CT111 Introduction to Communication Systems
Lecture 5: Fourier Transform

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Overview of Today's Talk

1. Correlation between Complex Phasors
2. Fourier Transform
3. Examples
4. Properties

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Time Average of Complex Phasors

- Time integral of a Complex Phasor $\exp\left(i\left(2\pi f t\right)\right)\, dt$ over a time interval T is zero, if $T = m \times \frac{1}{f} = m \times T_{cycle}$:

$$\frac{1}{T_{cycle}} \int_{t=-T_{cycle}/2}^{T_{cycle}/2} \exp\left(i\left(2\pi f t\right)\right)\, dt = 0$$

- Why?
  - → Complex phasor $\exp\left(i\left(2\pi f t\right)\right) = \cos\left(2\pi f t\right) + i \sin\left(2\pi f t\right)$ is comprised of two sinusoidal waveforms.
  - → When cycles are allowed to complete, the sinusoidal waveforms have equal positive and negative valued areas, which cancel out.

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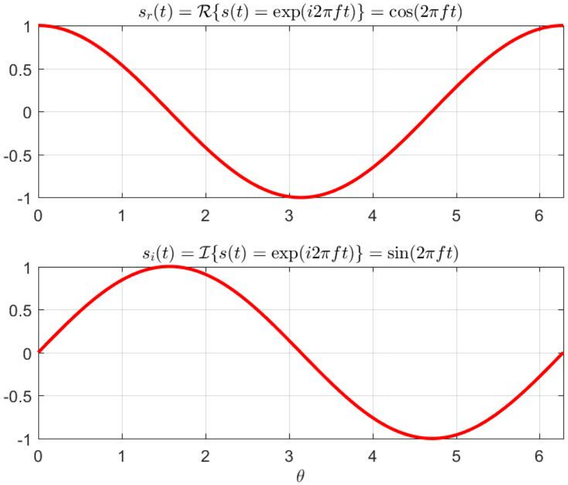

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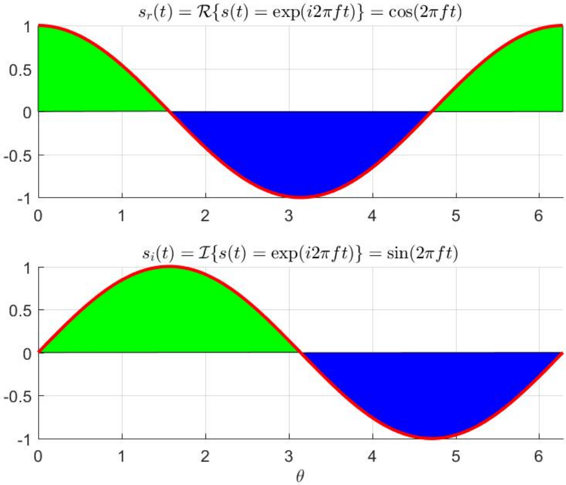
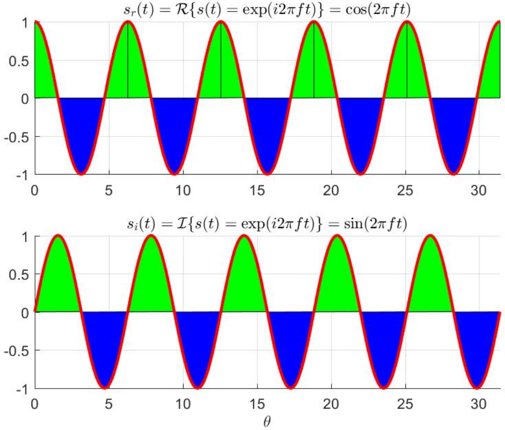


