A PROPERTY OF DIRAC DELTA FUNCTION :

Multiplication Property : (also known as the "sifting"* property)

→ $\int S(x) \cdot \delta(x - x_0)\, dx = S(x_0)$

x → either time t or frequency f

→ can be thought of as sampling $S(x)$ at $x = x_0$

Convolution Property :

→ $S(x) * \delta(x - x_0) = S(x - x_0)$

→ This is the "shiting"* property of $\delta(x)$.

The entire waverform $S(x)$ is shited by x_0.

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* Sift ≡ Examine by isolation, (sieve, screen, filter, etc.)
* Shift = move

Convolution Property (contd...) :

$$\mathcal{F}\{\delta(t-t_0)\} = \begin{cases} 1, \text{ (magnitude) for all } f \\ -2\pi f t_0, \text{ (phase) for all } f \\ e^{-j2\pi f t_0} \leftarrow \text{(complex representation)} \end{cases}$$

Consider :

$$s(t) * \delta(t-t_0) :$$

In time domain, we get $s(t-t_0)$

(Waveform $s(t)$ is shifted by t_0 seconds)

In frequency domain, we get

$$S(f) \cdot \mathcal{F}\{\delta(t-t_0)\}$$ ($\leftarrow$ Recall, convolution in time domain $\equiv$ multiplication in the frequency domain)

$$= S(f) \cdot e^{-j2\pi f t_0}$$

$\uparrow$ Proof of why time shift by t_0 sec. results in phase offset of $-2\pi f t_0$ radians

A Train of Impulses:

Let $$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

What is the Fourier Transform $P(f)$ of $p(t)$?

Notes:

(i) $p(t)$: is discrete-time (D-T) waveform with sample period of T_s second.

$\Rightarrow$ $P(f)$ has to be periodic with a period of $F_s = 1/T_s$ Hz.

(ii) $p(t)$: is periodic with a period of T_s seconds.

$\Rightarrow$ $P(f)$ has to have discretized frequencies with frequency sampling of F_s Hz.

Answer:

$$P(f) \propto \sum_{n=-\infty}^{\infty} \delta(f - nF_s)$$

Poisson Sum Formula. (Eqn 2-115 Couch 8th Ed.)

SUMMARY :

$$P(t) = \sum_n \delta(t - nT_s)$$

$$\underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \quad P(f) \propto \sum_n \delta(f - nF_s)$$

$$F_s = 1/T_s$$

Now, let us consider sampling in time domain.

We take a C-T signal $s(t)$ and discretize the time t.

(Eqn 2-171, Couch 8th Ed.)

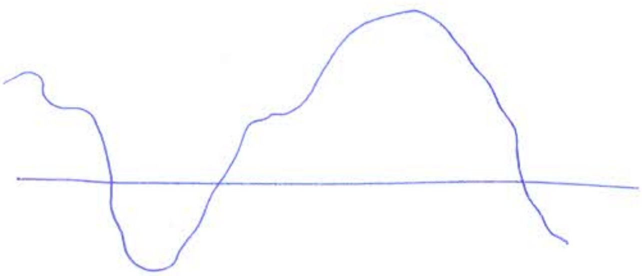
$$\tilde{s}(nT_s) = s(t) \cdot p(t) \qquad \text{(i)}$$

$$\Rightarrow \tilde{S}(f) = S(f) * P(f) \qquad \text{(ii)}$$

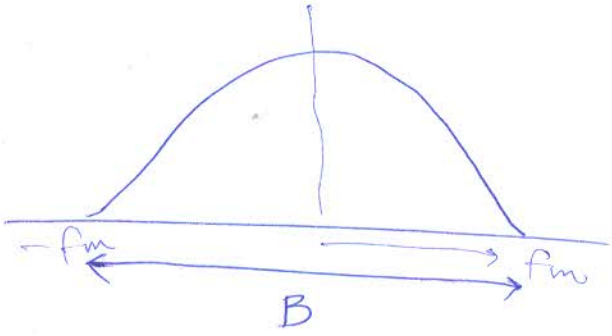
(Eq. 2-173 Couch 8th Ed.)

- Eq. 1 samples $s(t)$ in time domain.
- Eq. 2 says that as the result, $S(f)$ is shifted to multiple integers of F_s, and becomes periodic in frequency domain.

