The above admittance is then normalized with respect to the characteristic impedance of the stub, i.e.,

$$\tilde{y}_{\text{stub}} \equiv Y_{\text{stub}} \times Z_{0s} \equiv \frac{-j0.9}{150} \times 50 = -j0.3$$

(*g*) Finally, we have to determine the length of the short-circuited stub having characteristic impedance of $50\,\Omega$, which provides the normalized admittance of $\tilde{y}_{\text{stub}} = -j0.3$. Now, since the Smith chart is used as the admittance chart, hence the short circuit point is represented by the extreme right-hand side of the chart denoted by the point S, and the value read on the WTG scale at this point is 0.25. For calculating the length of the stub, we start at this point and move towards the generator until we reach the scale corresponding to the normalized admittance of $-j\ 0.3$ denoted by the point K on the chart with the value read on the WTG scale as 0.454. Hence, the length of the stub is given by

$$L_{\text{stub}} \equiv (0.454 - 0.25)\lambda = 0.204\lambda$$

D11.8 A normalized load, $z_L = 2 - j1$, is located at $z = 0$ on a lossless $50\ \Omega$ line. Let the wavelength be 100 cm. (*a*) A short-circuited stub is to be located at $z = -d$. What is the shortest suitable value for d? (*b*) What is the shortest possible length of the stub? Find s: (*c*) on the main line for $z < -d$; (*d*) on the main line for $-d < z < 0$; (*e*) on the stub.

Ans. 12.5 cm; 12.5 cm; 1.00; 2.62; ∞

11.14 TRANSIENT ANALYSIS

Animations

Throughout most of this chapter, we have considered the operation of transmission lines under steady-state conditions, in which voltage and current were sinusoidal and at a single frequency. In this section we move away from the simple time-harmonic case and consider transmission line responses to voltage step functions and pulses, grouped under the general heading of *transients*. These situations were briefly considered in Section 11.2 with regard to switched voltages and currents. Line operation in transient mode is important to study because it allows us to understand how lines can be used to store and release energy (in pulse-forming applications, for example). Pulse propagation is important in general since digital signals, composed of sequences of pulses, are widely used.

We will confine our discussion to the propagation of transients in lines that are lossless and have no dispersion, so that the basic behavior and analysis methods may be learned. We must remember, however, that transient signals are necessarily composed of numerous frequencies, as Fourier analysis will show. Consequently, the question of dispersion in the line arises, since, as we have found, line propagation constants and reflection coefficients at complex loads will be frequency-dependent. So in general, pulses are likely to broaden with propagation distance, and pulse shapes may change when reflecting from a complex load. These issues will not be considered in detail here, but they are readily addressed when the precise frequency dependences of β and Γ are known. In particular, $\beta(\omega)$ can be found by evaluating the imaginary part of γ, as given in Eq. (41), which would in general include the frequency dependences of R, C, G, and L arising from various mechanisms. For example, the skin effect (which affects both the conductor resistance and

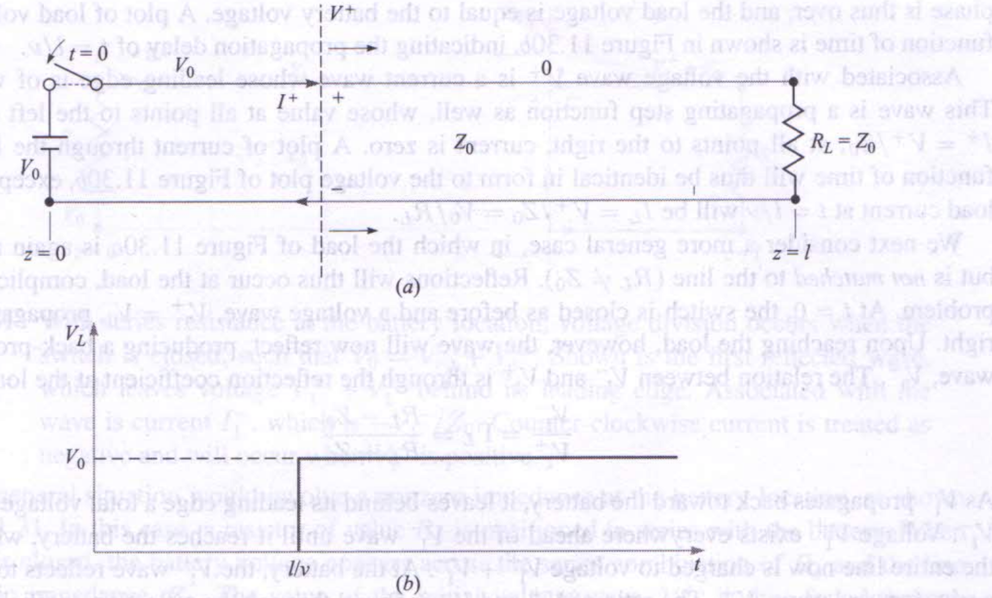


Fig. 11.30 (a) Closing the switch at time $t = 0$ initiates voltage and current waves V^+ and I^+. The leading edge of both waves is indicated by the dashed line, which propagates in the lossless line toward the load at velocity ν. In this case, $V^+ = V_0$; the line voltage is V^+ everywhere to the left of the leading edge, where current is $I^+ = V^+/Z_0$. To the right of the leading edge, voltage and current are both zero. Clockwise current, indicated here, is treated as positive and will occur when V^+ is positive. (b) Voltage across the load resistor as a function of time, showing the one-way transit time delay, l/ν.

the internal inductance) will result in frequency-dependent R and L. Once $\beta(\omega)$ is known, pulse broadening can be evaluated using the methods to be presented in Chapter 13.

We begin our basic discussion of transients by considering a lossless transmission line of length l terminated by a matched load, $R_L = Z_0$, as shown in Figure 11.30a. At the front end of the line is a battery of voltage V_0, which is connected to the line by closing a switch. At time $t = 0$, the switch is closed, and the line voltage at $z = 0$ becomes equal to the battery voltage. This voltage, however, does not appear across the load until adequate time has elapsed for the propagation delay. Specifically, at $t = 0$, a voltage wave is initiated in the line at the battery end, which then propagates toward the load. The leading edge of the wave, labeled V^+ in Figure 11.30, is of value $V^+ = V_0$. It can be thought of as a propagating step function, since at all points to the left of V^+, the line voltage is V_0; at all points to the right (not yet reached by the leading edge), the line voltage is zero. The wave propagates at velocity ν, which in general is the group velocity in the line.[4] The wave reaches the load at time $t = l/\nu$ and then does not reflect, since the load is matched. The transient

[4] Since we have a step function (composed of many frequencies) as opposed to a sinusoid at a single frequency, the wave will propagate at the group velocity. In a lossless line with no dispersion as considered in this section, $\beta = \omega\sqrt{LC}$, where L and C are constant with frequency. In this case we would find that the group and phase velocities are equal; i.e., $d\omega/d\beta = \omega/\beta = \nu = 1/\sqrt{LC}$. We will thus write the velocity as ν, knowing it to be both ν_p and ν_g.

phase is thus over, and the load voltage is equal to the battery voltage. A plot of load voltage as a function of time is shown in Figure 11.30*b*, indicating the propagation delay of $t = l/\nu$.

Associated with the voltage wave V^+ is a current wave whose leading edge is of value I^+. This wave is a propagating step function as well, whose value at all points to the left of V^+ is $I^+ = V^+/Z_0$; at all points to the right, current is zero. A plot of current through the load as a function of time will thus be identical in form to the voltage plot of Figure 11.30*b*, except that the load current at $t = l/\nu$ will be $I_L = V^+/Z_0 = V_0/R_L$.

