Fundamental theorem of calculus

If f is cont's on closed $[a,b]$ & if

$$F(x) = \int_a^x f(t)\,dt \quad (a \le x \le b)$$

then $F'(x) = f(x) \quad (a \le x \le b)$.

$\square$ For any fixed $x \in [a,b]$ we have if $h \neq 0$ & $x+h \in [a,b]$

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt \quad \text{——— (1)}$$

$\because$ f is cont's on $[x, x+h]$,

$\Rightarrow$ there are pts. of $[x, x+h]$ at which f attains a max. value M & a min. value m (assuming $h > 0$)

$$\Rightarrow \quad m \le f(t) \le M \qquad x \le t \le x+h$$

and $f(t_1) = m$, $f(t_2) = M$ for some t_1 & t_2 in $[x, x+h]$

$$\Rightarrow \quad \int_x^{x+h} m\,dt \le \int_x^{x+h} f(t)\,dt \le \int_x^{x+h} M\,dt$$

$$\Rightarrow \quad mh \le \int_x^{x+h} f(t)\,dt \le Mh$$

$$\Rightarrow \quad m \leq \theta \leq M$$

$$\theta = \frac{1}{h} \int_{x}^{x+h} f(t)\, dt$$

$\Rightarrow$ $\exists$ a pt. $c(h)$ in $[x, x+h]$ s.t.

$$f[c(h)] = \theta$$

From (1)

$$\frac{F(x+h) - F(x)}{h} = f[c(h)]$$

But $\lim\limits_{h \to 0} c(h) = x \quad (\because x \leq c(h) \leq x+h)$

$$\Rightarrow \quad F'(x) = f(x) \quad \blacksquare$$

Taylor's series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{--- (1)}$$

$|x-a|<R$

R = Radius of convergence

Let $$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

n^{th} degree Taylor poly.

Example for $f(x)=e^x$ & $a=0$

$T_1(x) = 1+x$ $\quad T_2(x) = 1+x+\frac{x^2}{2!}$ & so on

Now if the series in (1) converges then

$$\lim_{n\to\infty} T_n(x) = f(x)$$

Let $R_n(x) = f(x) - T_n(x)$

(remainder of the Taylor's series)

If we can show that $\lim_{n\to\infty} R_n(x) = 0$ we are done. Since then

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}\left(f(x) - R_n(x)\right) = f(x) - \lim_{n\to\infty} R_n(x)$$

$$= f(x) - 0 = f(x)$$

$\therefore$ If $f(x) = T_n(x) + R_n(x)$ & $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a|<R$

then $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Different form of remainder ~~...~~ $R_n(x)$ is known.

1. Cauchy form of remainder
2. integral form of remainder
3. Lagrange form of remainder

In each case it can be shown that

$$\lim_{n\to\infty} R_n(x) = 0$$ with some conditions on $f(x)$.

Taylor's inequality

If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$ then the remainder $R_n(x)$ of the Taylor's series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$ for $|x-a| \le d$

Proof: Let $n=1$

$\Rightarrow$ $|f''(x)| \le M \Rightarrow -M \le f''(x) \le M$

$\Rightarrow$ for $a \le x \le a+d$

$$\int_a^x f''(t)\,dt \le \int_a^x M\,dt$$

$\Rightarrow$ $f'(x) - f'(a) \le M(x-a)$

$\Rightarrow$ $f'(x) \le f'(a) + M(x-a)$

Thus

$$\int_a^x f'(t)\,dt \le \int_a^x [f'(a) + M(t-a)]\,dt$$

$$\Rightarrow f(x) - f(a) \le f'(a)(x-a) + M\frac{(x-a)^2}{2}$$

$$\Rightarrow f(x) - f(a) - f'(a)(x-a) \le \frac{M}{2}(x-a)^2$$

But $R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x-a)$

$$\Rightarrow \quad R_1(x) \le \frac{M}{2}(x-a)^2$$

Similarly using $f''(x) \ge -M$ we can show that $R_1(x) \ge -\frac{M}{2}(x-a)^2$

$$\Rightarrow \quad -\frac{M}{2}(x-a)^2 \le R_1(x) \le \frac{M}{2}(x-a)^2$$

$$\Rightarrow \quad \boxed{|R_1(x)| \le \frac{M}{2}|x-a|^2}$$

we have assumed $x > a$ similar calculation shows that this inequality is also ~~be~~ true for $x < a$

Thus we have proved it for $n=1$

We can prove it similarly for any n by integrating $n+1$ times.