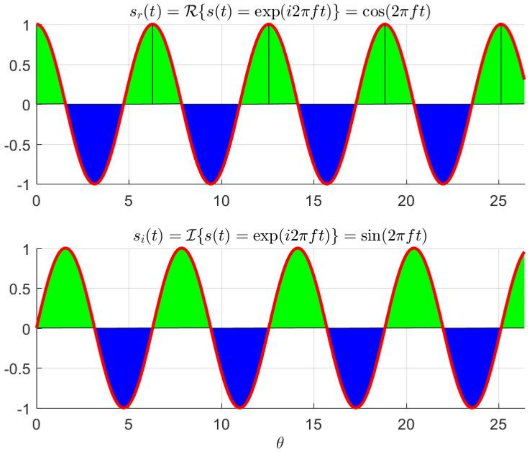
Time Average of Complex Phasors

- Time average remains zero with multiple cycles as well, as long as they're allowed to complete



Time Average of Complex Phasors

- Time average does become nonzero if the cycle is cut short



Time Average of Complex Phasors

- However, the time integral still approaches zero as $T \to \infty$

$$\frac{1}{T}\int_{t=-T/2}^{T/2} \exp\left(i\left(2\pi f t\right)\right)\, dt \to 0$$

- Why?
  - $\to$ Maximum value of the integral is limited by the area under *half* cycle.
  - $\to$ Let us call this constant q. Note that q does not increase with T.
  - $\to$ Therefore, as $T \to \infty$, the ratio $\frac{q}{T} \to 0$.

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Time Average of Complex Phasors

- A general conclusion: time integral of a Complex Phasor $\exp\left(i\left(2\pi ft\right)\right)\,dt$ over a time interval T approaches zero as $T \rightarrow \infty$

$$\frac{1}{T}\int_{t=-T/2}^{T/2} \exp\left(i\left(2\pi ft\right)\right)\,dt \rightarrow 0$$

- An exception to the above is when $f = 0$. In this case

$$\frac{1}{T}\int_{t=-T/2}^{T/2} \exp\left(i\left(2\pi 0t\right)\right)\,dt = \frac{1}{T}\int_{t=-T/2}^{T/2} 1\,dt = 1$$

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A Short Form Notation of Complex Phasors

Let us use the following short-form notation to denote a complex phasor:

$$\begin{aligned} s_{A,f} &\stackrel{\text{def}}{=} a \exp\left(i(2\pi f t + \theta\right) \\ &= a \exp\left(i\theta\right) \exp\left(i(2\pi f t)\right) \\ &= A \exp\left(i(2\pi f t)\right) \end{aligned}$$

Note that a is the real-valued amplitude, whereas $A \stackrel{\text{def}}{=} a \times \exp\left(i\theta\right)$ is complex-valued amplitude

Two Complex Phasors

- Let us denote two complex phasors as follows:
  1. $w(t) = s_{A,f_1}(t) = A\exp\left(i(2\pi f_1 t)\right)$
     - ▷ $w(t)$, in general, will represent the signal that is given to us (whose frequency content we are interested in evaluating)
  2. $s_{1,f}(t) = \exp\left(i(2\pi f t)\right)$
     - ▷ $s_{1,f}(t)$ is the signal that is locally (on our computer or using our hardware) generated. It has unit amplitude and a frequency f that is swept over a range of interest

Dot Product between Two Complex Phasors

Also Called Correlation

- Let us define the dot product between these two complex phasors as follows:

$$W(f) = \frac{1}{T} \int_{t=-T/2}^{T/2} w(t)\, s_{1,f}^{*}(t)\, dt$$

- Here $s_{1,f}^{*}(t)$ denotes the *complex-conjugate* of signal $s_{1,f}(t)$

→ Conjugate x^* of any complex number x is defined as follows:

$$|x^*| = |x|$$
$$\theta_{x^*} = -\theta_x$$

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Dot Product between Two Complex Phasors

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- The dot product can be calculated as follows:

$$W(f) = \frac{1}{T}\int_{t=-T/2}^{T/2} \underbrace{A\exp\left(i\left(2\pi f_1 t\right)\right)}_{w(t)} \underbrace{\exp\left(-i\left(2\pi f t\right)\right)}_{s_{1,f}(t)} \, dt$$

$$= \frac{1}{T}\int_{t=-T/2}^{T/2} A\exp\left(i\left(2\pi(f_1 - f)t\right)\right) \, dt$$

