$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Properties of $\mathbb{R}$

(1) $\mathbb{R}$ is a field $(+, -, \times, \div)$

(2) $\mathbb{R}$ is totally ~~ordered~~ ordered set (T.O.S.)
(linearly ordered set) (l.o.s.)

$\leq$ is a relation on $\mathbb{R}$

$x, y \in \mathbb{R}$

1° $x \leq x \ \forall x \in \mathbb{R}$

2° if $x \leq y$ & $y \leq x \Rightarrow x = y$

3° if $x \leq y$ & $y \leq z \Rightarrow x \leq z$

4° For any two $x, y \in \mathbb{R}$ either $x \leq y$ or $y \leq x$

(other example $\mathbb{Q}$ is also T.O.S.)

Example (1) $X = \{(m,n) \mid m, n \in \mathbb{N}\} \longrightarrow$ P.O.S. (Partially ordered set)

$(m,n) \leq (m',n')$ if $m \leq m'$ & $n \leq n'$

not T.O.S.

4° does not hold ($(1,2) \not\leq (2,1)$ & $(2,1) \not\leq (1,2)$)

Find other such examples.

Ex - (2) $X = \{(m,n) \mid m, n \in \mathbb{N}\}$

$(m,n) \leq (m',n')$ if either $m < m'$ or else $m = m'$ and $n \leq n'$

(lexicographic)

(Prove it!) X is T.O.S.

(3) $\mathbb{R}$ is ordered field. ($\mathbb{Q}$ is also O.F.)

1° $\mathbb{R}$ is T.O.S.

2° whenever $x \geq y$ then $x+z \geq y+z$ $\forall z \in \mathbb{R}$

3° whenever $x \geq 0$ & $y \geq 0$ then $x \cdot y \geq 0$

(4) In $\mathbb{R}$ if $x \cdot y = 0 \Rightarrow$ either $x = 0$ or $y = 0$

(Property of field)

(5) Every O.F. has the following properties

(1) If $a \geq 0$ and $x \geq y \Rightarrow ax \geq ay$

(2) If $a \leq 0$ & $x \geq y$ then $ax \leq ay$

(3) If $0 < x \leq y$ then $0 < \frac{1}{y} \leq \frac{1}{x}$

(6) $\mathbb{R}$ has l.u.b property ($\mathbb{Q}$ does not have l.u.b. property)

$\mathbb{N}$ has l.u.b. property

$\boxed{\text{l.u.b}}$ T.O.S. X has l.u.b. property if every subset ($\neq \phi$) of X that is bounded above has a sup. in X.

$\boxed{\text{l.u.b. of a set}}$ X T.O.S. & $S \subseteq X$

$a \in X$ called the sup. or l.u.b. of S

if (1) a is u.b. of S

(2) No element b of X (which is less than a) can be upper bound.

(7) Let $x \in \mathbb{R}$ $\exists\, n \in \mathbb{N}$ s.t. $n > x$

By #, suppose $n \leq x \;\forall\, n \in \mathbb{N}$

$\Rightarrow x$ is u.b. for $\mathbb{N}$ $\Rightarrow$ $\mathbb{N}$ is a $(\neq \phi) \subseteq \mathbb{R}$

which is bounded above & so by l.u.b. property it has a sup., say 's'.

now $s-1 < s$ $\Rightarrow s-1$ can not be u.b. for $\mathbb{N}$

$\Rightarrow$ there must be $n \in \mathbb{N}$ s.t. $n > s-1$

$\Rightarrow$ $n+1 > s$ (# that s is sup.) #.

(8) $x < y$ $(x, y \in \mathbb{R})$ then there is a rational no. q s.t. $x < q < y$

$\mathbb{Q}$ are dense in $\mathbb{R}$ $(\overline{\mathbb{Q}} = \mathbb{R})$

Absolute Value function

$x \in \mathbb{R}$ $$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Properties of $|x|$

(1) $|x| \geq 0$ & $|x| = 0$ iff $x = 0$

(2) $|xy| = |x||y|$, In particular $|-x| = |x|$

(3) $-|x| \leq x \leq |x|$

(4) $|x| \leq a \Leftrightarrow -a \leq x \leq a$

(5) $|x+y| \leq |x| + |y|$ (6) $|x-y| \geq ||x| - |y||$

Algebraic numbers

Number 'a' algebraic if $f(a)=0$, $f(x)$ is nonzero poly with rational coff.

Ex ① All rational nos ($x=\frac{a}{b} \Rightarrow bx-a=0$)

② $\sqrt{2}$ & $\sqrt[3]{3/2}$

($x^2-2=0$) ($8x^3-3=0$)

③ ϕ (Golden ratio) is alg. ($\because \phi^2-\phi-1=0$)

1° Set of algebraic nos. is countable

2° All algebraic nos are computable

3° Set of alg. no. forms a field.

~~Transcend~~ Transcendental numbers

A no. 'a' is called Transcendental if it is not algebraic

Ex ① π (1882 after 2500 years) ③ $\sin x$

② e ④ $\sum_{k=1}^{\infty} 10^{-k!} = 0.110001\ldots$

Set of Tran. Nos. is uncountably ∞ (Not $\mathbb{R}$ & $\mathbb{C}$ are uncountable)

⑤ e^{π} ⑥ Ω ⑦ W ⑧ $2^{\sqrt{2}}$ (Hilbert

⑨ $i^{i} = 0.207879576\ldots$

OPEN PROBLEMS

$\pi+e$, $\pi-e$, $\pi \cdot e$, π/e, π^{π}, e^{e}, π^{e} Not known

$$\boxed{\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \pi}$$