Lecture 23:
Calculus of Variations
Euler-Lagrange Equation

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.}$$

Using the Euler equation

For any variables

$$\int F(r,\theta,\theta')dr \quad \text{where} \quad \theta' = d\theta / dr,$$

$$\frac{d}{dr}\left(\frac{\partial F}{\partial \theta'}\right) - \frac{\partial F}{\partial \theta} = 0.$$

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.}$$

$$\int F(t,x,x')dt \quad \text{where} \quad x' = dx/dt,$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0.$$

Imp.

1. The first derivative is with respect to the integration variable in the integral.
2. The partial derivatives are with respect to the other variables and its derivatives.

Summary : finding an extremum

• Solutions to many physical problems → maximizing or minimizing some **parameter *I*.**

- Distance
- Time
- Surface Area

• **Parameter *I*** dependent on **selected path *u*** and domain of interest D.

$$I = \int_D F\left(x, u, u_x\right) dx$$

1. The parameter *I* to be maximized or minimized
2. Extremal →The solution path *u* that maximizes or minimizes *I*

Examples

Geodesic

- Geodesic: a curve for a shortest distance between two points along a surface
  1) On a plane, a straight line
  2) On a sphere, a circle with a center identical to the sphere
  3) On an arbitrary surface$\rightarrow$ we can use the calculus of the variation.

Because the geodesic is the shortest value, finding the geodesic is relevant to finding the max. or min. values.

Geodesics

A locally length-minimizing curve on a surface

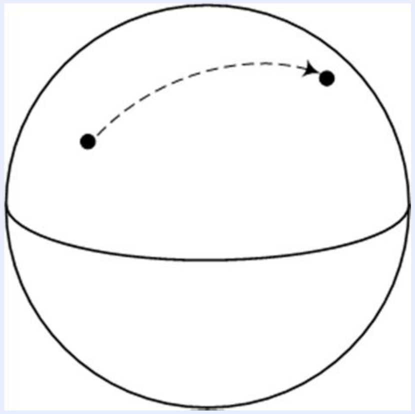
Find the equation $y = y(x)$ of a curve joining points (x_1, y_1) and (x_2, y_2) in order to minimize the arc length

$$ds = \sqrt{dx^2 + dy^2} \quad \text{and} \quad dy = \frac{dy}{dx}dx = y'(x)\,dx$$

so

$$ds = \sqrt{1 + y'(x)^2}\,dx$$

$$L = \int_C ds = \int_C \sqrt{1 + y'(x)^2}\,dx$$



Geodesics minimize path length

Shortest Path Between Two Points

- The problem of the shortest path between two points can be expressed as

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx.$$

- The integrand contains our function $f(y, y', x) = \sqrt{1 + y'(x)^2}$.
- The two partial derivatives in the Euler-Lagrange equation are:

$$\frac{\partial f}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}.$$

- Thus, the Euler-Lagrange equation gives us

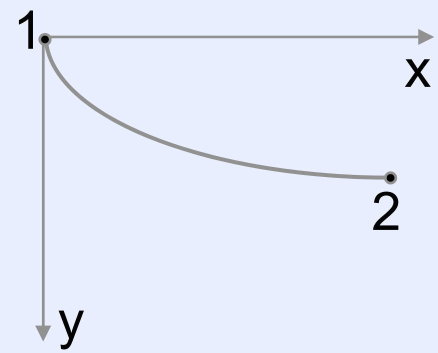
$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} = 0.$$

- This says that $\dfrac{y'}{\sqrt{1 + y'^2}} = C, \quad \text{or} \quad y'^2 = C^2(1 + y'^2).$
- The final result: $y'^2 = \text{constant}$ (call it m^2), so $y(x) = mx + b$. In other words, a straight line is the shortest path.

The Brachistochrone

- Statement of the problem:

Given two points 1 and 2, with 1 higher above the ground, in what shape could we build a track for a frictionless roller-coaster so that a car released from point 1 would reach point 2 in the shortest possible time? See the figure, which takes point 1 as the origin, with y positive downward.



