SC 216 : Calculus with Complex Variables

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Elementary functions

Now we will define analytic fn.s of a complex variable which reduces to elementary fn. on $\mathbb{R}$ when $z = x + i0$.

The Exponential fn.

$z \in \mathbb{C}$ $\quad e^z = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

— Note $e^z \neq 0$ for any $z \in \mathbb{C}$ ($\because |e^z| > 0$)

— e^z is periodic with a pure imaginary period $2\pi i$

($\because e^{z+2\pi i} = e^z$)

— e^x is never −ve while $\exists\, z$ s.t. $e^z = -1$

□ for if $e^x e^{iy} = 1 e^{i\pi}$ we get $e^x = 1$ & $y = \pi + 2n\pi$ $(n \in \mathbb{Z})$

$\Rightarrow x = 0 \Rightarrow z = (2n+1)\pi i$ $(n \in \mathbb{Z})$ for whic $e^z = -1$. ■

— $\because e^z = \rho e^{i\phi}$ where $\rho = e^x$ & $\phi = y$

we get $|e^z| = e^x$ & $\arg(e^z) = y + 2n\pi$ $(n \in \mathbb{Z})$

The Logarithmic fn.

We want to find for what $w \in \mathbb{C}$

& $z (\neq 0) \in \mathbb{C}$ $\quad e^w = z$ ------ (1)

If $z = re^{i\Theta}$ $(-\pi < \Theta \leq \pi)$ & $w = u + iv$

we get $e^u e^{iv} = re^{i\Theta}$

$\Rightarrow e^u = r$ and $v = \Theta + 2n\pi$ $(n \in \mathbb{Z})$

now $e^u = r \Rightarrow u = \ln r$ thus (1) is satisfied

iff $w = \ln r + i(\Theta + 2n\pi)$ $(n \in \mathbb{Z})$

So we write $\boxed{\log z = \ln r + i(\Theta + 2n\pi)}$ $(n \in \mathbb{Z})$ 1

s.t. $e^{\log z} = z$ $(z \neq 0)$ 1

Example If $z = -1 - \sqrt{3}\, i$

then $r = 2$ & $\Theta = -\frac{2\pi}{3}$

~~$\log z = \ln|z| + i \arg z$~~

$$\log(-1-\sqrt{3}\, i) = \ln 2 + i\left(-\frac{2\pi}{3} + 2n\pi\right)$$
$$= \ln 2 + 2\left(n - \frac{1}{3}\right)\pi i \quad (n \in \mathbb{Z})$$

— Note in general $\log(e^z) \neq z$.

For $\boxed{\log z = \ln|z| + i \arg z}$

now $\because$ $|e^z| = e^x$ & $\arg(e^z) = y + 2n\pi \ (n \in \mathbb{Z})$

$$\Rightarrow \log(e^z) = \ln|e^z| + i \arg(e^z)$$
$$= \ln(e^x) + i(y + 2n\pi) \quad (n \in \mathbb{Z})$$
$$= (x + iy) + 2n\pi i$$

$$\Rightarrow \boxed{\log(e^z) = z + 2n\pi i \quad (n \in \mathbb{Z})}$$

The principle value of $\log z$ is obtained from (1) by putting $n = 0$ & we denote it by

$$\boxed{\mathrm{Log}(z) = \ln r + i\Theta}$$

- $\mathrm{Log}(z)$ is well defined and single valued when $z \neq 0$

& also $\boxed{\log z = \mathrm{Log}(z) + 2n\pi i}$ $(n \in \mathbb{Z})$

— $\mathrm{Log}(z)$ reduced to natural log when $z = r$ (the real)

or $z = re^{i0}$ then $\mathrm{Log}\, r = \ln r$

— $\log(1) = \ln 1 + i(0 + 2n\pi) = 2n\pi i$ $(n \in \mathbb{Z})$

$\Rightarrow \log(1) \neq 0$

— $\mathrm{Log}(1) = 0$

— $\log(-1) = \ln 1 + i(\pi + 2n\pi) = (2n+1)\pi i$ $(n \in \mathbb{Z})$

& $\mathrm{Log}(-1) = \pi i$

— If $z = re^{i\theta}$ $(\neq 0) \in \mathbb{C}$

$$\theta = \Theta + 2n\pi \quad (n \in \mathbb{Z}), \quad \Theta = \operatorname{Arg}(z)$$

