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Flow Optimization on Traffic Networks

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This work has been motivated by the work of [1]. Here in traffic models for networks based on PDE's are considered. A simplified algebraic model is derived from PDE-based model. Optimization problem is to minimize cost functional measuring properties of network flows. We use a new approach to solve the minization problem for the reformulated algebraic model.

1 Network and Macroscopic model

Simulation and optimization of traffic flow in networks plays an important role in traffic management. A Fluid dynamic formulation was proposed in [2] for macroscopic modelling of vehicular traffic flow. We introduce the detailed model for the behavior of vehicles in the network.

Network model: A traffic flow network is a finite, connected, directed graph, where each edge models a road and each vertex denotes a junction. Roads $j = 1, \ldots, J$ are modelled as intervals $[a_j, b_j]$ where a_j or b_j can be $\pm\infty$.

Traffic model: Conservation of density $\rho_j(x,t) \in [0, \rho_{j_{max}}]$ on each road j:

$$\partial_t \rho_j + \partial_x f_j(\rho_j) = 0, \quad \rho_j(x,0) = \rho_{j,0}(x)\ \forall (x,t) \in I\!R \times [0,\infty) \tag{1}$$

where f is the flux function. The first assumption is that velocity u is a function of density only.
For example $u(\rho) = u_{\max}(1 - \rho/\rho_{\max}), \quad f(\rho(x,t)) = \rho(x,t)u(\rho(x,t))$.
Suitable conditions to govern the flow at junctions are given by flux conservation [3, 4]:

$$\sum_{j \in \delta^-} f_j(\rho_j(b,t)) = \sum_{j \in \delta^+} f_j(\rho_j(a,t)) \tag{2}$$

where δ^- are incoming and δ^+ are outgoing roads to a junction. To obtain unique solution of initial-boundary value problem for ρ_j additional boundary conditions need to be specified as in [3].

2 Algebraic model

Assign $\rho_{j,0}$ as an approximation of the density ρ_j and t_j as the evolution of traffic flow on each road j. We assume a given inflow ρ_0 on the inflow arc, time interval $[0,T]$ and $\rho_{j,0} = 0 \forall j$. We assume that no backward going shock waves appear, which means no traffic jam situation occurs. We use $0 \le \rho_{j,0} \le \sigma_j$ where $\sigma_j = argmax f_j(\rho_j)$.
We assume network with one incoming and one outgoing road and junctions only joining total of 3 roads as in Fig.1. We can determine $\rho_{j,0}$ solely by coupling conditions at junction. In case of a dispersing junction with an incoming road k, outgoing roads l, m and a merging junction with incoming roads k, l, an outgoing road m we have following expressions respectively,

$$\text{Dispersing:} \quad \rho_{l,0} = f_l^{-1}(\alpha f_k(\rho_{k,0})), \rho_{m,0} = f_m^{-1}((1-\alpha) f_k(\rho_{k,0})) \quad \text{Merging:} \quad \rho_{m,0} = f_m^{-1}(f_k(\rho_{k,0}) + f_l(\rho_{l,0})) \tag{3}$$

We determine $t_{j,0}$ by tracking a single shock front on road j (for further details refer to [1]). In case of dispersing and merging junctions with a geometry as before:

$$\text{Dispersing:} \quad t_l = t_{k,0} + \frac{b-a}{s_k} \quad \text{Merging:} \quad t_m = (t_{k,0} + \frac{b-a}{s_k})\frac{\rho_k}{\rho_k + \rho_l} + (t_{l,0} + \frac{b-a}{s_l})\frac{\rho_l}{\rho_k + \rho_l} \tag{4}$$

Suitable controls $0 < \alpha < 1$ are applied at the dispersing junctions steering the flux distribution. Thus the cost functional measuring minimal traveling time for cars in network is given as

$$J_2(\alpha; T, \rho_0) = \sum_j (T - t_{j,0}) L \rho_{j,0} - \frac{\rho_{j,0}}{2u_j(\rho_{j,0})} L^2 \quad \text{where} \quad L = b_j - a_j \tag{5}$$

Hence, the control problem leads to a nonlinear optimization problem $\mathrm{Min}_\alpha \mathrm{J}_2$ subject to bound constraints, $0 < \alpha < 1$.

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3 Reformulated Algebraic model

Neglecting the dynamics in the simplified model $t_{j,0} = 0$, we define the model in terms of flux q_j. The coupling conditions are rewritten as, linear constraints: $\sum_{j\in\delta^-} q_j = \sum_{j\in\delta^+} q_j$ and the flux restriction as, bound constraints : $0 \le q_j \le \text{const}_j$. The corresponding cost functional in terms of flux q_j is given by

$$J_2(q) = \sum_j \left(T - \frac{\tau(q_j)}{2}\right)\tau(q_j)q_j \qquad \tau(q_j) = \frac{1}{u_j(f_j^{-1}(q_j))}$$

Hence the resulting optimal problem is: $\mathrm{Minimize_q} J_2(q)$ subject to linear and bound constraints.

4 Optimization Algorithm

We describe how to solve general optimal problem:

$$\text{Minimize} \quad f(x) \quad \text{subject to} \quad h(x) = H^T x + h^0 = 0, \qquad 0 \le x \le const.$$

The algorithm PL2-penalty [5] is based on exact penalization. The nonsmooth exact penalty function is smoothened and then new problem is solved by an inexact Newton method based on Lanczos decomposition. This allows descent directions of negative curvature [5]. The smoothened exact penalty function is:

$$\Psi(x;\beta,\gamma) = f(x) + \beta\left(\sum_{j=1}^{p} \frac{1}{\gamma} ln(1 + cosh(\gamma h_j(x)))\right) \tag{6}$$

where γ is a smoothening-parameter and β is a penalty-parameter. β is fixed and is increased on detection of an infeasible minimizer of Ψ. γ has to be chosen to decrease the constraint violation. γ is used as increasing sequence $\gamma_0 < \gamma_1 \ldots$ with sequential minimization of Ψ.

5 Numerical Results

The algebraic model was derived to apply optimization schemes on large scale networks. We have tested the reformulated algebraic model and algorithm on networks with different number of controls. We have chosen inflow $q_0 = 0.96$, $T = 5.0$ and maximal flux on each road is 1.0. The optimal value for flux is $q_{inflow} = q_{outflow} = 0.96$, $q_k = 0.0$ for inner roads and $q_j = 0.48$ for boundary roads in sample network Fig.1. We have plotted difference of the current approximation of flux and the optimal flux vs. number of iterations of PL2-penalty algorithm in Fig.2. We observe fast convergence of algorithm in-spite of increasing number of controls. Also the computation time for mid-size networks is reasonable as tabulated below in Table1.

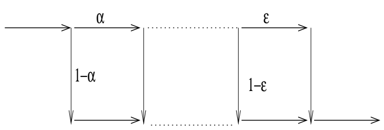

Fig.1 General layout of sample network.

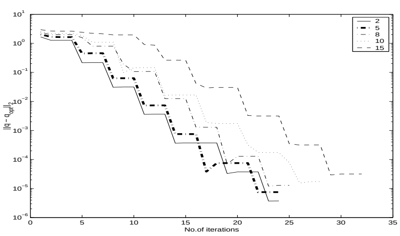

Fig.2 L_2 residual norm $||q - q_{opt}||_2$ on networks of different number of controls.

Controls(#)	Roads(#)	Total time
15	46	0.15sec
50	151	7.7sec
100	301	11.3sec
500	1501	11.8min

Table1. Computation time for mid size networks

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