Lecture -32



Recap:
Computing probability by conditioning

Markov's inequality:

$$P(X \geq a) \leq \frac{E[X]}{a}$$

---

<u>Uniform</u> random variable

$X \quad (0,10)$

$P(X \geq 9) = ?$

$$P(X \geq 9) \leq \frac{E[X]}{9} = \frac{5}{9} = 0.55$$

$$P(X \geq 9) = \int_9^{10} \frac{dx}{10} = \frac{1}{10} = 0.10$$

# Chebyshev's inequality.

$X$ is a random variable, mean $\mu$, variance $\sigma^2$, then for $k > 0$,

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

$$\Uparrow$$

Using $P(X \geq a) \leq \dfrac{E[X]}{a}$ <u>M.I.</u>

$$\downarrow$$

$$X \geq 0$$

starting point

$$P\Big(\underset{\substack{\equiv \\ X}}{(x-\mu)^2} \geq \underset{\substack{\equiv \\ a}}{k^2}\Big) \leq \frac{E[x-\mu]^2}{k^2}$$

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

C.I.

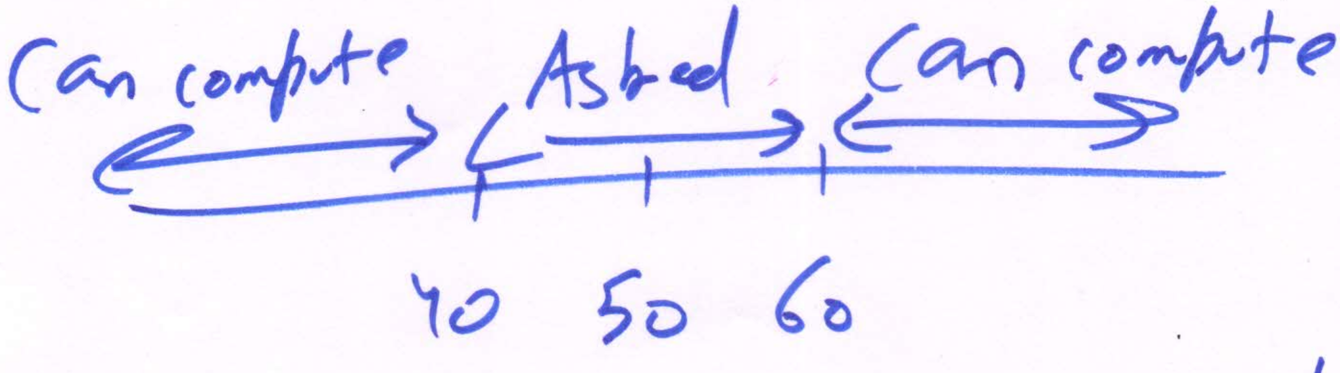
e.g.: no. of items produced in a factory in a week is a r.v. with

$\mu = 50$, $\sigma^2 = 25$.

What can be said about the ~~probability~~ that this week's production is between 40 & 60?

$$P(|X-\mu| \geq k)$$

$$P(|X-50| \geq k)$$




$$P(|X-50| \geq 10) = P(X \leq 40 \cup X \geq 60)$$

4

$$P(X \leq 40 \cup X \geq 60) \leq \frac{\sigma^2}{b^2}$$

$$= \frac{25}{100} = \frac{1}{4}$$

$$P(\bar{A}) \leq \frac{1}{4}$$

$$P(A) \geq 1 - \frac{1}{4} = \frac{3}{4}$$

---

X is uniformly distributed

$(0,10)$ $E[X] = \mu$

$$P(|X - 5| \geq 4) \leq \frac{\sigma^2}{4^2} = \frac{\cancel{100}^{25}}{12 \cdot \cancel{16}_{4}}$$