~~Verif~~ Prove it for $n=2$ case

Example

Prove that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x$

□ If $f(x) = e^x$

$\Rightarrow f^{(n+1)}(x) = e^x \quad \forall n$

If d is a +ve no. & $|x| \le d$

then $|f^{(n+1)}(x)| = e^x \le e^d$

Thus using Taylor's inequality with $a = 0$ & $M = e^d$

we get

$$|R_n(x)| \le \frac{e^d}{(n+1)!} |x|^{n+1} \quad \text{for } |x| \le d$$

Note $M = e^d$ works $\forall n$

But $\lim_{n\to\infty} \frac{e^d}{(n+1)!} |x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}$

$= 0$

$\boxed{\because \lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \forall x}$

Thus from Squeeze theorem

$$\lim_{n\to\infty} |R_n(x)| = 0$$

$$\Rightarrow \lim_{n\to\infty} R_n(x) = 0 \quad \forall x$$

Thus $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad R = \infty$

$\boxed{\text{Example}}$ Show that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \& \text{ find } R = ?$$

(Hint $M = 1$ in Taylor's inequality)

$\boxed{\text{Example}}$

$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n \quad \forall x$$

$\boxed{\text{Integration}}$

$F(x) \to$ anti derivative

$$\int f(x)\, dx = F(x) + c$$

Q1. Let $f(x) = \begin{cases} 0, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$

Find $\int f(x)\, dx = ?$

Q2: $\int x^x\, dx = ?$

Q3. $\int \sin x^3\, dx = ?$

Definite Integrals

$f(x)$ is cont. s on $[a,b]$

$m = \min\limits_{a \le x \le b} f(x)$ & $M = \max\limits_{a \le x \le b} f(x)$

Divide $[a,b]$ into $[x_{i-1}, x_i]$ $i = 1, \dots, n$

$x_0 = a$ & $x_n = b$

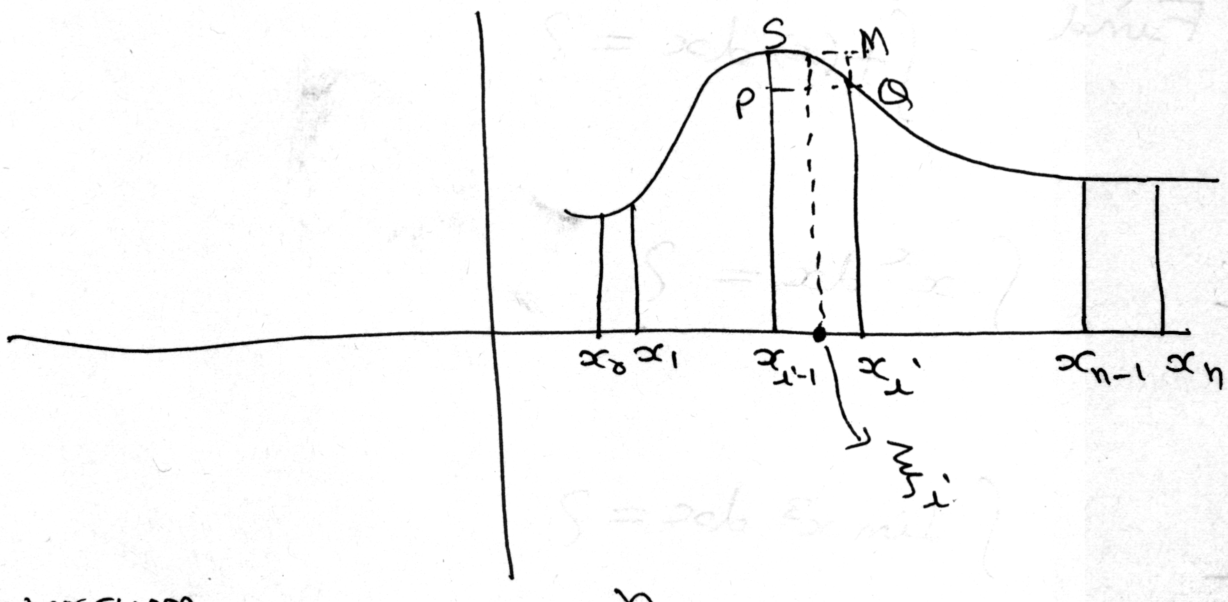
$a = x_0 < x_1 < \dots < x_n = b$

Let $\Delta x_i = x_i - x_{i-1}$ $i = 1, 2, \dots, n$

$m_i = \min\limits_{x_{i-1} \le x \le x_i} f(x)$ & $M_i = \max\limits_{x_{i-1} \le x \le x_i} f(x)$

