AC analysis

Now we begin looking at circuits with sinusoidal sources. We had a brief prior look at sinusoids, but now we add in capacitors and inductors, which make the picture much more interesting.

$$V_S(t) = V_m \cos \omega t \qquad\qquad I_S(t) = I_m \sin \omega t$$

Generally, it doesn't matter if we use sine or cosine – a sinusoid is a sinusoid.

V_m – amplitude (or magnitude or peak). In lab we typically use RMS values. However in doing calculations, we will often use the amplitude. Be sure to note which units are being used.

T – period. Time for one complete cycle.

f – frequency. Number of cycles in one second. $f = 1/T$.

ω – angular frequency. Number of radians in one second. $\omega = 2\pi f$.

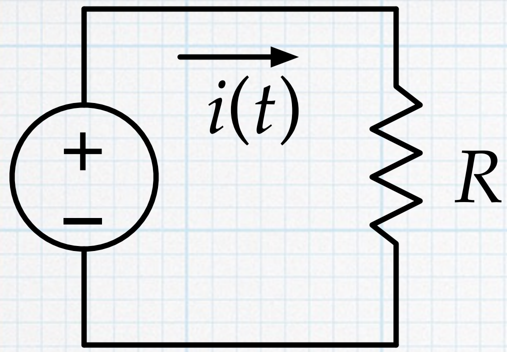
$$V_m \cos\left(\frac{2\pi}{T} t\right) = V_m \cos(2\pi f t) = V_m \cos(\omega t)$$

Why sinusoids

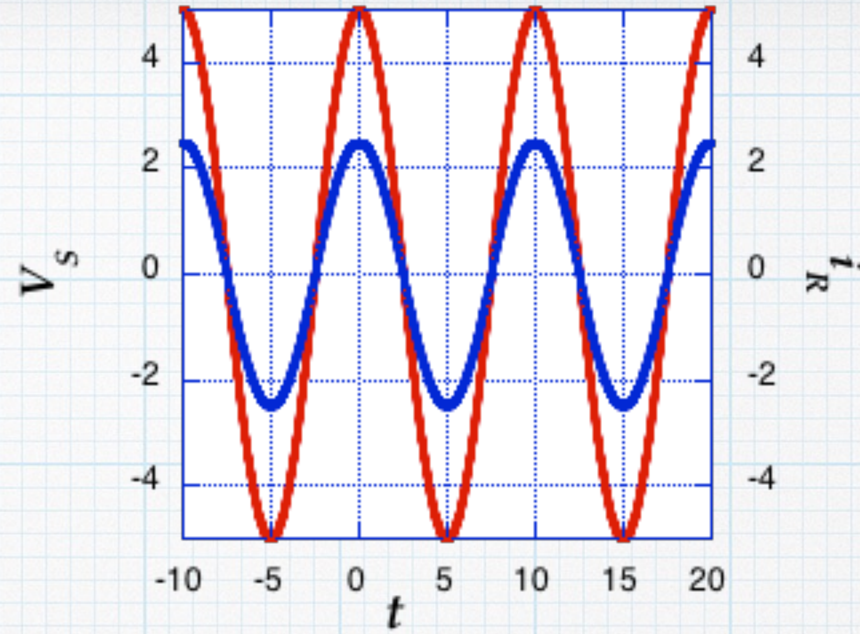
- Almost all electrical energy generated and transmitted in the world is in the form of sinusoid voltages and currents. (DC generation – solar cells – is on the rise, though.) Rotating machines (generators) naturally produce AC voltages. Also, it is easy to change voltage levels using transformers. Much of this is due to Tesla (the man, not the car).

- As you will learn later in your signal processing classes, all electrical waveforms can be described in terms of sinusoids. (Fourier series, Fourier transforms.) For instance, a square wave can be viewed to consist of a combination of many sinusoids of various frequencies and amplitudes. This idea is at the heart of communications theory. Therefore, if we know how a circuit responds to a sinusoid, we can know how it will respond to any kind of waveform.

