Example: Find the inverse of $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}$

We write the matrix in an augmented form as

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 2 & -2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad m_{21} = 1, \quad m_{31} = -2$$

Apply
$(E_2) - m_{21}(E_1)$
$(E_3) - m_{31}(E_1)$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 2 & 0 & 1 \end{array}\right] \quad m_{32} = 3$$

Apply
$(E_3) - m_{32}(E_2)$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right] \Rightarrow \begin{aligned} x_{11} + x_{21} - x_{31} &= 1 \\ x_{21} - x_{31} &= -1 \\ 2x_{31} &= 5 \end{aligned}$$

$$\Rightarrow \boxed{x_{31} = 5/2} \Rightarrow \boxed{x_{21} = 3/2} \quad \boxed{x_{11} = 1}$$

Similarly,

$$\begin{aligned} x_{12} + x_{22} - x_{32} &= 0 \\ x_{22} - x_{32} &= 1 \\ 2x_{32} &= -3 \end{aligned} \Rightarrow \begin{aligned} x_{32} &= -3/2 \\ x_{22} &= -1/2 \\ x_{12} &= -1 \end{aligned}$$

and also

$$\begin{aligned} x_{13} + x_{23} - x_{33} &= 0 \\ x_{23} - x_{33} &= 0 \\ 2x_{33} &= 1 \end{aligned} \Rightarrow \begin{aligned} x_{33} &= 1/2 \\ x_{23} &= 1/2 \\ x_{13} &= 0 \end{aligned}$$

$\therefore$ The inverse of the given matrix is

$$\begin{bmatrix} 2 & -1 & 0 \\ 3/2 & -1/2 & 1/2 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

This method can be extended to higher orders

Now $\boxed{\tilde{A}\tilde{x} = \tilde{I}}$ $\Rightarrow$ $\boxed{(\tilde{A}^{-1}\tilde{A})\tilde{x} = \tilde{A}^{-1}}$ $\Rightarrow$ $\boxed{\tilde{I}\tilde{x} = \tilde{A}^{-1}}$

or

Alternatively, x_3 can be eliminated from the first two equations and x_2 from by first equation, by row operations so that the left half of the augmented matrix becomes identity matrix.

Starting with the upper triangular augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right]$$

Eliminate x_3 from the first two rows, and x_2 from the first row.

Divide row 3 by 2.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right] \rightarrow$$

Add row 3 to row 1 and row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 7/2 & -3/2 & 1/2 \\ 0 & 1 & 0 & 3/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right]$$

Now, subtract row 2 from row 1 to get an identity matrix in the left half.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 3/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right]$$

The right half of the augmented matrix gives the inverse matrix $\tilde{A}^{-1}$.

Iteration Methods

Used to solve linear systems $\boxed{\tilde{A}\vec{x} = \vec{b}}$ of a very large order (approximate method)

$\boxed{\text{Two methods: Jacobi Method / Gauss-Seidel Method.}}$

Example (Jacobi Method):

$$\boxed{\begin{aligned} 9x_1 + x_2 + x_3 &= b_1 \\ 2x_1 + 10x_2 + 3x_3 &= b_2 \\ 3x_1 + 4x_2 + 11x_3 &= b_3 \end{aligned}} \qquad \boxed{\begin{aligned} b_1 &= 10 \\ b_2 &= 19 \\ b_3 &= 0 \end{aligned}}$$

$x_1 = \frac{1}{9}\left[b_1 - x_2 - x_3\right]$, $x_2 = \frac{1}{10}\left[b_2 - 2x_1 - 3x_3\right]$

$x_3 = \frac{1}{11}\left[b_3 - 3x_1 - 4x_2\right]$. Iterative solving starts with an initial guess of $\boxed{x_1^{(0)}, x_2^{(0)}, x_3^{(0)}}$.

$$\boxed{\begin{aligned} x_1^{(k+1)} &= \frac{1}{9}\left[b_1 - x_2^{(k)} - x_3^{(k)}\right] \\ x_2^{(k+1)} &= \frac{1}{10}\left[b_2 - 2x_1^{(k)} - 3x_3^{(k)}\right] \\ x_3^{(k+1)} &= \frac{1}{11}\left[b_3 - 3x_1^{(k)} - 4x_2^{(k)}\right] \end{aligned}}$$

This iteration method is also known as the method of simultaneous replacements.

Example (Gauss - Seidel Method):

$$\boxed{\begin{aligned} x_1^{(k+1)} &= \frac{1}{9}\left[b_1 - x_2^{(k)} - x_3^{(k)} \right] \\ x_2^{(k+1)} &= \frac{1}{10}\left[b_2 - x_1^{(k+1)} - 3x_3^{(k)} \right] \\ x_3^{(k+1)} &= \frac{1}{11}\left[b_3 - 3x_1^{(k+1)} - 4x_2^{(k+1)} \right] \end{aligned}}$$

This iteration is known as the method of successive replacements. Usually it converges faster. ($x_1 = 1$, $x_2 = 2$, $x_3 = -1$)

General Principle of Iterative Methods

For a linear system $\boxed{\tilde{A}\vec{x} = \vec{b}}$ decompose $\boxed{\tilde{A} = \tilde{N} - \tilde{P}}$.

