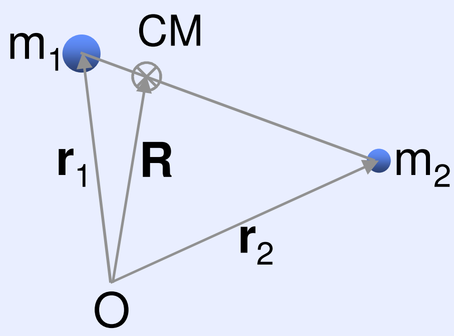
Central Force Motion

The motion of a system consisting of two bodies in the presence of a Central Force.

- Angular momentum is conserved
- 2 body problem → **2 masses** **m_1** & **m_2**: Need **6 coordinates.**
- **Lagrangian for such a system**

center of mass for a two particle system

$$R = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$



The distance of the CM from m_1 and m_2 is in the ratio m_2/m_1.

if $m_1 >> m_2$, then the CM will be very close to m_1.

The Gravitation 2-Body Problem

Two bodies of mass m_1 and m_2, at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. The potential energy is

$$U(\mathbf{r}_1, \mathbf{r}_2) = -\frac{G m_1 m_2}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|}.$$

It depends only on the magnitude $\left|\mathbf{r}_1 - \mathbf{r}_2\right|$

$$U(\mathbf{r}_1, \mathbf{r}_2) = U(\left|\mathbf{r}_1 - \mathbf{r}_2\right|).$$

a new variable, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, which is the position of body 1 relative to body 2.

$$U = U(r).$$

The Gravitation 2-Body Problem

In terms of Lagrangian mechanics, we have for the two-body problem:

$$\mathcal{L} = T - U = \tfrac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\mathbf{r}}_2^2 - U(r).$$

Write r_1 and r_2 in terms of the center of mass R.

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r} \quad \text{and} \quad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}.$$

Reduced Mass

$$T = \tfrac{1}{2} M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\mu\dot{\mathbf{r}}^2.$$

μ for the **reduced mass**: $\boxed{\mu = \dfrac{m_1 m_2}{m_1 + m_2}.}$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

Lagrangian

$$\mathcal{L} = T - U = \tfrac{1}{2} M\dot{\mathbf{R}}^2 + \left(\tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)\right)$$

$$= \mathcal{L}_{\mathrm{CM}} \quad + \quad \mathcal{L}_{\mathrm{rel}}.$$

CM and relative cords → generalized cords which split the problem into two parts.

- Lagrangian for **2 body problem**

$$L = L_{CM} + L_{rel}$$

Transformed the 2 body problem into ***2 one body problems!***

1. **Motion of the CM**

$$L_{CM} \equiv (½)M|\mathbf{R}|^2$$

2. **Relative Motion**

$$L_{rel} \equiv (½)\mu|\mathbf{r}|^2 - U(r)$$

Motion of CM → $L_{CM} \equiv (\frac{1}{2})M|\dot{R}|^2$

- *Assuming no external forces.*

R = (X,Y,Z) ⇒ 3 Lagrange Eqns:

$$(d/dt)(\partial[L_{CM}]/\partial\dot{X}) - (\partial[L_{CM}]/\partial X) = 0$$

$$(\partial[L_{CM}]/\partial X) = 0 \Rightarrow (d/dt)(\partial[L_{CM}]/\partial\dot{X}) = 0$$

$\ddot{X} = 0$, ***CM acts like a free particle!***

→ Solution: $\dot{X} = V_{x0}$ = constant; Determined by initial conditions!

⇒ $X(t) = X_0 + V_{x0}t$, ***exactly like a free particle!***

Similar eqns for **Y, Z**:

⇒ $R(t) = R_0 + V_0 t$, ***exactly like a free particle!***

CM Motion → trivial motion of a free particle.

The Equations of Motion

Lagrangian $$\mathcal{L} = T - U = \tfrac{1}{2} M\dot{\mathbf{R}}^2 + \left(\tfrac{1}{2} \mu\dot{\mathbf{r}}^2 - U(r) \right)$$

What are The equations of motion ?

The CM equation is : $M\ddot{\mathbf{R}} = 0$ or $\dot{\mathbf{R}} = \text{const.}$

The Lagrange equation for the relative position $\mathbf{r}$

$$\mu\ddot{\mathbf{r}} = -\nabla U(r),$$

This is the equation of motion for a single free particle of mass μ (reduced mass) subject to potential energy $U(r)$.

