TUTORIAL 01

1. In each case, find an open interval about x_0 on which the inequality $|f(x) - L| < \epsilon$ holds. Then give a value $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$, the inequality $|f(x) - L| < \epsilon$ holds.
   a. $f(x) = \sqrt{x+1},\ L = 1,\ x_0 = 0,\ \epsilon = 0.1$
   b. $f(x) = x^2 - 5,\ L = 11,\ x_0 = 4,\ \epsilon = 1$
   c. $f(x) = \frac{1}{x},\ L = -1,\ x_0 = -1,\ \epsilon = 0.1$

2. Using (ϵ, δ) definition, show that
   a. $\lim_{x\to 4}(9 - x) = 5$
   b. $\lim_{x\to -3} \frac{x^2-9}{x+3} = -6$
   c. $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$

3. For

$$f(x) = \begin{cases} \sin(\frac{1}{x}), & x > 0 \\ 0, & x \leq 0 \end{cases}$$

   a. Does $\lim_{x\to 0^+}$ exist? Why or why not?
   b. Does $\lim_{x\to 0^-}$ exist? Why or why not?

4. Use (ϵ, δ) approach to prove that
   a. $\lim_{x\to 0^-} \frac{x}{|x|} = -1$
   b. $\lim_{x\to 2^+} \frac{x-2}{|x-2|} = 1$

5. Using $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$, find the following
   a. $\lim_{x\to 0} \frac{x^2-x+\sin x}{2x}$
   b. $\lim_{t\to 0} \frac{\sin(1-\cos t)}{1-\cos t}$

6. For what values of a and b, the function

$$g(x) = \begin{cases} -2 & , x \leq -1 \\ ax + b & , -1 < x < 1 \\ 3 & , x \geq 1 \end{cases}$$

is continuous at every point?

7. Show that the function

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

is discontinuous at every point.
   a. Is $f(x)$ right continuous at any point?
   b. Is $f(x)$ left continuous at any point?

8. If functions $f(x)$ and $g(x)$ are continuous for $0 \leq x \leq 1$, could $\frac{f(x)}{g(x)}$ possibly be discontinuous at a point of $[0,1]$? Give reason for your answer.

9. Determine the values of a, b and c so that the function

$$f(x) = \begin{cases} \frac{\sin(a+1)x+\sin x}{x} & , x < 0 \\ c & , x = 0 \\ \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{\frac{3}{2}}} & , x > 0 \end{cases}$$

is continuous for all x.

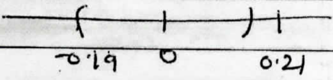
10 Let $f(x) = \lfloor x \rfloor + \sqrt{x - |x|}$. Test the continuity of $f(x)$ at $x = 1$.

TUTORIAL I SOLUTIONS

Q.1 (a) $f(x) = \sqrt{x+1}$, $L=1$, $x_0 = 0$, $\epsilon = 0.1$

$$|f(x) - L| < \epsilon$$
$$|\sqrt{x+1} - 1| < 0.1$$
$$-0.1 < \sqrt{x+1} - 1 < 0.1$$
$$0.9 < \sqrt{x+1} < 1.1$$
$$0.81 < x+1 < 1.21$$
$$-0.19 < x < 0.21$$
$$(-0.19, 0.21).$$

$$|x - x_0| < \delta$$
$$x_0 = 0$$
$$|x| < \delta$$
$$-\delta < x < \delta$$



We take $\delta = 0.19$
so that for all $x \in (0-\delta, 0+\delta)$
$|f(x) - L| < \epsilon$.

(b) $f(x) = x^2 - 5$ $L = 11$ $x_0 = 4$ $\epsilon = 1$.

$$|f(x) - L| < \epsilon$$
$$|x^2 - 5 - 11| < 1$$
$$|x^2 - 16| < 1$$
$$-1 < x^2 - 16 < 1$$
$$15 < x^2 < 17$$
$$\sqrt{15} < x < \sqrt{17}$$
$$|x - x_0| < \delta$$
$$|x - 4| < \delta$$

$-\delta + 4 < x < \delta + 4 \Rightarrow -\delta + 4 = \sqrt{15} \Rightarrow \delta = 4 - \sqrt{15}$
or $\delta + 4 = \sqrt{17} \Rightarrow \delta = \sqrt{17} - 4$.