- Force on the particle is constant, ignore friction.
- Field is conservative. Total energy is constant.
- KE=1/2mv^2; PE=-mgy

The Brachistochrone

- Solution:
  - *The time to travel from point 1 to 2 is* $\tau = \int_1^2 \frac{ds}{v},$ *where* $v = \sqrt{2gy}$ *from kinetic energy considerations.*
  - *Since this depends on* y*, we will take* y *as the independent variable, hence*

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'(y)^2 + 1}\, dy.$$

  - *Our integral now becomes:*

$$\tau = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{x'^2 + 1}}{\sqrt{y}}\, dy.$$

  - *From the Euler-Lagrange equation:*

Since we are using y as the independent variable, we swap x and y

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'}.$$

cont'd

- *Since* $f = \dfrac{\sqrt{x'^2 + 1}}{\sqrt{y}}$, *clearly* $\dfrac{\partial f}{\partial x} = 0$, *and so* $\dfrac{\partial f}{\partial x'} = \text{constant}.$
- *Evaluating this derivative and squaring it , we will have*

$$\frac{x'^2}{y(x'^2 + 1)} = \text{constant} = \frac{1}{2a}$$

where the constant is renamed $1/2a$ *for future convenience.*

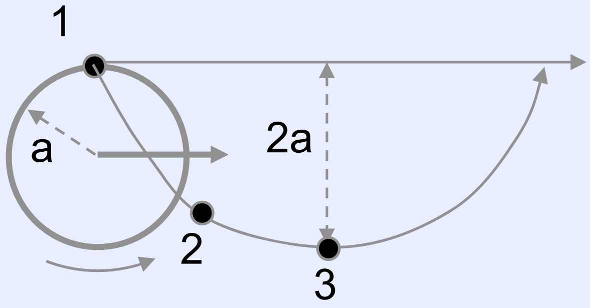
- *Solving for* x' *we have:* $x' = \sqrt{\dfrac{y}{2a - y}}$. *Finally, to get* x *we integrate:* $x = \int \sqrt{\dfrac{y}{2a - y}}\, dy.$
- *Change of variable, by the substitution* $y = a(1 - \cos\theta)$, *which gives dy*
- *The two equations that give the path are then:* $x = a(\theta - \sin\theta)$ *in terms of* θ.

$$y = a(1 - \cos\theta)$$

$$x = a\int (1 - \cos\theta)\, d\theta = a(\theta - \sin\theta) + \text{const.}$$

cont'd

- Solution, cont'd:
  - *This curve is called a cycloid, and is a very special curve.*
  - *it is the curve traced out by a wheel rolling (upside down) along the x axis.*
  - *Constant of integration →0*
  - *Another remarkable thing is that the time it takes for a cart to travel this path from 2→3 is the same, no matter where 2 is placed, from 1 to 3! Thus, oscillations of the cart along that path are exactly isochronous (period perfectly independent of amplitude).*



Fermat's Principle

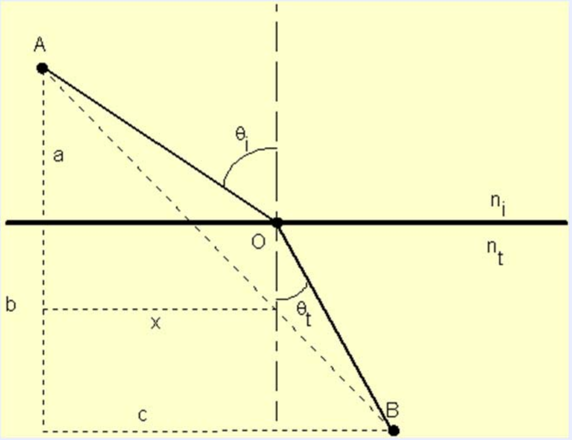
Refractive index of light in an inhomogeneous medium

$v = {}^{c}\!/_{n}$, where *v* = velocity in the medium and *n* = refractive index

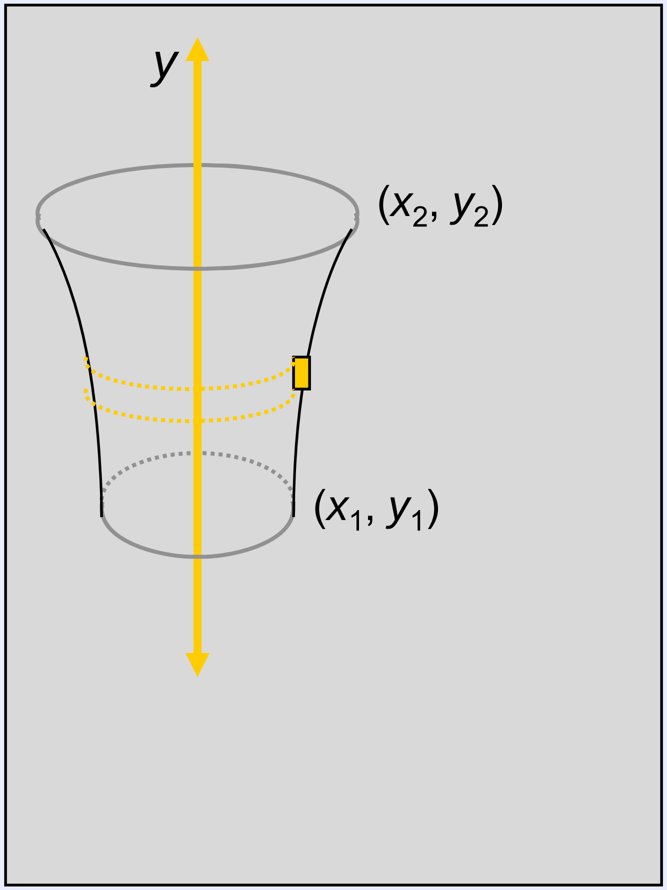
Time of travel = $T = \int_C dt = \int_C \frac{ds}{v} = \frac{1}{c}\int_C n\,ds$

