25% for 3 exams
10% - lab evaluation
10% - " exam
5% - attendance

Books: ① Bernard Sklar: Digital comm: Fundamentals & applications 2nd ed
② John Proakis, Digital comm, 4th ed

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Digital communication

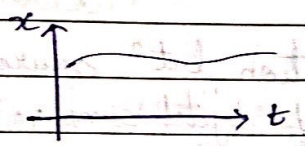
Noodle

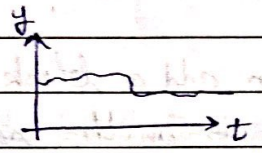
→ Most common assumption when noise is to be modelled in the system is that it is additive.

→ diff. b/t analog & digital?

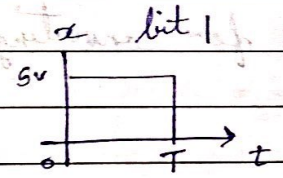
→ $y(t) = h(t) * x(t) + \eta(t)$

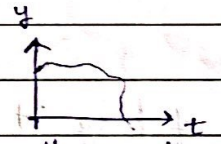
the comm. channel does not actually perform convolution, it is just an abstract mathematical representation for modelling input & output.

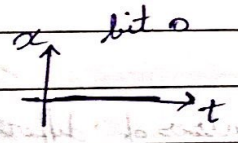




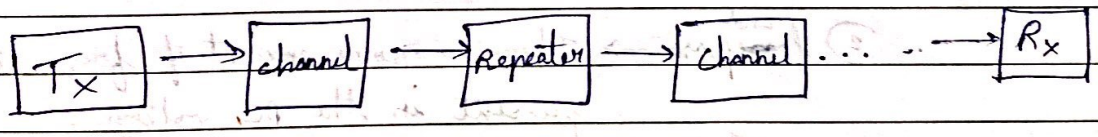
The reciever has no info. whether it is a valid signal or not (in case of analog).







The receiver only has to interpret that whether the signal is '1' or '0' & makes it easier than analog.



→ Advantages

① 1st advantage of digital comm. is that we can use repeaters (regenerative repeaters)

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② 2nd " " " " " " " " digital circuits which is more stable than analog circuit (eg:- temp. difference won't affect digital circuits much but it will affect analog circuits)

③ 3rd - - - - digital devices & circuits are reprogrammable.

④ Digital devices would be cheaper since they don't have to be resistant to temp. changes etc.

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→ Disadvantages

① Time synchronisation (we need efficient hardware to maintain synchronisation betn receiver & transmitter)

② If the error in transmission crosses a threshold, all the info. is lost (non-graceful degradation)

→ eg of source encoding - Morse code

→ channel encoding ?

→ we can add a block of encryption betn source & channel encoding which is another advantage of digital comm :- we have developed number theory.

→ Digital modulation is responsible for converting bits to wave-forms.

2 ways of doing it :-

① baseband → we send pulses of different amplitudes.

② band and pass → There is some amount of freq. component present in the DC voltage.

why would we want a freq. component ?

- To utilise free space for comm.

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→ In autocorrelation slide, D stands for delay. The definition of inner product will change if the signal is complex.

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→ $\|x\|_{L_2} = \int_{-\infty}^{\infty} |x(t)|^2 dt$ (energy of the signal)

→ For power signals, inner product may not exist, so we need to redefine for power signals ; $R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \langle (P_{-T}\, x)_T, x_T \rangle, \forall x \in P$.
power signal

→ To characterise a RV, means to find it's probability distribution.

→ To model IS, we need joint probability distributions, if the variable are not independent.

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→ stationarity implies that the joint prob. distribution is not going to change (it should be valid for all n & t)
eg:- Rolling a die.

$n=1 : p(x_{t_1}) = p(x_{t_1+t}), \forall t$
$n=2 : p(x_{t_1}, x_{t_2}) = p(x_{t_1+t}, x_{t_2+t}), \forall t$

→ Any stochastic process which satisfies the conditions,
$m_x(t) = m_x$
$R_x(t_1, t_2) = R_x(t_1 - t_2), \forall t_1, t_2$
is called as wide-sense-stationary (WSS)

→ The autocorrelation fun^n will be highest at 0.
$R_x(0) = E[x_t^2]$.
$X_{t_1} = X$, $X_{t_2} = Y$, $t_1 - t_2 = \tau$.

$R_x(0) = E[X^2]$
$R_x(\tau) = E[XY]$

Cauchy-schwartz inequality : $|\langle f, g \rangle| \leq \|f\| \|g\|$
Try to prove that $R_x(0) \geq R_x(\tau)$

→ Is this WSS?

$X(t) = A\cos(\omega_c t + \theta)$, $A \sim U[-1, 1]$.

→ $\therefore$ ~~$X(t+T) = A\cos(\omega_c(t+T)+\theta)$~~

~~$= A\cos(\omega_c t + \omega_c T + \theta)$~~

~~$= A\cos(\omega_c t + (\omega_c T + \theta))$~~

~~$= A[\cos\omega_c t \cos(\omega_c T + \theta) - \sin\omega_c t \sin(\omega_c T + \theta)]$~~

~~$= A[\cos\omega_c t(\cos\omega_c T \cos\theta - \sin\omega_c T \sin\theta)$~~

~~$- \sin\omega_c t(\sin\omega_c T + \theta \cos(\omega_c t) + \sin(\omega_c t)\cos(\omega_c T + \theta)]$~~

→ n RVs → $X(t_1) = A\cos(\omega_c t_1 + \theta)$

$X(t_2) = A\cos(\omega_c t_2 + \theta)$

$\vdots$

$X(t_n) = A\cos(\omega_c t_n + \theta)$

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~~$\int_{-1}^{1} A\cos(\omega_c t_n + \theta)\,da = \cos(\omega_c t_n + \theta)(2A)$~~

~~Mean $(m_x(t)) = 2A\cos(\omega_c t + \theta)$~~

$m_x(t) = E[x(t)]$

$= E[A\cos(\omega_c t + \theta)]$

$= E[A]\cos(\omega_c t + \theta)$

$= 0 \times \cos(\omega_c t + \theta)$

$= 0$

$R_x(t_1, t_2) = E[X_{t_1} \cdot X_{t_2}]$

$= E[x(t_1)\, x(t_2)]$

$= E[A^2 \cos(\omega_c t_1 + \theta)\cos(\omega_c t_2 + \theta)]$

$= E[A^2]\cos(\omega_c t_1 + \theta)\cos(\omega_c t_2 + \theta)$

$\therefore$ It is not WSS.

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→ Is this WSS?

$$X(t) = \cos(\omega_c t + \theta), \quad \theta \sim U[-\pi, \pi]$$

→ n R.Vs → $X(t_1) = \cos(\omega_c t_1 + \theta)$

$X(t_2) = \cos(\omega_c t_2 + \theta)$

$\vdots$

$X(t_n) = \cos(\omega_c t_n + \theta)$

$$m_X(t) = E[\cos(\omega_c t + \theta)]$$
$$= E[\cos\omega_c t \cos\theta - \sin\omega_c t \sin\theta]$$
$$= E[\cos\omega_c t]E[\cos\theta] - E[\sin\omega_c t]E[\sin\theta]$$
$$= 0 - 0$$
$$= 0$$

$$R_X(t_1, t_2) = E[X_{t_1} X_{t_2}]$$
$$= E[X(t_1)X(t_2)]$$
$$= E[\cos(\omega_c t_1 + \theta)\cos(\omega_c t_2 + \theta)]$$
$$= \frac{1}{2}E[\cos(\omega_c(t_1 - t_2)) - (\sin(\omega_c t_1 + \theta))\sin(\omega_c t_2 + \theta)]$$
$$= \frac{1}{2}E[\cos(\omega_c(t_1 - t_2)) - [(\sin\omega_c t_1 \cos\theta + \sin\theta\cos\omega_c t_1)(\ldots)]]$$

$$= \frac{1}{2}E[\cos(\omega_c(t_1 - t_2))]$$
$$= \frac{1}{2}\cos(\omega_c(t_1 - t_2))$$

∴ It is a WSS.

$X(t) = A, \quad A \sim [-1, 1]$

∴ it is stationary in the strict sense as well.

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→ Is this WSS?

Bernoulli process : $P(X_n = 1) = p$
$P(X_n = 0) = 1-p, \forall n$

→ mean $= p$

$R_X(X_{n_1}, X_{n_2}) = E[p \cdot p]$

$= p^2$
(R_X)

∴ It is WSS, ∵ it does not depend on n.

It is also stationary in the strict sense; ∵ prob. does not depend on n but rather on the no. of 1s & 0s.

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→ $X(t) = A(t)\cos(w_c t + \theta)$
$A(t) \sim U[-1, 1]$
what is autocorrelation?

→ 0, ∵ A(t) will be a different RV for different time instances.

→ X(t) is a Gaussian random process if $\forall t_1, t_2, \ldots, t_n$ $\forall n$

$$p(x(t_1), \ldots, x(t_n)) = \frac{1}{(2\pi)^{n/2} \det(C)} e^{-\frac{1}{2}(X-M)^T C^{-1}(X-M)}$$

where $X = \begin{pmatrix} x(t_1) \\ \vdots \\ x(t_n) \end{pmatrix}$, $M = \begin{pmatrix} m(t_1) \\ \vdots \\ m(t_n) \end{pmatrix}$ ∴

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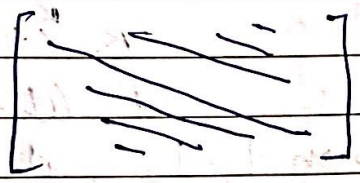
$$C(t_i, t_j) = C_{ij} = E\left[(X(t_i) - M(t_i))(X(t_j) - M(t_j))\right]$$

$\downarrow$

Covariance matrix

(by definition, it has to be symmetric)

For $x(t)$ to be WSS, the mean should be a constant & $C(t_i, t_j)$ should depend on $t_i - t_j$ & not t_i & t_j individually.

$C =$ 

Toeplitz matrix

For strict-sense stationarity,

$$p(x(t_{1+z}), \ldots, x(t_{n+z})) = p(x(t_1), \ldots, x(t_n))$$

$\therefore$ only in a Gaussian Random Process, WSS implies SSS.

→ Power spectral density

Assuming process is WSS,

$$G_x(f) = \int_{-\infty}^{\infty} R_x(z)\, e^{-j2\pi f z}\, dz$$

Power spectral density

$$R_x(z) = \int_{-\infty}^{\infty} G_x(f)\, e^{j2\pi f z}\, df$$

$$R_x(0) = \int_{-\infty}^{\infty} G_x(f)\, df = E[x_t^2]$$

$$= \int_{-\infty}^{\infty} x_t^2\, p(x_t)\, dx_t$$

→ no matter what the value of t, the integration would always be the same

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→ Autocorrelation of the output of an LTI system :-

$$E[\overset{Y_{t_1}\ Y_{t_2}}{\cancel{y(t_1)\,y(t_2)}}] = E\left[\left(\int_{-\infty}^{\infty} h(\tau_1)\,x(t_1-\tau_1)\,d\tau_1\right)\left(\int_{-\infty}^{\infty} h(\tau_2)\,x(t_2-\tau_2)\,d\tau_2\right)\right]$$

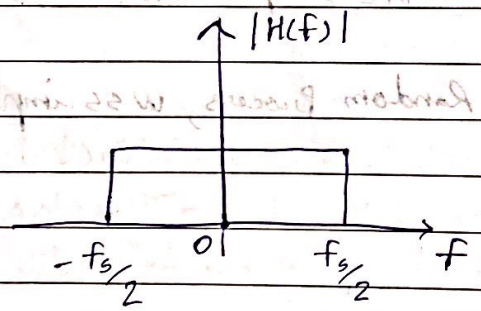
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\,h(\tau_2)\,\underbrace{E[x(t_1-\tau_1)\,x(t_2-\tau_2)]}_{R_x(t_1-\tau_1-t_2+\tau_2)}\,d\tau_1\,d\tau_2$$

$$= R_x((t_1-t_2)-\tau_1+\tau_2)$$

$\because E[(Y_{t_1}Y_{t_2})]$ depends on t_1-t_2, it is atleast WSS.

$\therefore$ an LTI system does not distort stationarity.

→ Ideal filters ?

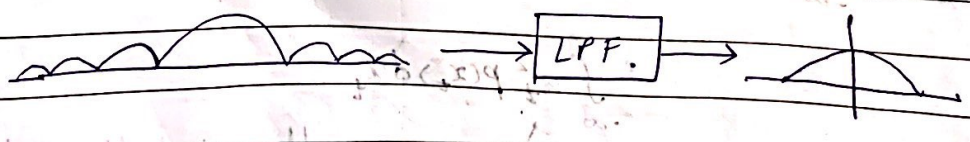
- 

ideal low pass filter

- How to construct them ?

→ Nyquist criteria is valid only for bandlimited signals

→ Anti-aliasing filter



→ Transition bandwidth ?

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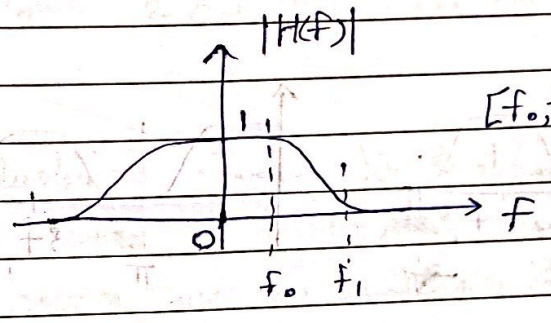
→ For an LTI system which is not causal, we can add delay to make it causal.

Can we do it for a low pass filter?

→ No ∵ sinc(x) = 0 for $x \to \infty$ or $x \to -\infty$, so, even if we shift it by any amount, it will remain non-causal.

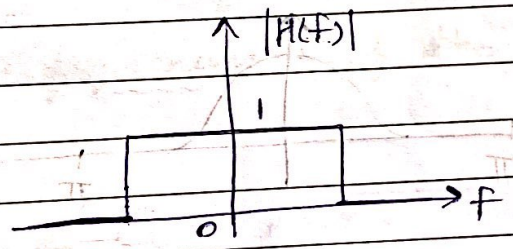
→ ~~we can never realize a discontinuous freq. transform~~.

non-ideal LPF –



$[f_0, f_1]$ → transition band width

ideal LPF –



f_0 should be higher than the min. freq. of the signal which we want to preserve.

The sampling rate of a CD is 44KHZ & not 40KHZ because of f_1.
∴ we should try to have f_1 as close to f_0 as possible which is possible if we use a higher-order LTI system ⟹ higher order polynomial fun$^{\underline{c}}$ ⟹ higher no. of zeros & poles.

→ If we oversample, we can use a high transition bandwidth filter & it is also less sensitive, so, it is cheaper.

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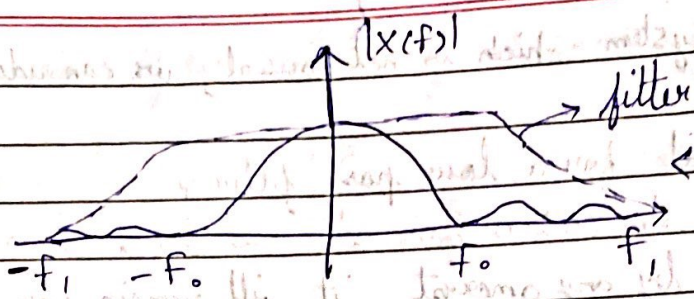
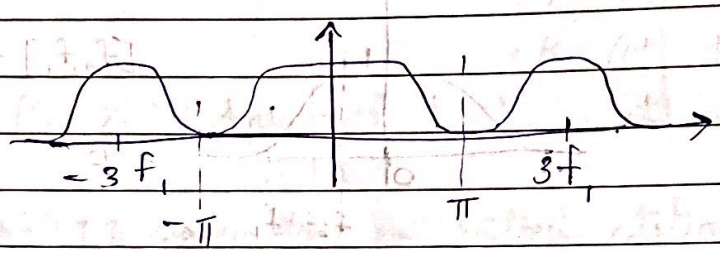
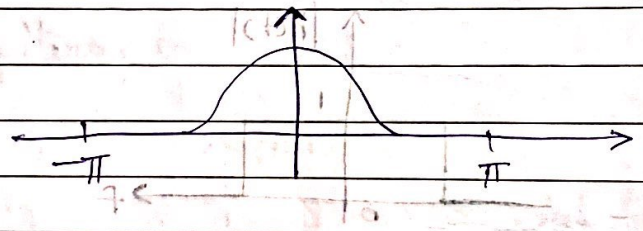
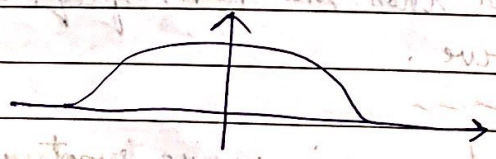


if it is sampled at $3f_1$,





after downsampling,



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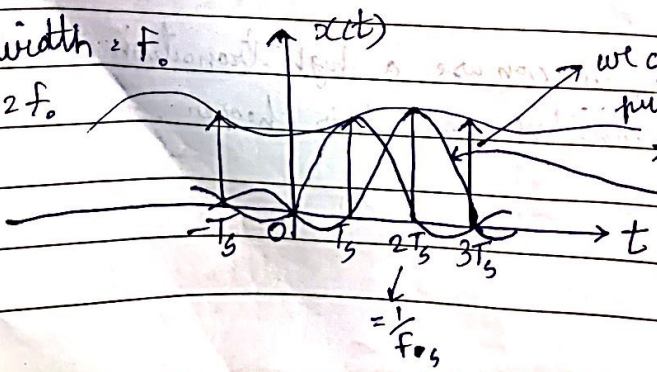
→ Smoother the func, smaller the bandwidth.

→ bandwidth = f_0

$f_s = 2f_0$



we cannot transmit these pulses; so, we transmit the waveforms

bandwidth of each is f_0

we can transmit the superposition of these waveforms. ∵ If the channel's bandwidth is f_0, the message ~~could be sent~~ is sent with ISI but, at the receiver, when we sample at every T_s seconds, we will not get ISI.
The rate at which we are transmitting the waveforms is $\frac{1}{f_s}$. ∴ for every hertz in the channel, we can transmit 2 symbols per sec.
∴ In order to transmit 2 symbols, the waveforms have to be sinc fun^{n}.

→ The basic idea of duobinary encoding is that to introduce ISI at the source, & since we know the ISI which we have introduced at the source, we can eliminate it at the receiver.

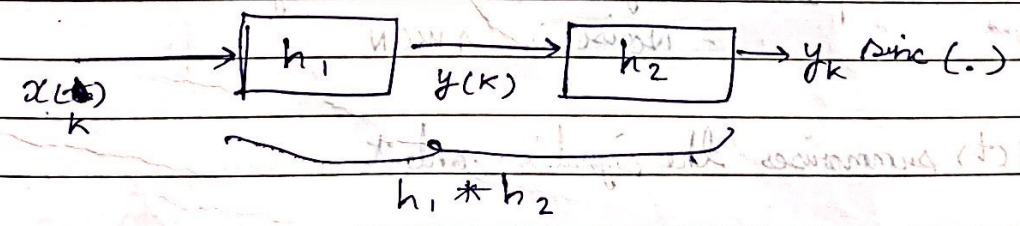
What will be the freq. response of y_k w.r.t x_k if $y_k = x_k + x_{k-1}$

$$\longrightarrow \quad y(KT_s) = x(KT_s) + x((K-1)T_s)$$

$$h(t) = \delta(t) + \delta(t - T_s)$$

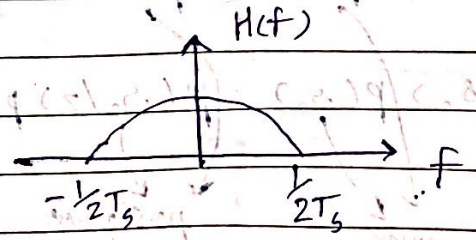
$$H(f) = 1 + e^{-j2\pi f T_s}$$

$$= e^{-j\pi f T_s}\left(e^{j\pi f T_s} + e^{-j\pi f T_s}\right)$$



$H_1 \cdot H_2$

2cos( ) rectangular pulse



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This can be realised to a very high accuracy with analog filters.

