Tutorial-01

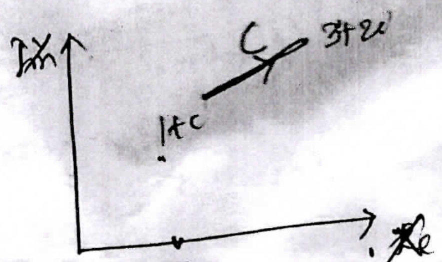
Q.1 Integrate the following:

(a) $\int_C \mathrm{Re}\, z \, dz$, where C is the shortest path from $1+i$ to $3+2i$

(b) $\int_C \bar{z}\, dz$, where C from 0 along the parabola $y = x^2$ to $1+i$

(c) $\int_C z e^{z^2} dz$, where C from 1 along the axes to i.

(d) $\int_C \sec^2 z \, dz$, where C is any path from $\frac{\pi i}{4}$ to $\pi/4$ in the unit disk.



Sol^n (a) $\int_C \mathrm{Re}\, z\, dz$

Parametric representation of C is

$r(t) = (1-t)(1+i) + t(3+2i), \quad 0 \le t \le 1$

$= 1 - t + 3t + (1-t)i + 2ti$

$= 1 + 2t + (1+t)i$

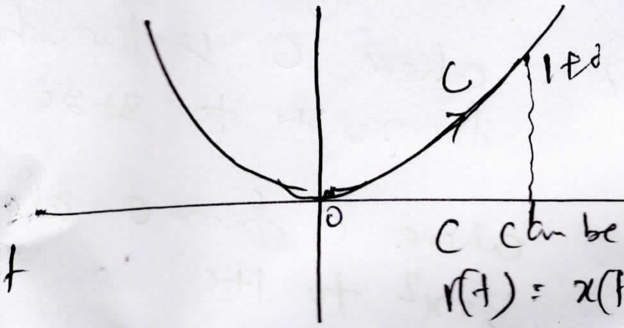
$r'(t) = 2 + i$

$$\int_C \mathrm{Re}\, z\, dz = \int_0^1 \mathrm{Re}(r(t))\, r'(t)\, dt$$

$$= \int_0^1 (1+2t)(2+i)\, dt = (2+i)\int_0^1 (1+2t)\, dt$$

$$= (2+i)\left[t + t^2\right]_0^1 = 2(2+i) = 4+2i$$

(b) $\int_C \bar{z}\,dz$, where C from 0 along the parabola $y=x^2$ to $1+i$



C can be represented as

$r(t) = x(t) + i\,y(t)$

$\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$

$r(t) = t + it^2$

$r'(t) = 1 + 2it$

$0 \le t \le 1$

$$\int_C \bar{z}\,dz = \int_0^1 \overline{r(t)}\, r'(t)\,dt$$

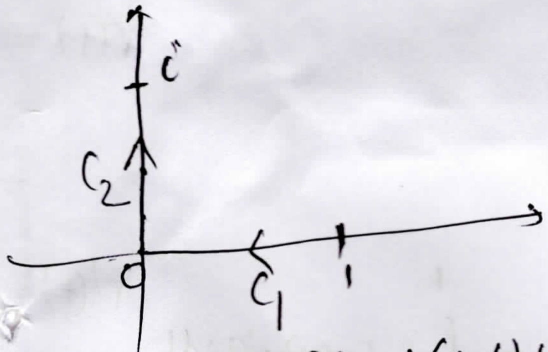
$$= \int_0^1 (t - it^2)(1+2it)\,dt$$

$$= \int_0^1 (t + 2it^2 - it^2 + 2t^3)\,dt$$

$$= \int_0^1 (t + it^2 + 2t^3)\,dt = \left[\frac{t^2}{2} + \frac{it^3}{3} + \frac{2t^4}{4}\right]_0^1$$

$$= \frac{1}{2} + \frac{i}{3} + \frac{1}{2} = 1 + \frac{i}{3}$$

(c) $\int_C z e^{z^2}\,dz$, where C from 1 to i along the axes.



