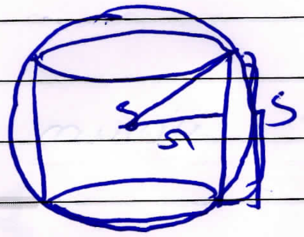
TUTORIAL - 03

Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius 5.

Let the radius of the cylinder be r cm.

Then the height $h = 2\sqrt{25 - r^2}$.



The volume of the cylinder,

$$V(r) = 2\pi r^2 \sqrt{25 - r^2}$$

$$\Rightarrow V'(r) = 2\pi r^2 \left(\frac{1}{2\sqrt{25 - r^2}}\right)(-2r) + 2\pi\sqrt{25 - r^2}\,(2r)$$

$$= \frac{-2\pi r^3 + 4\pi r(25 - r^2)}{\sqrt{25 - r^2}}$$

$$= \frac{2\pi r(-r^2 + 50 - 2r^2)}{\sqrt{25 - r^2}}$$

$$\therefore V'(r) = \frac{2\pi r(50 - 3r^2)}{\sqrt{25 - r^2}}$$

Now, $V'(r) = 0 \Rightarrow 2\pi r = 0$ or $50 - 3r^2 = 0$

$\therefore r = 0$ $\quad\quad$ $\therefore r = 5\sqrt{\frac{2}{3}}$

which is not possible

Now, $V'(r) > 0$ for $0 < r < 5\sqrt{\frac{2}{3}}$

and $V'(r) < 0$ for $5\sqrt{\frac{2}{3}} < r < 5$ ($\because r$ can be maximum 5)

$\Rightarrow r = 5\sqrt{\frac{2}{3}}$ (check for $v''(r)$)

$\therefore r = 5\sqrt{\frac{2}{3}} \Rightarrow h = 2\sqrt{25 - r^2} = 2\sqrt{25 - \frac{50}{3}}$

$\therefore h = 2\sqrt{\frac{25}{3}} = \frac{10}{\sqrt{3}}$

$\therefore$ volum $v = \pi r^2 h = \frac{22}{7} \times \frac{50}{3} \times \frac{10}{\sqrt{3}} = \frac{11000}{21\sqrt{3}}$

You operate a tour service that offers a following rates: Rs 200 per person if 50 people (the minimum number to book the tour) go on the tour. For each additional person, upto maximum of 80 people total, the rate per person is reduce by Rs 2. It costs Rs 6000 (a fixed cost) plus Rs 32 per person to conduct the tour. How many people does it take to maximize your profit?

let x represents the number of people over 50.

The profit function, $P(x) = (50+x)(200-2x) - 32(50+x) - 6000$

$(50+x)(200-2x)$ is total many you get.

$32(50+x) + 6000$ is expenditure you have to make for the trip.

$P(x) = -2x^2 + 68x + 2400$

$\Rightarrow P'(x) = -4x + 68$

$\Rightarrow P'(x) = 0$

$\therefore 4x + 68 = 0$

$\therefore x = 17$

$\Rightarrow P''(x) = -4 < 0.$

so, at $x = 17$, maximum profit can be made.
so, it would takes 67 people to maximize the profit.

Q.3 A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

- The trucker's average speed is 79.8 mph. By MVT, the trucker must have been going that speed at least once during the trip.

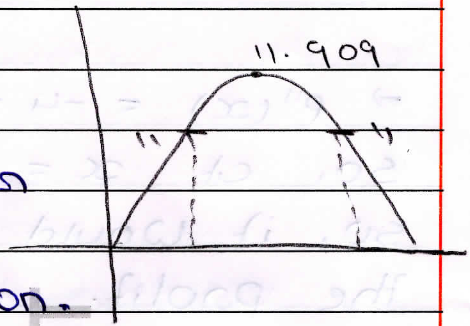
- That means she must had speed up from 65 mph to reach 79.8 mph.

(MVT says $\exists\ c \in (a,b)$ at which the instataneous speed of c is equal the average speed.)

- So the trucker was cited for speeding as the speed limit is 65 mph.

A marathoner ran the 26.2-mi marathon in 2.2 hours. Show that at least twice the marathoner was running at exactly 11 mph, assuming the initial and final speeds are zero.

- The runner's average speed for the marathon was approximately 11.909 mph. By the MVT, the runner must have been going that speed at least once during the marathon.



- Since the initial speed and final speed both are zero (0 mph) and the runner's speed is continue, by intermediate value theorem, the runner's speed must have been 11 mph at least twice.

If $f'(x) = 0$ for all $x \in (a,b)$, then $f(x)$ is constant. Prove it.

Let x_1 and x_2 be two points in (a,b) and $x_1 < x_2$.

Consider $[x_1, x_2]$

f is differentiable on (x_1, x_2)
f is continuous on $[x_1, x_2]$

$\Rightarrow \exists\, c \in (x_1, x_2)$ such that,

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_2 - x_1}$$

But $f'(c) = 0$ $(\because$ as $f'(x) = 0 \;\forall x \in (a,b))$

$\Rightarrow f(x_1) = f(x_2)$

Since x_1 and x_2 are arbitrary points in (a,b), $f(x)$ is constant throughout.

