AFTER INSEM-1: Till now we discussed analytical methods for solving non linear & non-linear problems. There are simpler methods for linear prob

⇒ **Linear Programming Problem**

It is an optimization problem of the form:

Minimize $C^T X$ subject to $\left.\begin{array}{l} AX = b \\ X \geq 0 \end{array}\right\}$

$C^T \rightarrow$ Cost vector

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ n components $C^T = (c_1, c_2, \ldots c_n)$

So $C^T X = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \rightarrow$ linear objective function

$AX = b \rightarrow$ linear constraints

$A_{m \times n} \; X_{n \times 1} = b_{m \times 1}$

$X \geq 0$ means that each component of X is non-negative.

Constraints may also be of $AX \leq b$ or $AX \geq b$ but they can be made into equality constraints by adding slack variables or subtracting surplus variables. They may also be mixed forms.

X is the solution of the problem or decision vector. $x_1, x_2 \ldots x_n$ are the decision variables and decide the value of f.

Scanned by CamScanner

DATE 09.09.19

The region satisfied by the constraints is called the feasible region. We search for X in the feasible region that will minimize the objective f^n $c^T X$.

Ex 1: A manufacturer produces 4 different types of products. There are 3 inputs for the production process. (1) No. of people (manpower) required
(2) Kilograms of raw material A.
(3) Units of raw material B.

Each product has a different input requirement.

Inputs	Product I	p.II	p.III	p.IV	Input available
Persons	1	2	1	2	20
Units of A	6	5	3	2	100
Units of B	3	4	9	12	75
Profit / unit →	6000/-	4000/-	7000	5000/-	

requirements of I/p

given these availabilities

How many units of each product should be manufactured to maximize profit?

A. Let x_1 be units of product I, $x_2 \to$ product II, $x_3 \to$ product III, $x_4 \to$ product IV to be manufactured (to be determined - decision variables).

Objective f^n: Maximize : $6000x_1 + 4000x_2 + 7000x_3 + 5000x_4$

$c^T = (6000, 4000, 7000, 5000)$

Subject to constraints :

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 &\le 20 \\ 6x_1 + 5x_2 + 3x_3 + 2x_4 &\le 100 \\ 3x_1 + 4x_2 + 9x_3 + 12x_4 &\le 75 \end{aligned}\right\} \leftarrow \text{linear functions of } X.$$

$$x_1, x_2, x_3, x_4 \ge 0.$$

Maximize $c^T X$ subject to $AX \le b$, ~~subject to~~ $X \ge 0$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 6 & 5 & 3 & 2 \\ 3 & 4 & 9 & 12 \end{bmatrix} \quad B = \begin{bmatrix} 20 \\ 100 \\ 75 \end{bmatrix}. \quad C = \begin{bmatrix} 1000 \\ 4000 \\ 7000 \\ 5000 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

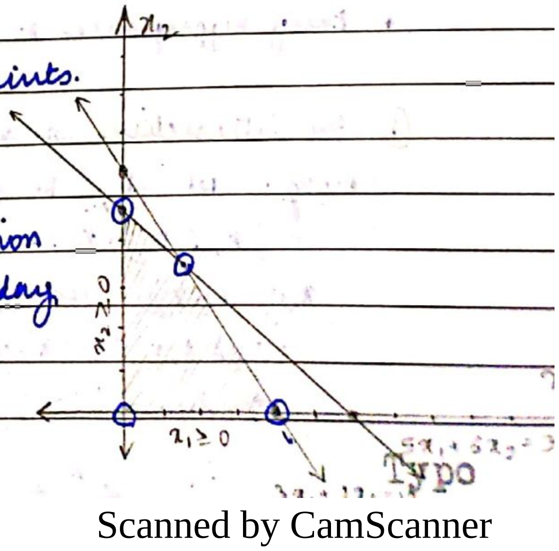
⇒ **Two Dimensional LP. problem**

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ Minimize $z = x_1 + 5x_2$ subject to $\left.\begin{aligned} 5x_1 + 6x_2 &= 30 \\ 3x_1 + 2x_2 &= 12 \\ x_1, x_2 &\ge 0 \end{aligned}\right\}$ feasible Region

We try to choose values of x_1 and x_2 and observe how large we can make z while the values of x_1 and x_2 should satisfy the constraints.

⇒ Solution using the graphical Method.

1. Plot the feasible region



Theorem: A continuous f^n on a closed bounded set/region obtained its maximum and minimum on the boundary

Corner points : $(0,0)$, $(0,5)$, $(4,0)$, $\left(\frac{3}{2}, \frac{15}{4}\right)$

Objective f^n : $z = 0$ $\quad z = 25$ $\quad z = 4$ $\quad z = 20\frac{1}{4} = 20.25$

Soln

⇒ **Convex set**: collection of points such that if x_1 and x_2 are in the collection, the line segment joining them is also in the collection. If $x_1, x_2 \in K$ then $X \in K$ where $X = \lambda x_1 + (1-\lambda) x_2$, $0 \le \lambda \le 1$

That is the line segment joining the point x & y lies entirely in K.

ϕ, $\{x\}$, $\mathbb{R}^n$ are known trivial examples of convex sets.

eg. $K = \{(x, y) \in \mathbb{R}^2 ;\ x + 2y = 3\}$

Let $(x_1, y_1) \in K$ and $(x_2, y_2) \in K$ → $x_1 + 2y_1 = 3$ and $x_2 + 2y_2 = 3$

Let $\lambda \in [0,1]$. $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)$

$\Rightarrow \lambda x_1 + (1-\lambda)x_2 + \lambda y_1 + (1-\lambda) y_2$

$\Rightarrow \lambda x_1 + (1-\lambda) x_2 + 2[\lambda y_1 + (1-\lambda) y_2]$

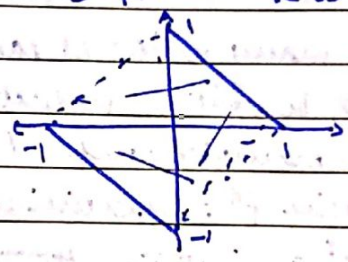
$\Rightarrow \lambda[x_1 + 2y_1] + (1-\lambda)(x_2 + 2y_2)$

$\Rightarrow 3\lambda + 3(1-\lambda) = \underline{3} \in K$ ∴ K is a convex set

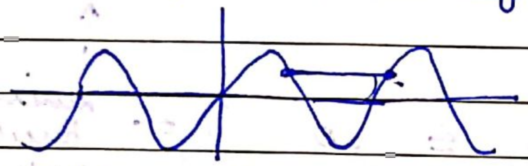
eg. $S = \{(x,y) \in \mathbb{R}^2 :\ |x| + |y| \le 1\}$

Entire line falls in the set

Taking any 2 points, join them so this is convex – prove mathematically.



eg.

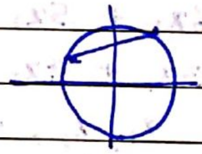


$S = \{(x, y) \in \mathbb{R}^2 ;\ y = \sin x\}$

entire line does not fall in the set ∴ Not convex

eg. $S = \{(x,y) \in \mathbb{R}^2 :\ x^2 + y^2 = 1\}$

This also not convex as entire line does not fall inside the set



⇒ **Hyperplane**: A hyperplane in $\mathbb{R}^n$ is defined as the set of points of $\mathbb{R}^n$ that satisfy an equation $c_1 x_1 + c_2 x_2 + \cdots c_n x_n = d$.

$c_1, c_2, \ldots c_n, d$ are real, constant and are not all zeros.

⇒ **Hyperspace**: A hyperplane in $\mathbb{R}^n$ is defined as the set of points which satisfy one of the following:

$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \le d$ $\qquad$ $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \ge d$

$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n < d$ $\qquad$ $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n > d$

* Every hyperplane & hyperspace is a convex set.

① The intersection of an arbitrary family of convex sets is a convex set.

Proof: Let $\{K_\alpha\}$ be a family of convex sets. Let $x, y \in \bigcap_\alpha K_\alpha$ $(K_1 \cap K_2 \cap K_3 \ldots \cap K_n)$ and $\lambda \in (0,1)$

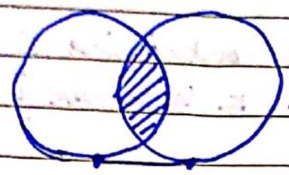
$x, y \in K_\alpha$ for each α.

→ $\lambda x + (1-\lambda) y \in K_\alpha$ (because K_α is convex)(for each α)

→ $\lambda x + (1-\lambda) y \in \bigcap_\alpha K_\alpha$

∴ $\bigcap_\alpha K_\alpha$ is convex.

Scanned by CamScanner

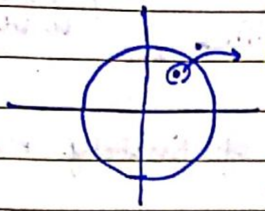


Intersection → convex set

Union → need not be convex

* Note 1. A convex set may be bounded or unbounded.

