Thursday

SIGNALS AND SYSTEMS (CT203)

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Grading Policy

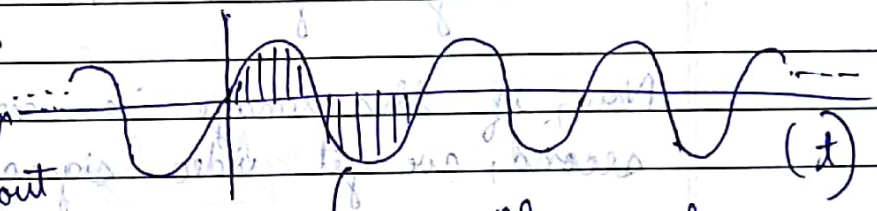
1st & 2nd SEM = 20% & 20%

Subject to at least 80% attendance ← Endsem = 40%

Tutorial = 10%, Attendance = 10%

Alan V. Oppenheim, Signals and Systems

Def^n of Signal :→



Without these dots, it's not a sinusoidal signal.

Sequentially samples are taken

If we randomly take the samples and connect the dots, we WILL NOT get a sinusoidal signal. The crosses are determined by amplitude of signal

→ Images are read in MATLAB by using the imread function.

↳ A matrix of pixel intensities is taken. Eg :- 512×512

↳ The order of the entries in the matrix is very important. Any sort of interchanging will tamper the image. Everything should be sequentially.

→ A signal is a function of 1, 2, 3 or 'n' independent variables. However, it is Not merely a mathematical function, rather it carries information. Many times,

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Info in a signal is contained in the pattern of vibrations form of some form

this information is carried in the sequence of samples of given signal.

Let image =

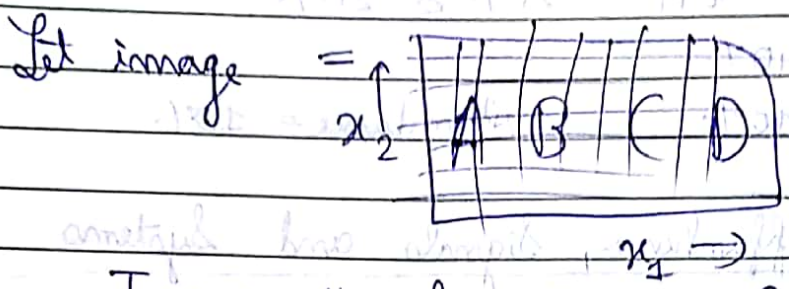
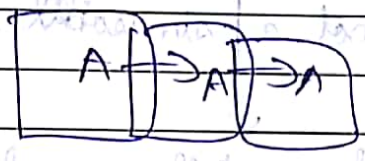


Image $= I(x_1, x_2)$

Image signals are 2D signals.

Now, if this image is moved by 30 frames per second, we get video signal



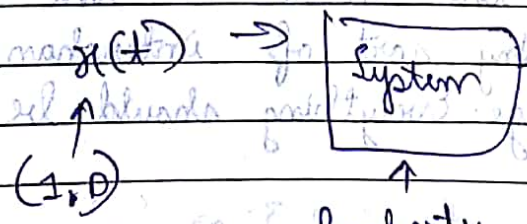
$\Rightarrow$ Video $= V(x_1, x_2, x_3)$

Video is a 3-D signal.

→ In human brain, auditory and visual perception mechanism process contextual and sequential information.

Def^n of Systems:- Can be hardware circuit, but not by itself. Software is connected.

→ A system takes an input signal and provides an O/P signal.



property of a system is embedded onto signal.

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Formally, a system is an entity which takes I/P signal and processes it according to its property and gives a desired O/P.

→ In Signals and systems, neither signal alone or system alone is important. Rather coupling b/w signal and system is important. Many a times, this coupling is non-linear and it is one of the key challenges to understand this coupling.

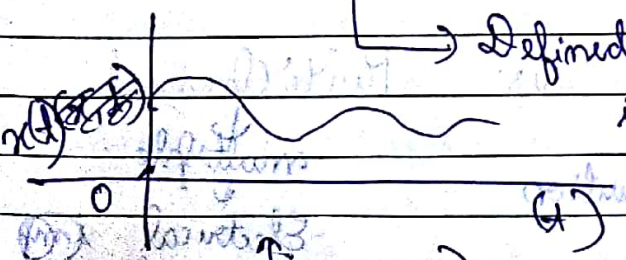
Signals and Systems
↑
Coupling
↓
Non-linear
↓
Difficult to quantify

There can't be any signal, without a system producing it. (∵ Why coupling? ans). Signal is never an individual quantity. There is always a system in the background, which may / may not be perceptible to us. Therefore, signals and systems are connected and not independent.

Classification of signals

① Continuous time (→ Analog signal.) vs discrete time.

→ Defined for each and every instant of time.



(1D signal)

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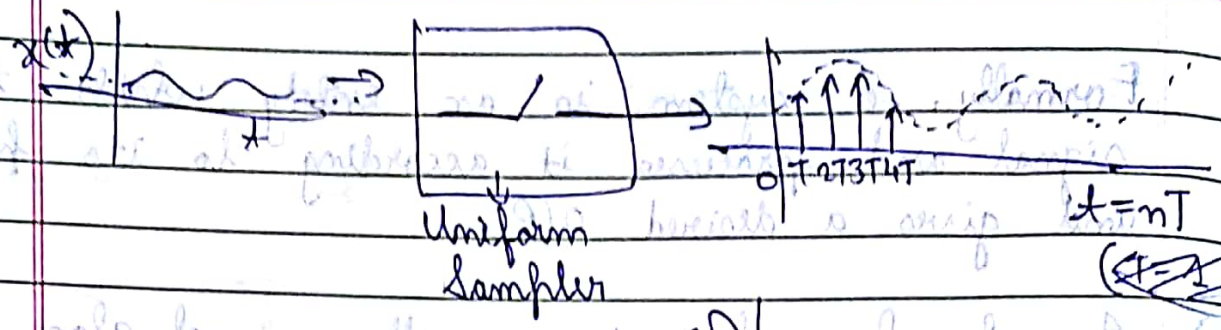
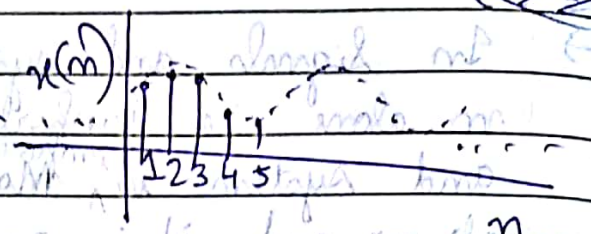


(~~t = 1 sec~~)



(for T = 1 sec)

Discrete time signal is defined at discrete instants of time

Since dots are to be shown to show that it's for infinite time. If we don't show the dots, alternative is :

$$\boxed{\begin{array}{ll} x(t) \text{ where } t \in \mathbb{R} & (\because \text{Real Nos}) \\ x(n) \text{ where } n \in \mathbb{Z} & (\because \text{Integers}) \end{array}}$$

(2) Single dimensional vs. Multi dimensional Signals

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⇒ Hearing lips and seeing voices ⇒ A research paper to read.

System produces a signal. Some source in the system produces it.

⇒ If the signal is the function of 1 independent variable, it is 1-D. If it is function of n different dimensions, it is n-D.

(3) Single Channel vs. Multi Channel

Source & production

generated by

Eg:- Signal ⇒ single source
Audio O/P of mono speaker

multiple

Electrical (m) stimulus in brain

↳ EEG, EGG, ECG

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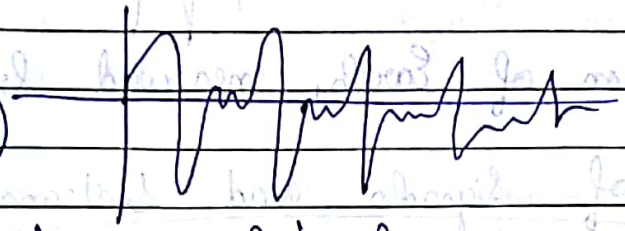
4) Deterministic vs Random / Stochastic

→ $x(t) = A \sin(2\pi 50 t)$ → sinusoid of 50Hz signal.

→ |a| (vowel a spoken)



(Human Speech)

→ Deterministic signals' values can be predicted by mathematical formula. But processes like speech produce stochastic / random signal. If we speak 'a' 5 times, each will be different. It can never be exactly the same in a lifetime.

↳ No mathematical formula can be defined for speech ⇒ random.

Representation of Signals

1) Functional Representation

$x(t) = A \sin(2\pi 100 t)$: Signals which can be represented by math formula.

→ $x(n) = A \sin(2\pi 100 n)$ $(\because n \in \mathbb{Z})$

2) IPA → Intelligent personal assistants. SIRI, Amazon Echo, Cortana, Alexa, Google assistant. All these devices operate on speech. They catch the frequencies in speech through microphone, called frequency response.

Since mathematical not possible, they are

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p) scientific reason why we can speak?
q) meaning of 'calculus'?

Tutorial

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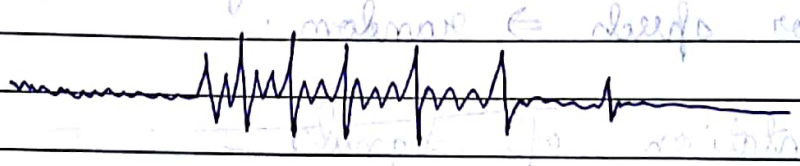
captured by graphical representation. ~~The~~ This is done by Acoustic pressure variation at lips.

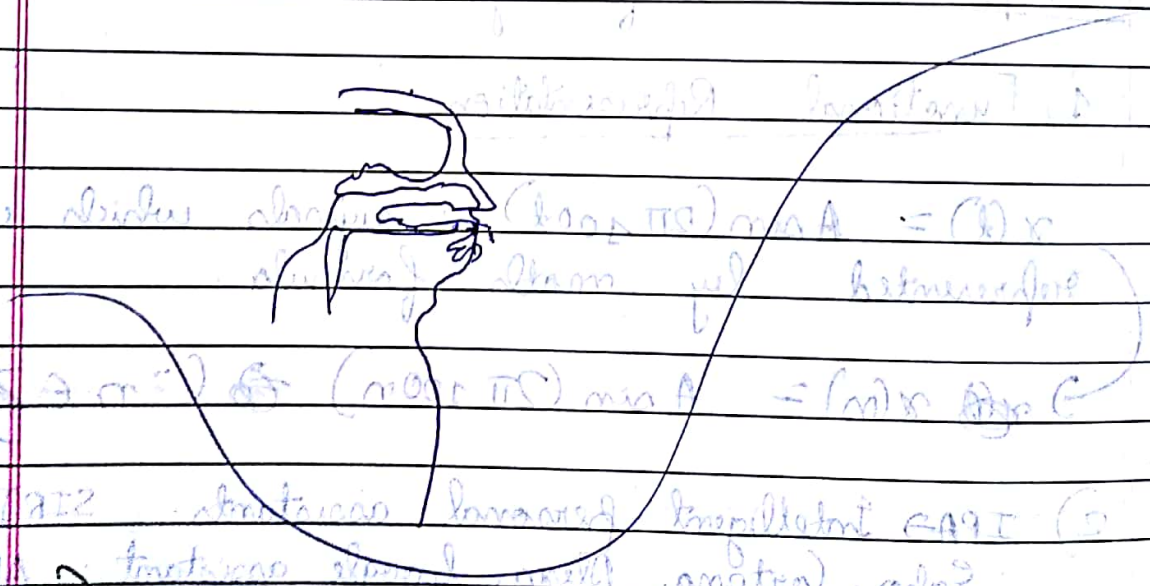
→ Vibration of Earth, measured by seismic signals.

Examples of Signals and Systems

1) Human speech Signal
Syllable has vowel as nucleus. There can't be syllables without vowels. Vowels are high amplitude periodic signals.

Shape





2) Humming Sounds by humans
↳ not at all similar to speech even though they are coming from similar speech production mechanism (∵ not same though)

⟹ In Signals & Systems, signals produced by natural systems capture the signature of the System that has produced it. In particular, any slight change in system config^n could make a significant change in signal production.

↳ Humming is different from speech because the speech production system's mechanism/ config^n is different. Both

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Query by Humming (QBH)

3] Seismic signal : → related to vibration of earth.

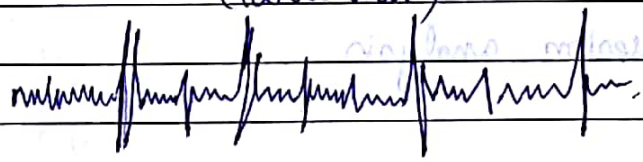
→ Vibration is deliberate for oil exploration.

→ Vibration is undesirable for mankind → Earthquake

→ " " also by humming sound from the earth. Natural but not an earthquake. Singing by the Earth.

4] Heart-Sound Signal

(Heart beat)



→ Heart is a system.

→ Some sort of input/ stimulus is needed to generate signal.

→ Rhythmic contraction and expansion of the walls of the heart (sudden closure & opening) creates signals.

Sudden closure ⟹ lub sound

Heart wall relaxation ⟹ dub sound

→ This lub-dub sound is called heart beat, plotted above.

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→ Heartbeats are person-specific. Can be used as identification.

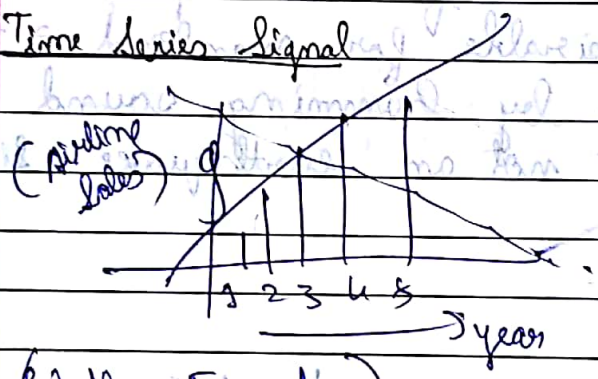
5] Trachea Sound

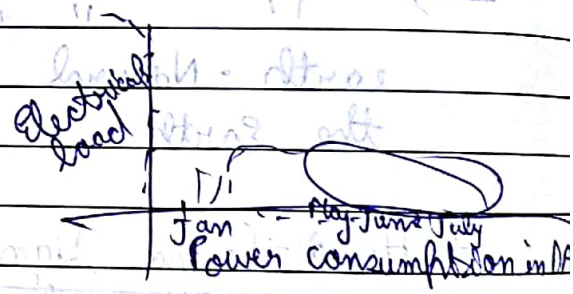
→ Carry sounds from not only lungs but also from other parts of the body.

6] Infant Cry Signal (Newborn's first cry)

7] SIDS
(Sudden Infant Death Syndrome)
death of infant without any understandable reason within 3 months of birth, it may happen.

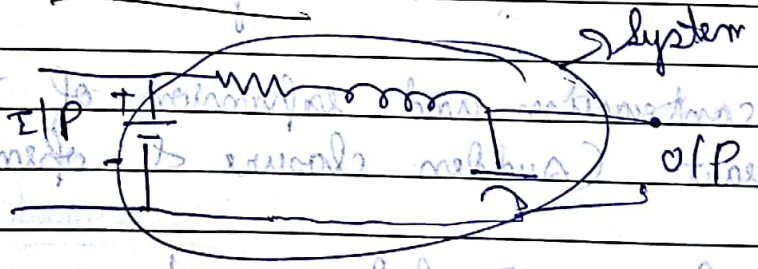
8) Time Series Signal





(Weather Forecasting)
↳ Regression analysis.

Electric Circuit



SHM (Spring)
Simple & Inverted pendulum
Used in space research.

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Monday

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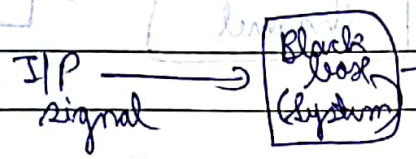
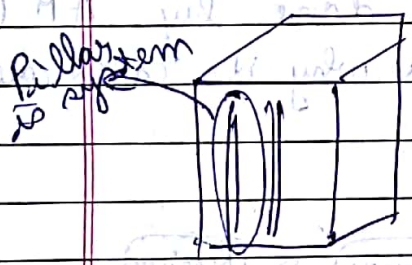
* Distinct Problems in Signals & Systems

1] Analysis of systems :-



Problem:- We need to analyze the O/P response of a black box system to investigate characterisation of system.

⇒ Vitthala Temple (Musical Pillars) Hampi / Karnataka (56 pillars)



⇒ Our tapping / excitation is I/P signal
→ Output response is music.

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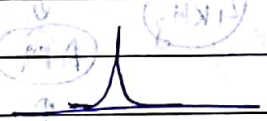
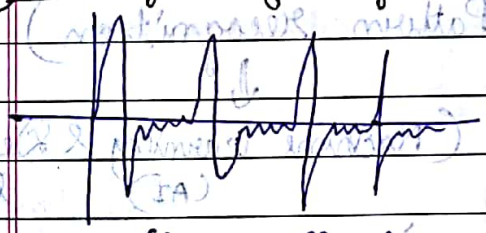
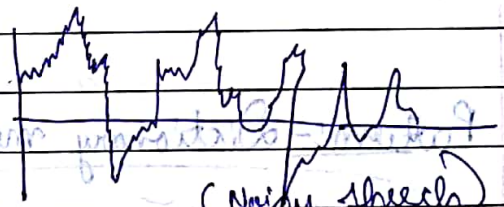
→ Impulse like excitation



2] Design of systems :-

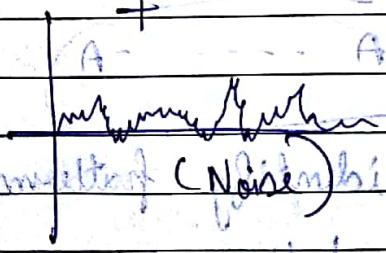


Clean Speech →



(Noisy speech)

+



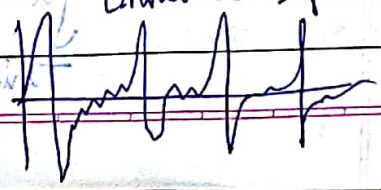
(Noise)

↓

Design filter

↓

Enhanced speech



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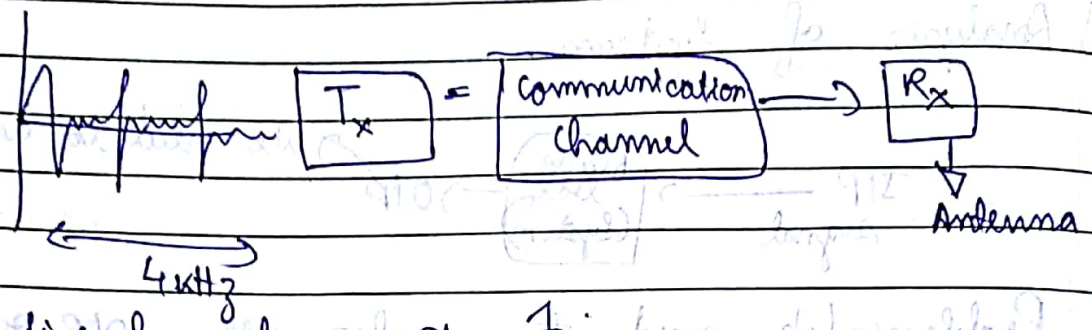
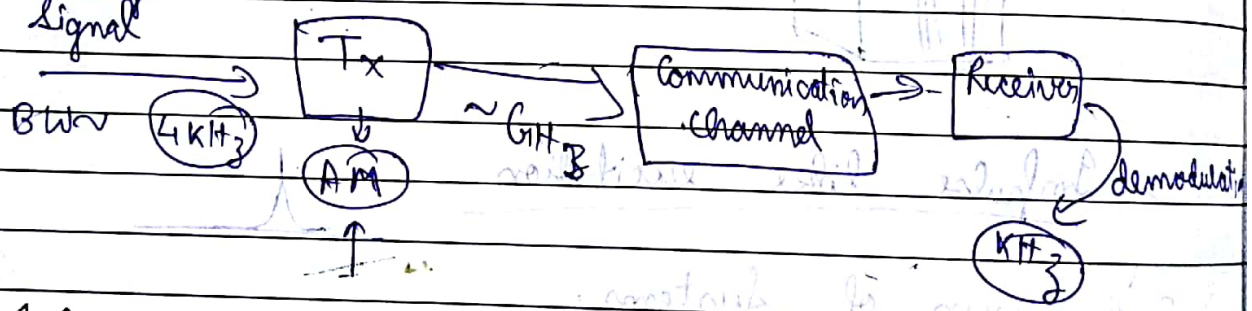
③ Design of signals:-

Analog communication & transmission line -



size of antenna $\propto \frac{1}{\text{Band width}}$

→ Aim is to make the bandwidth of signal artificially large, to reduce antenna cost (due to reduction in size). This is done by AM (Amplitude Modulation). That is why its called design of signals.



④ Information Extraction (Pattern recognition)

⇓

(Machine learning & Deep learning) (AI)

Pattern:- Dictionary meaning is repetitive signature

A A A A A A ---- A

We are trained to identify patterns - Training

This is called testing.

AI is imitating human intelligence to a certain extent

In any pattern recognition, there are two blocks-

I/P (1) Feature Extraction → (2) Pattern Classifier

Baye's Theorem

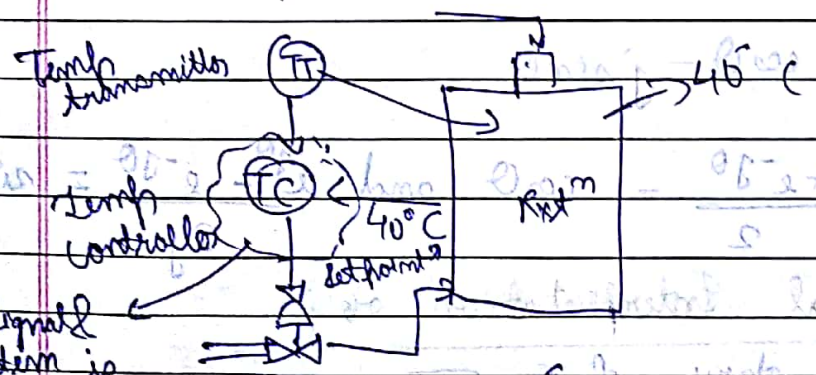
Contextual Knowledge:-

If one eye is here, another can't be at corner. Symmetry is required. This is a very challenging task to implement.

Scope of pattern recognition:- Face, finger prints, palm recognition, iris recognition, DNA, signature, walking pattern, "texture" segmentation, hand geometry

Feature Extraction → Handcrafted features (fixed algorithm)
→ Learned feature → Deep Learning

(5) Instrumentation and Control
↓
control system



system is exploited in design of TC & its algorithm.

(Imitation of feedback. It is a feedback control system)

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Wednesday

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Elementary Mathematical Background

→ In SAS, we model signals as vectors.

Signal → Vector → {magnitude, direction (phase)}

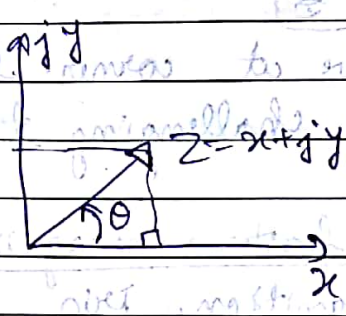
(Complex Numbers)

Complex Numbers :-

Cartesian representation ← $Z = x + jy$

$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$



(∵ Argand's diagram)

$j = \sqrt{-1}$ (∵ Imaginary unit / part)

$x = Re(Z)$, $y = Img(Z)$

Now,

polar, $\boxed{Z = re^{j\theta}}$ → polar representation

$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$ ⇒ $\boxed{Z = r\cos\theta + jr\sin\theta}$

⇒ $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$ (∵ Euler's formula) (B1)

Find out the motivation for trigonometry.

$e^{-j\theta} = \cos\theta - j\sin\theta$

$\therefore \frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta$ and $\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$

Geometrical Interpretation of j

The story of $j = \sqrt{-1}$

Paul, Stanford University (USA)

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Wednesday

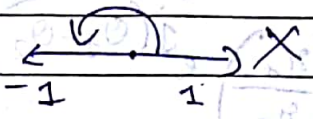
$j = l \angle \theta$

$j^2 = j \times j = l^2 \angle \theta + \theta$

$= l^2 \angle 2\theta$

$-1 = j^2 \angle 2\theta$

$(-1) = 1 \angle 180^\circ$



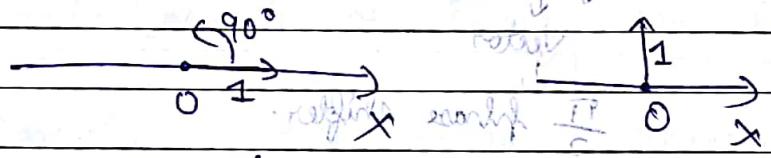
$\therefore \; 1 \angle 180^\circ = l^2 \angle 2\theta$

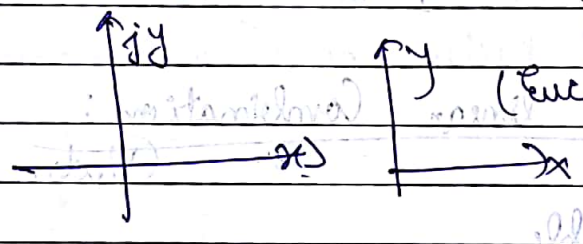
$l^2 = 1$ & $2\theta = 180^\circ$

$\therefore l^2 = 1$ & $\theta = 90^\circ$

$l = +1$

$\boxed{j = l \angle \theta = 1 \angle 90^\circ}$ ($\because$ Inference)





(Euclidean) $\Rightarrow$ Argand j and y are parallel so, we multiply.

$\therefore$ j is a vector having magnitude $= 1$ and an angle of 90° with +ve direction of x-axis.

Algebraic Operations on Complex Numbers 2/8/18

$z_1 = x_1 + j y_1, \quad z_2 = x_2 + j y_2$

$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

$z_1 - z_2 =$

$z_1 = r_1 e^{j\theta_1} \quad r_1 = \sqrt{x_1^2 + y_1^2}, \; \theta_1 = \tan^{-1}\left(\frac{y_1}{x_1}\right)$

$z_2 = r_2 e^{j\theta_2} \quad r_2 = \sqrt{x_2^2 + y_2^2}, \; \theta_2 = \tan^{-1}\left(\frac{y_2}{x_2}\right)$

$z_1 z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}$

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Inference:- When two complex nos are multiplied then, their corresponding mags are multiplied and their corresponding phases are added.

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{j(\theta_1 - \theta_2)}$$

Inference:- On division, mags are divided and phase angle of denominator is subtracted from phase angle of numerator.

→ $j = \sqrt{-1}$ → Imaginary unit
↓
Vector
↓
$\frac{\pi}{2}$ phase shifter.

Concept of linear Combination:
(Vector Algebra)

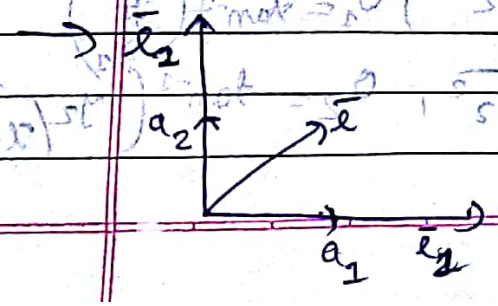
Scalar multiple

Newton's 2nd law of motion:-

Force (F) = mass (m) × accl (a)

$\underset{\text{vect}}{F} = \underset{\text{scal}}{m} \times \underset{\text{vector}}{\bar{a}}$

One vector can be written as a scalar multiple of another vector. We say "another" because F and $\bar{a}$ are not same.



$\bar{e} = a_1 \bar{e}_1 + a_2 \bar{e}_2$

For simplicity,

$e = a_1 e_1 + a_2 e_2$ (Linear Combination)

Linear Algebra

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Thursday

Calculus:- Counting with the help of stones → in ∫, it means area

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$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

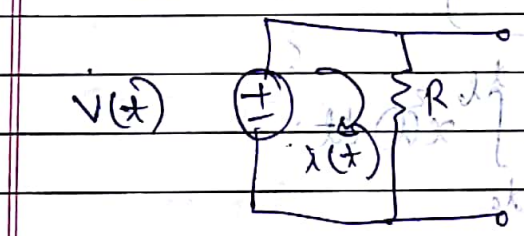
$$n \to \infty \Rightarrow a\left(\frac{1}{1-r}\right) = \frac{a}{1-r}$$

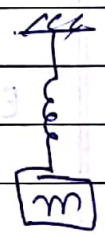
$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\int_{x=a}^{x=b} f(x)\,dx = \lim_{n \to \infty} \sum_{x=a}^{b} f(x)\,h$$

Concept of Energy and Power :-

Something which requires energy to operate (∵ electrical energy in this case) is called load (∵ electrical load)



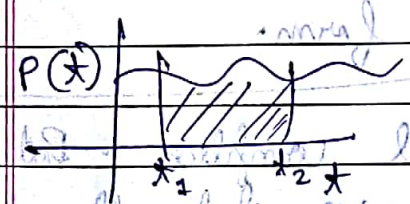


$E = \frac{1}{2} m A^2 \omega^2$

(a) Resistive Circuit

Energy = Power × time

$E = P \times t$



$$\Rightarrow E = \int_{t_1}^{t_2} P(dt)$$

$$\Rightarrow E = \lim_{h \to 0} \sum_{(n)} p(t_n) \cdot h = \int_{t_a}^{t_b} p(t)\,dt = \int_{t_a}^{t_b} v(t)\,i(t)\,dt$$

This represents all values of n.

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Thursday

$$= \int_{t_a}^{t_b} i^2(t) R\,dt = \int_{t_a}^{t_b} \frac{V^2(t)}{R}\,dt$$

→ Physical system has an attribute that it respond to energy.

Q) Why to study E and P ?

→ Since physical systems such as electric circuit, human ear, mass-spring damper system, etc respond to energy, to investigate the characteristics of this energy transfer, to find the energy of a system, to produce a signal or, in turn to find energy content in a signal.

$$\boxed{\text{Physical system}} \rightarrow x(t) \rightarrow \text{speech, sonic, electric voltage}$$

$$\text{Energy in } x(t) = K \int_{t_a}^{t_b} x^2(t)\,dt$$

In last page, the energy formula for electric circuit, was like that as there $x(t)$ was $v(t)$ or $i(t)$. Here $x(t)$. There const R. We don't know here. Hence, this form.

→ In SAS, $x(t)$ is in general complex. But in the formula, E is real. Hence, we take the magnitude of $x(t)$

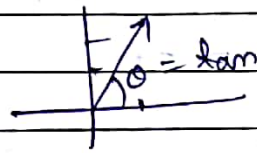
$$\Rightarrow E = \int_{t_a}^{t_b} |x(t)|^2\,dt$$

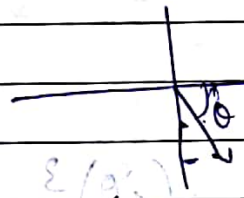
⟹ In SPS, we deal with infinite duration signals.

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt$$

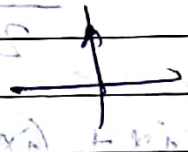
$$\boxed{E = \int_{-\infty}^{\infty} |x(t)|^2 dt}$$

2/8/18

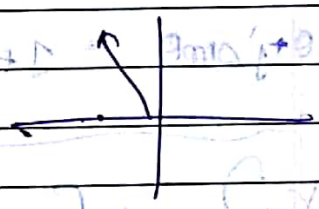
1) $z_1 =$ 

$z_2 =$ 

$z_3 = 2e^{j\pi/2}$

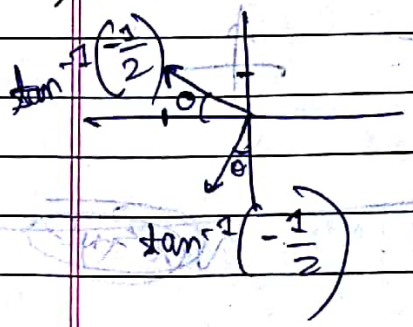
$= 2j$ 

$z_4 = \sqrt{3}\cos\left(\frac{-\pi}{3}\right)$

$= \frac{-\sqrt{3}}{2} + j\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)$ 

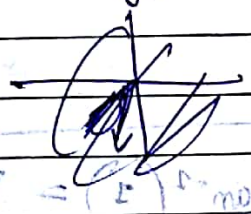
$= -\frac{\sqrt{3}}{2} + \frac{3j}{2}$

2) (a) $z = -4+2j$

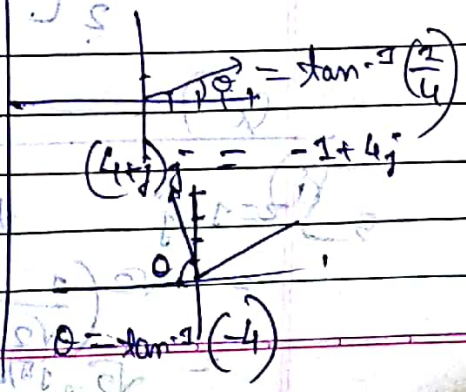


$(-4+2j)\,j$

$-4j - 2$



(b) $z = (4+j)$



$(4+j)j = -1+4j$

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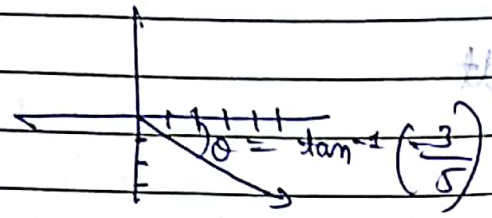
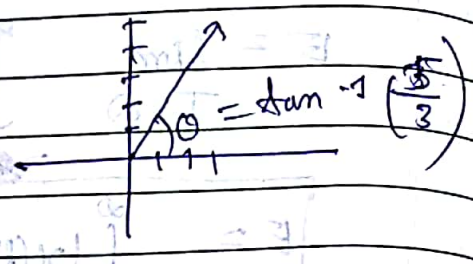
(a) $z = 5 - 3j$ $\qquad (5-3j)j = 5j + 3$





3] (a) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

$$\cos\theta = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\sin\theta = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

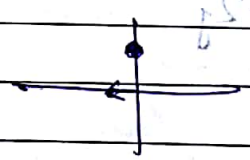
$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!}$$

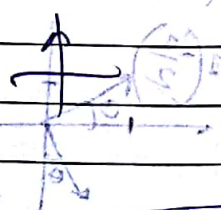
$$\theta \to x = 1 + jx + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!}$$

$$\cos\theta + j\sin\theta = 1 + jx + \frac{(jx)^2}{2!}$$ 1 by all.

4] (a) $\frac{1}{2}\left[\cos\pi + j\sin\pi\right]$ $\qquad$ (b) $\cos\frac{\pi}{2} + j\sin\pi/2$

$\frac{1}{2}\left[-1 + 0\right] =$



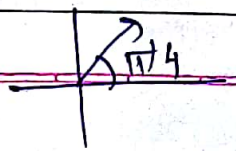


(c)

5] (i) $z = 1 + j$ $\qquad \tan^{-1}(1) = \pi/4$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right)$$

$$= \sqrt{2}\, e^{j\pi/4}$$



$\sqrt{x^2 + y^2}$

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Thursday

Learn how to find Taylor series for any function

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(b) $5+0j$

$\sqrt{5^2+0^2} = 5$

(c) $\tan^{-1}\left(\frac{\sqrt{3}}{2}/\frac{1}{2}\right)$

$= \tan^{-1}(\sqrt{3}) = 60^\circ$

$= 5\angle 60^\circ$

6] (a) $(x+iy)(x-jy) = x^2-(j)^2(y^2)$

$= x^2+y^2$

$= \left(\sqrt{x^2+y^2}\right)^2 = r^2$

(b) $\frac{z}{z^*} = e^{j2\theta}$

$= \frac{re^{i\theta}}{re^{-i\theta}} = e^{2\theta}$

(c) $z+z^* = x+iy+x-iy$

$= 2\,Re(z)$

(d) $z-z^* = x+iy$

$-x+iy$

$= 2j\,Im\{z\}$

(e) $\Rightarrow |z| = |z^*|$

$\sqrt{x^2+y^2}$, $\sqrt{x^2+(-y)^2}$

(f) $|z_1 z_2| = \cancel{|re^{j\theta_1} re^{j\theta_2}|}$

$= \cancel{r^2 e^{4\theta}}$

$= (x_1+iy_1)(x_2+iy_2)$

$= x_1x_2 + i(x_2y_1+y_2x_1) - y_1y_2$

$= (x_1x_2 - y_1y_2) + j(x_1y_2+x_2y_1)$

$|r_1e^{j\theta_1} r_2e^{j\theta_2}| = |r_1r_2e^{j(\theta_1+\theta_2)}| = r_1r_2$

$|r_1e^{j\theta_1}|\,|r_2e^{j\theta_2}| = r_1r_2\,|e^{j\theta_1}|\,|e^{j\theta_2}|$

$= r_1r_2$

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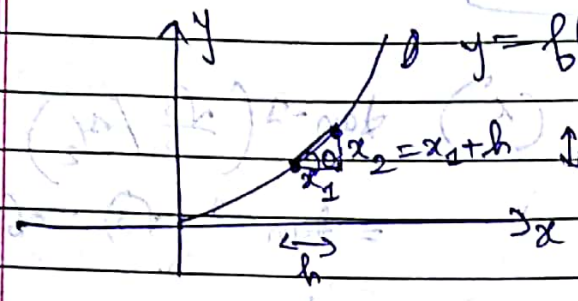
7]



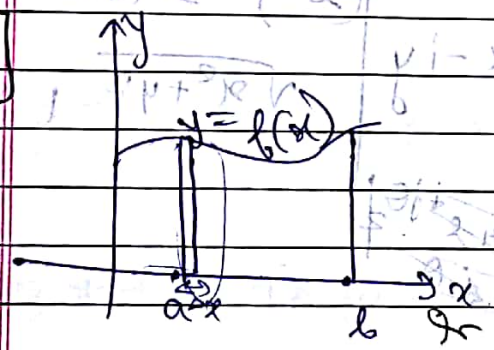
$$\tan\theta = \frac{f(x_1+h) - f(x_1)}{(x_1+h) - (x_1)}$$

$$\tan\theta = \frac{f(x_1+h) - f(x_1)}{h} = \frac{\Delta y}{\Delta x}$$

Now, when $\Delta x \to 0$, the chord of curve becomes tangent at point

$$\therefore \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{dy}{dx} = \text{slope of tangent}$$

8]



$$\Delta x = \frac{b-a}{n}$$

As $n \to \infty$, the parts become very small.

$$\therefore \lim_{n\to\infty}\left(\frac{b-a}{n}\right) = h$$

$\therefore$ Area of small rectangular strip length × breadth

$$f(x_1)\,h$$

Area b/w $y = a$ and $y = b$ (∵ Not $y = f(a)$ and $f(b)$)

$$\int_a^b f(x)\,dx = \lim_{h\to 0} f(x_1)\,dx + f(x_1+h)\,dx \cdots f(x_1+(n-1)h) + f(b)\,dx$$

$$= \lim_{h\to 0}\left[f(a)h + f(a+h)h + \cdots f(a+(n-1)h)h + f(b)h\right]$$

$$= \lim_{h\to 0} \sum_{n=1}^{N} f(x_n)h \quad \text{where } x_1 = a,\ x_2 = a+h$$

Thursday

Fundamental theorem of Integral Calculus. definite integral is nothing but area under curve, bounded by lines defined by limits of integral.

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9] $S_n = a + ar + ar^2 + \cdots ar^{n-1}$

$-rS_n = -ar - ar^2 + \cdots - ar^n - a^n r^{n}$

$S_n - rS_n = a - ar^n$

$S_n = \dfrac{a(1-r^n)}{(1-r)}$

$\lim_{r \to 0} \Rightarrow S_n = \left(\dfrac{a}{1-r}\right)$

10) $\int_{-\infty}^{\infty} f^*(t)\,dt = \left[\int_{-\infty}^{\infty} f(t)\,dt\right]^*$

~~LHS~~

$f(t) = \sum_{n=-\infty}^{\infty} r e^{j(\omega_n t)}$

$\int_{-\infty}^{\infty} \left(r e^{-j\omega_n t}\right) dt$

LHS

$f(t) = \sum_{n=-\infty}^{\infty} r_n e^{j(\omega_n t)}$

$\sum_{n=-\infty}^{\infty} r_n \int_{-\infty}^{\infty} e^{-j\omega_n t}\,dt$

$\sum_{n=-\infty}^{\infty} \left[\dfrac{r_n e^{-j\omega_n t}}{(-j\omega_n)t}\right]$

RHS

$\left[\sum r_n \int_{-\infty}^{\infty} e^{-j\omega_n t}\,dt\right]$

$\sum_{n=-\infty}^{\infty} r_n \left[\dfrac{e^{-j\omega_n t}}{(-j\omega_n t)}\right]$

$\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} (\cos\omega_n t + j\sin\omega_n t)\,dt$

even $\Rightarrow$ conjugate $\int_{-\infty}^{\infty} (\cos\omega_n t) + j\int_{-\infty}^{\infty} (\sin\omega_n t)\,dt$

$\Rightarrow 0$

Oscillating limit

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Thursday

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11) $\left|\int_{-\infty}^{\infty} f(t)\,dt\right| \le \int_{-\infty}^{\infty} |f(t)|\,dt$

$$f(t) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(r_n e^{j\omega_n t}\right) dt$$

$$\left| r_n \frac{e^{j\omega_n t}}{j\omega_n t} \right|_{-\infty}^{\infty}$$

$$\cancel{\sum_{n=-\infty}^{\infty} r_n e^{j\omega_n t}}$$

$$\cancel{\sum_{-\infty}^{\infty} r_n \left[\cos\omega_n t + j\sin\omega_n t\right]}$$

$$\sum_{n=-\infty}^{\infty} \left| \left(\frac{r_n}{j\omega_n t}\right) \right|_{-\infty}^{\infty}$$

$$\sum_{n=-\infty}^{\infty} \left| \frac{r_n}{j\omega_n t} \right| \le \sum_{n=-\infty}^{\infty}$$

$$\rightarrow \left| \int_{-\infty}^{\infty} \sum_{-\infty}^{\infty} c_n e^{j\omega_n t} \right|$$

$$= \left| \sum c_n \int_{-\infty}^{\infty} (\cos\omega_n t + j\sin\omega_n t) \right|$$

$$= \left| \sum c_n \int_{-\infty}^{\infty} \cos\omega_n t \right|$$

$$\sum c_n$$

$$= \left| C_{-\infty} + C_{-\infty+1} \quad C_{-1} + C_0 + C_1 + \quad C_{\infty} \right| \le |C_{\infty}| +$$

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Monday

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$$\boxed{E_\infty = \int_{-\infty}^{+\infty} |x(t)|^2 dt} \rightarrow e = \sum_{n=-N}^{N} |x(t)|^2$$

(∵ small 'e' is for discrete energy)

$$e_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(t)|^2$$

$$= \{x_n\}_{n\in Z}$$

Average Power

$$P_{a_n} = \left(\frac{1}{t_b - t_a}\right)\left[\int_{t_a}^{t_b} |x(t)|^2 dt\right]$$ ← Energy

(∵ a = average)

$$P_\infty = \lim_{T\to\infty}\left(\frac{1}{2T}\right)\left[\int_{-T}^{T} |x(t)|^2 dt\right]$$

(∵ Power of ∞ signal)

↑ Required if $x(t)$ is periodic.

$$P_\infty = \lim_{T\to\infty}\left(\frac{1}{2T}\right)(E_\infty)$$

$$P_{av} = \frac{1}{N_2 - N_1}\left[\sum_{n=N_1}^{N_2} |x(t)|^2\right]$$

(∵ small p for discrete)

$$P_\infty = \lim_{N\to\infty}\left(\frac{1}{2N+1}\right)\left[\sum_{n=-N}^{N} |x(t)|^2\right]$$

Classification of Signal

5] Based on concept of energy & power

Signal with finite $E_\infty = K < +\infty$

$$\Rightarrow P_\infty = \lim_{T\to\infty}\left(\frac{E_\infty}{2T}\right) = \lim_{T\to\infty}\frac{K}{2T} = \boxed{0}$$

(generally aperiodic) ← (∵ Such signals with finite E_∞ are called energy signals)

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→ Signals with finite avg power but ∞ energy are called Power signals

Signals with $P_\infty > 0$. but $P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T}$

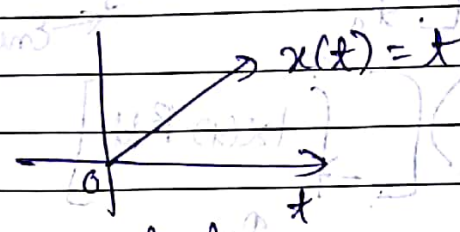
$$\Rightarrow E_\infty = \left(\lim_{T\to\infty} 2T\right) P_\infty$$

$$\approx \infty$$

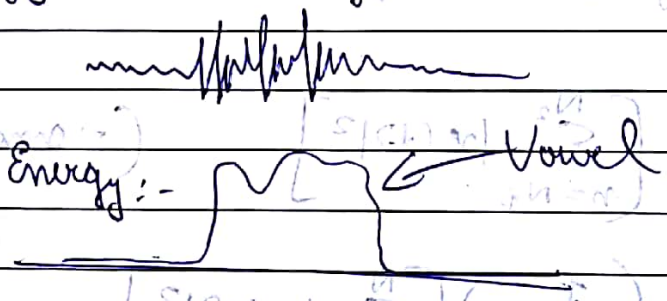
(∵ generally periodic signals)

→ There are signals with are neither energy nor power signals.

Eg:-

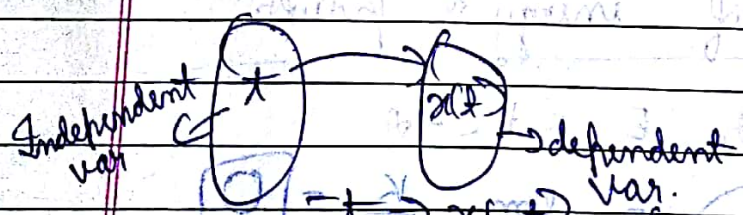


→ Energy of a such signal helps us to identify where there is a vowel as they have lot of energy. Thus, syllable identification uses this.



Transformation of Independent Variables

$x(t)$



P.T.O

$-t \Rightarrow x(-t)$ (∵ $-t \Rightarrow$ flipped)

If $t_2 \Rightarrow x(t/2)$

$2t \Rightarrow x(2t)$ and so on.

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Monday

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In discrete time too,

$n/2 \rightarrow x(n/2) \rightarrow ??$ (n to $\mathbb{Z}$)

$2n \rightarrow x(2n)$

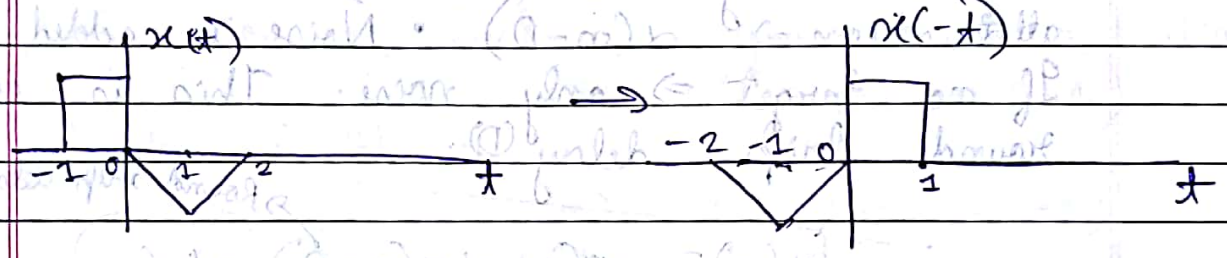
$n \rightarrow x(n)$

$-n \rightarrow x(-n)$ ...

Motivation:-

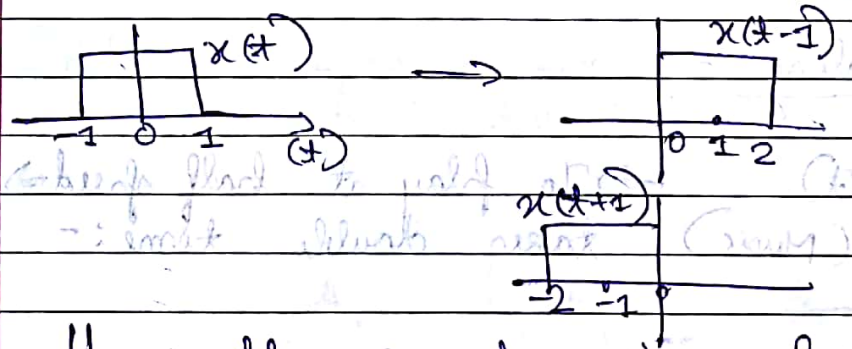
1) **Time Reversal:-**

$x(t) \rightarrow x(-t)$ ($\because 180^\circ$ phase distortion)



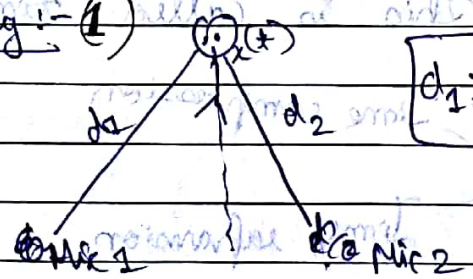
Playing music in fwd direction is $x(t)$ and in backward version, its called $x(-t)$.

2) **Time Shifting:-**



Il examples can be given for discrete time.

Eg:- (i)

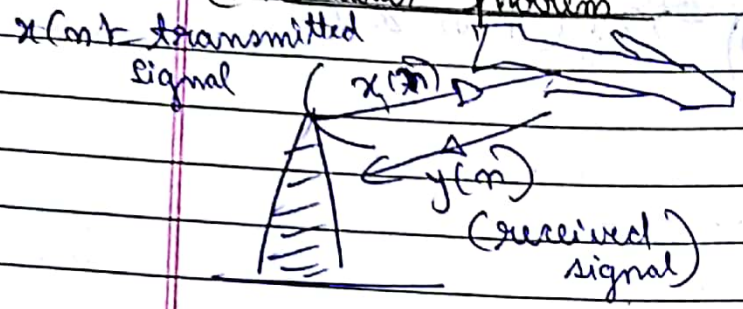


$\boxed{d_1 > d_2}$

(relative difference/ delay in signal)

This is important for distant speech recognition (Microphone arrays). Shifting is naturally encountered

(ii) Radar problem



→ If the target is a target

→ We need to find distance

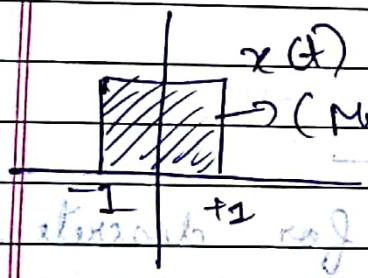
y(n) would depend on x(n)

→ Due to delay ⇒ $x(n-D)$. There is some attenuation ⇒ $\alpha(n-D)$. Noise is added $W(n)$. If no target ⇒ only noise. This is called round trip delay (D).

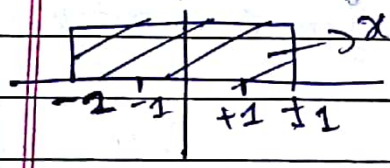
$$\therefore \boxed{y(n) = \underbrace{\alpha}_{\text{Attenuation factor}} x(n-\overbrace{D}^{\text{Round trip delay}}) + \underbrace{W(n)}_{\text{Noise}}}$$

Thus, D came naturally. We don't study it abstractly.

3) Time Scaling :-

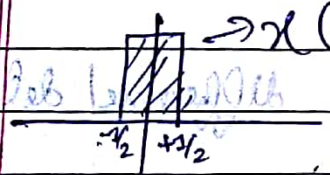


→ To play at half speed ⇒ it takes double time :-



(∵ This is called time scaling)

Time compression



Time expansion

Application :- Time scale modification of a speech, music, audio)

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Monday

DATE 5 8 18 (31)

Problem ⟹ Develop an algorithm such that a fast speaking person's voice, would sound like same person speaking slowly.

Challenge is keeping identity same.

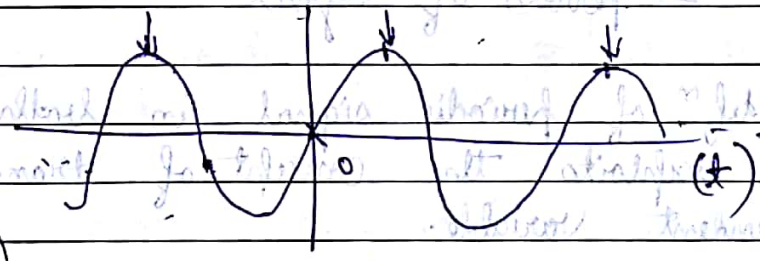
* Study of Periodic Signals:-

Motivation:-

1) Motion of planets.
2) Generation of electrical signal at Power Station
   ↳ $x(t) = A\sin(\omega t + \phi)$
3) Ideal Simple Harmonic Motion.
   No damping ↳ $x(t) = A\cos(\omega t + \phi)$
4) Response of ideal tank (LC) circuit.

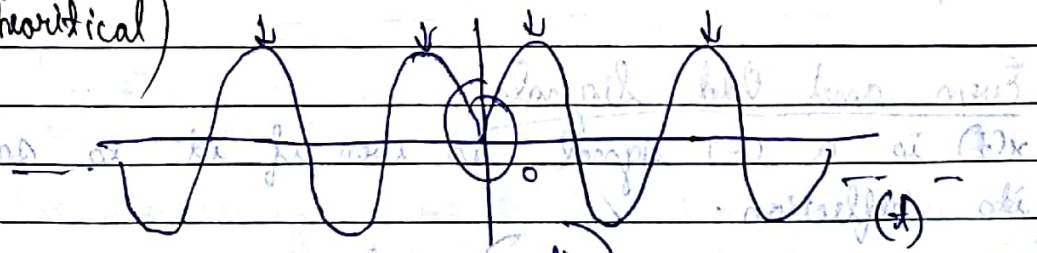
5) Ideal simple pendulum.

Defn:- A signal is said to be periodic if each and every feature of that signal, repeats over a certain interval of time.



The signature of the signal repeats.

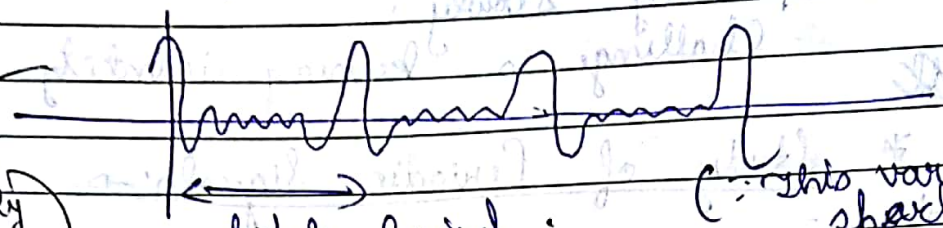
(Theoritical)



(Not periodic)

Monday

→ But in nature, not all periodic signals are like this. Say, vowel A



(practical signatures don't exactly repeat in reality) → (Hence called Quasi-periodic.)

(∵ this variability in speech is desirable)

8/8/18

→ If a signal is periodic for period T, then it is also periodic for 2T, 3T, ---- nT

$x(t) \rightarrow \{T, 2T, 3T, \ldots nT\}_{n \in Z}$

Periodic ⇒

$x(t) = x(t+T) = x(t+2T) = \ldots x(t+nT)$

→ fundamental period

||ly for discrete,

$x(n) = x(n+N) = x(n+2N) = \ldots = x(n+kN)$

→ fundamental period.

⇒ $\{N, 2N, 3N, \ldots kN\}$

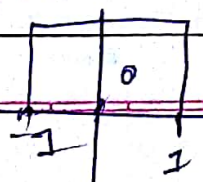
↳ periods of signal

↑ Here, def^n of periodic signal in both cont & discrete time, exploits the concept of transformation of independent variable.

Even and Odd Signals

$x(t)$ is a C-T signal is even if it is same as its reflection.

Eg:-



$x(t) = x(-t)$

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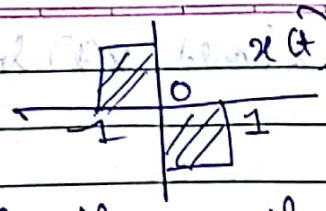
Wednesday

Eg:- 

$x(t) = -x(-t)$ ($\because$ Odd signals)

Exactly similar for D.T.

Even ~~path~~ part of a signal:-

$$Ev\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

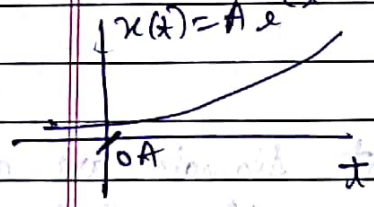
$$Odd\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

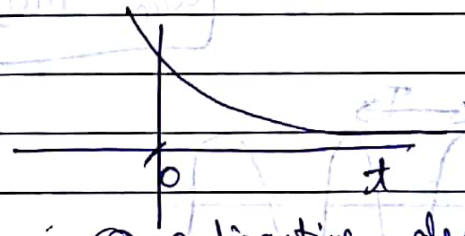
* Exponential and Sinusoidal Signals

1) Continuous Time Complex exponential and sinusoidal signals :-

$$\boxed{x(t) = Ae^{ct}}$$

$c > 0$ $\qquad e = 2.718182 \qquad$ $c < 0$





① Population growth
② Chain reaction

① Radioactive decay

2) Periodic Complex exponential and sinusoidal signals

$$\boxed{x(t) = A \cdot e^{ct}}$$

Case I: For simplicity, let $A = 1$, c = purely imaginary

$c = j\omega$

$$\Rightarrow \boxed{x(t) = Ae^{j\omega t}}$$

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Investigate "behaviour" of $x(t)$, should $x(t)$ be periodic

Since, $x(t)$ is periodic,

$$x(t) = x(t+T)$$

$$e^{j\omega t} = e^{j(t+T)\omega}$$

$$\therefore e^{j\omega t} = e^{j\omega t} e^{j\omega T}$$

$$\therefore e^{j\omega T} = 1 = e^{0}$$

Q.) For what value of ω, LHS of above eq^n is 1,

$\Rightarrow$ For $\omega = 0$ $\qquad \omega \propto \frac{1}{T \to \infty}$

$\Rightarrow$ it is periodic for arbitrary value of T.

Case II :-

$x(t) = e^{j\omega t}$ $\qquad Re(x(t)) = \cos(\omega t)$

$\boxed{x(t) = \cos(\omega t) + j\sin(\omega t)}$ $\qquad Img[x(t)] = \sin(\omega t)$

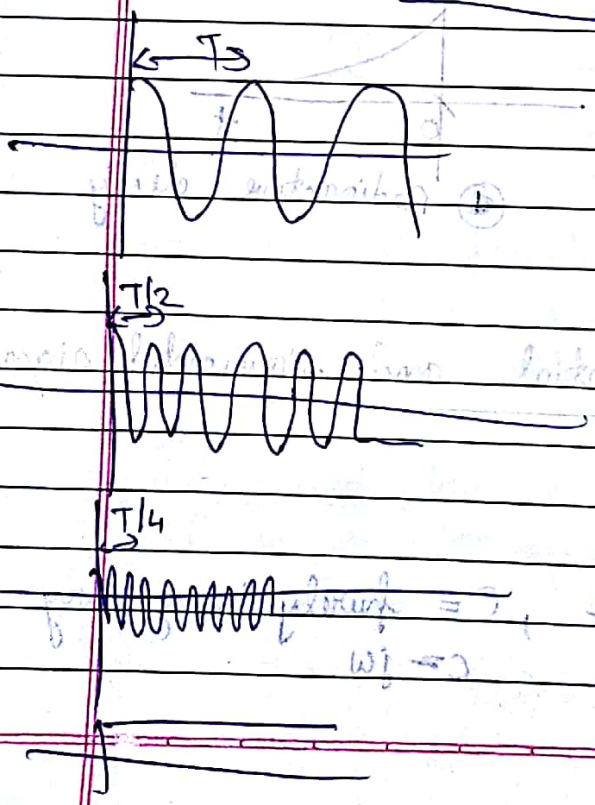
$$\omega t = 2\pi n \quad , n \in \mathbb{Z}$$

$$\omega = \underbrace{\left(\frac{2\pi}{T}\right)}_{\omega_0} n$$

$$\boxed{\omega_n = n\omega_0}$$

If $n = 0$

$\omega = 0$ ($\because$ case I)



$\Rightarrow$ Constant signals are also periodic, with arbitrary period.

$$\boxed{\omega_n = n\omega_0} \quad , n \in \mathbb{Z}$$

integral multiples of fundamental frequency

$\omega_0 = \frac{2\pi}{T}$ = fundamental frequency

T = fundamental period.

$\therefore$ ω_n are called "harmonics"

music literature.

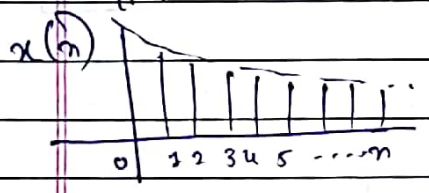
→ Harmonium generates harmonically related frequencies for all the notes.

Discrete-time Complex Exponential & Sinusoidal signal

$x(t) = Ae^{ct}$ (CT) $\longrightarrow$ $x(n) = A \cdot \beta^n$ (DT) $\quad (\because \beta = e^c)$

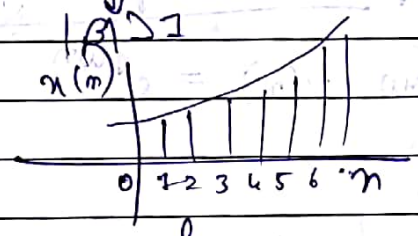
$$x(n) = A \cdot \beta^n$$

$|\beta| < 1$



Decaying exponential

$|\beta| > 1$



Growing exponential.

Now, $x(n) = Ae^{\alpha n}$ $\quad (\because \beta = e^{\alpha}) \quad \alpha = j\omega$

$A = 1$ and $\alpha = j\omega$

$$\boxed{x(n) = e^{j\omega n}}$$

Now, $\omega \to \omega + 2\pi$

$$\boxed{x(n) = e^{j\omega n}}$$

$$e^{j(\omega+2\pi)n} = e^{j\omega n} e^{j2\pi n}$$

$$e^{j2\pi n} = \cos 2\pi n + j\sin 2\pi n$$
$$= 1 + 0$$
$$= 1$$

$$\therefore \boxed{e^{j(\omega+2\pi)n} = e^{j\omega n}}$$

In D-T complex exponential, the signal repeats itself even if the frequency ω is shifted by integer multiple of 2π.

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But $e^{j(\omega+2\pi)t} \neq e^{j\omega t}$, $t \in R$

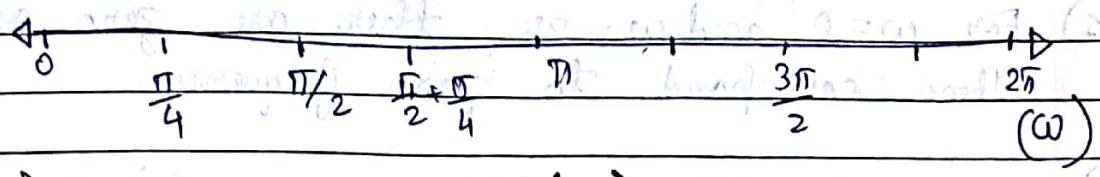
Only for $t = n$, but then its discrete.

$e^{j(\omega+2\pi)n} = e^{j\omega n}$

→ Let's investigate behaviour of D-T periodic complex exponential signal

(Q.) How $x(n) = e^{j\omega n}$

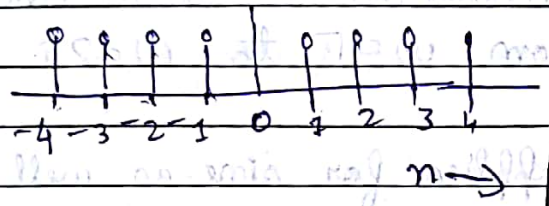
Thursday



1) $\omega = 0$

$x(n) = \cos(\omega n)$

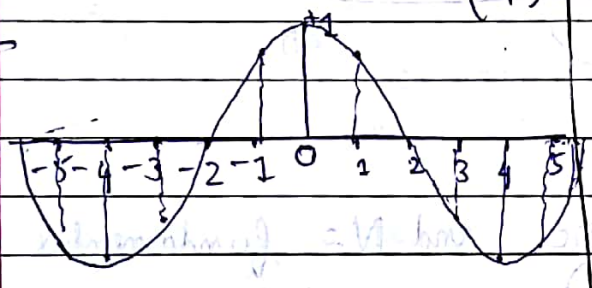
$= \mathrm{Re}\{e^{j\omega n}\}$



2) $\omega = \pi/4$

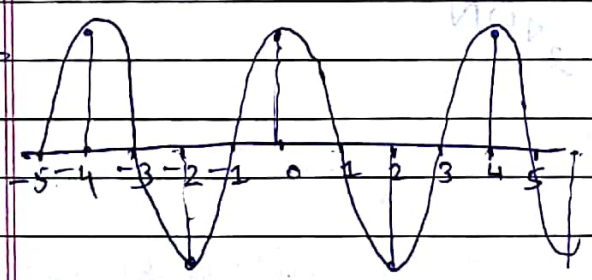
$x(n) = \cos(\omega \cdot n) = \cos\left(\frac{\pi}{4} n\right)$

cos curve

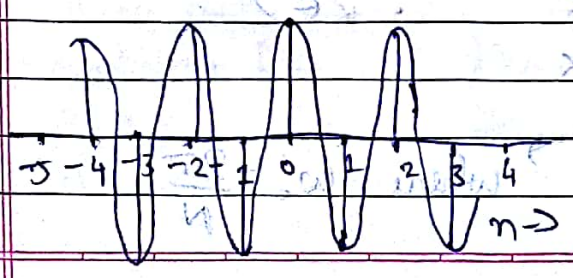


3) $\omega = \pi/2$ $\quad x(n) = \cos\left(\frac{\pi n}{2}\right)$

curve of cos

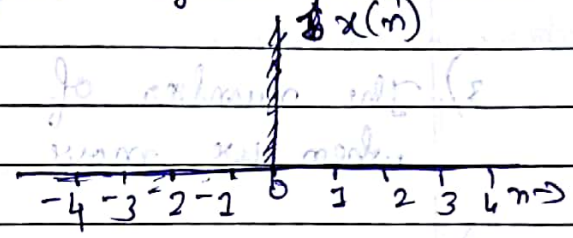


4) $\omega = \pi$ $\quad x(n) = \cos(\pi n)$



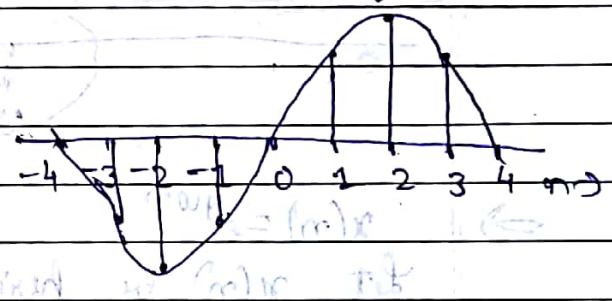
1) $\omega = 0$

$x(n) = j\sin(\omega n)$



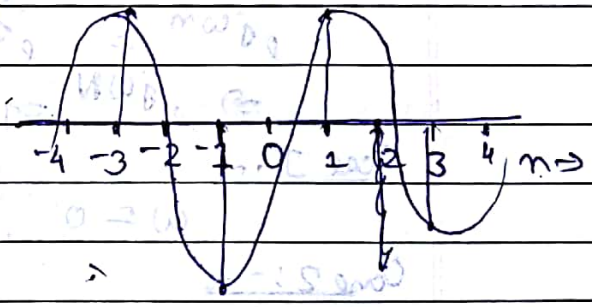
2) $\omega = \pi/4$

$x(n) = \sin\left(\pi/4\, n\right)$



3) $\omega = \pi/2$

$x(n) = \sin\left(\pi/2\, n\right)$



If we plot cos till $\omega = 2\pi$, then $\omega = 0$ repeats at $\omega = 2\pi$. Therefore, periodic in ω with period $= 2\pi$. $\omega \uparrow$ upto π and decreases from π to 2π

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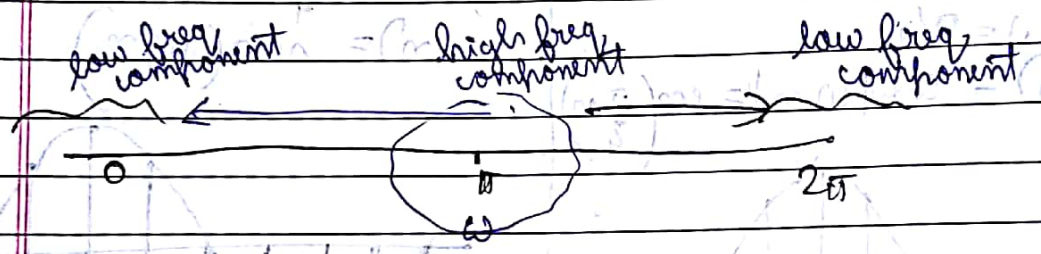
Observations:-

1) For $\omega = 0$ and $\omega = 2\pi$, there are zero oscillations. These correspond to zero frequency.

2) The number of oscillations in $x(n)$ increases, when we move from $\omega = 0$ to $\omega = \pi$

3) The number of oscillations in $x(n)$ decreases, when we move from $\omega = \pi$ to $\omega = 2\pi$

4) The same as above applies for sine as well.



$\Rightarrow$ $x(n) = e^{j\omega n}$

Let $x(n)$ be periodic and N = fundamental period.

$$x(n) = x(n+N)$$

$$\therefore e^{j\omega n} = e^{j\omega(n+N)}$$

$$e^{j\omega n} = e^{j\omega n} e^{j\omega N}$$

$$\Rightarrow e^{j\omega N} = 1$$

Case 1:-

$$\omega = 0 \quad , \checkmark$$

Case 2:-

$$\cos(\omega N) + j\sin(\omega N) = 1$$

$$\omega N = 2\pi k \quad ; k \in Z$$

$$\omega = \left(\frac{2\pi}{N}\right) k$$

harmonics $\leftarrow$ $\boxed{\omega_k = k\omega_0}$ $\rightarrow$ where $\omega_0 = \frac{2\pi}{N}$

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$$x_n(n) = \left\{ e^{j\omega_n t} \right\} = \left\{ e^{jn\omega_0 t} \right\}_{n \in Z}$$

$$x_k(n) = \left\{ e^{j\omega_k n} \right\} = \left\{ e^{jk\omega_0 n} \right\}_{k \in Z}$$

↳ Set of harmonically related complex exponential signals.

$$e^{j\omega t} \neq e^{j(\omega + 2\pi)t}$$

$$e^{j\omega n} = e^{j(\omega + 2\pi)n}$$

$$x_k(n) = e^{j k \cdot \omega_0 n}$$

$$x_{k+N}(n) = e^{j(k+N) \cdot \left(\frac{2\pi}{N}\right) n}$$

$$= e^{jk\left(\frac{2\pi}{N}\right)n} \; e^{j(2\pi)n}$$

$$= e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$\Rightarrow \boxed{x_{k+N}(n) = x_k(n)}$$

→ Mathematically intact.

$\Rightarrow$ $\omega_n = n\omega_0$; $n \in Z$

$\omega_1 = \omega_0$ and $\omega_{-1} = -\omega_0$ → negative freq

$\omega_1 \to 50Hz$, $\omega_{-1} \to -50Hz$

Concept of Negative freq

It is only because of reference. There is an origin and +ve freq ⇒ there has to be a -ve freq of 50Hz. +ve is in anti-clk dir & -ve is in clk dir.

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Excitation Source :-

Physical systems have an inertia. Inertia means that a physical system won't change its state unless an excitation source is applied.

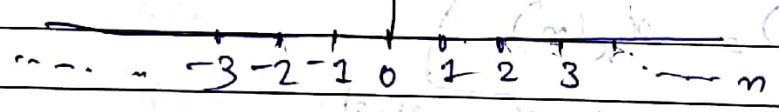
Excitation source → [Physical Systems] → O/P response

to overcome inertia

① Unit Impulse Signal
(Sudden change)
(∵ Behaviour)

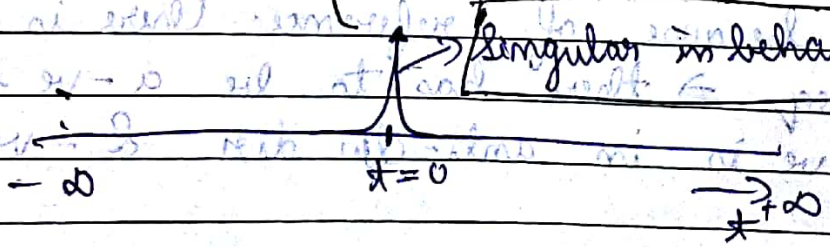
Discrete-time unit impulse signal

$$\delta(n) = \begin{cases} 1 & , n = 0 \\ 0 & , n \neq 0 \end{cases}$$



Continuous time unit impulse signal

$$\delta(t) = \begin{cases} \int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1 & (\text{area of under impulse} = 1) \\ 0 & t \neq 0 \end{cases}$$



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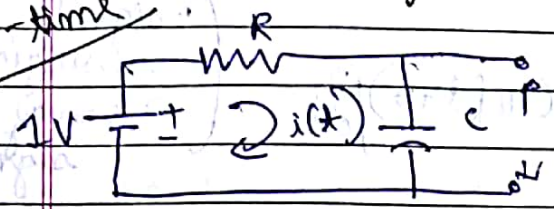
② Unit Step Signal

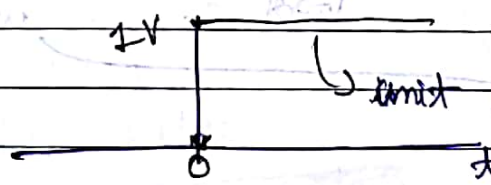
Cont-time



$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

(O/P taken across cap)

→ Switched on at $t = 0$ ⇒ $t=0$ is a reference
(∵ switching on of dc voltage battery)

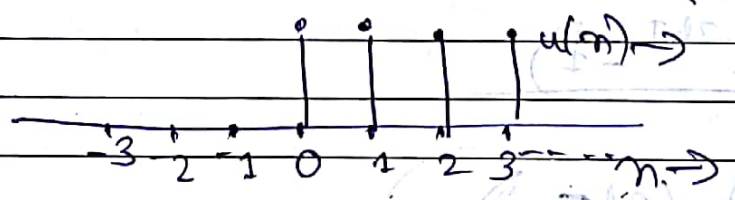


↳ unit step input to RC circuit

↳ (Sudden switching ON and OFF will give us Impulse signal)

Discrete time

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Now, if we do,

$$\boxed{u(n) = \delta(n) + \delta(n-1) + \delta(n-2) \ldots}$$

$$u(n) = \sum_{k=0}^{+\infty} \delta(n-k) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

⇒ Here, $n=0$ because in cont-time, there was discontinuity at $t=0$.