Q.6 Show that $|\cos x - 1| \leq |x|$ for all x-values.
(Hint : Consider $f(t) = \cos t$ on $[0, x]$).

$f(t) = \cos t$ on $[0, x]$
$\cos t$ is continuous on $[0, x]$,
$\cos t$ is differentiable on $(0, x)$.

$\Rightarrow \exists\, c \in (0, x)$ such that,

$$f'(c) = \frac{f(x) - f(c)}{x - 0}$$

$$\therefore -\sin c = \frac{\cos x - \cos 0}{x - 0} = \frac{\cos x - 1}{x}$$

Now, as $-1 \leq \sin c \leq 1$

$-1 \leq -\sin c \leq 1$

$\therefore -1 \leq \dfrac{\cos x - 1}{x} \leq 1$

if $x > 0$, $-1 \leq \dfrac{\cos x - 1}{x} \leq 1 \Rightarrow -x \leq \cos x - 1 \leq x$

$\Rightarrow \therefore |\cos x - 1| \leq |x|$

if $x < 0$, $-1 \leq \dfrac{\cos x - 1}{x} \leq 1 \Rightarrow -x \geq \cos x - 1 \geq x$

$\Rightarrow x \leq \cos x - 1 \leq -x$

$\Rightarrow -(-x) \leq \cos x - 1 \leq -x$

$\Rightarrow |\cos x - 1| \leq -x = |x|$

Q.7 If f has a finit 3rd derivative f''' on $[a,b]$ and if $f(a) = f(b) = f'(a) = f'(b) = 0$, prove that $f'''(c) = 0$ for some $c \in (a,b)$.

- f is continuous on $[a,b]$
  f is differentiable on (a,b).
  $f(a) = f(b)$

By Rolls theorem there exist $\varepsilon_1 \in (a,b)$ such that $f'(\varepsilon_1) = 0$.

Again applying Rolls theorem on f' in the interval $[a, \varepsilon_1]$

f' is continuous on $[a, \varepsilon_1]$
f' is differentiable on (a, ε_1).
$f'(a) = f'(\varepsilon_1)$

$\Rightarrow$ ~~$\exists\ \varepsilon_3 \in (\varepsilon_1, b)$~~.
$\Rightarrow \exists\ \varepsilon_2 \in (a, \varepsilon_1)$ such that $f''(\varepsilon_2) = 0$.

Again applying Rolls theorem on f' in the interval $[\varepsilon_1, b]$

f' is continuous on $[\varepsilon_1, b]$
f' is differentiable on (ε_1, b)
$f'(\varepsilon_1) = f'(b)$

$\Rightarrow \exists\ \varepsilon_3 \in (\varepsilon_1, b)$ such that $f''(\varepsilon_3) = 0$.

Now apply Rolle's theorem on f'' in the interval $[\xi_2, \xi_3]$,

f'' is continuous on $[\xi_2, \xi_3]$.
f'' is differentiable on (ξ_2, ξ_3).
$f''(\xi_2) = f''(\xi_3)$

$\Rightarrow \exists\ \xi_4 \in (\xi_2, \xi_3)$ such that $f'''(\xi_4) = 0$.

Hence, there exist $\xi_4 \in (a,b)$ such that $f'''(\xi_4) = 0$.

Q.8 Assume f has a finite derivation in (a,b) and is continuous on $[a,b]$ with $f(a) = f(b) = 0$. Prove that for every real λ there is some $c \in (a,b)$ such that $f'(c) = \lambda f(c)$.

Let us consider the function,
$g(x) = e^{-\lambda x} f(x)$; $x \in \mathbb{R}$, $\lambda \in \mathbb{R}$.

g is continuous on $[a,b]$
g is differentiable on (a,b).

and $g(a) = e^{-\lambda a} f(a) = 0$
$g(b) = e^{-\lambda b} f(b) = 0$.

