Cardan's Solution with All Real Roots

Consider the standard form of the cubic equation $\boxed{x^3 + px + q = 0}$ with the discriminant

$$\boxed{D = \frac{q^2}{4} + \frac{p^3}{27}}$$

. When $D < 0$ (possible when $p < 0$), write $\boxed{D \equiv -(-D)}$.

Now $\boxed{y = \left(-\frac{q}{2} + \sqrt{D}\right)^{1/3}}$, $\boxed{z = \left(-\frac{q}{2} - \sqrt{D}\right)^{1/3}}$.

Defining $\boxed{A = -\frac{q}{2}}$ and $\boxed{B = \sqrt{-D}}$, we get,

$\boxed{y = (A + iB)^{1/3}}$ and $\boxed{z = (A - iB)^{1/3}}$ when $D < 0$.

Now write $\boxed{r\cos\theta = A}$, $\boxed{r\sin\theta = B}$

$$\therefore \boxed{r^2(\cos^2\theta + \sin^2\theta) = r^2 = A^2 + B^2 = +\frac{q^2}{4} + (-D)}$$

$$\Rightarrow r^2 = \cancel{\frac{q^2}{4}} - \cancel{\frac{q^2}{4}} - \frac{p^3}{27} \Rightarrow \boxed{r^2 = -p^3/27 = \left(-\frac{p}{3}\right)^3}$$

Further $\boxed{\cos\theta = \frac{A}{r} = \frac{-q}{\sqrt{-4(p/3)^3}}}$ $\boxed{\theta = \arccos\left(\frac{A}{r}\right)}$

Also $\boxed{r^{1/3} = \sqrt{-\frac{p}{3}}}$

The solutions are $\boxed{x_1 = 2r^{1/3}\cos\left(\frac{\theta}{3}\right)}$,

$\boxed{x_2 = 2r^{1/3}\cos\left(\frac{\theta + 2\pi}{3}\right)}$, $\boxed{x_3 = 2r^{1/3}\cos\left(\frac{\theta + 4\pi}{3}\right)}$

These are set as for $\boxed{j = 1, 2, 3}$

$$\boxed{x_j = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{\theta + 2\pi(j-1)}{3}\right]}$$

All roots are real.