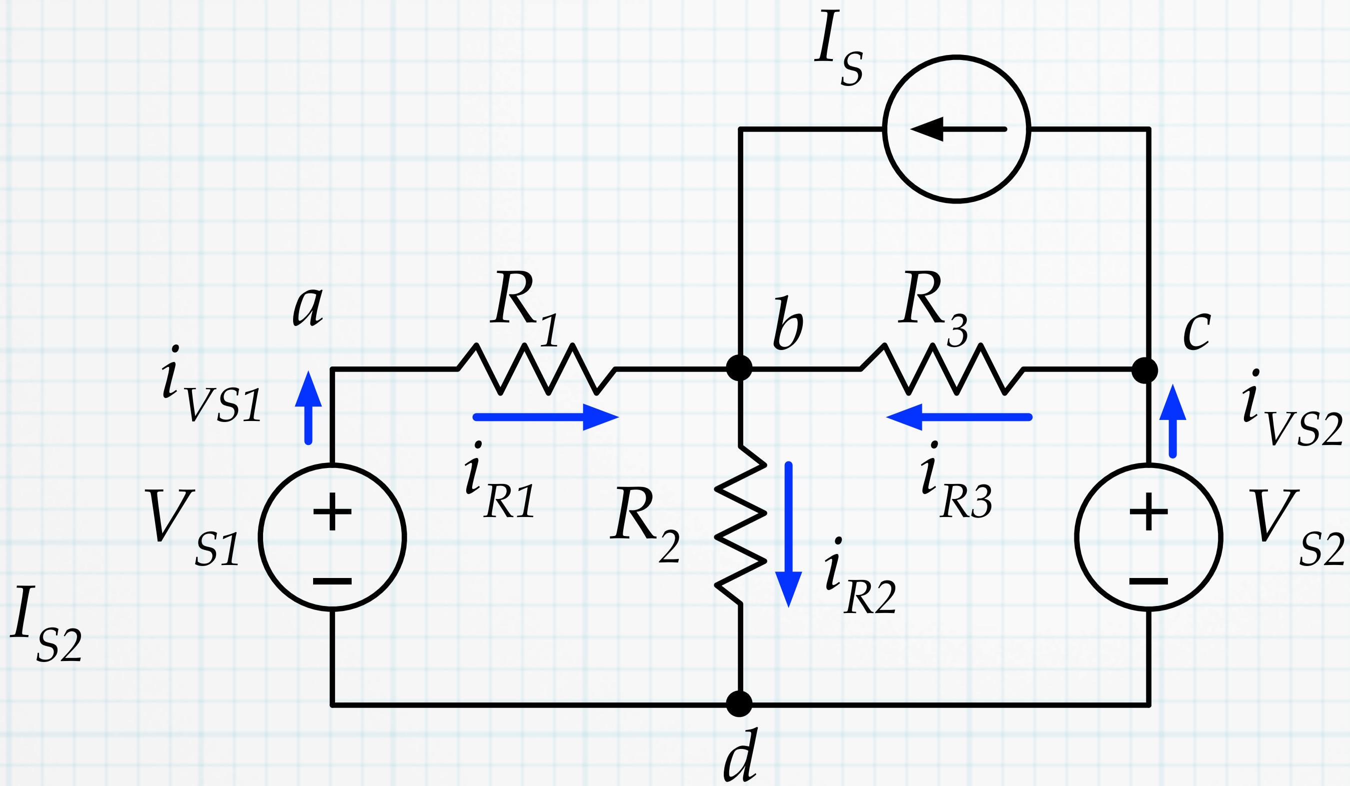
The node voltage method

- Equivalent resistance
- Voltage / current dividers
- Source transformations
- Node voltages
- Mesh currents
- Superposition

Not every circuit lends itself to “short-cut” methods. Sometimes we need a formal approach that does not rely on seeing a trick that can be used. The node-voltage is the first (and maybe most used) of our three formal methods.

The node-voltage method allows for the calculation of the voltages at each node of the circuit, relative to a reference node. Once the node voltages are known, all currents in the circuit can be determined easily. The method leads to a set of simultaneous that must be solved. Bigger circuits will have more nodes and require more equations (more math).

Recall:



Using KCL at each node:

a: $i_{VS1} = i_{R1}$

b: $i_{R1} + i_{R3} + I_S = i_{R2}$

c: $i_{VS2} = i_{R3} + I_S$

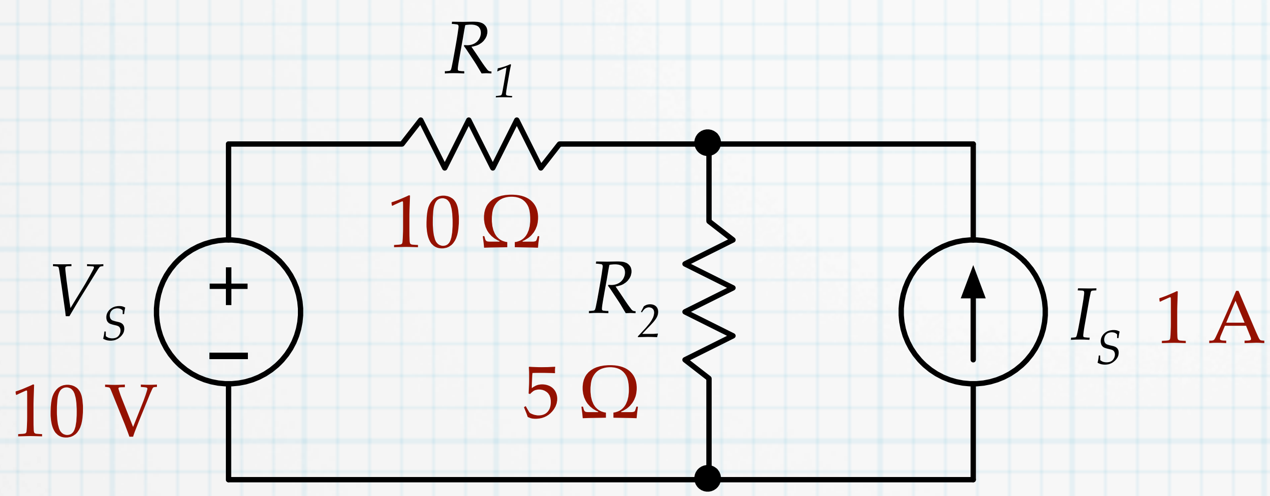
d: $i_{R2} = i_{VS1} + i_{VS2}$

4 equations, but 5 unknowns (i_{VS1}, i_{VS2}, i_{R1}, i_{R2}, and i_{R3}).