We next consider a more general case, in which the load of Figure 11.30*a* is again a resistor but is *not matched* to the line ($R_L \neq Z_0$). Reflections will thus occur at the load, complicating the problem. At $t = 0$, the switch is closed as before and a voltage wave, $V_1^+ = V_0$, propagates to the right. Upon reaching the load, however, the wave will now reflect, producing a back-propagating wave, V_1^-. The relation between V_1^- and V_1^+ is through the reflection coefficient at the load

$$\frac{V_1^-}{V_1^+} = \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \tag{115}$$

As V_1^- propagates back toward the battery, it leaves behind its leading edge a total voltage of $V_1^+ + V_1^-$. Voltage V_1^+ exists everywhere ahead of the V_1^- wave until it reaches the battery, whereupon the entire line now is charged to voltage $V_1^+ + V_1^-$. At the battery, the V_1^- wave reflects to produce a new forward wave, V_2^+. The ratio of V_2^+ and V_1^- is found through the reflection coefficient at the battery:

$$\frac{V_2^+}{V_1^-} = \Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1 \tag{116}$$

where the impedance at the generator end, Z_g, is that of the battery, or zero.

V_2^+ (equal to $-V_1^-$) now propagates to the load, where it reflects to produce backward wave $V_2^- = \Gamma_L V_2^+$. This wave then returns to the battery, where it reflects with $\Gamma_g = -1$, and the process repeats. Note that with each round trip the wave voltage is reduced in magnitude since $|\Gamma_L| < 1$. Because of this the propagating wave voltages will eventually approach zero, and steady state is reached.

The voltage across the load resistor can be found at any given time by summing the voltage waves that have reached the load and have reflected from it up to that time. After many round trips, the load voltage will be in general:

$$V_L = V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \cdots$$

$$= V_1^+\left(1 + \Gamma_L + \Gamma_g\Gamma_L + \Gamma_g\Gamma_L^2 + \Gamma_g^2\Gamma_L^2 + \Gamma_g^2\Gamma_L^3 + \cdots\right)$$

With a simple factoring operation, the preceding equation becomes

$$V_L = V_1^+(1 + \Gamma_L)\left(1 + \Gamma_g\Gamma_L + \Gamma_g^2\Gamma_L^2 + \cdots\right) \tag{117}$$

Allowing time to approach infinity, the second term in parentheses in (117) becomes the power series expansion for the expression $1/(1 - \Gamma_g\Gamma_L)$. Thus, in steady state we obtain

$$V_L = V_1^+\left(\frac{1 + \Gamma_L}{1 - \Gamma_g\Gamma_L}\right) \tag{118}$$

In our present example, $V_1^+ = V_0$ and $\Gamma_g = -1$. Substituting these into (118), we find the expected result in steady state: $V_L = V_0$.

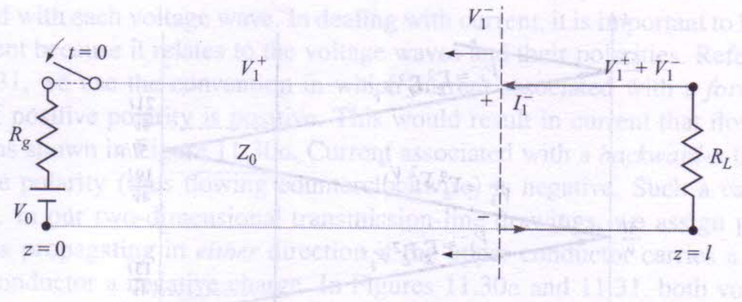


Fig. 11.31 With series resistance at the battery location, voltage division occurs when the switch is closed, such that $V_0 = V_{rg} + V_1^+$. Shown is the first reflected wave, which leaves voltage $V_1^+ + V_1^-$ behind its leading edge. Associated with the wave is current I_1^-, which is $-V_1^-/Z_0$. Counter-clockwise current is treated as negative and will occur when V_1^- is positive.

A more general situation would involve a nonzero impedance at the battery location, as shown in Figure 11.31. In this case, a resistor of value R_g is positioned in series with the battery. When the switch is closed, the battery voltage appears across the series combination of R_g and the line characteristic impedance, Z_0. The value of the initial voltage wave, V_1^+, is thus found through simple voltage division, or

$$\boxed{V_1^+ = \frac{V_0 Z_0}{R_g + Z_0}} \tag{119}$$

With this initial value, the sequence of reflections and the development of the voltage across the load occurs in the same manner as determined by (117), with the steady-state value determined by (118). The value of the reflection coefficient at the generator end, determined by (116), is $\Gamma_g = (R_g - Z_0)/(R_g + Z_0)$.

A useful way of keeping track of the voltage at any point in the line is through a *voltage reflection diagram*. Such a diagram for the line of Figure 11.31 is shown in Figure 11.32*a*. It is a two-dimensional plot in which position on the line, z, is shown on the horizontal axis. Time is plotted on the vertical axis and is conveniently expressed as it relates to position and velocity through $t = z/\nu$. A vertical line, located at $z = l$, is drawn which, together with the ordinate, defines the z axis boundaries of the transmission line. With the switch located at the battery position, the initial voltage wave, V_1^+, starts at the origin, or lower left corner of the diagram ($z = t = 0$). The location of the leading edge of V_1^+ as a function of time is shown as the diagonal line that joins the origin to the point along the right-hand vertical line that corresponds to time $t = l/\nu$ (the one-way transit time). From there (the load location), the position of the leading edge of the reflected wave, V_1^-, is shown as a "reflected" line which joins the $t = l/\nu$ point on the right boundary to the $t = 2l/\nu$ point on the ordinate. From there (at the battery location), the wave reflects again, forming V_2^+, shown as a line parallel to that for V_1^+. Subsequent reflected waves are shown, and their values are labeled.

The voltage as a function of time at a given position in the line can now be determined by adding the voltages in the waves as they intersect a vertical line drawn at the desired location. This addition is performed starting at the bottom of the diagram ($t = 0$) and progressing upward (in time). Whenever a voltage wave crosses the vertical line, its value is added to the total at that

