# Lecture 20:
# Hamilton's Principle – Lagrangian and Hamiltonian Dynamics.

# Force of Constraints



Force of constraint → forces exist which keeps the particle in contact with a surface or in a certain position.

Smooth horizontal surface, simple to define !!.

Many real life situations difficult to express analytically.

Newton's Equation → total forces acting on the body. (we must know all the forces).

Alternate method to deal with complicated problems.

Hamilton's principle and the application of this principle leads to Lagrange's equations of motion.

Equivalent to Newton's equations.

**We will limit ourselves to Conservative system !!**

# Conservative Forces



- If the forces acting on a particle are conservative, then the total energy of the particle, T+V is conserved.

# Constraints



- Some systems have constraints on the motion.
  - *Move along a wire*
  - *Swing on a pendulum*
  - *Stay on a surface*
  - *Roll without slipping*

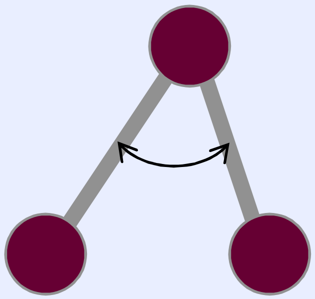
- Constraints → remove coordinates from the motion or link coordinates together.

# Degrees of Freedom



- Each particle begins with three *degrees of freedom* ($f$).
  - *Motion in 3 dimensions*
  - $N$ *free particles:* $f = 3N$

- The constraints reduce the number of degrees of freedom $f$.
  - *Number of constraints* $k$
  - $f = 3N - k$



$$N = 3$$

$$k = 2$$

$$f = 7$$

# Holonomic Constraints



- A *holonomic* constraint is a function of only the coordinates and time.
  - $\phi(x_1, x_2, x_3, t) = 0$
  - *Fixed constraint without time*
  - *Moving constraint with time*

- *Non-holonomic* constraints include terms like velocity or acceleration.

## Configuration Space



Dynamical system whose configuration evolves in time according to some deterministic law.
A particle is a dynamical system whose configuration space is the set of allowed positions of the particle.

Configuration Space – set of variables which characterizes what the system is doing at a given time.
A Continuous space; dimension is the number of degrees of freedom of the system.

Dynamical evolution –a curve in configuration space.

Generalized coordinates: A dynamical system with N degrees of freedom will have its configuration described by N generalized coordinates.

Generally denoted by $\mathbf{q}_{i}$.
These are coordinates on some space known as manifold and motion of the system is specified by a curve on this manifold.

Phase Space: space of possible initial conditions for the system; the set of conditions you need to uniquely specify a solution to the equations of motion.

Positions are points in some manifold, the velocities are tangent vectors along curves through the points. The set of all positions and possible tangent vectors is called the velocity phase space (space of possible initial conditions for the equations of motion).

# What is Action?



- Action = s = $\int$(KE – PE) dt      [ (from t1-t2) ]
- KE – PE is known as the Lagrangian
- Commonly written as:
  - *$L(x,v,t) = T(v) - V(x)$*

*Motion of a particle → the path that minimizes the action.*

***Nature follows the path where s is smallest***

# Action



- The time integral of the Lagrangian is the *action*.
  - *Action is a functional*
  - *Extends to multiple coordinates*

$$S = \int_{t_1}^{t_2} L(q, \dot{q}; t)\, dt$$

- The Euler-Lagrange equations are equivalent to finding the least time for the action.
  - *Multiple coordinates give multiple equations*

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

- This is ***Hamilton's principle***.

# Hamilton's Principle



Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energy.

How to construct a Lagrangian
How to determine the action
Find the true path nature takes.

Configuration space and Phase Space.

- Emphasis→Formalism of Lagrangian and Hamiltonian approaches
- Less emphasis on solving equations.

- Handle all kinds of dynamics: Classical and Quantum

## Lagrange Eqn of motion for the 1D Harmonbic Oscillator



- ***Write down the Lagrangian function***
- ***Apply Lagrange eqs. Of motion***

# Simple Harmonic Oscillator



$$T = \tfrac{1}{2} m\dot{x}^2$$

$$V = \tfrac{1}{2} kx^2$$

$$L = T - V = T = \tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} kx^2$$

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$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial x} = -kx$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} + kx = 0$$

- The 1-D simple harmonic oscillator has one force.
  - $F = -kx$
  - *Conservative force*
- Select $x$ as the generalized coordinate.
  - *$T$, $V$ in terms of generalized coordinate and velocity*
- Use Lagrange's EOM.
  - *Usual Newtonian equation*

