MISCELLANEOUS PROBLEMS

23. Let $f(z)$ be analytic in a region $\mathcal{R}$ bounded by two concentric circles C_1 and C_2 and on the boundary [Fig. 5-11]. Prove that if z_0 is any point in $\mathcal{R}$, then

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-z_0}\,dz$$

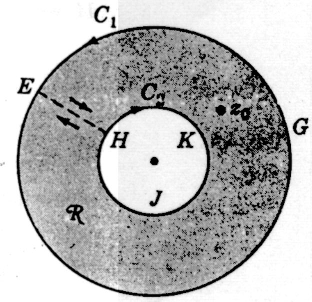

Fig. 5-11

Method 1.

Construct cross-cut EH connecting circles C_1 and C_2. Then $f(z)$ is analytic in the region bounded by $EFGEHKJHE$. Hence by Cauchy's integral formula,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\oint_{EFGEHKJHE}\frac{f(z)}{z-z_0}\,dz \\ &= \frac{1}{2\pi i}\oint_{EFGE}\frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\int_{EH}\frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{HKJH}\frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\int_{HE}\frac{f(z)}{z-z_0}\,dz \\ &= \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-z_0}\,dz \end{aligned}$$

since the integrals along EH and HE cancel.

Similar results can be established for the derivatives of $f(z)$.

Method 2. The result also follows from equation (3) of Problem 6 if we replace the simple closed curves C_1 and C_2 by the circles of Fig. 5-11.

24. Prove that $\displaystyle\int_0^{2\pi}\cos^{2n}\theta\,d\theta = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}2\pi$ where $n = 1, 2, 3, \ldots$.

Let $z = e^{i\theta}$. Then $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ or $d\theta = dz/iz$ and $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}(z+1/z)$. Hence if C is the unit circle $|z| = 1$, we have

$$\begin{aligned} \int_0^{2\pi}\cos^{2n}\theta\,d\theta &= \oint_C\left\{\frac{1}{2}\left(z+\frac{1}{z}\right)\right\}^{2n}\frac{dz}{iz} \\ &= \frac{1}{2^{2n}i}\oint_C\frac{1}{z}\left\{z^{2n} + \binom{2n}{1}(z^{2n-1})\left(\frac{1}{z}\right) + \cdots + \binom{2n}{k}(z^{2n-k})\left(\frac{1}{z}\right)^k + \cdots + \left(\frac{1}{z}\right)^{2n}\right\}dz \\ &= \frac{1}{2^{2n}i}\oint_C\left\{z^{2n-1} + \binom{2n}{1}z^{2n-3} + \cdots + \binom{2n}{k}z^{2n-2k-1} + \cdots + z^{-2n}\right\}dz \\ &= \frac{1}{2^{2n}i}\cdot 2\pi i\binom{2n}{n} = \frac{1}{2^{2n}}\binom{2n}{n}2\pi \\ &= \frac{1}{2^{2n}}\frac{(2n)!}{n!\,n!}2\pi = \frac{(2n)(2n-1)(2n-2)\cdots(n)(n-1)\cdots 1}{2^{2n}\,n!\,n!}2\pi \\ &= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}2\pi \end{aligned}$$

25. If $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a region $\mathcal{R}$, prove that u and v are harmonic in $\mathcal{R}$.

In Problem 6, Chapter 3, we proved that u and v are harmonic in $\mathcal{R}$, i.e. satisfy the equation $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$, under the *assumption* of existence of the second partial derivatives of u and v, i.e. the existence of $f''(z)$.

This assumption is no longer necessary since we have in fact proved in Problem 4 that if $f(z)$ is analytic in $\mathcal{R}$ then *all* the derivatives of $f(z)$ exist.