- We notice the integrand itself is just a complex phasor at a frequency $f_1 - f$. Therefore, we can write

$$W(f) = \begin{cases} A, & \text{when } f_1 - f = 0, \text{ i.e., } f = f_1 \\ 0, & \text{otherwise} \end{cases}$$

Dot Product between Two Complex Phasors

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- A short-hand notation for the above (involves a more deeper concept of Dirac Delta functions, which we will not study in this course):

$$W(f) = A\delta(f - f_1)$$

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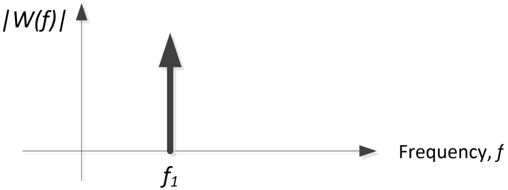
Fourier Transform

Analysis Equation

- Pictorial view of the Fourier Transform of $w(t) = s_{A,f_1}(t) = A\exp\left(i(2\pi f_1 t)\right)$:



- Note that the F.T. $W(f)$ is a *complex* number in general. It has both magnitude and phase (polar coordinates) or (in Cartesian coordinates) real and imaginary part

Fourier Transform

Analysis Equation

- Suppose a communication signal $w(t)$ is made up of two complex phasors, at different frequencies, and different complex-valued amplitudes:

$$w(t) = s_{A_1,f_1}(t) + s_{A_2,f_2}(t)$$

- The dot-product between $w(t)$ and $s_{1,f}(t)$ is given as follows:

$$W(f) = \frac{1}{T}\int_{t=-T/2}^{T/2} w(t)\, s^*_{1,f}(t)\, dt = \begin{cases} A_1, & f = f_1 \\ A_2, & f = f_2 \\ 0, & \text{otherwise} \end{cases}$$

$$= A_1\, \delta(f - f_1) + A_2\, \delta(f - f_2)$$

Fourier Transform

Analysis Equation

- Suppose any arbitrary communication signal $w(t)$ is made up of an infinite number of complex phasors, at different frequencies, and complex-valued amplitudes:

$$w(t) = \sum_{k=-\infty}^{\infty} s_{A_k, f_k}(t)$$

- The dot-product between $w(t)$ and $s_{1,f}(t)$ is given as follows:

$$W(f) = \frac{1}{T} \int_{t=-T/2}^{T/2} w(t)\, s_{1,f}^{*}(t)\, dt = \begin{cases} A_k, & f = f_k \\ 0, & \text{otherwise} \end{cases}$$

$$= \sum_{k=-\infty}^{\infty} A_k\, \delta(f - f_k)$$

Fourier Transform
Analysis Equation

- For technical reasons, when T is replaced by ∞, the Fourier Analysis equation becomes as follows:

$$W(f) = \int_{t=-\infty}^{\infty} w(t)\, s^*_{1,f}(t)\, dt$$
$$= \int_{t=-\infty}^{\infty} w(t) \exp\left(-i\left(2\pi f t\right)\right)\, dt$$

- Points to note:
  - $\rightarrow$ Frequency f is a variable, it is moved from a low value to a high value, and for each value of f, $W(f)$ is calculated as above
  - $\rightarrow$ $W(f)$ calculated at a given value of f gives the complex-valued amplitude of the complex phasor at f in the signal $w(t)$

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Inverse Fourier Transform

Also known as Synthesis Equation

- Suppose you're given the complex-valued amplitudes A_k and the frequencies f_k of the corresponding complex phasors that make up a signal $w(t)$
- Using this information, the signal $w(t)$ can be generated, or *synthesized*

$$w(t) = \sum_{k=-\infty}^{\infty} s_{A_k,f_k}(t) = \sum_{k=-\infty}^{\infty} A_k \exp\left(i(2\pi f_k t\right)$$