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Thursday

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Tutorial - 2

1.] (a) $g(t) = Ae^{j(2\pi f_0 t + \theta)}$

$\therefore$ Energy $= \int_0^T \left|Ae^{j(2\pi f_0 t+\theta)}\right|^2 dt$ (∵ signals converging ⇒ energy signal)

$= \int_0^T A^2 dt$

$= \boxed{A^2 T}$ ⇒ $\lim_{T\to\infty} \to \infty$ ⇒ Power signal with power

$\boxed{\text{Power} = A^2}$

(b) $\int_0^T |Ae^{-bt}|^2 dt$

$= \left[\frac{A^2 e^{-2bt}}{-2b}\right]_0^T$

$= \frac{A^2 e^{-2bT}}{-2b} - \frac{A^2}{-2b}$

$= \frac{A^2(e^{-2bT} - 1)}{-2b}$

$= \left(\frac{-A^2}{2b}\right)(e^{-2bT} - 1)$

$\lim_{T\to\infty} = \left(\frac{-A^2}{2b}\right)(0-1)$

$E = \boxed{\frac{A^2}{2b}}$ = finite ⇒ $\left(\frac{dE}{dT}\right) = 0$ ⇒ power = 0 ⇒ energy signal.

(c) $g(t) = t \; (t > 0$

Thursday

$$E = \int_0^T t^2 dt$$

$$= \frac{T^3}{3} \Rightarrow \lim_{T\to\infty} = \infty \quad (\because \text{Infinite energy})$$

$$\text{Power} = T^2 \Rightarrow \lim_{T\to\infty} = \infty \quad \text{Power} \quad (\text{Neither energy nor power signal})$$

(d) $g(t) = Kt^{-1/4}$

$$\text{Energy} = \int_0^\infty t^{-1/2} \quad \left[2Kt^{1/2}\right]_0^T$$

$$\lim_{T\to\infty} \Rightarrow \infty \text{ energy}$$

$$E = 2Kt^{1/2}$$

$$\cancel{\frac{dE}{dt} = \boxed{K t^{-1/2}}}$$

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt$$

$$\hookrightarrow = 0$$

But P must be greater than 0 for power signal
$\Rightarrow$ neither energy nor power.

2.] $x_1(t) = x_1(t + k_1 T_1)$
$x_2(t) = x_2(t + k_2 T_2)$

$$x_1(t) + x_2(t) = x_1(t + k_1 T_1) + x_2(t + k_2 T_2)$$

$$x_3(t) = x_1(t) + x_2(t)$$

$$x_3(t+T) = x_1(t + mT_1) + x_2(t + nT_2)$$

$$T = mT_1 = nT_2$$

$$\boxed{\frac{T_1}{T_2} = \frac{n}{m}} \Rightarrow T = \frac{LCM(a,p)}{HCF(b,q)}$$

($\because$ LCM of $\frac{a}{b}$ and $\frac{p}{q}$

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$\cos 60\pi\left(t + \frac{1}{30}\right)$

$x(t) = \cos 60\pi t + \sin 50\pi t$

$\left(\frac{1}{30}\right)$ $\left(\frac{1}{25}\right)$

$T = \frac{1}{5}$ sec

3] (a) $x_1(t) = e^{-5t} u(t)$

$$\int_{-\infty}^{\infty} \left|e^{-5t} u(t)\right|^2 dt$$

$$E_\infty = \int_0^\infty \left|e^{-5t}\right|^2 dt = \left[\frac{e^{-10t}}{-10}\right]_0^\infty$$

$$\Rightarrow \boxed{E = \frac{1}{10} \text{ unit}}$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|e^{-5t} u(t)\right|^2 dt$$

$$= \lim_{T\to\infty} \left(\frac{1}{2T}\right) \int_0^\infty \left|e^{-5t}\right|^2 dt$$

$$= \lim_{T\to\infty} \left(\frac{1}{2T}\right) \left[\frac{e^{-10t}}{-10}\right]$$

$$= \frac{\frac{1}{10} - 0}{\left(\lim_{T\to\infty} (2T)\right)}$$

$$= 0$$

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(b) $x_2(t) = e^{j(4t+\pi/4)}$

$$E_\infty = \int_{-\infty}^{\infty} \left| e^{j(4t+\pi/4)} \right|^2 dt$$

$$= \left[\frac{e^{j(8t+\pi/2)}}{8} \right]_{-\infty}^{\infty}$$

$$= \left(e^{j\pi/2}\right) \left[\frac{e^{j8T}}{8} \right]_{-\infty}^{\infty}$$

$$= j \left[\right.$$

$$= \int_{-\infty}^{\infty} dt$$

$$= \infty$$

$$\text{Power} = \lim_{T\to\infty} \left(\frac{1}{2T}\right) \int_{-T}^{T} dt$$

$$P = 1 \text{ unit}$$

(c) $x_1(n) = \left(\frac{1}{3}\right)^n u(n)$

$$\Rightarrow \sum_{-\infty}^{\infty} |x_1(n)|^2$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n}$$

$a = 1$ as $n \geq 0$

$$\left(\frac{1}{9}\right)^n = \frac{1}{1-\frac{1}{9}} = \frac{1}{8/9} = \frac{9}{8}$$

$$= 9/8$$

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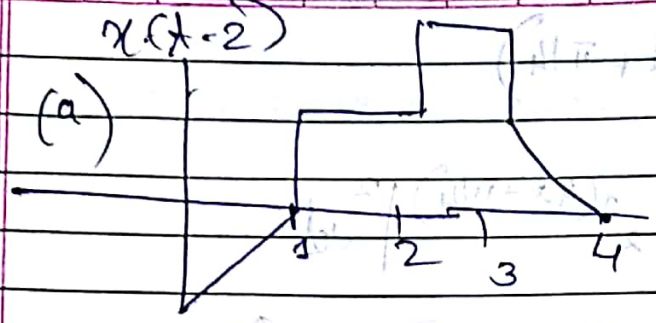
Order :- Shifting, scaling, reversal

4) (a) $x(t-2)$

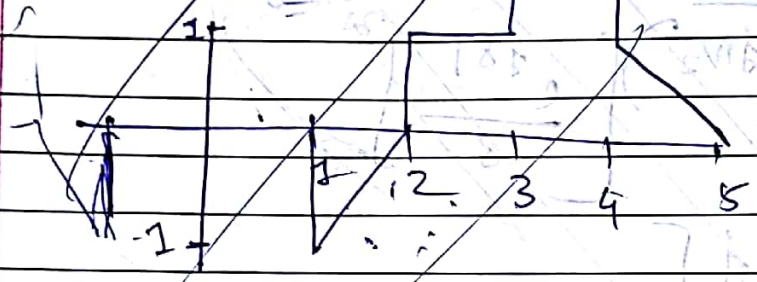


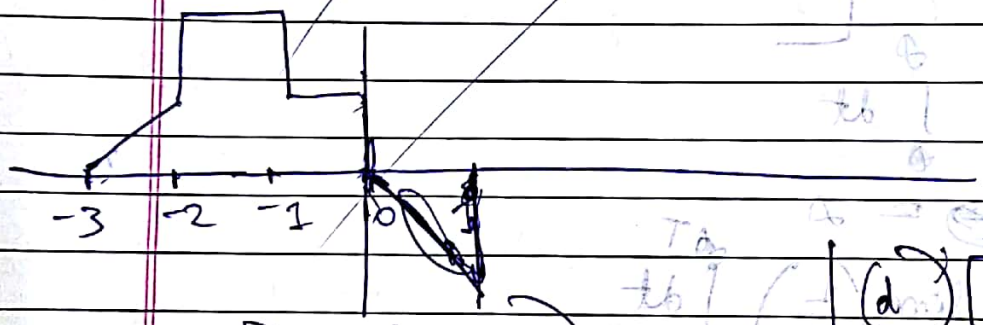
(b) $x(3-t)$

$x[-(t-3)]$

$x(t-3) = g(t)$

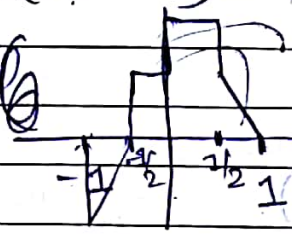
$g(-t)$

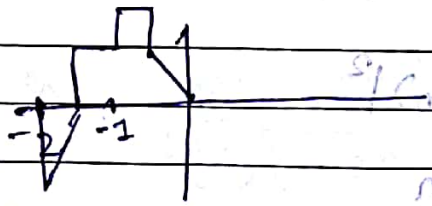




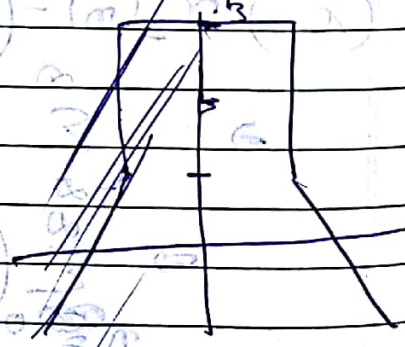
(c) $x(2t+1)$

$x(6$





(d) $[x(t)+x(-t)]u(t)$



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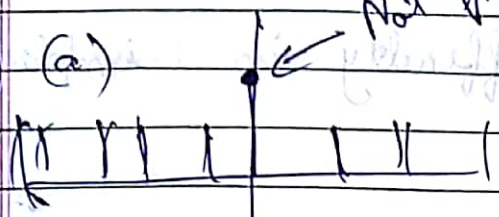
$e^{j\omega_0 t}$ $e^{j2\pi t} = e^{j\omega_0 t}$ $e^{j2\pi t} = 1$ $0+2n\pi$ $e^{j2\pi t} = 2n\pi$

6] (a) Not periodic

If $x_1(n) = x_1(-n)$



(b) $x_2(t) = 2e^{j(t+\pi/4)} u(t)$

$= \left[2e^{j\pi/4}\right]\left[e^{jt}\right] u(t)$

$e^{j2\pi t} = 1$

13/8/18

→ The excitation signal should be called "Impulse-like excitation" and not impulse excitation.

→ For step excitation, just switch on the battery.

→ Whenever we turn on, $t=0$.

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Now, if it was cont ⇒ integration

$\therefore \delta(n) = \frac{d}{dn}$ differentiate $(u(n))$

$= u(n) - u(n-1)$

In C-T,

$\delta(t) = \frac{d}{dt}[u(t)]$

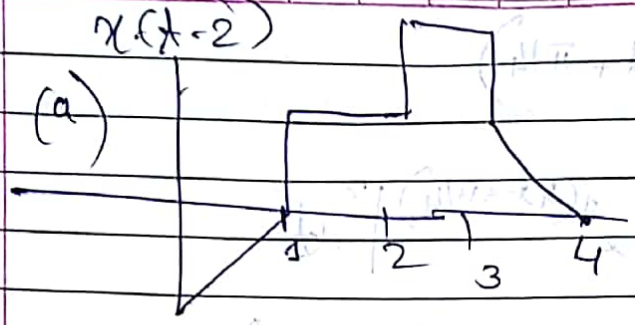
P.T.O

Thursday

Order :- Shifting, scaling, reversal

DATE 9 8 18 PAGE NO. 46

4] (a) $x(t-2)$

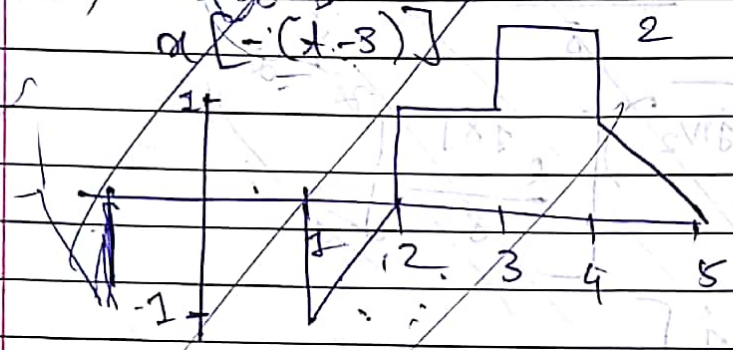


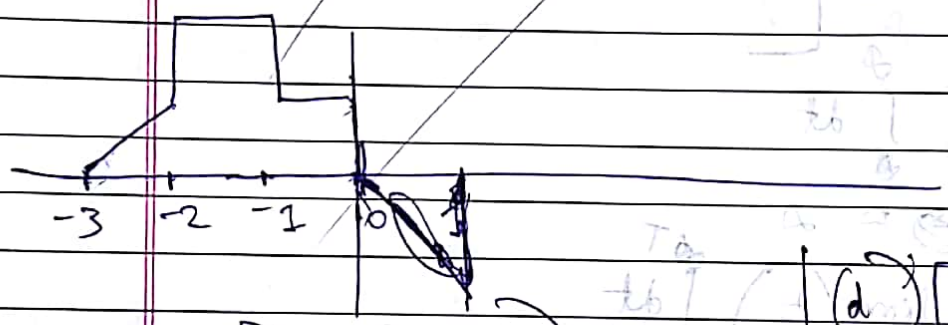
(b) $x(3-t)$

$x[-(t-3)]$

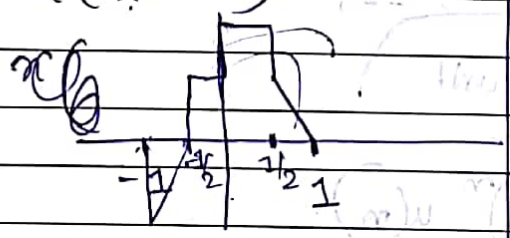
$x(t-3) = g(t)$

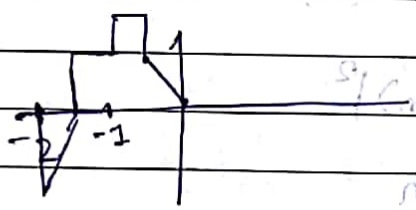
$g(-t)$



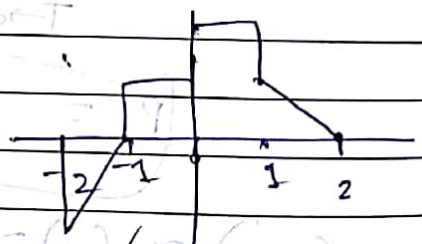


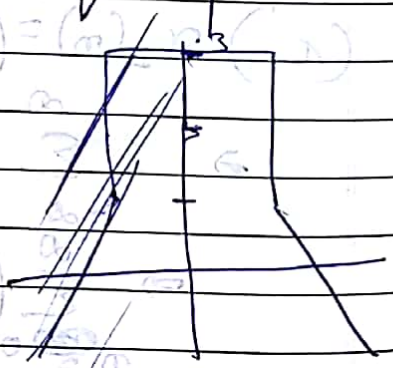
(c) $x(2t+1)$





(d) $[x(t)+x(-t)]u(t)$





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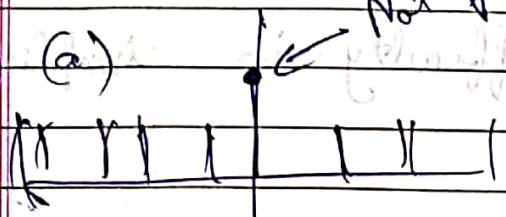
$e^{j\omega_0 t}$ $e^{j2\pi t} = e^{j\omega_0 t}$ $e^{j2\pi t} = 1$ $0+2n\pi$ $j2\pi t = 2n\pi$

6] (a) Not periodic If $x_1(n) = x_1(-n)$



(b) $x_2(t) = 2e^{j(t+\pi/4)} u(t)$

$= \left[2e^{j\pi/4}\right]\left[e^{jt}\right] u(t)$

$e^{j2\pi t} = 1$

13/8/18

→ The excitation signal should be called "Impulse-like" excitation and not impulse excitation.

→ For step excitation, just switch on the battery.

→ Whenever we turn on, $t=0$.

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Now, if it was cont ⇒ integration

$\therefore \delta(n) = \frac{d}{dn}$ differentiate $(u(n))$

$= u(n) - u(n-1)$

In C-T,

$\delta(t) = \frac{d}{dt}[u(t)]$

P.T.O

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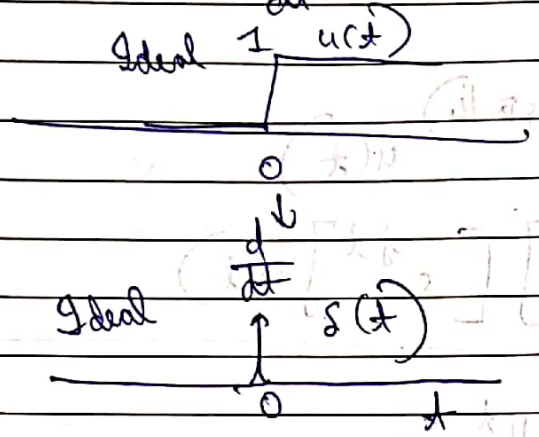
Monday

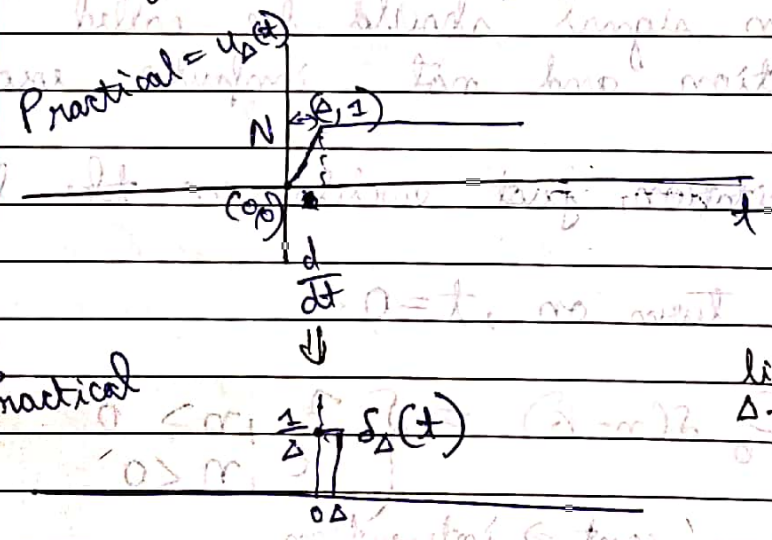
→ Now to understand difficulty in implementation

$$\delta(t) = \frac{d}{dt}[u(t)]$$

Ideal



→ Note that sudden rise to 1V can never physically happen. The time that battery takes to reach 1V from 0V = $u_\Delta(t)$



Clearly

$$\lim_{\Delta \to 0} u_\Delta(t) = u(t)$$

$$\lim_{\Delta \to 0} \delta_\Delta(t) = \delta(t)$$

→ The concept of limit was introduced to explain convergence and divergence.

$$\delta_\Delta(t) = \frac{d}{dt}[u_\Delta(t)]$$

Sampling Property of Impulse Signal

Case-I :- Discrete-time signal

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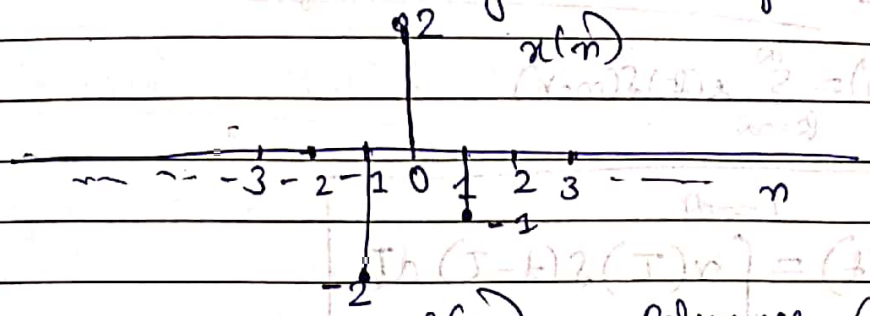
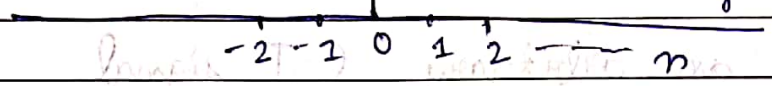
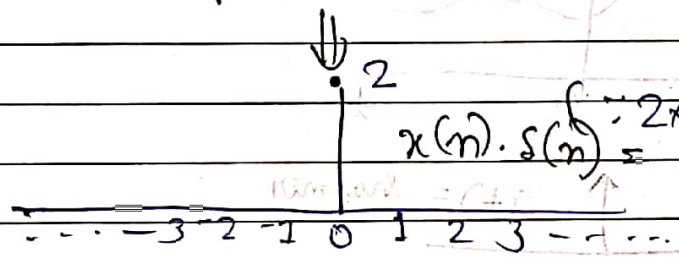
Consider an arbitrary D-T signal $x(n)$.





→ Reference signal (∵ Impulse signal)

→ To obtain sample at $n=0$, multiply $x(n)$ and $\delta(n)$

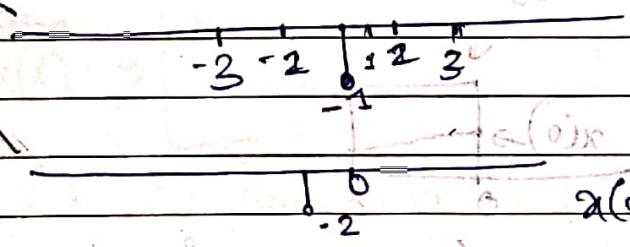


$x(n)\cdot\delta(n) = 2\cdot\delta(n)$ (∵ $2\times 1 = 2$) $\Rightarrow$ $\boxed{x(n)\cdot\delta(n) = x(0)\cdot\delta(n)}$

We could've got the same signal if we multiplied $\delta(n)$ by 2, i.e., applied scaling property of signals.

$$x(n)\cdot\delta(n-1) = x(1)\,\delta(n-1)$$

These signals are called constituent signals since an adder will give back $x(n)$



$x(n)\,\delta(n-1) = x(1)\,\delta(n-1)$

$x(n)\cdot\delta(n+1) = x(-1)\cdot\delta(n+1)$

∴ $\boxed{\delta(n) \text{ is called hammer}}$

∴ $x(n) = x(-1)\cdot\delta(n+1) + x(1)\cdot\delta(n-1) + x(0)\cdot\delta(n)$

(∵ sifting = breaking) ↓

→ $\delta(n)$ is called hammer

$$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)}$$

(∵ Sifting property of impulse signal)

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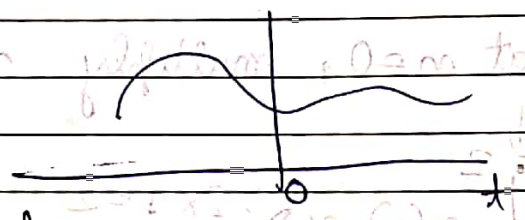
Monday

Case II : Continuous Time Signal

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

$$\boxed{x(t) = \int_{\tau=-\infty}^{\tau=+\infty} x(\tau)\,\delta(t-\tau)\,d\tau}$$

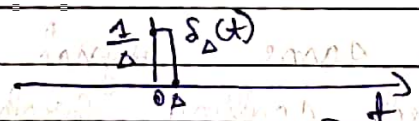
We take an arbitrary C-T signal

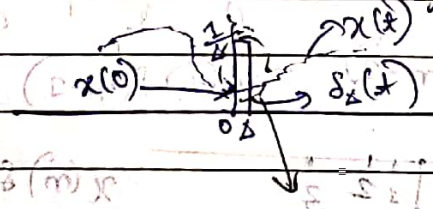


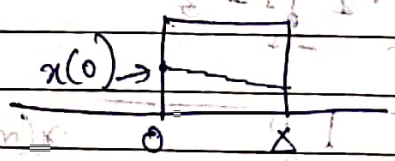
Ideal

$\delta(t)$ = hammer

Actual







size of stone in context of calculus

$$x(t)\cdot\delta_\Delta(t) \cong x(0)\,\delta_\Delta(t)$$

Now

$$\lim_{\Delta\to 0}\left[x(t)\,\delta_\Delta(t)\right] \cong \lim_{\Delta\to 0} x(0)\cdot\delta_\Delta(t)$$

$$\therefore x(t)\lim_{\Delta\to 0}\left[\delta_\Delta(t)\right] \cong x(0)\left[\lim_{\Delta\to 0}\delta_\Delta t\right]$$

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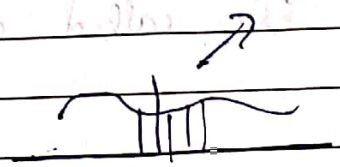
After applying limit,

$$\boxed{x(t)\cdot\delta(t) = x(0)\cdot\delta(t)}$$

Therefore $\hat{x}(t) = \ldots + x(-\Delta)\cdot\delta_\Delta(t+\Delta) + x(0\cdot\Delta)\cdot\delta_\Delta(t) + x(1\cdot\Delta)\cdot\delta_\Delta(t-\Delta) + \ldots$



$$\lim_{\Delta\to 0}\hat{x}(t) = \lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}\left[x(k\cdot\Delta)\,\delta(t-k\Delta)\right]\Delta x$$

$$\left(= \lim_{h\to 0}\sum_{\langle n\rangle} F(x_n)\cdot h\right) \quad (\because \text{Tutorial 7})$$

$(\because k\Delta = \tau)$

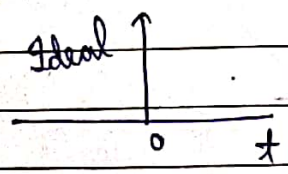
$$= \boxed{\int_{\tau=-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau}$$

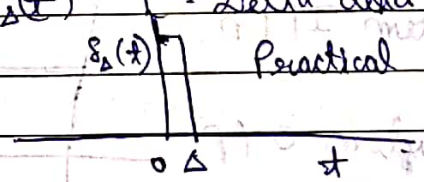
Impulse Excitation to Real Physical System (OR) Concept of Impulse Response. 16/8/18

I/P excitation $x(t) \longrightarrow$ [Real physical system] $\longrightarrow y(t)$ output Response

$x(t) = \delta(t)$ • Real life impulse signal is $\delta_\Delta(t)$ ($\because$ Delta delta t)

Ideal





Practical impulse = pulse of duration 'Δ' and height $1/\Delta$

$\Rightarrow$ Impulse input to a system is equivalent to giving a pulse input of arbitrary shape.

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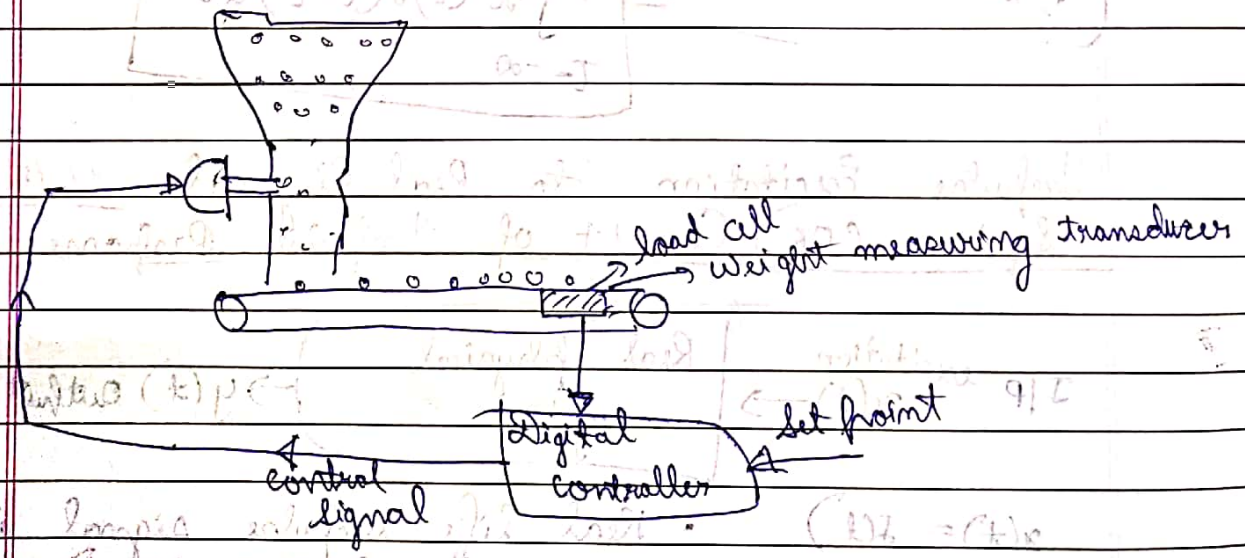
Thursday

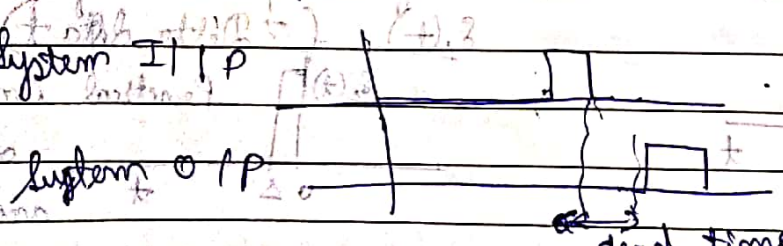
→ A real physical system has an attribute of system's inertia.

↳ The system will take 'some time' to respond to 'pulse input'.

↳ System is dead for some time. It doesn't instantly operate. This is called dead time of a signal.

→ By the time system responds to this pulse I/P, system will not respond to the individual effect of pulse width / pulse height / pulse shape, rather system will respond to its net integrating effect, i.e, area of the pulse.





Thursday

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Cont Time Vs. Discrete Time Systems

1) Cont-Time

$$\underset{\text{I/P}}{x(t)} \Rightarrow \underset{\text{system}}{\boxed{H\{\cdot\}}} \rightarrow \underset{\text{O/P}}{y(t)}$$

$y(t) = H\{x(t)\}$ where $H\{\cdot\}$ is system property.

$x(t)$ $\xrightarrow[\text{Operator / Mapping / Transformation / Function}]{H\{\cdot\}}$ $y(t) = H\{x(t)\}$

In this course, this (domain) is called I/P signal space

This is O/P signal space

Set vs. Space

Set → collection of objects

→ A set is equipped with some structure (constraint) rule that every element in the set must obey.

$\boxed{\text{Set} + \text{Structure} = \text{Space}}$

→ Space is distinctly different from a set and this space has a property, i.e, structure.

Structure → may be algebraic (+, ..) → Vector space,

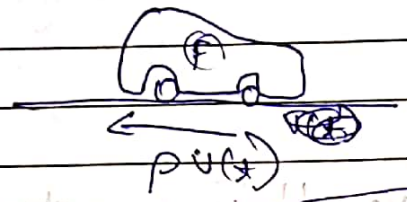
→ may be geometric (angles) → Inner product space.

→ 'A space is "closed" under an operation means all the elements satisfy'

P.T.O

Examples of Systems :-

1) Automobile Systems



$$F = ma$$

$$|f(t) - \rho v(t)| = m \frac{d}{dt} v(t)$$

$$\boxed{\frac{dv_t}{dt} + \frac{\rho}{m} v(t) = \frac{1}{m} f(t)}$$

μN N

friction

$m g \sin\theta$ N

In discrete, $\frac{dv(t)}{dt} = \frac{V(n\Delta) - V[(n-1)\Delta]}{\Delta}$

$$= \frac{1}{m}\left(f(t) - \rho v(t)\right]$$

$$\frac{V(n) - V(n-1)}{\Delta} = \frac{1}{m}\left[f(n\Delta) - \rho V[n]\right]$$

$$\frac{V[n]}{\Delta} - \frac{V[n-1]}{\Delta} = \frac{f(n)}{m} - \frac{\rho V[n]}{m}$$

$$V[n]\left[\frac{1}{\Delta} + \frac{\rho}{m}\right] - \frac{V[n-1]}{\Delta} = \frac{f[n]}{m}$$

$$V[n]\left[\frac{m + \rho\Delta}{\Delta m}\right] - \frac{V[n-1]}{\Delta} = \frac{f[n]}{m}$$

$$\therefore V[n] - \frac{(m)V[n-1]}{(m+\rho\Delta)} = \frac{\Delta}{(m+\rho\Delta)} f[n]$$

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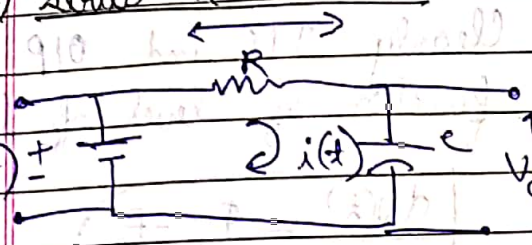
$q = CV$

$V = \frac{q}{C}$

2.) Series RC circuit



$$V_s(t) = R\,i(t) + \cancel{C\frac{di(t)}{dt}}\,v_c(t)$$

$$i(t) = \frac{V_s(t) - V_c(t)}{R} \quad \text{--- (1)}$$

$$i(t) = C\frac{dV_c(t)}{dt}$$

$$\therefore C\frac{dV_c(t)}{dt} = \frac{V_s(t) - V_c(t)}{R}$$

$$\therefore \boxed{\frac{dV_c(t)}{dt} + \left(\frac{1}{RC}\right)V_c(t) = \left(\frac{1}{RC}\right)\cdot V_s(t)} \longrightarrow B_1$$

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Thursday

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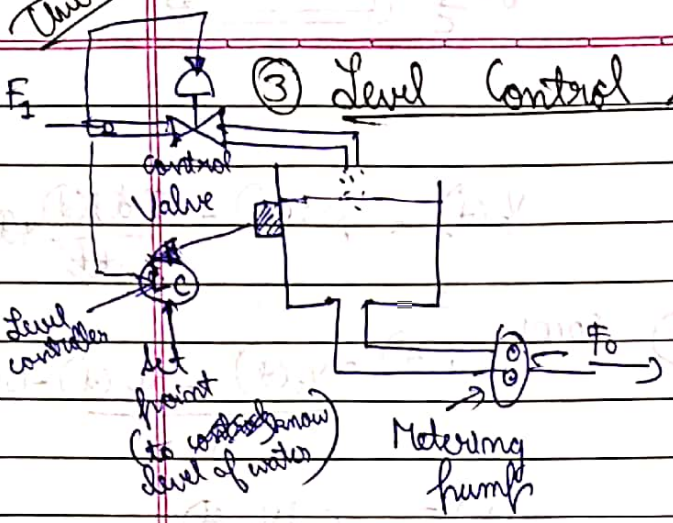
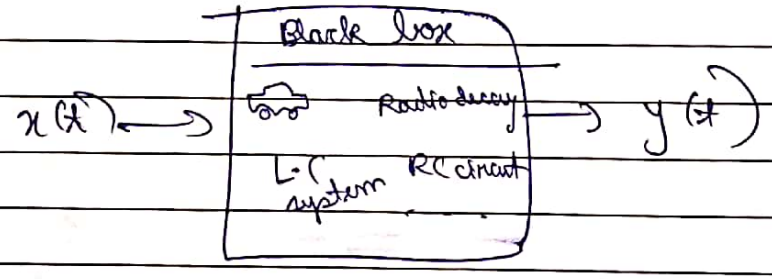
③ Level Control System



Clearly I/P and O/P are functions of level of liquid

$$\frac{d\,i(t)}{dt} = F_i - F_o \quad \rightarrow (1)$$

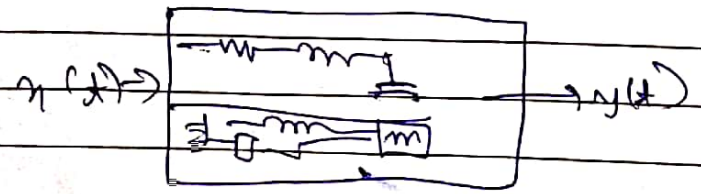
These three different systems from different fields of engg have a common feature that their dynamics is modeled by 1st order diff eqn.



→ This system can be modelled by 1st order diff eqn.

$$\frac{dy(t)}{dt} + a \cdot y(t) = b\,x(t)$$

$$a = \frac{B}{m}$$



↳ 2nd Order diff eqn.

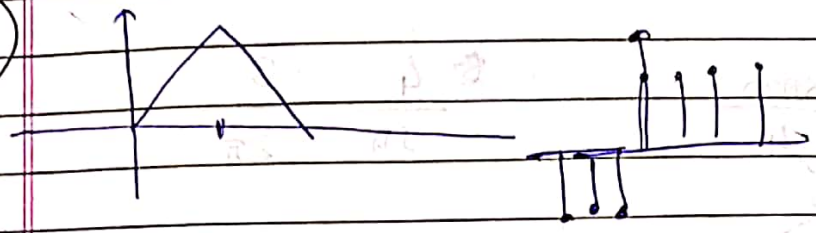
Goal of Signal and Systems

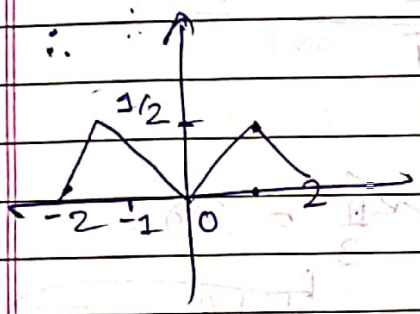
Tutorial

$x_e(t) = \frac{x(t) + x(-t)}{2}$

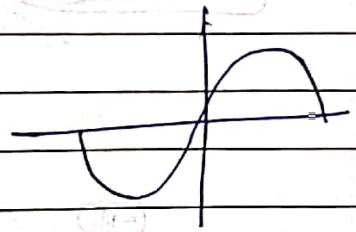
$x_o(t) = \frac{x(t) - x(-t)}{2}$

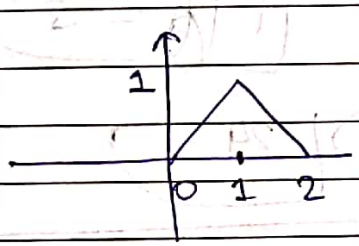
1)

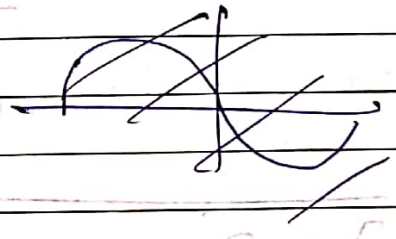


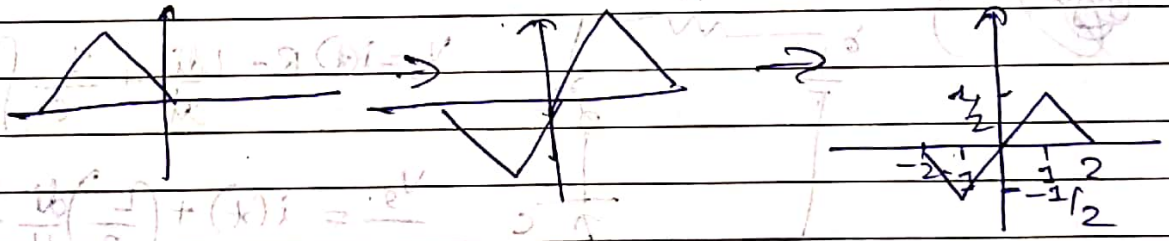


(even)

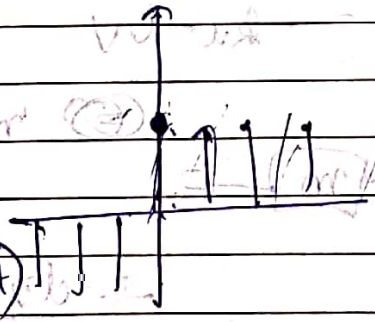


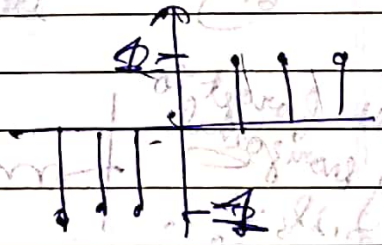






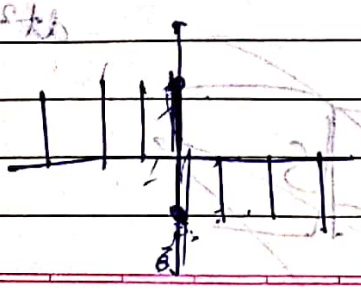
2.

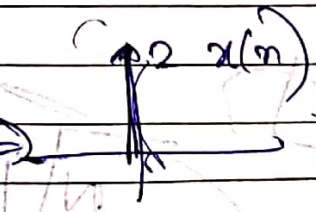




$x(-t)$

$x(t) = x(-t)$





Thursday

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$\omega = 2\pi\nu$

2] $x(n) = e^{j(2\pi/3)n} + e^{j(3\pi/4)n}$

$\frac{2\pi n}{3}, \frac{3\pi n}{4}$

$\frac{4}{3\pi} \quad \frac{3}{2\pi}$

$\boxed{T = 6\pi n}$

$\frac{12}{\pi} =$

$\frac{2\pi}{3} = 2\pi\nu$

2π

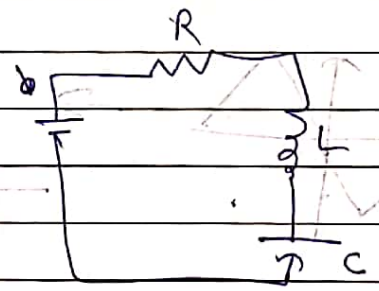
$\frac{3\pi}{4} = \frac{2\pi}{T}$

$\boxed{T = 3}$

$T = \frac{8}{3}$

$\Rightarrow \boxed{24}$

5]

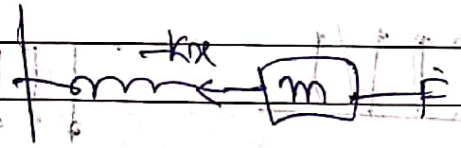


$V_s = i(t)R + L\frac{di}{dt} + \frac{1}{C}\int i(t)dt$

$\frac{V_s}{R} = i(t) + \left(\frac{L}{R}\right)\frac{di}{dt} + \frac{1}{RC}\int i(t)dt$

6] $F = -kx - bV$

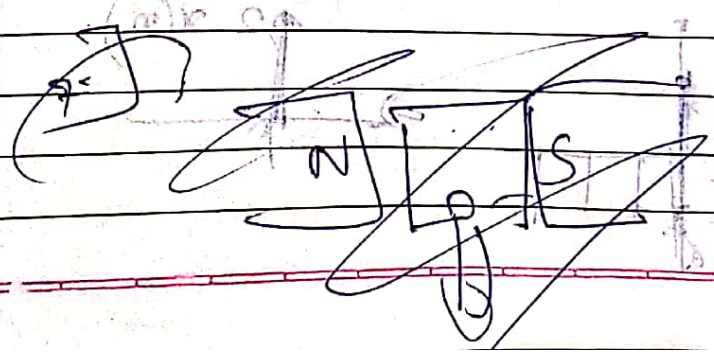
don't forget to specify horizontal system, if u've don't take mg



$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$

$-\frac{d^2x}{dt^2} + \left(\frac{b}{m}\right)\frac{dx}{dt} + \left(\frac{k}{m}\right)x = 0$

7]



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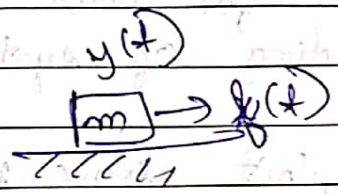
4) Kinetic friction ka koi general formula nai hai

$F_k = aV + bV^2 + cV^3 + \dots$

- Velocity
- SOC
- Mass
- Shape



$F_{net} = m a_{net}$

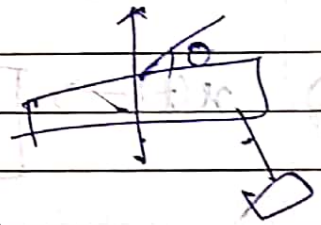
Make the eqn as complex as you wish, but define everything very clearly.

$f(t) - a\,V(t) = M \dfrac{d^2 y(t)}{dt^2}$

$f(t) - a \dfrac{dy(t)}{dt} = \dfrac{M d^2 y(t)}{dt^2}$

Ang mo $= I\omega$

$\tau =$



7) Torque $\tau = \dfrac{I d^2\theta}{dt^2}$

$F = ma$

$\tau = I\alpha$

$I \dfrac{d^2\theta}{dt^2}$

$\tau - k\dfrac{d\theta}{dt} = \dfrac{I d^2\theta}{dt^2}$

$\uparrow$ IP

Friday

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Goals of SAS

1) To analyze properties and characteristics of systems in the common class to develop mathematical tool for their analysis and to get deeper understanding of systems' behaviour.

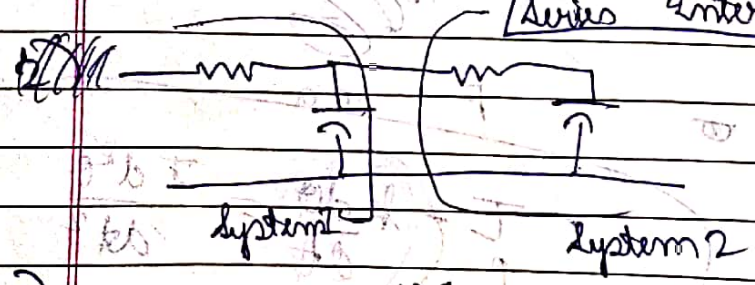
2) Exploit above mathematical tool to 'model' many physical systems of practical importance, such as
(i) Speech production system.
(ii) Seismic readers → vibration of Earth.
(iii) Process Control loops → (a) Weight control
(b) pressure control
(c) temp control, (d) flow control

System Interconnections

$x(t) \rightarrow \boxed{\text{System}} \rightarrow y(t)$

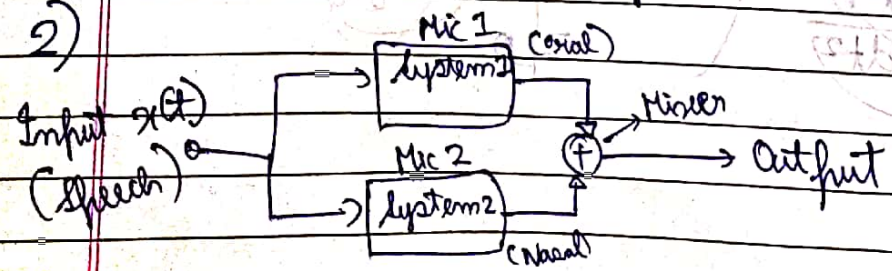
1) $x(t) \rightarrow \boxed{\text{System } 1} \xrightarrow{y(t)} \boxed{\text{System } 2} \rightarrow w(t)$

Series Interconnection



(∵ Cascade of filters)

2)



Parallel Interconnection.

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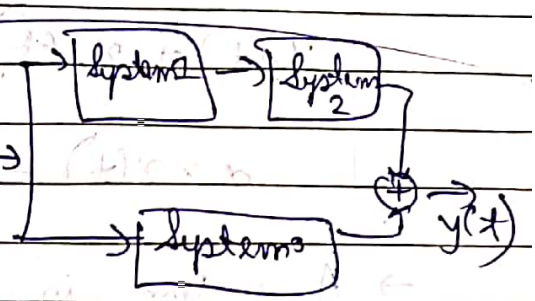
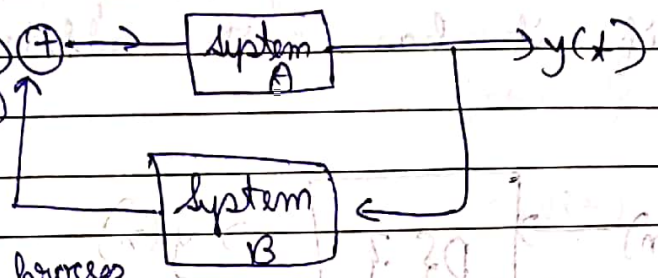
Nasal
Oral
Villus

3.) Series - Parallel Connection



4)



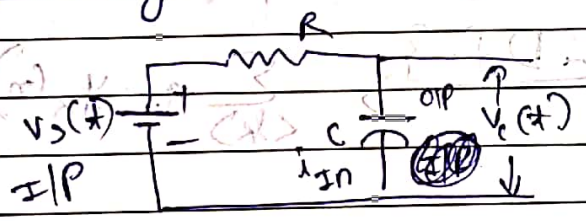
Feedback Interconnection

represents -ve feedback which is a process of regulation.

Properties of Systems

are functions of system itself and not of I/P excitations.

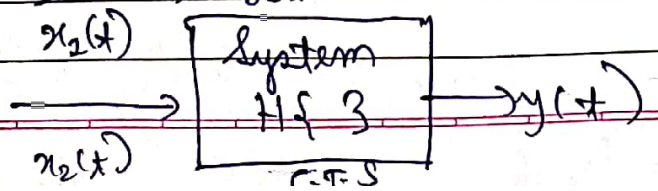
① Linearity :→



$2 \times 2V \rightarrow 2 \times 1A$
$4 \times 3V \rightarrow 4 \times 1.5A$
} Called principle of superposition.

If we have an I/P 1, there is output 1, we have I/P 1 + I/P 2 ⇒ constants multiplied O/P 1 + O/P 2. But if we take one I/P out, corresponding O/P is removed.

→ Capital H ⇒ Cont Time System
" D ⇒ Discrete " "



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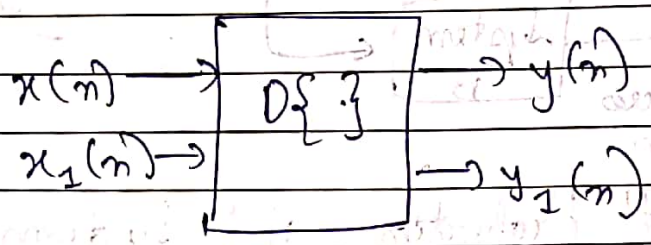
Friday

$x_1(t) + x_2(t) \rightarrow \quad \rightarrow y_1(t) + y_2(t)$ (Additivity)

$\alpha \cdot x(t) \rightarrow \alpha \cdot y(t)$ (Homogenity)

→ A system is said to follow principle of superposition if it has both additivity and Homogenity.



$x(n) + x_1(n) \rightarrow \rightarrow y(n) + y_1(n)$ (∵ Linearity)

$\alpha[x(n)] \rightarrow \rightarrow \alpha[y(n)]$ (∵ Scaled → Homogenity)

→ A system is linear if both props are satisfying.

$\therefore \sum_{\langle k \rangle} a_k x_k(n) \xrightarrow{D\{\cdot\}} \sum_{\langle k \rangle} a_k \cdot y_k(n)$

1ly for C-T

Q.] For the following system, check if linear or not

1] $y(t) = H\{x(t)\} = 4 \cdot x(t) + 3$

This relⁿ ship is called I/P, O/P relⁿ ship

$y_1(t) = 4x_1(t) + 3$
$y_2(t) = 4x_2(t) + 3$ $\Rightarrow y_1(t) + y_2(t) = 4[x_1(t) + x_2(t) + 6]$

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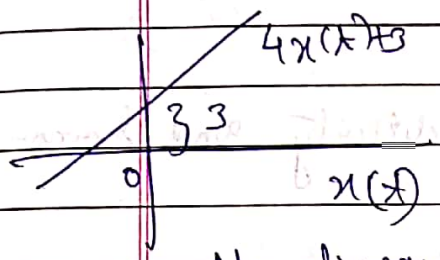
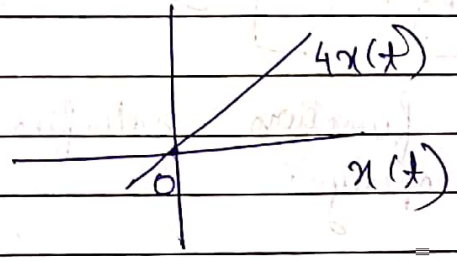
$\Rightarrow y(t) = \cancel{4x(t)}\ y_1(t) + y_2(t)$
$= 4x_1(t) + 4x_2(t)$
$= 4x(t)$ where $x(t) = x_1(t) + x_2(t)$

$\therefore$ If $y(t) = 4x(t) + 3$, principle of superposition doesn't hold $\Rightarrow$ non-linear from perspective of SAS

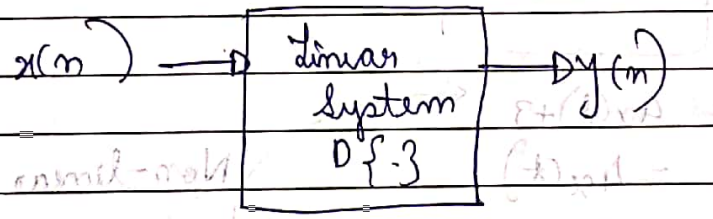


Non-linear



When bias is added (here 3), system is non-linear.

<u>Proposition 1.1</u>: Prove that for a D.T linear system zero I/P must give zero O/P. (holds for CT as well)



$y(n) = D\{x(n)\} \rightarrow$ I/O

<u>Given</u>: System $D\{\cdot\}$ is linear
$\Rightarrow D\{\}$ satisfies the principle of superposition.
$\Rightarrow D\{\cdot\}$ satisfies homogenity and additivity.

1) <u>Homogenity</u>

$y(n) = D\{x(n)\}$
$D\{x(n)\} = y(n)$
$D\{a\, x(n)\} = a\, D\{x(n)\}$ ($\because$ homogenity)

Now special $\Rightarrow a = 0$

$D\{0\, x(n)\} = 0 \cdot D\{x(n)\}$
$D\{0\} = 0$ ($\because$ If system is linear, if I/P is zero $\Rightarrow$ O/P response is zero)

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2) Additivity

$D\{x_1(n) + x_2(n)\} = D\{x_1(n)\} + D\{x_2(n)\}$

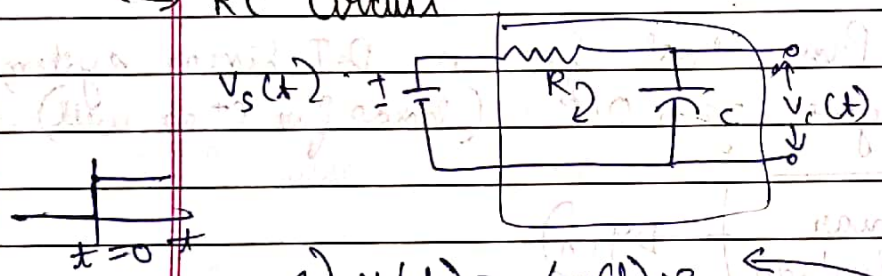
Let $\underset{2}{x_2(n)} = \underset{-2}{-x_1(n)}$ (∵ called anti-modal signals)

$D\{x_1(n) - x_1(n)\} = D\{x_1(n)\} + D\{-x_1(n)\}$
$= D\{x_1(n)\} - D\{x_1(n)\}$
$= y_1(n) - y_1(n)$

$\boxed{D\{0\} = 0}$ ✓

Thus, this function satisfies additivity and homogenity property.

→ RC Circuit



1) $y(t) = 4x(t)+3$
2) $y(t) = 4x(t)$

Non-linear RC circuit

Let $V_s(t)=0$, $V_c(t) \neq 0$

→ Battery is off

When grounded and hence initial $V_c(t)$ becomes zero for an input ⇒ Linear RC circuit

→ $y(t) = H\{x(t)\} = Im\{x(t)\}$

$H\{\cdot\}$ = linear or Non-linear

$y(t) = \cancel{x(t) - x(-t)}$ $Im\{x(t)\} = Im\{x_R(t) + jx_I(t)\}$
$= x_I(t)$

$y_1(t) = \cancel{Im}$ $\cancel{Im}$ $x_{I_1}(t)$

$y_2(t) = x_{I_2}(t)$

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$$I\{x_1(t) + x_2(t)\} = Im\left[x_{R_1}(t) + jx_{I_1}(t) + x_{R_2}(t) + jx_{I_2}(t)\right]$$

$$= x_{I_1}(t) + x_{I_2}(t)$$

$\Rightarrow$ Additivity satisfied.

Now for homogenity

$$y(t) = H(x(t))$$

(i) a = real Now, $H[a \cdot x(t)] = Im\left[a x_R(t) + a x_I(t)\right]$

$$= a x_I(t)$$

$$= a H[x(t)] \quad (\because \text{Homo satisfied})$$

(ii) a = complex $= 2j$ (say)

$$H[2j \cdot x(t)] = Im\left[2j x_R(t) + j(2j) x_I(t)\right]$$

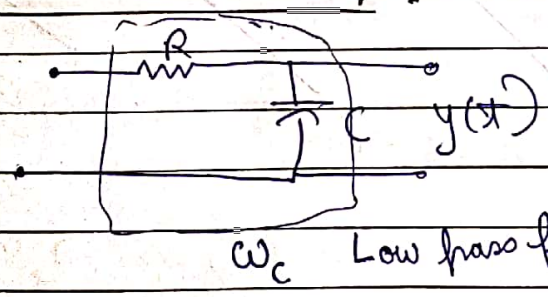
$$= Im\left[2x_R(t) j - 2x_I(t)\right]$$

$$= 2x_R(t)$$

$$\neq 2j\, H[x(t)] \quad (\because \text{Not satisfied})$$

Since both (i) and (ii) do not hold for homogenity the function is not homogeneous $\Rightarrow$ Non-linear.

2.] Time Invariance :- D (TI)



A system is time invariant if the O/P of system is NOT function of time when I/P is applied.

In engineering design, ideally we expect the O/P of a system should be consistent no matter when (time) input is applied. If this consistency is there, the circuit can be used for application.

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Monday

Steps to check TI vs TV

Step 1: Obviously before we check anything for any concept, we need an I/P O/P relationship.

→ Find I/P O/P rel^{n}ship for given system.

$$I/O \rightarrow y(t) = H\{x(t)\}, \quad y(n) = D\{x(n)\}$$

Step 2: Find the O/P of the system for I/P excitation applied today, i.e.,

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$$x(n) \longrightarrow \boxed{\text{Ideal Delay } (n_d) \text{ system}} \quad y(n) = T\{x(n)\} = x(n-n_d)$$

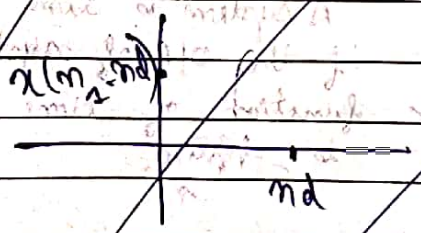
$$\cancel{y_1(n) = x_1(n-n_d)}$$

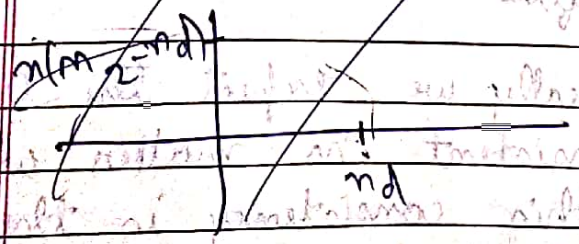
$$\cancel{y_2(n) = x_2(n-n_d)}$$

$$\cancel{y(n_1+n_2) = y(n_1) + y(n_2)}$$

$$\cancel{y(n_1+n_2) = x(n_1+n_2-n_d)}$$

$$\cancel{y(n_1)+y(n_2) = x(n_1-n_d) + x(n_2-n_d)}$$





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Theory

$(f(x_1+x_2) = f(x_1) + f(x_2))$

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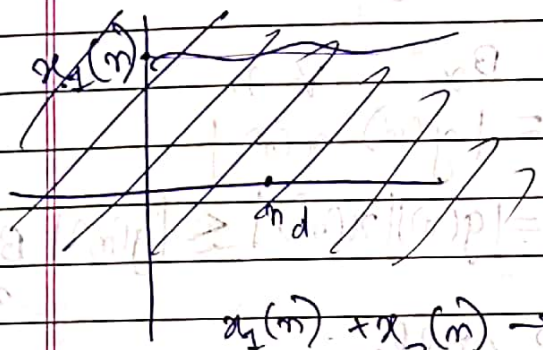
1. $y_1(n) = x_1(n-n_d)$

$y_2(n) = x_2(n-n_d)$

$D\{x_1(n) + x_2(n)\} = D\{x_1(n)\} + D\{x_2(n)\}$



$x_1(n) \rightarrow \boxed{D} \rightarrow x_1(n-n_d)$

$x_2(n) \rightarrow \boxed{D} \rightarrow x_2(n-n_d)$ $\oplus$ $y_1(n)$

$x_1(n) + x_2(n) \rightarrow \boxed{D} \rightarrow x_1(n-n_d) + x_2(n-n_d)$
$= y(n)$

$\alpha x(n) \rightarrow \boxed{D} \rightarrow \alpha x(n-n_d) = \alpha y(n)$

$\rightarrow$ $\cancel{\alpha x_1(n-n_d)}$

$\cancel{x_1(n-n_d)}$ $\alpha x_1(n) \rightarrow \boxed{} \rightarrow \alpha x_1(n-n_d)$

$x(n) \rightarrow \boxed{MA} \rightarrow y(n) = T\{x(n)\} = \frac{1}{N_1+N_2-1} \sum_{k=-N_1}^{N_2} x(n-k)$

$y_1(n) = \frac{1}{N_1+N_2-1} \sum_{k=-N_1}^{N_2} x_1(n-k)$

$y_2(n) = \frac{1}{N_1+N_2-1} \sum_{k=-N_1}^{N_2} x_2(n-k)$

$y_1(n) + y_2(n) = \frac{1}{N_1+N_2-1} \sum_{k=-N_1}^{N_2} [x_1(n-k) + x_2(n-k)]$

$T\{x_1(n)+x_2(n)\} = \frac{1}{N_1+N_2-1} \sum_{k=-N_1}^{N_2} [\quad \ldots \quad]$

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Thursday

2.7 (i) A stable system is one where bounded (finite) input results in bounded (finite) O/P.

If $|x(n)| \le B_x \quad \forall n$
then $|y(n)| \le B_y < \infty \quad , \forall n$

(a) Let $|x(n)| \le B_x \quad \forall n$
Now $|y(n)| = |g(n)\, x(n)|$
$= |g(n)||x(n)| \le |g(n)|\, B_x$

If $g(n)$ is bounded $\Rightarrow$ stable

(ii) If the O/P of a system depends on present and/or past or both I/P $\Rightarrow$ causal system.

(iii) Linear to

(b) 2nd is not stable. $x(n)$ must be bounded. n doesn't have to be bounded. But $x(n_0) + x(n_0+1) - - x(n_0+k)$ unbounded if $n \to \infty$

Linearity:-

$$y_1(n) = \sum_{k=n_0}^{n} x_1(n_0) + x_1(n_0+1) + - x_1(n)$$

$$y_2(n) = x_2(n_0) + x_2(n_0+1) + - - x_2(n)$$

$$= (x_1 + x_2)(n_0) + x_1(n_0+1) - -$$

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$y(n) = x(n) + x(n+1)$

$|x+y| \leq |x| + |y|$

$|xy| = |x||y|$

$y(t-2)$ $H\{x(t-2)\}$

$\delta(t+\frac{3}{4})$ $e^{-t}dt$

$e^{+3/4}$

CONTINUE

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$y(t) = H\{x(t)\}$

Step 3: Find the O/P of the system due to delayed / shifted I/P i.e., $y_1(t) = H\{x(t-1)\}$

delayed (I/P)

Step 4:- Find the O/P $y(t)$ by replacing $t \to t-1$

$\therefore y(t-1)$

If $y(t-1) = y_1(t) = H\{x(t-1)\}$

then $H\{\cdot\} = TI$

else $H\{\cdot\} = TV$

Q.) For the following systems check whether system is TI/TV.

1) $y(t) = H\{x(t)\} = t\,x(t)$ ($\because$ Step 1 I/O already given)

because this is system property

$H\{x(t-1)\} = (t)x(t-1)$ ($\because t_0$) — (1)

Replace t by $t-1$

$y(t-1) = (t-1)\,x(t-1)$ — (2)

$H\{\cdot\}$ is Time Variant (TV)

2) $y(t) = H\{x(t)\}$ $y(t) = kx(t) \Rightarrow$ TI

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3) Causality :- The system must have cause and effect relationship to be called causal.

→ If the O/P of system is either from of present I/P or past I/P or both but not the future I/P.

$\therefore \; y(n) = f[x(n)]$

Or $y(n) = f\{x(n), x(n-1), x(n-2) \ldots\}$

But $y(n) \neq F\{x(n+1), x(n+2) \ldots x(n+k)\}$

then $y(n) = D\{x(n)\}$

$D\{\;\} = \text{Causal}$

Eg 1) $y(n) = \frac{1}{3}\sum_{k=-1}^{+1} x(n-k) = D\{x(n)\}$

$y(n) = \frac{1}{3}\{x(n+1) + x(n) + x(n-1)\}$ ⇒ Not causal.

Since the operator $D\{\;\}$ is function of $x(n+1)$, it is not causal. (non-causal)

2) $y(n) = \frac{1}{3}\sum_{k=0}^{2} x(n-k) = D\{\cdot\}$ → Causal

Proposition 1.2 : For a D-T linear system, which is causal for any input, $x(n)=0$ for $n \leq n_0$, then the O/P $y(n)$

⇒ $y(n) = 0$ not for all n as it is causal

↳ for $n \leq n_0$.

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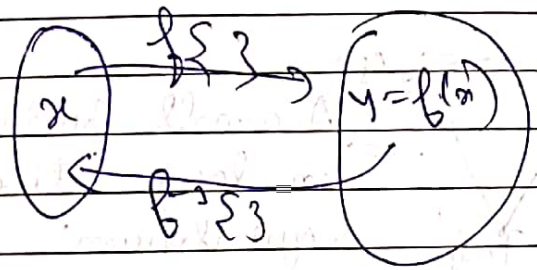
4.] **Invertibility:-** $y = f(x)$
(Inverse System) F has F^{-1} if F is one-one, onto



$$x(n) \xrightarrow{D\{\cdot\}} y(n) = D\{x(n)\}$$
$$y(n) \xrightarrow{D^{-1}\{\cdot\}} x(n)$$

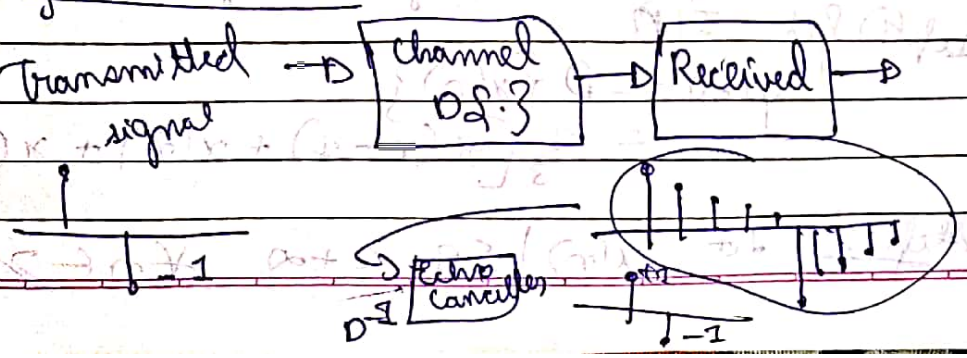
if Distinct I/P leads to distinct output via mapping function $D\{\cdot\}$

$$x(n) \rightarrow \boxed{D\{\cdot\}} \xrightarrow{y(n)} \boxed{D^{-1}\{\cdot\}} \rightarrow w(n) = x(n)$$

~~Identity operator~~

$$y(n) = D\{x(n)\}$$
$$w(n) = D^{-1}\{y(n)\}$$
$$w(n) = D^{-1}\{D\{x(n)\}\}$$
$$x(n) = \underbrace{(D^{-1}D)}_{I}\{x(n)\}$$
$$= I\,x(n) = x(n)$$

Eg:- Digital comm



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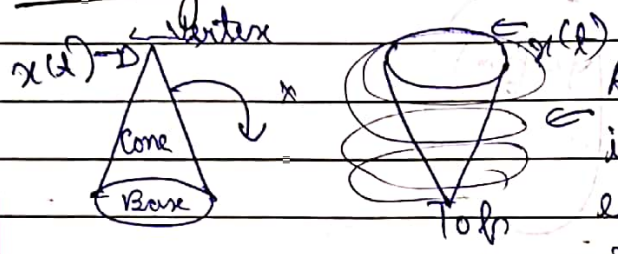
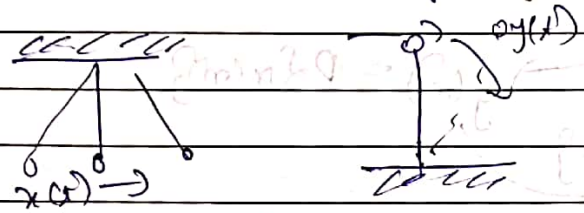
Q.) Is half wave & full-wave rectifier invertible or not?

5.) Stability



A small perturbation results in large diversions from equilibrium.



$x(t) \rightarrow \boxed{\text{System}} \rightarrow y(t)$

If $\underbrace{|x(t)| \le B_x < +\infty \quad \forall t \in R}_{\text{called bounded I/P}}$ ($\because$ Small perturbation)

$|y(t)| \le B_y < +\infty \quad \forall t \in R$ ($\because$ bounded O/P)

then this system is stable system

else unstable system.

$\Rightarrow$ Called Bounded I/P — Bounded O/P

BIBO Stability criteria

Eg:- 1.) $y(n) = \frac{1}{3}\sum_{k=-1}^{+1} x(n-k)$

called 3-point moving Average (MA) system

Solⁿ: Step (1) Get I/P

$$y(n) = D\{x(n)\}$$
$$= \frac{1}{3}\left[x(n-1) + x(n) + x(n+1)\right]$$

Step (2) Let $|x(n)| \le B_x < +\infty \quad \forall n \in Z$

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$\therefore$ For bounded I/P

$$|y(n)| = \left|\frac{1}{3}\sum x(n)\right|$$

$$= \frac{1}{3}\left\{|x(n+1)| + |x(n)| + |x(n-1)|\right\}$$

$$= \frac{1}{3}\left\{B(n) + B(n) + B(n)\right\}$$

$$= B_x < +\infty$$

$$|y(n)| \leq B_x < +\infty \quad \forall n \Rightarrow \text{Stable}$$

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6.] Memoryless or System with memory

Memoryless $\leftarrow$ $y(n) = 2x(n) + 3$

$y(n) = 2x(n) + 5x(n-1) + 7x(n+2)$

System with memory

Memory

Memory need not be only of past. It can also be of future

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Chapter 2: Linear Time-Invariant System (LTI)

Why LTI?

① Many practical systems of importance can be modelled reasonably well as LTI.

② To develop a mathematical tool to model many LTI systems.

Concept of Impulse systems

(i) Input excitation

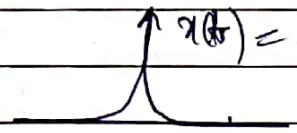
$$x(n) = \delta(n) \longrightarrow \boxed{\text{System } D\{\cdot\}} \longrightarrow y(n) = \text{Impulse response}$$

Written as $y(n) = h(n)$

$$y(n) = D\{x(n)\}$$

For impulse response, we set input excitation source as impulse, i.e, $x(n) = \delta(n)$

(ii) For C-T



$$x(t) = \delta(t) \longrightarrow \boxed{H\{t\}} \longrightarrow y(t)$$

$y(t) = h(t)$.

Q.) Find I/P response

1) 3 point MA system

$$x(n) \longrightarrow \boxed{3\text{ pt MA } D\{\cdot\}} \longrightarrow y(n)$$

Step 1:- Find I/P $\quad y(n) = \frac{1}{3}\sum_{k=-1}^{1} x(n-k)$

Step 2:- For impulse response, we set

$x(n) = \delta(n)$

$y(n) = D\{\delta(n)\}$

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$$\therefore h(n) = D\{\delta(n)\} = \frac{1}{3}\sum_{k=-1}^{+1} \delta(n-k)$$

$$h(n) = \frac{1}{3}\left[\delta(n+1) + \delta(n) + \delta(n-1)\right]$$

Now plotting ⇒ lets take values of n.