If we subtract (2) from (1), we find

$$f(z) = \frac{1}{2\pi i}\oint_C \left\{\frac{1}{w-z} - \frac{1}{w - R^2/\bar{z}}\right\} f(w)\,dw$$

$$= \frac{1}{2\pi i}\oint_C \frac{z - R^2/\bar{z}}{(w-z)(w-R^2/\bar{z})} f(w)\,dw \tag{3}$$

Now let $z = re^{i\theta}$ and $w = Re^{i\phi}$. Then since $\bar{z} = re^{-i\theta}$, (3) yields

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\{re^{i\theta} - (R^2/r)e^{i\theta}\}\, f(Re^{i\phi})\, iRe^{i\phi}\,d\phi}{\{Re^{i\phi} - re^{i\theta}\}\{Re^{i\phi} - (R^2/r)e^{i\theta}\}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - R^2)\, e^{i(\theta+\phi)}\, f(Re^{i\phi})\,d\phi}{(Re^{i\phi} - re^{i\theta})(re^{i\phi} - Re^{i\theta})}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, f(Re^{i\phi})\,d\phi}{(Re^{i\phi} - re^{i\theta})(Re^{-i\phi} - re^{-i\theta})}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, f(Re^{i\phi})\,d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

(b) Since $f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$ and $f(Re^{i\phi}) = u(R,\phi) + i\,v(R,\phi)$, we have from part (a),

$$u(r,\theta) + i\,v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\{u(R,\phi) + i\,v(R,\phi)\}\,d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,u(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2} + \frac{i}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,v(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2}$$

Then the required result follows on equating real and imaginary parts.

POISSON'S INTEGRAL FORMULAE FOR A HALF PLANE

22. Derive Poisson's formulae for the half plane [see Page 120].

Let C be the boundary of a semicircle of radius R [see Fig. 5-10] containing ζ as an interior point. Since C encloses ζ but does not enclose $\bar{\zeta}$, we have by Cauchy's integral formula,

$$f(\zeta) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\zeta}\,dz, \qquad 0 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - \bar{\zeta}}\,dz$$

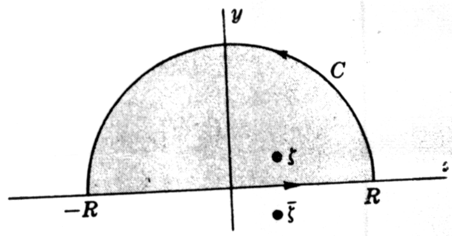

Fig. 5-10

Then by subtraction,

$$f(\zeta) = \frac{1}{2\pi i}\oint_C f(z)\left\{\frac{1}{z-\zeta} - \frac{1}{z-\bar{\zeta}}\right\}dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{(\zeta - \bar{\zeta})\,f(z)\,dz}{(z-\zeta)(z-\bar{\zeta})}$$

Letting $\zeta = \xi + i\eta$, $\bar{\zeta} = \xi - i\eta$, this can be written

$$f(\zeta) = \frac{1}{\pi}\int_{-R}^{R} \frac{\eta\, f(x)\,dx}{(x-\xi)^2 + \eta^2} + \frac{1}{\pi}\int_\Gamma \frac{\eta\, f(z)\,dz}{(z-\zeta)(z-\bar{\zeta})}$$

where Γ is the semicircular arc of C. As $R \to \infty$, this last integral approaches zero [see Problem 7] and we have

$$f(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta\, f(x)\,dx}{(x-\xi)^2 + \eta^2}$$

Writing $f(\zeta) = f(\xi + i\eta) = u(\xi,\eta) + i\,v(\xi,\eta)$, $f(x) = u(x,0) + i\,v(x,0)$, we obtain as required,

$$u(\xi,\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta\, u(x,0)\,dx}{(x-\xi)^2+\eta^2}, \qquad v(\xi,\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta\, v(x,0)\,dx}{(x-\xi)^2+\eta^2}$$

$$\left|\frac{g(z)}{f(z)}\right| = \frac{|a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}|}{|a_n z^n|}$$

$$\leqq \frac{|a_0| + |a_1|\,r + |a_2|\,r^2 + \cdots + |a_{n-1}|\,r^{n-1}}{|a_n|\,r^n}$$

