Tutorial - 7

Q.1 Find the volume of the region bounded above by the surface $z = 2\sin x \cos y$ and below by the rectangle $R: 0 \le x \le \pi/2$, $0 \le y \le \pi/4$.

Q.2 Double integrate $f(s,t) = e^s \ln t$ over the region in the 1st quadrant of the st-plane that lies above the curve $s = \ln t$ from $t=1$ to $t=2$.

Q.3 Sketch the region of integration and write an equivalent double integral with the order of integration reversed.

(i) $$\int_0^1 \int_1^{e^x} dy\,dx$$

(ii) $$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x\,dy\,dx$$

(iii) $$\int_1^e \int_0^{\ln x} xy\,dy\,dx$$

(iv) $$\int_0^{\sqrt{3}} \int_0^{\tan^{-1} y} \sqrt{xy}\,dx\,dy$$

Q.4 Sketch the region. ~~of integration of, evaluate the integral~~. Then change the order of integration and evaluate the new integral.

$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx$$

Observe that if you directly evaluate the integral without changing the order of integration then it is very difficult to evaluate. So sometimes changing the order of integration makes our task much easier..

Q.5 Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2+y^2=4$, and the plane $z+y=3$.

Q.6 Find the average value of $f(x,y)=x\cos xy$ over the rectangle $R:\ 0\le x\le \pi,\ 0\le y\le 1$.

(Hint: Average value of f over R $= \frac{1}{\text{area of } R}\iint_R f(x,y)\,dA$)

Q.7 Find the volume (Using triple integral) of the region between the cylinder $z=y^2$ and the xy-plane that is bounded by the planes $x=0, x=1, y=-1,$ & $y=1$

Solution Tutorial 7

Q.1

$$V = \iint_R f(x,y)\,dA$$

$$= \int_0^{\pi/2}\int_0^{\pi/4} 2\sin x\cos y \, dy\, dx$$

$$= \int_0^{\pi/2} \left[2\sin x \sin y\right]_0^{\pi/4} dx$$

$$= \int_0^{\pi/2} \left(2\sin x \times \frac{1}{\sqrt{2}}\right) dx$$

$$= \sqrt{2}\left[-\cos x\right]_0^{\pi/2} =$$

$$= \sqrt{2}\left[-0+1\right] = \sqrt{2}$$

Q.2

$$\iint_R e^s \ln t \, dA$$

$$= \int_{t=1}^{2}\int_{s=0}^{\ln t} e^s \ln t \, ds\, dt$$

$$= \int_{t=1}^{2} \left[e^s \ln t\right]_{s=0}^{\ln t} dt$$

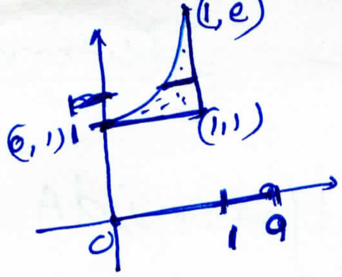
$$= \int_{t=1}^{2} \left(e^{\ln t}\ln t - e^0 \ln t\right) dt$$

$$= \int_{t=1}^{2} (t\ln t - \ln t)\, dt \quad \text{(By parts)} = \left[\frac{t^2}{2}\ln t - \frac{t^2}{4} - t\ln t + t\right]_1^2$$

$$= \frac{1}{4} \quad \text{(Ans)}$$

Q3 (i)

$$\int_0^1 \int_1^{e^x} dy\, dx$$

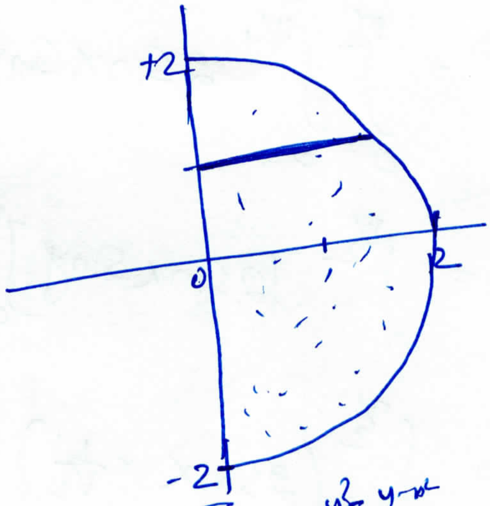


$y = e^x$
$x = \ln y$

$$= \int_{y=1}^{y=e} \int_{x=\ln y}^{x=1} dx\, dy$$

(ii)

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x\, dy\, dx$$



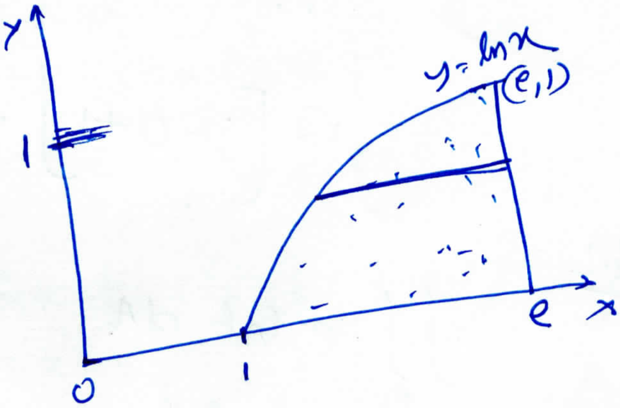
$y = \pm\sqrt{4-x^2}$

$y^2 = 4-x^2$
$\Rightarrow x^2+y^2 = 4$

$$= \int_{-2}^{2} \int_{x=0}^{\sqrt{4-y^2}} 6x\, dx\, dy$$

(iii)

