NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Theory of Differential Equations

Consider a first-order differential equation of one independent variable. The usual form of such an equation is

$$\boxed{\frac{dy}{dx} = y'(x) = f(x, y(x))}, \quad x \geq x_0 .$$

The above form is non-autonomous. If the right hand side has no explicit dependence on the independent variable x, then the form $f(y)$ is autonomous.

The closed-form solution of a simple equation like $\boxed{y'(x) = g(x)}$ can be obtained by integrating the equation.

$$\boxed{y(x) = \int g(x)\, dx + c}, \quad c \text{ is an arbitrary constant.}$$

The constant c can be determined by a condition $\boxed{y(x_0) = y_0}$.

Example: $\boxed{y'(x) = \sin(x)} \Rightarrow y(x) = -\cos(x) + c$

For $y(\pi/3) = 2$ we write $2 = -1/2 + c \Rightarrow c = 5/2$.

$\therefore$ The closed-form solution is $\boxed{y(x) = \frac{5}{2} - \cos(x)}$

Integrating Factors

Consider $\boxed{y'(x) = f(x, y(x)) = a(x)\, y(x) + b(x)}$,

a non-autonomous linear equation in which $a(x)$, $b(x)$ are continuous. This equation can be integrated by the use of an integrating factor.

Special case: $\boxed{a = \text{Constant}}$ $e^{-ax} \to$ integrating factor

$$\Rightarrow e^{-ax}\frac{dy}{dx} = a\, e^{-ax} y + b(x)\, e^{-ax}$$

$$\Rightarrow e^{-ax}\frac{dy}{dx} - a y e^{-ax} = b(x)\, e^{-ax}$$

The above equation has been multiplied throughout by an integrating factor e^{-ax}.

$$\Rightarrow \frac{d}{dx}\left(e^{-ax} y\right) = b(x)\, e^{-ax}$$

$$\Rightarrow y(x)\, e^{-ax} = \int_{x_0}^{x} b(x)\, e^{-ax}\, dx$$

$$\Rightarrow \boxed{y(x) = \left[c + \int_{x_0}^{x} b(x)\, e^{-ax}\, dx \right] e^{ax}}$$ in which

x_0 comes from the initial condition $\boxed{y(x_0) = y_0}$.

When $x = x_0$, $\boxed{c = y(x_0)\, e^{-ax_0}}$, which fixes c from the initial value.

The Initial-Value Problem

$\boxed{Y'(x) = f(x, Y(x))}$, $\boxed{Y(x_0) = Y_0}$, $x \geq x_0$

Example: $\boxed{\frac{dY}{dx} = Y - Y^2}$.

There is a trivial solution $Y(x) = 0$.

$\Rightarrow \int \frac{dY}{Y(1-Y)} = \int dx$

$\frac{1}{Y(1-Y)} \equiv \frac{A}{Y} + \frac{B}{1-Y}$ (partial fractions)

$\Rightarrow 1 \equiv A(1-Y) + BY$. When $Y = 0$, $A = 1$. When $Y = 1$, $B = 1$.

$\Rightarrow \int \frac{dY}{Y} - \int \frac{d(-Y)}{1-Y} = \int dx \Rightarrow \ln Y - \ln(1-Y) = x + \ln A$

$\Rightarrow \frac{Y}{1-Y} = Ae^x \Rightarrow Y = Ae^x - YAe^x$

$\Rightarrow Y(1 + Ae^x) = Ae^x \Rightarrow \boxed{Y = \frac{Ae^x}{1+Ae^x}}$.

$\Rightarrow \boxed{Y = \frac{1}{1+A^{-1}e^{-x}} = \frac{1}{1+Ce^{-x}}}$ $\boxed{A^{-1} = C}$.

When $x \to \infty$, $Y \to 1$. Initial value $\boxed{Y(0) = 4}$.

$\Rightarrow 4 = \frac{1}{1+C} \Rightarrow 1 + C = 0.25 \Rightarrow \boxed{C = -3/4 = -0.75}$.

