Tutorial-10

Q.1 Integrate the following :

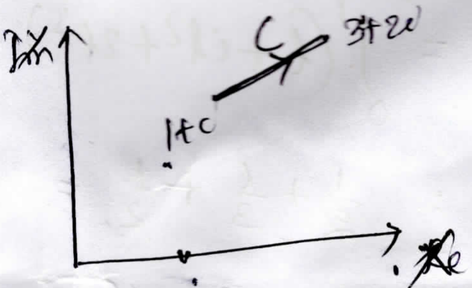
(a) $\int_C \text{Re}\, z \, dz$, where C is the shortest path from $1+i$ to $3+2i$

(b) $\int_C \bar{z} \, dz$, where C from 0 along the parabola $y = x^2$ to $1+i$

(c) $\int_C z e^{z^2} dz$, where C from 1 along the axes to i.

(d) $\int_C \sec^2 z \, dz$, where C is any path from $\frac{\pi i}{4}$ to $\pi/4$ in the unit disk.

Sol^n (a) $\int_C \text{Re}\, z \, dz$



Parametric representation of C is

$$r(t) = (1-t)(1+i) + t(3+2i), \quad 0 \le t \le 1$$

$$= 1-t+3t + (1-t)i + 2ti$$

$$= 1+2t + (1+t)i$$

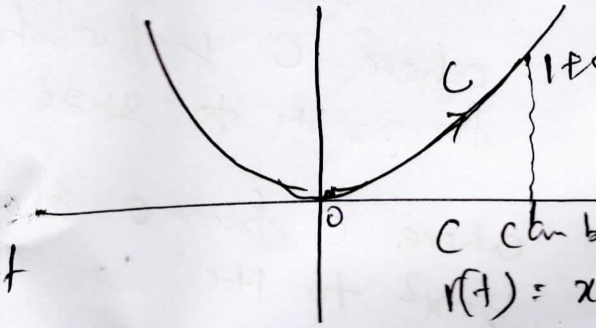
$$r'(t) = 2+i$$

$$\int_C \text{Re}\, z \, dz = \int_0^1 \text{Re}(r(t))\, r'(t)\, dt$$

$$= \int_0^1 (1+2t)(2+i)\, dt = (2+i)\int_0^1 (1+2t)\, dt$$

$$= (2+i)\left[t+t^2\right]_0^1 = 2(2+i) = 4+2i$$

(b) $\int_C \bar{z}\,dz$, where C from 0 along the parabola $y = x^2$ to $1+i$



C can be represented as
$r(t) = x(t) + i\,y(t)$

$\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$

$r(t) = t + it^2$

$r'(t) = 1 + 2it$

$0 \le t \le 1$

$$\int_C \bar{z}\,dz = \int_0^1 \overline{r(t)}\, r'(t)\,dt$$

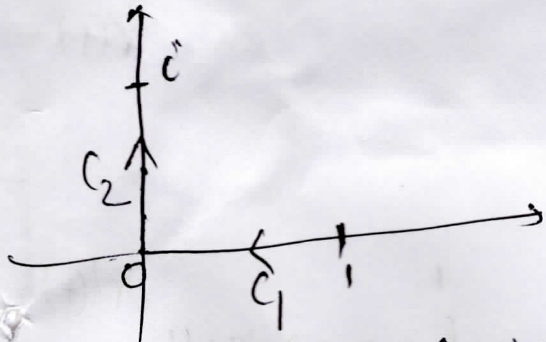
$$= \int_0^1 (t - it^2)(1+2it)\,dt$$

$$= \int_0^1 (t + 2it^2 - it^2 + 2t^3)\,dt$$

$$= \int_0^1 (t + it^2 + 2t^3)\,dt = \left[\frac{t^2}{2} + \frac{it^3}{3} + \frac{2t^4}{4}\right]_0^1$$

$$= \frac{1}{2} + \frac{i}{3} + \frac{1}{2} = 1 + \frac{i}{3}$$

(c) $\int_C z e^{z^2}\,dz$, where C from 1 to i along the axes.



