SC-216 Calculus with Complex Variables

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Laplace transform (L.T.)

$f(t)$ defined for $t \geq 0$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad \text{—— (1)}$$

Provided Integral exists as it is an improper integral

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \quad \text{(Inverse L.T.)} \quad \text{—— (2)}$$

Properties of L.T.

① Linearity $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$

□ (Verify All !)

② $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \dfrac{1}{s-a} \quad (s > a)$

③ $\mathcal{L}\{1\} = \dfrac{1}{s} \quad (s > 0) \Rightarrow \mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} = 1$

④ $\mathcal{L}\{t\} = \dfrac{1}{s^2} \quad (s > 0) \Rightarrow \mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t$

& so on

⑤ $\mathcal{L}\{\sinh \omega t\} = \dfrac{\omega}{s^2 - \omega^2} \quad (s > \omega)$

⑥ $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2 + a^2}$

— A fn. $f(t)$ is said to be of exponential order α if $\exists\ \alpha$ & $M > 0$ s.t. $|f(t)| \leq M e^{\alpha t}$, $t \geq 0$

— If $f(t)$ is a piecewise cont.s fn. on $[0, \infty)$ & is of exponential order α for $t \geq 0$ then $\mathcal{L}\{f(t)\}$ exists.

Type(2) — Give an example of a fn. $f(t)$ for which $\mathcal{L}\{f(t)\}$ does not exist

L.T. of Derivatives

① $\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)} = sF(s) - f(0)$

□ $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = \left[e^{-st} f(t)\right]_0^\infty$

$+ s\int_0^\infty e^{-st} f(t)\,dt$

$= -f(0) + s\,\mathcal{L}\{f(t)\}$, $s > \alpha$ ▨

② $\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$.

— Using L.T. of derivatives & inverse Laplace transform we can solve O.D.E.

Example $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 6$ (IVP)

□ Taking L.T. of the O.D.E.

$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{0\} = 0$

$\Rightarrow \mathcal{L}\{y''\} + 4\,\mathcal{L}\{y\} = 0$

$\Rightarrow s^2 Y(s) - s\,y(0) - y'(0) + 4Y(s) = 0$

using initial conditions we get

$(s^2+4)\,Y(s) = s + 6$

$\Rightarrow Y(s) = \dfrac{s+6}{s^2+4} = \dfrac{s}{s^2+4} + \dfrac{6}{s^2+4}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\dfrac{6}{s^2+4}\right\}$

$= \cos 2t + 3\sin 2t$ ▨

Using Tables write ~~down~~ them directly.

Exercise: Solve the IVP using L.T.

$$y'' - 5y' + 4y = e^{2t}$$

Ans. $y = -\frac{1}{2}e^{2t} + \frac{14}{9}e^{t} + \frac{19}{31}e^{4t}$

L.T. of Integral

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}}$$ $s>0$, $s>\alpha$

□ The result follows since $\frac{d}{dt}\left\{\int_0^t f(\tau)\,d\tau\right\} = f(t)$

$\int_0^t f(\tau)\,d\tau = \phi(t)$ (say)

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{\phi'(t)\}$$

$$= s\,\mathcal{L}\{\phi(t)\} - \phi(0) = s\,\mathcal{L}\{\phi(t)\}$$

$\phi(0) = 0$

$$\Rightarrow \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$ ■

Example: $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

□ $\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t \frac{1}{3}\sin 3\tau\,d\tau$

$$= \frac{1}{9}(1-\cos 3t)$$

since $\mathcal{L}\left\{\frac{\sin 3t}{3}\right\} = \frac{1}{s^2+9}$ ■

Shifting formula

If $\mathcal{L}\{f(t)\} = F(s)$ $s>\alpha\geq 0$, $\alpha\in\mathbb{R}$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad s > a+\alpha$$

$$\Rightarrow f(t) = e^{-at}\,\mathcal{L}^{-1}\{F(s-a)\} \quad \text{etc.}$$

Heaviside fn.

(Unit step fn.) $H(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t\geq 0 \end{cases}$

$$H(t-a) = u(t-a) := u_a(t) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t\geq a \end{cases}$$

(7) $\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}F(s)$ $\quad s>\alpha,\ a\geq 0$

where $\mathcal{L}\{f(t)\} = F(s)$

(8) $\mathcal{L}\{u_a(t)\} = \dfrac{e^{-as}}{s}$ & $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u_a(t)$

$\boxed{\text{Dirac - Delta fn.}}$

— δ- sequences

(1) $$w_k(x) = \begin{cases} k/2 & , |x| < \frac{1}{k} \\ 0 & , |x| \geq 1/k \end{cases}$$

then $$\int_{-\infty}^{\infty} w_k(x)\,dx = \int_{-1/k}^{1/k} \frac{k}{2}\,dx = 1$$

