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(1) An Introduction to optimization
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(2) Engineering Optimization Theory
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(3) Operation Research : An Introduction
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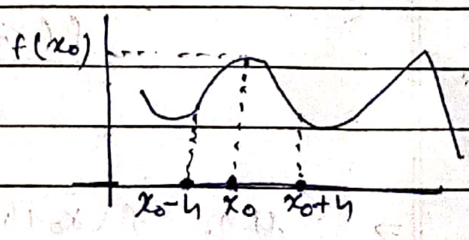
* Single Variable Optimization :

$$f : \mathbb{R} \to \mathbb{R}$$

x_0 is called a local maximum for f if

$$f(x_0) \geq f(x_0 + h)$$

for all sufficient small positive & negative values of h.



x_0 is called a local minimum for f if

$$f(x_0) \leq f(x_0 + h)$$

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→ To find Local Min / Max,

* sufficient conditions / Not necessary :
  - f continuous
  - f is defined on a closed bounded set

→ x_0 is said to be "Global Maximum" for f if,

$$f(x_0) \geq f(x) \quad \forall x \in R$$

* Necessary condition :

- Suppose x_0 is an extreme point (local max / min) of f.

- x_0 is an enterior point (Not on boundry) on the domain of f.

- $f(x)$ is differenciate at x_0, then $f'(x_0) = 0$

Proof: $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Suppose x_0 is local min.

$$\Rightarrow f(x_0+h) \geq f(x_0)$$

$$\frac{f(x_0+h) - f(x_0)}{h} \geq 0 \quad \text{if } h > 0$$

$$\frac{f(x_0+h) - f(x_0)}{h} \leq 0 \quad \text{if } h < 0$$

$$\Rightarrow \lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} \geq 0$$

$$\lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h} \leq 0$$

equal, $\lim_{h \to 0^+} = \lim_{h \to 0^-}$ because x_0 is differenciable.

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$$\lim_{h\to 0^+} (\quad) = \lim_{h\to 0^-} (\quad) = 0$$

$$f'(x_0) = 0$$

* Sufficient Condition :

let $f'(x_0) = f''(x_0) = \cdots\cdots = f^{(n-1)}(x_0) = 0$

but $f^{(n)}(x_0) \neq 0$

Then (1) $f(x_0)$ is local minima if $f^{(n)}(x_0) > 0$ and n is even

(2) x_0 is local maxima if $f^{(n)}(x_0) < 0$ and n is ~~odd~~ even

(3) If n is odd then x_0 is neither local max nor local min.
x_0 is called a point of inflection.

Proof :

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots$$

$$+ \frac{h^{(n-1)}}{(n-1)!} f^{(n-1)}(x_0) + \frac{h^n}{n!} f^{(n)}(x_0+\theta h)$$

$$0 < \theta < 1$$

$$\Rightarrow f(x_0+h) - f(x_0) = \frac{h^n}{n!} f^{(n)}(x_0+\theta h)$$

Since $f^{(n)}(x_0) \neq 0 \Rightarrow f^{(n)}(x_0+\theta h) \neq 0$

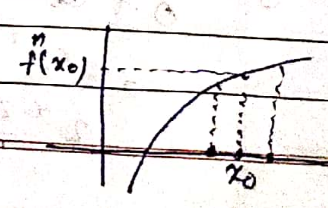
both have same symbol.

$\Rightarrow$ If $f^{(n)}(x_0) > 0$ at n is even

$f(x_0+h) - f(x_0) \geq 0$

$f(x_0+h) \geq f(x_0)$

x_0 is local min



exp : $f(x) = 12x^5 - 45x^4 + 40x^3 + 5$

$f'(x) = 0$.

$\therefore 60x^4 - 180x^3 + 120x^2 = 0$.

$\therefore 60x^2(x^2 - 3x + 2) = 0$

$\therefore x^2 = 0$ or $x^2 - 2x - x + 2 = 0$

$\boxed{\therefore x = 0}$ $\therefore x(x-2) - 1(x-2) = 0$

$\boxed{\therefore x = 2}$ $\boxed{x = 1}$

so, this is stationary points or critical points.

$f''(x) = 240x^3 - 540x^2 + 240x$

At $x=0$ $f''(x)\Big|_{x=0} = 0$ so go for 3rd derivatives.

$f'''(x)\Big|_{x=0} = 240\Big|_{x=0}$

here $n = 3$ is odd.

so $x=0$ is neither a local maxima or neither a le minima.

so it is point of inflection.

At $x=1$ $f''(x)\Big|_{x=1} = 240 - 540 + 240 = -60 < 0$.

so $x = 1$ is local maximum.

At $x=2$ $f''(x)\Big|_{x=2} = 240\times8 - 540\times4 + 240\times2$

$= 1920 - 2160 + 480$

$= 240 > 0$

$x = 2$ is local minima.

exp $f(x) = (x-3)^5$

$f'(x) = 5(x-3)^4 = 0$ at $x=3$

so $x=3$ is only stationary point.

$f''(x) = 20(x-3)^3$ here $x=3$ also zero.

$f'''(x) = 60(x-3)^2$ bees.

$f^{iv}(x) = 120(x-3)$

$f^{v}(x) = 120$

$n=5$ is odd.

so $x=3$ is point of inflection.

exp $f(x) = (x-3)^4 + 1$

$f'(x) = 4(x-3)^3 = 0$

$x=3$ is the stationary point

$f^{iv}(x)\Big|_{x=3} = 24 > 0$

$x=3$ is local minima.

* Multivariable optimization :-

$y = f(x)$

$\frac{dy}{dx} = f'(x)$

$dy = f'(x)\,dx$ (this is single variable)

$z = f(x, y)$

$\partial z = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$

$z = f(x, y, z)$

$\partial z = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial w} dw.$

$f(x_1, x_2, \ldots x_n)$

If all partial derivatives of f through order $r \geq 1$ exist and are continuous at a point x^*, then the r^{th} derivative of f at x^*

$$d^r f(x^*) = \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{n} \cdots \sum_{k=1}^{n}}_{r \text{ sumations.}} h_i h_j \cdots h_k \frac{\partial^r f(x^*)}{\partial x_i \partial x_j \cdots \partial x_k}$$

h_i

One h_i is associated with each sum $(dx, dy, \ldots)$

$r = 1$ & $n = 3$ $f(x_1, x_2, x_3)$

$$\partial f(x^*) = \sum_{i=1}^{3} \cancel{dx_1\, dx_2\, dx_3}\; h_i \frac{\partial f(x^*)}{\partial x_i}$$

It is vector

$$= h_1 \frac{\partial f(x^*)}{\partial x_1} + h_2 \frac{\partial f(x^*)}{\partial x_2} + h_3 \frac{\partial f(x^*)}{\partial x_3}$$

$$= dx_1 \frac{\partial f}{\partial x_1} + dx_2 \frac{\partial f}{\partial x_2} + dx_3 \frac{\partial f}{\partial x_3}$$

