$\boxed{\text{CCV}-1}$ SC216 Calculus with Complex Variables — Manish K. Gupta

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<u>Set</u> A collection of well defined objects

$A \cup B \qquad A \cap B$

$x \in B$

element x is a member of set B

$\boxed{\text{Number system}}$

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$ Set of Natural Numbers

$\downarrow \qquad x + 2 = 0$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ Set of Integers

$\downarrow \qquad 2x = 1$

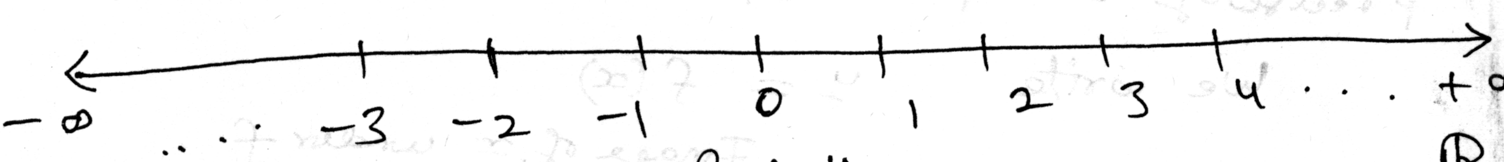
$\mathbb{Q}$ = Set of Rational Numbers $= \{p/q \qquad \}$

$\downarrow \qquad x^2 = 2$

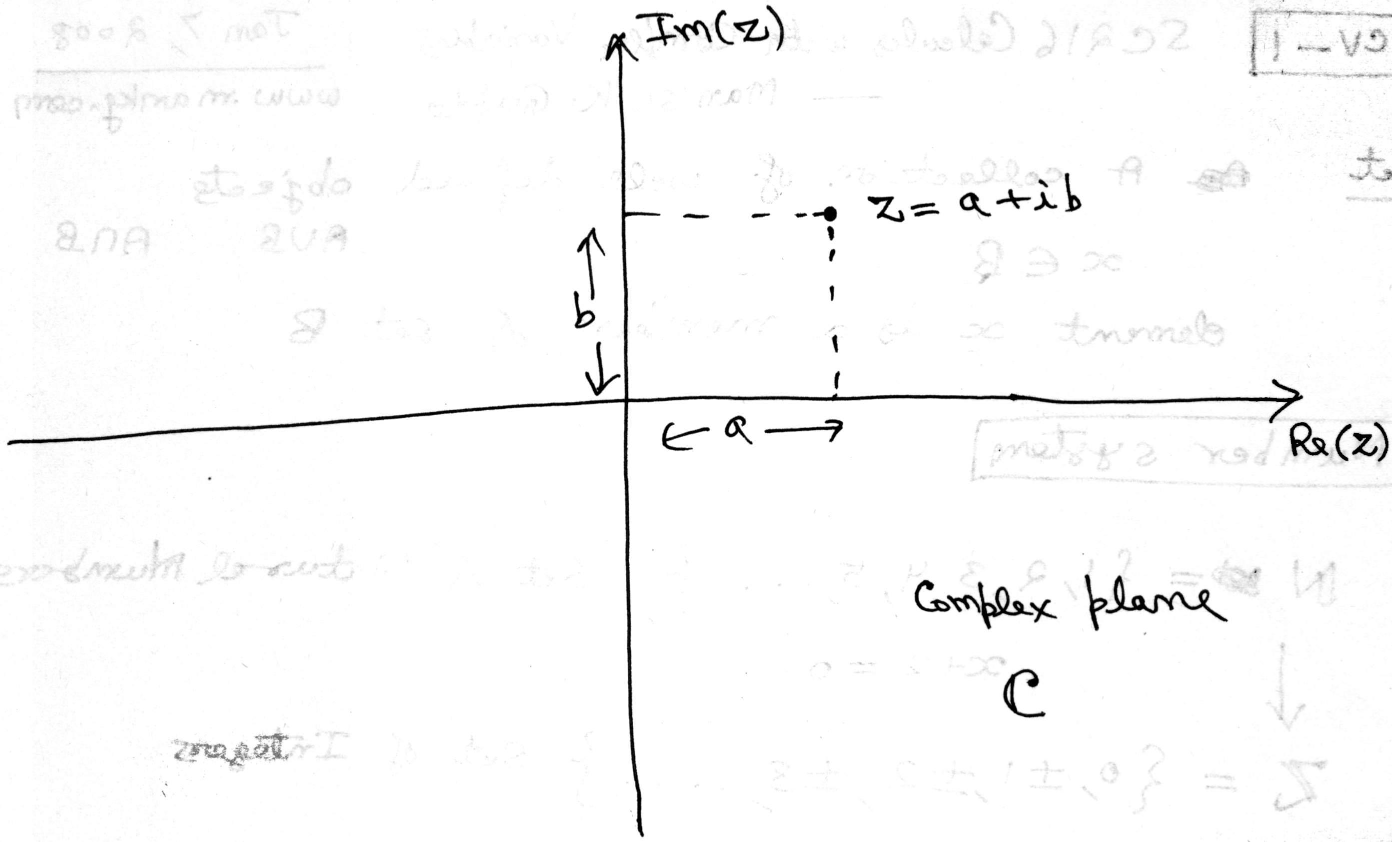
$\mathbb{R} = \overline{\mathbb{Q}}$ = Set of Real Numbers