$$\leqq \frac{|a_0|\,r^{n-1} + |a_1|\,r^{n-1} + |a_2|\,r^{n-1} + \cdots + |a_{n-1}|\,r^{n-1}}{|a_n|\,r^n}$$

$$= \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n|\,r}$$

Then by choosing r large enough we can make $\left|\frac{g(z)}{f(z)}\right| < 1$, i.e. $|g(z)| < |f(z)|$. Hence by Rouché's theorem the given polynomial $f(z) + g(z)$ has the same number of zeros as $f(z) = a_n z^n$. But since this last function has n zeros all located at $z = 0$, $f(z) + g(z)$ also has n zeros and the proof is complete.

20. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Consider the circle C_1: $|z| = 1$. Let $f(z) = 12$, $g(z) = z^7 - 5z^3$. On C_1 we have

$$|g(z)| = |z^7 - 5z^3| \leqq |z^7| + |5z^3| \leqq 6 < 12 = |f(z)|$$

Hence by Rouché's theorem $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z| = 1$ as $f(z) = 12$, i.e. there are no zeros inside C_1.

Consider the circle C_2: $|z| = 2$. Let $f(z) = z^7$, $g(z) = 12 - 5z^3$. On C_2 we have

$$|g(z)| = |12 - 5z^3| \leqq |12| + |5z^3| \leqq 60 < 2^7 = |f(z)|$$

Hence by Rouché's theorem $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z| = 2$ as $f(z) = z^7$, i.e. all the zeros are inside C_2.

Hence all the roots lie inside $|z| = 2$ but outside $|z| = 1$, as required.

POISSON'S INTEGRAL FORMULAE FOR A CIRCLE

21. (a) Let $f(z)$ be analytic inside and on the circle C defined by $|z| = R$, and let $z = re^{i\theta}$ be any point inside C. Prove that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, f(Re^{i\phi})\, d\phi$$

(b) If $u(r,\theta)$ and $v(r,\theta)$ are the real and imaginary parts of $f(re^{i\theta})$, prove that

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,u(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,v(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

The results are called *Poisson's integral formulae for the circle.*

(a) Since $z = re^{i\theta}$ is any point inside C, we have by Cauchy's integral formula

$$f(z) = f(re^{i\theta}) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - z}\, dw \qquad (1)$$

The *inverse of the point* z with respect to C lies outside C and is given by $R^2/\bar{z}$. Hence by Cauchy's theorem,

$$0 = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - R^2/\bar{z}}\, dw \qquad (2)$$

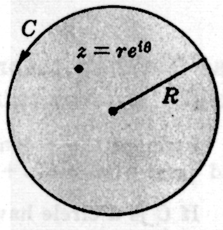

Fig. 5-9

number of zeros and poles of $f(z)$ inside C, counting multiplicities, prove that

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_j$ and $\beta_1, \beta_2, \ldots, \beta_k$ be the respective poles and zeros of $f(z)$ lying inside C [Fig. 5-8] and suppose their multiplicities are $p_1, p_2, \ldots, p_j$ and $n_1, n_2, \ldots, n_k$.

Enclose each pole and zero by non-overlapping circles $C_1, C_2, \ldots, C_j$ and $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$. This can always be done since the poles and zeros are isolated.

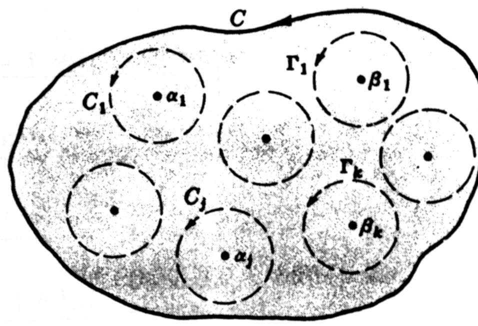

Fig. 5-8

Then we have, using the results of Problem 16,

$$\begin{aligned}
\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz &= \sum_{r=1}^{j}\frac{1}{2\pi i}\oint_{\Gamma_r}\frac{f'(z)}{f(z)}\,dz + \sum_{r=1}^{k}\frac{1}{2\pi i}\oint_{C_r}\frac{f'(z)}{f(z)}\,dz \\
&= \sum_{r=1}^{j} n_r - \sum_{r=1}^{k} p_r \\
&= N - P
\end{aligned}$$