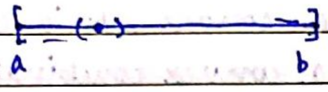
2. A convex set may be open, closed, both open & closed, neither open nor closed

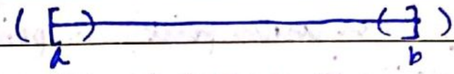


$x^2 + y^2 < 1$:

$(x, y) \in \mathbb{R}^2$ such that

If you find a disc with the point as a center and disc lies entirely in the set, the point is an interior point





→ a, b are not interior points. Any other points except a & b are interior

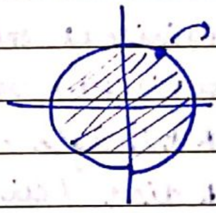
A set is said to be an open set if all the points are interior points

eg. $x^2 + y^2 < 1$: all points are interior ∴ open set

- Points can be interior or boundary points

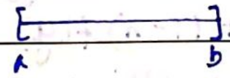
eg. $(x, y) \in \mathbb{R}^2$: $x^2 + y^2 \le 1$ →

This is not an open set



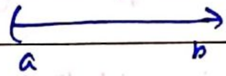
- A set is open if each of the points are interior points
- A set is closed if it contains all boundary points

$S = [a, b]$



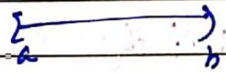
→ closed set

$S = (a, b)$



→ open set

$S = [a, b)$



→ neither open nor closed

$S = \phi$ (empty / null set) → both open and closed

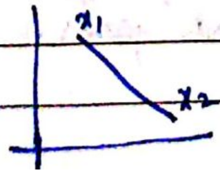
- A set is said to be closed if its complement is open

$S = \phi$ → $S' = \mathbb{R}^n$: open ∴ S is closed

$\mathbb{R}^n$ is open and closed ϕ is open and closed

→ Convex combination : A convex combination of vectors $x_1, x_2, \dots x_n$ means $\lambda_1 x_1 + \lambda_2 x_2 + \dots \lambda_n x_n$ where $0 < \lambda_i < 1$ and $\lambda_1 + \lambda_2 + \dots \lambda_n = 1$

A convex combination is a particular case of a linear combination.



$\lambda_1 x_1 + (1 - \lambda_1) x_2$

$1 - \lambda_1 = \lambda_2$

$0 < \lambda < 1$

Scanned by CamScanner

→ **Convex hull** · The smallest convex set ($S \subseteq \mathbb{R}^n$) containing S is called the convex hull of S. $\text{Conv}(S) = \bigcap \{K : S \subseteq K,\ K \text{ is convex}\}$

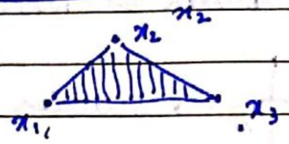
Exp. (i) $S = \{\{x\}\}$, $\text{conv}(S) = \{x\}$

(ii) Convex hull of two distinct points $S = \{x_1, x_2\}$

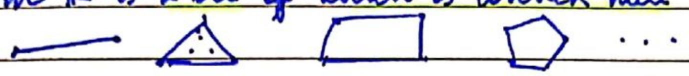
$\text{conv}(S) \rightarrow$ line from x_1 to x_2

(iii) Convex hull of three distinct points (circle would be bigger)

The triangle joining the three points.



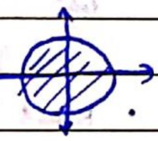
→ **Convex Polyhedron** in $\mathbb{R}^n$ is a set of which is convex hull of finitely many points in $\mathbb{R}^n$



A polyhedron is the intersection of finitely many hyperspaces

→ **Extreme point** (or corner point) a point of a convex set is called an extreme point if it cannot be expressed as a convex combination of points of the set. (will not lie in line joining two points)

* Point & line don't have extreme points
* Infinitely extended 1st quadrant - 1 extreme point
* infinitely extended strip in 1st Quadrant · 2 extreme points
* A triangle - 3 extreme points, Rectangle - 4 extreme points
* A disc (circle) - infinitely many extreme points



$(x, y) : x^2 + y^2 \leq r^2$

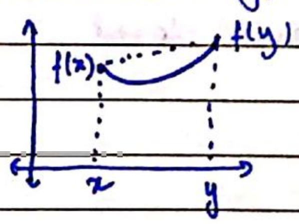
→ **Affine function** - $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called affine if

$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y)$ for $\lambda \in (0,1)$, $x, y \in \mathbb{R}^n$

* Equations of a line, plane are affine functions.

→ **Convex function** - $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a convex function if

$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for $\lambda \in (0,1)$, $x, y \in \mathbb{R}^n$



A function is said to be concave, if $-f$ is a convex function.

Strictly convex → strictly less than.

→ **Standard form of a Linear Programming problem.**

Minimize $c^T x$ subject to $Ax = b$, $x \geq 0$

$A_{m \times n}$ matrix with $m \leq n$, $\text{rank}(A) = m$

Without loss of generality, we assume $b \geq 0$ (Else multiply both sides by -1)

· If constraints are ≥ 0 type, subtract positive quantity $y_j \geq 0$, $j = 1 \ldots m$ from each of the equations to convert it to standard form.

Scanned by CamScanner

DATE

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n - y_1 = b_1$ $y_1 \ge 0$ $n+m$ variables now.
$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n - y_2 = b_2$ $y_2 \ge 0$
$\vdots$
$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n - y_m = b_m$ $y_m \ge 0$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & -1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & -1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 & 0 & \dots & -1 \end{bmatrix}}_{m \times (m+n)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}}_{(m+n)\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$m \times n$ $m \times m$

Minimize $c^T X$ subject $[A, -I_m]\begin{bmatrix} X \\ Y \end{bmatrix} = b$, $X \ge 0$, $Y \ge 0$

or $AX - I_m Y = b$

↳ $y_1, y_2 \dots y_m$ are called surplus variables.

- If constraints are $\le$ type, we add a positive quantity.
  Minimize $c^T X$ subject to $[A, I_m]\begin{bmatrix} X \\ Y \end{bmatrix} = b$, $X \ge 0$, $Y \ge 0$

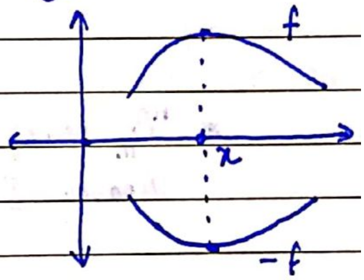
↳ $y_1, y_2 \dots y_m$ are slack variables.

- If some constraints are $\ge b$ and some $\le$. then ?

$\cancel{2x_1 + 3y_2 \le 5}$ $2x_1 + 3x_2 \le 5$
$\cancel{x_1 - y_2 \ge 3}$ $x_1 - x_2 \ge 3$ $\Rightarrow$ $2x_1 + 3x_2 + y_1 = 5$, $x_1 - x_2 - y_2 = 3$

here $y_1 \to$ slack variable and $y_2 \to$ surplus variable.

- conversion of a Maximize problem to Minimization.
  Maximize $c^T X$ $\longrightarrow$ $-$ Minimize $c^T X$



⇒ Basic Solution.

Standard form: Minimize $c^T X$ subject to $AX = b$, $x \ge 0$

$A_{m\times n}$ $c \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, $X \in \mathbb{R}^n$ Rank$(A) = m$

Since Rank$(A) = m$, we have m linearly independent columns of A.

Form the matrix B, by choosing m linearly independent columns of A.

$$A = [\underset{m\times m}{B} : \underset{m\times(n-m)}{D}]$$

Solve the system $BX = b$ $\rightarrow$ unique solution because $B_{m\times m}$ & columns
$X_B = B^{-1} b$ are linearly independent ↳ invertible

Let X be a vector where 1st m components are X_B and the remaining are zero

$$X = [X_B^T, 0^T]^T$$

X is called the Basic Solution to system $AX = b$, w.r.t B.

B is called the Basis Matrix.

The columns of B are called Basic columns.

The components of vector X_B are called Basic variables.

Typo

Scanned by CamScanner

DATE 16 · 09 · 2019

- **Degenerate Basic Solution** - If some of the basic values (X_B components) of the basic solution are zeros, then the basic solution is called degenerate basic solution.
- **Feasible Solution** - any vector X satisfying $AX = b$, $X \geq 0$ is said to be a feasible solution to the LP problem.
- **Basic Feasible solution** - any feasible solution that is also basic solution
- **Degenerate Basic Feasible Solution** - ~~if the basic solution has some components of the basic feasible solⁿ are~~ if the basic feasible solution is degenerate.

Ex: consider the system $AX = b$
4 variables, 2 equations

$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array}$$
$$\underset{2\times 4}{\begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 8 \\ 2 \end{bmatrix}}$$

Choose $[a_1, a_2] = B$

$BX = b$

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$x_1 + x_2 = 8$ — (1)
$x_1 - 2x_2 = 2$ — (2)
$\Rightarrow$ $x_1 = 6$, $x_2 = 2$ $\Rightarrow X_B = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$.

$X = \begin{bmatrix} 6 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ → Basic Solution corresponding to basis matrix B.
It is also a basic feasible solution.

* ${}^nC_m$ no. of basic solutions are possible

choose $[a_3, a_4] = B$

$$\begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$-x_3 + 4x_4 = 8$
$-x_3 + x_4 = 2$
$\rightarrow$ $x_3 = 0$, $x_4 = 2$ $\Rightarrow X_B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$ → Degenerate Basic Solution since $x_3 = 0$ (feasible)

given $X = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ is it a feasible solution?
Yes, but it is not Basic.

* In basic solution at most m components will be non-zero.
(At least n-m components will be zero.)

choose $B = [a_2\ a_3]$

$$\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

Scanned by CamScanner

DATE 18.09.19

$$\left.\begin{array}{l} x_2 - x_3 = 8 \\ -2x_2 - x_3 = 2 \end{array}\right\} \Rightarrow \begin{array}{l} x_2 = 2 \\ x_3 = -6 \end{array} \rightarrow x_B = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 \\ 2 \\ -6 \\ 0 \end{bmatrix}$$ - Basic solution but NOT feasible because $x_3 < 0$

- The Basic feasible solutions are nothing but the corner points. So we find all the basic feasible solutions, put them in the objective function to find maxima/minima.

→ Fundamental Theorem of LP : Consider an LP problem in standard form. If
(1) there exists a feasible solution, then there exists a basic feasible solution.
(2) If there exists an optimal feasible solution, then there exists an optimal basic feasible solution.