If $X^* = (x_1^*, x_2^*, x_3^*)$

$h_1 = (x_1 - x_1^*)$

$h_2 = (x_2 - x_2^*)$

$h_3 = (x_3 - x_3^*)$

$$d^2f(x^*) = \sum_{i=1}^{3}\sum_{j=1}^{3} h_i h_j \frac{\partial^2 f(x^*)}{\partial x_i \partial x_j}$$

$$= h_1^2 \frac{\partial^2 f(x^*)}{\partial x_1^2} + h_2^2 \frac{\partial^2 f(x^*)}{\partial x_2^2} + h_3^2 \frac{\partial^2 f(x^*)}{\partial x_3^2} + 2h_1 h_2 \frac{\partial^2 f(x^*)}{\partial x_1 \partial x_2}$$

$$+2h_1 h_3 \frac{\partial^2 f(x^*)}{\partial x_1 \partial x_3} + 2h_2 h_3 \frac{\partial^2 f(x^*)}{\partial x_2 \partial x_3}$$

ex

$f(x_1, x_2, x_3) = x_2^2 x_3 + x_1 e^{x_3}$ find df and d^2f at $x^* = (1, 0, -2)$

$$df(x^*) = dx_1 \frac{\partial f(x^*)}{\partial x_1} + dx_2 \frac{\partial f(x^*)}{\partial x_2} + dx_3 \frac{\partial f(x^*)}{\partial x_3}$$

$$= e^{x_3} dx_1 + (2x_2 x_3) dx_2 + (x_2^2 + x_1 e^{x_3}) dx_3$$

$dx_1 = (x_1 - 1)$ $\quad = e^{-2}(x_1 - 1) + 2(0)(-2)(x_2 - 0) + e^{-2}(x_3 + 2)$

$dx_2 = (x_2 - 0)$

$dx_3 = (x_3 + 2)$ $\quad = \cancel{e^{-2}(x_1 + x}$

$$df(x^*) = e^{-2}(x_1 - 1) + (x_3 + 2)e^{-2}$$

$d^2f(x^*) = (x_1 - 1)^2 (0) + (x_2)^2 ($

$-4x_2^2 + e^{-2}(x_3 + 2)^2 + 2$

* Tailor series :-

$f : \mathbb{R}^n \to R$

$$f(x_0+H) = f(x_0) + \frac{1}{1!} \underset{(df(x_0))}{f'(x_0)} + \frac{1}{2!} \underset{(d^2f(x_0))}{f''(x_0)} + \frac{1}{3!} f'''(x_0) + \cdots$$

⇒ To find the taylor series expansion for the function.
ex.

$f(x_1, x_2, x_3) = x_2^2 x_3 + x_1 e^{x_3}$ about the point $x_0^* = (1, 0, -2)$

~~f(x_0)~~ Upto 3 terms. what is the remainder term ?

* Necessary condition for existance of extreme values :-

let $f(x)$ has an extreme point at x_0 and x_0 is an interior point in the domain of f. Also if all the 1st partial derivati of f exist then

$$\frac{\partial f}{\partial x_1}(x_0) = \cdots\cdots \frac{\partial f}{\partial x_n}(x_0) = 0.$$

proof : Suppose one of the 1st partial derivative say $\frac{\partial f}{\partial x_k}(x_0) \neq 0$.

$$f(x_0+H) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + \frac{1}{2!} d^2 f(x_0 + \theta H) \quad [0 < \theta < 1]$$

$$f(x_0+H) - f(x_0) = h_k \frac{\partial f}{\partial x_k}(x_0) + \frac{1}{2!} d^2 f(x_0 + \theta H)$$

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Since $d^2 f(x_0 + \theta H)$ is of order h^2.

So, the sign of $f(x_0+H) - f(x_0)$ is determined by the term $h_k \frac{\partial f}{\partial x_k}(x_0)$

If $\frac{\partial f}{\partial x_k}(x_0) > 0$ then $\left.\begin{array}{l} f(x_0+H) - f(x_0) > 0 \\ f(x_0+H) - f(x_0) < 0 \end{array}\right\{ \begin{array}{l} \text{if } h_k > 0 \\ \text{if } h_k < 0 \end{array}$

$\therefore f(x_0+H) > f(x_0) \quad h_k > 0.$

or

$f(x_0+H) < f(x_0) \quad h_k < 0,$

So x_0 is neither local maxima nor local minima,
x_0 is not extreme point.
So, this is contradiction of our hypothesis that x_0 is an extreme point.
So, our assumption that $\frac{\partial f(x_0)}{\partial x_k} \neq 0$ is wrong.

$\Rightarrow$ All 1st partial derivatives ~~of~~ at f at x_0 are zero.

* **Sufficient Condition** :-

Continuous

let x_0 be stationary point at $f(x)$, All partition derivatives are
If The matrix of second order partial derivatives (Hessian matrix) of $f(x)$ evaluated at (x_0)

(I) positive definite then x_0 is a local minimum.
(II) negative definite then x_0 is a local maximum.
(III) semidefinite we have to look for higher order derivatives.

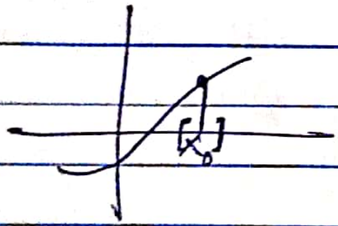
proof: $f(X_0+H) = f(X_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(X_0) + \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(X_0+\theta H)$

Since X_0 is stationary $\frac{\partial f}{\partial x_i}(X_0) = 0$ for all $i = 1, 2, \ldots n$

$\therefore f(X_0+H) - f(X_0) = \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} h_i h_j \frac{\partial^2 f(X_0+\theta H)}{\partial x_i \partial x_j}$

So, since the 2nd partial derivates are continuous in the neighbourhood of X_0,

The sign of $\frac{\partial^2 f}{\partial x_i \partial x_j}(X_0+\theta H)$ is same as $\frac{\partial^2 f}{\partial x_i \partial x_j}(X_0)$



The sign of $f(X_0+H) - f(X_0)$ is determined by the quantity

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} h_i h_j \frac{\partial^2 f(X_0)}{\partial x_i \partial x_j}$$

$$= \frac{1}{2} H^T \left[\frac{\partial^2 f(X_0)}{\partial x_i \partial x_j}\right] H$$ ex for 2 variable.

$$[h_1 \ h_2] \begin{bmatrix} \frac{\partial^2 f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$f(X_0-H) - f(X_0) > 0$ if

$$H^T \left[\frac{\partial^2 f(X_0)}{\partial x_i \partial x_j}\right] H > 0$$

Hessian $(f) = \left[\frac{\partial^2 f(x_0)}{\partial x_i \, \partial x_j}\right]$ x_0 is a local minimum.

