For Bhavesh Dharmani PhD Student

SC213

Assignment : 10

B Shukla

Bhavesh Sir, Solution 10

~~Find~~

Topics covered :-

1. Taylor's and Laurent's Series

Q-1 Find Laurent series about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of convergence of each series.

(a) $(z-3)\sin\frac{1}{(z+2)}$; $z=-2$.

(b) $\frac{z-\sin z}{z^3}$; $z=0$

(c) $\frac{z}{(z+1)(z+2)}$; $z=-2$

(d) $\frac{1}{z^2(z-3)^2}$; $z=3$

→ (a) Let $z+2=u \Rightarrow z=u-2$

$$(z-3)\sin\frac{1}{(z+2)} = (u-5)\sin\frac{1}{u}$$

$$= (u-5)\left\{\frac{1}{u}-\frac{1}{3!u^3}+\frac{1}{5!u^5}-\dots\right\}$$

$$= 1-\frac{5}{u}-\frac{1}{3!u^2}+\frac{5}{3!u^3}+\frac{1}{5!u^4}-\dots$$

$$= 1-\frac{5}{z+2}-\frac{1}{6(z+2)^2}+\frac{5}{6(z+2)^3}+\frac{1}{120(z+2)^4}-\dots$$

$z=-2$ is an essential singularity.

The series converges for all values of $z\neq -2$.

(b) $$\frac{z-\sin z}{z^3} = \frac{1}{z^3}\left\{ z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \right)\right\}$$

$$= \frac{1}{z^3}\left\{ \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \dots \right\}$$

$$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \dots$$

$z=0$ is a removable singularity.

The series converges for all values of z.

(c) Let $z+2=u$.

$$\frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-1)u} = \frac{2-u}{u}\cdot\left(\frac{1}{1-u}\right)$$

$$= \frac{2-u}{u}\left\{ 1+u+u^2+u^3+\dots \right\}$$

$$= \frac{2}{u} + 1 + u + u^2 + \dots$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \dots$$

$z=-2$ is a pole of order 1. or simple pole.

The series converges for all values of z s.t.

$0 < |z+2| < 1$

(d) ~~z~~ Let $z-3=u$. Then by the binomial th.,

$$\frac{1}{z^2(z-3)^2} = \frac{1}{u^2(u+3)^2} = \frac{1}{9u^2(1+u/3)^2}$$

$$= \frac{1}{9u^2}\left[1 + (-2)\frac{u}{3} + \frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3 + \dots \right]$$

$$= \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \dots$$

$$= \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4}{243}(z-3) + \dots$$

Q-2 Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent Series valid

for (a) $1 < |z| < 3$ (b) $|z| > 3$; (c) $0 < |z+1| < 2$

(d) $|z| < 1$.

$(1+n)^{-4}$

(a) $f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\left[\frac{1}{(z+1)} - \frac{1}{(z+3)}\right]$

If $|z| > 1$

$\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots\right)$

$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \dots$

If $|z| < 3$.

$\frac{1}{2(z+3)} = \frac{1}{6(1+z/3)} = \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \dots\right)$

$= \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \dots$

Then the reqd. Laurent expansion valid for both $|z| > 1$ and $|z| < 3$ i.e. $1 < |z| < 3$, is

$\dots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \dots$

(b) $|z| > 3$

$\frac{1}{2(z+3)} = \frac{1}{2z(1+3/z)} = \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \dots\right)$

The required Laurent expansion for both $|z| > 1$ and $|z| > 3$ i.e. $|z| > 3$ is by substraction

$\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \dots$

(c) $0 < |z+1| < 2$

$z+1 = u$ then

$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \dots\right)$

$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \dots$

(d) $|z| < 1$

$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \frac{1}{2}(1 - z + z^2 - z^3 + \dots)$

$= \frac{1}{2} - \frac{z}{2} + \frac{z^2}{2} - \frac{z^3}{2} + \dots$

If $|z| < 3$ we have in part (a).

$k(|z|<1) - k(|z|<3) = \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \dots$

Q-9 Show that $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \frac{\pi}{12}$.

$\Rightarrow$ $\cos\theta = \frac{z+z^{-1}}{2} \Rightarrow \cos 3\theta = \frac{e^{3i\theta}+e^{-3i\theta}}{2} = \frac{z^3+z^{-3}}{2}$

$z = e^{i\theta} \Rightarrow dz = iz\,d\theta$

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \oint_c \frac{(z^3+z^{-3})}{2\left(5-4\left(\frac{z+z^{-1}}{2}\right)\right)} \frac{dz}{iz}$$

$$= -\frac{1}{2i} \oint_c \frac{z^6+1}{z^3(2z-1)(z-2)}\, dz$$

poles within c are $z=0$ of order 3 and $z=1/2$.

Residue at $z=0$ is:

$$\lim_{z\to 0} \frac{1}{2!} \frac{d^2}{dz^2}\left[z^3 \cdot \frac{z^6+1}{z^3(2z-1)(z-2)}\right] = \frac{21}{8}$$

Residue at $z=\frac{1}{2}$ is:

$$\lim_{z\to 1/2} \left[(z-1/2) \frac{z^6+1}{z^3(2z-1)(z-2)}\right] = -\frac{65}{24}$$

Then $-\frac{1}{2i}\oint_c \frac{z^6+1}{z^3(2z-1)(z-2)}\, dz = -\frac{1}{2i}(2\pi i)\left[\frac{21}{8} - \frac{65}{24}\right]$

$$= \frac{\pi}{12}.$$

Solved Problems

RESIDUES AND THE RESIDUE THEOREM

1. Let $f(z)$ be analytic inside and on a simple closed curve C except at point a inside C.

(a) Prove that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \qquad \text{where} \qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz, \quad n = 0, \pm 1, \pm 2, \ldots$$

i.e. $f(z)$ can be expanded into a converging Laurent series about $z = a$.