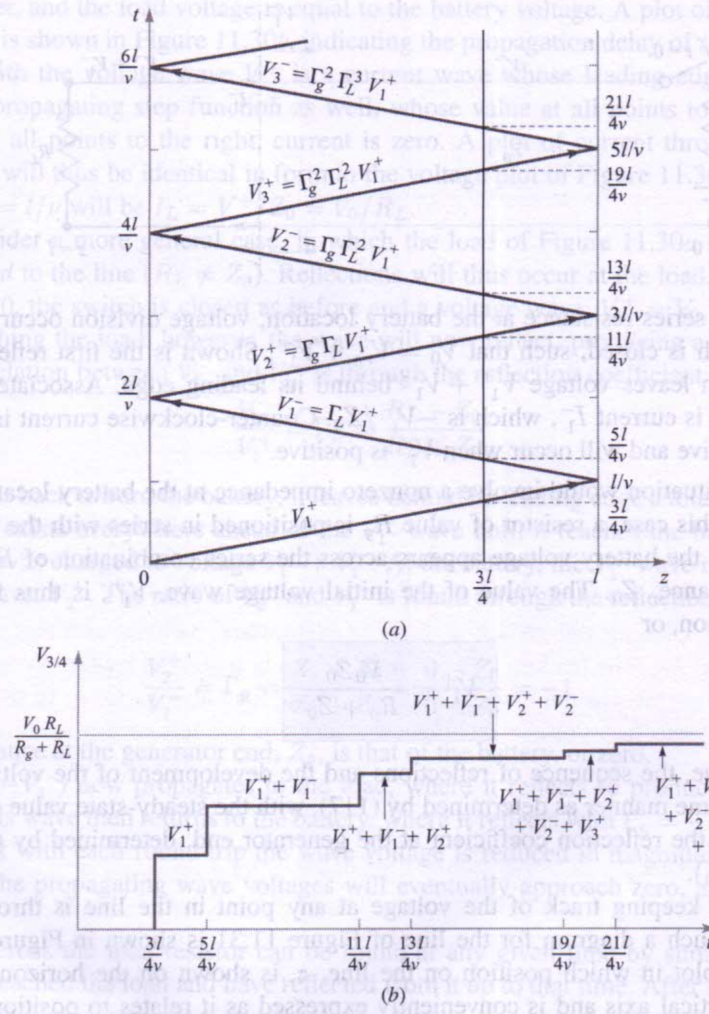


Fig. 11.32 (*a*) Voltage reflection diagram for the line of Figure 11.31. A reference line, drawn at $z = 3l/4$, is used to evaluate the voltage at that position as a function of time. (*b*) The line voltage at $z = 3l/4$ as determined from the reflection diagram of (*a*). Note that the voltage approaches the expected $V_0 R_L/(R_g + R_L)$ as time approaches infinity.

time. For example, the voltage at a location three-fourths the distance from the battery to the load is plotted in Figure 11.32*b*. To obtain this plot, the line $z = (3/4)l$ is drawn on the diagram. Whenever a wave crosses this line, the voltage in the wave is added to the voltage that has accumulated at $z = (3/4)l$ over all earlier times. This general procedure enables one to easily determine the voltage at any specific time and location. In doing so, the terms in (117) that have occurred up to the chosen time are being added, but with information on the time at which each term appears.

Line current can be found in a similar way through a *current reflection diagram*. It is easiest to construct the current diagram directly from the voltage diagram by determining a value for current that is associated with each voltage wave. In dealing with current, it is important to keep track of the *sign* of the current because it relates to the voltage waves and their polarities. Referring to Figures 11.30a and 11.31, we use the convention in which current associated with a *forward-z* traveling voltage wave of positive polarity is positive. This would result in current that flows in the clockwise direction, as shown in Figure 11.30a. Current associated with a *backward-z* traveling voltage wave of positive polarity (thus flowing counterclockwise) is negative. Such a case is illustrated in Figure 11.31. In our two-dimensional transmission-line drawings, we assign positive polarity to voltage waves propagating in *either* direction if the upper conductor carries a positive charge and the lower conductor a negative charge. In Figures 11.30a and 11.31, both voltage waves are of positive polarity, so their associated currents will be net positive for the forward wave and net negative for the backward wave. In general, we write

$$\boxed{I^+ = \frac{V^+}{Z_0}} \tag{120}$$

and

$$\boxed{I^- = -\frac{V^-}{Z_0}} \tag{121}$$

Finding the current associated with a backward-propagating voltage wave immediately requires a minus sign, as (121) indicates.

Figure 11.33a shows the current reflection diagram that is derived from the voltage diagram of Figure 11.32a. Note that the current values are labeled in terms of the voltage values, with the appropriate sign added as per (120) and (121). Once the current diagram is constructed, current at a given location and time can be found in exactly the same manner as voltage is found using the voltage diagram. Figure 11.33b shows the current as a function of time at the $z = (3/4)l$ position, determined by summing the current wave values as they cross the vertical line drawn at that location.

EXAMPLE 11.19

In Figure 11.31, $R_g = Z_0 = 50\,\Omega$, $R_L = 25\,\Omega$, and the battery voltage is $V_0 = 10$ V. The switch is closed at time $t = 0$. Determine the voltage at the load resistor and the current in the battery as functions of time.

Solution. Voltage and current reflection diagrams are shown in Figure 11.34a and b. At the moment the switch is closed, half the battery voltage appears across the 50-ohm resistor, with the other half comprising the initial voltage wave. Thus $V_1^+ = (1/2)V_0 = 5$ V. The wave reaches the 25-ohm load, where it reflects with reflection coefficient

$$\Gamma_L = \frac{25-50}{25+50} = -\frac{1}{3}$$

So $V_1^- = -(1/3)V_1^+ = -5/3$ V. This wave returns to the battery, where it encounters reflection coefficient $\Gamma_g = 0$. Thus, no further waves appear; steady state is reached.

Once the voltage wave values are known, the current reflection diagram can be constructed. The values for the two current waves are

$$I_1^+ = \frac{V_1^+}{Z_0} = \frac{5}{50} = \frac{1}{10}\ \text{A}$$

and

$$I_1^- = -\frac{V_1^-}{Z_0} = -\left(-\frac{5}{3}\right)\left(\frac{1}{50}\right) = \frac{1}{30}\text{ A}$$

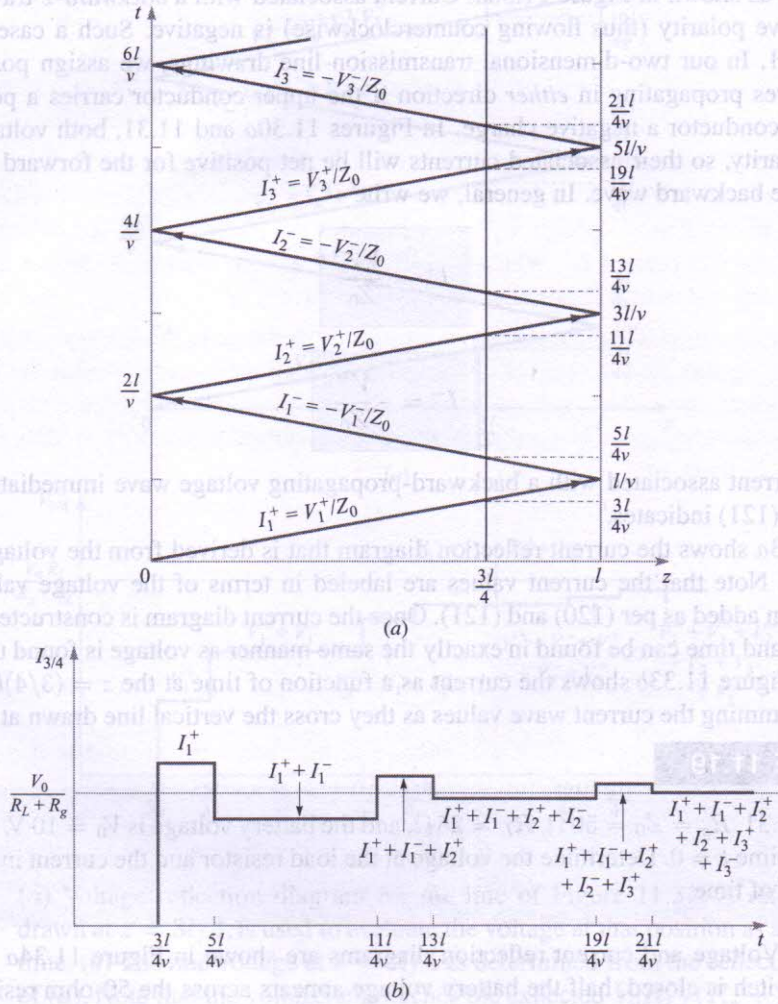


Fig. 11.33 (*a*) Current reflection diagram for the line of Figure 11.31 as obtained from the voltage diagram of Figure 11.32*a*. (*b*) Current at the $z = 3l/4$ position as determined from the current reflection diagram, showing the expected steady-state value of $V_0/(R_L + R_g)$.