$\Rightarrow [A]_{n\times n}$ (only real matrix)
A is said to be $\boxed{\text{positive definite}}$ if $x^TAx > 0$ for all $x \in \mathbb{R}^n$.

$\Rightarrow$ A is $\boxed{\text{Negative definite}}$ if $x^TAx < 0$

$\Rightarrow$ A is $\boxed{\text{positive semidefinite}}$ if $x^TAx \geq 0 \quad \forall x \in \mathbb{R}^n$

$\Rightarrow$ A is $\boxed{\text{negative semidefinite}}$ if $x^TAx \leq 0 \quad \forall x \in \mathbb{R}^n$

eg. $A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$ $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ $x^T = (x_1 \; x_2) \in \mathbb{R}^2$

$$\underset{1\times 2}{(x_1 \; x_2)} \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}} \underset{2\times 1}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = (x_1 \; x_2) \begin{bmatrix} 2x_1 + x_2 \\ 4x_2 \end{bmatrix}$$

$$= x_1(2x_1 + x_2) + 4x_2^2$$

$$= \underset{>0}{2x_1^2} + \underset{>0}{4x_2^2} + \boxed{x_1 x_2} \quad \boxed{>0}$$

positive definite.

$\Rightarrow$ characterizations :-

$\rightarrow$ A matrix A is positive definite if all its eigenvalues are positive (>0).

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→ A matrix A is negative definite if all its eigenvalues are negative (<0).

→ A is positive semi-definite if all its eigenvalues are non negative (≥ 0).

→ A is negative semi-definite if all its eigenvalues are non-positive (≤ 0).

→ If A is none . .

⇒ characterization - II

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}_{n \times n}$$

let $A_1 = a_{11}$

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

→ The matrix A is positive definite if $A_j > 0$ for all $j = 1, 2, \cdots n$.

$A_n = \det(A)$

→ The matrix A is negative definite if the sign of A_j is $(-1)^j$ for all $j = 1, 2, \cdots n$.

→ A is positive semi-definite if all the $A_j \geq 0$ for all $j = 1, 2, \cdots n$

→ A is negative semi-definite if $A_1 \leq 0$, $A_2 \geq 0$, $A_3 \leq 0$, $A_4 \geq 0$, ⋯ and so on.

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→ A is indefinite matrix if it is not in any of the above categaries.

$A = \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix} = 6 > 0$ positive definite

$A = \begin{bmatrix} 4 & 0 \\ 1 & 0 \end{bmatrix}$ positive Semidefinite

$-1 < 0$

$A = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix} = 2 > 0$ Negative definite

$A = \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}$ Negative Semidefinite

$A = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$ indefinite.

* Sufficient condition:-

$$f(x_0+H) - f(x_0) = \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j \frac{\partial^2 f(x_0+\theta H)}{\partial x_i \partial x_j} \quad 0 < \theta < 1$$

$$f(x_0+H) > f(x_0) \text{ If } \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j \frac{\partial^2 f(x_0+\theta H)}{\partial x_i \partial x_j} > 0.$$

↓

x_0 is local minimum if $\sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) > 0.$

that is $H^T \left[\frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \right] H > 0.$

If $\left[\frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \right]$ is positive definite

Conclusion : x_0 is local minimum if $\left[\frac{\partial^2 f(x_0)}{\partial x_i \partial x_j}\right]$ is positive definite.

$$\text{Hess}(f)|_{x_0} = \left[\frac{\partial^2 f(x_0)}{\partial x_i \partial x_j}\right]_{n\times n}$$

$\Rightarrow$ x_0 is local maximum if $\left[\frac{\partial^2 f(x_0)}{\partial x_i \partial x_j}\right]$ is negative definite

$\Rightarrow$ Semi definite case :-

let the partial derivatives of f of all orders (upto ($k \geq 2$)) be continuous in the nbd of a stationary point x_0 and $df|_{x_0} = 0$, $d^2f|_{x_0} = 0$, ... $d^{(k-1)}f|_{x_0} = 0$.

so that $d^k f|_{x_0}$ is the first non vanishing derivative.

① If k is even and $d^k f|_{x_0} > 0$, then x_0 is a local minimum

② If k is even and $d^k f|_{x_0} < 0$, then x_0 is a local maximum

③ If k is odd then x_0 is neigther a local maxima or local minima. [It is called a saddle point]

[2 dimensional case.

exp. $f(x,y) = x^2 - y^2$.

$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

$\frac{\partial f}{\partial y} = 0 \Rightarrow -2y = 0 \Rightarrow y = 0$.

(0,0) is stationary point.

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}_{\text{at } (0,0)} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

In definite matrix.

(0,0) neither ~~nor~~ local maxima ~~and~~ nor local minima

eg: $f(x,y) = x^3 + y^3 + 2x^2 + 4y^2 + 6$

Sol.

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$$\Downarrow \qquad \Downarrow$$

$$3x^2 + 4x = 0 \qquad 3y^2 + 8y = 0$$

$$\therefore x(3x+4) = 0 \qquad y(3y+8) = 0$$

$$\boxed{\therefore x = 0 \text{ or } x = -4/3} \qquad \boxed{y = 0 \text{ or } y = -8/3}$$

possible points that we investigate are
$(0,0), (0,-8/3), (-4/3, 0), (-4/3, -8/3)$

At $(0,0)$: $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = 6x + 4 = 4$ $\quad \left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = 6y + 8 = 8$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = 0 \qquad \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)} = 0$$

$$\text{Hess}(f) = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}_{\text{at } (0,0)}$$

Positive definite matrix

(0,0) is local minima.

At $(0, -8/3)$ = $\begin{bmatrix} 4 & \cancel{8}0 \\ 0 & -8 \end{bmatrix}$ = ~~negative definite~~ ~~$(0, -8/3)$ is local maxima~~.

Indefinite matrix
$(0, -8/3)$ neither local maxima nor local minima.
Saddle point.

At $(-4/3, 0)$ = $\begin{bmatrix} -4 & 0 \\ 0 & 8 \end{bmatrix}$ Indefinite matrix.
$(-4/3, 0)$ is saddle point.

At $\left(-\frac{4}{3}, -\frac{8}{3}\right)$ = $\begin{bmatrix} -4 & 0 \\ 0 & -8 \end{bmatrix}$ Negative definite matrix.
$\left(-\frac{4}{3}, -\frac{8}{3}\right)$ is local maxima.

As there are no other local maxima & local minima, so $(-4/3, -8/3)$ is global maxima, $(0,0)$ is global minima.

$f_{max} = 50/3$ $f_{min} = 6$.

* Multivariable optimization with equality constraints :-

$f : \mathbb{R}^n \to \mathbb{R}^m$
$g_j : \mathbb{R}^n \to \mathbb{R}$ $j = 1, 2, \ldots m.$

$$\begin{cases} \text{Minimize } f(x) \\ \text{subject to } g_j(x) = 0 \quad j = 1, 2, \ldots m. \end{cases}$$

$f(x)$ is objective function
$g_j(x) = 0$, $j = 1, 2, \ldots m$. are called constraints.