(i) $n=0 \Rightarrow h(n) = \frac{1}{3}\left[\delta(1) + \delta(0)\ \delta(-1)\right] = \frac{1}{3}(0+1+0) = \frac{1}{3}$

(ii) $h(1) = \frac{1}{3}\left[0+0+1\right) = \frac{1}{3}$

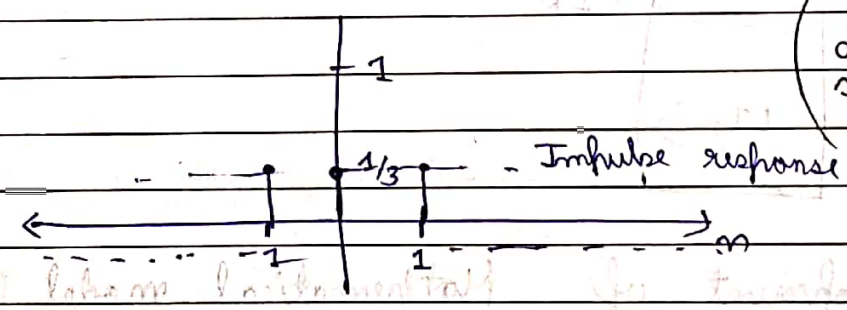
(iii) $h(2) = \frac{1}{3}(0) = 0$, $h(3)$, ... $= 0$

(iv) $h(-1) = \frac{1}{3}\left[1+0+0\right] = \frac{1}{3}$

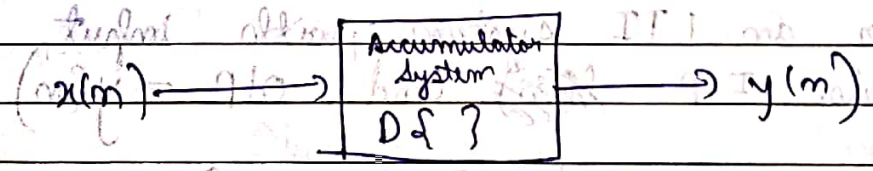
(v) $h(-2) = 0$ ..... $= 0$

(While plotting diagram, make sure to label axis)

⇒



2) Accumulator System



Step 1:- $y(n) = \sum_{k=-\infty}^{+n} x(k) \rightarrow I/O$

Step 2:- i.e. Impulse ⇒ $x(n) = \delta(n)$

$\therefore y(n) = D_n\{\delta(n)\}$

$h(n) = \sum_{k=-\infty}^{n} \delta(k)$

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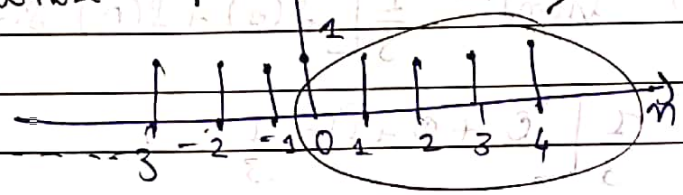
$h(n) = \delta(-\infty) + \cdots \delta(-1) + \delta(0) + \delta(1) + \cdots$

$\delta(-\infty) \to 0$, $\delta(-1) \to 0$, $\delta(0) \to 1$, $\delta(1) \to 0$

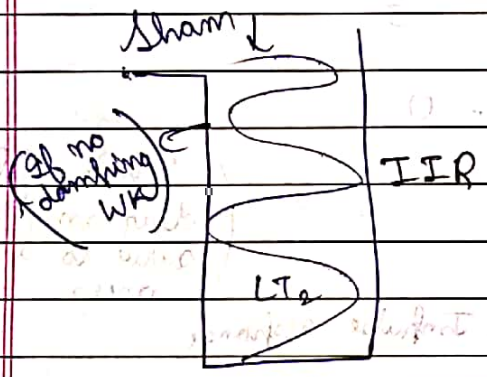
$h(0) = 1$	$h(-1) = 0$
$h(1) = 0$	$h(-2) = 0$
$h(2) = 0$	⋮
$h(3) = 0$	
⋮	
$h(n) = 0$	

$\Rightarrow$ Summation $= 1$

Systems whose Impulse response is finite, are called Finite duration Impulse response system (FIR)

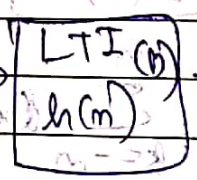


3) Impulse response of real physical system is of infinite duration



Development of Mathematical model (Analytical tool) for analysis of D-T LTI systems

Consider an LTI system, with input signal $x(n)$, impulse I/P and O/P $= y(n)$



Step 1:- $y(n) = D\{x(n)\}$ — (1) ($\Rightarrow$ D is dependent on system characteristics)

Step 2:- Using sifting property of Impulse signal,

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$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \quad \text{——— (2)}$$

Using this eqn,

$$y(n) = D\left\{ \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \right\}$$

$$\cancel{= D\{x}$$

$$y(n) = D\left\{ \sum_{k=-1}^{+1} x(k)\,\delta(n-k) \right\}$$

$$= D\{ \underbrace{x(-1)\,\delta(n+1)}_{x_1(n)} + \underbrace{x(0)\,\delta(n)}_{x_2(n)} + \underbrace{(1)\,\delta(n-1)}_{x_3(n)} \}$$

$$= D\{x_1(n) + x_2(n)\}$$ (∵ 3rd bhi hai but rehnde)

→ Now, LTI ⇒ linear

⇒ $y(n) = D\{x_1(n)\} + D\{x_2(n)\} + \cdots$

Now, $D\{x(-1)\,\delta(n+1)\} \Rightarrow$

$y(n) = D\{x(n)\}$

(∵ $D\{x(n) = \delta(n)\} = h(n)$

$D\{\delta(n)\} = h(n)$)

→ LTI ⇒ TI

$D\{\delta(n-k)\} = h(n-k)$ $\qquad$ $D\{\delta(n)\} = h(n)$

↓ Observed ... $\qquad$ ↑ One observed today

Now, → scalar, → signal

$D\{x(-1)\,\delta(n+1)\} = x(-1)\,D\{\delta(n+1)\}$

$= x(-1)\,h(n+1)$

$D\{x_2(n)\} = D\{\underset{\text{scalar}}{x(0)}\ \underset{\text{signal}}{\delta(n)}\}$

$= x(0)\,D\{\delta(n)\} = x(0)\,h(n)$

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$\rightarrow D\{x_3(n)\} = D\{x(1)\cdot\delta(n-1)\}$
$= x(1)\, h(n-1)$

$y(n) = \ldots x(-1)\, h(n+1) + x(0)\, h(n) + x(1)\, h(n-1) + \ldots$

$\therefore \boxed{y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k)}$ This eq^n is ONLY true for LTI systems, nor other than that (Linear TI)

$\Rightarrow \boxed{y(n) = x(n) * h(n)}$

Convolution

(Impulse response $h(n)$ is critical)

$x(n) \longrightarrow \boxed{\text{LTI } h(n)} \longrightarrow y(n)$

Impulse response

$\boxed{y(n) = x(n) * h(n)}$ $\rightarrow$ (This model is used to model production of speech)

$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$ ($\because$ Only for LTI)

Q.1) For an LTI system, Impulse response $h(n) = \begin{cases} 1/3 & -1 \le n \le 1 \\ 0 & \text{elsewhere} \end{cases}$

Find Output $y(n) = ??$

$x(n) \rightarrow \boxed{D\{\cdot\} \quad h(n) = \frac{1}{3}} \rightarrow y(n) = ?$

$\therefore D\{\}$ is LTI

Since, LTI $\Rightarrow$ $y(n) = x(n) * h(n)$
$= \sum_{k=-\infty}^{+\infty} x(k)\, \delta(n-k)$

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(Also note that convolution is commutative $x(n) * h(n) = h(n) * x(n)$)

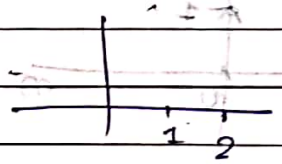
$$\therefore y(n) = \cancel{\cdots} + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + \cancel{\cdots}$$

$$= \frac{1}{3}x(n) + \frac{1}{3}x(n) + \frac{1}{3}x(n-1)$$

$$\boxed{y(n) = \frac{1}{3}\left[\sum_{k=-1}^{+1} x(n-k)\right]} \rightarrow \text{3 pt MA}$$

→ There is unique relation b/w LTI systems' impulse response and its I/O relationship, which is why Impulse response $h(n)$ of an LTI system characterizes an LTI system completely!

Problem 1) $x_2(t) = 0.5x_1(t) - 1.5x_1(t-2)$ | 27/8/18



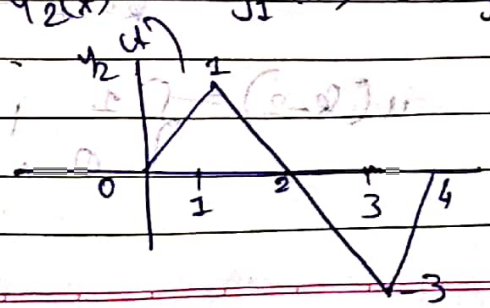
~~$y_1(t) =$~~ Since linear, we can have a linear relⁿ b/w x & y

∴ ~~$y_1(t)$~~ For linear, intercept must be 0

~~$\therefore \left(\frac{2-0}{1-0}\right)$ is slope for $t \in (0,1)$~~

~~$\left(\frac{0-2}{1-0}\right)$ " " " " $\in (1,2)$~~

~~$\therefore y_1(t) = 2x_1(t)$~~ $y_2(t) = 0.5y_1(t) - 1.5y_1(t-2)$



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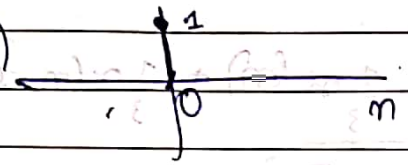
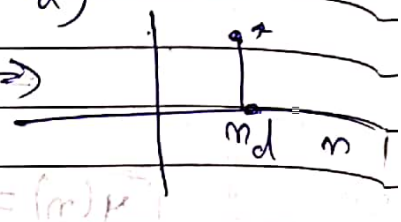
Problem 2: (a) $y(n) = T\{x(n)\}$

Impulse $\delta(n) \Rightarrow h(n) = T\{\delta(n)\}$
$= \delta(n - n_d)$

$x(n)$



$x(n-n_d) \Rightarrow$



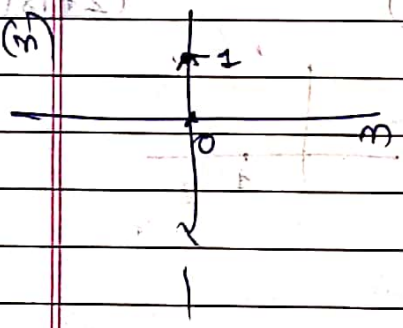
(b) $y(n) = T\{x(n)\}$

$$= \frac{1}{N_1+N_2-1} \sum_{k=-N_1}^{N_2} x(n-k)$$

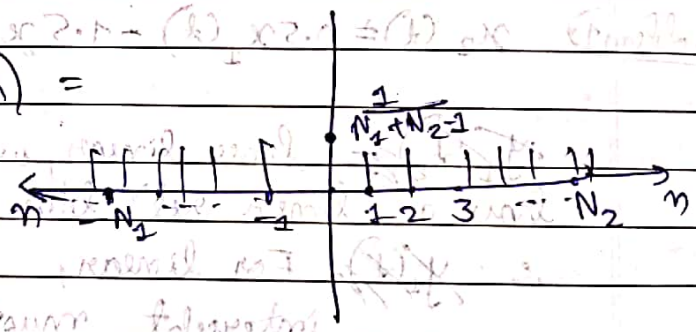
$h(n) = T\{\delta(n)\}$

$$= \frac{1}{N_1+N_2-1} \sum_{k=-N_1}^{N_2} \delta(n-k)$$

$\delta(n)$



$h(n) =$



Problem 3: $y(n) = x(n) * h(n)$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{k-2} u(k-2)\, u(n-k+2)$$

$$u(k-2) = \begin{cases} 1, & k \geq 2 \\ 0, & k < 2 \end{cases}$$

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Tutorial – 6

1] $h_1(n) = \left(\frac{1}{3}\right)^n u(n) \qquad h_2(n) = \left(\frac{1}{6}\right)^n u(n)$

$h(n) = h_1(n) * h_2(n)$

$\left(\frac{1}{3}\right)^n \left(\frac{1}{6}\right)^n u(n)\, u(n)$

$\therefore \cancel{h(n) = \sum_{k=0}^{\infty} \left(\frac{1}{18}\right)^k} \qquad \cancel{h(n) = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^{n-k}}$

$\cancel{= \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^n \left(\frac{1}{6}\right)^{-k}}$

$\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \cancel{u(k)}^{1} \left(\frac{1}{6}\right)^{n-k} u(n-k)$

$\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^{n-k} u(n-k)$

$k \geq 0$

$u(n-k) = \{1 \;,\; n-k > 0 \Rightarrow k < n$

$\left(\frac{1}{6}\right)^n \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^{-k}$

2^k

$\left(\frac{1}{6}\right)^n \left(\frac{2^{n+1}-1}{2-1}\right)$

2.] $\cancel{(a)\; y(n) = T\{x(n)\} = x(n-n_d)}$

$\cancel{\therefore y(n) = x(n) * h(n)}$

$\cancel{= \delta(n) *}$

2.] If $\sum_{k=-\infty}^{\infty} h(k)$ is finite, system is stable.

$h(n) = 0 \;\; \forall n < 0$ system is causal

Put $x \to \infty$ 8

P.T.O

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DATE 6 9 18 (82)

(a) $y(n) = T\{x(n)\}$

$y(n) = x(n-n_d)$

$= \delta(n-n_d) * x(n)$

$\therefore h(n) = \delta(n-n_d)$

Stability :- $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

$\sum_{n=-\infty}^{\infty} |\delta(n-n_d)| = 1$ = finite ⇒ stable

$h(n) = 0, \; n < 0$

$h(n) = \delta(n-n_d) = 1 \; ; \; n = n_d$

$h(n) = 0 \; ; \; n \neq n_d$

(b) $y(n) = T\{x(n)\} = \sum_{k=-\infty}^{\infty} x(k)$

$y(n) = x(n) * h(n)$

$= x(n) * \sum_{k=-\infty}^{n} \delta(k)$

$\cancel{h(n) = \sum_{k=-\infty}^{n} \delta(k) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}}$

$\cancel{h(n) = \sum_{k=-\infty}^{\infty} \delta(k)}$

$\cancel{\sum_{n=-\infty}^{\infty} h(n)}$

$\sum_{k=-\infty}^{\infty} \delta(k) = 1$

$\sum_{n=-\infty}^{\infty} 1 = \infty$ ⇒ unstable

Causality :-

$h(n) = \sum_{k=-\infty}^{n} \delta(k) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$

for causality ; we check for $n < 0$, $h(n) = 0 \; ; \; n < 0$

CAUSAL

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3.) $x(t) * \delta(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$

4.) For autocorrelation, $y = x$

$\Rightarrow r_{xy}(l) = \sum_{(l)} x(n)\, y(n-l)$

$x(n) = \{1,2,3,4\}$.

$x(m) = \{1,2,3,4\}$

$r_{xx}(0) = 1\times1 + 2\times2 + 3\times3 + 4\times4 = 30$ ($r_{xx}(0)$ same indices)

$r_{xx}(1) = 1\times2 + 2\times3 + 3\times4 + 4\times1 = 24$

$r_{xx}(2) = 1\times3 + 2\times4 + 3\times1 + 4\times2 = 22$

$r_{xx}(3) = 24 \quad r_{xx}(4) = 30$

$r_{xx}(5) = ?$ (24)

We can use r_{xx} autocorrelation on signals and if the period of autocorrelation $r_{xx} = A \Rightarrow$ periodity of signal is also A.

5.) $x(n) = \{\ldots 0, 0, 2, -1, 3, 7, 1, 2, -3, 0, 0 \ldots\}$

$y(n) = \{\ldots 0, 0, 1, -1, 2, -2, 4, 1, -2, 5, 0, 0 \ldots\}$

~~After 1 shift~~

$r_{xy}(0) = \sum x(m)\, y(m)$

6.) $r_{xy}(l) = \sum_{n=-\infty} x(n)\, y(n-l)$

Let $n-l = m$

$r_{yx}(l) = \sum_{n=-\infty} y(m)\, x(m-l) = \sum_{m=-\infty}^{\infty} y(m+l)\, x(m)$

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$$= \sum_{m=-\infty}^{\infty} x(m)\,y(m-(-l)) = r_{xy}(-l)$$

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$y(1) = \sum x(k) \cdot h(1-k)$

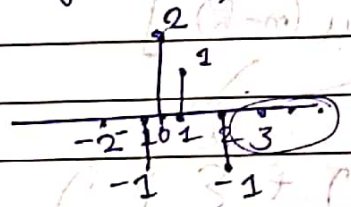
$y(-1) = \sum x(k) \cdot h(-1-k)$

$h(k) \rightarrow h(1-k) \rightarrow h(1-k)$, $h(-1-k)$

$h(-k) = h(0)$ $\quad$ $h(1-k) = h(0)$

$k=0$ $\quad$ value of k.

Shift $h(-k)$ to right by 1 sample to get $h(1-k)$



→ To get $h(-1-k)$, shift $h(-k)$ left by 1 sample.

Method IV Analytical Method

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$ → Convolution sum

$$y(0) = \sum_{(k)} x(k)\, h(-k)$$

$$= \cdots + x(-2)h(2) + x(-1)h(1) + x(0)h(0) + x(1)h(-1) \cdots + 0 \cdots$$

$y(0) =$

→ $y(1) = \sum x(k) h(1-k)$

$= x(-2)h(3) + x(-1)h(2) + x(0)h(1) + \cdots$

(∵ The sequences till now are called finite duration signals)

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$$x(n) = b^n u(n) \longrightarrow \boxed{\begin{array}{l} \text{LTI} \\ h(n) = a^n u(n) \end{array}} \longrightarrow y(n) = ?$$

$$y(n) = x(n) * h(n)$$

(LTI = convolution & convolution = LTI)

⟹ The 4 methods discussed aren't applicable for infinite duration signals.

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \left[b^k u(k)\right]\left[a^{n-k} u(n-k)\right]$$

$$= \cancel{\sum_{k=-\infty}^{-1} (\quad)} + \sum_{k=0}^{n} (\quad) + \cancel{\sum_{k=n+1}^{+\infty} (\quad)}$$

$$u(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases} \qquad u(n-k) = \begin{cases} 1 & n-k \geq 0 \\ 0 & n-k < 0 \end{cases}$$

$$\boxed{y(n) = \sum_{k=0}^{+n} b^k u(k)\, a^{n-k} u(n-k)}$$

$$\therefore y(n) = \sum_{k=0}^{n} b^k a^{n-k}$$

$$= a^n \left[\sum_{k=0}^{n} \left(\frac{b}{a}\right)^k\right]$$

First term of GP $\Rightarrow a_1 = \left(\frac{b}{a}\right)^0 = k$

$$r = \left(\frac{b}{a}\right)$$

$$= a^n \times 1 \left[\frac{1 - (b/a)^{n+1}}{1 - \left(\frac{b}{a}\right)}\right]$$

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Property of LTI Systems (means property of convolution)

$$LTI \Longleftrightarrow h(n) \qquad (\because LTI \text{ implies } h(n))$$

1) Commutative property

$$x(n) * h(n) = h(n) * x(n)$$

Proof:-

$$LHS = \sum_{\langle k \rangle} x(k)\, h(n-k)$$

$$n-k = m \Rightarrow k = n-m$$

$$= \sum_{m=-\infty}^{\infty} x(n-m)\, h(m)$$

$$= \sum_{m=-\infty}^{\infty} h(m)\, x(n-m)$$

$$= \cancel{h(m)} \; LHS = \cancel{h(n)\, h(n-k)} \; x(n) * h(n)$$

Inference

$$y(n) = x(n) * h(n)$$

$$x(n) \rightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \rightarrow y(n) = x(n) * h(n) = h(n) * x(n)$$

$$\Updownarrow$$

$$h(n) \rightarrow \boxed{\begin{matrix} LTI \\ x(n) \end{matrix}} \rightarrow y(n)$$

$\Rightarrow$ For LTI system, the O/P $y(n)$ remains unchanged even if the role of I/P excitation source $x(n)$ and system's impulse response $h(n)$ are interchanged.

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Welcoming

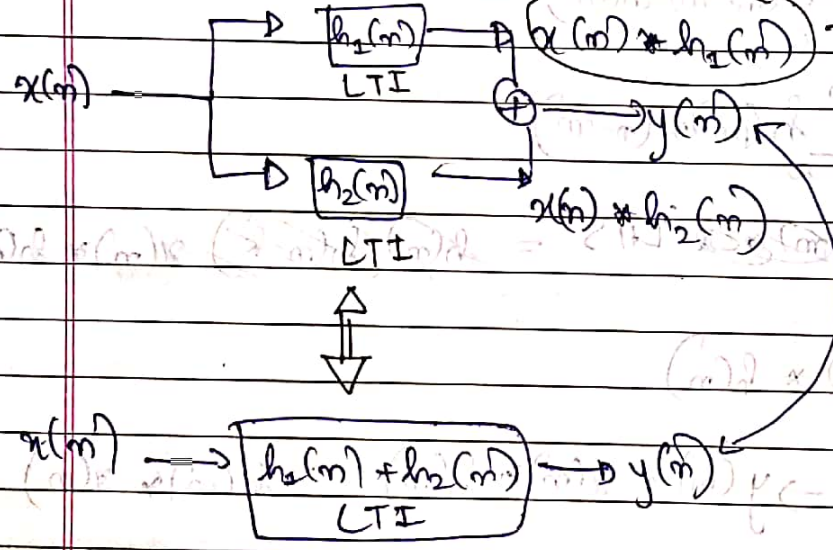
2) Distributive Property

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

Convolution operation distributes over addition.

Proof:

$$\text{LHS} = x(n) * [h_1(n) + h_2(n)]$$
$$= \sum_{\langle k \rangle} x(k) \cdot [h_1(n-k) + h_2(n-k)]$$
$$= \sum x(k) h_1(n-k) + \sum x(k) h_2(n-k)$$
$$= \text{RHS}$$



→ in hardware, its a multiply-add structure.

→ To save cost, prefer below one than upper one.

3) Associative property

$$x(n) * h_1(n) * h_2(n) = [x(n) * h_1(n)] * h_2(n)$$
$$= x(n) * [h_1(n) * h_2(n)]$$

Inference:-

$$x(n) \to \boxed{h_1(n)} \to \boxed{h_2(n)} \to y(n)$$

$$\cancel{x(n) \to \boxed{h_2(n)} \to \boxed{h_1(n)} \to y(n)}$$

$$x(n) \to \boxed{h_1(n) * h_2(n)} \to y(n)$$

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$\Updownarrow$ Commutative

$$x(n) \longrightarrow \boxed{h_2(n) * h_1(n)} \longrightarrow y(n)$$

$\Updownarrow$ Associative

$$x(n) \longrightarrow \boxed{h_2(n)} \longrightarrow \boxed{h_1(n)} \longrightarrow y(n)$$

→ For LTI systems, the order in which systems are cascaded, doesn't matter. It is only the strength of LTI systems.

* LTI systems coupled with additional properties

Q.) What if LTI system is stable

LTI $\Longleftrightarrow$ Convolution

↓

(by impulse response $h(n)$)

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Convolution

$$= h(n) * x(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

→ For ~~discrete~~ stable, LTI system BIBO stability criteria

For bounded I/p, let $|x(n)| \le B_x < \infty$ $\forall n \in \mathbb{Z}$

$$|y(n)| \le \sum_k |h(k)|\, |x(n-k)|$$

$B_x < \infty$ $\forall$ integers.

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$$|y(n)| \le \sum_{(k)} |h(k)| \, B_x$$

$$\downarrow \qquad < +\infty$$

$$< \infty$$

$\Rightarrow$ For stability, $\sum_{(k)} |h(k)| < +\infty$

$$\int_{\tau \in \mathbb{R}} h(\tau)\, d\tau \quad < \infty$$

Impulse response of an LTI system, characterizes it completely

Abhi tak method I

Method II

$$x(n) \rightarrow \boxed{\begin{matrix} \text{LTI} \\ h(n) \end{matrix}} \rightarrow y(n)$$

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{(k)} x(k)\, h(n-k)$$

By sifting property, $x(n) = \sum_{k=-\infty}^{\infty} x(k)\, \delta(n-k)$

$$\therefore \boxed{x(n) = x(n) * \delta(n)}$$

$$2 = 2 * 1$$

$\Rightarrow$ In SAS, impulse is identity i.e., impulse has no effect under convolution.

$\Rightarrow$ When I/P is $\delta(n)$, impulse response is $h(n)$. Impulse signal has no property in its own right to excite an LTI system. Thus $h(n)$ depends on property of system. Hence Impulse response of system

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characterizes it completely.

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LTI system & invertibility

$$x(n) \rightarrow \boxed{\begin{matrix}\text{LTI}\\ h_1(n)\end{matrix}} \xrightarrow{y(n)} \boxed{\begin{matrix}\text{LTI}\\ h_2(n)\end{matrix}} \rightarrow w(n) = (?)\,x(n)$$

$$h(n) = h_1(n) * h_2(n)$$

Since δ function doesn't change anything, $h(n)$ must be $\delta(n)$, as, $x(n) * \delta(n) = x(n)$

LTI system & causality

$$y(n) = x(n) * h(n) = h(n) * x(n)$$
$$= \sum x(k)\,h(n-k)$$

$$\Rightarrow y(n) = \cdots + h(-1)\,x(n+1) + h(0)\,x(n) + h(1)\,x(n-1) + \cdots$$

$y(n) \rightarrow$ Non-causal $(\because x(n+1))$

If we want to force this system to be causal

For Causal LTI system,

$$h(-1) = h(-2) = h(-3) = \cdots h(n) = 0 \quad \forall\, n < 0$$

$\therefore$ $\boxed{\text{Condition is } h(n) = 0 \quad \forall\, n < 0}$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \quad , \quad y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

LTI system with/without memory

$$y(n) = \cdots + h(-1)\,x(n+1) + h(0)\,x(n) + h(1)\,x(n-1) + \cdots$$

I/P

System is with memory

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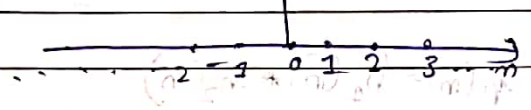
Clearly for memoryless, $h(-1) = h(1) = h(2) = h(-2) = \cdots = 0$
and $h(0) = k$

$$h(n) = \begin{cases} K & n = 0 \\ 0 & , n \neq 0 \end{cases}$$

$h(n) = k\delta(n)$



$\therefore y(n) = x(n) * K\delta(n)$
$= K\, x(n) * \delta(n)$
$= K\, x(n)$

} Impulse response of an LTI system characterizes it completely.

Unit Step Response

$$x(n) \longrightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \longrightarrow y(n)$$

Clearly, $x(n) = u(n)$ for step response.
step response, $x(n) = u(n)$
$\therefore y(n) = s(n)$ = step response

$s(n) = D\{u(n)\} = u(n) * h(n)$
$\therefore s(n) = h(n) * u(n)$
$= \sum_{k=-\infty}^{\infty} h(k)\, u(n-k)$
$= h(-1)\, u(n+1) + h(0)\, u(n) + h(1)\, u(n-1) + \cdots h(k)\, u(n-k)$

$$\boxed{s(n) = \sum_{k=-\infty}^{n} h(k)} \quad D\text{-}T$$

For C-T, $s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau \Rightarrow h(t) = \frac{d\, s(t)}{dt}$

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Ully, $h(n) = s(n) - s(n-1)$

Linear Constant Coefficient Differential Eqn (LCCDE)

Q.1) What is meaning of differential eqn?
Q.2) What does it tells us?
Q.3) Why should you solve differential eqn?
Q.4) Physical significance of DE.

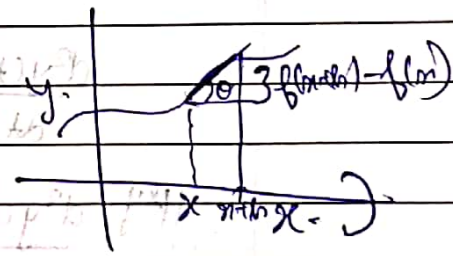
Numerical Simulation of DE in computer

Objective:- How to implement differential equation

$$\boxed{\frac{dy(t)}{dt} + ay(t) = bx(t)}$$

$$\cancel{y(n) - y(n-1)} + ay(n) = bx(n)$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



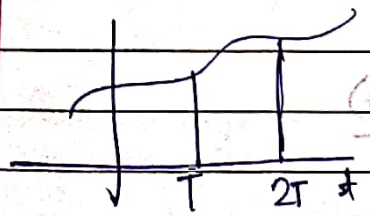
$\therefore \frac{dy}{dx} \approx \tan(\theta) = \text{Slope}$

inclination of line

$\tan\theta = \frac{f(x+h) - f(x)}{h}$

Geometrical importance of derivative is slope of curve at that point.

Now



$$\frac{dy(t)}{dt} = \frac{y(2T) - y(T)}{T}$$

Normalizing $T = 1$ sec

$$\frac{dy}{dt} \approx y(2) - y(1)$$ (∵ called backward difference)

Ully $y(n+1) - y(n)$ = forward difference

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→ Derivative can be approximated by several formulas. But we will use $y(n) - y(n-1)$

$$\therefore \; y(n) - y(n-1) + a\,y(n) = b\,x(n)$$

$$\therefore \; (1+a)\,y(n) - (1)\,y(n-1) = b_0\,x(n)$$

Difference Eqn :- $\boxed{a_0 y(n) - a_1 y(n-1) = b_0 x(n)}$

↳ Differential in C-T

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Now, $\dfrac{d^2y(t)}{dt^2} = \dfrac{d}{dt}\left(\dfrac{dy(t)}{dt}\right)$

$$= \frac{d}{dt}\left[y(n) - y(n-1)\right]$$

$$\simeq \frac{d}{dt}\left[y(n)\right] - \frac{d}{dt}\left[y(n-1)\right]$$

$$= y(n) - y(n-1) - \left[y(n-1) - y(n-2)\right]$$

$$\frac{d^2y(t)}{dt^2} \simeq y(n) - 2y(n-1) + y(n-2)$$

1/ly $\dfrac{d^3y(t)}{dt^3} \simeq a_0 y(n) + a_1 y(n-1) + a_2 y(n-2) + a_3 y(n-3)$

1/ly for input $\dfrac{d}{dt}x(t) \simeq b_0 x(n) + b_1 x(n-1)$

$$\frac{d^2x(t)}{dt} \simeq b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

Now, these a_ks are for differential eqn

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

N^{th} ~~order~~ order LCCDE

$$\therefore \; a_0 y(n) + a_1 y(n-1) + a_2 y(n-2) + \cdots + a_n y(n-1) = b_0 x(n) + b_1 x(n-1) + \cdots b_m x(n-m)$$

these a_ks are for difference eqns

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$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k)$$

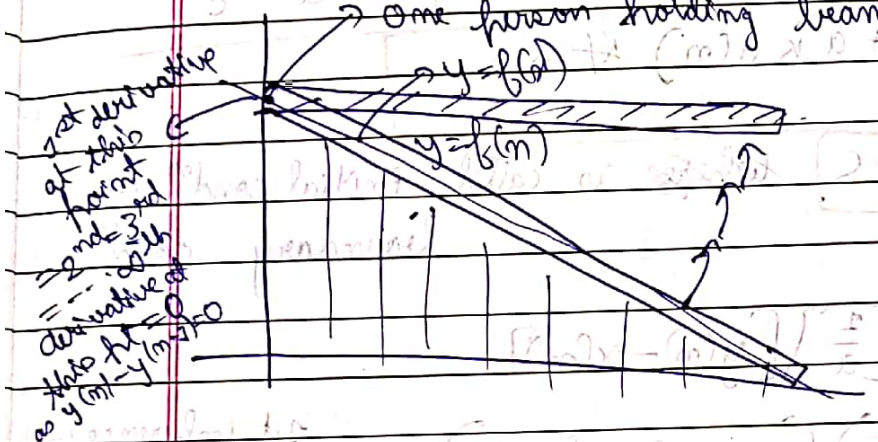
N^{th} order LCC Difference Equation.

→ Differential eq^n is a point property of system

(Eg:- Slope is a point property, whereas integration is a global property (area under curve))

slope

→ One person holding beam applies force at a point to make it "Ground"



Therefore, differential eq^n tells us about the point property of a system.

Q.)

$$x(n) \longrightarrow \boxed{\begin{array}{c}\text{System}\\ D\{\cdot\}\end{array}} \longrightarrow y(n)$$

$x(n) = K\delta(n)$ (scaled impulse)

Consider a system with I/O, $y(n) = D\{x(n)\}$ and it is described by difference eq^n,

$$y(n) = a\,y(n-1) + x(n)$$

Solution of a differential eq^n depends on boundary conditions.

Solve DE for $y(-1) = C$ → Initial Condition.

Sol^n:-

$n \geq 0$ | $n < 0$

$y(n) = a\,y(n-1) + x(n)$ ⟹ $y(n-1) = \frac{1}{a}[y(n) - x(n)]$

(1) $y(0) = a\,y(-1) + x(0)$ | (1) $y(-1) = C$

Now $x(n) = K\delta(n) = K\delta(0) = K$ | (2) $y(-2) = \frac{1}{a}[y(-1) - x(-1)]$

$\therefore y(0) = ac + K$ | $= \frac{c}{a}$

(2) $y(1) = a\,y(0) + x(1) = a[ac+k]$

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Note $y(-1)=0$ means that before I/P is given, O/P is

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(2) $y(2) = a\,y(1) + x(2)$
$= a^3 c + a^2 k$

$\vdots$

$y(n) = \underbrace{a^{n+1} c + a^n k}_{\text{Valid for } n \geq 0}$

(3) $y(-3) = \frac{1}{a}\left[y(-2) - x(-2)\right]$
$= \frac{c}{a^2}$

$\vdots$

$\cancel{y(-n) = \frac{c}{a^{n-1}}}$

$\cancel{y(-n) = a^{n+1}}$

$y(n) = a^{n+1} c$

$\therefore \boxed{y(n) = a^{n+1} c + a^n k\, u(n) \;\forall n}$

where $\boxed{y(-1)=c}$ ~~where~~ is called initial condⁿ or boundary condⁿ

Now,

$y(n-1) = \left[\frac{1}{a}\right]\left[y(n) - x(n)\right]$

$y(n) = \left(\frac{1}{a}\right)\left(y(n+1) - x(n+1)\right)$ $\Rightarrow$ 1st inference :- Non-Causal

(2) inference:- If $x(n)=0 \;\forall n$

$y(n) = a^{n+1} c \neq 0$

$c \neq 0,\; a \neq 0$

Clearly, zero I/P $\not\Rightarrow$ zero O/P $\Rightarrow$ Non-linear.

(3) TI: $y(n) = D\{x(n)\} = a^{n+1} c + a^n k\, u(n)$

$D\{x(n-n_0)\} \neq y(n-n_0)$ $\Rightarrow$ Time Variant.

Problem 2:- $y(-1) = C$ $\rightarrow$ system is initially at rest i.e., before I/P is applied, no O/P.

$y(-1) = 0$ rest same

$y(n) = a^n k\, u(n) \;\forall n$ $\Rightarrow$ Causal (i)

If $x(n) = 0 \;\forall n \Rightarrow y(n) = 0 \Rightarrow$ Linear (ii)

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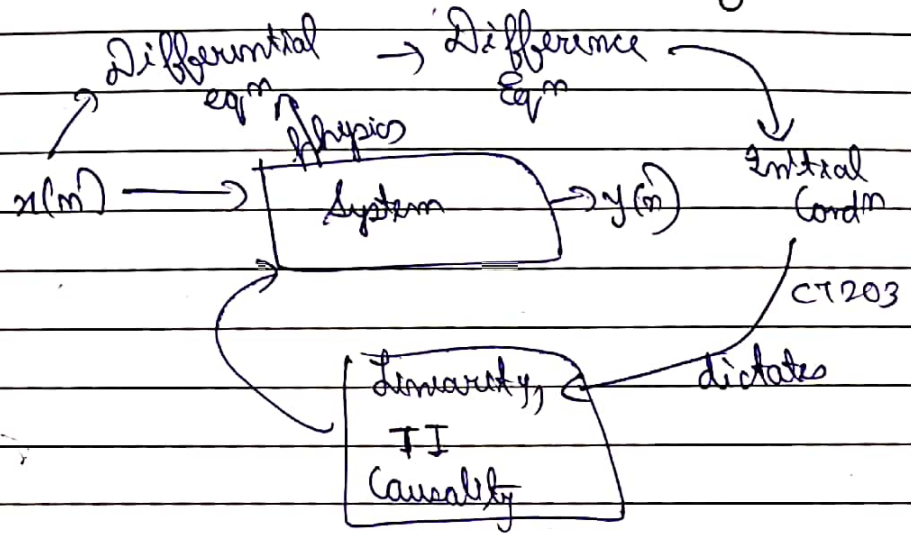
(iii) $D\{x(n-n_0)\} = a^{n-n_0} K u(n-n_0)$

Time Invariant

∴ Initial Cond^{n}s dictate system properties, like linearity, TI/TV, causality.



How to implement difference eq^n

Block diagram representation for difference eq^n

$$\boxed{y(n) + a\,y(n-1) = b\,x(n)}$$

We want to know cost of implementation in hardware

$$y(n) = b\,x(n) - a\,y(n-1)$$

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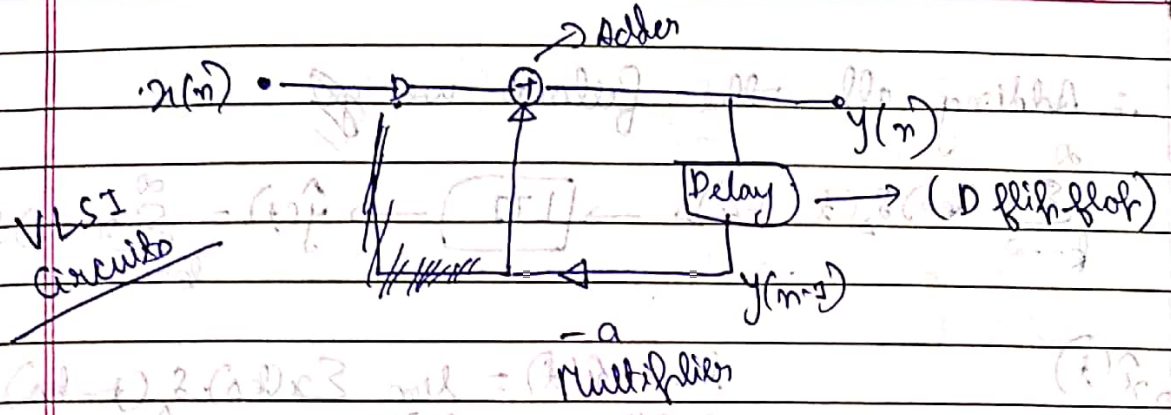
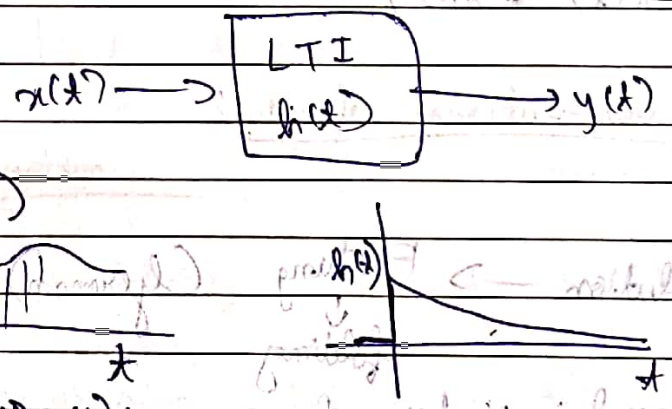


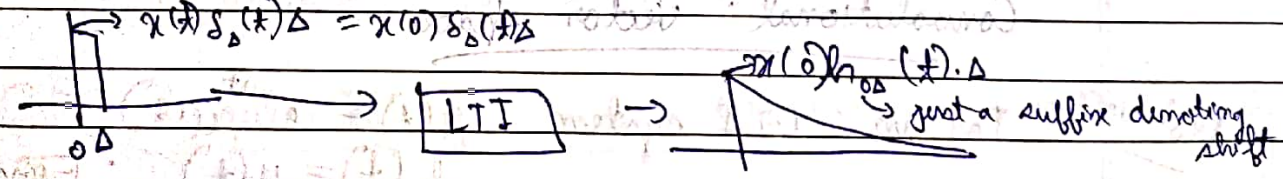
∴ To implement difference eqn in hardware, we need:-

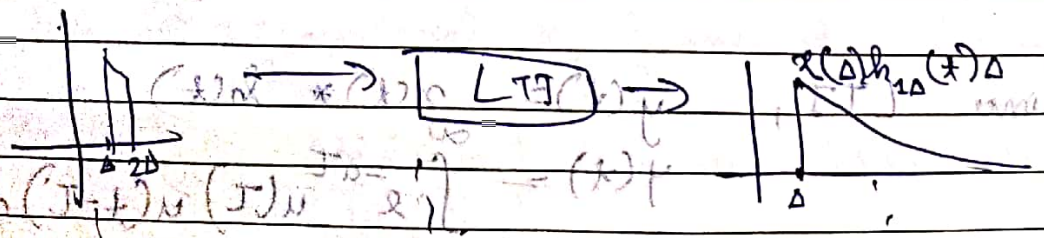
1) Delay element
2) Multiplier
3) Adder

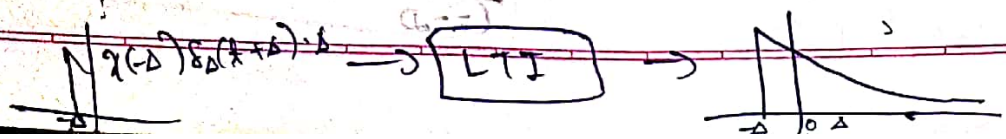
→ We should try to optimize the circuit, using as less multipliers as possible.

Development of Analytical tool for Continuous Time - LTI system









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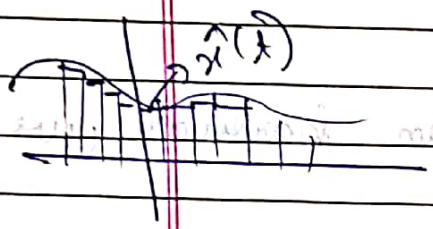
$\therefore$ Adding all the follows we get

$$\sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\Delta \longrightarrow \boxed{LTI} \longrightarrow \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,h(t-k\Delta)$$



$$\lim_{\Delta\to 0} \hat{x}(t) = \lim_{\Delta\to 0} \sum x(k\Delta)\cdot\delta_\Delta(t-k\Delta)\Delta$$

$$\boxed{LTI} \longrightarrow y(t) = \lim_{\Delta\to 0} \hat{y}(t)$$

$$= \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\,h(t-k\Delta)\,\Delta$$

$$\boxed{\therefore\ y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}$$ ← Convolution Integral

$$y(t) = x(t) * h(t)$$

History of Convolutional operator:-

D. Cruotar

Convolution → Faltung (German)
↓
folding

Convolutional vector space

Problem:- For an LTI system $x(t) = e^{-at}u(t)$
$h(t) = u(t)$. Find $y(t)$

Since LTI, $y(t) = x(t) * h(t)$

$$y(t) = \int_{\tau=-\infty}^{\infty} e^{-a\tau}u(\tau)\,u(t-\tau)\,d\tau$$

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$$u(t-\tau) = \begin{cases} 1 & t-\tau > 0 \\ 0 & t-\tau < 0 \end{cases}$$

$$= \begin{cases} 1 & t > \tau \\ 1 & t < \tau \end{cases}$$

$$\therefore y(t) = \frac{1}{a}(1-e^{-at})\, u(t)$$

→ Now, $x(t) * h(t) = h(t) * x(t)$

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

→ $x(t) * h_1(t) * h_2(t) = [x(t) * h_1(t)] * h_2(t)$

$$= x(t) * [h_1(t) * h_2(t)]$$

LTI system stable :-

$$\int_{-\infty}^{\infty} |h(t)|\, dt < +\infty$$

LTI system causal :- $h(t) = 0 \quad , t < 0$

LTI invertibility :- $h_1(t) * h_2(t) = \delta(t)$

Memoryless :- $h(t) = K\delta(t)$

2.5] Singularity functions (Behaviour analysis of impulse fxn)

Find out geometrical significance of determinant of a matrix.

$$\boxed{A^{-1} = \frac{Adj(A)}{det(A)}}$$

If $det(A) = 0 \Rightarrow$ singular matrix

$\Rightarrow A^{-1} \to \infty$

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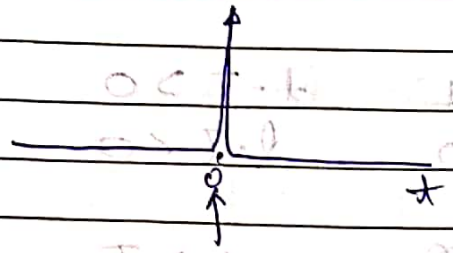


Singularity means that at a point, values of function go abnormally high, whereas in the LHL and RHL values are finite.

→ The behaviour of impulse is singular at a point.

→ In this impulse signal $\delta(t)$, there are infinite signals embedded into it.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

$$x(t) = x(t) * \delta(t) \quad \text{——— (1)}$$

Let $x(t) = \delta(t)$ in (1)

$$\Rightarrow \delta(t) = \delta(t) * \delta(t) \quad \text{——— (2)}$$

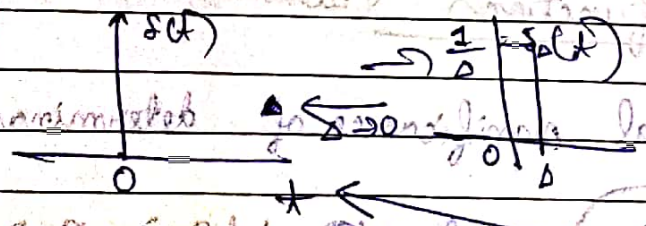
Using (2) in (1)

$$\therefore \quad x(t) = x(t) * \delta(t) * \delta(t)$$

Again $\delta(t) = \delta(t) * \delta(t) * \delta(t)$

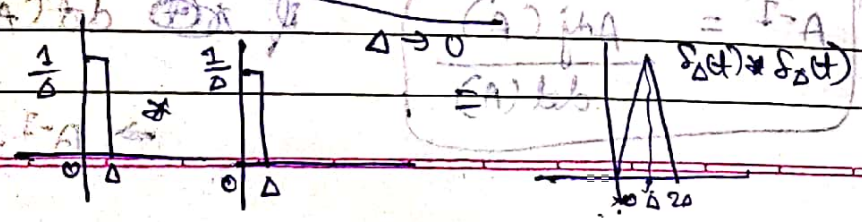
$$\delta(t) = \delta(t) * \delta(t) * \cdots \delta(t)$$

If we take an impulse signal and convolve it with itself infinite times, we still get the impulse signal.



$$\delta(t) * \delta(t) = \delta(t)$$

$$\lim_{\Delta \to 0} \delta_\Delta(t) = \delta(t)$$



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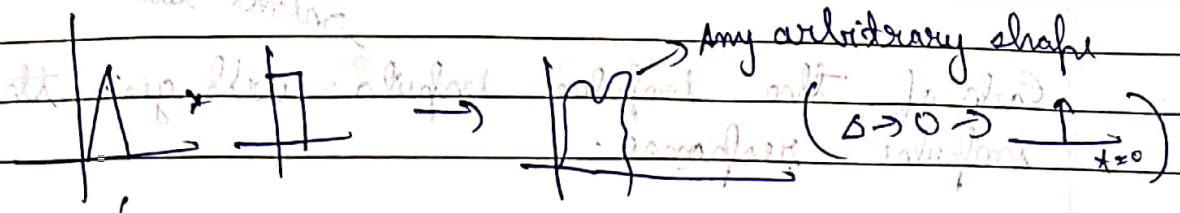
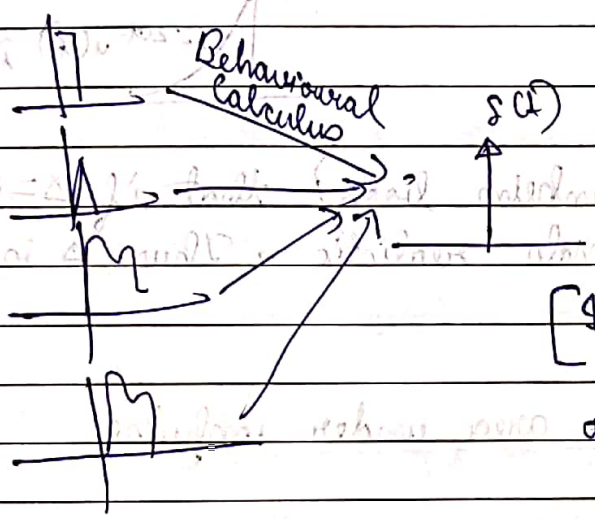


We get pulses of arbitrary shape, but all of them become ↑, when $\lim \Delta \to 0$.

$\therefore \lim_{\Delta \to 0} \delta_\Delta(t) * \delta_\Delta(t) = \delta(t)$

[Here we assume $\lim_{\Delta \to 0} \int_{\tau \in R} \delta_\Delta(\tau)\, \delta_\Delta(t-\tau)\, d\tau$
$\lim \int = \int \lim$
This is true in CT203 but not always]



[Impulse contains infinite number of dissimilar signals, which all BEHAVE like impulse, under convolution]

→ Impulse signal should not be studied as $y = f(x)$, a fn at each and every value of independent variable, but rather its behaviour under convolution.

Q) Consider an LTI system described by 1st order differential eqn,

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$x(t) \longrightarrow \boxed{\text{LTI } h(t)} \longrightarrow y(t)$

when $x(t) = \delta(t)$, we get $h(t)$

$\boxed{h(t) = e^{-2t} u(t)}$ (∵ proved later)

But $x(t)$ can be $\delta_\Delta(t)$, or $\delta_\Delta(t) * \delta_\Delta(t) = x_\Delta(t)$ or $\delta_\Delta(t) * x_\Delta(t)$ or $x_\Delta(t) * x_\Delta(t)$

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Each of the impulse inputs will give the same impulse response. (since each behaves like $\delta(t)$)

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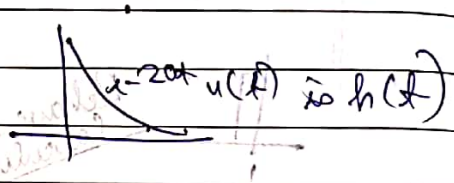
→ As delta becomes smaller and smaller, then the functions get closer and closer to the actual function.

Q.) Whether Δ is same for all the systems?

→ No. So, why? → The 2nd system $\frac{dy(t)}{dt} + 20y(t) = x(t)$



Here, we see (in Oppenheim book) that if $\Delta = 0.00025$, makes all the signals realistic. Thus Δ is a property of the system.

Proof of impulse ⇒ area under impulse

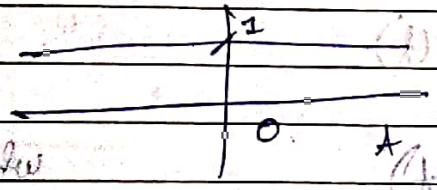
$$\int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1$$

Proof:- By sifting property $x(t) = x(t) * \delta(t)$

By commutative property,

$$x(t) = \delta(t) * x(t)$$

$$x(t) = \int_{-\infty}^{\infty} \delta(\tau)\, x(t-\tau)\, d\tau$$



Let $x(t) = 1 \;\forall t$

$$\therefore 1 = \int_{-\infty}^{\infty} \delta(\tau)(1)\, d\tau$$

$$\boxed{1 = \int_{-\infty}^{\infty} \delta(\tau)\, d\tau}$$ (proved)

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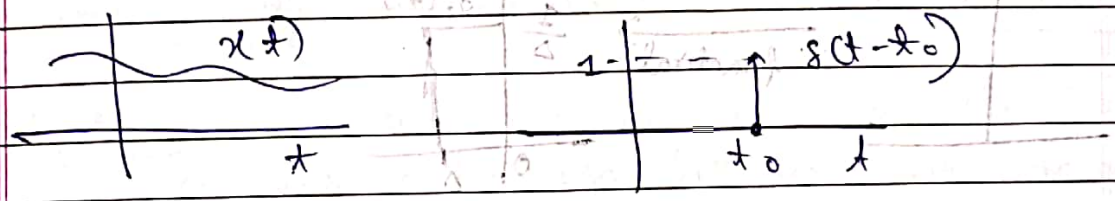
But
No matter what we choose $x(t)$, $\int \delta(t)$ will always be 1.

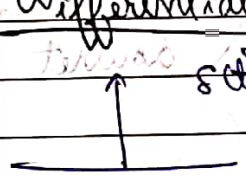
⇒ Prove sampling property of Impulse (pg 131-132)

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$



Unit doublet and other singularity functions

1) Differentiator :-



$$x(t) \longrightarrow \boxed{\frac{d}{dt}(\cdot)} \longrightarrow y(t)$$

$$y(t) = \frac{d}{dt}[x(t)]$$

If $x(t) = \delta(t)$, then $y(t) = \frac{d}{dt}[\delta(t)] = u(t)$

↳ Impulse I/P to differentiator.

$$x(t) \longrightarrow \boxed{\frac{d^2(\cdot)}{dt^2}} \longrightarrow y(t)$$

$$y(t) = \frac{d^2 y(t)}{dt^2} = \frac{d}{dt}\left(\frac{dy(t)}{dt}\right)$$

∴ We just need to cascade the two 1st order systems.

$$x(t) \longrightarrow \boxed{\frac{d(\cdot)}{dt}} \longrightarrow \boxed{\frac{d(\cdot)}{dt}} \longrightarrow y(t)$$

↳ $u_2(t) = u_1(t) * u_1(t)$

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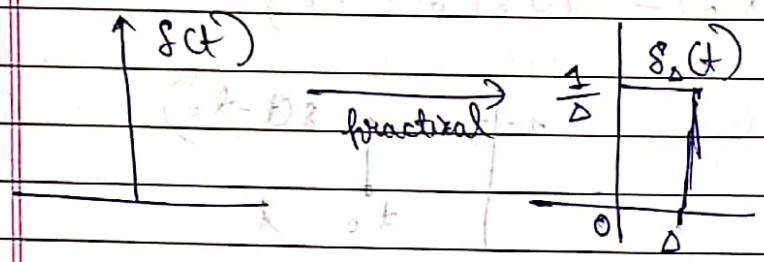
Clearly $u_3(t) = u_1(t) * u_1(t) * u_1(t)$

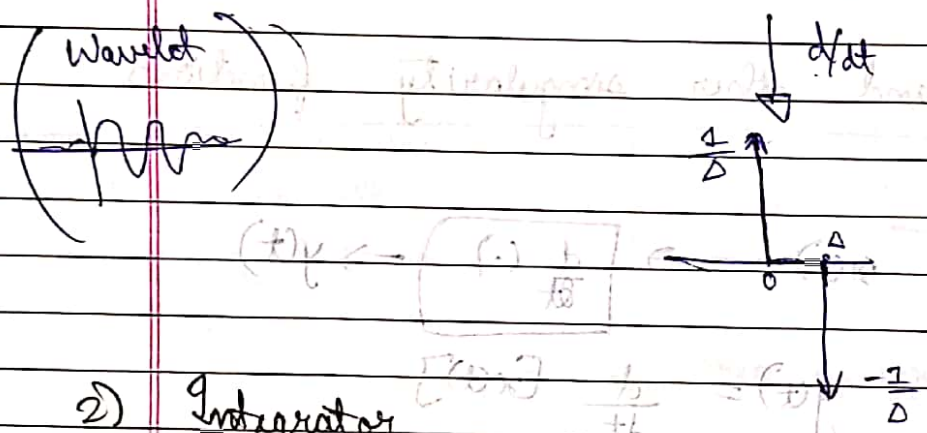
$\vdots$

$u_K(t) =$ K^{th} order derivative

$= u_1(t) * u_1(t) \ldots * u_1(t)$



(Wavelet)



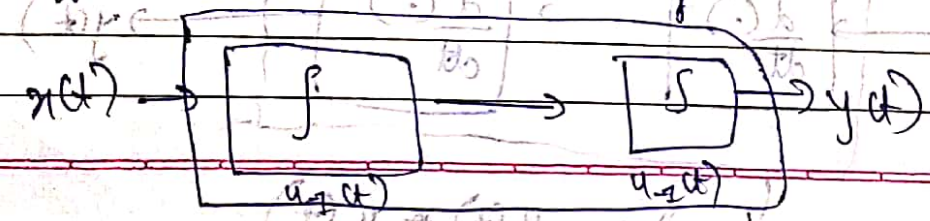
(Because of shape and nature, it is called unit doublet)

2) <u>Integrator</u>

$$x(t) \longrightarrow \boxed{\int \cdot \, d\tau} \longrightarrow y(t) = \int x(t)\,dt$$

$$\boxed{y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau} \rightarrow$$ (Same spirit of accumulator, which was integrator in discrete time)

If $x(t) = \delta(t)$ then $y(t) = u_{-1}(t)$. Minus 1 only represents that this was reverse process of differentiation. No other algebraic meaning.



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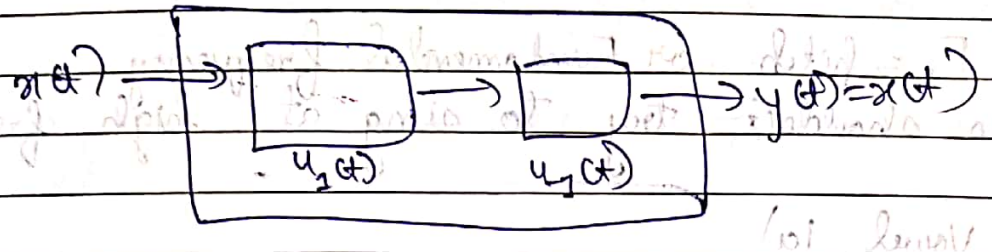
$u_2(t) = u_{-1}(t) * u_{-1}(t)$

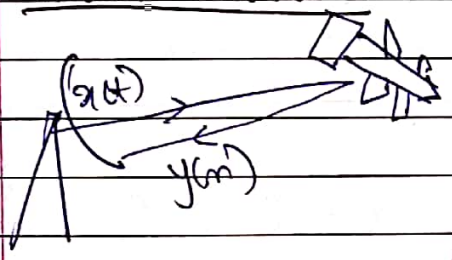


Concept of Auto correlation and Cross correlation

Objective:- These two concepts are used to find degree of closeness / match between two signals $x(n)$ and $y(n)$

few problems:-

① **Radar Problem:**



to detect as well as calc distance of object

$$y(n) = \underbrace{\alpha}_{\text{attenuation factor}}\, x(n-\underbrace{D}_{\text{Round trip delay}}) + \underbrace{W(n)}_{\text{Noise}}$$

Q1) Whether $y(n)$ contains $x(n)$

Q2) If yes, find D

(If some degree of match b/w $y(n)$ and $x(n)$, we can do it by auto & cross correlation)

② **Digital Communication**

$$\boxed{T_x} = \boxed{\text{Communication Channel}} = \boxed{R_x}$$

(1,0)

$x_1(n) = \{1,1,0,1\} \longrightarrow x_0(n) = \{1,0,0,0\}$ (Error in bit transfer)

If $x_1(n)$ and $x_0(n)$ match, we can use the concept.

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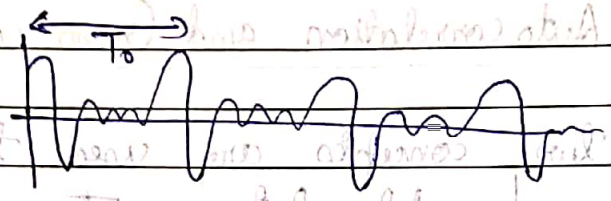
3] Speech & Music

F_0 = pitch or Fundamental frequency.
Males shouldn't try to sing at high frequency.

Let Vowel 'a'



$F_0 = ?$

T_0 = pitch period & $\frac{1}{T_0} = F_0$

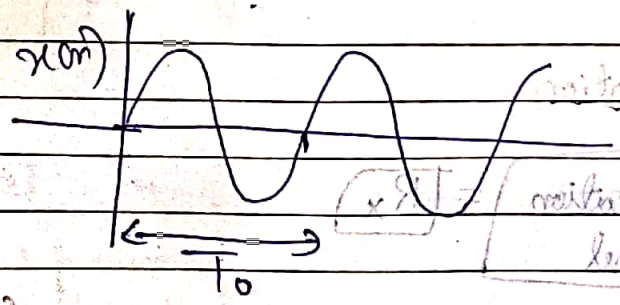
To find value of T_0 and hence F_0, autocorrelation can be used.

Cross Correlation :

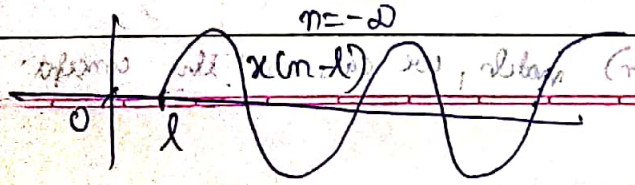
$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l)$$

$\therefore$ Autocor: $r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l)$

Q:) How to use autocorrelation to get F_0 from speech and Music?



$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l) \quad l \in \mathbb{Z}$$



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Tuesday

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When l is 0, energy is maximum. As we vary l, the matching will keep on increasing/decreasing. We get exact max value again at $l = T_0$. In computer, we can have a peak-picking algo to find period.

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(cross-correlation vs. convolution)

Q) Can we use computer program (computer fxn) to implement convolution for finding cross-correlation?

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l) \quad \text{--- (1)}$$

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad \text{--- (2)}$$

$$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(n-k) \quad \text{--- (3)}$$

$n \to l$

$$x(l) * y(l) = \sum_{k=-\infty}^{\infty} x(k)\, y(l-k)$$

$$x(l) * y(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(l-n) \quad \text{--- (4)}$$

~~If we~~ From eq^n (1) and (4), $\boxed{r_{xy}(l) = x(l) * y(-l)}$

$\boxed{r_{xx}(l) = x(l) * x(-l)}$

$x(n) \longrightarrow \boxed{\text{LTI System } h(n)} \longrightarrow y(n)$

We need to find out rel^n b/w $x(n)$ and $y(n)$ in terms of auto and cross-correlation

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Wednesday

$$y(n) = x(n) * h(n)$$

$$r_{xy}(l) = x(l) * y(-l)$$

$$= \left(x(l) * x(-l)\right) * h(-l)$$

$$\boxed{r_{xy}(l) = r_{xx}(l) * h(-l)}$$

$$\boxed{r_{yx}(l) = r_{yy}(l) * h(-l)}$$

$$\boxed{r_{yy}(l) = r_{xx}(l) * r_{hh}(l)}$$

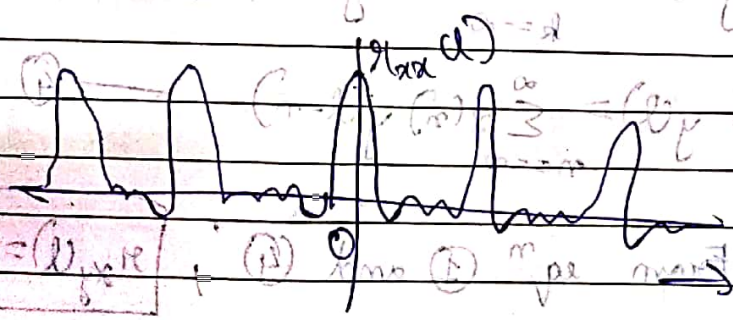
<u>Autocorrelation as an even function</u>

$$r_{xx}(l) = r_{xx}(-l)$$

LHS, $r(l) * r(-l)$ $= r(-l) * r(l)$ ①

$= r(-l) * r(l)$ ②

① ⇒ ②



<u>Advantage</u>:- In computer we don't need to calc the LHS as it is same ⇒ 50% computing time is saved.

Wednesday

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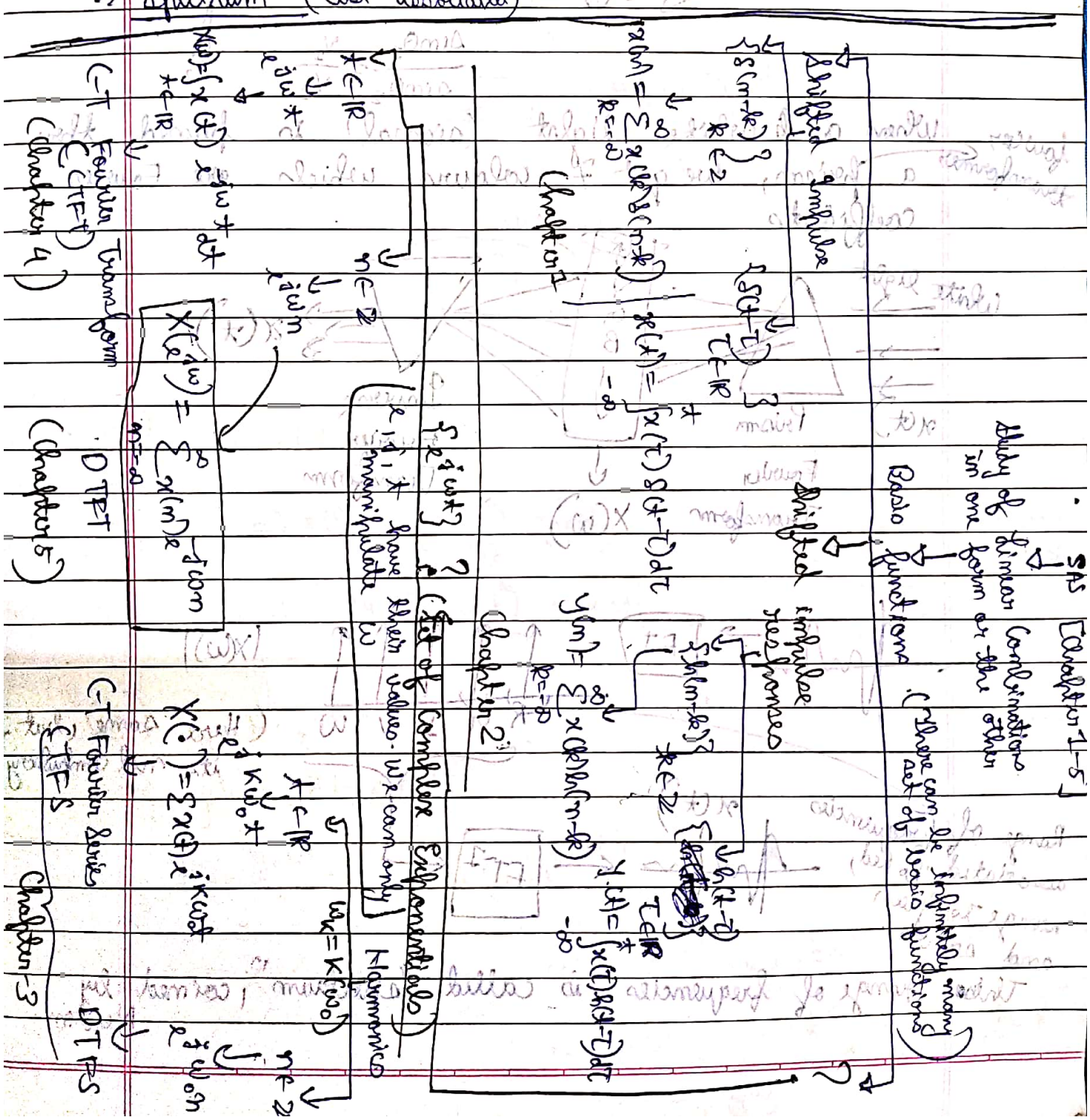
Fourier Kingdom

→ Fourier Transform and Fourier Series have ruled over signal processing and communication for past 150 years.

→ 50-60% work of electrical and computer engineering is related to this.

→ Spectrum (Cost associated)



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Wednesday

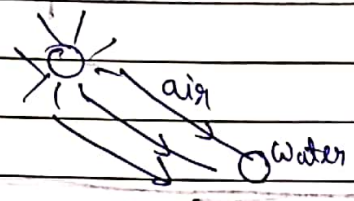
History of Fourier Analysis

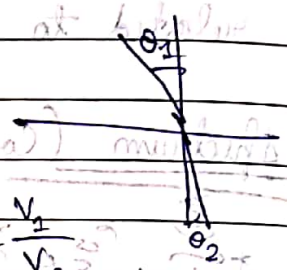
Motivation :-

① Phenomen of Rainbow 1000 years before

② Music and Acoustics → properties of sound.

①

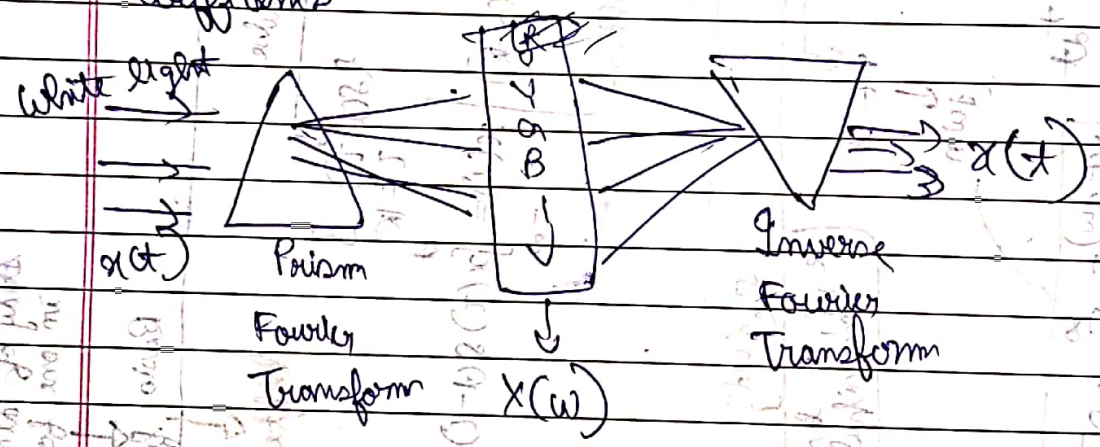


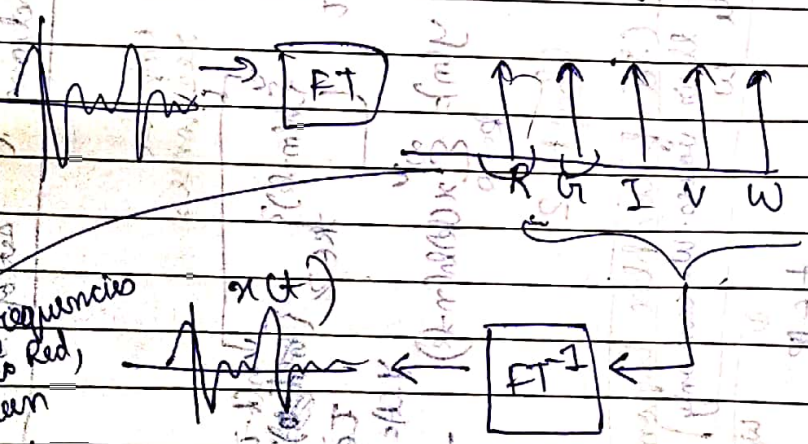


$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

fourier transformer

When a white light (signal) is passed thru a prism, we get 7 colours which are Fourier coefficients





(Here same, but its not compulsory)

Range of frequencies associated to Red, range to green and so on.

This range of frequencies is called "spectrum", coined by Newton.

Thursday

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Tutorial-8

Problem 1:- $x(t) \rightarrow \boxed{H} \rightarrow y(t) = H(x(t))$

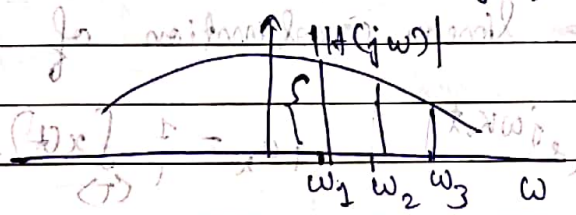
If $y(t) = H(x(t)) = c\,x(t \pm \alpha)$; $\alpha =$ const

Then $x(t)$ is Eigen function.
"c" is Eigen value } of the system

$c = H(j\omega)$ = Fourier transform of impulse response of system

$$e^{j\omega t} \rightarrow \boxed{LTI} \rightarrow H(j\omega)\, e^{j\omega t}$$

$$c = H(j\omega)$$



(a) $V_L = L\dfrac{dy(t)}{dt}$

$\Rightarrow x(t) = \dfrac{L}{R}\dfrac{dy(t)}{dt} + y(t)$

$\therefore \dfrac{dy(t)}{dt} + y(t) = x(t)$

(Voltage on resistor is $L\dfrac{dy(t)}{dt}$ $\Rightarrow$ current $= \dfrac{L}{R}\dfrac{dy(t)}{dt}$)

(b) $x(t) = e^{j\omega t}$

$y(t) = H(j\omega)\, e^{j\omega t}$

$H(j\omega)\, e^{j\omega t} + \dfrac{d}{dt}\left[H(j\omega)\, e^{j\omega t}\right] = e^{j\omega t}$

$\cancel{K + \dfrac{dK}{dt} = 1}$ $\quad \therefore K e^{j\omega t} + K\dfrac{d}{dt} e^{j\omega t} = e^{j\omega t}$

$\therefore \cancel{K e^{j\omega t}} + \cancel{e^{j\omega t}}(j\omega) = \cancel{e^{j\omega t}}$

$\therefore \left(K = \dfrac{1}{1 + j\omega}\right)$

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Dulguday

$$y e^{\int P dx} = \int Q \int P dx + c \qquad \frac{dy}{dx} + P \cdot y = Q(x)$$

General expression of $H(j\omega) = \frac{1}{1+j\omega}$

(2) $\cos\omega t = \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]$

$$\cancel{= \frac{1}{2}\left[\frac{1}{1+j\omega}\right]}$$

$$= \frac{1}{2} H(j\omega) e^{j\omega t} + \frac{1}{2} H(-j\omega) e^{-j\omega t}$$

$$= \frac{1}{2}\left(\frac{1}{1+j}\right) e^{j\omega t} + \frac{1}{2}\left(\frac{1}{1-j}\right) e^{-j\omega t}$$

2] Fourier series = linear combination of complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega k_0 t} \qquad a_k = \frac{1}{T}\int_{\langle T\rangle} x(t) e^{-jk\omega_0 t} dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$ = Four transform of $x(t)$

(a) $\cancel{x(t) * h(t) = \sum_{k=-\infty}^{\infty}}$ $\qquad h(t) = \begin{cases} e^{4t} & t \leq 0 \\ e^{-4t} & t > 0 \end{cases}$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt = \int_{-\infty}^{0} e^{4t} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-4t} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{(4-j\omega)t} dt + \int_{0}^{\infty} e^{-(4+j\omega)t} dt$$

$$= \left[\frac{e^{4-j\omega}}{4-j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(4+j\omega)}}{-(4+j\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{4-j\omega} + \frac{1}{4+j\omega}$$

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Thursday

$e^{j0\omega_0 t} = 1 \xrightarrow{T} H(j0)\cdot 1$

$e^{j\omega_0 t} \rightarrow H(j\omega_0)\, e^{j\omega_0 t}$

$e^{jn\omega_0 t} \rightarrow H(jn\omega_0)\, e^{jn\omega_0 t}$

$a_k e^{jwt} \rightarrow \boxed{} \rightarrow H(jw)\, e^{jwt}$

$x(t) = \sum \delta(t-n)$

$= \sum a_k e^{jwt}$

$y(t)$

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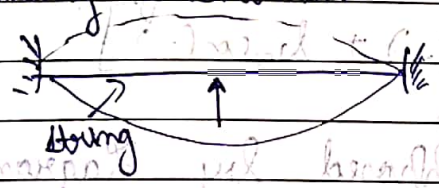
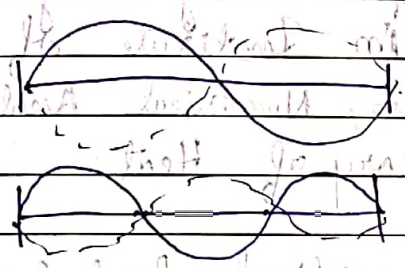
Modern History

(1700)

Music - Analysis of Vibrating String :-
String Instruments



When string is set into vibrations, called Normal Modes of Oscillations.