$i = 1, 2, \dots, n$

Let ξ_i be any pt. in $[x_{i-1}, x_i]$



lowersum $= l_n(f) = \sum\limits_{i=1}^{n} m_i \Delta x_i$

upper sum $= u_n(f) = \sum\limits_{i=1}^{n} M_i \Delta x_i$

let $S_n(f) = \sum_{i=1}^{n} f(\xi_i) \Delta x_i$

then

$$l_n(f) \leq S_n(f) \leq u_n(f)$$

If we make $n \to \infty$ s.t. $\max(\Delta x_i) \to 0$

If $\lim_{n\to\infty} l_n(f) = \lim_{n\to\infty} S_n(f) = \lim_{n\to\infty} u_n(f)$

for any choice of subintervals $= \int_a^b f(x)\,dx$

(1) $$m(b-a) \leq \int_a^b f(x) \leq M(b-a)$$

(2) $$\int_a^b f(x)\,dx = (b-a) f(\xi), \quad a < \xi < b$$

(3) $$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

(If both $f(x)$ & $|f(x)|$ are bounded & integrable on $[a,b]$)

(4) The Average value of an integrable fn. $f(x)$ on $[a,b]$ is given by, $\frac{1}{(b-a)} \int_a^b f(x)\,dx$

(5) Let $f(x)$ & $g(x)$ are integrable & $f(x) \leq g(x)$, $a \leq x \leq b$ then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

(6) $$\int_a^b f(x)\,dx = F(b) - F(a)$$

Calculus of several variables

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$$

n-dimensional vector space

Now we can study the following kind of functions

1. $f : \mathbb{R}^n \to \mathbb{R}$ ✓
2. $f : \mathbb{R} \to \mathbb{R}^n$
3. $f : \mathbb{R}^m \to \mathbb{R}^n$ ✓

First we focus on $n = 2$

$$f : \mathbb{R}^2 \to \mathbb{R}$$

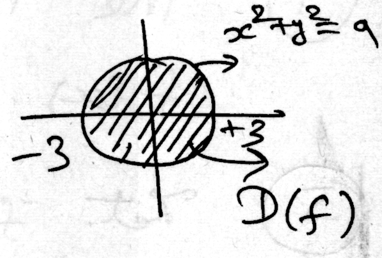
Example $f(x,y) = \sqrt{9 - x^2 - y^2}$

What is the domain & range?

☐ Domain of $f(x,y)$

$$= \{(x,y) \mid 9 - x^2 - y^2 \geq 0\}$$

$$= \{(x,y) \mid x^2 + y^2 \leq 9\}$$



Range of $f(x,y)$

$$= \{z \mid z = \sqrt{9 - x^2 - y^2},\ (x,y) \in D(f)\}$$

Since z is +ve square root, $z \geq 0$

also $9 - x^2 - y^2 \leq 9 \Rightarrow \sqrt{9-x^2-y^2} \leq 3$

$\therefore$ Range of $f(x,y) = \{ z \mid 0 \leq z \leq 3 \}$

$= [0,3]$

$\subset \mathbb{R}$

- Level curves of a fn. f of two variables are the curves with eq^n $f(x,y) = K$

(K - constant) in Range of f

- Draw the level curves for $f(x,y) = \sqrt{9-x^2-y^2}$ for $K = 0, 1, 2, 3$

$\boxed{\text{Example}}$ $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x,y,z) = \ln(z-y) + xy \sin z$$

Find the domain of f?