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $\boxed{m \leq n}$

here $x_1, x_2, \ldots x_n$ decision variable which we have to determined.

Assumption :-

If $m > n$ then the problem is overdefined or no solution

⇒ Solution by direct substitution :-

exp find the dimension of a box of largest volume that can be inscribed in a sphere of unit radius

let the origin be at the center of the sphere. let the sides of the box be $2x$, $2y$ and $2z$, respectively
The volume of the box is

$$f(x, y, z) = 8xyz.$$

The corner of the box lie on the surface of the sphere

one corner point (x, y, z), so it will satisfy $x^2+y^2+z^2=1$

$$\boxed{\text{maximize } 8xyz \text{ subject to } x^2+y^2+z^2=1}$$

Using the equality contraints.

$x = \sqrt{1-y^2-z^2}$ (only consider positive direction)
so, substituting in the objective function, we get

$$\boxed{\text{maximize } 8\sqrt{1-y^2-z^2}\, yz}$$

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$g(y,z) = 8\sqrt{1-y^2-z^2}\; yz.$

Now find the maximum value of $g(y,z)$.

$$\frac{\partial g}{\partial y} = 0 \Rightarrow \frac{8yz(-\cancel{2}y)}{\cancel{2}\sqrt{1-y^2-z^2}} + 8z\sqrt{1-y^2-z^2} = 0$$

$$-8y^2z + 8z(1-y^2-z^2) = 0$$

$$\cancel{8zy^2 \therefore -16y^2z + 8} \quad \therefore -8y^2z + 8z - 8y^2z - 8z^3 = 0$$

$$\boxed{\therefore 1 - 2y^2 - z^2 = 0} \quad \text{--- (1)}$$

$$\frac{\partial g}{\partial z} = 0 \Rightarrow \boxed{1 - y^2 - 2z^2 = 0} \quad \text{--- (2)}$$

$y^2 = 1 - 2z^2$ putting in (1).

$$\Rightarrow 1 - 2(1-2z^2) - z^2 = 0$$
$$\Rightarrow 1 - 2 + 4z^2 - z^2 = 0$$
$$\Rightarrow 3z^2 = 1$$
$$\Rightarrow z^2 = \frac{1}{3} \quad \Rightarrow z = \frac{1}{\sqrt{3}}$$

putting in (2) $\left(y = \frac{1}{\sqrt{3}}\right)$

so point $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\left.\frac{\partial^2 g}{\partial y^2}\right|_{(1/\sqrt{3},1/\sqrt{3})} = \left.-4y\right|_{(1/\sqrt{3},1/\sqrt{3})} = -4/\sqrt{3} \qquad \frac{\partial^2 g}{\partial z^2} = -4\left(\frac{1}{\sqrt{3}}\right) = -4/\sqrt{3}$$

$$\frac{\partial^2 g}{\partial y\,\partial z} = 0 \qquad \begin{bmatrix} -4/\sqrt{3} & 0 \\ 0 & -4/\sqrt{3} \end{bmatrix}$$

negative semidefine.

$(1/\sqrt{3}, 1/\sqrt{3})$ global (local) maxima.

$x = \sqrt{1-\frac{1}{3}-\frac{1}{3}} = \sqrt{1-\frac{2}{3}} = \sqrt{\frac{1}{3}}$

So, maximize volume is $8\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) = \boxed{\frac{8}{3\sqrt{3}}}$

⇒ Solution by the method of Constant variable :-

- The idea is to find a closed form expression for the 1st order differential df at all point at which $g_j(x) = 0$ $j = 1, 2, \ldots n$. Then set $df = 0$ to find a necessary condition.

minimize $f(x_1, x_2)$
subject to $g(x_1, x_2) = 0$ ——— ①

Necessary condition of f to have a local minimum ~~is~~ at ~~$df \neq 0$~~ (x_1^*, x_2^*) is $df\big|_{(x_1^*, x_2^*)} = 0$.

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 \Bigg|_{(x_1^*, x_2^*)} = 0 \quad \text{———} \quad ②$$

Since the constant ① is also to be satisfied at that point

$g(x_1^*, x_2^*) = 0$.

⇒ Admissible ~~variable~~ variation :-

Any variation dx_1 and dx_2 taken about the point (x_1^*, x_2^*) are called admissible variation if

$$g(x_1^* + dx_1, x_2^* + dx_2) = 0 \quad \text{———} \quad ③$$

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$$g(x_1^* + dx_1, x_2^* + dx_2) \approx g(x_1^*, x_2^*) + \frac{\partial g}{\partial x_1}(x_1^*, x_2^*)\,dx_1 + \frac{\partial g}{\partial x_2}(x_1^*, x_2^*)\,dx_2 = 0.$$

for admissible variations dx_1 & dx_2.

Already we have

$g(x_1^*, x_2^*) = 0$.

$$\Rightarrow \left.\frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2\right|_{(x_1^*, x_2^*)} = 0. \quad \text{—— } 5.$$

Assuming $\left.\frac{\partial g}{\partial x_2}\right|_{(x_1^*, x_2^*)} \neq 0.$

we can write $\boxed{dx_2 = -\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}} dx_1 \quad \text{at } (x_1^*, x_2^*)}$ —— 6

put eq. 6 in eq. 2.

$$df = \left.\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2}\left[-\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}}\right] dx_1\right|_{(x_1^*, x_2^*)} = 0.$$

$$\Rightarrow df = \left.\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} dx_1 - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1} dx_1\right|_{(x_1^*, x_2^*)} = 0$$

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}\right) dx_1 = 0$$

$$\therefore \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1} = 0 \quad \text{at } (x_1^*, x_2^*)$$

$$\begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{vmatrix}_{\text{at } (x_1^*, x_2^*)} = 0$$

$$J\left(\frac{f, g}{x_1, x_2}\right) = \begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{vmatrix} = 0 \quad \text{provided } \left(\frac{\partial g}{\partial x_2} \neq 0\right).$$

⇒ Necessary codition for a general problem :-

n - variables $\quad m \leq n$.
m - constaints

each constraint $g_j(x) = 0$, $j = 1, 2, \cdots m$ gives rise to a linear equation in the variations dx_i, $i = 1, 2, \cdots n$. There are m such equation having n variables

Hence any m variations can be expressed in terms of remaining $(n-m)$ variations.

These expression can be used to express the differential df in terms of the $(n-m)$ independent variables.

