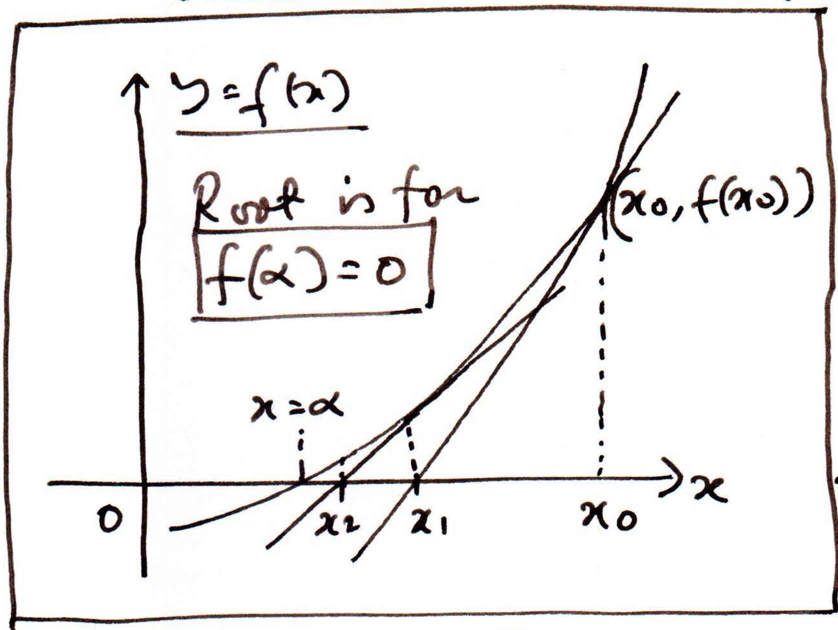
The Newton-Raphson Method

ONE-POINT METHOD



1/. Root is when $f(x)=0$ at $x=\alpha$

2/. Uses a tangent line as an approximation to find α. (ONE-POINT)

3/. Requires both $f(x)$ and $f'(x)$ at any x. (the tangent)

At $x=0$, the first-order Taylor polynomial is $\boxed{p_1(x) = f(x_0) + f'(x_0)(x-x_0)}$. Now $\boxed{p_1(x_1)=0}$.

$$\Rightarrow \quad x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)} \quad \Rightarrow \quad \boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}.$$

Once x_1 is known, iterate $\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$.

Iteration Formula: $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$.

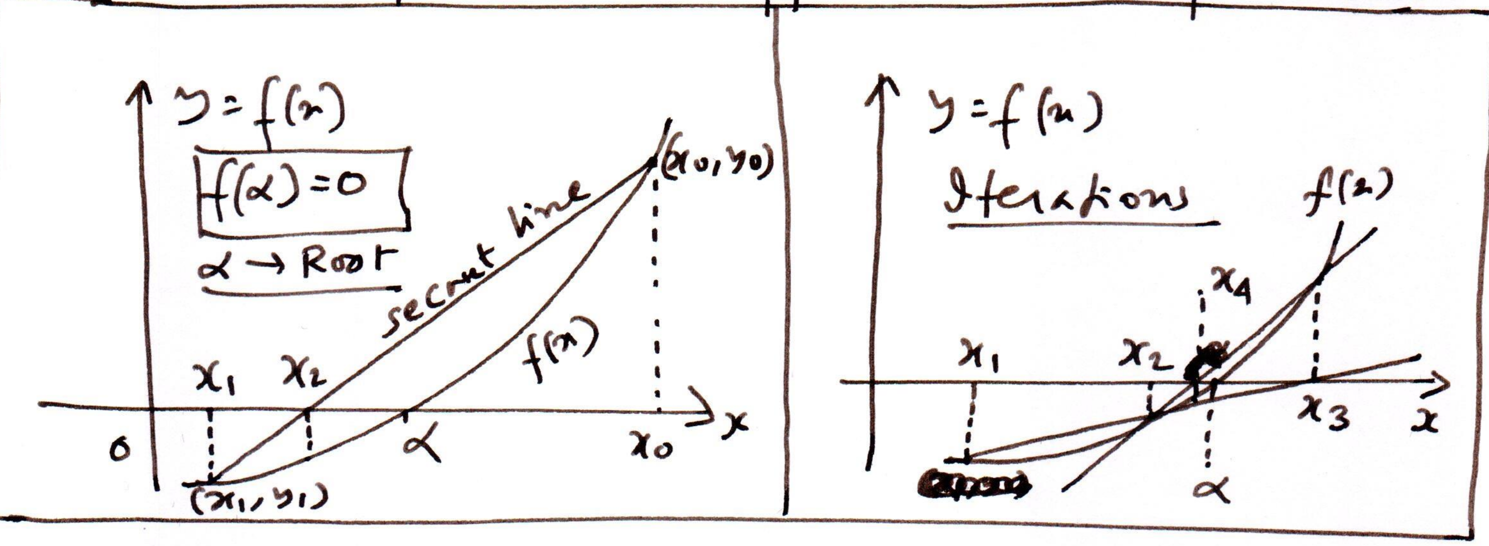
Example: Find the largest root of $\boxed{x^6 - x - 1 = 0}$ Used in Bisection Method

