Q-1 (a) Whether the quadratic form
$f: -x_1^2 + 4x_1x_2 - 9x_2^2 + 2x_1x_3 + 8x_2x_3 - 4x_3^2$ is
positive definite / negative definite / indefinite ?
Give reason.

Sol^n The matrix of the quadratic form is

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -9 & 4 \\ 1 & 4 & -4 \end{bmatrix}$$

(1)

$A_1 = -1 < 0$

$A_2 = \begin{vmatrix} -1 & 2 \\ 2 & -9 \end{vmatrix} = 9-4 = 5 > 0$

(1½)

$A_3 = \begin{vmatrix} -1 & 2 & 1 \\ 2 & -9 & 4 \\ 1 & 4 & -4 \end{vmatrix} = -1(36-16) - 2(-8-4) + 1(8+9)$

$= -20 + 24 + 17 = 21 > 0$

So the matrix A is indefinite matrix

Hence the quadratic form f is also indefinite.

Q-1 (b) Find the stationary points of $f = x^3 - 3xy^2$ and identify their nature

Sol^n

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 3y^2 = 0 \Rightarrow x = \pm y \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow -6xy = 0 \quad \text{--- (2)}$$

$$\Rightarrow x = 0 \quad \text{or} \quad y = 0 \qquad \frac{1}{2}$$

From (1) & (2) we have $x = y = 0$

$(0,0)$ is the stationary point.

$$\text{Hess } f \Big|_{(0,0)} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \text{ at } (0,0)$$

$$= \begin{bmatrix} 6x & -6y \\ -6y & -6x \end{bmatrix} \text{ at } (0,0) \qquad \frac{1}{2}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{which is semi-definite.}$$

We have to compute $d^3 f|_{(0,0)}$

$$d^3 f\Big|_{(0,0)} = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} h_i h_j h_k \frac{\partial^3 f\,(0,0)}{\partial x_i \partial x_j \partial x_k}$$

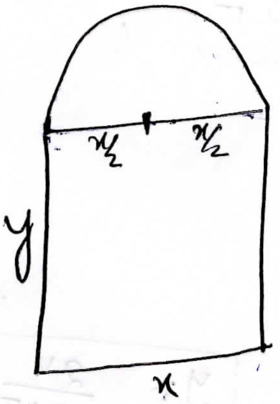
(1)
$$= h_1 h_1 h_1 \frac{\partial^3 f}{\partial x^3} + 2 h_1 h_1 h_2 \frac{\partial^3 f}{\partial x^2 \partial y} + h_1 h_2^2 \frac{\partial^3 f}{\partial x \partial y^2}$$

$$+ h_2 h_1 h_1 \frac{\partial^3 f}{\partial y \partial x^2} + 2 h_2 h_1 h_2 \frac{\partial^3 f}{\partial y^2 \partial x} + h_2 h_2 h_2 \frac{\partial^3 f}{\partial y^3}$$

As $\frac{\partial^3 f}{\partial x^3} = 6 \neq 0$ So $d^3 f|_{(0,0)} \neq 0$ 3 is odd. So $(0,0)$ is a saddle point.

Q.2 A window is being built, the bottom is a rectangle and the top is a semi-circle. If there are 12 meters of framing materials available. What will be the dimensions of the window so that it can allow maximum light?

Sol^n



We want to maximize the area to allow maximum light

$$\begin{cases} \text{Maximize } f = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = xy + \frac{\pi x^2}{8} & \text{--- (1)} \\ \text{subject to } \quad 2(x+y) + \pi \frac{x}{2} = 12 & \text{--- (2)} \end{cases}$$

$$\Rightarrow \frac{4x+4y+\pi x}{2} = 12 \Rightarrow (4+\pi)x + 4y = 24$$

$$\Rightarrow 4y = 24 - (4+\pi)x$$

$$\Rightarrow y = \frac{1}{4}(24 - 4x - \pi x) = 6 - x - \frac{\pi}{4}x$$

New objective function after putting the value of y in (1) becomes

$$f = x\left(6 - x - \frac{\pi}{4}x\right) + \frac{\pi x^2}{8}$$

$$= 6x - x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2 = 6x - x^2 - \frac{\pi}{8}x^2$$