$\Rightarrow \boxed{\boxed{Y = \frac{1}{1 - 0.75\,e^{-x}}} \;,\; x > 0}$

For a general initial value $\boxed{Y(x_0) = Y_0}$,

$\Rightarrow Y_0 = \frac{1}{1+C} \Rightarrow \boxed{C = \frac{1}{Y_0} - 1}$.

If $Y_0 > 0$, $C > -1$, and hence solution of $Y(x)$ exists for $\boxed{0 \leq x \leq \infty}$.

 (Trivial solution, $y=0$)

Example: $\boxed{\frac{dy}{dx} = -y^2} \Rightarrow \int y^{-2} dy = -\int dx$

$\Rightarrow -y^{-1} = -x - c \Rightarrow \frac{1}{y} = x+c \Rightarrow \boxed{y = \frac{1}{x+c}}$

i.) If $y(0) = 0 = y_0 \Rightarrow 0 = \frac{1}{0+c} \Rightarrow c \to \infty$
$\Rightarrow \boxed{y(x) = 0}$.

ii.) If $y_0 \neq 0 \Rightarrow y_0 = \frac{1}{c} \Rightarrow c = y_0^{-1} \Rightarrow \boxed{y = \frac{1}{x + y_0^{-1}}}$.

iii.) If $y_0 > 0$, solution exists for any $x \geq 0$

Example: $\boxed{\frac{dy}{dx} = \lambda y + e^{-x}}$ λ = constant ($\lambda \neq -1$)

This is in the form $\boxed{y'(x) = ay + b(x)}$.

Initial value is $\boxed{y(0) = 1}$ $b(x) = e^{-x}$

$\lambda \neq -1$

$$\Rightarrow \boxed{y = \left[c + \int e^{-\lambda x} e^{-x} dx\right] e^{\lambda x}}$$

$\boxed{\int e^{-(\lambda+1)x} dx = -\frac{e^{-(\lambda+1)x}}{\lambda+1}}$. $\rightarrow$ The integral.

$\Rightarrow y(x) = \left[c - \frac{e^{-(\lambda+1)x}}{\lambda+1}\right] e^{\lambda x}$. With $y(0) = 1$,

we get $1 = c - \frac{1}{\lambda+1} \Rightarrow \boxed{c = 1 + \frac{1}{\lambda+1}}$

$$\Rightarrow y(x) = \left[1 + \frac{1}{\lambda+1} - \frac{e^{-(\lambda+1)x}}{\lambda+1}\right] e^{\lambda x}$$

$$\Rightarrow \boxed{y(x) = \left[1 + \frac{1}{\lambda+1}\left\{1 - e^{-(\lambda+1)x}\right\}\right] e^{\lambda x}}$$

Examples of Non-Analytical Cases

1/. $\boxed{Y'(x) = \exp(-xY^4)}$ 2/. $\boxed{Y'(x) = 2xY + 1}$ $x > 0$, $Y(0) = 1$

Solvability

Given an initial-value problem

$\boxed{Y'(x) = f(x, Y(x)), \ Y(x_0) = Y_0, \ x \geq x_0}$ if $f(x,z)$ and $\frac{\partial f}{\partial z}$ be continuous functions of x and z at all points (x, z) in the neighbourhood of (x_0, Y_0), then there is a unique function defined on some interval $[x_0 - \alpha, x_0 + \alpha]$ satisfying $\boxed{Y'(x) = f(x, Y)}$, $\boxed{Y(x_0) = Y_0}$ over a range $\boxed{x_0 - \alpha \leq x \leq x_0 + \alpha}$.

With a continuous function $b(x)$ in $\boxed{Y'(x) = \lambda Y + b(x)}$ (λ is constant), $\alpha \to \infty$.
Other cases may have bounded intervals.