- Proof (1): suppose $x = [x_1, x_2, \ldots x_n]^T$ is a feasible solution & it has p positive components. Without loss of generality, we assume that the first p components are positive, whereas the remaining components are zeros.

$A = [a_1, a_2, a_3, \ldots a_p, \ldots a_n]$ ($a_1, a_2 \ldots$ are columns of A)

The solution satisfies $x_1 a_1 + x_2 a_2 + \cdots x_p a_p = b$ —— (1) [$x_{p+1} \ldots x_n$ are zeroes]

Case 1 : If $a_1, a_2, \ldots a_p$ are linearly independent

- If $p = m$ then the solution x is basic (m linearly independent columns, rest 0)

$\therefore$ x is a basic feasible solution.

Case 2 : If $p < m$, then since $\text{Rank}(A) = m$, we can find $(m-p)$ columns of A from the remaining $(n-p)$ columns so that the ~~remaining~~ resulting set of m columns form a basis $((p + (m-p))$ columns). Hence the solution X is a degenerate basic feasible solution corresponding to the above basis.

$$X = (\underbrace{x_1\ x_2\ x_3 \ldots x_p}_{p\text{-nonzeros}}, \underbrace{\overset{0}{x_{p+1}} \overset{0}{\ldots} \overset{0}{x_m}}_{m-p \text{ zeros}}, \underbrace{x_{m+1} \ldots x_n}_{m-n})$$

$\therefore$ some basic variables are zeroes

Case 2 : If $a_1, a_2, \ldots a_p$ are linearly dependent then $p > m$ (must be >)

There exists numbers y_i, $i = 1, 2, \ldots p$, not all zeroes, such that

$$a_1 y_1 + a_2 y_2 + \ldots + a_p y_p = 0 \text{ —— (2)} \quad \text{(Definition of L.I.)}$$

We can assume that there exists atleast one y_i that is positive.

Multiply eqn (2) by some scalar $\epsilon > 0$, and subtract the resulting eqn from (1) -

$$(x_1 - \epsilon y_1) a_1 + (x_2 - \epsilon y_2) a_2 + \cdots + (x_p - \epsilon y_p) a_p = b \text{ —— (3)}$$

If $Y = [y_1, y_2, \ldots y_p, 0, 0, 0 \ldots 0]^T$

(3) can be written as $A[X - \epsilon Y] = b$

$\therefore$ $X - \epsilon Y$ is also a solution to the system.

Let $\epsilon = \min\left\{ \frac{x_i}{y_i} \;\; i = 1, 2, \ldots p, \; y_i > 0 \right\}$ then $\forall (x_i - \epsilon y_i) \geq 0 \quad i = 1, 2, \ldots p$

The first p components of $X - \epsilon Y$ are non negative (≥ 0) and atleast one of these components is zero.

Typo

We have a feasible solution with at most $(p-1)$ positive components. We repeat this procedure until we get linearly independent columns of A, after which we are back to case 1.

- **Proof ②**: Suppose that $X = (x_1, x_2 \dots x_p)^T$ is an optimal feasible solution and the first p components are positive.

Case 1: If $a_1 a_2 \dots a_p$ are L.I., same argument as in part ①.

Case 2: If $a_1 a_2 \dots a_p$ are Dependent $(X - \epsilon Y)$

This follows the same argument as Case 2 of part ① but in addition we must show that $(X-\epsilon Y)$ is optimal for any ϵ.

Consider

If we can prove

Suppose $c^T Y \neq 0$

For ϵ of sufficiently small magnitude $\left(|\epsilon| \leq \min\left\{ \frac{x_i}{y_i} \right\} \right)$

The vector $X-\epsilon Y$ is feasible. $X - \epsilon Y \geq 0$

We can choose ϵ such that $\cancel{c^T(X-\epsilon Y) > 0} \Rightarrow \cancel{c^T X > \epsilon c^T Y}$

$$c^T X > c^T X - \epsilon c^T Y$$
$$= c^T(X - \epsilon Y)$$

- This contradicts to the fact that X is optimal. Our assumption that $c^T Y \neq 0$ is wrong. So $c^T Y = 0$.

- Geometric view of Basic Feasible Solution

$AX = b, \quad X \geq 0 \quad]$ convex set.

A point X in a convex set S is called an extreme point if there are no two distinct points $X_1 + X_2$ in S, such that

$$X = \lambda X_1 + (1-\lambda) X_2, \quad \text{for } \lambda \in (0,1)$$

If X is an extreme point and $X = \lambda X_1 + (1-\lambda) X_2$ for some $X_1, X_2 \in S$ and $\lambda \in (0,1)$

Then $X_1 = X_2$

Theorem: Let S be the convex set containing all feasible solutions that is all X satisfying $AX = b$, $X \geq 0$ where $A \in \mathbb{R}^{m\times n}$, $m < n$, rank $(A) = m$

Then X is an extreme point at S if and only if X is basic feasible solution to $AX=b$, $X \geq 0$.

Proof: Let X be extreme point of S. Suppose that X has p positive components

$X = (x_1, x_2 \dots x_p, 0, 0, 0 \dots)^T$

(W.L.G we can assume that the first p components are > 0)

$$x_1 a_1 + x_2 a_2 + \dots + x_p a_p = b \quad \text{— ①}$$

To prove X is a basic solution, i.e. To prove $a_1, a_2 \dots a_p$ are L.I.

Scanned by CamScanner

DATE 23·09·2019

Suppose $y_1 a_1 + y_2 a_2 + \dots + y_p a_p = 0$ —— ②

* We show $y = 0$ for all $i = 1, 2, \dots, p$

Multiply ② by $\epsilon\ (>0)$, then add & subtract the result from ①.

$(x_1 + \epsilon y_1) a_1 + \dots + (x_p + \epsilon y_p) a_p = b$ —— ③

$(x_1 - \epsilon y_1) \cdot a_1 + \dots + (x_p - \epsilon y_p) a_p = b$ —— ④

$x_i \geq 0$. $\epsilon > 0$ can be chosen such that $x_i + \epsilon y_i$ and $x_i - \epsilon y_i \geq 0$

$\therefore \epsilon = \min \left\{ \left| \frac{x_i}{y_i} \right| : i = 1, 2, \dots, p \right\}$

Let $Z_1 = X + \epsilon Y$

$Z_2 = X - \epsilon Y$

Such a choice of ϵ makes $X - \epsilon Y,\ X + \epsilon Y \geq 0$.

$Z_1 Z_2 \geq 0$

From ③, $A(X + \epsilon Y) = b$, $X + \epsilon Y \geq 0$

From ④, $A(X - \epsilon Y) = b$, $X - \epsilon Y \geq 0$

$X = \frac{1}{2}(Z_1 + Z_2) = \frac{1}{2}(X + \epsilon Y + X - \epsilon Y)$ which can be written as $\lambda Z_1 + (1-\lambda) Z_2$

But since X is an extreme point, $Z_1 = Z_2$

$\therefore X + \epsilon Y = X - \epsilon Y$, $\rightarrow$ True only when $Y = 0$

i.e. $x_i + \epsilon y_i = x_i - \epsilon y_i \quad \forall i = 1, 2, \dots p$

$\therefore a_1 a_2 \dots a_p$ are linearly independent

$\therefore X$ is a basic feasible solution

- Conversely, let $X \in S$ be a basic feasible solution. To prove that X is an extreme point. Let $Y, Z \in S$ be such that $X = \lambda Y + (1-\lambda) Z$ for $\lambda \in (0,1)$

To prove: $Y = Z$.

Since $AY = A(Z) = b$ $(\because Y, Z \in S)$

$y_1 a_1 + \dots + y_m a_m = b$ since Rank$(A) = m$, remaining components

$z_1 a_1 + \dots + z_m a_m = b$ can be taken as 0.

Subtracting gives: $(y_1 - z_1) a_1 + \dots + (y_m - z_m) a_m = 0$

Since $a_1, a_2, \dots a_m$ are L.I. $y_1 - z_1 = y_m - z_m = \dots = 0$

$\Rightarrow y_1 = z_1,\ y_2 = z_2 \dots y_m = z_m$

$\Rightarrow \underline{Y = Z}$ $\Rightarrow$ X is an extreme point

- Matrix, Elementary Row Operations -

① Interchanging two rows ② Multiplying one of it's rows by a non-zero real number. ③ Adding a scalar multiple of one row to another row

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Rows interchange $\nearrow$ $E_1 A$ $\qquad$ $E_1 \rightarrow$ interchanged I.

So all these operations can be accomplished by premultiplying some matrix

* A is invertible if and only if there exist elementary matrices E_i, $i = 1, 2, \dots t$ such that $E_t E_{t-1} \cdots E_2 E_1 A = I$.

Scanned by CamScanner

DATE

$A^{-1} = E_t E_{t-1} \dots E_2 E_1$

→ Canonical Augmented Matrix.

$AX = b$.

Canonical Representation of the system

$$\begin{cases} x_1 + 0 + 0 + \dots + 0 + y_{1m+1} x_{m+1} + \dots + y_{1n} x_n = b_1 \\ 0 + x_2 + 0 \dots + y_{2m+1} x_{m+1} + \dots + y_{2n} x_n = b_2 \\ \vdots \\ 0 + 0 + \dots + x_m + y_{mm+1} x_{m+1} + \dots + y_{mn} x_n = b_m \end{cases}$$

$$[I_m, Y_{m \times (n-m)}] X = b \qquad b = [b_1, b_2 \dots b_m]^T$$

If all y are zeros, $\begin{bmatrix} x_1 = b_1 \\ x_2 = b_2 \\ x_m = b_m \\ x_{m+1} = 0 \\ \vdots \\ x_n = 0 \end{bmatrix}$ i.e. $X = \begin{bmatrix} b \\ 0 \end{bmatrix}$

This is a basic feasible sol^n.