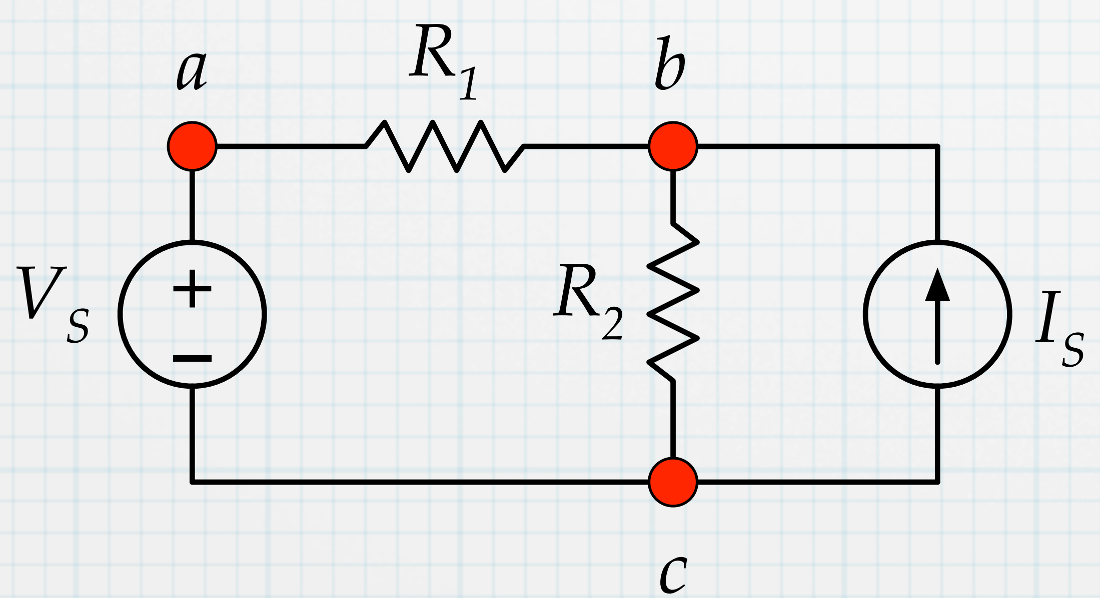
Need more info. Or a better method.

The node voltage method

Look at an example.



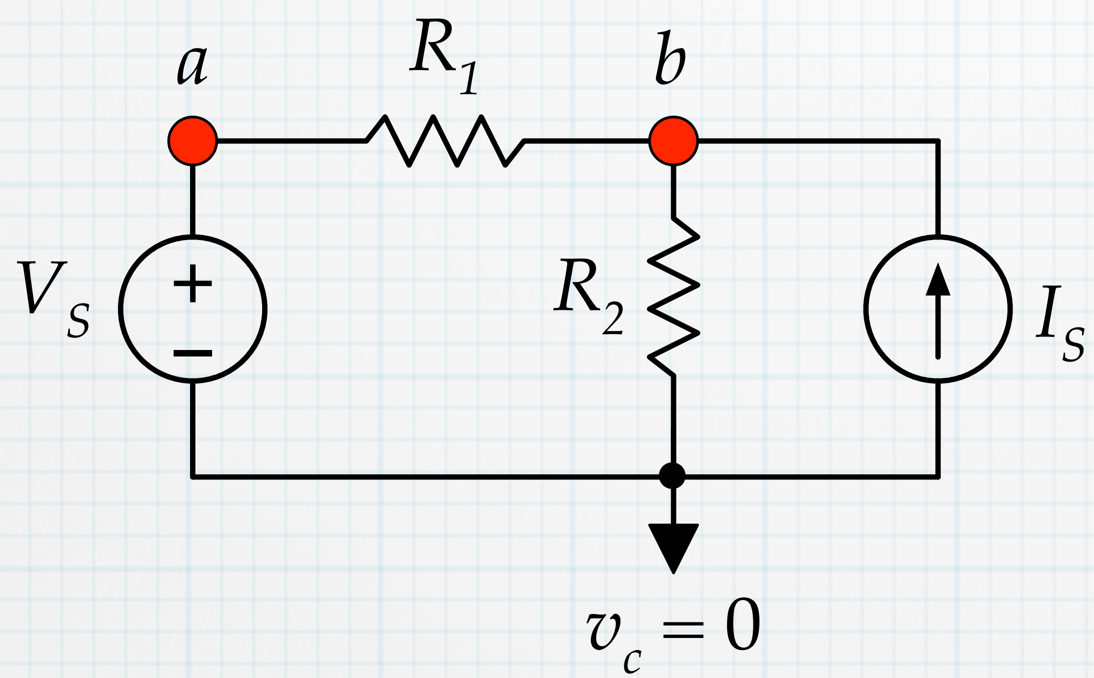
Step 1 - Identify all of the nodes in the circuit.



This small circuit has three nodes. In principle, three unknown voltages, but we will try to reduce this.

Step 2 - Choose one node to be the reference, usually called *ground*.