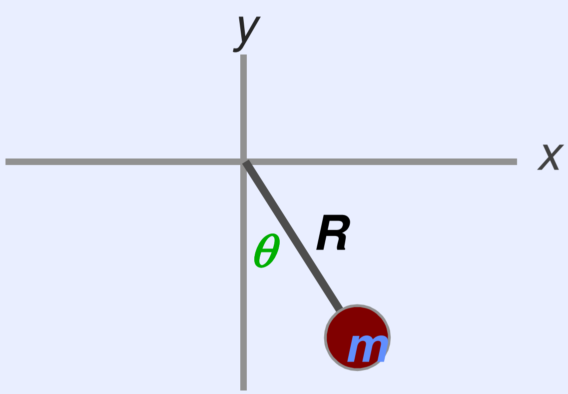
# Problem.

Generalized co-ordinates for a point particle moving on the surface of a hemisphere of radius R whose center is at the origin.

# Solution for Plane Pendulum



- The plane pendulum is a 2-D system.
  - *Two degrees of freedom*
  - *One constraint .*
  - *Angle $\theta$ as generalized coordinate*




**• Write down the Langrangian function**
**• Apply Lagrange eqs. Of motion**

# Constraints



- A plane pendulum (2D motion) swings freely from a fixed point. Write the constraint on the Cartesian coordinates.

- The pendulum length $L$ is constant. Eq. of Constraint:
  - $L^2 = x^2 + y^2$ *is the constraint*

Circular Path
Length is fixed.

Degrees of freedom?
How many Constraints?

We need eqns.
Depending on degrees of freedom.

**Generalized Coordinate**

# Constraints



- A plane pendulum (3D point of view) swings freely from a fixed point.
- Eqns. of constraint?

How many Constraints?

Degrees of freedom?

We need DEs depending on degrees of freedom.

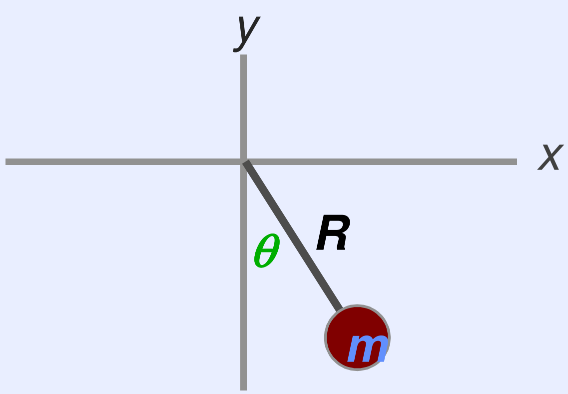
**Apply the Lagrangian on the generalized coordinate**

# Plane Pendulum

- The plane pendulum is a 2-D system.
  - *Two degrees of freedom*
  - *One constraint (R constant).*
  - *Angle $\theta$ as generalized coordinate*

$$x = R\sin\theta \qquad y = -R\cos\theta$$

$$\dot{x} = R\dot{\theta}\cos\theta \qquad \dot{y} = R\dot{\theta}\sin\theta$$




$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \tfrac{1}{2}mR^2\dot{\theta}^2$$

$$V = mgy = -mgR\cos\theta$$

$$L = \tfrac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta$$

$$\frac{\partial L}{\partial\dot{\theta}} = mR^2\dot{\theta} \qquad \frac{\partial L}{\partial\theta} = -mgR\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = mR^2\ddot{\theta} + mgR\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{R}\sin\theta = 0$$

# Using the Lagrangian



- Identify the degrees of freedom.
  - *One generalized coordinate for each*
  - *Velocities as functions of generalized coordinates*
- Find the Lagrangian
  - *Kinetic energy in terms of velocity components*
  - *Potential energy in terms of generalized coordinates*
- Write Lagrange's equations of motion.

