Another approach → ***differential eqn*** **for the orbit!**

- Usually, given **f(r),** we use the integral formulation.
- However, **differential eqn are most useful for**

The Inverse Problem:

≡ ***Given a known orbit $r(\theta)$ or $\theta(r)$,***

determine the force law $f(r)$.

Equation of the Orbit

Start with the equation of motion in terms of forces, and transform it using a couple of tricks. Radial eqn.

$$\mu\ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}.$$

First change variables from r to $u = 1/r$.

Second convert the differential operator d/dt in terms of $d/d\phi$:

Equation of the Orbit

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}.$$

Find $\ddot{r}$

Equation of the Orbit

$$\dot{r} = \frac{d}{dt}(r) = \frac{\ell u^2}{\mu} \frac{d}{d\phi} \frac{1}{u} = -\frac{\ell}{\mu} \frac{du}{d\phi}$$

Equation of the Orbit

$$\dot{r} = \frac{d}{dt}(r) = \frac{\boldsymbol{\ell} u^2}{\mu} \frac{d}{d\phi} \frac{1}{u} = -\frac{\boldsymbol{\ell}}{\mu} \frac{du}{d\phi}$$

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{\boldsymbol{\ell} u^2}{\mu} \frac{d}{d\phi}\left(-\frac{\boldsymbol{\ell}}{\mu} \frac{du}{d\phi}\right) = -\frac{\boldsymbol{\ell}^2 u^2}{\mu^2} \frac{d^2 u}{d\phi^2}.$$

Equation of the Orbit

$$\dot{r} = \frac{d}{dt}(r) = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\frac{1}{u} = -\frac{\ell}{\mu}\frac{du}{d\phi}$$

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$$\mu\ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}.$$

$$-\mu\frac{\ell^2 u^2}{\mu^2}\frac{\partial^2 u}{\partial\phi^2} = F(r) + \frac{\ell^2 u^3}{\mu} \qquad \text{or}$$

$$\boxed{u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}F(r).}$$

Equation of the Orbit

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$$\boxed{u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}F(r).}$$

we substituted $u = 1/r$.

The Kepler Orbits

A general equation for the path of a body in the 2-body central force problem:

$$\boxed{u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2} F(r).}$$

Change of variables → $u = 1/r$.

- True for any central force $F(r)$,
- For the gravitational case (the Kepler problem), using $\gamma = Gm_1m_2$, we have

$$F(r) = -\frac{\gamma}{r^2} = -\gamma u^2.$$

the simpler, linear equation $$u''(\phi) = -u(\phi) + \gamma\mu / \ell^2.$$

Another substitution → $w(\phi) = u(\phi) - \gamma\mu/\ell^2$

New form ?

The Kepler Orbits

$$\boxed{u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2} F(r).}$$

$$F(r) = -\frac{\gamma}{r^2} = -\gamma u^2.$$

$$\gamma = G m_1 m_2$$

$$u''(\phi) = -u(\phi) + \gamma\mu / \ell^2.$$

$$w(\phi) = u(\phi) - \gamma\mu/\ell^2 \quad \rightarrow \quad \boxed{w''(\phi) = -w(\phi),}$$

Solution → ?

The Kepler Orbits

$$\boxed{u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2} F(r).}$$

$$F(r) = -\frac{\gamma}{r^2} = -\gamma u^2.$$

$$\gamma = G m_1 m_2$$

$$u''(\phi) = -u(\phi) + \gamma\mu / \ell^2.$$

$w(\phi) = u(\phi) - \gamma\mu/\ell^2$ → $$w''(\phi) = -w(\phi),$$

Solution → $w(\phi) = A\cos(\phi - \delta)$.

for Choose coordinates for which $\delta = 0$, the final solution is

$$\boxed{u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi = \frac{1}{c}\left(1 + \varepsilon\cos\phi\right).}$$

$$\boxed{c = \frac{\ell^2}{\gamma\mu}} \qquad \boxed{\varepsilon = \frac{A\ell^2}{\gamma\mu}}$$

Final Kepler Path

- substituting $u = 1/r$, we have

$$\boxed{u(\phi) = \frac{1}{c}\left(1 + \varepsilon\cos\phi\right) \quad \Rightarrow \quad r(\phi) = \frac{c}{1 + \varepsilon\cos\phi}.}$$

Bounded Orbits

- The dimensionless constant $\varepsilon = \dfrac{A\ell^2}{\gamma\mu}$ → big role in the shape of the orbit, depending on whether it is greater or less than 1.