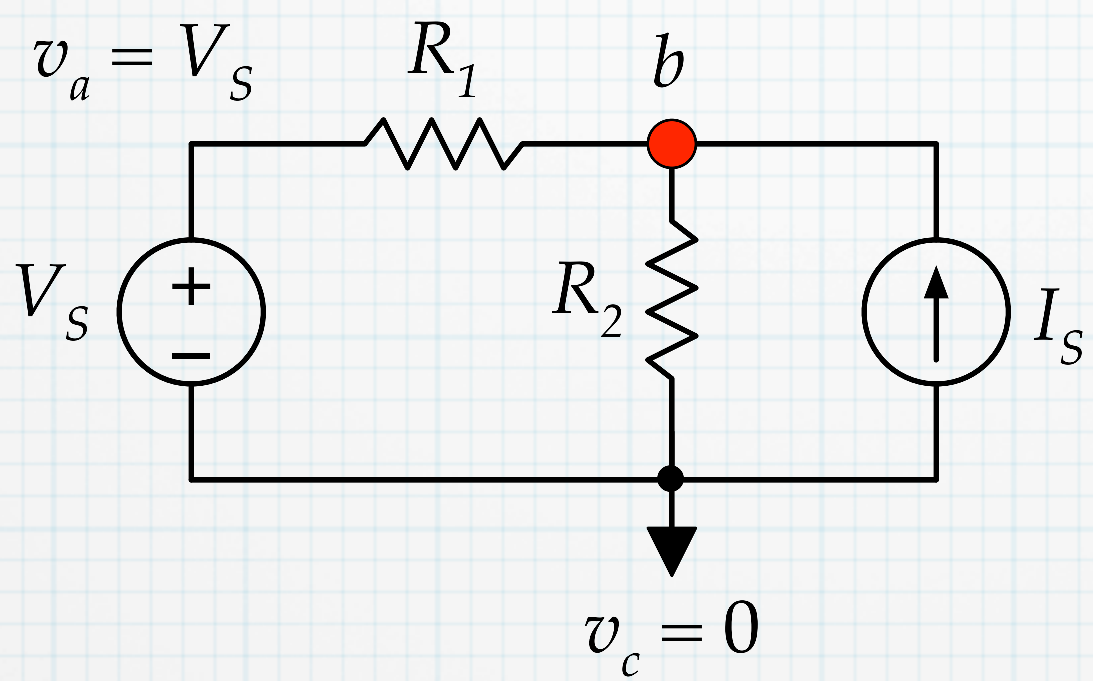
Since voltage is a relative quantity (only voltage differences matter), we can choose one point in the circuit to be $V = 0$. The voltages at the other nodes will be with respect this ground node.



In general, we can choose any node as the reference, but some choices are better than others. Typically, we should choose a node that is connected to the positive or negative terminal of a voltage source.

For this circuit, that implies node *a* or node *c*. This time, we choose node *c*, so we can now say $v_c = 0$.

Step 3 - Identify any other nodes for which the voltages (with respect to ground) are known.



For this circuit, the voltage source tells us that node a is V_S higher in voltage than the ground node. Therefore $v_a = V_S$.

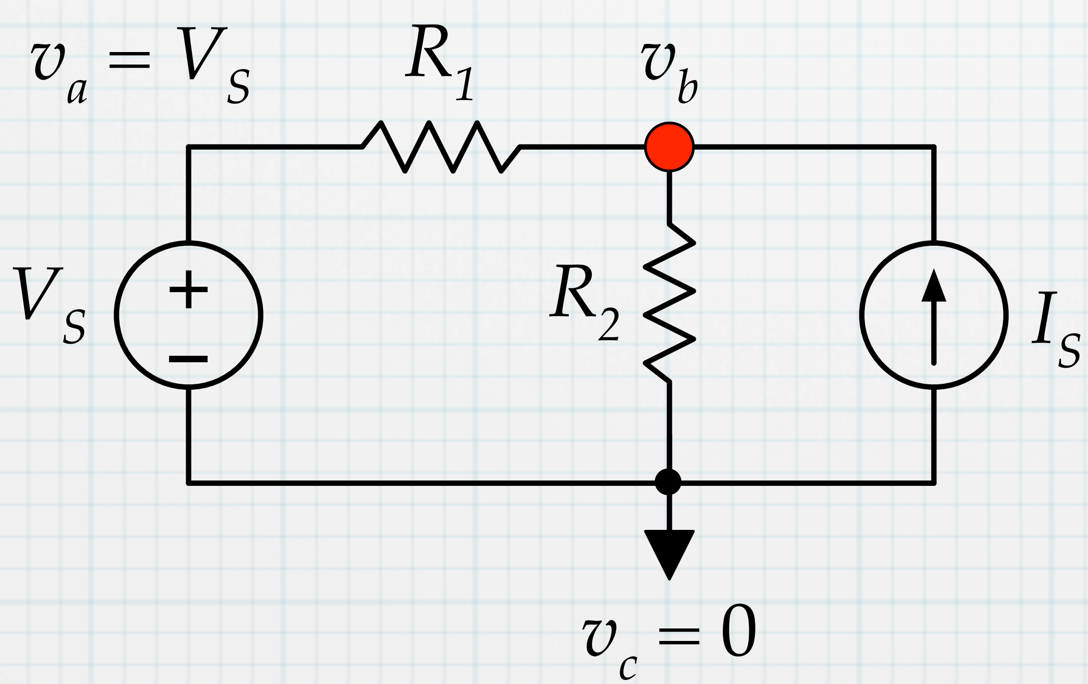
Since the voltage at a is now known, we have reduced the number of unknowns down to one – the voltage at node b.

Step 4 - Look for other ways (like resistor reductions) that could be used to reduce the number of unknown voltages further.

It is not necessary to calculate the voltage at every node. If we can eliminate non-essential nodes, we should do so.