ROUCHE'S THEOREM

18. Prove *Rouché's theorem*: If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

Let $F(z) = g(z)/f(z)$ so that $g(z) = f(z)F(z)$ or briefly $g = fF$. Then if N_1 and N_2 are the number of zeros inside C of $f+g$ and f respectively, we have by Problem 17, using the fact that these functions have no poles inside C,

$$N_1 = \frac{1}{2\pi i}\oint_C \frac{f'+g'}{f+g}\,dz, \qquad N_2 = \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

Then

$$\begin{aligned}
N_1 - N_2 &= \frac{1}{2\pi i}\oint_C \frac{f'+f'F+fF'}{f+fF}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz \\
&= \frac{1}{2\pi i}\oint_C \frac{f'(1+F)+fF'}{f(1+F)}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz \\
&= \frac{1}{2\pi i}\oint_C \left\{\frac{f'}{f} + \frac{F'}{1+F}\right\} dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz \\
&= \frac{1}{2\pi i}\oint_C \frac{F'}{1+F}\,dz = \frac{1}{2\pi i}\int_C F'(1-F+F^2-F^3+\cdots)\,dz \\
&= 0
\end{aligned}$$

using the given fact that $|F| < 1$ on C so that the series is uniformly convergent on C and term by term integration yields the value zero. Thus $N_1 = N_2$ as required.

19. Use Rouché's theorem (Problem 18) to prove that every polynomial of degree n has exactly n zeros (fundamental theorem of algebra).

Suppose the polynomial to be $a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$, where $a_n \neq 0$. Choose $f(z) = a_nz^n$ and $g(z) = a_0 + a_1z + a_2z^2 + \cdots + a_{n-1}z^{n-1}$.

If C is a circle having centre at the origin and radius $r > 1$, then on C we have

15. Give an example to show that if $f(z)$ is analytic inside and on a simple closed curve C and $f(z) = 0$ at some point inside C, then $|f(z)|$ need not assume its minimum value on C.

Let $f(z) = z$ for $|z| \leqq 1$, so that C is a circle with centre at the origin and radius one. We have $f(z) = 0$ at $z = 0$. If $z = re^{i\theta}$, then $|f(z)| = r$ and it is clear that the minimum value of $|f(z)|$ does not occur on C but occurs inside C where $r = 0$, i.e. at $z = 0$.

THE ARGUMENT THEOREM

16. Let $f(z)$ be analytic inside and on a simple closed curve C except for a pole $z = \alpha$ of order (multiplicity) p inside C. Suppose also that inside C $f(z)$ has only one zero $z = \beta$ of order (multiplicity) n and no zeros on C. Prove that

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = n - p$$

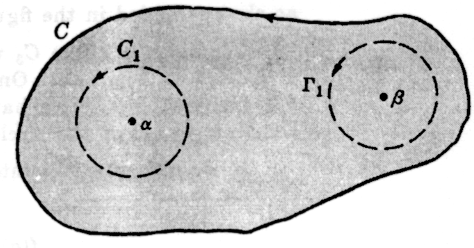

Fig. 5-7

Let C_1 and Γ_1 be non-overlapping circles lying inside C and enclosing $z = \alpha$ and $z = \beta$ respectively. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}\,dz \tag{1}$$

Since $f(z)$ has a pole of order p at $z = \alpha$, we have

$$f(z) = \frac{F(z)}{(z-\alpha)^p} \tag{2}$$

where $F(z)$ is analytic and different from zero inside and on C_1. Then taking logarithms in (2) and differentiating, we find

$$\frac{f'(z)}{f(z)} = \frac{F'(z)}{F(z)} - \frac{p}{z-\alpha} \tag{3}$$

so that

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{C_1} \frac{F'(z)}{F(z)}\,dz - \frac{p}{2\pi i}\oint_{C_1} \frac{dz}{z-\alpha} = 0 - p = -p \tag{4}$$

Since $f(z)$ has a zero of order n at $z = \beta$, we have

$$f(z) = (z-\beta)^n G(z) \tag{5}$$

where $G(z)$ is analytic and different from zero inside and on Γ_1.

