PHASE - 3

→ Newtonian mechanics — takes into account the external effects

→ Euler lagrange - only considers the inherent inbuilt properties of system (like energy).

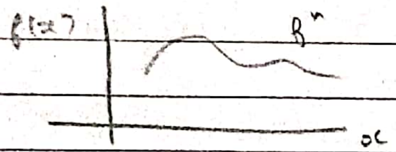
→ Also Euler langrange very helpful in quantum mechanics (Newtonian difficult here i.e difficult to analyse forces)

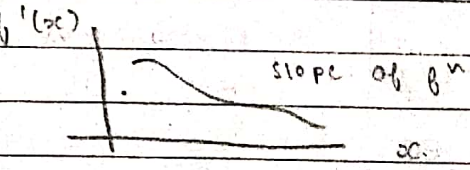
→ Also Euler lagrange only scalar so easier

Hamiltonian

→ $H(q, \underset{\text{momentum}}{p}, t)$: $H = T + U$

$L(q, \dot{q}, t)$: $L = T - U$





→ $f(x, y)$ with x, y independent variable

→ $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

→ suppose $g(X, y)$ where $\frac{\partial f}{\partial x} = X$ new independent variables are slopes

→ $dg = (A)dX + (B)dy$ — ①

$g = f - xX$ legendre transformation.

↓

If we have g of this form, we get something like in ①

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→ $dg = df - x\,dX - X\,dx$
$= X\,dx + Y\,dy - x\,dX - X\,dx$
$= Y\,dy - x\,dX$

Comparing $A = -x$ $B = Y$

→ so $f(x, y) \rightarrow g(X, y)$ then $g = f - xX$
$f(x, y) \rightarrow h(x, Y)$ then $h = f - yY$
$f(x, y) \rightarrow j(X, Y)$ then
$j = f - xX - yY$.

→ Using this for lagrangian → Hamiltonian
$L(q, \dot{q}, t) \rightarrow H(q, p, t)$ then
$H = L - \dot{q}p$ (Legendre transform)

Here too $\frac{\partial L}{\partial \dot{q}} = p$

→ $$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$ Euler lagrange

Challenge here is its second order
so we try to simplify it

→ Consider a close system with no external force

→ $L = T - U = T(\dot{q}) - U(q)$

$\frac{\partial L}{\partial \dot{q}} = p$ $\left[\frac{d}{dv}\frac{1}{2}mv^2 = mv\right]$

→ Taking $H(q, p, t) = \dot{q}p - L$
(reverse the terms to maintain positive solution — we can modify legendre transformation accordingly)

$$dH = p\,d\dot{q} + \dot{q}\,dp - dL$$

$$= \underbrace{p\,d\dot{q}} + \dot{q}\,dp - \frac{\partial L}{\partial q}dq - \underbrace{\frac{\partial L}{\partial \dot{q}}d\dot{q}} - \frac{\partial L}{\partial t}dt$$

$$= \dot{q}\,dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt$$

$$dH = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial t}dt$$

Comparing ① $\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}$

② $\dot{q} = \frac{\partial H}{\partial p}$

③ $-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$

→ System autonomous — does not change with time so we neglect equation ③

→ Equations ① and ② are Hamiltonian equations (cyclic)

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}(p) = \dot{p}$$

so now $\frac{\partial H}{\partial q} = -\dot{p}$, $\frac{\partial H}{\partial p} = \dot{q}$

are called canonical equations of motion. (2 FO ODE in H in place of 1 SO ODE in L)

→ Taking an autonomous system, $H = H(p,q)$

$\frac{dH}{dt} = 0$ → To prove $\frac{\partial H}{\partial t} = 0$ → given

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$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\frac{dq}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt} + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + 0$$

$$= \frac{\partial H}{\partial q}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial H}{\partial p}\left(-\frac{\partial H}{\partial q}\right) = 0.$$

(Hence verified)
[energy conserved]

→ Also, $\frac{dL}{dt} = \frac{\partial L}{\partial q}\frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt}$

[for closed system (autonomous) $\frac{\partial L}{\partial t} = 0$]

Does this imply ??

?!