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→ $\tilde{w}_k = x_k \oplus \tilde{w}_{k-1}$, $\tilde{w}_{-1} = 0$

$x_k =$	1	0	1	0	0	1	1	1	0
$\tilde{w}_k =$	1	1	0	0	0	1	0	1	1
$w_k =$	V	V	-V	-V	-V	V	-V	V	V
$y_k =$	0	2V	0	-2V	-2V	0	0	0	2V

H.W. → why is this the right decoding scheme? (why would it work for all examples?)

Chapter - 3

→ 1st assumption - System is an LTI system
2nd " - Noise is AWGN

→ $z(t)$ summarises the signal's content

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→ $p(z \,\&\, s_1) = p(z/s_1)\, p(s_1) = p(s_1/z)\, p(z)$

$p(z \,\&\, s_2) = p(z/s_2)\, p(s_2) = p(s_2/z)\, p(z)$

likelihood func^n — prior probability — posterior prob.

if $p(\omega_1/z) > p(\omega_2/z) \Rightarrow z \in \omega_1$

" " < " " " " ω_2

Maximum a posteriori classification (MAP)

if $p(z/\omega_1) > p(z/\omega_2) \Rightarrow z \in \omega_1$

" " < " " " " ω_2

Maximum Likelihood estimation / (ML) classification

here, we ignore the prior prob.

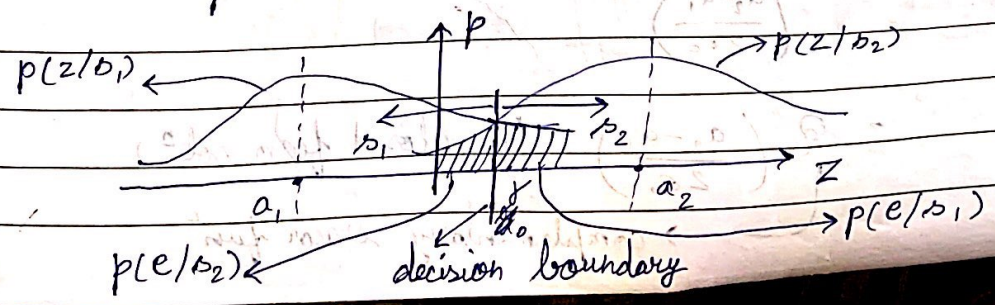
$$p(\omega_1/z) = \frac{p(z/\omega_1)\, p(\omega_1)}{p(z)}$$

$$p(\omega_2/z) = \frac{p(z/\omega_2)\, p(\omega_2)}{p(z)}$$

if $p(\omega_1) = p(\omega_2)$, MAP = ML

→ In our case,

$$p(z/\omega_1) = N(a_1, \sigma_0)$$
$$p(z/\omega_2) = N(a_2, \sigma_0)$$



if $p(z/s_1) \sim p(z/s_2)$ & $p(s_1) = p(s_2)$,

$$\gamma_0 = \left(\frac{a_1 + a_2}{2}\right)$$

→ Probability of misclassification,

if signal is classified as s_2,

$$p(e/s_1) = \int_{\gamma_0}^{\infty} p(z/s_1)\, dz$$

$$p(e/s_2) = \int_{-\infty}^{\gamma_0} p(z/s_2)\, dz$$

$$P_B = p(e/s_1)\, p(s_1) + p(e/s_2)\, p(s_2)$$

P_B → probability of misclassifying a bit

$$= p(e/s_1) \text{ or } p(e/s_2)$$

(if $p(s_1) = p(s_2)$ & $p(z/s_1) \sim p(z/s_2)$)

$$p(e/s_1) = \int_{\frac{a_1+a_2}{2}}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{1}{2}\left(\frac{z-a_1}{\sigma_0}\right)^2} dz$$

Let $y = \frac{z - a_1}{\sigma_0} \Longrightarrow dy = \frac{dz}{\sigma_0}$

$$P_B = \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{a_2-a_1}{2\sigma_0}\right)}^{\infty} e^{-y^2/2}\, dy$$

$$= Q\left(\frac{a_2 - a_1}{2\sigma_0}\right) \text{ (no closed form sol}^{n}\text{)}$$

Q → complementary error fun^{c}

→ we have to ~~decide~~ make a decision about ~~the design~~ freq. response of the receiving filter, so that the signal-to-noise ratio is maximised & the ~~error~~ prob. of misclassification is minimised.

→ $z(T) = a_i(T) + \eta_o(T)$

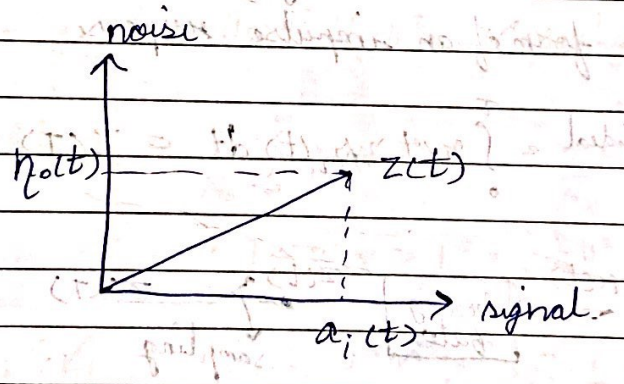
$z(T)$ → measurement, $a_i(T)$ → signal, $\eta_o(T)$ → noise

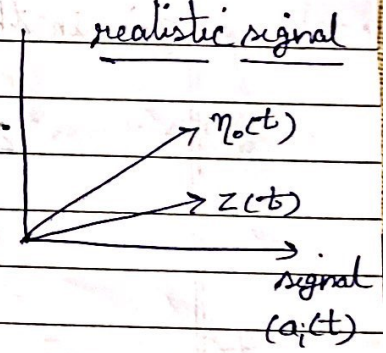
$r(t) \longrightarrow \boxed{H(f)} \longrightarrow z(t)$

$r(t)$ → 2 components, $s_i(t)$, $\eta(t)$ → AWGN

$z(t)$ → $a_i(t)$, $\eta_o(t)$

(we want to maximise the contribution of $a_i(t)$ in $z(t)$.)





$$\|z\|^2 = \|a_i\|^2 + \|\eta_o\|^2$$

$$SNR = \frac{\|a_i\|^2}{\|\eta_o\|^2}$$ → we want to maximise it. $\cancel{= \frac{N_o}{2}\int_{-\infty}^{\infty} |H(f)|^2 df}$

$\|\eta_o\|^2$ → noise power

$\eta_o(f) = H(f)\,\eta(f)$, $\cancel{a_i(t) = h(t) * s_i(t)}$

$\eta(f)$ → $\frac{N_o}{2}$ (amplitude)

$$\|\eta_0\|^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df$$

$$a_i(t) = h(t) * s_i(t)$$

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$\eta_0(t)$ noise

$r(t) \rightarrow$ received signal

$z(t)$

signal $(a_i(t))$

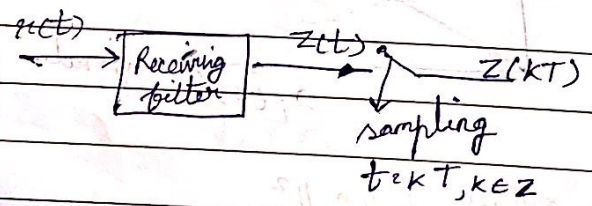
$(s_i(t))$

ideal $z(t)$

$$z^{ideal} = \langle r(t), s_i(t)\rangle s_i(t)$$

we also need time invariancy, so, we need to write the projection in the form of an impulse response

$$z^{ideal} = \int_0^T r(t)\, s_i(t)\, dt = Z(T)$$



write this using a convolution (we want to write a correlation using convolution

$$y = h * x$$

$$y(t_0) = \int_{-\infty}^{\infty} h(\tau)\, \underbrace{x(t-\tau)}_{s(\tau)}\, d\tau$$

we are interested only in a time period T.

$$\therefore y(T) = \int_0^T h(\tau)\, x(T-\tau)\, d\tau$$

to write z^{ideal} in terms of convolution,

$$a_i(T) = \int_{-\infty}^{\infty} H(f)\, S_i(f)\, e^{+j2\pi f T}\, df$$

$$\|a_i\|^2 = \left| \int_{-\infty}^{\infty} H(f)\, S_i(f)\, e^{j2\pi f T}\, df \right|^2$$

$$SNR = \left(\frac{\|a_i\|^2}{\|\eta_0\|^2} \right) = \frac{\left| \int_{-\infty}^{\infty} H(f)\, S_i(f)\, e^{j2\pi f T}\, df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}$$

Let $x = H(f)$, $y = \overline{S_i(f)}\, e^{-j2\pi f T}$

$$SNR = \frac{|\langle x, y \rangle|^2}{\frac{N_0}{2} \|x\|^2}$$

$$\langle u, v \rangle := \int_{-\infty}^{\infty} u(f)\, \overline{v(f)}\, df$$

using, $|\langle x, y \rangle|^2 \le \|x\|^2 \|y\|^2$ (Cauchy-Schwartz inequality)

$$\therefore (SNR)_{max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |S_i(f)|^2 df = \frac{2E}{N_0} \rightarrow \text{signal energy.}$$

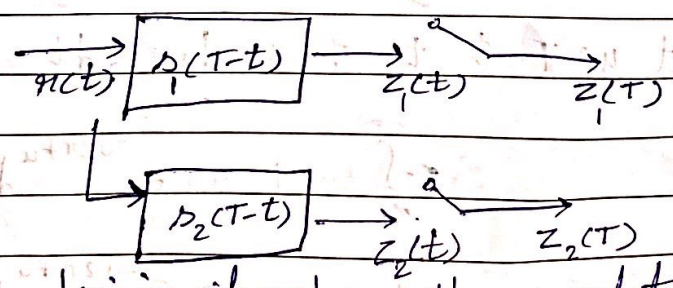
$(\because y = x^* \Longrightarrow |\langle x, y \rangle|^2 = \|x\|^2)$

~~$H(f) = \overline{S}(f) e^{-j2\pi f T}$ (ideal)~~ $\longrightarrow$ correction on next page

~~$h(t) = s(T-t)$, $0 \le t \le T$~~

~~$= 0$, elsewhere~~

This is known as matched filter response



we make a decision based on the correlation b/t the impulse response & $r(t)$.

$$\cancel{\therefore z(t) = \int_{-\infty}^{\infty} h(t-\tau)\, r(\tau)\, d\tau}$$

$$\cancel{h(t) = s(T-t), \quad 0 \le t \le T}$$
$$\cancel{0, \quad \text{elsewhere}}$$

→ in correction

$$= \int_{-\infty}^{\infty} s(T-(t-\tau))\, r(\tau)\, d\tau$$

$$z(t) = \int_{-\infty}^{\infty} s(T-(t-\tau))\, r(\tau)\, d\tau$$

$$z(T) = \int_{-\infty}^{\infty} s(\tau)\, r(\tau)\, d\tau$$

$$z(T) = \int_{0}^{T} s(\tau)\, r(\tau)\, d\tau$$

<u>Correction</u> → $(SNR)_{max} = \frac{2E}{N_0}$

This max SNR is achieved for $x = ky$. Setting $k=1$, ∵ the value of k does not affect the SNR, we get,

$$SNR = \frac{2|\langle y, y\rangle|^2}{N_0 \|y\|^2} = \frac{2\|y\|^4}{N_0 \|y\|^2} = \frac{2\|y\|^2}{N_0} = \frac{2E}{N_0} = SNR_{max}$$

$\therefore H(f) = \overline{S_i(f)}\, e^{-j2\pi f T}$, now,

$$\overline{S_i(f)}\, e^{-j2\pi f T} = \left(\int_{-\infty}^{\infty} s_i(t)\, e^{j2\pi f t}\, dt\right) e^{-j2\pi f T} \quad (\because s_i(t) \in \mathbb{R})$$

$$= \int_{-\infty}^{\infty} s_i(t)\, e^{(j2\pi f (t-T))}\, dt$$

Let $u = T-t$, $t = T-u$, $du = -dt$

$$= -\int_{\infty}^{-\infty} s_i(T-u)\, e^{-j2\pi f u}\, du$$

$$= \int_{-\infty}^{\infty} s_i(T-u)\, e^{-j2\pi f u}\, du = F(s_i(T-u))$$

$\therefore h(t) = s_i(T-t), \forall t \in [0, T]$

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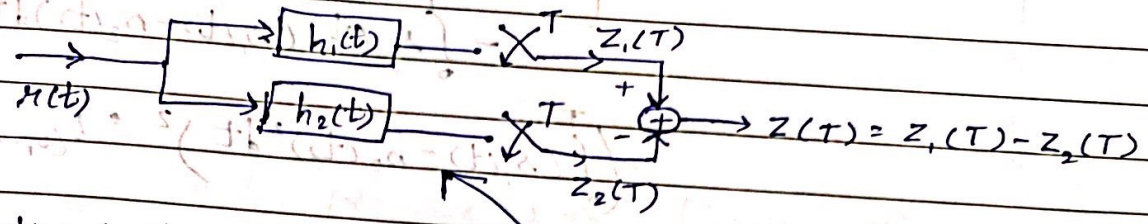
→ Binary PCM :

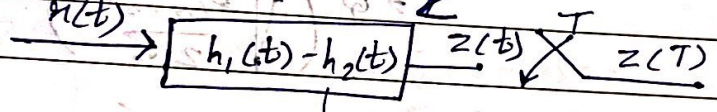


$r(t) = r_i(t) + \eta(t)$
↳ AWGN

Same

$h_1 : SNR_{max} = \frac{2E_{s_1}}{N_0}$

$h_2 : SNR_{max} = \frac{2E_{s_2}}{N_0}$



→ given the impulse response, which signal is going to maximise the output?

$r(t) = r_i(t) + \eta(t)$
↳ AWGN

$s_1(t) - s_2(t)$ will maximise the output.

$\therefore SNR_{max} = \frac{2E_{s_1 - s_2}}{N_0}$

$P_B = Q\left(\frac{a_2 - a_1}{2\sigma_0}\right)$

$a_1 = \int_0^T s_1(t)\,(s_1(t) - s_2(t))\,dt$

$a_2 = \int_0^T s_2(t)\,(s_1(t) - s_2(t))\,dt$

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$$(a_1-a_2)^2 = \left(\int_0^T s_1(t)\,(s_1(t)-s_2(t))\,dt - \int_0^T s_2(t)\,(s_1(t)-s_2(t))\,dt\right)^2$$

$$= \left(\int_0^T (s_1(t)-s_2(t))^2\,dt\right)^2 = E^2_{(s_1-s_2)}$$

$$SNR_{max} = \frac{(a_1-a_2)^2}{\sigma_0^2} = \frac{E^2_{(s_1-s_2)}}{\sigma_0^2}$$

$$P_B = Q\left(\frac{a_2-a_1}{2\sigma_0}\right) = Q\left(\frac{1}{2}\sqrt{SNR_{max}}\right)$$

$$= Q\left(\frac{1}{2}\sqrt{\frac{2}{N_0}E_{(s_1-s_2)}}\right)$$

$$= Q\left(\sqrt{\frac{E_{(s_1-s_2)}}{2N_0}}\right)$$

eg:- NRZ-L bipolar

$s_1(t) = +A, \quad 0 \le t \le T$

$s_2(t) = -A, \quad 0 \le t \le T$

$$P_B = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

unipolar

$s_1(t) = +A, \quad 0 \le t \le T$

$s_2(t) = 0, \quad 0 \le t \le T$

$$P_B = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right)$$

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→ P_B only depends on difference in energy betn 2 signals & not on the shape of the signal.

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→ $E_d = E_{(s_1 - s_2)} = \int_0^T (s_1(t) - s_2(t))^2 dt = \int_0^T s_1^2(t)dt + \int_0^T s_2^2(t)\,dt - 2\int_0^T s_1(t)s_2(t)dt$

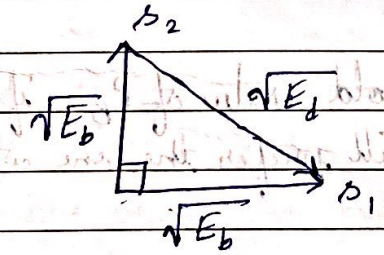
assumption → E_b (bit energy) $= \int_0^T s_1^2(t)\,dt = \int_0^T s_2^2(t)\,dt$

$$\rho = \frac{1}{E_b}\int_0^T s_1(t)s_2(t)\,dt = \cos\theta_{s_1 s_2}$$

$$E_d = 2E_b - 2E_b\cos\theta = 2E_b(1-\cos\theta)$$
$$= 2E_b(1-\rho)$$

$$\therefore P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

examples : ① $\rho = 0$



$E_d = \|s_1 - s_2\|^2$

$E_d = 2E_b$

$$P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \quad \text{— ①}$$

② $\rho = -1$

$E_d = 4E_b$

$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad \text{— ②}$$

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③ unipolar NRZ-L

$$s_1(t) = +A, \quad 0 \le t \le T$$
$$s_2(t) = 0, \quad 0 \le t \le T$$

$$s_2 \xrightarrow{A\sqrt{T}} s_1$$

$$P_B = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right)$$

$$E_b = P(s_1).E_{s_1} + P(s_2).E_{s_2}$$
$$= \frac{1}{2}A^2T \implies P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \quad ③$$

normalized correlation

$$C_{s_1 s_2} = \frac{\int_0^T s_1(t)s_2(t)dt}{\|s_1\|\,\|s_2\|} = \frac{\int_0^T s_1(t)s_2(t)dt}{\sqrt{E_{s_1}}\sqrt{E_{s_2}}}$$

from ①, ② & ③,

$$P_B = f\left(\frac{E_b}{N_0}\right)$$

→ In, $\left(\frac{E_b}{N_0}\right)$, $E_b \to$ bit energy, $N_0 \to$ Noise PSD → dimensionless

→ Given a certain threshold value of P_B, if we achieve it by 2 values of $\frac{E_b}{N_0}$, we will prefer the one which has a lower value of $\frac{E_b}{N_0}$.

→ Reasons for using $\frac{E_b}{N_0}$ in place of SNR :-

① In digital comm., we model waveforms as energy signals whereas in analog comm., we model waveforms as power signals.

② In digital comm., we want to recover the bit & not the waveform. we are concerned with the bits & not the waveforms. We compare the performances of different algorithms / encoding schemes ~~using~~ based on the recovery of bits.

→ M-ary signalling :

s_i, $i = 1, \dots, M$ → waveforms

each waveform represents $\log_2 M$ bits

How to decode these waveforms ?

① use M matched filters.

② Let $S = \text{span}\{s_1, \dots, s_M\}$

Let $B = \{\psi_1, \dots, \psi_N\}$ be an orthonormal basis of S.

$N \leq M$

$$s_i = \sum_{j=1}^{N} c_{ij} \psi_j$$

what if we use N matched filters corresponding to the bases?

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$s_i(t)$, $i = 1, \dots, M$, $0 \leq t \leq T$

subspace ← $S = \text{span}\{s_1(t), \dots, s_M(t)\}$

bases ← $B = \{\psi_1(t), \dots, \psi_N(t)\}$ $N \leq M$

(orthonormal basis (ONB))

$$s_i(t) = \sum_{j=1}^{N} a_{ij}\, \psi_j(t) \quad ; \; i = 1, \dots, M$$

$$s_i(t) \Longrightarrow \begin{pmatrix} a_{i1} \\ \vdots \\ a_{iN} \end{pmatrix} \Longrightarrow a_i \in \mathbb{R}^N$$

$$r(t) = s_i(t) + \underbrace{\eta(t)}_{\text{AWGN}}$$

Does $r(t) \in$ 'S' ?
$\hookrightarrow$ signal space

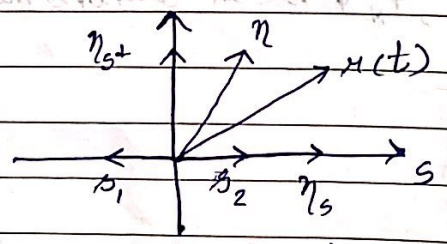
$\longrightarrow$ No, it could be the case that $\eta(t)$ has a component not in the direction of the vectors of signal space.

$$\eta(t) = \eta_S(t) + \eta_{S^\perp}(t)$$

$\eta_S(t)$: component of noise in signal basis

$\eta_{S^\perp}(t)$: component of " " " $\perp^r$ to signal basis

$\eta_{S^\perp}(t) \longrightarrow$ It's projection in the signal plane $= 0$.
$\therefore$ It will not affect it.