n	x_n	$f(x) = x^6 - x - 1$	$f'(x) = 6x^5 - 1$	$x_n - x_{n-1}$	x_{n+1}	$x_{n+1} - x_n$
0	1.5	8.8906	44.5625		1.3005	−0.1995
1	1.3005	2.5375	21.3205	−0.1995	1.1815	−0.119
2	1.1815	0.5387	12.8140	−0.119	1.1395	−0.042
3	1.1395	0.0497	10.5272	−0.042	1.1348	-4.7×10^{-3}
4	1.1348	7.806×10^{-4}	10.2914	-4.7×10^{-3}	1.1347	-1×10^{-4}
5	1.1347	-2.483×10^{-4}	10.2864	-1.0×10^{-4}	1.1347	0 (STOP)
				$\boxed{\text{ROOT}} \rightarrow$	1.1347	Root = 1.1347

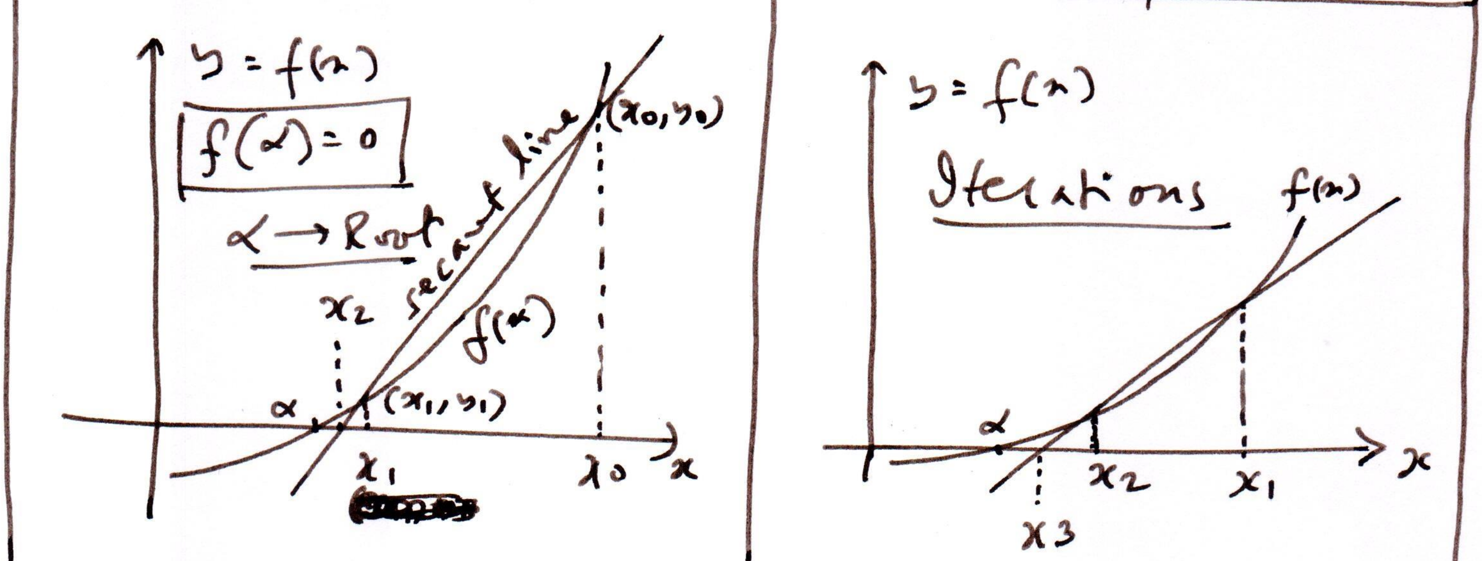
The Secant Method | TWO-POINT METHOD

1/. Requires a secant line to find the root.
2/. Requires $f(x)$ at two points. (TWO-POINT METHOD)

Initial points on opposite sides of the root.