Note that no attempt is made here to derive I_1^- from I_1^+. They are both obtained independently from their respective voltages.

The voltage at the load as a function of time is now found by summing the voltages along the vertical line at the load position. The resulting plot is shown Figure 11.35*a*. Current in the battery

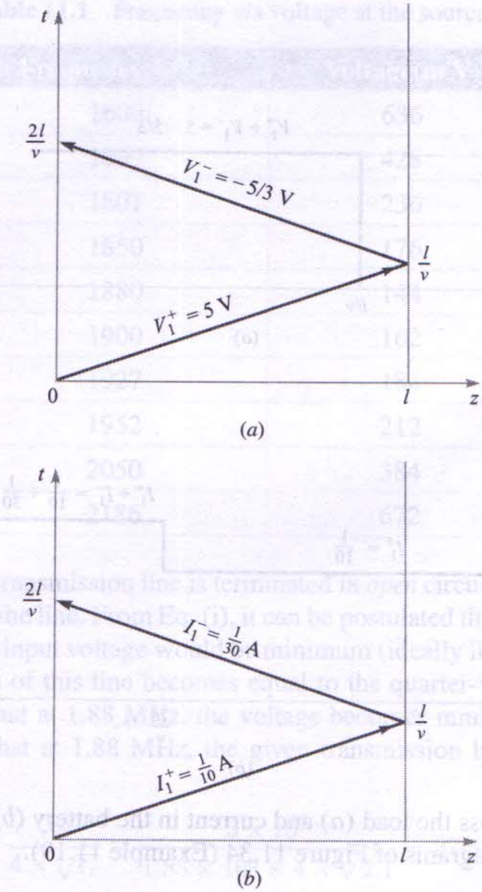


Fig. 11.34 Voltage (a) and current (b) reflection diagrams for Example 11.19.

is found by summing the currents along the vertical axis, with the resulting plot shown as Figure 11.35b. Note that in steady state, we treat the circuit as lumped, with the battery in series with the 50- and 25-ohm resistors. Therefore, we expect to see a steady-state current through the battery (and everywhere else) of

$$I_B(\text{steady state}) = \frac{10}{50+25} = \frac{1}{7.5}\text{ A}$$

This value is also found from the current reflection diagram for $t > 2l/\nu$. Similarly, the steady-state load voltage should be

$$V_L(\text{steady state}) = V_0\frac{R_L}{R_g+R_L} = \frac{(10)(25)}{50+25} = \frac{10}{3}\text{ V}$$

which is found also from the voltage reflection diagram for $t > l/\nu$.

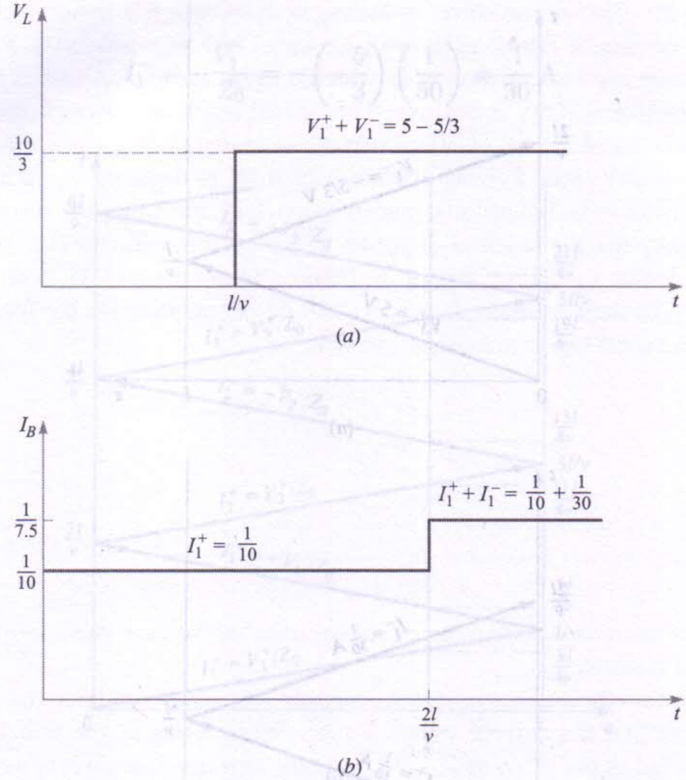


Fig. 11.35 Voltage across the load (*a*) and current in the battery (*b*) as determined from the reflection diagrams of Figure 11.34 (Example 11.19).

EXAMPLE 11.20

A quarter-wavelength-long transmission line, often known as the quarter-wave transformer, is quite often being used for matching a load with a line having different characteristic impedance. However, this concept can also be used to determine the length of a long line with the help of a variable-frequency voltage oscillator and an oscilloscope. Suppose, you are given a transmission line of length l, which is open-circuited at its far end. You apply a sinusoidal voltage of fixed amplitude, and measure the voltage observed at the input of the line by changing the frequency of the oscillator as shown in Table 11.1. You can assume the line to be filled with Teflon ($\epsilon_r = 2.1$). Determine the length of the line 'l'. Explain all the intermediate steps.

Solution. If the characteristic impedance of transmission line is assumed to be Z_0 then the input impedance of this quarter-wave line terminated in the load Z_L is given by the following expression:

$$Z_{\text{in}} = \frac{Z_0^2}{Z_L}$$

Table 11.1 Frequency v/s voltage at the source end

Frequency(kHz)	Voltage(m V)
1600	636
1699	428
1801	236
1850	176
1880	144
1900	162
1927	184
1952	212
2050	384
2186	672

Now, it is given that the transmission line is terminated in *open* circuit ($Z_L \equiv \infty$), and the voltage is measured at the input of the line. From Eq. (i), it can be postulated that for the open circuited line, the input impedance or the input voltage would be minimum (ideally it would be zero) at the center frequency when the length of this line becomes equal to the quarter-wavelength. From the above table, it can be observed that at 1.88 MHz, the voltage becomes minimum for the open circuited line. Hence, we can say that at 1.88 MHz, the given transmission line acts like a quarter-wave transformer, which gives:

$$l = \frac{\lambda_0}{4 \times \sqrt{\epsilon_r}} = \frac{3 \times 10^8}{1.88 \times 10^6 \times 4 \times \sqrt{2.1}} = 27.529 \text{ m} \quad \text{(ii)}$$

where, λ_0 is the free space wavelength at the frequency of 1.88 MHz, and it is given that $\epsilon_r = 2.1$.

EXAMPLE 11.21

A time-domain reflectometer (TDR) is an instrument, which is based on transient analysis. It is mainly used to locate faults and damages on long transmission lines by sending a step voltage down the line, and observing the voltage at any specific position along the line as a function of time. Suppose a TDR having source impedance equal to the characteristic impedance of the line is connected to the input of a 100 Ω matched transmission line filled with Polystyrene ($\epsilon_r = 2.25$) as shown in Figure 11.36(a). The resistance R_d in this figure represents the additional shunt resistance, which might occur at this position due to some damage of the line at a distance l from the source point.