$\square$ $f(x,y,z)$ is defined for $z - y > 0$

$\therefore$ Domain of f is

$$D(f) = \{ (x,y,z) \in \mathbb{R}^3 \mid z > y \}$$

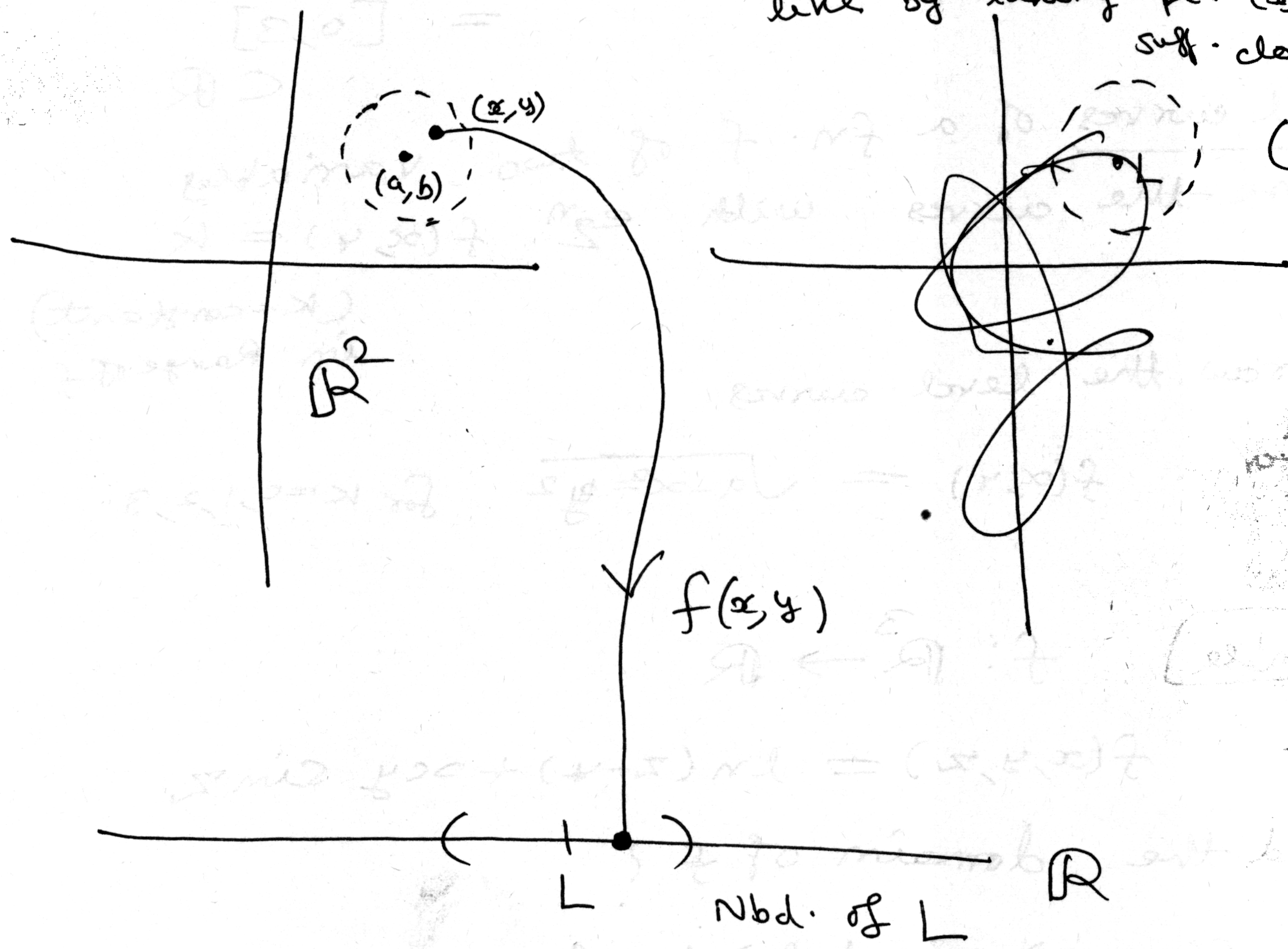
- $\boxed{\text{Level of surfaces}}$

Surfaces with eq^n $f(x,y,z) = K$

K - constant.

Limit of a fn. $f(x,y)$

$\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ if we can make the values of $f(x,y)$ close to L as we like by taking pt. (x,y) suff. close to (a,b) (but $\neq (a,b)$)



~~...~~ $f(x,y) \to L$ as $(x,y) \to (a,b)$ ~~along~~ along any path in the Domain of f

$(x,y) \to (a,b)$ ~~in~~ from several directions.

