

Power Series

$$\sum_{n=0}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$$

- Series converges for $z=a$
- If \exists a real no. R s.t. series converges for $|z-a| < R$ \rightarrow Radius of convergence.
- series diverges for $|z-a| > R$ & for $|z-a| = R$ it may or may not converge.

Example

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$$

~~$a-1$~~ ~~z^{n+1}~~

$$(-1)^{n-1} \frac{z^n}{n} + \dots \quad |z| < 1$$

$\therefore R = 1 \text{ & } a = 0$

$$(1+z)^b = 1 + b z + \frac{b(b-1)}{2!} z^2 + \dots + \frac{b(b-1)\dots(b-n+1)}{n!} z^n$$

$R = 1 \quad |z| < 1$

Taylor's Theorem

Let $f(z)$ be analytic inside & on a simple closed curve C . Let a & $a+h$ be two points inside C , then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2 f''(a)}{2!} + \dots + \frac{h^n f^{(n)}(a)}{n!} + \dots$$

or writing $z = a+h \Rightarrow h = z-a$

$$\Rightarrow f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots$$

Taylor's series

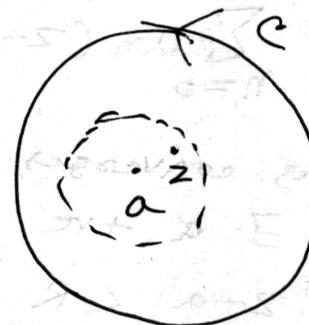
- Region of convergence of this is $|z-a| < R$.

- For $a = 0$ it is called as MacLaurin series.

Proof:

Let z be any pt. inside C .

Construct a circle C_1 with center
at a & enclosing z then
by Cauchy's integral formula



$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)}{\omega-z} d\omega \quad \text{--- (1)}$$

$$\begin{aligned} \text{Now } \frac{1}{\omega-z} &= \frac{1}{(\omega-a)-(z-a)} = \frac{1}{\omega-a} \left\{ \frac{1}{1 - \frac{(z-a)}{\omega-a}} \right\} \\ &= \frac{1}{\omega-a} \left\{ 1 + \left(\frac{z-a}{\omega-a} \right) + \left(\frac{z-a}{\omega-a} \right)^2 + \dots + \left(\frac{z-a}{\omega-a} \right)^{n-1} \right. \\ &\quad \left. + \left(\frac{z-a}{\omega-a} \right)^n \frac{1}{1 - \frac{(z-a)}{\omega-a}} \right\} \\ \Rightarrow \frac{1}{\omega-z} &= \frac{1}{\omega-a} + \frac{z-a}{(\omega-a)^2} + \frac{(z-a)^2}{(\omega-a)^3} + \dots + \frac{(z-a)^{n-1}}{(\omega-a)^n} \\ &\quad + \frac{(z-a)^n}{(\omega-a)^n} \frac{1}{\omega-z} \end{aligned}$$

Multiplying (2) by $f(\omega)$ & using (1) we have — (2)

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)}{\omega-a} d\omega + \frac{z-a}{2\pi i} \oint_{C_1} \frac{f(\omega)}{(\omega-a)^2} d\omega \\ &\quad + \dots + \frac{(z-a)^{n-1}}{2\pi i} \oint_{C_1} \frac{f(\omega)}{(\omega-a)^n} d\omega + U_n \\ &\quad \text{--- (3)} \end{aligned}$$

where $U_n = \frac{1}{2\pi i} \oint_{C_1} \left(\frac{z-a}{w-a} \right)^n \frac{f(w)}{w-z} dw$

Now using Cauchy's integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw, n=0,1,2,3,\dots$$

(3) becomes

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!} (z-a)^2 + \dots$$

$$+ \frac{f^{(n-1)}(a)}{(n-1)!} (z-a)^{n-1} + U_n$$

\therefore Now we only want to show that as $n \rightarrow \infty$ $U_n \rightarrow 0$.

To do this we note that since w is on C_1

we have $\left| \frac{z-a}{w-a} \right| = n < 1$

Also, we have $|f(w)| < M$ M is a const.

$$\& |w-z| = |(w-a) - (z-a)| \geq r_1 - |z-a|$$

where r_1 is radius of C_1

$$\Rightarrow |U_n| = \frac{1}{2\pi} \left| \oint_{C_1} \left(\frac{z-a}{w-a} \right)^n \frac{f(w)}{w-z} dw \right|$$

$$\leq \frac{1}{2\pi} \frac{n^n M}{r_1 - |z-a|} \cdot 2\pi r_1 = \frac{n^n M r_1}{r_1 - |z-a|}$$

$\therefore n \rightarrow \infty$
 $\therefore n < 1$

(3)

