DA-IICT, Gandhinagar
END Term Examination
Subject: Computational and Numerical Methods (CS374)

Date: 17/12/2020 | **Start Time: 02:30 PM**
Time Duration: 60 minutes | **Max. Marks: 60**

Instructions:

1. Scientific Calculator is allowed.
2. Negative Marking -1.25.
3. All questions carry equal marks.

1. Let p be a polynomial of degree $\leq n-1$ that interpolates the function $f(x) = \sinh x$ at any set of n nodes in the interval $[-1, 1]$, subject only to the condition that one of the nodes is 0. Then, the relative error $\frac{|p(x)-f(x)|}{|f(x)|}$ is not greater than

   (a) $\frac{2^n}{n!}$. (b) $\frac{2^{n-1}}{(n-1)!}$. (c) $\frac{2^n}{(n-1)!}$. (d) $\frac{2^{n-1}}{n!}$.

2. The Cubic-Hermite interpolation determines a polynomial of the form

$$p(x) = c_0 + c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^2(x-2)$$

   that takes the values

$$p(1) = 2,\ p'(1) = 3,\ p(2) = 6,\ p'(2) = 7.$$

   The values of the parameters $c_0,\ c_1,\ c_2$ and c_3, respectively, are

   (a) $2, 3, 6, 7$. (c) $2, 3, 1, 2$.
   (b) $2, 6, 3, 7$ (d) $2, 6, 7, 3$.

3. Let $f(x) = x^4 + 1$ for $x \in [0, 1]$. In order to achieve accuracy of 10^{-6} on [0,1], the step size h for piecewise quadratic interpolating polynomial will be

   (a) < 0.04.
   (b) < 0.01.
   (c) < 0.02.
   (d) < 0.03.

4. The values of the parameters a, b, c, d, e are such that the following function becomes a cubic spline:

$$f(x) = \begin{cases} a(x-2)^2 + b(x-1)^3 & x \in [0,1] \\ c(x-2)^2 & x \in [1,3] \\ d(x-2)^2 + e(x-3)^3 & x \in [3,4]. \end{cases}$$

   Find the values of the parameters so that the cubic spline interpolates the following table:

x	0	1	4
f(x)	26	7	25

(a) $a = c = d = 7, b = 2, e = -3$.

(b) $a = c = d = 7, b = -2, e = -3$.

(c) $a = c = d = 7, b = 3, e = -2$.

(d) $a = c = d = 7, b = -3, e = 2$.

5. The values of the parameters A, B, C in the following formula

$$\int_0^2 xf(x)dx = Af(0) + Bf(1) + Cf(2)$$

are such that the formula is exact for all polynomials of degree as high as possible. If $f(x) = \frac{2}{\pi x}\sin\left(\frac{\pi x}{2}\right)$, then the value of above integral equals to

(a) $\frac{4}{3\pi}$. (b) $\frac{8}{3\pi}$. (c) $\frac{2}{3\pi}$. (d) $\frac{16}{3\pi}$.

6. Consider the following Gauss 3-point rule which is exact for polynomials of degree ≤ 5:

$$\int_{-1}^1 f(x)dx \approx \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right).$$

Using this formula the value of the integral $\displaystyle\int_0^4 \frac{\sin t}{t}dt$ equals:

(a) 0.879. (b) 1.7581. (c) 1.003. (d) 2.006.

7. For Trapezoidal rule, the minimum number of subintervals needed to approximate

$$\int_1^2 \left(x + e^{-x^2}\right) dx$$

to an accuracy of at least $\frac{1}{2} \times 10^{-3}$ will be

(a) 10. (b) 11. (c) 12. (d) 13.

8. Let y be solution of the following differential equation

$$y' = -x^2 y, \;\; y(0) = 2.$$

Using Euler's Method with step size 0.1, the value of $y(0.4)$ will be

(a) 2. (b) 1.99. (c) 1.9721. (d) 1.998.