If the voltage waveform shown in Figure 11.36(b) was observed at a distance d from the source end of the TDR, determine (a) the position of the observation point d, (b) the source voltage V_0, (c) the position l at which the damage occurs on the line, and (d) the shunt resistance R_d resulting from the damage of the line. (e) Draw the waveform seen at the observation point of the TDR if no fault or damage is detected on the line.

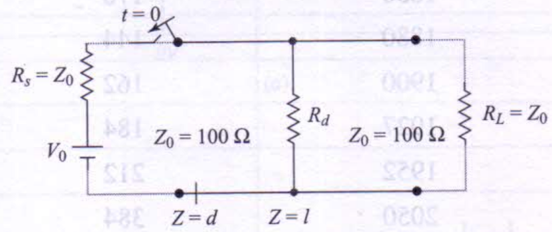


Fig. 11.36(a) Transmission line

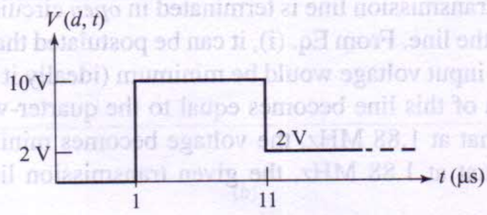


Fig. 11.36(b) Voltage waveform

Solution.

(a) From Figure 11.36(b), it seen that the voltage recorded at the position d is zero till $t = 1$ μs. It means that the voltage wave starting from the source end ($d = 0$) would take $t = 1$ μs to reach the observation point. The location of the observation point 'd' can hence be calculated using the following expression:

$$d = v \times t_1 \equiv \frac{t_1 \times c}{\sqrt{\epsilon_r}} = \frac{10^{-6} \times 3 \times 10^8}{\sqrt{2.25}} = 200 \text{ m} \quad \text{(i)}$$

where, v is the velocity along the transmission line filled with the Polystyrene material $t_1 = 1$ μs is time taken by the voltage wave to travel the distance d from source to the observation point, and c is the free-space velocity.

(b) It is given that the line is fully matched at both the source and the load end when there is no fault or damage along the line. It can also be observed from Figure 11.36(b) that the damage on the line occurs only at $t = 11$ μs when seen at the observation point. Hence, at $t_1 = 1$ μs, the transmission line at the observation point would appear to be fully matched and the voltage at this location would be simply the initial voltage wave V_1^+, which is given as 10 V as seen in Figure 11.36(b). Meanwhile, the initial voltage wave can be related to the source voltage V_0

using the following expression:

$$V_1^+ = V_0\left(\frac{Z_0}{R_s+Z_0}\right) \equiv\equiv V_0\left(\frac{Z_0}{Z_0+Z_0}\right) = \frac{V_0}{2} \equiv 10\ \text{V} \tag{ii}$$

From above equation, it is deduced that the source voltage $V_0 = 20$ V.

(*c*) When there is some damage or fault on the transmission line, then it would not appear to be matched as some additional shunt resistance appears on the line due to this damage. From Figure 11.36(*b*), it is seen that the damage on the line changes the level of the voltage at the observation point at $t = 11$ μs. Now, if the damage on the line is assumed to be at position l, then the voltage wave would travel a total distance of $2(l - d)$ so that the effect of this damage appears on the observation point $z = d$. Hence, the position of the disturbance on the transmission line can be determined using the following relationship:

$$\Delta t \equiv (t_2 - t_1) = \frac{2(l-d)}{v} \equiv \frac{2(l-d)}{c/\sqrt{\epsilon_r}} \equiv \frac{2(l-d)\times\sqrt{2.25}}{3\times 10^8} \equiv (11-1)\times 10^{-6} \tag{iii}$$

From above equation, one obtains $(l - d) \equiv \dfrac{10\times 10^{-6}\times 3\times 10^8}{2\times 1.5} = 1000$ m

Hence, the position l at which the damage occurs along the line is given by

$$l = 1000 + 200 = 1200\ \text{m}$$

(*d*) The shunt resistance R_d resulting from the damage of the line can be obtained by taking note of the fact that the total voltage recorded at the observation point after the occurrence of damage $t \geq 11\mu s$ is the sum of the initial voltage wave, and the reflected wave which appears due to the line mismatch at $t = 11\mu s$ in Figure 11.37 i.e.,

$$V(d, t = 11\mu s) \equiv V_1^+ + \Gamma_{Ld}V_1^+ \equiv 10[1 + \Gamma_{Ld}] = 2\ V \tag{iv}$$

where Γ_{Ld} is the reflection coefficient occurring due to mismatch at $z = l = 1200$ m shown in Figure 11.36(*a*). The value of this reflection coefficient from the above equation may be determined as

$$\Gamma_{Ld} = \frac{1}{5} - 1 \equiv -\frac{4}{5} = -0.8 \tag{v}$$

The reflection coefficient Γ_{Ld} may be related to the characteristic impedance of the line Z_0 and the resultant load impedance occurring due to damage at $z = l$ in Figure 11.37(*a*) using the following expression:

$$\Gamma_{Ld} = \left(\frac{Z_{Ld} - Z_0}{Z_{Ld} + Z_0}\right) \equiv \left(\frac{Z_{Ld} - 100}{Z_{Ld} + 100}\right) = -0.8 \quad\Rightarrow\quad Z_{Ld} = \frac{100}{9}\Omega \tag{vi}$$

Now, it is to be noted that the impedance Z_{Ld} defined in the above equation represents the parallel combination of the shunt resistance R_d and the characteristic impedance Z_0 as can be seen from Figure 11.36(*a*), i.e.,

$$Z_{Ld} \equiv \frac{R_d\, Z_0}{R_d + Z_0} \equiv \frac{R_d \times 100}{R_d + 100} = \frac{100}{9}\Omega \tag{vii}$$

The above equation can be solved to obtain the shunt resistance occurring due to damage, i.e.,

$$R_d = 12.5\ \Omega$$

(*e*) If no damage were detected on the transmission line then the shunt resistance R_d would not be present indicating that its value would be given by $R_d = \infty$. Under this situation, the line would be fully matched, and the waveform seen at the observation point of the TDR would like as shown in Figure 11.36(*c*). We can infer from Figure 11.36(*c*) that when no damage is detected on the transmission line then no reflection is present anywhere on the line.

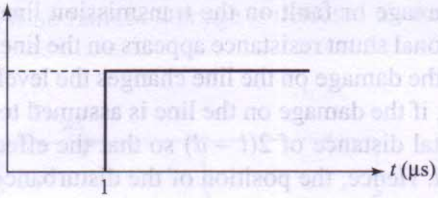


Fig. 11.36(*c*) Waveform.

Another type of transient problem involves lines that are *initially charged*. In these cases, the manner in which the line discharges through a load is of interest. Consider the situation shown in Figure 11.37, in which a charged line of characteristic impedance Z_0 is discharged through a resistor of value R_g when a switch at the resistor location is closed.[5] We consider the resistor at the $z = 0$ location; the other end of the line is open (as would be necessary) and is located at $z = l$.

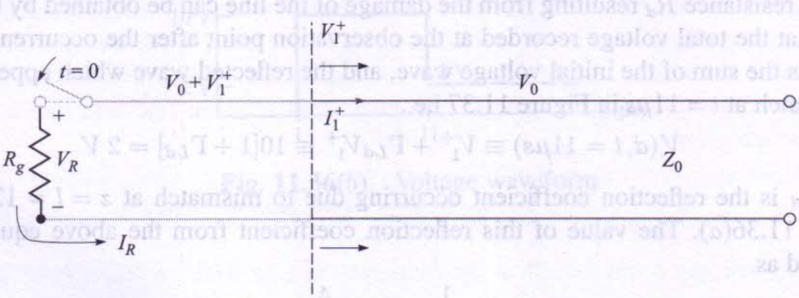


Fig. 11.37 In an initially charged line, closing the switch as shown initiates a voltage wave of opposite polarity to that of the initial voltage. The wave thus depletes the line voltage and will fully discharge the line in one round trip if $R_g = Z_0$.

