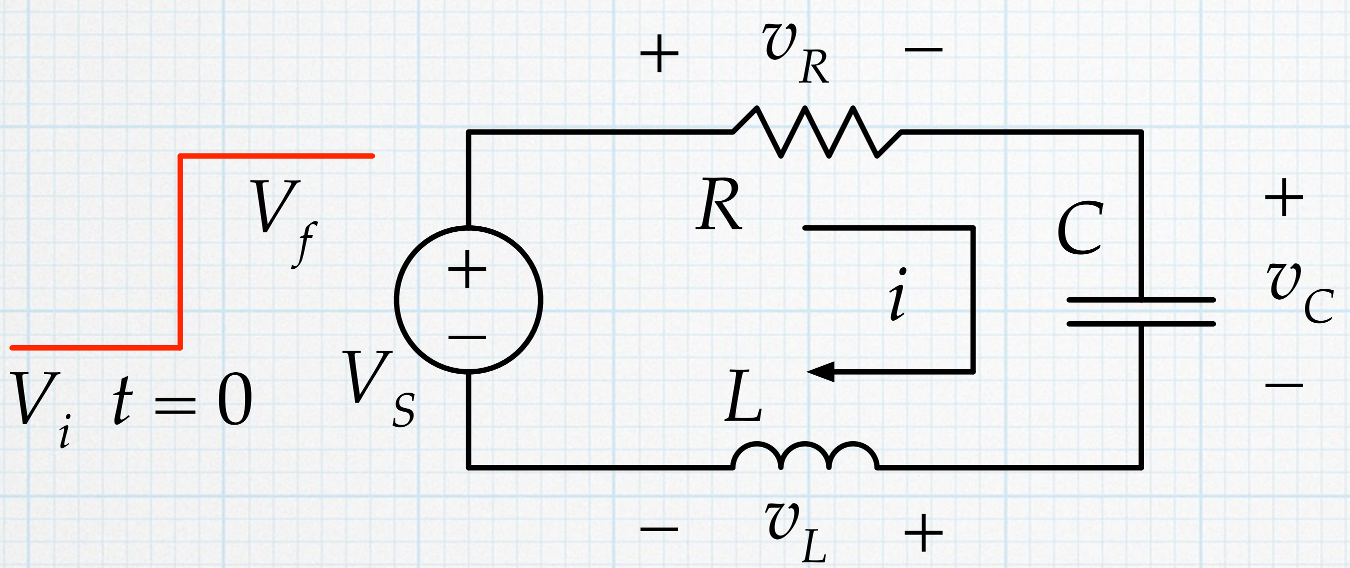
RLC transients

When there is a step change (or switching) in a circuit with capacitors and inductors together, a transient also occurs. With some differences:

- Energy stored in capacitors (electric fields) and inductors (magnetic fields) can trade back and forth during the transient, leading to possible "ringing" effects.
- The transient waveform can be quite different, depending on the exact relationship of the values of C, L, and R.
- The math is more involved.