- In continuous-frequency domain, A_k become $W(f)$, f_k is simply the integral variable f and the summation is replaced by integration

$$w(t) = \int_{-\infty}^{\infty} W(f) \exp\left(i(2\pi f t)\right) df$$

- The above is called the Synthesis equation or Inverse Fourier Transform

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Fourier Transform

Summary

- Fourier Transform (Analysis Equation):
  - $\rightarrow$ $W(f) = \int_{t=-\infty}^{\infty} w(t) \exp\left(-i\left(2\pi f t\right)\right) dt$
  - $\rightarrow$ Moves the viewpoint of looking at the signal from time domain to frequency domain
- Inverse Fourier Transform (Synthesis Equation):
  - $\rightarrow$ $w(t) = \int_{f=-\infty}^{\infty} W(f) \exp\left(i(2\pi f t)\right) df$
  - $\rightarrow$ Moves the viewpoint back to time domain from the frequency domain

An Example: A Sinusoidal Signal

- Let $w(t) = a\sin(2\pi f_c t)$
- This is also written as

$$\begin{aligned} w(t) &= \frac{a}{2i}\left(\exp(i2\pi f_c t) - \exp(-i2\pi f_c t)\right) \\ &= \frac{a}{2}\exp(-i\pi/2)\exp(i2\pi f_c t) + \frac{a}{2}\exp(i\pi/2)\exp(-i2\pi f_c t) \end{aligned}$$

- Therefore, its Fourier Transform is given as

$$W(f) = A_1\,\delta(f - f_c) + A_2\,\delta(f + f_c)$$

where $A_1 = a\exp(-i\pi/2)/2$ and $A_2 = a\exp(i\pi/2)/2$

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- Let $w(t) = a \sin(2\pi f_c t)$
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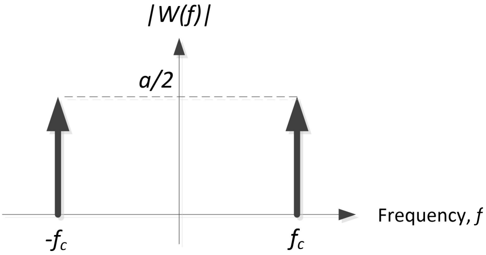
Fourier Transform
for a Sinusoidal Signal

- Pictorial view of the Fourier Transform of $w(t) = a\ \sin(2\pi f_c t)$:

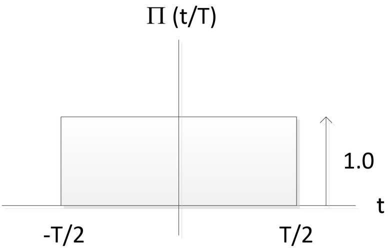


- Above is the magnitude of the F.T.
  - $\rightarrow$ Need also to specify the phase of the F.T. to define it completely.

An Example: A Rectangular Signal

- A rectangular pulse is denoted by function $\Pi(\cdot)$:

$$\Pi\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| \leq \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$



An Example: A Rectangular Signal
Fourier Transform

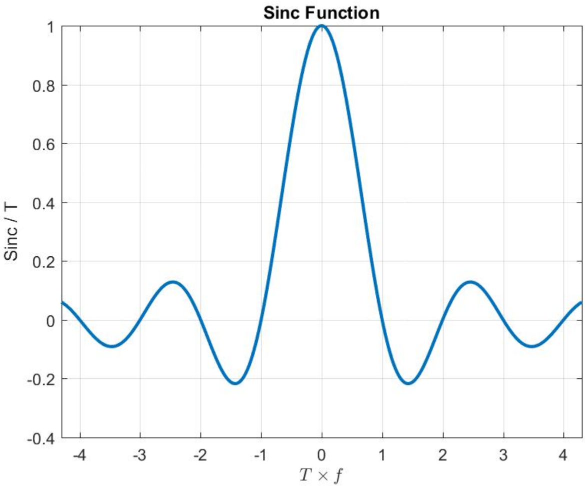
- Fourier Transform of the rectangular pulse is easy to calculate:

$$
\begin{aligned}
W(f) &= \int_{-T/2}^{T/2} 1 \exp\left(-i2\pi f t\right) dt \\
&= \frac{\exp\left(-i2\pi f T/2\right) - \exp\left(i2\pi f T/2\right)}{-j2\pi f} \\
&= T \operatorname{sinc}\left(T f\right)
\end{aligned}
$$