Necessary conditions

$$J\left(\frac{f, g_1, g_2, \cdots g_m}{x_k, x_1, x_2 \cdots x_m}\right)$$

$$= \begin{vmatrix} \frac{\partial f}{\partial x_k} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_m} \\ \frac{\partial g_1}{\partial x_k} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \vdots & \vdots & & & \\ \frac{\partial g_m}{\partial x_k} & \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_m} \end{vmatrix}_{(m+1)\times(m+1)} = 0$$

$k = m+1, m+2, \ldots, n$

provided $J\begin{pmatrix} g_1, g_2, \ldots g_m \\ x_1, x_2, \ldots, x_m \end{pmatrix} \neq 0$

ex.

minimize $f = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2 + y_4^2)$

subject to $g_1(y) = y_1 + 2y_2 + 3y_3 + 5y_4 - 10 = 0$. —— (A)

$g_2(y) = y_1 + 2y_2 + 5y_3 + 6y_4 - 15 = 0$. —— (B)

sol^n $n = 4$ $\quad$ y_1, y_2 $\quad$ [y_3 & y_4 are dependent]

$m = 2$

$$J\begin{pmatrix} g_1 & g_2 \\ y_1 & y_2 \end{pmatrix} = \begin{vmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} \end{vmatrix} = \cancel{\begin{vmatrix} 3 & 5 \\ 5 & 6 \end{vmatrix}}$$

now y_3, y_4.

$$J\begin{pmatrix} g_1 & g_2 \\ y_3 & y_4 \end{pmatrix} = \begin{vmatrix} \frac{\partial g_1}{\partial y_3} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_4} & \frac{\partial g_2}{\partial y_4} \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 5 & 6 \end{vmatrix} = (-7)$$

$$= \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$ (so, we can't pick y_1, y_2)

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③

If we choose $\boxed{y_2 \text{ \& } y_4}$ are independent

$$J\begin{pmatrix} g_1 & g_2 \\ y_1 & y_3 \end{pmatrix} = \begin{vmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_3} \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 2 \neq 0.$$

Necessary conditions:

w.r.t. → This is independents.

$K = m+1 = 3$ $(y_2 \text{ \& } y_4)$

w.r.t y_2

$$\begin{vmatrix} \frac{\partial f}{\partial y_2} & \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_3} \\ \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_2} & \frac{\partial g_2}{\partial y_1} & \frac{\partial g_1}{\partial y_3} \end{vmatrix} = 0 \text{ --- } ①$$

w.r.t y_4

$$\begin{vmatrix} \frac{\partial f}{\partial y_4} & \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_3} \\ \frac{\partial g_1}{\partial y_4} & \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_1}{\partial y_4} & \frac{\partial g_2}{\partial y_4} & \frac{\partial g_2}{\partial y_3} \end{vmatrix} = 0 \text{ --- } ②.$$

now ① becomes.

$$\begin{vmatrix} y_2 & y_1 & y_3 \\ \cancel{2} & 1 & 3 \\ \cancel{2} & 1 & 5 \end{vmatrix} = 0 \Rightarrow y_2(2) - y_1(\cancel{4}) + y_3(0) = 0$$

$$\Rightarrow 2y_2 - 4y_1 = 0 \text{ --- } ③$$

now (2) becomes.

$$\begin{vmatrix} y_4 & y_1 & y_3 \\ 5 & 1 & 3 \\ 6 & 1 & 5 \end{vmatrix} = 0 \Rightarrow y_4(2) + y_1(7) + y_3(-1) = 0$$

$$\Rightarrow 2y_4 - 7y_1 - y_3 = 0 \quad \text{——— (4)}$$

from (3) & (4)

$$\left.\begin{aligned} y_1 &= \tfrac{1}{2} y_2 \\ y_3 &= 2y_4 - 7y_1 = 2y_4 - \frac{7}{2} y_2 \end{aligned}\right\} \text{——— (C)}$$

putting these values in (A) &

$$\frac{1}{2} y_2 + 2y_2 + 6y_4 - \frac{21}{2} y_2 + 5y_4 = 10$$

$$\Rightarrow -8y_2 + 11y_4 = 10 \quad \text{——— (5)}$$

putting these values in (B)

$$\frac{1}{2} y_2 + 2y_2 + 10y_4 - \frac{35}{2} y_2 + 6y_4 = 15$$

$$\Rightarrow -15y_2 + 16y_4 = 15 \quad \text{——— (6)}$$

solving (5) & (6) we get.

$$\boxed{y_2 = \frac{-5}{37}}, \quad \boxed{y_4 = \frac{30}{37}}$$

from (C)

$$\boxed{y_1 = \frac{-5}{74}} \quad \boxed{y_3 = \frac{155}{74}}$$

Sufficient condition for a general problem :-

By eliminating the first m variables, using the m equally constraints, the objective function can be made to depend on the variables

$x_{m+1}, x_{m+2}, \ldots x_n$

Then the Taylor's series expansion of f about an extreme point x^* is

$$f(x^* + dx) = f(x^*) + df(x^*) + \frac{1}{2} d^2 f(x^*) \ldots$$

$$= f(x^*) + \cancel{\sum_{i=m+1}^{n} \frac{\partial f}{\partial x_i}\Big|_{x^*} dx_i}^{\,0} + \frac{1}{2!} \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_{x^*} dx_i dx_j \right)$$

$$f(x^* + dx) - f(x^*) = \underbrace{\frac{1}{2!} \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)\Big|_{x^*} dx_i dx_j}_{Q}$$

$\Rightarrow$ $f(x^* + dx) - f(x^*) > 0$
if Q is positive

$\Rightarrow$ x^* is a local minimum if Q is positive

$\Rightarrow$ x^* is a local minimum
if $Hess(f)\big|_{x^*} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{x^*}$ is positive definite

Similary x^* is local maxima
if $Hess(f)\big|_{x^*} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{x^*}$ is negative definite

problem continues.

$$\begin{vmatrix} \frac{\partial^2 f}{\partial y_2^2} & \frac{\partial^2 f}{\partial y_2 \partial y_4} \\ \frac{\partial^2 f}{\partial y_4 \partial y_2} & \frac{\partial^2 f}{\partial y_4^2} \end{vmatrix}_{\text{at } x^*} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0 \text{ positive definite}$$

$\therefore x^* = \left(\frac{-5}{74}, \frac{-5}{37}, \frac{155}{74}, \frac{30}{37}\right)$ is local minima.