$C_1 : r_1(t) = 1(1-t) + 0\cdot t = 1-t$

$0 \le t \le 1$

$C_2 : r_2(t) = 0(1-t) + t\cdot i = it$

$0 \le t \le 1$

$$= \int_{C_1} z e^{z^2}\,dz + \int_{C_2} z e^{z^2}\,dz$$

$$= \int_0^1 f(r_1(t))\, r_1'(t)\,dt + \int_0^1 f(r_2(t))\cdot r_2'(t)\,dt$$

$$= \int_0^1 (1-t)\, e^{(1-t)^2} (-1)\,dt$$

$$+ \int_0^1 it\, e^{(it)^2}\cdot i\,dt$$

$$= \frac{1}{2}\int_1^0 e^u\,du + \frac{1}{2}\int_0^{-1} e^v\,dv$$

$$= \frac{1}{2}\left[e^u\right]_1^0 + \frac{1}{2}\left[e^v\right]_0^{-1} = \frac{1}{2}(1-e) + \frac{1}{2}(e^{-1}-1)$$

$$= \frac{1}{2}(e^{-1} - e)$$

$$= -\sinh 1$$

$(1-t)^2 = u \qquad 1 \le u \le 0$

$2(1-t)(-1)\,dt = du$

$\Rightarrow -(1-t)\,dt = \frac{1}{2}du$

$(it)^2 = v \qquad 0 \le v \le -1$

$2(it)\cdot i\,dt = dv$

$\Rightarrow (it)\,i\,dt = \frac{dv}{2}$

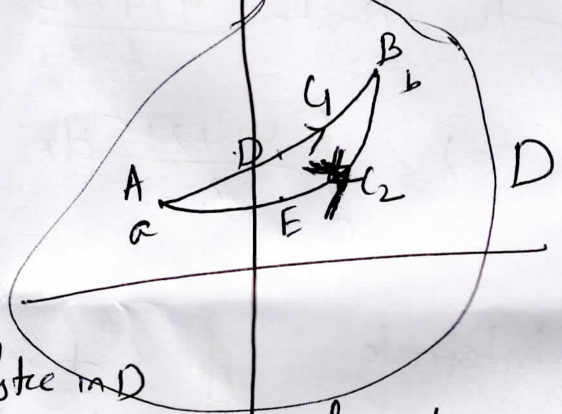
(d) $\int_C \sec^2 z \, dz$, C is any path from $\frac{\pi}{4}$ to $\frac{\pi i}{4}$ in the unit disc.

Sol^n $\int_C \sec^2 z \, dz = \left[\tan z\right]_{\pi/4}^{\pi i/4}$

$= \tan \frac{\pi}{4} - \tan i\frac{\pi}{4} = 1 - i \tanh \frac{\pi}{4}$

Q.2 If $f(z)$ is analytic in a simply connected domain D. Prove that $\int_a^b f(z)dz$ is independent of the path in D joining any two points a and b in D.

Sol^n



By Cauchy's Theorem

$$\int_{ADBEA} f(z)dz = 0$$

as $f(z)$ is analytic in D
as ADBEA is a simple closed path

$$\Rightarrow \int_{ADB} f(z)dz + \int_{BEA} f(z)dz = 0$$

$$\Rightarrow \int_{ADB} f(z)dz = \int_{-BEA} f(z)dz = \int_{AEB} f(z)dz$$

$$\Rightarrow \int_{C_1} f(z)dz = \int_{C_2} f(z)dz = \int_a^b f(z)dz$$

Hence the integral is independent of the path followed.