series RLC

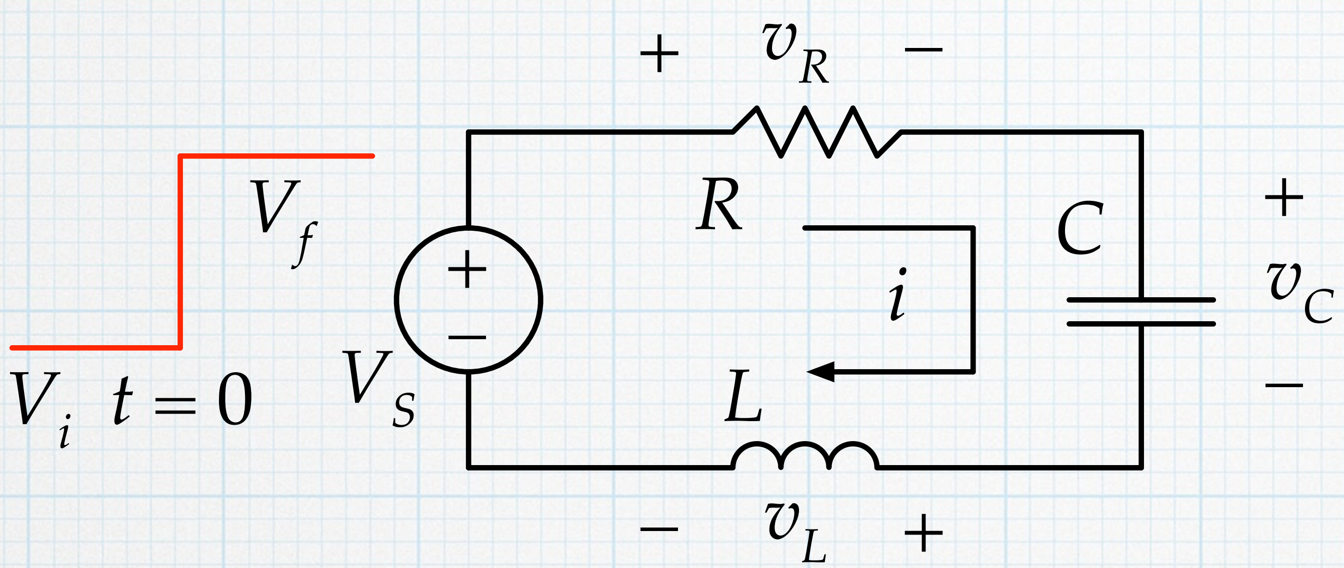


$$V_S = v_R + v_C + v_L$$

Voltage source makes an abrupt change from V_i to V_f at t = 0.

t < 0:	$i = 0.$	t >> τ:	$i = 0.$
	$v_R = 0.$		$v_R = 0.$
	$v_L = 0.$		$v_L = 0.$
	$v_C = V_i.$		$v_C = V_f.$

What is τ? What happens in between?



$$t > 0: \quad V_f = iR + v_c + L\frac{di}{dt}$$

$$i = C\frac{dv_C}{dt}$$

$$V_f = RC\frac{dv_C}{dt} + v_C + LC\frac{d^2v_C}{dt^2}$$

$$\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{V_f}{LC}$$

Second-order differential equation.

Diff. Eq. – a bit of a review (or preview)

$$\frac{d^2 f(x)}{dx^2} + a\frac{df(x)}{dx} + bf(x) = g(x)$$

g(x) is a "forcing function"

$$f(x) = f_{tr}(x) + f_{ss}(x)$$

$f_{tr}(x)$ → transient solution (homogenous)

$f_{ss}(x)$ → steady-state solution (particular)

$f_{ss}(x)$ is any function that you can find that is a solution to the full differential equation. Usually find it with trial-and-error. The form of *g(x)* suggests what to try.

$f_{tr}(x)$ is the solution to the homogenous differential equation.

$$\frac{d^2 f_{tr}(x)}{dx^2} + a\frac{df_{tr}(x)}{dx} + bf_{tr}(x) = 0$$

The solution will have two unknown constants that will be specified using initial conditions, *f(0)* and *df(0)/dx*.

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{V_f}{LC}$$

$$v_C(t) = v_{tr}(t) + v_{ss}(t)$$

Since the forcing function is a constant, try using $v_{ss}(t)$ = constant = V_{ss}.

$$\frac{d^2 V_{ss}}{dt^2} + \frac{R}{L}\frac{dV_{ss}}{dt} + \frac{V_{ss}}{LC} = \frac{V_f}{LC}$$

$V_{ss} = V_f$. (Easy!)

(We could have determined this from the physics of the circuit, also.)

$$v_C(t) = v_{tr}(t) + V_f$$

Now we must find the transient part.

homogenous solution

$$\frac{d^2 v_{tr}}{dt^2} + \frac{R}{L}\frac{dv_{tr}}{dt} + \frac{v_{tr}}{LC} = 0$$

Guess: $v_{tr} = A\exp(st)$ Will need to determine s.

$$s^2\left(Ae^{st}\right) + \frac{R}{L}s\left(Ae^{st}\right) + \frac{1}{LC}\left(Ae^{st}\right) = 0$$

$$\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)\left(Ae^{st}\right) = 0$$

$Ae^{st} \neq 0$ so $\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$v_{tr}(t) = Ae^{s_1 t} + Be^{s_2 t}$$

Initial conditions

$$v_C(t) = Ae^{s_1 t} + Be^{s_2 t} + V_f$$

Need to determine A and B. Use initial conditions, $v_c(0)$ and $dv_c(0)/dt$.