$\delta = \min\{4 - \sqrt{15}, \sqrt{17} - 4\}$ $\Rightarrow \delta = \sqrt{17} - 4$.

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(c) $f(x) = \frac{1}{x}$, $L = -1$, $x_0 = -1$, $\epsilon = 0.1$

$|f(x) - L| < \epsilon$. $\qquad |x - x_0| < \delta$

$\left|\frac{1}{x} + 1\right| < 0.1$. $\qquad |x - (-1)| < \delta$

$\qquad |x+1| < \delta$.

$-0.1 < \frac{1}{x} + 1 < 0.1$. $\qquad -\delta < x+1 < \delta$

$-1.1 < \frac{1}{x} < -0.9$. $\qquad -1-\delta < x < -1+\delta$.

$-\frac{11}{10} < \frac{1}{x} < -\frac{9}{10}$ $\qquad$ By comparing both.

$-\frac{10}{9} < x < -\frac{10}{11}$ $\qquad \delta = 1/9$ or $\delta = 1/11$.

$\Rightarrow \delta = \frac{1}{11}$.

$\delta = \operatorname{Min}\left\{\frac{1}{9}, \frac{1}{11}\right\} = \frac{1}{11}$

2.

(a) $\lim_{x \to 4} (9 - x) = 5$.

$f(x) = 9 - x$, $L = 5$, $x_0 = 4$.

$|f(x) - L| < \epsilon$ $\qquad |x - x_0| < \delta$

$|(9-x) - 5| < \epsilon$ $\qquad |x - 4| < \delta$

$|4 - x| < \epsilon$ $\qquad |4 - x| < \delta$

$-\epsilon < 4 - x < \epsilon$ $\qquad -\delta < 4 - x < \delta$.

Comparing both.

$\delta = \epsilon$.

So for each $\epsilon > 0$, $\exists\ \delta > 0$ such that

for all $0 < |x - 4| < \delta$, $(\delta = \epsilon)$

$|f(x) - L| < \epsilon$.

(b) $\lim_{x \to -3} \frac{x^2-9}{x+3} = -6$

$f(x) = \frac{x^2-9}{x+3}$, $L = -6$, $x_0 = -3$

$|f(x) - L| < \epsilon$

$\left| \frac{x^2-9}{x+3} - (-6) \right| < \epsilon$

$|x - 3 + 6| < \epsilon$

$|x+3| < \epsilon$

$-\epsilon < x+3 < \epsilon$

$-\epsilon - 3 < x < \epsilon - 3$

$|x - x_0| < \delta$

$|x - (-3)| < \delta$

$|x+3| < \delta$

$-\delta < x+3 < \delta$

$-\delta - 3 < x < \delta - 3$

Comparing both, $\delta = \epsilon$

So for Each $\epsilon > 0$, $\exists\ \delta > 0$ such that for all

$0 < |x - (-3)| < \delta = \epsilon$

$|f(x) - L| < \epsilon$

(c) $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$

$f(x) = x \sin\frac{1}{x}$, $L = 0$, $x_0 = 0$

$|f(x) - L| < \epsilon$

$\left| x \sin\frac{1}{x} - 0 \right| < \epsilon$

$\left| x \sin\frac{1}{x} \right| < \epsilon$

We know that $\left|\sin\frac{1}{x}\right| \le 1$

$\therefore |x| \left|\sin\frac{1}{x}\right| < \epsilon$

$\therefore |x| < \epsilon$

$-\epsilon < x < \epsilon \to (1)$

$|x - x_0| < \delta$

$|x - 0| < \delta$

$|x| < \delta$

$-\delta < x < \delta \to (2)$

Comparing (1) and (2), $\delta = \epsilon$

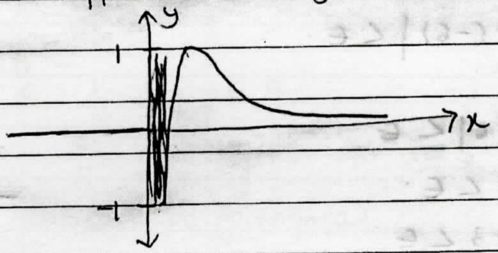
So for each $\epsilon > 0$, $\exists\ \delta > 0$ such that for all