Q-3 Integrate $\oint_C \frac{4z^2+z+5}{z-3.5}\,dz$ where C is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Sol^n $z=3.5$ is the only singular point (that is a point where function is not analytic & analytic at every where else)

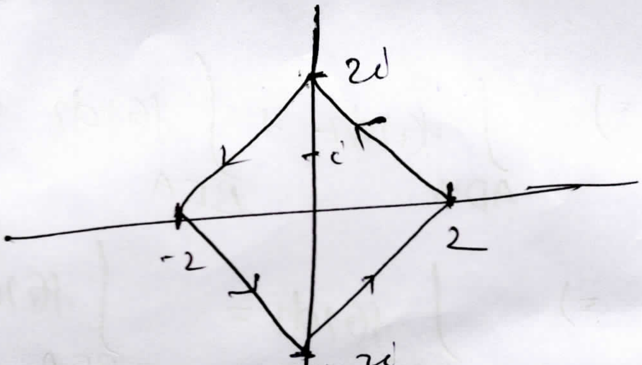
Also $z=3.5$ is outside the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

i.e $\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$

So by Cauchy's theorem the funct $\frac{4z^2+z+5}{z-3.5}$ is analytic on an inside the curve C.

$$\Rightarrow \int_C \frac{4z^2+z+5}{z-3.5}\,dz = 0.$$

Q-4 Integrate $\oint \frac{z^3+\sin z}{(z-i)^3}\,dz$, where C is the boundary of the square with vertices $\pm 2, \pm 2i$



Sol $f(z)=z^3+\sin z$ is analytic

The internal $\oint \frac{f(z)}{(z-i)^3}\,dz$

The point i is inside the simple closed curve C.

By Cauchy integral formula

$$\oint \frac{f(z)}{(z-i)^3}\,dz = \frac{2\pi i}{2!} f^{(2)}(i) = \pi i f^{(2)}(i)$$

$f(z)=z^3+\sin z$

$f'(z)=3z^2+\cos z$

$f''(z)=6z-\sin z$

$f''(i)=6i-\sin i$

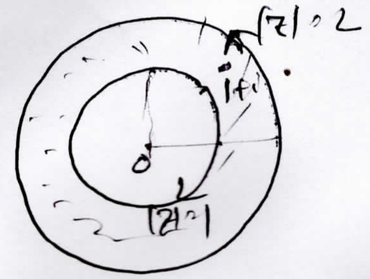
$$=\pi i(6i-\sin i)$$

$$=-6\pi - \pi i\sin i$$

Q.5 Integrate $\oint_C \frac{2z^3-3}{z(z-1-i)^2}\,dz$ $\quad C: |z|=2$ anticlockwise and $|z|=1$ clockwise.

Sol^n

~~Here $f(z) = 2z^3-3$~~



$z=0$ & $z=1+i$ are the points to be considered.

$z=0$ is outside the region of consideration

$z=1+i$ is inside the region of consideration

We take here $f(z) = \frac{2z^3-3}{z}$ which is analytic inside the region.

So by Cauchy's integral formula

$$\oint \frac{\frac{2z^3-3}{z}}{(z-1-i)^2}\,dz = \frac{2\pi i}{1!} f'(1+i)$$

$$= 2\pi i\, f'(1+i)$$

$$= 2\pi i\,\frac{(8i-5)}{2i}$$

$$= \pi(8i-5)$$

$$= -5\pi + 8\pi i$$

(Ans)

$$f(z) = \frac{2z^3-3}{z}$$

$$f'(z) = \frac{6z^2 \times z - (2z^3-3)\cdot 1}{z^2}$$

$$= \frac{6z^3 - 2z^3 + 3}{z^2}$$

$$f'(1+i) = \frac{4(1+i)^3+3}{(1+i)^2}$$

$$= \frac{4(1+i^3+3i+3i^2)+3}{1+i^2+2i}$$

$$= \frac{4(1-i+3i-3)+3}{1-1+2i}$$

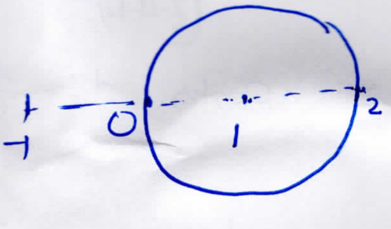
$$= \frac{4(2i-2)+3}{2i} = \frac{8i-5}{2i}$$

Q-6 Evaluate $\int_C \frac{3z^2+z}{z^2-1}\,dz$, where C is the circle $|z-1|=1$

Sol^n The integrand is not analytic at $z=1$ and $z=-1$.