(2) $$w_k(x) = \frac{k}{\pi(1+k^2x^2)}, \quad k>0$$

then also
$$\int_{-\infty}^{\infty} w_k(x) = \frac{k}{\pi}\int_{-\infty}^{+\infty} \frac{dx}{1+k^2x^2} = 1$$

Now we can define the Dirac Delta fn using these δ- sequences as follows

$$\boxed{\delta(x) = \lim_{k\to\infty} w_k(x)}$$

(3) Let $$\delta_k(t) = \begin{cases} 0 & t<0 \\ 1/k & 0\leq t<k \\ 0 & t\geq k \end{cases}$$

$$\delta_k(t) = \frac{1}{k}\left[u_0(t) - u_k(t)\right]$$
$$= \frac{1}{k}\left[H(t) - H(t-k)\right]$$

Let
$$\delta(t) = \lim_{k\to 0^+} \delta_k(t).$$ $\Rightarrow \delta(t) = 0$ for $t \neq 0$, at $t=0$ it is ∞

If we take $\kappa \to 0$ in ~~the~~ $\delta_\kappa(t)$
we arrive at $\boxed{H'(t) = \delta(t)}$

~~Filte~~ $\boxed{\text{Filtering property of } \delta\text{- fn.}}$

— $\boxed{\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)}$

— $\mathcal{L}\{H'(t)\} = \mathcal{L}\{\delta(t)\} = 1$

— $\mathcal{L}\{t\,f(t)\} = -F'(s)$

— $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$ $\quad s > \alpha$

— $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s^*)\,ds^*$, $s > \alpha$
$\mathcal{L}\{f(t)\} = F(s)$

$\boxed{\text{Convolution Th.}}$

If $f(t)$, $g(t)$ defined in $[0, \infty)$ then
$$(f * g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau \quad (t \geq 0)$$

— $\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}$
$= F(s)\cdot G(s)$

— If $f(t)$ is piecewise cont.s on $[0, \infty)$, is of exp. order & periodic with period ~~@~~ T ~~then~~
$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt, \quad s > 0$$

— Please refer your book for a Table of L.T. of standard fns. ~~a~~ !

$\boxed{\text{Application of L.T. on P.D.E.}}$

— $\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial x}\right\} = \int_0^\infty \frac{\partial u}{\partial x} e^{-st}\, dt$

$$= \frac{\partial}{\partial x}\int_0^\infty e^{-st} u(x,t)\, dt = \frac{d}{dx}\{\mathcal{L}\{u(x,t)\}\}$$

— Also we write $U(x,s) = \mathcal{L}\{u(x,t)\}$

$\boxed{\text{Example}}$ Using L.T. find the soln. of the IVP.

$$x\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = xt, \quad u(x,0)=0, \; u(0,t)=t \qquad \text{— (1)}$$

□ Let $\mathcal{L}\{u(x,t)\} = U(x,s)$

Taking L.T. of (1) w.r. to t & the B.C. $u(0,t)=t$

we get

$$x[sU(x,s) - u(x,0)] + \frac{d}{dx}U(x,s) = \frac{x}{s^2}$$

$$U(0,s) = \frac{1}{s^2}$$

$$\Rightarrow \quad \frac{dU}{dx} + sxU = \frac{x}{s^2}$$

I.F. of the ODE is $e^{sx^2/2}$ ∴ soln

$$e^{sx^2/2}U = \int \frac{x}{s^2} e^{sx^2/2}\, dx + a(s) \;\rightarrow \text{arbitrary fn. of } s$$

$$= \frac{1}{s^3}e^{sx^2/2} + a(s)$$

$$\Rightarrow \quad U(x,s) = \frac{1}{s^3} + a(s)\, e^{-sx^2/2}$$

using $U(0,s) = \frac{1}{s^2}$ we get $a(s) = \frac{1}{s^2} - \frac{1}{s^3}$

$$\Rightarrow U(x,s) = \frac{1}{s^3} + \left(\frac{1}{s^2} - \frac{1}{s^3}\right) e^{-sx^2/2}$$

$$\Rightarrow u(x,t) = \frac{t^2}{2!} + \mathcal{L}^{-1}\left\{\left(\frac{1}{s^2} - \frac{1}{s^3}\right)e^{-sx^2/2}\right\}$$

using shift ~~the~~ theorem

$$u(x,t) = \frac{t^2}{2!} + \left[\left(t - \frac{x^2}{2}\right) - \frac{1}{2}\left(t - \frac{x^2}{2}\right)^2\right] u_{x^2/2}(t)$$

$$= \begin{cases} \frac{t^2}{2}, & t < \frac{x^2}{2} \\ \frac{t^2}{2} + \left(t - \frac{x^2}{2}\right) - \frac{1}{2}\left(t - \frac{x^2}{2}\right)^2, & t \geq \frac{x^2}{2} \end{cases}$$

Simily one ~~w~~ can ~~s~~ use F.T. to solve P.D.E ~~E~~
(See Example 9.36 page 664 of the text book)