The CM Reference Frame

Since the velocity of the CM is constant, we can change to a frame moving with this constant velocity → alternate inertial frame,

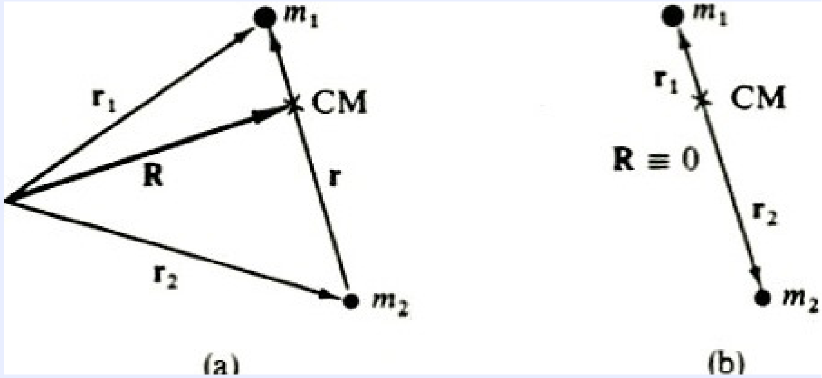
$$\dot{\mathbf{R}} = 0.$$

In the CM frame, the Lagrangian is just

$$\mathcal{L} = \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)$$

and the problem is reduced to a one-body problem.
Kind of pseudo-single body system.

Origin at CM, Path relative to CM



Relative Motion

Relative Motion is

$$L_{rel} \equiv (½)\mu|\dot{r}|^2 - U(r)$$

- *Assuming no external forces. And* $L_{rel} \equiv L$
- ***origin of coordinates at CM:*** $\Longrightarrow$ $R = 0$

$$r_1 = (\mu/m_1)r; \qquad r_2 = -(\mu/m_2)r$$

$$\mu \equiv (m_1 m_2)/(m_1+m_2)$$

$$(\mu)^{-1} \equiv (m_1)^{-1} + (m_2)^{-1}$$

The 2 body, central force problem has been reduced to an ***EQUIVALENT ONE BODY PROBLEM*** in which the motion of a "particle" of mass **μ** in **U(r)** is to be determined! Get **r(t)**, → get $\mathbf{r_1(t)}$ **&** $\mathbf{r_2(t)}$

- **System:** "Particle" of mass **μ** (**μ** $\rightarrow$ **m**) moving in a force field described by potential **U(r)**.
- Now, **conservative Central Forces**:

$$\mathbf{U} \rightarrow \mathbf{V} \qquad\qquad \text{where } \mathbf{V} = \mathbf{V(r)}$$

  - ***V(r)*** *depends only on* $\mathbf{r} = |\mathbf{r}_1 - \mathbf{r}_2|$ *= distance of particle from force center. No orientation dependence.* $\Rightarrow$ ***System has spherical symmetry***
  - *Rotation about any fixed axis can't affect eqns of motion.*
  - ***The angle representing such a rotation be cyclic***
  - ***the corresponding generalized momentum (angular momentum) will be conserved.***

Angular Momentum of the system ?

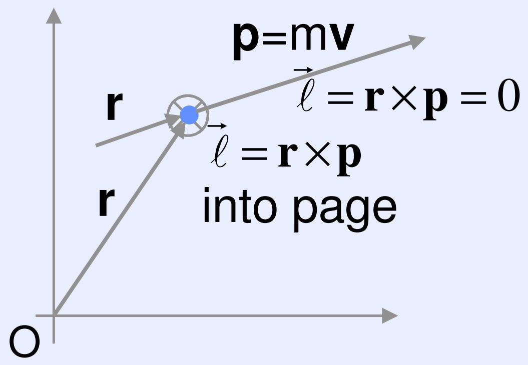
Angular Momentum for a Single Particle

Law of conservation of angular momentum.

The angular momentum $\vec{\ell}$ of a single particle is defined as the vector

$$\vec{\ell} = \mathbf{r} \times \mathbf{p}$$

particle's position vector $\mathbf{r}$, relative to the chosen origin O, and its momentum $\mathbf{p}$.



Angular Momentum of the system ?

Spherical symmetry

The Angular Momentum of the system is conserved:

$\mathbf{L} = \mathbf{r} \times \mathbf{p}$ = constant (magnitude & direction!)

Angular momentum conservation!