C is the circle $|z-1|=1$

The singular point 1 is inside C where as the singular point -1 is outside C.



Also $f(z)=3z^2+z$ is analytic inside and on the curve C.

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1}-\frac{1}{z+1}\right)$$

$$\int_C \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2}\int_C \frac{3z^2+z}{z-1}\,dz + \frac{1}{2}\int_C \frac{3z^2+z}{z+1}\,dz$$

By Cauchy integral formula

$$\int_C \frac{3z^2+z}{z-1}\,dz = 2\pi i\, f(1) = 2\pi i\,(3+1) = 8\pi i$$

$$\int_C \frac{3z^2+z}{z+1}\,dz = 0$$ as by Cauchy's theorem $\frac{3z^2+z}{z+1}$ is analytic inside and on C.

So $$\int_C \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2}\times 8\pi i + \frac{1}{2}\times 0 = 4\pi i$$

Q.7 Evaluate $\oint_C \frac{e^{zt}}{(z^2+1)^2}dz$ if $t>0$ and C is the circle $|z|=3$.

Sol^n The singular points of the integrand are $+i$ and $-i$.

As $\frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2} =$

Both $+i$ and $-i$ are inside the circle $|z|=3$.

$$\frac{1}{(z+i)(z-i)} = \frac{-\frac{1}{2i}}{z+i} + \frac{\frac{1}{2i}}{z-i}$$ By partial fraction

$$\text{So } \frac{1}{(z+i)^2(z-i)^2} = \frac{-\frac{1}{4}}{(z+i)^2} + \frac{-\frac{1}{4}}{(z-i)^2} + \frac{1}{2}\left(\frac{1}{(z+i)(z-i)}\right)$$

$$= \frac{-\frac{1}{4}}{(z+i)^2} + \frac{-\frac{1}{4}}{(z-i)^2} + \frac{1}{2}\left[\frac{-\frac{1}{2i}}{z+i} + \frac{\frac{1}{2i}}{z-i}\right]$$

$$= \frac{-\frac{1}{4}}{(z+i)^2} + \frac{-\frac{1}{4}}{(z-i)^2} + \frac{-\frac{1}{4i}}{z+i} + \frac{\frac{1}{4i}}{z-i}$$

$$\text{So } \oint_C \frac{e^{zt}}{(z^2+1)^2}dz = \oint_C \frac{e^{zt}}{(z+i)^2(z-i)^2}dz$$

$$= -\frac{1}{4}\oint_C \frac{e^{zt}}{(z+i)^2}dz + -\frac{1}{4}\oint_C \frac{e^{zt}}{(z-i)^2} + -\frac{1}{4i}\oint_C \frac{e^{zt}}{z+i}dz + \frac{1}{4i}\oint_C \frac{e^{zt}}{z-i}dz$$

$$= -\frac{1}{4}\left[\frac{2\pi i}{1!}f'(-i)\right] - \frac{1}{4}\left[\frac{2\pi i}{1!}f'(i)\right] - \frac{1}{4i}\left[\frac{2\pi i}{0!}f(-i)\right] + \frac{1}{4i}\left[\frac{2\pi i}{0!}f(i)\right]$$

$$= -\frac{1}{4}\left[2\pi i\, t e^{-it}\right] - \frac{1}{4}\left[2\pi i\, t e^{it}\right]$$
$$-\frac{1}{4i}\left[2\pi i\, e^{-it}\right] + \frac{1}{4i}\left[2\pi i\, e^{it}\right]$$

$$= -\frac{1}{2}\pi i t e^{-it} - \frac{1}{2}\pi i t e^{it} - \frac{\pi}{2}e^{-it} + \frac{\pi}{2}e^{it}$$