- Here $\operatorname{sinc}(x) \stackrel{\text{def}}{=} \frac{\sin(\pi x)}{\pi x}$

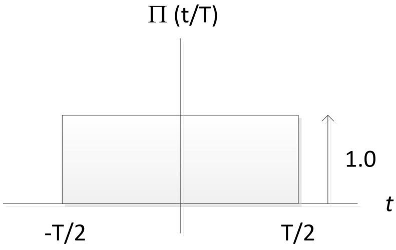
An Example: A Rectangular Signal

Magnitude of F.T.

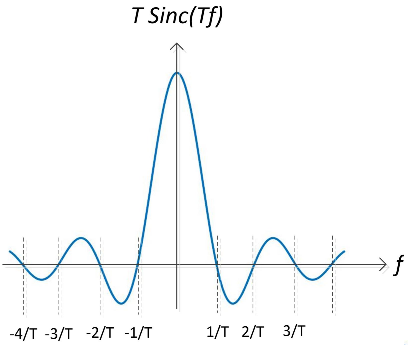


Rectangular and Sinc Signals Form F.T. Pair

Time Domain



Frequency Domain



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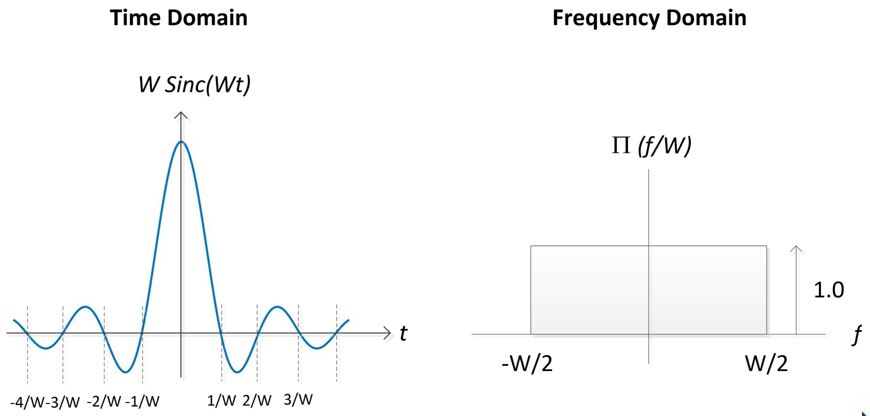

Rectangular and Sinc Signals Form F.T. Pair

- Both the rectangular and Sinc signals are used in the communication system design and modeling
  - → These are ideal signals, hard to implement them in the real life
  - → In actual implementation, their practically feasible versions are instead used
- These functions allow us to appreciate the concepts of spectrum and bandwidth
  - → It is seen, for example, that enlarging the spectral bandwidth implies making the time domain pulse narrower, and vice versa

Several Other Examples

- There are several other signals whose F.T. can be easily calculated:
  - $\rightarrow$ A decaying exponential pulse
  - $\rightarrow$ A triangular pulse
- We will not go over these in the class, but you're *encouraged* to refer to Couch book, Chapter 2, Section 2.2
  - $\rightarrow$ Gain familiarity with Tables of Fourier Transform Pairs