$$= \int_{C_1} z e^{z^2}\,dz + \int_{C_2} z e^{z^2}\,dz$$

$$= \int_0^1 f(r_1(t))\, r_1'(t)\,dt + \int_0^1 f(r_2(t))\cdot r_2'(t)\,dt$$

$C_1 : r_1(t) = 1(1-t) + 0\cdot t = 1-t$, $0 \le t \le 1$

$C_2 : r_2(t) = 0(1-t) + t\cdot i = it$, $0 \le t \le 1$

$$= \int_0^1 (1-t) e^{(1-t)^2} (-1)\,dt$$

$$+ \int_0^1 it\, e^{(it)^2}\cdot i\,dt$$

$$= \frac{1}{2}\int_1^0 e^u\,du + \frac{1}{2}\int_0^{-1} e^v\,dv$$

$$= \frac{1}{2}\left[e^u\right]_1^0 + \frac{1}{2}\left[e^v\right]_0^{-1} = \frac{1}{2}(1-e) + \frac{1}{2}(e^{-1}-1)$$

$$= \frac{1}{2}(e^{-1} - e)$$

$$= -\sinh 1$$

$(1-t)^2 = u$ $\quad 1 \le u \le 0$

$2(1-t)(-1)\,dt = du$

$\Rightarrow -(1-t)\,dt = \frac{1}{2}du$

$(it)^2 = v$ $\quad 0 \le v \le -1$

$\Rightarrow 2(it)\cdot i\,dt = dv$

$(it)\,i\,dt = \frac{dv}{2}$

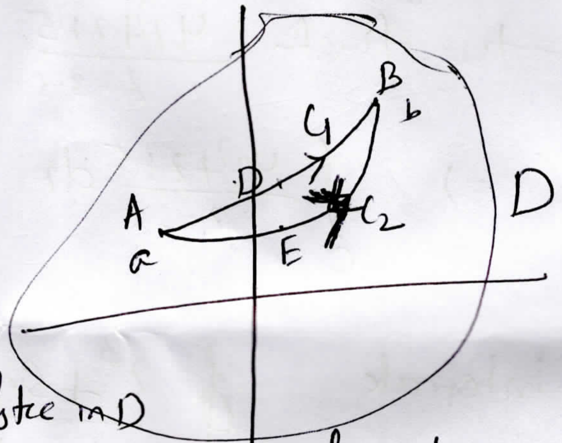
(a) $\int_C \sec^2 z\, dz$, C is any path from $\frac{\pi}{4}$ to $\frac{i\pi}{4}$ entire complex plane.

Sol^n $\int_C \sec^2 z\, dz = \left[\tan z\right]_{\pi/4}^{i\pi/4}$

$= \tan \frac{\pi}{4} - \tan i\frac{\pi}{4} = 1 - i \tanh \frac{\pi}{4}$

Q.2 If $f(z)$ is analytic in a simply connected domain D. Prove that $\int_a^b f(z)dz$ is independent of the path in D joining any two points a and b in D.

Sol^n



By Cauchy's Theorem

$$\int_{ADBEA} f(z)dz = 0$$

as $f(z)$ is analytic in D

as ADBEA is a simple closed path.

$$\Rightarrow \int_{ADB} f(z)dz + \int_{BEA} f(z)dz = 0$$

$$\Rightarrow \int_{ADB} f(z)dz = \int_{-BEA} f(z)dz = \int_{AEB} f(z)dz$$

$$\Rightarrow \int_{C_1} f(z)dz = \int_{C_2} f(z)dz = \int_a^b f(z)dz$$

Hence the integral is independent of the path followed.