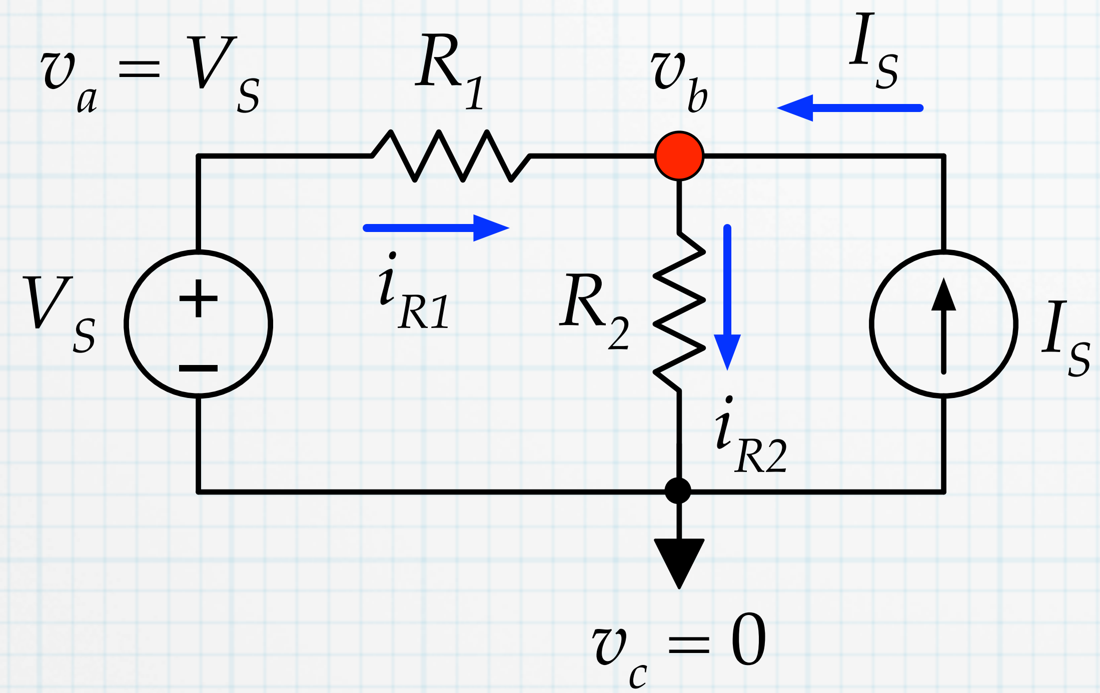
In this example, we are already down to one node – not much more we can do.

Step 5 - Assign voltage variables to each of the remaining unknown nodes.



For this example, there is only one unknown node voltage.

Step 6 - Assign currents to all the branches connected to each of the unknown nodes. You are at liberty to choose whatever directions you want for the current arrows.



Step 7 - Use KCL to write equations balancing the currents at each unknown node.

$$i_{R1} + I_S = i_{R2}$$

Step 8 - Use Ohm's law to express the resistor currents in terms of the node voltages on either side of the resistor. Pay attention to polarity!

$$i_{R1} = \frac{V_S - v_b}{R_1} \qquad i_{R2} = \frac{v_b - 0}{R_2}$$

Step 9 - Substitute the Ohm's law expressions for the resistor currents into the KCL equations to form the *node-voltage equations*.

$$\frac{V_S - v_b}{R_1} + I_S = \frac{v_b}{R_2}$$

The circuit analysis is done. The rest is just math.

Step 10 - Solve the equation(s).

$$V_S - v_b + R_1 I_S = \frac{R_1}{R_2} v_b$$

$$V_S + R_1 I_S = \left(1 + \frac{R_1}{R_2}\right) v_b$$

$$v_b = \frac{V_S + R_1 I_S}{1 + \frac{R_1}{R_2}}$$

$$v_b = \frac{10\text{V} + (10\Omega)(1\text{A})}{\left(1 + \frac{10\Omega}{5\Omega}\right)} = \boxed{6.67\text{V}}$$

Finally, the resistor voltages and currents (using Ohm's law) can be calculated.

$$v_{R1} = V_S - v_b = 10\text{ V} - 6.67\text{ V} = 3.33\text{ V}$$

$$v_{R2} = v_b - 0 = 6.67\text{ V}$$

$$i_{R1} = \frac{v_{R1}}{R_1} = \frac{3.33\text{ V}}{10\Omega} = 0.333\text{ A}$$