$$f'(x) = 6 - 2x - \frac{\pi}{4}x = 0 \Rightarrow \left(2 + \frac{\pi}{4}\right)x = 6$$

$$\Rightarrow x = \frac{6}{(2+\pi/4)} = \frac{6}{\frac{8+\pi}{4}} = \frac{24}{8+\pi} \quad (2)$$

is the stationary or critical point

$f''(x) = -2 - \frac{2\pi}{8} < 0$

So $x = \frac{24}{8+\pi}$ corresponds to maximum value to the objective function f

$\boxed{x = \frac{24}{8+\pi}}$, $y = 6 - x - \frac{\pi x}{4}$

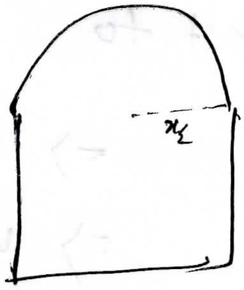
$= 6 - \frac{24}{8+\pi} - \frac{\pi}{4}\frac{24}{(8+\pi)}$

$= \frac{24(8+\pi) - 96 - 24\pi}{4(8+\pi)}$

$= \frac{192 + \cancel{24\pi} - 96 - \cancel{24\pi}}{4(8+\pi)} = \frac{96}{4(8+\pi)} = \frac{24}{8+\pi}$

$\boxed{y \text{ is } \frac{24}{8+\pi}}$

$\boxed{r = \frac{x}{2} = \frac{12}{8+\pi}}$



OR

Maximize $f = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

$= xy + \frac{\pi x^2}{8}$ —(1)

subject to $2y + x + \pi\frac{x}{2} = 12$

$\Rightarrow 2y = 12 - x - \frac{\pi x}{2} = \frac{24 - 2x - \pi x}{2}$

$= 12 - x - \frac{\pi}{2}x$

$\Rightarrow y = 6 - \frac{x}{2} - \frac{\pi}{4}x$

putting the value of y in (1)

$f = x\left(6 - \frac{x}{2} - \frac{\pi}{4}x\right) + \frac{\pi x^2}{8} = 6x - \frac{x^2}{2} - \frac{\pi}{4}x^2 + \frac{\pi x^2}{8}$

$= 6x - \frac{x^2}{2} - \frac{\pi}{8}x^2$

Q-2 OK continued:- - -

$f'(x) = 6 - x - \frac{\pi}{8}\cdot 2x = 0$

$\Rightarrow 6 - x - \frac{\pi}{4}x = 0 \Rightarrow x(1+\pi/4) = 6$

$\Rightarrow x = \frac{6}{1+\pi/4} = \frac{6}{\frac{4+\pi}{4}} = \frac{24}{4+\pi}$

stationary point

2

$f''(x) = -1 - \pi/4 < 0$

1/2

So $x = \frac{24}{4+\pi}$ corresponds to maximum value to f.

So $\boxed{x = \frac{24}{4+\pi}}$

$y = 6 - \frac{x}{2} - \frac{\pi}{4}x = 6 - \frac{24}{2(4+\pi)} - \frac{\pi}{4}\cdot\frac{24}{4+\pi}$

1/2

$= 6 - \frac{12}{4+\pi} - \frac{6\pi}{4+\pi} = \frac{6(4+\pi)-12-6\pi}{4+\pi}$

$= \frac{24+\cancel{6\pi}-12-\cancel{6\pi}}{4+\pi} = \frac{12}{4+\pi}$

$\boxed{y = \frac{12}{4+\pi}}$ 1.68

$\boxed{r = \frac{x}{2} = \frac{12}{4+\pi}}$

Q.3 Using KKT conditions find the value(s) of β for which the point $x_1^* = 1$, $x_2^* = 2$ will be an optimal solution to the problem