And
$$\begin{bmatrix} 1 & 0 & \dots & 0 & y_{1m+1} & \dots & y_{1n} & \vdots & b_1 \\ 0 & 1 & \dots & 0 & y_{2m+1} & \dots & y_{2n} & \vdots & b_2 \\ \vdots \\ 0 & \dots & & 1 & y_{mm+1} & \dots & y_{mn} & \vdots & b_m \end{bmatrix}$$
This is the canonical augmented matrix

basic columns — non-basic

$m \times n$ → transformed A.

Simplex method

$AX = b \quad X \geq 0$ → feasible region

converted to canonical augmented matrix

$$\left[\begin{array}{cccc cccc|c} 1 & 0 & \dots & 0 & y_{1m+1} & y_{1m+2} & \dots & y_{1n} & y_{10} \\ 0 & 1 & \dots & 0 & y_{2m+1} & y_{2m+2} & \dots & y_{2n} & y_{20} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & y_{mm+1} & y_{mm+2} & & y_{mn} & y_{m0} \end{array}\right] \qquad Y_0 = b = \begin{bmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{bmatrix}$$

$$= [I_{m \times m} \; Y_{m \times (n-m)} \; Y_0]$$

$$b = y_{10} a_1 + y_{20} a_2 + \dots + y_{m0} a_m$$

for $m < j \leq n$ → non-basic columns – can be written as linear combination of basic colu

$$a_j = y_{1j} a_1 + y_{2j} a_2 + \dots + y_{mj} a_m$$

Initial basic feasible sol^n $[y_{10}, y_{20}, \dots y_{m0}, 0, 0, 0 \dots 0]^T$

$a_1, a_2, \dots a_m$ are basic columns $\qquad y_{i0} \geq 0$

$a_{m+1}, a_{m+2}, \dots a_n$ are non-basic col^n

Scanned by CamScanner

DATE 25.09.19

Suppose we decide to make a_q, $q > m$ enter into the basis.

$$y_{10} a_1 + y_{20} a_2 + \dots + y_{m0} a_m = b \quad \text{--- (1)}$$

$$a_q = y_{1q} a_1 + \dots + y_{mq} a_m \quad \text{--- (2)}$$

multiply $\epsilon > 0$ to (2)

$$\epsilon a_q - \epsilon (y_{1q} a_1 + \dots + y_{mq} a_m) = 0 \quad \text{--- (3)}$$

Adding (1) & (3)

$$(y_{10} - \epsilon y_{1q}) a_1 + (y_{20} - \epsilon y_{2q}) a_2 + \dots + (y_{m0} - \epsilon y_{mq}) a_m + \epsilon a_q = b \quad \text{--- (4)}$$

$$[\underset{\substack{\uparrow \\ a_1}}{y_{10} - \epsilon y_{1q}}, \underset{\substack{\uparrow \\ a_2}}{y_{20} - \epsilon y_{2q}}, \dots, \underset{\substack{\uparrow \\ a_m}}{y_{m0} - \epsilon y_{mq}}, 0 \dots 0, \underset{\substack{\uparrow \\ a_q}}{\epsilon}, 0, \dots 0]^T$$

corresponding to — solution

Not a basic solution since $> m$ components are ≥ 0.

In basic solution, atmost m components > 0.

If $\epsilon = \min_i \left\{ \frac{y_{i0}}{y_{iq}} : y_{iq} > 0 \right\}$ then one of the components in the above new solution will become zero, giving a basic feasible solution

Suppose $p = \arg\min_i \left\{ \frac{y_{i0}}{y_{iq}} : y_{iq} > 0 \right\}$ {the min index i is p}

$$\epsilon = \frac{y_{p0}}{y_{pq}}$$

$$[y_{10} - \epsilon y_{1q}, \dots, 0, y_{p+1\,0} - \epsilon y_{(p+1)q}, \dots, y_{m0} - \epsilon y_{mq}, 0, 0 \dots \epsilon, 0 \dots 0]^T$$

↳ new basic feasible solution

$[a_1, a_2, \dots a_{p-1}, a_{p+1}, a \dots a_m, a_q]$ (a_p not included)

(provided these are linearly independent)

Initial basic feasible soln $[y_{10}, y_{20}, \dots y_{m0}, 0, 0 \dots 0]^T$

Objective function value for initial basic feasible soln

$$z_0 = c_1 y_{10} + c_2 y_{20} + \dots + c_m y_{m0} = C_B^T X_B$$

New basic feasible soln $x_1 = [y_{10} - \epsilon y_{1q}, \dots, y_{p-1} - \epsilon y_{p-1\,q}, 0, y_{p+1\,0} - \epsilon y_{p+1\,q}, \dots y_{m0} - \epsilon y_{mq}, 0, \dots 0, \epsilon, 0, \dots 0]$

Objective fn value for new soln

$$z = c_1 (y_{10} - \epsilon y_{1q}) + \dots + c_m (y_{m0} - \epsilon y_{mq}) + c_q \epsilon$$

$$= c_1 y_{10} + \dots + c_m y_{m0} + (c_q - [c_1 y_{1q} + \dots c_m y_{mq}]) \epsilon$$

$$= z_0 + (c_q - z_q) \epsilon$$

where $z_q = c_1 y_{1q} + \dots + c_m y_{mq}$

$$z - z_0 = (c_q - z_q) \epsilon$$

$\because (\epsilon > 0)$ $\quad z - z_0 \geq 0$ if $c_q - z_q \geq 0$

$$z \geq z_0 \text{ if } (c_q - z_q) \geq 0$$

This solution will not work since it is a minimization problem.

Typo

Scanned by CamScanner

DATE 27·09·19

If $(c_j - z_j) \geq 0 \quad \forall j = m+1, m+2, \dots n$

solution doesn't improve, so z_0 is the optimal solution

→ stopping criteria

$z_j = c_1 y_{1j} + c_2 y_{2j} \dots + c_m y_{mj}$

If $c_q - z_q < 0$, $\quad z - z_0 < 0 \quad \rightarrow \quad z < z_0$

solution has improved by choosing the q^{th} column.

Select q such that $c_q - z_q < 0$ (for multiple -ve columns, we choose most negative column)

⇒ **Problems**

Q. Maximize $z = 2x_1 + 3x_2$ subject to $x_1 + x_2 \leq 1$

$3x_1 + x_2 \leq 4$, $\quad x_1, x_2 \geq 0$

Sol^n ① Bring it into standard form.

Minimize $-2x_1 - 3x_2$ subject to $x_1 + x_2 + s_1 = 1$

$x_1, x_2, s_1, s_2 \geq 0$ $\qquad 3x_1 + x_2 + s_2 = 4$

s_1, s_2 → slack variables

② Convert to canonical ~~augmented~~ matrix (Canonical Matrix)

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \rightarrow \text{canonical form}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & s_1 & s_2 \\ y_1 & y_2 & \underbrace{y_3 \quad y_4}_{2\times2\ I} & \end{array}$$

↳ R.H.S must be positive.

③ Simplex table - I

$Y_B \quad X_B \quad C_B \quad Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \quad b$

Y_B → Basic Variables, X_B → Basic variables, C_B → cost of basic variables

	Y_B	X_B	C_B	Y_1 (−2)	Y_2 (−3)	Y_3 (0)	Y_4 (0)	b	ratio (w.r.t to Y_2)
R_1	y_3	s_1	0	1	1	1	0	1	$1/1 = 1$
R_2	y_4	s_2	0	3	1	0	1	4	$4/1 = 4$

→ pivot element → pivot row

Compute $c_j - z_j$ → $-2 - (1\times0 + 3\times0)$.

$c_1 - z_1 = -2 \qquad c_2 - z_2 = -3$

$c_3 - z_3 = 0 \qquad c_4 - z_4 = 0$

If $\forall c_j - z_j \geq 0$, solution doesn't improve; which is not the case here. We choose the most -ve value & replace the corresponding column.

Y_2 will enter into the basis.

choose $\epsilon = \left\{ \min \left(\frac{y_{i0}}{y_{iq}} \right) \right\}$ so Y_3 will leave the basis.

Use elementary row operations to make $Y_2 \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Scanned by CamScanner

DATE 27.09.19

Table - 2. ⟶ operations must be throughout the system

Y_B	X_B	C_B	-2 Y_1	-3 Y_2	0 Y_3	0 Y_4	b
R_1' Y_2	x_2	-3	1	1	1	0	1
R_2' Y_4	S_2	0	2	0	-1	1	3

$\underline{R_2' = R_2 - R_1'}$

compute $C_j - Z_j$

$C_1 - Z_1 = (-2) - (-3+0) = 1$ $\qquad C_3 - Z_3 = 0 - (-3+0) = 3$

$C_2 - Z_2 = (-3) - (-3+0) = 0$ $\qquad C_4 - Z_4 = 0 - (0+0) = 0$

$\because \forall j \;\; C_j - Z_j \geq 0$, optimal condition is reached.

current solution (basic feasible solution) $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}$

$x_2 = 1 \;;\; S_2 = 3 \;,\; x_1, S_1 = 0.$

And objective function value will be,

Minimum of $-2x_1 - 3x_2 \rightarrow \underline{-3}$ $\quad (-Z)$

Maximum value $\rightarrow \underline{3}$.