# Problem



Equation of motion of a single particle moving (near the earth) under gravity. (three dimensional motion)

# Soln.

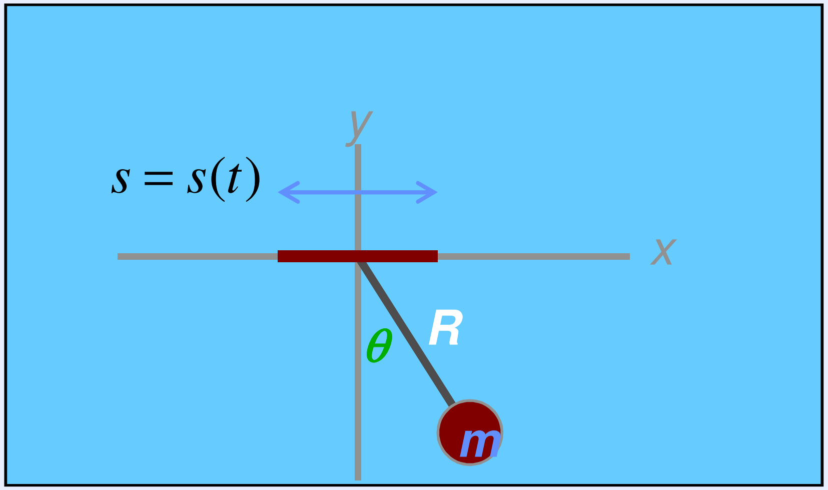
$$T = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right), \qquad V = mgz$$
$$L = T - V = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0. \quad \text{Lagrange's equation}$$

$$\begin{cases} \dfrac{d}{dt}(m\dot{x}) = 0, \\ \dfrac{d}{dt}(m\dot{y}) = 0, \\ \dfrac{d}{dt}(m\dot{z}) + mg = 0, \end{cases} \qquad \text{or} \qquad \begin{cases} \dot{x} = \text{const.}, \\ \dot{y} = \text{const.}, \\ \dot{z} = -g. \end{cases}$$

# Oscillating Support

- The moving support depends only on time.



# Oscillating Support

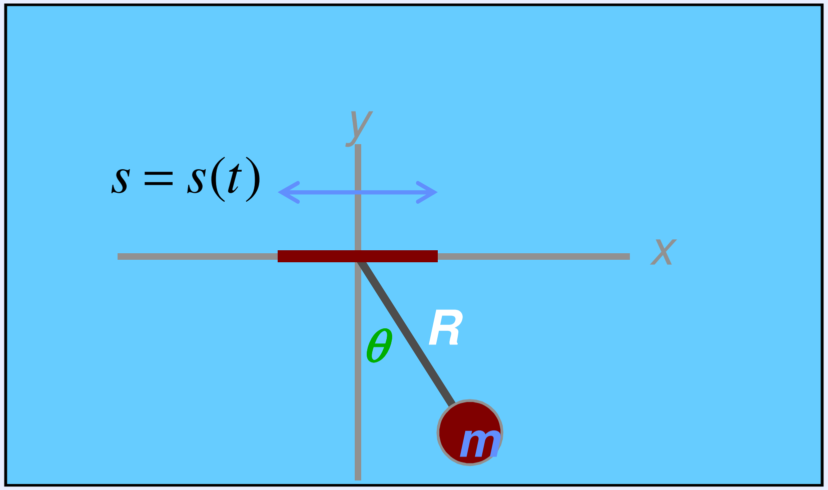
- The moving support depends only on time.
  - *Not a new degree of freedom*
  - *add to* $x$
  - *Angle* $\theta$ *still the generalized coordinate*

$$x = R\sin\theta + s(t)$$

$$\dot{x} = R\dot{\theta}\cos\theta + \dot{s}(t) = \dot{u} + \dot{s}$$

$$y = -R\cos\theta$$

$$\dot{y} = R\dot{\theta}\sin\theta$$




$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$T = \tfrac{1}{2}m\left(\dot{u}^2 + 2\dot{u}\dot{s} + \dot{s}^2 + \dot{y}^2\right)$$

$$T = \tfrac{1}{2}m\left(R^2\dot{\theta}^2 + 2R\dot{\theta}\dot{s}\cos\theta + \dot{s}^2\right)$$

$$V = mgy = -mgR\cos\theta$$

# Forced Oscillator



- The support term is time dependent.
  - *Provides a driving force*
- The Lagrangian method gives the equation of motion.

$$L = T - V =$$
$$\tfrac{1}{2} m\left(R^2\dot{\theta}^2 + 2R\dot{\theta}\dot{s}\cos\theta + \dot{s}^2\right) + mgR\cos\theta$$

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$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} + mR\dot{s}\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mR^2\ddot{\theta} + mR\ddot{s}\cos\theta - mR\dot{\theta}\dot{s}\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -mR\dot{\theta}\dot{s}\cos\theta - mgR\sin\theta$$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = mR^2\ddot{\theta} + mgR\sin\theta + mR\ddot{s}\cos\theta = 0}$$