→ The mannerisms of these oscillations are called Normal Modes of Vibrating strings

These have a time and space parameters.

→ $f(t,x)$: The problem was how to express $f(t,x)$ in terms of closed form solution.

Bernoulli and Euler → said that $f(t,x)$ could be represented as a linear combination of sines and cosines.

$$f(t,x) = \sum a_n \sin(\omega_n t) + b_n \cos(\omega_n t)$$

Lagrange opposed this, as there were some discontinuities which could not be represented by this sin, because sin and cos are smooth functions (Smooth means infinitely differentiable)

P.T.O

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Wednesday

Thermodynamics
(Heat transfer)

Heat diffusion process

Can we use maths to explain heat diffusion process in a body, i.e, analytically

Analytical = closed form mathematical representation.

Jean Baptiste Joseph Fourier

who also told to use summation of sins and cosines to explain heat transfer (independent of other fields)

$$f(\cdot) = \sum \left[a_x \sin(\cdot) + b_x \cos(\cdot) \right]$$

His research paper was opposed by Lagrange, so he published a book, in Institute of France, which talked of Physics, Numerical Analysis, PDE, etc. called Analytical theory of Heat.

These were then commercialized, first by IBM

Cooley & Tukey developed an algorithm called Fourier Transfer (FT)

$$X(e^{j\omega n}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

They came up with a computationally efficient algorithm called Fast Fourier transform (FFT). All companies were waiting for such an algorithm

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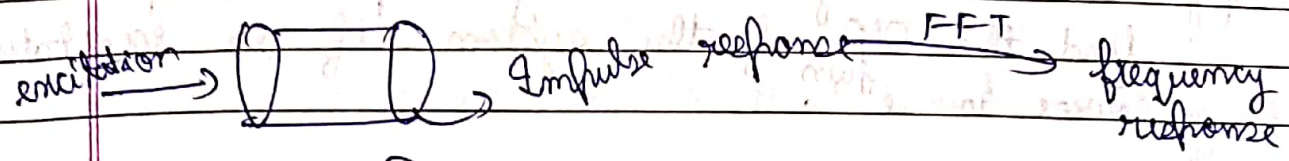
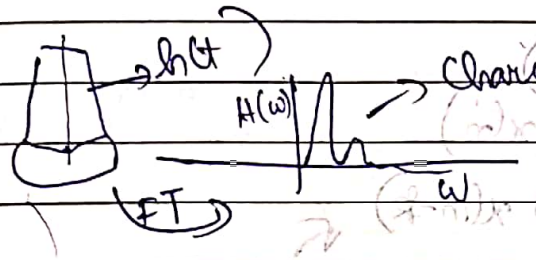
→ We consider musical instrument as system. Sir C.V Raman did his research on Tabla.

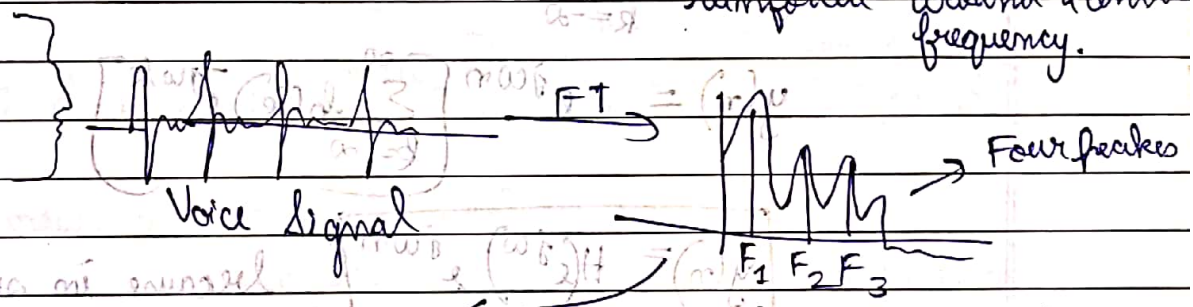




Characteristic of bell like sounds.

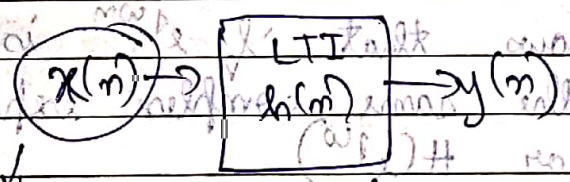
So, Fourier Transform tells us that when we take FT of a signal, we get the ~~frequies~~ resonance.

spectral frequency is reinforced around a central frequency.



This proves that impulse response of bell is different from speech signal.

LTI system and Fourier Analysis



Instead of shifted (simple impulse response to system, we consider x(n) as complex exponentials.

→ # It takes time to understand Fourier Analysis.

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Wednesday

Let $x(n) = e^{j\omega n}$. Find $y(n)$ is the problem.

Q.) For an LTI system having impulse response $h(n)$, find the O/P of the system if the excitation source is $e^{j\omega n}$.

$$y(n) = x(n) * h(n)$$
$$y(n) = h(n) * x(n)$$
$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

Now, $x(n) = e^{j\omega n}$

$\Rightarrow x(n-k) = e^{j\omega(n-k)}$

$$\therefore y(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)}$$

$$y(n) = e^{j\omega n} \left[\sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k} \right]$$

$$\boxed{y(n) = H(e^{j\omega})\, e^{j\omega n}}$$ because in summation k is gone and we get a fn of $j\omega$ only.

→ In labs, for input to LTI system, in place of $e^{j\omega n}$, we take sinusoidal signals.

→ This equation shows that if $e^{j\omega n}$ is input, the output is also the same complex exponential, scaled by factor $H(e^{j\omega})$

$\Rightarrow H(e^{j\omega})$ is transformation of impulse response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

very simple for loop. This eqn is only

governed by $h(n)$. Again we see therefore that impulse response of an LTI system characterizes the system completely.

→ That is why it is $H(e^{j\omega})$, not anything else.

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$$x(n) = e^{j\omega n} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h(n)\end{matrix}} \longrightarrow y(n) = \underbrace{H(e^{j\omega})}_{\text{eigen value}}\,\underbrace{e^{j\omega n}}_{\text{eigenfunction}}$$

Eigen means: $\boxed{\text{proper / characteristic}}$

→ If I/P is $x(n) = a_1 e^{j\omega_1 n} \Rightarrow y(n) = a_1 H(e^{j\omega_1})\, e^{j\omega_1 n}$ (Linear & TI)

→ If I/P is $x(n) = a_2 e^{j\omega_2 n} \Rightarrow y(n) = a_2 H(e^{j\omega_2})\, e^{j\omega_2 n}$

If now, $x(n) = x_1(n) + x_2(n)$
$= a_1 e^{j\omega_1 n} + a_2 e^{j\omega_2 n}$

$\Rightarrow y(n) = y_1(n) + y_2(n)$
$= a_1 H(e^{j\omega_1})\, e^{j\omega_1 n} + a_2 H(e^{j\omega_2})\, e^{j\omega_2 n}$

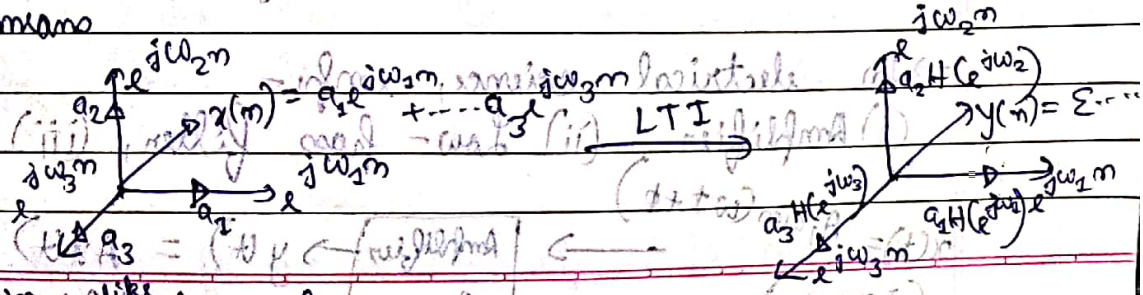
If $x(n) = \sum_{\langle k \rangle} x_k(n) \Rightarrow \sum_{\langle k \rangle} a_k e^{j\omega_k n}$

This need not be real. It represents amplitude & phase of signal

$\left[\omega \in [-\pi, \pi] \text{ or } \omega \in [0, 2\pi]\right]$

$\Rightarrow \boxed{y(n) = \sum_{\langle k \rangle} a_k H(e^{j\omega_k})\, e^{j\omega_k n}}$

⇒ This means



$e^{j\omega n}$ are like vectors here

Thursday

→ The coefficient of the eigen function when passed through an LTI system, gets scaled by the Eigen values.

⟹ From above analysis it is clear that, if input to an LTI system is represented as linear combination of complex exponentials, $\{e^{j\omega_k n}\}_{k\in\mathbb{Z}}$ with corresponding coefficients of linear combinations a_k, then O/P of the LTI system is also linear combination of same complex exponentials $\{e^{j\omega_k n}\}_{k\in\mathbb{Z}}$. However, the coefficient of linear combination in the O/P is equal to coeff of linear combo of I/P scaled by corresponding Eigen value of the LTI operator, i.e. $H(e^{j\omega_k})$

(We use concept of linear combinations, which is a concept made for vectors in maths, in CT203 because signals are modelled as vectors)

→ If there are n equations & n unknowns, their representation is n-dimensional vector.

Importance (beauty) of Sinusoidal Excitation Source:-

$e^{j\omega n}$ ⟶ In real life we have ⟶ $Re\{e^{j\omega n}\}$, $Im\{e^{j\omega n}\}$ since cosine and sin are similar, except one is shifted by 90° degrees

⟹ If vectors are shifted by 90°, we get same vector

→ In electrical science, imp:-
(i) Amplifier (ii) Low-pass filter, (iii) high pass filter

$x(t) = A_i \sin(\omega t + \phi)$ ⟶ [Amplifier, A] ⟶ $y(t) = A\,x(t)$
(x = input) (A: gain factor) $= A A_i \sin(\omega t + \phi)$

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Tuesday

Q.) Find out architecture of opamp

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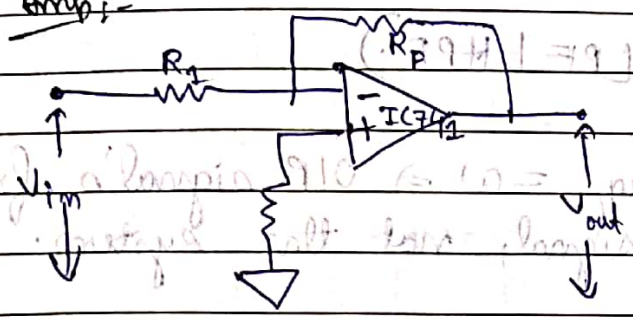
Amp :-

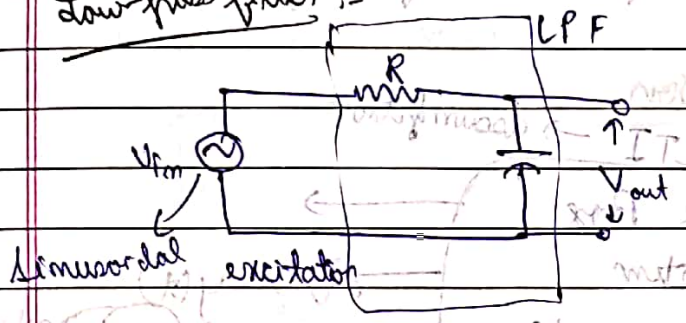


⇒ Inverting Amplifier

⇒ Output $= \left(\frac{-R_F}{R_1}\right) x(t)$

$= \boxed{AA_i \sin(\omega t + \phi)}$ — (A)

Low-pass filter :-



→ LPF is mathematically integration

$\therefore V_{out} = \int V_{in} + c$

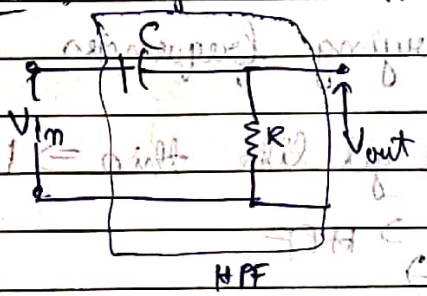
$= \int A_i \sin(\omega t + \phi) dt + c$

(DC level of signal)

$= A_i \left[\frac{-\cos(\omega t + \phi)}{\omega}\right] + c$

$\boxed{V_{out} = \left(\frac{-A_i}{\omega}\right) \sin\left[\frac{\pi}{2} - (\omega t + \phi)\right] + c}$ — (B)

HPF :- (just interchange C & R)



Differentiator Circuit

$V_{out} = \frac{d}{dt}[V_{in}]$

$= \frac{d}{dt}$

$= \omega(A_i)\cos(\omega t + \phi)$

$= \boxed{(A_i\omega) \sin\left(\frac{\pi}{2} - (\omega t + \phi)\right)}$ — (C)

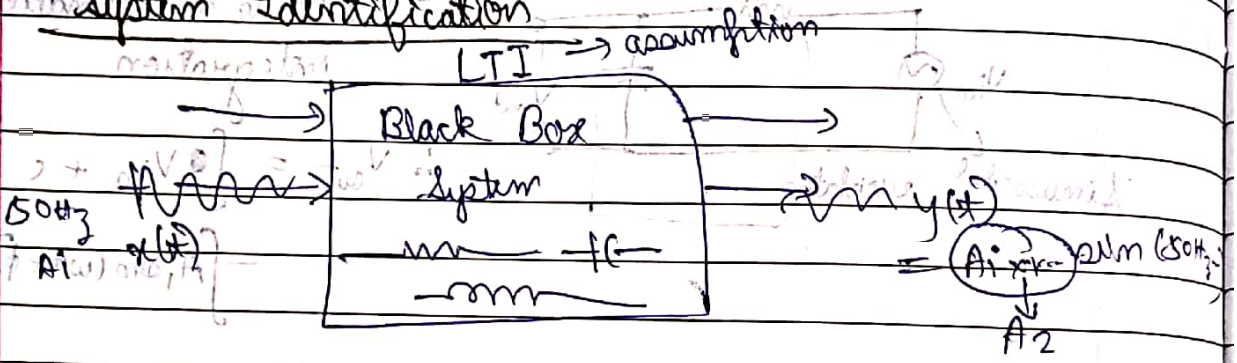
→ Low pass filter boosts the low frequency components and hpf boosts high frequencies. Just do $\omega \to 0$ ⇒ LPF → ∞, $\omega \to \infty$, HPF → ∞

→ (The scaling factor can be real if it is amp and it is

complex, if it is LPF | HPF.)

→ If I/P signal's freq = ω ⇒ O/P signal's freq = ω
ω is property of signal, not the system.
↳ But there is a phase shift of ±90°

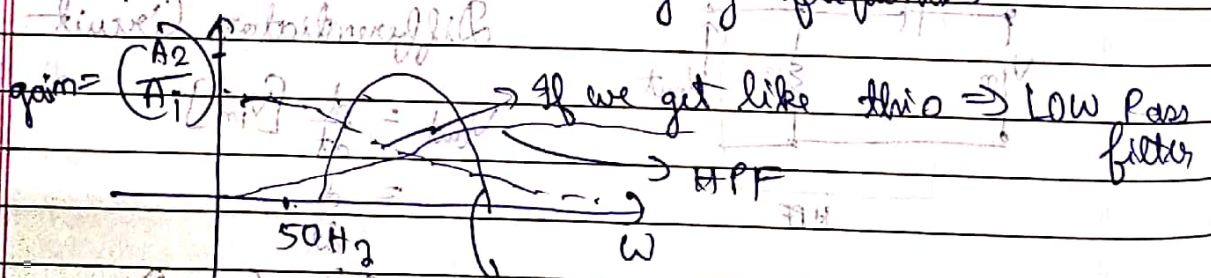
System Identification

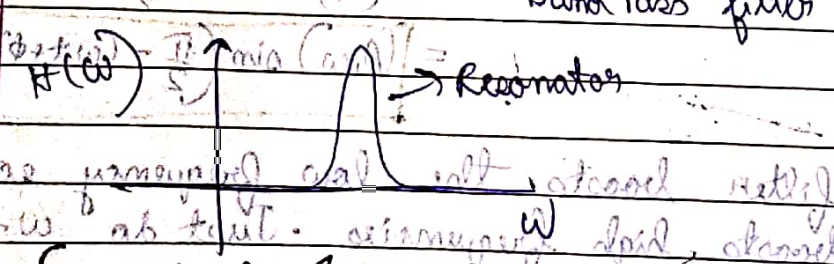


2-port Network

(The mannerism of connection is not known, but it must be combin of resistor, inductor, capacitor)

→ We pass sinusoids with varying frequencies





Concept of Inner Product

(Generalization of idea of dot product)

Thursday

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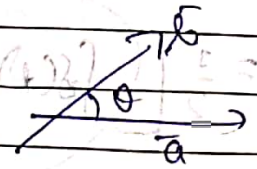
$\bar{a} = a_1 i + a_2 j + a_3 k$

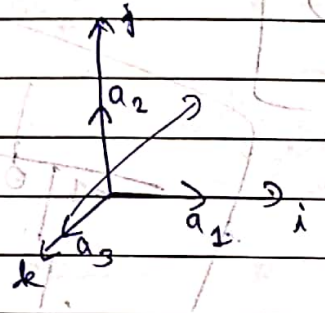
$\bar{b} = b_1 i + b_2 j + b_3 k$



$\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta$

If $\theta = 90^\circ \Rightarrow \bar{a}\cdot\bar{b} = 0$, If $\theta = 0^\circ \Rightarrow \bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|$

$\Rightarrow$ (Dot product represents the degree of closeness b/w two signals)



$$\int (\quad) = \int_{-\infty}^{0} + \int_{0}^{\infty}$$

Tutorial - 8

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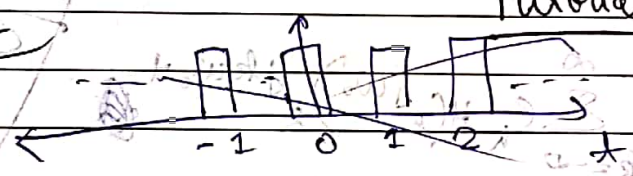
Problem 2:-



~~(a)~~ (b)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$c_k = \frac{1}{T}\int_{-0.5}^{+1.5} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$\int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$$

$$H(e^{j\omega}) = \int e^{-4t} e^{-j\omega_0 t}\, dt$$

$t < 0 \Rightarrow \int_{-\infty}^{\infty} e^{(4+j\omega_0)t}\, dt = \cancel{\frac{2}{-(4+j\omega_0)}}$

$t \geq 0 \Rightarrow \int_{-\infty}^{\infty} e^{(4-j\omega_0)t}\, dt \quad \frac{1}{4-j\omega_0} + \frac{1}{4+j\omega_0}$

P.T.O

$$= \frac{1}{2}\left[\int \delta(t) e^{-jk\omega_0 t}\,dt - \int \delta(t-1) e^{-jk\omega_0 t}\,dt\right]$$

$$= \frac{1}{2}\left[\int_{-0.5}^{0.5} 1\,dt - \int_{0.5}^{1.5} e^{-jk\omega_0 t}\,dt\right]$$

$$= \frac{1}{2}\left[1 - e^{-jk\left(\frac{2\pi}{2}\right)}\right]$$

$$C_k = \frac{1}{2}\left[1 - e^{-jk\pi}\right]$$

$$\therefore\ x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}\,dt$$

$$\Rightarrow y(t) = \sum_{k=-\infty}^{\infty} C_k H(e^{j\omega_0}) e^{jk\omega_0 t}$$

$$C_k = \begin{cases} 0, & \{\text{even}\} \\ 1, & \text{odd} \end{cases}$$

$\omega_0 = \frac{2\pi}{T}$

$= \frac{2\pi}{2}$

$= \pi$

(c) $$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t}\,dt$$

$$= \left(\frac{1}{1}\right)\int_{-1/2}^{1/2} (1) e^{-jk\omega_0 t}\,dt$$

$$= \int_{-1/4}^{1/4} e^{-jk\omega_0 t}\,dt$$

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Thursday

$\cos\left(\frac{k\pi}{2}\right) - \sin\left(\frac{k\pi}{2}\right)$ $= \cos\frac{k\pi}{2} - \sin\left(\frac{k\pi}{2}\right)$

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k^2.

$$= \left(\frac{1}{-jk\omega_0}\right)\left[e^{\left(-\frac{jk\omega_0}{4}\right)} - e^{\frac{jk\omega_0}{4}}\right], \quad \omega_0 = \frac{2\pi}{1} = 2\pi$$

$$= \left(\frac{1}{-j\omega_0 k}\right)\left[e^{-\frac{jk\pi}{2}} - e^{\frac{jk\pi}{2}}\right]$$

Case 1) $k = 2n \quad n \in \mathbb{Z}$

$$\cancel{\frac{1}{(-j\omega_0 k)} - \sin} \quad 0$$

Case 2:- $k = 2n+1 \quad n \in \mathbb{Z}$

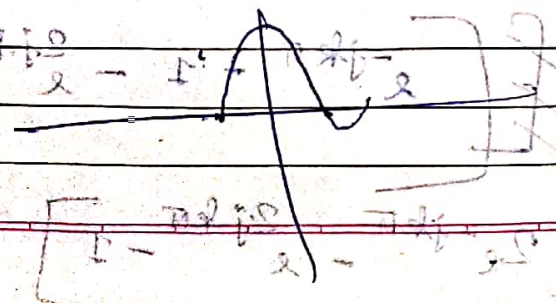
$$\cos\left(-\frac{k\pi}{2}\right) - j\sin\left(\frac{k\pi}{2}\right) - \left[\cos\left(\frac{k\pi}{2}\right) + j\sin\left(\frac{k\pi}{2}\right)\right]$$

$$= -2\sin\left(\frac{k\pi}{2}\right)$$

$$\cancel{= \frac{-2j}{-(j\omega_0)k}, \frac{+2}{-j}, -2, +2}$$

$$\frac{-2j}{-j\omega_0(1)}, + \frac{2j}{-j\omega_0(3)}, + \frac{2j}{-(j\omega_0 5)}$$

$$= \frac{1}{\pi}, -\frac{1}{3\pi}, +\frac{1}{5\pi} \cdots$$



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Writing in terms of sinc,

$$\frac{2A}{T}\left[\frac{e^{jk\omega_0\delta/2} - e^{-kj\omega_0\delta/2}}{2jk\omega_0}\right]$$

$$= \frac{A\,2\,\delta \sin k\omega_0\delta/2}{T\;\frac{\delta}{2}k\omega_0} = \frac{A\delta}{T}\,\mathrm{sinc}\left(\frac{k\omega_0\delta}{2}\right)$$

$$C_k = \begin{cases} 0, & k = \text{even} \\ \frac{A\delta}{T}\,\mathrm{sinc}\left(\frac{k\delta\omega_0}{2}\right), & k \text{ is odd} \end{cases}$$

3] $x(t) = f(t) = \begin{cases} 1 & 0 \le t < 4 \\ -1 & 4 \le t < 8 \end{cases}$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{8}\int_0^4 e^{-jk\omega_0 t}\, dt - \frac{1}{8}\int_4^8 e^{-jk\omega_0 t}\, dt$$

$T = 8$

$\Rightarrow \omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}$

$$= \frac{1}{8}\left[\frac{e^{-jk\frac{\pi}{4}\times 4}}{-jk\omega_0} - \frac{1}{-jk\omega_0}\right]$$

$$= \frac{1}{8}\left[\frac{e^{-jk\left(\frac{\pi}{4}\right)8}}{-jk\omega_0} - \frac{e^{-jk\left(\frac{\pi}{4}\right)4}}{-jk\omega_0}\right]$$

$$= \frac{1}{-8jk\omega_0}\left[e^{-jk\pi} - 1 - e^{-2jk\pi} + e^{-jk\pi}\right]$$

$$= \frac{1}{8jk\omega_0}\left[2e^{-jk\pi} - e^{-2jk\pi} - 1\right]$$

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Thursday

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Odd :-

$$\cancel{\frac{-1}{\left(\frac{80\pi}{4}\right)(1)j}\left[-1-1-1\right] = \frac{3}{2\pi j}}$$

Even :-

0

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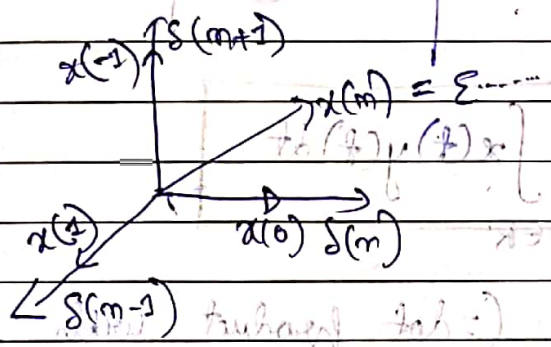
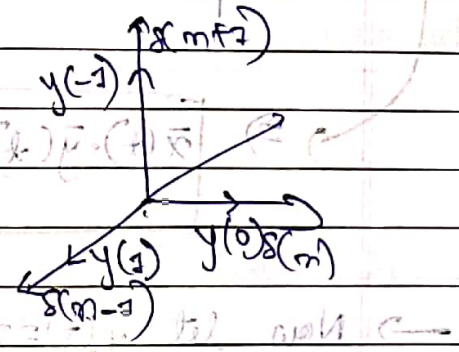
We can write this by sifting property

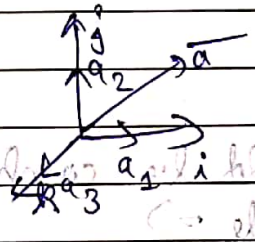


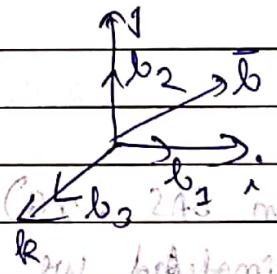


$$x(n) = \sum_{k=-1}^{+1} x(k)\delta(n-k)$$

$$y(n) = \sum_{k=-1}^{1} y(k)\delta(n-k)$$

$$|||$$





Clearly replacing i, j, k by δ functions, we can get the signal. Hence, signals are modelled as vectors.

∴ Just like $\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\bar{x}(n)\cdot\bar{y}(n) = x(-1)y(-1) + x(0)y(0) + x(1)y(1)$$

$$\parallel$$

$$\therefore \sum_{k=-1}^{1} x(k)y(k)$$

But sifting is valid $\forall k \Rightarrow$ $\boxed{\bar{x}(n)\cdot\bar{y}(n) = \sum_{k=-\infty}^{\infty} x(k)y(k)}$

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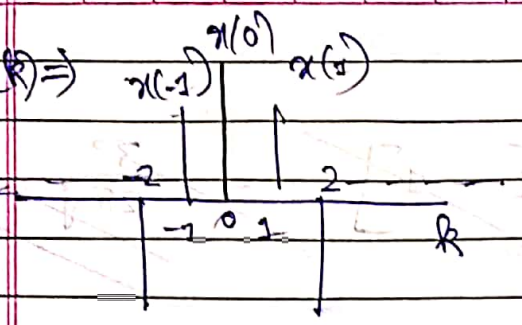
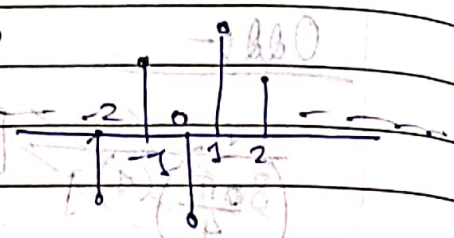
Friday

$x(k) \Rightarrow$

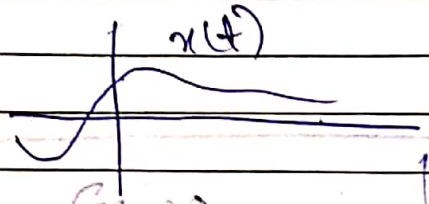
Discrete time

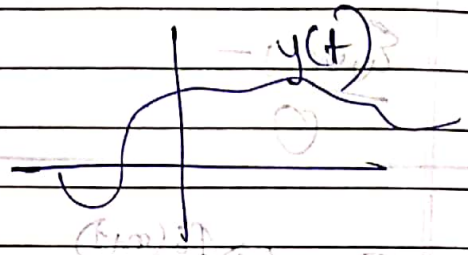


$y(k) \Rightarrow$



Cont. time





$$\Rightarrow \boxed{\bar{x}(t)\cdot\bar{y}(t) = \int_{t\in R} x(t)\,y(t)\,dt}$$

→ Now let $y(t) = x(t)$ (∵ dot product with itself)

$$\bar{x}(t)\cdot\bar{x}(t) = \int_{t\in R} x(t)\,x(t)\,dt$$

$$= \int_{t\in R} x^2(t)\,dt$$

In SAS, $x(t)$ in general could be complex. But if instead we took magnitude ⇒

$$\int_{t\in R} |x(t)|^2\,dt \rightarrow \text{energy of signal}$$

Now we do back substitution

$$\bar{x}(t)\cdot\bar{x}(t) = \int_{t\in R} x(t)\,x^*(t)\,dt$$

One multiply structure

$x(t) = y(t)$

$$\langle x(n), y(n)\rangle = \sum_{\langle k\rangle} x(k)\,y^*(k)$$

$$\Rightarrow \boxed{\bar{x}(t)\cdot\bar{y}(t) = \int_{t\in R} x(t)\,y^*(t)\,dt}$$

$\langle x(t)\cdot y(t)\rangle$

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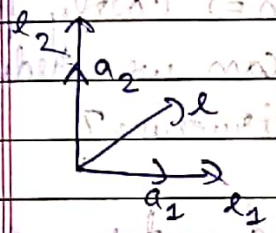
This is a new structure developed by creating a new geometric structure for inner product space, from vector space (also an algebraic structure)

New Notation for inner product: $\boxed{\langle x(t), y(t)\rangle = \int_{t\in R} x(t)\, y^*(t)\, dt}$

Concept of linear Combination in terms of Inner Product



$$\bar{e} = a_1 e_1 + a_2 e_2 \quad \text{—— (1)}$$

Take inner product on both sides of eqn (1) wrt e_1

$$\therefore \langle e, e_1\rangle = \langle (a_1 e_1 + a_2 e_2), e_1\rangle$$

$$(\bar{a}+\bar{b})\cdot\bar{c} = \bar{a}\cdot\bar{c} + \bar{b}\cdot\bar{c}$$

$$= \langle a_1 e_1, e_1\rangle + \langle a_2 e_2, e_1\rangle$$

Now $k\bar{a}\cdot\bar{b} = k(\bar{a}\cdot\bar{b})$

$$= a_1\langle e_1, e_1\rangle + a_2\langle e_2, e_1\rangle$$

Let e_1 mutually perpendicular with $e_2 \Rightarrow \langle e_2, e_1\rangle = 0$

$$\therefore \boxed{a_1 = \frac{\langle e, e_1\rangle}{\langle e_1, e_1\rangle}} \Rightarrow \boxed{a_2 = \frac{\langle e, e_2\rangle}{\langle e_2, e_2\rangle}}$$

If in (1), we had $+a_3 e_3$

$$\Rightarrow a_3 = \frac{\langle e, e_3\rangle}{\langle e_3, e_3\rangle}$$

If we had $a_k e_k$ $\Rightarrow a_k = \dfrac{\langle e, e_k\rangle}{\langle e_k, e_k\rangle}$

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$$\therefore \boxed{e = \frac{\langle e, e_1\rangle}{\langle e_1, e_1\rangle} e_1 + \frac{\langle e, e_2\rangle}{\langle e_2, e_2\rangle} e_2 + \cdots \frac{\langle e, e_K\rangle}{\langle e_K, e_K\rangle} e_K}$$

Here $\{e_K\}_{K \in Z}$ are called mutually orthogonal vectors

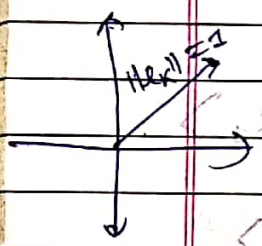
$$\boxed{e = \sum_{\langle k\rangle} \frac{\langle e, e_K\rangle}{\langle e_K, e_K\rangle} e_K}$$

($\because$ k could be finite or infinite, anything)

($\because$ developed on premise that e_Ks are mutually perpendicular)

→ As inner product has multiplication ⇒ costly procedure. For cost effective implementation, can we reduce the burden of one Multiply Add Structure?

→ We use $\boxed{\|e_K\| = 1}$ called norm of a vector. Norm is notion of generalization of length of vector, from origin



$$\boxed{\|e_K\| = 1 \quad \forall K}$$

$\langle e_K, e_K\rangle$ which was very costly (nearly ∞), by making $\|e_K\| = 1 \Rightarrow \langle e_K, e_K\rangle = \|e_K\|\|e_K\|\cos 0$
$= 1$

coefficient of linear combination

$$\Rightarrow \boxed{e = \sum \langle e, e_K\rangle e_K}$$

signal ← e; Basis (or) Representative Vectors

We saved one multiply add structure, per k. If k = 100, we saved equivalent to 100 multiply add structure

$$\boxed{e = \sum \langle e, e_K\rangle e_K}$$

For our course, this is called transform coefficient.

→ Used in transform-based signal processing.

After normalizing, $\{e_K\}$ → ~~Orthogonal Vector~~ → orthonormal vectors

(meaning 1 and normalized)

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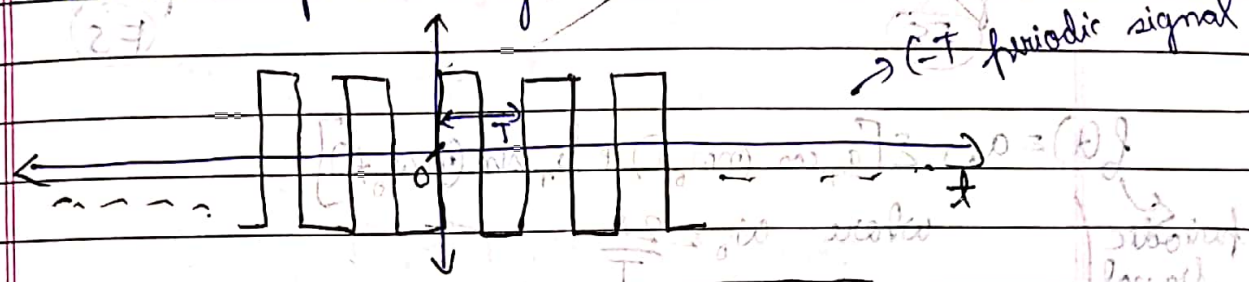
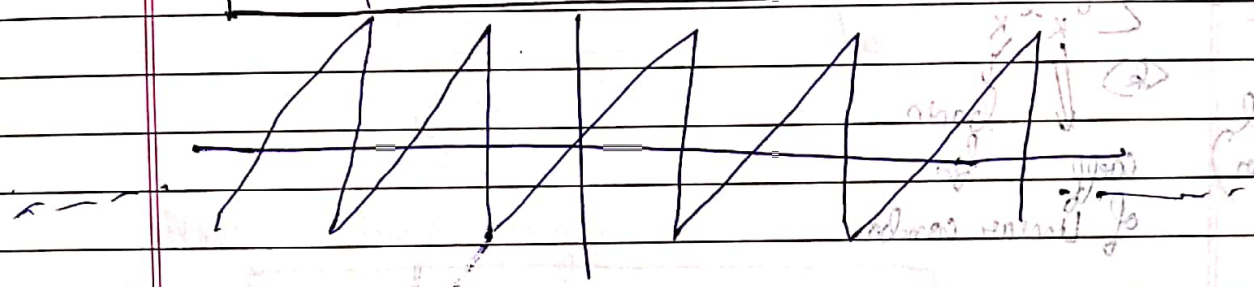
Fourier Series

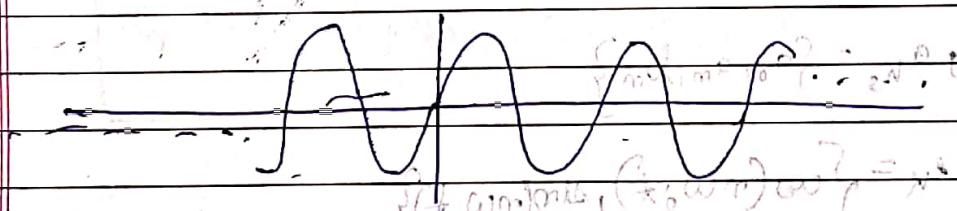
Objective:- To do frequency domain analysis of periodic signal.

Consider a periodic signal:-



$$\boxed{\begin{array}{l} T = \text{fundamental period} \\ \omega_0 = \frac{2\pi}{T} = \text{fundamental frequency} \end{array}}$$





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→ Can we find coefficient of linear combination of $f(t) = \sum a_k e_k$ just by looking at signal? Answer:- No. That is why we need Fourier Series.

→ Sinusoids and complex exponentials are related.

- For the above three periodic signals, if T is same, then the e_k (basis functions) are same but coefficients are different.

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Continuous Time Fourier Series

Representation

- Trigonometric (FS)
- Exponential (FS)

$$f(t) = a_0 + \sum \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right]$$

where $\omega_0 = \frac{2\pi}{T}$

$f(t)$ → periodic signal

$$e = \sum_{(k)} a_k e_k$$

e → signal (vector)

a_k → coeff of linear combo

e_k → basis fns

$$f(t) = e \quad ; \quad a_{k_s} = \{a_0, a_m, b_m\}$$

$$e_k = \{\cos(n\omega_0 t), \sin(n\omega_0 t)\}$$

$$a_k = \frac{\langle e, e_k \rangle}{\langle e_k, e_k \rangle}$$

$$\Rightarrow a_n = \frac{\langle f(t), \cos(n\omega_0 t) \rangle}{\langle \cos(n\omega_0 t), \cos(n\omega_0 t) \rangle}$$

where inner product

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t) g^*(t)\, dt$$

$f(t)$ → periodic

$$\Rightarrow \langle f(t), \cos(n\omega_0 t) \rangle = \int_{-\infty}^{\infty} f(t) \cos(n\omega_0 t)\, dt$$

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First find out integral for one T

$$\Rightarrow \int_0^T f(t)\cos(n\omega_0 t)\,dt$$

$$\langle \cos(n\omega_0 t), \cos(n\omega_0 t)\rangle = \int_0^T \cos^2(n\omega_0 t)\,dt$$

$$\omega_0 = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega_0}$$

$$= \int_0^{\frac{2\pi}{\omega_0}} \left[\frac{1}{2} + \frac{\cos(2n\omega_0 t)}{2}\right] dt$$

$$= \left[\frac{T}{2}\right] + \cancel{\left[\frac{\sin(2n\omega_0 t)}{2}\right]_0^{\frac{2\pi}{\omega_0}}} \to 0$$

$$= \frac{T}{2}$$

Now, substituting,

$$\boxed{a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt}$$

$$\text{i}^{\text{ly}},\ b_n = \frac{\langle f(t), \sin(n\omega_0 t)\rangle}{\cancel{\langle \cos(n\omega_0 t), \cos(n\omega_0 t)\rangle}\ \langle \sin(n\omega_0 t), \sin(n\omega_0 t)\rangle}$$

$$\Rightarrow b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt$$

$$\Rightarrow a_0 = \frac{2}{T}\int_0^T f(t)\,dt$$

P.T.O

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② Exponential FS representation

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$e = \sum_{\langle k \rangle} a_k e_k \quad \text{where} \quad F_n = \frac{\langle f(t), e^{jn\omega_0 t} \rangle}{\langle e^{jn\omega_0 t}, e^{jn\omega_0 t} \rangle}$$

Nothing different from before essentially. That is benefit of a good base.

$$\int_{-\infty}^{\infty} f(t) e^{-jn\omega_0 t}$$

$$\int_{-\infty}^{T} \left(e^{jn\omega_0 t} \cdot e^{-jn\omega_0 t} \right) dt = T$$

$$\boxed{F_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt}$$

∴ Summing up,

basis for representative fns.

Since these synthesize $f(t)$, it is called synthesis eqn.

$$\boxed{\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \\ F_n &= \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt \end{aligned}}$$

periodic signal

Fourier Series Coefficients

Analysis eqn

This is much better and compact than trig FS and it is mathematically sound as sin & cos are related to $e^{jn\omega_0 t}$

(We are analysing the coeffs of LC)

⇒ For following signals, find Exponential FS representation.

Q.1)

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(1) $f(t) = \sin(\omega_0 t)$

$$\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\cancel{= \left(\frac{-j}{2}\right) e^{j\omega_0 t} + \left(\frac{j}{2}\right) e^{-j}}$$

$$F_n = \frac{1}{T}\int_0^{\frac{2\pi}{\omega_0}} \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right) e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^{\frac{2\pi}{\omega_0}} \left(\frac{e^{0}}{2j}\right) dt - \frac{1}{T}\int_0^{\frac{2\pi}{\omega_0}} e^{-2j\omega_0 t}\, dt$$

$$\left.\begin{aligned} F_0 &= 0 \\ F_1 &= \frac{1}{2j} \\ F_{-1} &= \frac{-1}{2j} \end{aligned}\right\} \text{FS coefficients.}$$

$F_n = 0 \quad \forall n \in \{0, 1, -1\}$

NOTE:- FS coefficients are in general, complex numbers. This implies there are two representations, mag and phase.

$|F_n| \Rightarrow$ magnitude

$\measuredangle F_n \Rightarrow$ phase

$$|F_1| = \left|\frac{1}{2j}\right| = \frac{1}{2}, \quad |F_{-1}| = \left|\frac{-1}{2j}\right| = \frac{1}{2}$$

$$\measuredangle F_1 = \tan^{-1}\left(\frac{-\frac{1}{2}}{0}\right) = \frac{-\pi}{2}, \quad \measuredangle F_{-1} = \frac{\pi}{2}$$

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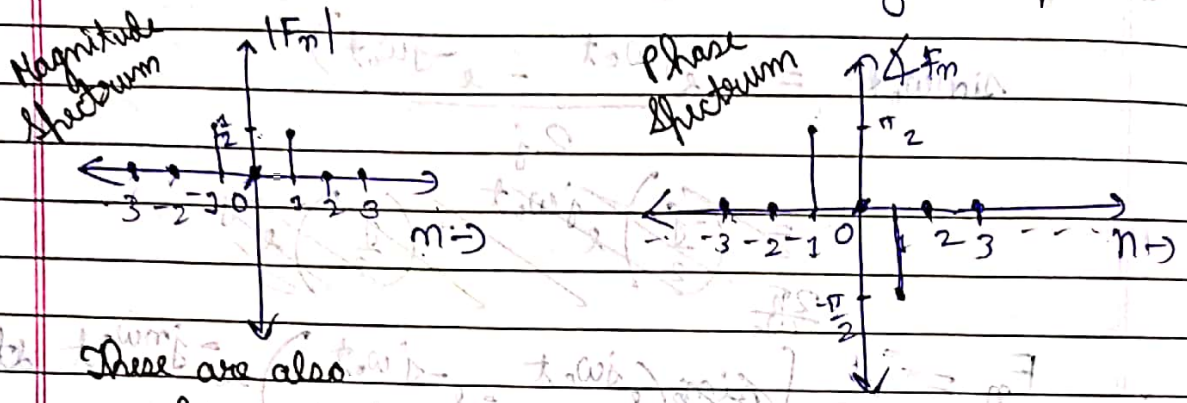
→ X axis and Y axis labels are very important



These are also line spectrum

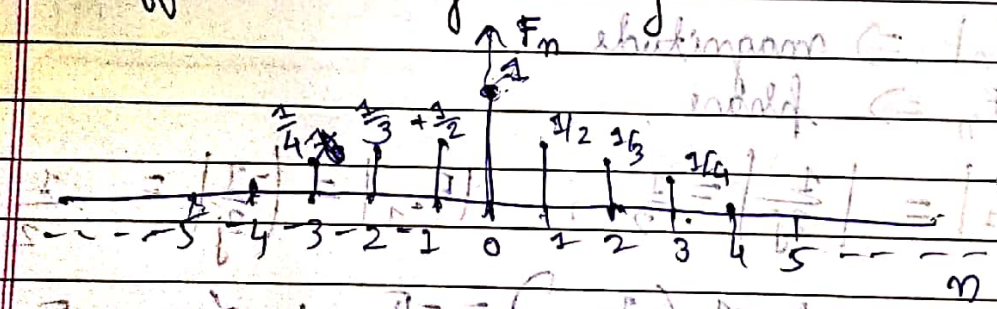
$n \rightarrow$ integer multiple of $\omega_0 \Rightarrow$ harmonics. Don't confuse this with x axis n of 1st chapter, Here it is multiplier of of fundamental freq and hence fundamentally different.

Find FS rep of :-

1) $\cos \omega_0 t$
2) $\cos \omega_0 t + \sin \omega_0 t$
3) $\sin 2\omega_0 t + \cos 2\omega_0 t$

1) Understanding Fourier Series
2) Proof as to why eqn ① is called Synthesis Eqn.

Problem : For a periodic signal $f(t)$, Fourier Series coefficients are given by



$F_0 = 1$, $F_1 = F_{-1} = \frac{1}{2}$, $F_2 = F_{-2} = \frac{1}{3}$, $F_3 = F_{-3} = \frac{1}{4}$ ....

F_n need not be real but we took so for

Memory

simplicity.

$$F(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$= \sum_{-3}^{3} F_n e^{jn\omega_0 t}$$ (Fn from -4 to -∞ and 4 to ∞ are all zero)

$$= F_{-3} e^{-j3\omega_0 t} + F_{-2} e^{-2j\omega_0 t} + F_{-1} e^{-j\omega_0 t} + F_0 + F_1 e^{j\omega_0 t} + F_2 e^{2j\omega_0 t} + F_3 e^{3j\omega_0 t}$$

$$= \left(\frac{1}{4}\right) e^{-3j\omega_0 t} + \left(\frac{1}{4}\right) e^{3j\omega_0 t} + \left(\frac{1}{3}\right) e^{-2j\omega_0 t} + \left(\frac{1}{3}\right) e^{2j\omega_0 t} + \left(\frac{1}{2}\right) e^{-j\omega_0 t} + \left(\frac{1}{2}\right) e^{j\omega_0 t} + 1$$

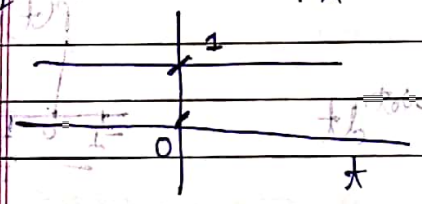
$$= 1 + \cos(\omega_0 t) + \left(\frac{2}{3}\right)\cos(2\omega_0 t) + \frac{1}{2}\cos(3\omega_0 t)$$

We'll try to construct a plot

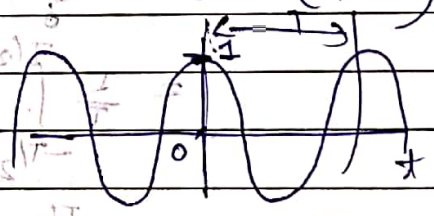
$$\rightarrow f(t) = F_0(t) + F_1(t) + F_2(t) + F_3(t)$$

Now (Let's take $\omega_0 = 2\pi$ (say))

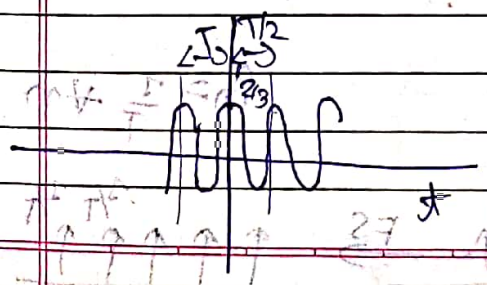
1) $F_0(t) = 1 \ \forall t$



$F_1(t) = 1\cos(2\pi t)$



$F_2(t) = \left(\frac{2}{3}\right)\cos(2\times 2\pi t)$



$F_3(t) = \frac{1}{2}\cos(3\times 2\pi t)$

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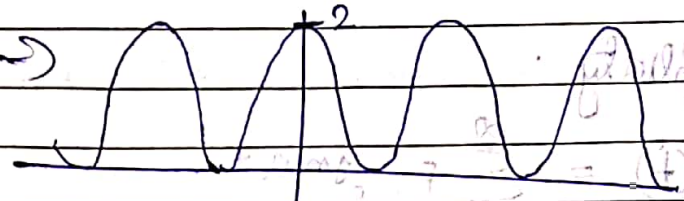
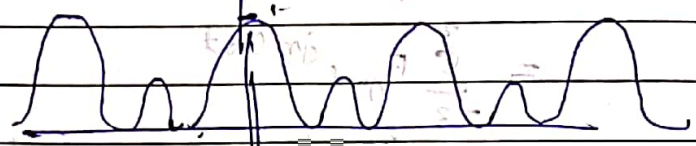
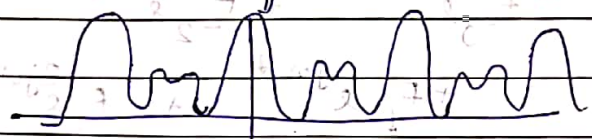
$F_0(t) + F_1(t) \rightarrow$



$F_0(t) + F_1(t) + F_2(t) \rightarrow$



$F_0(t) + F_1(t) + F_2(t) + F_3(t) \rightarrow$



When we add periodic signals, we get periodic signal.

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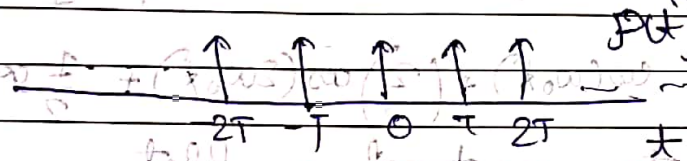
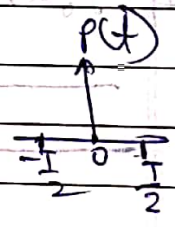
Problem 4: Impulse train signal



$$p(t) = \sum_{n=-\infty}^{\infty} P_n e^{jn\omega_0 t}$$

$$P_n = \frac{1}{T}\int_0^T p(t) e^{-j\omega_0 t}\, dt$$



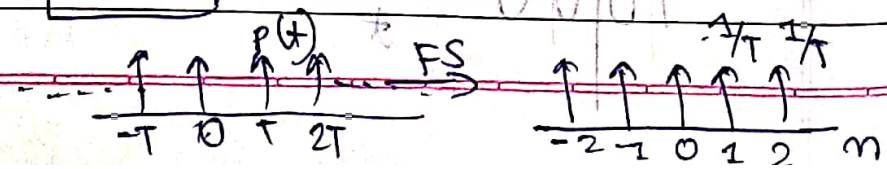
$$= \frac{1}{T}\int_{-T/2}^{T/2} p(t) e^{-j\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-j\omega_0 t}\, dt$$

$$= \boxed{\frac{1}{T} \times 1}$$

$$P_n = \frac{1}{T} \quad \forall n$$

Inference:



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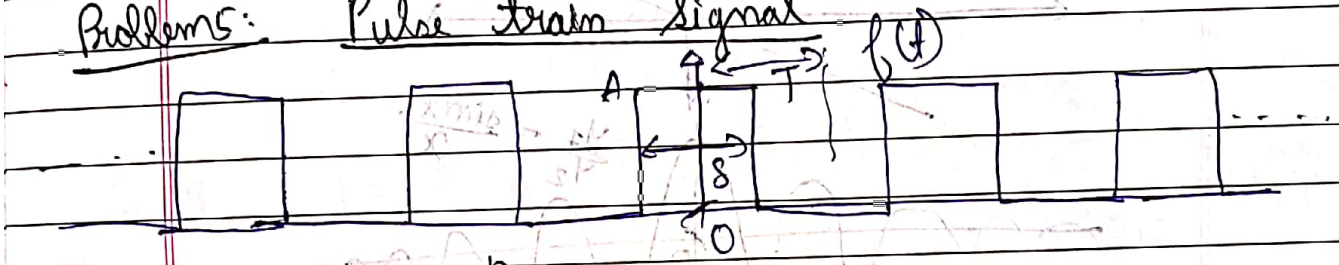
1) ∴ FS representation of pulse-train signal is also a pulse-train signal in frequency domain, $\omega_0 = \frac{2\pi}{T}$

2) If amp of input signal is 1, then the amplitude in the frequency domain signal is $1/T$.

3) time and frequency domain signals are inversely related in the sense that if time domain signals come closer, freq signals separate farther & vice versa. Thus, resolution of one ↑ ⇒ other ↓.

Problems: Pulse train signal



$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_0 t}\,dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\,dt$$

$$= \left(\frac{1}{T}\right)\int_{-\delta/2}^{\delta/2} A e^{-jn\omega_0 t}\,dt$$

$$F_n = \frac{A}{T}\int_{-\delta/2}^{\delta/2} e^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T}\left.\frac{e^{-jn\omega_0 t}}{(-jn\omega_0)}\right|_{-\delta/2}^{\delta/2}$$

$$= \frac{A}{T}\,\frac{e^{-jn\omega_0 \delta/2} - e^{jn\omega_0 \delta/2}}{(-jn\omega_0)}$$

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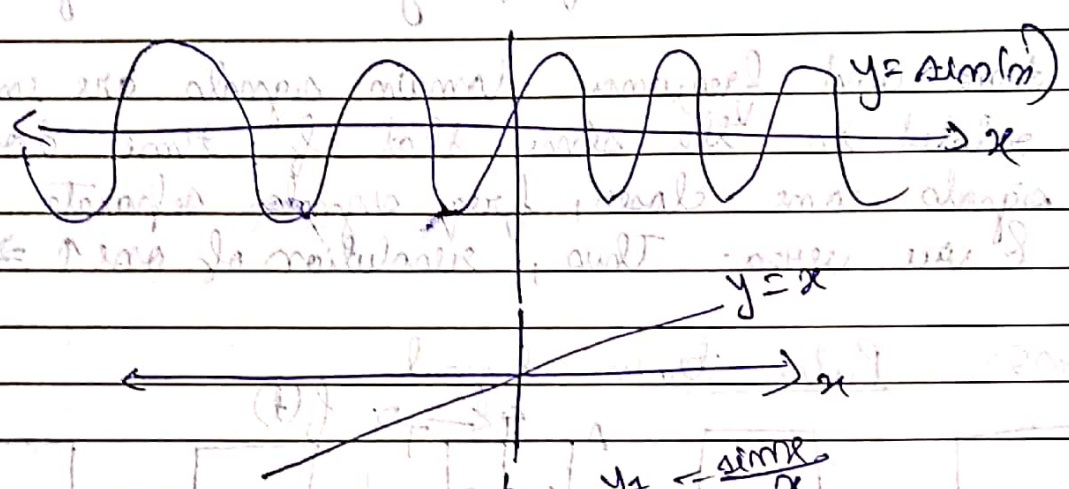
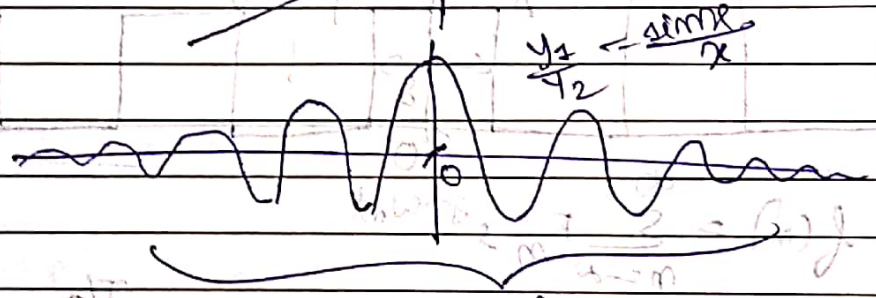
$$= \frac{2A}{T} \frac{\sin(n\omega_0 \delta/2)}{n\omega_0(\delta/2)} \times \delta/2$$

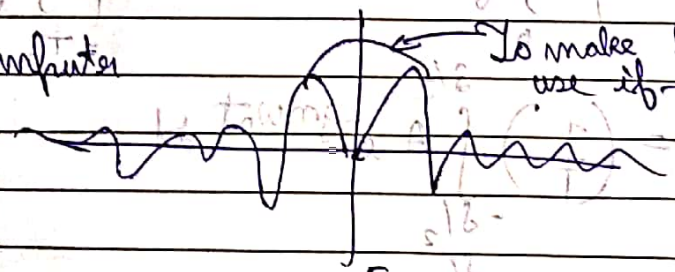
$$F_n = \left(\frac{A\delta}{T}\right) \frac{\sin[X]}{[X]}$$



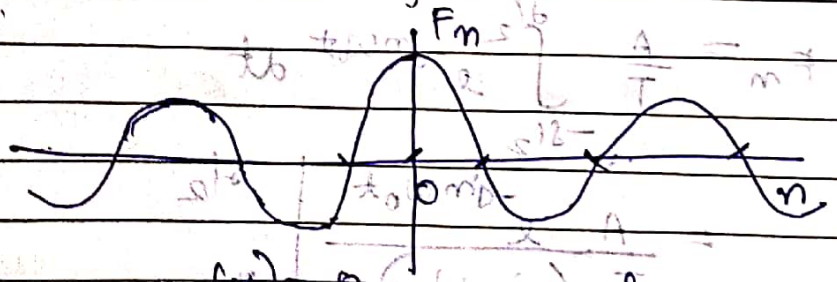


Shape of Sinc function.

In computer



To make like this, we need to use if-else condition to define at this point.



$\sin(x) = 0$; $x = k\pi$

$x = k\pi$

Wednesday

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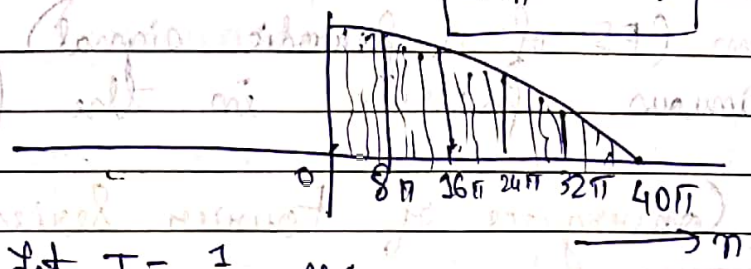
$\therefore n\omega_0 \frac{\delta}{2} = k\pi$

If $\delta = \frac{1}{20} \Rightarrow \omega_0 n = 40\pi k$; $k \in \mathbb{Z}$

$\therefore \boxed{\omega_n = 40\pi k}$ ⇒ matlab ki har 40π ke baad zero hoga since δ zero



Case I:- Let $T = \frac{1}{4}$ sec

$\omega_n = n\omega_0$

$= n\left(\frac{2\pi}{T}\right) = n\left(\frac{2\pi}{\frac{1}{4}}\right) = 8\pi n$

$\therefore$ ω's are separated by 8π

$\therefore$ In $40\pi k$, there are 5

Note ki FS nikalne ke baad graph me n axis is of $n\omega_0$ (frequency-dom) Eg:- $n = 1$ matlab $1(\omega_0) = \omega_0$ freq pe amp kya hai, $n = 2 \Rightarrow 2\omega_0$ pe amp. yahan pe $n = 5$ pe hi amp 0 hoga

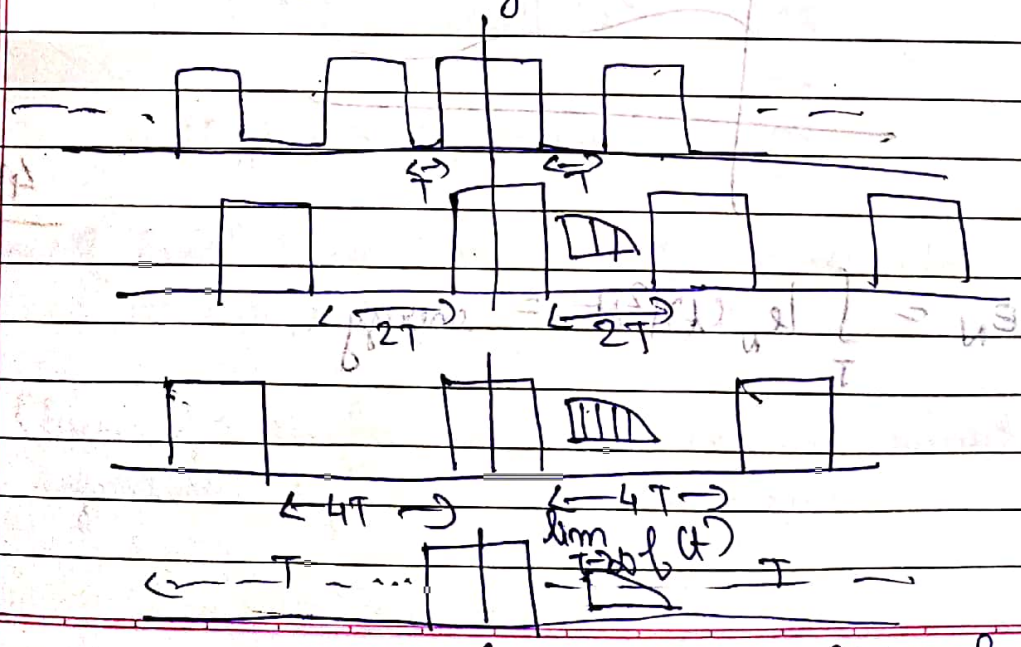
Aisa mat sochna ki x-axis here tabtak hi hai. Signals come close (for kya matlab) hahle $n = 1, 2, 3, 4, 5$ pe jaake 40π pe gaya → 0. Aur me $n = 10$ pe zero

Case II:- $T = \frac{1}{2}$ sec

$\Rightarrow \omega_n = 4\pi n$ ⇒ 10 b/w 0 & 40π

Case III:- $T = 1$ sec, $\omega_n = 2\pi n$

In terms of diagram,



$\lim_{T \to \infty} f(t)$

↳ periodic signal with period $T = \infty$.

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Wednesday

∴ An aperiodic signal can be considered as periodic signal with period $T = \infty$.

→ When we make $T \to \infty$, then the discrete line spectrum (FS of a periodic signal) tends to be a continuous spectrum in the frequency domain.

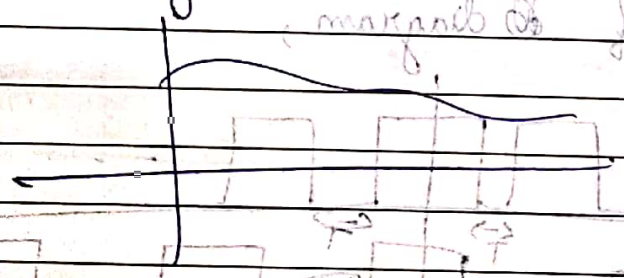
Convergence of Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \rightarrow \text{theoritical}$$

$$f_N(t) = \sum_{n=-N}^{+N} F_n e^{jn\omega_0 t} \rightarrow \text{practical (computer)}$$

Reason - We can never define infinite dimensional/ infinite length in array.

→ This is the point of departure as we can picture $-\infty$ to ∞ in brain but not in computer, it gives NAN. Also, in computer, having a reference is compulsory.



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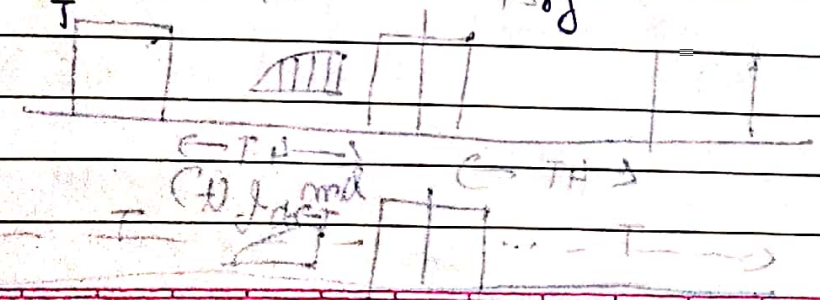
$$E_N = \int_T |e_N(t)|^2 dt = \text{energy}$$



Thursday

→ Even if the function in time domain has discontinuities, can any periodic signal always have a Fourier Series representation? Answer → NO

∴ FS representation doesn't exist ∀ signals.

If signals have finite energy, $\boxed{\int_T |x(t)|^2 dt < +\infty}$ ⇒

FS representation is Possible.

Dirichlet's Condition :-

Condition 1: Over any period, signal $f(t)$ must be absolutely integrable.

$$\boxed{\int_T f(t)\,dt < +\infty} \rightarrow \text{Why?}$$

Proof:-

$$F_n = \frac{1}{T}\int_0^T f(m) e^{-jm\omega_0 t}\,dt \quad \text{(Synthesis)}$$

$$\therefore |F_n| \le \frac{1}{T}\int_0^T |f(t)|\,\underbrace{|e^{-jm\omega_0 t}|}_{1}\,dt$$

$$\therefore |F_n| \le \frac{1}{T}\int_0^T |f(t)|\,dt \quad \text{—— (1)}$$

If Cond^n 1 is satisfied (1) $< +\infty$ ∀n

⇒ $|F_n| < +\infty$ ∀n.

(Even if a single F_n is ∞, when we substitute in the synthesis eq^n, it is meaningless)

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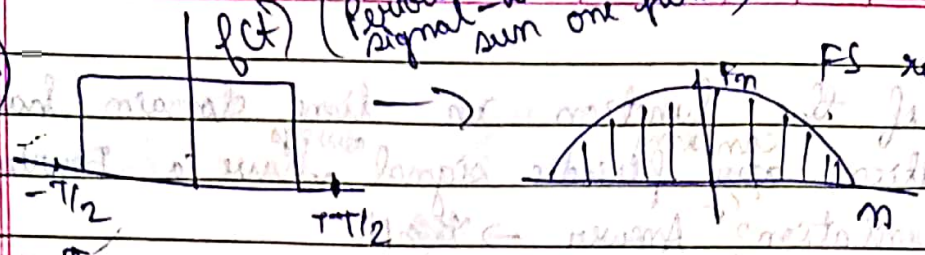
Thursday

(Michelson)

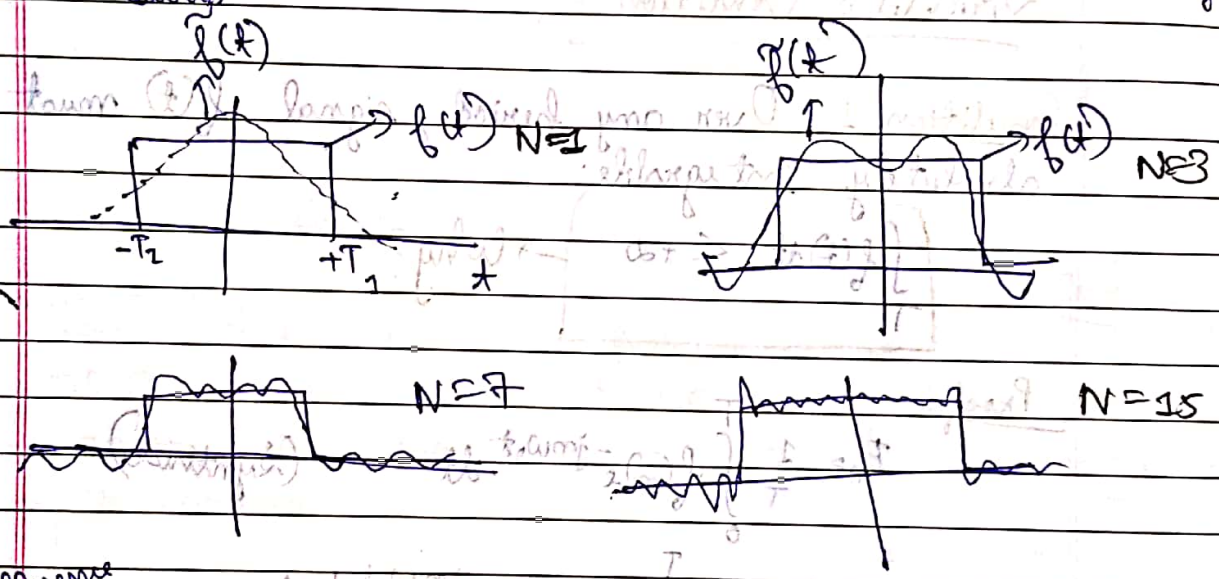


First analyzed by American Physicist in 1898. He made a device to take in signal & give out FS rep.

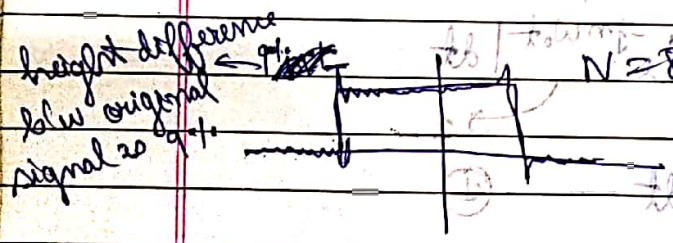
$$\tilde{f}_N(t) = \sum_{-N}^{N} F_n e^{jn\omega_0 t}$$

N = number of FS coefficients.

→ When he passed such a square wave, surprising results.



Used in design of filters



Gibbs phenomena (Albert Michelson Michelson's effect)

Clearly as N ↑, $\tilde{f}(t)$ matches $f(t)$ more and more. However, we see "ripples" at discontinuities. However, the ripples are more prominent (evident) at lower FS coeffs.

P.T.O

Thursday

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Properties of Fourier Series (C-T)

1) $x(t) \xrightarrow{FS} X_n$

$y(t) \longrightarrow Y_n$

Let $y(t) = x(t-t_0)$ = Time Shift, what is Y_n in terms of X_n

$$Y_n = \frac{1}{T}\int_0^T y(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^T x(t-t_0)\, e^{-jn\omega_0 t}\, dt = \boxed{X_n e^{-jn(\omega_0)t_0}}$$

$(|Y_n| = |X_n|)$

$t - t_0 = t_1$

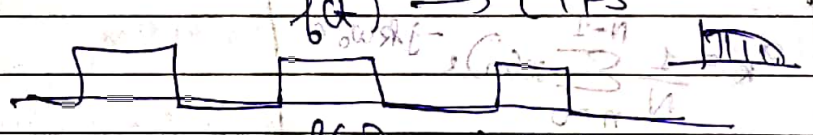
$$= \frac{1}{T}\int x(t)\, e^{-jn\omega_0 t_0} \ldots$$

∴ If we shift the signal in time domain by t_0, magnitude spectrum is unaltered while phase spectrum of each of FS coeffs is affected by a factor $e^{-jn\omega_0 t_0}$.

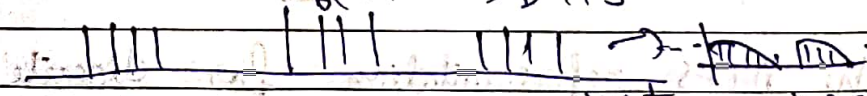
(Remaining properties are H.W)

D-T FS representation

$f(t) \longrightarrow$ CTFS



$f(n) \longrightarrow$ DTFS



Means to understand $e^{j\omega t}$ vs $e^{j\omega n}$

($e^{j\omega t}$ → for every ω, there is T_0 : $e^{j\omega n}$, for values of ω, separated by 2π, the signal is same as original → cue to understand CTFS vs DTFS)

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Thursday

→ Whatever the values of FS coefficients in DTFS it will be repeated after an ~~extra~~ interval of 2π.

$$x(n) = x(n+N)$$ → fundamental period.

$$\phi_k(n) = e^{jk\omega_0 n}$$
$$= e^{jk\left(\frac{2\pi}{N}\right)n} \quad , k = 0, \pm 1, \pm 2, \ldots$$
$$k \in \mathbb{Z}$$

$$\phi_k(n) = \phi_{k+rN}$$

$$\rightarrow x(n) = \sum_{k=-\infty}^{\infty} a_k \phi_k(n)$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 n} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$\boxed{x(n) = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n}}$$

$$\|$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

Now, as $F_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_0 t}\, dt$

$$\boxed{a_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-jk\omega_0 n}}$$

Q.1) Find the DTFS representation for discrete time periodic signal

$$x(n) = \sin(\omega_0 n)$$
$$x(n) = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

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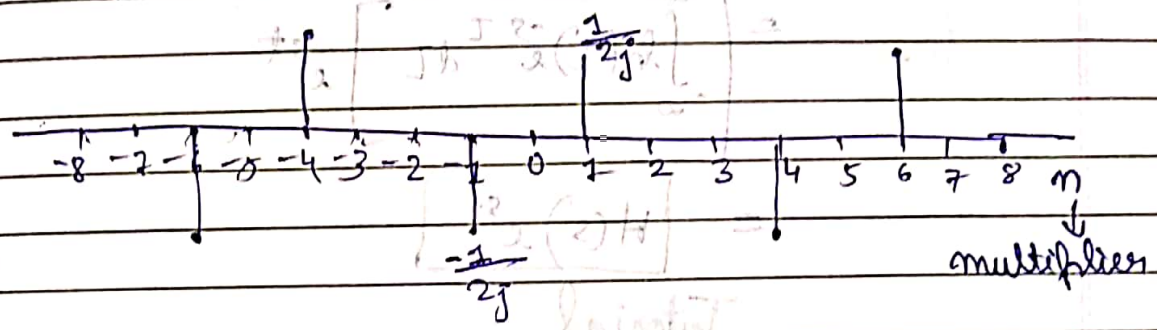
$$x(n) = \sum_{k=0}^{N-1} a_k e^{jk(\omega_0)n}$$

$a_1 = \frac{1}{2j}$, $a_0 = 0$, $a_2 = 0$ . . .

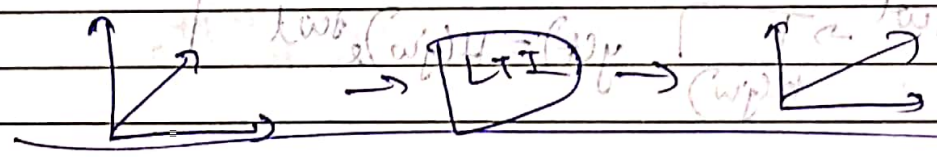
$a_{-1} = \frac{-1}{2j}$, $a_{-2} = 0$ . . . . . $N=5$



H.W Eq 3.11 pg 216
↳ properties of DTFS

Fourier Series and LTI Systems

Not new ~~just start of~~ just beginning of Chap 3.