$\downarrow \qquad x^2 + 1 = 0$

$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R},\ i = \sqrt{-1}\}$
= Set of Complex Numbers



Real line



Complex plane

$\mathbb{C}$

<u>Def 1:</u> A & B are sets

$$A \times B = \{ (a, b) \mid a \in A, b \in B \}$$

Cartesian product of A & B

Example:

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{ (a, b) \mid a \in \mathbb{R}, b \in \mathbb{R} \}$$

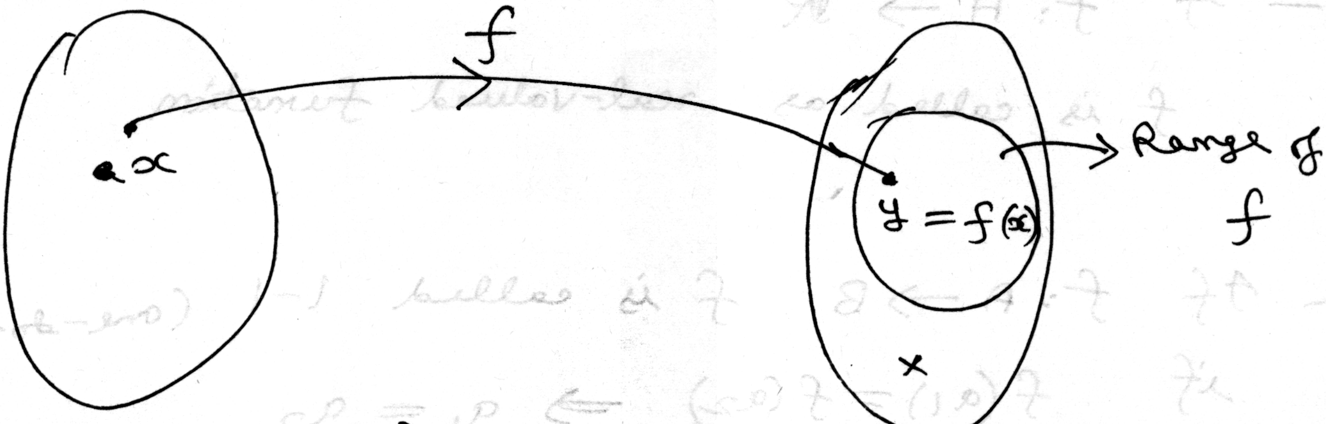
Plane

<u>Def 2:</u> Function Sets A & B

$f : A \to B$ is a subset of $A \times B$ with the property that each $a \in A$ belongs to precisely one pair (a, b)

We write $y = f(x)$

Image of x under f



A (Domain)

B

A <u>function</u> is a rule which associates to every member $x \in A$ giving a unique ! image in B

Domain of f : Set A

Range of f: $\{b \in B \mid b = f(a) \text{ for some } a\}$

<u>Example:</u>

$$f = \{(x, x^2) \mid -\infty < x < \infty\}$$

$$f(x) = x^2 \quad (-\infty < x < \infty)$$

$\text{Domain}(f) = \mathbb{R}$

$\text{Range}(f) = [0, \infty) = \{0 \leq x < \infty\}$

$f(2) = 4 \qquad f^{-1}(4) = \{-2, 2\}$

$f^{-1}(-7) = \phi$

$\cancel{f(x) = x^2}$ $f(\{x \mid x^2 = 9\}) = \{9\}$

$f([0, 3)) = [0, 9)$

- If $f : A \to \mathbb{R}$

  f is called as real-valued function

- If $f : A \to B$ f is called 1-1 (one-to-one)

  if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

- f is 1-1 if $f^{-1}(b)$ contains precisely one element $\forall b \in$ Range of f.

<u>Example</u>:

$$f(x) = x^2 \quad (-\infty < x < \infty)$$

is not 1-1 (Why?)

$$g(x) = x^2 \quad (0 \leq x < \infty) \text{ is 1-1}$$

(Why?)