The details will depend on s_1 and s_2, and, as we will see, there are three possible cases.

In any case:

$$v_C(0) = V_i$$

$$\frac{dv_C(t)}{dt} = \frac{i_C(t)}{C}$$

But don't necessarily know $i_C(0)$ since capacitor current can change abruptly. However, in the series circuit, $i_C(t) = i_L(t)$. Inductor current cannot change abruptly. So

$$\frac{dv_C(t)}{dt} = \frac{i_L(t)}{C}$$

$$\left.\frac{dv_C(t)}{dt}\right|_{t=0} = \frac{i_L(0)}{C} = 0$$

For this particular case. May not be true in every case.

roots of characteristic equation

Transient behavior depends on the values of s_1 and s_2.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Rename things slightly: $R/2L = \alpha$ and $1/LC = \omega_o$.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

α is the *damping factor* or *decay constant* [s^{-1}]

ω_o is the *resonant frequency* or *undamped natural frequency* [radian/s].

Overdamped response

If $\dfrac{R}{2L} > \dfrac{1}{\sqrt{LC}}$ $(\alpha > \omega_o)$ then s_1 and s_2 are both real and negative.

The capacitor voltage consists of two decaying exponentials.

$$v_C(t) = Ae^{s_1 t} + Be^{s_2 t} + V_f$$

A fast, one-way trip from V_i to V_f.

Finally, use initial conditions to find A and B

$$v_c(0) = V_i = A + B + V_f$$

$$\left.\frac{dv_C}{dt}\right|_{t=0} = 0 = s_1 A + s_2 B$$

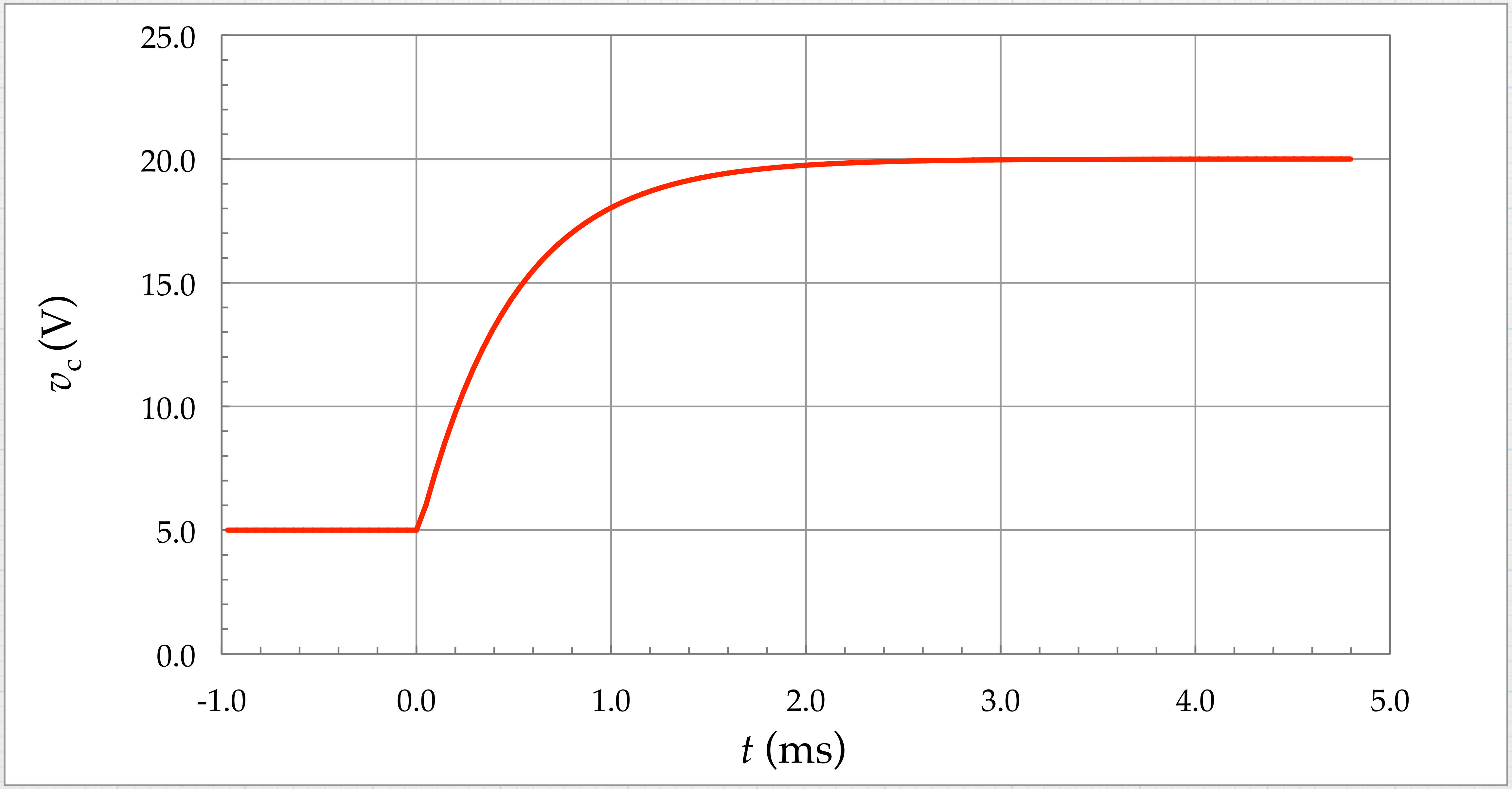
solve to give:

$$A = \frac{V_i - V_f}{1 - \frac{s_1}{s_2}}$$

$$B = \frac{V_i - V_f}{1 - \frac{s_2}{s_1}}$$

$$v_C(t) = \left(V_i - V_f\right)\left(\frac{e^{s_1 t}}{1 - \frac{s_1}{s_2}} + \frac{e^{s_2 t}}{1 - \frac{s_2}{s_1}}\right) + V_f$$

Over-damped response: $V_i = 5$ V, $V_f = 20$ V, $R = 1$ kΩ, $L = 15$ mH, and $C = 0.5$ μF.



Underdamped response

If $\dfrac{R}{2L} < \dfrac{1}{\sqrt{LC}}$ $(\alpha < \omega_o)$ then s_1 and s_2 are complex!

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Write as complex numbers. $j = \sqrt{-1}$ (For EE/CprEs only.)

$$s_1 = -\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \qquad s_2 = -\frac{R}{2L} - j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$= -\alpha + j\sqrt{\omega_o^2 - \alpha^2} \qquad = -\alpha - j\sqrt{\omega_o^2 - \alpha^2}$$

$$= -\alpha + j\omega_d \qquad = -\alpha - j\omega_d$$

$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$ (ω_d is the *damped frequency*)

$$v_C(t) = Ae^{-\alpha t}e^{j\omega_d t} + Be^{-\alpha t}e^{-j\omega_d t} + V_f$$

$$v_C(t) = e^{-\alpha t}\left[Ae^{j\omega_d t} + Be^{-j\omega_d t}\right] + V_f$$

$$e^{jx} = \cos x + j\sin x \qquad \text{(Euler's identity.)}$$

$$v_C(t) = e^{-\alpha t}\left[A\left(\cos\omega_d t + j\sin\omega_d t\right) + B\left(\cos\omega_d t - j\sin\omega_d t\right)\right] + V_f$$

$$= e^{-\alpha t}\left[(A+B)\cos\omega_d t + j(A-B)\sin\omega_d t\right] + V_f$$

$$\boxed{v_C(t) = e^{-\alpha t}\left[A'\cos\omega_d t + B'\sin\omega_d t\right] + V_f}$$

There is an oscillation in the response. The voltage changes from V_i to V_f, but wiggles back and forth a few times in the process. The oscillation dies out according to the damping factor over about 5 time constants, where the time constant $\tau = 1/\alpha$.