$\Rightarrow \boxed{\tilde{N}\vec{x} = \vec{b} + \tilde{P}\vec{x}} \Rightarrow \boxed{\tilde{N}x^{(k+1)} = \vec{b} + \tilde{P}x^{(k)}}$ $k = 0, 1, 2 \ldots$

1/. Jacobi Method: $\tilde{N} \equiv \begin{bmatrix} 9 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 11 \end{bmatrix}$, $\tilde{P} \equiv \begin{bmatrix} 0 & -1 & -1 \\ -2 & 0 & -3 \\ -3 & -4 & 0 \end{bmatrix}$

$\tilde{N}$ is chosen to be diagonal

2/. Gauss-Seidel Method: $\tilde{N} \equiv \begin{bmatrix} 9 & 0 & 0 \\ 2 & 10 & 0 \\ 3 & 4 & 11 \end{bmatrix}$, $\tilde{P} \equiv \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$\tilde{N}$ is chosen to be lower triangular.

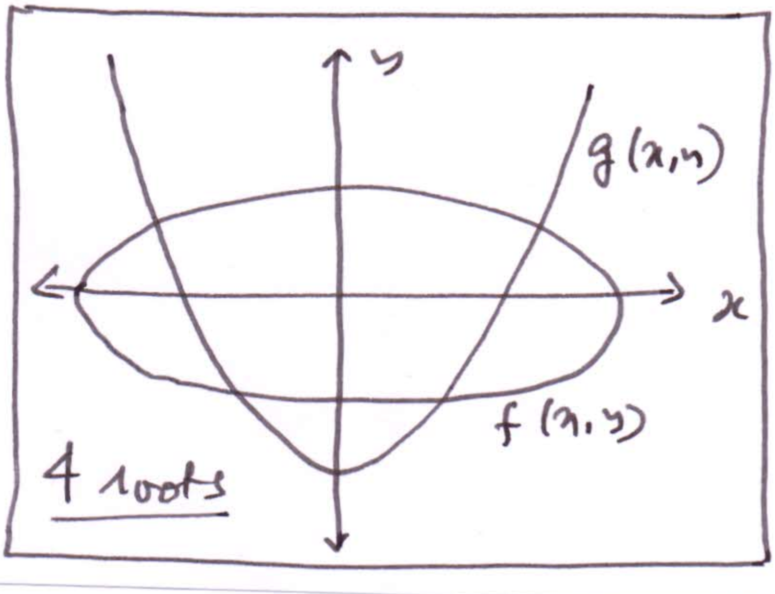
Nonlinear Systems

Solutions of a system of two nonlinear equations are extracted from $\boxed{f(x,y)=0}$ and $\boxed{g(x,y)=0}$ in a general form.

Example: $$\boxed{\begin{aligned} f(x,y) &\equiv x^2+4y^2-9=0 \\ g(x,y) &\equiv 18y-14x^2+45=0 \end{aligned}}$$

For a function $\boxed{z=f(x,y)}$ in the xyz-space, its zero curve is obtained when $\boxed{z=0}$, i.e. on the x-y plane. Hence, the intersection of $\boxed{z=f(x,y)}$ and $\boxed{z=0}$ is the zero curve. Example: The zero curve of the equation of a sphere. $\boxed{x^2+y^2+z^2=r^2}$ is a circle for $\boxed{z=0}$.

The zero curve of $\boxed{f(x,y)=x^2+4y^2-9=0}$ is an ellipse and of $\boxed{g(x,y)=18y-14x^2+45=0}$ is a parabola.



For the general system $\boxed{f(x,y) = g(x,y) = 0}$, let (x_0, y_0) be an initial guess for the actual solution $\boxed{\alpha \equiv (\xi, \eta)}$. Now $\boxed{z = f(x,y)}$ is ~~the~~ a surface in xyz-space. An approximation of it is a tangent plane at $(x_0, y_0, f(x_0, y_0))$. The equation of the tangent plane is obtained by a Taylor expansion up to first order.

$$\therefore \boxed{z = p(x,y) = f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0)}$$

Here $\boxed{f_x = \frac{\partial f}{\partial x}}$ and $\boxed{f_y = \frac{\partial f}{\partial y}}$.

If $f(x_0, y_0)$ is sufficiently close to zero, then the zero curve of $p(x,y)$ will approximate the zero curve of $f(x,y)$ in the neighbourhood of (x_0, y_0). The zero curve of $\boxed{z = p(x,y) = 0}$ is a straight line.

For $\boxed{f(x,y) = x^2 + 4y^2 - 9}$ $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 8y$.

At $(1, -1)$, $\boxed{f(1,-1) = -4}$, $\boxed{f_x = 2}$, $\boxed{f_y = -8}$.

Hence, $\boxed{p(x,y) = -4 + 2(x-1) - 8(y+1) = z}$.