Example: $\boxed{Y'(x) = 2xY^2}$ $Y(0) = 1$.

$$\Rightarrow \int Y^{-2}\, dY = 2\int x\, dx \Rightarrow \frac{Y^{-1}}{-1} = 2\frac{x^2}{2} - C$$

$$\Rightarrow \boxed{\frac{1}{Y} = C - x^2}$$

When $x = 0$, $Y = 1$. $\Rightarrow C = 1$.

$$\Rightarrow \boxed{Y = \frac{1}{1 - x^2}} \quad \text{(bounded interval)}.$$

$\boxed{f(x, z) = 2xz^2}$ and $\boxed{\frac{\partial f}{\partial z} = 4xz}$. Both $f(x,z)$ and $\partial f/\partial z$ are continuous, but the integral has a discontinuity at $x = \pm 1$.

Stability of the Initial-Value Problem

Given an initial-value problem $\boxed{Y'(x) = f(x, Y)}$ $\boxed{Y(x_0) = Y_0}$, make a <u>small change</u> in the initial condition, $\boxed{Y_0 \to Y_0 + \epsilon}$

Hence the new solution $Y_\epsilon(x)$ is given by

$\boxed{Y_\epsilon'(x) = f(x, Y_\epsilon(x))}$, $\boxed{Y_\epsilon(x_0) = Y_0 + \epsilon}$ $(x_0 \leq x \leq b)$

A desirable property for stability is that a small change in the initial value Y_0 leads to small changes in the solution $Y(x)$.

<u>Example:</u> $\boxed{Y'(x) = 1 - Y}$ $\underline{Y(0) = 1}$

$\Rightarrow \int \frac{d(-Y)}{1-Y} = -\int dx \Rightarrow \boxed{\ln(1-Y) = -x + \ln C}$

$\Rightarrow \boxed{1 - Y = Ce^{-x}}$ when $\boxed{x = 0, Y = 1}$ $\Rightarrow C = 0 \Rightarrow \boxed{Y = 1}$

Now $\boxed{Y_\epsilon(0) = 1 + \epsilon}$ $\Rightarrow \boxed{Y_\epsilon'(x) = 1 - Y_\epsilon(x)}$

$\Rightarrow \boxed{1 - Y_\epsilon = Ce^{-x}}$. When $\boxed{x = 0, Y_\epsilon(0) = 1 + \epsilon}$

$\Rightarrow 1 - 1 - \epsilon = C \Rightarrow \boxed{C = -\epsilon}$ $\therefore \boxed{Y_\epsilon = 1 + \epsilon e^{-x}}$

Now $\boxed{Y - Y_\epsilon = -\epsilon e^{-x}}$ As <u>x becomes large</u> <u>$Y_\epsilon \to Y$</u>. Hence, the initial-value problem is stable.

> If the difference between the two solution becomes large, then the problem is <u>ill-conditioned</u>.

Example: $\boxed{y'(x) = \lambda(y(x)-1)}$ $\boxed{y(0)=1}$ $(0 \le x \le b)$

$\Rightarrow \int \frac{dy}{y-1} = \lambda \int dx \Rightarrow \ln(y-1) = \lambda x + \ln c.$

$\Rightarrow y-1 = c e^{\lambda x} \Rightarrow \boxed{y = 1 + c e^{\lambda x}}.$

When $x=0$, $y=1$, $\Rightarrow \boxed{c=0} \Rightarrow \boxed{y=1}$ for all x.

Now $\boxed{y'_\epsilon(x) = \lambda[y_\epsilon(x) - 1]} \Rightarrow y_\epsilon = 1 + c e^{\lambda x}$

When $x=0$, $y_\epsilon(0) = 1+\epsilon \Rightarrow 1+\epsilon = 1+c$

$\Rightarrow \boxed{c = \epsilon}$.

$\Rightarrow y_\epsilon = 1 + \epsilon e^{\lambda x} \Rightarrow \boxed{y - y_\epsilon = -\epsilon e^{\lambda x}}$

As x increases stability is ensured if $\lambda < 0$.

