# Lecture 14

# Newtonian Mechanics : Oscillations

# Summary



**<u>Equation of Motion of Single Particle</u>**

- Position r(t)
- Velocity v(t)=dr(t)/dt
- Acceleration $a(t)=dv(t)/dt=d^2r(t)/dt^2$
- Momentum p(t)=mv(t)

**Newton's second Law (Inertial Frame) → the equation of motion that is position of the particle as a function of time.**

F(r,v,t)=dp(t)/dt

# SHM

$$F = -k\ x = m\ \ddot{x} \rightarrow \boxed{\ddot{x} + \omega^2 x = 0}$$

This is a homogenous, linear DE

We solve it in general by inserting

a solution of the form: $x = B\exp(rt)$

$$\ddot{x} + \omega^2 x = r^2 B\exp(rt) + \omega^2 B\exp(rt) = 0$$

This leads to an auxillary equation of

form $\boxed{r^2 + \omega^2 = 0}$ which has two roots:

$r_{1,2} = \pm\sqrt{-\omega^2} = \pm i\omega$. We construct the

general solution as the sum of terms

$$\boxed{x(t) = B_1 \exp(i\omega t) + B_2 \exp(-i\omega t)}$$

X is real by demanding x=x*

$$x = B_1 \exp(i\omega t) + B_2 \exp(-i\omega t)$$

$$x^* = B_1^* \exp(-i\omega t) + B_2^* \exp(i\omega t)$$

x* = x if and only if $\boxed{B_2 = B_1^*}$

$$x(t) = B_1 \exp(i\omega t) + B_1^* \exp(-i\omega t)$$

Choose $B_1 = \alpha\exp(-i\delta)$ where $\alpha$ and $\delta$ are real numbers corresponding to the phase and modulus of $B_1$.

$$x(t) = \alpha e^{-i\delta}\exp(i\omega t) + \alpha e^{i\delta}\exp(-i\omega t)$$

$$= \alpha\exp\left[i\left(\omega t - \delta\right)\right] + \alpha\exp\left[-i\left(\omega t - \delta\right)\right]$$

$$= 2\alpha\frac{\exp\left[i\left(\omega t - \delta\right)\right] + \exp\left[-i\left(\omega t - \delta\right)\right]}{2}$$

$$x = 2\alpha\cos\left(\omega t - \delta\right) \rightarrow x = A\cos\left(\omega t - \delta\right)$$

