Lecture - 34 P 1

Recap: Central Limit Theorem

Information Theory

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started by a paper called

"A Mathematical Theory of Communication" by

Claude Shannon, 1948

"**The** Mathematical theory of Communication"

Ahmedabad	Delhi
Sun 95: 95% 100	Sun: 50%
Rain: 5 5% 0	Rain: 50%
$0.95 \log_2\left(\frac{1}{0.95}\right) +$	$\left[0.5 \log_2\left(\frac{1}{0.5}\right)\right] * 2$
$0.05 \log_2\left(\frac{1}{0.05}\right)$	$= 1$ bit
$= 0.286$ bits.	

Information: amount of uncertainity.

Entropy:

$$H(X) = \sum_{i=1}^{n} p_i \log_2\left(\frac{1}{p_i}\right)$$

X: discrete random variable

$$= E\left(\log_2\left(\frac{1}{p_i}\right)\right) \text{ bits}$$

e.g.

Throwing a dice

1	2	3	4	5	6	← X
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	← $p(x)$

$\log\left(\frac{1}{p_i}\right)$ = Random variable

$$H(X) = \sum_{i=1}^{6} p_i \log\left(\frac{1}{p_i}\right)$$

$$= \frac{1}{6} 6 \log_2(6)$$

$$= \log_2 6 = 2.59 \text{ bits.}$$

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e.g. A fair coin is tossed until you get the first head. Let X denote the no. of tosses required. Compute $H(X)$.

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X	$P(X)$	
$\rightarrow$ 1	$1/2$	H
2	$\frac{1}{2}\cdot\frac{1}{2}$	$T \cdot H$
3	$\left(\frac{1}{2}\right)^3$	TTH
4	$\left(\frac{1}{2}\right)^4$	$TTTH$
$\vdots$		
i	$\left(\frac{1}{2}\right)^i$	$\underbrace{T\ldots T}_{i-1} H$

$$H(X) = \sum_{i=1}^{\infty} p_i \log \frac{1}{p_i}$$

A.G.P.

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \log\left(2^i\right) = \sum_{i=1}^{\infty} i \cdot 2^{-i}$$

$$= 2 \text{ bits.}$$

e.g.

$$X = \begin{cases} A & \frac{1}{2} & p \\ B & \frac{1}{2} & 1-p \\ & & 0 \le p \le 1 \end{cases}$$

$$H_p(X) = p \log \frac{1}{p} + (1-p) \log\left(\frac{1}{1-p}\right)$$

Maximize $\underline{\underline{H_p(X)}}$ over p.

