \cdot \vec{r}) - k_z^2 \sin(\vec{k} \cdot \vec{r}) \\ &= -k^2 \sin(\vec{k} \cdot \vec{r})\end{aligned}$$

(iii)

$$\begin{aligned}\nabla^2 \left(\frac{1}{r} \right) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{1}{r} \right) \\ \frac{\partial^2}{\partial x^2} \left(\frac{1}{r} \right) &= \frac{\partial}{\partial x} \left(-\frac{1}{r^2} \frac{\partial r}{\partial x} \right) \\ &= \frac{\partial}{\partial x} \left(-\frac{1}{r^2} \frac{1}{2r} \cdot 2x \right) \\ &= \frac{\partial}{\partial x} \left(-\frac{x}{r^3} \right) \\ &= -x \left(-\frac{3}{r^4} \frac{1}{2r} \cdot 2x \right) - \frac{1}{r^3} \\ &= \frac{3x^2}{r^5} - \frac{1}{r^3}\end{aligned}$$

$$\therefore \nabla^2 \left(\frac{1}{r} \right) = \frac{3}{r^5} (x^2 + y^2 + z^2) - \frac{3}{r^3} = 0$$

This is valid only for $r \neq 0$. At $r = 0$ the function is not differentiable.

2. Evaluate $(\hat{r} \cdot \vec{\nabla})r$ and $(\hat{r} \cdot \vec{\nabla})\hat{r}$ **soln:**

$$\begin{aligned}(\hat{r} \cdot \vec{\nabla})r &= \left(\frac{x}{r} \frac{\partial}{\partial x} + \frac{y}{r} \frac{\partial}{\partial y} + \frac{z}{r} \frac{\partial}{\partial z} \right) r \\ &= \frac{x}{r} \frac{\partial r}{\partial x} + \frac{y}{r} \frac{\partial r}{\partial y} + \frac{z}{r} \frac{\partial r}{\partial z} \\ &= \frac{x}{r} \frac{2x}{2r} + \frac{y}{r} \frac{2y}{2r} + \frac{z}{r} \frac{2z}{2r} \\ &= \frac{x^2 + y^2 + z^2}{r^2} = 1\end{aligned}$$

$$\begin{aligned}
(\hat{r} \cdot \vec{\nabla})\hat{r} &= \left(\frac{x}{r} \frac{\partial}{\partial x} + \frac{y}{r} \frac{\partial}{\partial y} + \frac{z}{r} \frac{\partial}{\partial z} \right) \left[\hat{i} \frac{x}{r} + \hat{j} \frac{y}{r} + \hat{k} \frac{z}{r} \right] \\
&= \hat{i} \left(\frac{x}{r} \frac{\partial}{\partial x} + \frac{y}{r} \frac{\partial}{\partial y} + \frac{z}{r} \frac{\partial}{\partial z} \right) \left(\frac{x}{r} \right) + \hat{j} \left(\frac{x}{r} \frac{\partial}{\partial x} + \frac{y}{r} \frac{\partial}{\partial y} + \frac{z}{r} \frac{\partial}{\partial z} \right) \left(\frac{y}{r} \right) + \hat{k} \left(\frac{x}{r} \frac{\partial}{\partial x} + \frac{y}{r} \frac{\partial}{\partial y} + \frac{z}{r} \frac{\partial}{\partial z} \right) \left(\frac{z}{r} \right) \\
&= \hat{i}0 + \hat{j}0 + \hat{k}0 \\
&= 0
\end{aligned}$$

3. Find the volume of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the tripple integral $\int \int \int dx dy dz$ with appropriate limits.

soln:

We will calculate the volume in the first quadrant of the coordinate system which is $1/8$ the volume of the ellipsoid. We will first integrate over z at a fixed (x, y) . The lower limit is 0 while the upper limit is decided by the equation of the ellipsoid and is given as a function of (x, y) . This will be $c\sqrt{1 - x^2/a^2 - y^2/b^2}$. The ellipsoid cuts the xy plane along an ellipse whose equation is obtained by putting $z = 0$ in the equation of the ellipsoid. This gives the equation of the ellipse as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Next we do the integration over y . The limits will be given as 0 and $b\sqrt{1 - x^2/a^2}$. Finally the limits on x will be from 0 to a .