$$T = \int_C n(x, y)\sqrt{1 + y'(x)^2}\,dx$$

Fermat's principle states that the path must minimize the time of travel.

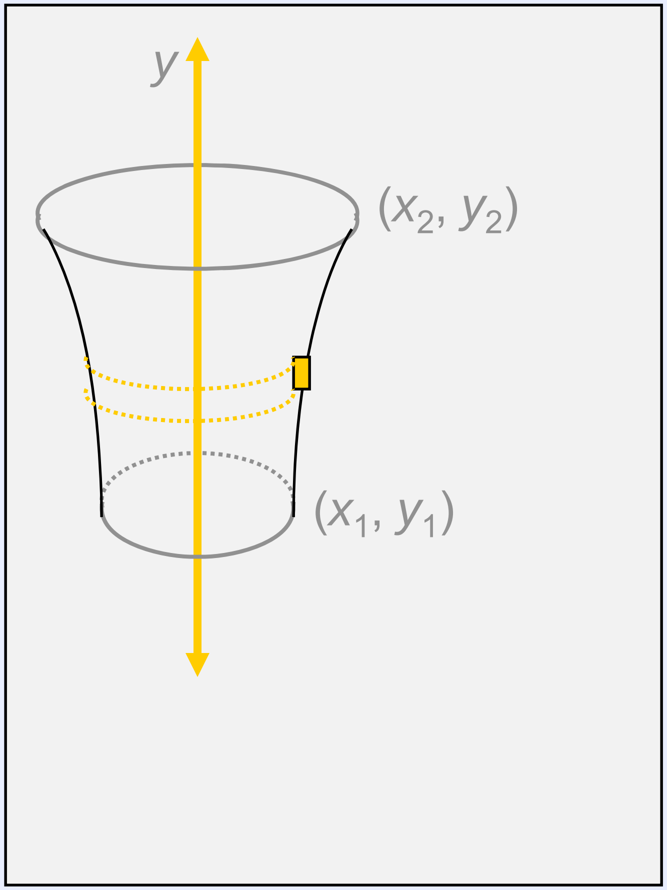


Soap Film – Find the solution



- A soap film forms between two horizontal rings that share a common vertical axis. Find the curve that defines a film with the minimum surface area.

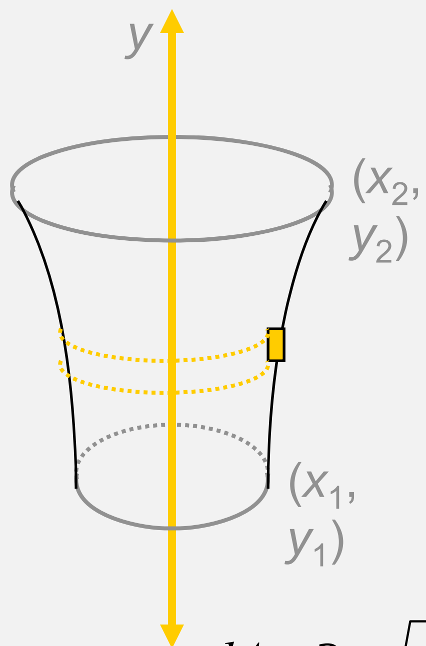
Soap Film



- A soap film forms between two horizontal rings that share a common vertical axis. Find the curve that defines a film with the minimum surface area.

- Define a function y.
- The area A can be found as a surface of revolution.