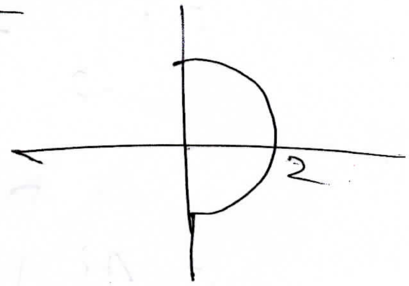
$$= -i\pi t\cos t + i\pi\sin t$$ (Ans)

$$\left[\begin{array}{l} f(z) = e^{zt} \\ f'(z) = te^{zt} \\ f'(i) = te^{it} \\ f'(-i) = te^{-it} \\ f(i) = e^{it} \\ f(-i) = e^{-it} \end{array}\right.$$

(8) Let C denote the right-hand half of the circle $|z|=2$ in the counterclockwise direction. Show that the two parametric representation for C are

(a) $z = \gamma(\theta) = 2e^{i\theta}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

(b) $z = \sqrt{4-y^2} + iy, \quad -2 \le y \le 2$



Find the value of the integral

$$I = \int_C \bar{z}\, dz$$

using both parametric representation.

Sol^n 1. (a) The paramedric representation of $|z|=2$ is $z = 2e^{i\theta}$.

$$I = \int_C f(\gamma(\theta))\, \gamma'(\theta)\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \overline{2e^{i\theta}}\, (2e^{i\theta})'\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 2e^{-i\theta} \cdot 2ie^{i\theta}\, d\theta$$

$$= 4i \int_{-\pi/2}^{\pi/2} d\theta = 4\pi i$$

(b) Note that C is the right hand half of the circle $x^2+y^2=4$. So on C, $x = \sqrt{4-y^2}$. This suggest the parametric representation $C: z = \sqrt{4-y^2} + iy \quad (-2 \le y \le 2)$

So
$$I = \int_C \bar{z}\,dz = \int_{-2}^{2}\left(\sqrt{4-y^2} - iy\right)\left(\frac{-y}{\sqrt{4-y^2}} + i\right)dy$$

$$= \int_{-2}^{2}(\cancel{-y}\cancel{+y})\,dy + i\int_{-2}^{2}\left(\frac{y^2}{\sqrt{4-y^2}} + \sqrt{4-y^2}\right)dy$$

$$= i\int_{-2}^{2}\frac{y^2+4-y^2}{\sqrt{4-y^2}}\,dy = 4i\int_{-2}^{2}\frac{dy}{\sqrt{4-y^2}}$$

$$= 4i\left[\sin^{-1}\left(\frac{y}{2}\right)\right]_{-2}^{2} = 4i\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= 4i\left[\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right] = 4\pi i$$

Note that the answer is same in both cases. because the path is same.

(2) Let C_R denote the upper half of the circle $|z| = R$ $(R>2)$, taken in the counterclockwise direction. Show that

$$\left|\int_{C_R}\frac{2z^2-1}{z^4+5z^2+4}\,dz\right| \le \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$$

<u>Sol^n</u> Note that $|z| = R$ $(R>2)$ then

$$|2z^2-1| \le 2|z|^2+1 = 2R^2+1$$

and
$$|z^4+5z^2+4| = |z^2+1|\,|z^2+4|$$
$$\ge \left||z|^2-1\right|\left||z|^2-4\right| = (R^2-1)(R^2-4)$$

($\cancel{|a+b| \le |a|+|b|}$)