$V_S(t) = V_m \cos \omega t$

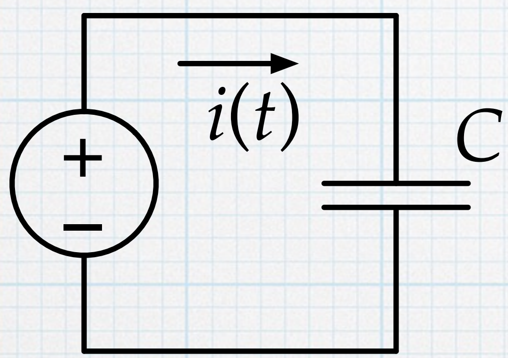


$$i\,(t) = \frac{V_m}{R} \cos \omega t$$

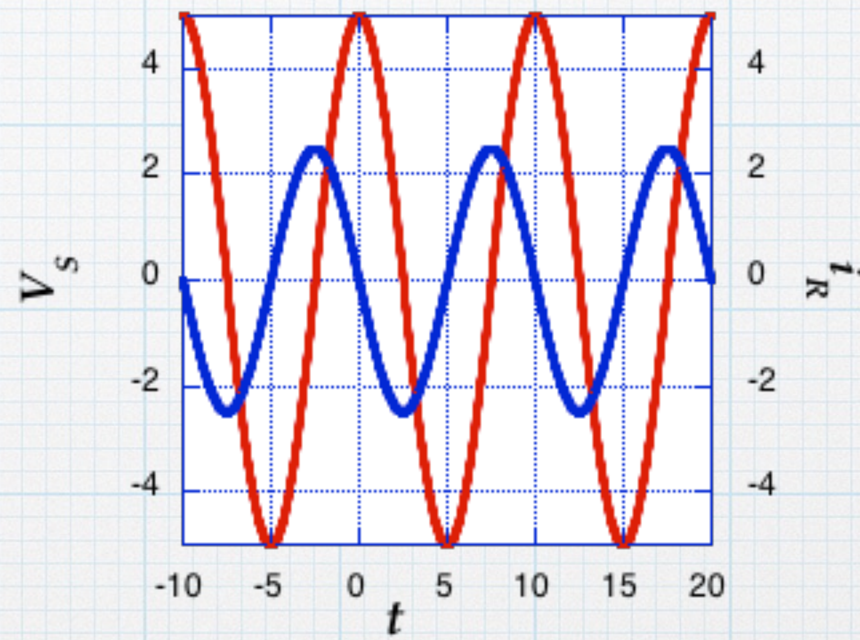


Current in phase with voltage.

$V_S(t) = V_m \cos \omega t$

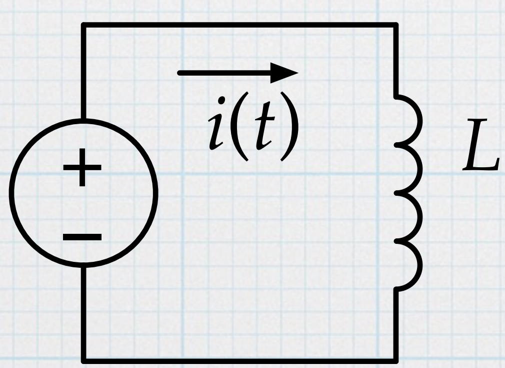


$$i\,(t) = -\omega C V_m \sin \omega t$$

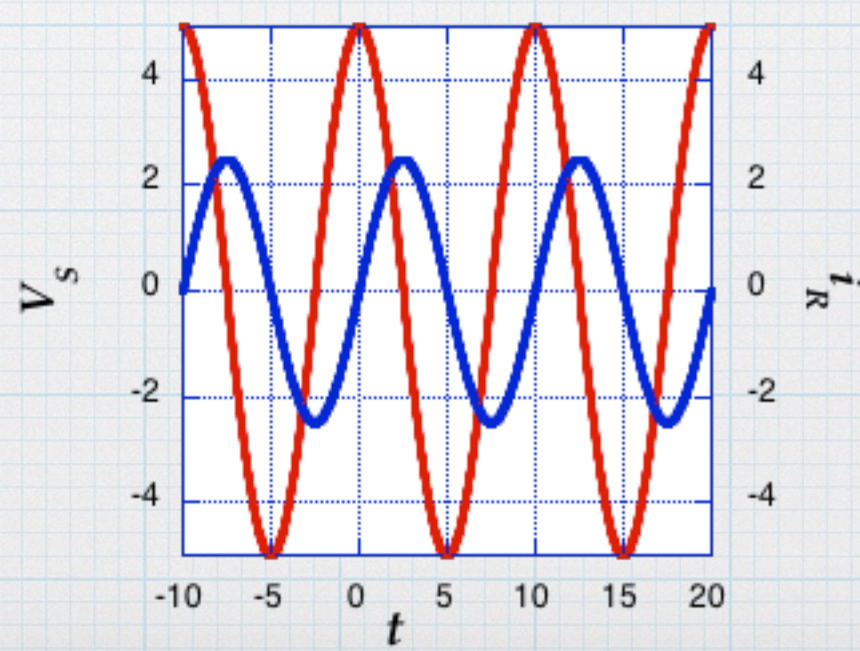


Current *leads* voltage by 90°.

$V_S(t) = V_m \cos \omega t$



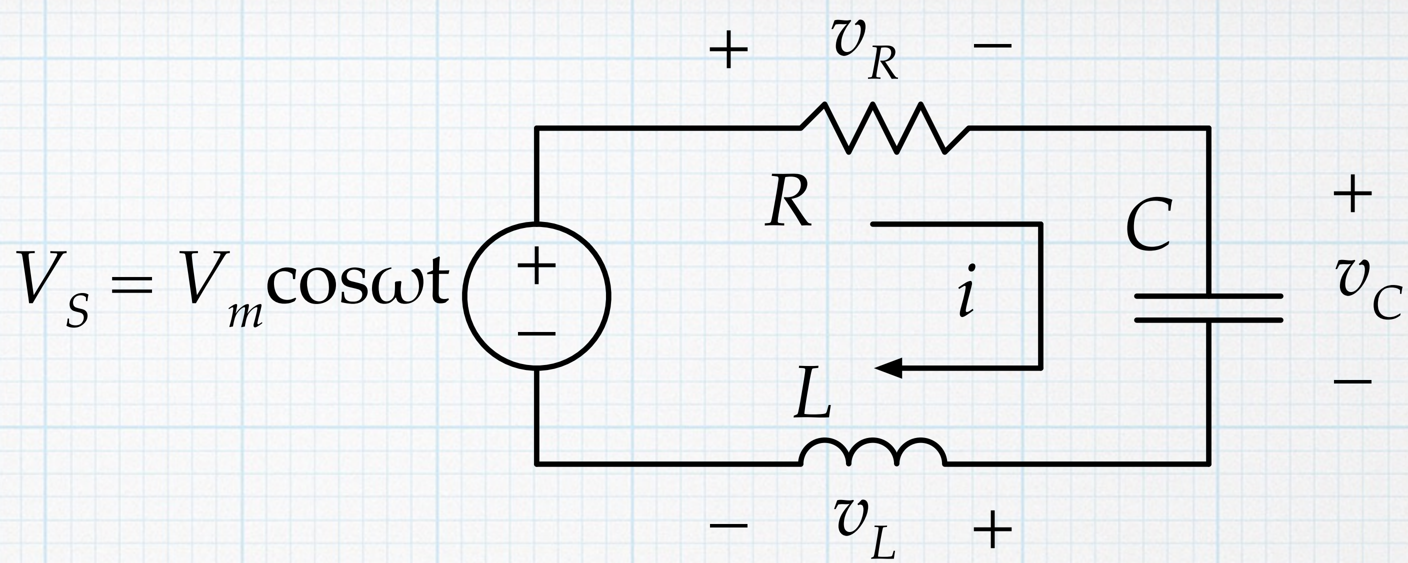
$$i\,(t) = \frac{V_m}{\omega L} \sin \omega t$$