Then by logarithmic differentiation, we have

$$\frac{f'(z)}{f(z)} = \frac{n}{z-\beta} + \frac{G'(z)}{G(z)} \tag{6}$$

so that

$$\frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}\,dz = \frac{n}{2\pi i}\oint_{\Gamma_1} \frac{dz}{z-\beta} + \frac{1}{2\pi i}\oint \frac{G'(z)}{G(z)}\,dz = n \tag{7}$$

Hence from (1), (4) and (7), we have the required result

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}\,dz = n - p$$

17. Let $f(z)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside C. Suppose that $f(z) \neq 0$ on C. If N and P are respectively the inside C with centre at a (see Fig. 5-6). If we exclude $f(z)$ from being a constant inside C_1, then there must be a point inside C_1, say b, such that $|f(b)| < M$ or, what is the same thing, $|f(b)| = M - \epsilon$ where $\epsilon > 0$.

Now by the continuity of $|f(z)|$ at b, we see that for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$||f(z)| - |f(b)|| < \tfrac{1}{2}\epsilon \quad \text{whenever} \quad |z - b| < \delta \qquad (1)$$

i.e.,

$$|f(z)| < |f(b)| + \tfrac{1}{2}\epsilon = M - \epsilon + \tfrac{1}{2}\epsilon = M - \tfrac{1}{2}\epsilon \qquad (2)$$

for all points interior to a circle C_2 with centre at b and radius δ, as shown shaded in the figure.

Construct a circle C_3 with centre at a which passes through b (dashed in Fig. 5-6). On part of this circle [namely that part PQ included in C_2] we have from (2), $|f(z)| < M - \frac{1}{2}\epsilon$. On the remaining part of the circle we have $|f(z)| \leqq M$.

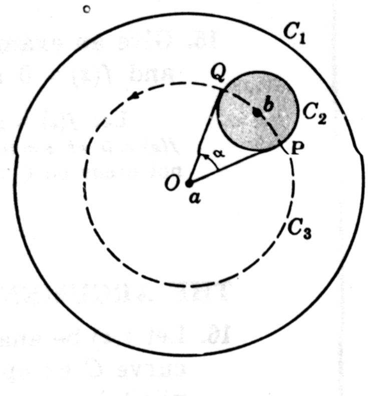

Fig. 5-6

If we measure θ counterclockwise from OP and let $\angle POQ = \alpha$, it follows from Problem 12 that if $r = |b - a|$,

$$f(a) = \frac{1}{2\pi}\int_0^\alpha f(a + re^{i\theta})\,d\theta + \frac{1}{2\pi}\int_\alpha^{2\pi} f(a + re^{i\theta})\,d\theta$$

Then

$$\begin{aligned}
|f(a)| &\leqq \frac{1}{2\pi}\int_0^\alpha |f(a + re^{i\theta})|\,d\theta + \frac{1}{2\pi}\int_\alpha^{2\pi} |f(a + re^{i\theta})|\,d\theta \\
&\leqq \frac{1}{2\pi}\int_0^\alpha (M - \tfrac{1}{2}\epsilon)\,d\theta + \frac{1}{2\pi}\int_\alpha^{2\pi} M\,d\theta \\
&= \frac{\alpha}{2\pi}(M - \tfrac{1}{2}\epsilon) + \frac{M}{2\pi}(2\pi - \alpha) \\
&= M - \frac{\alpha\epsilon}{4\pi}
\end{aligned}$$

i.e. $|f(a)| = M \leqq M - \dfrac{\alpha\epsilon}{4\pi}$, an impossible situation. By virtue of this contradiction we conclude that $|f(z)|$ cannot attain its maximum at any interior point of C and so must attain its maximum on C.