— If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path c_1 & $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path c_2 where $L_1 \neq L_2$ then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$$

Example

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{ does not exist}$$

$\square$ $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

If $(x,y) \to (0,0)$ along x-axis then $y=0$

$\Rightarrow f(x,0) = \dfrac{x^2}{x^2} = 1 \quad \forall x \neq 0$

$\therefore f(x,y) \to 1$ as $(x,y) \to (0,0)$ along x-axis

If $(x,y) \to (0,0)$ along y-axis then $x=0$

$\Rightarrow f(0,y) = -\dfrac{y^2}{y^2} = -1 \quad \forall y \neq 0$

$\therefore f(x,y) \to -1$ as $(x,y) \to (0,0)$ along the y-axis

$\therefore$ limit does not exist. ▨

Example $f(x,y) = \dfrac{xy}{x^2+y^2}$

$\lim_{(x,y)\to(0,0)}$

□ If $y=0$, $f(x,0)=0$

$\therefore f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x-axis

If $x=0$ then $f(0,y)=0$

$\therefore f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the y-axis

Now let $(x,y)\to(0,0)$ along the line $y=x$

then $\forall x\neq 0$ $f(x,y) = \dfrac{x^2}{x^2+x^2} = \dfrac{1}{2}$

$\therefore f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$ along $y=x$

$\therefore$ limit is different along different paths

$\therefore$ limit does not exist. ■

Example $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ Does $\lim_{(x,y)\to(0,0)} f(x,y)$

□ Let $(x,y)\to(0,0)$ along the line $y=mx$

$$f(x,y) = f(x,mx) = \frac{m^2x}{1+m^4x^2}$$

$\therefore f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the line $y=mx$

Now let $(x,y)\to(0,0)$ along the parabola $x=y^2$

we get $f(x,y) = f(y^2,y) = \frac{1}{2}$

$\therefore f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$ along $x=y^2$

$\therefore$ limit does not exist ■

Example Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exist

□ We can see that lim $(x,y)\to(0,0)$ along any line $y = mx$ is 0

Also we can see that along the parabolas $y = x^2$ or $x = y^2$ it turns out to be 0

$\therefore$ We suspect that limit is 0 but we need to prove that.

$\therefore$ Consider $\left| \frac{3x^2y}{x^2+y^2} - 0 \right| = \left| \frac{3x^2y}{x^2+y^2} \right| = \frac{3x^2|y|}{x^2+y^2}$

Now $x^2 \leq x^2+y^2$ $(\because y^2 \geq 0)$

$\therefore \frac{x^2}{x^2+y^2} \leq 1$

$\Rightarrow 0 \leq \left| \frac{3x^2y}{x^2+y^2} - 0 \right| \leq \frac{3x^2|y|}{x^2+y^2} \leq 3|y|$

Now since $\lim_{(x,y)\to(0,0)} 0 = 0$ and $\lim_{(x,y)\to(0,0)} 3|y| = 0$

By sandwich principle

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

$\boxed{\text{Continuity}}$

A fn. $f : \mathbb{R}^2 \to \mathbb{R}$ is called cont.s at (a,b) if
$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

- We say that f is cont.s on D if f is cont.s at every pt. (a,b) in D.

— Sum, differences, products & quotients of cont.s fns are cont.s on their domains.

— All polynomials are cont.s on $\mathbb{R}^2$

$$f(x,y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$ is a polynomial

— A rational fn.
$$g(x,y) = \frac{2xy+1}{x^2+y^2}$$

$\boxed{\text{Example}}$ The fn. $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

☐ This fn. is not cont.s at $(0,0)$ $\because$ $f(0,0)$ is not defined. It is cont.s on D
$$D = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$$

$\boxed{\text{Example}}$
$$g(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

☐ Here $g(0,0) = 0$ but $\lim\limits_{(x,y)\to(0,0)} g(x,y)$ does not exist.

