Additional Discussions

1/. Refer to the example in Page 8: $\boxed{y'(x) = -y^2}$

The integral solution is $\boxed{y = \frac{1}{x+c}}$, which for $\boxed{y(0) = 1}$, gives $\boxed{y = \frac{1}{x+1}}$. We now use the initial condition $\boxed{y_\epsilon(0) = 1+\epsilon}$ on $\boxed{y'_\epsilon(x) = -[y_\epsilon(x)]^2}$. As before, the integral solution is $\boxed{y_\epsilon = \frac{1}{x+c}}$, which, for the given initial condition becomes $\boxed{y_\epsilon = \frac{1}{x+\frac{1}{1+\epsilon}}}$.

$$\therefore\ y - y_\epsilon = \frac{1}{x+1} - \frac{1}{x+\frac{1}{1+\epsilon}} = \frac{1}{x+1} - \frac{1+\epsilon}{1+x+x\epsilon}$$

$$\Rightarrow\ y - y_\epsilon = \frac{x+1+x\epsilon-(1+\epsilon)(1+x)}{(x+1)(1+x+x\epsilon)} = \frac{\cancel{x}+\cancel{1}+\cancel{x\epsilon}-\cancel{x}-\cancel{1}-\epsilon(\cancel{x}+1)}{(x+1)^2\left[1+\frac{x\epsilon}{x+1}\right]}$$

$$\Rightarrow\ \boxed{y - y_\epsilon = \frac{-\epsilon}{(x+1)^2\left[1+\frac{x\epsilon}{x+1}\right]} \simeq \frac{-\epsilon}{(x+1)^2}}$$

To a first order in ϵ, matching the known result.

2/. Numerical Stability and Implicit Methods

i) The Euler Method: $\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$

ii) The Backward Euler Method: $\boxed{y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})}$

The former is an explicit method, the latter is an implicit method. To test their stability consider now an example $\boxed{y'(x) = \lambda y}$ $\boxed{y(0) = 1}$

(P.T.O.)

The integral solution is $\boxed{y = e^{\lambda x}}$ for $y(0)=1$.
Under the restriction $\boxed{\lambda < 0}$ and the range $\boxed{x > 0}$, we see that as $\boxed{x \to \infty,\ y \to 0}$. By the Euler method $\boxed{y_{n+1} = y_n + h\lambda y_n}$.

$\therefore \boxed{y_{n+1} = (1+h\lambda)y_n} \Rightarrow \boxed{y_{n+2} = (1+h\lambda)y_{n+1}}$

Now $\boxed{y_0 = 1}$ (initial condition). $\therefore \boxed{y_1 = (1+h\lambda)y_0}$
$\Rightarrow y_1 = (1+h\lambda).1$. Further, $y_2 = (1+h\lambda)y_1 = (1+h\lambda)^2$

Hence, we can write $\boxed{y_n = (1+\lambda h)^n}$.

Now, $\boxed{x_n = x_0 + nh}$ $\because \boxed{x_0 = 0}$, $\boxed{x_n = nh}$.

For stable convergence we require $\boxed{\begin{array}{l} x_n \to \infty,\ n \to \infty. \\ y_n \to 0. \end{array}}$

In $\boxed{y_n = (1+\lambda h)^n}$ this is possible only when $\boxed{|1+\lambda h| < 1}$ $\therefore$ The stable convergence is violated when i) $\boxed{\lambda h > 0}$ OR ii) $\boxed{\lambda h < -2}$. Hence, the range for stability is $\boxed{-2 < \lambda h < 0}$.

$\Rightarrow 2 > (-\lambda)h > 0 \Rightarrow \boxed{0 < h < 2/(-\lambda)}$. For stability h is to be restricted in this range.

By the backward Euler method, we can write

$\boxed{y_{n+1} = y_n + h\lambda y_{n+1}} \Rightarrow y_{n+1}(1-h\lambda) = y_n$

$\therefore \boxed{y_{n+1} = (1-h\lambda)^{-1} y_n}$, $\boxed{y_{n+2} = (1-h\lambda)^{-1} y_{n+1}}$

(P.T.O.)

