Tutorial 4

6. $Y_1 = X_1^2 - X_2^2$

$Y_2 = X_1^2 + X_2^2$

$Y_3 = X_3$

$X_1 = \pm\sqrt{\dfrac{Y_1 + Y_2}{2}}$

$X_2 = \pm\sqrt{\dfrac{Y_2 - Y_1}{2}}$

$X_3 = Y_3$

$\Rightarrow$

$x_1^{(1)} = \sqrt{(y_1+y_2)/2}$

$x_2^{(1)} = \sqrt{(y_2-y_1)/2}$

$x_3^{(1)} = y_3$

$x_1^{(2)} = \sqrt{(y_1+y_2)/2}$

$x_2^{(2)} = -\sqrt{(y_2-y_1)/2}$

$x_3^{(2)} = y_3$

$x_1^{(3)} = -\sqrt{(y_1+y_2)/2}$

$x_2^{(3)} = \sqrt{(y_2-y_1)/2}$

$x_3^{(3)} = y_3$

$x_1^{(4)} = -\sqrt{(y_1+y_2)/2}$

$x_2^{(4)} = -\sqrt{(y_2-y_1)/2}$

$x_3^{(4)} = y_3$

These are real roots

$\Rightarrow y_2 > y_1$ and $y_1 > 0$

$$f_{Y_1Y_2Y_3}(y_1, y_2, y_3) = \left\{ \frac{f_{X_1X_2X_3}(x_1^{(1)}, x_2^{(1)}, x_3^{(1)})}{|J_1|} + \frac{f_{X_1X_2X_3}(x_1^{(2)}, x_2^{(2)}, x_3^{(2)})}{|J_2|} + \frac{f_{X_1X_2X_3}(x_1^{(3)}, x_2^{(3)}, x_3^{(3)})}{|J_3|} + \frac{f_{X_1X_2X_3}(x_1^{(4)}, x_2^{(4)}, x_3^{(4)})}{|J_4|} \right\} \times u(y_2 - y_1) \times u(y_1)$$

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_1 & -2x_2 & 0 \\ 2x_1 & 2x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|J| = |2x_1(2x_2) - 2x_2(-2x_1)|$$
$$= |8x_1x_2|$$

$$|J_1| = \left| 8 \times \left(\frac{y_1+y_2}{2}\right)^{1/2} \times \left(\frac{y_2-y_1}{2}\right)^{1/2} \right| = 4\sqrt{y_2^2 - y_1^2}$$

||ly $|J_2| = |J_3| = |J_4| = 4\sqrt{y_2^2 - y_1^2}$

$$f_{Y_1Y_2Y_3}(y_1, y_2, y_3) = \left\{ \frac{f_{X_1X_2X_3}(x_1^{(1)}, x_2^{(1)}, x_3^{(1)})}{4\sqrt{y_2^2 - y_1^2}} + \ldots \right\}$$

Here, $f_{X_1X_2X_3}(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2}\left(\frac{y_1+y_2}{2} + \frac{y_2-y_1}{2} + y_3^2\right)}$

$$= \frac{1}{(2\pi)^{3/2}} e^{-1/2(y_2 - y_3^2)}$$

$$f_{Y_1Y_2Y_3}(y_1, y_2, y_3) = \frac{1}{(2\pi)^{3/2}} \cdot \frac{e^{-1/2(y_2-y_3^2)}}{4(y_2^2-y_1^2)^{1/2}} \times 4$$ for each root

$$= e^{-1/2(y_2-y_3^2)} \Big/ (2\pi)^{3/2} \cdot (y_2^2 - y_1^2)^{1/2}$$

5. $Z = X + Y$

$W = X$

$$\begin{cases} z = x + y \\ w = x \end{cases} \Rightarrow \begin{matrix} x = w \\ y = z - w \end{matrix}$$

$$J = \left| \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \right|$$

$$= \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = |-1| = 1$$

$$f_{ZW}(z, w) = \frac{f_{XY}(x, y)}{1}$$

$$= f_{xy}(w, z-w)$$

$$= f_X(w) \cdot f_Y(z-w)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w) \cdot f_Y(z-w)\, dw.$$

$$= f_X(z) * f_Y(z)$$

1. $\Rightarrow$ $f_X(x) = \int_0^1 4xy \, dy = 4x\left[\frac{y^2}{2}\right]_0^1 = 2x$

$$f_Y(y) = \int_0^1 4xy \, dx = 4y\left[\frac{x^2}{2}\right]_0^1 = 2y$$

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y).$$

$\Rightarrow$ Independent.