$$\text{Maximize } f(x_1, x_2) = 2x_1 + \beta x_2$$

$$\text{subject to } \quad x_1^2 + x_2^2 \le 5,$$

$$x_1 - x_2 \le 2.$$

Solⁿ $(x_1^*, x_2^*) = (1, 2)$

$$g_1(x_1, x_2) = x_1^2 + x_2^2 - 5 \le 0$$

$$g_2(x_1, x_2) = x_1 - x_2 - 2 \le 0$$

$g_1(1,2) = 0$, so g_1 is active at $(1,2)$

$g_2(1,2) \neq 0$, so g_2 is not active at $(1,2)$

So KKT conditions are

$$\frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g_1}{\partial x_1} = 0 \Rightarrow 2 + 2\lambda_1 x_1 = 0 \quad \text{— (1)}$$

$$\frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial g_1}{\partial x_2} = 0 \Rightarrow \beta + 2\lambda_1 x_2 = 0 \quad \text{— (2)}$$

from (1) $\lambda_1 x_1 = -1 \Rightarrow \lambda_1 = -1$ at $(1,2)$

from (2) $\beta + 2\lambda_1 x_2 = 0 \Rightarrow \beta + 2(-1) \times 2 = 0$

$$\Rightarrow \boxed{\beta = 4}$$

$\lambda_1 < 0$.

Q.4 Solve the following problem by the method of Lagrange multipliers

Minimize $f(X) = \frac{1}{2}(x_1^2+x_2^2+x_3^2)$

subject to $g_1(X) = x_1 - x_2 = 0$

$g_2(X) = x_1+x_2+x_3-1 = 0$

Sol^n

$$L(x_1,x_2,x_3,\lambda_1,\lambda_2) = f+\lambda_1 g_1+\lambda_2 g_2$$

$$= \frac{1}{2}(x_1^2+x_2^2+x_3^2)+\lambda_1(x_1-x_2)+\lambda_2(x_1+x_2+x_3-1)$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow x_1+\lambda_1+\lambda_2 = 0 \quad \text{--- (1)}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow x_2-\lambda_1+\lambda_2 = 0 \quad \text{--- (2)}$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow x_3+\lambda_2 = 0 \quad \text{--- (3)}$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1-x_2 = 0 \quad \text{--- (4)}$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow x_1+x_2+x_3-1 = 0 \quad \text{--- (5)}$$

From eq^n (3), $\lambda_2 = -x_3$

From eq^n (1) + (2), $x_1+x_2+2\lambda_2 = 0$

$$\Rightarrow \lambda_2 = -\frac{(x_1+x_2)}{2}$$

$$\Rightarrow -x_3 = -\frac{(x_1+x_2)}{2}$$

$$\Rightarrow x_3 = \frac{x_1+x_2}{2}$$

$$\Rightarrow x_1+x_2-2x_3 = 0 \quad \text{--- (6)}$$

From eqn (4), (5) and (6)

$$x_1 - x_2 = 0 \quad -(4)$$
$$x_1 + x_2 + x_3 - 1 = 0 \quad -(5)$$
$$x_1 + x_2 - 2x_3 = 0 \quad -(6)$$

$(5)-(6) \Rightarrow \quad 3x_3 - 1 = 0 \Rightarrow 3x_3 = 1 \Rightarrow x_3 = \frac{1}{3}$

From eqn (4) and (6)

$$x_1 - x_2 = 0 \quad -(7)$$
$$x_1 + x_2 = \frac{2}{3} \quad -(8)$$

$(7)+(8) \Rightarrow \quad 2x_1 = \frac{2}{3} \Rightarrow x_1 = \frac{1}{3}$

2.5

From (7) $\quad x_2 = \frac{1}{3}$

$$\boxed{x_1 = x_2 = x_3 = \frac{1}{3}}$$

Sufficient condition:

$$\begin{vmatrix} L_{11}-\alpha & L_{12} & L_{13} & g_{11} & g_{21} \\ L_{21} & L_{22}-\alpha & L_{23} & g_{12} & g_{22} \\ L_{31} & L_{32} & L_{33}-\alpha & g_{13} & g_{23} \\ g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & g_{23} & 0 & 0 \end{vmatrix} = 0 \quad —— (A)$$

$L_{11} = \frac{\partial^2 L}{\partial x_1^2} = 1 \qquad L_{12} = \frac{\partial^2 L}{\partial x_1 \partial x_2} = 0 = L_{21}$