With $\boxed{y_0 = 1}$ (initial condition), $\boxed{y_1 = (1-\lambda h)^{-1} \cdot y_0}$

$\Rightarrow y_1 = (1-\lambda h)^{-1} \cdot 1$. Further, $y_2 = (1-\lambda h)^{-1} y_1 = (1-\lambda h)^{-2}$.

Hence, we can write $\boxed{y_n = (1-\lambda h)^{-n}}$.

Since $\boxed{h > 0}$ and $\boxed{\lambda < 0}$, $\boxed{1 - \lambda h > 1}$.

For $x_n = nh$ when $x_n \to \infty$, $n \to \infty$. This ensures $y_n \to 0$ for $\boxed{y_n = \dfrac{1}{(1-\lambda h)^n}}$. Hence, stable convergence is assured without any restriction on h in the implicit method.

Test numerically: $\boxed{\lambda = -100}$, $\boxed{x_n = nh}$, $\boxed{x_n = 0.2}$

No.	h	$n = \dfrac{x_n}{h}$	Euler method $y_n = (1+\lambda h)^n$	Backward Euler method $\cdot$ $y_n = (1-\lambda h)^{-n}$	Actual value
1.	0.1	2	81	8.26×10^{-3}	$y_n = e^{\lambda x_n}$
2.	0.05	4	256	7.72×10^{-4}	$= e^{-100 \times 0.2}$ $= e^{-20}$
3.	0.02	10	1	1.69×10^{-5}	$= 2.06 \times 10^{-9}$
4.	0.01	20	0	9.54×10^{-7}	Backward Euler
5.	0.001	200	7.06×10^{-10}	5.27×10^{-9}	gives stable convergence

i. Too small a value of h (step size) decreases the efficiency and the speed of the numerics.

ii. The optimisation is to maintain efficiency and stability even when h is NOT too small.

iii. Solve for y_{n+1} in the implicit method, by a root-finding method, such as bisection (generally).

Implicit Derivatives: Eulerian and Lagrangian Derivatives

Consider a function, $\boxed{\psi = \psi(t, x, y, z)}$.

$$\therefore d\psi = \frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial z} dz$$

$$\Rightarrow \frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x}\frac{dx}{dt} + \frac{\partial \psi}{\partial y}\frac{dy}{dt} + \frac{\partial \psi}{\partial z}\frac{dz}{dt}$$

$$\Rightarrow \frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \left(\hat{x}\frac{\partial \psi}{\partial x} + \hat{y}\frac{\partial \psi}{\partial y} + \hat{z}\frac{\partial \psi}{\partial z}\right) \cdot \left(\hat{x}\frac{dx}{dt} + \hat{y}\frac{dy}{dt} + \hat{z}\frac{dz}{dt}\right)$$

Noting that $\hat{x}, \hat{y}, \hat{z}$ are constant unit vectors, $\boxed{\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}}$, and $\boxed{\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$

we can write $\boxed{\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + (\vec{\nabla}\psi)\cdot\frac{d\vec{r}}{dt}}$.

which is similar to $\boxed{\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{dz}{dx}}$ in Page 16.

Recasting $\boxed{\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \frac{d\vec{r}}{dt}\cdot\vec{\nabla}\psi}$, we get

an operator $\boxed{\underbrace{\frac{d}{dt}}_{\text{Lagrangian derivative}} = \underbrace{\frac{\partial}{\partial t} + \frac{d\vec{r}}{dt}\cdot\vec{\nabla}}_{\text{Eulerian derivative}}}$

In Fluid Mechanics, treating a small fluid element as a particle, the variation is studied by the Lagrangian derivative. The variation of the fluid variables as a continuum is studied by the Eulerian derivative.

4. More Examples of Coupled Systems of Equations Like the Predator-Prey Model and Competitive Exclusion (Refer to Page 22).