Q. Minimize $Z = x_1 - 3x_2 + 2x_3$ subject to
$$\begin{cases} 3x_1 - x_2 + 2x_3 \leq 7 \\ -2x_1 + 4x_2 \leq 12 \\ -4x_1 + 3x_2 + 8x_3 \leq 10 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Soln Since constraints are of $\leq$ type we make -

$3x_1 - x_2 + 2x_3 + s_1 = 7$

$-2x_1 + 4x_2 + s_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

$$\begin{array}{c} \begin{bmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} y_1 & y_2 & y_3 & \underbrace{y_4 \;\; y_5 \;\; y_6} \end{matrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}$$

Y_B	X_B	C_B	1 Y_1	-3 Y_2	2 Y_3	0 Y_4	0 Y_5	0 Y_6	b
y_4	s_1	0	+3	-1	2	1	0	0	7
y_5	s_2	0	-2	4	0	0	1	0	12
y_6	s_3	0	-4	3	8	0	0	1	10

Scanned by CamScanner

⇒ **PENALTY METHOD** (a type of simplex method for $\geq$ type constraints)

Exp: Minimize $z = 4x_1 + 2x_2$ subject to $3x_1 + x_2 \geq 27$

$x_1 + x_2 \geq 21$, $x_1 + 2x_2 \geq 30$, $x_1, x_2 \geq 0$

Soln: Standard form:

Minimize $z = 4x_1 + 2x_2$ subject to

$3x_1 + x_2 - s_1 = 27$

$x_1 + x_2 - s_2 = 21$

$x_1 + 2x_2 - s_3 = 30$

surplus variables

$x_1, x_2, s_1, s_2, s_3 \geq 0$

Matrix form:

$$\begin{bmatrix} 3 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 27 \\ 21 \\ 30 \end{bmatrix}$$

But we do not get an Identity matrix. To get I, we modify the form by adding some artificial variables with high penalty.

$3x_1 + x_2 - s_1 + A_1 = 27$

$x_1 + x_2 - s_2 + A_2 = 21$

$x_1 + 2x_2 - s_3 + A_3 = 30$

A_1, A_2, A_3 are called artificial variables

surplus (or slack) variables have zero cost, but A_1 A_2 A_3 will have high penalty (cost).

$$\begin{bmatrix} 3 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 27 \\ 21 \\ 30 \end{bmatrix}$$

canonical representation

I

Now we do the simplex method.

- Table-1

	Y_B	X_B	C_B	$c_j \rightarrow$ 4 Y_1	2 Y_2	0 Y_3	0 Y_4	0 Y_5	M Y_6	M Y_7	M Y_8	b	b/Y_1
R_1	Y_6	A_1	M	$\boxed{3}$	1	-1	0	0	1	0	0	27 ←	9
R_2	Y_7	A_2	M	1	1	0	-1	0	0	1	0	21	21
R_3	Y_8	A_3	M	1	2	0	0	-1	0	0	1	30	30

$c_j - z_j \rightarrow$ $(4-5M)$ ↓ $(2-4M)$ $0+M$ M M 0 0 0

since M is large this will be the most negative

Y_1 will enter into the basis

Minimum ratio → 9 → corresponds to Y_6 so this will leave the basis

$$\begin{matrix} 3 \\ 1 \\ 1 \end{matrix} \Rightarrow \begin{matrix} 1 \\ 0 \\ 0 \end{matrix}$$

Scanned by CamScanner

$\frac{1}{5}\left(1-\frac{1}{2}\right)$ $-\frac{2}{5}-\frac{1}{10}$ $\frac{1}{3}-\frac{2}{3}\left(\frac{1}{5}\right)$ $\frac{-4-1}{10}=\frac{-5}{10}=-\frac{1}{2}$ $2-\frac{4}{3}-\frac{7M}{3}$ $\frac{4}{3}M-\frac{M}{3}-\frac{M}{3}$

DATE 30.09.2019

- **Table - 2** (Y_6 removed)

	Y_B	X_B	C_B \ c_j	Y_1 (4)	Y_2 (2)	Y_3 (0)	Y_4 (0)	Y_5 (0)	Y_7 (M)	Y_8 (M)	b	b/Y_2	
R_1'	Y_1	x_1	4	1	1/3	-1/3	0	0	0	0	9	27	$R_1 \to R_1/3$
R_2'	Y_7	A_2	M	0	2/3	1/3	-1	0	1	0	12	18	$R_2' = R_2 - R_1'$
R_3'	Y_8	A_3	M	0	$\boxed{5/3}$	1/3	0	-1	0	1	21	63/5	$R_3' = R_3 - R_1'$
			$c_j - z_j$	0	$\left(\frac{2}{3}-\frac{7M}{3}\right)$	$\left(\frac{4}{3}-\frac{2M}{3}\right)$	+M	M	0	0	another basic feasible sol^n		

Most Negative

$1/3 \to 0$
$2/3 \to 0$
$5/3 \quad 1$

Y_2 enters & Y_8 leaves the basis

- **Table - 3**

	Y_B	X_B	C_B \ c_j	Y_1 (4)	Y_2 (2)	Y_3 (0)	Y_4 (0)	Y_5 (0)	Y_7 (M)	b	b/Y_5	
R_1''	Y_1	x_1	4	1	0	-2/5	0	1/5	0	24/5	24	$R_3'' = R_3' \times \frac{3}{5}$
R_2''	Y_7	A_2	M	0	0	1/5	-1	$\boxed{2/5}$	1	18/5	9	$R_1'' = R_1' - \frac{1}{3}R_3''$
R_3''	Y_2	x_2	2	0	1	1/5	0	-3/5	0	63/5		$R_2'' = R_2' - \frac{2}{3}R_3''$
			$c_j - z_j$	0	0	$\left(\frac{7}{5}-\frac{M}{5}\right)$	M	$\left(\frac{2}{5}-\frac{2M}{5}\right)$	0			

we don't consider -ve Y value

Y_5 will enter the basis, Y_7 will leave.

$1/5 \quad 0$
$2/5 \to 1$
$-3/5 \quad 0$

- **Table - 4.**

	Y_B	X_B	C_B	Y_1 (4)	Y_2 (2)	Y_3 (0)	Y_4 (0)	Y_5 (0)	b	
R_1'''	Y_1	x_1	4	1	0	-1/2	1/2	0	3	$R_2''' = R_2'' \times \frac{5}{2}$
R_2'''	Y_5	s_3	0	0	0	1/2	-5/2	1	9	$R_1''' = R_1'' - \frac{1}{5}R_2'''$
R_3'''	Y_2	x_2	2	0	1	1/2	-3/2	0	18	$R_3''' = R_3'' + \frac{3}{5}R_2'''$
			$c_j - z_j$	0	0	1	1	0		

All $c_j - z_j \geq 0$ $\therefore$ This is the optimal sol^n

$x_1 = 3,\ x_2 = 18,\ S_3 = 9,\ S_1 = S_2 = A_1 = A_2 = A_3 = 0$

Optimal (Minimum) value of $z = 4x_1 + 2x_2 = 12 + 36 = \underline{\underline{48}}$.

$\Rightarrow$
$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &\leq b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &\geq b_3 \end{aligned}\right\} \text{Mixed type of constraints.}$$

$+A_1 \to$ add artificial variable !

Identity Matrix ?

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \ {+A_1} &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + S_1 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 - S_2 \ {+A_2} &= b_3 \end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & -1 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccccccc} & & & A_1 & S_1 & A_2 & S_2 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ A_1 \\ S_1 \\ A_2 \\ S_2 \end{bmatrix} = \end{array}$$

Scanned by CamScanner

DATE 04.10.2019

Ex Minimize $z = x_1 + x_2 + x_3$ subject to $x_1 - x_2 + 2x_3 = 2$

$x_1 + 2x_2 - x_3 = 1$

$x_1, x_2, x_3 \geq 0$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Problem already in standard form, but we don't have an identity matrix so we add artificial variables with high penalty.

$x_1 - x_2 + 2x_3 + A_1 = 2$

$x_1 + 2x_2 - x_3 + A_2 = 1$, $x_1, x_2, x_3, A_1, A_2 \geq 0$

$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 1 & 2 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\therefore$ $=$ constraint $\rightarrow$ artificial variable

$\geq$ constraint $\rightarrow$ subtract surplus variable, add artificial variable

$\leq$ constraint $\rightarrow$ add slack variable.

→ **TWO PHASE METHOD**

- Phase-I convert to standard form - Minimization problem

construct the objective function.

$$z^* = 0 \cdot x_1 + 0 \cdot x_2 + \dots + 0 \cdot x_n + 1 \cdot x_{n+1} + 1 \cdot x_{n+2} + \dots + 1 \cdot x_{n+m}.$$

$x_{n+1}, x_{n+2}, \dots x_{n+m}$ are the artificial variables

New objective function having zero cost for variables (also slack & surplus) and cost 1 for artificial variables.

With this new objective function, solve the problem in simplex method.

1. Minimum $z^* \geq 0$ (cannot be negative)

- If $z^* = 0$ — Case 1: No artificial variables appear in the basis, in this case we get a basic feasible solution.

  Case 2: One or more artificial variables appear in the basis, at zero level (i.e. zero value) then we get a basic feasible s. to proceed for phase 2.

  (All basic feasible soln's obtained are carried forward to phase-2)

- If $z^* > 0$ - clearly one or more artificial variables appear in the basis with some positive value. Because the soln is optimal, there is no basic feasible solution to the original problem ($z = 0$)

- Phase-II - If phase-I provides a soln which gives min $z^* = 0$, we go to phase II for optimal soln of the original problem. For this we start with the last table (of phase-II) with the original objective function and then apply the simplex method.