Method 2.

From Problem 12, we have

$$|f(a)| \leqq \frac{1}{2\pi}\int_0^{2\pi} |f(a + re^{i\theta})|\,d\theta \qquad (3)$$

Let us suppose that $|f(a)|$ is a maximum so that $|f(a + re^{i\theta})| \leqq |f(a)|$. If $|f(a + re^{i\theta})| < |f(a)|$ for one value of θ then, by continuity of f, it would hold for a finite arc, say $\theta_1 < \theta < \theta_2$. But in such case the mean value of $|f(a + re^{i\theta})|$ is less than $|f(a)|$, which would contradict (3). It follows therefore that in any neighbourhood of a, i.e. for $|z - a| < \delta$, $f(z)$ must be a constant. If $f(z)$ is not a constant, the maximum value of $|f(z)|$ must occur on C.

For another method, see Problem 57.

MINIMUM MODULUS THEOREM

14. Prove the *minimum modulus theorem:* Let $f(z)$ be analytic inside and on a simple closed curve C. Prove that if $f(z) \neq 0$ inside C, then $|f(z)|$ must assume its minimum value on C.

Since $f(z)$ is analytic inside and on C and since $f(z) \neq 0$ inside C, it follows that $1/f(z)$ is analytic inside C. By the maximum modulus theorem it follows that $1/|f(z)|$ cannot assume its maximum value inside C and so $|f(z)|$ cannot assume its minimum value inside C. Then since $|f(z)|$ has a minimum, this minimum must be attained on C.

Another method. Letting $n = 1$ in Problem 8 and replacing a by z we have,

$$|f'(z)| \leqq M/r$$

Letting $r \to \infty$, we deduce that $|f'(z)| = 0$ and so $f'(z) = 0$. Hence $f(z) = \text{constant}$, as required.

FUNDAMENTAL THEOREM OF ALGEBRA

10. Prove the *fundamental theorem of algebra*: Every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$, where the degree $n \geqq 1$ and $a_n \neq 0$, has at least one root.

If $P(z) = 0$ has no root, then $f(z) = \dfrac{1}{P(z)}$ is analytic for all z. Also $|f(z)| = \dfrac{1}{|P(z)|}$ is bounded (and in fact approaches zero) as $|z| \to \infty$.

Then by Liouville's theorem (Problem 9) it follows that $f(z)$ and thus $P(z)$ must be a constant. Thus we are led to a contradiction and conclude that $P(z) = 0$ must have at least one root or, as is sometimes said, $P(z)$ has at least one *zero*.

11. Prove that every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$, where the degree $n \geqq 1$ and $a_n \neq 0$, has exactly n roots.

By the fundamental theorem of algebra (Problem 10), $P(z)$ has at least one root. Denote this root by α. Then $P(\alpha) = 0$. Hence

$$\begin{aligned} P(z) - P(\alpha) &= a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n - (a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n) \\ &= a_1(z-\alpha) + a_2(z^2-\alpha^2) + \cdots + a_n(z^n - \alpha^n) \\ &= (z-\alpha)\,Q(z) \end{aligned}$$

where $Q(z)$ is a polynomial of degree $(n-1)$.

Applying the fundamental theorem of algebra again, we see that $Q(z)$ has at least one zero which we can denote by β [which may equal α] and so $P(z) = (z-\alpha)(z-\beta)\,R(z)$. Continuing in this manner we see that $P(z)$ has exactly n zeros.

GAUSS' MEAN VALUE THEOREM

12. Let $f(z)$ be analytic inside and on a circle C with centre at a. Prove *Gauss' mean value theorem* that the mean of the values of $f(z)$ on C is $f(a)$.

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \qquad (1)$$

If C has radius r, the equation of C is $|z-a| = r$ or $z = a + re^{i\theta}$. Thus (1) becomes

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})\,ire^{i\theta}}{re^{i\theta}}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

which is the required result.