Laurent's theorem

If $f(z)$ is analytic ~~and~~ inside and on the boundary of the ring-shaped region R bounded by two concentric circles C_1 & C_2 with center at a & respective radii r_1 & r_2 ($r_1 > r_2$) then $f(z) \in R$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

where

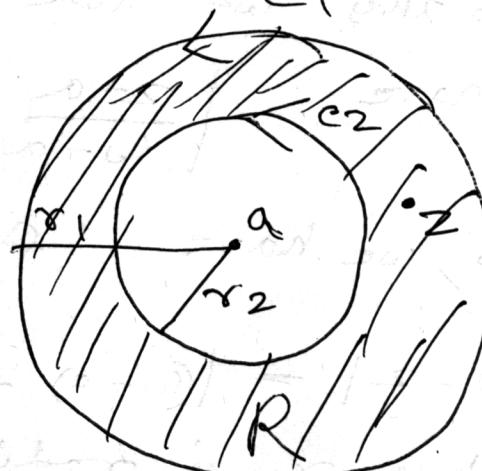
$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)}{(\omega-a)^{n+1}} d\omega, \quad n=0,1,2,\dots$$

$$a_{-n} = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\omega)}{(\omega-a)^{n+1}} d\omega, \quad n=1,2,3,\dots$$

□ By Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)}{\omega-z} d\omega$$

$$- \frac{1}{2\pi i} \oint_{C_2} \frac{f(\omega)}{\omega-z} d\omega$$



Now the proof is similar to Taylor's series proof by expanding $\frac{1}{w-z}$ etc.

— The part $a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$ is called the analytic part & the part with inverse power of $(z-a)$ is called principal part.

Example

$$f(z) = \ln(1+z)$$

Expand $f(z)$ in Taylor series about $z=0$

$$\begin{aligned} \square \quad f(z) &= \ln(1+z) = f(0) + f'(0)z + \frac{f''(0)z^2}{2!} \\ &= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots \end{aligned}$$

Alternative method:

$$\therefore \text{for } |z| < 1, \frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots$$

integrating from 0 to z yields

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

$$\therefore R = 1 \quad (\because |z| < 1)$$

Classification of Singularity

- Poles If Laurent's expansion ~~of~~ of $f(z)$ has finite no. of terms having negative powers of $(z-a)$. Eg.

if $f(z)$ has

$$\frac{a_1}{(z-a)} + \frac{a_2}{(z-a)^2} + \dots + \frac{a_n}{(z-a)^n}$$

$\Rightarrow z=a$ is a pole of order n .

Removable Singularity

If $f(z)$ is not defined at $z=a$ but $\lim_{z \rightarrow a} f(z)$ exists $\Rightarrow z=a$ called removable singularity

Example Find Laurent series about $z=1$ and name the singularity

$$\frac{e^{2z}}{(z-1)^3}$$

~~at $z=1$~~

Let $z-1=u \Rightarrow z=1+u$

$$\Rightarrow \frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2}{u^3} \cdot e^{2u}$$

$$= \frac{e^2}{u^3} \left\{ 1 + 2u + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \dots \right\}$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \frac{2e^2}{3(z-1)} + \dots$$

$\Rightarrow z=1$ is a pole of order 3.

Residues

$f(z)$ single valued & analytic inside & on C except at $z=a$

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-a)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz, n=0,\pm 1, \pm 2, \dots$$

For $n=-1$

$$\oint_C f(z) dz = 2\pi i a_1$$

a_1 = Residue of $f(z)$ at $z=a$

Calculation of Residues

$$a_1 = \lim_{z \rightarrow a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left\{ (z-a)^k f(z) \right\}$$

if $z=a$ is a pole of order k .

$$\therefore f(z) = \frac{a_k}{(z-a)^k} + \frac{a_{k+1}}{(z-a)^{k-1}} + \dots + \frac{a_1}{(z-a)} + a_0 + a_1(z-a) + \dots$$

as $z=a$ is a pole of
order k

$$+ a_0 + a_1(z-a) + \dots \quad (1)$$

Multiply both sides by $(z-a)^k$ & then differentiating both sides of (1)
 $(k-1)$ times

we get

$$\frac{d^{k-1}}{dz^{k-1}} \left\{ (z-a)^k f(z) \right\} = \cancel{(k-1)! a_1} + k(k-1) \dots 2 a_0 + \dots$$

let $z \rightarrow a$

$$\lim_{z \rightarrow a} \frac{d^{k-1}}{dz^{k-1}} \left\{ (z-a)^k f(z) \right\} = \cancel{(k-1)! a_1}$$

Example

$$f(z) = \frac{z}{(z-1)(z+1)^2}$$

$z=1$ is a simple pole

\therefore Residue at $z=1$ is

$$\lim_{z \rightarrow 1} (z-1) \left\{ \frac{z}{(z-1)(z+1)^2} \right\} = \frac{1}{4}$$

$z=-1$ is a pole of order 2

\Rightarrow Residue at $z=-1$ is

$$\lim_{z \rightarrow -1} \frac{1}{1!} \frac{d}{dz} \left\{ (z+1)^2 \left(\frac{z}{(z-1)(z+1)^2} \right) \right\} = -\frac{1}{4}$$

If $z=a$ is an essential singularity
the residue can be found by using known
series expansions.

Example

$$f(z) = e^{\frac{1}{z}}$$

$z=0$ is essential singularity

$$\& \therefore e^{\frac{1}{z}} = 1 - \frac{1}{2} + \frac{1}{2!2^2} - \frac{1}{3!3^3} + \dots$$

$$\therefore \text{coeff of } \frac{1}{z} = -1 = \text{Residue at } z=0$$