$$\begin{aligned}
V &= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz dy dx \\
&= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-x^2/a^2-y^2/b^2} dy dx \\
&= c \int_0^a \int_0^{b\alpha} \sqrt{\alpha^2 - y^2/b^2} dy dx \quad \text{where } \alpha = \sqrt{1-x^2/a^2} \\
&= c \int_0^a \alpha \int_0^{b\alpha} \sqrt{1 - \left(\frac{y}{b\alpha}\right)^2} dy dx \\
&= bc \int_0^a \alpha^2 \frac{\pi}{4} dx \\
&= \frac{\pi bc}{4} \int_0^a (1 - x^2/a^2) dx \\
&= \frac{\pi bc}{4} (a - a/3) \\
&= \frac{\pi abc}{6}
\end{aligned}$$

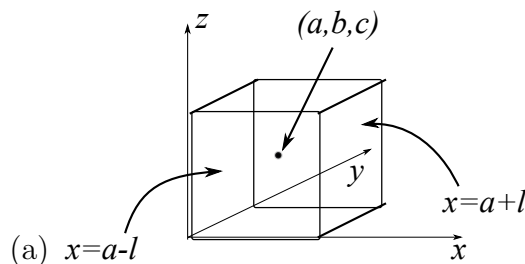
The volume of the whole ellipsoid is $8 \times \frac{\pi abc}{6} = \frac{4}{3}\pi abc$.

4. Consider $\vec{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

- (a) Evaluate $\oint_S \vec{A} \cdot d\vec{a}$ where S is a cubical surface given by the planes $x = a \pm l$; $y = b \pm l$; $z = c \pm l$.
- (b) Verify that at the point (a, b, c) ,

$$\vec{\nabla} \cdot \vec{A} = \lim_{l \rightarrow 0} \frac{1}{8l^3} \oint_S \vec{A} \cdot d\vec{a}$$

soln:



The surface of the cube consists of 6 planes. Let S_1 be the surface $x = a + l$. Over S_1 , $\vec{A} = (a + l)^2 \hat{i} + y^2 \hat{j} + z^2 \hat{k}$ and $d\vec{a} = \hat{i} dy dz$.

$$\begin{aligned} \therefore \int_{S_1} \vec{A} \cdot d\vec{a} &= \int_{c-l}^{c+l} \int_{b-l}^{b+l} (a + l)^2 dy dz \\ &= 4l^2 (a + l)^2 \end{aligned}$$

Over the surface S_2 : $x = a - l$,
 $\vec{A} = (a - l)^2 \hat{i} + y^2 \hat{j} + z^2 \hat{k}$ and $d\vec{a} = -\hat{i} dy dz$

$$\begin{aligned} \therefore \int_{S_2} \vec{A} \cdot d\vec{a} &= - \int_{c-l}^{c+l} \int_{b-l}^{b+l} (a - l)^2 dy dz \\ &= -4l^2 (a - l)^2 \end{aligned}$$

\therefore net flux from S_1 and S_2 is $4l^2[(a + l)^2 - (a - l)^2] = 16al^3$.

Similarly from the other two pair of surfaces we will have $16bl^3$ and $16cl^3$.

So the total flux of the vector field \vec{A} through the given cube is $16l^3(a + b + c)$.

- (b) The volume of the cube is $8l^3$.

$$\begin{aligned} \therefore \lim_{l \rightarrow 0} \frac{1}{V} \oint_S \vec{A} \cdot d\vec{a} &= \lim_{l \rightarrow 0} \frac{1}{8l^3} 16l^3 (a + b + c) \\ &= 2(a + b + c) \end{aligned}$$

This is same as the value of $\vec{\nabla} \cdot \vec{A}$ at (a, b, c) .

This limit will be true for volume of any shape enclosing the point (a, b, c) .

5. Evaluate $\int_P^Q \vec{A} \cdot d\vec{l}$ for $\vec{A} = y\hat{i} - x\hat{j}$ along the following paths : $P \equiv (-a, 0)$; $Q \equiv (a, 0)$.

- (a) $(-a, 0) \rightarrow (0, a) \rightarrow (a, 0)$

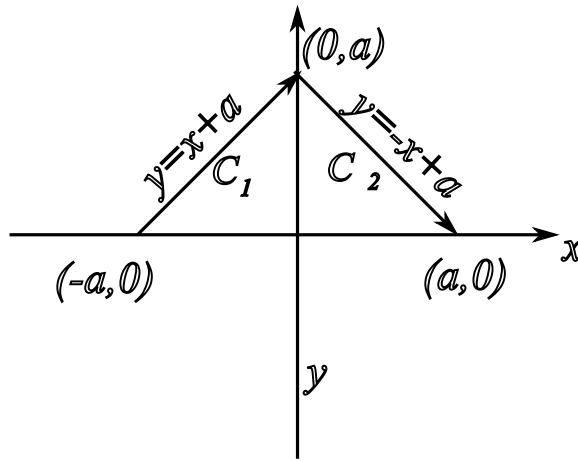
- (b) $(-a, 0) \rightarrow (0, -a) \rightarrow (a, 0)$
 (c) a loop, forward along (a) and backward along (b)
 (d) Let I be the value of the loop integral evaluated in (c). Let S be the flat area enclosed by the loop. Verify that at the origin

$$(\vec{\nabla} \times \vec{A}) = \left[\lim_{a \rightarrow 0} \frac{I}{S} \right] (-\hat{k})$$