now with only indipendent variables

$$f = \frac{1}{2}\left(\frac{y_2^2}{4} + y_2^2 + \left(2y_4 - \frac{7}{2}y_2\right)^2 + y_4^2\right).$$

$$\frac{\partial f}{\partial y_2} = \frac{1}{2}\left(\frac{\cancel{2}y_2}{\cancel{4}2} + 2y_2 - \cancel{2}\left(2y_4 - \frac{7}{2}y_2\right)\cdot\frac{7}{\cancel{2}}\right)$$

$$= \frac{y_2}{4} + y_2 - 7y_4 + \frac{49}{4}y_2$$

$$= \frac{25}{2}y_2 + y_2 - 7y_4 = \frac{27y_2}{2} - 7y_4$$

$$\frac{\partial^2 f}{\partial y_2^2} = \frac{27}{2} \qquad \frac{\partial^2 f}{\partial y_2 \partial y_4} = -7$$

$$\frac{\partial f}{\partial y_4} = \frac{1}{\cancel{2}}\left(0 + 0 + \cancel{2}\left(2y_4 - \frac{7}{2}y_2\right)2 + \cancel{2}y_4\right)$$

$$= 4y_4 - \frac{7}{\cancel{2}}y_2 + y_4$$

$$= 5y_4 - 7y_2 \qquad \frac{\partial^2 f}{\partial y_4^2} = 5 \qquad \frac{\partial^2 f}{\partial y_4 \partial y_2} = -7$$

$$\begin{vmatrix} \frac{27}{2} & -7 \\ -7 & 5 \end{vmatrix}$$

$$= \frac{27\times 5 + 49}{2}$$

$$= > 0$$

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$$Hes(f) = \begin{vmatrix} 13.5 & -7 \\ -7 & 5 \end{vmatrix}$$

$= 67.5 - 49 > 0$

Positive definite matrix

So, point $x^a = \left(\frac{-5}{74}, -\frac{5}{37}, \frac{155}{74}, \frac{30}{37}\right)$ is a local minimum. Since there are no other stationary point so x^a is global minimum.

Method of Lagrange Multipliers.

- Minimize $f(x_1, x_2)$ subject to $g(x_1, x_2) = 0$.

Necessary condition in constraint variation method is.

$$\frac{\partial f}{\partial x_1} - \frac{\partial f / \partial x_2}{\partial g / \partial x_2} \frac{\partial g}{\partial x_1} = 0 \Bigg|_{at\ x'} \quad \text{--- (1)}$$

define a quantity

lagrange multiplier $\lambda = \dfrac{-\partial f / \partial x_2}{\partial g / \partial x_2}\Bigg|_{at\ x^*}$ --- (2)

from (1),

$$\frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1}\Bigg|_{x^a} = 0 \quad \text{--- (3)}$$

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From ② $\frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2}\Big|_{x^*} = 0$ —— ④

along with $g(x_1, x_2) = 0$ —— ⑤

so, eq. ③, ④ & ⑤ are necessary condition for x^* to be a minimum or maximum for the given problem.

→ Define a f^n (lagrange f^n)

$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda \cdot g(x_1, x_2)$

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= \frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} = 0 \\ \frac{\partial L}{\partial x_2} &= \frac{\partial f}{\partial x_2} + \lambda \cdot \frac{\partial g}{\partial x_2} = 0 \\ \frac{\partial L}{\partial \lambda} &= g(x_1, x_2) = 0 \end{aligned}\right\}$$

this are same as ③, ④ & ⑤ —— ⑥

So, ⑥ is necessary condition for x^* to be a local max. / local min.

EXP Minimize $f(x, y) = k x^{-1} y^{-2}$ subject to ~~gf~~ $g(x, y) = x^2 + y^2 - a^2 = 0$

circle of a radius 'a'.

→ $L(x, y, \lambda) = f(x, y) + \lambda \cdot g(x, y)$

$= k x^{-1} y^{-2} + \lambda \cdot (x^2 + y^2 - a^2)$

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Necessary condition,

$\frac{\partial L}{\partial x} = -k x^{-2} y^{-2} + \lambda \cdot 2x = 0$ —— ①

$\frac{\partial L}{\partial y} = -2k x^{-1} y^{-3} + \lambda \cdot (2y) = 0$ —— ②

$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - a^2 = 0$ —— ③

from eq - ①

$2\lambda = \frac{k}{x^3 y^2}$

from eq. ②

$2\lambda = \frac{2k}{x y^4}$

$\frac{k}{x^3 y^2} = \frac{2k}{x y^4}$

$\frac{1}{x^2} = \frac{2}{y^2}$

$y^2 = 2x^2$ —— ④

→ from ③

$\frac{\partial L}{\partial \lambda} = x^2 + 2x^2 - a^2 = 0$

$3x^2 = a^2$

$x = a/\sqrt{3}$ $\quad y = \sqrt{\frac{2}{3}}\, a$

$\lambda = \frac{k}{\frac{a}{\sqrt{3}} \cdot \frac{4}{9} a^4} = \frac{9\sqrt{3}\,k}{4a^5}$

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☆ Necessary condition for general problem.

$f: R^n \to R$

Problem - Minimize or maximize $f(x)$
subject to $g_j(x) = 0$
$x = x_1, x_2, \ldots, x_n$
$j = 1, 2, \ldots, m$ $m <= n$. constraints

define lagrange f^n

$$L(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m)$$
$$= f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x) + \ldots + \lambda_m g_m(x)$$

- necessary conditions are

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0, \quad i = 1, 2, \ldots, n. \qquad (A)$$

$$\frac{\partial L}{\partial \lambda_j} = g_j(x) = 0, \quad j = 1, 2, \ldots, m \qquad (B)$$

This system has $n+m$ variables and $n+m$ equations.

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- Necessary condition.

$$\frac{\partial L}{\partial x_i} = 0 \qquad i = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial \lambda_j} = 0 \qquad j = 1, 2, \ldots, m$$

$n+m$ variables
$n+m$ equations.

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x^* , λ^* ↑ vector m components.
↑ vector n component

$x^* = (x_1^*, x_2^*, \ldots\ldots, x_n^*)$ — extreme point
$\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots\ldots, \lambda_m^*)$ — sensitivity information.

• Sufficient condition:

→ A sufficient condition for f(x) to have a local minimum at x^* is that the quadratic

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 L}{\partial x_i \partial x_j} dx_i\, dx_j \Big|_{x^*} > 0$$

that is $Hes(L)\big|_{x^*} = \left[\frac{\partial^2 L}{\partial x_i \partial x_j}\right]_{x^*}$ is positive definite.

→ local maximum at x^* if

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 L}{\partial x_i \partial x_j} dx_i\, dx_j \Big|_{x^*} < 0$$

that is $Hes(L)\big|_{x^*} = \left[\frac{\partial^2 L}{\partial x_i \partial x_j}\right]_{x^*}$ is negative definite

→ x^* is neither local max nor local min if $Hes(L)\big|_{x^*}$ is indefinite matrix.

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local ~~minimum~~ maximum at x^* if

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 L}{\partial x_i \partial x_j} dx_i dx_j \Big|_{x^*} < 0$$

that is $\text{Hess}(L)\big|_{x^*} = \left[\frac{\partial^2 L}{\partial x_i \partial x_j}\right]_{x^*}$ is negative definite.

x^* is neither a local maximum nor a local minimum if the $\text{Hess}(L)\big|_{x^*} = \left[\frac{\partial^2 L}{\partial x_i \partial x_j}\right]_{x^*}$ is indefinite.