If $\lambda > 0$, the problem is ill-conditioned.

General Formulation of Stability

$$\boxed{y(x) - y_\epsilon(x) \simeq -\epsilon \exp\left[\int_{x_0}^{x} g(t)\,dt\right]}$$

where $\boxed{y'(x) = f(x,y)}$, $f(x,z)$ in which $y \equiv z$.

and $\boxed{g(t) = \frac{\partial f(t,z)}{\partial z}\Big|_{z=y(t)}}$ Integral over $t(x)$, Derivative over $z(y)$.

For $f(x,z) = \lambda(z-1)$ (in the previous example), $\frac{\partial f}{\partial z} = \lambda$.

$\Rightarrow g(t) = \lambda \Rightarrow \int_0^x g(t) = \int_0^x \lambda\, dt = \lambda x.$

$\Rightarrow y(x) - y_\epsilon(x) \simeq -\epsilon e^{\lambda x}$

(This result was found earlier)

Example: $\boxed{Y'(x) = -Y^2}$, $Y(0) = 1$

$$\Rightarrow \int Y^{-2}\,dY = \int dx \Rightarrow -Y^{-1} = -x - C \Rightarrow \boxed{Y = \frac{1}{x+C}}$$

When $x = 0$, $Y = 1$. $\Rightarrow C = 1$ $\therefore \boxed{Y = \frac{1}{x+1}}$.

New initial value is $Y_\epsilon(0) = 1+\epsilon$,

and $Y_\epsilon'(x) = -[Y_\epsilon(x)]^2$. $f(x,z) = -z^2$.

$$\Rightarrow \boxed{\frac{\partial f}{\partial z} = -2z} \Rightarrow \boxed{g(t) = -2Y = \frac{-2}{t+1}} \quad \left(g(t) = \frac{\partial f}{\partial z}\right)$$

Hence, $$\int_0^x g(t)\,dt = -2\int_0^x \frac{dt}{t+1} = -2\ln(t+1)\Big|_0^x$$

$$= -2\ln(1+x) = \ln(1+x)^{-2}.$$

Hence, $$\exp\left[\int_0^x g(t)\,dt\right] = e^{\ln(1+x)^{-2}} = \frac{1}{(1+x)^2}$$

$\therefore \boxed{Y(x) - Y_\epsilon(x) \simeq -\frac{\epsilon}{(1+x)^2}}$ for $x \geq 0$. This result shows stability.

In general, if $\boxed{\frac{\partial f}{\partial z} \leq 0}$, $x_0 \leq x \leq b$, then the initial-value problem is stable and well-conditioned.

Euler's Method

Given $\boxed{Y'(x) = f(x,Y),\ Y(x_0) = Y_0,\ x_0 \leq x \leq b}$.

obtain approximate solutions $y(x)$ at a set of discrete nodes $x_0 < x_1 < x_2 < \cdots < x_N \leq b$.

Define $\boxed{x_n = x_0 + nh}$, $n = 0, 1, \ldots, N$ (evenly spaced).

At the nodes, the approximate solutions are $\underline{y_n \equiv y(x_n)}$. The derivative is