(b) Prove that

$$\oint_C f(z)\,dz = 2\pi i\, a_{-1}$$

(a) This follows from Problem 25 of Chapter 6.

(b) If we let $n = -1$ in the result of (a), we find

$$a_{-1} = \frac{1}{2\pi i}\oint_C f(z)\,dz, \qquad \text{i.e.} \qquad \oint_C f(z)\,dz = 2\pi i\, a_{-1}$$

We call a_{-1} the *residue* of $f(z)$ at $z = a$.

2. Prove the *residue theorem*. If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of points $a, b, c, \ldots$ inside C at which the residues are $a_{-1}, b_{-1}, c_{-1}, \ldots$ respectively, then

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots)$$

i.e. $2\pi i$ times the sum of the residues at all singularities enclosed by C.

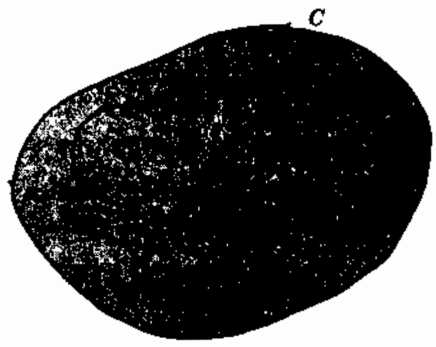

Fig. 7-4

With centres at $a, b, c, \ldots$ respectively construct circles $C_1, C_2, C_3, \ldots$ which lie entirely inside C as shown in Fig. 7-4. This can be done since $a, b, c, \ldots$ are interior points. By Theorem 5, Page 97, we have

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \cdots \tag{1}$$

But by Problem 1,

$$\oint_{C_1} f(z)\,dz = 2\pi i\, a_{-1}, \qquad \oint_{C_2} f(z)\,dz = 2\pi i\, b_{-1}, \qquad \oint_{C_3} f(z)\,dz = 2\pi i\, c_{-1}, \qquad \ldots \tag{2}$$

Then from (1) and (2) we have, as required,

$$\oint_C f(z)\,dz = 2\pi i\,(a_{-1} + b_{-1} + c_{-1} + \cdots) = 2\pi i\,(\text{sum of residues})$$

The proof given here establishes the residue theorem for simply-connected regions containing a finite number of singularities of $f(z)$. It can be extended to regions with infinitely many isolated singularities and to multiply-connected regions (see Problems 96 and 97).

3. Let $f(z)$ be analytic inside and on a simple closed curve C except at a pole a of order m inside C. Prove that the residue of $f(z)$ at a is given by

$$a_{-1} = \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$$

Method 1. If $f(z)$ has a pole a of order m, then the Laurent series of $f(z)$ is

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \tag{1}$$

Then multiplying both sides by $(z-a)^m$, we have

$$(z-a)^m f(z) \quad = \quad a_{-m} + a_{-m+1}(z-a) + \cdots + a_{-1}(z-a)^{m-1} + a_0(z-a)^m + \cdots \qquad (2)$$

This represents the Taylor series about $z=a$ of the analytic function on the left. Differentiating both sides $m-1$ times with respect to z, we have

$$\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} \quad = \quad (m-1)!\,a_{-1} + m(m-1)\cdots 2a_0(z-a) + \cdots$$

Thus on letting $z \to a$,

$$\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} \quad = \quad (m-1)!\,a_{-1}$$

from which the required result follows.

Method 2. The required result also follows directly from Taylor's theorem on noting that the coefficient of $(z-a)^{m-1}$ in the expansion (2) is

$$a_{-1} \quad = \quad \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}\bigg|_{z=a}$$

Method 3. See Problem 28, Chapter 5, Page 132.

4. Find the residues of (a) $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ and (b) $f(z) = e^z\csc^2 z$ at all its poles in the finite plane.

(a) $f(z)$ has a double pole at $z=-1$ and simple poles at $z=\pm 2i$.

Method 1.

Residue at $z=-1$ is

$$\lim_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2\cdot\frac{z^2-2z}{(z+1)^2(z^2+4)}\right\} \quad = \quad \lim_{z\to -1}\frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2} \quad = \quad -\frac{14}{25}$$

Residue at $z=2i$ is

$$\lim_{z\to 2i}\left\{(z-2i)\cdot\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\} \quad = \quad \frac{-4-4i}{(2i+1)^2(4i)} \quad = \quad \frac{7+i}{25}$$

Residue at $z=-2i$ is

$$\lim_{z\to -2i}\left\{(z+2i)\cdot\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\} \quad = \quad \frac{-4+4i}{(-2i+1)^2(-4i)} \quad = \quad \frac{7-i}{25}$$

Method 2.

Residue at $z=2i$ is

$$\lim_{z\to 2i}\left\{\frac{(z-2i)(z^2-2z)}{(z+1)^2(z^2+4)}\right\} \quad = \quad \left\{\lim_{z\to 2i}\frac{z^2-2z}{(z+1)^2}\right\}\left\{\lim_{z\to 2i}\frac{z-2i}{z^2+4}\right\}$$

$$= \quad \frac{-4-4i}{(2i+1)^2}\cdot\lim_{z\to 2i}\frac{1}{2z} \quad = \quad \frac{-4-4i}{(2i+1)^2}\cdot\frac{1}{4i} \quad = \quad \frac{7+i}{25}$$

using L'Hospital's rule. In a similar manner, or by replacing i by $-i$ in the result, we can obtain the residue at $z=-2i$.

