# Tutorial - 11

**Q.1** Find the Laurent series of the function

$$f(z) = \frac{2z-3i}{z^2-3iz-2}$$

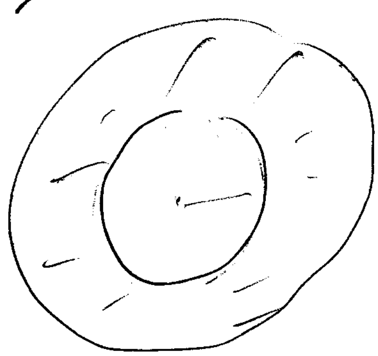
in the following domains

(a) $0 < |z| < 1$

(b) $1 < |z| < 2$

(c) $|z| > 2$

(d) $0 < |z+i| < 2$



**Sol^n** By doing the partial fractions we can write

$$f(z) = \frac{2z-3i}{z^2-3iz-2} = \cancel{\frac{2z-3i}{z^2-3iz}} = \frac{2z-3i}{(z-i)(z-2i)}$$

$$= \frac{1}{z-i} + \frac{1}{z-2i}$$

(a) $0 < |z| < 1$

$$\frac{1}{z-i} = \frac{1}{-i(1+iz)} = i(1+iz)^{-1} = i\sum_{n=0}^{\infty}(iz)^n(-1)^n = \sum_{n=0}^{\infty}(-1)^n i^{n+1} z^n \quad \text{—(1)}$$

The above series (1) is valid for $|iz| < 1 \Rightarrow |z| < 1$

$$\frac{1}{z-2i} = \frac{1}{-2i\left(1+\frac{iz}{2}\right)} = \frac{i}{2}\left(1+\frac{iz}{2}\right)^{-1} = \frac{i}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{iz}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{i}{2}\right)^{n+1} z^n \quad \text{—(2)}$$

The above series (2) is valid for $\left|\frac{iz}{2}\right| < 1$ that is $\left|\frac{z}{2}\right| < 1 \Rightarrow |z| < 2$

So the series (2) is also valid for $|z| < 1$ as it is valid for $|z| < 2$

Hence both series (1) and (2) are valid in $|z|<1$.

Hence $f(z)$ can be written as

$$f(z) = \frac{2z-3i}{z^2-3iz-2} = \sum_{n=0}^{\infty} (-1)^n i^{n+1} z^n + \sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{2}\right)^{n+1} z^n$$

(b) $1<|z|<2$

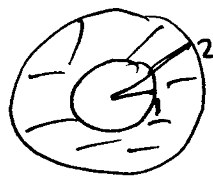
$$f(z) = \frac{1}{z-i} + \frac{1}{z-2i}$$

$$\frac{1}{z-i} = \frac{1}{z\left(1-\frac{i}{z}\right)} = \frac{1}{z}\left(1-\frac{i}{z}\right)^{-1} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n = \sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}} \quad \text{—— (3)}$$

This series is valid for $\left|\frac{i}{z}\right|<1 \Rightarrow \frac{1}{|z|}<1$
$\Rightarrow |z|>1$

From (2) $\displaystyle\frac{1}{z-2i} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{2}\right)^{n+1} z^n$
is valid for $|z|<2$

So in the intersection region $1<|z|<2$ both the series (2) & (3) are valid.




$$\text{So } f(z) = \sum_{n=0}^{\infty} \frac{i^n}{z^{n+1}} + \sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{2}\right)^{n+1} z^n$$

(c) $|z|>2$

$$f(z) = \frac{1}{z-i} + \frac{1}{z-2i}$$

From (3) $\displaystyle\frac{1}{z-i} = \sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}}$ valid for $|z|>1$
So it is also valid for $|z|>2$.

$$\frac{1}{z-2i} = \frac{1}{z\left(1-\frac{2i}{z}\right)} = \frac{1}{z}\left(1-\frac{2i}{z}\right)^{-1} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2i}{z}\right)^n \quad \text{—— (4)}$$

valid for $\left|\frac{2i}{z}\right|<1 \Rightarrow \frac{2}{|z|}<1 \Rightarrow |z|>2$.