$$\boxed{y'(x) \simeq \frac{y(x+h) - y(x)}{h} = D_h(y)}$$

But $y'(x_n) = f(x_n, y(x_n))$

$$\Rightarrow \quad \frac{y(x_{n+1}) - y(x_n)}{h} \simeq f(x_n, y(x_n))$$

$$\Rightarrow \quad y(x_{n+1}) \simeq y(x_n) + h f(x_n, y(x_n))$$

<u>Note:</u> $x_{n+1} = x_0 + (n+1)h$ and $x_n = x_0 + nh$

$\Rightarrow x_{n+1} - x_n = h \Rightarrow \boxed{x_{n+1} = x_n + h}$

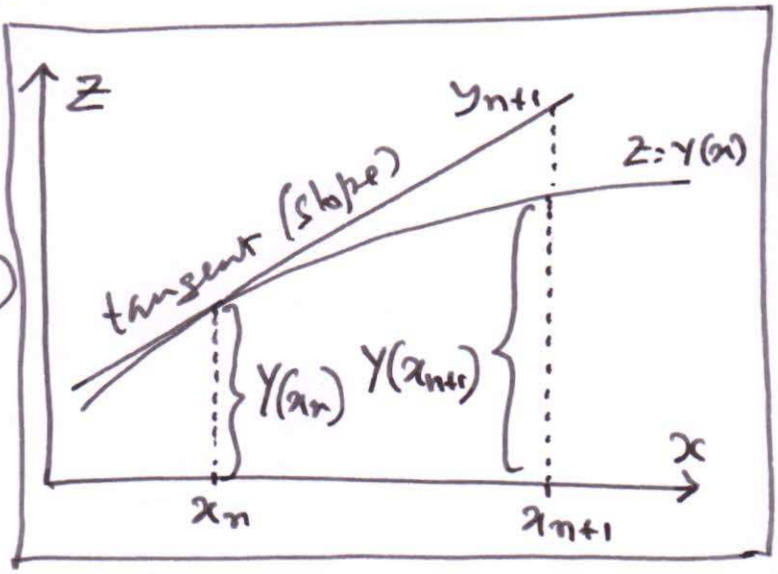
The numerical solution is approximated as $\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$ $\quad 0 \le n \le N-1$

The initial $y_0 = Y_0$ is determined by a close guess approximation.

Define $\boxed{z = Y(x)}$.

$$Y'(x_n) = \underbrace{f(x_n, Y(x_n))}_{\text{(slope at } x_n)}$$

$$\therefore \frac{y(x_{n+1}) - y(x_n)}{h} = f(x_n, Y(x_n))$$



Example: $\boxed{Y'(x) = -Y(x)}$ $\quad Y(0) = 1$

$$\Rightarrow \int \frac{dY}{Y} = -\int dx \Rightarrow \boxed{Y(x) = ce^{-x}}$$

when $x = 0$, $Y = 1$ $\Rightarrow \boxed{C = 1}$.

Now $\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$ $\quad x_0 = 0,\ y_0 = 1$. $\Rightarrow \boxed{x_n = nh}$

$\Rightarrow \boxed{y_{n+1} = y_n - hy_n}$ choose $h = 0.1$.

$\Rightarrow y_1 = 1 - 0.1 \times 1 = 0.9$ at $x_1 = 0.1$.

$y_2 = 0.9 - 0.1 \times 0.9 = 0.81$ at $x_2 = 0.2$.

Error $(x_1) = e^{-0.1} - 0.9 = 0.9048 - 0.9 = 4.837 \times 10^{-3}$

Error $(x_2) = e^{-0.2} - 0.81 = 0.8187 - 0.81 = 8.731 \times 10^{-3}$

Example: $\boxed{Y'(x) = \frac{Y(x) + x^2 - 2}{x + 1}}$, $Y(0) = 2$

$$\Rightarrow \frac{dY}{dx} = \frac{Y-1}{x+1} + \frac{x^2 - 1}{x+1} = \frac{Y-1}{x+1} + x - 1$$

define $\boxed{u = Y - 1}$, $\boxed{v = x + 1}$ $\Rightarrow \boxed{\frac{dY}{dx} = \frac{du}{dv} = \frac{u}{v} + (v - 2)}$

Further define $z = uv^{-1} \Rightarrow v^{-1}\frac{du}{dv} + u \times \frac{-1}{v^2} = \frac{dz}{dv}$

$$\Rightarrow \frac{du}{dv} - \frac{u}{v} = v\frac{dz}{dv} = v - 2 \Rightarrow \boxed{\frac{dz}{dv} = 1 - \frac{2}{v}}$$

$$\Rightarrow z = v - 2\ln v + c \Rightarrow \frac{u}{v} = v - 2\ln v + c$$

$$\Rightarrow Y - 1 = v^2 - 2v\ln v + cv = (x+1)^2 + 2(x+1)\ln(x+1) + c(x+1)$$

when $x = 0$, $Y = 2$ $\Rightarrow 0 = 1 + 2 \times 1 \times 0 + c \times 1 \Rightarrow c = 0$.

