$$
\begin{aligned}
P[11 < X < 21] &= P[|X - 16| < 5] \\
&= P[|X - 16| < 1.25\sigma] \\
&\geq 1 - \frac{1}{1.25^2} \\
&= 0.36
\end{aligned}
$$

2

$$C_Y = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$|A - \lambda I| = 0.$$

$$\left| \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0.$$

$$\left| \begin{bmatrix} 1-\lambda & 0.5 \\ 0.5 & 1-\lambda \end{bmatrix} \right| = 0.$$

$(1-\lambda)^2 - 0.25 = 0.$

$1 - 2\lambda + \lambda^2 - 0.25 = 0.$

$\lambda^2 - 2\lambda + 1 - 0.25 = 0.$

$\lambda^2 - 2\lambda + 0.75 = 0.$

$$\lambda = \frac{-(-2) \pm \sqrt{4 - 4(1)(0.75)}}{2}$$

$$= \frac{2 \pm \sqrt{1}}{2}$$

$$= \frac{2 \pm 1}{2}$$

$\boxed{\lambda_1 = 1.5}$ $\boxed{\lambda_2 = 0.5}$

$\lambda = 1.5$

$$\begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -0.5 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x + y = 0.$ $y = 1$ $x = 1$

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ after normalization $v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 0.5$

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$x + y = 0. \quad y = 1 \quad x = -1.$

$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

$A = V \Sigma^{\frac{1}{2}}$

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}^{1/2}.$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1.22 & 0 \\ 0 & 0.707 \end{bmatrix}.$$

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1.22 & -0.707 \\ 1.22 & 0.707 \end{bmatrix}$$

$C_Y = A C_X A^T$

$$C_Y = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1.22 & -0.707 \\ 1.22 & 0.707 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1.22 & 1.22 \\ -0.707 & 0.707 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1.22 & -0.707 \\ 1.22 & 0.707 \end{bmatrix} \begin{bmatrix} 1.22 & 1.22 \\ -0.707 & 0.707 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1.98 & 0.988 \\ 0.988 & 1.98 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

x_2

(3)

$$Y_1 = \frac{X_1}{X_2} \qquad \Rightarrow \qquad X_1 = Y_1 Y_2$$

$$Y_2 = X_2 \qquad\qquad X_2 = Y_2$$

range $0 < x_1 < x_2 < 1 \Leftrightarrow 0 < y_1 y_2 < y_2 < 1$

Here we can have ~~that~~

$0 < y_2 < 1$ & $0 < y_1 < 1$

Jacobian

$$D_y = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix}$$

$\text{Det}(D_y) = y_2.$

$$D_x = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{x_2} & \frac{x_1}{x_2^2} \\ 0 & 1 \end{bmatrix}$$

$\text{Det}(D_x) = \frac{1}{x_2} \qquad |D_y| = \frac{1}{|D_x|}$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 y_2, y_2) \times y_2$$

$$= 2y_2 \;, \; 0 < y_1 < 1 \;, \; 0 < y_2 < 1.$$

& 0 otherwise.

(4) Find the joint density function

Soln:

$$Y_1 = X_1 + X_2 \qquad Y_2 = \frac{X_1}{X_1 + X_2}$$

$$X_1 = Y_1 - X_2$$

$$\Rightarrow Y_2 = \frac{Y_1 - X_2}{Y_1 - X_2 + X_2}$$

$$\Rightarrow y_2 = \frac{y_1 - x_2}{y_1}$$

$$\Rightarrow y_1 y_2 = y_1 - x_2$$

$$\Rightarrow x_2 = y_1 - y_1 y_2 = y_1(1 - y_2)$$

$$\Rightarrow x_1 = y_1 - (y_1 - y_1 y_2) = y_1 y_2$$

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix}$$

range

$y_1 \geq 0$

$0 \leq y_2 \leq 1$

$$= \begin{vmatrix} y_2 & y_1 \\ (1-y_2) & -y_1 \end{vmatrix}$$

$$|J| = -y_1 y_2 - y_1(1 - y_2)$$

$$= -y_2 y_1 - y_1 + y_1 y_2$$

$$|J| = -y_1$$

$$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(y_1 y_2, y_1(1-y_2)).$$

$$= e^{-(\cancel{y_1 y_2} + y_1 - \cancel{y_1 y_2})} |-y_1|$$

$$f_{Y_1 Y_2}(y_1, y_2) = y_1 e^{-y_1}$$

$$, \quad y_1 \geq 0 \quad \& \quad 0 \leq y_2 \leq 1$$