$0 < |x| < \delta = \epsilon$

$|f(x) - L| < \epsilon$

3. $f(x) = \begin{cases} \sin(1/x) & x > 0 \\ 0 & x \le 0 \end{cases}$

(a) $\lim_{x \to 0^+} \sin \frac{1}{x}$ does not exist because $\sin \frac{1}{x}$ does not approach a single value as $x \to 0^+$ and it keeps oscillating between 1 and -1.



(b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0$

4. Use (ϵ, δ) approach to prove that

(a) $\lim_{x \to 0^-} \frac{x}{|x|} = -1.$

$f(x) = \frac{x}{|x|}$, $L = -1$, $x_0 = 0.$

$|f(x) - L| < \epsilon$

$\left| \frac{x}{|x|} - (-1) \right| < \epsilon.$

$\left| \frac{x}{|x|} + 1 \right| < \epsilon.$

Since $x \to 0^-$, $|x| = -x$

$\left| \frac{x}{-x} + 1 \right| < \epsilon.$

$|-1+1| < \epsilon.$

$0 < \epsilon$ which is always true independent of the value of x

Hence we can chose a $\delta > 0$ with $-\delta < x < 0$ so that $\lim_{x \to 0^-} \frac{x}{|x|} = -1.$

(b) $\lim_{x \to 2^+} \frac{x-2}{|x-2|} = 1$

$f(x) = \frac{x-2}{|x-2|}$, $L = 1$, $x_0 = 2$.

$|f(x) - L| < \epsilon$.

$\left| \frac{x-2}{|x-2|} - 1 \right| < \epsilon$.

$x \to 2^+$, so $|x-2| = x-2$.

$\left| \frac{x-2}{x-2} - 1 \right| < \epsilon$.

$0 < \epsilon$. which is always true independent of the value of x

Hence we can choose any $\delta > 0$, $2 < x < 2+\delta$

$\Rightarrow \left| \frac{x-2}{|x-2|} - 1 \right| < \epsilon$.

5. Using $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

(a) $\lim_{x \to 0} \frac{x^2 - x + \sin x}{2x} = \frac{1}{2} \lim_{x \to 0} \left[x - 1 + \frac{\sin x}{x} \right]$

$= \frac{1}{2} (0 - 1 + 1) = 0$.

(b) $\lim_{t \to 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$.

Let $y = 1 - \cos t$.

As $t \to 0$ $y = 1 - \cos t$

$y \to 1 - 1 = 0$

$\therefore \lim_{y \to 0} \frac{\sin y}{y} = 1$.

6. For what values of a and b, the function is continuous at every point

$$g(x) = \begin{cases} -2 & x \leq -1 \\ ax+b & -1 < x < 1 \\ 3 & x \geq 1. \end{cases}$$

$$\lim_{x \to -1^-} g(x) = -2.$$

$$\lim_{x \to -1^+} g(x) = a(-1)+b = b-a$$

$$\lim_{x \to 1^-} g(x) = a+b.$$

$$\lim_{x \to 1^+} g(x) = 3.$$

$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} g(x)$ so that $g(x)$ is continuous at $x=1$.

$$a+b = 3. \quad \rightarrow (1)$$

$$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^+} g(x)$$

$$b-a = -2. \quad \rightarrow (2)$$

Solving equations (1) and (2)

$$a = \frac{5}{2} \quad b = \frac{1}{2}$$

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7\. Show that the function

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

is discontinuos at every point

(a) Is $f(x)$ right continuos at any point ?