Current *lags* voltage by 90°.

AC analysis

Consider the RLC circuit with a sinusoidal source.



This is similar to the to the series *RLC* circuit we saw earlier in looking at transients, except that the forcing function in this case is a sinusoid, not a simple DC voltage. As done in the previous case, we can use KVL, along with the *i-v* relationships for the resistor, capacitor, and inductor to arrive at a 2nd-order differential equation.

$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{1}{LC}V_m \cos\omega t$$

This is identical to the equation we derived for the step change transient, except that the forcing function (right-hand side) is a time-varying sinusoid, rather than a constant.

$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC} v_C(t) = \frac{1}{LC} V_m \cos \omega t$$

As in the previous case, we look for two solutions – the steady-state (particular) and the homogeneous (transient).

$$v_C(t) = v_{tr}(t) + v_{ss}(t)$$

The transient function will be the solution to the homogeneous equation:

$$\frac{d^2 v_{tr}(t)}{dt^2} + \frac{R}{L}\frac{dv_{tr}(t)}{dt} + \frac{1}{LC} v_{tr}(t) = 0$$

Since this is the same homogeneous equation as in the step-function problem, the solution here will be the same, which can be over-damped or under-damped, depending on the amount of resistance in the circuit. In either case, the transient function will go to zero after a few time constants, leaving only the steady-state (forced) response.

Sinusoidal steady-state analysis

In the step-change problem, the transient function was the only thing that was really interesting, since the forced response was just a constant.

But in the sinusoid case, the constantly changing voltage of the source will cause all of the voltages and currents in the rest of the circuit to continue changing in response. This goes on forever. Thus, the steady-state response is a very interesting part of an AC problem. In fact, it is much more interesting that the transient response, which will die away within a few moments after the circuit has started. Therefore, for the time being, we will ignore the transients and focus entirely on the steady-state solution.

An added benefit of ignoring the transient response and looking only at the steady-state response is that we do not need to worry about initial conditions.

This approach of ignoring the transient function and looking only at the forced response is known in the textbooks as *sinusoidal steady-state analysis*. But we will call it *AC analysis* for short.

$$v_C(t) = v_{ss}(t) \text{ only}$$

So we need to find the steady-state function.

$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{1}{LC}V_m \cos\omega t$$

The usual approach to finding the particular function is to use a trial function, and a first guess might be to try $v_c(t) = A\cos\omega t$. But when we insert that into the differential equation and try to determine A, we find that we can't do it.

$$-\omega^2 A\cos\omega t - \omega\frac{R}{L}A\sin\omega t + \frac{1}{LC}A\cos\omega t = \frac{1}{LC}V_m\cos\omega t$$

No joy.

$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{1}{LC}V_m \cos\omega t$$

So let's try two terms in the trial function: $v_C(t) = A\cos\omega t + B\sin\omega t$

$$-\omega^2 \left[A\cos\omega t + B\sin\omega t\right]$$

$$+\frac{\omega R}{L}\left[-A\sin\omega t + B\cos\omega t\right]$$

$$+\frac{1}{LC}\left[A\cos\omega t + B\sin\omega t\right] = \frac{1}{LC}V_m\cos\omega t$$

Combining like terms:

$$\left[-\omega^2 A + \frac{\omega R}{L}B + \frac{1}{LC}A\right]\cos\omega t + \left[-\omega^2 B - \frac{\omega R}{L}A + \frac{1}{LC}B\right]\sin\omega t = \frac{V_m}{LC}\cos\omega t$$