Initial points on the same side of the root



The Equation of the Secant Line

Initially the secant line joins (x_0, y_0) and (x_1, y_1).

The slope of the line, $\boxed{m = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{y_0 - y_1}{x_0 - x_1}}$.

The equation of a line with this slope, and passing through (x_1, y_1) is $\boxed{\dfrac{y-y_1}{x-x_1} = m}$.

$\Rightarrow$ $\boxed{y = y_1 + m(x-x_1)}$ Writing $y = p(x)$, $y_1 = f(x_1)$ and $y_0 = f(x_0)$ we can get from $y = y_1 + \dfrac{(x-x_1)(y_1-y_0)}{x_1-x_0}$ the ~~secant~~ equation of the secant line,

$$\boxed{p(x) = f(x_1) + (x-x_1) \times \left[\frac{f(x_1)-f(x_0)}{x_1-x_0}\right]}.$$

The ~~[illegible]~~ iteration formula is to be obtained from the condition $\boxed{p(x_2)=0}$.

$$\therefore \quad x_2 - x_1 = -f(x_1)\left[\frac{x_1-x_0}{f(x_1)-f(x_0)}\right].$$

$$\Rightarrow x_2 = x_1 - f(x_1)\left[\frac{x_1-x_0}{f(x_1)-f(x_0)}\right]$$

The General Iteration Formula is:

$$\boxed{x_{n+1} = x_n - f(x_n)\cdot\frac{x_n - x_{n-1}}{f(x_n)-f(x_{n-1})}}$$

Two points on the right hand side.

Alternatively the initial secant line passes through (x_0, y_0). $\therefore$ Its equation is $\boxed{\dfrac{y-y_0}{x-x_0} = m}$, leading to $y = y_0 + (x-x_0)\left(\dfrac{y_1-y_0}{x_1-x_0}\right)$. The equation of the secant line is $p(x) = f(x_0) + (x-x_0)\left[\dfrac{f(x_1)-f(x_0)}{x_1-x_0}\right]$, which for $p(x_2)=0$, gives as before,

$$\boxed{x_2 = x_0 - f(x_0)\left[\frac{x_1-x_0}{f(x_1)-f(x_0)}\right]} \quad \text{OR} \quad \boxed{x_{n+1} = x_{n-1} - f(x_{n-1})\frac{x_n - x_{n-1}}{f(x_n)-f(x_{n-1})}}$$

Example:

Find the largest root of $\boxed{x^6 - x - 1 = 0}$. From previous methods

n	x_n	$f(x_n)$	$x_n - x_{n-1}$	x_{n+1}	Position
0	2	61.0			
1	1	-1.0	-1	1.01613	Short
2	1.01613	-0.9154	0.01613	1.19066	Overshoot
3	1.19066	0.6586	0.17453	1.11763	short
4	1.11763	-0.1690	-7.303×10^{-2}	1.13254	short
5	1.13254	-2.235×10^{-2}	0.01491	1.13481	Overshoot
6	1.13481	8.835×10^{-4}	2.27×10^{-3}	1.13472	Convergence
7	1.13472	-4.2574×10^{-5}	-9×10^{-5}	1.13472	ROOT

The root is $\boxed{1.13472}$ $\boxed{\alpha = 1.13472}$.

In the previous approach the 2 initial guess values $\boxed{x_0 = 2}$ and $\boxed{x_1 = 1}$ on opposite sides of the root $\alpha = 1.13472$. Now we take $\boxed{x_0 = 2,\ x_1 = 1.5}$ on the same side of the root. THERE IS NO OVERSHOOT.

n	x_n	$f(x_n)$	$x_n - x_{n-1}$	x_{n+1}	Position
0	2	61.0			
1	1.5	8.8906	-0.5	1.41469	Short
2	1.41469	5.6015	-0.08531	1.26940	Short
3	1.26940	1.9146	-0.14529	1.19395	Short
4	1.19395	0.7028	-0.07545	1.15019	Short
5	1.15019	0.1652	-0.04376	1.13674	Short
6	1.13674	0.0208	-0.01345	1.13480	Short
7	1.13480	7.8058×10^{-4}	-0.00194	1.13472	Convergence
8	1.13472	-4.2574×10^{-5}	-0.8×10^{-5}	1.13472	ROOT