$x(t) \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = ?$

$x(t) = e^{j\omega t}$ $\quad y(t) = H(j\omega) e^{j\omega t}$

$x(n) = e^{j\omega n}$ $\quad y(n) = H(j\omega) e^{j\omega n}$

Now, we take general representation,

$x(t) = e^{st}$, $s = \sigma + j\omega$ = complex number.

$x(n) = z^n$, z = complex

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Now, $y(t) = x(t) * h(t)$

$= h(t) * x(t)$

$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$

$= \left[\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau\right] e^{st}$

$= \boxed{H(s)\, e^{st}}$

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Tutorial

1) $x(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t}$

$x_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$

Fourier transform

$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$e^{j\omega t} \rightarrow \boxed{} \; y(t) = H(j\omega)\, e^{j\omega t}$

$H(j\omega)$

$y(t) = x(t) * h(t)$

$Y(\omega) = X(\omega)\, H(j\omega)$

If $x(t) = e^{-at} u(t) \Rightarrow X(\omega) = \frac{1}{a + j\omega}$

$F\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$

$t - t_0 = \tau$

$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega t}\, e^{-j\omega t_0}\, d\tau$

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$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= e^{-j\omega t_0}\, X(\omega)$$

$\therefore$ $X(\omega)$ $\quad e^{-at}u(t) \longleftrightarrow \dfrac{1}{a+j\omega}$

$$e^{-a(t-1)}\, u(t-1) \longleftrightarrow \frac{e^{-j\omega}}{a+j\omega}$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= 1$$

1- (a) $\delta(t+1) + \delta(t-1)$

$$\int_{-\infty}^{\infty} \left[\delta(t+1) + \delta(t-1)\right] e^{-j\omega t}\, dt$$

$$= e^{+j\omega} + e^{-j\omega}$$

(b) $e^{-2t}\, u(t)$

$$= \frac{1}{2+j\omega}$$

(c) $e^{-3(t-1)}\, u(t-1)$

$$= \frac{e^{j\omega}}{3+j\omega}$$

2.) $\cancel{e^{-4t}(u(t))}$ $\quad H(j\omega) = \dfrac{1}{4+j\omega}$

$$y(t) = e^{-3t}u(t) - e^{-4(t)}u(t)$$

$$Y(\omega) = \frac{1}{3+j\omega} - \frac{1}{4+j\omega}$$

$$\frac{Y(\omega)}{H(\omega)} = X(\omega) \rightarrow \frac{\left(\frac{1}{3+j\omega}\right) - \left(\frac{1}{4+j\omega}\right)}{\frac{1}{4+j\omega}} = \frac{\cancel{4+j\omega} - \cancel{3-j\omega}}{3+j\omega}$$

$$= \frac{1}{3+j\omega}$$

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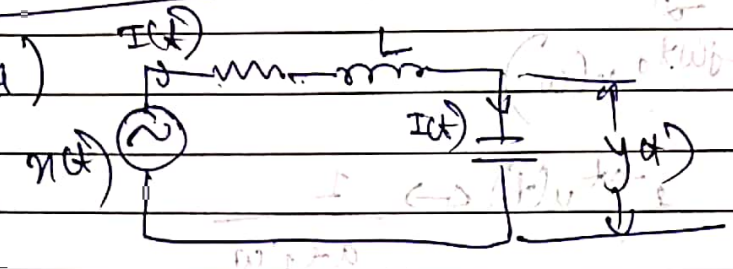
$\therefore x(t) = e^{-3t} u(t)$

Q.3) (a)



$$I = C\frac{dv}{dt}$$

$$x(t) - I(t)R - L\frac{dI(t)}{dt} - y(t) = 0$$

$$I(t) = C\frac{dy(t)}{dt} \quad \text{—— (2)}$$

$$x(t) - RC\frac{dy(t)}{dt} - LC\frac{d^2y(t)}{dt^2} - y(t) = 0$$

$$x(t) = \frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t)$$

Property :-

$$x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\frac{dx}{dt} = \int_{-\infty}^{\infty} X(\omega) \left\{\frac{d}{dt} e^{j\omega t}\right\} d\omega$$

$$\frac{dx(t)}{dt} = (j\omega) \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) \longleftrightarrow X(\omega)$$

$$\frac{dx(t)}{dt} \longleftrightarrow j\omega X(\omega)$$

$$\frac{d^n x(t)}{dt^n} \longleftrightarrow (j\omega)^n X(\omega)$$

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$$y(t) = x(t) * h(t)$$

$$Y(\omega) = X(\omega)\, H(j\omega)$$

$$H(j\omega) = \frac{Y(\omega)}{X(j\omega)}$$

$$X(\omega) = (j\omega)^2 Y(\omega) + (j\omega)\, Y(\omega) + Y(\omega)$$

$$X(\omega) = Y(\omega)\left[(j\omega)^2 + j\omega + 1\right] \Rightarrow \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + j\omega + 1}$$

$$\therefore H(j\omega) = \frac{1}{(j\omega)^2 + (j\omega) + 1}$$

Let $j\omega = x$

$$H(x) = \frac{1}{x^2 + x + 1} \quad \begin{matrix} \to p \\ \to q \end{matrix}$$

$$= \frac{1}{(x-p)(x-q)}$$

$$x^2 + x + 1 = 0 \quad \therefore p, q = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm j\sqrt{3}}{2}$$

$$= \left(\frac{1}{(x-p)} - \frac{1}{x-q}\right)\left(\frac{1}{p-q}\right)$$

$$= \left[\frac{1}{j\omega - p} - \frac{1}{j\omega - q}\right]\left(\frac{1}{p-q}\right)$$

$$h(t) = \left[e^{pt} u(t) - e^{qt} u(t)\right]\frac{1}{(p-q)}$$

4] $\phi_{xy} = \int_{-\infty}^{\infty} x(t+\tau)\, y(\tau)\, d\tau$

$\cancel{x(t) * y(t) = \int_{-\infty}^{\infty} x(t+\tau)\, y(\tau)\, d\tau}$

$x(t) * y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, y(\tau)\, d\tau$

$\therefore \phi_{xy}(t) = x(t) * y(-t) \Rightarrow \Phi_{xy}(\omega) = [X(\omega) Y^*(\omega)]$

$\phi_{yx}(t) = y(t) * x(-t) \Rightarrow \Phi_{yx}(\omega) = [Y(\omega) X^*(\omega)]$

This is complex conjugate of above.

$\Phi_{xx}(\omega) = X(\omega) X^*(\omega)$

$= |X(\omega)|^2$

(d) $\phi_{xy}(\omega) = X(\omega) Y^*(\omega)$

$= X(\omega) [X(\omega) H(\omega)]^*$

$= X(\omega) X^*(\omega) H^*(\omega)$

$= \phi_{xx}(\omega) H^*(\omega)$

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Laplace Transform

$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt \quad \mathcal{L}\{h(t)\}$

$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \mathcal{L}\{x(t)\}$ = Laplace transform.

$||^{ly}$, $H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} = Z\{h(n)\}$

(Z-transform of $h(n)$)

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Friday

In general,

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \text{Z-transform}$$

(Chapter 10)

$\Rightarrow$ Now on, $x(n) = \sum_{\langle k \rangle} x_k(n)$ is periodic

Let's now consider LTI system as filter.

Periodic signal $x(t)$ $\rightarrow$ $\boxed{\text{LTI filter } h(t)}$ $\rightarrow y(t)$

$\rightarrow H(j\omega)$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega t}$$

LC of complex exponentials

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

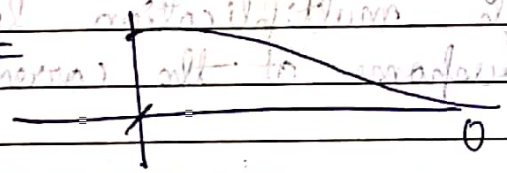
$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

Special case: $s = \sigma + j\omega$

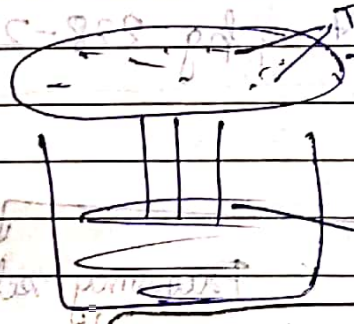
$\sigma = 0 \Rightarrow s = j\omega$

$H(j\omega)$ = frequency response of LTI system

$H(j\omega) =$



Tea filter



(i) In our course, the tea particles are selected frequency components

(ii) filtered signal.

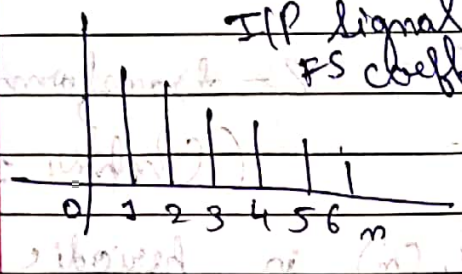
$$\boxed{y(t) = \sum_{n=-\infty}^{\infty} X_n H(jn\omega_0) e^{jn\omega_0 t}}$$

Clearly this is same as $x(t)$ (scaled) $\Rightarrow y(t)$ is periodic as well.

PTO

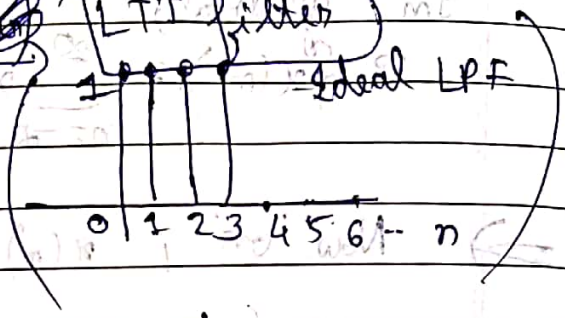
Friday

I/P signal FS coeffs

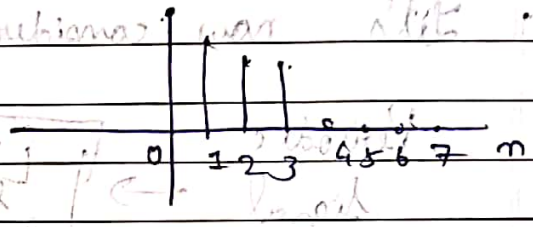


LTI filter

Ideal LPF



Ideal LPF, low frequency are made to same amplitudes & high freqs filtered out.



(Here, even if we alter even if one frequency component through the LTI system, its still a filter)

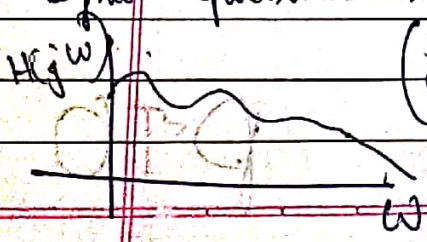
Final statement :- The effect of LTI filter is to modify individually each of the FS coeffs of the I/P through multiplication by the value of frequency response at the corresponding frequency.

(Problem 3.16 – 3.17, pg 228–231)

LTI filtering

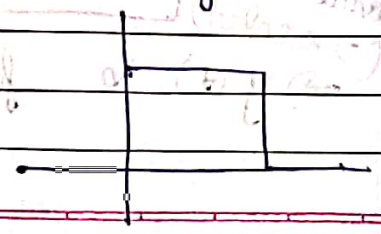
Frequency shaping filters

job is to shape of input signal spectrum is altered (even if only one component is altered)



Frequency selective filters

Selects a particular frequency band (range in spectrum)



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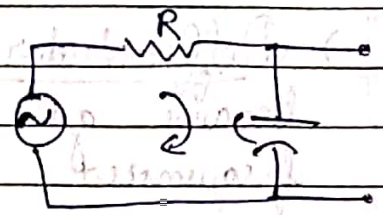
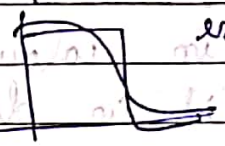
Music is a signal which is time varying spectrum.

Winamp.

(Bass and Treble are examples of frequency shaping filters)



(Approximate LPF.
Approx because not exactly brick wall filtering)



1) High pass filter
↳ Differentiator

Proof:-

$x(t) \longrightarrow \boxed{\text{Differentiator } \frac{d}{dt}(\cdot)} \longrightarrow y(t)$ = filtered signal (Tea)

$y(t) = \frac{d}{dt}[x(t)]$ (I/O)

Let $x(t) = e^{j\omega t}$, $y(t) = H(j\omega)e^{j\omega t}$

$\therefore y(t) = \frac{d}{dt}[e^{j\omega t}]$

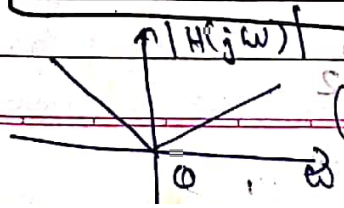
$\boxed{y(t) = (j\omega)e^{j\omega t}} \Rightarrow \boxed{H(j\omega) = j\omega}$

(very imp)

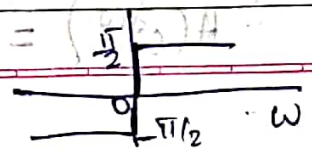
$|H(j\omega)| = |j\omega| = |j||\omega| = |\omega|$

$\measuredangle H(j\omega) = \tan^{-1}\left(\frac{\omega}{0}\right) = \begin{cases} \pi/2, & \omega > 0 \\ -\pi/2, & \omega < 0 \end{cases}$

Magnitude Spectrum



(a)



Phase spectrum.
(b)

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Friday

→ Differentiator is a High pass filter. This is because gain of differentiator is more at lower frequencies (multiplied with ω)

→ In HPF, edges are more distinct in image processing. This is because, at the edges the difference in colour & intensity is maximum and since it is differentiator, this is more distinct.

2) Integrator System (H.W)

Low Pass filter, $H(j\omega) = \frac{1}{j\omega}$

3) Integrator in D-T

$$y(n) = \frac{1}{2}\left[x(n) + x(n-1)\right]$$

2 point MA system

$x(n) = e^{j\omega n}$

$$x(n) \longrightarrow \boxed{\begin{matrix}\text{2pt MA} \\ h(n) \\ \text{LTI}\end{matrix}} \longrightarrow y(n)$$

$$y(n) = H(e^{j\omega})\, e^{j\omega n} \quad \text{—— (2)}$$

$$y(n) = \frac{1}{2}\left[e^{j\omega n} + e^{j\omega(n-1)}\right] \quad (\omega_c, H)$$

$$= \frac{1}{2}\left[1 + e^{-j\omega}\right] \times e^{j\omega n}$$

$$= e^{-j\omega/2}\left[\frac{e^{j\omega/2} + e^{-j\omega/2}}{2}\right] \times e^{j\omega n}$$

$$\therefore H(e^{j\omega}) = \cos\left(\frac{\omega}{2}\right) e^{-j\omega/2}$$

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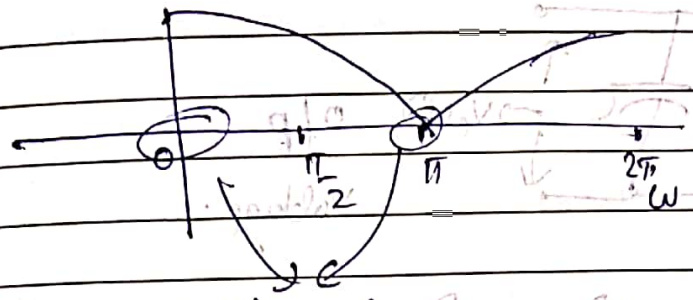
$$|H(e^{j\omega})| = |\cos(\omega/2)|$$



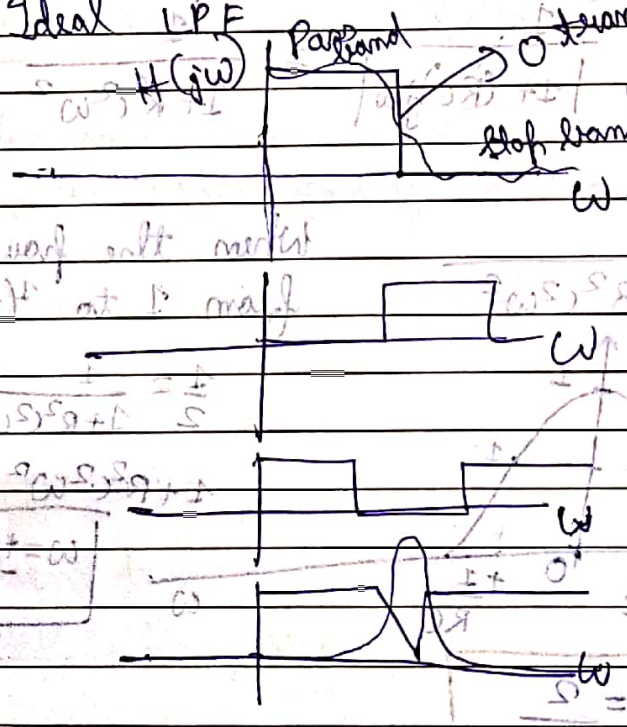
From chapter 1, gain is higher at lower frequency i.e., around 0 and less at around π. Therefore, it is ~~digital~~ discrete real-life LPF filter.

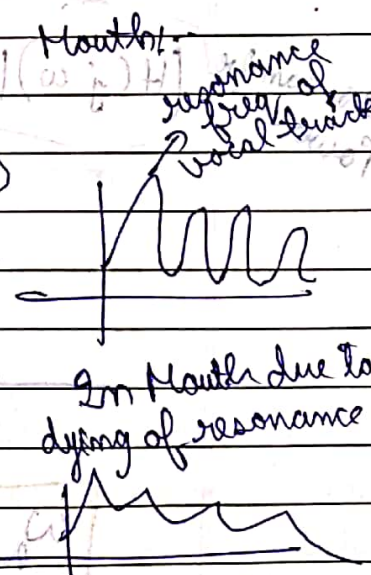
4.) Differentiator in D-T

$y(n) = x(n) - x(n-1)$ (HPF)

Frequency Selection filter

Ideal LPF





Vocal LTI $h(t) \to H(j\omega)$

What our mouth does to the speech at the vocal chord, is spectrally colour it. Some freqs are boosted, some not.

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Friday

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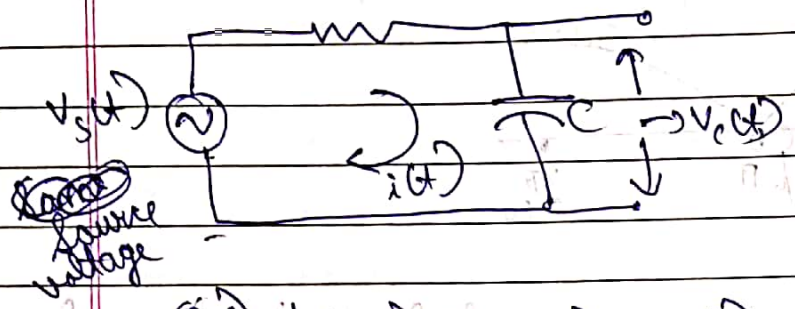
Example of Cont-Time filter designed by differential Eq^n



$$(RC)\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

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$v_s(t) = e^{j\omega t}$ (Eigen function)

$v_c(t) = H(j\omega)\, e^{j\omega t}$

$\therefore H(j\omega) = \dfrac{1}{1+(RC)j\omega}$ Frequency response of RC LPF

$H(\omega)$ Find the freq response of RC HPF

$$\therefore |H(j\omega)| = \frac{1}{|1+(RC)j\omega|} = \frac{1}{1+R^2C^2\omega^2}$$

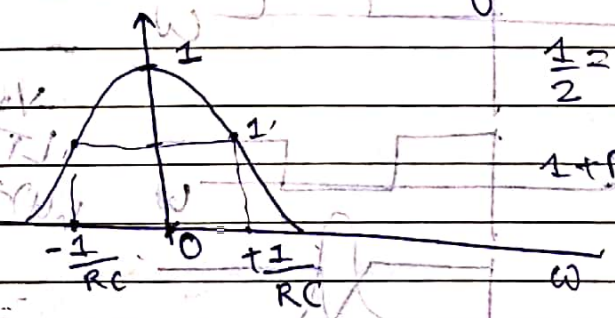
This can be power $\leftarrow$ $|H(j\omega)|^2 = \dfrac{1}{1+R^2C^2\omega^2}$

When the power falls from 1 to 1/2,

$$\frac{1}{2} = \frac{1}{1+R^2C^2\omega^2}$$

$$1+R^2C^2\omega^2 = 2$$

$$\boxed{\omega = \pm\frac{1}{RC}}$$



$$\boxed{\omega_{half\text{-}power} = \frac{2}{RC}}$$

$B = \dfrac{2}{RC}$ radian/sec ($\because$ analog filter)

$\therefore \boxed{B \propto \dfrac{1}{RC}}$

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Monday

This means that if we want to design resonant circuits, bandwidth must be smaller ⇒ RC must be higher.
↳ narrower

$$\text{Quality factor} \propto \frac{1}{\text{Bandwidth}}$$

→ But using analog filters has a caveat, i.e, increasing the cost of implementation due to more value of R and C.

To address this problem, we design digital filters.

$$\frac{dy(t)}{dt} + a\,y(t) = x(t)$$

$$\Rightarrow y(n) - a\,y(n-1) = x(n) \quad (\because \text{difference equation})$$

<u>Motivation for digital filter</u>

We saw that we can't design ideal low pass filter with analog at low cost.

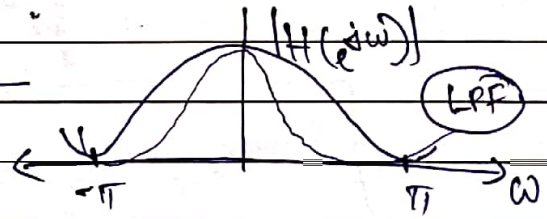
Eigen function

Let $x(n) = e^{j\omega n}$

$$\therefore y(n) = H(e^{j\omega})\, e^{j\omega n}$$

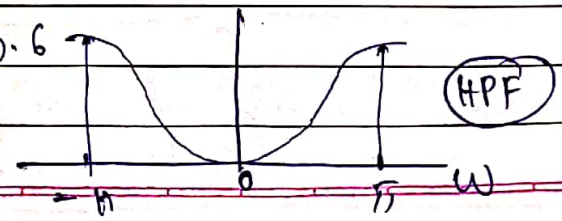
$$H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

Magnitude response :—



(Absolute value of a will control -3dB/half power bandwidth)

If $a = -0.6$ (sign flipped)



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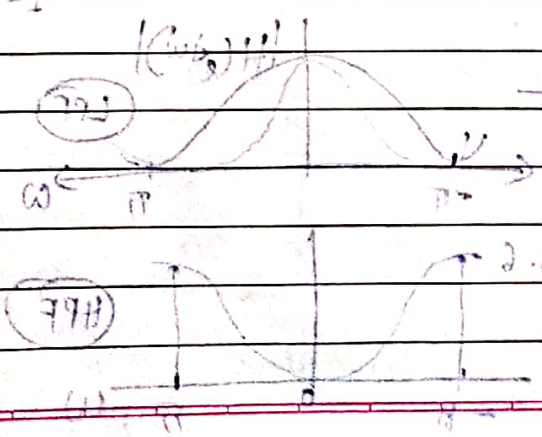
→ Clearly here, nearly no money is required.

→ This filter is completely flexible and very easy conversion from LPF to HPF and vice versa, while in analog filter we have to flip positions.



Monday

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Chapter -4 : Development of Fourier Transform

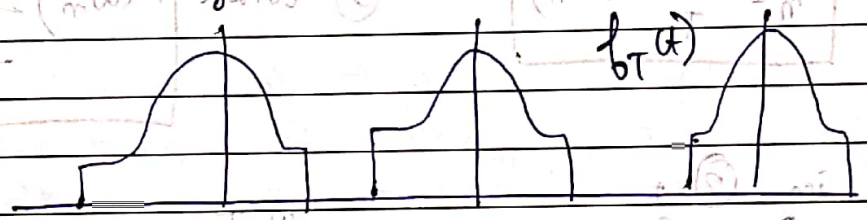
Motivation

→ Extend 'Chapter 3' for frequency domain analysis of aperiodic signals. (∵ Hum FS likhte hai periodic ke liye)

CTFS ⟶ CTFT (Chapter 4)
DTFS ⟶ DTFT (Chapter 5)

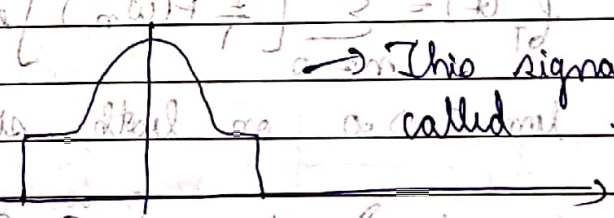
Fourier's original contribution :- Thought process

Fourier thought and claimed that an aperiodic signal can be considered to be periodic with period ∞. This changed the entire understanding of subject.



(∵ $f(t)$ with period T)

$\lim_{T\to\infty} f_T(t)$



→ This signal can be called $f(t)$

$$\boxed{\therefore \lim_{T\to\infty} f_T(t) = f(t)} \quad \text{—— (1)}$$

Since $f_T(t)$ is a periodic signal, we can express it in terms of exponential FS (∵ assuming Dirichlet cond^{n}s satisfied)

$$\boxed{f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{j\omega_n t}} \quad \text{—— (2)}$$

when $$\boxed{F_n = \frac{1}{T}\int_0^T f_T(t)\, e^{-j\omega_n t}\, dt} \quad \text{—— (3)}$$

$$\boxed{n\omega_0 = \omega_n} \quad \text{—— (4)}$$

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Using ④ in ② & ③

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{j\omega_n t}$$

$$F_n = \frac{1}{T}\left[\int_{-\frac{T}{2}}^{+T/2} f_T(t) e^{-j\omega_n t} dt\right]$$

Now, WKT $\int_a^b f(x)\,dx = [g(x)]_a^b = g(b) - g(a)$

On integrating, the integral is no longer a fn of independent variable t

$$\Rightarrow \boxed{F_n = \frac{1}{T} F(\omega_n)} \quad \text{—— ⑤ where} \quad \boxed{F(\omega_n) = \int_{-T/2}^{T/2} f_T(t) e^{-j\omega_n t} dt} \quad \text{—— ⑥}$$

Use ⑤ in ②,

$$f_T(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} F(\omega_n)\right] e^{j\omega_n t}$$

Take limit $T \to \infty$ on both sides

$$\therefore \lim_{T\to\infty} f_T(t) = \lim_{T\to\infty} \left\{ \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} F(\omega_n)\right] e^{j\omega_n t} \right\}$$

Now, $\frac{1}{T} = \frac{\omega_0}{2\pi}$

$$\therefore \lim_{T\to\infty} f_T(t) = \left(\frac{1}{2\pi}\right) \left\{ \lim_{T\to\infty} \sum \left[F(\omega_n) e^{j\omega_n t} \omega_0\right] \right\}$$

(When $T \to \infty$, $\omega_0 \to 0 \Rightarrow \omega_n$ becomes continuous function ω)

$\therefore$ Now,

$$f(t) = \frac{1}{2\pi} \left\{ \lim_{\omega_0 \to 0} \sum_{n=-\infty}^{\infty} \left(F(\omega_n) e^{j\omega_n t}\right) \omega_0 \right\}$$

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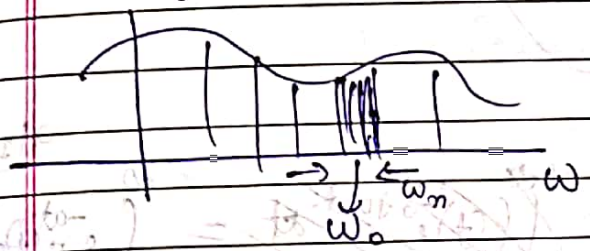
This is nothing but fundamental theorem of integration



$$\Rightarrow f(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$\cancel{F(\omega_n) = \int_{-T/2}}$$

$$\therefore F(\omega_n) = \int_{-T/2}^{+T/2} f_T(t)\, e^{-j\omega_n t}\, dt$$

$$\lim_{T\to\infty} F(\omega_n) = \lim_{T\to\infty}\int_{-T/2}^{T/2} f_T(t)\, e^{-j\omega_n t}\, dt$$

$\Rightarrow$ As $T\to\infty$, $\omega_n \to \omega$ $\quad$ $\left(\because \lim \int \neq \int \lim\right)$

$$\lim_{T\to\infty} F(\omega_n) = \lim_{T\to\infty}\int_{-T/2}^{+T/2}\left[\lim_{T\to\infty} f(t)\right]\left[\lim_{T\to\infty} e^{-j\omega_n t}\right] dt$$

$\downarrow$ (Applying limit)

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt} \quad \text{— (B)}$$

Let $f(t)$ = aperiodic signal

$$\therefore \boxed{\begin{aligned} F\{f(t)\} = F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \quad \text{— (I)} \\ f(t) = F^{-1}\{F(\omega)\} &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=+\infty} F(\omega)\, e^{j\omega t}\, d\omega \quad \text{— (II)} \end{aligned}}$$

Analysis eqn

Synthesis eqn

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Monday

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Time domain (because independent variable is t) → frequency domain (because independent variable $= \omega$)

$$f(t) \xrightarrow{F(\cdot)} F(\omega)$$

$$f(t) \xleftarrow{F^{-1}(\cdot)} F(\omega)$$

I/P signal space

9/10/18

① $F(\omega) = F(f(t)) = \cancel{\int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt} = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$

$$F(\omega) = \int_{0}^{\infty} e^{-(a+j\omega)t} dt = \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} \quad (a>0)$$

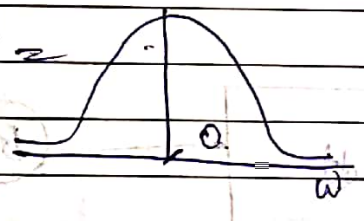
$$\boxed{F(\omega) = \frac{1}{a+j\omega}}$$

In general, Fourier transform of signal $f(t)$ is a complex number

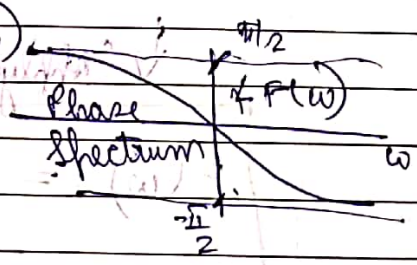
$$|F(\omega)| = \left|\frac{1}{a+j\omega}\right| = \frac{1}{\sqrt{a^2+\omega^2}}$$

and $\angle F(\omega) =$

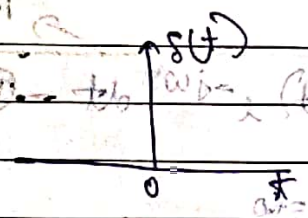
$|F(\omega)|$



and $\angle F(\omega)$



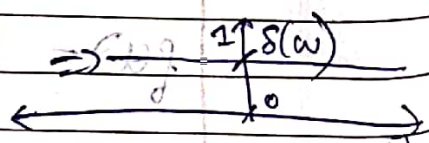
② Find the F.T of impulse signal



$$F(\delta(t)) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

$$= 1 \quad \forall \omega$$

$\delta(\omega) = 1 \ \forall \omega \Rightarrow$



Inference:- Impulse signal contains all possible frequencies. In terms of Newton's laws & physics, it is white light.

↓

$\delta(\omega)$

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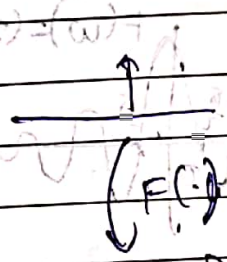
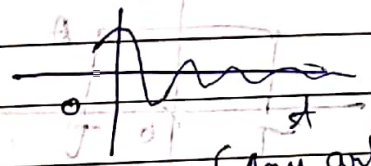
Tuesday

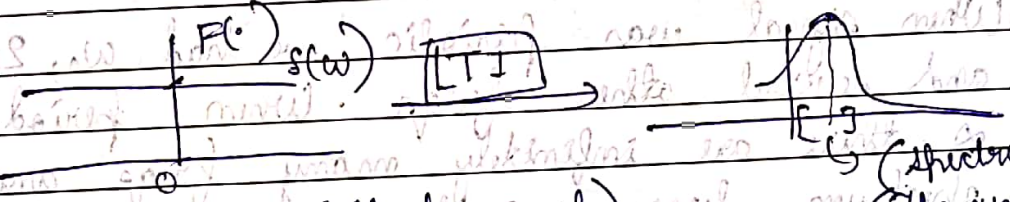
Problem:- Prove that impulse response of a system characterizes the system completely, using F-T method

$$r(t) = \delta(t) \longrightarrow \boxed{\begin{matrix} LTI \\ h(t) \end{matrix}} \longrightarrow y(t) = h(t)$$





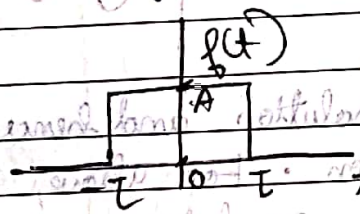
(Any arbitrary signal say)



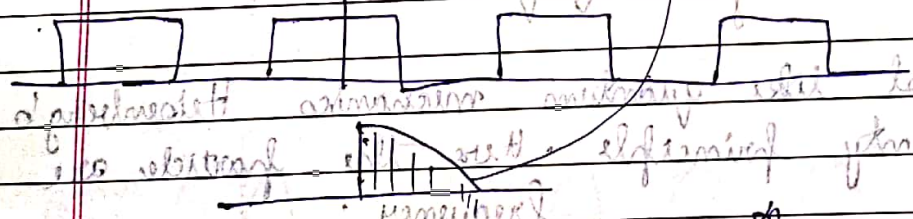
(Spectral colouring. ~~The system charac-~~ teristic centralizes the colour)

The system (filter/resonator) will select a particular freq depending on the properties of the system.

3)



$\cancel{u(t-\tau) - u(t+\tau)}$



$$\therefore F(\omega) = F\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\tau}^{\tau} A e^{-j\omega t} dt$$

$$= \left(\frac{A}{j\omega}\right)\left[e^{-j\omega t}\right]_{-\tau}^{\tau}$$

$$= \frac{A}{-j\omega}\left[e^{-j\omega\tau} - e^{j\omega t}\right]$$

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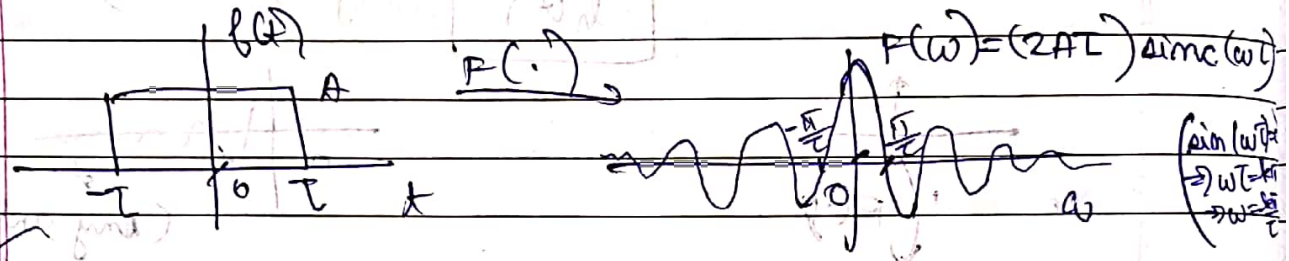
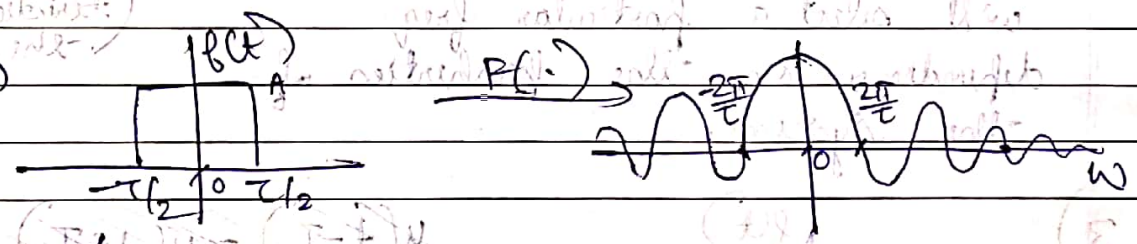
$$= \left(\frac{2A}{j\omega}\right) \frac{e^{j\omega\tau} - e^{-j\omega\tau}}{2j}$$

$$= \frac{2A}{\omega} \sin(\omega\tau) = 2A\tau \left[\frac{\sin(\omega\tau)}{\omega\tau}\right]$$

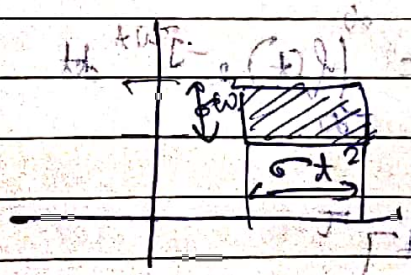


When signal was periodic, we had $\omega_0, 2\omega_0, 3\omega_0$ and several other freqs. When period becomes ∞, there are infinitely many freqs and line spectrum becomes continuous spectrum.



This had good time resolution and hence we get poor frequency resolution. For above, poor time resolution ⇒ good freq resolution.

↳ just like quantum mechanics Heisenberg's uncertainty principle. Here the particle are frequency.



$$\sigma_\omega^2 \sigma_t^2 \geq \frac{1}{4}$$

(called Heisenberg's principle in signal processing)

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Understanding | Reading of Fourier Transform (CTFT)

1) Graphical Interpretation

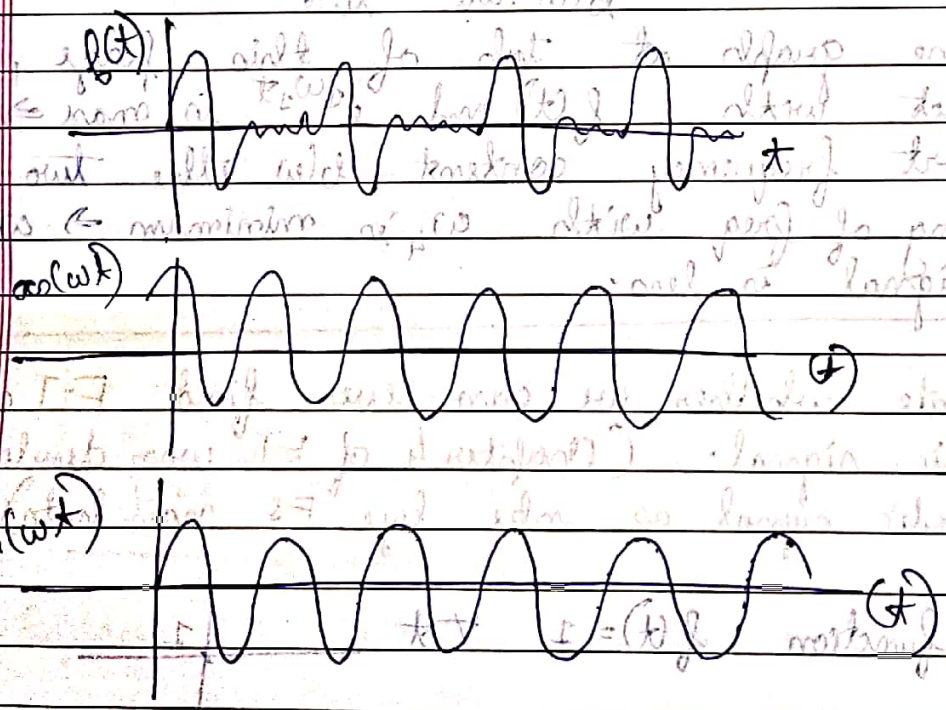
Reading $F(\omega)$ vs. ω

$$F(\omega) = F\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\left[\cos(\omega t) - j\sin(\omega t)\right] dt$$

$$= \underbrace{\int_{-\infty}^{\infty} f(t)\cos(\omega t)\, dt}\; - j \underbrace{\int_{-\infty}^{\infty} f(t)\sin(\omega t)\, dt}$$ (∵ Integral is linear operator)

$$\Rightarrow F(\omega) = F_R(\omega) + j\, F_I(\omega)$$ (∵ Proof that FT is in general a complex number)



We multiply ft by ft $\cos(\omega t)$ and $f(t)$. If $f(t)$ has a particular freq $\omega \Rightarrow$ on multiplying $f(t)$ with that freq $\cos(\omega t)$, and adding them, we get max value. If ω is not present $\Rightarrow$ ideally zero.

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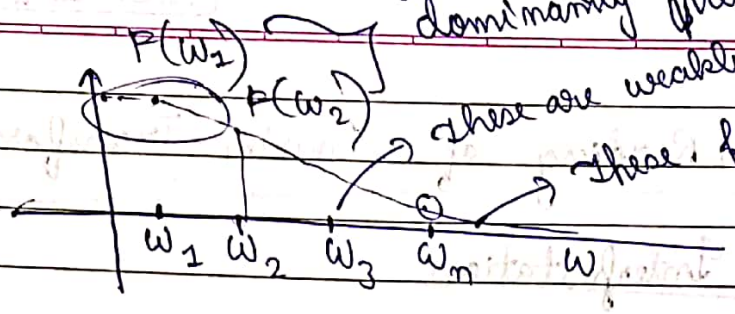


Another Interpretation :-

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt \quad \text{--- (1)}$$

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \quad \text{--- (2)}$$

$$\therefore \boxed{F(\omega) = \langle f(t), e^{j\omega t} \rangle}$$

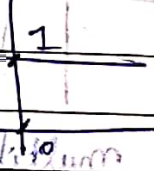
$f(t)$

$e^{j\omega t}$ The inner product of these two shows the degree of closeness b/w the two.

→ For the graph at top of this page, inner product with $f(t)$ and $e^{j\omega_1 t}$ is max ⇒ matching is wrt frequency content b/w the two vectors. Matching of freq with ω_4 is minimum → ω_4 content in signal is less.

4.) Investigate whether we can ever find F-T of periodic signal. (Chapter 4 of FT was developed for aperiodic signal as unke liye FS nai hota)

Let a function $f(t) = 1 \quad \forall t$



$F(\omega) = ??$

Proof :- $F(\omega) = \int_{-\infty}^{\infty} 1\, e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-\infty}^{\infty}$

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$$= \cancel{\frac{1}{-j\omega}} = \int_{-\infty}^{\infty} \cos\omega t \, dt - j \cancel{\int_{-\infty}^{\infty} \sin\omega t \, dt} \to 0$$

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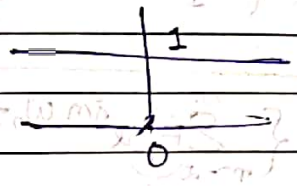
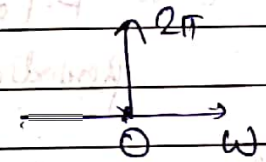
5) $f(t) = 1 \quad \forall t$



$F(\omega) = ??$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-j\omega t} dt = ??$$

Claim :- $2\pi\delta(\omega) = F(\omega) \Rightarrow$



$\therefore \cancel{F\{1\} = 2\pi\delta(\omega)}$

If on F^{-1} we get $1 \Rightarrow$ valid claim

$$\therefore f(t) = F^{-1}\{F(\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega) e^{j\omega t} d\omega$$

$$= \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega$$

$$= \boxed{1 \ \forall t} \Rightarrow \text{Correct}$$

$$\boxed{\begin{aligned} &\therefore F\{1\} = 2\pi\delta(\omega) \\ &\int_{-\infty}^{\infty} 1 e^{-j\omega t} dt = 2\pi\delta(\omega) \end{aligned}}$$

Replacing $\omega \to \omega - n\omega_0$ on both sides, & multiplying a const

$$\int_{-\infty}^{\infty} F_n e^{-j(\omega - n\omega_0)t} dt = 2\pi F_n \delta(\omega - n\omega_0)$$

$$\therefore \int_{-\infty}^{\infty} [F_n e^{jn\omega_0 t}] e^{-j\omega t} dt = 2\pi F_n \delta(\omega - n\omega_0)$$

$$\Rightarrow \boxed{F\{F_n e^{jn\omega_0 t}\} = 2\pi F_n \delta(\omega - n\omega_0)}$$

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$\|^{ly}$, $F\{F_1 e^{j\omega_0 t}\} = 2\pi F_1 \delta(\omega - \omega_0)$

$F\{F_2 e^{j2\omega_0 t}\} = 2\pi F_2 \delta(\omega - 2\omega_0)$

$F\{F_1 e^{j1\omega_0 t} + F_2 e^{j2\omega_0 t}\} = 2\pi F_1 \delta(\omega-\omega_0) + 2\pi F_2 \delta(\omega - 2\omega_0)$

$$\therefore \boxed{F\left\{\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}\right\} = \sum_{n=-\infty}^{\infty} 2\pi F_n \delta(\omega - n\omega_0)}$$

Exponential F.S representation

F.S coeff

F.T of periodic signal

$\therefore$ <u>Inference</u>:- From above expression, it is clear that, we can indeed find F.T of a periodic signal $f(t)$, having F.S coefficients F_n. That is,

$$F(\omega) = \sum_{n=-\infty}^{\infty} 2\pi F_n \delta(\omega - n\omega_0)$$

provided, impulses are allowed to represent in the frequency domain, i.e, $\delta(\omega - n\omega_0)$ $n \in \mathbb{Z}$ is allowed.

Q.1) Find the F.T of impulse train signal shown below

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad (\because \text{This is a periodic signal})$$

$F(\omega) \Rightarrow$ Since $f(t)$ is periodic, we can write it in terms of exponential F.S

$$\Rightarrow f(t) = \sum_{n=-\infty}^{\infty} P_n e^{jn\omega_0 t} \quad \text{where } P_n = \frac{1}{T} \ \forall n \ (\text{Chapter } 3)$$

$$\therefore \boxed{F(\omega) = \sum_{n=-\infty}^{\infty} 2\pi P_n \delta(\omega - n\omega_0) = \frac{2\pi}{T} \sum \delta(\omega - n\omega_0)}$$

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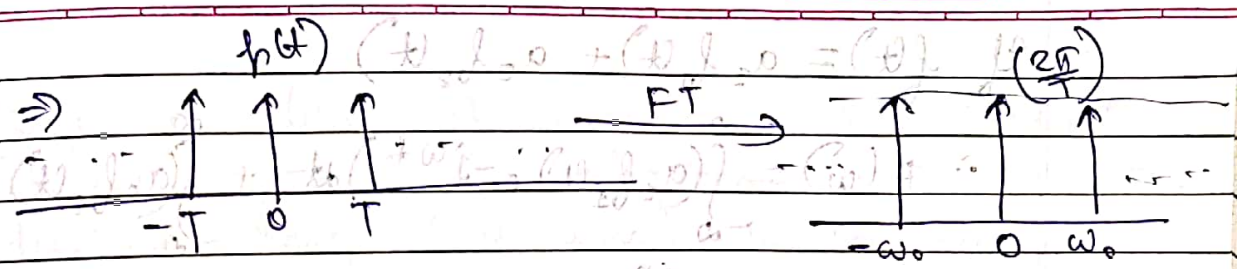
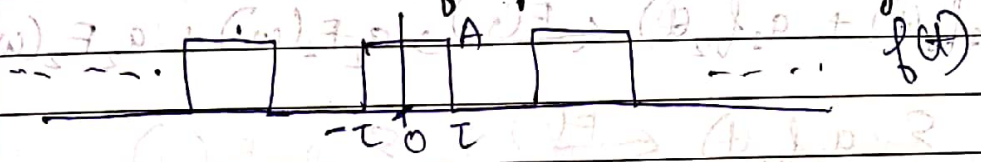
Q|P ⇒



∴ F.T of impulse train signal is impulse train signal.

HW :- Find the F.T of pulse train signal.



Side result :- $P(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

$$= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} \delta(t-nT) \right] e^{-j\omega t} dt$$

Assuming $\int \Sigma = \Sigma \int$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \delta(t-nT) e^{-j\omega t} dt \right]$$

$$P(\omega) = \sum_{n=-\infty}^{\infty} e^{-j\omega nT} \Rightarrow P(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

Poisson's formula →
$$\boxed{\sum_{n=-\infty}^{\infty} e^{-j\omega nT} = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)}$$

Properties of F.T

① **Linearity** :-

Fourier Transform is a Fourier Transform pair.

$$f(t) = e^{-at} u(t) \xleftrightarrow{F(\cdot)} \frac{1}{a+j\omega}$$

Signal → F.T
F.T pair.

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If $f(t) = a_1 f_1(t) + a_2 f_2(t)$

$$\therefore F(\omega) = \int_{-\infty}^{\infty} \left(a_1 f_1(t) e^{-j\omega t}\right) dt + \int_{-\infty}^{\infty} \left(a_2 f_2(t) e^{-j\omega t}\right) dt$$

$$= a_1 \int_{-\infty}^{\infty} f_1(t) e^{-j\omega t} dt + a_2 \int_{-\infty}^{\infty} f_2(t) e^{-j\omega t} dt$$

$$= a_1 F\{f_1(t)\} + a_2 F\{f_2(t)\}$$

$$\therefore a_1 f_1(t) + a_1 f_2(t) \xleftrightarrow{F(\cdot)} a_1 F_1(\omega) + a_2 F_2(\omega)$$

$$\therefore \sum_{\langle k \rangle} a_k f_k(t) \xleftrightarrow{F(\cdot)} \sum_{\langle k \rangle} a_k F_k(\omega)$$

<u>Inference:-</u> This is why F.T is a linear transform.

Now,

<u>Eg:-</u> $e^{-at} u(t) \xleftrightarrow{F} \dfrac{1}{a+j\omega}$

$$e^{-bt} u(t) \xleftrightarrow{F} \frac{1}{b+j\omega}$$

$$b_1 e^{-at} u(t) + b_2 e^{-bt} u(t) \xleftrightarrow{F} \left(\frac{1}{a+j\omega}\right) b_1 + \left(\frac{1}{b+j\omega}\right) b_2$$

(2) Time Shifting

If $f(t) \longleftrightarrow F(\omega)$, then $f(t-t_0) \longleftrightarrow ??$

<u>Proof:-</u>

$$F\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(t-t_0) e^{-j\omega t} dt$$

Using method of substitution, let $t - t_0 = t_1$

$\Rightarrow t = t_0 + t_1$

$$= \int_{-\infty}^{\infty} f(t_1) e^{-j\omega(t_0+t_1)} dt$$

$$\therefore \boxed{F\{f(t-t_0)\} = e^{-j\omega t_0} F(\omega)}$$

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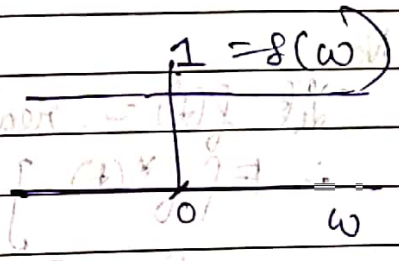
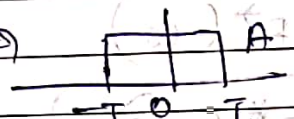
∴ **Inference:-** When the signal in time domain is shifted by t_0, the magnitude spectrum of $F(\omega)$ does not change. Only phase is changed.

1) $\int_{-\infty}^{\infty} \delta(t - nT) e^{-j\omega t} dt = e^{-j\omega nT}$

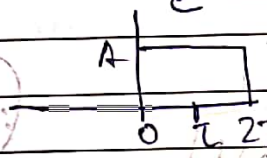
$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$



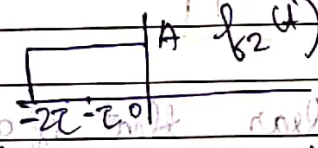
2) $f(t) \Rightarrow$



$\longleftrightarrow 2A\tau \,\text{sinc}(\omega\tau)$ $[F(\omega)]$



$\xleftrightarrow{F_1(\cdot)} e^{-j\omega\tau} F(\omega)$



$\xleftrightarrow{F_2(\cdot)} e^{j\omega\tau} F(\omega)$

(3) **Conjugation and Symmetry**

means complex conjugate

fn will have some symmetry either in time or freq domain.

Statement:- If $f(t) \xleftrightarrow{F} F(\omega)$

then $f^*(t) \xleftrightarrow{F} ?\ F^*(-\omega)$

What will happen to $F(\omega)$ if we take complex conjugate of signal $f(t)$

Proof

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Taking complex conjugation on both sides,

$$F^*(\omega) = \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt\right]^*$$

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$$= \int_{-\infty}^{\infty} f^*(t)\, e^{j\omega t}\, dt$$ (∵ done in Tutorial 1)

$(\because \omega \to -\omega)$

$$\therefore F^*(-\omega) = \int_{-\infty}^{\infty} [f^*(t)]\, e^{+j\omega t}\, dt = F\{f^*(t)\}$$

Now,

Let $f(t) = \text{real} \Rightarrow f^*(t) = f(t)$

$\therefore F\{f^*(t)\} = F^*(-\omega)$

$\therefore F\{f(t)\} = F^*(-\omega)$

$$\Rightarrow \boxed{F(\omega) = F^*(-\omega)} \quad \text{—— (I)}$$

$$|F(\omega)| = |F^*(-\omega)|$$

$$\boxed{|F(\omega)| = |F(-\omega)|}$$

(∵ WKT $|f(x)|$ and $|f^*(x)|$ is same but phase is different)

From above eq^n, it is clear that if a signal is real in time domain, then its magnitude spectrum is symmetric in frequency domain.

Now,

$$F(\omega) = F_R(\omega) + j F_I(\omega) \longrightarrow A_1$$

$$\omega \to -\omega$$

$$F(-\omega) = F_R(-\omega) + j F_I(-\omega)$$

$$F^*(-\omega) = F_R(-\omega) - j F_I(-\omega) \longrightarrow A_2$$

If $f(t) = \text{real} \Rightarrow F(\omega) = F^*(-\omega)$

$$\therefore F_R(\omega) = F_R(-\omega)$$

$$F_I(\omega) = -F_I(-\omega)$$

∴ If signal is real in time domain, the real part

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of & its F-T is even fcn while the imaginary part of its F-T is an odd fcn of frequency.

Proposition 4.1:- If a signal $f(t)$ is real and even, then its F.T is also real and even.

$$f(t) \xleftrightarrow{F} F(\omega)$$

real & even real & even

Proof:-

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\infty}^{\infty} f(\tau) e^{-j\omega \tau} d\tau$$

Now let $\tau = -t_1$

$$\therefore F(\omega) = \int_{+\infty}^{-\infty} f(-t_1) e^{j\omega t_1} (-dt_1)$$

$$= \underbrace{\int_{-\infty}^{\infty} f(t_1) e^{j\omega t_1} dt_1}_{F(\omega)}$$

$$\left(\because F\{f(-t)\} = \int_{-\infty}^{\infty} f(-t) e^{j\omega t} dt = \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt \right)$$

$F(-\omega) = F(\omega)$

$\Rightarrow F(\omega)$ is even function.

Since $f(t)$ is real

$$F^*(-\omega) = F(\omega)$$

$$F(-\omega) = F^*(-\omega)$$

$$-\omega \to \omega$$

$$\boxed{F(\omega) = F^* \omega}$$

$F(\omega)$ = real function.

P.T.O

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Proposition 4.2 :- If $f(t)$ = real and odd $\xleftrightarrow{F(\cdot)}$ odd & purely imaginary.

$$f(t) = f_e(t) + f_o(t) \qquad (\because \text{even} + \text{odd})$$

$$F\{f(t)\} = F\{f_e(t)\} + F\{f_o(t)\}$$

$$F(\omega) = F_e(\omega) + F_o(\omega) \quad \text{——— (1)}$$

e = even $\qquad o$ = odd

Now, WKT

$$F(\omega) = F_R(\omega) + jF_I(\omega) \quad \text{——— (2)}$$

From last proposition,

$$\boxed{F_e(\omega) = F_R(\omega) \quad \text{and} \quad F_o(\omega) = jF_I(\omega)}$$

~~Property~~ 4] Differentiation in time domain

If $f(t) \xleftrightarrow{F(\cdot)} F(\omega)$, then $\frac{d}{dt}[f(t)] \longleftrightarrow ?$

Proof :-

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$$

differentiate both sides wrt 't' under integral sign.

$$\frac{d}{dt}\{f(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\frac{d}{dt}[e^{j\omega t}]\,d\omega$$

$$\cancel{= \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega F(\omega)]e^{j\omega t}\,d\omega$$

F^{-1} of $j\omega F(\omega)$

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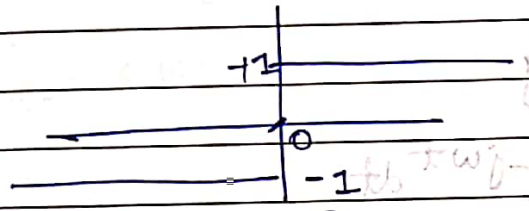
$$\therefore \boxed{F\left\{\frac{d}{dt}f(t)\right\} = j\omega F(\omega)}$$

If $\dfrac{d^2 f(t)}{dt^2} \xleftrightarrow{F} (j\omega)^2 F(\omega)$

$\vdots$

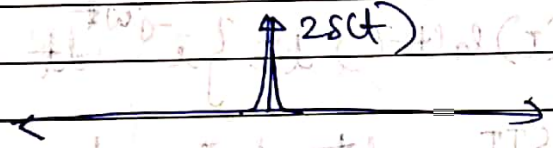
$\dfrac{d^n f(t)}{dt^n} \longleftrightarrow (j\omega)^n F(\omega)$

Q.) Find the F.T of signum function.



$\text{sgn}(t) = f(t)$

$\frac{d}{dt}(\cdot\cdot)$



$\Rightarrow \frac{d}{dt}\{f(t)\} = 2\delta(t)$

$\therefore F\left\{\frac{d}{dt}f(t)\right\} = F\{2\delta(t)\}$

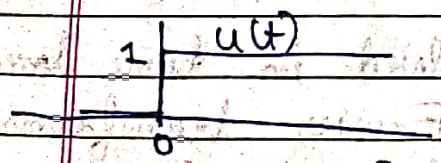
$\downarrow$

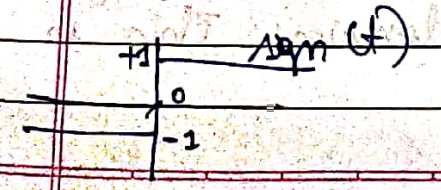
$j\omega F(\omega) = 2\times 1$

$\therefore \boxed{F(\omega) = \frac{2}{j\omega}}$

Q.) Find the F.T of $u(t)$ ($\because$ unit step fxn)

$\therefore f(t) = u(t)$





$u(t) = \frac{1}{2}\{1 + \text{sgn} f(t)\}$

$F\{u(t)\} = F\left[\frac{1}{2}\{1 + \text{sgn} f(t)\}\right]$

$= \frac{1}{2}\left[F(1) + F\{\text{sgn}(t)\}\right]$

$= \frac{1}{2}\left[2\pi\delta(\omega) + \frac{2}{j\omega}\right]$

$\therefore \boxed{F[u(t)] = \frac{1}{j\omega} + \pi\delta(\omega)}$

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5] Convolution Theorem

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t)\end{matrix}} \longrightarrow y(t) = x(t) * h(t)$$

$$x(t) * h(t) \xleftrightarrow{F(\cdot)} = ?$$

Method I:- (Method of Substitution)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\therefore Y(\omega) = F\{y(t)\}$$

$$= \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right\} e^{-j\omega t}\, dt$$

Using method of STT, let $t-\tau = t_1$

$$= \int_{-\infty}^{\infty} x(\tau)\, d\tau \left\{ \int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt \right\}$$

$$= \left[\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega \tau}\, d\tau \right] \left[\int_{-\infty}^{\infty} h(t_1)\, e^{-j\omega t_1}\, dt_1 \right]$$

$$= X(\omega)\, H(\omega)$$

Inference:-

When two signals are convolved in time domain, then their corresponding Fourier transforms are multiplied in frequency domain. This is convolution theorem.

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$$x(t) * h(t) \xleftarrow{F(\cdot\,\omega)} X(\omega)H(\omega)$$

$$\xrightarrow{L(\cdot)} X(s)H(s)$$

$$\xrightarrow{Z(\cdot)} X(z)H(z)$$

time-domain

frequency domain

[$s = \sigma + j\omega$ called Laplace transform variable]

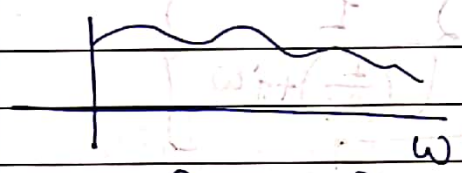
Application of Convolution Theorem :-

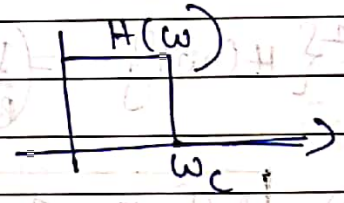
1) LTI filtering

I/P signal $x(t)$ → $\boxed{\text{LTI filter } h(t) \to H(\omega)}$ → $y(t)$ filtered signal

↓ Let's say $x(t)$ has spectrum $X(\omega)$

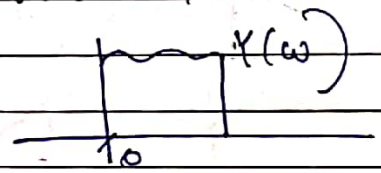
↓ Spectrum $Y(\omega)$



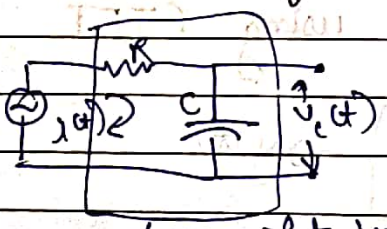


(LPF)

$\therefore Y(\omega) = X(\omega)\,H(\omega)$



Q.] Using FT method, find the freq response of RC low pass filter.



$v_c(t) = v_s(t) * h(t)$

$V_c(\omega) = V_s(\omega)\,H(\omega)$

LTI. Let impulse response be $h(t) \to H(\omega)$

$$\frac{dv_c(t)}{dt} + \left(\frac{1}{RC}\right)v_c(t) = \left(\frac{1}{RC}\right)v_s(t)$$

$$F\left\{\frac{d}{dt}v_c(t)\right\} + F\left\{\frac{1}{RC}v_c(t)\right\} = F\left\{\frac{1}{RC}v_s(t)\right\}$$

$$(j\omega)V_c(\omega) + \frac{1}{RC}V_c(\omega) = \frac{1}{RC}V_s(\omega)$$

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$$H(\omega) = \frac{V_c(\omega)}{V_s(\omega)} = \frac{\left(\frac{1}{RC}\right)}{\left(\frac{1}{RC}\right) + j\omega}$$

$H(\omega)$ Using FT method, find freq response of ideal RC HPF.

→ Side result :-
find impulse response of RC LPF

$$\therefore H(\omega) = \frac{\left(\frac{1}{RC}\right)}{\left(\frac{1}{RC}\right) + j\omega}$$

$$\therefore F^{-1}\{H(\omega)\} = \left(\frac{1}{RC}\right) F^{-1}\left\{\frac{1}{\left(\frac{1}{RC}\right) + j\omega}\right\}$$

$$\Rightarrow \boxed{h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)}$$

Q.] $\frac{dy(t)}{dt} + 10y(t) = x(t)$

$$\Rightarrow h(t) = e^{-10t} u(t).$$

2] Analysis of LTI system using CTFT

$$x(t) \longrightarrow \boxed{\text{LTI system } h(t)} \longrightarrow y(t)$$

relⁿ b/w $x(t)$ & $y(t)$ is governed by LCCDE

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

P.T.O

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$$F\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = F\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

$$\Rightarrow H(\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

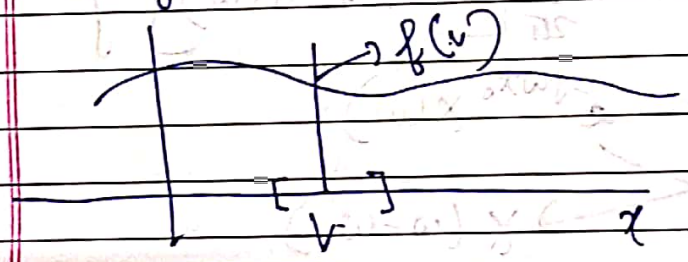
freq response of LTI system is a function (ω)

$$\boxed{H(\omega) = \frac{b_0 + b_1(j\omega)^1 + b_2(j\omega)^2 + \cdots + b_M(j\omega)^m}{a_0 + a_1(j\omega) + a_2(j\omega)^2 + \cdots + a_N(j\omega)^N}}$$

$\hookrightarrow$ ratio of 2 polynomials $\Rightarrow \frac{\text{polynomial in}(\omega)}{\text{polynomial in}(\omega)}$

$\therefore H(\omega)$ = polynomial in 'ω'

[Only 1 condition, $H(\omega)$ is continuous]



A continuous fn can be approximated very well by n^{th} order polynomial.

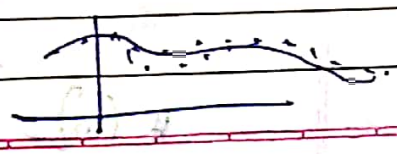
$$\boxed{f(x) = \lim_{n\to\infty} P_n(x)}$$

Weierstrass – Stone approximation.

$P_1(x)\ P_2(x)\ P_3(x) \cdots P_n(x) \rightarrow$ converge to $f(x)$ as $n \to \infty$

$f(x) = 2x^2+3x+5 \Rightarrow [2\ 3\ 5]$ (x^2, x^1)

$g(x) = 7x^2+8x+3 \quad [7\ 8\ 3]$



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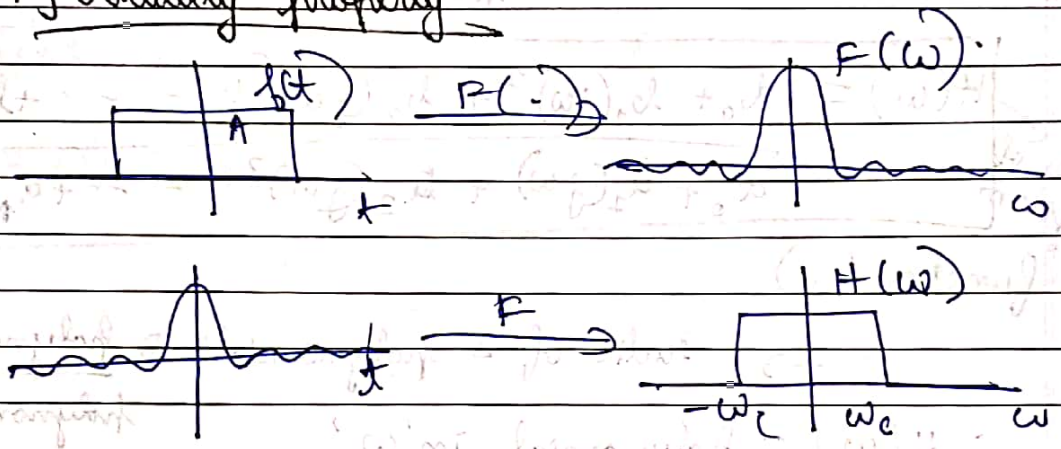
6.] Modulation Theorem

$$x(t)\,h(t) \xleftrightarrow{F(\cdot)} \frac{1}{2\pi} X(\omega) * F(\omega)$$

Multiplication of two signals in time domain ⇒ convolving of frequency responses in freq domain.

7.] Duality property



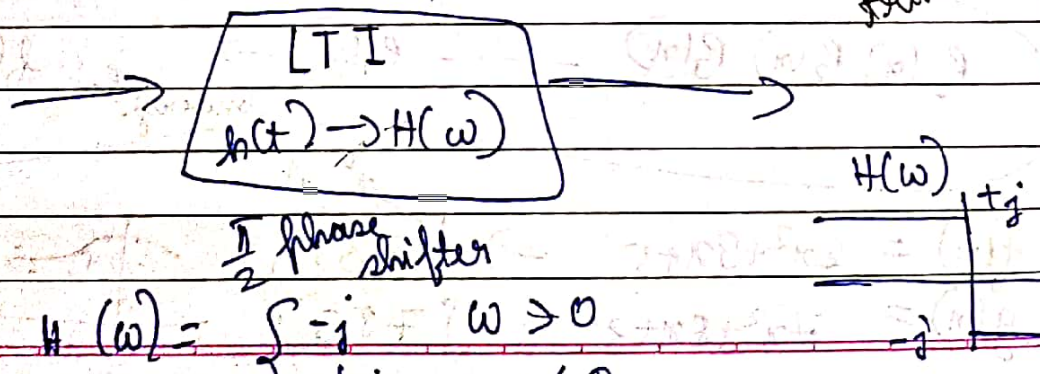
$$x(t) * h(t) \xleftrightarrow{F(\cdot)} X(\omega)\,H(\omega)$$
$$x(t)\,h(t) \xleftrightarrow{} \frac{1}{2\pi}\left[X(\omega) * H(\omega)\right]$$
} duality

$$x(t-t_0) \xleftrightarrow{F(\cdot)} e^{-j\omega t_0} X(\omega)$$
$$e^{j\omega t_0} x(t) \longleftrightarrow X(\omega-\omega_0)$$

8.] Time Scaling

$$F\{x(at)\} = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Hilbert's transform



$\frac{\pi}{2}$ phase shifter

$$H(\omega) = \begin{cases} -j & \omega > 0 \\ +j & \omega < 0 \end{cases}$$

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(9) $\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$ } Parseval's theorem

Energy of signal in time domain = energy of signal in freq domain. Hence, Parseval's theorem is manifestation of energy conservation theorem.

→ gives us a guarantee for existence of Inverse F.T.

Tutorial-10

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1) $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$

$f(t) = e^{-at}u(t) \Rightarrow F(\omega) = \frac{1}{a+j\omega}$

Now, $\int_{-\infty}^{\infty} |A^2 e^{-2bt}| u(t)\, dt$

$= A^2 \int_{0}^{\infty} e^{-2bt}\, dt$

$E_g = \frac{A^2}{2b}$

% of $E_g \Rightarrow \left(\frac{x}{100}\right)\left(\frac{A^2}{2b}\right)$

We need range of frequencies. Hence we do in freq domain.

$g(t) = Ae^{-bt}u(t)$

$G(\omega) = \frac{A}{b+j\omega}$

$|G(\omega)| = \frac{A}{\omega^2+b^2} \Rightarrow E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$

$= \frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} \frac{A^2}{\omega^2+b^2} d\omega$

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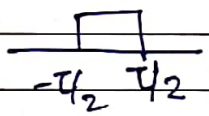
Now $\frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} \frac{A^2}{\omega^2+b^2}\,d\omega = \frac{0.01\times A^2}{2b}$

$\Rightarrow \frac{1}{2\pi}\int_{-\omega_1}^{\omega_1}\left(\frac{1}{\omega^2+b^2}\right)d\omega = \frac{0.01\times}{2b}$

$\frac{1}{}$

2] $x(t) \longleftrightarrow X(f)$

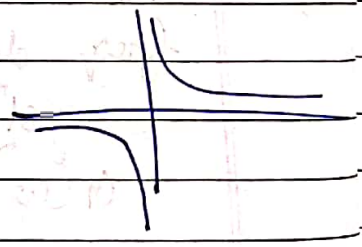
$X(t) \longleftrightarrow x(-f)$

$A\Pi\left(\frac{t}{\tau}\right)$ 

$A\Pi\left(\frac{t}{\tau}\right) \longleftrightarrow A\tau\,\mathrm{sinc}(f\tau)$

$\left(\frac{1}{2\omega}\right) 2\omega A\,\mathrm{sinc}(2\omega\tau) \Leftrightarrow \frac{1}{2\omega} A\Pi\left(\frac{-f}{2\omega}\right)$

3] $\hat{x}(t) = \frac{1}{\pi t} * x(t)$



$\hat{X}(f) = (-j\,\mathrm{sgn}(f))X(f)$

$|\hat{X}(f)|^2 = |(-j\,\mathrm{sgn}(f)X(f)|^2$

$= |X(f)|^2$

Now $H\{\hat{x}(t)\} = \frac{1}{\pi t} * \hat{x}(t) = \frac{1}{\pi t} * \frac{1}{\pi t} * x(t)$

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$$F\{H\{\hat{x}(t)\}\} = (-j\,\mathrm{sgn}(f))(-j\,\mathrm{sgn}(f))$$
$$= -1\,X(f)$$

$$\Rightarrow H\{\hat{x}(t)\} = -x(t)$$

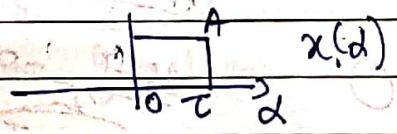
$$\therefore \int_{-\infty}^{\infty} x(t)x^*(t)\,dt = \int_{-\infty}^{\infty} X(f)X^*(f)\,df$$

$$\Rightarrow \int_{-\infty}^{\infty} x(t)\,y^*(t)\,dt = \int_{-\infty}^{\infty} X(f)\,Y^*(f)\,df$$

$$\int_{-\infty}^{\infty} x(t)\,[\hat{x}(t)]^*\,dt = \int_{-\infty}^{\infty} X(f)\,[\hat{X}(f)]^*\,df$$

$$= \int_{-\infty}^{\infty} X(f)\,[-j\,\mathrm{sgn}(f)X(f)]^*$$

$$= \int_{-\infty}^{\infty} j\,X(f)X^*(f)\,\mathrm{sgn}(f)\,df$$

4.] 

$$\hat{x}(t) = y(t) * x(t)$$
$$= \int_{-\infty}^{\infty} y(\alpha)\,x(t-\alpha)\,d\alpha$$

Case 1:- $t < 0$

$$\hat{x}(t) = \int_{t-\tau}^{t} A\left(\frac{1}{\pi\alpha}\right)d\alpha = \frac{A}{\pi}\left[\ln(\alpha)\right]_{t-\tau}^{t}$$

$$= \frac{A}{\pi}\ln\left(\frac{t}{t-\tau}\right)$$

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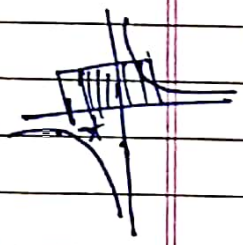
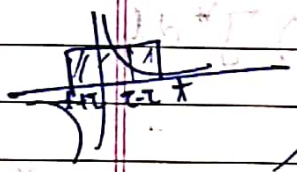
Case 2:- $\hat{x}(t) = \int_{t-\tau}^{t} \frac{A}{\pi\alpha}\, d\alpha = \frac{A}{\pi} \ln\left(\frac{t}{t-\tau}\right)$

Case 3:- 0

Case 4:- $\int_{t-\tau}^{-t} \frac{A}{\pi\alpha}\, d\alpha = \frac{A}{\pi} \ln\left(\frac{-t}{t-\tau}\right)$



Case 5:- $\int_{-t+\tau}^{t} \left(\frac{A}{\pi\alpha}\, d\alpha\right) = \frac{A}{\pi} \ln\left(\frac{t}{-t+\tau}\right)$



$$\hat{x}(f) = \begin{cases} \frac{A}{\pi}\ln\left(\frac{t}{t-\tau}\right) & t<0 \\ \frac{A}{\pi}\ln\left(\frac{t}{t-\tau}\right) & t-\tau>0,\ t>\tau \\ 0 & \frac{t+t-\tau}{2}=0,\ \frac{2t-\tau}{2}=0,\ t=\frac{\tau}{2} \\ \frac{A}{\pi}\ln\left(\frac{-t}{t-\tau}\right) & 0<t\le\frac{\tau}{2} \\ \frac{A}{\pi}\ln\left(\frac{t}{-t+\tau}\right) & \tau/2<t\le\tau \end{cases}$$

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Convolution Theorem

$$x(t) * h(t) \xleftrightarrow{F} X(\omega)H(\omega)$$

Proof :-

Method 2 (Using Eigen function theory)

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t)\end{matrix}} \longrightarrow y(t) = x(t) * h(t)$$

Logic : Express input and impulse response in terms of eigen functions.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega = F^{-1}(X(\omega))$$

$$= \frac{1}{2\pi}\left\{\lim_{\omega_0 \to 0}\sum_{n=-\infty}^{\infty}\left[X(\omega_n)e^{j\omega_n t}\right]\omega_0\right\}$$

$$= \lim_{\omega_0 \to 0}\sum\left(\frac{X(\omega_n)\cdot\omega_0}{2\pi}\right)e^{j\omega_n t}$$

$$\omega_n = n\omega_0$$

Thus, input signal $x(t)$ has been represented as linear combination of complex exponentials, i.e, Eigen functions.