$\therefore$ Matched filter is just a projection. $r(t)$ is projected to signal space.
Hence $\eta_{S^\perp}$ will not affect o/p of matched filter.

☆ whatever the noise we consider, we will consider that noise is in the signal space.

$$\therefore \; r(t) = s_i(t) + \eta(t) \; ; \; \eta(t) \in S$$
$\hookrightarrow \because$ it lies in signal space,

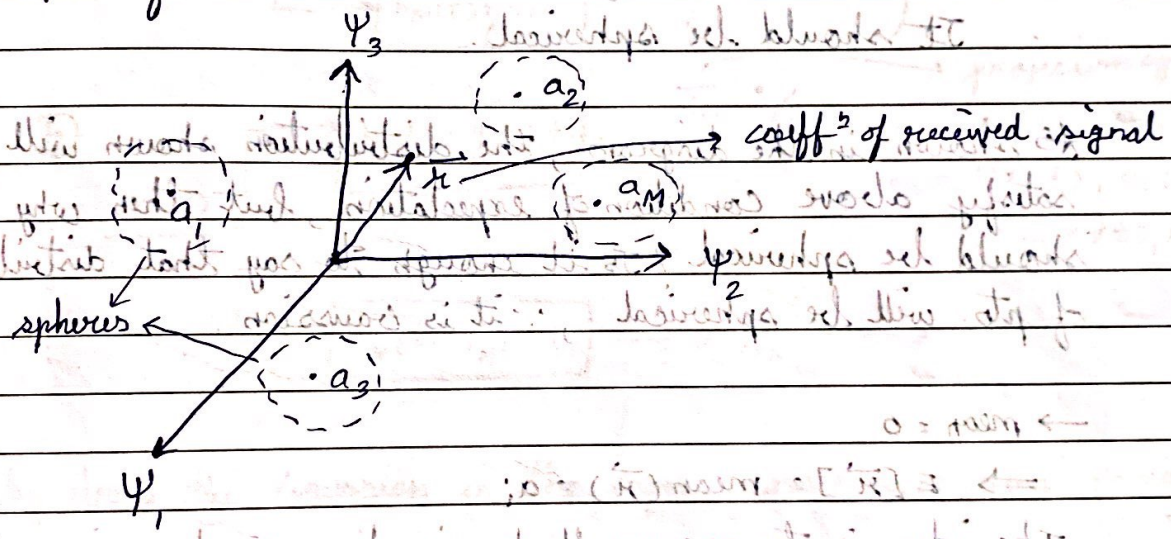
$$\therefore r(t) = \sum_{i=1}^{N} r_i \psi_i(t)$$

$$r(t) \Longrightarrow \begin{pmatrix} r_1 \\ \vdots \\ r_N \end{pmatrix} = r$$

$s_i(t) \longrightarrow$ to be transmitted

$r(t) \longrightarrow$ received signal

we are now representing $s_i(t)$ in 3-D plane in coordinate space of 'r'.



① Let $s_i(t)$ be transmitted,

in ideal case $\Longrightarrow$ no noise,

$$r(t) = s_i(t)$$

$$\& \ \vec{r} = a_i$$

② $r(t) = s_i(t) + \eta(t)$; $\eta(t) \in S$

If we transmit more no. of times, how does $\vec{r}$ look like?

$$r(t) = s_i(t) + \eta(t) \ ; \ \eta(t) \in S$$

$\hookrightarrow$ zero-mean Gaussian

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$$r_1(t) \leftarrow s_i(t) + \eta_1(t)$$

we are sending the same signal, but since $\eta(t)$ is Gaussian, we will get different vectors everytime $\Rightarrow$ we have different realisation everytime & $\therefore$ we are trying to plot all those vectors.

vectors will be equally & symmetrically distributed about (a_i), where,

$$E[\vec{r}] = a_i$$

It should be spherical.

As shown in the diagram, the distribution shown will also satisfy above condition of expectation, but then why it should be spherical? Is it enough to say that distribution of pts. will be spherical, $\because$ it is Gaussian.

$\rightarrow$ mean $= 0$

$\Rightarrow E[\vec{r}] = \text{mean}(\vec{r}) = a_i$

The idea is to assume that noise has equal variance in all direction (ψ_1, ψ_2, ψ_3).

Hence, pts. (a_i) have spherical symmetry around them.

But, can we ~~simply~~ imply that radii at every (a_i) is same?

$\Rightarrow$ Yes, it has to be the same.

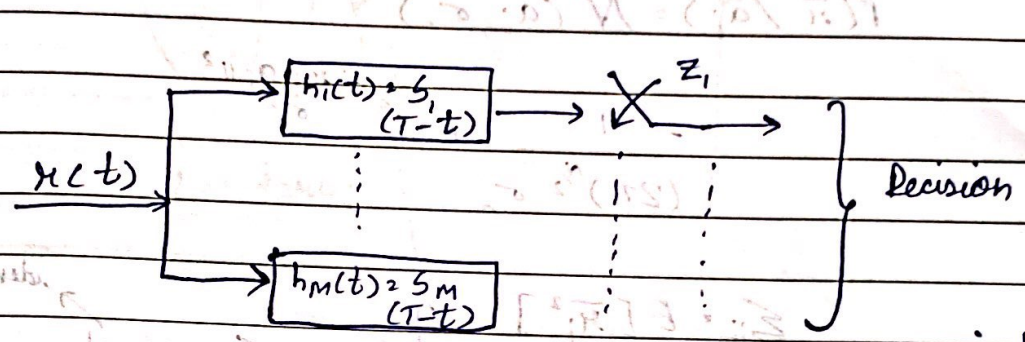
Let $\vec{r} \rightarrow$ represent a particular waveform

→ classification

<u>step 1</u> : Matched filter :

↳ What sort of matched filters we will have here ?

ⓐ



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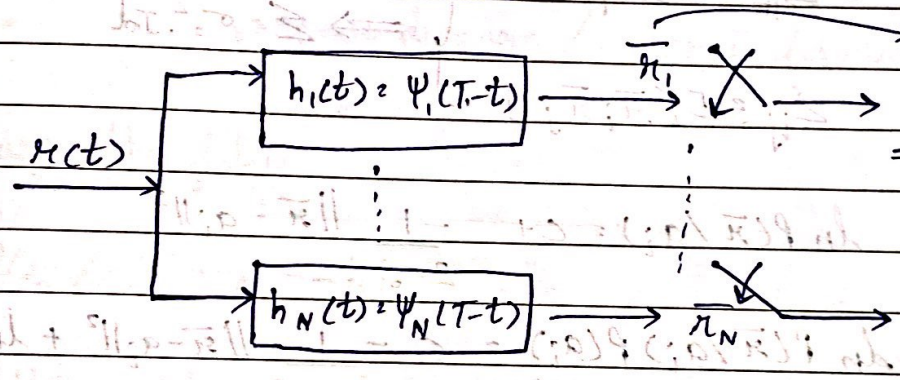

→ projection of $r_1(t)$ along ψ_1

$\Rightarrow \int \psi_1(t) \cdot r(t)\,dt$

$= \langle \psi_1(t), r(t) \rangle$

→ Why does the Gaussian in the signal space have equal variance ?

→ Equivariance of the Gaussian noise in the signal space is due to the "whiteness" of the Gaussian.

$$\eta(t) = \sum n_i \psi_i(t)$$

$$E[n_i^2] = E[\langle \eta(t), \psi_i(t) \rangle^2]$$

$$= E\left[\left(\int_0^T \eta(t)\, \psi_i(t)\, dt\right)^2\right]$$

$$E[n_i n_j] = E[\langle \eta, \psi_i \rangle \langle \eta, \psi_j \rangle]$$

$$= E\left[\int_0^T \eta(t)\, \psi_i(t)\, dt \int_0^T \eta(p)\, \psi_j(p)\, dp\right]$$

$$= E\left[\int_0^T \int_0^T \eta(t)\, \eta(p)\, \psi_i(t)\, \psi_j(p)\, dt\, dp\right]$$

$$= \int_0^T \int_0^T E[\eta(t)\, \eta(p)]\, \psi_i(t)\, \psi_j(p)\, dt\, dp$$

$$\int_0^T \psi_i(t)\, \psi_j(t)\, dt = N_0 \delta(i-j)$$

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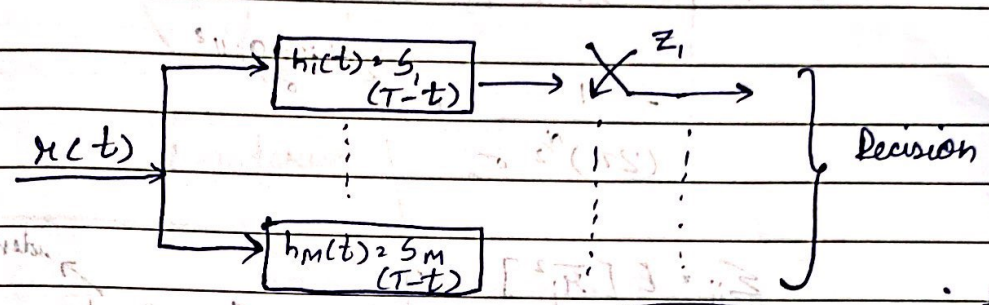
→ classification

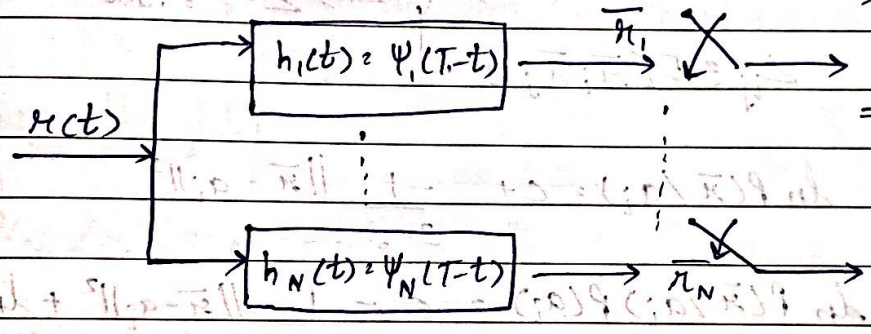
<u>step 1</u> : Matched filter :

↳ what sort of matched filters we will have here ?

(a)



(b)



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$$E[n_i n_j] = E[\langle \eta, \psi_i \rangle \langle \eta, \psi_j \rangle]$$

$$= E\left[\int_0^T \eta(t)\, \psi_i(t)\, dt \int_0^T \eta(p)\, \psi_j(p)\, dp\right]$$

$$= E\left[\int_0^T \int_0^T \eta(t)\, \eta(p)\, \psi_i(t)\, \psi_j(p)\, dt\, dp\right]$$

$$= \int_0^T \int_0^T E[\eta(t)\, \eta(p)]\, \psi_i(t)\, \psi_j(p)\, dt\, dp$$

$$= \int_0^T \int_0^T N_0\, \delta(t-p)\, \psi_i(t)\, \psi_j(p)\, dt\, dp = \int_0^T N_0\, \psi_i(t)\, \psi_j(t)\, dt = N_0\, \delta(i-j)$$

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→ MAP criterion for classification :

$$P(\omega_i/x) = P(a_i/\bar{x}) = \frac{P(\bar{x}/a_i)P(a_i)}{P(\bar{x})}$$

(Posterior prob.) → likelihood → prior

$$P(\bar{x}/a_i) = N(a_i, \sigma_0)$$

$$= \frac{1}{(2\pi)^{N/2}\sigma_0} e^{-\frac{1}{2}\frac{\|\bar{x}-a_i\|^2}{\sigma_0^2}}$$

$$\Sigma_{ii} = E[\bar{x}_i^2]$$

$$\Sigma_{ij} = E[\bar{x}_i \bar{x}_j]$$

$$\Longrightarrow \Sigma = \sigma_0^2 . I_d$$

identity matrix

$$\ln P(\bar{x}/a_i) = c + \frac{-1}{2\sigma_0^2}\|\bar{x}-a_i\|^2$$

$$\ln P(\bar{x}/a_i)P(a_i) = c' - \frac{1}{2\sigma_0^2}\|\bar{x}-a_i\|^2 + \ln P(a_i)$$

$$\sim -\frac{1}{2\sigma_0^2}\|\bar{x}-a_i\|^2 + \ln P(a_i)$$

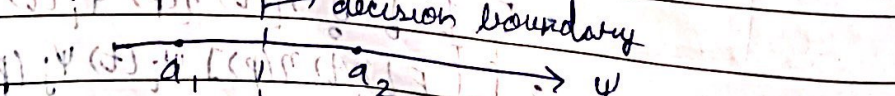
$$\sim \underbrace{2\sigma_0^2 \ln P(a_i)}_{K_i} - \|\bar{x}-a_i\|^2$$

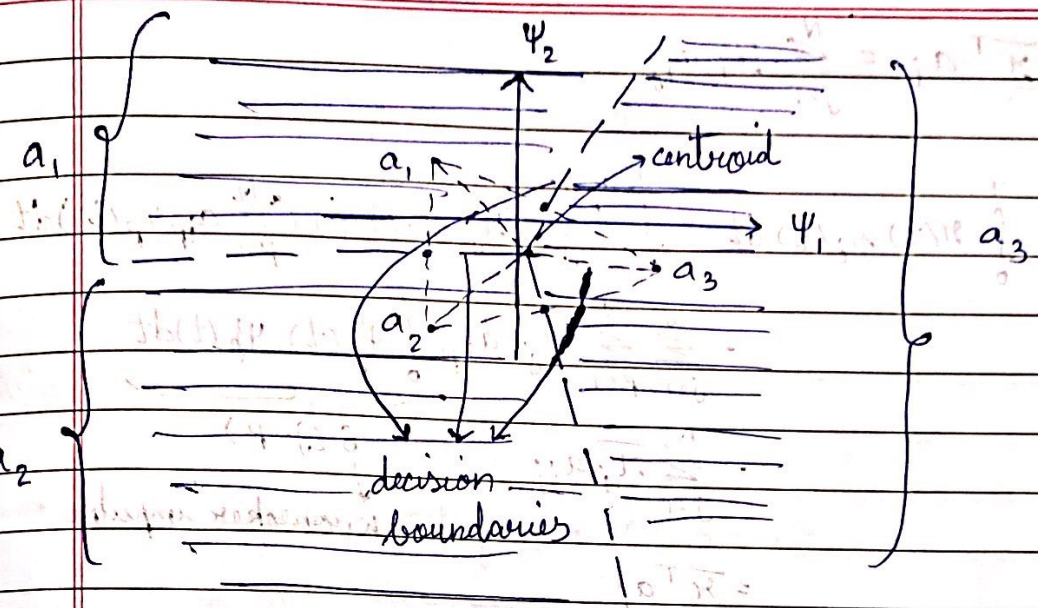
Assuming $P(a_i) = \frac{1}{M}$

$$\boxed{\text{MAP decision} : i^* = \arg\min \|\bar{x}-a_i\|^2, \quad i = 1, \ldots, M}$$

if the prior probabilities are not same, we will need to consider K_i

$$\ln P(\omega_i/x) \sim K_i - \|\bar{x}-a_i\|^2$$





voronoi diagram / voronoi tesselation

→ Designing the filter

$$\ln P(s_i / r) \sim K_i - \|\bar{r} - a_i\|^2$$

$$\|\bar{r} - a_i\|^2 = \|\bar{r}\|^2 - \|a_i\|^2 + 2\bar{r}^T a_i$$

we can ignore this, ∵ it will be same for all s_i

$$\therefore \ln P(s_i/r)\, P(s_i) \sim K_i - \|a_i\|^2 + 2\bar{r}^T a_i$$

$$\sim \underbrace{\frac{K_i - \|a_i\|^2}{2}}_{q_i} + \bar{r}^T a_i$$

$$\sim q_i + \bar{r}^T a_i$$

known at the receiver, ∵ it only depends on the symbols

$$\bar{r}^T a_i = \sum_{j=1}^{N} \bar{r}_j a_{ij}$$

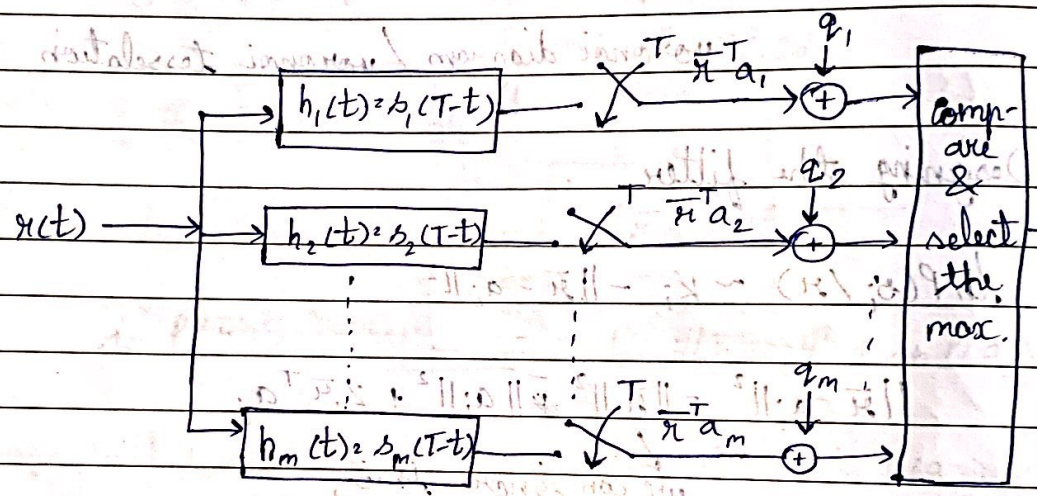
$$\int_0^T r(t)\, s_i(t)\, dt = \int_0^T \left(\sum_{j=1}^{N} \bar{r}_j \psi_j(t) \right) \left(\sum_{p=1}^{N} a_{ip} \psi_p(t) \right) dt$$

$$= \sum_{j=1}^{N} \sum_{p=1}^{N} \bar{r}_j a_{ip} \underbrace{\int_0^T \psi_j(t)\, \psi_p(t)\, dt}_{\delta(j-p)}$$

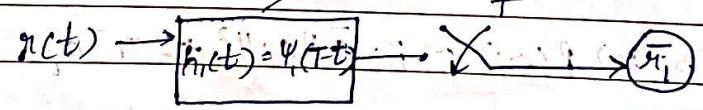
$\delta(j-p)$ → Kronecker impulse

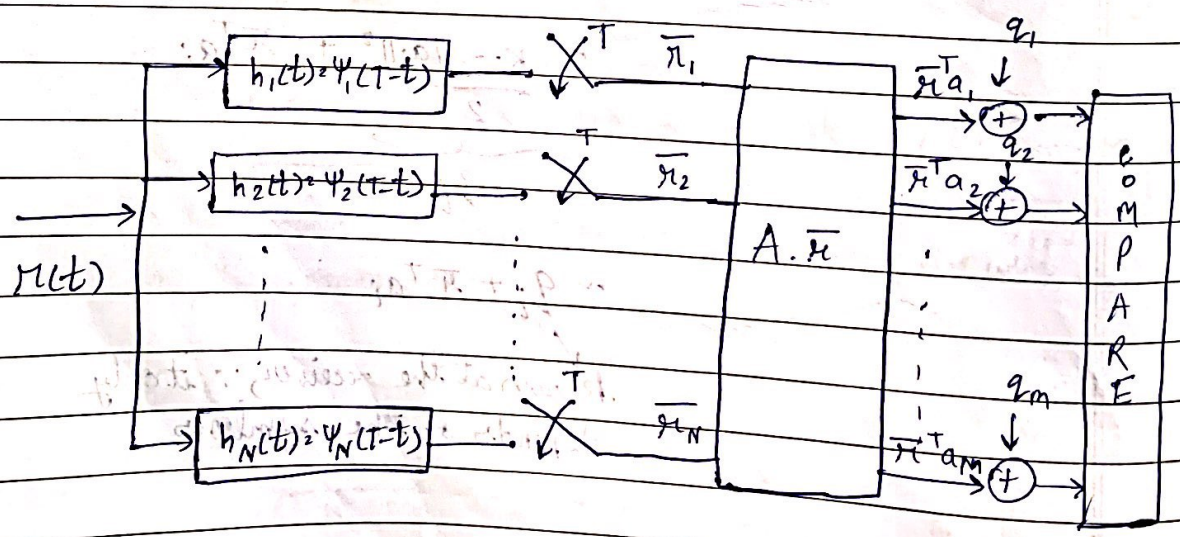
$$= \sum_{j=1}^{N} \bar{r}_j a_{ij}$$

$$= \bar{r}^T a_i$$



filter is just projecting the received signal onto an axis.