Similarly, for the surface $\boxed{z = g(x,y)}$.
The tangent plane at $(x_0, y_0, g(x_0, y_0))$ is
$z = q(x,y)$ obtained after a Taylor expan-sion up to the linear order.

$$\therefore \boxed{z = q(x,y) = g(x_0,y_0) + (x-x_0)\, g_x + (y-y_0)\, g_y}$$

in which $\boxed{g_x \equiv \left.\frac{\partial g}{\partial x}\right|_{x_0,y_0}}$ and $\boxed{g_y \equiv \left.\frac{\partial g}{\partial y}\right|_{x_0,y_0}}$.

For $\boxed{g(x,y) = 18y - 14x^2 + 45}$ $\frac{\partial g}{\partial x} = -28$, $\frac{\partial g}{\partial y} = 18$

At $(1,-1)$, $\boxed{g(1,-1) = 13}$, $\boxed{g_x = -28}$, $\boxed{g_y = 18}$.

Hence, $\boxed{q(x,y) = 13 - 28(x-1) + 18(y+1) = z}$.

Solutions of $\boxed{f(x,y) = 0}$ and $\boxed{g(x,y) = 0}$ are the intersections of the zero curves of $\boxed{z = f(x,y)}$ and $\boxed{z = g(x,y)}$. Approximate by the zero curves of the tangent planes, $\boxed{p(x,y) = 0}$ and $\boxed{q(x,y) = 0}$. This gives

$$\boxed{\begin{aligned} f(x_0,y_0) + (x-x_0)\, f_x(x_0,y_0) + (y-y_0)\, f_y(x_0,y_0) &= 0 \\ g(x_0,y_0) + (x-x_0)\, g_x(x_0,y_0) + (y-y_0)\, g_y(x_0,y_0) &= 0 \end{aligned}}$$

- A set of two linear equations.

$$\Rightarrow \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} = \begin{bmatrix} -f(x_0,y_0) \\ -g(x_0,y_0) \end{bmatrix}$$

Define $\boxed{\delta_x = x - x_0}$ and $\boxed{\delta_y = y - y_0}$. Let the solution of the system,

$$\boxed{\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} = \begin{bmatrix} -f \\ -g \end{bmatrix}}$$

at $\underline{(x_0, y_0)}$ be $\underline{(x_1, y_1)}$.

$\therefore$ The <u>iterated result</u> is $\boxed{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}}$

Now use (x_1, y_1) to <u>iterate</u> and get (x_2, y_2).

The <u>general iteration</u> formula is

$$\begin{bmatrix} f_x(x_k,y_k) & f_y(x_k,y_k) \\ g_x(x_k,y_k) & g_y(x_k,y_k) \end{bmatrix} \begin{bmatrix} \delta_{x,k} \\ \delta_{y,k} \end{bmatrix} = \begin{bmatrix} -f(x_k,y_k) \\ -g(x_k,y_k) \end{bmatrix}$$

and $\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} \delta_{x,k} \\ \delta_{y,k} \end{bmatrix}$. $k = 0, 1, \ldots, n$

<u>The General Newton Method</u>

For $\boxed{F_1(x_1, x_2) = 0}$ and $\boxed{F_2(x_1, x_2) = 0}$.

Define <u>vectors</u> $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{F}(x) = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$

(P. T. O.)

$$\tilde{F}'(x) \equiv \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix}$$ → This matrix of derivatives is known as the <u>FRECHET derivative.</u>

For $\boxed{\vec{F}(\vec{x}) = \vec{0}}$, <u>the solution</u> is α. Iterate close to α according to (with $\underline{k = 0, 1, \cdots}$)

$$\boxed{\tilde{F}'(\vec{x}^{(k)})\,\vec{\delta}^{(k)} = -\vec{F}(\vec{x}^{(k)})} \quad \boxed{\vec{x}^{(k+1)} = \vec{x}^{(k)} + \vec{\delta}^{(k)}}.$$

$$\Rightarrow \vec{x}^{(k+1)} - \vec{x}^{(k)} = \vec{\delta}^{(k)} = -\left[\tilde{F}'(\vec{x}^{(k)})\right]^{-1} \vec{F}(\vec{x}^{(k)})$$

$$\Rightarrow \boxed{\vec{x}^{(k+1)} = \vec{x}^{(k)} - \left[\tilde{F}'(\vec{x}^{(k)})\right]^{-1} \vec{F}(\vec{x}^{(k)})}.$$ This formula is to be <u>implemented for practical computations.</u> Also it is the <u>generalisation of the Newton-Raphson method to multiple variables.</u>

<u>A System of n Nonlinear Equations</u>

$$\boxed{\begin{matrix} F_1(x_1, \ldots, x_n) = 0 \\ \vdots \\ F_n(x_1, \ldots, x_n) = 0 \end{matrix}} \quad \vec{x} \equiv \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \vec{F} \equiv \begin{bmatrix} F_1(x_1, \ldots, x_n) \\ \vdots \\ F_n(x_1, \ldots, x_n) \end{bmatrix}$$

we can <u>generalise</u>,

$$\tilde{F}'(x) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}.$$

The <u>Newton-Raphson method</u> can be <u>extended</u> to <u>any higher order n.</u>