(b) $f(z) = e^z\csc^2 z = \dfrac{e^z}{\sin^2 z}$ has double poles at $z = 0, \pm\pi, \pm 2\pi, \ldots$, i.e. $z = m\pi$ where $m = 0, \pm 1, \pm 2, \ldots$.

Method 1.

Residue at $z=m\pi$ is

$$\lim_{z\to m\pi}\frac{1}{1!}\frac{d}{dz}\left\{(z-m\pi)^2\frac{e^z}{\sin^2 z}\right\}$$

$$= \quad \lim_{z\to m\pi}\frac{e^z[(z-m\pi)^2\sin z + 2(z-m\pi)\sin z - 2(z-m\pi)^2\cos z]}{\sin^3 z}$$

Letting $z - m\pi = u$ or $z = u + m\pi$, this limit can be written

$$\lim_{u\to 0} e^{u+m\pi}\left\{\frac{u^2 \sin u \ + \ 2u \sin u \ - \ 2u^2 \cos u}{\sin^3 u}\right\}$$

$$= \quad e^{m\pi}\left\{\lim_{u\to 0}\frac{u^2 \sin u \ + \ 2u \sin u \ - \ 2u^2 \cos u}{\sin^3 u}\right\}$$

The limit in braces can be obtained using L'Hospital's rule. However, it is easier to first note that $\lim_{u\to 0}\frac{u^3}{\sin^3 u} = \lim_{u\to 0}\left(\frac{u}{\sin u}\right)^3 = 1$ and thus write the limit as

$$e^{m\pi}\lim_{u\to 0}\left(\frac{u^2 \sin u \ + \ 2u \sin u \ - \ 2u^2 \cos u}{u^3}\cdot\frac{u^3}{\sin^3 u}\right)$$

$$= \quad e^{m\pi}\lim_{u\to 0}\frac{u^2 \sin u \ + \ 2u \sin u \ - \ 2u^2 \cos u}{u^3} \quad = \quad e^{m\pi}$$

using L'Hospital's rule several times. In evaluating this limit we can instead use the series expansions $\sin u = u - u^3/3! + \cdots$, $\cos u = 1 - u^2/2! + \cdots$.

Method 2 (using Laurent's series).

In this method we expand $f(z) = e^z \csc^2 z$ in a Laurent series about $z = m\pi$ and obtain the coefficient of $1/(z - m\pi)$ as the required residue. To make the calculation easier let $z = u + m\pi$. Then the function to be expanded in a Laurent series about $u = 0$ is $e^{m\pi + u}\csc^2(m\pi + u) = e^{m\pi}e^u\csc^2 u$. Using the Maclaurin expansions for e^u and $\sin u$, we find using long division

$$e^{m\pi}e^u\csc^2 u \quad = \quad \frac{e^{m\pi}\left(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right)}{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right)^2} \quad = \quad \frac{e^{m\pi}\left(1 + u + \frac{u^2}{2} + \cdots\right)}{u^2\left(1 - \frac{u^2}{6} + \frac{u^4}{120} - \cdots\right)^2}$$

$$= \quad \frac{e^{m\pi}\left(1 + u + \frac{u^2}{2!} + \cdots\right)}{u^2\left(1 - \frac{u^2}{3} + \frac{2u^4}{45} + \cdots\right)} \quad = \quad e^{m\pi}\left(\frac{1}{u^2} + \frac{1}{u} + \frac{5}{6} + \frac{u}{3} + \cdots\right)$$

and so the residue is $e^{m\pi}$.

5. Find the residue of $F(z) = \dfrac{\cot z \coth z}{z^3}$ at $z = 0$.

We have as in Method 2 of Problem 4(*b*),

$$F(z) \quad = \quad \frac{\cos z \cosh z}{z^3 \sin z \sinh z} \quad = \quad \frac{\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)}{z^3\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)}$$

$$= \quad \frac{\left(1 - \frac{z^4}{6} + \cdots\right)}{z^5\left(1 - \frac{z^4}{90} + \cdots\right)} \quad = \quad \frac{1}{z^5}\left(1 - \frac{7z^4}{45} + \cdots\right)$$

and so the residue (coefficient of $1/z$) is $-7/45$.

Another method. The result can also be obtained by finding

$$\lim_{z\to 0}\frac{1}{4!}\frac{d^4}{dz^4}\left\{z^5\,\frac{\cos z \cosh z}{z^3 \sin z \sinh z}\right\}$$

but this method is much more laborious than that given above.

6. Evaluate $\dfrac{1}{2\pi i}\displaystyle\oint_C \frac{e^{zt}}{z^2(z^2 + 2z + 2)}\,dz$ around the circle C with equation $|z| = 3$.

The integrand $\dfrac{e^{zt}}{z^2(z^2 + 2z + 2)}$ has a double pole at $z = 0$ and two simple poles at $z = -1 \pm i$ [roots of $z^2 + 2z + 2 = 0$]. All these poles are inside C.