Use initial conditions to find A' and B'.

$$v_c(0) = V_i = A' + V_f$$

$$\left.\frac{dv_C}{dt}\right|_{t=0} = 0 = -\alpha A' + \omega_d B'$$

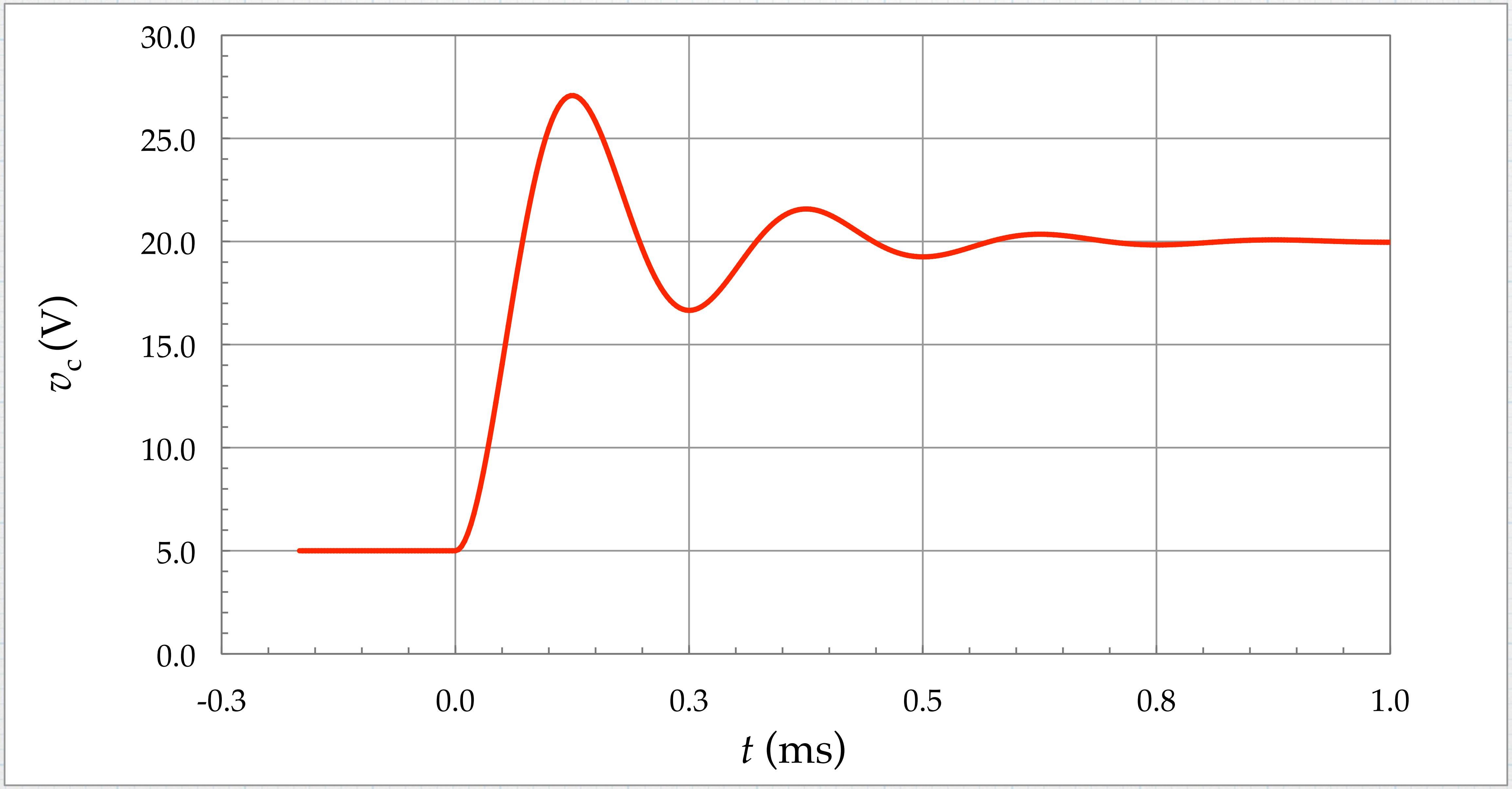
solve to give:

$$A' = \left(V_i - V_f\right)$$

$$B' = \frac{\alpha}{\omega_d}\left(V_i - V_f\right)$$

$$v_C(t) = \left(V_i - V_f\right) e^{-\alpha t}\left[\cos \omega_d t + \frac{\alpha}{\omega_d} \sin \omega_d t\right] + V_f$$

