$\boxed{\text{Change of Variables}}$ $f(x, y)$ and $x = g(s, t)$
$y = h(s, t)$

by chain rule

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \quad \text{--- (1)}$$

and
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \quad \text{--- (2)}$$

— If we want to determine $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ in terms of $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ then we can solve the equations (1) & (2)

$$\frac{\partial f/\partial x}{\frac{\partial f}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial f}{\partial t}\frac{\partial y}{\partial s}} = \frac{\partial f/\partial y}{\frac{\partial f}{\partial t}\frac{\partial x}{\partial s} - \frac{\partial f}{\partial s}\frac{\partial x}{\partial t}}$$

(How?)

$$= \frac{1}{\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}}$$

— If we define

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \frac{\partial(x, y)}{\partial(s, t)} \quad \text{(say)}$$

$\boxed{\text{Jacobian of the variables}}$

— and
$$\frac{\partial f}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial f}{\partial t}\frac{\partial y}{\partial s} = \frac{\partial(f, y)}{\partial(s, t)} = \begin{vmatrix} \partial f/\partial s & \partial f/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{vmatrix}$$

&
$$\frac{\partial f}{\partial t}\frac{\partial x}{\partial s} - \frac{\partial f}{\partial s}\frac{\partial x}{\partial t} = \frac{\partial(x, f)}{\partial(s, t)} = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial f/\partial s & \partial f/\partial t \end{vmatrix}$$

$$= -\frac{\partial(f, x)}{\partial(s, t)}$$

$\therefore$ we get

$$\boxed{\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y)}{\partial(s, t)}\right] \quad \& \quad \frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x)}{\partial(s, t)}\right]} \quad \text{--- (3)}$$

A similar formula can be obtained for 3-variably

- If $f(x, y, z)$ and $x = F(u, v, w)$

$y = G(u, v, w)$

$z = H(u, v, w)$

Establish a similar formula (Exercise!)

$\boxed{\text{Example}}$ If $z = f(x, y)$, $x = r\cos\theta$

$y = r\sin\theta$

show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

□

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = r$$

$$\frac{\partial(f, y)}{\partial(r, \theta)} = r\cos\theta \frac{\partial f}{\partial r} - \sin\theta \frac{\partial f}{\partial \theta}$$

$$\frac{\partial(f, x)}{\partial(r, \theta)} = -r\sin\theta \frac{\partial f}{\partial r} - \cos\theta \frac{\partial f}{\partial \theta}$$

Using eq^n (3) we get

$$\frac{\partial f}{\partial x} = \cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r} \frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial y} = \sin\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r} \frac{\partial f}{\partial \theta}$$

Substitute in L.h.s. of the expression above we get the result.

Implicit fn.

$F(x, y) = 0$ defines y implicitly as a fn. of x.

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \qquad \text{(chain rule (1))}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}} \qquad \left(\text{provided } \partial F/\partial y \neq 0\right)$$

Example Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

$$\therefore \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

— If z is given implicitly as a fn. $z = f(x, y)$ by an eq^n $F(x, y, z) = 0$ then again using chain rule

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}}$$

Simly

$$\boxed{\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}}$$

Ex.
$y^3 - y = x$

Homogeneous fns

A fn. $f(x,y)$ is called homogeneous of degree n in x & y if it can be written in any one of the following form

① $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

② $f(x,y) = x^n g(y/x)$

③ $f(x,y) = y^n g(x/y)$

Example

$$f(x,y) = x^2 + xy + y^2$$

$$f(\lambda x, \lambda y) = \lambda^2 (x^2 + xy + y^2) = \lambda^2 f(x,y)$$

$\therefore$ f is homo. of deg 2.

Example $f(x,y) = \dfrac{x^2+y}{x+y^2}$

This fn. is not homo.

Euler's Theorem

If $f(x,y)$ is a homo. fn. of degree n in x & y and has conts first & second order partial derivatives then

① $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

② $$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

Since $f(x,y)$ is homo. of deg n

$\Rightarrow f(x,y) = x^n g(y/x)$

$\Rightarrow \dfrac{\partial f}{\partial x} = n x^{n-1} g\left(\dfrac{y}{x}\right) + x^n g'\left(\dfrac{y}{x}\right)\left(-\dfrac{y}{x^2}\right)$

$\dfrac{\partial f}{\partial x} = n x^{n-1} g\left(\dfrac{y}{x}\right) - y x^{n-2} g'(y/x)$

& $\dfrac{\partial f}{\partial y} = x^n g'\left(\dfrac{y}{x}\right)\left(\dfrac{1}{x}\right) = x^{n-1} g'\left(\dfrac{y}{x}\right)$