Match up similar coefficients:

$$-\omega^2 A + \frac{\omega R}{L} B + \frac{1}{LC} A = \frac{V_m}{LC}$$

$$-\omega^2 B - \frac{\omega R}{L} A + \frac{1}{LC} B = 0$$

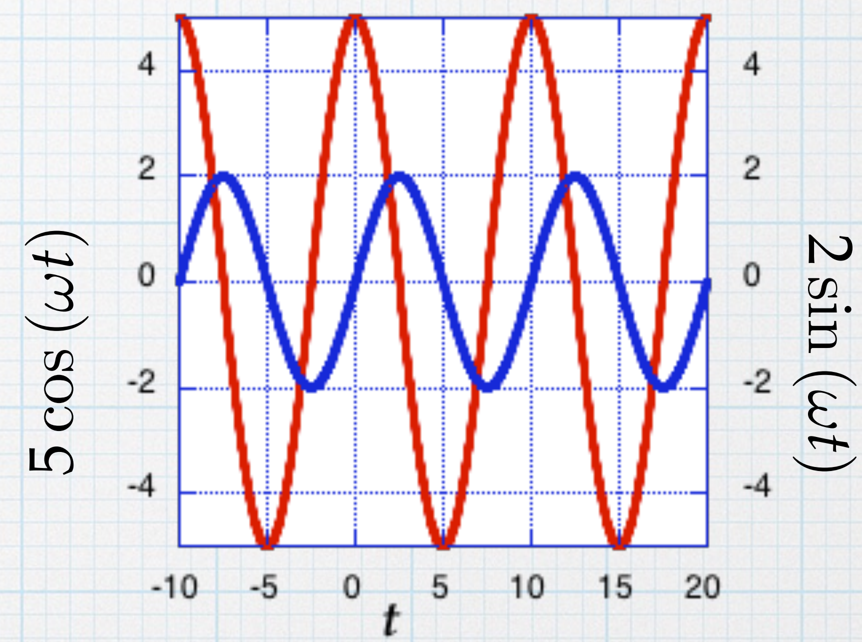
solve for A and B:

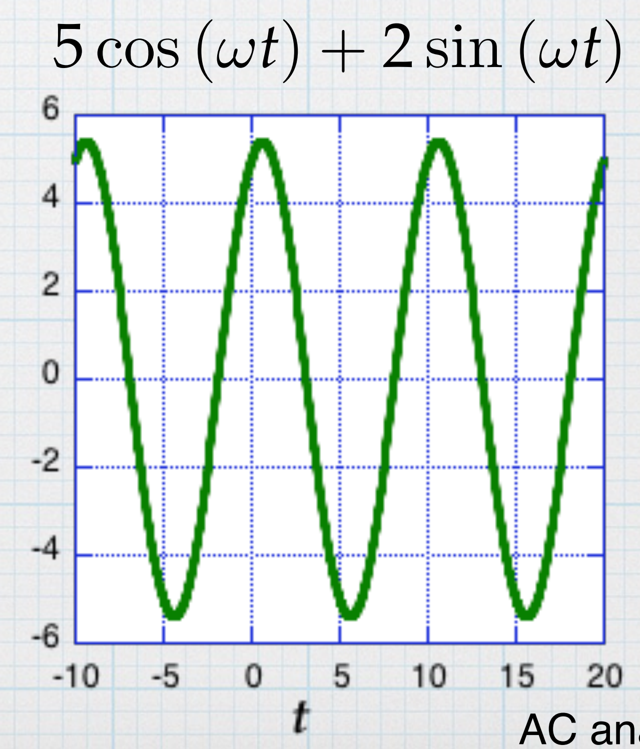
$$A = \frac{(1 - \omega^2 LC)}{(1 - \omega^2 LC)^2 + (\omega RC)^2} V_m$$

$$B = \frac{(\omega RC)}{(1 - \omega^2 LC)^2 + (\omega RC)^2} V_m$$

$v_C(t) = A\cos\omega t + B\sin\omega t$

Sum of sinusoids, both have frequency ω. In this form it is hard to visualize the resulting capacitor voltage.

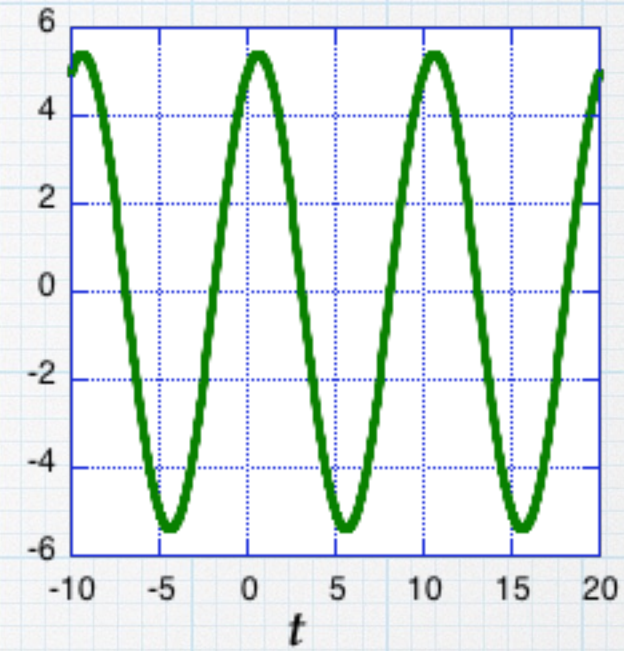




Use the (somewhat unfamiliar) trig. identity:

$$A\cos x + B\sin x = C\cos(x - \delta x)$$

$$C = \sqrt{A^2 + B^2} \qquad \delta x = \arctan\left(\frac{B}{A}\right)$$



Applying this to the capacitor voltage expression:

$$v_C(t) = V_{Cm}\cos(\omega t - \theta)$$

$$V_{Cm} = \frac{V_m}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}} \qquad \theta = \arctan\left(\frac{\omega RC}{1 - \omega^2 LC}\right)$$

We see that $v_C(t)$:

1. has time dependence of cos(ωt), just like the source,
2. has amplitude, V_{Cm}, (different from the source), and
3. has a phase shift, θ, relative to the source.