So, applying Rolle's theorem on g,

there exists $c \in (a,b)$ such that $g'(c) = 0$

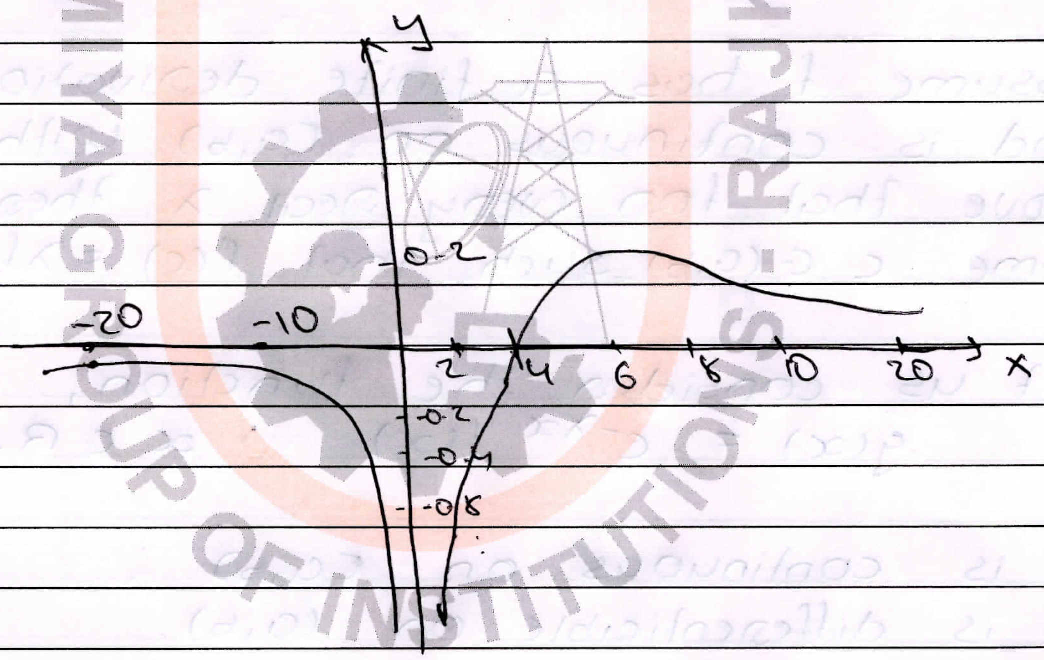
$$g'(c) = (-\lambda)e^{-\lambda c}f(c) + e^{-\lambda c}f'(c) = 0$$

$$\Rightarrow e^{-\lambda c}(f'(c) - \lambda f(c)) = 0$$

But $e^{-\lambda c} \neq 0 \Rightarrow f'(c) - \lambda f(c) = 0$

$$f'(c) = \lambda f(c)$$

9 Plot the graph of the function $f(x) = \frac{x-4}{x^2}$ and then use derivative to discuss any significant features of the plot.



A plot of the function on the interval $-20 \leq x \leq 20$. is given above.

→ As $\lim_{x \to 0^-} f(x) = -\infty$ $\lim_{x \to 0^+} f(x) = -\infty$

So, $x=0$ i.e., y-axis is a vertical asymptote.

→ As $\lim_{x \to -\infty} f(x) = 0^-$, $\lim_{x \to +\infty} f(x) = 0^+$

So, $y=0$ i.e, x-axis is a horizontal asymptote.

→ Since, $f(x)=0$ when $x=4$, the graph cross the x-axis at $x=4$.

now, $f'(x) = \dfrac{x^2(1) - (x-4)2x}{x^4}$

$= \dfrac{8-x}{x^3}$

$\Rightarrow \dfrac{8-x}{x^3} = 0$

$\therefore$ $x=8$ and $x=0$ are the critical points.

Since $f(0)$ not defined so $x=0$ is not a critical point.

So,	$x < 8$	$x > 8$
Sign of f'	+	−
behavior of f'	increasing	decreasing

$\Rightarrow x = 8$ is a local maximum point.

Now, $f''(x) = \dfrac{x^3(-1) - (8-x)(3x^2)}{x^6}$

$$= \frac{2(x-12)}{x^4}$$

$$\Rightarrow \frac{2(x-12)}{x^4} = 0$$

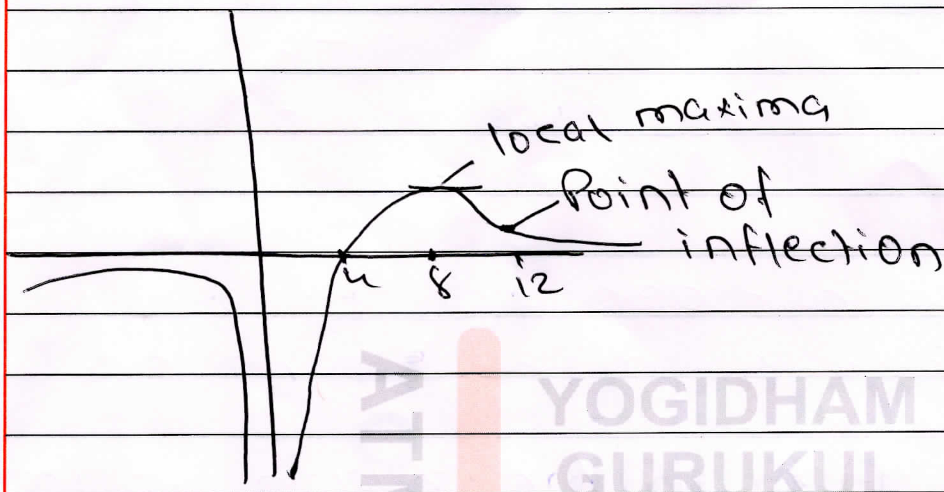
$f''(x)$ is not defined at $x=0$, but as $f(0)$ is not defined also at $x=0$. So we don't consider the point $x=0$.

	$x < 12$	$x > 12$
Sign of f'	$-$	$+$
behavior of f'	concave down	concave up

Summarizing the information from above two table we get,

$x < 8$	$8 < x < 12$	$x > 12$
increasing	decreasing	decreasing
concave down	concave down	concave up

So, $x = 12$ is an inflection point.



- x-axis is horizontal asymptote.
- y axis is vertical asymptote.
- $f(x)$ crosses x-axis at $x = 4$.