Scanned by CamScanner

DATE 04.10.[illegible]

Exp. Minimize $z = 4x_1 + x_2$ subject to $x_1 + 2x_2 \le 3$

$4x_1 + 3x_2 \ge 6$

$3x_1 + x_2 = 3$, $x_1, x_2 \ge 0$

Sol: Convert to standard form

Minimize $z = 4x_1 + x_2$ subject to $x_1 + 2x_2 + s_1 = 3$

$4x_1 + 3x_2 - s_2 + A_1 = 6$

$3x_1 + x_2 + A_2 = 3$

$x_1, x_2, s_1, s_2, A_1, A_2 \ge 0$

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & A_1 & A_2 \end{array}$$
$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & -1 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$$

(1) Phase 1 - $z^* = 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 1 \cdot A_1 + 1 \cdot A_2$

→ Table-1

	Y_B	X_B	C_B	Y_1 (0)	Y_2 (0)	Y_3 (0)	Y_4 (0)	Y_5 (1)	Y_6 (1)	b	b/Y_1
R_1	Y_3	S_1	0	1	2	1	0	0	0	3	3
R_2	Y_5	A_1	1	4	3	0	-1	1	0	6	3/2
R_3	Y_6	A_2	1	$\boxed{3}$	1	0	0	0	1	3	1
		$c_j - z_j$		-7	-4	0	1	0	-1		

Reduced cost coefficients

$\begin{matrix} 1 & \to 0 \\ 4 & \to 0 \\ 3 & \to 1 \end{matrix}$

Y_1 will enter into the basis, Y_6 will leave the basis.

→ Table-2

	Y_B	X_B	C_B	Y_1 (0)	Y_2 (0)	Y_3 (0)	Y_4 (0)	Y_5 (1)	b	b/Y_2
R_1'	Y_3	S_1	0	0	$\boxed{5/3}$	1	0	0	2	6/5
R_2'	Y_5	A_1	1	0	5/3	0	-1	+1	2	6/5
R_3'	Y_1	x_1	0	1	1/3	0	0	0	1	3
		$c_j - z_j =$		0	$-\frac{5}{3}$	0	1	0		

$R_3' \to R_3/3$

$R_2' = R_2 - 4R_3$

$R_1' = R_1 - R_3$

any can be chosen.

Y_2 will enter the basis, Y_3 will leave the basis.

(keep Y_3 in table because it is not a artificial variable)

→ Table-3

	Y_B	X_B	C_B	Y_1 (0)	Y_2 (0)	Y_3 (0)	Y_4	Y_5 (1)	b
R_1''	Y_2	x_2	0	0	1	3/5	0	0	6/5
R_2''	Y_5	A_1	1	0	0	-1	-1	1	0
R_3''	Y_1	x_1	0	1	0	-1/5	0	0	3/5
		$c_j - z_j$		0	0	1	0		

$R_1'' = R_1' / (5/3)$

$R_2'' = R_2' - \frac{5}{3} R_1''$

$R_3'' = R_3' - \frac{1}{3} R_1''$

This is the optimal solution & artificial variable appears in basis with zero value (also $z^* = 0$). Now we proceed for phase 2.

(2) Phase 2.

Y_B	X_B	C_B	Y_1 (4)	Y_2 (1)	Y_3 (0)	Y_4 (0)	Y_5 (0)	b
Y_2	x_2	1	0	1	3/5	0	0	6/5
Y_5	A_1	0	0	0	-1	-1	1	0
Y_1	x_1	4	1	0	-1/5	0	0	3/5
	$c_j - z_j$		0	0	1/5	0	0	

optimal solution

Scanned by CamScanner

DATE 07.10.19

⇒ **Duality** useful for problems with larger no. of constraints - reduces computation cost. The dual problem is constructed from the cost and constraints of the original problem. Used to reduce computations along with simplex method.

Symmetric form of Duality:
Suppose the Primal problem is:
Minimize $c^T x$ subject to $Ax \geq b$, $x \geq 0$ → (A)
The corresponding dual problem is defined as
Maximize $\lambda^T b$ subject to $\lambda^T A \leq c^T$, $\lambda \geq 0$ → (B)
$\lambda \in R^n$ is called the dual variable.

→ Cost vector c in the primal has moved to the R.H.S of constraints of the dual.

→ The vector b on the R.H.S of the primal ~~objective~~ constraints becomes cost of the dual.

(A), (B) is called the symmetric form of duality.

Step 1: Convert the LP problem in the form of (A).
Step 2: Then using the symmetric form of duality, construct the dual.

Exp! Minimize $2x_1 + 3x_2$ subject to

$$x_1 + x_2 \geq 1$$
$$3x_1 + x_2 \geq 2$$
$$x_1 - x_2 \geq \frac{1}{2}$$
$$x_1, x_2 \geq 0$$

Dual problem:
Maximize $1 \cdot \lambda_1 + 2\lambda_2 + \frac{1}{2} \cdot \lambda_3$ (3 constraints → 3 dual variables)
subject to

$$\lambda_1 + 3\lambda_2 + \lambda_3 \leq 2$$
$$\lambda_1 + \lambda_2 - \lambda_3 \leq 3$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

$$\underset{1\times 3}{[\lambda_1\ \lambda_2\ \lambda_3]} \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 1 & -1 \end{bmatrix}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T$$

Symmetric form:
If constraints are of equality type, the primal problem -
Minimize $c^T x$ subject to $Ax = b$, $x \geq 0$

$Ax = b \rightarrow Ax \geq b$, $Ax \leq b \Rightarrow -Ax \geq -b$

The problem becomes,
Minimize $c^T x$ subject to

$$Ax \geq b$$
$$-Ax \geq -b$$
$$x \geq 0$$

or $\begin{bmatrix} A \\ -A \end{bmatrix} x \geq \begin{bmatrix} b \\ -b \end{bmatrix}$, $x \geq 0$

converted to symmetric form & corresponding dual problem -
Maximize $[u^T\ v^T] \begin{bmatrix} b \\ -b \end{bmatrix}$ subject to $[u^T\ v^T] \begin{bmatrix} A \\ -A \end{bmatrix} \leq c^T$

$$u, v \geq 0$$

& this can again be formulated as.

Scanned by CamScanner

DATE

Maximize: $(u-v)^T b$ subject to $(u-v)^T A \leq c^T$, $u, v \geq 0$

If $u-v = \lambda$ then it is -

Maximize $\lambda^T b$ subject to $\lambda^T A \leq c^T$, ($u, v \geq 0$) λ is unrestricted

Therefore

→ Symmetric form of duality.

Primal problem	Dual
Minimize $c^T X$	Maximize $\lambda^T b$
subject to $AX \geq b$, $X \geq 0$.	subject to $\lambda^T A \leq c^T$, $\lambda \geq 0$

→ Asymmetric form of duality.

Primal problem	Dual
Minimize $c^T X$	Maximize $\lambda^T b$
subject to $AX = b$, $X \geq 0$	subject to $\lambda^T A \leq c^T$

EX: Find the dual of - Minimize $c^T X$ subject to $AX \leq b$ (No X restriction)

* Dual of the dual is primal

Primal: Minimize $c^T X$ subject to $AX \geq b$, $X \geq 0$ — Dual: Maximize $\lambda^T b$ subject to $\lambda^T A \leq c^T$, $\lambda \geq 0$

To connect dual to it's we need to bring it into primal form.

→ Minimize $-\lambda^T b$ subject to $\lambda^T(-A) \geq -c^T$, $\lambda \geq 0$

Dual of this -

Maximize $(-c^T) X$ subject to $(-A) X \leq -b$, $X \geq 0$

$\equiv$ Minimize $c^T X$ subject to $AX \geq b$, $X \geq 0$ = primal

Ex → Dual of dual (previous example)

Maximize $\lambda_1 + 2\lambda_2 + \frac{1}{2}\lambda_3$ subject to
$$\lambda_1 + 3\lambda_2 + \lambda_3 \leq 2$$
$$\lambda_1 + \lambda_2 - \lambda_3 \leq 3$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

$\equiv$ Minimize $-\lambda_1 - 2\lambda_2 - \frac{1}{2}\lambda_3$ subject to
$$(-1)(\lambda_1 + 3\lambda_2 + \lambda_3) \geq -$$
$$(-1)(\lambda_1 + \lambda_2 - 3\lambda_3) \geq -3$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

Dual:

Maximize $\begin{bmatrix} -2 \\ -3 \end{bmatrix}^T X$ subject to
$$-1(x_1 + x_2) \leq -1$$
$$-1(3x_1 + x_2) \leq -2$$
$$-1(x_1 - x_2) \leq -\frac{1}{2}$$
$$x_1, x_2 \geq 0$$

$\equiv$ Minimize $2x_1 + 3x_2$ subject to
$$x_1 + x_2 \geq 1$$
$$3x_1 + x_2 \geq 2$$
$$x_1 - x_2 \geq -1/2 \quad ; x_1, x_2 \geq 0$$

Scanned by CamScanner

DATE 09.10.19

Ex. Find the dual of Minimize $c^T x$ subject to $Ax \leq b$

Since dual of dual is primal, looking at asymmetric case,

Dual of this problem will be Maximize $\lambda^T b$ subject to $\lambda^T A = c^T$

$\lambda \geq 0$

⇒ Properties of dual problems

- Weak Duality lemma: suppose x & λ are feasible solutions to the primal and dual problems, respectively. Then $c^T x \geq \lambda^T b$

Proof: Because x and λ are feasible (we prove it for asymmetric form)

we have $Ax = b$, $x \geq 0$ and $\lambda^T A \leq c^T$

(Primal) (Dual)

Post multiply both sides of $\lambda^T A \leq c^T$ by $x \geq 0$

$\lambda^T A x \leq c^T x$

$\Rightarrow \lambda^T b \leq c^T x$

- Suppose x_0 and λ_0 are feasible solutions to the primal and Dual respectively. (Both symmetric & asymmetric form). If $c^T x_0 = \lambda_0^T b$, then x_0 and λ_0 are optimal solutions to their respective problems.