Under-damped response: $V_i = 5$ V, $V_f = 20$ V, $R = 300\ \Omega$, $L = 25$ mH, and $C = 60$ nF.



Changing R, L, and C changes the oscillation frequency and the damping rate.

Critically damped response

If $\dfrac{R}{2L} = \dfrac{1}{\sqrt{LC}}$ ($\alpha = \omega_o$) then $s_1 = s_2$ (double root).

Note: This will almost *never* happen. It will be the wildest fluke if the components have *exactly* the correct ratios to meet the above requirement. For the most part, critical damping is only of academic interest.

$$v_C(t) = Ae^{s_1 t} + Be^{s_1 t} + V_f = Ce^{s_1 t} + V_f$$

$$= C\exp\left(-\frac{t}{{}^{2L}/_{R}}\right) + V_f$$

This causes a bit of a problem, because we are left with only one term in the general solution, and hence only one coefficient – not enough to satisfy the initial conditions.

This suggests that there must be another solution lurking around in the math. In the special circumstances for the critically damped case, the homogeneous equation can be written:

$$\frac{d^2 v_{tr}}{dt^2} + 2\alpha\frac{dv_{tr}}{dt} + \alpha^2 v_{tr} = 0$$

$$\frac{d^2 v_{tr}}{dt^2} + 2\alpha \frac{dv_{tr}}{dt} + \alpha^2 v_{tr} = 0$$

$$\frac{d}{dt}\left[\frac{dv_{tr}}{dt} + \alpha v_{tr}\right] + \alpha \left[\frac{dv_{tr}}{dt} + \alpha v_{tr}\right] = 0$$

$$\frac{dy}{dt} + \alpha y = 0 \qquad \text{where} \qquad y = \frac{dv_{tr}}{dt} + \alpha v_{tr}$$

$$\downarrow$$

$$y = Ae^{-\alpha t} \qquad \longrightarrow \qquad Ae^{-\alpha t} = \frac{dv_{tr}}{dt} + \alpha v_{tr}$$

$$A = e^{\alpha t}\frac{dv_{tr}}{dt} + \alpha e^{\alpha t} v_{tr}$$

$$A = \frac{d}{dt}\left[e^{\alpha t} v_{tr}\right]$$

$$At + B = e^{\alpha t} v_{tr}$$

$$\boxed{v_{tr}(t) = (At + B)\, e^{-\alpha t}}$$

Now there are two constants.

$$v_C(t) = (At + B)\, e^{-\alpha t} + V_f$$

Also a fast trip from V_i to V_f.

Use initial conditions to find A and B.

