NUMERICAL SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Polynomial Functions : $\boxed{y = f(x)}$

[Names derive from the highest degree.]

1/. Linear : $\boxed{f(x) = ax + b}$

2/. Quadratic : $\boxed{f(x) = ax^2 + bx + c}$

3/. Cubic : $\boxed{f(x) = ax^3 + bx^2 + cx + d}$

4/. Quartic : $\boxed{f(x) = ax^4 + bx^3 + cx^2 + dx + e}$

5/. Quintic : $\boxed{f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f}$

A General n-order Polynomial

$$\boxed{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_2 x^2 + a_1 x + a_0}$$

OR $\boxed{f(x) = \sum_{i=0}^{n} a_i x^i}$ is a summation notation.

i) This implies that polynomials have a finite-order series.

ii) Analytical roots of a polynomial $f(x) = 0$ can be obtained only up to a quartic.

Transcendental Functions $\boxed{y = f(x)}$

1/. $\boxed{f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$

2/. $\boxed{f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots}$

3/. $\boxed{f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$

4/. $\boxed{f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$

All transcendental functions (like some of the examples shown above) are given by an infinite series.

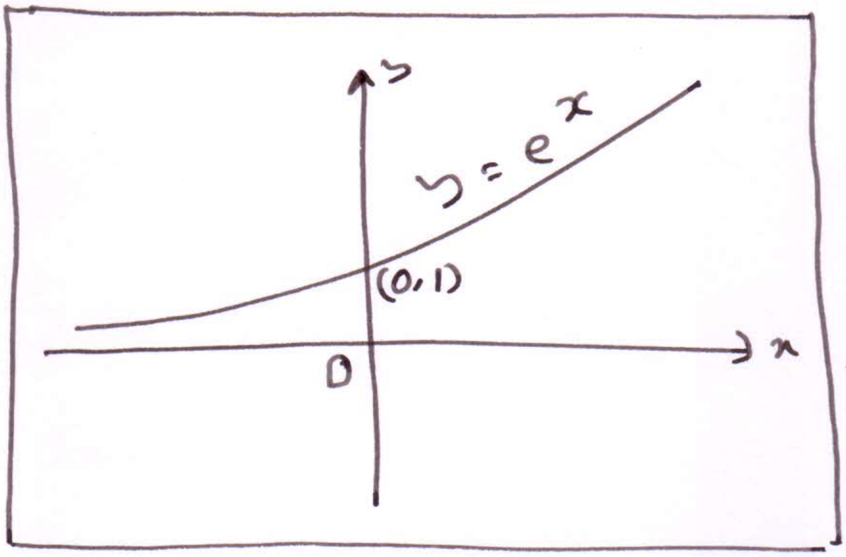
Plotting Techniques | On some Transcendental Functions

1/. $\boxed{y = e^x}$. i.) When $x > 0$, $y > 0$ and when $x < 0$, $y > 0$.
Hence the function is in the first and second quadrants.

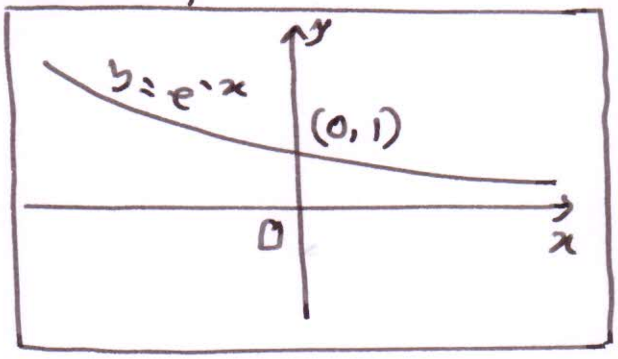
ii.) The function crosses the y-axis ($x = 0$). Hence the point through which the function passes is $(0, 1)$, while changing quadrants.

iii.) $\frac{dy}{dx} = e^x = 0 \Rightarrow x \to -\infty$. The function does not have a turning point, i.e. it is a monotonic ~~[illegible]~~ function.

iv.) When $x \to \infty$, $y \to \infty$ and when $x \to -\infty$, $y \to 0$. (The asymptotic behaviour.)



Also for $y = e^{-x}$, ~~[illegible]~~ plot by $\boxed{x \to -x}$.

