Lecture - 31 P 1

Recap:

$$-1 \le \rho(X,Y) \le 1$$

$$E[X] = E[E[X|Y]]$$

Treasure Hunt

1	2	3	4 -	-	-	100
10 lacs	1 Cr	100	50,000	100 Cr	. . .	

10 20

→ Reject first k prizes

→ Then choose the one that's better than all till now

$P_k(\text{best})$ = the probability ②
of choosing the best prize, after rejecting the 1^{st} k prizes, and then choosing the first prize that is better than all these k prizes.

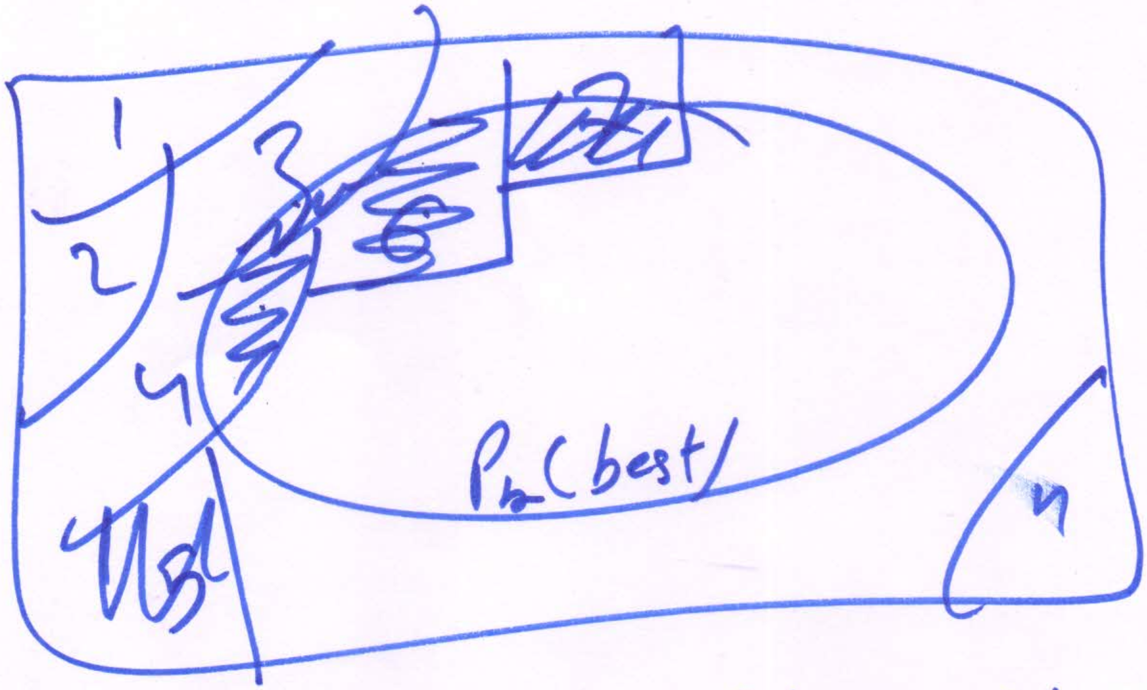
$$P_k(\text{best}) = \sum_{i=1}^{n} P_k(\text{best} \mid X=i)\, P(X=i)$$

X is the position of the best prize $X \in \{1, 2, \cdots, n\}$.

Condition on X.