Final Kepler Path

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Bounded Orbits

- The dimensionless constant $\varepsilon = \dfrac{A\ell^2}{\gamma\mu}$ → big role in the shape of the orbit, depending on whether it is greater or less than 1.

- If $\varepsilon < 1$, then the denominator is always positive for any value of ϕ.
- If $\varepsilon > 1$, there is a range of values of ϕ for which the denominator vanishes, and r blows up (the object is unbound).
- $\varepsilon = 1$ is the demarcation between bound and unbound orbits.

Final Kepler Path

$$u(\phi) = \frac{1}{c}\left(1 + \varepsilon\cos\phi\right) \quad \Rightarrow \quad r(\phi) = \frac{c}{1 + \varepsilon\cos\phi}. \qquad \varepsilon = \frac{A\ell^2}{\gamma\mu}$$

- **first take $\varepsilon < 1$.**

In the above equation, as $\cos\phi$ oscillates between –1 and 1, the orbital distance r varies between

$$r_{\min} = \frac{c}{1+\varepsilon} \quad \text{and} \quad r_{\max} = \frac{c}{1-\varepsilon}.$$

$r_{\min}$ = *perihelion (perigee)*

$r_{\max}$ = *aphelion (apogee)*

Bounded Orbits

- The shape of the orbit, looks like ellipse

$$r(\phi) = \frac{c}{1 + \varepsilon \cos\phi}$$

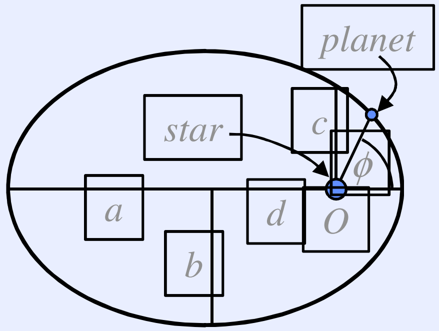
Can be written in the form:

$$\frac{\left(x+d\right)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This shape is an ellipse,

Bounded Orbits

$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi} \quad \rightarrow \quad \frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1.$$



$$a = \frac{c}{1-\varepsilon^2}; \quad b = \frac{c}{\sqrt{1-\varepsilon^2}}; \quad d = a\varepsilon.$$

a is called the semi-major axis (half the longer axis) and b is the semi-minor axis.

- The constant ε is the **eccentricity** of the ellipse, and can be determined from

$$\frac{b}{a} = \sqrt{1-\varepsilon^2}.$$

- **As $\varepsilon \to 0$, d goes to zero, a & b become equal, and the ellipse becomes a circle.**
- **As $\varepsilon \to 1$, $d \to a$, $a \to \infty$ and $b/a \to 0$, and the ellipse grows long and skinny (i.e. very eccentric).**

Halley's Comet

Halley's comet follows a very eccentric orbit, with $\varepsilon = 0.967$. Given that the closest approach to the Sun (perihelion) is 0.59 AU (astronomical units), what is its greatest distance from the Sun?