# Summary of (SHM : ideal case)



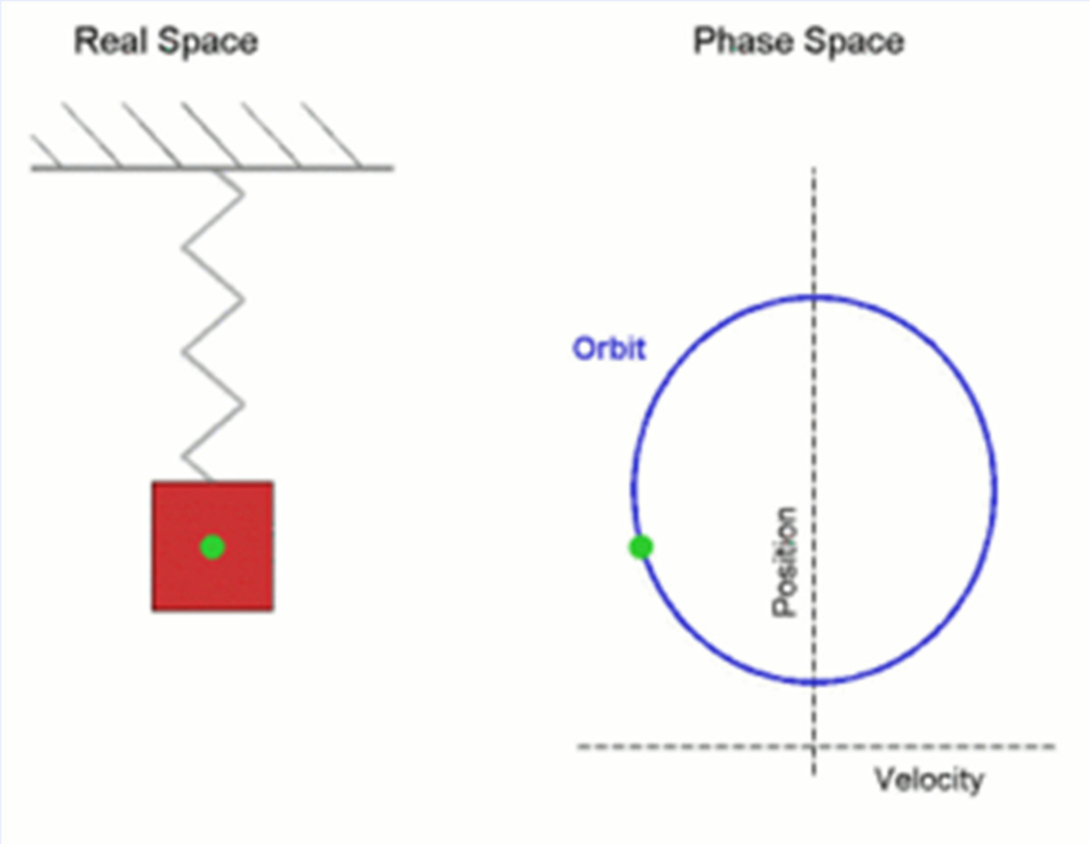
- Oscillations→ perturbation from equilibrium.
- Simplest approximation of restoring force F=-kx
- (Hookes Law within elastic limit)
- So if we make "x" n times larger , F will be n times larger.
- Spring constant k=del F/del x.
- Equation of motion of SHM→ $\mathbf{d^2x/dt^2+w^2x=0;}$ where $w^2=k/m$
- General Solution → $x=A\cos(wt-\phi)$
- Total energy is proportional to square of the amplitude. $E=.5\ kA^2$
- Time period does not depend on amplitude.
- Phase Space behavior.

# SHM



$$F = -k(x - x_e) = -k\ x'$$

$$U(x) = \frac{k}{2} x'^2 \ ; T = \frac{m}{2} \dot{x}'^2$$



# Linear Drag Force



$m\,\ddot{x} = -k\;x - b\dot{x}$ or writing

$\omega = \sqrt{\mathrm{k/m}}$ and $\beta = \mathrm{b/2m}$ we have

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega^2 x = 0}$$

Auxillary eq by substituting

$$x = B\exp(rt)$$

$$B\exp(rt)\left(r^2 + 2\beta r + \omega^2\right) = 0$$

$$r^2 + 2\beta r + \omega^2 = 0$$

$$\boxed{r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega^2}}$$

# 3 possible cases

$$r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega^2}$$

$$x = B_1 \exp(r_1 t) + B_2 \exp(r_2 t)$$

There are three cases:

$$\boxed{\begin{array}{l} \beta^2 - \omega^2 < 0 \text{ under damped} \\ \beta^2 - \omega^2 = 0 \text{ critically damped} \\ \beta^2 - \omega^2 < 0 \text{ over damped} \end{array}}$$

# Underdamp

Define $\omega_1 = \sqrt{\omega^2 - \beta^2} \quad r_{1,2} = -\beta \pm i\omega_1$

$$x = e^{-\beta t}\left(B_1 \exp(i\omega_1 t) + B_2 \exp(-i\omega_1 t)\right)$$

We force a real x via x*=x which

$$x^* = e^{-\beta t}\left(B_1{}^* \exp(-i\omega_1 t) + B_2{}^* \exp(i\omega_1 t)\right)$$

$$\therefore x = e^{-\beta t}\left(B_1 \exp(i\omega_1 t) + B_1{}^* \exp(-i\omega_1 t)\right)$$