$$\Rightarrow \log z = \ln r + i(\Theta + 2n\pi) \quad (n \in \mathbb{Z})$$

· a multiple valued fn. can be written as

$$\log z = \ln r + i\theta$$

If $\alpha \in \mathbb{R}$ and let $\alpha < \theta < \alpha + 2\pi$

the fn. $\boxed{\log z = \ln r + i\theta \quad (r > 0, \alpha < \theta < \alpha + 2\pi)}$

with $u(r,\theta) = \ln r$ & $v(r,\theta) = \theta$ — (2)

is single-valued & cont.s in the ~~sated~~ stated domain. and also analytic in the domain $r > 0$ $\alpha < \theta < \alpha + 2\pi$ ($\because$ partial derivatives of u & v are cont.s & satisfy $r u_r = v_\theta$ & $u_\theta = -r v_r$)

— A <u>branch</u> of a multiple valued fn. f is any single valued fn. F that is analytic in some domain at each pt. z of which the value $F(z)$ is one of the values $f(z)$

— $\boxed{\operatorname{Log} z = \ln r + i\Theta \quad (r > 0, -\pi < \Theta < \pi)}$ is called a <u>principal branch</u>. — (3)

— A <u>branch cut</u> is a portion of a line or curve that is introduced in order to define a branch F of a multiple-valued fn. f. Points on the branch cut for F are singular pts of F and any pt. that is common to all branch cuts of f is called a <u>branch pt.</u>

$\boxed{\text{Example}}$ The origin & ray $\theta = \alpha$ make up the branch cut for the branch (2) of the logarithmic fn. and the origin & ray $\Theta = \pi$ are the branch cut for (3). $z = 0$ is branch pt.

① $\log(z_1 z_2) = \log z_1 + \log z_2$

② $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log}(z_2)$

COMPLEX EXPONENTS

$$z^c = e^{c \log z} \qquad (z \neq 0 \in \mathbb{C}) \quad c \in \mathbb{C}$$

$\log z$ is the multiple valued logarithmic fn.

Example $(i)^{-2i}$ are all real nos.

$$(i)^{-2i} = \exp(-2i \log i)$$

but $\log i = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = i\left(2n + \frac{1}{2}\right)\pi \quad n \in \mathbb{Z}$

$$\Rightarrow (i)^{-2i} = \exp((4n+1)\pi) \quad n \in \mathbb{Z} \quad \blacksquare$$

TRIGONOMETRIC FN.

Note since $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

& $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

we define in the same way

① $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

② $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

— Note $\sinh y = \dfrac{e^{y} - e^{-y}}{2}$ & $\cosh y = \dfrac{e^{y} + e^{-y}}{2}$

$\Rightarrow \sin(iy) = i \sinh y$

& $\cos(iy) = \cosh y$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$
$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

(3) $$\sinh z = \frac{e^{z} - e^{-z}}{2}$$
$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

(4) $$\sin^{-1} z = w \text{ (say)}$$
$$\Rightarrow z = \sin w$$
$$\Rightarrow z = \frac{e^{iw} - e^{-iw}}{2i}$$
$$\Rightarrow e^{iw} = iz + (1-z^2)^{1/2}$$
→ double valued fn. of z.

$$\Rightarrow \sin^{-1} z = -i \log\left[iz + (1-z^2)^{1/2}\right]$$
is a multiple valued fn.

$\boxed{\text{Example}}$ $$\sin^{-1}(-i) = -i \log(1 \pm \sqrt{2})$$

but $\log(1+\sqrt{2}) = \ln(1+\sqrt{2}) + 2n\pi i \quad (n \in \mathbb{Z})$

& $\log(1-\sqrt{2}) = \ln(\sqrt{2}-1) + (2n+1)\pi i \quad (n \in \mathbb{Z})$

but $\ln(\sqrt{2}-1) = \ln \frac{1}{(1+\sqrt{2})} = -\ln(1+\sqrt{2})$

$\Rightarrow$ the numbers
$$(-1)^n \ln(1+\sqrt{2}) + n\pi i \quad (n \in \mathbb{Z})$$
constitute the set of values of $\log(1 \pm \sqrt{2})$

$$\Rightarrow \sin^{-1}(-i) = n\pi + i(-1)^{n+1} \ln(1+\sqrt{2})$$
$n \in \mathbb{Z}$