$L_{13} = \frac{\partial^2 L}{\partial x_1 \partial x_3} = 0 = L_{31} \qquad L_{22} = \frac{\partial^2 L}{\partial x_2^2} = 1$

$L_{23} = \frac{\partial^2 L}{\partial x_2 \partial x_3} = 0 = L_{32} \qquad L_{33} = \frac{\partial^2 L}{\partial x_3^2} = 1$

$$g_{11} = \frac{\partial g_1}{\partial x_1} = 1 \qquad g_{21} = \frac{\partial g_2}{\partial x_1} = 1$$

$$g_{12} = \frac{\partial g_1}{\partial x_2} = -1 \qquad g_{22} = \frac{\partial g_2}{\partial x_2} = 1$$

$$g_{13} = \frac{\partial g_1}{\partial x_3} = 0 \qquad g_{23} = \frac{\partial g_2}{\partial x_3} = 1$$

(A) becomes

$$\left|\begin{array}{cc} \begin{bmatrix} 1-\alpha & 0 & 0 \\ 0 & 1-\alpha & 0 \\ 0 & 0 & 1-\alpha \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \\ \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}\right| = 0$$

$$\cancel{\det \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}}$$

$$(1-\alpha)\left|\begin{array}{cc} \begin{pmatrix} 1-\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} & \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}\right| - 1 \begin{vmatrix} 0 & 1-\alpha & 0 & 1 \\ 0 & 0 & 1-\alpha & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$$+1 \begin{vmatrix} 0 & 1-\alpha & 0 & -1 \\ 0 & 0 & 1-\alpha & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0$$

$$\cancel{(1-\alpha) \det (AD - BC)}$$

$$\Rightarrow (1-\alpha)\left[(1-\alpha)\begin{vmatrix}1-\alpha&0&1\\0&0&0\\1&0&0\end{vmatrix} - 1\begin{vmatrix}0&1-\alpha&1\\-1&0&0\\1&1&0\end{vmatrix} + 1\begin{vmatrix}0&1-\alpha&0\\-1&0&0\\1&1&0\end{vmatrix}\right]$$

$$-1\left[-(1-\alpha)\begin{vmatrix}0&1-\alpha&1\\1&0&0\\1&1&0\end{vmatrix} - 1\begin{vmatrix}0&0&1-\alpha\\1&-1&0\\1&1&1\end{vmatrix}\right]$$

$$+1\left[-(1-\alpha)\begin{vmatrix}0&1-\alpha&0\\1&0&0\\1&1&0\end{vmatrix} - (-1)\begin{vmatrix}0&0&1-\alpha\\1&-1&0\\1&1&1\end{vmatrix}\right] = 0$$

$$\Rightarrow (1-\alpha)\left[(-1)\left[-(1-\alpha)(0)+1(-1-0)\right]-1[0]\right]$$

$$+\cancel{(1-\alpha)}$$

$$-1\left[-(1-\alpha)\{1(1-0)\} - 1\{(1-\alpha)(1+1)\}\right]$$

$$+1\left[-(1-\alpha)\times 0 + 1\left((1-\alpha)(1+1)\right)\right] = 0$$

$$\Rightarrow (1-\alpha)(+1) - 1\left[-(1-\alpha) - 2(1-\alpha)\right] + 1\left[2(1-\alpha)\right] = 0$$

$$\Rightarrow (1-\alpha)+(1-\alpha)+2(1-\alpha)+2(1-\alpha)=0$$

$$\Rightarrow 6(1-\alpha)=0 \Rightarrow 1-\alpha=0 \Rightarrow \alpha=1 > 0$$

(1.8) So $x_1=\frac{1}{3}=x_2=x_3$ corresponds to minimum value

$$\text{Min} f = \frac{1}{2}\left(\frac{1}{9}+\frac{1}{9}+\frac{1}{9}\right)=\frac{1}{2}\left(\frac{3}{9}\right)=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$$

(Ans)

Q.5 Minimize $f = (x_1-2)^2+(x_2-1)^2$

subject to $2 \geq x_1+x_2$

$x_2 \geq x_1^2$

Using KKT conditions, find which of the following vectors is/are local minima.