2. $f_X(x) = \int_x^2 \frac{1}{2}\, dy$

$= \frac{1}{2}[y]_x^2$

$= \frac{1}{2}[2-x]$

$f_Y(y) = \int_0^y \frac{1}{2}\, dx$

$= \frac{1}{2}[x]_0^y$

$= \frac{1}{2} y$

$f_X(x) \cdot f_Y(y) = \frac{1}{2}(2-x) \cdot \frac{1}{2} y$

$= \frac{1}{4} y(2-x)$

$\neq f_{XY}(x,y)$

$\Rightarrow$ Not independent.

$\hookrightarrow f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)}$

$= \frac{1/2}{1/2\,(2-x)}$

$= \frac{1}{2-x}, \quad 0 \le x \le y,\ 0 \le y \le 2$

|ly $f_{X/Y}(x/y) = \frac{1/2}{1/2\, y} = 1/y, \quad 0 \le x \le y,\ 0 \le y \le 2$

3. $$I = \int_0^{\infty}\int_0^{x} c e^{-x} e^{-y}\, dy\, dx$$

$$= \int_0^{\infty} c e^{-x}(1 - e^{-x})\, dx$$

$$= c\left[-e^{-x} - \frac{e^{-2x}}{2}\right]_0^{\infty}$$

$$1 = \frac{c}{2}$$

$$\therefore \boxed{c = 2}$$

$$f_x(x) = \int_0^{\infty} f_{xy}(x, y).\, dy$$

$$= \int_0^{\infty} 2 e^{-x} e^{-y}\, dy$$

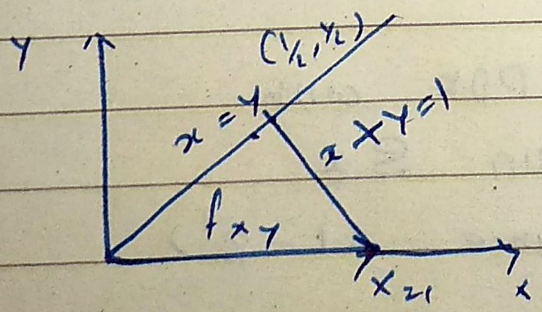
$$= 2e^{-x}(1 - e^{-x}) \quad , 0 < x < \infty$$

$$f_y(y) = 2e^{-y} \quad , 0 < y < \infty$$

$$f_x(x).\, f_y(y) = 4e^{-x}.e^{-y}(1 - e^{-x})$$

$$\neq f_{xy}(x, y)$$

$\Rightarrow$ Not Independent.



$$P[x + y \leq 1]$$

$$= \int_0^{1/2}\int_y^{1-y} 2e^{-x}.e^{-y}\, dx.dy$$

$$= 1 - 2e^{-1}$$

4. $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) \, . \, dy$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}} dy$$

$$= \frac{e^{-\frac{x^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{(y^2 - 2\rho xy)}{2(1-\rho^2)}} dy$$

$$y^2 - 2\rho xy + \rho^2 x^2 - \rho^2 x^2$$

$$= (y - \rho x)^2 - \rho^2 x^2$$

$$= \frac{e^{-\frac{x^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}} + e^{\frac{-\rho^2 x^2}{2(1-\rho^2)}} dy$$

$$= \frac{e^{\frac{-x^2(1-\rho^2)}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}} . dy$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}} dy \right)$$

↓

Gaussian PDF with mean ρx variance $(1-\rho^2)$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

||ly $f_Y(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}}$

⇒ Not independent

if $\rho = 0$ then only possible.