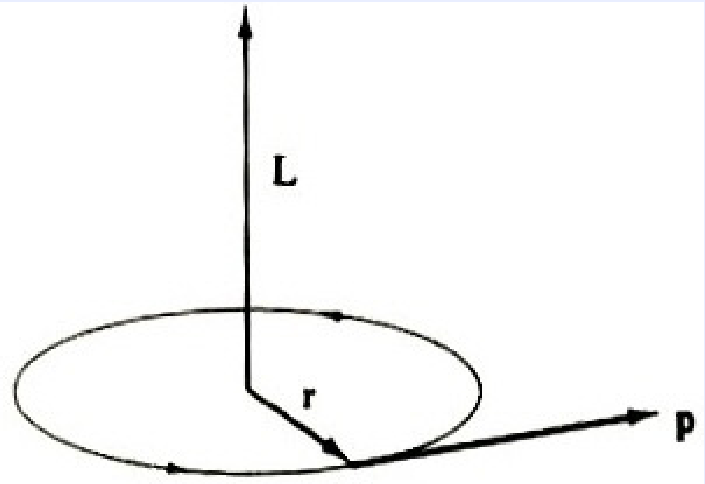
r & **p** (the particle motion!) always **lie in a plane** $\perp$ **L**,

which is fixed in space.

The problem is reduced

from 3d to 2d

(particle motion in a plane)!



Motion in a Plane

3d motion in spherical coordinates **(r,θ,ψ).**

θ = angle in the plane (plane polar coordinates).

ψ **=** azimuthal angle.

L ***is fixed,*** Choose the polar (**z**) axis along **L**.

ψ **= (½)π** ***& drops out of the problem.***

$\Rightarrow$ **The motion is in a plane.** Effectively reducing the 3d problem to a 2d.

Conservation of angular momentum **L**

- Started with **6d, 2 body problem.**
- Reduced it to **2, 3d 1 body problems,** one (CM motion) of which is trivial.
- **Angular momentum conservation** reduces 2nd 3d problem (relative motion) **from 3d to 2d** (motion in a plane)!
- **Lagrangian** ($\mu \rightarrow$ **m** , conservative, central forces):

$$L = (1/2) m |\dot{r}|^2 - V(r)$$

- **Motion in a plane**

plane polar coordinates to do the problem:

$$L = (1/2) m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$$

The Lagrangian is cyclic in $\boldsymbol{\theta}$

$\Rightarrow$ **The generalized momentum $\mathbf{p_\theta}$ is conserved:**

$$\mathbf{p_\theta} \equiv (\partial L/\partial\dot{\theta}) = mr^2\dot{\theta}$$

Lagrange's Eqn: $\quad (d/dt)[(\partial L/\partial\dot{\theta})] - (\partial L/\partial\theta) = 0$

$\Rightarrow \qquad \dot{p}_\theta = 0, \qquad p_\theta = \text{constant} = mr^2\dot{\theta}$

- $\mathbf{p_\theta = mr^2\dot{\theta}}$ = angular momentum about an axis $\perp$ the plane of motion. ***Conservation of angular momentum***!

- The problem symmetry has allowed to integrate one eqn of motion. $\mathbf{p_\theta}$ $\equiv$ a **"1st Integral"** of motion.

  Let us define: $\qquad \ell \equiv \mathbf{p_\theta} \equiv \mathbf{mr^2\dot{\theta}} = \text{constant}.$

  (interpretation!!)

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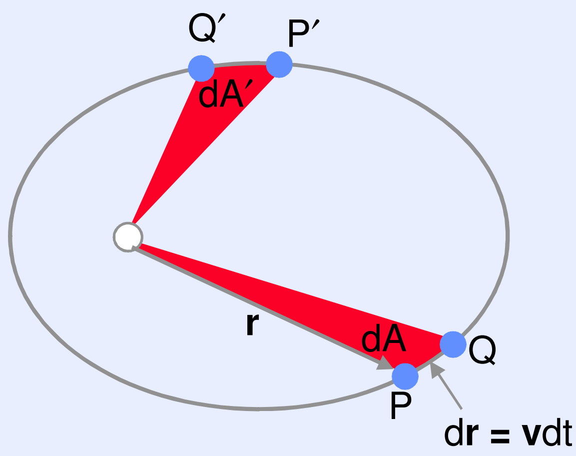
Kepler's Second Law

- Kepler's second law states that

> As each planet moves around the Sun, a line drawn from the planet to the Sun sweeps out equal areas in equal times.

- The two segments of the orbit that can be approximated as triangles (the approximation becomes exact in the limit as the width of the triangles goes to zero).



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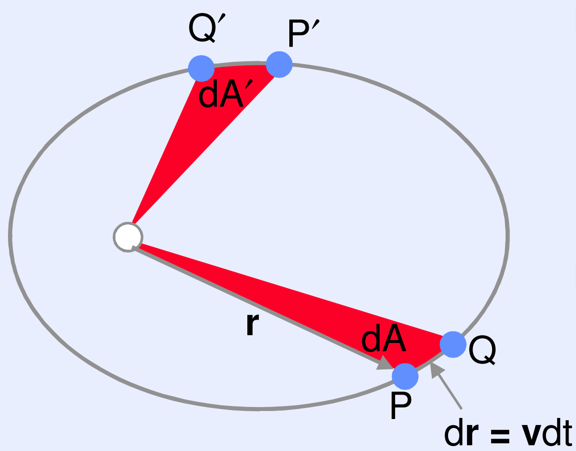
- Kepler's 2nd law is equivalent to saying that so long as the elapsed time dt for the planet to go from P to Q is the same as for it to go from P' to Q', then the areas of these two triangles must be equal. Equivalently, $\mathrm{dA/dt}$ = constant.