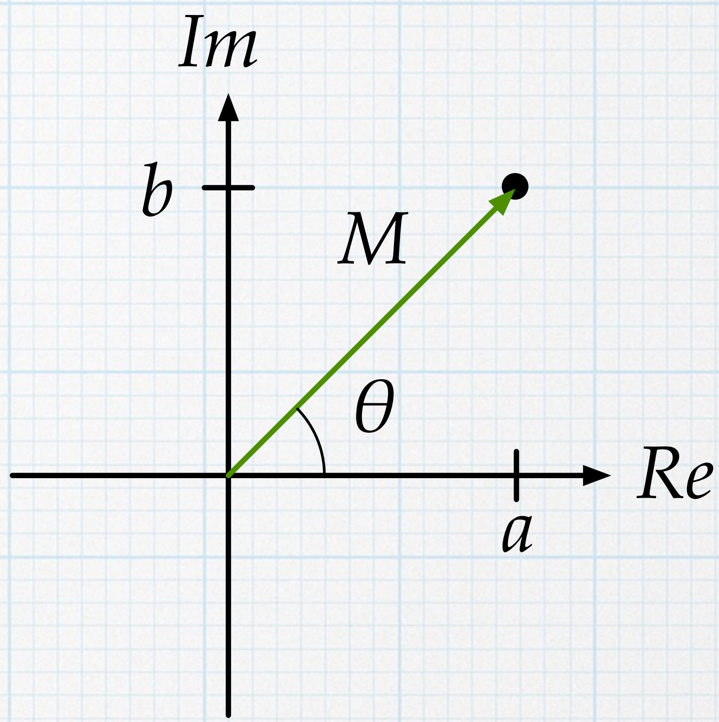
This is typical in sinusoidal circuits. All of the voltages and currents in the circuit will vary sinusoidally with the same angular frequency as the source. The problem reduces to finding the amplitude and phase shift. It would be nice to have a simpler way to find these.

complex numbers

$z = a + jb$

$a \rightarrow$ real part

$jb \rightarrow$ imaginary part $\qquad j = \sqrt{-1}$



polar description: (M, θ)

$$M = \sqrt{a^2 + b^2} \qquad a = M\cos\theta$$

$$\theta = \arctan\left(\frac{b}{a}\right) \qquad b = M\sin\theta$$

Amplitude and phase angle are built into the description of complex numbers.

$$e^{j\theta} = \cos\theta + j\sin\theta$$ Euler's identity

Complex numbers and sinusoids are intimately connected.

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad j\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2}$$

Write sines and cosines in term of complex exponentials.

complex voltages and currents

Try this: Use complex numbers to describe the sinusoidal forcing functions in the circuits. The resulting voltages and currents will be complex quantities as well, each having a magnitude and phase. Instead of a lots of trigonometry, we can go back to using algebra, although the algebra will involve complex numbers.

$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{1}{LC}V_m \exp(j\omega t)$$

Trial solution: $v_C(t) = Ae^{(j\omega t)}$

$$(j\omega)^2 Ae^{j\omega t} + \frac{R}{L}(j\omega) Ae^{j\omega t} + \frac{1}{LC}Ae^{j\omega t} = \frac{1}{LC}V_m e^{j\omega t}$$

$$\left[-\omega^2 LC + j\omega RC + 1\right] A = V_m$$

$$A = \frac{V_m}{1 - \omega^2 LC + j\omega RC}$$

Complex voltage for the capacitor.

$$A = \frac{V_m}{1 - \omega^2 LC + j\omega RC}$$

$$= \frac{1 - \omega^2 LC}{(1 - \omega^2 LC)^2 + (\omega RC)^2} V_m - j \frac{\omega RC}{(1 - \omega^2 LC)^2 + (\omega RC)^2} V_m$$

Even better, write it in magnitude and phase form:

$$A = Me^{j\theta}$$

$$M = \frac{V_m}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}}$$

$$\theta = \arctan\left(\frac{-\omega RC}{1 - \omega^2 LC}\right) = -\arctan\left(\frac{\omega RC}{1 - \omega^2 LC}\right)$$

The quantity A is a complex voltage, with a magnitude that matches the amplitude of the capacitor sinusoid and an angle that matches the phase angle of the capacitor sinusoid. Using complex numbers, we can obtain the magnitudes and phases without messing with sinusoids and trig. identities.

The complex voltages and currents are sometimes known as *phasors*.