$$i_{R2} = \frac{v_{R2}}{R_2} = \frac{6.67\text{ V}}{5\Omega} = 1.33\text{ A}$$

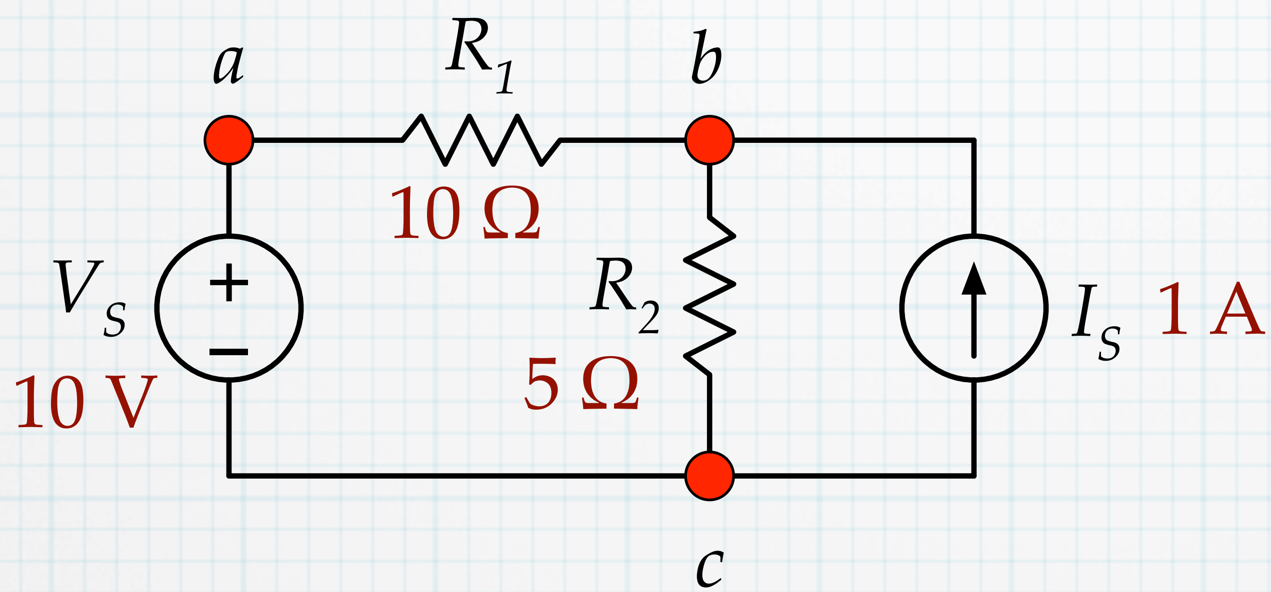
node-voltage method – summary

1. Identify all of the nodes in the circuit.
2. Choose one node to be ground.
3. Identify nodes for which the voltage is known due to sources.
4. Use resistor reductions to eliminate any other non-essential nodes.
5. Assign variables for the voltages at the remaining unknown nodes.
6. Assign currents to all of the branches connected to the nodes.
7. Write KCL equations balancing the currents at each of the nodes.
8. Use Ohm's law to express resistor currents in terms of the (unknown) node voltages on either side of the resistor. (Be sure to get the correct polarity!)
9. Substitute the resistor currents into the KCL equations to form the node-voltage equations. (Set of equations relating the unknown node voltages.)
10. Do the math to solve the equations and determine the node voltages. Determine currents, powers, etc., if needed.

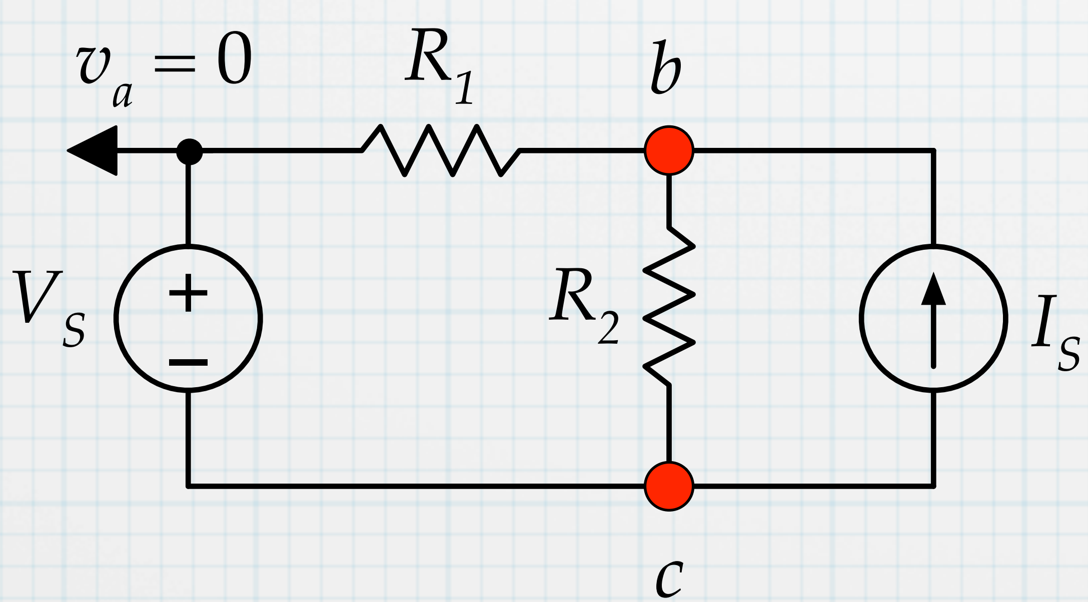
Example

Same circuit, but choose a different ground.