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n x^n g\left(\frac{y}{x}\right) - y x^{n-1} g'\left(\frac{y}{x}\right) + y x^{n-1} g'\left(\frac{y}{x}\right)$$

$$= n x^n g(y/x) = nf$$

$\therefore$ 1 is proved

Simly verify the 2 (Exercise!) ■

$\boxed{\text{Example}}$ $f(x,y) = xy^2 - x^3$ (Monkey saddle)

$\boxed{\text{Example}}$ $f(x,y) = xy^3 - yx^3$ (dog saddle)

$\boxed{\text{Taylor's Series}}$

$$\int_a^x f'(x)\,dx = f(x) - f(a) \quad --- \quad (1)$$

$$\Rightarrow f(x) = f(a) + \int_a^x f'(x)\,dx \quad ---- \quad (2)$$

replace $f(x)$ by $f'(x)$ in (2)

$$f'(x) = f'(a) + \int_a^x f''(x)\,dx \quad ---- \quad (3)$$

Substitute this in (2) we get

$$f(x) = f(a) + \int_a^x \left\{ f'(a) + \int_a^x f''(x)\,dx \right\} dx$$

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x \int_a^x f''(x)\,dx\,dx \quad ---- \quad (4)$$

Simly if we replace f' in (3) by f'' we get

$$f''(x) = f''(a) + \int_a^x f'''(x)\,dx$$

Put this in (4)

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x \int_a^x \left[f''(a) + \int_a^x f'''(x)\,dx \right] dx\,dx$$

$$\Rightarrow f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \int_a^x \int_a^x \int_a^x f'''(x)\,dx\,dx\,dx$$

& so on

$$\Rightarrow \boxed{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + R_n(x)}$$

where $R_n(x) = \int_a^x \cdots \int_a^x f^{(n)}(x)\,(dx)^n$

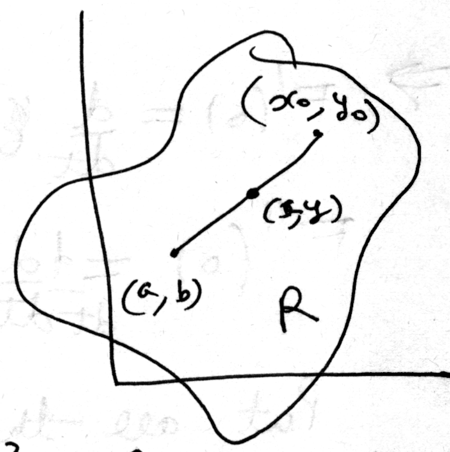
Integral form of Remainder

For single variable

- Knowing all about $f(x)$ at $x=a$ ($f(a), f'(a), f''(a), \ldots$ etc) we tell what $f(x)$ will be at some other pt. x.

Now in 2-variables

- $f(x,y)$ is defined in an open region R in the (x,y) plane & suppose that all the values of f, $f_x, f_y, f_{xx}, f_{xy}, f_{yy}, \ldots$, up to n^{th} order are known at a pt. (a,b) in R. Can we find f at some other pt. (x,y) in R?



YES!

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[f_x(a,b)(x-a) + f_y(a,b)(y-b)\right]$$
$$+ \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right] + \cdots$$

$\boxed{\text{Proof:}}$ Suppose (a,b) is an initial pt. and (x_0,y_0) a final pt. and (x,y) is a generic pt. on the line (see fig)

$$\Rightarrow \quad \begin{aligned} x &= a + (x_0 - a)\,t \\ y &= b + (y_0 - b)\,t \end{aligned} \qquad \text{(Parametric eqn of line)}$$

t is a parameter and for $t=0$ $(x,y) = (a,b)$
& for $t=1$ $(x,y) = (x_0,y_0)$

$$\Rightarrow f(x,y) = f(a + (x_0-a)t,\ b + (y_0-b)t) \equiv F(t)$$

is a fn. of single variable t for $0 \le t \le 1$

$\therefore$ if F is suff. differentiable we can write (using Taylor's formula for single variable)

$$F(t) = F(0) + F'(0)\,t + \frac{F''(0)}{2!}t^2 + \dots + \frac{F^{(n-1)}(0)}{(n-1)!}t^{n-1} + R_n(t) \quad \text{--- (1)}$$

Now by chain rules

$$\frac{d}{dt} = \frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} = (x_0-a)\frac{\partial}{\partial x} + (y_0-b)\frac{\partial}{\partial y} \equiv D \text{ (say)}$$

$$\Rightarrow F'(0) = \frac{d}{dt}F(t)\Big|_{t=0} = Df(x,y)\Big|_{a,b}$$

$$F''(0) = \frac{d}{dt}\frac{d}{dt}F(t)\Big|_{t=0} = D^2 f(x,y)\Big|_{a,b}$$

& so on . . .