(a) We have the Model of Conflict.

$$\boxed{\frac{dY_1}{dx} = AY_2 - BY_1 + G}, \quad \boxed{\frac{dY_2}{dx} = CY_1 - DY_2 + H}.$$

Y_1 and Y_2 are the war-waging (capacity) potential of two countries in conflict. A, G, C, H are the hawk parameters (aggressiveness), and B, D are the dove parameters (promotes peace).

All $\boxed{A, B, C, D, G, H > 0}$ (the dove parameters have negative signs in front.)

(b) The Models of Combat (Isolated combat with NO operational loss)

i.) If two conventional armies with sizes Y_1 and Y_2 are in a conventional-conventional combat, then

$$\boxed{\frac{dY_1}{dx} = -AY_2}, \quad \boxed{\frac{dY_2}{dx} = -BY_1}$$

A, B are combat effectiveness parameters.

ii.) If a guerilla army of strength Y_1 fights a conventional army of strength Y_2, then

$$\boxed{\frac{dY_1}{dx} = -CY_1Y_2}, \quad \boxed{\frac{dY_2}{dx} = -DY_1}.$$

C, D are combat effectiveness parameters

5/. Partial Differential Equations

In which a dependent variable depends on more than one independent variable.

Consider a simple second order partial Differential equation on $\boxed{u = u(x,y)}$,

$$\boxed{A u_{xx} + B u_{xy} + C u_{yy} = F(x, y, u, u_x, u_y)}$$

in which A, B, C are constants and F is a function. $\boxed{u_x = \frac{\partial u}{\partial x}}$, $\boxed{u_y = \frac{\partial u}{\partial y}}$

$\boxed{u_{xx} = \frac{\partial^2 u}{\partial x^2}}$, $\boxed{u_{yy} = \frac{\partial^2 u}{\partial y^2}}$, $\boxed{u_{xy} = \frac{\partial^2 u}{\partial x \partial y}}$

Denote a discriminant, $\boxed{\Delta = B^2 - 4AC}$.

i.) If $\boxed{\Delta < 0}$, equation is elliptic.
ii.) If $\boxed{\Delta = 0}$, equation is parabolic.
iii.) If $\boxed{\Delta > 0}$, equation is hyperbolic.

Example: $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)}$

$\Rightarrow A = 1,\ B = 0,\ C = 1.$ $\Rightarrow$ $\Delta = -4 < 0$.
$\therefore$ Equation is elliptic.

The above equation is Poisson's equation in two spatial dimensions. For $\boxed{f(x,y) = 0}$, we get Laplace's equation in two dimensions.

Example: $\boxed{a \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} \pm f(x,t)}$ $\boxed{a > 0}$.

$\Rightarrow$ $A = a,\ B = 0,\ C = 0 \Rightarrow \Delta = 0 \therefore$ Equation is parabolic.

The above equation is the heat equation or the diffusion equation in one spatial dimension.

Example: $\boxed{a \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial t^2} = \pm f(x,t)}$ $\boxed{a > 0}$.

$\Rightarrow$ $A = a,\ B = 0,\ C = -1 \Rightarrow \Delta = 4a > 0 \therefore$ Equation is hyperbolic.

The above equation is the wave equation in one spatial dimension ($\sqrt{a} \rightarrow$ velocity of the wave).

Some Test Cases

i.) $4 \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x,y)$. $\boxed{B = 0}$ $\boxed{A = 4,\ C = 1}$ $\boxed{\Delta = -16}$ (Elliptic)

ii.) $4 \dfrac{\partial^2 u}{\partial x^2} + 4 \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = f(x,y)$ $\boxed{A = 4,\ B = 4,\ C = 1}$. $\boxed{\Delta = 0}$ (Parabolic)

iii.) $4 \dfrac{\partial^2 u}{\partial x^2} - 8 \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = f(x,y)$ $\boxed{A = 4,\ B = -8,\ C = 1}$ $\boxed{\Delta = 48}$ (Hyperbolic)

I.) If u is prescribed on the boundary, then that gives the Dirichlet boundary condition.

II.) If for a second-order differential equation, the boundary conditions involve the first partial derivative then that gives Neumann boundary condition.