- two sides of a triangle are given by vectors $\mathbf{a}$ and $\mathbf{b}$, then the area is $A = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$ (area = ½ base × height). Thus, the area of triangle OPQ is $dA = \frac{1}{2}|\mathbf{r}\times\mathbf{v}dt|$.
- This can be rearranged to get: $\dfrac{dA}{dt} = \dfrac{1}{2m}|\mathbf{r}\times\mathbf{p}| = \dfrac{\ell}{2m}$ since the angular momentum $\ell = \text{constant}$ implies that Kepler's law holds.

$\mathbf{(dA/dt) = (½)(\ell/m)}$ = constant!

$\Rightarrow$ Areal velocity is constant in time!

- First derived empirically by Kepler for planetary motion.

Conservation of areal velocity → ***General result for central forces!***

Not limited to the gravitational force law ($\mathbf{r^{-2}}$).

$$L = (\tfrac{1}{2})m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

Remove θ from this equation.

$$L = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$\ell \equiv mr^2\dot{\theta}$ = constant, the Lagrangian is:

$$L = (½)m\dot{r}^2 + [\ell^2/(2mr^2)] - V(r)$$

- **Symmetry** & the conservation of angular momentum has reduced the effective 2d problem (2 degrees of freedom) to an effective **1d problem!**

1 degree of freedom, one generalized coordinate r!

Solve the problem using the above Lagrangian. ?

Lagrange's Eqtn for r

- In terms of $\ell \equiv mr^2\dot{\theta}$ = const, the **Lagrangian** is:

$$L = (½)m\dot{r}^2 + [\ell^2/(2mr^2)] - V(r)$$

- **Lagrange's Eqtn** for **r**:

$$(d/dt)[(\partial L/\partial \dot{r})] - (\partial L/\partial r) = 0$$

$$\Rightarrow \quad m\ddot{r} - [\ell^2/(mr^3)] = - (\partial V/\partial r) \equiv f(r)$$

(**f(r)** ≡ force along **r**)

Energy Conservation.

Energy

- **Total mechanical energy is also conserved** since the central force is conservative:

$$\mathbf{E = T + V = constant}$$

$$\mathbf{E = (\tfrac{1}{2})m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)}$$

- **angular momentum** is:

$$\ell \equiv \mathbf{mr^2\dot{\theta}} = \text{const}$$

$$\Rightarrow \qquad \mathbf{\dot{\theta} = [\ell/(mr^2)]}$$

$$\Rightarrow \qquad \mathbf{E = (\tfrac{1}{2})m\dot{r}^2 + (\tfrac{1}{2})[\ell^2/(mr^2)] + V(r) =}\text{const}$$

Another **"1st integral"** of the motion

r(t) & θ(t)

$$E = (½)m\dot{r}^2 + [\ell^2/(2mr^2)] + V(r) = \text{const}$$

- **Energy Conservation** allows us to get solutions to the eqns of motion in terms of **r(t)** & **θ(t)** and **r(θ)** or **θ(r)** ≡ **The orbit of the particle!**
  - *Eqn of motion to get* ***r(t)****: One degree of freedom*

  ⇒ *Very similar to a 1 d problem!*

- Solve for $\dot{r}$ = **(dr/dt)** :

$$\dot{r} = \pm\left(\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)]\right)^{½}$$

This gives $\dot{r}(r)$, the phase diagram for the relative coordinate & velocity.

Solve for **dt** & formally integrate to to get **r(t).**

Get **θ(t)** in terms of **r(t)** using **conservation of angular momentum. Find θ(r)** .

Get **θ(t)** in terms of **r(t)** using **conservation of angular momentum. Find θ(r)** .

$$\left(\frac{d\theta}{dr}\right) = \pm (\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}$$

Integrating this gives the eqn for the orbit

$$\theta(r) = \pm \int (\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}\, dr$$

- Once the central force is specified, we know **V(r)** & can, in principle, do the integral & get the orbit **θ(r),** or, (if this can be inverted!) **r(θ)**.

⇒ Assuming only a central force law & nothing else:

We have reduced the original 6 d problem of 2 particles to a 2 d problem with only 1 degree of freedom. The solution for the orbit can be obtained simply by doing the above (1d) integral!