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$$A = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \vdots \\ a_{M1} & \cdots & a_{MN} \end{pmatrix}$$

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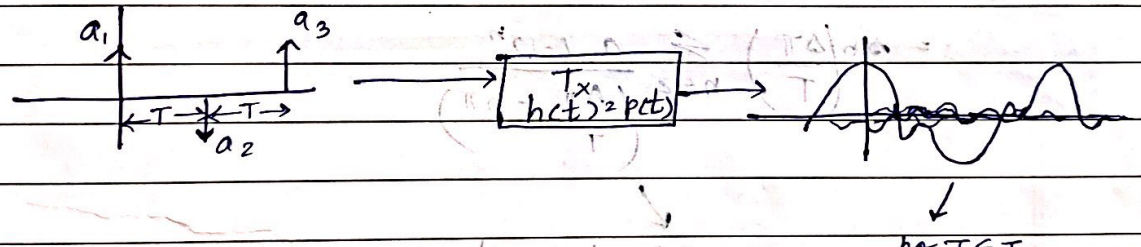
→ Inter Symbol - Interference (ISI):

Theoretically, we want to transmit symbols at the rate $\frac{1}{T}$ / sec
B.W. req. → min. → $\frac{R}{2}$ Hz.

$$p(t) = \text{sinc}\left(\frac{t}{T}\right) \text{ or } \text{sinc}(tT)$$



no ISI
(under the assumption that B.W. = $R/2$ & no noise)

∵ The receiver is only concerned about the amplitude of the waveforms at every T seconds, only the amplitude of the sinc wave at that moment matters, we will keep the amplitudes of other sinc waves at 0 at that moment.

$$x(t) = \sum_{n \in \mathbb{Z}} a_n \, \text{sinc}(t - nT)$$

$$x(nT) = a_n$$

but, one of the problems of digital comm. is synchronization.

$$x(\Delta) = \sum_{n \in Z} a_n \operatorname{sinc}(\Delta - nT)$$

$$= \sum_{n \in Z} a_n \frac{\sin\left(\left(\frac{\Delta - nT}{T}\right)\pi\right)}{\left(\frac{\Delta - nT}{T}\right)\pi}$$

$$= \sum_{n \in Z} a_n \left[\frac{\sin\left(\frac{\Delta\pi}{T}\right)\cos(n\pi) - \cos\left(\frac{\Delta\pi}{T}\right)\sin(n\pi)}{\left(\frac{\Delta - nT}{T}\right)\pi}\right]$$

$$= \sum_{n \in Z} a_n \left[\frac{(-1)^n \sin\left(\frac{\Delta\pi}{T}\right)}{\left(\frac{\Delta - nT}{T}\right)\pi}\right]$$

$$= \sin\left(\frac{\Delta\pi}{T}\right) \sum_{n \in Z} \frac{a_n (-1)^n}{\left(\frac{\Delta\pi}{T} - n\pi\right)}$$

$\because \sum \frac{1}{n}$ does not converge, $\therefore$ we need to keep a_n within a certain limit to avoid ISI, also, there is no limit on the value of Δ.

$\because \sum \frac{1}{n^2}$ converges, we can try to convert the expression to such a form.

$\sum \frac{1}{n}$ is due to sinc( ),

$$\because \operatorname{sinc}(t) = \frac{\sin(t)}{t}$$

$\propto \frac{1}{t} \longrightarrow$ we want a fun^c which decays faster $\left(\frac{1}{t^2}\right)$

$\therefore$ we want, $p(t) \propto \frac{1}{t^2}, \frac{1}{t^3}, \ldots$

$p(t) \longrightarrow$ no ISI, $t = nT$

$$\left.\begin{aligned} p(t) &= 1, \quad t = 0 \\ &= 0, \quad t = nT \end{aligned}\right\}$$

$$\boxed{p(t) = \sum_{n \in \mathbb{Z}} \delta(t - nT) = \delta(t)}$$

$$\text{F.T.} \rightarrow \boxed{\sum_{n \in \mathbb{Z}} P(f - nR) = T}, \quad R = \frac{1}{T} \text{ Hz}$$

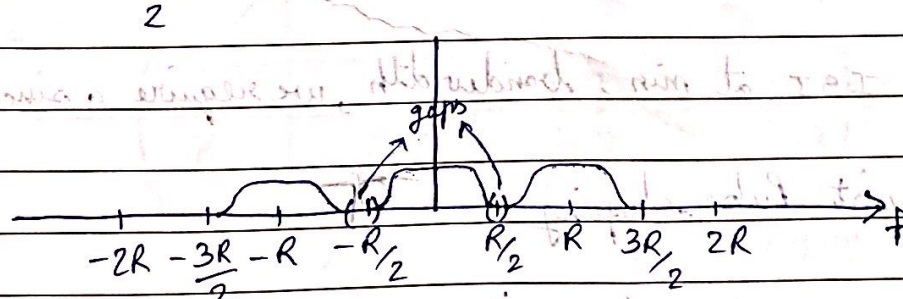
this is the eq^n which the F.T. of the pulse selected should look like if we want no ISI (Nyquist Pulse shape / Filter eq^n)

$\rightarrow \sum_{n \in \mathbb{Z}} P(f - nR) = T, \quad R = \frac{1}{T}$

$P(f - nR) \begin{cases} |T| & \leq \frac{R}{2} \\ 0, & \end{cases}$

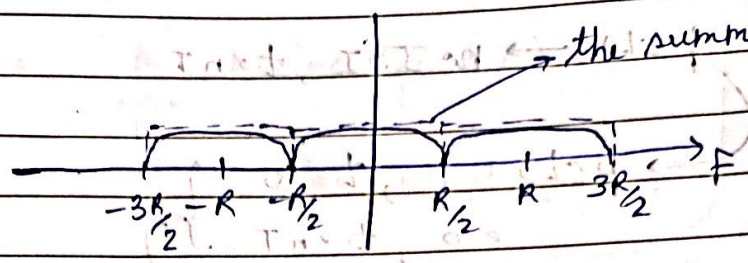
$\rightarrow$ Let B.W. of $P(f) = B$ Hz

① $B < \frac{R}{2}$



$\Rightarrow B < R/2$ not possible

② $B = \frac{R}{2}$



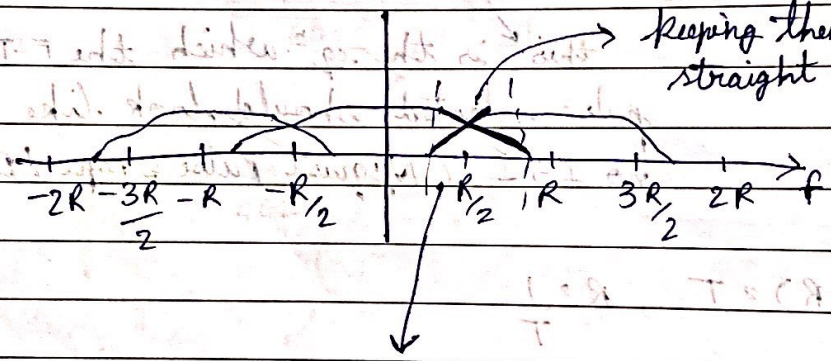
the summation will equal T when it will be a rectangular pulse.

$$\therefore P(f) = \text{rect}\left(\frac{f}{R}\right)$$

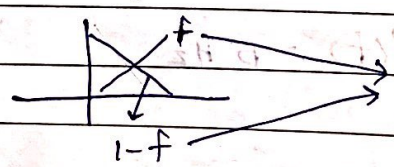
$B \geq R/2$

$$P(t) = \text{sinc}\left(\frac{t}{T}\right) \text{ or } \text{sinc}(Tt)$$

③ $B > \frac{R}{2}$



Keeping them to be straight lines will allow the summation to equal 1 (there are many other possibilities)



sum to 1.

$\therefore$ for no ISI at min. bandwidth, we require a sinc funⁿ.

→ eg:- Nyquist Pulse shaping

① Raised cosine pulse / filter

$$p(t) = R\,\text{sinc}(Rt)\,\frac{\cos\left(2\pi\left(\bar{R} - R/2\right)t\right)}{1 - \left[4\left(\bar{R} - R/2\right)t\right]^2}$$

it will decay by a rate of t^3.

$$P(f) = \begin{cases} 1, & |f| < R - \bar{R} \\ \cos^2\left(\frac{\pi}{4}\left(\frac{|f| + \bar{R} - R}{\bar{R} - R/2}\right)\right), & R - \bar{R} \le |f| \le \bar{R} \\ 0, & |f| \ge \bar{R} \end{cases}$$

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$\bar{R}$ = absolute bandwidth, $R = \frac{1}{2T}$ → min. Nyquist bandwidth,

$\bar{R} - R$ = excess bandwidth

→ $\bar{R}$ = B.W. of $p(t)$

According to the above 3 conditions,

$\bar{R} \ge \frac{R}{2}$ (to satisfy Nyquist pulse shaping fun$^{\underline{n}}$).

$R \ge \bar{R}$

↓

Symbol rate

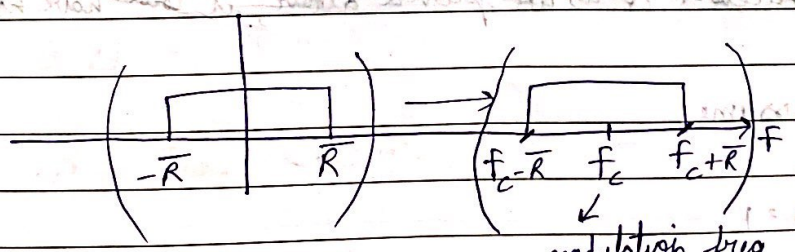
r → roll-off factor / ratio

$$r = \frac{\bar{R} - R/2}{R/2}$$

$$\underline{B.W.} \text{ of } p(t) = \bar{R} = r\frac{R}{2} + \frac{R}{2} = \frac{R}{2}(1+r)$$

↓

Baseband B.W.



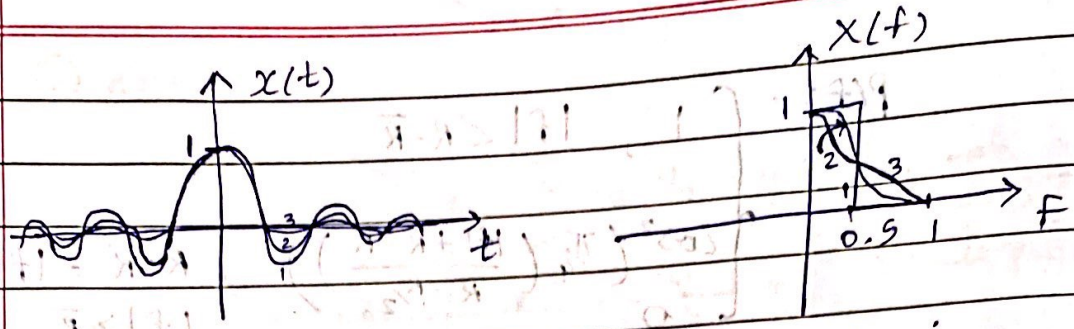
B.W. = $\bar{R}$

↓

modulation freq.

B.W. = $2\bar{R}$

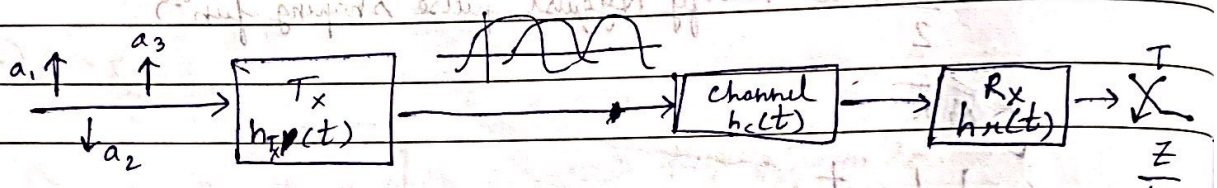
∴ Bandpass B.W. ($p(t)$) = $R(1+r)$

→ 

Raised ~~cosu~~ cosine (time domain)

Raised cosine (freq. domain)

3 is best ∵ the decay is fastest and ~~which~~ it will make it more resilient to time jitters, but, the tradeoff is that we will require higher bandwidth.

→ 

we want that,

we have to ensure no ISI is here

$$H_{T_x}(f)\, H_c(f)\, H_R(f) = H_{RC}(f)$$

raised cosine

If it is achieved, we will have only raised cosine waveforms at the receiver & as we proved above, it will have no ISI.

Let's assume,

$$H_c(f) = 1$$

$$\therefore H_{T_x}(f)\, H_{R_x}(f) = H_{RC}(f)$$

we can assume, $|H_{T_x}(f)| = |H_{R_x}(f)|$

$$\therefore \underbrace{|H_{T_x}(f)| = |H_{R_x}(f)|}_{\text{square root RC filters/waveforms.}} = \sqrt{|H_{RC}(f)|}$$

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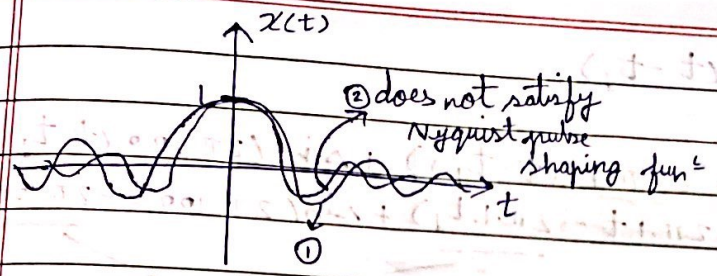

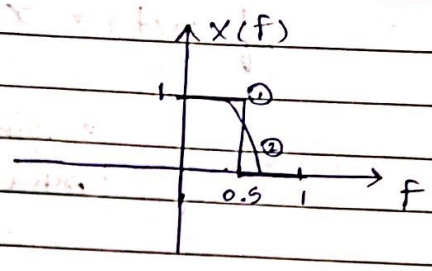

Root - RC
(time domain)

Root - RC
(freq. domain)

→ **Channel:**

Assuming, we can model the channel as an LTI system,

$$H_c(f) = \underbrace{|H_c(f)|}_{\text{magnitude response}} \, e^{j\underbrace{\theta_c(f)}_{\text{phase response}}}$$

In the ideal case,

① $|H_c(f)| = K, \forall f \longrightarrow$ ideal case

$\neq K, \longrightarrow$ amplitude distortion

② $\theta_c(f)$:

Let us take $|H_c(f)| = 1$

$$x(t) = \sin(2\pi.1.t) + \sin(2\pi.100.t)$$

$$y(t) \sim \sin(2\pi.1.t + \theta_c(1)) + \sin(2\pi.100.t + \theta_c(100))$$

what sort of $y(t)$ would be acceptable?

- we will see how $\theta_c(f)$ affects each signal (phase-freq. x time)

if $y(t) = x(t - t_d)$,

$$= \sin(2\pi \cdot 1 \cdot (t - t_d)) + \sin(2\pi \cdot 100 \cdot (t - t_d))$$

$$= \sin(2\pi \cdot 1 \cdot t - \underline{2\pi \cdot 1 \cdot t_d}) + \sin(2\pi \cdot 100 \cdot t - \underline{2\pi \cdot 100 \cdot t_d})$$

$\theta_c(f)$

$\therefore \theta_c(\omega) = -2\pi f t_d$

$\theta_c(f) = -f t_d$

linear

$\therefore$ Systems which have linear phase responses would not cause any distortion, but only a delay. (delay distortion)

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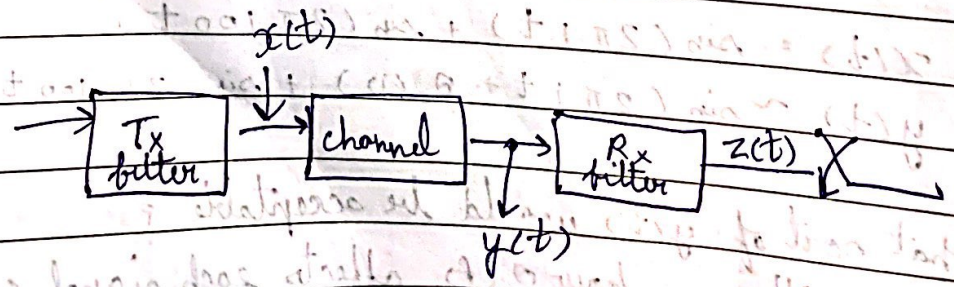
→ Non-linear phase response results in phase distortion. Human ears are not much sensitive to phase distortion & non-linear phase response results in a better magnitude response than a linear phase response.
Human eye is very sensitive to phase distortion.

→ $\therefore$ 3 kinds of distortions can be produced by the channel:

1. Amplitude
2. Phase
3. Delay.

→



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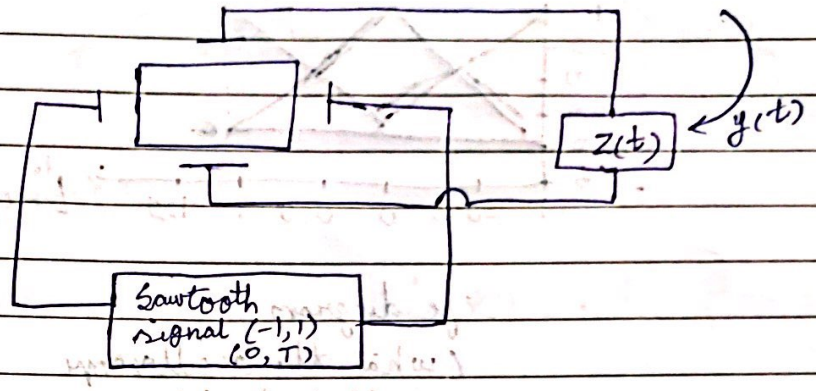
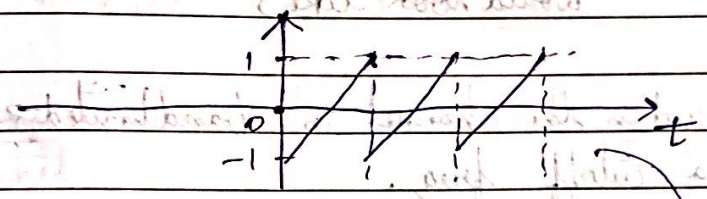


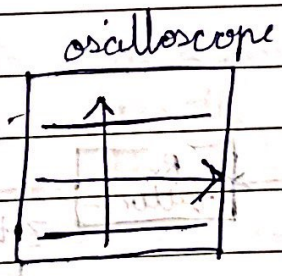


① $H_c(f) = 1, \forall f$

② $x(t) \longrightarrow$ Binary PCM, NRZ-L

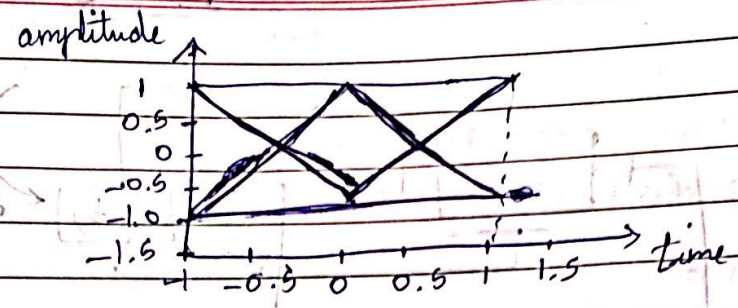
③ No noise

For $y(t)$,



→ Eye diagram

- Binary PCM, NRZ-L
- No Noise.



eye diagram
(what the oscilloscope would look like)

Eye diagram when the channel is bandlimited?
i.e., it has a cutoff freq.