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\,e^{-j\omega t}\,dt$$

$$H(\omega_n) = \int_{-\infty}^{\infty} h(t)\,e^{-j\omega_n t}\,dt$$

Using Eigen function

$$y(t) = H(\omega_n)$$

$$y(t) = \lim_{\omega_0 \to 0}\sum_{n=-\infty}^{\infty}\left(\frac{X(\omega_n)\,\omega_0}{2\pi}\right)H(\omega_n)\,e^{j\omega_n t}$$

$$y(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=+\infty} X(\omega)H(\omega)\,e^{j\omega t}\,d\omega$$

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Comparing with $y(t) = F^{-1}\{Y(\omega)\}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega) e^{j\omega t}\, d\omega$$

$$\Rightarrow \boxed{Y(\omega) = X(\omega)\cdot H(\omega)}$$

Problem:-

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h(t)\end{matrix}} \longrightarrow y(t)$$

$e^{-at}u(t)$ $\quad h(t) = e^{-bt}u(t)$

$$y(t) = x(t) * h(t)$$

$$Y(\omega) = \left(\frac{1}{a+j\omega}\right)\left(\frac{1}{b+j\omega}\right)$$

$f(x) = \frac{1}{(x+a)}\frac{1}{(x+b)} = \frac{K_1}{x+a} + \frac{K_2}{x+b}$

yhan hum x ko $-a$, $-b$ karte the. Here it is $Y(j\omega) \Rightarrow j\omega$ ko STI karo

$$= \frac{b+j\omega + a+j\omega}{ab + bj\omega + aj\omega + (j\omega)^2}$$

$= \cancel{(a+b)}$ Using method of partial fractions

$$Y(\omega) = \frac{K_1}{a+j\omega} + \frac{K_2}{b+j\omega}$$

$$= \frac{1}{(b-a)}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right]$$

$$y(t) = F^{-1}(Y(\omega)) = \frac{1}{b-a}\left[e^{-at}u(t) - e^{-bt}u(t)\right]$$

Proof of ~~Convolut~~ Modulation thm

$$y(t) = m(t)\cdot c(t)$$

then $Y(\omega) = \frac{1}{2\pi}\left[M(\omega) * C(\omega)\right]$

Proof:-

$$\text{RHS} = \left(\frac{1}{2\pi}\right) M(\omega) * C(\omega)$$

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$$= \frac{1}{2\pi}\left\{\int_{\xi=-\infty}^{+\infty} M(\xi)\, C(\omega-\xi)\, d\xi\right\}$$

(Volkswagen Jetta $= \xi$ huh kido!)

running variable for frequency

$$M(\xi) = F\{m(t)\} = \int_{-\infty}^{\infty} m(t)\, e^{-j\xi t}\, dt$$

$$C(\omega-\xi) = \int_{-\infty}^{\infty} c(t)\, e^{-j(\omega-\xi)t}\, dt$$

$$\therefore \text{RHS} = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} m(t)\, e^{-j\xi t}\, dt\right]\left[\int_{-\infty}^{\infty} c(t)\, e^{-j(\omega-\xi)t}\, dt\right] \times d\xi\right\}$$

$$\iint f(x,y)\, dx\, dy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\, dx\right] dy$$

(Fubini's theorem)

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\, dy\right] dx$$

$$\text{RHS} = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} m(t)\, c(t)\, e^{-j\xi t}\, dt \left[\underbrace{\iint e^{-j(\omega-\xi)t}\, dt}_{2\pi\delta(\omega-\xi)}\right] d\xi\right\}$$

$$= \int_{-\infty}^{\infty} m(t)\, c(t)\, e^{-j\xi t}\, dt \left[\int_{-\infty}^{\infty} \delta(\omega-\xi)\, d\xi\right]$$

$$\left[\int_{-\infty}^{\infty} \delta(\xi-\omega)\, d\xi\right] \quad (\because \delta(\omega) = \delta(-\omega))$$

$$\text{R.H.S} = \int_{-\infty}^{\infty} m(t)\, c(t)\, e^{-j\xi t}\, dt$$

$$F\{y(m)\} = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt \quad \Rightarrow y(t) = x(t)\, c(t)$$

$$= F^{-1}\{Y(\omega)\}$$

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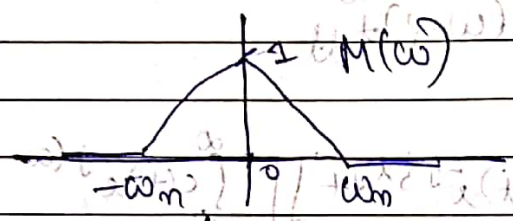
Application of Modulation Th^m

(Amplitude Modulation)

(message signal) $m(t)$ — T_x = Communication channel = R_x

Cost of Antenna Design



Spectrum of voice range (4kHz)

frequency of signal is inversely proportional to length of antenna.

$f \propto \frac{1}{l}$

In computer, we can artificially raise frequency to MHz or GHz

$$y(t) = m(t)\,c(t)$$

$$Y(\omega) = \frac{1}{2\pi} M(\omega) * C(\omega)$$

Let $c(t) = \cos(\omega_c t) = \frac{1}{2} e^{j\omega_c t} + \frac{1}{2} e^{-j\omega_c t}$

$$C(\omega) = F\{c(t)\} = \frac{1}{2} F\{e^{j\omega_c t}\} + \frac{1}{2} F\{e^{-j\omega_c t}\}$$

$$= \frac{1}{2}\left[2\pi\delta(\omega-\omega_c) + 2\pi\delta(\omega+\omega_c)\right]$$

$$= \pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)$$

$$Y(\omega) = \frac{1}{2\pi} M(\omega) * C(\omega)$$

$$= \frac{1}{2\pi}\left[M(\omega) * \pi\left[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)\right]\right]$$

$$= \frac{1}{2}\left[M(\omega) * \delta(\omega-\omega_c) + M(\omega) * \delta(\omega+\omega_c)\right]$$

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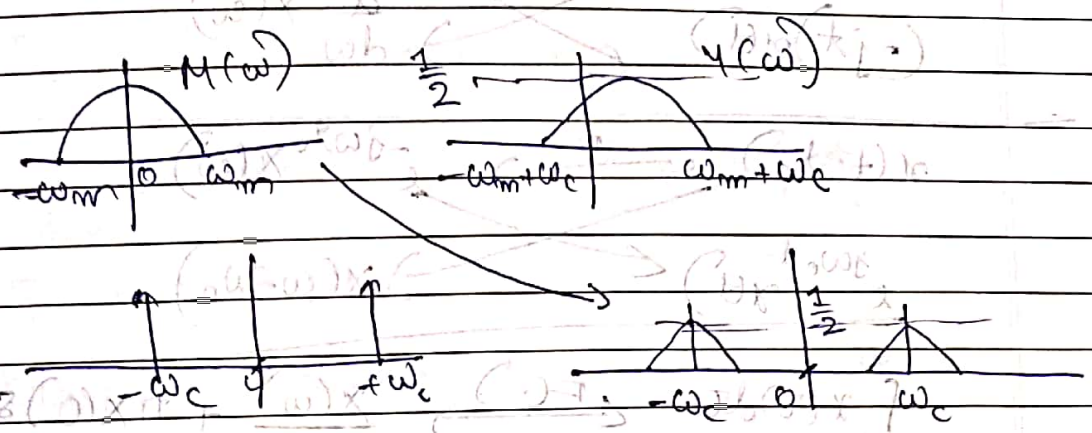
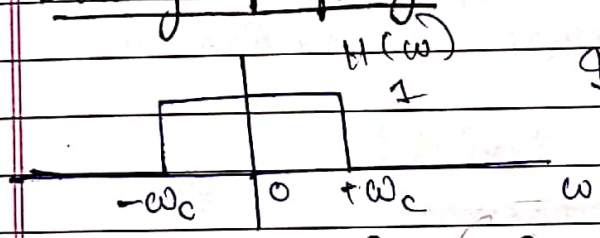
$$Y(\omega) = \frac{1}{2}\left[M(\omega-\omega_c) + M(\omega+\omega_c)\right]$$



Duality property



Ideal LPF

$$h(t) = F^{-1}\{H(\omega)\} = \left[\int_{\omega=-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega\right]\frac{1}{2\pi}$$

$$= \left[\int_{-\omega_c}^{\omega_c} 1\,e^{j\omega t}\,d\omega\right]\frac{1}{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-\omega_c}^{\omega_c}$$

$$\Rightarrow \left[\frac{e^{jt\omega_c}}{jt} - \frac{e^{-j\omega_c t}}{jt}\right]\left(\frac{1}{2\pi}\right)$$

$$= \frac{2}{t}\left[\frac{e^{j(t\omega_c)} - e^{-j(t\omega_c)}}{2j}\right]\left(\frac{1}{2\pi}\right)$$

$$= \left(\frac{2\omega_c}{t}\,\frac{\sin(\omega_c t)}{\omega_c}\right)\left(\frac{1}{2\pi}\right)$$

$$= \frac{2\omega_c}{2\pi}\left[\frac{\sin(\omega_c t)}{\omega_c t}\right]$$

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$$\frac{d}{dt}(x(t)) \xleftrightarrow{F(\cdot)} j\omega X(\omega)$$

$$(-jt)x(t) \xleftrightarrow{F(\cdot)} \frac{d}{d\omega}X(\omega)$$

$$x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0}X(\omega)$$

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X(\omega-\omega_0)$$

$$\int_{\tau=-\infty}^{\infty} x(\tau)d\tau \xleftrightarrow{F(\cdot)} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

Integration property

If $f(t) \xleftrightarrow{F} F(\omega)$

then $\int_{-\infty}^{t} f(\tau)d\tau \xleftrightarrow{F(\cdot)}$

$$f(t) \longrightarrow \boxed{\begin{array}{c}\text{Integrator}\\ h(t)=u(t)\end{array}} \longrightarrow y(t) = \int_{-\infty}^{t} f(\tau)d\tau$$

$$y(t) = f(t) * h(t) = f(t) * u(t)$$

$$Y(\omega) = F\{f(t) * u(t)\}$$

$$= F(\omega)U(\omega)$$

$$= F(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= \frac{F(\omega)}{j\omega} + \pi F(\omega)\delta(\omega)$$

$$= \frac{F(\omega)}{j\omega} + \pi F(0)\cdot\delta(\omega)$$

where $F(0)$ is value at $\omega=0$, i.e., the dc value of signal.

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Wednesday

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Development of Hilbert Transform

Case I:- Continuous Time Case

it is application of duality property.

Motivation

1) $\frac{\pi}{2}$ phase shifter
2) frequency modulation ($F(\omega)$)
3) Signal Sideband Modulation (SSB)
4) As a coupled application, algo for estimation of instantaneous frequency (IF)

$$x(t) \longrightarrow \boxed{\text{Hilbert Transform } h(t)} \longrightarrow y(t)$$

$\therefore y(t) = x(t) * h(t)$

For Hilbert transform, $h(t) = \frac{1}{\pi t}$

$$y(t) = \text{Hilbert}\{x(t)\} = \boxed{\frac{1}{\pi t} * x(t)}$$

We will call Hilbert transform as 22/10/18

$$\hat{x}(t) = \text{Hilbert}\{x(t)\} = x(t) * \frac{1}{\pi t}$$

Harmonic analysis $\longleftarrow$ $= \int_{-\infty}^{\infty} x(\tau)\frac{(1)}{(t-\tau)}\,d\tau$ $\quad (\because \tau \in \text{Real line})$ $t \in \mathbb{R}$

→ We see that when $t=\tau$, integrand is ∞ ⇒ basically we are integrating infinity. Such integrals are called singular integrals.

$$F\{\hat{x}(t)\} = F\left\{x(t) * \frac{1}{\pi t}\right\}$$
$$= X(\omega)\, F\left\{\frac{1}{\pi t}\right\}$$

$\because y(t) = F\{x(t)\}$

$\hat{x}(t) = \text{Hilbert}\{x(t)\}$
$= x(t) * \frac{1}{\pi t}$

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$F\left(\frac{1}{\pi t}\right) = ?$

$\Rightarrow F(u(t)) = \frac{1}{j\omega} + \pi\delta(\omega)$

$F\{2u(t) - 1\} = 2F\{u(t)\} - F\{1\}$

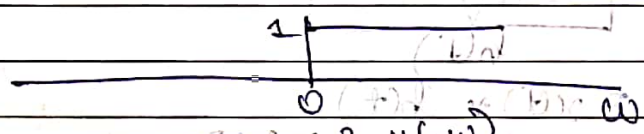
$= 2\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - 2\pi\delta(\omega)$

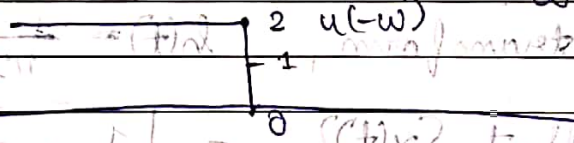
$2u(t) - 1 \xleftrightarrow{F} \frac{2}{j\omega}$

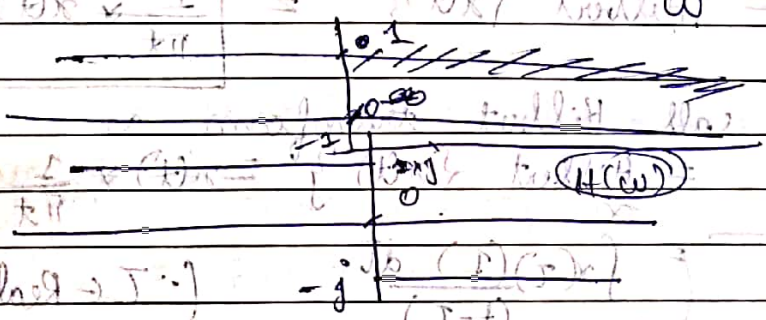
$\frac{2}{jt} \xleftrightarrow{F} 2\pi\left[2u(-\omega) - 1\right]$

$\Rightarrow h(t) = \frac{1}{\pi t} \xleftrightarrow{F} j\left[2u(-\omega) - 1\right] = H(\omega)$

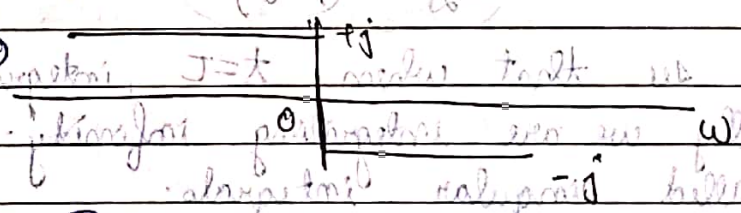
$\therefore \boxed{H(\omega) = j\left[2u(-\omega) - 1\right]}$







$\therefore H(\omega) \Rightarrow$



$= \begin{cases} +j, & \omega < 0 \\ -j, & \omega > 0 \end{cases}$

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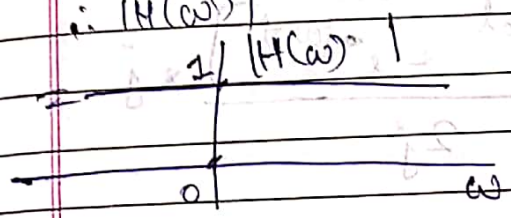
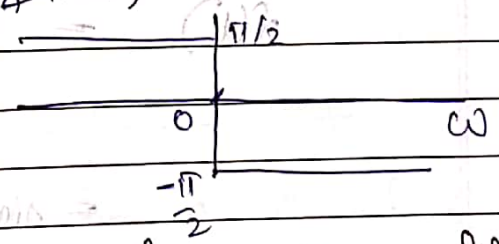
Now, $|H(\omega)| = \begin{cases} |j| = 1 & \omega < 0 \\ |-j| = 1 & \omega > 0 \end{cases}$

$\measuredangle H(\omega) = \begin{cases} \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2} & , \omega < 0 \\ \tan^{-1}\left(\frac{-1}{0}\right) = -\frac{\pi}{2} & , \omega > 0 \end{cases}$

$\therefore |H(\omega)|$



$\measuredangle H(\omega)$



Q.) So, can we say Hilbert transform is a filter? Phase contributes to shaping the signal. A new definition of filtering. It is an all-pass filter. But phases above and below $\omega = 0$ are shifted by -90 and $+90$ respectively

Q.) Find the Hilbert transform of signal $\cos(\omega_c t)$

$\cos(\omega_c t) \longrightarrow \boxed{\text{Hilbert Transform}} \longrightarrow y(t) = \hat{x}(t)$

$\hat{x}(t) = y(t) = \frac{1}{\pi t} * \cos(\omega_c t)$

$= \sin(\omega_c t)$

($\because \frac{\pi}{2}$ phase shifter)

$F\{\hat{x}(t)\} = F\{y(t)\} = F\left(\frac{1}{\pi t}\right) F(\cos\omega_c t)$

$= \cancel{j[2\pi(\omega) \cdot 1]} \cancel{[\pi]}$

$\hat{X}(\omega) = \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] \times \begin{cases} -j & , \omega > 0 \\ j & , \omega < 0 \end{cases}$

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$$\hat{X}(\omega) = \pi\left[-j\delta(\omega-\omega_c) + j\delta(\omega+\omega_c)\right]$$

$$\hat{X}(\omega) = \frac{j}{2}\left[2\pi\delta(\omega+\omega_c) - 2\pi\delta(\omega-\omega_c)\right]$$

$$F^{-1}\{\hat{X}(\omega)\} = \frac{j}{2}\left[F^{-1}\{2\pi\delta(\omega+\omega_c)\} - F^{-1}\{2\pi\delta(\omega-\omega_c)\}\right]$$

$$\downarrow$$

$$\hat{x}(t) = j\,\frac{\left[e^{-j\omega_c t} - e^{j\omega_c t}\right]}{2j} \times j$$

$$= \sin\omega_c t$$

Applications of H-T

1) Generation of "analytic" signal:

from complex variable theory

Analytic function is one which is differentiable at each and every point of the independent variable z. Cauchy-Rieman conditions hold of course.

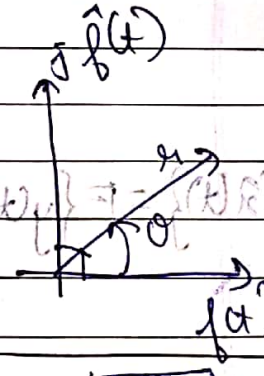
Let $f(t)$ = real signal

this is analytic function

$$f_a(t) = f(t) + j\hat{f}(t)$$

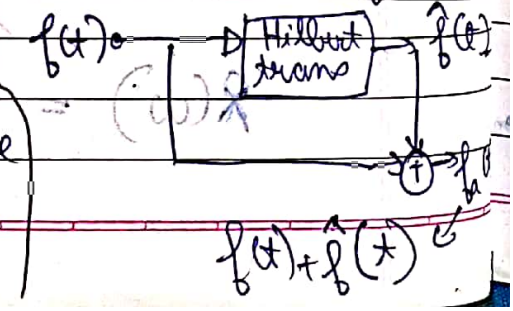
$$r = \sqrt{|f(t)|^2 + |\hat{f}(t)|^2}$$

Hilbert envelope



$$\theta = \tan^{-1}\left(\frac{\hat{f}(t)}{f(t)}\right)$$

called analytic phase



(i) Find F.T of $f_a(t)$

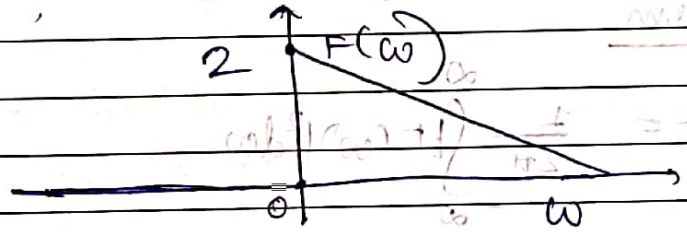
$$\therefore F\{f_a(t)\} = F\{f(t)\} + jF\{\hat{f}(t)\}$$

$$= F(\omega) + j[F(\omega)H(\omega)]$$

$$= F(\omega) + jF(\omega) \times \begin{cases} -j & \omega > 0 \\ +j & \omega < 0 \end{cases}$$

$$\boxed{F_a(\omega) = \begin{cases} 2F(\omega) & , \omega > 0 \\ 0 & , \omega < 0 \end{cases}}$$

Analytic function thus has a special property that it is zero for $\omega < 0$



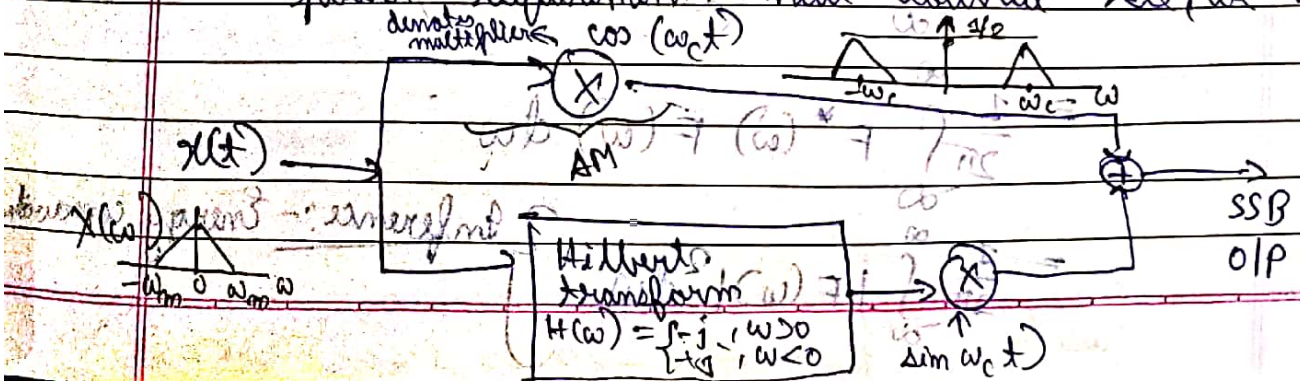
$h(t) = 0 \quad , t < 0$

$\Rightarrow$ Analytic function's impulse response is causal in freq domain

$\Rightarrow$ Using Conjugation symmetry, since the frequency response is asymmetric in freq domain $\Rightarrow$ the corresponding signal in time domain, has to be complex

2) <u>Single Sideband Modulation (SSB)</u>

AM $\rightarrow$ Limitation :- Since it became 2ω, the power requirement had doubled. So, we use SSB.



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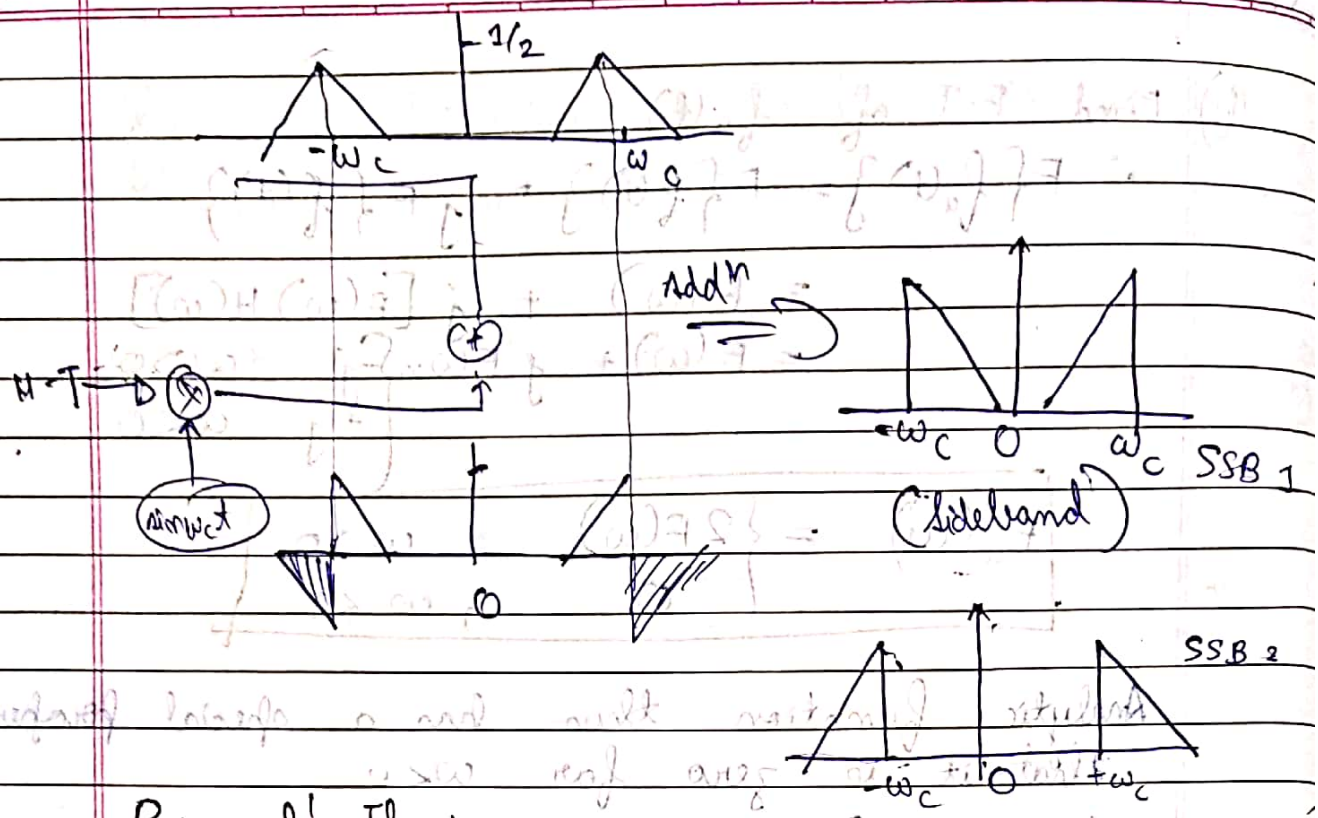

Parseval's Theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Proof:-

LHS: $\int_{-\infty}^{\infty} f(t)\, f^*(t)\, dt$

$$= \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega\right]^* dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} F^*(\omega)\, e^{-j\omega t} d\omega\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega) \left(\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\, F(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Inference:- Energy conservi

Monday.

The def^n of F-T theory rests on conservation of energy, i.e, Parseval's theorem.

Application of Parseval's Theorem

Signal

- Aperiodic
  - $\int |f(t)|^2 dt < +\infty$
  - $t \in R$ (= finite energy signal)
  - Energy Spectral Density (ESD)
- Periodic
  - LHS can't hold here, as it to infinite. Hence, the idea here is to extend the concept of finite energy, using the theorem.
  - $P_{av} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt$
  - Power Spectral Density (PSD)

⇒ ESD and PSD are used to determine the amount of energy and power transmission via communication channel.

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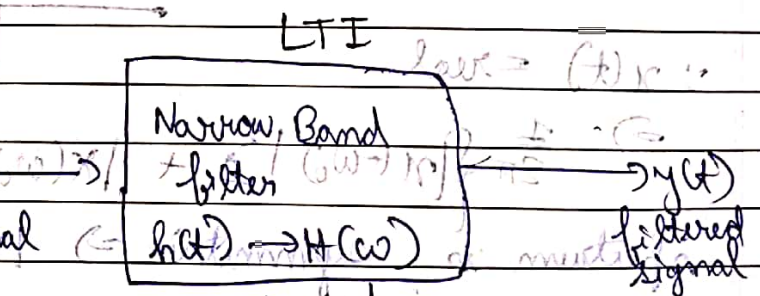
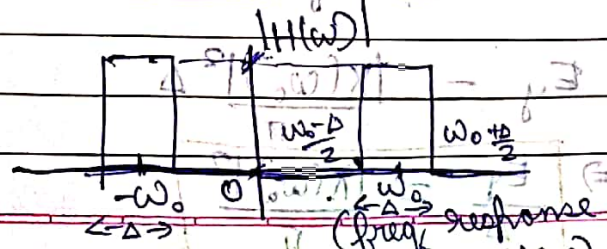
Concept of ESD :-

Problem :-



= real-valued signal

Narrow band means Δ is small



(freq response is narrow)

(Used in communication applications system)

Figure Narrowband filter

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$\therefore y(t) = x(t) * h(t)$

$Y(\omega) = X(\omega)\, H(\omega)$ —— (1)

$Y^*(\omega) = X^*(\omega)\, H^*(\omega)$ —— (2)

Multiplying,

$$\boxed{|Y(\omega)|^2 = |X(\omega)|^2\, |H(\omega)|^2} \quad \text{—— (3)}$$

The energy of O/P signal $y(t)$ is given by:

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt$$

By Parseval's thm,

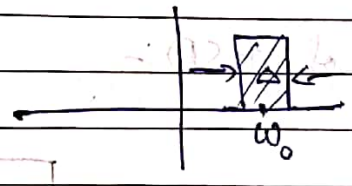
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 |H(\omega)|^2 d\omega$$ (And $|H(\omega)|$ only exists for the given ω range)

value is 1

$$= \frac{1}{2\pi}\left\{\int_{-\omega_0-\frac{\Delta}{2}}^{-\omega_0+\frac{\Delta}{2}} |X(\omega)|^2 |H(\omega)|^2 d\omega + \int_{\omega_0-\frac{\Delta}{2}}^{\omega_0+\frac{\Delta}{2}} |X(\omega)|^2 |H(\omega)|^2 d\omega\right\}$$

Value is 1



$\because x(t) = \text{real}$

$$\Rightarrow \frac{1}{2\pi}\left\{|X(-\omega_0)|^2 \Delta + |X(\omega_0)|^2 \Delta\right\}$$

spectrum is symmetric $\Rightarrow |X(-\omega_0)| = |X(\omega_0)|$

$$\therefore E_y = \frac{|X(\omega_0)|^2 \Delta}{\pi}$$

$$\Rightarrow \boxed{\frac{E_y}{\Delta} = \frac{|X(\omega_0)|^2}{\pi}}$$

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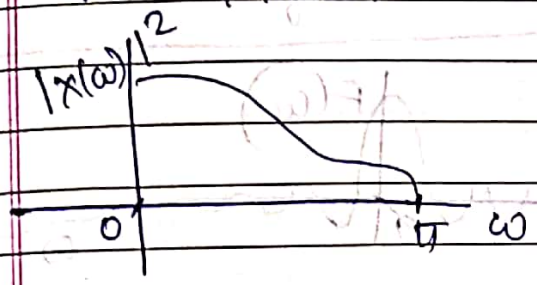
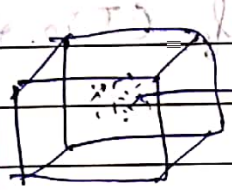
Let $|X(\omega)|^2$ ($\because$ considering only +ve side)



$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega$$

$$= \int_{0}^{\infty} \frac{|Y(\omega)|^2}{\pi} d\omega$$

(Energy transmission through a narrow band communication channel)



$\rightarrow$ density of fluid is a localized phenomena. (This cube is collapsed into a line and mass becomes instantaneous amplitude)

$\therefore$ This $\frac{E_y}{\Delta}$, the Δ plays role of the cube,

$$\Rightarrow \boxed{\frac{E_y}{\Delta} = \frac{|X(\omega_0)|^2}{\pi}} \rightarrow \text{Called Energy Spectral Density (ESD)}$$

H.W:- Study PSD

Proposition 4.3:- If a function $f(t)$ is k times continuously differentiable, then $F(\omega)$ decays.

$$(T_n - t)^2 \sum_{n=-\infty}^{\infty} (t) f = (t) f (t) f = (t) f$$

$$(T_n - t)^2 (t) f \sum_{n=-\infty}^{\infty} =$$

$$\boxed{(T_n - t)^2 (T_n) f \sum_{n=-\infty}^{\infty} = (t) f}$$

$$\text{as is steel no } T \text{ period}$$

$$t_0^{t\omega_0} (t) f \sum_{\omega_0} = (t) f \Rightarrow \therefore$$

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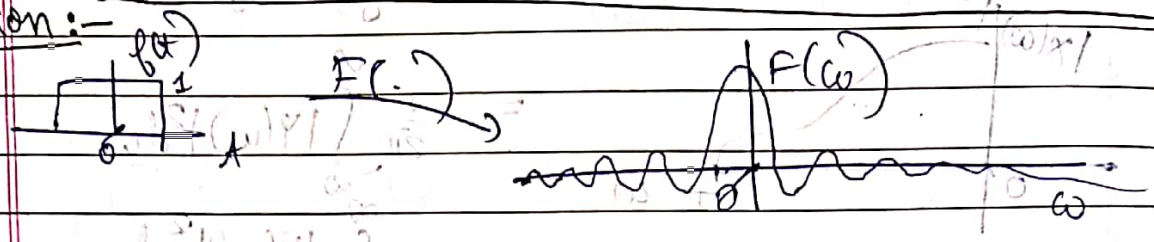
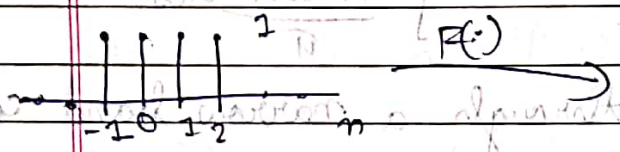
Chapter-5: "Development of Discrete Time Fourier Transform (DTFT"

Motivation:-

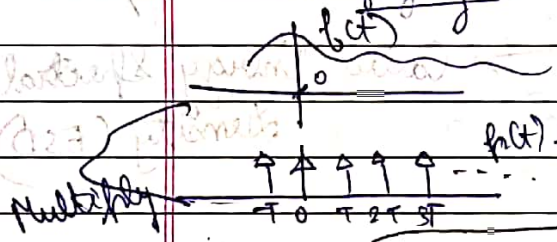




What will be results in chapter 4 if CT is replaced by DT?

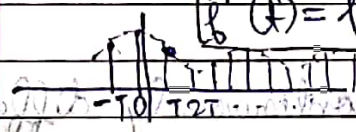
(Q.) How to get $f(n)$ from $f(t)$?

$\Rightarrow$ Sampling $\Rightarrow$ Chapter 7.



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$\boxed{\hat{f}(t) = f(t)\,p(t)} \rightarrow \text{called discrete time signal}$$



called Impulse train sampling.

$$\therefore \hat{f}(t) = f(t)\,p(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} f(t)\,\delta(t - nT)$$

$$\boxed{\hat{f}(t) = \sum_{n=-\infty}^{\infty} f(nT)\,\delta(t - nT)}$$ = discrete-time signal obtained by impulse train sampling.

Taking FT on both sides,

$$\therefore F\{\hat{f}(t)\} = \int_{-\infty}^{\infty} \hat{f}(t)\, e^{-j\omega t}\, dt$$

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$$F\{\hat{f}(t)\} = \int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} f(nT)\,\delta(t-nT) \right\} e^{-j\omega t}\,dt$$

Now, assuming $\int \Sigma = \Sigma \int$

$$F\{\hat{f}(t)\} = \sum_{n=-\infty}^{\infty} f(nT) \left[\int_{-\infty}^{\infty} \delta(t-nT)\, e^{-j\omega t}\,dt \right]$$

$$= \sum_{n=-\infty}^{\infty} f(nT)\, e^{-j\omega nT} \qquad (\because \text{Poisson's})$$

$$\hat{f}(t) = f(nT)$$

~~Let~~ $T = 1$ sec

$$F\{f(n)\} = \boxed{F(e^{j\omega}) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-j\omega n}} \rightarrow \text{Analysis equation DTFT}$$

$\therefore$ ~~$F(e^{j\omega}) = \sum_{n=-\infty}$~~ Now, multiplying both sides and integrate

$$\int_{\omega_0}^{\omega_0+2\pi} F(e^{j\omega})\, e^{j\omega m}\,d\omega = \int_{\omega_0}^{\omega_0+2\pi} \sum_{n=-\infty}^{\infty} f(n)\, e^{-j\omega n}\, e^{j\omega m}\,d\omega$$

Assuming $\int \Sigma = \Sigma \int$

$$\Rightarrow \quad = \sum_{n=-\infty}^{\infty} f(n) \left\{ \int_{\omega_0}^{\omega_0+2\pi} e^{-j\omega(n-m)}\,d\omega \right\}$$

Now integral $I = \int_{\omega_0}^{\omega_0+2\pi} e^{-j\omega(n-m)}\,d\omega = \begin{cases} 2\pi, & n = m \\ 0, & n \neq m \end{cases}$

Since $\forall\, n \neq m$ its 0 and only at one point $2\pi \Rightarrow$ it collapses to 1.

$$\int_{\omega_0}^{\omega_0+2\pi} F(e^{j\omega})\, e^{j\omega m}\,d\omega = f(m) \times 2\pi$$

$$\Rightarrow \boxed{f(n) = \frac{1}{2\pi} \int_{\omega_0}^{\omega_0+2\pi} F(e^{j\omega})\, e^{j\omega n}\,d\omega} \rightarrow \text{Synthesis eq}^{n}.$$

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Case I :-

$$f(n) = \frac{1}{2\pi}\int_{0}^{2\pi} F(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad (\omega_0 = 0)$$

Case 2 :-

$$\omega_0 = -\pi \Rightarrow f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})\, e^{j\omega n}\, d\omega$$

DTFT pair

$$\boxed{\begin{aligned} F(e^{j\omega}) &= DTFT\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n)\, e^{-j\omega n} \\ f(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})\, e^{j\omega n}\, d\omega \end{aligned}}$$

Q.) Find DTFT of :-

1) $x(n) = a^n u(n)$

$X(e^{j\omega}) = ??$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$\boxed{X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}}$$

$$X(e^{j\omega}) = X_R(e^{j\omega}) + j X_I(e^{j\omega})$$

$$|X(e^{j\omega})| = \sqrt{X_R^2 + X_I^2}$$

$$\angle X = \tan^{-1}\left(\frac{X_I}{X_R}\right)$$

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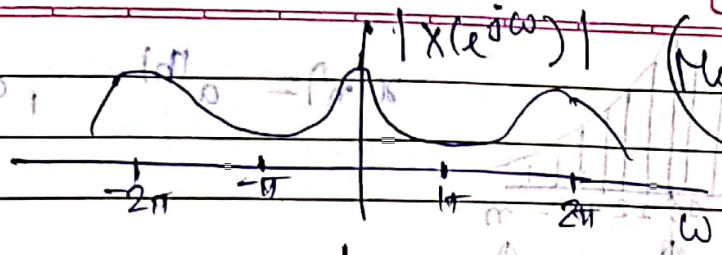
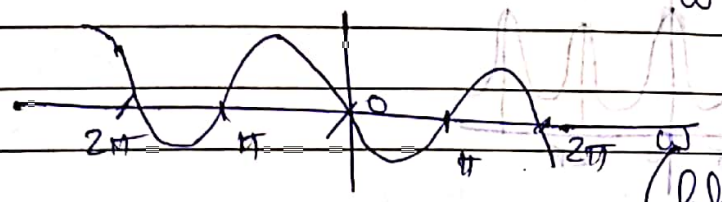
All these plots are for $a>0$



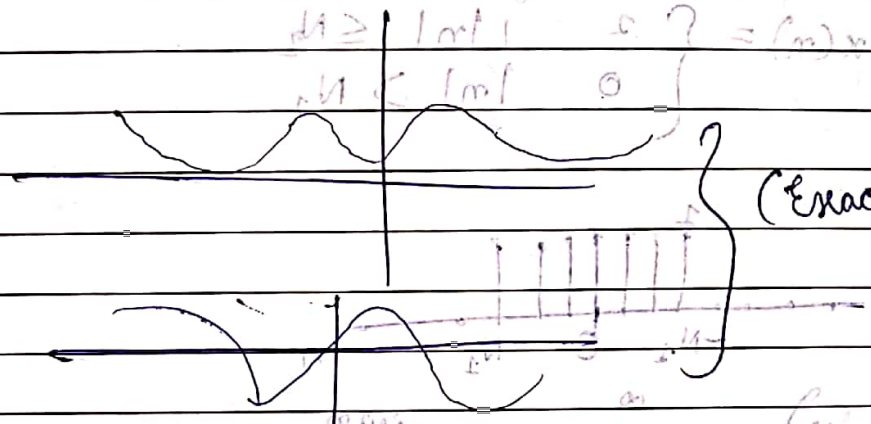
(Magnitude Spectrum)



$\measuredangle X(e^{j\omega})$

(Phase Spectrum)

If $a<0 \Rightarrow$



(Exactly reversed of above)

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Problem 2:- Find DTFT

$$x(t) = a^{|n|}, \quad |a|<1, \quad X(e^{j\omega}) = ?$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$|n| = \begin{cases} n, & n \geq 0 \\ -n, & n<0 \end{cases}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

$$= \sum_{m=\infty}^{1} a^{m} e^{j\omega m} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

$-n = m$

m

$$\cancel{= \frac{1}{1-(ae^{j\omega})} + \frac{1}{1-ae^{-j\omega}}}$$

$$= \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} = \frac{1-a^2}{1-2a\cos\omega+a^2}$$

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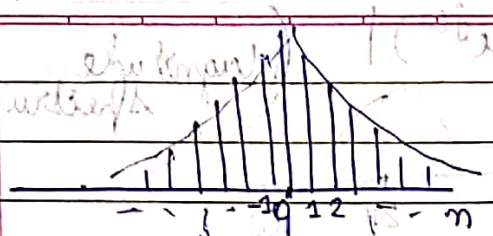
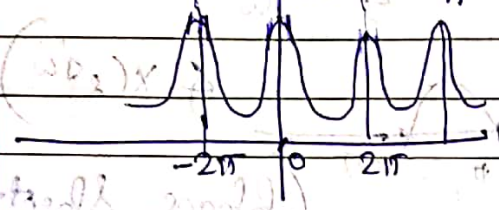
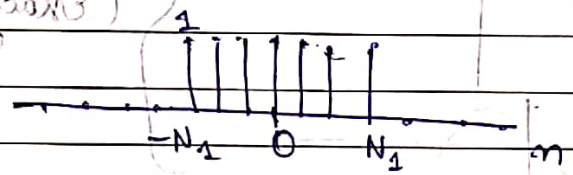


$x(n) = a^{|n|}, \quad a<1$



Problem 3:- Find DTFT of

$$x(n) = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases}$$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

$$= \sum_{n=-N_1}^{-1} e^{-j\omega n} + \sum_{n=0}^{N_1} e^{-j\omega n}$$

$$\cancel{= \sum_{m=-N_1}^{1} e^{-j\omega}}$$

$\Rightarrow$ Let $m = n + N_1$

$$= \sum_{m=0}^{2N_1} e^{-j\omega(m - N_1)}$$

$$= e^{-j\omega N_1} \left\{ \sum_{m=0}^{2N_1} e^{-j\omega m} \right\}$$

$$= e^{-j\omega N_1} \frac{1\left(e^{-j\omega}\right)^{2N_1+1} - 1}{\left(e^{-j\omega} - 1\right)}$$

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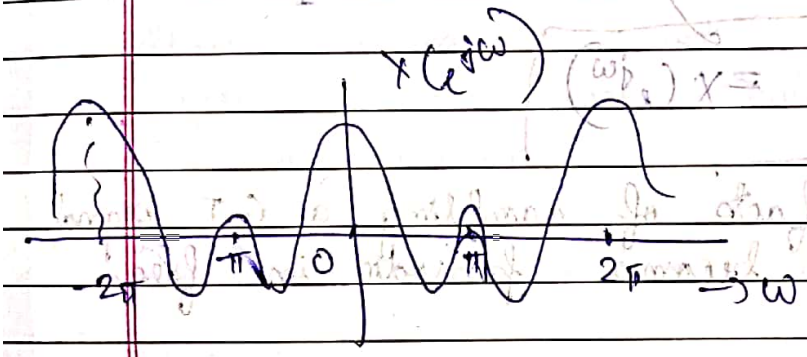
$$= e^{j\omega N_1}\left(\frac{\left(e^{-j\omega}\right)^{2N_1+1} - 1}{e^{-j\omega} - 1}\right)$$

$$\cancel{= \left(e^{-j\omega}\right)^{N+1} - 1 \; e^{-j\omega N_1}}$$

$$= \frac{\left(e^{-j\omega}\right)^{N_1+1} - e^{-j\omega N_1}}{\left(2j\right)}\left(\frac{2j}{e^{-j\omega} - 1}\right)$$

$$=$$

Answer is $\dfrac{\sin\left[\omega\left(N_1+\frac{1}{2}\right)\right]}{\sin\left(\frac{\omega}{2}\right)}$



Convergence issue for DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

DTFT expression will converge if discrete time signal $x(t)$ is absolutely summable or it has finite energy.

$x(t) \in L^2(R)$

$$\sum_{\langle n \rangle} |x(n)| < +\infty$$

$$\sum_{\langle n \rangle} |x(n)|^2 < +\infty$$

Such signals are called belonging to $\ell^2(z)$ (space of finite energy signals) → z means discrete

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Section 5.2 (H.W)

Properties of DTFT

1) Periodicity (in freq domain)

$$X(e^{j\omega}) = \sum x(n) e^{-j\omega n}$$

$\omega \to \omega + 2\pi$

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+2\pi)n}$$

$$= \sum x(n) e^{-j\omega n} \underbrace{e^{-j2\pi n}}_{1}$$

$$\boxed{X(e^{j(\omega+2\pi)}) = X(e^{j\omega})}$$

→ One of the artifacts of sampling a C-T signal, is that signal becomes periodic in freq domain.

2) $x(n) * h(n) \xleftrightarrow{DTFT} X(e^{j\omega}) H(e^{j\omega})$ (∵ Convolution theorem)

$$x(n) \longrightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \longrightarrow y(n)$$

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

3) Conjugation and Symmetry

If $x(n)$ has DTFT pair, i.e.,

$$x(n) \xleftrightarrow{F(\cdot)} X(e^{j\omega})$$

then $x^*(n) \xleftrightarrow{F(\cdot)} X^*(e^{-j\omega})$

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mag → even
phase → odd

If $x(n)$ = real ⇒ real part of F-T is even function and imaginary part of " " odd function.

Development of Hillbert Transform

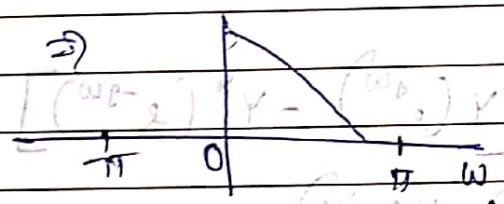
Case II (Discrete time signal) (∵ Case 1 in Chp 4)

↳ Motivation:- Analytic signals

Let $y(n)$ is analytic signal, then DTFT $Y(e^{j\omega})$

$Y(e^{j\omega}) = 0$ for -ve freq

⇒



(Causal in freq domain)

Since spectrum of analytic signal is asymmetric in freq domain, the analytic signal $y(n)$ is complex in time domain.

∴ Let $y(n) = x(n) + j\hat{x}(n)$ — ①

(∵ $\hat{x}(n)$ is Hillbert transform)

$= y_R(n) + j\, y_I(n)$

$x(n) \longrightarrow$ $\boxed{\text{D-T Hillbert Transform } h(n) \downarrow H(e^{j\omega})} \longrightarrow \hat{x}(n)$

$Y(e^{j\omega}) = X(e^{j\omega}) + j\,\hat{X}(e^{j\omega})$ — ②

Now taking complex conjugate on both sides,

$(u+v)^* = u^* + v^*$

∴ $Y^*(e^{j\omega}) = X^*(e^{j\omega}) - j\,\hat{X}^*(e^{j\omega})$ — ③

To understand symmetry

$\omega \to -\omega$

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$$\therefore Y^*(e^{-j\omega}) = X^*(e^{-j\omega}) - j\hat{X}^*(e^{-j\omega}) \quad \text{--- (4)}$$

Since, $x(n)$ is a real signal. By using conjugation and symmetry,

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

$\therefore$ eq^n (4) $\Rightarrow$

$$Y^*(e^{-j\omega}) = X(e^{j\omega}) - j\hat{X}(e^{j\omega}) \quad \text{--- (5)}$$

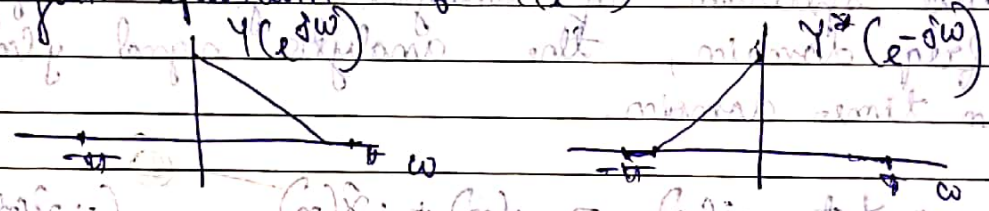
Now, (2) + (5) gives

$$X(e^{j\omega}) = \frac{1}{2}\left[Y(e^{j\omega}) + Y^*(e^{-j\omega})\right]$$

(2) − (5) gives,

$$j\hat{X}(e^{j\omega}) = \frac{1}{2}\left[Y(e^{j\omega}) - Y^*(e^{-j\omega})\right]$$

Again spectrum of $Y(e^{j\omega})$



(We write $-\pi$ to π as it is periodic)

Case 1:- $\omega \in [0, \pi]$

$$Y^*(e^{-j\omega}) = 0$$

$$\Rightarrow j\hat{X}(e^{j\omega}) = X(e^{j\omega})$$

$$\therefore \boxed{\hat{X}(e^{j\omega}) = (-j)X(e^{j\omega})}$$

Case 2:- $\omega \in [0, -\pi]$

$$Y(e^{j\omega}) = 0$$

(Easy)

$$\boxed{\hat{X}(e^{j\omega}) = jX(e^{j\omega})}$$

Now, combining

$$\hat{X}(e^{j\omega}) = \begin{cases} -j, & \omega \in [0, \pi] \\ j, & \omega \in [0, -\pi] \end{cases}$$

$$\hat{X}(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

Now, $H(e^{j\omega}) = \begin{cases} -j, & \omega \in [0, \pi] \\ j, & \omega \in [0, -\pi] \end{cases}$ | $H(\omega) = \begin{cases} -j & \omega > 0 \\ +j & \omega < 0 \end{cases}$

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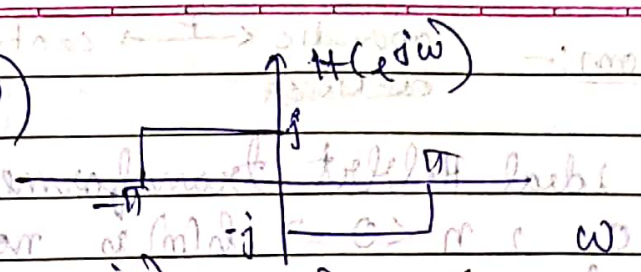
$\therefore H(e^{j\omega})$



(frequency response of D-T Hilbert transformer)

Now given $H(e^{j\omega})$, $h(n) \Rightarrow$ Inverse DTFT,

$$h(n) = F^{-1}\{H(e^{j\omega})\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$\therefore h(n) = \frac{1}{2\pi}\left\{\int_{-\pi}^{0} H(e^{j\omega})\, e^{j\omega n}\, d\omega + \int_{0}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega\right\}$$

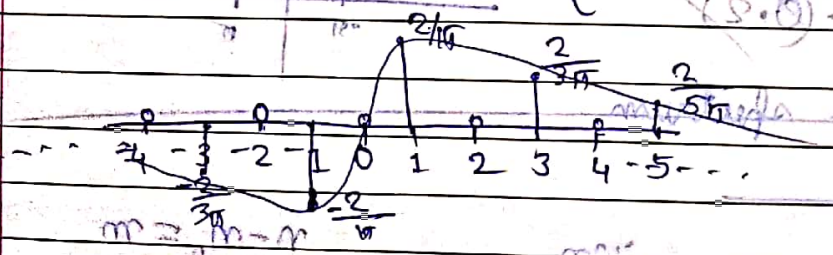
$$= \frac{1}{2\pi}\left\{\int_{-\pi}^{0} (j)\, e^{j\omega n}\, d\omega + \int_{0}^{\pi} (-j)\, e^{j\omega n}\, d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\left(\frac{\cancel{j}\, e^{j\omega n}}{\cancel{j}\, n}\right)_{-\pi}^{0} + \left(\frac{-\cancel{j}\, e^{j\omega n}}{-\cancel{j}\, n}\right)_{0}^{\pi}\right\}$$

$$= \boxed{\frac{1-\cos(\pi n)}{\pi n}} = \boxed{\frac{2\sin^2\left(\frac{\pi n}{2}\right)}{\pi n}}$$

$h(n)$

$$h(n) = \frac{2\sin^2\left(\frac{\pi n}{2}\right)}{\pi n} = \begin{cases} 0 & , n \in \text{even} \\ \frac{2}{\pi n} & , n \in \text{odd} \end{cases}$$



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Discrete $\xleftrightarrow{\mathcal{F}}$ aperiodic

Periodic $\xleftrightarrow{\mathcal{F}}$ Discrete

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aperiodic $\xleftrightarrow{\mathcal{F}}$ contin

Observations:-

1) $h(n)$ for ideal Hilbert transformer, is IIR
2) $h(n) \neq 0$, $n < 0 \Rightarrow h(n)$ is non-causal
3) In practical environment, we can multiply by window to get forced FIR, OR shift it to get causal.

Two Properties (basaki are simple algebraic manipulation)

1) Time Scaling property $\rightarrow$ (377-379)
2) Duality property $\rightarrow$ (5.7) (390-396)

In CTFT,

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

But doing so in discrete, we run into trouble.

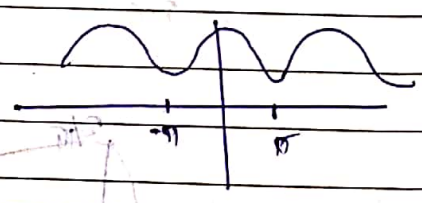
$\neq \mathcal{F}\{x(an)\}$ when (a is fraction an an problem)

Tutorial-10

1.] (a) $\sum_{n=-\infty}^{\infty} (0.2)^n u(n) e^{-j\omega n}$

$$\sum_{0}^{\infty} (0.2)^n e^{-j\omega n} = \left| \frac{1}{1-(0.2)e^{-j\omega}} \right|$$

2.] $\cancel{\Lambda_c \frac{1}{\sqrt{1+(0.2)^2}}}$



phase spectrum

2.] 2nd sorry

3.] $\sum_{n=-\infty}^{\infty} x(n-n_0) e^{-j\omega n}$

$n - n_0 = m$

$n = n_0 + m$

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time shifting gives phase shifting in freq dom

$$\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega(n_0+n)} = e^{-j\omega n_0} X(e^{j\omega})$$

1ly, 2nd ly duality

4.) $\sum_{n=-\infty}^{\infty} x(n)\, x^*(n)\, e$

5.) $y(n) = x(n) * h(n)$

$$\therefore Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} [x(n) * h(n)]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, h(n-k)\right] e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} h(n-k)\, e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, e^{-j\omega k} \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

$H(e^{j\omega})$

$$= X(e^{j\omega}) H(e^{j\omega})$$

6.) $y(n) = x(n) h(n)$ $\qquad Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n}$

$$\sum_{n=-\infty}^{\infty} x(n) h(n)\, e^{-j\omega n}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi} X(e^{j\theta})\, e^{j\theta n}\, d\theta\; h(n)\, e^{-j\omega n}$$

$$\frac{1}{2\pi} \int X(e^{j\theta})\, e^{j\theta n}\, d\theta \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

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Use frequency shifting,

7] $y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = 2x(n)$

$$Y(e^{j\omega}) - \frac{3}{4} Y(e^{j\omega}) e^{-j\omega} + \frac{1}{8} Y(e^{j\omega}) e^{-j\omega(2)} = 2X(e^{j\omega})$$

$$\therefore Y(e^{j\omega}) \left[1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{-j\omega(2)} \right] = 2X(e^{j\omega})$$

$$\therefore \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{-j\omega(2)}} = H(e^{j\omega})$$

Put $e^{-j\omega} = x \Rightarrow \frac{2}{1 - \frac{3}{4}x + \frac{1}{8}x^2}$

$$= \frac{16}{(x-2)(x-4)}$$

$$= \frac{16}{2} \left[\frac{1}{(x-4)} - \frac{1}{(x-2)} \right]$$

$$\Rightarrow \frac{8}{(x-4)} - \frac{8}{(x-2)}$$

$$= \frac{8}{-4\left(1 - \frac{x}{4}\right)} + \frac{8}{-2\left(1 - \frac{x}{2}\right)} = \frac{-2}{\left(1 - \frac{x}{4}\right)} + \frac{-4}{\left(1 - \frac{x}{2}\right)}$$

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Friday

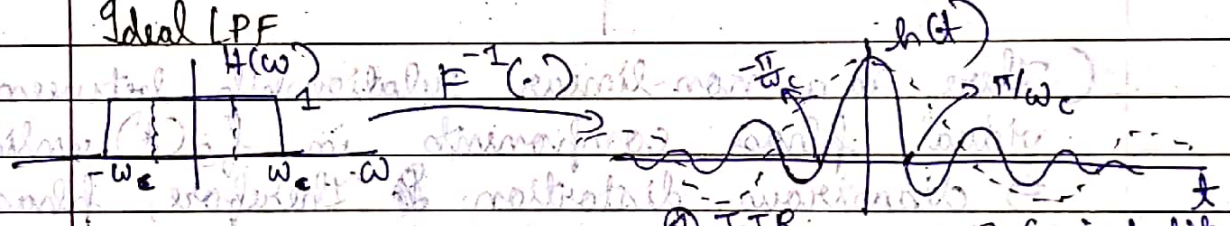
Chapter-6: Time and frequency Characterization of Signals and System

(important for application of signals & systems)
(to study significantly phase of Fourier Transform)

On the phase of the Fourier Transform

Motivation :-
1) To analyze Fourier Transform Phase
2) To understand Time-frequency trade offs for design of LTI filters.

Ideal LPF



① IIR
② Non-causal } (just like Hilbert transformer)

What is the consequence of change of a parameter of frequency response of LTI filter into corresponding impulse response?

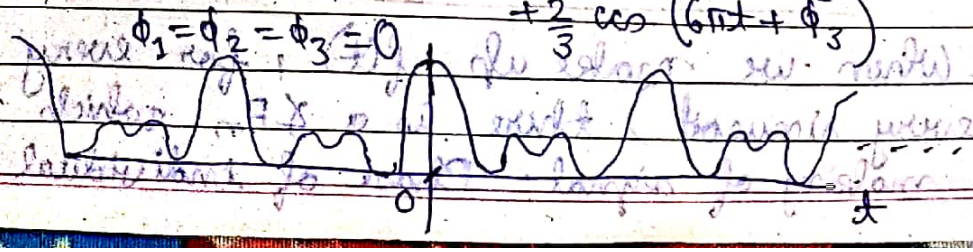
Phase of Fourier Transform

(What could be contribution of phase in making-up of the signal?)

(Chap 3 me kiya tha kido)

① $f(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + 1\cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3)$

$\phi_1 = \phi_2 = \phi_3 = 0$



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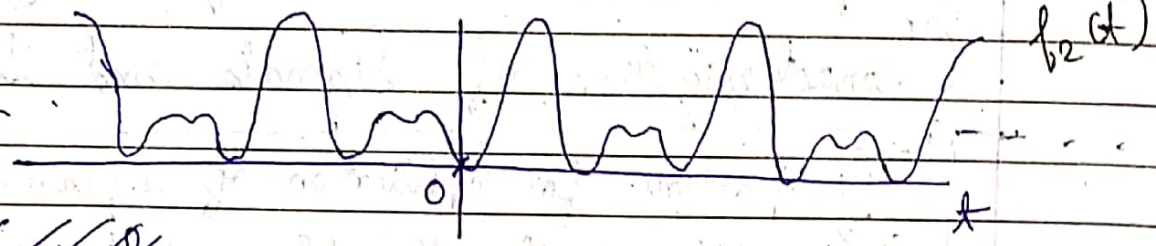
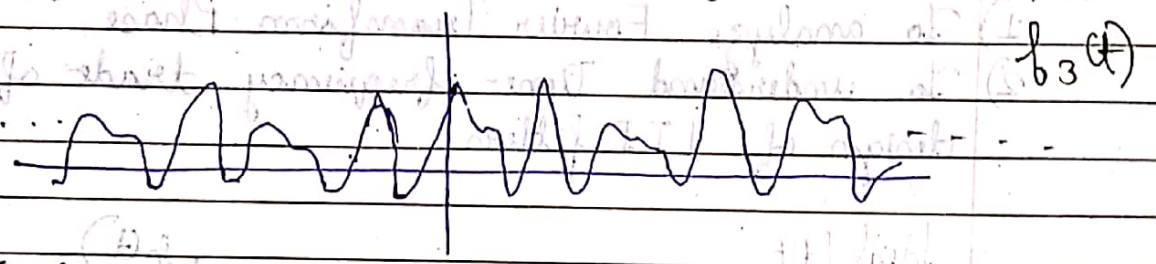
(2) Let $\phi_1 = 4$ rad, $\phi_2 = 8$ rad, $\phi_3 = 12$ rad



~~If we ch~~

$f_2(t)$ is shifted version of $f_1(t)$, if we choose $\phi_1 = 4, \phi_2 = 8, \phi_3 = 12$.

(3) $\phi_1 = 6$ rad, $\phi_2 = -2.7$ rad, $\phi_3 = 0.93$ rad



(There is a non-linear relationship between individual phase components in $f_3(t)$ unlike $f_2(t)$ ⇒ disastrous distortion. Therefore, phase of the signals contribute SIGNIFICANTLY to the shape of signal)

⇒ Let $f(t) =$ periodic

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$= \sum |F_n| e^{j\angle F_n} e^{jn\omega_0 t}$$ ($\because F_n$ is in general complex)

$$\therefore \boxed{f(t) = \sum_{n=-\infty}^{\infty} |F_n| e^{j(n\omega_0 t + \angle F_n)}}$$

∴ When we make up $f(t)$, for every n (i.e. for every sinusoid) there is a $\angle F_n$ which contributes to making of signal. Phase of individual matters and

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it is not just $|F_n|$ which is important.

→ Now, if $f(t)$ = aperiodic

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{\omega=-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega$$

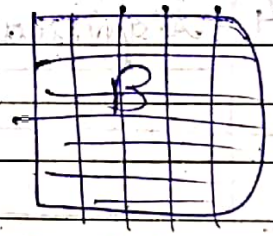
$$= \frac{1}{2\pi}\left\{\lim_{\omega_0 \to 0} \sum_{n=-\infty}^{\infty} F(\omega_n) e^{j\omega_n t} \omega_0\right\}$$

(∵ using fundamental th^m of calculus)

$$= \lim_{\omega_0 \to 0} \sum_{n=-\infty}^{\infty} |F(\omega_n)| e^{j\measuredangle F(\omega_n)} e^{j\omega_n (t)} \left(\frac{\omega_0}{2\pi}\right)$$

(2) Thanks to prof Oppenheim, we got importance of phase in image processing.

Image signal → 2-D
$f(t_1, t_2)$



$$F(\omega_1, \omega_2) = \mathcal{F}\{f(t_1, t_2)\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1, t_2) e^{-j\omega_1 t_1} e^{-j\omega_2 t_2} dt_1 dt_2$$

(∵ Fourier transform is for n-D)

6.2 images :- The phase is more imp than mag spectrum.

(3) Significance of phase in speech and audio applications :-

'Ohm' → known for research in Acoustics, more than electrics.
→ said, "Human Ear is phase-deaf".

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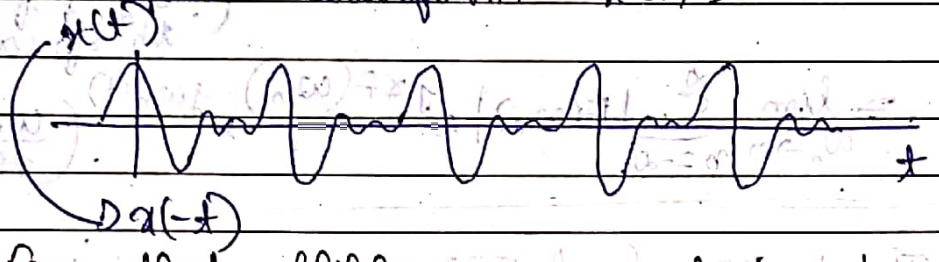
Friday

→ For 100 years, there was a stop on research in phase aspect of speech.

↳ But Indian institute researchers from TIFR (Tata Institute of Fundamental research) changed this. Prof Kuldip Paliwal, Prof Saberi (MIT)
(APSIPA ASC 2018, NOV 12-15)

Let speech waveform $x(t)$:-



Prove that flipping a signal in time-domain corresponds to 180° phase shift.

4) Significance of phase in Ocean Acoustics

6.2 Magnitude - Phase representation

C-T System | D-T system

$x(t) \longrightarrow \boxed{\text{LTI filter}} \longrightarrow y(t)$

$h(t) \rightarrow H(\omega)$

$y(t) = x(t) * h(t)$

$Y(\omega) = X(\omega) H(\omega)$

$|Y(\omega)| e^{j\measuredangle Y(\omega)}$

$= |X(\omega)| e^{j\measuredangle X(\omega)} \times |H(\omega)| e^{j\measuredangle H(\omega)}$

mag of $H(\omega)$ called gain of LTI filter

$$\boxed{\begin{aligned} |Y(\omega)| &= |X(\omega)| |H(\omega)| \quad (1) \\ \measuredangle Y(\omega) &= \measuredangle X(\omega) + \measuredangle H(\omega) \quad (2) \end{aligned}}$$

called phase shift of LTI filter

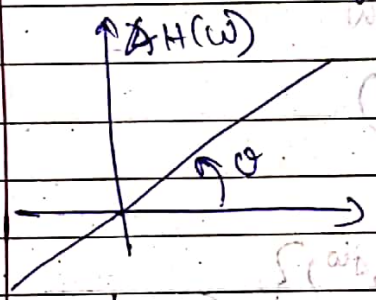
Friday

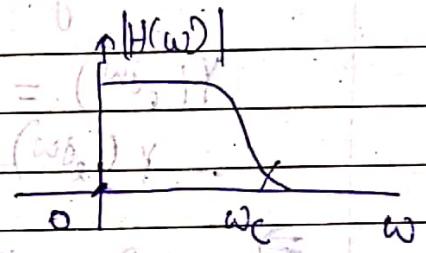
Digital Signal Processing (DSP)

Design of digital filter (LTI)
(50% of course) → Design of filter ⇒ design of impulse response

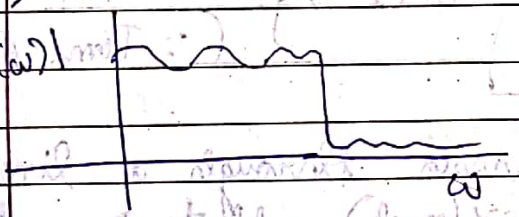
FIR filters → discrete time-dom
(only digital)
↓
Gift of DSP field (not heard in analog domain)
↓
Linear Phase filter

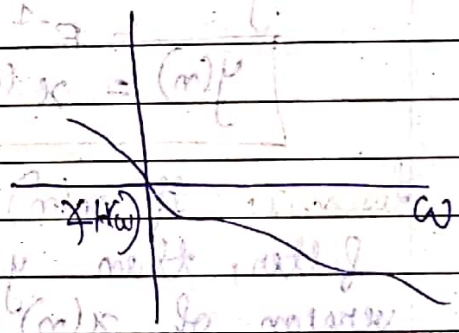
IIR filter (designed in analog & mapped to digital later)
29/10/18
(Both analog & digital)
↓
Flat magnitude Response





Non-flat magnitude response





Effect of linear phase filter on the filtered signal

Input signal $x(n)$ → [Linear Phase filter $h(n) \rightarrow H(e^{j\omega})$] → $y(n)$ → filtered signal

$$H(e^{j\omega}) = \underbrace{|H(e^{j\omega})|}_{} \, e^{j\measuredangle H(e^{j\omega})}$$

Let us consider this as 1, i.e., only as passband & mag is normalized, because we aren't concerned about mag in this chapter.

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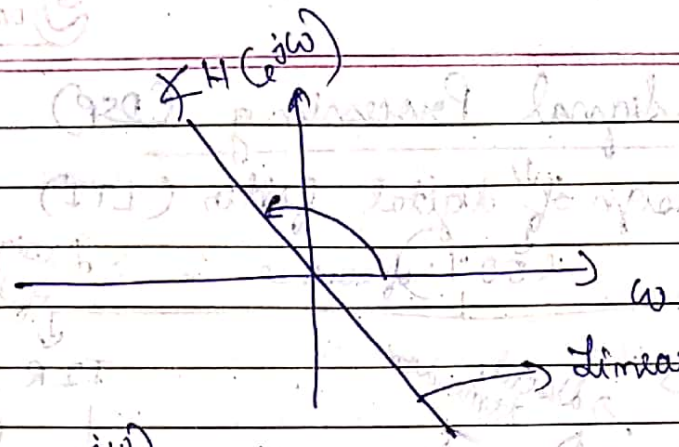


$\measuredangle H(e^{j\omega}) = -K \cdot \omega$

Now, since LTI, $y(n) = x(n) * h(n)$

By Convolution theorem, $Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$

(Substituting $H(e^{j\omega})$ from last page bottom)

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot 1 \cdot e^{-jK\omega}$$

$$\therefore Y(e^{j\omega}) = e^{-jK\omega} X(e^{j\omega})$$

$$\Rightarrow y(n) = F^{-1}\{Y(e^{j\omega})\}$$

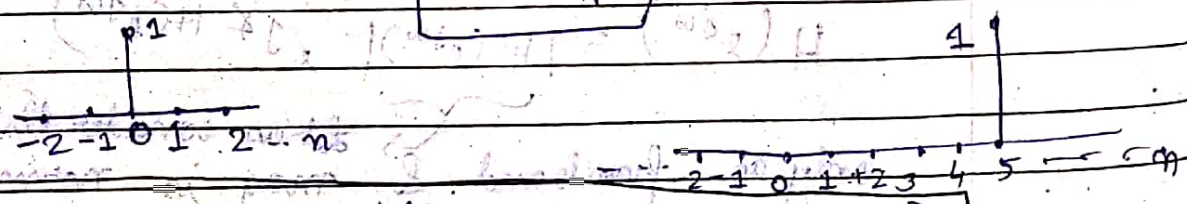
$$= F^{-1}\{e^{-jK\omega} X(e^{j\omega})\}$$

$$\boxed{y(n) = x(n-k)}$$ (∵ Time shift)

Lesson 1:- If $x(n)$ is passed through a linear phase filter, then $y(n) = x(n-k)$, that is, delayed version of $x(n)$. (Thus, we get exactly same replica, only with some delay)

Lesson 2:- The amount of delay, i.e., 'k' is controlled by shape of phase spectrum.