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

$$= \dot{q}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) + \frac{\partial L}{\partial q}\ddot{q}$$

$$\frac{dL}{dt} = \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial q}\right)$$

$$\frac{d}{dt}\left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right) = 0$$

So $L - \dot{q}\frac{\partial L}{\partial \dot{q}}$ must be constant w.r.t t

Hence $H = L - \dot{q}p$ is justified

→ $\frac{\partial L}{\partial \dot{q}} = \frac{\partial (T-U)}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}}$ [because $U(q)$]

→ $L - \dot{q}\frac{\partial L}{\partial \dot{q}} = T - U - \dot{q}\frac{\partial T}{\partial \dot{q}} = T - U - 2T$

$v\frac{d}{dv}\left(\frac{1}{2}mv^2\right) = mv^2$

$$= -T - U$$

$$= -(T+U) = -H.$$

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→ $L = T - U$

$= \frac{1}{2}\sum_i m(\dot{x}_i^2) - U(x_i, t)$ → Taking rectangular coordinates as generalised so $x_1 = x$, $x_2 = y$, $x_3 = z$

$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i = p_i$

Derivative wrt a particular x_i, suppose x

so $p_i = p_x$

$\frac{\partial L}{\partial x_i} = -\frac{\partial U}{\partial x_i} = F_i$

suppose U does not change with x_i

Then $\frac{\partial U}{\partial x_i} = 0$

→ So $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} = 0$

Hence $\frac{d}{dt} p_i = 0$

→ Thus momentum is conserved if L does not depend explicitely on q_j (x_i) such q_j are called cyclic

→ $p_j = \frac{\partial L}{\partial \dot{q}_j}$ → generalized momentum

(The generalised momentum associated with cyclic coordinate is conserved.)

→ If we do a space transformation and lagragian does not change, then linear momentum is conserved. Such systems are symmetric wrt. space transformation.

→ $(x, y, z) \rightarrow (x + \Delta x, y + \Delta y, z + \Delta z)$.
L does not change (invariant)

→ Derivatives of L

$$\frac{d}{dt} L(q, \dot{q}, t) = \frac{\partial L}{\partial q}\frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt} + \frac{\partial L}{\partial t}.$$

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt}} - \frac{dL}{dt} + \frac{\partial L}{\partial t} = 0$$

$$\frac{d}{dt}\left[\dot{q}\frac{\partial L}{\partial \dot{q}}\right] - \frac{dL}{dt} + \frac{\partial L}{\partial t} = 0$$

$$\frac{d}{dt}\underbrace{\left[\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right]} + \frac{\partial L}{\partial t} = 0$$

H conserved if $\partial L / \partial t = 0$.

→ Consider a simple particle moving in 1D.

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\dot{q}p \equiv \dot{x}\frac{\partial L}{\partial \dot{x}} = m(\dot{x})^2$$

$$H = m\dot{x}^2 - L \qquad (H \equiv \dot{q}p - L)$$

$$= m(\dot{x})^2 - \left(\frac{1}{2} m(\dot{x})^2 - U(x)\right)$$

$$= \frac{1}{2} m(\dot{x})^2 + U(x)$$

$$H = T + U$$

(Hence, since L did not explicity change with time, the energy is conserved).

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Projectile in 3D

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$L(q, \dot{q}, t)$ $[T - U]$ $\qquad$ $H(q, p, t)$ $[T + U]$

→ $\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p_x \qquad p_y = m\dot{y} \qquad p_z = m\dot{z}$

→ $\underset{\dot{q}p - L}{H} = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + mgz$

Here we have 3 generalised coordinates, so total 6 equations needed (2 for each coordinate).

→ $\frac{\partial H}{\partial p_x} = \frac{p_x}{m} = \dot{x}$

→ $\dot{y} = \frac{p_y}{m}$, $\dot{z} = \frac{p_z}{m}$.