$s(t)$



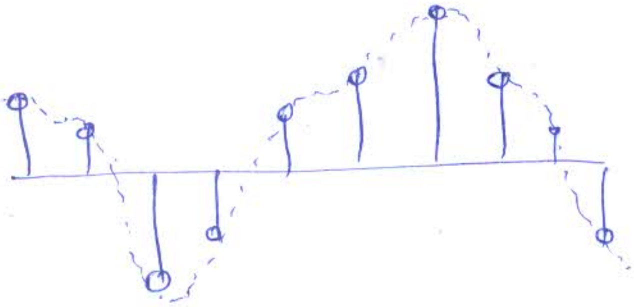
$S(f)$



Eq. (i) of previous page]

$$\tilde{S}(nT_s) = s(t) \cdot p(t)$$

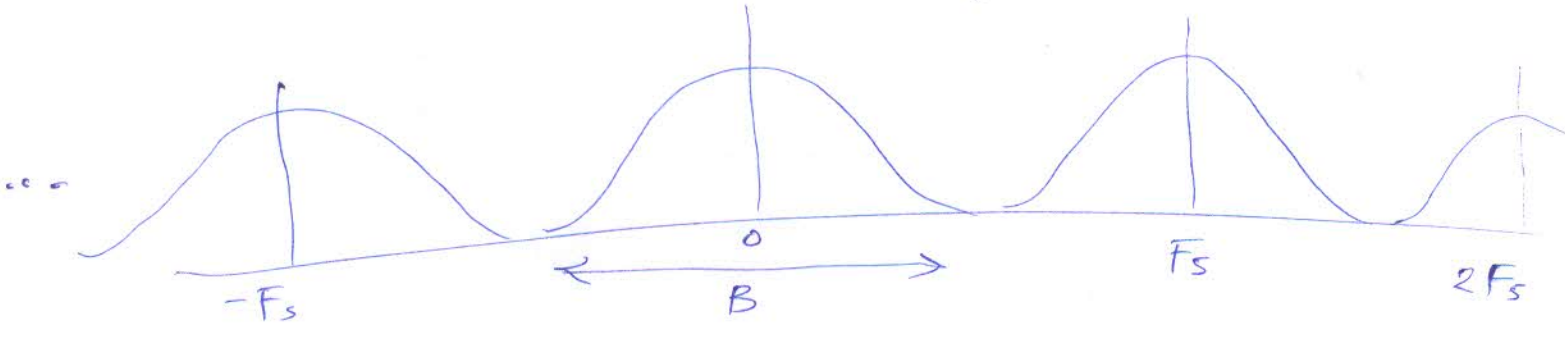
$$\tilde{S}(f) = S(f) * P(f)$$



Periodic Version of $s(f)$

(Eq. (ii) of previous page)

(See also Fig 2-18 Couch 8^{th} Ed.)



SHANNON – NYQUIST SAMPLING THEOREM :

→ Frequency domain view :

If $F_s > B$, it is possible to recover $S(f)$ from $\tilde{S}(f)$

- How ? Just multiply $S(f)$ by ideal Low Pass Filter, i.e., a rectangular gate centered at $f = 0$ and of width B Hz.
- This will remove all replicas (periods) and retain only the fundamental period which is exactly the original $S(f)$.

→ Time domain View :

If $T_s < 1/B$ seconds, it is possible to reconstruct $s(t)$ from its samples $\tilde{s}(nT_s)$.

► How ? Just ~~multiply~~ convolve $\tilde{s}(nT_s)$ by ideal LPF which is a SINC function in time-domain. This SINC is continuous-time function, and it will fill up the discarded waveform in between two samples. (this is also sometimes called interpolation operation].

See Fig 2-17, Couch 8^{th} Ed., for a plot of the interpolation by an LPF.

Ideal Versus Practical Sampling :

The scheme described so far is called ideal sampling, or impulse sampling.

→ Uses Dirac Delta functions, which are impossible to generate in an electronic circuit.

Practical Samplers :

→ Replace The Dirac Delta functions by Thin pulses.

→ Two schemes :

→ NATURAL Sampling

→ FLAT Top sampling.

A Reading Assignment

Read about These in The textbooks.