$|z_1 - z_2| \le |z_1| + |z_2|$

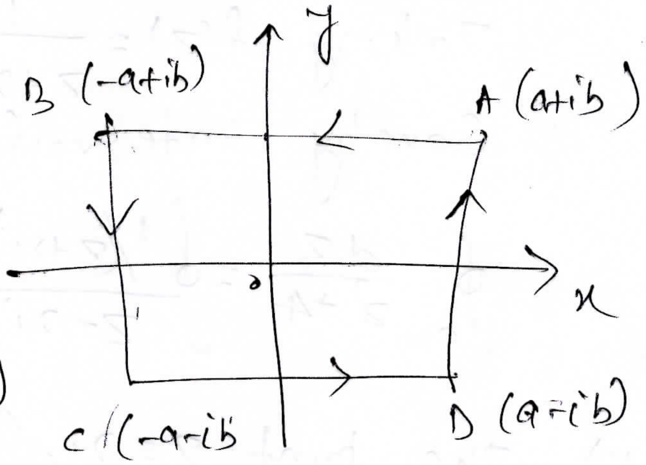
$|z_1 - z_2| \ge \left||z_1| - |z_2|\right|$

(9) Evaluate $\int_c (z+1)^2 \, dz$ where c is the boundary of the rectangle in anticlockwise direction with vertices at points $a+ib$, $-a+ib$, $-a-ib$, $a-ib$.

Sol^n Given that $I = \int_c (z+1)^2 dz$

where c is the boundary of the rectangle, with vertices at points $a+ib$, $-a+ib$, $-a-ib$, $a-ib$ as shown in the fig.

$I = \int_c (z+1)^2 \, dz = \int_{ABCD} (z+1)^2 dz$



$= \int_{ABCD} (x+iy+1)^2 (dx+idy)$

$= \int_{AB} (x+iy+1)^2 (dx+idy) + \int_{BC} (x+iy+1)^2 (dx+idy)$

$+ \int_{CD} (x+iy+1)^2 (dx+idy) + \int_{DA} (x+iy+1)^2 (dx+idy)$

Along AB $y = ib$ $dy = 0$

Along BC, $x = -a$, $dx = 0$

Along CD, $y = -ib$, $dy = 0$

Along DA, $x = a$ $dx = 0$

Thus

$I = \int_{AB} (x+i^2b+1)^2 dx + \int_{BC} (-a+iy+1)^2 i\,dy$

$+ \int_{CD} (x-i^2b+1)^2 dx + \int_{DA} (a+iy+1)^2 i\,dy.$

$= \int_a^{-a} (x-b+1)^2 dx + \int_{ib}^{-ib} (-a+iy+1)^2 i\,dy + \int_{-a}^{a} (x+b+1)^2 dx + \int_{-ib}^{ib} (a+iy+1)^2 i\,dy$ = Integrate

(10) Evaluate the integral $\oint_C \frac{dz}{z^2+4}$ where (4)

(i) $C: |z-2i|=1$, (ii) $C: |z+2i|=1$, (iii) $C: |z|=4$.

Sol^n The given integral is not analytic at $z=\pm 2i$

as $\frac{1}{z^2+4} = \frac{1}{(z-2i)(z+2i)}$

(i) The point $z=-2i$ lies outside C.
Taking $f(z)=\frac{1}{z+2i}$, $z_0=2i$ and using the Cauchy integral formula, we obtain

$$\oint_C \frac{dz}{z^2+4} = \oint \frac{1/(z+2i)}{z-2i}\,dz = 2\pi i\left[\frac{1}{z+2i}\right]_{z=2i} = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}$$

(ii) The point $z=2i$ lies outside C. Taking $f(z)=\frac{1}{z-2i}$, $z_0=-2i$ and using Cauchy's integral formula, we obtain

$$I = \oint_C \frac{1/(z-2i)}{z+2i}\,dz = 2\pi i\left(\frac{1}{z-2i}\right)\Big|_{z=-2i} = 2\pi i\left(\frac{-1}{4i}\right) = -\frac{\pi}{2}$$

