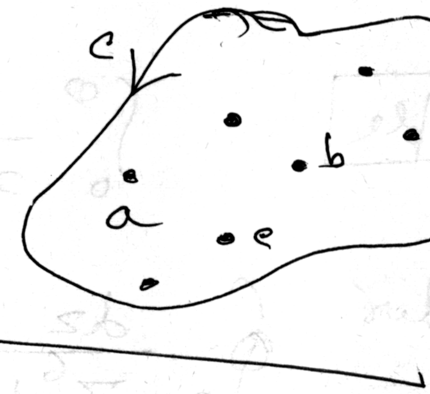
THE RESIDUE THEOREM

Let $f(z)$ be single-valued and analytic inside & on a simple closed curve c except at the singularities $a, b, c, \ldots$ inside c which have residues given by $a_{-1}, b_{-1}, c_{-1}, \ldots$ then the ~~resi~~ residue theorem states that

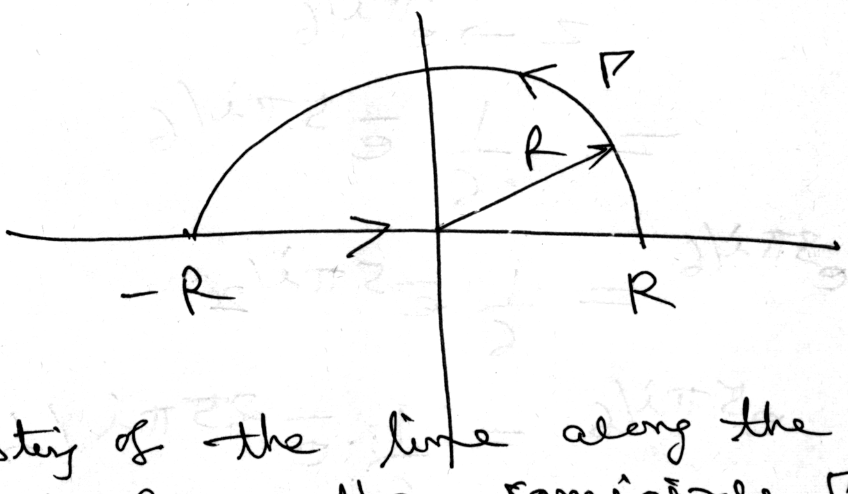


$$\oint_c f(z)\,dz = 2\pi i\,(a_{-1} + b_{-1} + c_{-1} + \cdots)$$

Evaluation of Definite Integrals

(1) $\displaystyle\int_{-\infty}^{+\infty} F(x)\,dx$, $F(x)$ is a rational fn.

Consider $\displaystyle\oint_c F(z)\,dz$ along a contour c



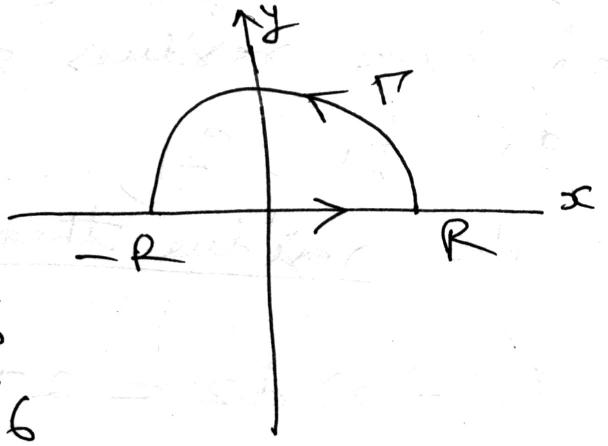
consisting of the line along the x-axis from $-R$ to R & the semicircle Γ above the x-axis having the line as diameter. then let $R \to \infty$

– If $F(x)$ is an even fn. this can be used to evaluate $\int_0^\infty F(x)\,dx$.

$\boxed{\text{Example}}$ $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$

□

Consider $\displaystyle\oint_C \frac{dz}{z^6+1}$ where C as shown



$\because z^6+1=0$ gives

$z = e^{\pi i/6}$ $\quad z=e^{7\pi i/6}$

$z = e^{3\pi i/6}$ $\quad z=e^{9\pi i/6}$

$z = e^{5\pi i/6}$ $\quad z=e^{11\pi i/6}$

as simple poles but only the poles $e^{\pi i/6}$, $e^{3\pi i/6}$ & $e^{5\pi i/6}$ lie within C

$\Rightarrow$ Residue at $e^{\pi i/6} = \displaystyle\lim_{z\to e^{\pi i/6}} \left\{ (z-e^{\pi i/6}) \frac{1}{z^6+1} \right\}$

$$= \frac{1}{6}\, e^{-5\pi i/6}$$

Residue at $e^{3\pi i/6} = \frac{1}{6}\, e^{-5\pi i/2}$

Residue at $e^{5\pi i/6} = \frac{1}{6}\, e^{-25\pi i/6}$

$$\Rightarrow \oint_C \frac{dz}{z^6+1} = 2\pi i\left\{\frac{1}{6}e^{-5\pi i/6} + \frac{1}{6}e^{-5\pi i/2} + \frac{1}{6}e^{-25\pi i/6}\right\}$$