Soap Film



$$dA = 2\pi\sqrt{dx^2 + dy^2}$$

$$A = \int_{x_1}^{x_2} 2\pi x\sqrt{1 + y'^2}\,dx$$

- A soap film forms between two horizontal rings that share a common vertical axis. Find the curve that defines a film with the minimum surface area.

- Define a function y.
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Euler Applied

- The area is a functional of the curve.
  - *Define functional*
- Use Euler's equation to find a differential equation.
  - *Zero derivative implies constant*
  - *Select constant a*
- The solution is a hyperbolic function.

$$A = \int_{x_1}^{x_2} 2\pi x\sqrt{1+y'^2}\,dx = \int_{x_1}^{x_2} f(y, y'; x)\,dx$$

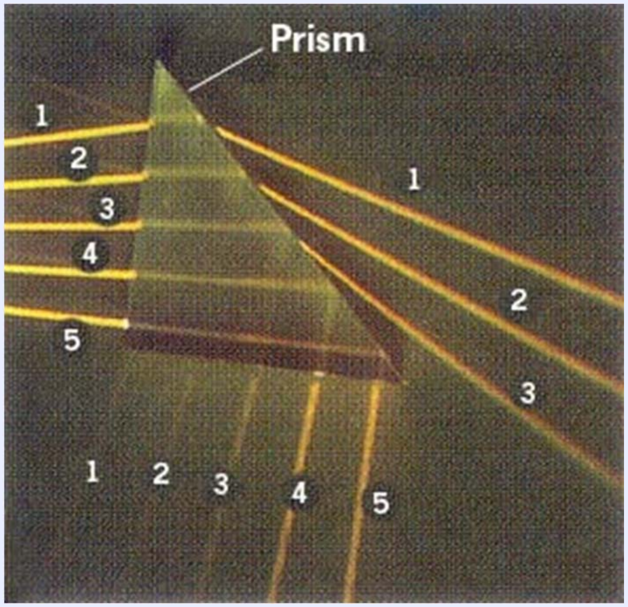
$$f = 2\pi x\sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 - \frac{d}{dx}\left(\frac{-xy'}{\sqrt{1+y'^2}}\right) = 0$$

$$\frac{xy'}{\sqrt{1+y'^2}} = a$$

$$y' = \frac{a}{\sqrt{x^2-a^2}} \qquad x = a\cosh\left(\frac{y-b}{a}\right)$$

Least Action



In optics, light is seen to take the minimum time path between two points (Fermat's principle).

What is Action?

- Action = s = $\int$(KE – PE) dt   [(from t1-t2)]
- KE – PE is known as the Lagrangian
- Commonly written as:
  - *L(x,v,t) = T(v) – V(x)*

Motion of a particle → the path that minimizes the action.

<u>Nature follows the path where s is smallest</u>

Lagrangian

Summary → Euler and Lagrange reformulated classical mechanics in terms of least action. The most important quantity is the **Lagrangian** which is simply the kinetic energy minus the potential energy.

<u>Consider a object moving vertically in a gravitational field, then; Write the Lagrangian</u>

Euler-Lagrange Equation

$$L = \frac{1}{2}m\dot{y}^2 - mgy$$

$$\dot{y} = \frac{dy}{dt}$$

Euler and Lagrange showed that the least action path obeys the Euler-Lagrange equation;

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$$

Object in a gravitational field, this is

$$-mg - m\frac{d\dot{y}}{dt} = 0$$

$$\ddot{y} = a_y = -g$$

Action

- The time integral of the Lagrangian is the *action*.
  - *Action is a functional*
  - *Extends to multiple coordinates*

$$S = \int_{t_1}^{t_2} L(q, \dot{q}; t)\,dt$$

- The Euler-Lagrange equations are equivalent to finding the least time for the action.
  - *Multiple coordinates give multiple equations*

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

- This is ***Hamilton's principle***.

Hamilton's Principle

Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energy.

Lagrange Eqn of motion for the 1D Harmonbic Oscillator

- ***Write down the Lagrangian function***
- ***Apply Lagrange eqs. Of motion***

Simple Harmonic Oscillator

$$T = \tfrac{1}{2} m\dot{x}^2$$

$$V = \tfrac{1}{2} kx^2$$

$$L = T - V = T = \tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} kx^2$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial x} = -kx$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} + kx = 0$$

- The 1-D simple harmonic oscillator has one force.
  - $F = -kx$
  - *Conservative force*
- Select x as the generalized coordinate.
  - *T, V in terms of generalized coordinate and velocity*
- Use Lagrange’s EOM.
  - *Usual Newtonian equation*