$X_1 = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}$, $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Sol^n Consider $X_1 = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}$

$f = (x_1-2)^2 + (x_2-1)^2$

$g_1 = x_1+x_2-2 \leq 0$

$g_2 = x_1^2 - x_2 \leq 0$

At the point $X_1 = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}$, g_1 is active. g_2 is inactive.

$L = f + \lambda_1 g_1 = (x_1-2)^2 + (x_2-1)^2 + \lambda_1(x_1+x_2-2)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2(x_1-2)+\lambda_1 = 0$ — (1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2(x_2-1)+\lambda_1 = 0$ — (2)

$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1+x_2-2 = 0$ — (3)

From (1) and (2)

$\lambda_1 = -2(x_1-2) = -2(x_2-1)$

$\Rightarrow x_1 - 2 = x_2 - 1 \Rightarrow x_1 = x_2+1$

Putting in (3)

$x_2+1+x_2-2 = 0 \Rightarrow 2x_2 = 1 \Rightarrow \boxed{x_2 = \frac{1}{2}}$

$\boxed{x_1 = \frac{3}{2}}$

$\lambda_1 = -2(x_1-2) = -2(\tfrac{3}{2}-2) = -2(-\tfrac{1}{2}) = +1 \geq 0$

~~But~~ this $(x_1, x_2) = (1.5, 0.5)$ ~~corresponds to maximum value of f.~~ does not satisfy the constraint g_2

2.6

$X_1 = \begin{pmatrix}1.5\\0.5\end{pmatrix}$ is not a local minimum.

Next point $X_2 = \begin{pmatrix}1\\1\end{pmatrix}$

At $\begin{pmatrix}1\\1\end{pmatrix}$ both g_1 and g_2 are active.

$L = f + \lambda_1 g_1 + \lambda_2 g_2$

$= (x_1-2)^2 + (x_2-1)^2 + \lambda_1(x_1+x_2-2) + \lambda_2(x_1^2 - x_2)$

2.5

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2(x_1-2) + \lambda_1 + 2\lambda_2 x_1 = 0$ — (1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2(x_2-1) + \lambda_1 - \lambda_2 = 0$ — (2)

$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1 + x_2 - 2 = 0$ — (3)

$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow x_1^2 - x_2 = 0$ — (4)

At the point $(1,1)$

(1) $\Rightarrow -2 + \lambda_1 + 2\lambda_2 = 0 \Rightarrow \lambda_1 + 2\lambda_2 = 2$ — (5)

(2) $\Rightarrow \lambda_1 - \lambda_2 = 0 \Rightarrow \lambda_1 - \lambda_2 = 0$ — (6)

Solving (5) & (6), we get

$\lambda_1 + 2\lambda_1 = 2 \Rightarrow 3\lambda_1 = 2$

$\Rightarrow \lambda_1 = \frac{2}{3} > 0$

$\lambda_2 = \frac{2}{3} > 0$

Also All the constraints are satisfied

(1, 1) corresponds to local minimum.

Q.6 Find the admissible and constrained variations at the point $X = (0,4)^T$ for the following problem

$$\text{Minimize } f = x_1^2 + (x_2 - 1)^2$$

subject to $-2x_1^2 + x_2 = 4$.

Sol^n

$$g(x_1, x_2) = -2x_1^2 + x_2 - 4 = 0$$

$$x^* = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

dx_1 and dx_2 will be admissible and constrained variations at x^* if

$$g(x^* + dx) = 0$$

By Taylor's series

$$g(x^* + dx) = g(x^*) + dg(x^*)$$

$$= 0 + \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 \Big|_{(0,4)}$$

$$= -4x_1\, dx_1 + dx_2 \Big|_{(0,4)}$$

$g(x^* + dx)$ will be 0 when $dx_2 = 0$

and dx_1 is any arbitrary value.

as $x_1^* = 0$

Admissible variations

$$\begin{cases} dx_1 & \text{arbitrary} \\ dx_2 = 0 \end{cases}$$