→ $\frac{\partial H}{\partial x} = 0 - \dot{p}_x$

→ $\frac{\partial H}{\partial y} = 0 = -\dot{p}_y$

→ $\frac{\partial H}{\partial z} = mg = -\dot{p}_z$

$p_x \rightarrow x \quad p_y \rightarrow y \quad p_z \rightarrow z$ gives a 6D plot phase space trajectory which tells everything about the system

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Simple Harmonic Oscillator in 1D

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H = \dot{x} p - L$$
$$= m\dot{x}^2 - \frac{1}{2} m \dot{x} + \frac{1}{2} k x^2$$
$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$
$$= \frac{p^2}{2m} + \frac{1}{2} k x^2$$

Here we will get 2 equation as there is just one generalised coordinate.

$$\frac{\partial H}{\partial p} = \dot{x} = \frac{p}{m} \quad , \quad \dot{p} = -kx = -\frac{\partial H}{\partial q}$$

Concepts (consider vector addition)

$$ds = dx + dy + dz \qquad \text{(Cartesian)}$$
$$v = \dot{x} + \dot{y} + \dot{z}$$
$$v^2 = (\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2$$

$$ds = dr + r\,d\theta + r\sin\theta\,d\phi \qquad \text{(spherical)}$$
$$v = \dot{r} + r\dot{\theta} + r\sin\theta\,\dot{\phi}$$
$$v^2 = (\dot{r})^2 + r^2(\dot{\theta})^2 + r^2\sin^2\theta\,(\dot{\phi})^2$$

$$ds = dr + r\,d\theta + dz \qquad \text{(Cylindrical)}$$
$$v = \dot{r} + r\dot{\theta} + \dot{z}$$
$$v^2 = (\dot{r})^2 + r^2(\dot{\theta})^2 + (\dot{z})^2$$

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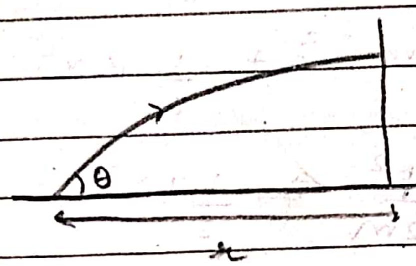
Projectile in 2D — spherical coordinate system

→ The 2 generalised coordinates are r and θ

$$T = \frac{1}{2} m \left[(\dot{r})^2 + r^2 (\dot{\theta})^2 \right]$$

$$U = mgr\sin\theta$$



$$L = T - U$$
$$= \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - mgr\sin\theta$$

→ $$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$$

$$mr(\dot{\theta})^2 - mg\sin\theta - \frac{d}{dt}(m\dot{r}) = 0$$

$$r(\dot{\theta})^2 - g\sin\theta - \ddot{r} = 0$$

Here, potential energy is a function of both G.C., whereas in cartesian it was simpler, hence in this case its better to choose that (cartesian) system.

→ For θ,

$$-gr\cos\theta - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta} = 0$$

(Here too cartesian coordinate system easier).

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→ For cylindrical coordinate

$$T = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2\right]$$

$$T = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right]$$

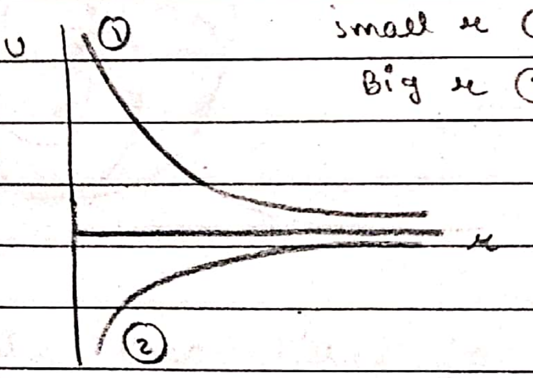
(For Hamiltonian)

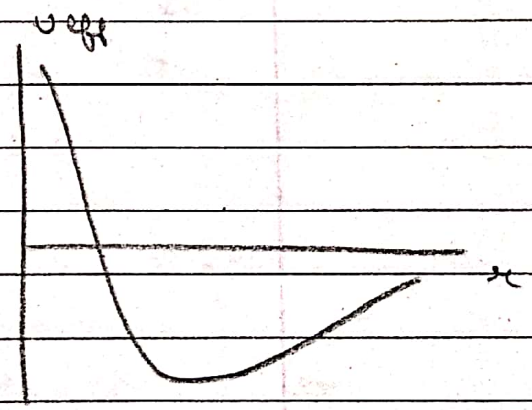
→ For Hamiltonian for spherical

$$T = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right]$$

Gravitation

$$V_{eff} = -\frac{G m_1 m_2}{r} + \frac{l^2}{2\mu r^2}$$





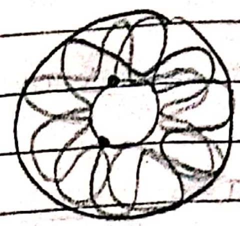
Motion of particle with energy $E_1 > 0$.