1. Identify the nodes.

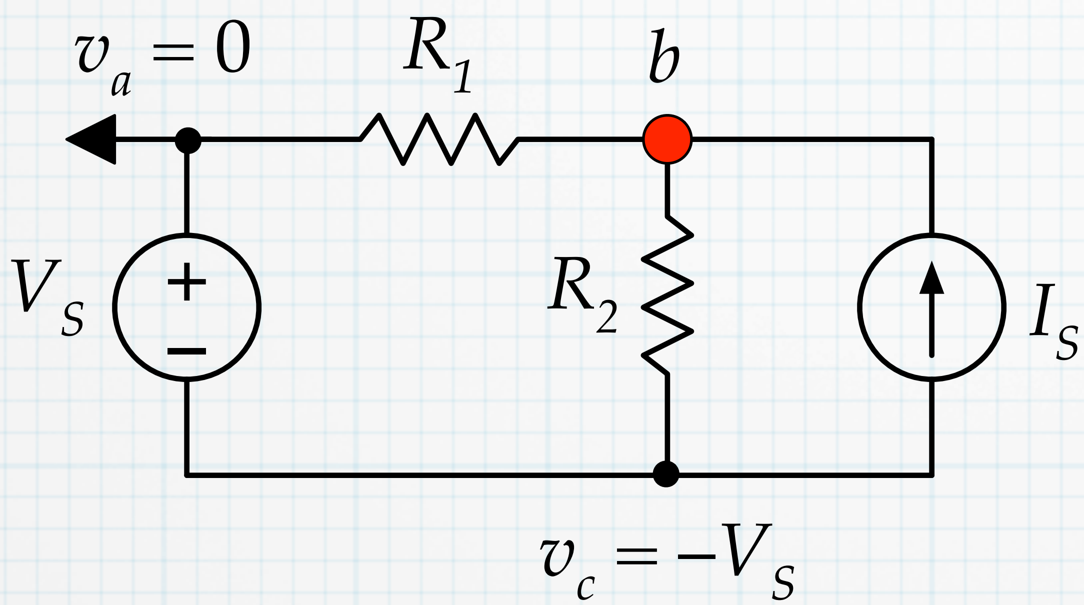


2. Choose one to be ground. This time, choose node *a*.

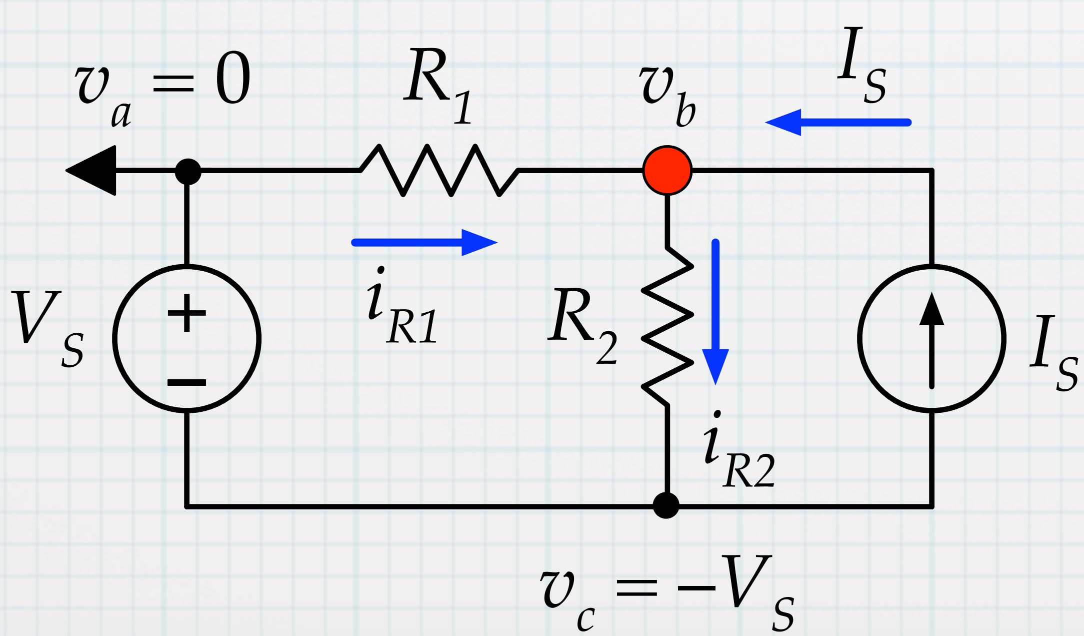


3. Identify nodes for which the voltage is known.

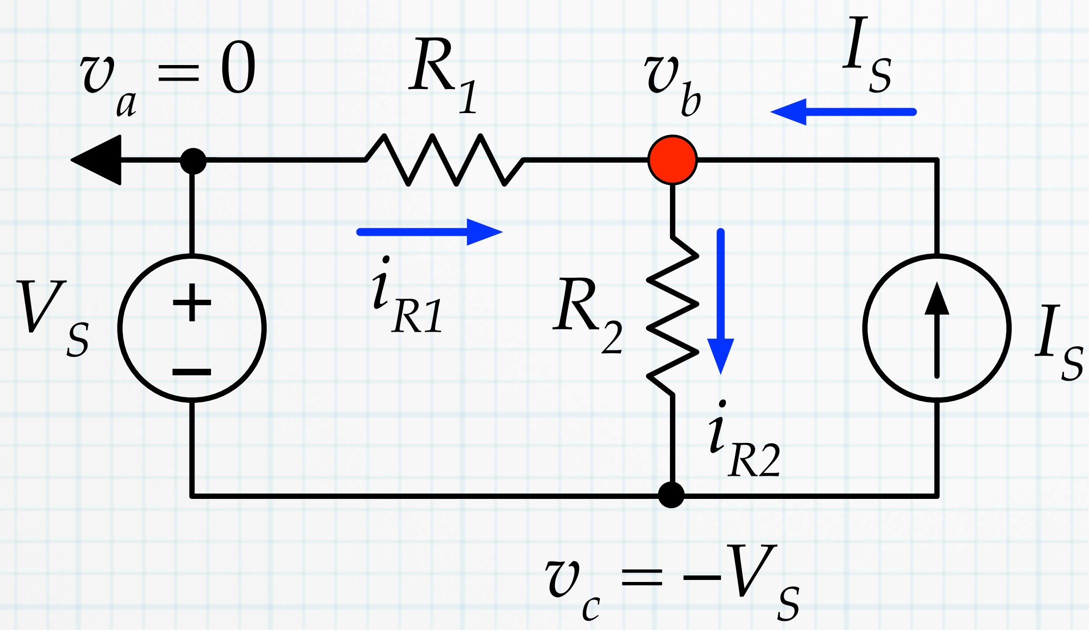
Clearly, $v_a - v_c = V_S$. Therefore, $v_c = -V_S$.



5&6. Assign variables for the unknown voltages. Assign currents for each branch connected to the nodes.