So both (3) & (4) are valid for $|z|>2$.

$$\text{Hence } f(z) = \sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}} + \sum_{n=0}^{\infty}(2i)^n \frac{1}{z^{n+1}}$$

(d) $0 < |z+i| < 2$

$$f(z) = \frac{1}{z-i} + \frac{1}{z-2i}$$

$$\frac{1}{z-i} = \frac{1}{z+i-2i} = \frac{-1}{2i\left(1+\frac{z+i}{-2i}\right)} = \frac{-1}{2i}\left(1+\frac{z+i}{-2i}\right)^{-1}$$

$$= \frac{-1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z+i}{-2i}\right)^n \quad \text{— (5)}$$

(5) is valid for $\left|\frac{z+i}{-2i}\right| < 1 \Rightarrow \frac{|z+i|}{2} < 1 \Rightarrow |z+i| < 2$

$$\frac{1}{z-2i} = \frac{1}{z+i-3i} = \frac{1}{-3i\left(1+\frac{z+i}{-3i}\right)} = \frac{-1}{3i}\left(1+\frac{z+i}{-3i}\right)^{-1}$$

$$= \frac{-1}{3i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z+i}{-3i}\right)^n \quad \text{— (6)}$$

(6) is valid for $\left|\frac{z+i}{-3i}\right| < 1 \Rightarrow \frac{|z+i|}{3} < 1$

$\Rightarrow |z+i| < 3$

As (6) is valid for $|z+i| < 3$, it is also valid for $|z+i| < 2$.

Hence both the series (5) & (6) are valid for $|z| < 2$

So $f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(-2i)^{n+1}}(z+i)^n + \sum_{n=0}^{\infty}(-1)^n \frac{1}{(-3i)^{n+1}}(z+i)^n$

Result-1 If $f$ has an isolated singularity at $z_0$, then $z = z_0$ is a removable singularity iff one of the following conditions holds:

(1) $f$ is bounded in a deleted nbd of $z_0$.

(2) $\lim_{z \to z_0} f(z)$ exists ~~and finite~~.

(3) $\lim_{z \to z_0} (z - z_0) f(z) = 0$

Result-2 If $z_0$ is an isolated singularity of $f(z)$ then $z_0$ is a pole of order $m$ if

$\lim_{z \to z_0} (z - z_0)^m f(z)$ exists.

Q.2 Determine the location and types of singularities of the following functions

(1) $z^2 - \frac{1}{z^2}$

(iv) $\frac{2}{z^3} - \frac{1}{z}$

(iii) $\tan z$

(v) $\sin \frac{1}{z}$

(iv) $z^{-2} \sin^2 z$

Sol$^n$ (1) $z^2 - \frac{1}{z^2}$

$z = 0$ is a singularity

It is a pole of order 2, because we can get it as a Laurent series about $z = 0$.

(ii) $\frac{2}{z^3} - \frac{1}{z}$

$z=0$ is the singularity and it is a pole of order 3.

(iii) $\tan z = \frac{\sin z}{\cos z}$

$\cos z = 0 = \cos(2n+1)\frac{\pi}{2}$ $\quad n = 0, 1, 2 \ldots$

$\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ all are singularity.

Let us consider $\frac{\pi}{2}$. We use the result -2

$$\tan z = \frac{\sin z}{\cos z} = -\frac{\cos(z-\frac{\pi}{2})}{\sin(z-\frac{\pi}{2})}$$

$$\lim_{z\to\frac{\pi}{2}} (z-\tfrac{\pi}{2}) \times \frac{-\cos(z-\frac{\pi}{2})}{\sin(z-\frac{\pi}{2})} = \lim_{z\to\frac{\pi}{2}} \frac{-\cos(z-\frac{\pi}{2})}{\frac{\sin(z-\frac{\pi}{2})}{(z-\frac{\pi}{2})}} = -1$$

So $\lim_{z\to\frac{\pi}{2}} (z-\frac{\pi}{2}) f(z)$ exists.