<u>Note</u> (Hancock) proved that the quadratic Q is positive (negative) for all admissible variations dx if each root of the polynomial α_j of the following equations are positive (negative).

$$\begin{bmatrix} L_{11}-\alpha & L_{12} & \cdots & L_{1n} & g_{11} & g_{21} & \cdots & g_{m1} \\ L_{21} & L_{22}-\alpha & \cdots & L_{2n} & g_{12} & g_{22} & \cdots & g_{m2} \\ \vdots & & & & & & & \\ L_{n1} & L_{n2} & \cdots & L_{nn}-\alpha & g_{1n} & g_{2n} & \cdots & g_{mn} \\ g_{11} & g_{12} & \cdots & g_{1n} & 0 & 0 & & 0 \\ g_{21} & g_{22} & \cdots & g_{2n} & 0 & & & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mn} & 0 & \cdots & & 0 \end{bmatrix} = 0$$

where $L_{ij} = \left[\frac{\partial^2 L}{\partial x_i \partial x_j}\right]_{(x^*, \lambda^*)}$ $\quad i = 1, 2, \ldots n$, $\quad j = 1, 2, \ldots n$

$g_{ij} = \frac{\partial g_i}{\partial x_j}\Big|_{x^*}$ → eigen values.

If all $\alpha_j > 0$
then x^* is a local minimum
If all $\alpha_j < 0$
then x^* is a local maximum

If some α_j are +ve and some are -ve, then x^* is neither local max non local min.

exp Find the dimension of a cylindrical tin (with top and bottom) made up of sheet metal to maximize its volume such that the total surface area is 24π.

Sol^n let r & h be the radius and height of the tin
Volume = $\pi r^2 h$

objective function
maximize $V = \pi r^2 h$

Subject to $2\pi r h + 2\pi r^2 = 24\pi$

$g = 2\pi r h + 2\pi r^2 - 24\pi$

$$L(r,h,\lambda) = \pi r^2 h + \lambda(2\pi r h + 2\pi r^2 - 24\pi)$$

$$=$$

$$\frac{\partial L}{\partial r} = 0 \Rightarrow 2\pi r h + \lambda(2\pi h + 4\pi r) = 0$$

$$\Rightarrow 2\pi h(r+\lambda) + 4\pi r\lambda = 0 \quad \text{——— (1)}$$

$\frac{\partial L}{\partial h} = 0 \Rightarrow \pi r^2 + 2\pi \lambda r = 0$ —— (2)

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 2\pi r h + 2\pi r^2 - 24\pi = 0$ —— (3)

from (1) $rh + 2\lambda r + \lambda h = 0$

$\Rightarrow \lambda = \frac{-rh}{2r+h}$ —— (4)

from (2) we have

$\lambda = \frac{-r}{2}$ —— (5)

from (4) & (5) $\frac{-rh}{2r+h} = \frac{-r}{2}$

$2h = 2r + h$

$\boxed{h = 2r}$ —— (6)

~~2r~~ $4\pi r^2 + 2\pi r^2 - 24\pi = 0$

$\cancel{6}\cancel{\pi} r^2 = \cancel{24}^{4}\cancel{\pi}$

$\boxed{r = 2}$ $\boxed{h = 4}$ $\boxed{\lambda = -1}$

so, we have to check sufficient condition :- Only two variable r & h and one λ.

$$\begin{vmatrix} L_{11} - \alpha & L_{12} & g_{11} \\ L_{21} & L_{22} - \alpha & g_{12} \\ g_{11} & g_{12} & 0 \end{vmatrix} = 0.$$

now $L_{11}\Big|_{(2,4,-1)} = \frac{\partial^2 L}{\partial r^2} = \frac{\partial}{\partial r}\left(2\pi r h + 2\pi h \lambda + 4\pi r \lambda\right)$

$= 2\pi h + 4\pi\lambda$

$= 8\pi - 4\pi = 4\pi$

$$L_{12}\Big|_{(2,4,-1)} = \frac{\partial^2 L}{\partial r\,\partial h} = \frac{\partial}{\partial r}\left(\pi r^2 + 2\pi r\lambda\right)$$

$$= 2\pi r + 2\pi\lambda$$
$$= 4\pi - 2\pi$$
$$= 2\pi$$

$$L_{21}\Big|_{(2,4,-1)} = \frac{\partial^2 L}{\partial h\,\partial r} = 2\pi r + 2\pi\lambda$$
$$= 4\pi - 2\pi$$
$$= 2\pi$$

$$L_{22}\Big|_{(2,4,-1)} = \frac{\partial^2 L}{\partial h^2} = 0$$

$$g_{11}\Big|_{(2,4,-1)} = \frac{\partial g}{\partial r}\Big|_{(2,4,-1)} = 2\pi h + 4\pi r = 8\pi + 8\pi$$
$$= 16\pi$$

$$g_{12}\Big|_{(2,4,-1)} = \frac{\partial g}{\partial h}\Big|_{(2,4,1)} = 2\pi r = 4\pi$$

$$\begin{vmatrix} 4\pi-\alpha & 2\pi & 16\pi \\ 2\pi & -\alpha & 4\pi \\ 16\pi & 4\pi & 0 \end{vmatrix} = 0$$

$$\therefore\ \cancel{(4\pi-\alpha)[-64\pi] - 2\pi[128\pi] + 16\pi[16\pi + 16\alpha\pi] = 0}$$

$$\cancel{\therefore -128\pi^2 + 64\pi\alpha + 256\pi^2 + 256\pi^2 + 256\pi^2\alpha = 0}$$

$$\therefore (4\pi-\alpha)\,[-16\pi^2] - 2\pi\,[-64\pi^2] + 16\pi\,[8\pi^2 + 16\pi\alpha] = 0$$

$$\therefore -64\pi^3 + 16\alpha\pi^2 + 128\pi^3 + 128\pi^3 + 256\pi^2\alpha = 0$$

$192\pi^3 + 272\pi^2 \alpha = 0$

$\therefore \alpha = \dfrac{-192\pi^3}{272\pi^2} = \dfrac{-12}{17}\pi < 0$

$\therefore (r=2, h=4)$ is ~~maxima~~ local maxima. it gives maximum volume of the cylinder.

max $V = \pi r^2 h = 16\pi$ ✓

* Interpretation of Langrange's Multiplier :-

Consider the problem ;

minimize $f(x)$

Subject to $\tilde{g}(x) = b$ —— ①

$g(x) = b - \tilde{g}(x) = 0$ —— ②

$L = f + \lambda g$

Necessary condition are

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0 \quad , \quad i = 1, 2, \ldots n. \quad \text{—— ③}$$

$$\frac{\partial L}{\partial \lambda} = g(x) \quad \text{—— ④}$$

let the solution of (3) and (4) be

x^*, λ^* at optimal value $f^* = f(x^*)$.

We want to find the effect of a small change in b on f^*.