7. Write KCL equations balancing the currents at each node.



$$i_{R1} + I_S = i_{R2}$$

8. Use Ohm's law to express the resistor currents in terms of the node voltages.

$$i_{R1} = \frac{0 - v_b}{R_1} \qquad i_{R2} = \frac{v_b - (-V_S)}{R_2}$$

9. Substitute resistor current equations into the KCL equations to form the node-voltage equations.

$$\frac{-v_b}{R_1} + I_S = \frac{v_b + V_S}{R_2}$$

10. Solve it.

$$\frac{-v_b}{R_1} + I_S = \frac{v_b + V_S}{R_2}$$

$$-v_b \frac{R_2}{R_1} + R_2 I_S = v_b + V_S$$

$$R_2 I_S - V_S = v_b \left(1 + \frac{R_2}{R_1}\right)$$

$$v_b = \frac{R_2 I_S - V_S}{\left(1 + \frac{R_2}{R_1}\right)}$$

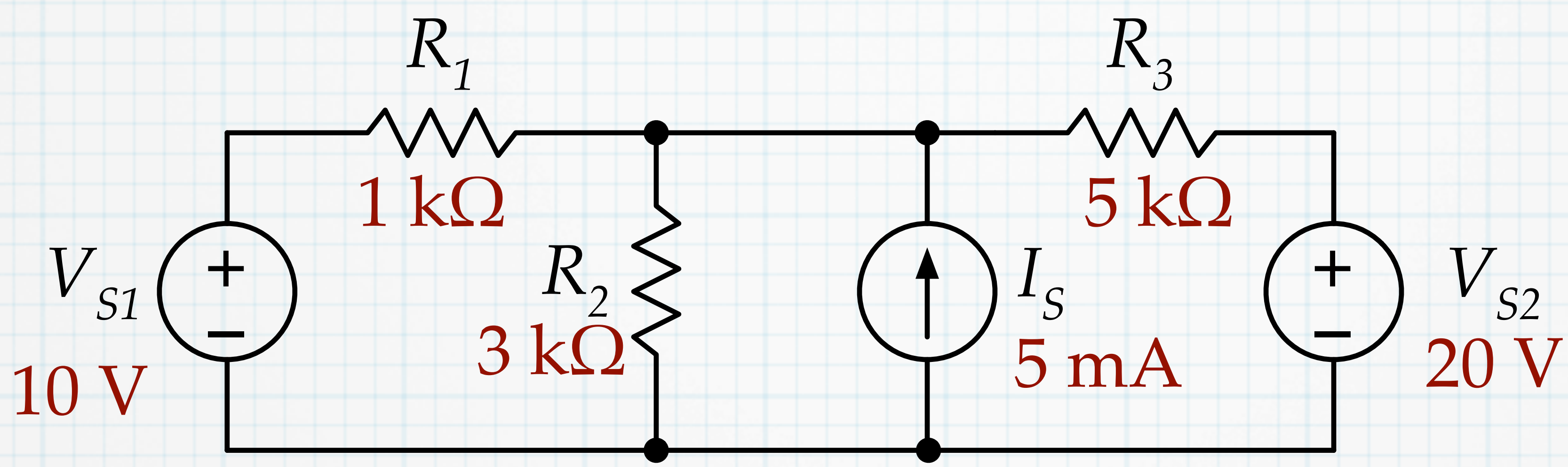
$$v_b = \frac{(5\Omega)\,(1\text{A}) - 10\text{V}}{\left(1 + \frac{5\Omega}{10\Omega}\right)} = \boxed{-3.33\text{V}}$$

At first glance, this seems wrong – v_b here is different from its value in the previous calculation.

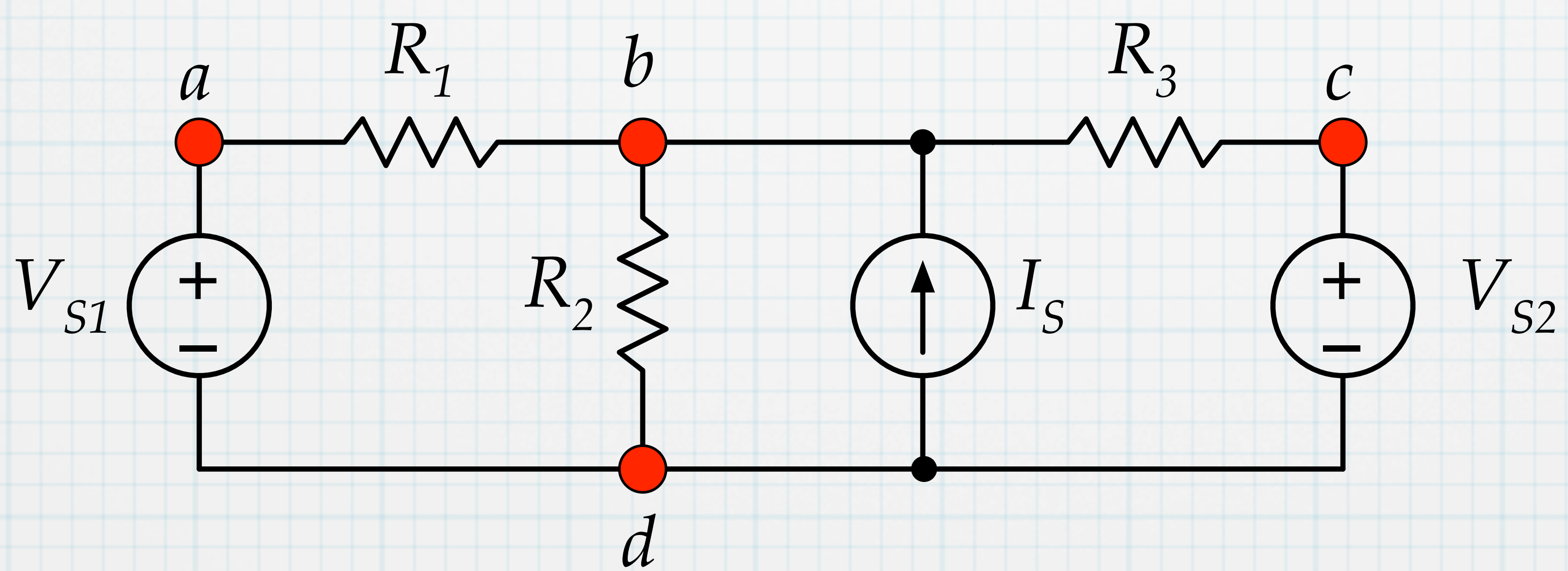
But remember that only voltage differences matter. In choosing a different reference node, all other node voltages will be shifted accordingly.

$v_{R1} = V_S - v_b = 3.33$ V and $v_{R2} = v_b - 0 = 6.67$ V, in both cases. The corresponding currents will be the same, as well.