9. Let y be solution of the following differential equation

$$5xy' + y^2 = 2, \;\; y(4) = 1.$$

Using Runge-Kutta Method of order 4 with step size 0.1, the value of $y(4.1)$ will be

(a) 1. (b) 1.0049. (c) 1.049. (d) 1.49.

10. If y'' and y' are approximated by using central difference operator in the following boundary value problem with step size $h = 0.5$:

$$y'' + 2y' + 10y = 0, \ y(0) = 1, \ y(1) = 2,$$

then the solution y at $x = 0.5$ will be

(a) 7. (b) -7. (c) 0.7. (d) -0.7.

Answer Key:

1) Option (a).
2) Option (c).
3) Option (b).
4) Option (a).
5) Option (b).
6) Option (b).
7) Option (d).
8) Option (c).
9) Option (b).
10) Option (b).

Sol 1 Error in polynomial interpolation

$$\Rightarrow |f(x) - p(x)| = \left| \frac{f^n(\xi_x)}{n!} \prod_{i=0}^{n-1} (x - x_i) \right|$$

$\because$ one of the nodes is 0, say $x_j = 0$

$$\Rightarrow |f(x) - p(x)| = \left| \frac{x f^n(\xi_x)}{n!} \prod_{\substack{i=0 \\ i \neq j}}^{n-1} (x - x_i) \right|$$

$$\Rightarrow \frac{|f(x)-p(x)|}{|f(x)|} = \left| \frac{x f^{(n)}(\xi_x)}{f(x)} \prod_{\substack{i=0 \\ i \neq j}}^{n-1} (x-x_i) \right|$$

Now, $f^{(n)}(x) = \begin{cases} \sinh x & \text{if } n \text{ even} \\ \cosh x & \text{if } n \text{ odd} \end{cases}$

& $f^{(n)}$ is increasing function.

$$\Rightarrow \|f^{(n)}\|_\infty = \max\{\sinh 1, \cosh 1\}$$
$$= 1.5431$$

Also $|x - x_i| \le 2 \quad \forall\, i = 0, 1, 2, \dots, n \quad [\because x, x_i \in [-1, 1]]$

$$\Rightarrow \frac{|f(x) - p(x)|}{|f(x)|} \le \left|\frac{x}{f(x)}\right| \frac{\|f^{(n)}\|_\infty}{n!} 2^{n-1}$$

Also $\left|\frac{x}{\sinh x}\right| < 1$

$$\Rightarrow \frac{|f(x) - p(x)|}{|f(x)|} < \frac{(1.5431)\, 2^{n-1}}{n!} < \frac{2^n}{n!}$$

Sol. 2 For cubic Hermite interpolation:

$$p(x) = f(x_0) + (x-x_0) f'(x_0) + (x-x_0)^2 f[x_0,x_0,x_1] + (x-x_0)^2 (x-x_1) f[x_0,x_0,x_1,x_1]$$

$$= c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^2(x-x_1)$$

x	f			
x_0	$f(x_0)$			
		$f'(x_0)$		
x_0	$f(x_0)$		$f[x_0,x_0,x_1]$	
		$f[x_0,x_1]$		$f[x_0,x_0,x_1,x_1]$
x_1	$f(x_1)$		$f[x_0,x_1,x_1]$	
		$f'(x_1)$		
x_1	$f(x_1)$			

Given $p(1) = 2$, $p'(1) = 3$,
$p(2) = 6$, $p'(2) = 7$

	c_0	c_1	c_2	c_3
1	2			
		3		
1	2		1	
		4		2
2	6		3	
		7		
2	6			

Sol. 3.