So $z=\frac{\pi}{2}$ is a pole of order 1

Similarly all other singularities of $\tan z$ are also poles of order 1.

(iv) $\frac{1}{z^2}\sin^2 z$

$z=0$ is the singular point. We use result -2.

$$\lim_{z\to 0} (z-0)^2 f(z) = \lim_{z\to 0} (z-0)^2 \frac{1}{z^2}\sin^2 z = \lim_{z\to 0} \sin^2 z = 0$$

So $z=0$ is a pole of order 2.

(v) $\sin \frac{1}{z}$

$z=0$ is the singular point.

$$\sin \frac{1}{z} = \frac{1}{z} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \cdots$$

Laurent series expansion about $z=0$.

The Laurent series has infinite no. of terms in the principal part.

So $z=0$ is an essential singularity.

## Residues

If $z_0$ is a simple pole of $f(z)$ of order 1.

Then
$$\operatorname*{Res}_{\text{at } z=z_0} f(z) = \lim_{z \to z_0} (z-z_0) f(z)$$

If $z_0$ is a pole of order $m$, then

$$\operatorname*{Res}_{\text{at } z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[ (z-z_0)^m f(z) \right] \right\}$$

## Residue Theorem

If $f(z)$ is analytic inside a simple closed curve $C$ and on $C$, except for finitely many singular points $z_1, z_2, \cdots z_k$ inside $C$. Then

$$\oint_C f(z)\, dz = 2\pi i \left( \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z) \right)$$

Q.3 Integrate $\oint_C \frac{e^z}{\cos z}\,dz \qquad C: |z|=3$

The singular points of $f(z) = \frac{e^z}{\cos z}$ are

$\frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \text{---}$

All are simple poles.

Only $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ are inside the circle $|z|=3$.

So By Residue theorem

$$\oint_{|z|=3} \frac{e^z}{\cos z}\,dz = 2\pi i \left[ \operatorname*{Res}_{z=\frac{\pi}{2}} f(z) + \operatorname*{Res}_{z=-\frac{\pi}{2}} f(z) \right]$$

$$\operatorname*{Res}_{z=\frac{\pi}{2}} f(z) = \lim_{z\to\frac{\pi}{2}} \left(z-\frac{\pi}{2}\right)\frac{e^z}{\cos z}$$

$$= \lim_{z\to\frac{\pi}{2}} \left(z-\frac{\pi}{2}\right)\frac{e^z}{-\sin\left(z-\frac{\pi}{2}\right)}$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{-e^z}{\frac{\sin\left(z-\frac{\pi}{2}\right)}{z-\frac{\pi}{2}}} = \frac{-e^{\frac{\pi}{2}}}{1} = -e^{\frac{\pi}{2}}$$

$$\operatorname*{Res}_{z=-\frac{\pi}{2}} f(z) = \lim_{z\to-\frac{\pi}{2}} \left(z+\frac{\pi}{2}\right)\frac{e^z}{\cos z} = \lim_{z\to-\frac{\pi}{2}} \frac{\left(z+\frac{\pi}{2}\right)e^z}{\sin\left(z+\frac{\pi}{2}\right)}$$

$$= \lim_{z\to-\frac{\pi}{2}} \frac{e^z}{\frac{\sin\left(z+\frac{\pi}{2}\right)}{z+\frac{\pi}{2}}} = \frac{e^{-\frac{\pi}{2}}}{1} = e^{-\frac{\pi}{2}}$$

So $\oint_{|z|=3} \frac{e^z}{\cos z}\,dz = 2\pi i\left[-e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}\right] = -4\pi i \sinh\frac{\pi}{2}$