Halley's Comet …

Solution:

- $r_{max}/r_{min} = (1 + \varepsilon)/(1 - \varepsilon)$.

$$r_{\max} = \frac{1+\varepsilon}{1-\varepsilon} r_{\min} = \frac{1.967}{0.033} r_{\min} = 60\ r_{\min} = 35 \text{ AU}.$$

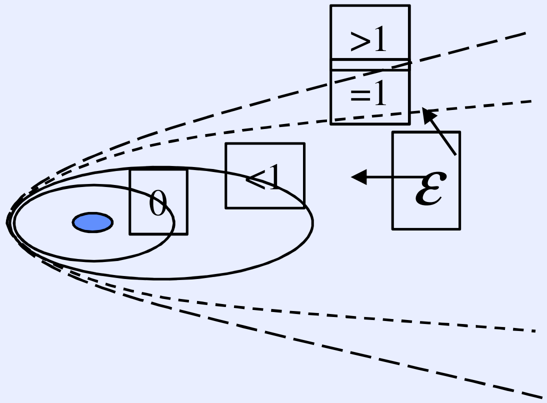
The Unbound Kepler Orbits

- **For $\varepsilon > 1$, the denominator blows up for some other value of ϕ, such that**

$$\varepsilon \cos\phi_{\max} = -1.$$

- **In this case, it can be shown that the cartesian form is a hyperbola:**

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1,$$



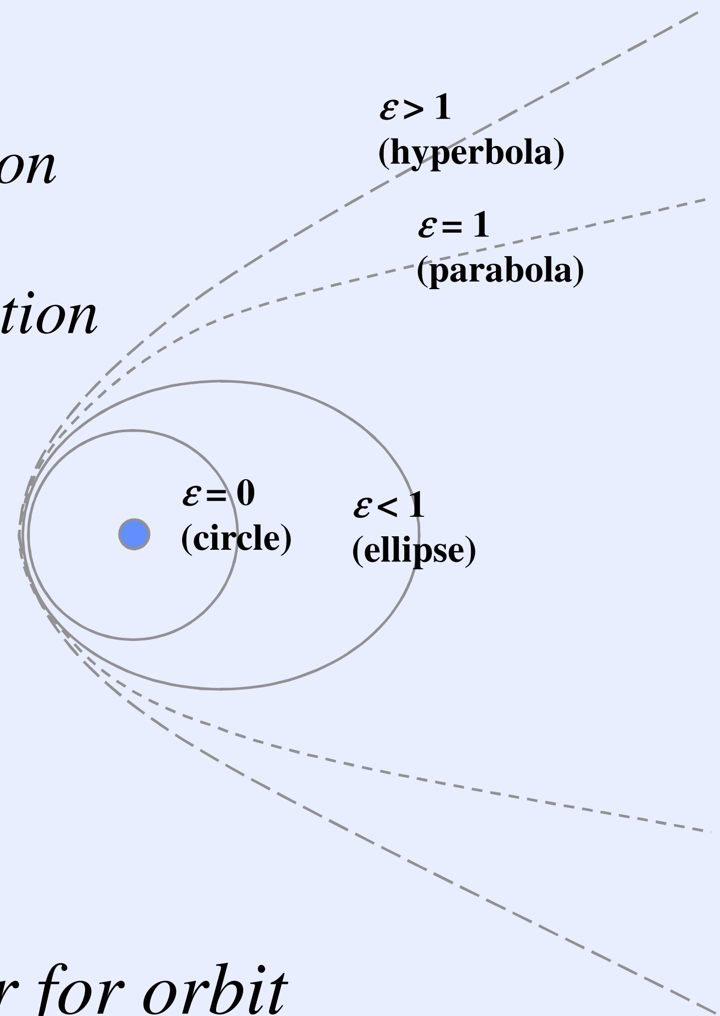
Summary of Kepler Orbits

- Important relations of Kepler orbits are:

$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi} \qquad \textit{path equation}$$

$$E = \frac{\gamma^2\mu}{2\ell^2}\left(\varepsilon^2 - 1\right). \qquad \textit{energy equation}$$

eccentricity	energy	orbit
$\varepsilon = 0$	E < 0	circle
$0 < \varepsilon < 1$	E < 0	ellipse
$\varepsilon = 1$	E = 0	parabola
$\varepsilon > 1$	E > 0	hyperbola



$$c = \frac{\ell^2}{Gm_1m_2\mu} \qquad \textit{scale factor for orbit}$$

Practice Examples/Problems of Marion and Thornton (chapter 8).