Proof: Let x be any arbitrary solution to the primal problem. Since λ_0 is a feasible solution to the dual, by weak duality lemma,

$c^T x \geq \lambda_0^T b$. If $c^T x_0 = \lambda_0^T b$

$\Rightarrow c^T x \geq \lambda_0^T b = c^T x_0$

Similarly for λ_0

$\Rightarrow c^T x \geq c^T x_0$ (x is arbitrary)

$\Rightarrow x_0$ is optimal solution to the primal.

Let λ be any arbitrary solution to the dual. Since x_0 is a feasible sol^n to the prim by Weak Duality lemma, $\lambda^T b \leq c^T x_0$

Since $c^T x_0 = \lambda_0^T b$ $\Rightarrow \lambda^T b \leq c^T x_0 = \lambda_0^T b$ $\Rightarrow \lambda^T b \leq \lambda_0^T b$

λ is arbitrary, so λ_0 gives maximum value to the dual → It is optimum.

- $\lambda^T b \leq c^T x$ - We want to Maximize λ and Minimize x so both try to approach each other since at optimum $\lambda^T b = c^T x$
- Using duality, we can directly arrive at the solution without the simplex method for some problems.
- Duality theorem: If the primal problem has an optimal solution, then so does the dual, and the objective function values of their respective objective functions are equal.
- Complementary slackness condition - The feasible solutions x and λ to the primal and Dual pair of problems are optimal if and only if (i) $(c^T - \lambda^T A)x = 0$ & (ii) $\lambda^T(Ax - b) = 0$

Scanned by CamScanner

Proof: In Asymmetric case, Primal : Minimize $c^T x$ subject to $Ax = b$, $x \geq 0$

Dual : Maximize $\lambda^T b$ subject to $\lambda^T A \leq c^T$

As $Ax = b$ proof for (ii) is trivial.

Let the two solutions x, and λ be optimal. Then, by duality theorem,

$\lambda^T b = c^T x$. But $Ax = b \Rightarrow c^T x = \lambda^T A x \Rightarrow (c^T - \lambda^T A) x = 0$.

Conversely, if $(c^T - \lambda^T A) x = 0 \rightarrow c^T x = \lambda^T A x \Rightarrow c^T x = \lambda^T b$

and by duality theorem, x and λ are optimal sol^ns to the primal & dual.

Exp: Suppose you have 26 \$ and you wish to purchase silver. You have a choice of 4 vendors with prices $\frac{1}{2}, 1, \frac{1}{7} \& \frac{1}{4}$ per gram, respectively. You wish to spend your entire 26 \$ by purchasing silver from these vendors, where x_i are the \$ spent on vendor i. $(i = 1, 2, 3, 4)$

a) Formulate the L.P. problem that reflects your desire to obtain the maximum amount of silver.

b) Write down the dual problem and find the optimal sol^n

c) Use the complementary slackness condition together with part (b) to find the optimal values of x_i. $i = 1, 2, 3, 4$.

Sol^n: a) Maximize $2x_1 + x_2 + 7x_3 + 4x_4$ subject to $x_1 + x_2 + x_3 + x_4 = 26$

$x_1, x_2, x_3, x_4 \geq 0$

b) To find dual, primal must be written as -

Minimize $-2x_1 - x_2 - 7x_3 - 4x_4$ subject to $x_1 + x_2 + x_3 + x_4 = 26$, $x_i \geq 0$

Dual - Maximize 26λ subject to $\lambda \leq -2$, $\lambda \leq -1$, $\lambda \leq -7$, $\lambda \leq -4$.

(No. of variables in dual = no. of constraints in primal) (λ unrestricted)

c) $\therefore$ Dual is optimal at $\lambda = -7$. $26(\lambda)_{max} = -182$.

So primal is optimal (minimum) at same value.

$\therefore$ Max. of $2x_1 + x_2 + 7x_3 + 4x_4 = 182$

$(c^T - \lambda^T A) x = 0$ at optimum. $c^T = (-2 \ -1 \ -7 \ -4)$

$\{-(2 \ 1 \ 7 \ 4) - (-7)(1 \ 1 \ 1 \ 1)\} x = 0$

$\{-(2 \ 1 \ 7 \ 4) + (7 \ 7 \ 7 \ 7)\} x = 0$

$(5 \ 6 \ 0 \ 3) x = 0$

$\cancel{5x_1 + 6x_2}$ $5x_1 = 0$, $6x_2 = 0$, $3x_4 = 0$ $\therefore x_1, x_2, x_4 = 0$

and from $x_1 + x_2 + x_3 + x_4 = 26 \Rightarrow x_3 = 26$

Scanned by CamScanner

DATE 11·10·19

→ Dual simplex Method.

Exp: Minimize $x+y$ subject to $\begin{cases} 2x+3y \le 1 \\ x-y \ge 5 \end{cases}$

Bring it to symmetric form $\Rightarrow -2x-3y \ge -1$ → but we want the b

$x-y \ge 5$

R.H.S vector becomes negative when we try to find the dual.

- So the Dual simplex Method is used when the R.H.S vector is/are −ve
- It starts with an optimal solution which is not feasible and we move towards the feasible region. The main advantage of this method is we do not need to use artificial variables. The disadvantage is that we start with a more or than optimal solⁿ & move towards feasible rg.

- Algorithm:

Step1: Convert the problem into standard form by introducing only slack variables (Max → Min).

Step2: Find the canonical representation and obtain an initial basic solⁿ

Step3: → If $c_j - z_j \ge 0$ $\forall j$ and $x_{Bi} \ge 0$ for each i, then optimal basic feasible solⁿ is obtained.

→ If $c_j - z_j \ge 0$ $\forall j$ and atleast one basic variable say x_{Bi} is negative then go to Step 4.

→ If atleast one $c_j - z_j$ is negative, the method is not applicable.

Step4: Select the most negative x_{Bi}. The corresponding basic vector then leaves the basis. Let x_{Br} be the most negative one, so that Y_r leaves.

Step5: Test the nature of y_{rj}, $j = 0,1,2 \ldots n$

i) If all $y_{rj} \ge 0$, there does not exist any feasible solution

ii) If atleast one y_{rj} is negative, compute the replacement ratio

$$\left\{ \frac{c_j - z_j}{y_{rj}},\ y_{rj} < 0 \right\} \quad j = 1,2,\ldots n$$

Choose the maximum of these ratios. & the corresponding column vector enters into the basis

Step6: Make the new Dual-Simplex table. Test the new iterated dual simplex table for optimality. The procedure is to be repeated until an optimal basic feasible solution is obtained or there is an indication for the non existence of feasible solution.

Exp: Use dual simplex method to solve:

Maximize $z = -3x_1 - x_2$ subject to $x_1 + x_2 \ge 1$

$2x_1 + 3x_2 \ge 2$, $x_1, x_2 \ge 0$

Solⁿ: Minimize $3x_1 + x_2$ subject to $-x_1 - x_2 \le -1$

$-2x_1 - 3x_2 \le -2$, $x_1, x_2 \ge 0$

Scanned by CamScanner

DATE 11·10·19

$-x_1 - x_2 + s_1 = -1$ (Only slack variables)

$-2x_1 - 3x_2 + s_2 = -2$, $x_1, x_2, s_1, s_2 \geq 0$

Canonical Representation
$$\begin{bmatrix} -1 & -1 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Dual simplex Table-1

	Y_B	X_B	C_B	Y_1 (3)	Y_2 (1)	Y_3 (0)	Y_4 (0)	b
R_1	Y_3	S_1	0	-1	-1	1	0	-1
R_2	Y_4	S_2	0	-2	$\boxed{-3}$	0	1	-2
			$c_j - z_j$	3	1	0	0	

← most negative Y_4 leaves basis.

$\forall j$ $c_j - z_j \geq 0$ but x_{bi} are negative so we go to next step.

Most negative value ($s_2 = -2$) is chosen & corresponding vector Y_4 leaves basis

$$\max\left\{ \frac{c_j - z_j}{y_{4j}} : y_{4j} < 0 \right\} = \max\left\{ \frac{3}{-2}, \frac{1}{-3} \right\} = \frac{-1}{3}$$ → corresponds to Y_2 so it will enter the basis.

$\begin{matrix} -1 \\ -3 \end{matrix} \Rightarrow \begin{matrix} 0 \\ 1 \end{matrix}$

Dual simplex Table-2

	Y_B	X_B	C_B	Y_1 (3)	Y_2 (1)	Y_3 (0)	Y_4 (0)	b
R_1'	Y_3	S_1	0	-1/3	0	1	$\boxed{-1/3}$	-1/3
R_2'	Y_2	x_2	1	2/3	1	0	-1/3	2/3
			$c_j - z_j$	7/3	0	$0\cancel{7/3}$	1/3	

$R_2' \rightarrow R_2 / -3$

$R_1' \rightarrow R_1 + R_2'$

→ most negative.

One value became +ve so we are moving towards the feasible region.

most negative value corresponds to Y_3 so it will leave the basis

$$\max\left\{ \frac{c_j - z_j}{y_{3j}} : y_{3j} < 0 \right\} = \max\{-7, -1\} = -1$$ → corresponds to Y_4

$\begin{matrix} -1/3 \\ -1/3 \end{matrix} \rightarrow \begin{matrix} 1 \\ 0 \end{matrix}$

Dual simplex Table-3

	Y_B	X_B	C_B	Y_1 (3)	Y_2 (1)	Y_3 (0)	Y_4 (0)	b
R_1''	Y_4	S_2	0	1	0	-3	1	1
R_2''	Y_2	x_2	1	1	1	-1	0	1
			$c_j - z_j \rightarrow$	2	0	1	0	

$R_1'' = R_1' \times -3$

$R_2'' = \frac{1}{3}R_1'' + R_2'$

↳ All $c_j - z_j \geq 0$ and $\forall$ $x_{bi} \geq 0$ ∴ This is the optimal basic feasible solutions.