- If $f : A \to B$ and Range of $f = B$

  <u>onto fn.</u>

- A 1-1 & onto fn. is called a bijection.

- ~~Cout~~ $\boxed{\text{Countable Set}}$

  The set A is said to be countable

  if $\exists$ 1-1 fn. f from $\mathbb{N}$ onto A

  $$A = \{f(1), f(2), \dots\}$$

Example Set of all integers $\mathbb{Z}$ is countable

$$f(n) = \frac{n-1}{2} \quad (n = 1, 3, 5, \dots)$$

$$f(n) = -\frac{n}{2} \quad (n = 2, 4, 6, \dots)$$

(Exercise)
Prove that the set of rational numbers $\mathbb{Q}$ is countable.

Example: The set of all rational numbers in $[0,1]$ is countable

$$Q[0,1] = \{x \mid 0 \le x \le 1 \text{ and } x \text{ is a rational number}\}$$

$\boxed{\text{Real Numbers}}$ $\mathbb{R}$

$$x \in \mathbb{R}$$

$$x = b.a_1 a_2 a_3 \ldots$$

$$= b_1 + \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots$$

$0 \le a_i \le 9$ a_i are integers

$$x \in [0,1] = \{x \mid 0 \le x \le 1\}$$

$$x = 0.a_1 a_2 a_3 \ldots\ldots$$

$\boxed{\text{The set } [0,1] = \{x \mid 0 \le x \le 1\} \text{ is uncountable}}$

Remark: $a, b \in \mathbb{R}$ and $a < b$ then there is a rational no. $x \in \mathbb{Q}$ & an irrational no. $y \notin \mathbb{Q} \subset \mathbb{R}$ s.t.

$$a < x < b \quad \& \quad a < y < b$$

$\boxed{\text{The set of all real numbers } \mathbb{R} \text{ is uncountable}}$

— (Bounded Above)

The subsets $A \subset \mathbb{R}$ is said to be bounded above if there is a number $N \in \mathbb{R}$ s.t.

$x \leq N \;\; \forall x \in A$ $\qquad N \to$ upper bound (u.b.)

— (Bounded Below)

The subset $A \subset \mathbb{R}$ is said to be bounded below if there is a number $M \in \mathbb{R}$ s.t.

$M \leq x \;\; \forall x \in A$ $\qquad M \to$ lower bound (l.b.)

— (Bounded)

If A is both bounded below & above we call it bounded

~~$\Rightarrow$ A is bounded $\Leftrightarrow$ A C~~

Examples:

— $\mathbb{N}$ is bounded below but not above
$\Rightarrow \mathbb{N}$ is not bounded

— $[0,1]$ is bounded

— l.b. $\{\mathbb{N}\} = -7$

— u.b. $\left\{\left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots, \frac{2^n-1}{2^n}, \ldots\right\}\right\} = 1$

$\boxed{\text{l.u.b}}$ Suppose $A \subset \mathbb{R}$ is bounded above.
A no. L is called l.u.b. for A if

1. L is an upper bound of A
2. No number smaller ~~to~~ than L is an upper bound for A

Similarly g.l.b (greatest lower bound)

~~Eg~~ Eg.

Examples

$$B = \left\{ \frac{1}{2}, \frac{3}{4}, \ldots, \frac{(2^n-1)}{2^n}, \ldots \right\}$$

$$\underset{x \in B}{\text{g.l.b.}} \{x\} = \frac{1}{2}$$

$$\underset{x \in B}{\text{l.u.b.}} \{x\} = 1 \quad \text{(Verify!)}$$

— $\text{g.l.b.} \{\mathbb{N}\} = 1$

— $\text{l.u.b.} \{0\} = 0 = \text{g.l.b.} \{0\}$

$\boxed{\text{Important property of } \mathbb{R}}$

If A is any $(\neq \phi)$ subset of $\mathbb{R}$ that is bounded above, then A has a least upper bound in $\mathbb{R}$.

(Roughly this says $\mathbb{R}$ has no holes in it)

— $\mathbb{Q}$ does have ~~a~~ ~~whole~~ holes in it.

e.g. $A = \{1, 1.4, 1.41, 1.414, \ldots\}$

then $\text{l.u.b} (A) = \sqrt{2} \notin \mathbb{Q} \subset \mathbb{R}$

\#\# If A is any $(\neq \phi) \subset \mathbb{R}$ that is bounded below, then A has a g.l.b. in $\mathbb{R}$.

Sequences of real Numbers

$S = \{s_i\}_{i=1}^{\infty}$ $s_i \in \mathbb{R}$ is a fn. from $\mathbb{N}$ into $\mathbb{R}$

$s(i)$

$\{s_i\}_{i=1}^{n}$ finite seq.