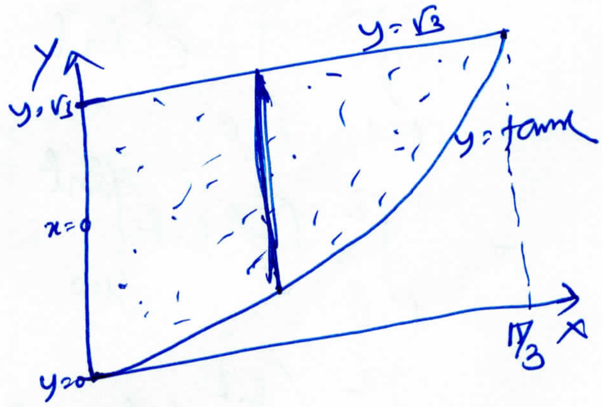
$$\int_1^e \int_0^{\ln x} xy\, dy\, dx$$



$$= \int_{y=0}^{1} \int_{x=e^y}^{x=e} xy\, dx\, dy$$

(iv)

$$\int_0^{\sqrt{3}} \int_0^{\tan^{-1} y} \sqrt{xy}\, dx\, dy$$

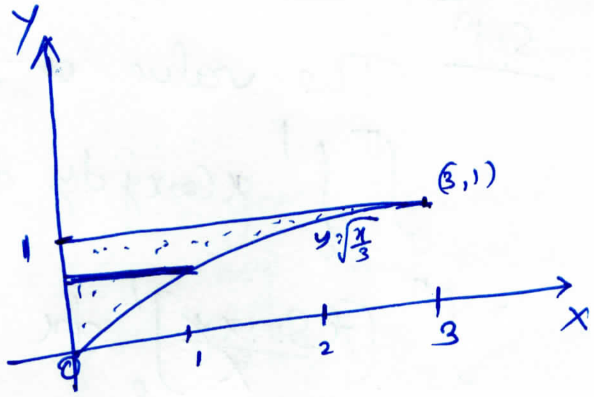


~~x=0~~ $\Rightarrow x = \tan^{-1} y$
$\Rightarrow y = \tan x$

$$= \int_{x=0}^{x=\pi/3} \int_{y=\tan x}^{y=\sqrt{3}} \sqrt{xy}\, dy\, dx$$

Sol^n

$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx$$



$$= \int_{y=0}^{y=1} \int_{x=0}^{x=3y^2} e^{y^3}\, dx\, dy$$

$y=\sqrt{x/3} \Rightarrow y^2 = \frac{x}{3}$

$\Rightarrow x = 3y^2$

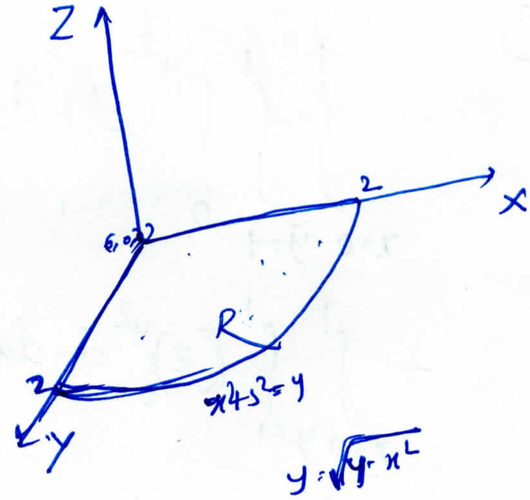
$$= \int_{y=0}^1 \left[x e^{y^3}\right]_{x=0}^{3y^2} dy$$

$$= \int_0^1 3y^2 e^{y^3}\, dy = \left[e^{y^3}\right]_0^1 = e^1 - e^0 = e-1$$

Q.5

Sol^n

$$z = f(x,y) = 3-y$$



$$V = \iint_R f(x,y)\, dA$$

$$= \int_{x=0}^{2} \int_{y=0}^{\sqrt{4-x^2}} (3-y)\, dy\, dx$$

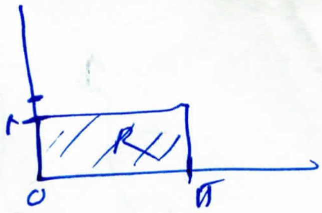
$$= \int_0^2 \left[3y - \frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx = \int_0^2 \left(3\sqrt{4-x^2} - \frac{4-x^2}{2}\right) dx$$

$$= \left[\frac{3}{2} x\sqrt{4-x^2} + 6\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^3}{6}\right]_0^2$$

$$= 3\pi - \frac{16}{6} = \frac{9\pi - 8}{3}$$

Q.6 Sol^n The value of the integral of f over R is

$$\int_0^{\pi}\int_0^1 x\cos xy\,dy\,dx$$



$$= \int_0^{\pi}\left[\cancel{x}\frac{\sin xy}{\cancel{x}}\right]_0^1 dx = \int_0^{\pi} \sin x\,dx$$

~~Area~~

$$= \left[-\cos x\right]_0^{\pi} = 1+1 = 2$$

The area of $R = \pi$

So average value of f over $R = \frac{2}{\pi}$.

Q.7 Sol^n

between $z = y^2$ & xy-plane $(z=0)$

So z-limit is $\int_0^{y^2}$

$$V = \int_{x=0}^{1}\int_{y=-1}^{1}\int_0^{y^2} dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=-1}^{1}\left[z\right]_0^{y^2} dy\,dx = \int_{x=0}^{1}\int_{y=-1}^{1} y^2\,dy\,dx$$

$$= \int_{x=0}^{1}\left[\frac{y^3}{3}\right]_{-1}^{1} dx = \int_{x=0}^{1}\left(\frac{1}{3} - \frac{(-1)}{3}\right) dx$$

$$= \int_0^1 \frac{2}{3}\,dx = \frac{2}{3}(x)_0^1 = \frac{2}{3}$$ (A)