As the bandwidth of the channel decreases, smaller is the "opening" of the eye & more is the ISI.
Noise margin?

More the noise, smaller is the opening of the eye.
The width of the eye diagram around amplitude value 0 ~~st~~ indicates the amount of jitter that can be introduced.

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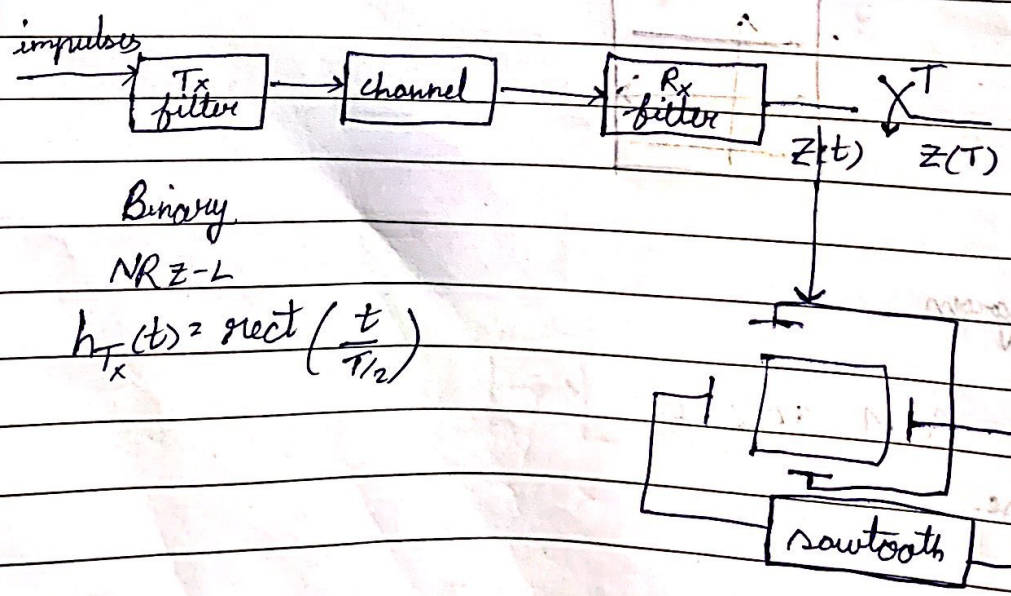
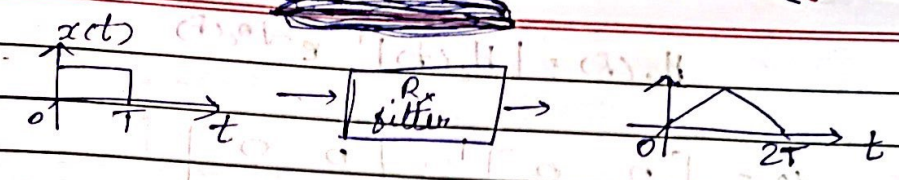
→



Binary
NRZ-L

$h_{T_x}(t) = \text{rect}\left(\frac{t}{T/2}\right)$



→ channel equalization :

$H_{Tx}(f) \cdot H_{Rx}(f) = H_{RC}(f)$, $t = kT$, $k \in \mathbb{Z}$ (NO ISI)

$H_e(f) H_{Tx}(f) H_c(f) H_{Rx}(f) = H_{RC}(f)$

↓

channel equalisation filter, we hope to filter out the noise from the channel.

$$H_c(f) = |H_c(f)| e^{+j\theta_c(f)}$$

$$H_e(f) \approx \frac{1}{|H_c(f)|} e^{-j\theta_c(f)}$$

How to design the channel equalization filter ?
- given that :
The amplitude & freq. response of the channel is known.

→ If $H_c(f)$ is 0 at a particular value, it means that the eigenvector is 0 & the determinant would be 0, which mean that $H_e(f)$ would not be defined.
It will rarely be the case that the channel ($H_c(f)$) would be invertible.

$Ax = b$, A is not invertible
∴ we will find the pseudoinverse.

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$$H_e(f) = |H_e(f)|^{+} e^{-j\theta_e(f)}$$

eg:- $\begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}^{+} = \begin{bmatrix} 0 & 0 \\ 0 & 1/10 \end{bmatrix}$

pseudoinverse will not be a good sol^{n} $\because$ if one of the elements of the matrix is huge, & we multiply it by b, we will get an even bigger amount of noise.

→ ill-posed problems

$Ax = b$

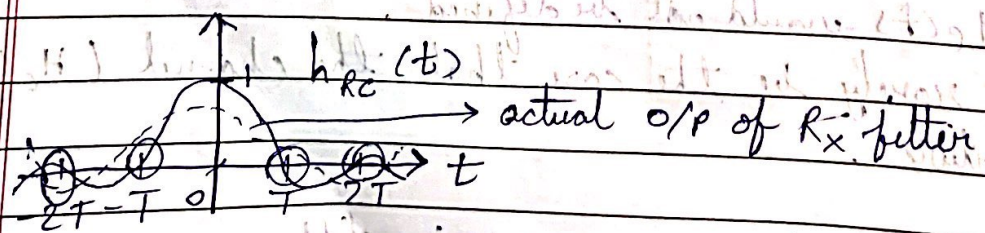
→ well-posed problem

1. There is a sol^{n}
2. The sol^{n} is unique
3. The sol^{n} continuously varies with the given data.

$Ax = b$ does not satisfy this for bigger values when it is non-invertible.

→ Regularization methods

channel equalization



at multiples of T, we needed the value to be 0 & at $t=0$ to be 1.

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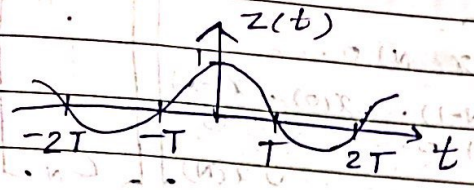
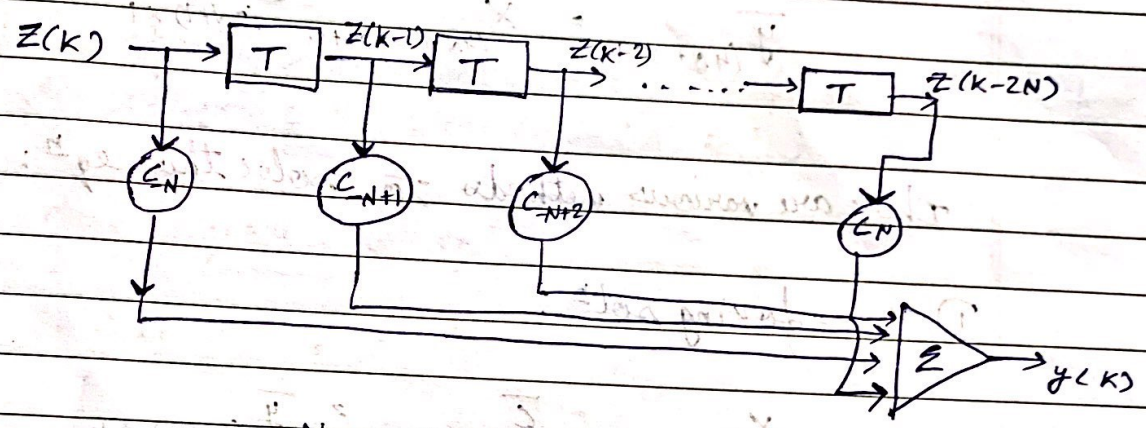
→



$z(k) := z(kT)$

$\hookrightarrow \delta \longrightarrow$ filter (transversal)



$$y(k) = \sum_{n=-N}^{N} C_n \, x(k-n)$$ (we can say that it's a discrete convolution)

$y(k)$ → desired output

$x(k-n)$ → corrupted with ISI (o/p of MF)

C_n → determine C_n such that we get the desired output (C_n = filter coeff^{n})

Let's solve it by imposing it as a problem of the form,

$$Ax = b$$

$$y(-2N) \propto \{x(-3N), \ldots, x(-N)\}$$

we will assume that after a certain 'N', there would be no ISI.

$$x(k) = 0, \quad k \notin [-N, N]$$

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$$\begin{bmatrix} y(-2N) \\ \vdots \\ y(0) \\ \vdots \\ y(2N) \end{bmatrix} = \underbrace{\begin{bmatrix} x(-N) & 0 & \cdots & & 0 \\ x(-N+1) & x(-N) & 0 & \cdots & 0 \\ \vdots & & & & \\ x(N) & x(N-1) & \cdots & x(0) \cdots & x(-N) \\ 0 & x(N) & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & 0 & x(N) \end{bmatrix}}_{\text{Toeplitz matrix}} \begin{bmatrix} c_{-N} \\ \vdots \\ c_N \end{bmatrix}$$

$$\bar{y}_{(4N+1)\times 1} = X_{(4N+1)\times(2N+1)} \; \bar{c}_{(2N+1)\times 1}$$

There are various methods to solve this eq^n:

① Zero-forcing sol^n:

$$X_{(4N+1)\times(2N+1)} \; \bar{c}_{(2N+1)\times 1} = \bar{y}_{(4N+1)\times 1}$$

In the toeplitz matrix, we ignore the top & bottom N eq^{n}s

$$X_{1\,(2N+1)\times(2N+1)} \; \bar{c} = \bar{y}_{1\,(2N+1)\times 1}$$

$$\bar{y}_1 = \begin{bmatrix} 0 \leftarrow (-N) \\ \vdots \\ 1 \leftarrow (0) \\ \vdots \\ 0 \leftarrow (N) \end{bmatrix}$$

$$\bar{c} = X_1^{-1} \bar{y}_1$$

Here, we have completely ignored the noise. If we take a new measurement, we will get new values of $\bar{c}$. We want the values to be stable atleast for a certain amount of time.

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$\{x(k)\} \leftarrow \underset{T_x \text{ filter}}{p(t)}, \underset{R_x \text{ filter}}{p(t)}, \underset{\text{channel IR}}{h_c(t)}$, $H_c(f_0) = 0$

Spectral Nulls

Let's say that X_1 is diagonalizable by the F.T. matrix,

$$X_1 = F \Lambda F^{-1}$$

$$X_1^{-1} = F \Lambda^{-1} F^{-1}$$

$\therefore$ If the channel has spectral nulls, coeff^n of $\bar{c}$ will be huge if the eigenvalues are close to 0 ($\Lambda^{-1} \approx \frac{1}{0}$).

(2) Min. mean-square error eq^n.

$$\bar{y}_{(4N+1)\times 1} = X_{(4N+1)\times(2N+1)} \, \bar{c}_{(2N+1)\times 1}$$

$$X^T \bar{y} = X^T X \bar{c}$$

$$X^T X = \underbrace{\begin{bmatrix} \sum_{k=-N}^{N} x_k^2 & & & \\ & \sum_{k=-N}^{N} x_k^2 & & \\ \sum_{k=-N+1}^{N+1} x_k x_{k-1} & & & \\ \sum x_k x_{k-2} & & & \\ & & & \\ & & & \sum_{k=-N}^{N} x_k^2 \end{bmatrix}}_{(2N+1)\times(2N+1)}$$

this matrix is R_{xx} under the assumption that

$X^T \bar{y} = r_{x\bar{y}}$ (cross correlation)

$\therefore r_{x\bar{y}} = R_{xx} \bar{c}$

$\bar{c} = R_{xx}^{-1} r_{x\bar{y}}$

we are still relying on just 1 measurement, but by making certain assumptions, R_{xx} gives us a kind of statistical summary.

$\therefore C_{-N}, \ldots, C_N$ are adapted rather than being constant.

1/10/19

Chapter - 4

→ we need to use high freq. as:

1) Length of antenna is directly proportional to wavelength.

2) Freq. multiplexing can be done.

$c(t) = A\cos(\omega_c t)$

carrier signal

We can modulate amplitude, phase & freq.

- amplitude → Amplitude shift keying (ASK).
- phase → Phase Shift Keying (PSK)
- freq. → Freq. shift Keying (FSK)

Amplitude Phase shift Keying (APK).

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$X^T \bar{y} = r_{xy}$ (cross correlation)

$\therefore r_{x\bar{y}} = R_{xx} \bar{c}$

$\bar{c} = R_{xx}^{-1} r_{x\bar{y}}$

we are still relying on just 1 measurement, but by making certain assumptions, R_{xx} gives us a kind of statistical summary.

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1/10/19

Chapter - 4

→ we need to use high freq. as:

1. Length of antenna is directly proportional to wavelength.
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Amplitude Phase shift Keying (APK)

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$$c(t) = \underset{\text{peak amp.}}{A} \cos(\omega_0 t)$$

$$= \sqrt{2 A_{rms}^2} \cos(\omega_0 t)$$

$$P = \frac{A_{rms}^2}{1}$$

$$\therefore c(t) = \sqrt{2P} \cos(\omega_0 t)$$

$$= \sqrt{\frac{2E}{T}} \cos(\omega_0 t)$$

(i) ASK

$$s_i(t) = \sqrt{\frac{2E_i}{T}} \cos(\omega_0 t), \quad 0 \le t \le T, \quad i = 1, \ldots, M$$

(ii) FSK

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t), \quad 0 \le t \le T, \quad i = 1, \ldots, M$$

(iii) PSK

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \phi_i(t)), \quad 0 \le t \le T, \quad i = 1, \ldots, M$$

$$\phi_i = \frac{2\pi}{M}(i-1)$$

eg:- $M = 4$

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_0 t + \frac{2\pi}{M}(i-1)\right), \quad 0 \le t \le \pi$$

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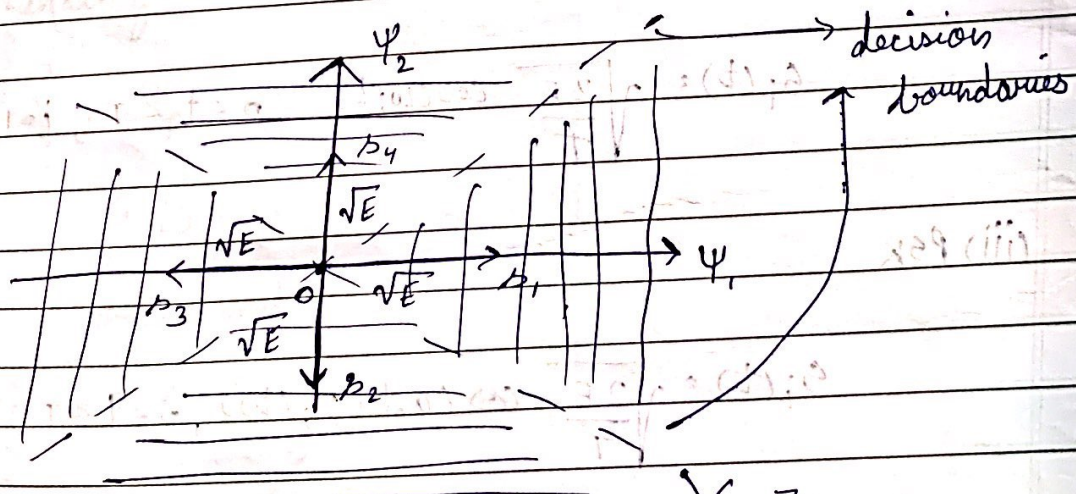
$$S_i(t) = \sqrt{\frac{2E}{T}}\left(\cos(\omega_0 t)\cos\left(\frac{2\pi}{M}(i-1)\right) - \sin(\omega_0 t)\sin\left(\frac{2\pi(i-1)}{M}\right)\right), \quad 0 \le t \le T$$

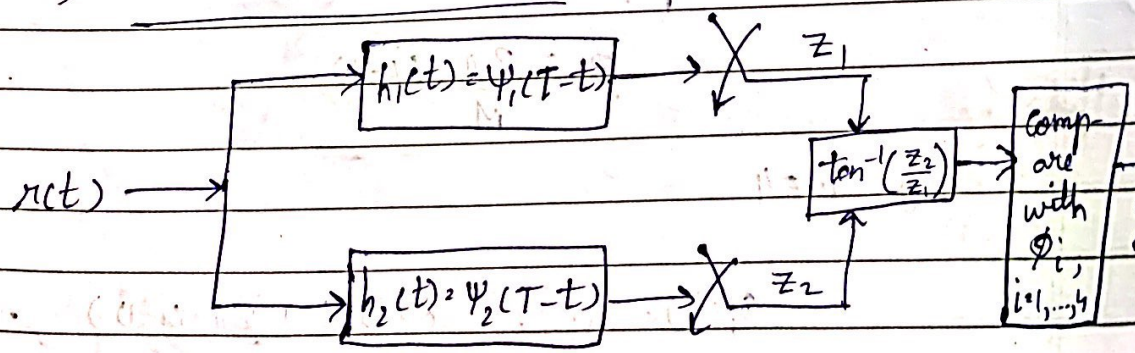
we can take these 2 as bases ∵ these 2 would be constant for a particular i

$\therefore$ Let $\psi_1(t) = \sqrt{\frac{2}{T}}\cos(\omega_0 t)$

$\psi_2(t) = \sqrt{\frac{2}{T}}\sin(\omega_0 t)$

$$S_i(t) = \sqrt{E}\left(\cos\left(\frac{2\pi}{M}(i-1)\right)\psi_1(t) - \sin\left(\frac{2\pi}{M}(i-1)\right)\psi_2(t)\right)$$





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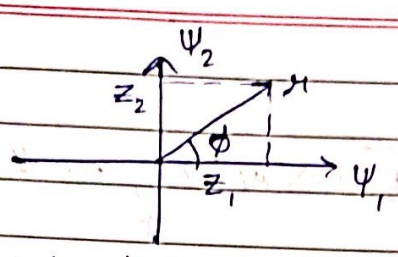
$d = 5t$

$3 \times 10^8 \times 0.25 \times 10^{-1}$

0.75×10^{-1}

$7.5 \quad 0.075$



we only need to find $\tan^{-1}\left(\frac{z_2}{z_1}\right)$ / angle to determine the type of signal & not the amplitude.

PSK: $s(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t)$, $\omega_0 = 2\pi f_0$ $\left(h(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t)\right)$

$$r(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 (t - t_d))$$

$$= \sqrt{\frac{2E}{T}} \cos(\omega_0 t - \omega_0 t_d)$$

let's say, $f_0 = 10^9$ Hz

if $\omega_0 t_d = \pi/2$,

$$2\pi f_0 t_d = \frac{\pi}{2}$$

$$t_d = 0.25 \times 10^{-9} \text{ sec.}$$

dist ~ 7 cm

$\therefore$ To distort the received signal by $\pi/2$, we only need to move about 7cm (eg:- while talking on phone).

$\therefore$ Instead of keeping the impulse response of the receiving filters fixed, we can keep track of the phase change happening due to the movement / relative distance b/t receiver & transmitter, & change the impulse response accordingly.

→ How do we keep track of the changes?
– Phase locked loop.