When the switch is closed, current I_R begins to flow through the resistor, and the line discharge process begins. This current does not immediately flow everywhere in the transmission line but begins at the resistor and establishes its presence at more distant parts of the line as time progresses. By analogy, consider a long line of automobiles at a red light. When the light turns green, the cars at the front move through the intersection first, followed successively by those further toward the rear. The point which divides cars in motion and those standing still is in fact a wave which propagates toward the back of the line. In the transmission line, the flow of charge progresses in a similar way. A voltage wave, V_1^+, is initiated and propagates to the right. To the left of its leading edge, charge is in motion; to the right of the leading edge, charge is stationary and carries its original

[5] Even though this is a load resistor, we will call it R_g because it is located at the front (generator) end of the line.

density. Accompanying the charge in motion to the left of V_1^+ is a drop in the charge density as the discharge process occurs, and so the line voltage to the left of V_1^+ is partially reduced. This voltage will be given by the sum of the initial voltage, V_0, and V_1^+, which means that V_1^+ must in fact be negative (or of opposite sign to V_0). The line discharge process is analyzed by keeping track of V_1^+ as it propagates and undergoes multiple reflections at the two ends. Voltage and current reflection diagrams are used for this purpose in much the same way as before.

Referring to Figure 11.37, we see that for positive V_0 the current flowing through the resistor will be counterclockwise and hence negative. We also know that continuity requires that the resistor current be equal to the current associated with the voltage wave, or

$$I_R = -I_1^+ = -\frac{V_1^+}{Z_0}$$

Now the resistor voltage will be

$$V_R = V_0 + V_1^+ = I_R R_g = -I_1^+ R_g = -\frac{V_1^+}{Z_0} R_g$$

We solve for V_1^+ to obtain

$$\boxed{V_1^+ = \frac{-V_0 Z_0}{Z_0 + R_g}} \tag{122}$$

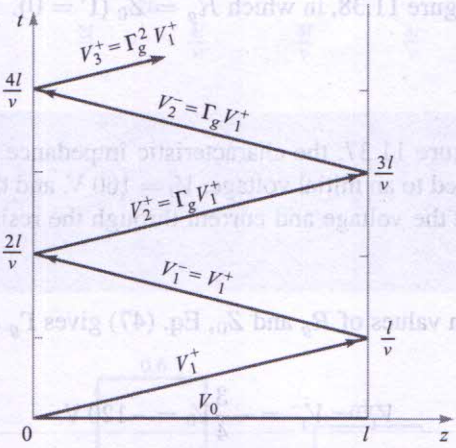


Fig. 11.38 Voltage reflection diagram for the charged line of Figure 11.37, showing the initial condition of V_0 everywhere on the line at $t = 0$.

Having found V_1^+, we can set up the voltage and current reflection diagrams. That for voltage is shown in Figure 11.38. Note that the initial condition of voltage V_0 everywhere on the line is accounted for by assigning voltage V_0 to the horizontal axis of the voltage diagram. The diagram is otherwise drawn as before, but with $\Gamma_L = 1$ (at the open-circuited load end). Variations in how the line discharges thus depend on the resistor value at the switch end, R_g, which determines the reflection coefficient, Γ_g, at that location. The current reflection diagram is derived from the voltage diagram in the usual way. There is no initial current to consider.

A special case of practical importance is that in which the resistor is matched to the line, or $R_g = Z_0$. In this case, Eq. (122) gives $V_1^+ = -V_0/2$. The line fully discharges in one round trip of V_1^+ and produces a voltage across the resistor of value $V_R = V_0/2$, which persists for time $T = 2l/\nu$. The resistor voltage as a function of time is shown in Figure 11.39. The transmission line in this application is known as a *pulse-forming line*; pulses that are generated in this way are well formed and of low noise, provided the switch is sufficiently fast. Commercial units are available that are capable of generating high-voltage pulses of widths on the order of a few nanoseconds, using thyratron-based switches.

When the resistor is not matched to the line, full discharge still occurs, but does so over several reflections, leading to a complicated pulse shape.

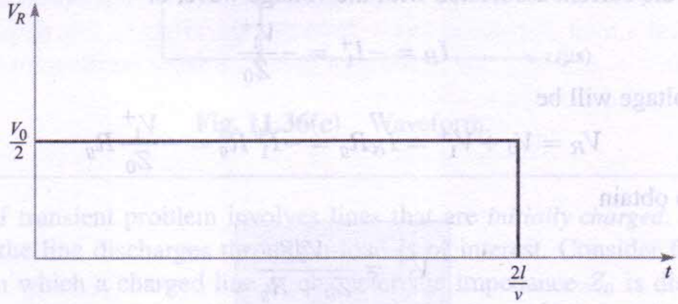


Fig. 11.39 Voltage across the resistor as a function of time, as determined from the reflection diagram of Figure 11.38, in which $R_g = Z_0$ ($\Gamma = 0$).

EXAMPLE 11.22

In the charged line of Figure 11.37, the characteristic impedance is $Z_0 = 100\ \Omega$, and $R_g = 100/3\ \Omega$. The line is charged to an initial voltage, $V_0 = 160$ V, and the switch is closed at time $t = 0$. Determine and plot the voltage and current through the resistor for time $0 < t < 8l/\nu$ (four round trips).

Solution. With the given values of R_g and Z_0, Eq. (47) gives $\Gamma_g = -1/2$. Then, with $\Gamma_L = 1$, and using (122), we find

$$V_1^+ = V_1^- = -\frac{3}{4}V_0 = -120\text{ V}$$

$$V_2^+ = V_2^- = \Gamma_g V_1^- = +60\text{ V}$$

$$V_3^+ = V_3^- = \Gamma_g V_2^- = -30\text{ V}$$

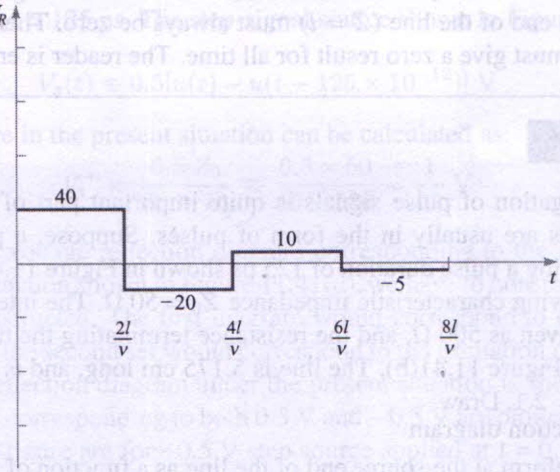
$$V_4^+ = V_4^- = \Gamma_g V_3^- = +15\text{ V}$$

Using these values on the voltage reflection diagram, we evaluate the voltage in time at the resistor location by moving up the left-hand vertical axis, adding voltages as we progress, and beginning with $V_0 + V_1^+$ at $t = 0$. Note that when we add voltages along the vertical axis, we are encountering the intersection points between incident and reflected waves, which occur (in time) at each integer multiple of $2l/\nu$. So, when moving up the axis, we add the voltages of *both* waves to our total at each occurrence. The voltage within each time interval is thus:

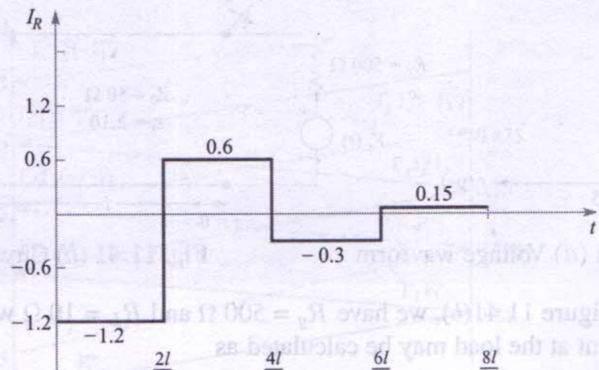
$$
\begin{aligned}
V_R &= V_0 + V_1^+ = 40\,\text{V} && (0 < t < 2l/\nu)\\
&= V_0 + V_1^+ + V_1^- + V_2^+ = -20\,\text{V} && (2l/\nu < t < 4l/\nu)\\
&= V_0 + V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ = 10\,\text{V} && (4l/\nu < t < 6l/\nu)\\
&= V_0 + V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + V_4^+ = -5\,\text{V} && (6l/\nu < t < 8l/\nu)
\end{aligned}
$$

The resulting voltage plot over the desired time range is shown in Figure 11.40*a*.

The current through the resistor is most easily obtained by dividing the voltages in Figure 11.40*a* by $-R_g$. As a demonstration, we can also use the current diagram of Figure 11.33*a* to obtain this



(*a*)



(*b*)

Fig. 11.40 Resistor voltage (*a*) and current (*b*) as functions of time for the line of Figure 11.37, with values as specified in Example 11.33.

result. Using (120) and (121), we evaluate the current waves as follows:

$$I_1^+ = V_1^+/Z_0 = -1.2\text{ A}$$
$$I_1^- = -V_1^-/Z_0 = +1.2\text{ A}$$
$$I_2^+ = -I_2^- = V_2^+/Z_0 = +0.6\text{ A}$$
$$I_3^+ = -I_3^- = V_3^+/Z_0 = -0.30\text{ A}$$
$$I_4^+ = -I_4^- = V_4^+/Z_0 = +0.15\text{ A}$$

Using these values on the current reflection diagram, Figure 11.33*a*, we add up currents in the resistor in time by moving up the left-hand axis, as we did with the voltage diagram. The result is shown in Figure 11.40*b*. As a further check to the correctness of our diagram construction, we note that current at the open end of the line ($Z = l$) must always be zero. Therefore, summing currents up the right-hand axis must give a zero result for all time. The reader is encouraged to verify this.

EXAMPLE 11.23

The study of propagation of pulse signals is quite important part of the transient analysis as the digital signals are usually in the form of pulses. Suppose, a pulse signal source of amplitude 0.5 V having a pulse duration of 125 ps shown in Figure 11.41(a) is connected to a transmission line having characteristic impedance $Z_0 = 50\ \Omega$. The internal impedance of the pulse generator is given as 500 Ω, and the resistance terminating the transmission line is $R_L = 10\ \Omega$ as shown in Figure 11.41(b). The line is 5.175 cm long, and is filled with a dielectric material having $\epsilon_r = 2.1$. Draw

(a) The voltage reflection diagram

(b) The voltage waveform at the source end of the line as a function of time up to 1 ns

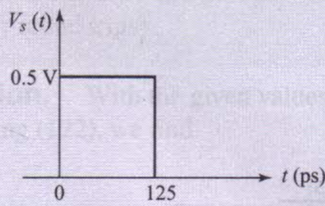


Fig. 11.41 (a) Voltage wavform

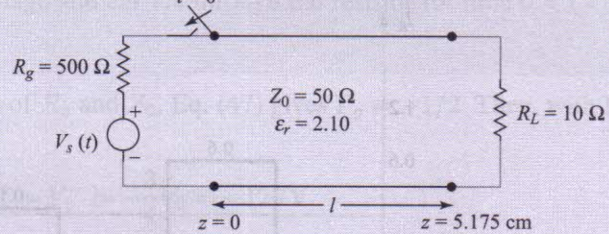


Fig. 11.41 (b) Circuit

Solution. From Figure 11.41(b), we have $R_g = 500\ \Omega$ and $R_L = 10\ \Omega$ with $Z_0 = 50\ \Omega$. Hence, the reflection coefficient at the load may be calculated as

$$\Gamma_L \equiv \frac{R_L - Z_0}{R_L + Z_0} = \frac{10-50}{10+50} = \frac{-2}{3} = -0.6667 \tag{i}$$

Similarly, the reflection coefficient at the source end may be calculated as

$$\Gamma_g \equiv \frac{R_g - Z_0}{R_g + Z_0} = \frac{500-50}{500+50} = \frac{45}{55} = 0.818 \tag{ii}$$

The velocity of pulse along the line would be given by

$$v = \frac{c}{\sqrt{\epsilon_r}} \equiv \frac{3 \times 10^8}{\sqrt{2.1}} = 2.07 \times 10^8 \text{ m/s} \tag{iii}$$

The time taken by the wave to travel from the source to the load end along the line would be given by

$$T = \frac{l}{v} \equiv \frac{5.175 \times 10^{-2}}{2.07 \times 10^8} = 0.25 \text{ ns} \tag{iv}$$

Now, the pulse signal shown in Figure 11.41(*a*) can be expressed as the summation of two step functions. The first step function of 0.5 V is appearing at $t = 0$ ps, while the second step function of -0.5 V is appearing at $t = 125$ ps. The step-signal source shown in Figure 11.41(a) can then be expressed as

$$V_s(t) \equiv 0.5[u(t) - u(t - 125 \times 10^{-12})] \text{ V} \tag{v}$$

The initial voltage wave in the present situation can be calculated as:

$$V_1^+ = \frac{0.5\, Z_0}{Z_0 + R_g} \equiv \frac{0.5 \times 50}{50 + 500} = \frac{1}{22} \text{ V} \tag{vi}$$

(*a*) Now, for drawing the voltage reflection diagram corresponding to the line of Figure 11.41(*b*) excited by the pulse function shown in Figure 11.41(*a*), we have to note that two sets of diagrams would have to be constructed. The first diagram would correspond to the excitation of 0.5 V applied at $t = 0$, while the second set would correspond to the excitation of -0.5 V applied at $t =$ 125 ps. The voltage reflection diagram under the present situation is shown in Figure 11.41(*c*), where the sets of lines corresponding to both 0.5 V and -0.5 V are plotted on the same diagram. The solid lines in this figure are for +0.5 V step source applied at $t = 0$, while the dashed lines represent the response due to the step signal of -0.5 V applied at $t = 0.125$ ns.

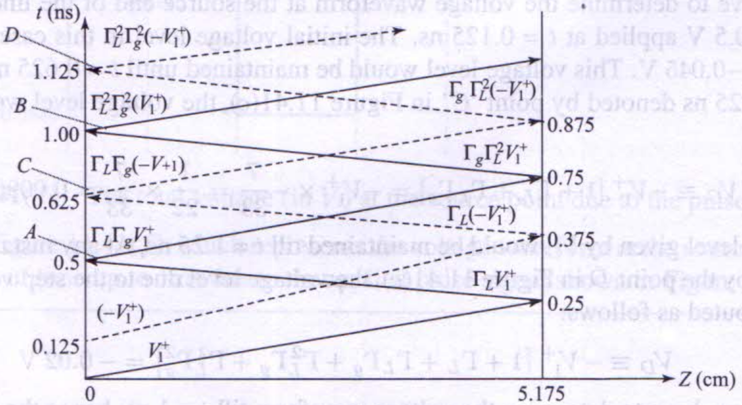


Fig. 11.41(*c*) Voltage reflection diagram.