Choose $\mathrm{B}_1 = \frac{A}{2}\exp(-i\delta)$ real amp & phase

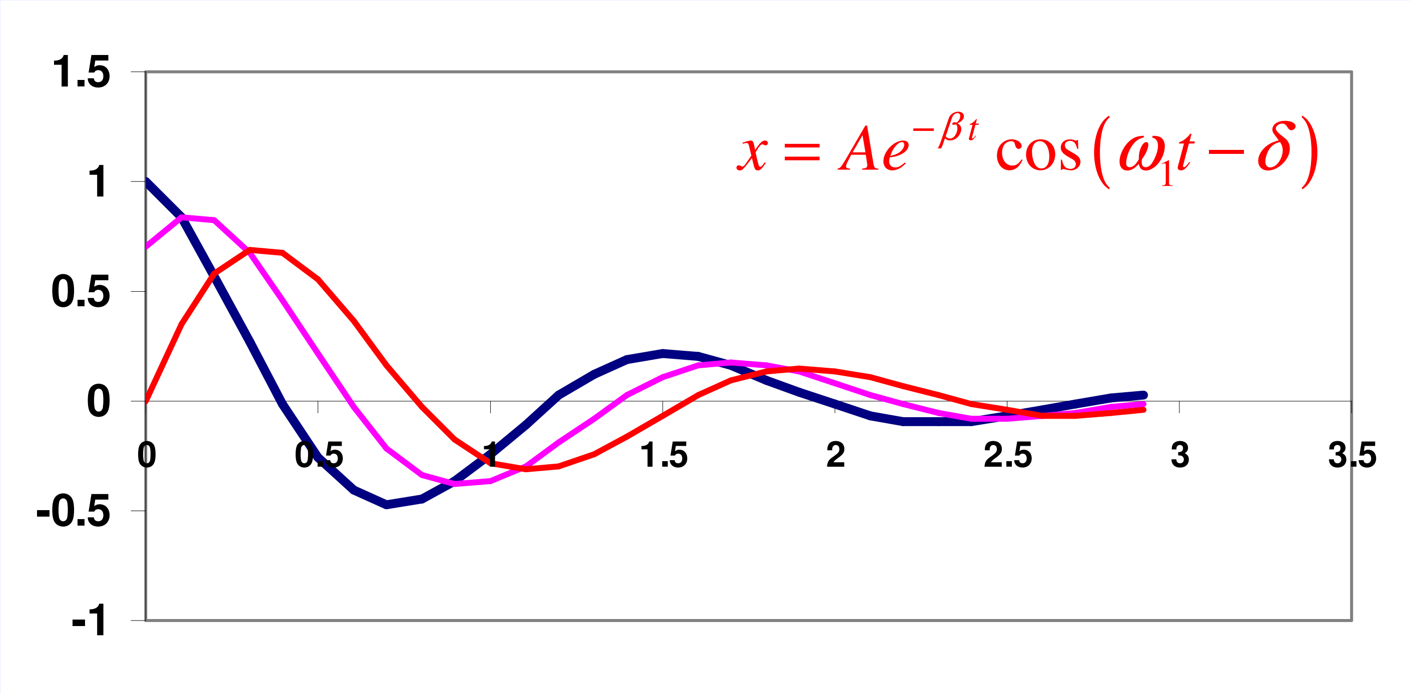
$$x = e^{-\beta t}\,\frac{A}{2}\left(e^{i(\omega_1 t-\delta)} + e^{-i(\omega_1 t-\delta)}\right)$$

$$\boxed{x = Ae^{-\beta t}\cos\left(\omega_1 t - \delta\right)} \text{ or } e^{-\beta t}\sin\left(\omega_1 t - \delta\right)$$

**The under damped oscillator has two constants (phase and amplitude) to match initial position and velocity.**

**The solution dies away while oscillating but with a frequency other than $\omega = (k/m)^{1/2}$.**

# Under damped




$$\boxed{\omega_1 = 4\beta}$$

# Over damped

$$r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega^2}$$

$$x = B_1 \exp(r_1 t) + B_2 \exp(r_2 t)$$

$$\omega_2 = \sqrt{\beta^2 - \omega^2} \qquad r_{1,2} = -\beta \pm \omega_2$$

$$x = B_1 \exp\left(-\left(\beta + \omega_2\right) t\right) + B_2 \exp\left(-\left(\beta - \omega_2\right) t\right)$$

Defining $\lambda_1 = \beta + \omega_2$ and $\lambda_2 = \beta - \omega_2$

$$x = B_1 \exp(-\lambda_1 t) + B_2 \exp(-\lambda_2 t)$$