- In the circuit, the sinusoidal source will be described as:

$$V_s(t) = V_m \cos \omega t \rightarrow \tilde{V}_s = V_m e^{j\omega t}$$

- The voltages and currents the rest of the circuit will also be complex, having the corresponding magnitudes and phase angles *with respect to* the source.

$$\tilde{v_C} = V_{Cm} e^{j(\omega t - \theta_C)} \rightarrow v_C(t) = V_{Cm} \cos(\omega t - \theta_C)$$

$$\tilde{i_L} = I_{Lm} e^{j(\omega t + \theta_L)} \rightarrow i_L(t) = I_{Lm} \cos(\omega t + \theta_L)$$

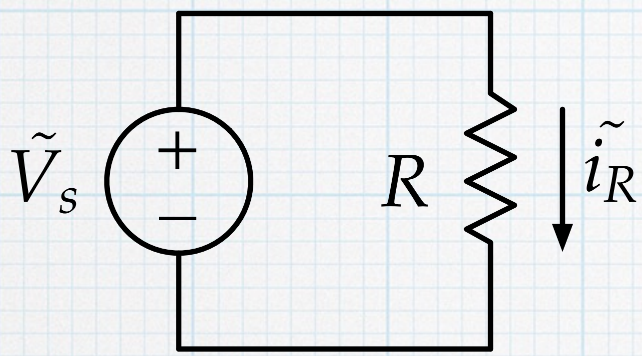
- Like voltage, phase angle is a relative quantity – only differences are meaningful. Usually define the phase of the source to be 0°. Usually, we will express the angle in terms of degrees, but we must be able to switch back and forth between degrees and radians as we do calculations. Note that phase angle also represents a time shift.

$$\theta = \frac{\pi}{3} \qquad \theta = 60^\circ \qquad \Delta t = \frac{T}{6}$$

- Finally, we will drop the $e^{j\omega t}$ factor, since all voltages and currents in the circuit oscillate with the same time dependence. The same factor shows up in every single term, and so we can always divide it. Since there is no need to carry the exp($j\omega t$) through every step in a calculation, there is really no need to include it at the beginning.

$$V_s\left(t\right) = V_m \cos \omega t \rightarrow \tilde{V}_s = V_m e^{j0^\circ}$$

impedances

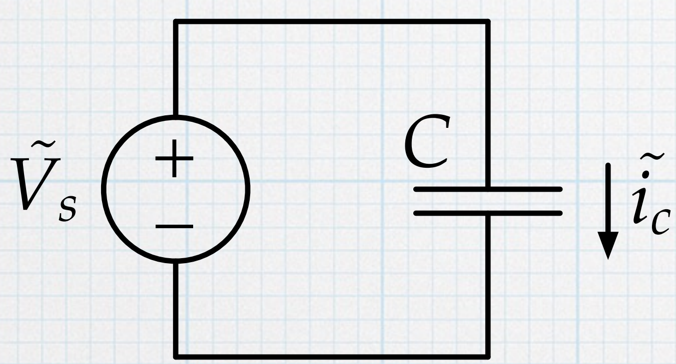


$$i_R = \frac{v_R}{R}$$

$$\tilde{v}_R = \tilde{V}_S = V_m e^{j\omega t}$$

$$\tilde{i}_R = \frac{V_m e^{j\omega t}}{R} = \frac{\tilde{v}_R}{R}$$

$$\frac{\tilde{v}_R}{\tilde{i}_R} = R$$

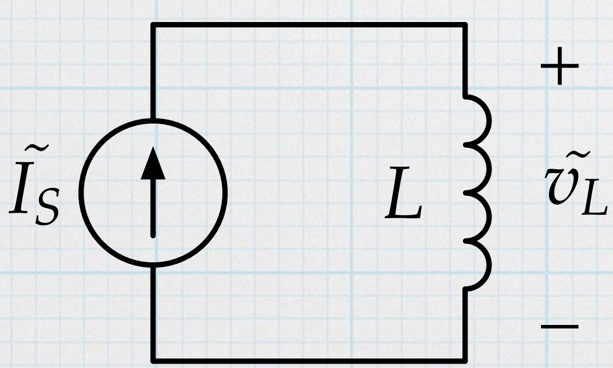


$$i_C = C\frac{dv_C}{dt}$$

$$\tilde{v}_C = \tilde{V}_S = V_m e^{j\omega t}$$

$$\tilde{i}_C = j\omega C V_m e^{j\omega t} = (j\omega C)\,\tilde{v}_C$$

$$\frac{\tilde{v}_C}{\tilde{i}_C} = \frac{1}{j\omega C}$$



$$v_L = L\frac{di_L}{dt}$$

$$\tilde{i}_L = \tilde{I}_S = I_m e^{j\omega t}$$

$$\tilde{v}_L = j\omega L I_m e^{j\omega t} = (j\omega L)\,\tilde{i}_L$$

$$\frac{\tilde{v}_L}{\tilde{i}_L} = j\omega L$$

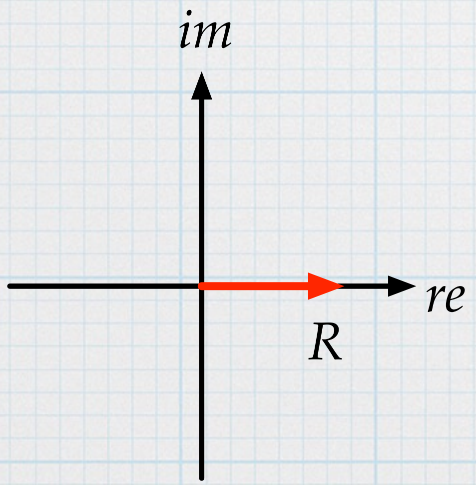
For each of the components, the ratio of complex voltage to complex current is a number. It is purely real in the case of the resistor and purely imaginary in the cases of the capacitor and inductor.