Example $f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$\because \lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ this fn. is cont.s at $(0,0)$ & so it is cont.s on $\mathbb{R}^2$. ▨

Example $g(x,y) = \arctan(y/x)$

is cont.s except at $x = 0$

For n variables

$$\lim_{(x_1,x_2,\ldots,x_n)\to(a_1,a_2,\ldots,a_n)} f(x_1,x_2,x_3,\ldots,x_n) = f(a_1,a_2,a_3,\ldots,a_n)$$

Derivatives of several variables

Let f be a fn. of two variables its partial derivatives are defined as

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

(derivative w.r. to x, keeping y constant)

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notations for partial derivatives

If $z = f(x,y)$

- $f_x(x,y) = f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x} f(x,y) = \dfrac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$

Rate of change of z w.r. to x when y is fixed

- $f_y(x,y) = f_y = \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y} f(x,y) = \dfrac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$

Example $f(x,y) = x^3 + x^2 y^3 - 2y^2$

find $f_x(2,1)$ & $f_y(2,1)$

$\square$ $f_x(x,y) = 3x^2 + 2xy^3 \Rightarrow f_x(2,1) = 16$

$f_y(x,y) = 3x^2y^2 - 4y \Rightarrow f_y(2,1) = 8$

Several Variables

If $u = f(x_1, x_2, \ldots, x_n)$

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

Higher derivatives

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$

Clairaut's Theorem

Suppose f is defined on a disk D that contains the pt. (a, b). If the ~~fns~~ functions f_{xy} & f_{yx} both are cont.s on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Tangent planes

Suppose f has cont.s partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the pt. $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example: The tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at $(1, 1, 3)$ is given by $z = 4x + 2y - 3$

Exercise 1: Plot the elliptic paraboloid & this tangent plane on MATLAB

- The linear fn. whose graph is the tangent plane is called the linearization of f at (a, b) & the approximation

$$f(x, y) \cong f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called tangent plane approximation

— For single variables we know that if f is differentiable at a then

$$\Delta y = f'(a)\,\Delta x + \epsilon\,\Delta x$$ where $\epsilon \to 0$ as $\Delta x \to 0$

— For 2 variables $z = f(x,y)$ if x changes from a to $a+\Delta x$ & y from b to $b+\Delta y$ then

$$\Delta z = f(a+\Delta x,\, b+\Delta y) - f(a,b)$$

Differentiable fn.

If $z = f(x,y)$ then f is differentiable at (a,b) if Δz can be written as

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 & $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

THEOREM If the partial derivatives f_x & f_y exist near (a,b) & are cont.s at (a,b), then f is differentiable at (a,b).

Example The fn. $f(x,y) = x e^{xy}$ is differentiable at $(1,0)$.

□ $f_x(x,y) = e^{xy} + xy\,e^{xy}$ $\qquad f_y(x,y) = x^2 e^{xy}$

$f_x(1,0) = 1$ $\qquad \Rightarrow f_y(1,0) = 1$

Both f_x & f_y are cont.s $\therefore$ f is differentiable

The linearization of f will be

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$$

$$= x + y \qquad \therefore x e^{xy} \cong x + y$$

Differentials

$z = f(x, y)$

we define dx & dy to be independent variables

dz (total differential) $= f_x(x,y)\,dx + f_y(x,y)\,dy$

$= \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$

$\therefore$ The linear app. will be

$f(x,y) \cong f(a,b) + dz$

For 3 variables

$w = f(x, y, z)$

$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz$

The chain Rule

We know for single variable

$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$

Rule ① Suppose $z = f(x, y)$ is a differentiable fn. of x & y where $x = g(t)$ & $y = h(t)$ are both diff. fns. of t. then z is a differentiable fn. of t and

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

Rule ② Suppose $z = f(x, y)$ is a diff. fn. of x & y where $x = g(s,t)$ & $y = h(s,t)$ are diff. fns. of s & t then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example If $z = e^x \sin y$

$x = st^2$ & $y = s^2 t$

find $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$

Rule ③ (General chain rule)

Suppose u is a differentiable fn. of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a diff. fn. of the m variables $t_1, t_2, \ldots, t_m$ then u is a fn. of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

$\forall\ i = 1, 2, \ldots, m$

Example If $u = x^4 y + y^2 z^3$ where $x = rse^t$, $y = rs^2 e^{-t}$ & $z = r^2 s \sin t$ find $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$ & $t = 0$

<u>Ans.</u> $\frac{\partial u}{\partial s} = 192$