(By Residue theorem) $= \frac{2\pi}{3}$

i.e.,

$$\int_{-R}^{R}\frac{dx}{x^6+1} + \underbrace{\int_{\Gamma}\frac{dz}{z^6+1}}_{\to 0} = \frac{2\pi}{3}$$

as $R \to \infty$ it can be shown that $\int_{\Gamma}\frac{dz}{z^6+1} \to 0$

& we get $\int_{-\infty}^{\infty}\frac{dx}{x^6+1} = \frac{2\pi}{3}$

$$\Rightarrow 2\cdot\int_0^{\infty}\frac{dx}{x^6+1} = \frac{2\pi}{3}$$

$$\Rightarrow \int_0^{\infty}\frac{dx}{x^6+1} = \frac{\pi}{3}$$

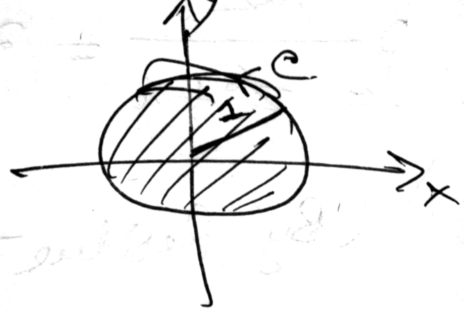
② $\int_0^{2\pi} G(\sin\theta, \cos\theta)$, where $G(\sin\theta, \cos\theta)$ is a rational fn. of $\sin\theta$ & $\cos\theta$

Let $z = e^{i\theta}$ & $\sin\theta = \frac{z - z^{-1}}{2i}$

$\cos\theta = \frac{z+z^{-1}}{2}$

& $dz = i e^{i\theta} d\theta$

$\oint_c F(z)\, dz$ where c is given as



Example

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$$

Let $z = e^{i\theta}$, $\sin\theta = \frac{z-z^{-1}}{2i}$

$\cos\theta = \frac{z+z^{-1}}{2}$

$dz = iz\, d\theta$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_c \frac{2\,dz}{(1-2i)z^2+6iz-1-2i}$$

Now poles of integrand are simple poles $z = 2-i$

$\frac{2-i}{5}$

Only $\frac{2-i}{5}$ lies inside C

& Residue at $\frac{2-i}{5} = \frac{1}{2i}$

$$\Rightarrow \oint_c \frac{2\,dz}{(1-2i)z^2+6iz-1-2i} = 2\pi i\left(\frac{1}{2i}\right)$$

$$= \pi$$

(3) $\displaystyle\int_{-\infty}^{+\infty} F(x) \begin{Bmatrix} \cos mx \\ \sin mx \end{Bmatrix}$ $F(x)$ is a rational fn.

Here we consider $\displaystyle\oint_C F(z)\, e^{imz}\, dz$

$\boxed{\text{Example}}$ $\displaystyle\int_0^{\infty} \frac{\cos mx}{x^2+1}\,dx = \frac{\pi}{2} e^{-m}$ $(m>0)$

□ $\displaystyle\oint_C \frac{e^{imz}}{z^2+1}\,dz$

The integrand has poles at $z=\pm i$

& only $z=i$ lies inside C

Residue at $z=i$ $= \dfrac{e^{-m}}{2i}$

$$\oint_C \frac{e^{imz}}{z^2+1} = 2\pi i\left(\frac{e^{-m}}{2i}\right) = \pi e^{-m}$$

$$\downarrow$$

$$\int_{-R}^{R} \frac{e^{imx}}{x^2+1}\,dx + \underbrace{\int_{\Gamma} \frac{e^{imz}}{z^2+1}\,dz}_{\to 0} = \pi e^{-m}$$

$$\downarrow$$

$$2\int_0^{R} \frac{\cos mx}{x^2+1}\,dx = \pi e^{-m}$$

as $R\to\infty$

SUMMATION OF SERIES

1. $$\sum_{-\infty}^{+\infty} f(n) = -\left\{\text{Sum of Residues of } \pi \cot \pi z\, f(z) \text{ at all the poles of } f(z)\right\}$$

2. $$\sum_{-\infty}^{+\infty} (-1)^n f(n) = -\left\{\text{sum of residues of } \pi \csc \pi z\, f(z) \text{ at all the poles of } f(z)\right\}$$

3. $$\sum_{-\infty}^{+\infty} f\left(\frac{2n+1}{2}\right) = \left\{\text{Sum of residues of } \pi \tan \pi z\, f(z) \text{ at all the poles of } f(z)\right\}$$

4. $$\sum_{-\infty}^{+\infty} (-1)^n f\left(\frac{2n+1}{2}\right) = \left\{\text{Sum of residues of } \pi \sec \pi z\, f(z) \text{ at all the poles of } f(z)\right\}$$

Example

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

□ $$F(z) = \frac{\pi \cot \pi z}{z^2} = \frac{\pi \cos \pi z}{z^2 \sin \pi z}$$

$$\Rightarrow F(z) = \frac{\left(1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - \cdots\right)}{z^3\left(1 - \frac{\pi^2 z^2}{3!} + \frac{\pi^4 z^4}{5!} - \cdots\right)}$$

$$= \frac{1}{z^3}\left(1 - \frac{\pi^2 z^2}{2!} + \cdots\right)\left(1 + \frac{\pi^2 z^2}{3!} + \cdots\right)$$

$$= \frac{1}{z^3}\left(1 - \frac{\pi^2 z^2}{3} + \cdots\right)$$

$\Rightarrow$ Residue at $z = 0$ is $-\frac{\pi^2}{3}$.

$\Rightarrow$

$$\oint_{C_N} \frac{\pi \cot \pi z}{z^2}\, dz = \sum_{n=-N}^{-1} \frac{1}{n^2} + \sum_{n=1}^{N} \frac{1}{n^2} - \frac{\pi^2}{3}$$

$$= 2\sum_{n=1}^{N} \frac{1}{n^2} - \frac{\pi^2}{3}$$

as $N \to \infty$ L.H.S. $\to 0$

$$\Rightarrow 2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$\square$