Differentiating (2)

$$db - d\tilde{g} = 0$$

$$\Rightarrow db = d\tilde{g}$$

$$\boxed{db = \sum_{i=1}^{n} \frac{\partial \tilde{g}}{\partial x_i} dx_i} \quad \text{——— (5)}$$

eq (3) can be written as

$$\frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = \underbrace{\frac{\partial f}{\partial x_i} - \lambda \frac{\partial \tilde{g}}{\partial x_i}}_{} = 0. \quad \text{——— (6)}$$

$$\Rightarrow \frac{\partial \tilde{g}}{\partial x_i} = \frac{1}{\lambda} \frac{\partial f}{\partial x_i} \qquad i = 1, 2, \ldots n$$

putting the value of $\frac{\partial \tilde{g}}{\partial x_i}$ in (5)

$$db = \sum_{i=1}^{n} \frac{1}{\lambda} \frac{\partial f}{\partial x_i} dx_i$$

$$\Rightarrow \boxed{db = \frac{1}{\lambda} \sum_{i=1}^{n} \frac{df}{\partial x_i} dx_i}$$

$$\Rightarrow \boxed{db = \frac{1}{\lambda} df}$$

$$\Rightarrow df = \lambda db$$

At the optimal point

$$\boxed{df^* = \lambda^* db}$$

λ^* denotes the sensitivity of f with respect to b.

Case 1 when $\lambda^* > 0$

Unit decrease in b contributes

$$df^* = \lambda^* db = \lambda^*(-1) = -\lambda^*$$

$$df^* = -\lambda^*$$

Unit increase in b contributes.

$$df^* = \lambda^* db = \lambda^*(+1) = \lambda^*$$

$$df^* = \lambda^*$$

Case 2 when $\lambda^* < 0$

Unit decrease in b contributes.

$$df^* = \lambda^* db = \lambda^*(-1) = -\lambda^* \geq 0$$

increase the optimal value of f^* by λ^*,

Unit increase in b contributes.

$$df^* = \lambda^* db = \lambda^*(1) = \lambda^* < 0$$

so decrease the optimal value f^* by λ^*.

Case 3: when $\lambda^* = 0$

There is no change in the optimal value of the objective function

$$df^* = \lambda^* db = 0 \quad \text{always.}$$

ex. find the maximum value of $f(x) = 2x_1 + x_2 + 10$
subject to $g(x) = x_1 + 2x_2^2 - 3 = 0$.

Using langrange multiplier method

find the effect of changing the right way side of constant on the optimal value of f.

$\Rightarrow$ Minimize $f(x)$
subject to $\tilde{g}(x) = b$ — ①

$$g(x) = \tilde{g}(x) - b = 0 \quad \text{— ②}$$

$L = f + \lambda g$
Necessary condition:

$$\begin{cases} \dfrac{\partial L}{\partial x_i} = \dfrac{\partial f}{\partial x_i} + \lambda \dfrac{\partial g}{\partial x_i} = 0 & \text{— ③} \quad i = 1, 2, \ldots, n \\[2ex] \dfrac{\partial L}{\partial \lambda} = g(x) = 0 & \text{— ④} \end{cases}$$

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Solving we get x^*, λ^*

Differentiating (2)

$$d\tilde{g} - db = 0$$

$$\Rightarrow db = d\tilde{g} = \sum_{i=1}^{n} \frac{d\tilde{g}}{\partial x_i} dx_i \quad \text{—— (5)}$$

eq. (3) $\Rightarrow$ $$\frac{\partial f}{\partial x_i} + \frac{\lambda \dot{g}}{\partial x_i} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x_i} + \lambda \frac{\partial \tilde{g}}{\partial x_i} = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{g}}{\partial x_i} = -\frac{1}{\lambda} \frac{\partial f}{\partial x_i}}$$

putting the value in (5)

$$db = \sum_{i=1}^{n} -\frac{1}{\lambda} \frac{\partial f}{\partial x_i} dx_i$$

$$db = -\frac{1}{\lambda} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

$$\boxed{db = -\frac{1}{\lambda} df}$$

$$\Rightarrow \boxed{df = -\lambda db}$$

At the optimal point

$$\boxed{df^* = -\lambda^* db}$$

case 1: when $\lambda^* < 0$

Then unit increase in b

$df^* = -\lambda^* (+1)$

$df^* = -\lambda^* \geq 0$

So function value increase.

Then unit decrease in b

$df^* = -\lambda^* (-1)$

$= \lambda^* \leq 0$.

So function value decrease.

Case 2: when $\lambda^* > 0$

Then unit increase in b

$df^* = -\lambda^* (+1)$

$= -\lambda^* \leq 0$

So function value decrease

Then unit decrease in b

$df^* = -\lambda^* (-1)$

$= \lambda^* > 0$

So function value increase.

case 3: when $\lambda^* = 0$.

$df^* = \lambda^* db = 0$ always.

There is no change in the optimal value of the objective function.

* Multivariable optimization with inequality constaints.

Minimize $f(x)$
subject to $g_j(x) \le 0$ —— ① $j = 1, 2, \dots m$.

We can convert the inequality constraint ① to equality constraint by adding some positive ~~value~~ variables.

$$g_j(x) + y_j^2 = 0 \quad j = 1, 2, \dots m \quad —— ②$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$ is called the vector of slack variable

for subtrage variable we call subslack variable

Minimize $f(x)$
subject to $G_j(x, Y) = g_j(x) + y_j^2 = 0$
$j = 1, 2, \dots m$.

Now, we can use the method of Langrange multiplier.
Langrange function $\lambda g_1(x) + \lambda y_2^2$

$$L(X, Y, \lambda) = f(x) + \lambda_1 G_1(X, Y) + \dots \lambda_m G_m(X, Y)$$

$$= f(x) + \sum_{j=1}^{m} \lambda_j G_j(X, Y)$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \quad \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix}$$

$2y$

$$= f(x) + \sum_{j=1}^{m} \lambda_j (g_j(x) + y_j^2)$$

Necessary condition

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{n} \lambda_j \frac{\partial G_j}{\partial x_i} = 0.$$

$$\Rightarrow \frac{\partial f}{\partial x_i} + \sum_{j=1}^{n} \lambda_j \frac{\partial g_j(x)}{\partial x_i} = 0, \quad \text{--- (5)}$$

$i = 1, 2, \dots n$

$$\frac{\partial L}{\partial \lambda_j} = G_j(x, y)$$

$$= \left(g_j(x) + y_j^2\right) = 0 \quad \text{--- (6)}$$

$j = 1, 2, \dots m$

$$\frac{\partial L}{\partial y_j} = 2\lambda_j y_j \quad j = 1, 2, \dots m \quad \text{--- (7)}$$

$$\begin{Bmatrix} n + 2m \text{ equations} \\ n + 2m \text{ variables} \end{Bmatrix}$$