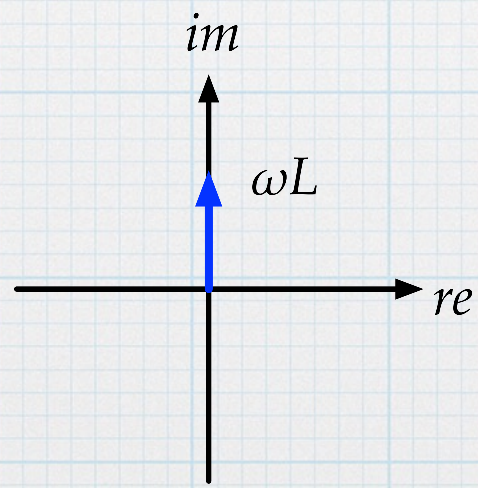
The ratios are called *impedances* and can be used as complex versions of Ohm's Law.

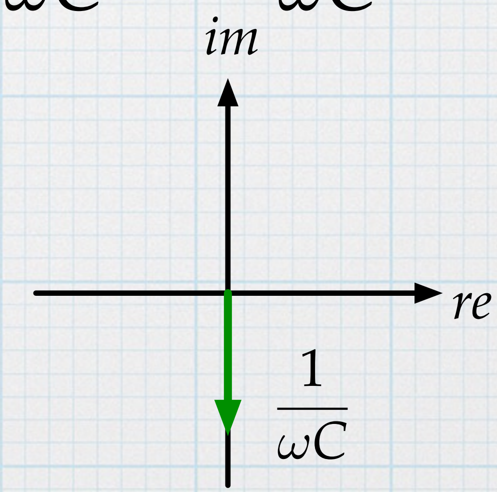
This allows us to get away from solving differential equations entirely and go back to using the circuit analysis techniques that we learned earlier. The difference is that since we are discussing sinusoidal circuits, the impedances are complex and we must use complex algebra.

$$Z_R = R$$

$$Z_C = \frac{1}{j\omega C} = -\frac{j}{\omega C} = \left(\frac{1}{\omega C}\right) e^{-j90^\circ}$$



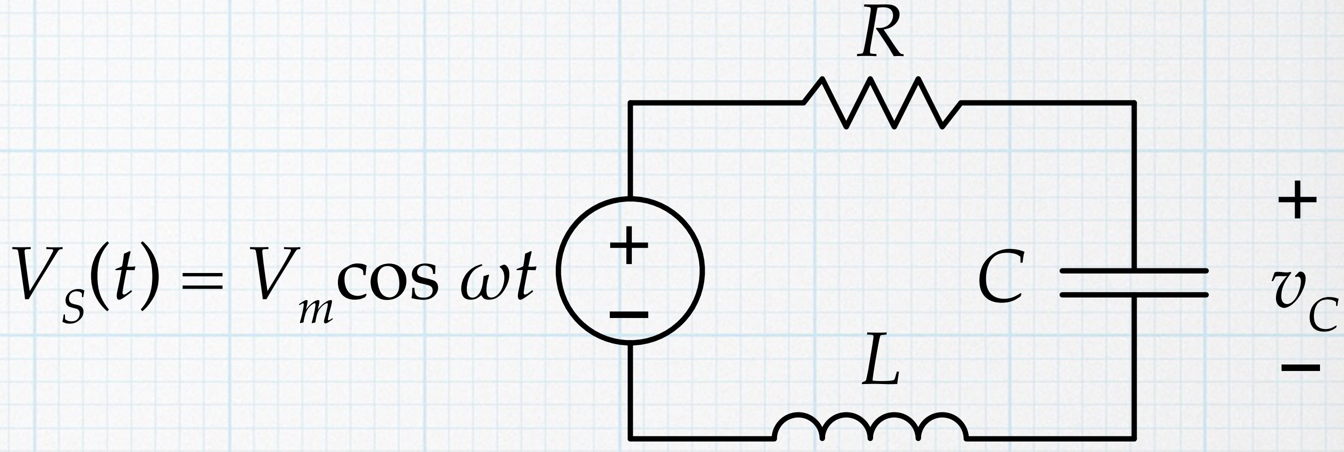




$$Z_L = j\omega L = (\omega L)\, e^{j90^\circ}$$

AC analysis - the method

Let's work the series RLC problem one more time, using the method of complex impedances. Find the capacitor voltage in the circuit at right.



1. Describe the sinusoidal sources as complex numbers. Usually, we assign one source to have 0° phase. (Note that $e^{j0°} = 1$.) If there is more than one source, it will be necessary to keep track of phase differences between the sources.

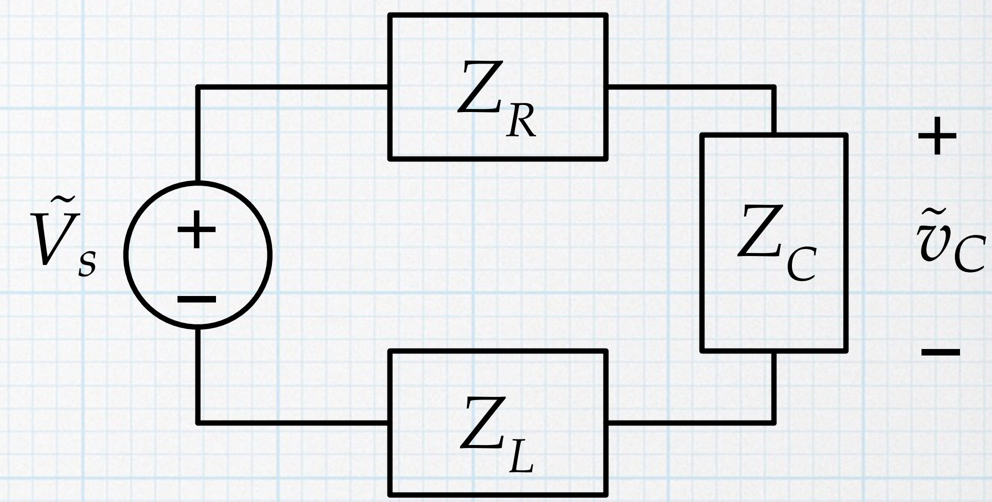
$$V_s(t) \rightarrow \tilde{V}_S = V_m e^{j0°} = V_m$$