$x_1 = 0$, $x_2 = 1$, $s_1 = 0$, $s_2 = 1$

Min $(-z) = 3x_1 + x_2 = 1$

Max $(z) = \underline{-1}$ (Optimal function value)

✓ Transportation problem
✓ Game theory
· Queue theory

Project submission : 15[th] November & presentation immediately after.

Scanned by CamScanner

DATE 14.10.19

TRANSPORTATION PROBLEM (A special case of simplex)

→ Origins Destinations

We have m origins and n destinations

$a_i \to$ availability at origin
(eg amount of goods)

$b_j \to$ requirement at destination j.

There is a transport cost associated with transporting goods from a_i to b_j, i.e. $c_{ij} \to$ cost of 1 unit to be transported from source i to dest j.
We need to find how much of the goods need to be transported from each source to destination, while minimizing the cost.
$x_{ij} \to$ no. of units to be transported from origin i to dest. j

Exp: A company has facilities at A, B, C that supply goods at warehouses X, Y, Z, W.
Determine the optimum distribution to minimize the transportation cost.

c_{ij}	A	B	C	b_j
X	50	30	15	150
Y	35	62	19	120
Z	12	24	49	90
W	17	6	32	250
a_i	500	650	450	

→ LPP formulation

Minimize $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} c_{ij}$

subject to $\displaystyle\sum_{j=1}^{n} x_{ij} \le a_i \quad \forall\, i = 1, 2, \dots m$ (Source requirement)

$\displaystyle\sum_{i=1}^{m} x_{ij} \le b_j \quad \forall\, j = 1, 2, \dots n$ (Destination requirement)

$x_{ij} \ge 0 \quad \forall\, i = 1, 2, \dots m \;;\; j = 1, 2, \dots n.$

Since these are special problems, even though we can solve it using simplex methods, there are easier ways to solve this.

→ 'Balanced' transportation problem: $\displaystyle\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

All inequalities in the LPP formulation can be converted to equalities

Minimize $Z = \sum\sum x_{ij} c_{ij} = c^T x$ subject to $Ax = b$, $x \ge 0$

$$C = (c_{11}\, c_{12}, \dots c_{1n}, c_{21}, c_{22}, \dots c_{mn})^T \quad (mn)$$
$$x = (x_{11}, x_{12}, \dots x_{1n}, x_{21}, x_{22}, \dots x_{mn})^T \quad (mn)$$
$$b = (a_1, a_2, \dots a_m, b_1, b_2, \dots b_n)^T \quad (m+n)$$

$$A = \begin{bmatrix} \underbrace{1\,1\,1 \cdots 1}_{n} & \underbrace{0\,0\,0 \cdots 0}_{n} & \underbrace{0\,0 \cdots 0}_{n} & 0\,0 \cdots 0 & \cdots \\ 0\,0\,0 \cdots 0 & 1\,1\,1 \cdots 1 & 0\,0 \cdots 0 & 0\,0 \cdots 0 & \cdots \\ 0\,0\,0 \cdots 0 & 0\,0\,0 \cdots 0 & 1\,1 \cdots 1 & 0\,0 \cdots 0 & \cdots \\ \vdots & & & & \\ 0\,0\,0 \cdots 0 & 0\,0\,0 \cdots 0 & 0\,0 \cdots 0 & 0\,0 \cdots 0 & 1 \cdots 1 \end{bmatrix}$$

for $\sum x_{ij} \le a_i$

Scanned by CamScanner

DATE

$$\begin{bmatrix} \begin{array}{c|cccc} 100\cdots0 & 100\cdots0 & 100\cdots0 & 100\cdots0 & \cdots \\ 010\cdots0 & 010\cdots0 & 010\cdots0 & 010\cdots0 & \cdots \\ \vdots & & & & \\ 000\cdots1 & 000\cdots1 & 000\cdots1 & 000\cdots1 & \cdots \end{array} \end{bmatrix} \rightarrow \text{for } \sum x_{ij} = b_j$$

$(m+n)\times(mn)$

$$A = \begin{bmatrix} 1' & 0' & \cdots & 0' \\ 0' & 1' & \cdots & 0' \\ \vdots & & & \\ 0' & 0' & \cdots & 1' \\ I_n & I_n & \cdots & I_n \end{bmatrix}$$

$1' = [1\ 1 \cdots 1]$ (m ones)

$0' = [0 \cdots 0]$ (m zeroes)

$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ $m \times n$ Identity matrix

Rank $(A) = m+n-1$

Claim: Rank $(A) < m+n$

Divide matrix into two parts - one formed by $1'$ & $0'$ and the other with I_n's.
Each column of the matrix will sum up to 2 (1 from 1^{st} part & 1 from 2^{nd})
Also, sum of first m rows = sum of last n rows

Rank $(A^*) = m+n-1$ where A^* - a submatrix of A formed by deleting first row and selecting columns numbers $1, 2, \ldots n, n+1, 2n+1, \ldots (m-1)n+1$

$$A^* = \left[\begin{array}{c|cccc} & 1 & 0 & 0 & \cdots 0 \\ O & 0 & 1 & 0 & \cdots 0 \\ & 0 & 0 & 0 & \cdots 0 \\ & \vdots & \vdots & \vdots & \vdots \\ & 0 & 0 & 0 & \cdots 1 \\ \hline & 1 & 1 & 1 & \cdots 1 \\ I_n & 0 & 0 & 0 & \cdots 0 \\ & 0 & 0 & 0 & \cdots 0 \end{array}\right] = \begin{bmatrix} 0 & I_{m-1} \\ & 1' \\ I_n & 0' \\ & \vdots \\ & 0' \end{bmatrix}$$

$\neq 0 \rightarrow$ Not Singular

$\therefore$ Rank $(A^*) = m+n-1$

which means, that on solving we will get $m+n-1$ basic variables.

⇒ Book: Linear Programming & Game Theory by Deepak Chatterjee.

Methods to find Basic Feasible Sol^n

1) Row Minima Method
2) North-West corner Rule
3) Vogel's approximation.

$$\begin{array}{c|cccc|c} & d_1 & d_2 & \cdots & d_n & \\ \hline O_1 & c_{11} & c_{12} & \cdots & c_{1n} & a_1 \\ O_2 & c_{21} & c_{22} & \cdots & c_{2n} & a_2 \\ \vdots & \vdots & \vdots & & \vdots & a_3 \\ O_m & c_{m1} & c_{m2} & \cdots & c_{mn} & a_m \\ \hline & b_1 & b_2 & \cdots & b_n & \end{array}$$

↳ requirement availability

DATE 16·10·2019

1) Row Minima Method

Exp.

c_{ij}				a_i
4	2	5	3	6
5	4	3	2	13
1	4	6	5	19
b_j: 7	8	5	8	28

— balanced transportation problem.

4	2 [6]	5	3	6 (0)
5	4	3 [5]	2 [8]	13 (8) (5)
1 [7]	4 [2]	6	5	19 (12) (10)
7 (0)	8 (2) (0)	5 (0)	8 (0)	

choose the smallest value in the first row → 2

corresponding to that assign min (a_i, b_j) → 6

∴ assign 6 to x_{12}.

This row is consumed & can be removed.

For second row choose 2 (minimum)

assign 8 to x_{24}. This row can't be removed but the column is removed. Next for x_{23} we assign 5. This row & column get deleted

Repeat for x_{31} and x_{32}.

We get 5 basic variables:

$x_{12} = 6$, $x_{23} = 5$, $x_{24} = 8$, $x_{31} = 7$, $x_{32} = 2$

$M+N-1 = 6$ (there should be 6 basic variables)

so this is a degenerate basic feasible solution. (Not Optimal).

Cost $Z = 2\times6 + 3\times5 + 2\times8 + 1\times7 + 2\times4 = 58$

similarly column minima can also be carried out, as well as Matrix Minima. We do this to minimize the cost of transportation.

2) North-West Corner Rule.

Exp.

4 [12]	6 [4]	9	5	16 (4) (0)
2	6 [10]	4 [2]	1	12 (2) (0)
5	7	2 [7]	9 [8]	15 (8) (0)
12 (0)	14 (10) (0)	9 (7) (0)	8 (0)	43 / 43

Rule: Assign maximum possible value to the north-west corner cell.

$x_{11} \to 12$ (column is consumed)

Delete the first column

$x_{12} \to 4$

Delete first row

$x_{22} \to 10$

Delete second column

$x_{23} \to 2$

Delete second row.

$x_{33} \to 7$

Delete third column

$x_{34} \to 8$.

cost $= 12\times4 + 4\times6 + 10\times6 + 2\times4 + 7\times2 + 8\times9$.

This is a balanced transportation problem and it satisfies that the no. of basic variables $= n+m-1$. (✓)

Scanned by CamScanner

DATE

3) Vogel's Approximation

$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	9/9/40/40 — 7 (2) (0)
70	30	$\boxed{7}$ 40	$\boxed{2}$ 60	10/20/20/20 — 9 (2) (0)
40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	12/20/50 — 18 (10) (0)
5 / 21 (0)	8 / 22 (0)	7 / 10/10 (0)	14 / 10/50 (4) (2) (0)	

We will calculate the different between the minimum & next minimum for each row & each column.

Pick row/column that has maximum difference. Second column in this case.

Minimum value is 8 & $x_{32} = 8$

This column is exhausted.

Repeat the process again by finding differences. (since column was deleted, compute only row difference again)

$x_{11} = 5$, $x_{14} = 2$, $x_{23} = 7$, $x_{24} = 2$, $x_{32} = 8$, $x_{34} = 10$

Cost (z) = $5 \times 19 + 2 \times 10 + 7 \times 40 + 2 \times 60 + 8 \times 8 + 10 \times 20$

Scanned by CamScanner