Residue at $z = 0$ is

$$\lim_{z\to 0} \frac{1}{1!}\frac{d}{dz}\left\{z^2 \frac{e^{zt}}{z^2(z^2+2z+2)}\right\} \quad = \quad \lim_{z\to 0} \frac{(z^2+2z+2)(te^{zt}) - (e^{zt})(2z+2)}{(z^2+2z+2)^2} \quad = \quad \frac{t-1}{2}$$

Residue at $z = -1+i$ is

$$\lim_{z\to -1+i}\left\{[z-(-1+i)]\,\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} \quad = \quad \lim_{z\to -1+i}\left\{\frac{e^{zt}}{z^2}\right\} \lim_{z\to -1+i}\left\{\frac{z+1-i}{z^2+2z+2}\right\}$$

$$= \quad \frac{e^{(-1+i)t}}{(-1+i)^2}\cdot\frac{1}{2i} \quad = \quad \frac{e^{(-1+i)t}}{4}$$

Residue at $z = -1-i$ is

$$\lim_{z\to -1-i}\left\{[z-(-1-i)]\,\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} \quad = \quad \frac{e^{(-1-i)t}}{4}$$

Then by the residue theorem

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz \quad = \quad 2\pi i \text{ (sum of residues)}$$

$$= \quad 2\pi i\left\{\frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}\right\}$$

$$= \quad 2\pi i\left\{\frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t\right\}$$

i.e.,

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz \quad = \quad \frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t$$

DEFINITE INTEGRALS OF THE TYPE $\int_{-\infty}^{\infty} F(x)\,dx$

7. If $|F(z)| \leqq M/R^k$ for $z = Re^{i\theta}$ where $k > 1$ and M are constants, prove that $\lim_{R\to\infty}\int_\Gamma F(z)\,dz = 0$ where Γ is the semicircular arc of radius R shown in Fig. 7-5.

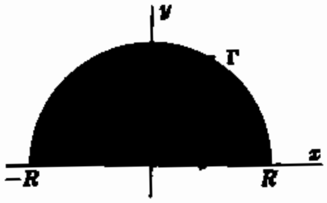

Fig. 7-5

By Property 5, Page 93, we have

$$\left|\int_\Gamma F(z)\,dz\right| \quad \leqq \quad \frac{M}{R^k}\cdot\pi R \quad = \quad \frac{\pi M}{R^{k-1}}$$

since the length of arc $L = \pi R$. Then

$$\lim_{R\to\infty}\left|\int_\Gamma F(z)\,dz\right| \quad = \quad 0 \qquad \text{and so} \qquad \lim_{R\to\infty}\int_\Gamma F(z)\,dz \quad = \quad 0$$

8. Show that for $z = Re^{i\theta}$, $|f(z)| \leqq \dfrac{M}{R^k}$, $k > 1$ if $f(z) = \dfrac{1}{z^6+1}$.

If $z = Re^{i\theta}$, $|f(z)| = \left|\dfrac{1}{R^6e^{6i\theta}+1}\right| \leqq \dfrac{1}{|R^6e^{6i\theta}|-1} = \dfrac{1}{R^6-1} \leqq \dfrac{2}{R^6}$ if R is large enough (say $R > 2$, for example) so that $M = 2$, $k = 6$.

Note that we have made use of the inequality $|z_1+z_2| \geqq |z_1| - |z_2|$ with $z_1 = R^6e^{6i\theta}$ and $z_2 = 1$.

9. Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$.

Consider $\displaystyle\oint_C \frac{dz}{z^6+1}$, where C is the closed contour of Fig. 7-5 consisting of the line from $-R$ to R and the semicircle Γ, traversed in the positive (counterclockwise) sense.

Since $z^6+1=0$ when $z=e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$, these are simple poles of $1/(z^6+1)$. Only the poles $e^{\pi i/6}, e^{3\pi i/6}$ and $e^{5\pi i/6}$ lie within C. Then using L'Hospital's rule,

$$\text{Residue at } e^{\pi i/6} = \lim_{z\to e^{\pi i/6}} \left\{(z-e^{\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/6}$$

$$\text{Residue at } e^{3\pi i/6} = \lim_{z\to e^{3\pi i/6}} \left\{(z-e^{3\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{3\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/2}$$

$$\text{Residue at } e^{5\pi i/6} = \lim_{z\to e^{5\pi i/6}} \left\{(z-e^{5\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-25\pi i/6}$$

Thus
$$\oint_C \frac{dz}{z^6+1} = 2\pi i\left\{\tfrac{1}{6}e^{-5\pi i/6} + \tfrac{1}{6}e^{-5\pi i/2} + \tfrac{1}{6}e^{-25\pi i/6}\right\} = \frac{2\pi}{3}$$

i.e.,
$$\int_{-R}^{R}\frac{dx}{x^6+1} + \int_\Gamma \frac{dz}{z^6+1} = \frac{2\pi}{3} \tag{1}$$

Taking the limit of both sides of (1) as $R\to\infty$ and using Problems 7 and 8, we have

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^6+1} = \int_{-\infty}^{\infty}\frac{dx}{x^6+1} = \frac{2\pi}{3} \tag{2}$$

Since $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x^6+1} = 2\int_0^{\infty}\frac{dx}{x^6+1}$, the required integral has the value $\pi/3$.

10. Show that $\displaystyle\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$.

The poles of $\dfrac{z^2}{(z^2+1)^2(z^2+2z+2)}$ enclosed by the contour C of Fig. 7-5 are $z=i$ of order 2 and $z=-1+i$ of order 1.