2. Convert the resistors, capacitors, and inductors into their respective impedances.

resistor → $Z_R = R$ capacitor → $Z_C = \dfrac{1}{j\omega C} = \dfrac{-j}{\omega C}$

inductor → $Z_L = j\omega L$

3. Form the complex representation of the circuit. The impedances play the same role as resistors did in our circuit earlier in the semester. Of course, all of the math is complex and the currents and voltages will be complex quantities (phasors)



4. Use any of the circuit analysis techniques that we learned earlier to solve for the desired currents and voltages in the circuit. In this case, a voltage divider would seem to appropriate.

$$\tilde{v}_C = \frac{Z_C}{Z_C + Z_R + Z_L}\tilde{V}_S$$

That's it for the circuit analysis. The rest is working through the complex math to interpret the results.

5. Work out the complex algebra in find the desired complex voltages and currents. Start by inserting the complex expressions for the impedances and sources.

$$\tilde{v}_C = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R + j\omega L} V_m$$

We could begin inserting numbers at this point. However, it often useful to do a bit more algebra in order to simplify the expression.

$$\tilde{v}_C = \frac{V_m}{1 + j\omega RC - \omega^2 LC} = \frac{V_m}{[1 - \omega^2 LC] + j\,[\omega RC]}$$

6. Express the result into magnitude and phase form (usually): $\tilde{v}_C = |\tilde{v}_C| e^{j\theta_c}$

$$|\tilde{v}_C| = \frac{V_m}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}} \qquad \theta_c = -\arctan\left(\frac{\omega RC}{1 - \omega^2 LC}\right)$$

7. Re-express the voltage and currents as sinusoids. (Often un-necessary.)

$$v_C(t) = |\tilde{v}_C| \cos(\omega t + \theta_c)$$

For the RLC example, plug in some numbers.

V_S = 5 V with $\omega = 2\pi$(2500 Hz) =15,700 rad/s. R = 1 kΩ, C = 0.1 μF, and L = 33 mH.

$$\left|v_C\right| = \frac{5\text{ V}}{\sqrt{\left[1-(15,700\text{ rad/s})^2(0.033\text{ H})(10^{-7}\text{ F})\right]^2+\left[(15,700\text{ rad/s})(1\text{ k}\Omega)(10^{-7}\text{ F})\right]^2}} = 3.16\text{ V}$$

$$\theta_c = -\arctan\left[\frac{(15700\text{ rad/s})(1000\ \Omega)\left(10^{-7}\text{ F}\right)}{1-(15700\text{ rad/s})^2(0.033\text{ H})(10^{-7}\text{ F})}\right] = -83.2^\circ$$

Using the same methods, we can find the resistor and inductor voltages.

$$\tilde{v}_R = \frac{R}{R + \frac{1}{j\omega C} + j\omega L} V_m = \frac{V_m}{1 + j\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)} = \frac{V_m}{\sqrt{1 + \left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)^2}} \exp\left(j\theta_R\right)$$

$$\theta_R = -\arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)$$

$$\tilde{v}_L = \frac{j\omega L}{R + \frac{1}{j\omega C} + j\omega L} V_m = \frac{V_m}{\left(1 - \frac{1}{\omega^2 LC}\right) + \left(\frac{R}{j\omega L}\right)} = \frac{V_m}{\sqrt{\left(1 - \frac{1}{\omega^2 LC}\right)^2 + \left(\frac{R}{\omega L}\right)^2}} \exp\left(j\theta_L\right)$$

$$\theta_L = \arctan\left(\frac{R}{\omega L - \frac{1}{\omega C}}\right)$$

As an exercise, you could show that the three complex voltages add up to the complex source voltage (= V_m.)

Since the capacitor and inductor impedances are frequency dependent, we expect different results (magnitudes and phases) at different frequencies. Using the expressions for the three complex voltages, we can calculate the magnitude and phase of each voltage – and the current – at some different frequencies to see the effects. The results are tabulated below. You should check some of these numbers for yourself.

f (Hz)	$\lvert v_C \rvert$	θ_C	$\lvert v_R \rvert$	θ_R	$\lvert v_L \rvert$	θ_L	$\lvert i \rvert$	θ_i
500	4.92 V	–18.0°	1.54 V	72.0°	0.16 V	–18.0°	1.54 mA	72.0°
2500	3.16 V	–83.3°	4.97 V	6.7°	2.57 V	–83.3°	4.97 mA	6.7°
12500	0.24 V	22.1°	1.88 V	–67.9°	4.87 V	22.1°	1.88 mA	–67.9°