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→ FSK:

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t), \quad 0 \le t \le T, \quad i = 1, \dots, M$$

$$\psi_j(t) = \sqrt{\frac{2}{T}} \cos(\omega_j t), \quad j = 1, \dots, N, \quad N \le M$$

is this basis orthogonal?

$$\langle \psi_i, \psi_j \rangle = \frac{2}{T} \int_0^T \cos(\omega_i t) \cos(\omega_j t)\, dt$$

$$= \frac{2}{T} \int_0^T \cos((\omega_i + \omega_j) t) + \underline{\cos((\omega_i - \omega_j) t)}\, dt$$

want it to be 0 if $i \neq j$ ~~($\omega_{i+1} - \omega_i$ is assumed to be an integral multiple of T)~~

$$= \left. \frac{\sin(\omega_i - \omega_j) t}{\omega_i - \omega_j} \right|_0^T$$

$$\sin((\omega_i - \omega_j) T) = 0$$

$$\omega_i - \omega_j = \frac{K\pi}{T}$$

$$\therefore \omega_{i+1} - \omega_i = \frac{K\pi}{T}, \quad \forall i \quad \text{—— (1)}$$

$\therefore$ (1) needs to be satisfied to have the basis as,

$$\psi_j(t) = \sqrt{\frac{2}{T}} \cos(\omega_j t), \quad j = 1, \dots, M, \quad 0 \le t \le T$$

$$s_i(t) = \sqrt{E}\, \psi_i(t), \quad i = 1, \dots, M$$

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$$d(s_i, s_j) = \sqrt{2E_i} \quad , i \neq j$$
$$= 0 \quad , i = j$$

→ $s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t)$

$r(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \phi)$

$$\langle r, \psi_i \rangle = \int_0^T \sqrt{\frac{2E}{T}} \cos(\omega_i t + \phi) \sqrt{\frac{2}{T}} \cos(\omega_i t) \, dt$$

$$= \frac{\sqrt{E}}{T} \int_0^T \left(\frac{\cos(2\omega_i t + \phi) + \cos(\phi)}{2} \right) dt$$

$$= \frac{\sqrt{E}}{T} T \cos\phi = \sqrt{E} \cos\phi$$

we have the same problem here as we had with PSK.
To solve it, we can change the basis to,

$$\psi_j(t) = \sqrt{\frac{2}{T}} \cos(\omega_j t + \hat{\phi}) \quad , j = 1, \ldots, M, \quad 0 \leq t \leq T$$

This particular problem is solved by PCL (Phase clocked Loop).

→ Non-coherent detection of PSK/FSK.

PSK: DPSK (Differential PSK).

In PSK, each symbol was encoded using an absolute ~~symbol~~ phase. Differential means w.r.t the previous instance. If we want to transmit i^{th} symbol / message,

original → $A \cos(\omega_0 t + \phi)$

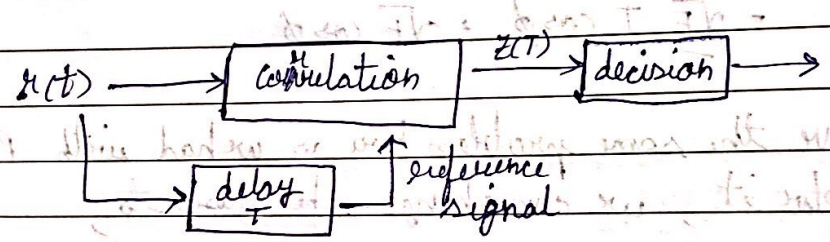
i^{th} message → $A \cos\left(\omega_0 t + \phi + \frac{2\pi}{M}(i-1)\right)$

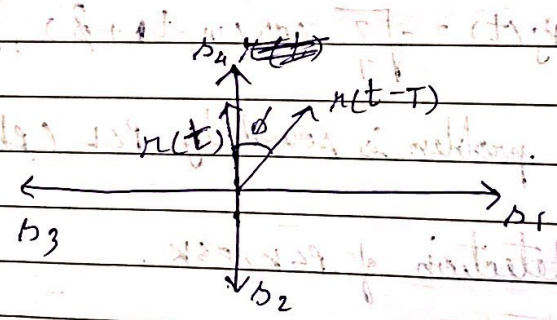
At the receiver, we don't need to detect the absolute phase, we only need to find the difference bt^n the present & previous symbols, ∴ we don't need to use phase detection methods.

$$r(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \phi + \alpha)$$

$$r(t-T) = \sqrt{\frac{2E}{T}} \cos\left(\omega_0 t + \phi + \frac{2\pi}{M}(i-1) + \alpha\right)$$

Here, we are making the assumption that α remains approx. same & does not change much in 'T' sec.

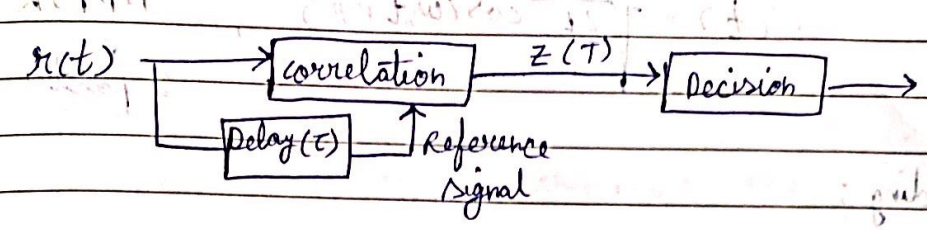




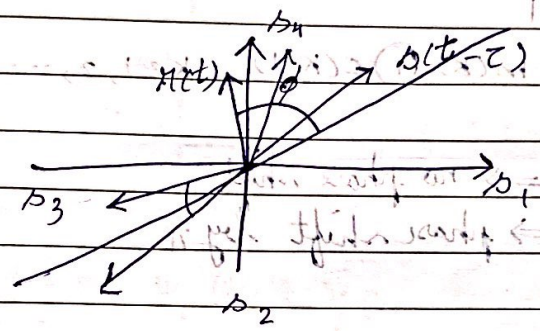
There might be a lot more errors even if noise has not taken the signal across the decision boundary, ∵ we are comparing the received signal with a signal received previously which also had noise.

∴ If we make an error in decision, it may propagate since the next signal's phase is decided from the previous one's.

∴ 2 disadvantages
- Higher error
- error propagation



what if



Phase difference is too small, ∴ it will cause more errors.
If there is an error at time t, then all decisions corresponding to $t+\tau$, $t+2\tau$, ... also may go wrong.

→ Binary DPSK

$m(K) \rightarrow$ binary bit stream / message

$$w(t) = \sqrt{\frac{2}{T}} \cos(\omega_0 t), \quad 0 \le t \le T$$

→ carrier signal

Encode (BDPSK), let $s(t)$ be the current waveform for $m(K-1)^{th}$ bit exists from $(K-2)T \le t \le (K-1)T$

$m(K) \Rightarrow$ if $m(K) = 0$, then $s'(t) = s(t-T)$
no phase shift w.r.t previous signal.

If $m(K) = 1$, $s'(t) = s(t-T)$, phase shifted by π,

$$s'(t) = \sqrt{\frac{2E}{T}} \cos \omega_0 t$$

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$$s'(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \pi)$$, BDPSK ↓ 2 possible waveforms

Precoding :

$m(k) \rightarrow$ bit stream , $k = 1, 2, \ldots$

c : $c(0) = 1$

$c(k) = m(k) \oplus c(k-1)$, $k = 1, 2, \ldots$

if $c(k) = 0 \Longrightarrow$ no phase shift

" $c(k) = 1 \Longrightarrow$ phase shift by π.

→ BPSK vs. BDPSK

BPSK :

① no phase delay

② Phase delay

BDPSK

① Phase delay

→ Non-coherent FSK demodulation

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t) , \quad 0 \le t \le \pi$$

$i = 1, 2, \ldots, M$

$$\psi_i(t) = \sqrt{\frac{2}{T}} \cos(\omega_i t) , \quad i = 1, 2, \ldots, M$$

$$\omega_{i+1} - \omega_i = \frac{k\pi}{T} \Longrightarrow \text{ONB}$$

orthogonal FSK

If signals are not orthogonal, errors would increase.

M signals $\rightarrow$ M basis $\rightarrow$ M matched filters

If not orthogonal, then more than 1 matched filter would give non-zero output. otherwise, when ONB, then just 1.

$$r(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \phi)$$

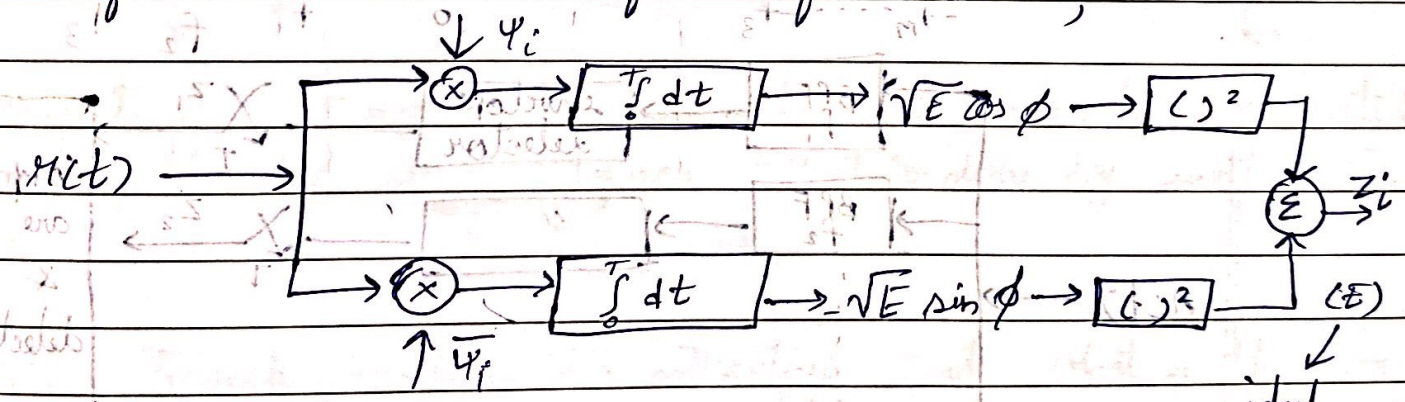
$$= \sqrt{\frac{2E}{T}} \left(\cos(\omega_i t) \cos\phi - \sin(\omega_i t) \sin\phi \right)$$

$$= \sqrt{E} \left(\cos\phi \, \psi_i(t) - \sin\phi \, \bar{\psi}_i(t) \right)$$

where $\psi_i(t) = \sqrt{\frac{2}{T}} \sin(\omega_i t)$

$$\cos\phi \propto \frac{1}{\sin\phi}$$

If we use 2 matched filters for detection,

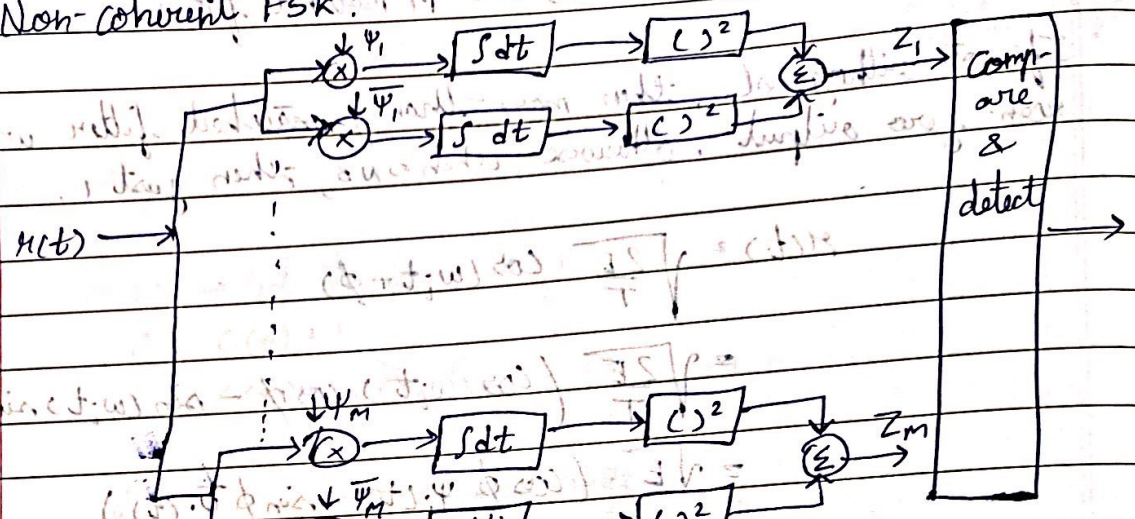


ideal case

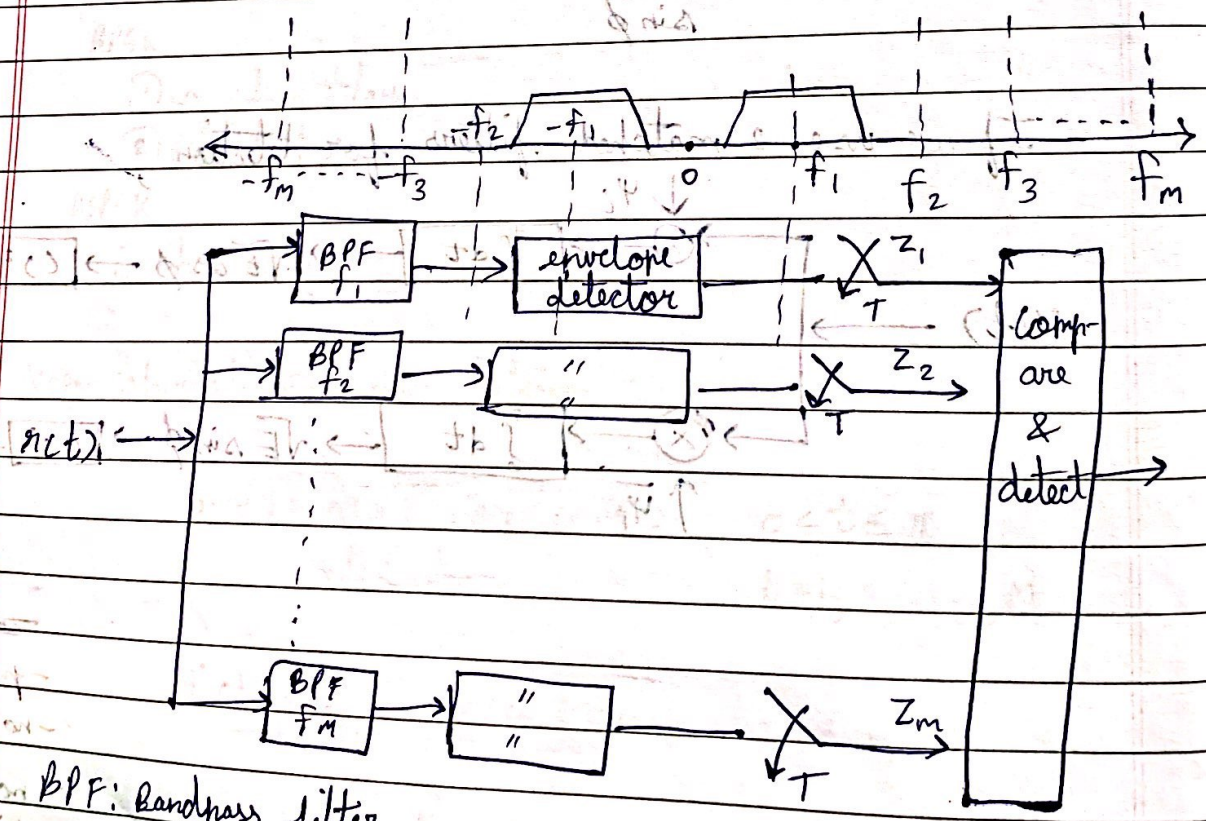
~~no phase~~

- phase delay
- no noise
- no amplitude distortion

→ Non-coherent FSK:

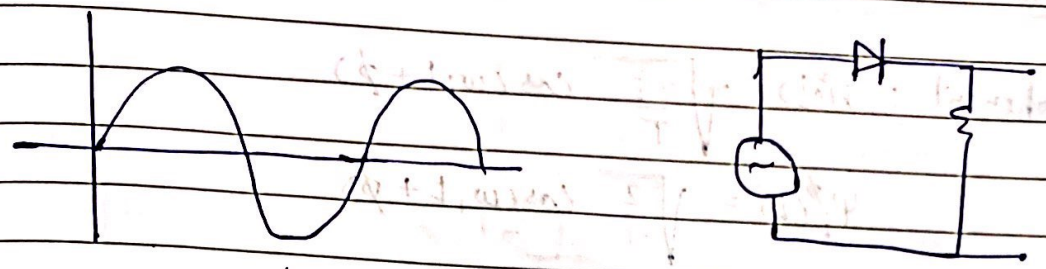


or



BPF: Bandpass filter
Centre freq. f_i & $-f_i$

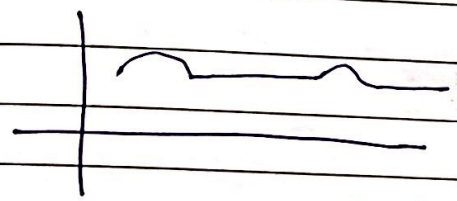
→ Rectifier :



Half-wave rectifier

↓ Half-wave

↓ LPF



Half-wave + LPF = envelope detector

output of B.P.F would be sinusoids. They should be converted into pulse by using rectifiers. That is done by envelope detector.

only 1 branch should contain an output. But, what is the guarantee that other branches would be non-zero. we can guarantee that the one particular branch would be non-zero.

we need to pass through LPF, so that we don't get 0 while sampling.

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→ Coherent OFSK & non-coherent OFSK.