Residue at $z=i$ is
$$\lim_{z\to i}\frac{d}{dz}\left\{(z-i)^2\frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\} = \frac{9i-12}{100}.$$

Residue at $z=-1+i$ is
$$\lim_{z\to -1+i}(z+1-i)\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}.$$

Then
$$\oint_C\frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i\left\{\frac{9i-12}{100}+\frac{3-4i}{25}\right\} = \frac{7\pi}{50}$$

or
$$\int_{-R}^{R}\frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} + \int_\Gamma\frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50}$$

Taking the limit as $R\to\infty$ and noting that the second integral approaches zero by Problem 7, we obtain the required result.

DEFINITE INTEGRALS OF THE TYPE $\int_0^{2\pi} G(\sin\theta,\cos\theta)\,d\theta$

11. Evaluate $\displaystyle\int_0^{2\pi}\frac{d\theta}{3-2\cos\theta+\sin\theta}$.

Let $z=e^{i\theta}$. Then $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{z-z^{-1}}{2i}$, $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{z+z^{-1}}{2}$, $dz = iz\,d\theta$ so that

$$\int_0^{2\pi}\frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_C\frac{dz/iz}{3-2(z+z^{-1})/2+(z-z^{-1})/2i} = \oint_C\frac{2\,dz}{(1-2i)z^2+6iz-1-2i}$$

where C is the circle of unit radius with centre at the origin (Fig. 7-6).

The poles of $\dfrac{2}{(1-2i)z^2+6iz-1-2i}$ are the simple poles

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1-2i)(-1-2i)}}{2(1-2i)}$$
$$= \frac{-6i \pm 4i}{2(1-2i)} = 2-i,\ (2-i)/5$$

Only $(2-i)/5$ lies inside C.

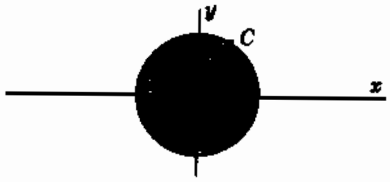

Fig. 7-6

$$\text{Residue at } (2-i)/5 = \lim_{z\to(2-i)/5} \{z-(2-i)/5\}\left\{\frac{2}{(1-2i)z^2+6iz-1-2i}\right\}$$
$$= \lim_{z\to(2-i)/5} \frac{2}{2(1-2i)z+6i} = \frac{1}{2i} \quad \text{by L'Hospital's rule.}$$

Then $$\oint_C \frac{2\,dz}{(1-2i)z^2+6iz-1-2i} = 2\pi i\left(\frac{1}{2i}\right) = \pi, \quad \text{the required value.}$$

12. Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ if $a > |b|$.

Let $z = e^{i\theta}$. Then $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{z-z^{-1}}{2i}$, $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ so that

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \oint_C \frac{dz/iz}{a+b(z-z^{-1})/2i} = \oint_C \frac{2\,dz}{bz^2+2aiz-b}$$

where C is the circle of unit radius with centre at the origin, as shown in Fig. 7-6.

The poles of $\dfrac{2}{bz^2+2aiz-b}$ are obtained by solving $bz^2+2aiz-b=0$ and are given by

$$z = \frac{-2ai \pm \sqrt{-4a^2+4b^2}}{2b} = \frac{-ai \pm \sqrt{a^2-b^2}\,i}{b}$$
$$= \left\{\frac{-a+\sqrt{a^2-b^2}}{b}\right\}i,\ \left\{\frac{-a-\sqrt{a^2-b^2}}{b}\right\}i$$

Only $\dfrac{-a+\sqrt{a^2-b^2}}{b}\,i$ lies inside C, since

$$\left|\frac{-a+\sqrt{a^2-b^2}}{b}\,i\right| = \left|\frac{\sqrt{a^2-b^2}-a}{b}\cdot\frac{\sqrt{a^2-b^2}+a}{\sqrt{a^2-b^2}+a}\right| = \left|\frac{b}{(\sqrt{a^2-b^2}+a)}\right| < 1 \quad \text{if } a > |b|$$

$$\text{Residue at } z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}\,i = \lim_{z\to z_1}(z-z_1)\frac{2}{bz^2+2aiz-b}$$
$$= \lim_{z\to z_1}\frac{2}{2bz+2ai} = \frac{1}{bz_1+ai} = \frac{1}{\sqrt{a^2-b^2}\,i}$$

by L'Hospital's rule.

Then $$\oint_C \frac{2\,dz}{bz^2+2aiz-b} = 2\pi i\left(\frac{1}{\sqrt{a^2-b^2}\,i}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}, \quad \text{the required value.}$$

13. Show that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \frac{\pi}{12}$.

If $z = e^{i\theta}$, then $\cos\theta = \dfrac{z+z^{-1}}{2}$, $\cos 3\theta = \dfrac{e^{3i\theta}+e^{-3i\theta}}{2} = \dfrac{z^3+z^{-3}}{2}$, $dz = iz\,d\theta$ so that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \oint_C \frac{(z^3+z^{-3})/2}{5-4(z+z^{-1})/2}\,\frac{dz}{iz} = -\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)}\,dz$$

where C is the contour of Fig. 7-6.

The integrand has a pole of order 3 at $z=0$ and a simple pole $z=\frac{1}{2}$ inside C.

Residue at $z = 0$ is $\lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\left\{z^3 \cdot \frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = \frac{21}{8}$.

Residue at $z = \frac{1}{2}$ is $\lim_{z\to 1/2}\left\{\left(z-\frac{1}{2}\right)\cdot\frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = -\frac{65}{24}$.