Given $f(x) = x^4 + 1$ on $[0, 1]$

$\Rightarrow a = 0, \ b = 1$

We know that the error in piecewise quadratic interpolation will be

$$|E| \le \frac{\|f'''\|_\infty}{72\sqrt{3}} h^3 \quad , \quad \boxed{h = \frac{b-a}{n} = \frac{1}{n}}$$

We wish to find h such that

$$|E| < 10^{-6}$$

Now, $f'''(x) = 24x$

$\Rightarrow \quad \|f'''\|_\infty = 24$

$\therefore \quad |E| < 10^{-6}$

if $\quad \dfrac{\|f'''\|_\infty}{72\sqrt{3}} h^3 < 10^{-6}$

$\Rightarrow \quad h < 0.017.$

Sol 4

The given function f is cubic spline on $[0,4]$, if all of the following are true

(i) f is cont. at 1 and 3

(ii) f' is cont. at 1 and 3

(iii) f'' is cont. at 1 and 3

The above implies that

$a = 2 = d$ and b, e are arbitrary.

Next, it is given that f is cubic spline that fits the table

x	0	1	4
f	26	7	25

Then, $f(0) = 26$, $f(1) = 7$ & $f(4) = 25$

$\Rightarrow a = c = d = 7, \quad b = 2, \quad e = -3.$

Sol. 5

The formula

$$\int_0^2 x\, f(x)\, dx = A f(0) + B f(1) + C f(2) \quad -(1)$$

is exact for polynomials of degree as high as possible.

For $f(x) = 1$, $A + B + C = 2$

For $f(x) = x$, $B + 2C = 8/3$

For $f(x) = x^2$

$B + 4C = 4$

These 3 equation have unique solution

$$A = 0, \; B = \frac{4}{3}, \; C = \frac{2}{3}$$

It can be check that (1) will not be exact for $f(x) = x^3$

Now, we wish to evaluate (1) for

$$f(x) = \frac{2 \sin\left(\frac{\pi x}{2}\right)}{\pi x}$$

$\Rightarrow \; f(0) = 1, \quad f(1) = \frac{2}{\pi}, \quad f(2) = 0$

$\therefore$ (1) $\Rightarrow$

$$\int_0^2 x f(x)\, dx = 0 \times 1 + \frac{4}{3} \times \frac{2}{\pi} + \frac{2}{3} \times 0$$

$$= \frac{8}{3\pi}$$

Sol. 6

Gauss 3-point Rule

$$\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$ (1)

$\phi : [-1,1] \to [0,4]$ be defined as

$$\phi(t) = \frac{0+4}{2} + \left(\frac{4-0}{2}\right) t = 2+2t$$

which will be bijective.

Now, $\int_0^4 g(x)\,dx = \int_{-1}^{1} g(\phi(t))\,\phi'(t)\,dt$

$$= 2\int_{-1}^{1} g(\phi(t))\,dt \quad - (2)$$

Using (1) in (2), we get

$$\int_a^b g(x)\,dx = 2\left[\frac{5}{9} g\left(\phi\left(-\sqrt{\frac{3}{5}}\right)\right) + \frac{8}{9} g(\phi(0)) + \frac{5}{9} g\left(\phi\left(\sqrt{\frac{3}{5}}\right)\right)\right]$$

$$\Rightarrow \int_0^4 g(x)\,dx = 2\left[\frac{5}{9}\,g\left(2-2\sqrt{\frac{3}{5}}\right)+\frac{8}{9}\,g(2)+\frac{5}{9}\,g\left(2+2\sqrt{\frac{3}{5}}\right)\right] \quad -(2)$$

Using (2), we get

$$\int_0^4 \frac{\sin x}{x}\,dx = 1.7581.$$

Sol. 7 $a = 1,\ b = 2,\ f(x) = x + e^{-x^2}$

For composite Trapezoidal rule,

$$E_T = -\left(\frac{b-a}{12}\right) h^2 f''(\xi)$$

$$\Rightarrow |E_T| \le \frac{\|f''\|_\infty (b-a)}{12} h^2 = \frac{\|f''\|_\infty h^2}{12}; \quad \boxed{\begin{aligned} h &= \frac{b-a}{n} \\ &= \frac{1}{n} \end{aligned}}$$