The value of the voltage wave is shown along each line. For the solid line corresponding to the step source of 0.5 V, it can be observed that in the beginning, the initial voltage wave is just equal to V_1^+, which gets multiplied by the factor of Γ_L at the load end, and by the factor of Γ_g

at the source end, and this process repeats whenever the voltage wave reaches these two ends as shown in the figure. Similarly, it can be observed from this figure that for the step source of −0.5 V applied at $t = 0.125$ ns, the initial voltage wave is equal to $-V_1^+$, which keeps on multiplying by the factor of Γ_L and Γ_g at the load and the source ends, respectively.

(*b*) To obtain the voltage waveform at the source end of the line due to applied pulse of Figure 11.41(*a*), we can take help of the voltage reflection diagram shown in Figure 11.41(*c*). Our approach would be to plot the voltage waveform at the source end of the line due to two-step voltages of +0.5 V and −0.5 V separately, and then obtain the overall plot due to the pulse signal by adding these two plots.

It should be noted here that for the step signal of 0.5 V applied at $t = 0$ ns, the change in the signal level at the input side is at $t = 0.5$ ns, and $t = 1$ ns denoted by points A and B respectively. However, for the step signal of −0.5 V applied at $t = 0.125$ ns, the change in the signal level at the input side is at $t = 0.675$ ns, and $t = 1.125$ ns denoted by points 'C' and 'D' respectively. The exact value of the voltage levels at these points can now be calculated as follows:

For a step signal of +0.5 V applied at $t = 0$ ns, in the beginning, the voltage level would be given by $V_1^+ = \frac{1}{22}V \equiv 0.045$ V. This voltage level would be maintained until $t = 0.5$ ns. However, at any instant after $t = 0.5$ ns denoted by point 'A' in Figure 11.41(*c*), the voltage level would be modified as follows:

$$V_A \equiv V_1^+ + \Gamma_L V_1^+ + \Gamma_L \Gamma_g V_1^+ = V_1^+ \times -\frac{7}{33} = -\frac{1}{22} \times \frac{7}{33} = -0.00964 \text{ V}$$

Similarly, the voltage level at any instant after $t = 1$ ns denoted by the point B in Figure 11.41(*c*) due to the step voltage of +0.5 V can be computed as follows:

$$V_B \equiv V_1^+ \left[1 + \Gamma_L + \Gamma_L \Gamma_g + \Gamma_L^2 \Gamma_g^+ \Gamma_L^2 \Gamma_g^2\right] \equiv V_A + V_1^+ \Gamma_L^2 \Gamma_g [1 + \Gamma_g] = V_A + 0.03 = 0.02 \text{ V}$$

The voltage waveform at the source end of the line up to 1 ns due to the applied step voltage of +0.5 V is shown in Figure 11.41(*d*).

Now, we have to determine the voltage waveform at the source end of the line due to the step signal of −0.5 V applied at $t = 0.125$ ns. The initial voltage level in this case would be given by $-V_1^+ \equiv -0.045$ V. This voltage level would be maintained until $t = 0.625$ ns. At any instant after $t = 0.625$ ns denoted by point 'C' in Figure 11.41(*c*), the voltage level would be modified as follows:

$$V_C \equiv -V_1^+ \left[1 + \Gamma_L + \Gamma_L \Gamma_g\right] = -V_1^+ \times -\frac{7}{33} = \frac{1}{22} \times \frac{7}{33} = 0.00964 \text{ V}$$

The voltage level given by V_C would be maintained till $t = 1.25$ ns. At any instant after $t = 1.125$ ns denoted by the point D in Figure 11.41(*c*), the voltage level due to the step voltage of −0.5 V can be computed as follows:

$$V_D \equiv -V_1^+ \left[1 + \Gamma_L + \Gamma_L \Gamma_g + \Gamma_L^2 \Gamma_g + \Gamma_L^2 \Gamma_g^2\right] = -0.02 \text{ V}$$

In our case, we have to determine the voltage waveform till $t = 1$ ns, hence the voltage level V_D would appear after this time. The voltage waveform at the source end of the line up to 1 ns due to the applied step voltage of −0.5 V is shown in Figure 11.41(*e*).

Once the voltage waveforms at the source end due to the step voltages of +0.5 V and −0.5 V have been obtained, the total voltage at the source end due to the pulse signal can be determined by adding these two individual waveforms shown in Figs. 11.41(*d*) and 11.41(*e*). The final result

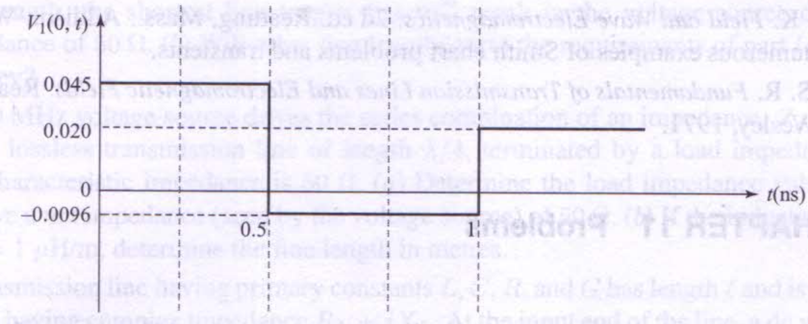


Fig. 11.41(*d*) The voltage (in *V*) at the source point due to the step signal of +0.5 V.

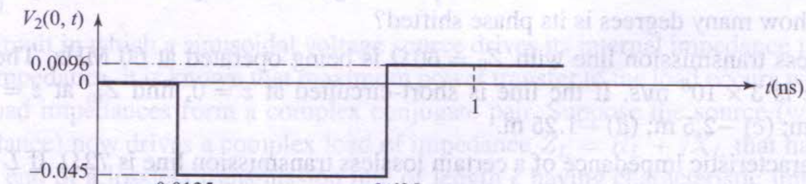


Fig. 11.41(*e*) The voltage (in *V*) at the source point due to the step signal of −0.5 V.

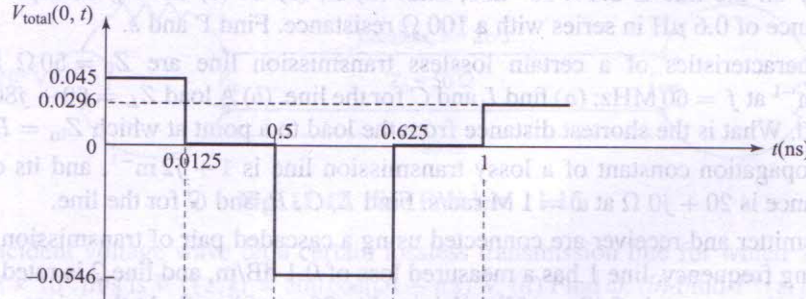


Fig. 11.41(*f*) The total voltage (in *V*) at the source point due to the pulse signal.

is shown in Figure 11.41(*f*), which represents the voltage waveform at the source end of the line as a function of time up to 1 ns due to the applied pulse signal shown in Figure 11.41(*a*).

References

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