Then $$-\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)}\,dz = -\frac{1}{2i}(2\pi i)\left\{\frac{21}{8}-\frac{65}{24}\right\} = \frac{\pi}{12}$$ as required.

14. Show that $\displaystyle\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2} = \frac{5\pi}{32}$.

Letting $z = e^{i\theta}$, we have $\sin\theta = (z-z^{-1})/2i$, $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ and so

$$\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2} = \oint_C \frac{dz/iz}{\{5-3(z-z^{-1})/2i\}^2} = -\frac{4}{i}\oint_C\frac{z\,dz}{(3z^2-10iz-3)^2}$$

where C is the contour of Fig. 7-6.

The integrand has poles of order 2 at $z = \dfrac{10i\pm\sqrt{-100+36}}{6} = \dfrac{10i\pm 8i}{6} = 3i,\ i/3$. Only the pole $i/3$ lies inside C.

$$\text{Residue at } z = i/3 = \lim_{z\to i/3}\frac{d}{dz}\left\{(z-i/3)^2\cdot\frac{z}{(3z^2-10iz-3)^2}\right\}$$

$$= \lim_{z\to i/3}\frac{d}{dz}\left\{(z-i/3)^2\cdot\frac{z}{(3z-i)^2(z-3i)^2}\right\} = -\frac{5}{256}.$$

Then $$-\frac{4}{i}\oint_C\frac{z\,dz}{(3z^2-10iz-3)^2} = -\frac{4}{i}(2\pi i)\left(\frac{-5}{256}\right) = \frac{5\pi}{32}$$

Another method.

From Problem 12, we have for $a > |b|$,

$$\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Then by differentiating both sides with respect to a (considering b as constant) using Leibnitz's rule, we have

$$\frac{d}{da}\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \int_0^{2\pi}\frac{\partial}{\partial a}\left(\frac{1}{a+b\sin\theta}\right)d\theta = -\int_0^{2\pi}\frac{d\theta}{(a+b\sin\theta)^2}$$

$$= \frac{d}{da}\left(\frac{2\pi}{\sqrt{a^2-b^2}}\right) = \frac{-2\pi a}{(a^2-b^2)^{3/2}}$$

i.e., $$\int_0^{2\pi}\frac{d\theta}{(a+b\sin\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

Letting $a = 5$ and $b = -3$, we have

$$\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2} = \frac{2\pi(5)}{(5^2-3^2)^{3/2}} = \frac{5\pi}{32}$$

DEFINITE INTEGRALS OF THE TYPE $\displaystyle\int_{-\infty}^{\infty} F(x)\begin{Bmatrix}\cos mx\\ \sin mx\end{Bmatrix}dx$

15. If $|F(z)| \leqq \dfrac{M}{R^k}$ for $z = Re^{i\theta}$ where $k > 0$ and M are constants, prove that

$$\lim_{R\to\infty}\int_\Gamma e^{imz}F(z)\,dz = 0$$

where Γ is the semicircular arc of Fig. 7-5 and m is a positive constant.

If $z = Re^{i\theta}$, $\displaystyle\int_\Gamma e^{imz}F(z)\,dz = \int_0^\pi e^{imRe^{i\theta}}F(Re^{i\theta})\,iRe^{i\theta}\,d\theta$. Then

$$\begin{aligned}\left|\int_0^\pi e^{imRe^{i\theta}} F(Re^{i\theta})\, iRe^{i\theta}\, d\theta\right| &\leq \int_0^\pi \left| e^{imRe^{i\theta}} F(Re^{i\theta})\, iRe^{i\theta}\right| d\theta \\ &= \int_0^\pi \left| e^{imR\cos\theta - mR\sin\theta} F(Re^{i\theta})\, iRe^{i\theta}\right| d\theta \\ &= \int_0^\pi e^{-mR\sin\theta}\, |F(Re^{i\theta})|\, R\, d\theta \\ &\leq \frac{M}{R^{k-1}}\int_0^\pi e^{-mR\sin\theta}\, d\theta \quad = \quad \frac{2M}{R^{k-1}}\int_0^{\pi/2} e^{-mR\sin\theta}\, d\theta\end{aligned}$$

Now $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$, as can be seen geometrically from Fig. 7-7 or analytically from Prob. 99.

Then the last integral is less than or equal to

$$\frac{2M}{R^{k-1}}\int_0^{\pi/2} e^{-2mR\theta/\pi}\, d\theta \quad = \quad \frac{\pi M}{mR^k}(1 - e^{-mR})$$

As $R \to \infty$ this approaches zero, since m and k are positive, and the required result is proved.

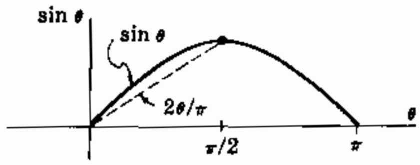

Fig. 7-7

16. Show that $\displaystyle\int_0^\infty \frac{\cos mx}{x^2+1}\, dx \;=\; \frac{\pi}{2}e^{-m}, \quad m > 0.$

Consider $\displaystyle\oint_C \frac{e^{imz}}{z^2+1}\, dz$ where C is the contour of Fig. 7-5. The integrand has simple poles at $z = \pm i$, but only $z = i$ lies inside C.