Several Properties
of Fourier Transform

- Duality:
  - $\rightarrow$ If $w(t) \Longleftrightarrow W(f)$, then $W(t) \Longleftrightarrow w(-f)$
- Time or Frequency Shifts:
  - $\rightarrow$ Time Shift or Delay:
    - $\triangleright$ $w(t - T_d) \Longleftrightarrow W(f) \exp(-i2\pi f T_d)$
  - $\rightarrow$ Frequency Shift or Translation (also known as Modulation Property):
    - $\triangleright$ (Complex): $w(t) \exp(i2\pi f_c t) \Longleftrightarrow W(f - f_c)$
    - $\triangleright$ (Real): $w(t) \cos(2\pi f_c t + \theta) \Longleftrightarrow \frac{1}{2}\left[e^{i\theta} W(f - f_c) + e^{-i\theta} W(f + f_c\right]$
- Spectral symmetry of real-valued signals:
  - $\rightarrow$ If $w(t)$ is real: $W(-f) = W^*(f)$
  - $\rightarrow$ If $w(t)$ is real and symmetric: $W(f)$ is real-valued and symmetric

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    - ▷ Communication channels introduce a delay. This property tells us what to expect in frequency domain, i.e., a linearly changing phase as a function of frequency. Alternatively, if the communication channel introduces nonlinear phase shift, that tells us that it is introducing a time distortion in the signal instead of a simple time delay
  - → Frequency Shift or Translation (also known as Modulation Property):
    - ▷ (Complex): a message signal $w(t)$ is typically centered at 0 Hz. Its frequency is translated to Radio Frequency or RF in the manner shown by the property of complex frequency shift.
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- Convolution and Multiplication:
  - $\rightarrow \; w_1(t) * w_2(t) = \int_{-\infty}^{\infty} w_1(\tau) w_2(t-\tau)\, d\tau \Longleftrightarrow W_1(f) W_2(f)$
  - $\rightarrow \; w_1(t) w_2(t) \Longleftrightarrow \int_{-\infty}^{\infty} W_1(\nu) W_2(f-\nu)\, d\nu$
- Scale Change:
  - $\rightarrow \; w(m\,t) \Longleftrightarrow \frac{1}{|m|} W\left(\frac{f}{m}\right)$
- Parseval's Theorem:
  - $\rightarrow \; \int_{-\infty}^{\infty} w_1(\tau) w_2^*(\tau)\, d\tau \Longleftrightarrow \int_{-\infty}^{\infty} W_1(f) W_2^*(f)\, df$
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  - → Convolution in time domain is used to perform filtering in frequency domain
  - → Effect of time domain sampling can be thought of convolution in frequency domain, which makes the frequency spectra periodic
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Sampling in Time Domain
Sampling Theorem

→ If a continuous-time (C-T) signal $w(t)$ is sampled to obtain a discrete-time (D-T) signal $w(t_n)$, where $t_n = n \times T_s$ are the samples in time obtained at a duration of T_s seconds, the effect in frequency domain is that the original spectrum $W(f)$ becomes periodic in frequency domain with a period equal to $F_s = 1/T_s$ Hz

- ▷ Alternatively, a signal whose spectrum is periodic in frequency domain with a period of F_s Hz has to be discrete-valued in time-domain with samples that are spaced T_s seconds apart
- ▷ Nyquist Sampling Theorem: requires the sample duration to be equal to or greater than the bandwidth W of the signal. With samples that are spaced no greater than $1/W$ seconds apart in the time domain, Nyquist Theorem guarantees that the actual C-T signal can be reconstructed from the D-T signal

Sampling in Frequency Domains

Fourier Series

→ If an aperiodic C-T signal $w(t)$ is turned into a periodic signal with a period T seconds, the effect in frequency domain is that the original spectrum $W(f)$ gets sampled, with inter-sample separation of $\Delta F = 1/T$ Hz

- Alternatively, if the signal in the frequency domain, instead of being a continuous-frequency spectrum, has spectral samples (also known as line spectrum) that are spaced $1/T$ Hz apart, the signal in the time domain has to be periodic with a period equal to T seconds
- A periodic-time domain signal has line spectrum (and not continous-frequency spectrum), which is evaluated by using Fourier Series
- Relation between Fourier Transform and Fourier Series: the latter is obtained by sampling the former. The spectral samples of the F.T. remain (i.e., F.T. turns into a Fourier Series) when the time domain aperiodic signal is made periodic.