$P(X=i) =$ Probability that the best prize is at position i, all $i=1,\cdots,n$

$= \frac{1}{n}$

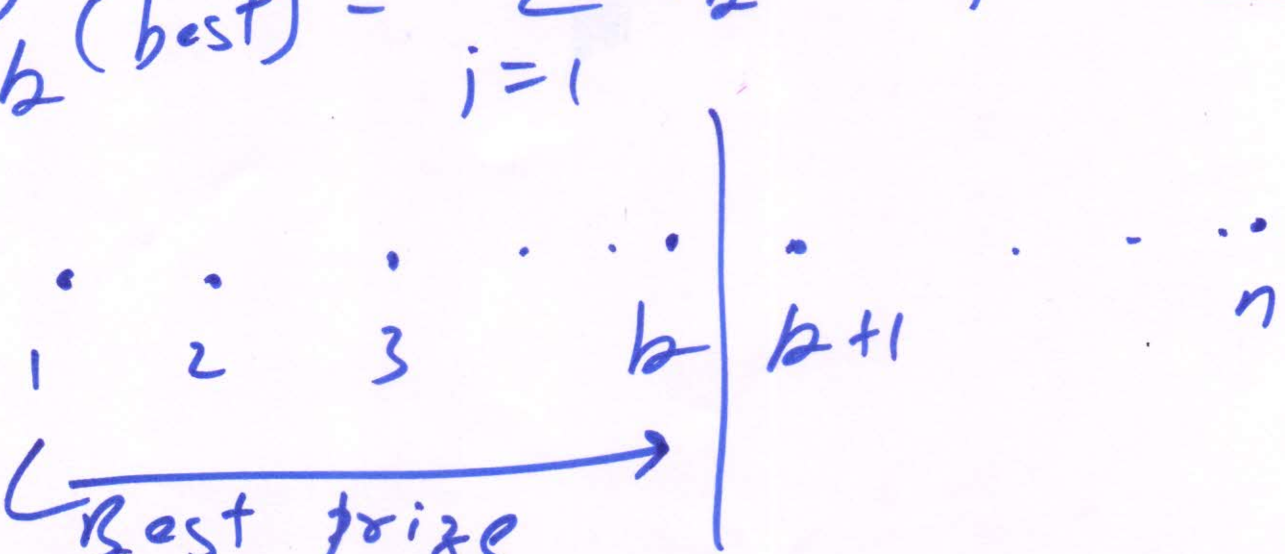


$$P_b \cancel{(best) \cap P(X=i)}$$

$$P_b(best \cap X=i)$$

$$= P_b(best \mid X=i)\, P(X=i)$$

$$P_b(best) = \sum_{i=1}^{n} P_b(best \mid X=i)\, \underline{\underline{P(X=i)}}$$



$$P_k(\text{best}) = \sum_{i=1}^{k} \underbrace{P_k(\text{best} \mid X=i)}_{=0} P(X=i) + \sum_{i=k+1}^{n} \underbrace{P_k(\text{best} \mid X=i)}_{=\frac{k}{i-1}} \underbrace{P(X=i)}_{=\frac{1}{n}}$$

$$= \frac{k}{n} \sum_{i=k+1}^{n} \frac{1}{i-1}$$

$$\sum_{i=k+1}^{n} \frac{1}{i-1} = \int_{k+1}^{n} \frac{dx}{x-1} = \log\left(\frac{n-1}{k}\right)$$

$$= \frac{k}{n} \log\left(\frac{n-1}{k}\right) \simeq \frac{k}{n} \log\left(\frac{n}{k}\right)$$

maximize this over k.

$$k = \frac{n}{e}$$

$$P_k(\text{best}) \simeq \frac{n}{e \cdot n} \log\left(\frac{n \cdot e}{n}\right)$$

$$= \frac{1}{e}$$

So, the probability of choosing the best prize is $\frac{1}{e} \simeq 0.37$

Markov's inequality.

<u>Thm</u>:
if X is a random variable which is non-negative, then for any $a \geq 0$

$$P(X \geq a) \leq \frac{E[X]}{a}$$

<u>Proof</u>: for $a \geq 0$, we define an Indicator random variable.

$$I = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{if } X < a \end{cases}$$

$$I \le \frac{X}{a}$$

$$I = 1 \Rightarrow \frac{X}{a} \ge 1$$

$$I = 0 \Rightarrow \frac{X}{a} < 1$$

$$E[I] \le E\left[\frac{X}{a}\right]$$

$$P(X \ge a) \le \frac{E[X]}{a} \quad \text{— M.I.}$$

$I = 1$ w.p. $P(X \ge a)$

$I = 0$ w.p. $P(X < a)$

$$E[I] = 1 \cdot P(X \ge a) + 0 \cdot P(X < a)$$
$$= P(X \ge a)$$

e.g.

No. of items produced in a factory in a week is a r.v. with mean

$E[X] = 50.$

What can be said about the probability of producing more than 75 items this week?

$$P(X \geq 75) \leq \frac{E[X]}{a} = \frac{50}{75} = \frac{2}{3}$$