Residue at $z = i$ is $\displaystyle\lim_{z\to i}\left\{(z-i)\frac{e^{imz}}{(z-i)(z+i)}\right\} = \frac{e^{-m}}{2i}$. Then

$$\oint_C \frac{e^{imz}}{z^2+1}\, dz \quad = \quad 2\pi i\left(\frac{e^{-m}}{2i}\right) \quad = \quad \pi e^{-m}$$

or

$$\int_{-R}^{R} \frac{e^{imx}}{x^2+1}\, dx + \int_\Gamma \frac{e^{imz}}{z^2+1}\, dz \quad = \quad \pi e^{-m}$$

i.e.,

$$\int_{-R}^{R} \frac{\cos mx}{x^2+1}\, dx + i\int_{-R}^{R} \frac{\sin mx}{x^2+1}\, dx + \int_\Gamma \frac{e^{imz}}{z^2+1}\, dz \quad = \quad \pi e^{-m}$$

and so

$$2\int_0^R \frac{\cos mx}{x^2+1}\, dx + \int_\Gamma \frac{e^{imz}}{z^2+1}\, dz \quad = \quad \pi e^{-m}$$

Taking the limit as $R \to \infty$ and using Problem 15 to show that the integral around Γ approaches zero, we obtain the required result.

17. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin\pi x}{x^2+2x+5}\, dx.$

Consider $\displaystyle\oint_C \frac{ze^{i\pi z}}{z^2+2z+5}\, dz$ where C is the contour of Fig. 7-5. The integrand has simple poles at $z = -1 \pm 2i$, but only $z = -1+2i$ lies inside C.

Residue at $z = -1+2i$ is $\displaystyle\lim_{z\to -1+2i}\left\{(z+1-2i)\cdot\frac{ze^{i\pi z}}{z^2+2z+5}\right\} = (-1+2i)\frac{e^{-i\pi-2\pi}}{4i}$. Then

$$\oint_C \frac{ze^{i\pi z}}{z^2+2z+5}\, dz \quad = \quad 2\pi i(-1+2i)\left(\frac{e^{-i\pi-2\pi}}{4i}\right) \quad = \quad \frac{\pi}{2}(1-2i)e^{-2\pi}$$

or

$$\int_{-R}^{R} \frac{xe^{i\pi x}}{x^2+2x+5}\, dx + \int_\Gamma \frac{ze^{i\pi z}}{z^2+2z+5}\, dz \quad = \quad \frac{\pi}{2}(1-2i)e^{-2\pi}$$

i.e.,

$$\int_{-R}^{R} \frac{x\cos\pi x}{x^2+2x+5}\, dx + i\int_{-R}^{R} \frac{x\sin\pi x}{x^2+2x+5}\, dx + \int_\Gamma \frac{ze^{i\pi z}}{z^2+2z+5}\, dz \quad = \quad \frac{\pi}{2}(1-2i)e^{-2\pi}$$

Residue of $\pi \cot \pi z \cdot f(z)$ at $z = n$, $n = 0, \pm 1, \pm 2, \ldots$, is

$$\lim_{z \to n} (z - n)\pi \cot \pi z\, f(z) \quad = \quad \lim_{z \to n} \pi \left(\frac{z-n}{\sin \pi z}\right) \cos \pi z\, f(z) \quad = \quad f(n)$$

using L'Hospital's rule. We have assumed here that $f(z)$ has no poles at $z = n$, since otherwise the given series diverges.

By the residue theorem,

$$\oint_{C_N} \pi \cot \pi z\, f(z)\, dz \quad = \quad \sum_{n=-N}^{N} f(n) \; + \; S \tag{1}$$

where S is the sum of the residues of $\pi \cot \pi z\, f(z)$ at the poles of $f(z)$. By Problem 24 and our assumption on $f(z)$, we have

$$\left| \oint_{C_N} \pi \cot \pi z\, f(z)\, dz \right| \quad \leqq \quad \frac{\pi A M}{N^k}(8N + 4)$$

since the length of path C_N is $8N + 4$. Then taking the limit as $N \to \infty$ we see that

$$\lim_{N \to \infty} \oint_{C_N} \pi \cot \pi z\, f(z)\, dz \quad = \quad 0 \tag{2}$$

Thus from (1) we have as required,

$$\sum_{-\infty}^{\infty} f(n) \quad = \quad -S \tag{3}$$

Case 2: $f(z)$ has infinitely many poles.

If $f(z)$ has an infinite number of poles, we can obtain the required result by an appropriate limiting procedure. See Problem 103.

26. Prove that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a$ where $a > 0$.

Let $f(z) = \dfrac{1}{z^2 + a^2}$ which has simple poles at $z = \pm ai$.

Residue of $\dfrac{\pi \cot \pi z}{z^2 + a^2}$ at $z = ai$ is

$$\lim_{z \to ai} (z - ai) \frac{\pi \cot \pi z}{(z - ai)(z + ai)} \quad = \quad \frac{\pi \cot \pi a i}{2ai} \quad = \quad -\frac{\pi}{2a} \coth \pi a$$

Similarly the residue at $z = -ai$ is $\dfrac{-\pi}{2a} \coth \pi a$, and the sum of the residues is $-\dfrac{\pi}{a} \coth \pi a$. Then by Problem 25,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} \quad = \quad -\text{(sum of residues)} \quad = \quad \frac{\pi}{a} \coth \pi a$$

27. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth \pi a - \frac{1}{2a^2}$ where $a > 0$.

The result of Problem 26 can be written in the form

$$\sum_{n=-\infty}^{-1} \frac{1}{n^2 + a^2} + \frac{1}{a^2} + \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} \quad = \quad \frac{\pi}{a} \coth \pi a$$

or

$$2 \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} + \frac{1}{a^2} \quad = \quad \frac{\pi}{a} \coth \pi a$$

which gives the required result.