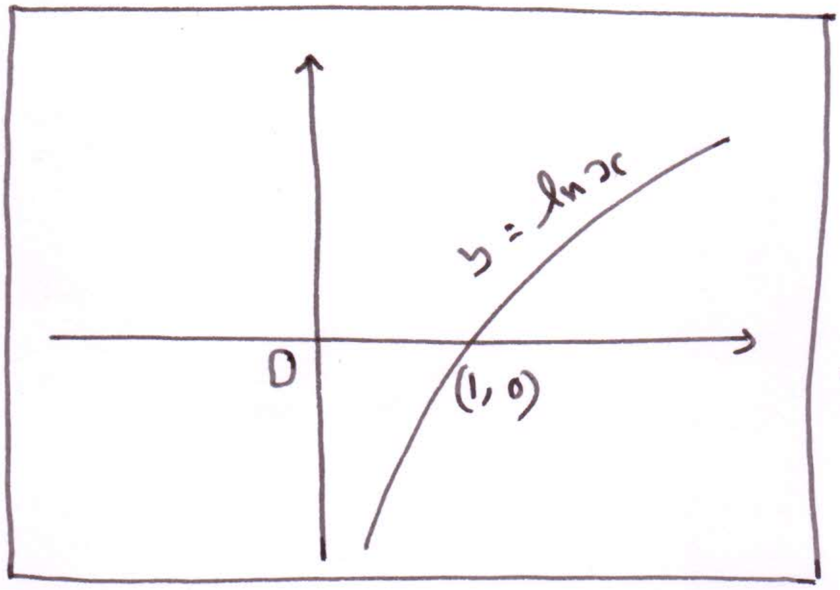


2/ $\boxed{y = \ln x}$ i.) x is always > 0. When $x > 1$, $y > 0$ and when $x < 1$, $y < 0$. The function is in the first and fourth quadrants.

ii.) The function crosses the x axis ($y = 0$). The point of crossing is for $\boxed{0 = \ln 1}$, i.e., $(1, 0)$.

iii.) $\frac{dy}{dx} = \frac{1}{x} = 0 \Rightarrow x \to \infty$, i.e. this function is also monotonic without any turning point at finite values of x.

iv.) When $x \to \infty$, $y \to \infty$ and when $x \to 0$, $y \to -\infty$.



I/. On large scales of x, the growth of e^x is much greater than $\ln x$.

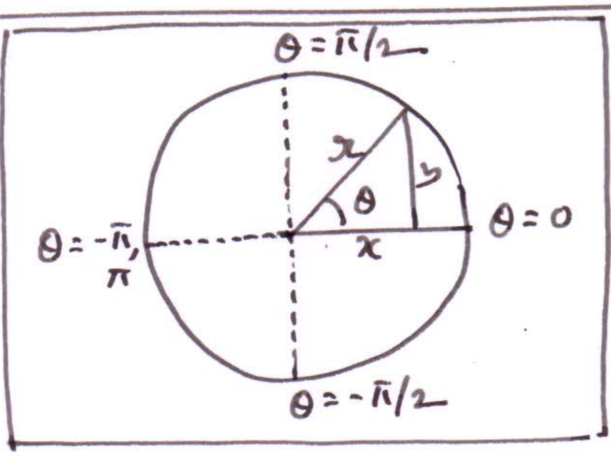
II/. Similarly x also grows much faster than $\ln x$ on large x.

Example : 1/. $\boxed{f(x) = e^x + \ln x}$. When $x \to \infty$, $f(x) \sim e^x$ and when $x \to 0$, $f(x) \sim \ln x$.

2/. $\boxed{f(x) = x + \ln x}$. When $x \to \infty$, $f(x) \sim x$ and when $x \to 0$, $f(x) \sim \ln x$.

3/. $\boxed{y = \sin x}$

(on an open linear scale)



On the circular path,

$\boxed{\sin\theta = y/r}$.

i.) When $\theta = 0$, $y = 0 \Rightarrow \sin\theta = 0$.
When $\theta = \pm\pi/2$ $y = \pm r \Rightarrow \sin\theta = \pm 1$.

$\Rightarrow$ for any argument x, $\sin x$ (on the open scale varies between ± 1 $(-1 < \sin x < 1)$

$\therefore$ For all x, $y = \sin x$ lies in all the four quadrants.

ii.) $\boxed{\dfrac{dy}{dx} = \cos x = \pm\sqrt{1-\sin^2 x}}$, because

$\boxed{\cos\theta = x/1}$, and by the Pythagoras theorem, $\boxed{x^2 + y^2 = 1^2}$ on the circular path.