- (e) Can we find a scalar function F such that $\vec{\nabla} F = y\hat{i} - x\hat{j}$?

soln:

- (a) The path is made up of two straight curves C_1 and C_2 .



Along C_1 , $y = x + a$.

$$\therefore dy = dx.$$

$$\vec{A} \cdot d\vec{l} = (y\hat{i} - x\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) = ydx - xdy.$$

Along C_1 we have $\vec{A} \cdot d\vec{l} = (x + a)dx - xdx = adx$.

$$\therefore \int_{C_1} \vec{A} \cdot d\vec{l} = \int_{-a}^0 adx = a^2$$

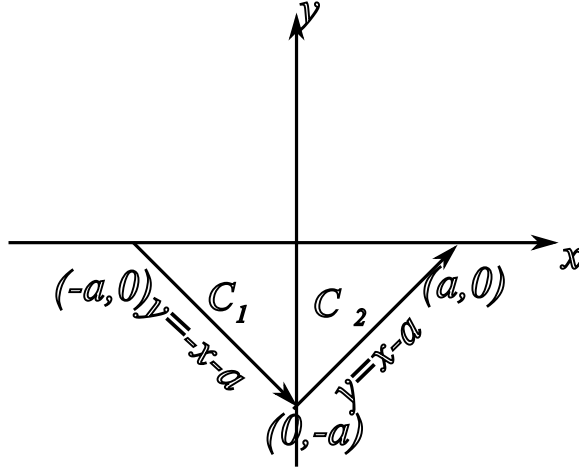
Along C_2 , $y = -x + a$.

$$\therefore dy = -dx.$$

So we have $\vec{A} \cdot d\vec{l} = (-x + a)dx + xdx = adx$.

$$\therefore \int_{C_2} \vec{A} \cdot d\vec{l} = \int_0^a adx = a^2$$

$$\therefore \int_P^Q \vec{A} \cdot d\vec{l} = \int_{C_1} + \int_{C_2} = 2a^2$$



- (b) The path is made up of two straight curves C_1 and C_2 .

Along C_1 , $y = -x - a$.

$$\therefore dy = -dx.$$

$$\vec{A} \cdot d\vec{l} = (y\hat{i} - x\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) = ydx - xdy.$$

Along C_1 we have $\vec{A} \cdot d\vec{l} = (-x - a)dx + xdx = -adx$.

$$\therefore \int_{C_1} \vec{A} \cdot d\vec{l} = \int_{-a}^0 (-a)dx = -a^2$$

Along C_2 , $y = x - a$.

$$\therefore dy = dx.$$

So we have $\vec{A} \cdot d\vec{l} = (x - a)dx - xdx = -adx$.

$$\therefore \int_{C_2} \vec{A} \cdot d\vec{l} = \int_0^a (-a)dx = -a^2$$

$$\therefore \int_P^Q \vec{A} \cdot d\vec{l} = \int_{C_1} + \int_{C_2} = -2a^2$$

- (c) Along the loop the value of the integral will be $2a^2 - (-2a^2) = 4a^2$.

- (d) We have $I = 4a^2$ and $S = 2a^2$

$$\therefore \frac{I}{S} = 2.$$

$$\therefore \lim_{a \rightarrow 0} \frac{I}{S} = 2.$$

$\vec{\nabla} \times \vec{A} = -2\hat{k}$ everywhere.

So we have $\vec{\nabla} \times \vec{A} = [\lim_{a \rightarrow 0} \frac{I}{S}] (-\hat{k})$ at the origin.

- (e) $\int_a^b \vec{\nabla} F \cdot d\vec{l} = \int_a^b dF = F(b) - F(a)$.

This is independent of any path we take from a to b .

We just saw in the previous part that for $\vec{A} = y\hat{i} - x\hat{j}$ the integral $\int_a^b \vec{A} \cdot d\vec{l}$ is path dependent.

So we can't have any scalar function F such that $\vec{\nabla} F = y\hat{i} - x\hat{j}$.