$$= \frac{25}{48} \simeq 0.5$$

$$P(X - 5 > 4 \cup X - 5 \leq -4)$$

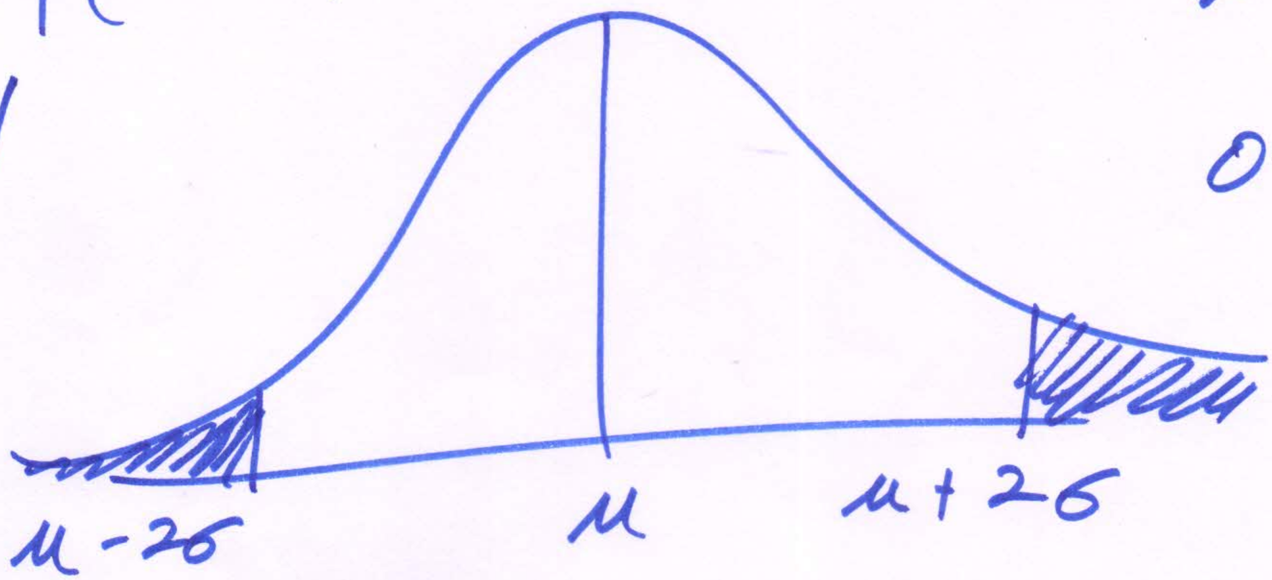
$$P(X > 9 \cup X < 1) = \int_0^1 \frac{dx}{10} + \int_9^{10} \frac{dx}{10} = 0.2$$

e.g.

5

$N(\mu, \sigma^2)$ Normal

$P(|X-\mu| > 2\sigma) \leq \frac{\sigma^2}{b^2} = \frac{1}{4} = 0.25$




$P(X-\mu > 2\sigma \cup X-\mu < -2\sigma)$

$P(X > 2\sigma + \mu \cup X < +\mu - 2\sigma)$

$P(X > \mu + 2\sigma) + P(X < \mu - 2\sigma)$

$P\left(\frac{X-\mu}{\sigma} > \frac{\mu + 2\sigma - \mu}{\sigma}\right) + P\left(\frac{X-\mu}{\sigma} < \frac{\mu - 2\sigma - \mu}{\sigma}\right)$

6

$P(Z > 2) + P(Z < -2)$

$1 - \Phi(2) + \Phi(-2)$

$= 2(1 - \Phi(2))$

$= 0.046 \leq 0.25$

C.I.

e.g.: if $\sigma^2 = 0$, then

~~with probability 1~~

$P(X = E[X]) = 1$

Prove this using Chebyshev's inequality..

7

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2} = 0$$

$$P(|X-\mu| \geq k) \leq 0$$

$$P(|X-\mu| \geq k) = 0$$

$k = \frac{1}{n}$, $n = 1, 2, 3 \dots$

$$P\left(|X-\mu| \geq \frac{1}{n}\right) = 0$$

what happens when $n \to \infty$?

$$P(|X-\mu| \neq 0) = 0$$

$$P(X \neq \mu) = 0 \Rightarrow$$

$$P(X = \mu) = 1$$

Weak law of large numbers.

$X_1, X_2, \ldots, X_n$ are independent & identically distributed i.i.d. each having a finite mean $E[X_i] = \mu$.

Then for any $\epsilon > 0$,

$$P\left\{ \left| \frac{X_1 + X_2 + \cdots + X_n}{n} - \mu \right| \geq \epsilon \right\} \to 0$$

as $n \to \infty$

Proof:

Assume same variance for each $X_i$, $Var(X_i) = \sigma^2$.

$$E\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right] = E\left[\frac{X_1}{n}\right] + E\left[\frac{X_2}{n}\right] + \cdots + E\left[\frac{X_n}{n}\right]$$

$$= \frac{1}{n} E[X_1] + \frac{1}{n} E[X_2] + \cdots + \frac{1}{n} E[X_n]$$

$$= \frac{1}{n} \cdot \mu \cdot n = \mu$$

$$Var\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = \sum_{i=1}^{n} Var\left(\frac{X_i}{n}\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) = \frac{n \cdot \sigma^2}{n^2}$$

$$= \frac{\sigma^2}{n}$$

$$Z = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$E[Z] = \mu, \quad Var(Z) = \frac{\sigma^2}{n}$$

$$P(|Z - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$$

$$P\left(\left|\frac{X_1 + X_2 \cdots + X_n}{n} - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2}$$

What happens when $n \to \infty$?

RHS $\to 0$