Q-3 Integrate $\oint_C \frac{4z^2+z+5}{z-3\cdot5} dz$ where C is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Sol^n

$z=3\cdot5$ is the only singular point (that is a point where function is not analytic & analytic at every where else)

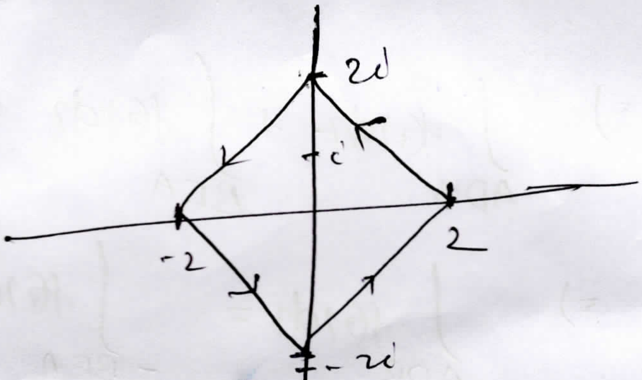
Also $z=3\cdot5$ is outside the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

i.e $\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$

So by Cauchy's theorem the funct $\frac{4z^2+z+5}{z-3\cdot5}$ is analytic on an inside the curve C.

$$\Rightarrow \int_C \frac{4z^2+z+5}{z-3\cdot5} dz = 0.$$

Q-4 Integrate $\oint \frac{z^3+\sin z}{(z-i)^3} dz$, where C is the boundary of the square with vertices $\pm2, \pm2i$



Sol

$f(z)=z^3+\sin z$ is analytic

The integral $\oint \frac{f(z)}{(z-i)^3} dz$

The point i is inside the simple closed curve C.

By Cauchy integral formula

$$\oint \frac{f(z)}{(z-i)^3} dz = \frac{2\pi i}{2!} f^{(2)}(i) = \pi i f^{(2)}(i)$$

$f(z)=z^3+\sin z$

$f'(z)=3z^2+\cos z$

$f''(z)=6z-\sin z$

$f''(i)=6i-\sin i$

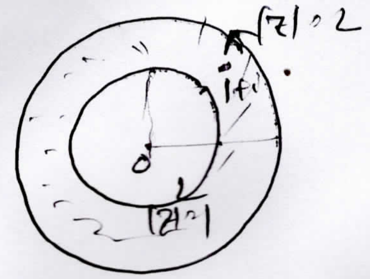
$$=\pi i(6i-\sin i)$$

$$=-6\pi - \pi i \sin i$$

Q.5 Integrate $\oint_C \frac{2z^3-3}{z(z-1-i)^2}\,dz$ C: $|z|=2$ anticlockwise and $|z|=1$ clockwise.

Sol^n

~~Here $f(z) = 2z^3-3$~~



$z=0$ & $z=1+i$ are the points to be considered.

$z=0$ is outside the region of consideration

$z=1+i$ is inside the region of consideration

We take here $f(z) = \frac{2z^3-3}{z}$ which is analytic inside the region.

So by Cauchy's integral formula

$$\oint \frac{\frac{2z^3-3}{z}}{(z-1-i)^2}\,dz = \frac{2\pi i}{1!} f'(1+i)$$

$$= 2\pi i\, f'(1+i)$$

$$= 2\pi i \frac{(8i-5)}{2i}$$

$$= \pi(8i-5)$$

$$= -5\pi + 8\pi i$$

(Ans)

$$f(z) = \frac{2z^3-3}{z}$$

$$f'(z) = \frac{6z^2 \times z - (2z^3-3)\cdot 1}{z^2}$$

$$= \frac{6z^3-2z^3+3}{z^2}$$

$$f'(1+i) = \frac{4(1+i)^3+3}{(1+i)^2}$$

$$= \frac{4(1+i^3+3i+3i^2)+3}{1+i^2+2i}$$

$$= \frac{4(1-i+3i-3)+3}{1-1+2i}$$

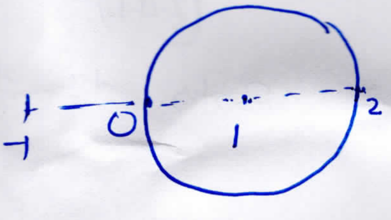
$$= \frac{4(2i-2)+3}{2i} = \frac{8i-5}{2i}$$

Q-6 Evaluate $\int_C \frac{3z^2+z}{z^2-1}\,dz$, where C is the circle $|z-1| = 1$

Sol^n The integrand is not analytic at $z=1$ and $z=-1$.