MAXIMUM MODULUS THEOREM

13. Prove the *maximum modulus theorem*: If $f(z)$ is analytic inside and on a simple closed curve C, then the maximum value of $|f(z)|$ occurs on C, unless $f(z)$ is a constant.

Method 1.

Since $f(z)$ is analytic and hence continuous inside and on C, it follows that $|f(z)|$ *does* have a maximum value M for at least one value of z inside or on C. Suppose this maximum value is not attained on the boundary of C but is attained at an interior point a, i.e. $|f(a)| = M$. Let C_1 be a circle

MORERA'S THEOREM

7. Prove *Morera's theorem* (the converse of Cauchy's theorem): If $f(z)$ is continuous in a simply-connected region $\mathcal{R}$ and if

$$\oint_C f(z)\,dz = 0$$

around every simple closed curve C in $\mathcal{R}$, then $f(z)$ is analytic in $\mathcal{R}$.

If $\oint_C f(z)\,dz = 0$ independent of C, it follows by Problem 17, Chapter 4, that $F(z) = \int_a^z f(z)\,dz$ is independent of the path joining a and z, so long as this path is in $\mathcal{R}$.

Then by reasoning identical with that used in Problem 18, Chapter 4, it follows that $F(z)$ is analytic in $\mathcal{R}$ and $F'(z) = f(z)$. However, by Problem 2, it follows that $F'(z)$ is also analytic if $F(z)$ is. Hence $f(z)$ is analytic in $\mathcal{R}$.

CAUCHY'S INEQUALITY

8. If $f(z)$ is analytic inside and on a circle C of radius r and centre at $z = a$, prove *Cauchy's inequality*

$$|f^{(n)}(a)| \leqq \frac{M \cdot n!}{r^n} \qquad n = 0, 1, 2, 3, \ldots$$

where M is a constant such that $|f(z)| < M$.

We have by Cauchy's integral formulae,

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0, 1, 2, 3, \ldots$$

Then by Problem 3, Chapter 4, since $|z - a| = r$ on C and the length of C is $2\pi r$,

$$|f^{(n)}(a)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right| \leqq \frac{n!}{2\pi}\cdot\frac{M}{r^{n+1}}\cdot 2\pi r = \frac{M \cdot n!}{r^n}$$

LIOUVILLE'S THEOREM

9. Prove *Liouville's theorem*: If for all z in the entire complex plane, (i) $f(z)$ is analytic and (ii) $f(z)$ is bounded [i.e. we can find a constant M such that $|f(z)| < M$], then $f(z)$ must be a constant.

Let a and b be any two points in the z plane. Suppose that C is a circle of radius r having centre at a and enclosing point b (see Fig. 5-5).

From Cauchy's integral formula, we have

$$f(b) - f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-b}\,dz - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$
$$= \frac{b-a}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-b)(z-a)}$$

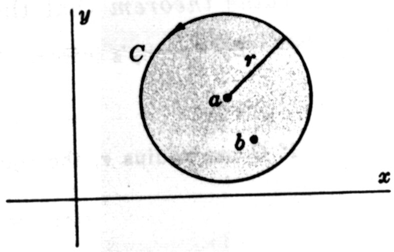

Fig. 5-5

Now we have

$$|z-a| = r, \qquad |z-b| = |z-a+a-b| \geqq |z-a| - |a-b| = r - |a-b| \geqq r/2$$

if we choose r so large that $|a-b| < r/2$. Then since $|f(z)| < M$ and the length of C is $2\pi r$, we have by Problem 3, Chapter 4,

$$|f(b) - f(a)| = \frac{|b-a|}{2\pi}\left|\oint_C \frac{f(z)\,dz}{(z-b)(z-a)}\right| \leqq \frac{|b-a|\,M(2\pi r)}{2\pi(r/2)r} = \frac{2\,|b-a|\,M}{r}$$

Letting $r \to \infty$ we see that $|f(b) - f(a)| = 0$ or $f(b) = f(a)$, which shows that $f(z)$ must be a constant.