Put all this in eq^n (1) & put $t=1$ we get ($\because F(1) = f(x_0,y_0)$)

$$\boxed{f(x_0,y_0) = f(a,b) + \frac{1}{1!}Df\Big|_{a,b} + \dots + \frac{1}{(n-1)!}D^{n-1}f\Big|_{a,b} + R_n(t)}$$

$$Df = \left[(x_0-a)\frac{\partial}{\partial x} + (y_0-b)\frac{\partial}{\partial y}\right]f = (x_0-a)f_x + (y_0-b)f_y$$

$$D^2 f = D(Df)$$

$$= (x_0-a)^2 f_{xx} + 2(x_0-a)(y_0-b)f_{xy} + (y_0-b)^2 f_{yy}$$

etc.

$\Rightarrow$

$$f(x_0, y_0) = f(a,b) + \frac{1}{1!}\left[f_x(a,b)(x-a) + f_y(a,b)(y-b)\right] + \cdots$$

(Here we have replace $x_0 \to x$ & $y_0 \to y$ etc.)

$\boxed{\text{Example}}$ $f(x,y) = e^{xy}$ Find the Taylor's series at $(1,2)$

$f_x = y e^{xy}$ $f_y = x e^{xy}$ $f_{xx} = y^2 e^{xy}$

$f_{xy} = (1+xy)e^{xy}$ & $f_{yy} = x^2 e^{xy}$ etc.

$\Rightarrow$

$$e^{xy}\Big|_{1,2} = e^2 + 2e^2(x-1) + e^2(y-2) + 2e^2(x-1)^2 + 3e^2(x-1)(y-2) + \frac{e^2}{2}(y-2)^2 + \cdots$$

Maxima, Minima & Saddle pt

$$u = f(x_1, x_2, \ldots, x_n)$$

let $\frac{\partial f}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} = 0, \ldots, \frac{\partial f}{\partial x_n} = 0$ at a pt. X in Domain(f). If f is diff. in some nbd. of X

define
$$A = \begin{bmatrix} f_{x_1x_2}(X) & f_{x_1x_2}(X) & \cdots & f_{x_1x_n}(X) \\ f_{x_2x_1}(X) & f_{x_2x_2}(X) & \cdots & f_{x_2x_n}(X) \\ \vdots & & & \vdots \\ f_{x_nx_1}(X) & f_{x_nx_2}(X) & \cdots & f_{x_nx_n}(X) \end{bmatrix}_{n\times n}$$

Let $\det(A) \neq 0$

1. If A is +ve definite then f has a local min. at X
2. If A is −ve " then f has a local max at X
3. If A has at least one +ve & one −ve eigen value then f has a saddle pt. at X.

$\boxed{\text{Example}}$ $\quad f(x,y) = \ln\left[2x(y-1)+1\right]$

Put $f_x = \dfrac{2(y-1)}{2x(y-1)+1} = 0$

& $f_y = \dfrac{2x}{2x(y-1)+1} = 0$ gives critical pt $x=0 \quad y=1$

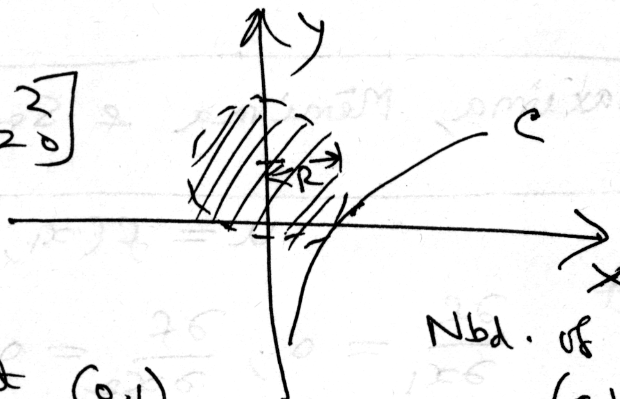
Now $f_{xx} = \dfrac{-4(y-1)^2}{(2x(y-1)+1)^2}$, $f_{yy}(x,y) = \dfrac{-4x^2}{(2x(y-1)+1)^2}$

$f_{xy} = \dfrac{2}{(2x(y-1)+1)^2}$

Partial derivatives are cont.s everywhere in (x,y)-plane except along the curve $2x(y-1)+1=0$ along which they are

Thus f is C^2 in any nbd. about $(0,1)$ of radius R

$A = \begin{bmatrix} f_{xx}(0,1) & f_{xy}(0,1) \\ f_{xy}(0,1) & f_{yy}(0,1) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$



Nbd. of $(0,1)$

$\therefore \lambda = +2, -2$

$\therefore f$ has a saddle at $(0,1)$.

$\boxed{\text{Example}}$ $\quad f(x,y,z) = \sin(x^2+y^2+z^2) + xy + xz + yz$

Verify that critical pt. is $(0,0,0)$

and $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow \lambda = 1, 1, 4$

$\therefore$ all eigen values are +ve

f has a local min. at $(0,0,0)$