Coherent : $r^{\phi}(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \phi)$

$$\psi_i^{\phi}(t) = \sqrt{\frac{2}{T}} \cos(\omega_i t + \phi)$$

$$\langle r^{\phi}, \psi_i^{\phi} \rangle = \langle r, \psi_i \rangle = \sqrt{E}$$

$$i \neq j, \langle r, \psi_j \rangle = 0 \quad , \quad \omega_{i+1} - \omega_i = \frac{K\pi}{T}$$

$$\min(\omega_{i+1} - \omega_i) = \frac{\pi}{T} \quad \text{—— (1)}$$

Non-coherent OFSK :

$$r^{\phi}(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \phi)$$

$$\langle r^{\phi}, \psi_i \rangle = \sqrt{E} \cos\phi \, , \quad \langle r^{\phi}, \bar{\psi}_i \rangle = -\sqrt{E} \sin\phi$$

$$\langle r^{\phi}, \psi_j \rangle \approx f(\omega_i - \omega_j)$$

$$= \int_0^T \left(\sqrt{\frac{2E}{T}} \cos\phi \cos\omega_i t - \sqrt{\frac{2E}{T}} \sin\phi \sin\omega_i t \right) \sqrt{\frac{2}{T}} \cos(\omega_j t) \, dt$$

$$= \sqrt{\frac{2}{T}} \sqrt{\frac{2E}{T}} \frac{\cos\phi}{2} \int_0^T \left[\cos((\omega_i + \omega_j)t) + \cos((\omega_i - \omega_j)t) \right] dt$$

$$- \frac{2}{T} \frac{\sin\phi}{2} \sqrt{E} \int_0^T \left[\sin((\omega_i + \omega_j)t) + \sin((\omega_i - \omega_j)t) \right] dt$$

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$$= \frac{1}{T}\sqrt{E}\cos\phi\left[\frac{\sin(\omega_i+\omega_j)t}{\omega_i+\omega_j} + \cancel{\frac{\sin(\omega_i-\omega_j)t}{\omega_i-\omega_j}}\right]_0^T$$

0, if $\omega_i-\omega_j=\frac{K\pi}{T}$

$$+ \frac{\sqrt{E}}{T}\sin\phi\left[\frac{\cos(\omega_i+\omega_j)t}{\omega_i+\omega_j} + \frac{\cos(\omega_i-\omega_j)t}{\omega_i-\omega_j}\right]_0^T$$

0, since $T \approx 10^{-6}$

$$= \frac{\sqrt{E}}{T}\cos\phi\left[\cancel{\frac{\sin((\omega_i+\omega_j)T)}{\omega_i+\omega_j}}\right] + \frac{\sqrt{E}}{T}\sin\phi\left[\cancel{\frac{\cos((\omega_i+\omega_j)T)-1}{\omega_i+\omega_j}} + \frac{\cos((\omega_i-\omega_j)T)-1}{\omega_i-\omega_j}\right]$$

since $T \approx 10^{-6}$

to be 0, $\omega_i-\omega_j = \frac{2K\pi}{T}$

$$\therefore \min(\omega_{i+1}-\omega_i) = \frac{2\pi}{T} \quad \text{—— (2)}$$

$\therefore$ (1) & (2) $\Longrightarrow$ B.W. for non-coherent OFSK > B.W. for coherent OFSK

→ Modulator / demodulator for PSK/FSK.

$$s(t) = \sqrt{\frac{2E}{T}}\cos(\omega_0 t+\phi)$$

$$= \sqrt{\frac{2E}{T}}\left(\cos(\omega_0 t)\cos\phi - \sin\phi\sin(\omega_0 t)\right)$$

$$= \underbrace{\sqrt{\frac{2E}{T}}\cos\phi}_{x(K)}\,\underbrace{\cos\omega_0 t}_{\text{carrier}} - \underbrace{\sqrt{\frac{2E}{T}}\sin\phi}_{y(K)}\,\underbrace{\sin\omega_0 t}_{\text{carrier}}$$

message

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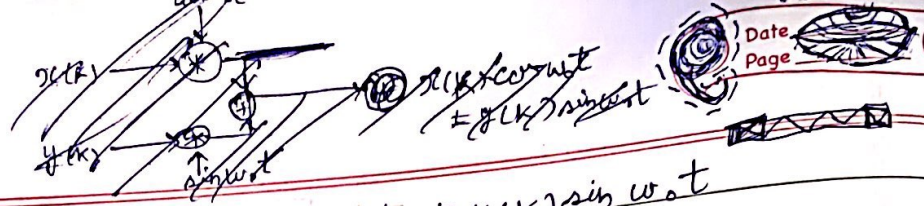
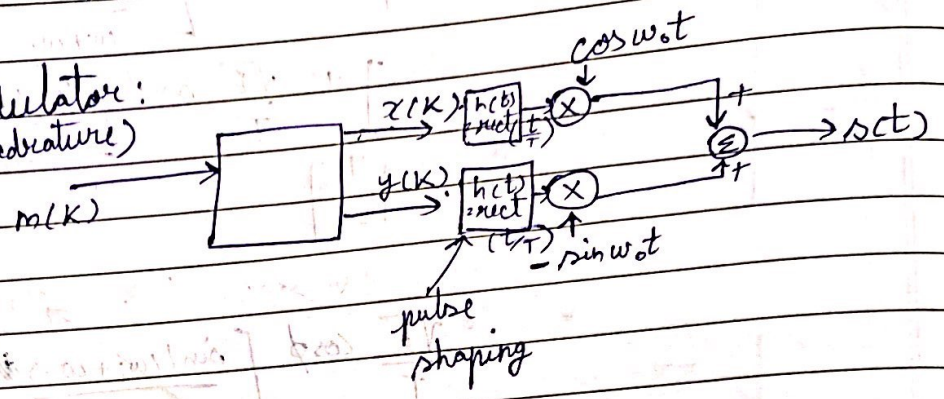


$\therefore s(t) = x(k)\cos w_0 t \pm y(k)\sin w_0 t$

modulator:
(quadrature)



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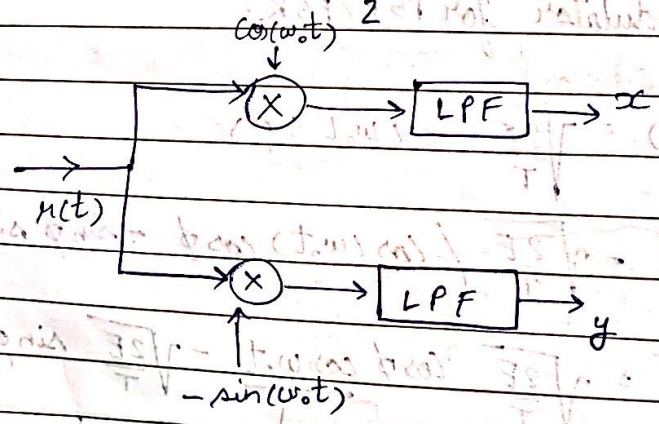
→ Quadrature demodulation:

$$s(t) = \sqrt{\frac{2E}{T}}\cos(w_0 t + \phi), \quad 0 \le t \le T$$

$$r(t) = s(t) = x(k)\cos(w_0 t) - y(k)\sin(w_0 t)$$

$$\int_0^T r(t)\cos(w_0 t)\,dt = \langle r, \cos(w_0 t)\rangle$$
$$= \frac{x(k)}{2} \cdot K \approx \bar{K}\, x(k)$$

$$\int_0^T r(t)\sin(w_0 t)\,dt = \langle r, \sin(w_0 t)\rangle$$
$$= -\frac{y(k)}{2} \cdot K \approx \tilde{K}\, y(k)$$



LPF is used as an integrator

if phase delay is present, we will use differential PSK.

$s(t)$ $0-T \rightarrow \phi_1$

$T-2T \rightarrow \phi_1 + \phi_0$

$r(t) \rightarrow \phi_1 + \tilde{\phi}(0,T)$

$\phi_1 + \tilde{\phi} + \phi_0$

difference $= \phi_0$

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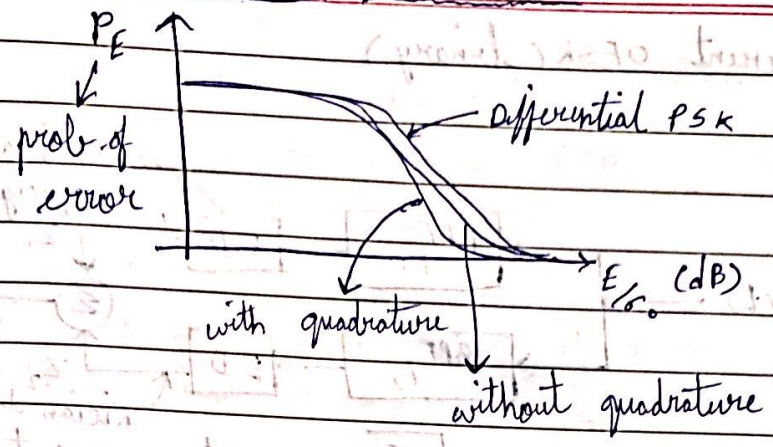
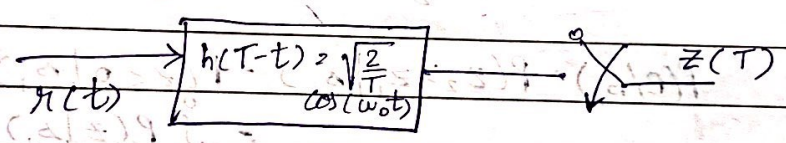
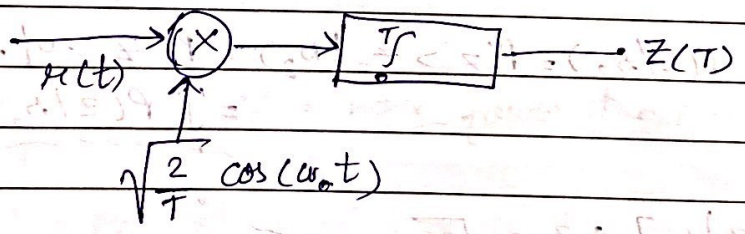


→ Error performance (P_B)

① Coherent BPSK

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + 2\pi(i-1)), \quad i = 1, 2, \quad 0 < t \le T$$





$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

② coherent OFSK

$$P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

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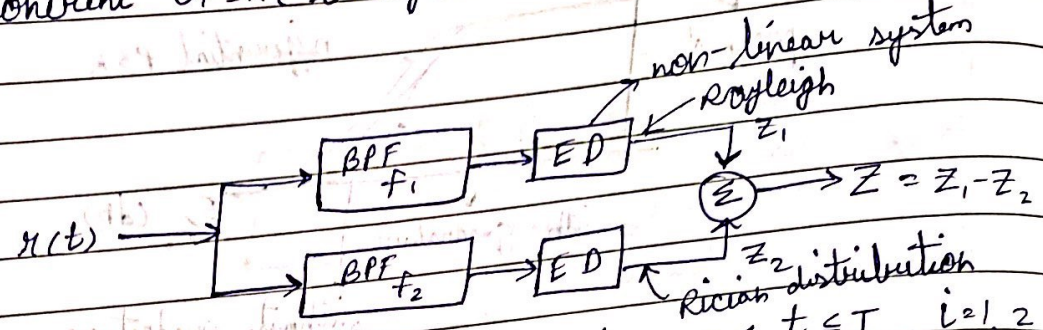
③ Non coherent OFSK (binary)



$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t), \quad 0 \le t \le T, \quad i = 1, 2$$

$$r(t) = s_i(t) + n(t)$$

$$P_B = P(e) = P(s_1) P(e/s_1) + P(s_2) P(e/s_2)$$

$$P(e) = \frac{1}{2} P(e/s_1) + \frac{1}{2} P(e/s_2)$$

$$P(e/s_1) = P(z_2 > z_1 | s_1) = P(z < 0 | s_1)$$
$$= \int_{-\infty}^{0} P(z|s_1)\, dz$$

$$P(e/s_2) = P(z_1 > z_2 | s_2) = P(z > 0 | s_2)$$
$$= \int_{0}^{\infty} P(z|s_2)\, dz$$

$$\left.\begin{aligned} E[z|s_1] &= z_1 = \sqrt{\frac{2E}{T}} \\ E[z|s_2] &= -\sqrt{\frac{2E}{T}} \end{aligned}\right\}$$ under the assumption that there was no noise

$$P(z|s_1) = P(-z|s_2)$$

$$P(e) = \frac{1}{2}\int_{-\infty}^{0} P(-z|s_2)\, dz + \frac{1}{2}\int_{0}^{\infty} P(z|s_1)\, dz$$

$$P_B = P(e) = \int_{0}^{\infty} P(z|s_2)\, dz.$$
$$= P(z_1 > z_2 | s_2)$$

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For $P(z|s_2)$,

$$r(t) = s_2(t) + \eta(t) \rightarrow \text{AWGN}$$

After passing this through BPF(f_1), there would only be noise component left

↳ GN (Gaussian noise)

∵ this noise passes through a non-linear system (ED), the output is not necessarily a Gaussian process/output

$$P(z_1|s_2) = \frac{z_1}{\sigma_0^2} e^{-(z_1^2/2\sigma_0^2)}, \quad z_1 \geq 0$$

$$= 0, \quad z_1 < 0$$

After passing $r(t)$ through BPF(f_2), the noise would be GN.

$$P(z_2|s_2) = \frac{z_2}{\sigma_0^2} e^{-(z_2^2+A^2)/2\sigma_0^2} I_0\left(\frac{z_2 A}{\sigma_0^2}\right), \quad z_2 \geq 0$$

$$= 0, \quad z_2 < 0$$

$A = \sqrt{\frac{2E}{T}}$, σ_0^2 = noise power at o/p of BPF

$I_0(x)$ = Modified Bessel fun^n of the first kind.

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{x\cos\theta} d\theta$$

$$P_B = P(z_1 > z_2 / s_2)$$

$$= \int_0^{\infty} P(z_1|s_2) \left[\int_0^{z_1} P(z_2|s_2)\, dz_2 \right] dz_1$$

$$= \frac{1}{2} e^{\left(-\frac{A^2}{4\sigma_0^2}\right)}, \quad A = \sqrt{\frac{2E}{T}}$$

$$\sigma_0^2 = N_0 . B$$

bandwidth of BPF (assuming both BPFs have same B)

$$\therefore P_B = \frac{1}{2} e^{\left(-\frac{E_b R}{2 N_0 B}\right)}$$ → data rate

$$B = R(1+r)$$
$$B_{min} = R$$
$$\therefore P_B = \frac{1}{2} e^{\left(\frac{-E_b}{2N_0}\right)}$$

17/10/19

④ BDPSK

$$\psi_1(t) = \cos(\omega_0 t)$$
$$\psi_2(t) = \cos(\omega_0 t + \pi)$$

Bit 0 : (ψ_1, ψ_1) or (ψ_2, ψ_2) , $0 \le t \le 2T$
Bit 1 : (ψ_1, ψ_2) or (ψ_2, ψ_1) , $0 \le t \le 2T$

$$\langle (\psi_1, \psi_1), (\psi_1, \psi_2) \rangle_{[0,2T]} = \langle \psi_1, \psi_1 \rangle_{[0,T]} + \langle \psi_1, \psi_2 \rangle_{[T,2T]}$$
$$= \langle \psi_1, \psi_1 \rangle - \langle \psi_1, \psi_1 \rangle = 0$$

For any pair of signals, we would have their product as 0.
∴ It is an orthogonal scheme over $[0, 2T]$ (over 2 pulse waveforms).

$$\therefore P_B = \frac{1}{2} e^{\left(-\frac{E_b}{N_0}\right)}$$ → there won't be a factor of 2 because we are taking energy over 2 waveforms at a time.

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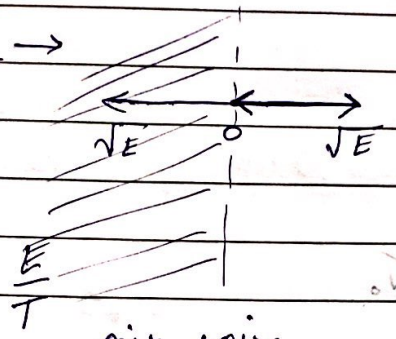
→ M-ary signalling

① M-ary PSK

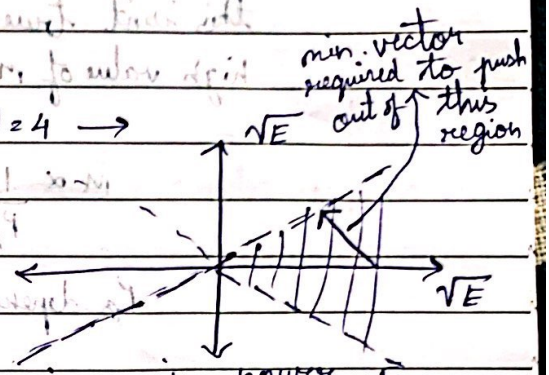
$M = 2^k$, k bits / waveform

Binary →

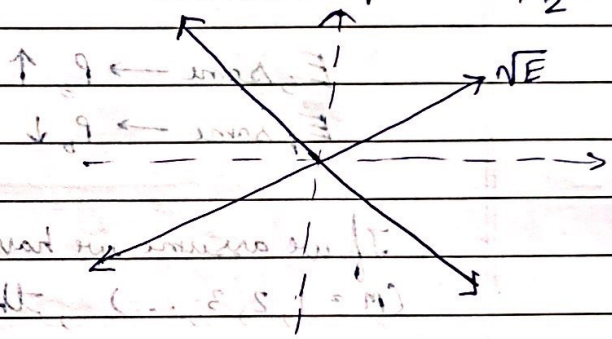


min. noise power = E

M = 4 →



min. noise power = E/2

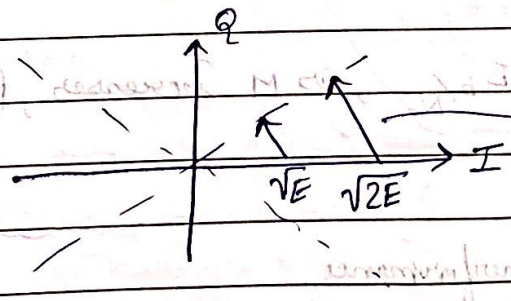


∴ As M inc., P_B inc.

So, why do we need more M, if P_B inc.?

→ ∵ we need more bit rate

→ If we inc. energy of a bit, P_B dec.



→ a larger value of noise is required to push a vector out of this region, ∴ less errors.

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② M-ary OFSK.

Even though M inc., the distance betⁿ any 2 signals would remain the same. Thus, this fact won't affect the P_B. But, this isn't true. As M inc., P_B dec. Better performance at high value of M.

$$M \propto \frac{1}{P_B}$$

P_B depends on $\frac{E_b}{N_0}$

E_s same $\rightarrow P_s \uparrow$

E_b same $\rightarrow P_b \downarrow$

If we assume we have the same energy E for all waveforms (M = 1, 2, 3, ...), then E_b would decrease, thus P_B dec.

As M inc., prob. of symbol error would inc. The distance remains the same, but the noise has more freedom to introduce misclassification

In binary, if it mis-classifies, it has just the other class to go.

For the same E_b/N_0, as M increases, P_B would increase.

4/11/19 $\rightarrow$ M-ary signalling performance

① M-ary FSK:

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as $M\uparrow$, $P_B\downarrow$, $BW\uparrow$, assuming that we are comparing for same $\frac{E_b}{N_0}$ ratio

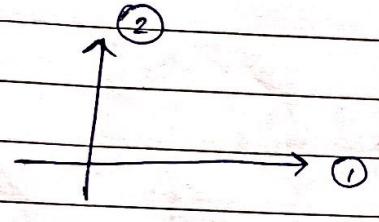
why ?

$P_s\uparrow$ as $M\uparrow$

$d(s_i, s_j) = K, \quad i \neq j$

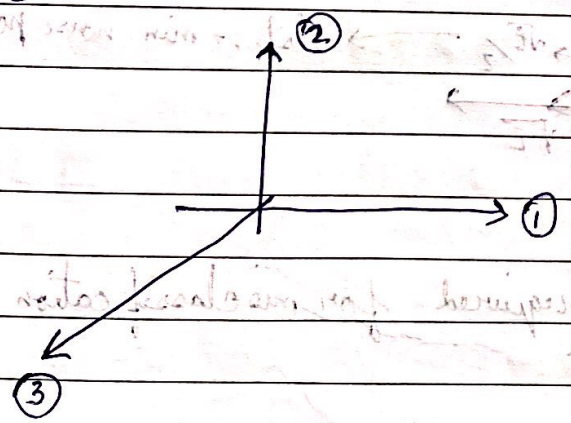
easier for the noise to add variation or produce misclassification in many more ways.

$M = 2$



only misclassification could be $1 \rightarrow 2$ / $2 \rightarrow 1$

$M = 3$



Here there are more ways of misclassification as $M\uparrow$, with the same power of noise

assuming looking at same SNR value ;

Each of the waveforms is encoding $\log_2 M = k$ bits.

Symbol energy $E_s = kE_b$

$E_b = \frac{E_s}{k}$, as $k\uparrow$, $E_b\downarrow$ assuming constant E_s.

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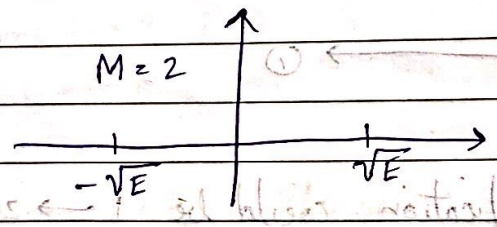
Compare bit error at the same $\frac{E_b}{N_0}$. Need to know many bits have been misclassified.

To have the same $\frac{E_b}{N_0}$ ratio, we need to increase E_s, as we increase E_s, $d(s_i, s_j) = k \uparrow$

As we increase M, $P_B \downarrow$, but trade-off is that of higher BW.