$\Rightarrow \boxed{Y(x) = x^2 + 2x + 2 - 2(x+1)\ln(x+1)}$ $\quad \boxed{y_{n+1} = y_n + h\frac{(y_n + x_n^2 - 2)}{x_{n+1}}}$

← (true solution)

Truncation Error

$$Y(x_{n+1}) \simeq Y(x_n) + h f(x_n, Y(x_n))$$

Now $\boxed{x_{n+1} = x_n + h}$ $\Rightarrow Y(x_n+h) \simeq Y(x_n) + h f(x_n, Y(x_n))$

By Taylor expansion we also get,

$$Y(x_{n+1}) = Y(x_n) + Y'(x_n)h + \frac{Y''(x_n)h^2}{2} + \cdots$$

Error $\equiv Y(x_{n+1}) - Y(x_n) - h f(x_n, Y(x_n)) \simeq \frac{Y''(x_n)h^2}{2}$

The truncation error $\simeq \frac{Y''(x_n)h^2}{2} = O(h^2)$

The Backward Euler Method

$$\boxed{Y'(x) \simeq \frac{Y(x+h) - Y(x)}{h}}$$ Transform $h \longrightarrow -h$.

$\Rightarrow Y'(x) \simeq \frac{Y(x-h) - Y(x)}{-h} \Rightarrow \boxed{Y' \simeq \frac{Y(x) - Y(x-h)}{h}}$

Now $Y'(x) = f(x, y) \Rightarrow Y(x) - Y(x-h) \simeq h f(x, Y(x))$

$\Rightarrow y_n - y_{n-1} = h f(x_n, y_n)$ Now $\boxed{\begin{matrix} n \to n+1 \\ n-1 \to n \end{matrix}}$

$\Rightarrow y_{n+1} - y_n = h f(x_{n+1}, y_{n+1})$

$\Rightarrow \boxed{y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})}$ This is an implicit method.

(Backward Euler Method)

$\boxed{\text{The Euler Method is an explicit method.}}$

The Midpoint Method

The Central Difference Formula is

$$\boxed{D_h(f) = \frac{f(x+h) - f(x-h)}{2h}} \qquad \boxed{\begin{aligned} x_{n+1} &= x_n + h \\ x_{n-1} &= x_n - h \end{aligned}}$$

The error is $\boxed{D_h(f) - f'(x) \simeq f'''(x)\frac{h^2}{6}}$

Now $y'(x_n) \simeq \dfrac{y(x_n+h) - y(x_n-h)}{2h} - \dfrac{y'''(x_n)h^2}{6}$

Also $\boxed{y'(x) = f(x, y(x))}$ Using this we get,

$$\frac{y(x_{n+1}) - y(x_{n-1})}{2h} - \frac{y'''(x_n)h^2}{6} \simeq f(x_n, y(x_n))$$

$$\Rightarrow y(x_{n+1}) = y(x_{n-1}) + 2h f(x_n, y(x_n)) + \frac{2y'''(x_n)h^3}{6}$$

Using $\boxed{x_n + h = x_{n+1}}$ and $\boxed{x_{n-1} = x_n - h}$.

The truncation error:

$$y(x_{n+1}) - y(x_{n-1}) - 2h f(x_n, y(x_n)) \simeq \frac{2y'''(x_n)h^3}{6}$$

$$\boxed{\text{The truncation error} = O(h^3)}$$

For numerical integration, we write

$\boxed{y_{n+1} = y_{n-1} + 2h f(x_n, y_n)}$ This gives greater accuracy