$$v_c(0) = V_i = B + V_f$$

$$\left. \frac{dv_C}{dt} \right|_{t=0} = 0 = A - \alpha B$$

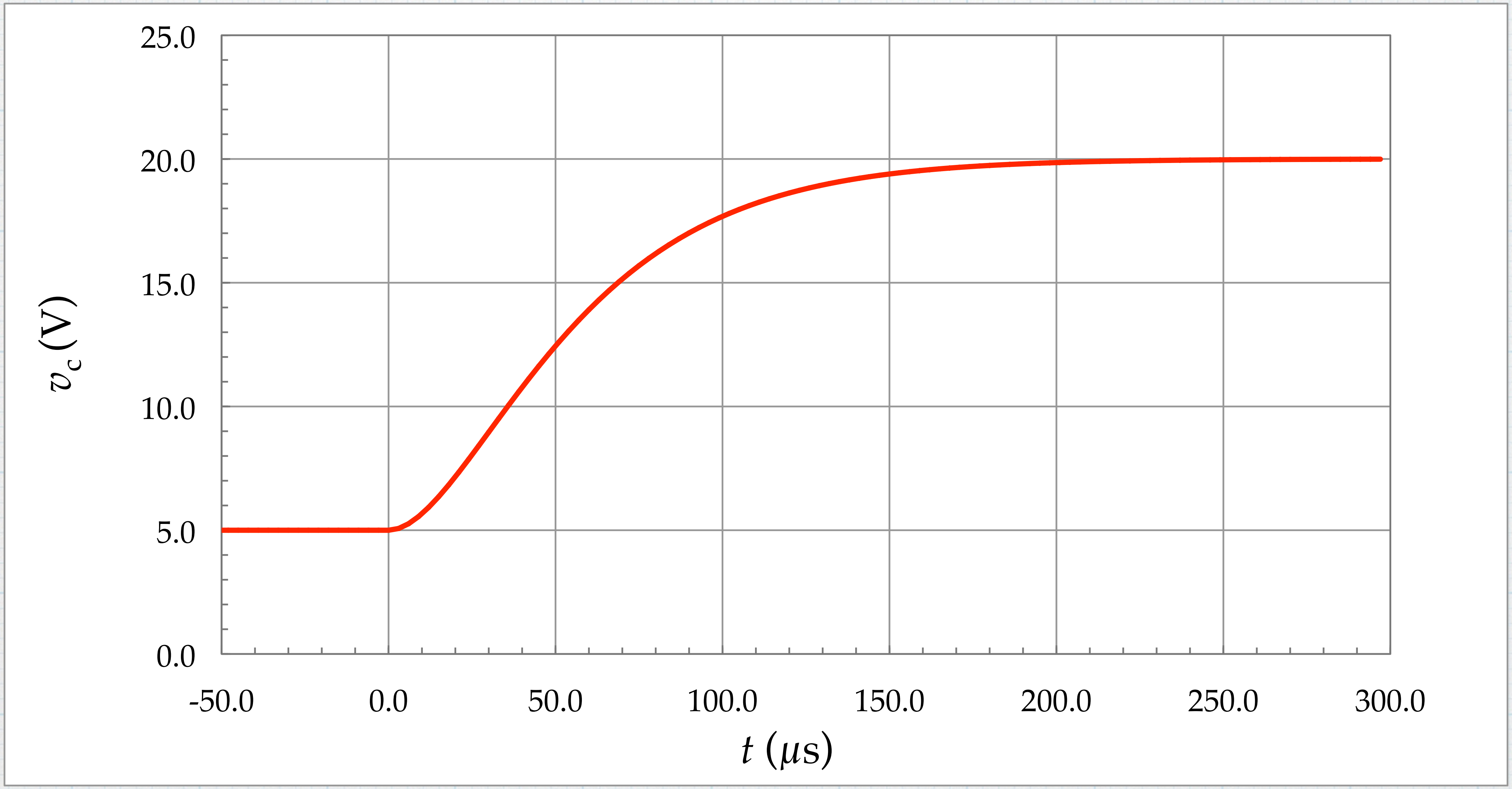
solve to give:

$$A = \alpha \left(V_i - V_f \right)$$

$$B = \left(V_i - V_f \right)$$

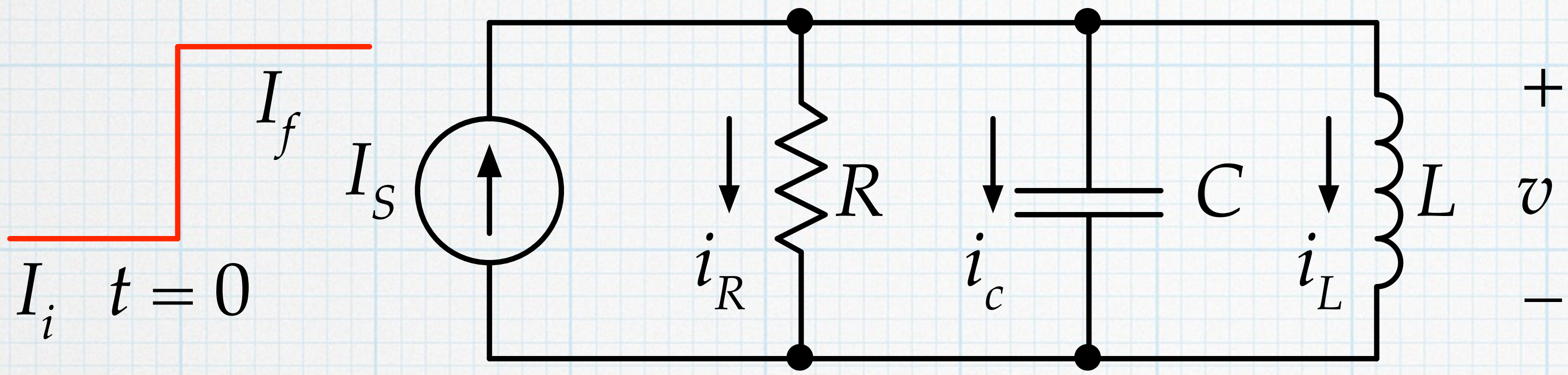
$$v_C(t) = \left(V_i - V_f \right) (1 + \alpha t)\, e^{-\alpha t} + V_f$$

Critically-damped response: $V_i = 5$ V, $V_f = 20$ V, $R = 1$ kΩ, $L = 15$ mH, and $C = 60$ nF. (Same as overdamped plot of slide 10, except C is smaller.)



Note the significantly faster rise time compared to the overdamped response.

parallel RLC



$t < 0$:	$t >> \tau$:
$v = 0.$	$v = 0.$
$i_R = 0.$	$i_R = 0.$
$i_C = 0.$	$i_C = 0.$
$i_L = I_i.$	$i_L = I_f.$

Current source makes an abrupt change from I_i to I_f at t = 0.

$$I_S = i_R + i_C + i_L$$

$$= \frac{v}{R} + C\frac{dv}{dt} + i_L \qquad v = L\frac{di_L}{dt}$$

$t > 0$:

$$I_f = \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} + i_L$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I_f}{LC}$$

initial conditions ($t = 0$)

$$i_L(0) = I_i$$

$$\left.\frac{di_L(t)}{dt}\right|_{t=0} = \frac{v_L(0)}{L} = \frac{v_C(0)}{L} = 0$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I_f}{LC}$$

This has exactly the same form as the series case, with inductor current replacing capacitor voltage. The steady-state function will be $i_{ss} = I_f$, and the transient function will have the general form:

$$i_{tr}(t) = Ae^{s_1 t} + Be^{s_2 t}$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \qquad = -\alpha + \sqrt{\alpha^2 - \omega_o^2}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \qquad = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

$\alpha = \frac{1}{2RC}$ damping factor - note the difference from series case

$\omega_o = \frac{1}{\sqrt{LC}}$ resonant frequency

We will obtain the exact same set of results for the parallel case:

$$\frac{1}{2RC} > \frac{1}{\sqrt{LC}} \qquad (\alpha > \omega_o)$$

overdamped – both roots are real and negative.

$$i_L(t) = \left(I_i - I_f\right)\left(\frac{e^{s_1 t}}{1 - \frac{s_1}{s_1}} + \frac{e^{s_2 t}}{1 - \frac{s_2}{s_1}}\right) + I_f$$

$$\frac{1}{2RC} < \frac{1}{\sqrt{LC}} \qquad (\alpha < \omega_o)$$

underdamped – roots are complex conjugates.

$$i_L(t) = \left(I_i - I_f\right) e^{-\alpha t}\left[\cos\omega_d t + \frac{\alpha}{\omega_d}\sin\omega_d t\right] + I_f \qquad \omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

$$\frac{1}{2RC} = \frac{1}{\sqrt{LC}} \qquad (\alpha = \omega_o)$$

critically damped – repeated root.

$$i_L(t) = \left(I_i - I_f\right)(1 + \alpha t)\, e^{-\alpha t} + I_f$$

(Virtually impossible to have critical damping.)