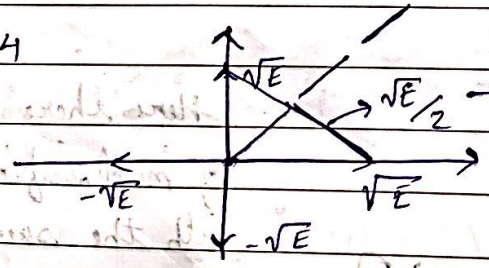
② M-ary PSK :

$M \uparrow$, $P_B \uparrow$



min. noise power is $(\sqrt{E})^2 = E$

$M = 4$



$\longrightarrow$ dist. = min. noise power required

min. noise power required for misclassification $\downarrow$ as $M \uparrow$, $\therefore P_B \uparrow$

But, bandwidth of the waveforms remains the same. As all the waveforms are of the same freq.

one thing we could do is increase f_s, in order to decrease P_B.

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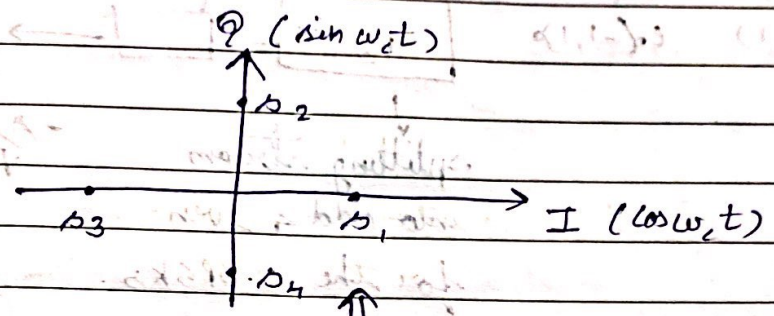
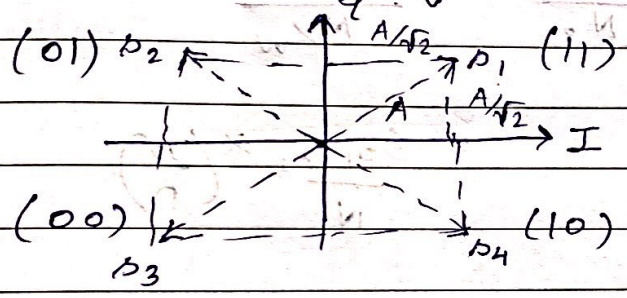
→ **Lab-10 : QPSK,**

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_c t + \frac{\pi}{2}(i-1)\right), \quad 0 \le t \le T, \quad i = 1, 2, 3, 4$$



Constellation diagram

QPSK₂



It doesn't matter which system we use

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_c t + \frac{\pi}{4} + \frac{\pi}{2}(i-1)\right), \quad 0 \le t \le T, \quad i = 1, 2, 3, 4$$

Each signal will either have $+\frac{A}{\sqrt{2}}$ of I axis or $-\frac{A}{\sqrt{2}}$ of I axis.

Similarly $+\frac{A}{\sqrt{2}}$ of Q axis or $-\frac{A}{\sqrt{2}}$ of Q axis

∴ If we look at each axis, we are performing BPSK.

QPSK scheme can be seen as 2 BPSK schemes

& the 2 schemes are orthogonal.
∴ They are not going to interfere with the others.

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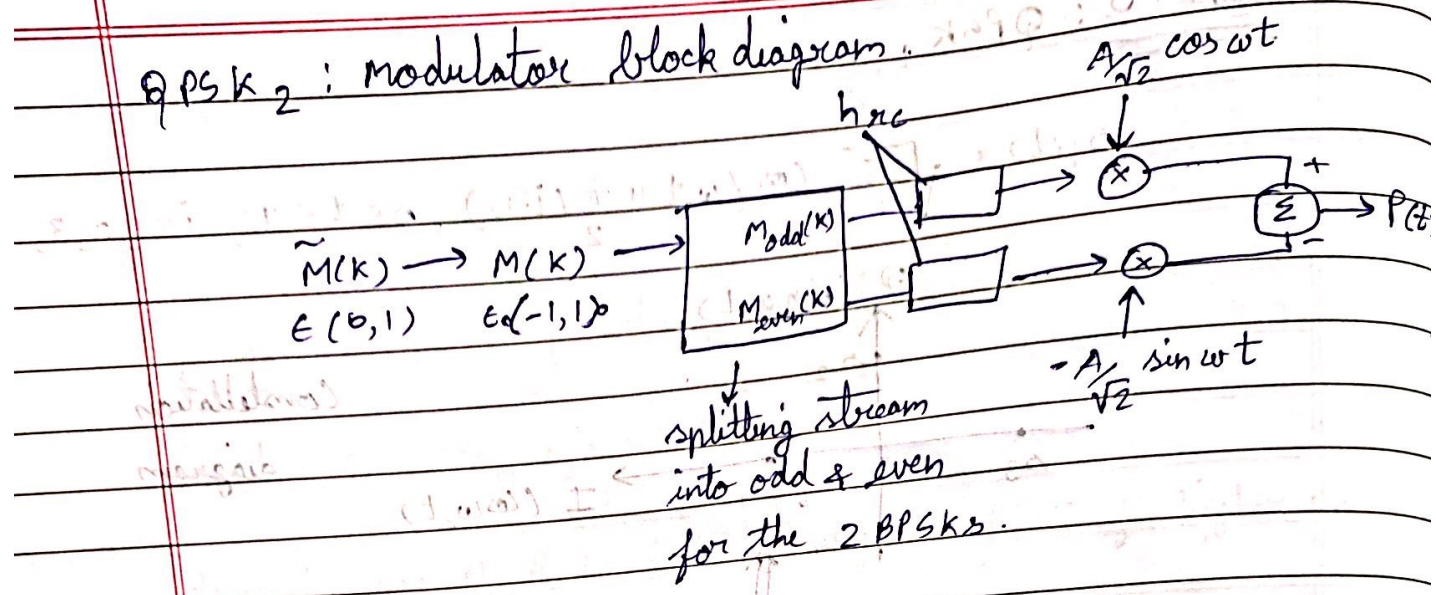
QPSK$_2$: modulator block diagram.



$$\frac{E_{QPSK_1}}{N_0}, \quad \frac{E_{QPSK_2}}{N_0} = \frac{S_{QPSK_2} \times T}{N/W}$$

$$= \frac{S_{QPSK_2}}{N} \times \left(\frac{W}{R}\right)$$

In QPSK$_1$, signal power is A^2

" " $_2$, " " " $\left(\frac{A}{\sqrt{2}}\right)^2 = \frac{A^2}{2}$

$$\therefore S_{QPSK_1} = 2 S_{QPSK_2}$$

$$\frac{E_{QPSK_2}}{N_0} = \frac{S_{QPSK_1}}{2N}\left(\frac{W}{R}\right) = \frac{S_{QPSK_1}}{\cancel{2}N} \times \frac{W}{\frac{R_{QPSK_1}}{\cancel{2}}} = \frac{E_{QPSK_1}}{N_0}$$

For every 2 bits of stream, we get 1 bit in even stream & 1 in odd stream.

$$\therefore R_{QPSK_1} = 2 R_{QPSK_2}$$

assuming using coherent detection, cos & sin don't interfere.

The 2 modulation schemes are same, $\frac{E_{QPSK_1}}{N_0} = \frac{E_{QPSK_2}}{N_0}$

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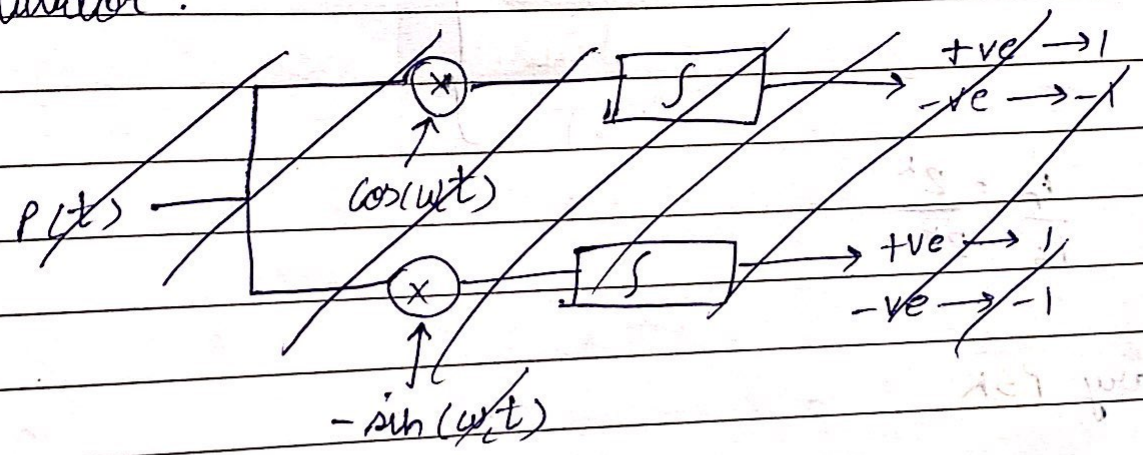
what about P_B?

2 BPSK, modulation $\longrightarrow$ 2 BPSK at the detection

$\therefore P_B(QPSK_1) = P_B(QPSK_2) = P_B(BPSK)$ at that $\frac{E_b}{N_0}$ ratio

but decided to reduce the rate by 2.
if not, $\frac{E_{QPSK_2}}{N_0}$ will go down, $\therefore$ larger error.

Demodulator:



in lab.

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M-ary FSK

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t), \quad 0 \le t \le T, \quad i = 1, \dots, M$$

$$d(s_i, s_j) = K, \quad i \ne j$$

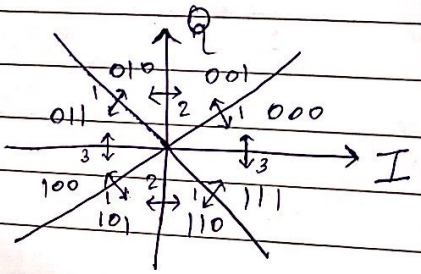
→ All symbol misclassifications are equally likely.

$M = 2^K$

$$\left.\begin{matrix} 0000 \\ \vdots \\ 1111 \end{matrix}\right\} K = 4$$

$$\frac{P_B}{P_S} = \frac{2^{K-1}}{2^K - 1}$$

→ M-ary PSK

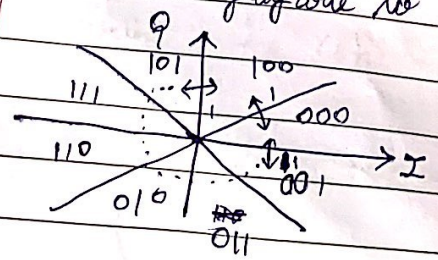


$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_0 t + \frac{2\pi}{M}(i-1)\right),$$

$$0 \le t \le T, \quad i = 1, \dots, M$$

Assuming P_s is small,

we can use gray code to minimise the error,



$$\frac{P_B}{P_S} = \frac{1}{K}$$

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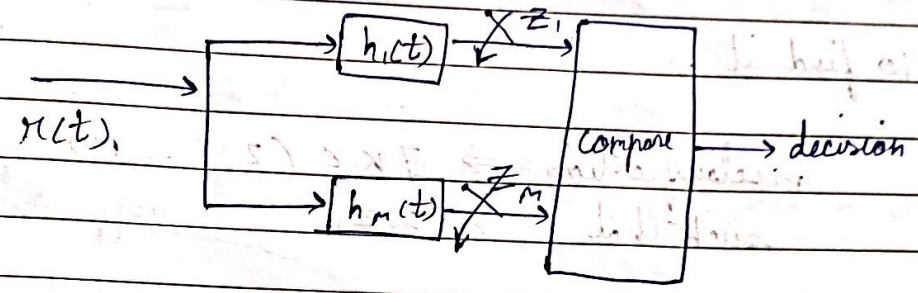
→ Symbol error rate (P_s) :

① MFSK

The classifier which we are going to model is coherent matched filter detector.



Assume s_1 was transmitted,

$$p_{z_1}(z_1) \sim N(\sqrt{E}, \sigma_0)$$

$$p_{z_k}(z_k) \sim N(0, \sigma_0), \quad k \neq 1$$

$$P_c = \int_{-\infty}^{\infty} P(z_1 > z_2, z_1 > z_3, \ldots, z_1 > z_M \mid z_1) P(z_1)\, dz_1$$

$$P(z_1 > z_2, \ldots, z_1 > z_M \mid z_1) = \prod_{k=2}^{M} P(z_1 > z_k \mid z_1)$$

7/11/19

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t), \quad 0 \le t \le T, \quad i = 1, \ldots, M$$

o/p of M correlators $\longrightarrow z_1, \ldots, z_M$

$$P_s = 1 - P_c$$

$$P(z_1 > z_k \mid z_1) = \int_{-\infty}^{z_1} N_{z_k}(0, \sigma_0)\, dz_k$$

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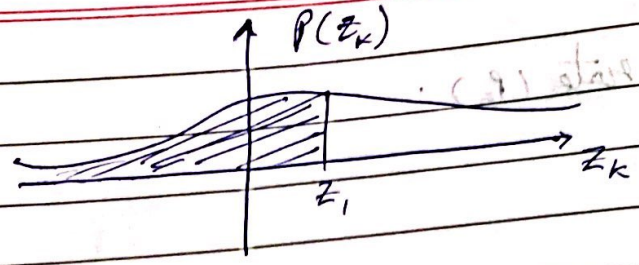

upper bounds for P_s :

To find it,

misclassification $\Longrightarrow \exists\, k \in (2, \ldots, M)$
such that, $z_k > z_1$

E_k event for $z_k > z_1$, $k = 2, \ldots, M$

$$P_s = P(\text{one of } E_k \text{ has occurred}) = P\left(\bigcup_{k=2}^{M} E_k\right) \leq \sum_{k=2}^{M} P(E_k)$$

Prob. of misclassification error in BFSK
($\because$ we only compare it with z_1)

$$\therefore P_s \leq (M-1)\, Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$E_s = k E_b$$

$$P_s \leq (M-1)\, Q\left(\sqrt{\frac{k E_b}{N_0}}\right)$$

$$\Longrightarrow k \to \infty,\ P_s \to 0$$

$$P_s < M\, Q\left(\sqrt{\frac{k E_b}{N_0}}\right), \quad Q \leq e^{-x^2/2}$$

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$$\therefore P_s < 2^k e^{-kE_b/2N_0}$$

$$< e^{k(\ln 2 - E_b/2N_0)}$$

$$< e^{k/2(2\ln 2 - E_b/N_0)}$$

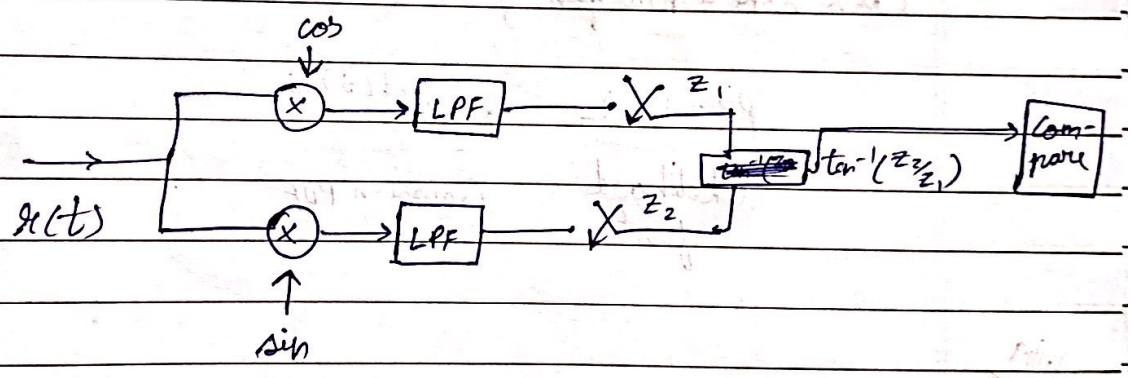
if $\frac{E_b}{N_0} > 2\ln(2)$, increasing k will result in smaller P_s. ($k \to \infty$, $P_s \to 0$)

Tighter upper bound $\longrightarrow$ $P_s \sim e^{-k(\sqrt{E_b/N_0} - \sqrt{\ln 2})^2}$

if $\frac{E_b}{N_0} > \ln 2$ (Shannon's limit),

$k \to \infty$, $P_s \to 0$

② MPSK



suppose s_1 was transmitted,

$z_1 \sim N(\quad)$

$z_2 \sim N(\quad)$

we want to convert $p(z_1, z_2)$ to $p(m, \phi)$,

$$m = \sqrt{z_1^2 + z_2^2}$$

$$\phi = \tan^{-1}(z_2/z_1)$$

$$P_s = 1 - \int_{-\pi/M}^{\pi/M} \left(\int_0^{\infty} p(m,\phi)\, dm \right) d\phi$$

15/11/19

→ $L(\hat{s}) = \bar{r}^T \bar{r} - 2 r^T \bar{h}$ ~~$+ \bar{h}^T \bar{h} \bar{r}$~~ ....

$\min(L(\hat{s})) \Rightarrow \max(r^T \bar{h})$

↑ Maximum / Projection or correlation or matched filter

($\because$ $\bar{r}^T \bar{r}$ remains unaffected & in M-ary PSK, b^2 for all points is same; all are on the circumference of circle)

Let's take a new fun^c,

$$\underbrace{P(\hat{s} \mid \bar{r})}_{\text{likelihood fun}^c} = \underbrace{\alpha\, e^{-L(\hat{s}).K}}_{\text{Gaussian PDF}}$$

18/11/19

→ M-PSK / MFSK

① P_b
② Bandwidth efficiency

M-PSK : $T \rightarrow$ symbol duration

$$R_s = \frac{1}{T}$$

$$W = R_s = \frac{R_b}{(\log_2 M)}$$

$\frac{R_b}{W}$ (bits/s/Hz) $= \log_2 M$

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$\uparrow M \longrightarrow \uparrow$ B.W. efficiency

M-FSK: $W_c = \frac{M}{2T} = \frac{M R_s}{2} = \frac{M R_b}{2 \log_2 M}$

$\therefore \frac{R_b}{W_c} = \frac{2 \log_2 M}{M}$

$\uparrow M \longrightarrow \downarrow$ B.W. efficiency

For the same bit-error rate (P_b),

M-PSK: As $M \uparrow$, $BW \uparrow$ efficiency, $\frac{E_b}{N_0} \uparrow$

M-FSK: As $M \uparrow$, $BW \downarrow$ efficiency, $\frac{E_b}{N_0} \downarrow$

→ Shannon-Hartley capacity theorem:

Assumption - AWGN channel

$$C = W \log_2 \left(1 + \frac{S}{N}\right)$$

C → channel capacity (max. bit rate)

W → bandwidth of channel

$\frac{S}{N}$ → avg. SNR received over W Hz.

$$\frac{E_b}{N_0} = \frac{ST}{N/W} = \frac{S}{N}\left(\frac{W}{R}\right)$$

$$C = R, \quad \therefore \frac{R}{W} = \log_2\left(1 + \frac{E_b}{N_0}\left(\frac{R}{W}\right)\right)$$

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$$\frac{E_b}{N_0} = \frac{2^{R/W} - 1}{R/W}$$

$$\frac{E_b}{N_0} = \lim_{R/W \to 0} \frac{2^{R/W} - 1}{R/W} = \ln 2 \text{ (shannon's limit)}$$

Recap

1. Analog vs. digital comm.
2. stochastic processes → stationarity, ergodicity
3. Analog to digital : Quantization → uniform / non-uniform
   Sampling → Anti aliasing filter
4. Modulation → Baseband
   → Bandpass
5. Demodulation (bandpass) → matched filter (maximize SNR)
   → Detection → MLE
   → MAP
   → Performance Criteria : $P_b \propto \frac{E_b}{N_0}$
   → ISI : Nyquist Pulse shaping (RC, RRC, ...) eye diagram
   → equalization filters : ZF / MMSE
6. Bandpass Modulation / demodulation - detection
   - PSK, FSK, ASK, APK
   - PSK / FSK / DPSK demodulation → coherent
     → non-coherent
   - Quadrature modulation / demodulation
   - Error performance : P_b, P_s
     - what happens as $M \uparrow$?
   - B.W. efficiency

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