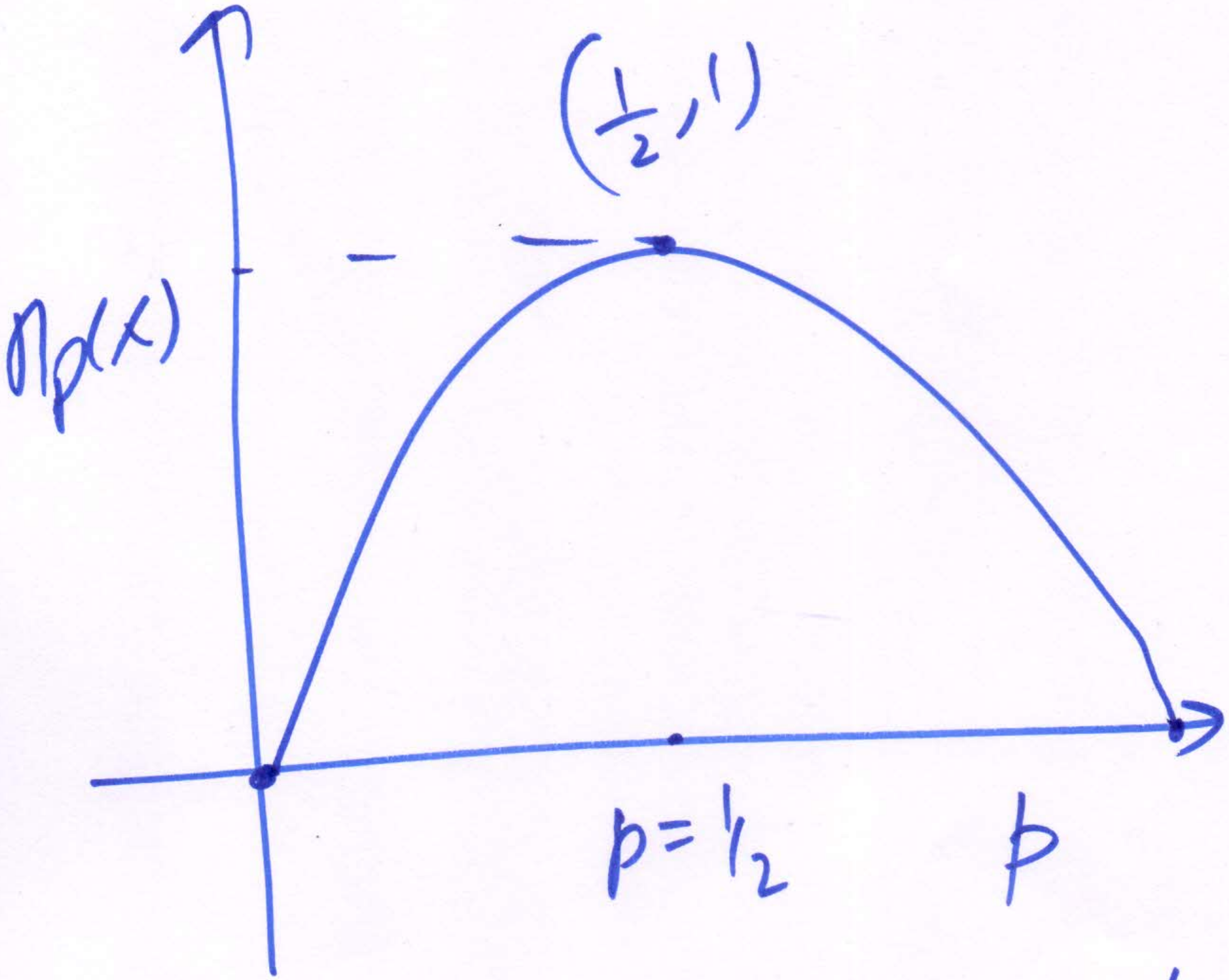
$$\frac{d}{dp}\left(H_p(X)\right) = 0$$

$$H_p(X) = -p\log p - (1-p)\log(1-p)$$

$$H'_p(X) = -\left[\frac{p}{p} + \log p - \frac{(1-p)}{(1-p)} - \log(1-p)\right]$$

$$\Rightarrow \log p = \log(1-p)$$

$$\Rightarrow p = 1-p \Rightarrow p = \frac{1}{2}$$



$$p \log \frac{1}{p} + (1-p) \log \frac{1}{1-p}$$

$p \to 0, \ H_p(X) \to 0$

$p \to 1, \ H_p(X) \to 0$

$X = $	A	0	1
	B	1	0

$H(X) = 0$

Joint Entropy

$$H(X,Y) =$$

$$-\sum_{X}\sum_{Y} p(x,y) \log p(x,y)$$

Conditional Entropy

$$H(Y|X) = \sum_{X} p(x)\, H(Y \mid X=x)$$

$$= \sum_{X}\sum_{Y} p(x,y) \log \frac{1}{p(y|x)}$$

e.g.

$y \backslash x$	1	2	3	4	
1	$1/8$	$1/16$	$1/32$	$1/32$	$1/4$
2	$1/16$	$1/8$	$1/32$	$1/32$	$1/4$
3	$1/16$	$1/16$	$1/16$	$1/16$	$1/4$
4	$1/4$	0	0	0	$1/4$
	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	

$H(X): -\sum p(X=x) \log p(X=x) = \frac{7}{4}$ bits

$H(Y)$: 2 bits

$H(X|Y)$:

$H(Y|X)$: $\sum_{x} p(x) H(Y|X=x)$

$H(X,Y)$:

Given $X=1$, what is the entropy of Y? 9

Given $X=1$, whats the conditional distribution of Y? H.W.