(b) " " left " " " " "

Suppose we want to check the continuity at x_0

Let x_0 be rational $\Rightarrow f(x_0) = 1$.

Choose $\epsilon = 1/2$.

For any $\delta > 0$, there is an irrational number x in the interval $(x_0 - \delta, x_0 + \delta) \Rightarrow f(x) = 0$.

Then $0 < |x - x_0| < \delta$ but $|f(x) - f(x_0)| = |0 - 1| = 1 > \frac{1}{2} = \epsilon$.

So $\lim_{x \to x_0} f(x)$ does not exist

f is discontinuos at rational

If x_0 is irrational $\Rightarrow f(x_0) = 0$, then there is a rational number x in $(x_0 - \delta, x_0 + \delta)$

$\Rightarrow f(x) = 1$.

Choose $\epsilon = 1/2$

$|f(x) - f(x_0)| = 1 > 1/2 = \epsilon$.

So $\lim_{x \to x_0} f(x)$ does not exist.

f is discontinuos at irrational

$\therefore$ f is discontinuos at every real number.

f is neither right/left continuos at any point x_0 because every interval $(x_0 - \delta, x_0)$ or $(x_0, x_0 + \delta)$ there exists both rational and irrational numbers.

8. If $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, would $\frac{f(x)}{g(x)}$ possibly be discontinuous at a point $[0,1]$? Give Reason.

Assume $f(x) = x$ and $g(x) = x - \frac{1}{2}$

Both $f(x)$ and $g(x)$ are continuous in $[0,1]$

But $\lim_{x \to \frac{1}{2}} \frac{f(x)}{g(x)} = \lim_{x \to \frac{1}{2}} \frac{x}{x - \frac{1}{2}}$ Does not Exist

9. Determine a, b, c

$$f(x) = \begin{cases} \frac{\sin((a+1)x) + \sin x}{x} & x < 0 \\ c & x = 0 \\ \frac{\sqrt{x + 5x^2} - \sqrt{x}}{bx^{3/2}} & x > 0 \end{cases}$$

* $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin((a+1)x) + \sin x}{x}$

$= \lim_{x \to 0^-} (a+1) \frac{\sin((a+1)x)}{(a+1)x} + \lim_{x \to 0^-} \frac{\sin x}{x}$

$= a+1+1 = a+2$

* $f(0) = c$

* $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}} = \lim_{x \to 0^+} \frac{x^{1/2}\sqrt{1 + bx} - 1}{bx \cdot x^{1/2}}$

$= \lim_{x \to 0^+} \frac{\sqrt{1+bx} - 1}{bx} \times \frac{\sqrt{1+bx} + 1}{\sqrt{1+bx} + 1} = \lim_{x \to 0} \frac{1 + bx - 1}{bx} \times \frac{1}{\sqrt{1+bx} + 1}$

$= \lim_{x \to 0} \frac{1}{\sqrt{1+bx} + 1} = \frac{1}{2}$

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$a+2 = c = \frac{1}{2} \Rightarrow c = \frac{1}{2}, \ a = -\frac{3}{2}$.

b is any non-zero real number.

10. $f(x) = \lfloor x \rfloor + \sqrt{x - |x|}$. Test continuity at $x=1$.

$$f(1) = \lfloor 1 \rfloor + \sqrt{1-|1|} = 1+0 = 1$$

$$\lim_{x \to 1^-} \lfloor x \rfloor + \sqrt{x-|x|} = 0 + \lim_{x \to 1^-} \sqrt{x-|x|}$$

$$= \lim_{h \to 0^+} \sqrt{(1-h) - |1-h|}$$

$$= 0$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \lfloor x \rfloor + \lim_{x \to 1^+} \sqrt{x - |x|}$$

$$= 1 + \lim_{h \to 0} \sqrt{(1+h) - |1+h|} = 1+0 = 1$$

LHL ≠ RHL

So not continuous at $x=1$.