→ turning point $r = r_1$
→ Min distance of approach $= r_1$
Max distance of approach = infinity
→ unbounded orbit

Bound and closed

start end point meet.

Bound but not closed
↓
Eg. double pendulum



Start end point never meet

E3 – r changes so motion not closed (its just bound)

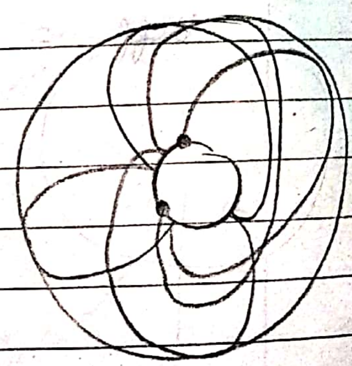
E4 – r does not change (so bound and closed – circular motion)

E3 – elliptical ⟶

E4 – circle

E2 – parabola

E1 – hyperbola.



→ ε decides shape of orbit and also type of motion – bounded / unbounded

$\varepsilon \to 0$: close to ellipse orbit

$\varepsilon \to 1$: skinny orbit

EQ Find force law for central force field that allows a particle to move in a spiral orbit given by $r = k\theta^2$, where k is constant.

Find Hamiltonian in spherically symmetric form $U(r) = -\frac{k}{r}$.

To find Hamiltonian, first find the Lagrangian in sperical coordinate.

$$L = \frac{1}{2} m \left(\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\,\dot{\phi})^2 \right) + \frac{k}{r}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr\dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\dot{r}\sin\theta\,\dot{\phi}$$

Now we will replace the r with the generalised momenta. Then use the canonical equations

$$H = \frac{m}{2} \left[\frac{p_r^2}{m^2} + \right.$$

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$$\frac{p_z^2}{m\left[1+\frac{r^2}{k^2}\right]}$$

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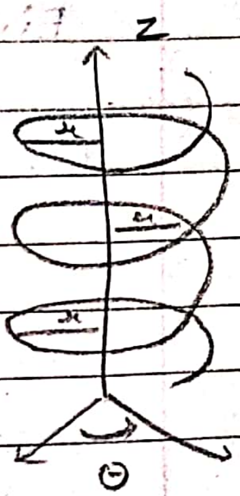
EQ. Particle of mass m follows a helical path. $z = k\theta$. r = constant.



Obtain H and equation of motion.

$$L = \frac{1}{2} m \left(\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2 \right) - mgz$$

(Here we choose cylindrical)

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr\dot{\theta}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$p_z \quad \frac{p_z}{m\left(\frac{r^2}{k^2}+1\right)}$$

$$-\frac{p_z^2}{2m(\quad)} + mgz$$

Alt way of finding H, legendre transformation. $H = p_z \dot{z} - L$

(Note here r is constant, so we don't need an equation for r. We want equation of z as that is changing)

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1-D motion of particle under force $F(x,t) = \frac{K}{x^2} e^{-t/\tau}$. Find L and H

$$L = \frac{1}{2} m \dot{x}^2 - U.$$

Now $F = -\frac{\partial U}{\partial x}.$

$$U = -\int F(x,t)\, dx.$$

$$= -\int \frac{K}{x^2} e^{-t/\tau}\, dx$$

$$= + \frac{K e^{-t/\tau}}{x}.$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{K e^{-t/\tau}}{x}.$$

$$H = \frac{p_x^2}{2m} + \frac{K e^{-t/\tau}}{x}$$ → depends on t so total energy not conserved.

Revise

→ $d(\frac{1}{2}mv^2) = F\,dx$.

→ $dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial t}dt$.

Now $F = -\nabla U$

$F\,dr = -\nabla U\,dr$

$= -dU + \frac{\partial U}{\partial t}dt$.

$d\left(\frac{1}{2}mv^2 + U(r)\right) = \frac{\partial U}{\partial t}dt$.

→ So if U will be a function of t, then total energy will not be conserved.

→ Thus TE conserved only if U is not a function of t