(iii) Both the points $z=\pm 2i$ lie inside C.
Enclose the points $z=-2i$ and $z=2i$ by circle $C_1: |z+2i|=\delta_1$ and $C_2: |z-2i|=\delta_2$ respectively s.t. C_1 and C_2 don't intersect each other and lie inside C. Then by Cauchy's integral th^m for multiply connected domain

$$\oint_C \frac{dz}{z^2+4} = \oint_{C_1} \frac{dz}{z^2+4} + \oint_{C_2} \frac{dz}{z^2+4} = \frac{\pi}{2} - \frac{\pi}{2} = 0.$$

(using results (1) & (2) above)

By partial fractions

(iii) $\dfrac{1}{z^2+4} = \dfrac{1}{(z+2i)(z-2i)} = \dfrac{i/4}{z+2i} + \dfrac{-i/4}{z-2i}$

So $\displaystyle\int_C \frac{dz}{z^2+4} = \frac{i}{4}\int_C \frac{1 \cdot dz}{z+2i} - \frac{i}{4}\int_C \frac{1 \cdot dz}{z-2i}$

$(f(z)=1)$ $(f(z)=1)$

By Cauchy's integral thm.

$\displaystyle\int_C \frac{dz}{z^2+4} = \frac{i}{4} \cdot 2\pi i \cdot 1 - \frac{i}{4} \cdot 2\pi i \cdot 1 = 0.$

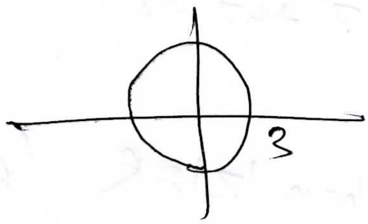
Sol^n

(6)

Let C be $|z|=3$ in anticlockwise direction.

Consider the function

$$g(w) = \int_C \frac{z^3+2z}{(z-w)^3}\, dz$$

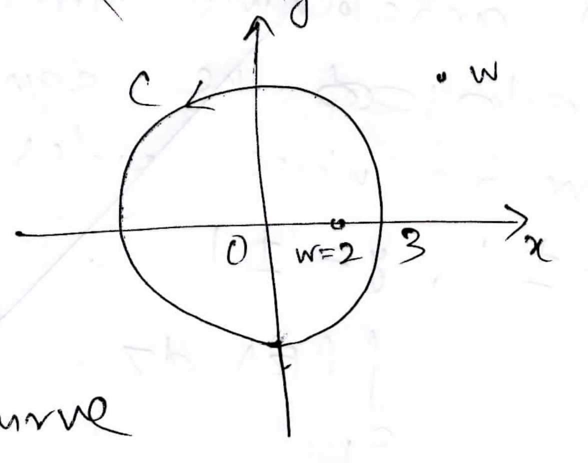


We wish to find $g(w)$ when $w=z_0$ and when $|w|>3$.

We observe that

$$g(2) = \int_C \frac{z^3+2z}{(z-w)^3}\, dz = 2\pi i\left[\frac{z^3+2z}{2}\right]\Bigg|_{z=w} = 2\pi i(4) = 8\pi i$$

On the other hand, when $|w|>3$, By Cauchy's thm $g(w) = 0$.



(11)

Let Γ be a smooth curve

(a) Suppose $\Gamma = [a,b]$. Prove that for any integrable function $f : [a,b] \to \mathbb{C}$

$$\overline{\int_a^b f(t)\, dt} = \int_a^b \overline{f(t)}\, dt$$

What can you deduce about the integrability of $\overline{f}$ if f is integrable?

Sol^n Let $f(t) = u(t) + i v(t)$

$$\overline{\int_a^b f(t)\, dt} = \overline{\int_a^b u(t)\, dt + i\int_a^b v(t)\, dt} = \int_a^b u(t)\, dt - i\int_a^b v(t)\, dt$$

$$= \int_a^b [u(t) - i v(t)]\, dt = \int_a^b \overline{f(t)}\, dt$$

Hence if f is integrable then so is $\overline{f}$.