It can be shown that f'' will be max. at $\sqrt{3/2}$ by equating $f'''(x) = 0$.

$$\Rightarrow \|f''\|_\infty = f''\left(\sqrt{\tfrac{3}{2}}\right) \approx 0.8925$$

We wish to find 'n' s.t.

$$|E_T| < \frac{1}{2}\times 10^{-3} \quad \text{--- (1)}$$

Now, $|E_T| \le \dfrac{\|f''\|_\infty}{12} h^2 \lesssim \dfrac{0.8925}{12} \cdot \dfrac{1}{n^2}$

(1) will be true, if

$$\frac{0.8925}{12n^2} < \frac{1}{2}\times 10^{-6} \Rightarrow n > 12.19$$

$$\Rightarrow n \ge 13.$$

Sol. 8

Given IVP $y' = -x^2 y$

$y(0) = 2$

$x_0 = 0$, $y_0 = 2$, $f(x,y) = -x^2 y$

$h = 0.1$

$x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$

$(x_i = x_0 + ih \;,\; i = 0, 1, 2, 3, 4)$

Euler's Method implies

$$y_{i+1} = y_i + h f(x_i, y_i) \quad , \quad i = 0, 1, 2, 3, 4.$$

$$= y_i - h x_i^2 y_i = (1 - h x_i^2) y_i$$

$\therefore$ $y_1 = (1 - h x_0^2) y_0 = 2$

$y_2 = (1 - h x_1^2) y_1 = 1.998$

$y_3 = (1 - h x_2^2) y_2 = 1.99$

$y_4 = (1 - h x_3^2) y_3 = 1.9721$

Sol. 9

Given IVP.

$$5xy' + y^2 = 2 \quad , \quad y(4) = 1$$

$$f(x, y) = \frac{2 - y^2}{5x} \quad , \quad x_0 = 4, \quad y_0 = 1, \quad h = 0.1$$

We wish to approximate $y(4.1)$ by using RK-4.

Runge-Kutta method of order 4

$$y_{i+1} = y_i + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$K_1 = h f(x_i, y_i)$$

$$K_2 = h f\left(x_i + \frac{h}{2}, y_i + \frac{K_1}{2}\right) \quad , i = 0, 1, 2, \ldots$$

$$K_3 = h f\left(x_i + \frac{h}{2}, y_i + \frac{K_2}{2}\right)$$

$$K_4 = h f(x_i + h, y_i + K_3)$$

For $i = 0$,

$$K_1 = 0.005$$

$$K_2 = 0.0049$$

$$K_3 = 0.0049$$

$$K_4 = 0.0048$$

$$\Rightarrow y_1 = 1.0048$$

Sol 10

Given BVP

$$\left.\begin{aligned} y'' + 2y' + 10y = 0 \\ y(0) = 1,\ y(1) = 2 \end{aligned}\right\} \quad (1)$$

$$(2) \quad \left\{\begin{aligned} y'' &\approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \\ y'_i &\approx \frac{y_{i+1} - y_{i-1}}{2h} \end{aligned}\right. \quad , \quad \begin{aligned} y_i &= y(x_i) \\ &= y(x_0 + ih) \end{aligned}$$

Using (2) in (1), we get

$$\left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right)+2\left(\frac{y_{i+1}-y_{i-1}}{2h}\right)+10y_i=0$$

On simplifying, we get

$$(1-h)y_{i-1}+(-2+10h^2)y_i+(1+h)y_{i+1}=0 \quad \text{--- (3)}$$

Given $h = 0.5$.

$$\left[\begin{array}{lll} x_0 = 0, & x_1 = 0.5, & x_2 = 1 \\ y_0 = 1, & y_1 = ?, & y_2 = 2 \end{array}\right]$$

For $i = 1$ in (3), we get $(h = 0.5)$

$$0.5\, y_0 + (-2 + 2.5)\, y_1 + 1.5\, y_2 = 0$$

$$y_1 = \frac{-3.5}{0.5} = -7.$$