$x(n) = \delta(n) \longrightarrow$ [Linear Phase, K=5 samples] $\longrightarrow y(n) = \delta(n-5)$



(Amount of delay = Slope of phase function)

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Monday

$\measuredangle H(e^{j\omega}) = -k\omega$

(to get slope k

$\frac{d}{d\omega} \measuredangle H(e^{j\omega}) = \frac{d}{d\omega}(-k\omega)$

$= (-k) \times 1$

$\therefore \boxed{-\frac{d}{d\omega} \measuredangle H(e^{j\omega}) = k \text{ samples}}$

Note ki yahan difference eqn nai hai, kyuki ik in t-dom, it is discrete, on FT, it is periodic & cont & hence derivative & not difference eqn

$\measuredangle H(\omega) = -t_0\omega$

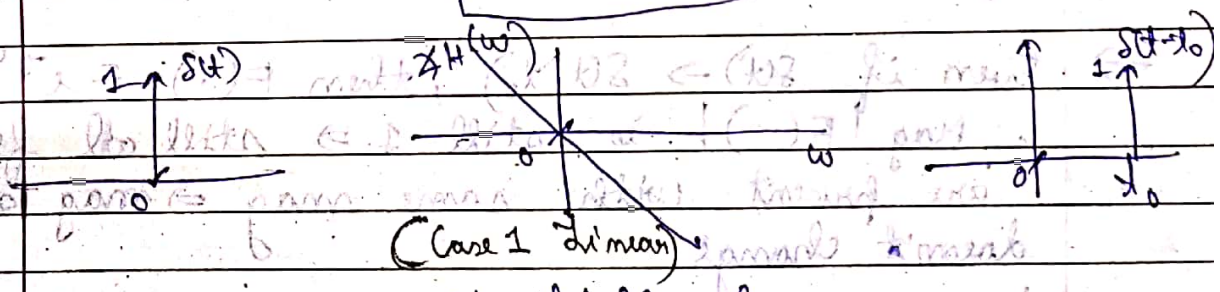
$\boxed{-\frac{d}{d\omega} \measuredangle H(\omega) = t_0 \text{ seconds}}$

(negative derivative of phase function wrt frequency is called group delay function) $= \tau(\omega) = -\frac{d}{d\omega} \measuredangle H(\omega)$

Why $\tau(\omega)$ is a called Group Delay function? (Justification)

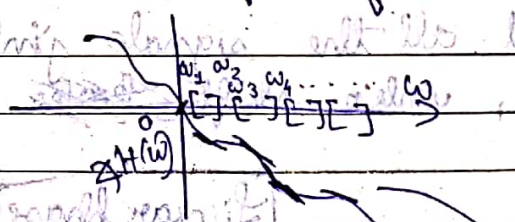
imp word

$x(t) = \delta(t) \longrightarrow \boxed{\begin{array}{c}\text{Linear Phase} \\ h(t) \to H(\omega)\end{array}} \longrightarrow y(t) = \delta(t - t_0)$



(Case 1 Linear)

Now let phase fxn:-



(Case 2 Nonlinear phase)

In this way we made linear approximations for certain ranges of frequencies

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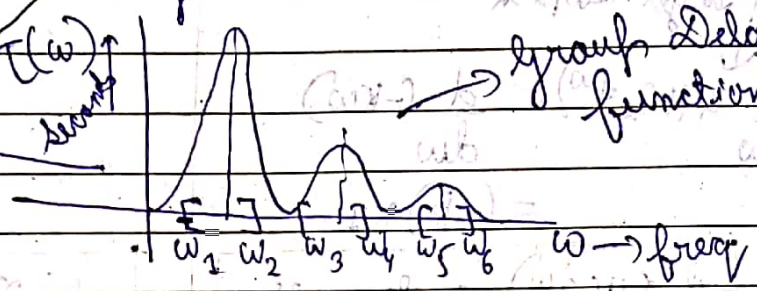
Taking derivative and negative

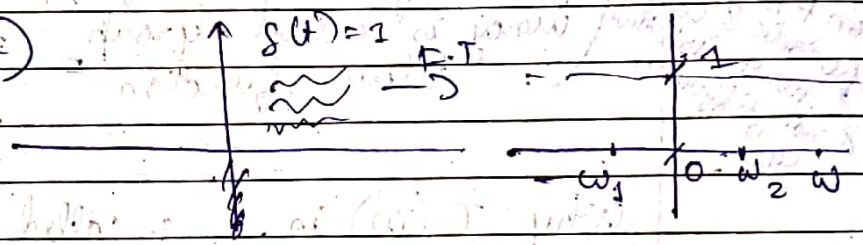
height represents amount of delay the localized freq response experience



→ Group Delay function

<u>Note</u>:- for linear phase case (Case 1), $\tau(\omega)$ is constant, but even if it is t_0 seconds (constant), it is still a function.

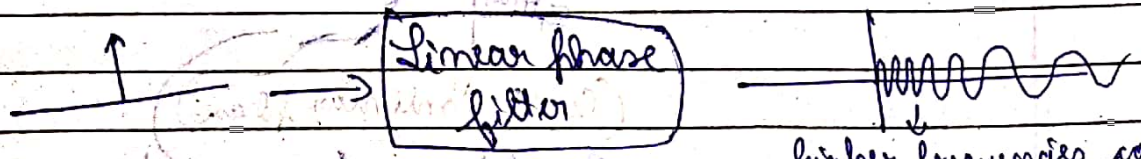
Now, $x(t) = \delta(t)$



→ Thus, taking all possible sinusoids of infinitesimally ~~small~~ closely spaced frequencies and dubbing them together, we get $\delta(t)$. When these infinite sinusoids are place one below other, they coherently form $\delta(t)$.

→ Even if $\delta(t) \rightarrow \delta(t-t_0)$, then $F(\cdot) = e^{-j\omega t_0}$ (1)
∴ Mag $|F(\cdot)|$ is still 1 ⇒ still all frequencies are present with same mag ⇒ mag spectrum doesn't change.

↳ But all the signals jinko club karne ~~ko~~ set $\delta(t)$ mila, unko ~~shift~~ ~~kaise~~ club kiya tab $\delta(t-t_0)$ mila



higher frequencies come out first & lower frequencies come out later, as higher frequencies have less $\tau(\omega)$.

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Monday

Clearly, different groups of frequencies experience different amounts of delay. Hence $\tau(\omega)$ is called group delay fn. Hence $\tau(\omega)$ guides infinite number of sinusoids to wait for the filter. The relative group of frequencies are sent out.

<u>Problem:-</u> Consider the impulse response of an all-pass system, with a group delay that varies with frequency. It is given by

$$H(j\omega) = \prod_{i=1}^{3} H_i(j\omega)$$

where each function $H_i(j\omega) = \dfrac{1 + \left(\frac{j\omega}{\omega_i}\right)^2 - 2j\xi_i(\omega/\omega_i)}{1 + \left(\frac{j\omega}{\omega_i}\right)^2 + 2j\xi_i(\omega/\omega_i)}$

Here ω is in analog domain,

$\omega_i = 2\pi f_i$

$f_1 = 50\,Hz$, $f_2 = 150\,Hz$, $f_3 = 300\,Hz$

(It is like $H(j\omega) = \dfrac{a+jb}{a-jb} \rightarrow |H(j\omega)| = 1 \Rightarrow$ This is also hence an all pass, just like Hilbert transform. Hence, Hilbert transform is not the only all pass filter)

$\cancel{\measuredangle H_i \measuredangle H_i(j\omega)}$

$$\measuredangle H(j\omega) = \sum_{i=1}^{3} \measuredangle H_i(j\omega)$$

$$= -2\tan^{-1}\left[\frac{2\xi(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right]$$

(∴ phase of numerator & phase of denominator in the eqn)

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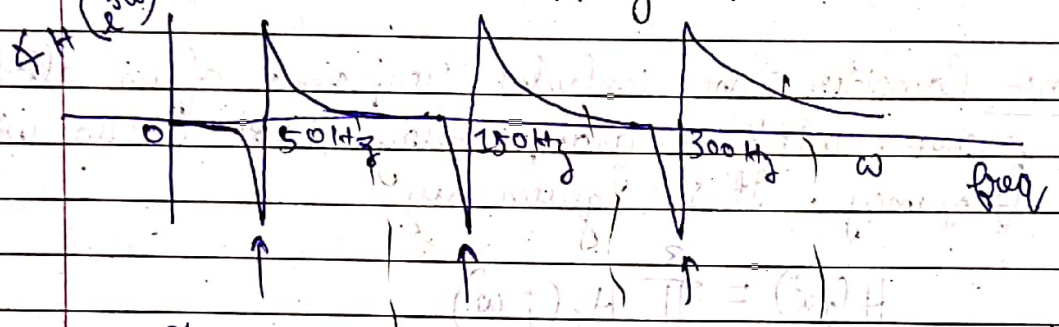
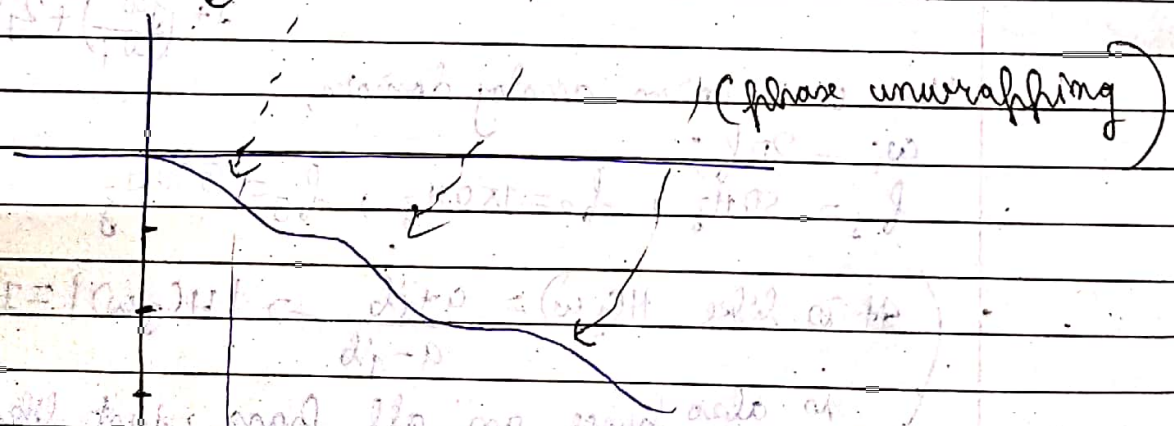
Since $\tan(2\pi+\theta) = \tan\theta \Rightarrow$ in $(0, 2\pi)$ we have a behaviour, while in $(2\pi, 4\pi)$ it is same, ie, $(2\pi, 4\pi)$ gets wrapped onto 0 to 2π, similarly $(-4\pi, -2\pi)$ & $(4\pi, 6\pi)$ and so on.

— This creates a wrapping problem.

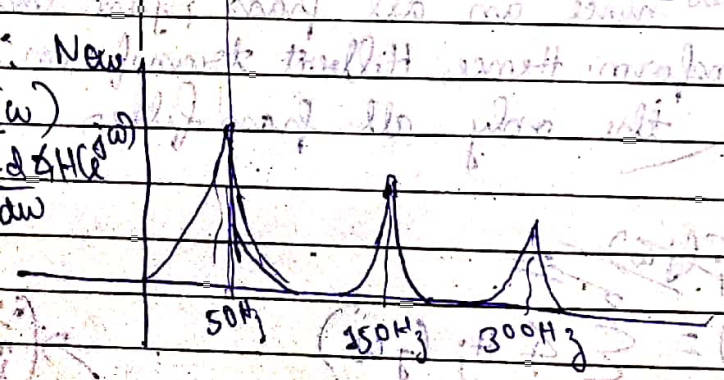


To get group delay, differentiate $\measuredangle H(e^{j\omega})$. But it is discont. We do unwrapping.



∴ New

$\tau(\omega) = -\frac{d\measuredangle H(e^{j\omega})}{d\omega}$



($\tau(\omega)$) Study example 6.2, pg 432-436

Monday

2) How group delay is used to assess communication networks?

Applications of Group delay function

1) Music Algorithm
   ↳ Microphone arrays

2) VOCODER → ~~Straig~~ STRAIGHT VOCODER
   ↳ H. Kawahara

3) Speech Application
   ↳ ASR, ASV

Log-Magnitude and Bode Plots

$$x(t) \longrightarrow \boxed{\begin{matrix} LTI \\ h(t) \end{matrix}} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

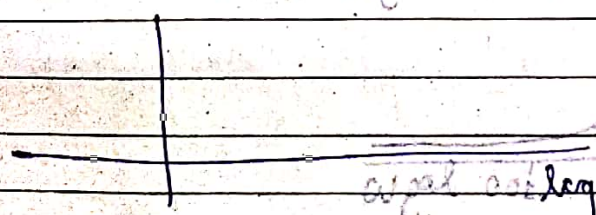
$$Y(j\omega) = X(j\omega)\, H(j\omega)$$

$$|Y(j\omega)| = |X(j\omega)|\,|H(j\omega)| \quad \text{—— (1)}$$

$$\angle Y(j\omega) = \angle X(j\omega) + \angle H(j\omega) \quad \text{—— (2)}$$

To convert (1), in additive form,

$$\log|Y(j\omega)| = \log|X(j\omega)| + \log|H(j\omega)|$$



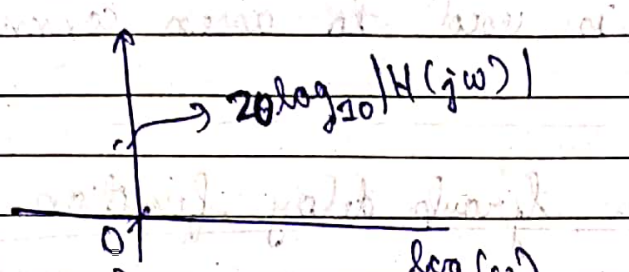
$\log(\omega)$ ⇒ Bode plots.

P.T.O

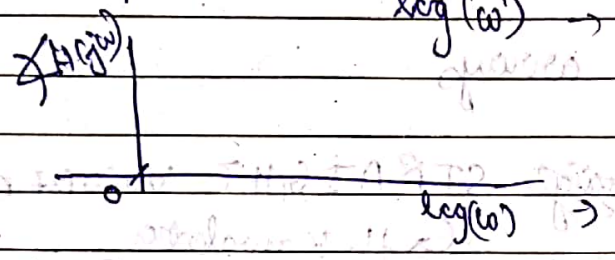
Wednesday

Bode Plot

(log is taken to convert multiplicative nature to additive nature)



(read pg 437 footnotes)



$$Y(\omega) = X(\omega)\,H(\omega) \Rightarrow |H(\omega)| = \frac{|Y(\omega)|}{|X(\omega)|}$$

$\therefore$ Now squaring, $|H(\omega)|^2 = \dfrac{|Y(\omega)|^2}{|X(\omega)|^2}$

Now, we know that $|Y(\omega)|^2 = \dfrac{|Y(\omega)|^2}{1}$ = energy per unit frequency = power

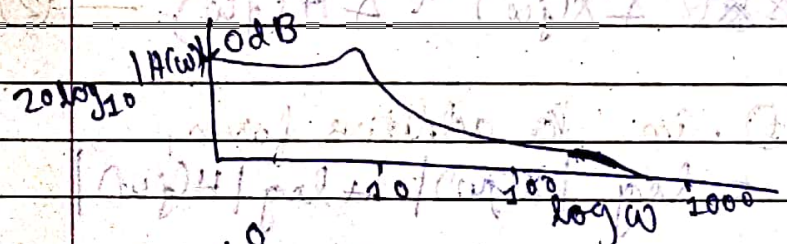
|||ly $|X(\omega)|^2$ = power

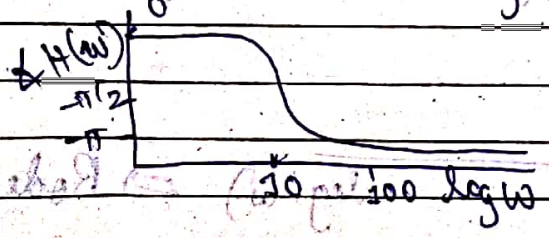
$\therefore$ $\dfrac{|Y(\omega)|^2}{|X(\omega)|^2}$ is the power ratio

(Bell)

decibel $\rightarrow$ $10\log_{10}|H(\omega)|^2$

$= 20\log_{10}|H(\omega)|$

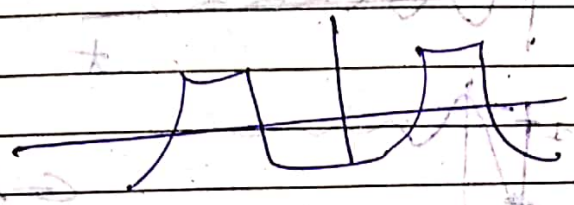




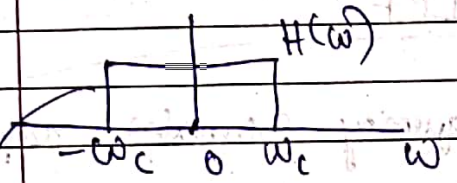
Now, will this type of plot be required for DTFT

Wednesday

→ Answer is NO. Bode plot is not used for DTFT



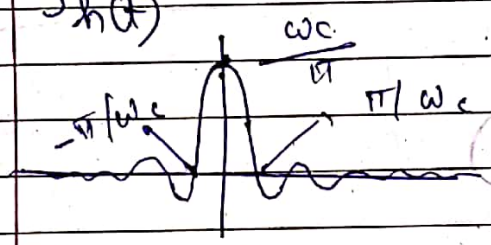
6.3 Time-domain properties for ideal frequency-selective filters



$$H(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

def^n of ideal LPF

→ h(t)



} Time-frequency trade-off in selecting the cut-off frequency.

As ω_c ↑ →

→ Ideal means brick wall characteristics ⇒ Almost zero of transition band.

features:- h(t)

1) IIR
2) Non-causal

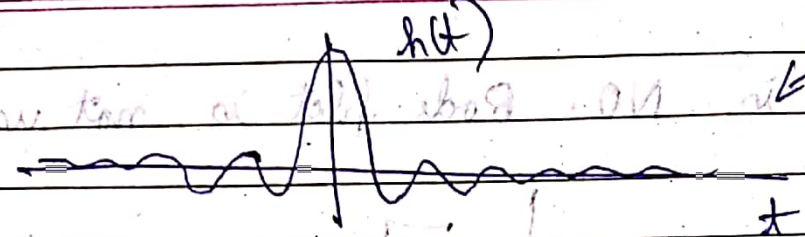
(Y-W) Study similar characteristics, if it is DTFT.

Now, for step response, we integrate impulse.

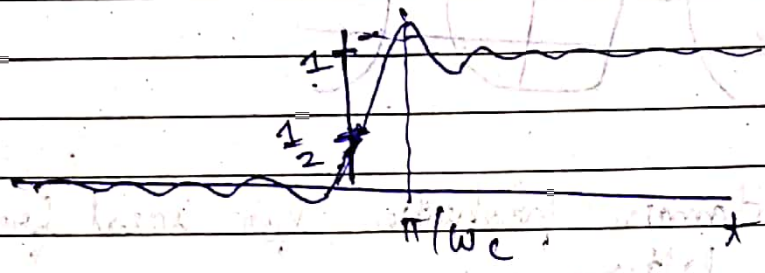
$$\therefore s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau = \text{Step response}$$

P.T.O

Wednesday



← impulse response of ideal LPF



← step response of ideal LPF

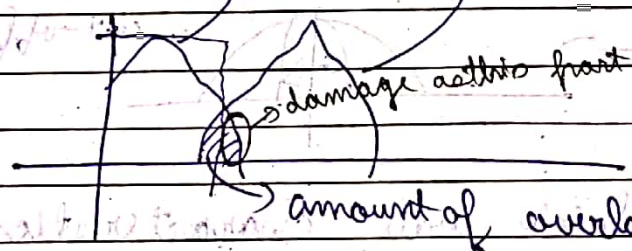
(H(ω) plot for ideal DT LPF

6.4 Time-domain and frequency domain aspects of Non-ideal filters

$$x(t) = x_1(t) + x_2(t)$$

$$\downarrow$$

$$X(\omega) = X_1(\omega) + X_2(\omega)$$



But it is good as almost completely recovered

But implementation is very costly as we need large number of networks of capacitors, inductors and resistors, to obtain steeper pass curve.

→ In practical situations, this kind of strictness is not usually required.

⇒ Instead of brick wall characteristics, we have some rippling as well as a transition band.

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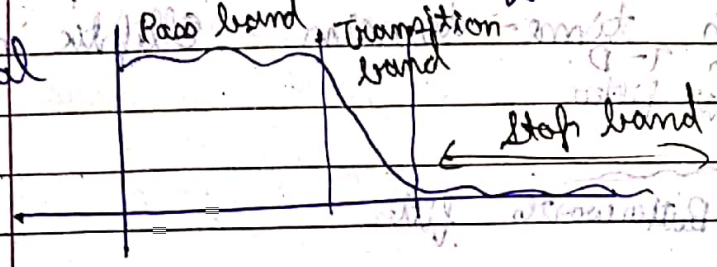
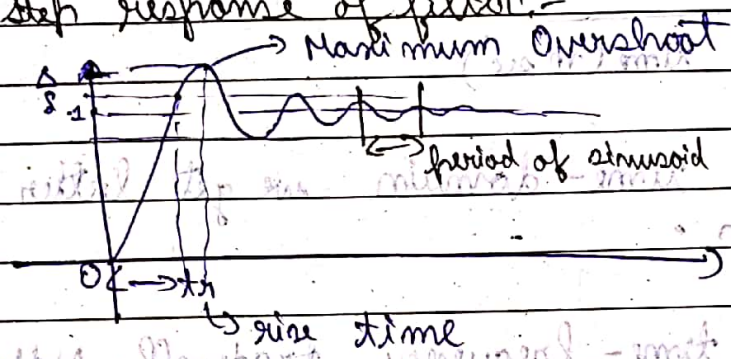
Wednesday

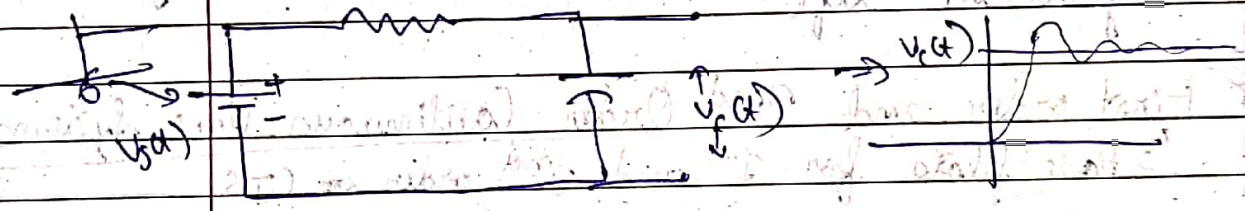
It is also cost effective.

Practical



⟹ Step response of filter:-

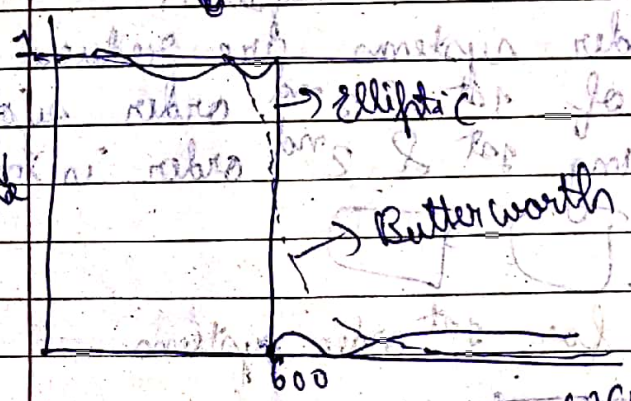




⟹ Now, in analog filter:

1) Butterworth Filter

2) Elliptic filter



→ With respect to transition characteristics, Elliptic filter is closer to ideal than Butterworth.

→ Elliptic has steep curve.

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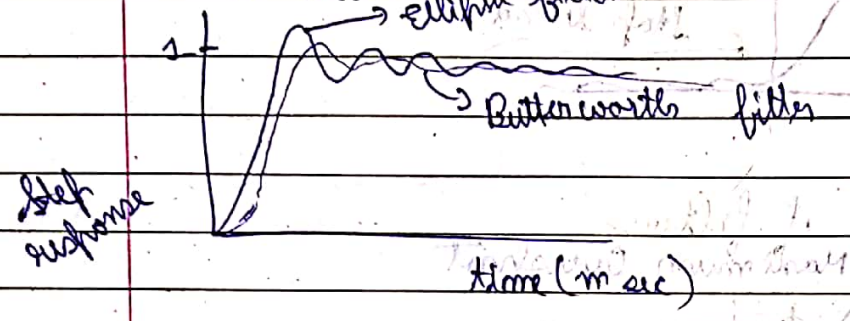
→ To settle down within limits, elliptic takes more time in time-domain. Elliptic is sluggish to settle in T-D.



→ Thus, in time-domain we get better case in Butterworth.

→ This was time-frequency trade-off. Nothing is all-good in life

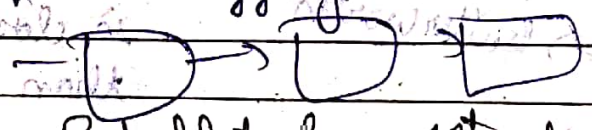
6.5 First order and 2nd Order Continuous Time Systems

↳ Bode plots for 1st and 2nd order CTS

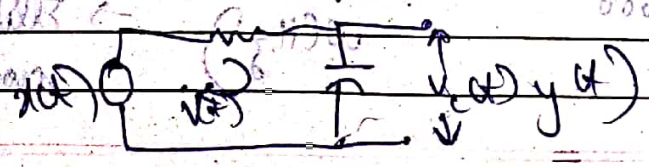
Higher Order
1) DSP
2) Electronics (Electrical Networks)
3) Control

Describe dynamics of the system using differential equation and order of eqn is order of system.

→ The higher-order systems are reduced to cascading of 1st & 2nd order systems. Therefore, analyzing 1st & 2nd order is important.



Case I :- Bode plot for 1st order systems



$$(RC)\frac{dy(t)}{dt} + y(t) = x(t)$$

Now, let $RC = \tau$

$$\therefore \tau \frac{dy(t)}{dt} + y(t) = x(t)$$

$$\therefore \tau F\left(\frac{dy(t)}{dt}\right) + F(y(t)) = F(x(t))$$ (∵ Taking FT on both sides)

$$\therefore \tau j\omega [Y(\omega)] + Y(\omega) = X(\omega)$$

$$\Rightarrow \boxed{H(\omega) = \frac{1}{1+\tau j\omega}}$$

$\therefore$ Impulse response $h(t) = F^{-1}\{H(\omega)\}$

$$= F^{-1}\left\{\frac{1}{1+j\omega\tau}\right\}$$

$$= F^{-1}\left\{\left(\frac{1}{\tau}\right) \times \frac{1}{j\omega + \frac{1}{\tau}}\right\}$$

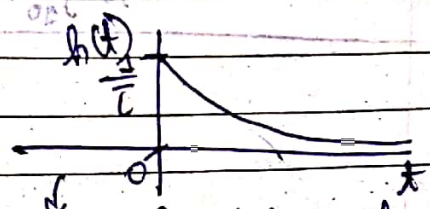
$$\boxed{h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)}$$

This is the impulse response of 1st order system. Any system and need not be RC circuit.

$\therefore$ Step response:

$$s(t) = \int_{\tau=-\infty}^{t} \frac{1}{\tau} e^{-\frac{t}{\tau}} u(\tau)\, d(\tau) \qquad s(t) = \left[1 - e^{-\frac{t}{\tau}}\right] u(t)$$

$$= \int_{\tau=0}^{t} \frac{1}{\tau} e^{-\frac{t}{\tau}}\, d\tau$$



(In general, step response)

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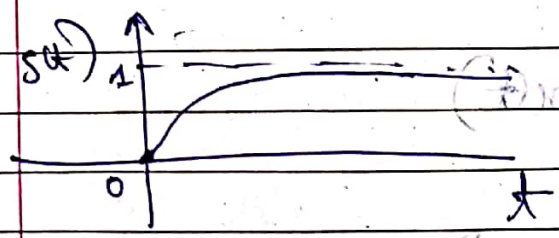
Wednesday

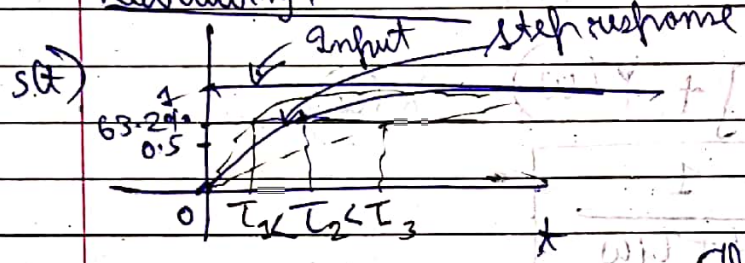


This is nature of step response of 1st order systems (generic) for any such system.

When $t = \tau$, $= 1 - \frac{1}{e} = \frac{e-1}{e} = \frac{2.7181-1}{2.718182}$

$= 0.632$

Redrawing :-



In $t = \tau$, it reaches 63.2% of the input.

The time required for 63.2% fill up, i.e, τ is called Time Constant.

→ τ controls the response speed of system. If τ is lesser ⇒ faster response. Here, τ_1 is fast and τ_3 is sluggish.

Now, Bode plot for 1st order system,

$$H(j\omega) = \frac{1}{1+j\omega\tau}$$

$$|H(j\omega)|^2 = \frac{1}{1+\omega^2\tau^2}$$

Now, $10\log_{10}|H(j\omega)|^2 = 10\log_{10}\left[\frac{1}{1+\omega^2\tau^2}\right]$

$$10\log_{10}|H(j\omega)|^2 = -10\log_{10}[1+\omega^2\tau^2]$$

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Thursday

$20\log_{10}|H(j\omega)| = -10\log_{10}(1+(\omega\tau)^2)$

Case I:

$\omega\tau >>> 1 \qquad (1+\omega\tau)^2 \approx (\omega\tau)^2$

$20\log_{10}|H(j\omega)| = -10\log(\omega\tau)^2$

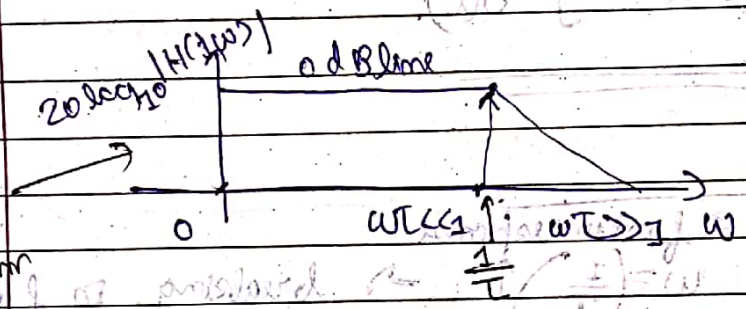
$20\log|H(j\omega)| = -20\log(\omega) - 20\log\tau$

Case II:

$\omega\tau << 1, \quad (1+\omega\tau)^2 \approx 1$

$20\log_{10}|H(j\omega)| = -10\log_{10}(1) = 0$

Bode Plot of 1st order system



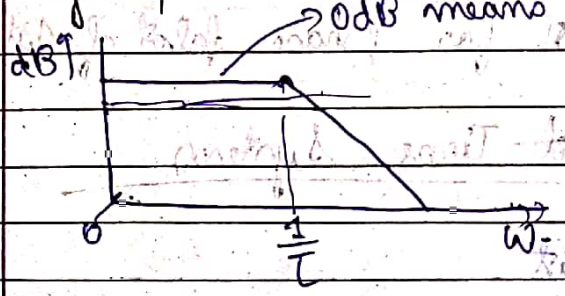
The frequency at which 0 dB line changes to 20 dB line is called break frequency
↳ cutoff

To find the point,

$0 = -20\log(\omega) - 20\log(\tau)$

$\Rightarrow \omega = \frac{1}{\tau}$

Again,



0dB means power gain is 1 here

What is the range when power falls from 1 to 1/2

$= 10\log_{10}\left(\frac{1}{2}\right) = -3\,dB$

Now, we are

$10\log\left[|H(\omega)|^2\right] = 10\log\left[\frac{1}{1+\omega^2\tau^2}\right]$

$= 10\log\left[\frac{1}{1+1}\right] \quad \left(\omega = \frac{1}{\tau}\right)$

$= 10\log\left(\frac{1}{2}\right)$

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Thursday

⇒ Break frequency is also the half power frequency.

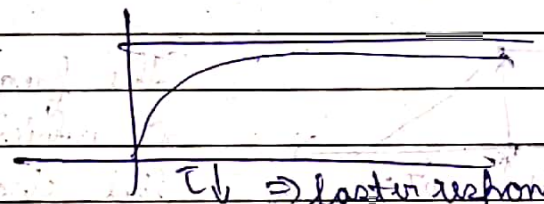
$\Longrightarrow h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$

$$\boxed{\begin{aligned} \tau h(t) &= e^{-t/\tau} u(t) \\ H(j\omega) &= \frac{1}{1+j\omega\tau} \end{aligned}}$$

Aim of this chapter is to understand time-freq trade-offs.

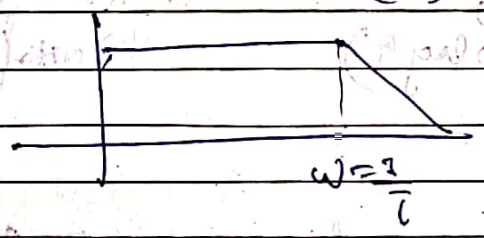
$x(2t)$ vs $x(t/2)$

$s(t) = \left[1 - e^{-t/\tau}\right] u(t)$



$\tau\downarrow \Rightarrow$ faster response

But then $\omega = \left(\frac{1}{\tau\downarrow}\right)\uparrow \Rightarrow$ breaking is far away



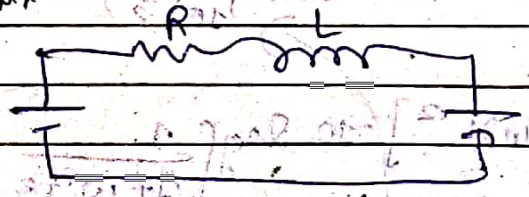
(H.W) Study Bode plot for phase plot of 1st order system

2nd Order Cont-Time System

Examples:-

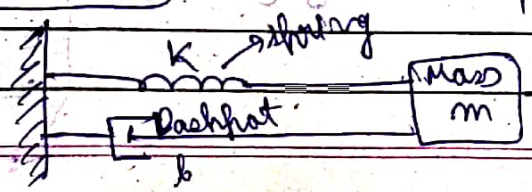
1) Series RLC circuit

Circuit



2) SHM

Mechanical system



→ displacement $y(t)$

→ Applied force

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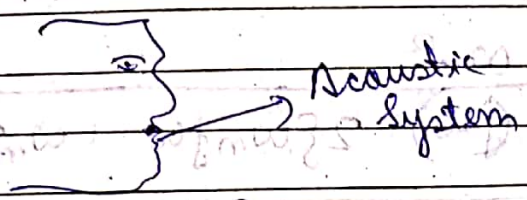
3) Your Mouth (Not Mine)



4) Analog Communication & Transmission line.

For each system, dynamics can be modelled. say Mass-spring

$$m\frac{d^2y(t)}{dt^2} = x(t) - Ky(t) - b\frac{dy(t)}{dt}$$

$$\boxed{\frac{d^2y(t)}{dt^2} + \left(\frac{b}{m}\right)\frac{dy(t)}{dt} + \left(\frac{K}{m}\right)y(t) = \left(\frac{1}{m}\right)x(t)}$$

We know that solution is a sinusoid if $b=0$, just like 12th physics

$$b=0 \Rightarrow \frac{d^2y(t)}{dt^2} + \left(\frac{K}{m}\right)y(t) = \left(\frac{1}{m}\right)x(t)$$

general mathematical form for 2nd order systems

$$\boxed{\frac{d^2y(t)}{dt^2} + (2\xi\omega_n)\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)}$$

$$\omega_n^2 = \frac{K}{m} \Rightarrow \boxed{\omega_n = \sqrt{\frac{K}{m}}}$$

$$2\xi\omega_n = \frac{b}{m}$$

$$\Rightarrow 2\xi\omega_n = \frac{b}{m} \Rightarrow \xi = \frac{b}{2\omega_n}$$

$$2\xi\sqrt{\frac{K}{m}} = \frac{b}{m} \Rightarrow \boxed{\xi = \frac{b}{2\sqrt{Km}}}$$ damping ratio $\xi \propto \frac{b}{\sqrt{Km}}$

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$$(j\omega)^2 Y(\omega) + (2\xi\omega_n)(j\omega)Y(\omega) + \omega_n^2 Y(\omega) = \omega_n^2 X(\omega)$$

$$\Rightarrow \boxed{\frac{Y(\omega)}{X(\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n j\omega + \omega_n^2} = H(j\omega)}$$

$$\therefore h(t) = F^{-1}\{H(j\omega)\}$$

$$= \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)}$$

$$c_1, c_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(use quadratic formula)}$$

$$\Rightarrow \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$= \frac{-2\xi\omega_n \pm 2\omega_n\sqrt{\xi - 1}}{2}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$= \omega_n\left[\pm\sqrt{\xi^2 - 1} - \xi\right]$$

$\therefore$ Now, $\dfrac{A_1}{j\omega - c_1} + \dfrac{A_2}{j\omega - c_2}$

$$h(t) = F^{-1}\{A(j\omega)\} = A_1 F^{-1}\left\{\frac{1}{j\omega - c_1}\right\} + A_2 F^{-1}\left\{\frac{1}{j\omega - c_1}\right\}$$

$$= A_1 e^{c_1 t} u(t) + A_2 e^{c_2 t} u(t)$$

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We get $A_2 = -A_1 = \dfrac{\omega_n}{\sqrt{1-\xi^2}}$

$$h(t) = A_1 \left[e^{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})t} - e^{(-\xi\omega_n - \omega_n\sqrt{\xi^2-1})t} \right] u(t)$$

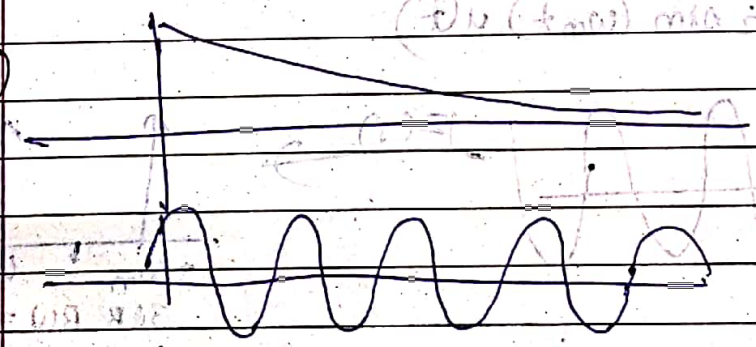
(ω_n^2 was a const and got absorbed in A_1)

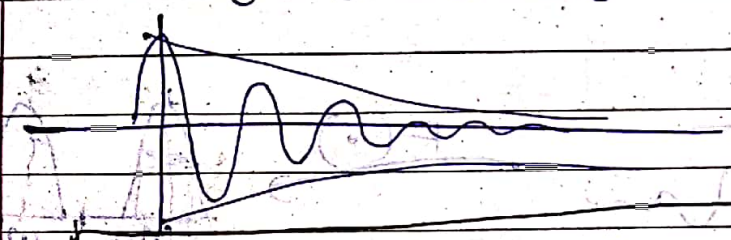
$$h(t) = A e^{-\xi\omega_n t} \frac{\left\{ e^{-j\omega_n\sqrt{1-\xi^2}\,t} - e^{-j\omega_n\sqrt{1-\xi^2}\,t} \right\}}{2j} u(t) \times 2j$$

$$= A e^{-\xi\omega_n t} \frac{\omega_n}{\sqrt{1-\xi^2}} \sin\left[\omega_n\sqrt{1-\xi^2}\,t\right] u(t)$$

$$\boxed{h(t) = K_1 e^{-\xi\omega_n t} \sin\left[\omega_n\sqrt{1-\xi^2}\,t\right] u(t)}$$ (∵ ξ is +ve no)

(1)





∴ Impulse response of a 2nd order system, in a realistic scenario, is a damped sinusoid

Thursday

2.) The number ξ, i.e, damping ratio controls amount of damping.

If ξ is high, the sinusoid dies down fast.

3.] $h(t)$ is IIR in nature.

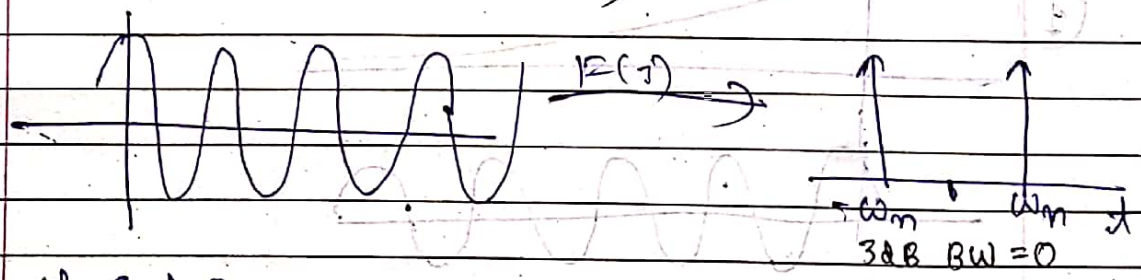
(1st insem last question - impulse response of real physical system is in general IIR)

4.] $\sin[\omega_n\sqrt{1-\xi^2}\,t] = \sin\omega t$
where $\omega = \omega_n\sqrt{1-\xi^2}$

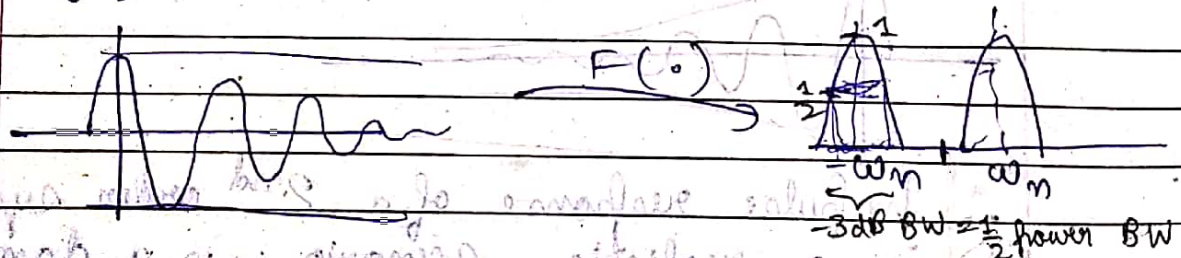
For ideal SHM,
$\xi = 0 \Rightarrow \boxed{\omega = \omega_n} \longrightarrow$ undamped natural frequency

5] If $\xi = 0$
$h(t) = (K_2)\, e^{-0\omega_n t} \sin[\omega_n\sqrt{1-0}\,t]\,u(t)$

$= K_2 \sin(\omega_n t)\, u(t)$



If $\xi \neq 0$



$e^{-j\omega t} \Rightarrow F(\cdot)$ will give inverse parabola. Now $e^{-j\cdots}$ is multiplied in T.D $\Rightarrow$ parabola will convolve with impulse in freq domain $\Rightarrow$ such graph is obtained.

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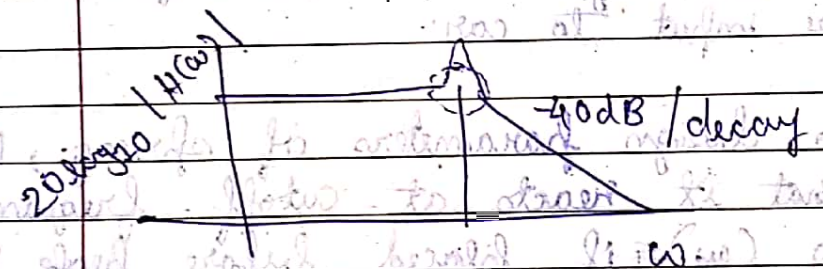
(Q) = Quality factor = $\frac{\text{Centre frequency}}{\text{bandwidth}}$

When $\xi = 0 \Rightarrow$ bandwidth is $0 \Rightarrow$ quality factor $= \infty$

$\xi \neq 0 \Rightarrow$ half power (3dB) bandwidth $\neq 0$.

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

$$10\log_{10}|H(j\omega)|^2 = -10\log_{10}\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2\right\}$$



$$\frac{\partial}{\partial\omega}F(\omega) = \frac{\partial}{\partial\omega}\log(j\omega)$$

$$= \frac{1}{\omega}\frac{\partial}{\partial\omega}(g(\omega))$$

$$\frac{d}{d\omega}\left[\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2\right\}\right] = 0$$

$$\Rightarrow 2\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] \times \frac{d}{d\omega}\left[1-\frac{\omega^2}{\omega_n^2}\right] + 4\xi^2 \times \frac{2\omega}{\omega_n^2}$$

$$2\left[1-\frac{\omega^2}{\omega_n^2}\right] \times \left[0 - \frac{2\omega}{\omega_n^2}\right] + 4\xi^2 \times \frac{2\omega}{\omega_n^2} = 0$$

$$\boxed{\omega = \omega_n\sqrt{1-2\xi^2}}$$ $\Rightarrow$ The frequency at which we get peak ~~frequency~~.

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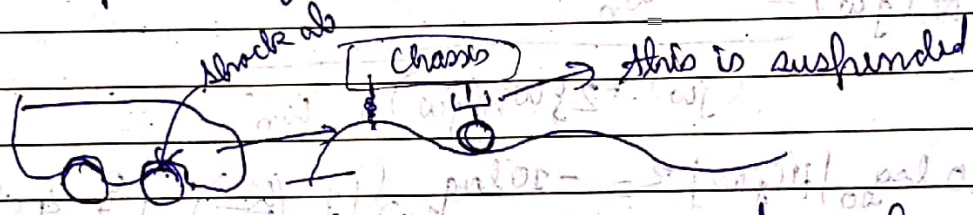
→ The fact that this freq response has a peak at a particular freq ω_{max} is used in design of filters, amplifiers.

Design of Automobile Suspension System → (Shock Absorber)

Design is fun of cost.



When sudden change in road surface, it is like impulse input to car.

We have to design parameters of spring, k and b such that it reacts at cutoff frequencies. So the freq (ω) is placed before peak vs exactly at peak.

∴ The design is such that ω_n is before the peak, at cut-off freq

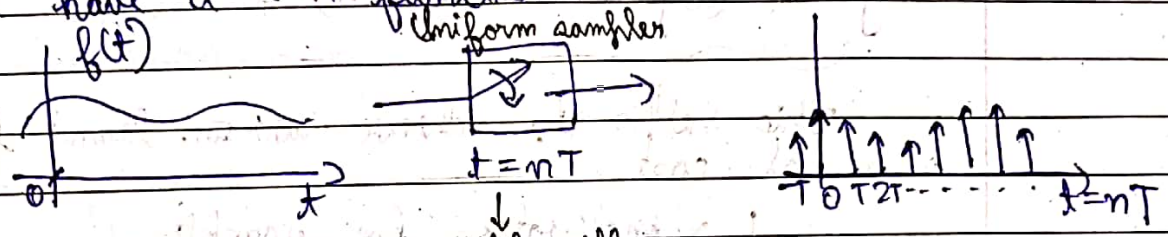
Because it will act as narrow pass.

Thursday

Chapter 7: Sampling Theory

Motivation :-

I] We have a C-T function :-



switch will close every T seconds & value is taken

Sampling

- Regular (Uniform): when information obtained at regular intervals of time ⇒ regular sampling
- Irregular [Non-uniform]: when information is obtained at irregular intervals

We don't do this just because real life me computer hota hai

→ We do because of -3dB BW → Ability to achieve narrow -3dB bandwidth in digital filter than in analog filter.

-3dB BW $\boxed{B_{\downarrow} = \frac{2}{RCT}}$

3dB BW in a system signifies the damping energy loss in system. The wider it is, more is energy loss.

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→ We want these filters to have narrow 3dB bandwidth. Ideally spikes, but it is not possible.

absolute value controls 3dB BW

$$\underbrace{y(n) - a\,y(n-1) = b\,x(n)}$$

This helps to achieve narrow-band filter at minimal cost.

↳ motivation for sampling.

2] Our ability to delay the signal

This simple element is a game changer

$$y(n) \longrightarrow \boxed{\text{Delay}} \longrightarrow y(n-1)$$

D-flip flop

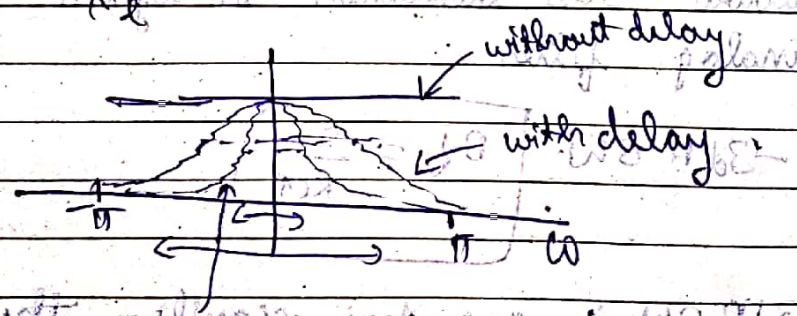
$$y(n) - a\,y(n-1) = b\,x(n) \quad \text{—— (1)}$$

$$\boxed{H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}}$$

Suppose now in hardware, delay element is not possible to fuit; then

$$y(n) = b\,x(n)$$

$$\Rightarrow \frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega}) = b$$



→ We can do this by tweaking a. Without delay, half-power (-3dB BW) = ∞ ⇒ Quality = 0

Thursday

Thus, a simple D-flip flop is central to communication.

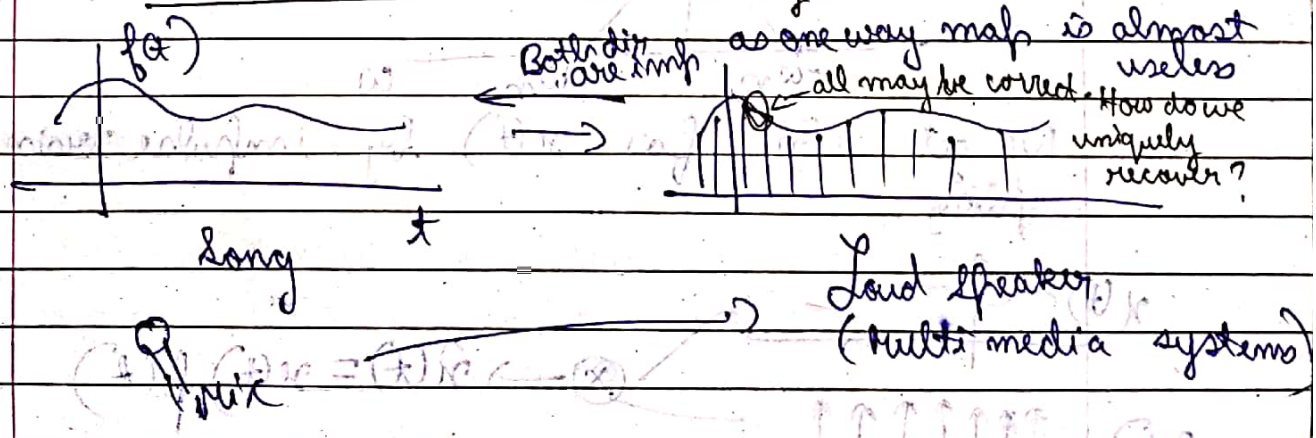
3] Ability to do multiplexing
Spectrum directly associated to cost.

4] If we store data in magnetic tape, over time it decays. But if we sample the data, then using properly sampling frequency we can store the data very compactly and it is permanent & safe.

Applications :- - Compressive Sensing
- Frame Theory — design of filters.
- Digital Communication

1) Y. Eldar
2) Frame Research Institute (New York)

Research Issue in Sampling



⇒ But how to get original from samples. How do we literally collect the dots? Connecting two dots is called interpolation and getting next one from two, is ~~extr~~ extrapolation.

⇒ No unique way to recover. $N \to \infty$ ⇒ There is dilemma

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as to which interpolator, we get back accurate signal.

Issue :- What is the guarantee that we can recover original analog signal $x(t)$, exactly from its discrete-time component?

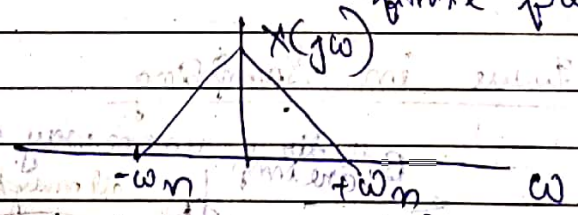
2] $x(t)$ from $x(nT)$?

Prof. Claude E. Shannon – father of digital communication.

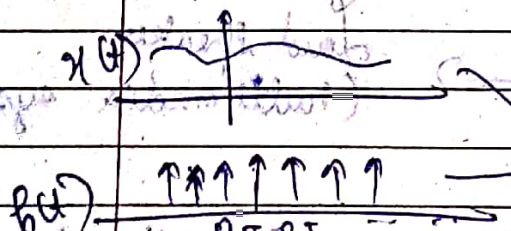
In Maths, J.M. Whittaker proposed same theory in 1936.

Development of Sampling Theorem

Let $f(t)$ = analog, bandlimited signal
finite frequencies



We get $x(nT)$ for $x(t)$ by impulse train sampling.



$\otimes \longrightarrow \hat{x}(t) = x(t)\, f(t)$

$$\hat{X}(\omega) = \frac{1}{2\pi}\left[X(\omega) * P(\omega)\right]$$

$$P(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0)$$

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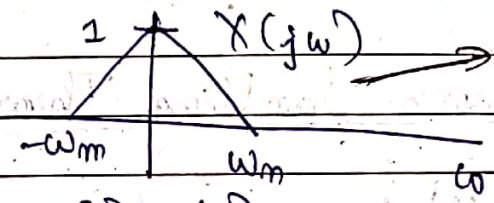
Thursday

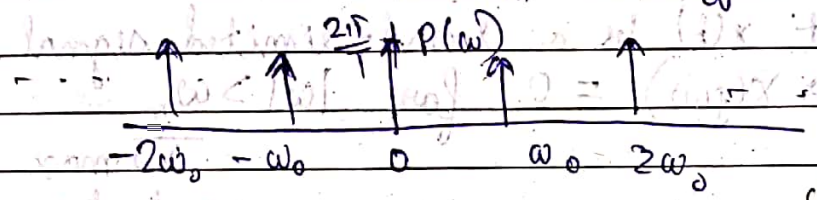
$$\Rightarrow \hat{X}(\omega) = \frac{1}{T}\left[X(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)\right]$$

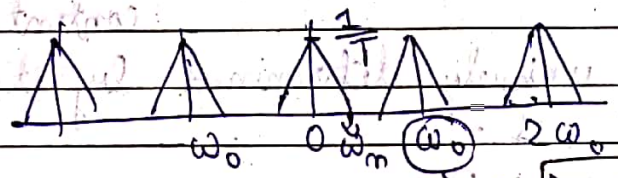
$$\boxed{\hat{X}(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_0)}$$

In real life, analog signals F.T is aperiodic, but in computer it becomes periodic



→ characteristic of signal at i/p



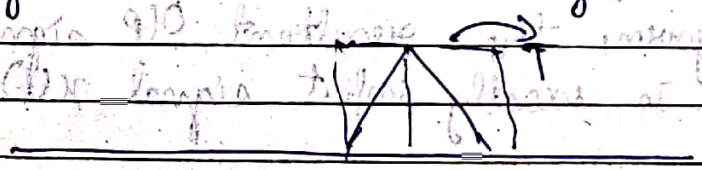


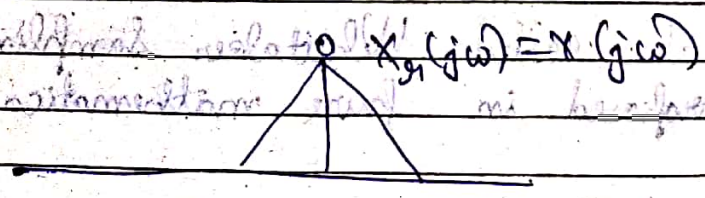
→ characteristic of signal in computer.

→ $\boxed{\text{is now called sampling frequency } \omega_s}$

→ The spectrum in analog was aperiodic. But after sampling, it became periodic. One of the important artifacts of sampling is that signal spectrum becomes periodic in discrete time.

To isolate original, lets use a low pass filter. Gain must be T to get original





X_r denotes spectrum of reconstructed signal

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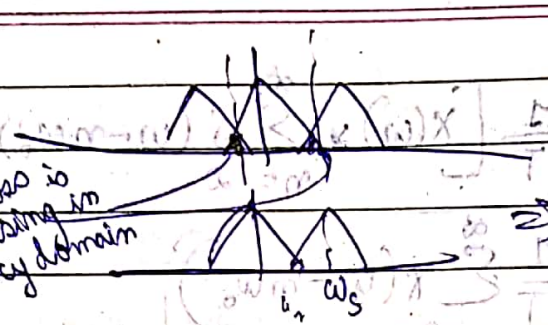
Thursday



This loss is called aliasing in frequency domain

$\Rightarrow \omega_s - \omega_m \geq \omega_m$

$\rightarrow \omega_s \geq 2\omega_m$

This eq^n is called Shannon's Sampling Theorem.

Shannon's Sampling Theorem 20/11/18

Statement :- Let $x(t)$ be a band-limited signal with $X(\omega)$ (or $X(j\omega)$) $= 0$ for $|\omega| > \omega_m$

ω_m → max frequency content in signal.

Sampling

Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \ldots$ if $\omega_s > 2\omega_m$

where $\omega_s = \frac{2\pi}{T}$, T = sampling interval

reconstruction

→ Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal LPF with gain T and cut-off frequency $\omega_m < \omega_c < \omega_s - \omega_m$

If this is given, the resultant O/P signal of ideal LPF is exactly input signal $x(t)$.

→ This is also called Whittaker Sampling theorem, who proposed in pure mathematics.

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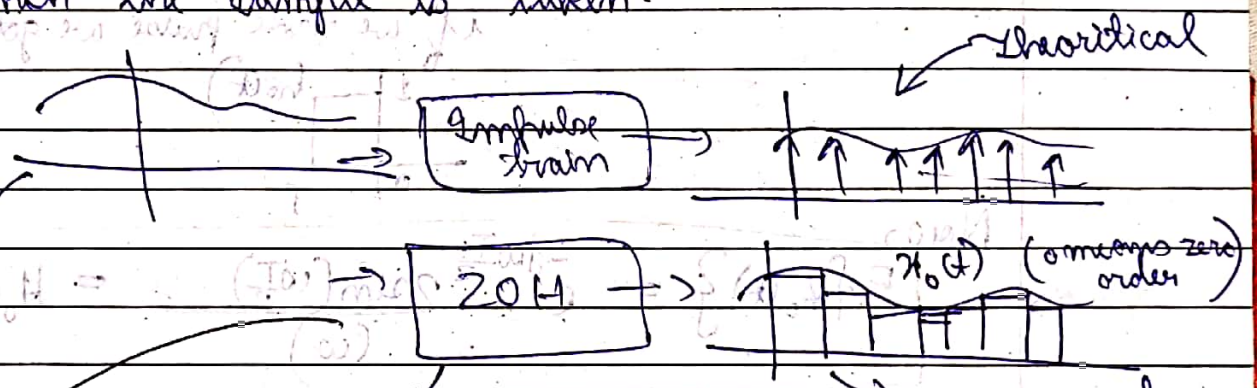
Tuesday

Issues with impulse-train sampling

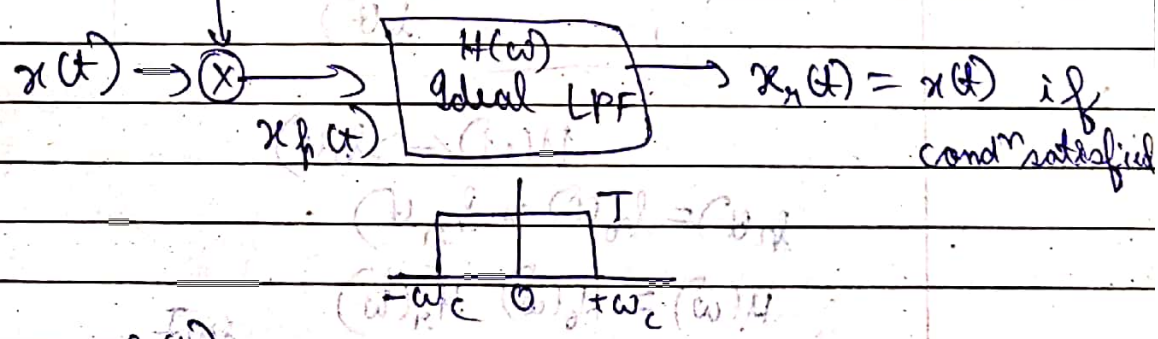
In practice, narrow large amplitude pulses - which approximate impulses, are very difficult to generate and transmit. Therefore, we used Zero-order hold (ZOH).

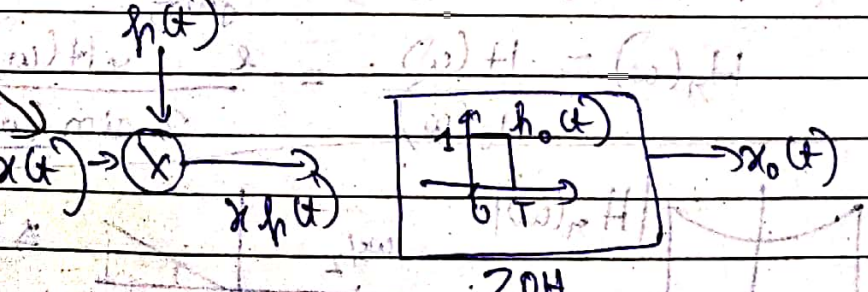
What ZOH does?
It holds sample value until the next instant at which the sample is taken.



Sample and hold

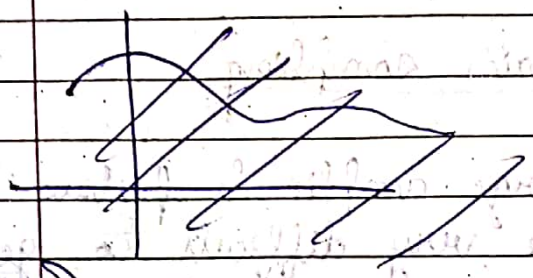
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$



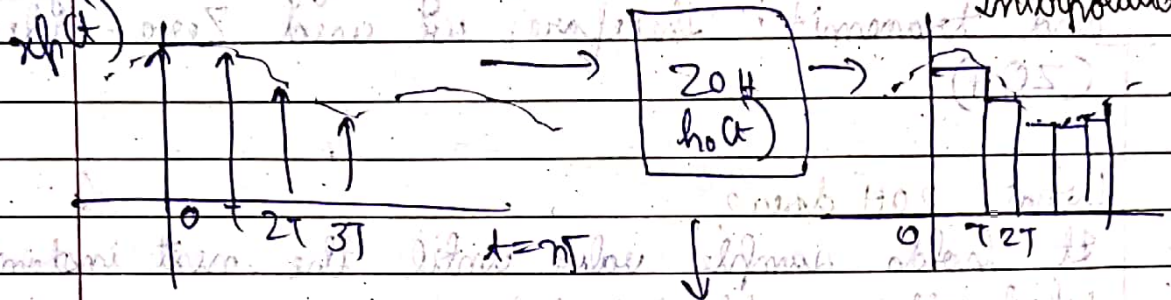


P.T.O

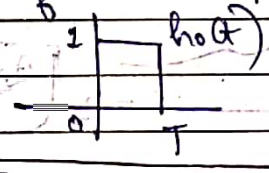
Tuesday



Interpolation

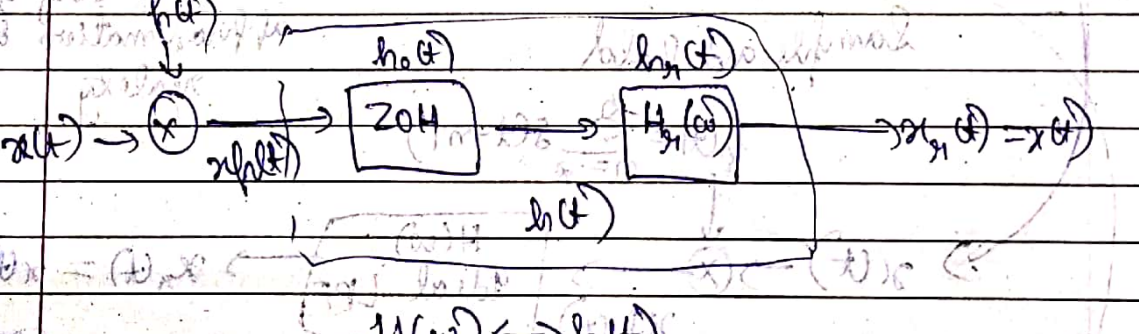


if we take pulse we get RHS



Now,

$$\boxed{F\{h_0(t)\} = e^{-j\omega\frac{T}{2}} \frac{2\sin\left(\frac{\omega T}{2}\right)}{(\omega)} = H_0(\omega)}$$

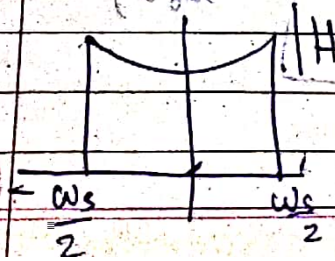


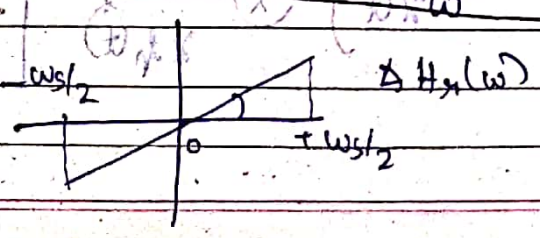
$H(\omega) \longleftrightarrow h(t)$

$h(t) = h_0(t) * h_r(t)$

$H(\omega) = H_0(\omega)\, H_r(\omega)$

$$H_r(\omega) = \frac{H(\omega)}{H_0(\omega)} = \frac{e^{j\omega\frac{T}{2}} \times H(\omega)}{2\sin\left(\frac{\omega T}{2}\right)/\omega}$$





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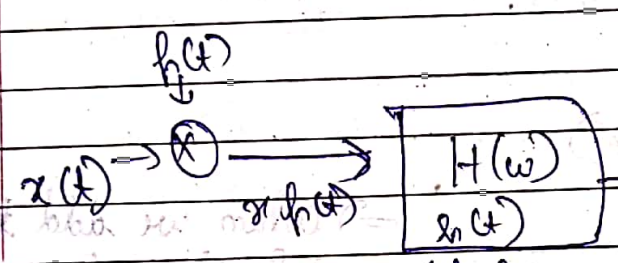
Tuesday

→ Job of ZOH is to hold sample value till the end of the interval

→ ZOH involves discontinuities at the bases for every sampling interval. To have smooth sampling, to get smooth approximation, we use 1st order, 2nd order curves.

Section 7.2 Reconstruction of a signal from its samples, using interpolation;

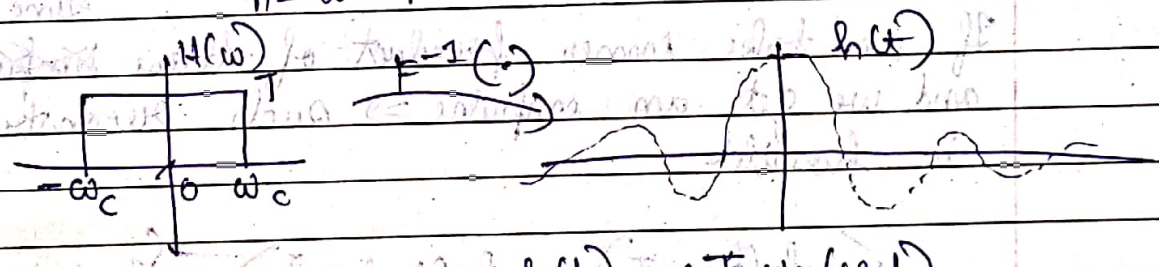
↳ (Given samples, estimate the signal b/w samples)



(Ideal LPF ~~(Includes ZOH filter)~~)
↳ LTI filter

$\therefore x_r(t) = x_p(t) * h(t)$ $\quad (\because h_0(t) * h_1(t) = h(t))$

$$= \sum_{n=-\infty}^{\infty} x_p(nT)\, h(t-nT)$$



$$\therefore h(t) = \omega_c T \frac{\sin(\omega_c t)}{\omega_c \pi t}$$

$$= \frac{\omega_c T}{\pi} \frac{\sin(\omega_c t)}{(\omega_c t)}$$

$$\Rightarrow \boxed{h(t) = \left(\frac{\omega_c T}{\pi}\right) \frac{\sin(\omega_c t)}{\omega_c t}}$$

$$\therefore x_r(t) = \sum_{n=-\infty}^{\infty} \underbrace{x(nT)}\, \underbrace{\left(\frac{\omega_c T}{\pi}\right) \frac{\sin[\omega_c (t-nT)]}{\omega_c [t-nT]}}_{h(t-nT)}$$

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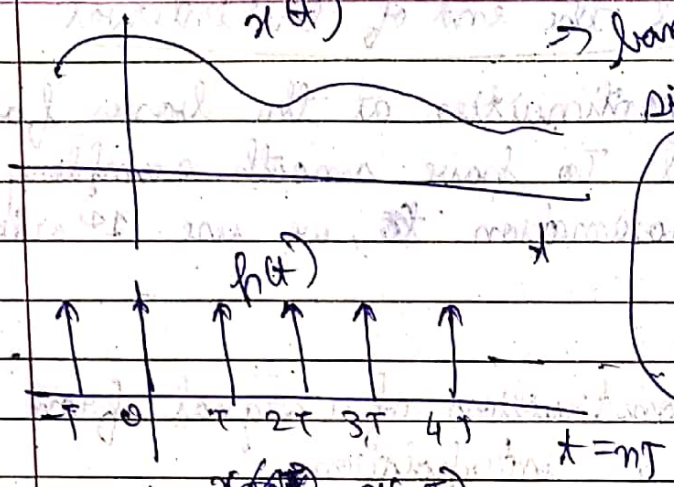
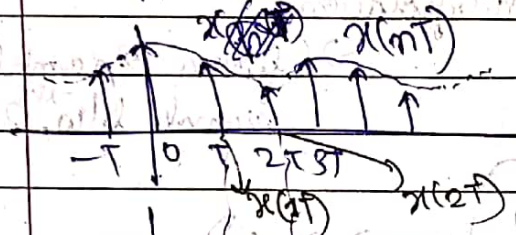
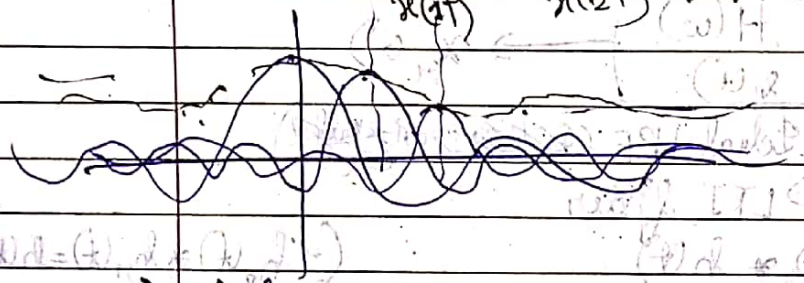
Tuesday



→ band limited sinc interpolation

→ Note :- Any mutually orthogonal fns can be used. Sinc is not unique





⇒ When we add these infinite lobes, we get exactly x(t)

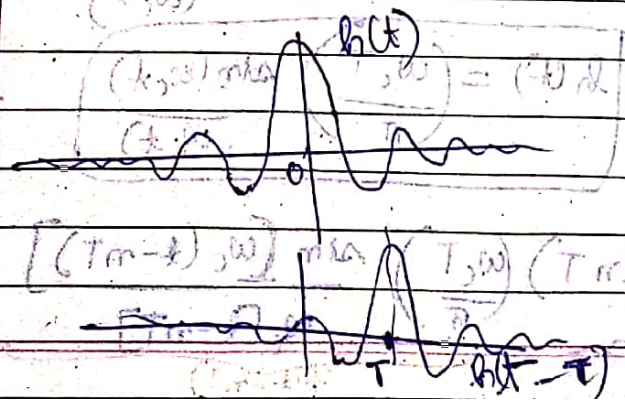
Q.) What is the guarantee that reconstruction is possible?

If we take inner product of these ~~impulses~~ sinc fns and we get an impulse ⇒ such reconstruction is possible

$$\langle h(t-nT), h(t-pT) \rangle = \begin{cases} \text{Impulse}, & n=p \\ 0, & n \neq p \end{cases}$$

$\downarrow$ sinc at n $\quad$ $\downarrow$ sinc at p $\quad$ $n, p \in \mathbb{Z}$

20/11/18



These sinc functions are basis fns for sampling.

i.e., $e_k = \{h(t-nT)\}$

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Tuesday

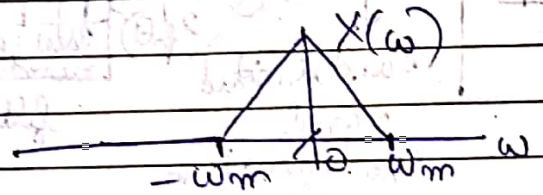
→ In a 2-dimensional function, sampling is done in 2 directions (X & Y)

→ Mosaic Effect:- ~~When~~ after reconstruction for various lengths of interval, we see that as T ↓, the clarity ↑.

Limitations of Shannon's Sampling Paradigm

1) Shannon's sampling paradigm assumes that the underlying continuous signal is bandlimited.

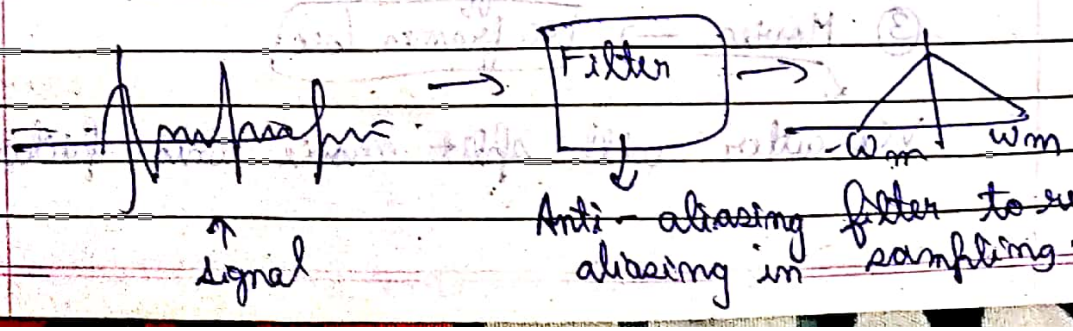
$x(t)$ = bandlimited.



But for most signals in nature, this band-limitedness doesn't hold. Rather, we actually force bandlimiting for transmission.

Eg:- Voice is not bandlimited. But by and large it is 4 KHz. Frigatives and Plosives have more than 4 KHz. Hence, ω_m is taken 4 KHz. For telecom operators and phones, ω_s is taken as 8 KHz.

∴ What is done:-



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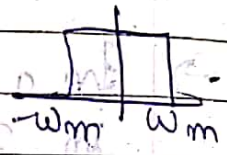
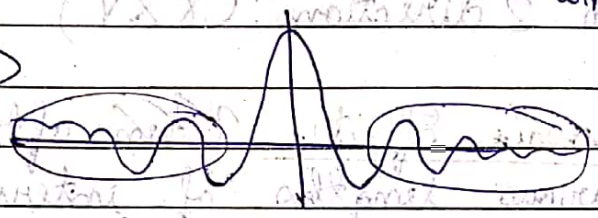
Tuesday

(2) → Anti-aliasing filter is ideal LPF



$\Rightarrow F^{-1} \Rightarrow$



IIR filter [Non-causal

→ The side lobes in time domain create whole lot of problems, hence not desirable.