31. Prove that if $a \neq 0, \pm 1, \pm 2, \ldots$, then

$$\frac{a^2+1}{(a^2-1)^2} - \frac{a^2+4}{(a^2-4)^2} + \frac{a^2+9}{(a^2-9)^2} - \cdots \quad = \quad \frac{1}{2a^2} - \frac{\pi^2 \cos \pi a}{2 \sin^2 \pi a}$$

The result of Problem 30 can be written in the form

$$\frac{1}{a^2} - \left\{\frac{1}{(a+1)^2} + \frac{1}{(a-1)^2}\right\} + \left\{\frac{1}{(a+2)^2} + \frac{1}{(a-2)^2}\right\} + \cdots \quad = \quad \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$$

or

$$\frac{1}{a^2} - \frac{2(a^2+1)}{(a^2-1)^2} + \frac{2(a^2+4)}{(a^2-4)^2} - \frac{2(a^2+9)}{(a^2-9)^2} + \cdots \quad = \quad \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$$

from which the required result follows. Note that the grouping of terms in the infinite series is permissible since the series is absolutely convergent.

32. Prove that $\dfrac{1}{1^3} - \dfrac{1}{3^3} + \dfrac{1}{5^3} - \dfrac{1}{7^3} + \cdots = \dfrac{\pi^3}{32}$.

We have

$$F(z) \quad = \quad \frac{\pi \sec \pi z}{z^3} \quad = \quad \frac{\pi}{z^3 \cos \pi z} \quad = \quad \frac{\pi}{z^3(1 - \pi^2 z^2/2! + \cdots)}$$

$$= \quad \frac{\pi}{z^3}\left(1 + \frac{\pi^2 z^2}{2} + \cdots\right) \quad = \quad \frac{\pi}{z^3} + \frac{\pi^3}{2z} + \cdots$$

so that the residue at $z = 0$ is $\pi^3/2$.

The residue of $F(z)$ at $z = n + \frac{1}{2}$, $n = 0, \pm 1, \pm 2, \ldots$ [which are the simple poles of $\sec \pi z$], is

$$\lim_{z \to n + \frac{1}{2}} (z - (n + \tfrac{1}{2}))\frac{\pi}{z^3 \cos \pi z} \quad = \quad \frac{\pi}{(n+\frac{1}{2})^3} \lim_{z \to n+\frac{1}{2}} \frac{z - (n+\frac{1}{2})}{\cos \pi z} \quad = \quad \frac{-(-1)^n}{(n+\frac{1}{2})^3}$$

If C_N is a square with vertices at $N(1+i)$, $N(1-i)$, $N(-1+i)$, $N(-1-i)$, then

$$\oint_{C_N} \frac{\pi \sec \pi z}{z^3}\, dz \quad = \quad -\sum_{n=-N}^{N} \frac{(-1)^n}{(n+\frac{1}{2})^3} + \frac{\pi^3}{2} \quad = \quad -8\sum_{n=-N}^{N} \frac{(-1)^n}{(2n+1)^3} + \frac{\pi^3}{2}$$

and since the integral on the left approaches zero as $N \to \infty$, we have

$$\sum_{-\infty}^{\infty} \frac{(-1)^n}{(2n+1)^3} \quad = \quad 2\left\{\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right\} \quad = \quad \frac{\pi^3}{16}$$

from which the required result follows.

MITTAG-LEFFLER'S EXPANSION THEOREM

33. Prove Mittag-Leffler's expansion theorem (see Page 175).

Let $f(z)$ have poles at $z = a_n$, $n = 1, 2, \ldots$, and suppose that $z = \zeta$ is not a pole of $f(z)$. Then the function $\dfrac{f(z)}{z - \zeta}$ has poles at $z = a_n$, $n = 1, 2, 3, \ldots$ and ζ.

Residue of $\dfrac{f(z)}{z-\zeta}$ at $z = a_n$, $n = 1, 2, 3, \ldots$, is $\displaystyle\lim_{z \to a_n} (z - a_n)\frac{f(z)}{z - \zeta} \; = \; \frac{b_n}{a_n - \zeta}$.

Residue of $\dfrac{f(z)}{z-\zeta}$ at $z = \zeta$ is $\displaystyle\lim_{z \to \zeta} (z - \zeta)\frac{f(z)}{z-\zeta} \; = \; f(\zeta)$.

Then by the residue theorem,

$$\frac{1}{2\pi i}\oint_{C_N} \frac{f(z)}{z - \zeta}\, dz \quad = \quad f(\zeta) + \sum_n \frac{b_n}{a_n - \zeta} \qquad (1)$$

where the last summation is taken over all poles inside circle C_N of radius R_N (Fig. 7-14).

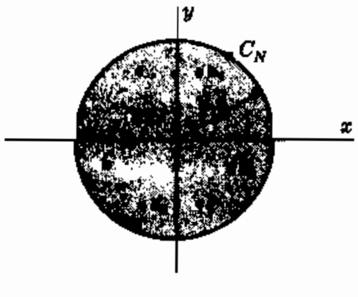

Fig. 7-14

Suppose that $f(z)$ is analytic at $z = 0$. Then putting $\zeta = 0$ in (1), we have

$$\frac{1}{2\pi i}\oint_{C_N} \frac{f(z)}{z}\, dz \quad = \quad f(0) + \sum_n \frac{b_n}{a_n} \qquad (2)$$