$s_1, s_2, s_3, \ldots\ldots s_i, s_{i+1}, \ldots\ldots$

Subseq.

$s_{n_1}, s_{n_2}, s_{n_3}, \ldots$

Example

$s_1 = 1$ Fibonacci seq.

$s_2 = 1$

$s_{n+1} = s_n + s_{n-1} \quad (n \geq 3)$

Limit of a seq.

$\{s_n\}_{n=1}^{\infty}$ has a limit L if

$s_n - L$ is "small" for sufficiently large values of n.

Example:

$1, 1, 1, 1, \ldots\ldots$ has limit $L = 1$

$1, \frac{1}{2}, \frac{1}{3}, \ldots$ has limit 0

$1, -2, 3, -4, \ldots\ldots$ does not have a limit

Def: $\{s_n\}_{n=1}^{\infty}$ Seq. of real numbers

$s_n \to L$ (as $n \to \infty$) if $\forall \epsilon > 0$, there is a +ve integer N s.t.

$$|s_n - L| < \epsilon \quad (\forall n \geq N)$$

$$\lim_{n\to\infty} s_n = L$$

$\boxed{\text{Example:}}$ $s_n = \frac{1}{n}$

Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$

□ We need to show that ~~Given~~ given $\epsilon > 0$ we must find N so that $\left|\frac{1}{n} - 0\right| < \epsilon$ $\forall n \geq N$

$\Rightarrow \quad \frac{1}{n} < \epsilon \quad (n \geq N)$

$\Rightarrow$ if we choose N so that $1/N < \epsilon$ then this will hold.

$\because \frac{1}{n} \leq \frac{1}{N}$ if $n \geq N$

but $1/N < \epsilon \Leftrightarrow N > \frac{1}{\epsilon}$

$\therefore$ if we choose $N \in \mathbb{N}$ s.t. $N > \frac{1}{\epsilon}$

then $\frac{1}{n} < \epsilon \quad \forall n \geq N$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} = 0$$

$\boxed{\text{Ex-2}}$ $s_n = 1 \quad \{n \in \mathbb{N}\}$

$s_1 = 1$
$s_2 = 1$
$\vdots$

$s_n - L = 1 - 1 = 0$ so ~~∀~~ for any $\epsilon > 0$

If we take $N = 1$ for any $\epsilon > 0$

$$|s_n - L| < \epsilon \quad (n \geq 1)$$

$$\Rightarrow \lim_{n \to \infty} \{s_n\} = 1$$

$\boxed{\text{Ex-3}}$ Show that the seq. $s_n = n$ does not have a limit

□ Suppose on the contrary,

$$\lim_{n \to \infty} s_n = L \text{ for some } L \in \mathbb{R}$$

$\Rightarrow$ for any ϵ there is N for which

$$|s_n - L| < \epsilon \quad (n \geq N)$$

Take $\epsilon = 1$

$$\Rightarrow |s_n - L| < 1 \quad (n \geq N)$$

$$\Rightarrow -1 < s_n - L < 1 \quad (n \geq N)$$

$$\Rightarrow L - 1 < n < L + 1 \quad (n \geq N)$$

This says all values of n > ~~than~~ N lie between $L-1$ & $L+1$ which is false ∴ a contradiction. ▨

Exercise: Show that $s_n = (-1)^n$ does not have a limit.

Convergent seq.

If $\lim_{n\to\infty} s_n = L$ seq. s_n is convergent

Example: $\frac{1}{n}$ is conv. seq. $(\because \frac{1}{n} \to 0$ as $n \to \infty)$

$(-1)^n$ is <u>divergent seq.</u>

Bounded seq. $\{s_n\}_{n=1}^{\infty}$ is bounded iff $\exists M \in \mathbb{R}$ s.t. $|s_n| \le M \quad (n \in \mathbb{N})$

Theorem Every conv. seq. is bounded.

Cauchy seq $\{s_n\}_{n=1}^{\infty}$ a seq. of real ~~no~~ numbers. is called Cauchy seq. if for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|s_m - s_n| < \epsilon \quad (m, n \ge N)$$

- Every convergent seq. is Cauchy. & vice versa
- A Cauchy seq. of real numbers is bounded
-

Series of real numbers

$$\lim_{n\to\infty}(a_1+a_2+\dots+a_n) = a_1+a_2+\dots\dots$$

infinite series

$\{a_n\}_{n=1}^{\infty}$ seq. of real nos.

If $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}\{s_n = a_1+a_2+\dots+a_n\}$ partial sum

exist. we say series $\sum_{n=1}^{\infty} a_n$ is convergent.

Example

— $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ is convergent

Hint Use $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ & then compute partial sum

— $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent

— $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges

— $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.