(3) Ideal LPF is difficult to implement in practice

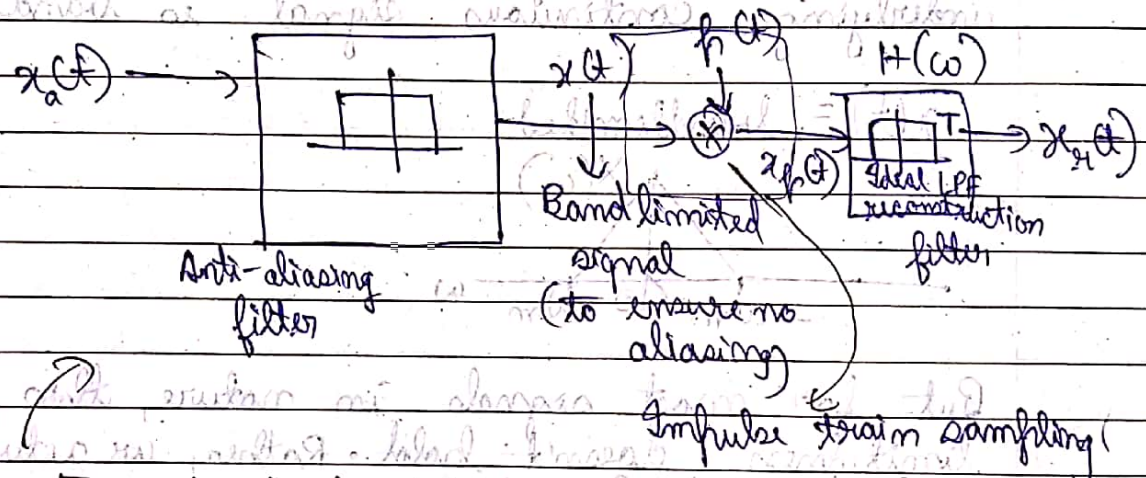


Fig. Standard three-step setup for Shannon's sampling paradigm.

~~Section 7.4~~ Section 7.3

(1) Effect of undersampling (Solved problem)

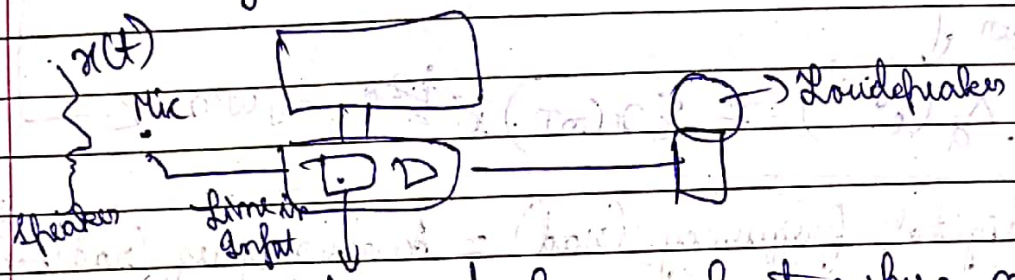
(2) Application of undersampling (application) Strobe effect

(3) Movies → $\boxed{\text{30 frames/sec}}$

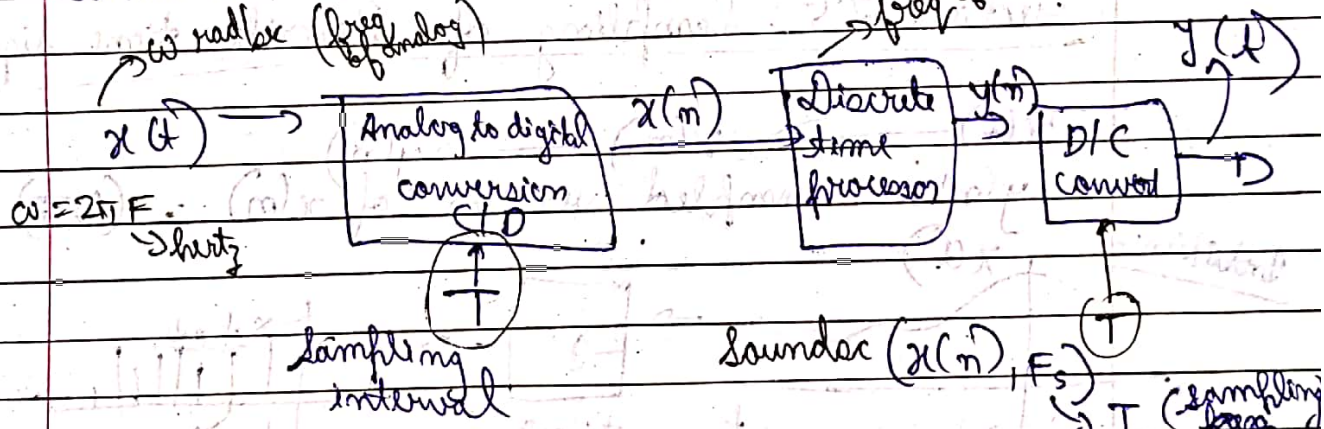
VCD cutter will split movie into picture frames

Section 7.4: Discrete-Time processing of CT. Signals :-

Record your speech & hear it



Sound card has a port where mic output is input. Here analog converted to digital via ADC converter. PC to Loudspeaker DAC



Soundsc is a fn in Matlab to reconstruct

To find relⁿ b/w ω and Ω

T = sampling interval (seconds)

$$x_p(t) = x(t)\,p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

$$F\{x_p(t)\} = F\left\{\sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)\right\}$$

$$= \sum x(nT)\,F\{\delta(t-nT)\}$$

$$X_p(\omega) = \sum x(nT)\,(1)\,e^{-j\omega nT}$$

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Tuesday

discrete sequence of samples

$$X_p(\omega) = \sum_{n=-\infty}^{\infty} \{x(nT)\} e^{-j\Omega nT}$$

$$\{x(nT)\} \xrightarrow{F(\cdot)} \text{DTFT} \qquad \omega = \Omega T$$

$n \in Z$

frequency of DT signal in computer

$$X_d(e^{j\omega}) = \sum x(nT)\, e^{-j\omega n}$$

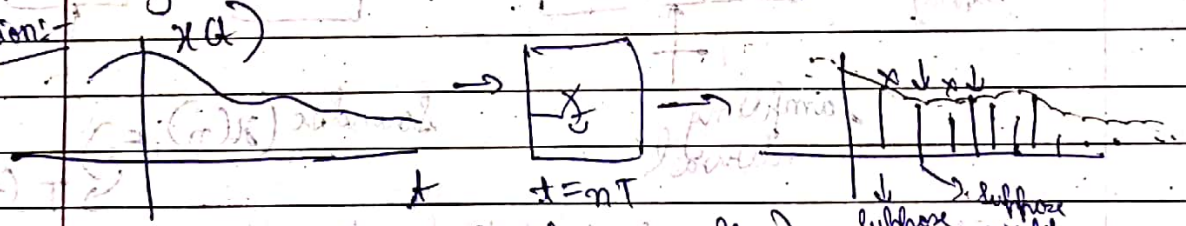
$$\boxed{\text{digital frequency (rad)} = \text{Analog freq (rad/sec)} \times \text{sampling interval}}$$

Section 7.5 : Sampling of Discrete time Signal

$x(t) \longrightarrow x(n) \longrightarrow y(n)$

$y(n)$ is sampled version of $x(n)$ .. (WTF)

Intuition:-

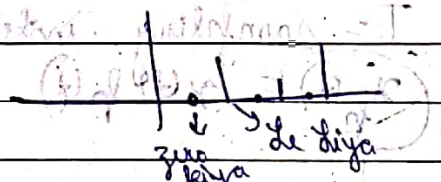


(Uniform sampler only in course)

suppose we reject this

suppose accept

This mannerism is just like sampling.



Mimicry of $x(2t)$ and $x(t/2)$ in computer

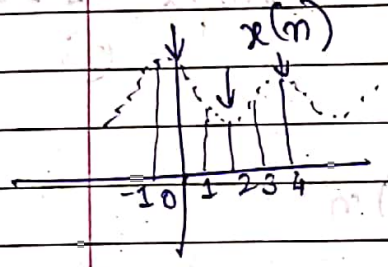
time compression time expansion

Note:- Section 2-7.5 is connected to DTFT (Chapter 5)

$$F\{x(at)\} = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$F\{x(2t)\} = \frac{1}{2} X\left(\frac{\omega}{2}\right)$$

① Decimation [mimicri of time compression]



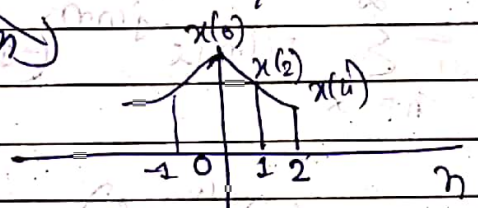
$\rightarrow$ $\boxed{\downarrow 2 \text{ down-sampling by } 2 \text{ (decimation)}}$ $\rightarrow y(n) = x(2n)$

$\{\ldots, y(-1), y(0), y(1), y(2) \ldots\}$

$\downarrow$

Time Compression

$= \{\ldots, x(-2), x(0), x(2), x(4)\}$



This process of taking only alternate sampling, is called decimation by factor of 2.

Now, $y(n) = x(2n)$

Find FT of time compressing to see effects of compress & expand

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(2n) e^{-j\omega n}$$

~~2n = n~~ $\omega \rightarrow 2\omega$

$$Y(e^{j2\omega}) = \sum_{n=-\infty}^{\infty} x(2n) e^{-j2\omega n}$$

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Tuesday

$$= \cdots + x(-2)e^{+j2\omega 1} + x(0) + x(2)e^{-j2\omega 1} + x(4)e^{-j4\omega 1} \cdots \quad (1)$$

Now, $x(n) \rightarrow X(e^{j\omega})$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$X(e^{j\omega}) = \cdots + x(-1)e^{j\omega} + x(0) + x(1)e^{-j\omega} + x(2)e^{-j2\omega} + \cdots \quad (2)$$

Now, Let a case $\omega \rightarrow \omega + \pi$

$$X(e^{j(\omega+\pi)}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} e^{-j\pi n}$$

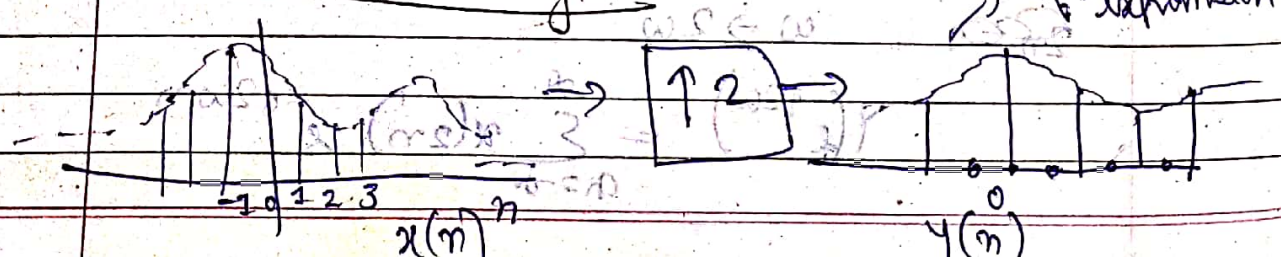
$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} (-1)^n$$

$$= \cdots - x(-1)e^{j\omega} + x(0) - x(1)e^{-j\omega} + x(2)e^{-j2\omega} \cdots \quad (3)$$

$$\frac{\text{Eqn } (2) + \text{Eqn } (3)}{2} = \text{Eqn } (1) \quad \text{(just see)}$$

$$\therefore Y(e^{2j\omega}) = \text{DTFT}\{x(2n)\} = \frac{1}{2}\left[X(e^{j\omega}) + X(e^{j(\omega+\pi)})\right]$$

(2) Upsampler by 2



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Let's define $y(n) = \begin{cases} x(p) & \text{if } n = 2p \\ 0 & \text{, if } n = 2p+1 \end{cases}$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n) e^{-j\omega n}$$

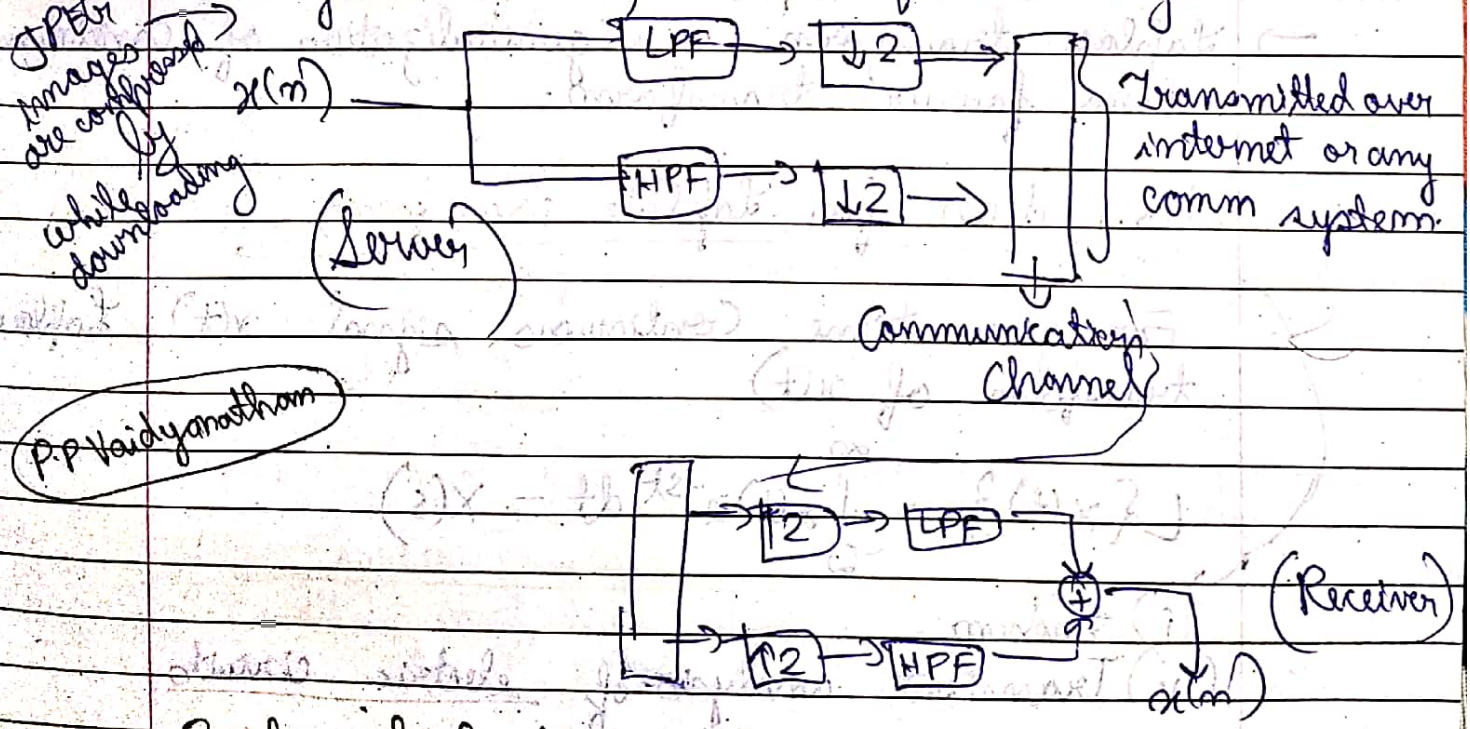
$$= \sum_{p=-\infty}^{\infty} x(p) e^{-j\omega(2p)} + 0 \text{ (and zero } \forall \text{ other } p)$$

$$\boxed{Y(e^{j\omega}) = \sum^{p \to n} x(n) e^{-j(2\omega)n} = X(e^{j2\omega})}$$

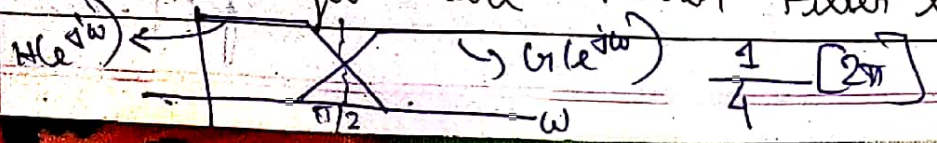
Case-Study

Filter bank = Bank of filters

In digital comm, we transfer $x(n)$ by :-



2-channel Quadrature Mirror Filter bank



Tuesday

The Laplace Transform

Generalization of Eigen function of LTI system

$$x(t) = e^{j\omega t} \rightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t)\end{matrix}} \rightarrow y(t) = \underbrace{H(j\omega)}_{\text{system's eigen value}} e^{j\omega t}$$

Eigen function

proper

$$x(t) = e^{st} \rightarrow \boxed{} \; y(t) = H(s)\, e^{st}$$

where $H(s) = \int_{t \in R} h(t) \cdot e^{-st}\, dt = L\{h(t)\}$

Laplace transform $\{h(t)\}$

where $\boxed{s = \sigma + j\omega}$

s = Laplace transform variable

→ Laplace transform is a generalization of continuous time fourier transform.

Motivation for Laplace transform

For given time continuous signal $x(t)$, Laplace transform of $x(t)$

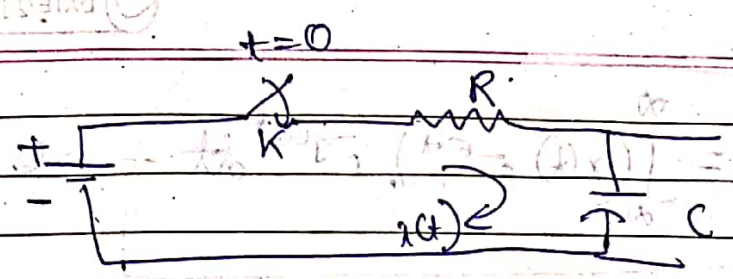
$$L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = X(s)$$

(i) Endsem

(ii) Transient analysis of electric circuits

P.T.O

Wednesday



(Network Analysis) (GATE)

(iii) Eigen analysis of LTI System

$x(t) = e^{st}$, then $\boxed{y(t) = H(s)e^{st}}$

$$H(s) = L\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

(iv) Solution of differential equation

(v) It is useful for analysis of both stable and unstable LTI system

[Control system, Electrical Networks]

(vi) It is useful for analysis of LTI
cont-time

(vii) Let $x(t) = u(t)$

We can't find $X(j\omega)$ directly as it is not directly integrable.

But L.T exists.

Relationship b/w Laplace Transform and C-T Fourier Transform

Let $x(t)$ = cont-time signal

Then CTFT, $F\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = X(j\omega) = X(\omega)$

Now, L.T = $L\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt = X(s)$ where $s = \sigma + j\omega$

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Wednesday

$$\therefore \boxed{X(s) = \int_{-\infty}^{\infty} \left(x(t)\, e^{-\sigma t}\right) e^{-j\omega t}\, dt} \quad \text{—— (2)}$$

Laplace transform is F.T of $x(t)e^{-\sigma t}$, i.e,

$$\cancel{X(\omega)} = \boxed{X(s) = F\{x(t)\, e^{-\sigma t}\}} = L\{x(t)\}$$

Problems:- Find L.T of following:-

1) $x(t) = e^{-at} u(t)$ $\quad a > 0$ $\quad \boxed{X(j\omega) = \frac{1}{a+j\omega} = F\{x(t)\}}$ (1)

$$L\{x(t)\} = \int_{-\infty}^{\infty} \left(e^{-at} u(t)\, e^{-\sigma t}\right) e^{-j\omega t}\, dt$$

Exam me 1st step, $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$

Now, substitute,

$$X(s) = \int_{-\infty}^{\infty} \left[e^{-at} u(t)\right] e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-st}\, dt = \int_{0}^{\infty} e^{-(a+s)t}\, dt$$

$$\Rightarrow 0 - \frac{e^{-(a+s)(t)}}{-(a+s)} = \boxed{\frac{1}{a+s}}$$

$$\therefore \boxed{X(s) = X(\sigma + j\omega) = \frac{1}{a+\sigma+j\omega}} \quad \text{—— (2)}$$

If this should be finite and should converge

If $(a+\sigma) < 0 \Rightarrow$ the integrand is +ve and it is diverging $\Rightarrow$ not absolutely integrable and hence Laplace Transform won't exist.

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Wednesday

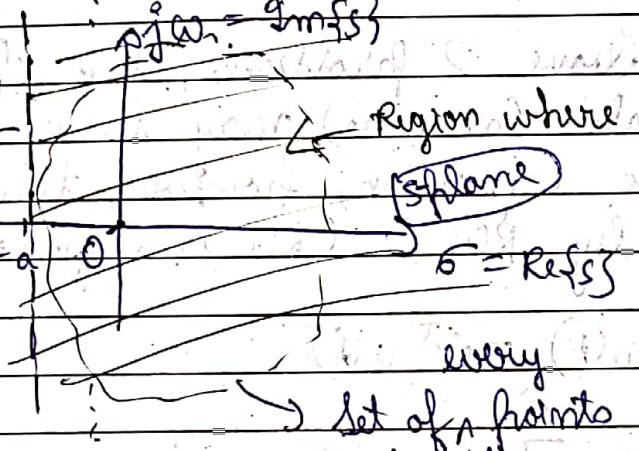
$\therefore$ $\boxed{\text{For } X(\sigma + j\omega) \text{ to converge, } (a+\sigma) > 0}$

Now, $a + \sigma > 0$ Since $s = \sigma + j\omega$

$\Rightarrow \sigma > -a$ $\sigma = Re\{s\}$

$\Rightarrow \boxed{Re\{s\} > -a}$ → This inequality is a game-changer for L-T

(i) $\sigma = -a$



Dotted line kyunki at $\sigma = -a$ pe marginality chalegi strict inequality

→ Set of every points s, satisfying $a + \sigma > 0$ leads to convergence & finite answer of Laplace Transform, is called ROC :- Region of Convergence (is the locus of s under given condⁿ)

2) Find L-T of $x(t) = -e^{-at} u(-t)$.

$$X(s) = \int_{-\infty}^{\infty} -e^{-at} u(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} -e^{-at} e^{-st}\, dt$$

$$= -\left[\int_{-\infty}^{0} e^{-(a+s)t}\, dt\right] = \left[-\frac{e^{-(a+s)t}}{-(a+s)}\right]_{-\infty}^{0}$$

$$= \left(\frac{1}{a+s}\right)$$

$a + \sigma < 0$

$\boxed{Re\{s\} < -a}$

$\frac{-1}{-(a+s)} - \frac{e^{-(a+s)\infty}}{-(a+s)}$ → $a+s<0$ $\Rightarrow \frac{1}{a+s}$

$\frac{-1}{-(a+s)} - 0$

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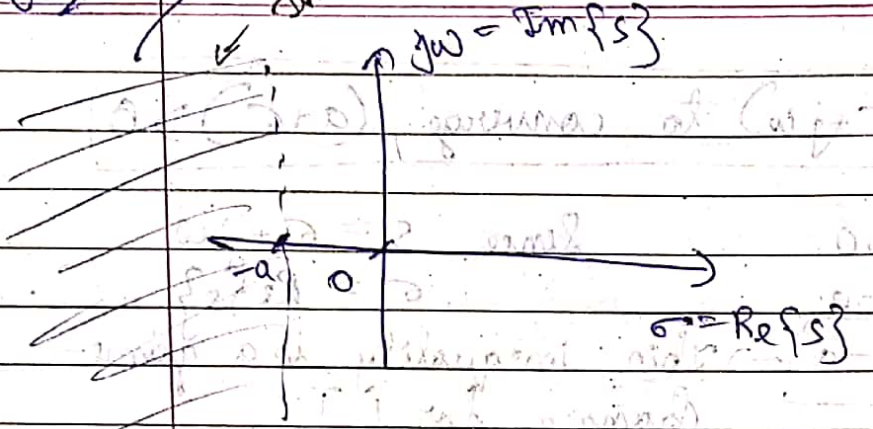
Wednesday

ROC

No region can't have ⊗ but ⊗ can have ⊙



Now, from above 2 problems, it is clear that 2 signals (or more) may have exactly same algebraic expression, however, they are distinguished by ROC (ROC is a game changer)

3) $x(t) = 2e^{-2t}u(t) - e^{-t}u(t)$

$$X(s) = 2\left[\frac{1}{s+2}\right] - \frac{1}{s+1}$$

$$= \frac{s}{(s+1)(s+2)}$$ (This is a rational transform)

$$= \frac{N(s)}{D(s)}$$ (In most of engg. problems, we get rational)

Now, roots of Numerator $N(s) \Rightarrow s = 0$

" " Deno. $D(s) \Rightarrow s = -1, s_2 = -2$

$X(s)\Big|_{s=0} = 0 = \boxed{\text{Zero of } X(s)}$ → value of s for which $X(s) = 0$

$X(s)\Big|_{s_1, s_2 = -1, -2} = \infty = \boxed{\text{poles of } X(s)}$ → values of s for which $X(s) \to \infty$

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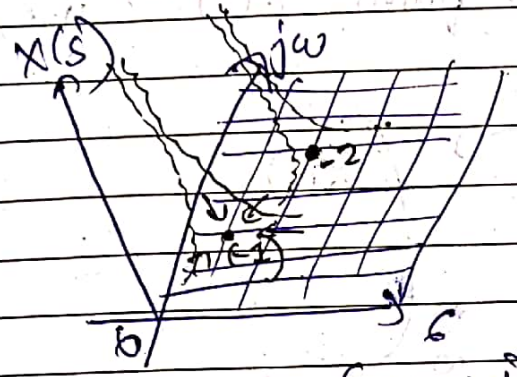
Wednesday

Why poles?

→ Proof :- $s = \sigma + j\omega$



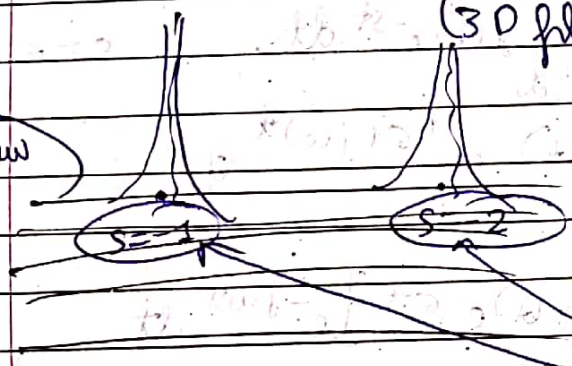
$$X(s) = \frac{s}{(s+1)\,(s+2)}$$

$s = -1,\ s = -2$

The moment we approach -1 from any direction, the value shoots to ∞. Same for -2

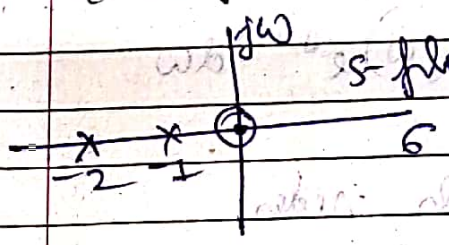
(3D plot)

(Side-view)



→ sharp structure like electricity pole. That's why these locations are called poles.

⟹ ⊙ → zero (where LT = 0)
× → pole (where L-T not difined ⇒ 0)



s-plane ⟹ $\boxed{\text{Called a Pole-Zero Plot}}$

→ By looking into defn of ROC and poles, it is clear that $\boxed{\text{ROC can't contain poles.}}$

↳ No × in ROC, but ⊙ is ok as ⊙ is finite value.

Inverse Laplace Transform

$$F\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = X(j\omega)$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega = F^{-1}\{X(j\omega)\}$$

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Wednesday

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$$

- Can we get $x(t)$ from $X(s)$? i.e., $L^{-1}\{X(s)\}$
  ↳ Yes

$$x(t) \xrightarrow{L\{\cdot\}} X(s)$$

$$x(t) \xleftarrow{L^{-1}\{\cdot\}} X(s)$$

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt \qquad s = \sigma + j\omega$$

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[x(t) e^{-\sigma t}\right] e^{-j\omega t}\, dt$$

$$= F\{x(t) e^{-\sigma t}\}$$

$$\therefore x(t) e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{j\omega t}\, d\omega$$

Multiplying $e^{\sigma t}$ on both sides,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{j\omega t} e^{\sigma t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{(\sigma + j\omega)t}\, d\omega$$

Now, put $s = \sigma + j\omega \rightarrow \frac{ds}{d\omega} = 0 + j(1) \Rightarrow \boxed{d\omega = \frac{ds}{j}}$

$\omega = -\infty \Rightarrow s = \sigma - j\infty$

$\omega = +\infty \Rightarrow s = \sigma + j\infty$

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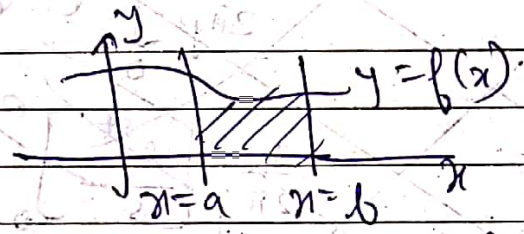
$$\Rightarrow x(t) = \frac{1}{2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} \frac{ds}{j}$$

Most of time this is rational fn.

$$\boxed{L^{-1}\{X(s)\} = x(t) = \left(\frac{1}{2\pi j}\right) \int_{\sigma - j\infty}^{\sigma + j\infty} [X(s)] e^{st} ds}$$

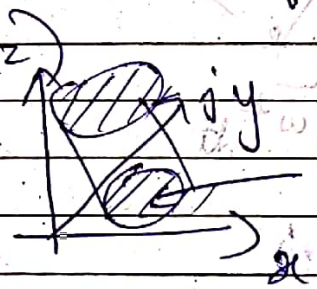
Since s is complex variable, $\sigma - j\infty$, $\sigma + j\infty$ is called contour of integration.

Why contour? $y = f(x)$ $\int_{x \in R} f(x) dx$ ↓ Line integral



But when instead of $f(s)$, $f(z)$ where $z = a + ib$

$\Rightarrow$



Contour (region over which integration is done)

21/11/18

Q) Consider a signal $X(s) = \frac{1}{(s+1)(s+2)}$, $ROC > -1$

$x(t) = ?$

Express by method of partial fractions

$$X(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$A = 1$, $B = -1$ $\Rightarrow X(s) = \frac{1}{s+1} - \frac{1}{s+2}$

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Wednesday

Take $\mathcal{L}^{-1}\{.\}$

$$\Rightarrow \mathcal{L}^{-1}\{X(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= e^{-t}u(t) - e^{-2t}u(t)$$

If ROC > -1 and ROC > -2 $\Rightarrow$ intersection of the 2, i.e., ROC > -1 $(\because R_1 \cap R_2)$

ROC dictates Inverse $\mathcal{L}$-T look

Q.) Find LT of $x(t) = u(t)$

$$\cancel{\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} u(t)e^{st}\,dt = \frac{1}{2\pi j}\int_{0}^{\sigma+j\infty} e^{st}\,dt}$$

$$\cancel{= \frac{1}{2\pi j}\left[\frac{e^{st}}{s}\right]_0^{\sigma+j\infty}}$$

$$\cancel{= \frac{1}{2\pi j s}\left[e\right.}$$

$$\int_{-\infty}^{\infty}\left(u(t)e^{-\sigma t}\right)e^{-j\omega t}\,dt = \frac{1}{s}$$

Q.) $x(t) = \delta(t)$

$$\Rightarrow X(s) = \int_{-\infty}^{\infty}\left(\delta(t)e^{-\sigma t}\right)e^{-j\omega t}\,dt$$

$$= 1 \quad \text{(obviously)}$$

Q.) $x(t) = \cos(\omega t)u(t)$

$$= \frac{1}{2}e^{-j\omega t}u(t) + \frac{1}{2}e^{j\omega t}u(t)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{-j\omega t}e^{-\sigma t}\right)e^{-j\omega t}u(t)\,dt + \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{j\omega t}e^{-\sigma t}\right)e^{-j\omega t}u(t)\,dt$$

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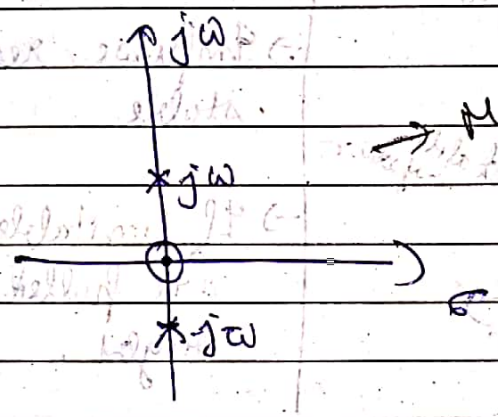
$$= \frac{1}{2}\left[\int_0^\infty e^{-j\omega t} e^{-\sigma t}\,dt + \int_0^\infty e^{-\sigma t}\,dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+j\omega} + \frac{1}{s-j\omega}\right]$$

$$= \boxed{\frac{s}{s^2+\omega^2}} = \frac{s}{(s+j\omega)(s-j\omega)}$$

$\Rightarrow$ Zero at $s=0$, pole at $-j\omega$, $+j\omega$



$\rightarrow$ Marginally stable (unstable hi hai)

Q.) $x(t) = \sin(\omega t)\,u(t)$

$\Rightarrow X(s) = \dfrac{\omega}{s^2+\omega^2}$

Properties of Laplace Transform

1.) Linearity

If $x(t) \xleftrightarrow{L\{\cdot\}} X(s)$ L-T pair

then $a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{L\{\cdot\}} a_1 X_1(s) + a_2 X_2(s)$

2.) Time Shifting

$x(t-t_0) \xleftrightarrow{L\{\cdot\}} e^{-st_0} X(s)$

3) Shifting in Laplace domain (s-domain)

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Wednesday

$$e^{s_0 t} x(t) \xleftrightarrow{L\{\cdot\}} X(s-s_0)$$

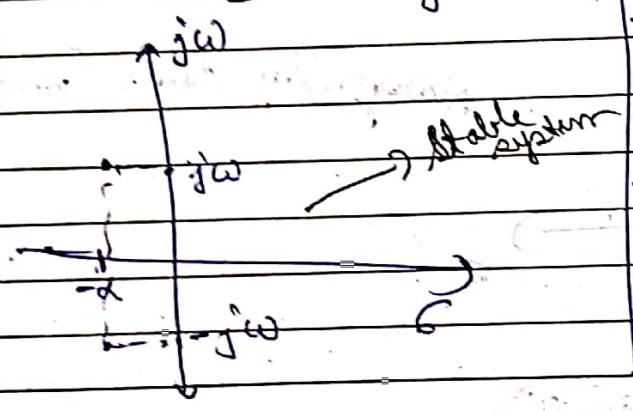
Q.) Find L.T of $e^{-\alpha t}\cos(\omega t)\,u(t)$

$$= \frac{s+\alpha}{(s+\alpha)^2+\omega^2} = \frac{(s+\alpha)}{(s+\alpha+j\omega)(s+\alpha-j\omega)}$$

Zero $\Rightarrow s = -\alpha$

Poles $s_1 = -\alpha + j\omega$

$s_2 = -\alpha - j\omega$



→ Impulse response is stable

→ If unstable, poles are pulled towards right.

Q.) $L\{e^{-\alpha t}\sin(\omega t)\,u(t)\}$

$$\Rightarrow \frac{\omega}{(s+\alpha)^2+\omega^2}$$

4] Convolution Theorem

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI System}\\ h(t)\end{matrix}} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t) \xleftrightarrow{L\{\cdot\}} Y(s) = X(s)\,H(s)$$

(By exactly same method as F.T)

System function or Transfer fn $\Rightarrow H(s) = \dfrac{Y(s)}{X(s)}$

→ This $H(s)$ is the game-changer.

<u>Def^n:-</u> Transfer function $H(s)$ is defined as

$$H(s) = \frac{L\{O/P\}}{L\{I/P\}} = \frac{L\{y(t)\}}{L\{x(t)\}}$$ given zero initial conditions. ~~Valid if~~

Q.) $x(t) = e^{-at}u(t)$; $h(t) = u(t)$
LTI
find $y(t) = ?$

$y(t) = x(t) * h(t)$

But we do, $X(s) = \frac{1}{s+a}$, $H(s) = \frac{1}{s}$

$\therefore Y(s) = X(s)\,H(s)$

$$Y(s) = \left(\frac{1}{s}\right)\left(\frac{1}{s+a}\right)$$

$$= \frac{A}{s} + \frac{B}{s+a}$$

$$\Rightarrow A = \frac{-1}{a},\ B = \frac{1}{a}$$

$$\Rightarrow \left(\frac{-1}{a}\right)\left(\frac{1}{s}\right) + \left(+\frac{1}{a}\right)\left(\frac{1}{s+a}\right)$$

$$\Rightarrow y(t) = \frac{1}{a}\left[1 - e^{-at}\right]u(t)$$

5.] <u>Differentiation property</u>

$$\frac{d}{dt}[x(t)] \xleftrightarrow{L} sX(s)$$

$$\frac{d^2}{dt^2}\{x(t)\} \xleftrightarrow{L} s^2X(s)$$

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Wednesday

Proof :- $x(t) = L^{-1}\{X(s)\}$

$$= \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$$

Diff both sides wrt t under integral sign.

$$\therefore \frac{d}{dt}[x(t)] = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\frac{d}{dt}[e^{st}]\, ds$$

$$= \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} (sX(s))\, e^{st}\, ds$$

$$L\left\{\frac{d}{dt}\{x(t)\}\right\} = = sX(s)$$

6.) Integration property

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{L} \frac{X(s)}{s}$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$$

$$Y(s) = X(s)\, L\{u(t)\} = \frac{X(s)}{s}$$

Q.) For the following series RL circuit, find out expression of i(t) when (i) switch is closed at t = 0

P.T.O

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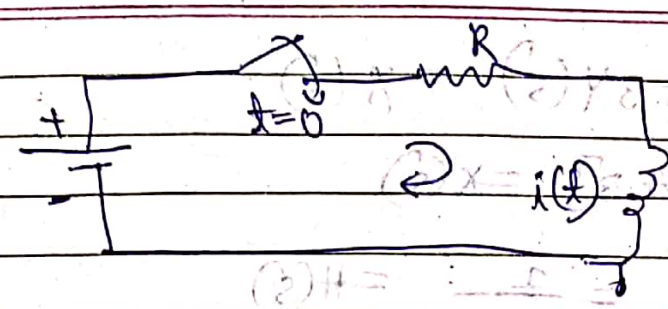
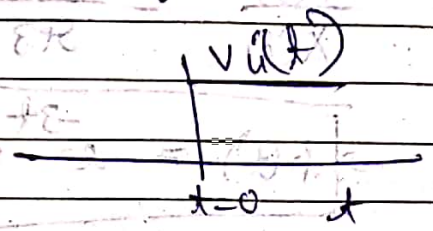
Wednesday

Van Valkenberg

Switch &



$V = V u(t)$



Now,

$$V \cdot u(t) = R\, i(t) + L \frac{di(t)}{dt}$$

Goal $i(t) = ?$

Take L.T on both sides,

$$L\{V u(t)\} = L\{R\, i(t)\} + L\left\{L \frac{di(t)}{dt}\right\}$$

$$\frac{V}{s} = R\, I(s) + L s\, I(s)$$

$$\therefore I(s) = \frac{\frac{V}{s}}{R + Ls} = \frac{\frac{V}{L}}{s\left(s + \frac{R}{L}\right)}$$

$$\mathcal{L}^{-1}\{I(s)\} = i(t) = \frac{V}{L} \times \mathcal{L}^{-1}\left\{\frac{A_1}{s} + \frac{A_2}{s + \frac{R}{L}}\right\}$$

$$= \boxed{\frac{V}{R}\left[1 - e^{-R/L\, t}\right] u(t)}$$

diff eqn :- $\frac{dy(t)}{dt} + 3y(t) = x(t)$

LTI

Find impulse response of system

$$\therefore L\left\{\frac{dy(t)}{dt}\right\} + 3L\{y(t)\} = L\{x(t)\}$$

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Wednesday

$\therefore sY(s) + 3Y(s) = X(s)$

$\Rightarrow Y(s)[s+3] = X(s)$

$\therefore \frac{Y(s)}{X(s)} = \frac{1}{s+3} = H(s)$

$\Rightarrow \boxed{h(t) = e^{-3t} u(t)}$ → $Re\{s\} > -3$

ROC > -3

$= -e^{-3t} u(-t)$

ROC < -3

Analysis of LTI system using L.T

$x(t) \longrightarrow \boxed{\begin{matrix} LTI \\ h(t) \end{matrix}} \longrightarrow y(t)$

$y(t) = x(t) * h(t)$

$Y(s) = X(s) H(s)$

$\boxed{H(s) = \frac{Y(s)}{X(s)}}$ ← Rational in most cases

Systems & Control

$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{b_0 + b_1 j\omega + b_2 (j\omega)^2 + \cdots + b_m (j\omega)^m}{a_0 + a_1 j\omega + a_2 (j\omega)^2 + \cdots a_m (j\omega)^m}$

Weirstrass-Stone approximation

$H(s) = \frac{Y(s)}{X(s)} = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \cdots a_m s^m}$

LCCDE

We can represent a fxn by polynomial.

(i) Roots of Numerator = Zeros (ii) Roots of Deno = Poles

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Wednesday

Analysis of Stability

Chapter 1 BIBO $\longrightarrow$ Chapter 2

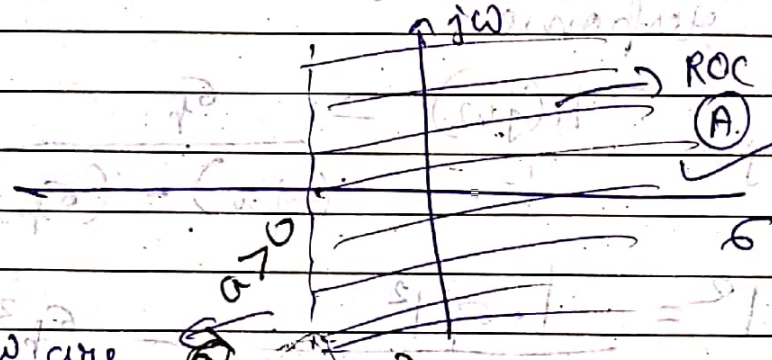
$$\int_{t \in R} |h(t)|\, dt < +\infty$$

(Impulse response Must be absolutely integrable)

$$L\{h(t)\} = H(s) = \frac{N(s)}{D(s)} = \text{(rational + causal)}$$

($\because$ given)

If $h(t) = e^{-at} u(t) \rightarrow$ ROC

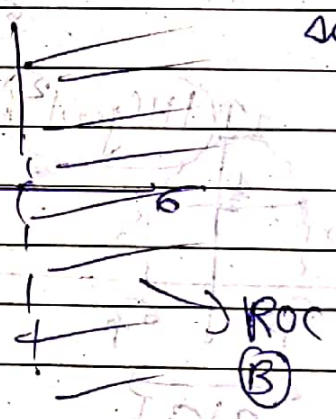


If $h(t)$ is stable
$\Rightarrow$ absolutely integrable
$\Rightarrow$ Dirichlet Cond^n is satisfied
$\downarrow$
If Dirichlet satisfied $\Rightarrow$ F-T exists $\Rightarrow X(j\omega)$ is finite

jω are included (a)

$e^{+at} u(t)$

But ROC here has no jωs



$\rightarrow$ For a causal, stable, rational fxn, the poles must lie towards left side of s-plane, as ROC can't have any poles.

$\Rightarrow$ If any one pole ~~cross~~ crosses jω axis $\Rightarrow$ unstable guaranteed.

P.T.O

Wednesday

* Expression for −3dB Bandwidth in s-domain

In general, system function of 1st order filter,

System function $\boxed{H(s) = \dfrac{-\sigma_p}{s-(s_p)}}$ (Order of a system is no of poles)

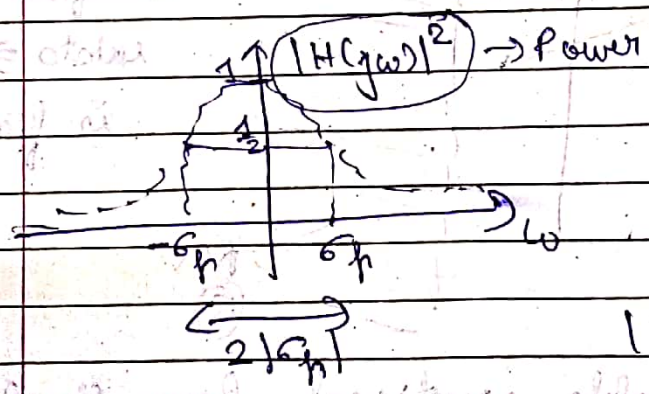
$s = \sigma + j\omega$ = Laplace Transform Variable

$s_p = \sigma_p + j\omega_p$

When we substitute, $s = j\omega$ in s-domain, we get frequency response

$$H(s)\Big|_{s=j\omega} = H(j\omega) = \frac{-\sigma_p}{(j\omega) - (\sigma_p + j\omega_p)}$$

$$|H(j\omega)|^2 = \frac{|-\sigma_p|^2}{\sigma_p^2 + (\omega-\omega_p)^2} = \frac{\sigma_p^2}{\sigma_p^2 + (\omega-\omega_p)^2}$$



$$|H(j\omega)|^2 = \frac{\sigma_p^2}{\sigma_p^2 + \omega^2}$$

$$|H(j\omega)|^2\Big|_{j\omega=0} = 1$$

Find range of freq when power falls from 1 to 1/2

$$\frac{\sigma_p^2}{\sigma_p^2+\omega^2} = \frac{1}{2} \Rightarrow \omega = \pm\sigma_p$$

∴ Half Power Bandwidth (−3dB bandwidth),

rad/sec ← $\omega_{-3dB} = 2|\sigma_p|$

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To get in Hertz,

$$2\pi B(\text{in hertz}) = 2|\sigma_p|$$

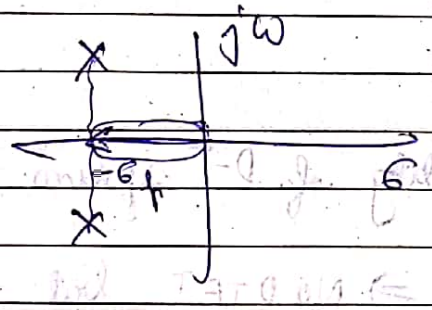
$$\Rightarrow \boxed{B(\text{in hertz}) = \frac{|\sigma_p|}{\pi}}$$

As -3dB BW ↑, quality ↓

$$\Rightarrow \text{Quality } (Q) \propto \frac{1}{B}$$

$$\boxed{Q \uparrow = \frac{F_c}{B\downarrow}}$$

(∵ As B↓ ⇒ less energy loss in system ⇒ quality high)



If $\sigma_p \uparrow \Rightarrow B \uparrow \Rightarrow Q$

σ_p moves more towards left ⇒ low quality filter and vice versa

Thursday

Z-Transform

Connection to LTI System

$x(n) = 2^n$

$$x(n) \longrightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \longrightarrow y(n) = ??$$

$z = x + jy$ $\qquad$ $y(n) = H(z) z^n$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} = Z\{h(n)\}$$

Defn:-

For discrete time signal $x(n)$, $Z\{x(n)\}$ is defined as $\boxed{\sum_{n=-\infty}^{\infty} x(n) z^{-n}}$

Motivation for Z-transform:-

1) L.T → C.T system
   Z.T → D.T system

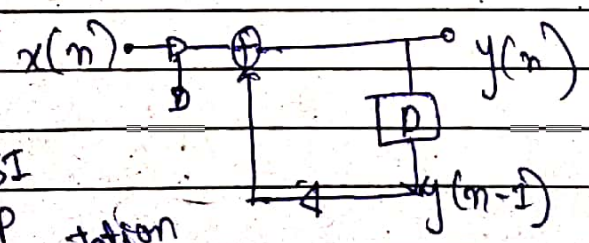
2) To characterize stability of D-T system.

3) If $x(n) = u(n)$ ⇒ No DTFT but we can find their Z transform.

4) Design of digital filters.

Keshav K. Parhi

$y(n) - a y(n-1) = b x(n)$ → Z-transform will tell us how to optimize the structure of filter



VLSI DSP Implementation

→ For cost-effective & low power implementation.

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5] In chapter

$$x(n) \Rightarrow DTFT\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = X(e^{j\omega})$$

$$Z\{x(n)\} \Rightarrow \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$ ⇐ This has notational convenience

$$\boxed{X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega})}$$

6] Solution of difference equation.

7] We can use powerful results from complex variable theory – Cauchy's Residue thm.

$$\oint X(z)\, dz = 0$$

<u>Find Z-trans of:-</u>

1) $x(n) = a^n u(n)$

$$\downarrow$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots \left(\frac{a}{z}\right)^n$$

$$= \frac{1}{1 - az^{-1}} \qquad \boxed{r = \frac{a}{z} \text{ and } \left|\frac{a}{z}\right| < 1}$$

$$X(z) = Z\{a^n u(n)\} = \frac{1}{1 - az^{-1}}$$

$$\Rightarrow Z^{-1}\left\{\frac{1}{1 - az^{-1}}\right\} = a^n u(n)$$

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Thursday

The above Z-transform exists, if $|r| = |az^{-1}| < 1$

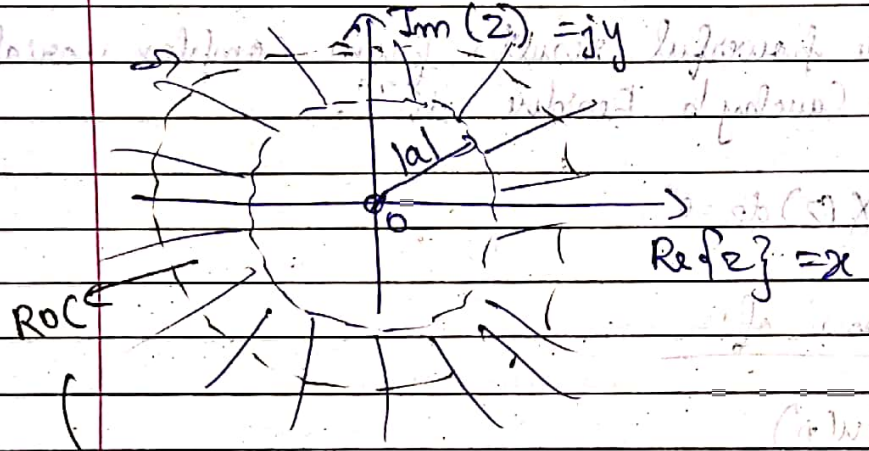
$$\therefore \boxed{\left|\frac{a}{z}\right| < 1 \Rightarrow |z| > |a|}$$

Now, let $z = x + jy$

$$|z| = \sqrt{x^2+y^2}$$

$$\sqrt{x^2+y^2} > |a|$$

$$\Rightarrow \boxed{x^2+y^2 > a^2}$$



→ Set of points of z, for which X(z) exists and converges.

2) $x(n) = -a^n u(-n-1)$

Inverse Z-transform

$$x(n) \underset{Z^{-1}\{\cdot\}}{\overset{Z\{\cdot\}}{\rightleftarrows}} X(z)$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

$$x(n) = \frac{1}{2\pi j}\oint X(z)\,z^{n-1}\,dz$$

Closed contour integral

In most of integrations, $X(z)$ is rational $\Rightarrow$ integration is not needed.

Properties of Z-transform

$$\boxed{x(n) \xleftrightarrow{Z\{\cdot\}} X(z)}$$ → Z-T pair

1) **Linearity**

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{Z} a_1 X_1(z) + a_2 X_2(z)$$

Z-transform is linear transform.

2) **Time Shifting**

$$x(n-k) \xleftrightarrow{Z\{\cdot\}} z^{-k} X(z)$$

$$x(n) \longrightarrow \boxed{\text{Delay}} \longrightarrow y(n) = x(n-1)$$

is same as $x(n) \longrightarrow \boxed{z^{-1}} \longrightarrow y(z) = z^{-1} X(z)$

A delay of 1 corresponds to one z^{-1} multiplier

3) **Convolution theorem**

$$x(n) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h(n)\end{matrix}} \longrightarrow y(n)$$

$$y(n) = x(n) * h(n) \xleftrightarrow{Z\{\cdot\}} X(z)H(z) = Y(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{Z\{y(n)\}}{Z\{x(n)\}}$$

Now, as in s-domain, this is a ~~transform~~ transfer function / system fcn in Z-domain.

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Now, application n^{th} order LCCDE

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$\therefore$ Taking Z-T on both sides

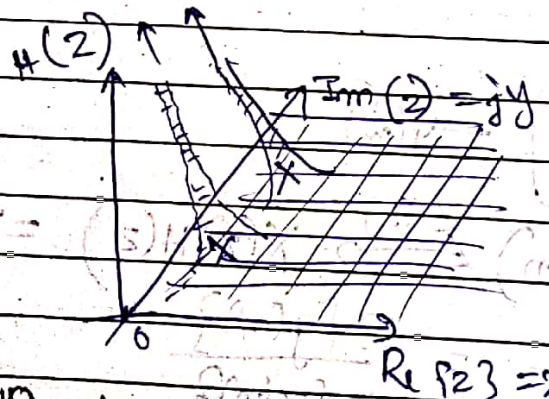
$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = H(z)$$

Same Weierstrass-Stone approximation can be used here too.

$$\therefore H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_m z^{-m}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}}$$

$$H(z) = \frac{N(z)}{D(z)} = \frac{\text{Roots in } N(z) \rightarrow \text{zeros}}{\text{Roots in } D(z) \rightarrow \text{poles}}$$



~~These locations of poles will tell~~

$\rightarrow$ These locations will tell us filter is speaking and what is speaking.

Relationship b/w Z-transform and F.T

$$\boxed{X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega})}$$

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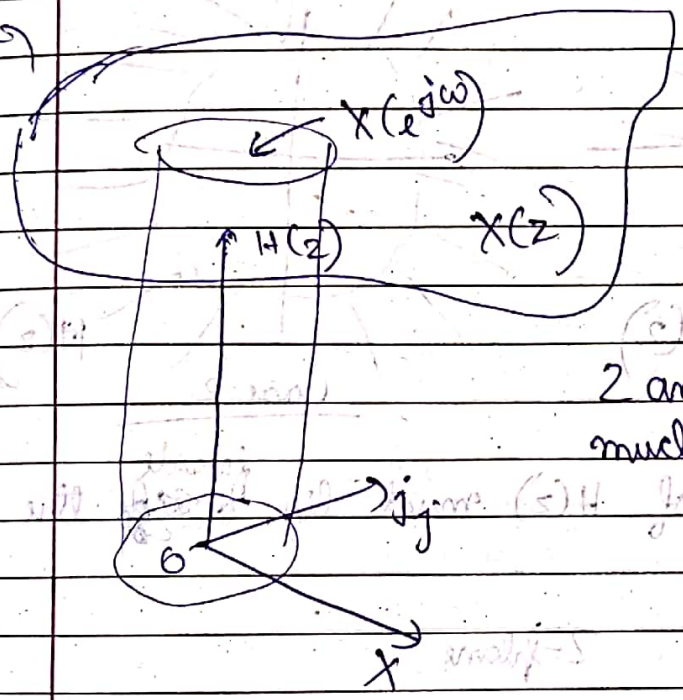
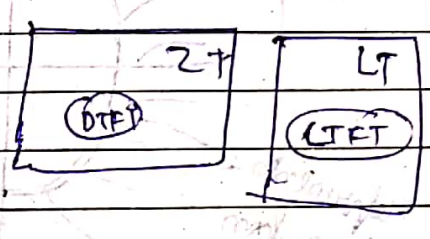
→ $z = e^{j\omega}$ is a unit circle (contour)

$z = re^{j\omega}$ vs $(1)e^{j\omega}$. Thus, only unit circle is F-T, while Z-T is the entire Z-domain of infinite circles.

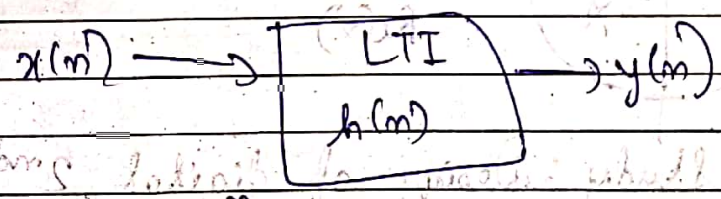
→





Z and L Transforms are much bigger representations.

Analysis of LTI System using Z-T



→ If $h(n) = a^n u(n)$ ⇒ ROC outside unit circle.

→ It is causal as $h(n) = 0 \ \forall n < 0$

→ Claim:- $h(n)$ is stable.

If $h(n)$ is stable ⇒ impulse response is absolutely summable.

$$\Rightarrow \sum_{\langle n \rangle} |h(n)| < +\infty$$

Dirichlet cond^n satisfied, DTO

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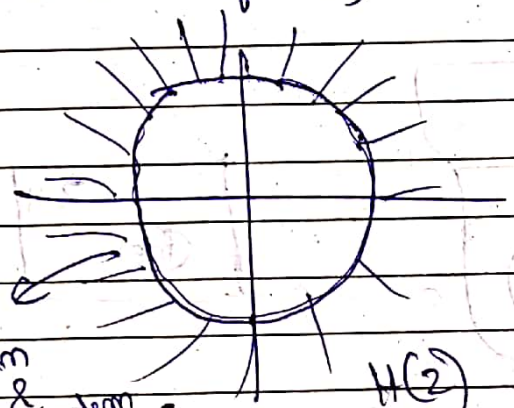
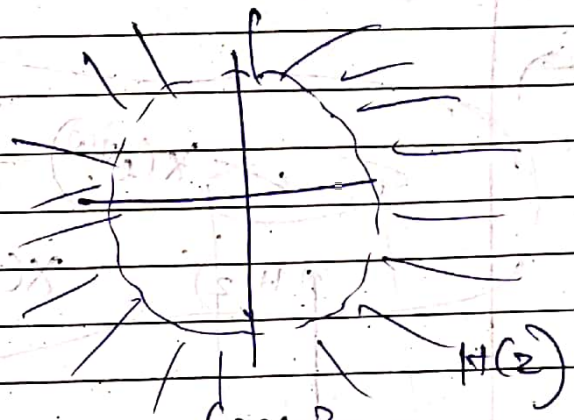
↓

DTFT exists and $H(e^{j\omega})$ is finite and converges.

→ We know ROC can't have any pole at all.
→ ROC includes unit circle (as unit circle for Z.T = DTFT is finite)



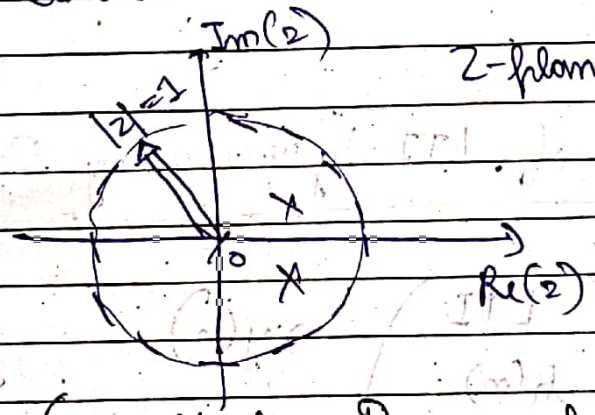


represents system form of stable & causal LTI system

Case 1 H(z)

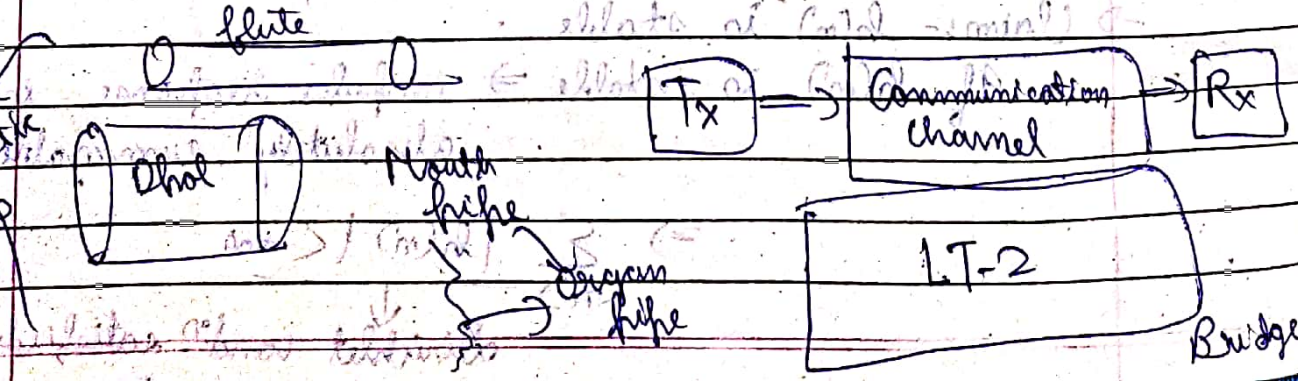
Case 2 H(z)

→ Therefore poles of H(z) must lie ~~inside~~ inside the unit circle.



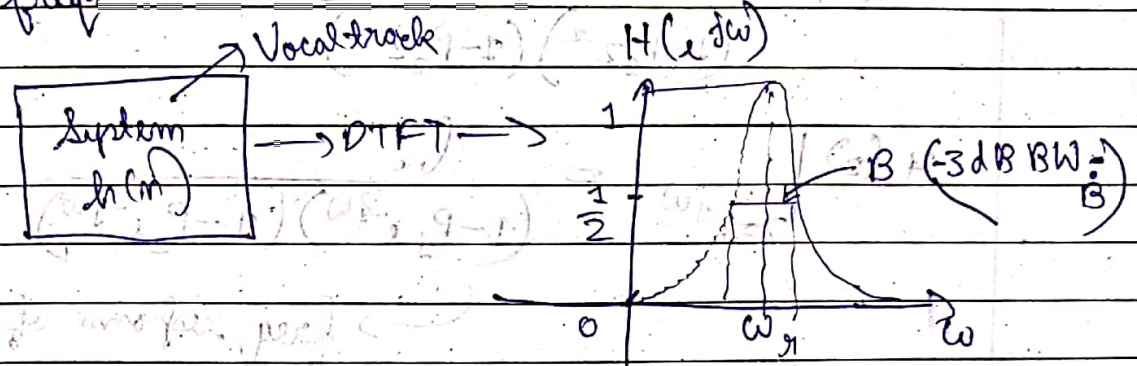
Case-Study : Design of digital 2nd Order Resonator

Q.) What is a resonance?



In RF & Microwave, we study Electrical transmission

→ For each of these systems, spectrum energy of these ~~energy~~ systems is reinforced at or around a single freq.



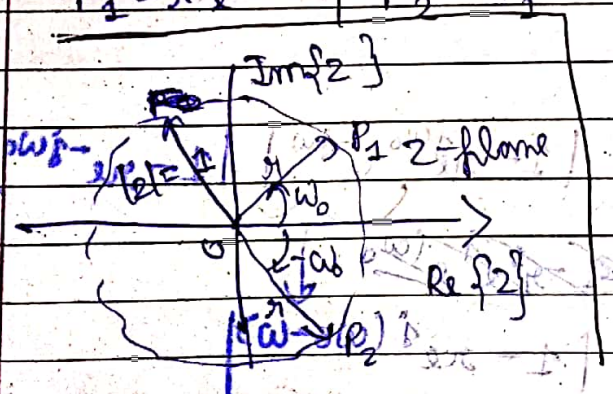
ω_r = resonant frequency of resonator.

B = -3dB Bandwidth ($\frac{1}{2}$ power BW) of resonator.

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}}$$ → 2nd Order resonator system.

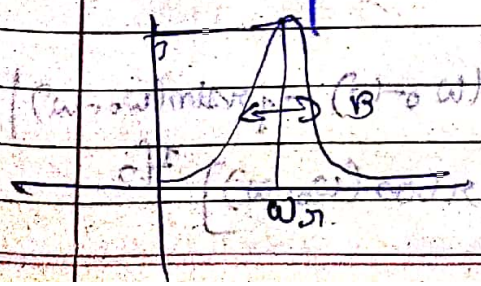
$$= \frac{b_0}{(1 - P_1 z^{-1})(1 - P_2 z^{-2})}$$

$P_1 = r e^{j\omega_0}$, $P_2 = P_1^* = (r e^{j\omega_0})^*$



$H(z)$ = causal & stable resonator

⇒ P_1 and P_2 lie inside the unit circle.



Q.1) How resonant freq ω_r is related to pole angle ω_0?

i.e., $\boxed{\omega_r = F(\omega_0)}$ $F(\cdot)$ =>

Q2) How 0-3dB BW in Hertz is related to pole radius 'r'?

$$B = g(r) \rightarrow g(\cdot) = ?$$

$$H(z) = \frac{b_0}{(1-P_1 z^{-1})(1-P_2 z^{-2})}$$

freq response

$$H(z)\Big|_{z=e^{j\omega}} = \frac{b_0}{(1-P_1 e^{-j\omega})(1-P_2 e^{-j\omega})} = H(e^{j\omega})$$

→ freq response of resonator ke liye z me $e^{j\omega}$ daaldo.

$$\therefore H(e^{j\omega}) = \frac{b_0}{(1-re^{+j\omega_0}e^{-j\omega})(1-re^{-j\omega_0}e^{-j\omega})}$$

$$|H(e^{j\omega})| = \frac{|b_0| \rightarrow \text{constant}}{U_1(\omega)\,U_2(\omega)}$$ → this is important

→ For resonance, we need to minimize $U_1(\omega)$ and $U_2(\omega)$

$$U(\omega) = U_1(\omega)U_2(\omega)$$

$$\therefore \frac{\partial U(\omega)}{\partial \omega} = 0$$

Now, $U(\omega) = |1-re^{j\omega_0}e^{-j\omega}|\,|1-re^{-j\omega_0}e^{-j\omega}|$

$$= \cancel{|1-re^{j(\omega_0}e^{(\omega}}$$

$$= |1-re^{j(\omega_0-\omega)}|$$

$$= |1-r\cos(\omega_0-\omega) - jr\sin(\omega_0-\omega)|$$

$$= [1+r^2-2r\cos(\omega_0-\omega)]^{1/2}$$

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$$\text{ly}, \quad \left|1 - r e^{-j(\omega_0+\omega)}\right| = \left[1+r^2 - 2r\cos(\omega_0+\omega)\right]^{1/2}$$

$$= \left\{\left[(1+r^2)^2 - 2r(1+r^2)\cos(\omega_0+\omega)\right] - 2r(1+r^2)\cos(\omega_0-\omega) + 4r^2\cos(\omega_0-\omega)\cos(\omega_0+\omega)\right]^{1/2}$$

$$\frac{\partial\sqrt{x}}{\partial x} = 0 \Rightarrow \frac{1}{2\sqrt{x}} = 0$$

$$\frac{d}{d\omega}(\cdot) = 0$$

$$\Rightarrow 0 - 2r(1+r^2) \times -\sin(\omega_0+\omega) \times 1 - 2r(1+r^2) \times -\sin(\omega_0-\omega) \times -1 + 4r^2\left[\cos(\omega_0-\omega) \times -\sin(\omega_0+\omega) + \cos(\omega_0+\omega) \times -\sin(\omega_0-\omega) \times -1\right] = 0$$

$$\cos(\omega_r) = \boxed{\frac{1+r^2}{2r}\cos(\omega_0)}$$

This is ω_r as we obtained by diff.

$$\therefore \boxed{\omega_r = \cos^{-1}\left\{\left(\frac{1+r^2}{2r}\right)\cos(\omega_0)\right\}}$$ resonant freq of resonator

↳ answer of Q.1

When $r = 1$

$$\Rightarrow \omega_r = \cos^{-1}\left\{\frac{1+1}{2\times 1}\cos(\omega_0)\right]$$

$\rightarrow \boxed{\omega_r = \omega_0}$ $\Rightarrow$ As pole radius $r \to 1$, i.e. unit circle, then resonant freq approximates well the pole angle ω_0

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Now, Q.2:-

$B = f\, g(r)$

B in Hz $= \frac{|\sigma_p|}{\pi}$

$\Rightarrow$ (a) $\boxed{\sigma_p = \pm \pi B}$

(b) $z = e^{sT}$ is a mapping b/w z-plane and s-plane

called Inverse-invariant transformation (IIT)

This ensures that stable poles in s-domain are mapped to stable poles in z-dom.

$s_p = \sigma_p + j\omega$

$z_p = e^{\sigma_p T} e^{j\omega_p T}$

$z_p = re^{j\omega} \Rightarrow \boxed{r = e^{\sigma_p T}}$

Only if we substitute σ_p as negative, we get $r<1$ which is stable pole

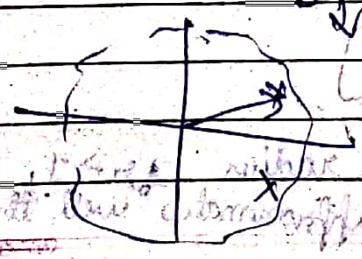
$\therefore r = e^{\pm \pi B T}$

✓ $e^{-\pi B T}$ $\quad$ $e^{+\pi B T}$ ✗

$\therefore$ For stable resonator, $\boxed{r = e^{-\pi B T}}$

$r \propto \frac{1}{B\downarrow}$

$Q\uparrow \propto \frac{1}{B\downarrow} \Rightarrow Q\uparrow \Rightarrow B\downarrow \Rightarrow r\uparrow$



$\therefore r$ must be close to unit circle to give better quality, but r can't cross 1

Thursday

Q.) Find the impulse response of 2nd order resonator?

$$H(z) = \frac{b_0}{(1-P_1 z^{-1})(1-P_2 z^{-1})}$$

$$h(n) = Z^{-1}\{H(z)\}$$

$$H(z) = \frac{A_1}{1-P_1 z^{-1}} + \frac{A_2}{1-P_2 z^{-1}}$$

z^{-1} on both sides

$$\therefore h(n) = A_1 P_1^n u(n) + A_2 P_2^n u(n)$$

$$A_1 = \frac{b_0 P_1}{P_1 - P_2}, \quad A_2 = \frac{b_0 P_2}{P_2 - P_1}$$

$$\therefore h(n) = \left[\frac{b_0 (re^{j\omega_0})}{re^{j\omega_0} - re^{-j\omega_0}}\right](re^{j\omega_0})^n u(n)$$

$$- \left[\frac{b_0 re^{-j\omega_0}}{re^{j\omega_0} - re^{-j\omega_0}}\right](re^{-j\omega_0})^n u(n)$$

$$\therefore \boxed{h(n) = \frac{b_0 r^n}{\sin[\omega_0]} \cdot \sin[\omega_0(n+1)]\, u(n)}$$

just like analog 2nd order system in 6th chapter.

impulse response of 2nd order digital system resonator.

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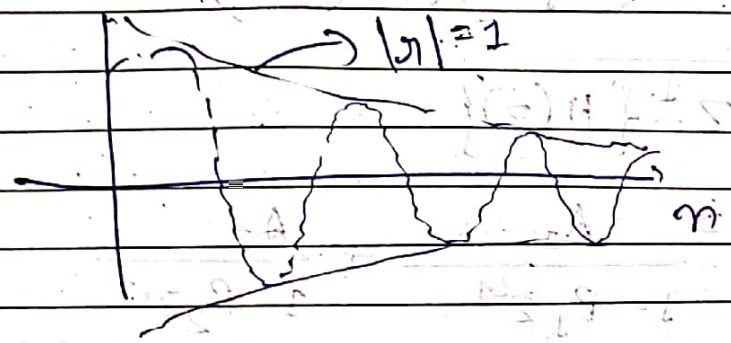
Inferences :-

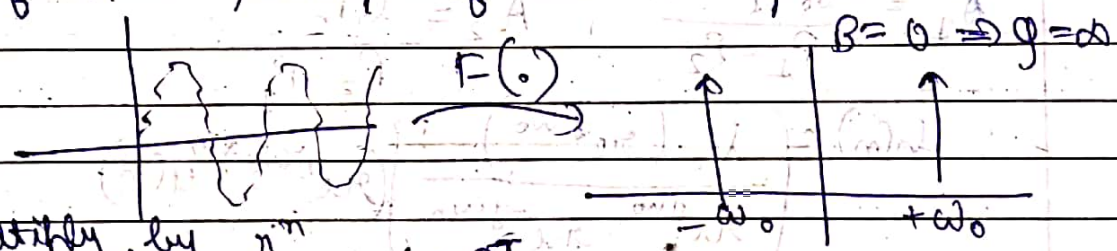
1) Impulse response $h(n)$ of 2nd order resonator is damped sinusoid if $|r| < 1$ (stable filter)



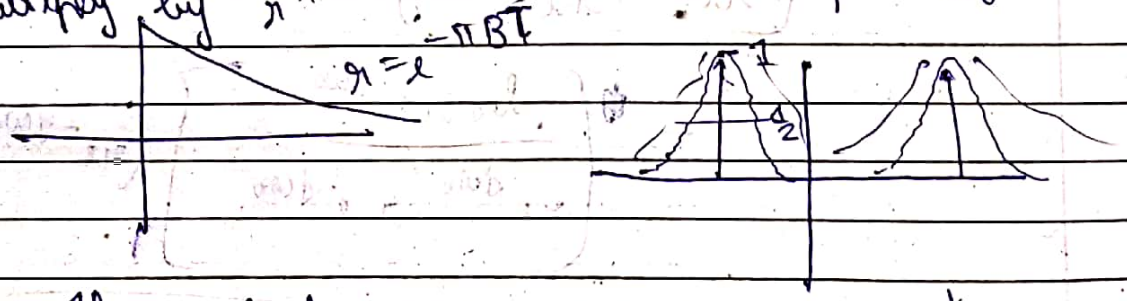
2) If $|r| > 1$ ⇒ unstable

3) $h(n)$ is causal.

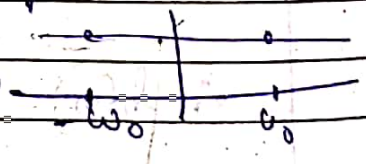
4) If $|r| = 1$, shape of $h(n)$ = pure sinusoid



Multiply by r^n



If we keep on increasing β ⇒



⇒ as $\beta = \infty$, $g = 0$

This means that the concept of resonance breaks down.

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