$\because$ $\boxed{x = 1\cos\theta}$ and $\boxed{y = 1\sin\theta}$

$\Rightarrow x^2 + y^2 = \cancel{1^2}\cos^2\theta + \cancel{1^2}\sin^2\theta = \cancel{1^2}$

$\Rightarrow \boxed{\cos^2\theta + \sin^2\theta = 1} \Rightarrow \boxed{\cos x = \pm\sqrt{1-\sin^2 x}}$.

On the linear scale we plot $\sin x$ over $[-\pi, \pi]$.

a.) When $x = -\pi$, $\boxed{\sin x = 0}$, and $\boxed{\cos x = -1}$.

b.) When $x = -\pi/2$ $\boxed{\sin x = -1}$, and $\boxed{\cos x = 0}$.

$\therefore \dfrac{dy}{dx} = 0$ at $x = -\pi/2$ (a <u>turning point</u>).

c.) When $x = 0$, $\boxed{\sin x = 0}$, and $\boxed{\cos x = +1}$.

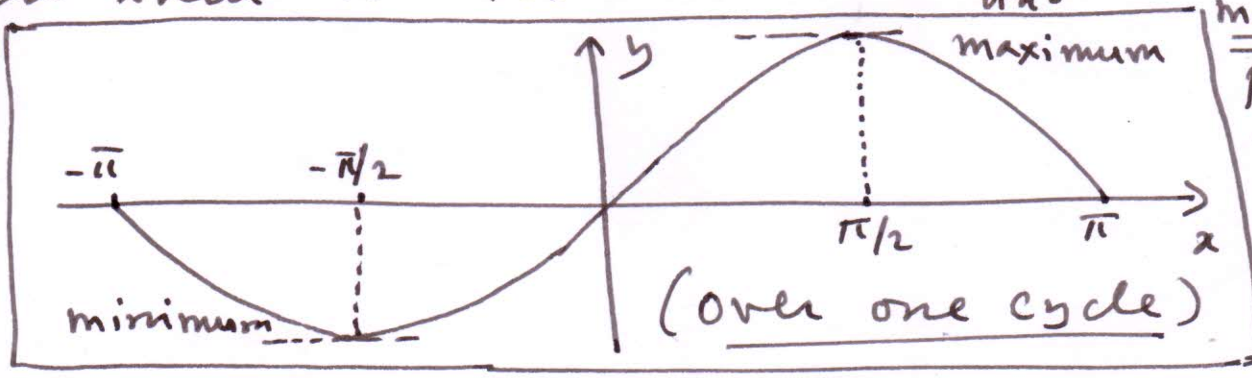
d.) When $x = \pi/2$, $\boxed{\sin x = 1}$ and $\boxed{\cos x = 0}$.

$\therefore \dfrac{dy}{dx} = 0$ at $x = \pi/2$ (a <u>turning point</u>)

e.) $\boxed{\dfrac{d^2y}{dx^2} = -\sin x}$. When $x = -\pi/2$, $\sin x = -1$.

$\Rightarrow \dfrac{d^2y}{dx^2} = 1 > 0$ (<u>stable minimum</u>)

And when $x = \pi/2$, $\sin x = 1 \Rightarrow \dfrac{d^2y}{dx^2} = -1 < 0$ (<u>unstable maximum</u>)



<u>Repeat this form for all cycles.</u>

 (<u>Gaussian function</u>)

4/. $\boxed{y = e^{-x^2}}$ i.) When $x > 0$, $y > 0$ and when $x < 0$, $y > 0$.

Also $\boxed{y(x) = y(-x)}$. This is an even function lying in the first and fourth quadrants.

ii.) The function crosses the y axis ($x = 0$), at the point $(0, 1)$.

iii.) $\boxed{\frac{dy}{dx} = e^{-x^2}.(-2x) = 0}$. The solutions are $x = 0$ and ~~[illegible]~~ $x \to \pm\infty$.

There is a turning point at $x = 0$.

$\boxed{\frac{d^2y}{dx^2} = -2\left[e^{-x^2} + x\, e^{-x^2}. -2x\right]}$ At $x = 0$, $\frac{d^2y}{dx^2} = -2 < 0$.

$\Rightarrow$ The turning point is a maximum.

iv.) When $x \longrightarrow \pm\infty$, $y \longrightarrow 0$.

v.) $\boxed{y = \frac{1}{e^{x^2}} \simeq \frac{1}{1 + x^2}}$ $\rightarrow$ Going up to <u>the first order</u>.

$\boxed{y = \frac{1}{1+x^2}}$ is known as <u>the Lorentz function</u>.