C is the circle $|z-1|=1$

The singular point 1 is inside C where as the singular point -1 is outside C.



Also $f(z) = 3z^2+z$ is analytic inside and on the curve C.

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$

$$\int_C \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2}\int_C \frac{3z^2+z}{z-1}\,dz + \frac{1}{2}\int_C \frac{3z^2+z}{z+1}\,dz$$

By Cauchy integral formula

$$\int_C \frac{3z^2+z}{z-1}\,dz = 2\pi i\, f(1) = 2\pi i\,(3+1) = 8\pi i$$

$$\int_C \frac{3z^2+z}{z+1}\,dz = 0$$ as by Cauchy's theorem $\frac{3z^2+z}{z+1}$ is analytic inside and on C.

So $\int_C \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2} \times 8\pi i + \frac{1}{2} \times 0 = 4\pi i$

Q.7 Evaluate $\oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz$ if $t>0$ and C is the circle $|z|=3$.

Sol^n The singular points of the integrand are $+i$ and $-i$.

As $\frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2} =$

Both $+i$ and $-i$ are inside the circle $|z|=3$.

$$\frac{1}{(z+i)(z-i)} = \frac{-\frac{1}{2i}}{z+i} + \frac{\frac{1}{2i}}{z-i}$$ By partial fraction

$$\text{So } \frac{1}{(z+i)^2(z-i)^2} = \frac{-\frac{1}{4}}{(z+i)^2} + \frac{-\frac{1}{4}}{(z-i)^2} + \frac{1}{2}\left(\frac{1}{(z+i)(z-i)}\right)$$

$$= \frac{-\frac{1}{4}}{(z+i)^2} + \frac{-\frac{1}{4}}{(z-i)^2} + \frac{1}{2}\left[\frac{-\frac{1}{2i}}{z+i} + \frac{\frac{1}{2i}}{z-i}\right]$$

$$= \frac{-\frac{1}{4}}{(z+i)^2} + \frac{-\frac{1}{4}}{(z-i)^2} + \frac{-\frac{1}{4i}}{z+i} + \frac{\frac{1}{4i}}{z-i}$$

$$\text{So } \oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz = \oint_C \frac{e^{zt}}{(z+i)^2(z-i)^2}\,dz$$

$$= -\frac{1}{4}\oint_C \frac{e^{zt}}{(z+i)^2}\,dz + -\frac{1}{4}\oint_C \frac{e^{zt}}{(z-i)^2} + -\frac{1}{4i}\oint_C \frac{e^{zt}}{z+i}\,dz + \frac{1}{4i}\oint_C \frac{e^{zt}}{z-i}\,dz$$

$$= -\frac{1}{4}\left[\frac{2\pi i}{1!} f'(-i)\right] \cancel{+} -\frac{1}{4}\left[\frac{2\pi i}{1!} f'(i)\right] - \frac{1}{4i}\left[\frac{2\pi i}{0!} f(-i)\right] + \frac{1}{4i}\left[\frac{2\pi i}{0!} f(i)\right]$$

$$= -\frac{1}{4}\left[2\pi i\, t e^{-it}\right] - \frac{1}{4}\left[2\pi i\, t e^{it}\right]$$
$$-\frac{1}{4i}\left[2\pi i\, e^{-it}\right] + \frac{1}{4i}\left[2\pi i\, e^{it}\right]$$

$$= -\frac{1}{2}\pi i t e^{-it} - \frac{1}{2}\pi i t e^{it} - \frac{\pi}{2}e^{-it} + \frac{\pi}{2}e^{it}$$ (Ans)

$$= -i\pi t \cos t + i\pi \sin t$$

$f(z) = e^{zt}$
$f'(z) = te^{zt}$
$f'(i) = te^{it}$
$f'(-i) = te^{-it}$
$f(i) = e^{it}$
$f(-i) = e^{-it}$