another example

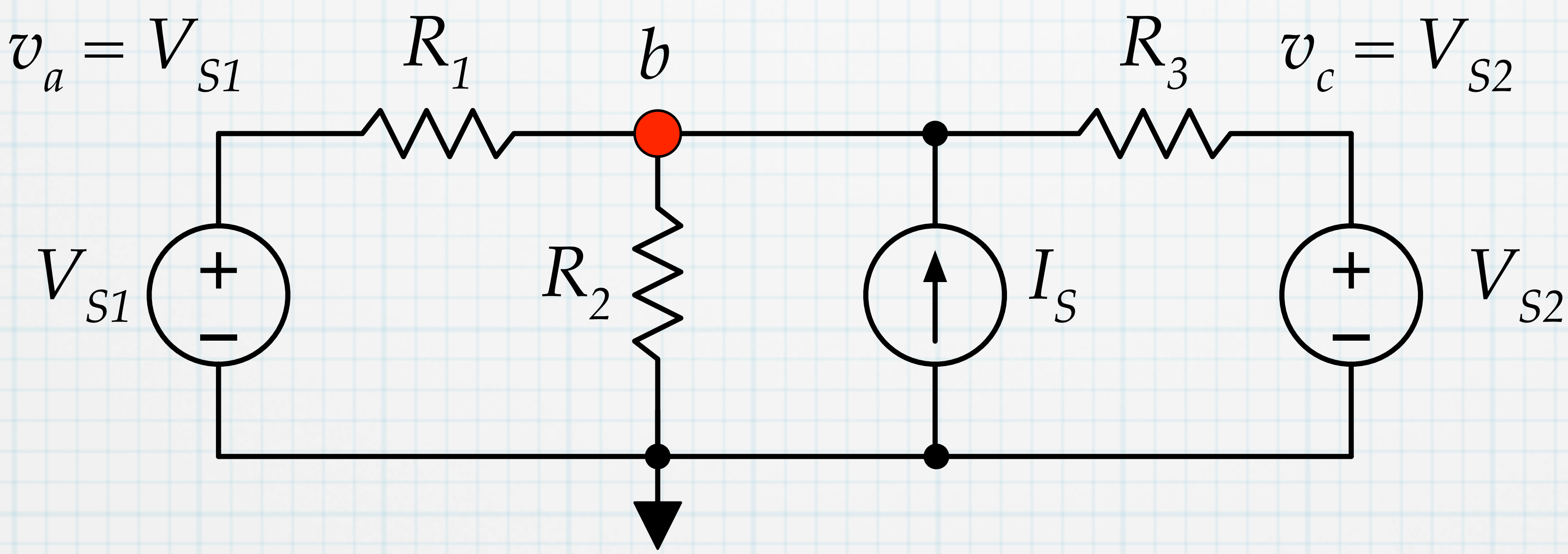


1. Identify the nodes.



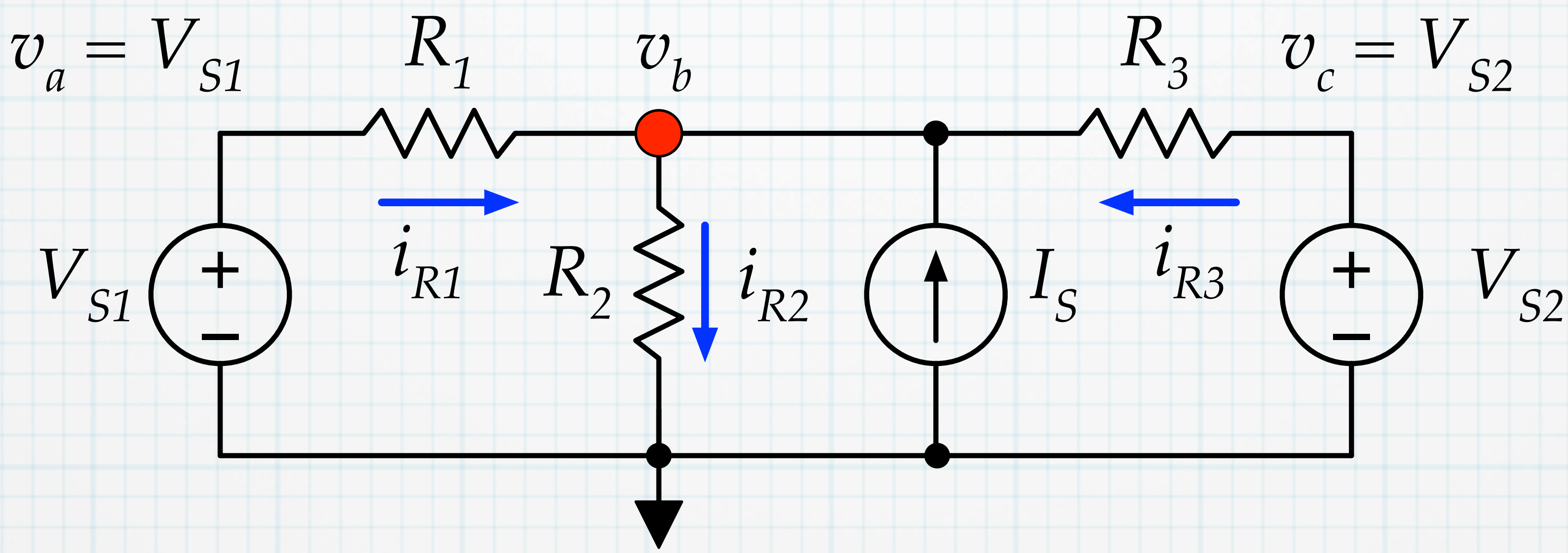
2. Choose one to be ground. Since there are voltage sources connected to *d*, that would seem to be a good choice.

3. Identify nodes for which the voltage is known. With *d* as ground, then we see that $v_a = V_{S1}$ and $v_c = V_{S2}$.



Only one node left, so only one unknown voltage to be found.

5&6. Assign a variable for the unknown voltage. Assign currents for each branch connected to the node.



7. Write KCL equations balancing the currents at the node.

$$i_{R1} + i_{R3} + I_S = i_{R2}$$

8. Express the resistor currents in terms of the node voltage.

$$i_{R1} = \frac{V_{S1} - v_b}{R_1} \qquad i_{R2} = \frac{v_b - 0}{R_2} \qquad i_{R3} = \frac{V_{S2} - v_b}{R_3}$$

9. Substitute resistor currents into the KCL equation to form the node-voltage equation.

$$\frac{V_{S1} - v_b}{R_1} + \frac{V_{S2} - v_b}{R_3} + I_S = \frac{v_b}{R_2}$$

10. Solve it.

$$\frac{R_3}{R_1} V_{S1} - \frac{R_3}{R_1} v_b + V_{S2} - v_b + R_3 I_S = \frac{R_3}{R_2} v_b$$

$$\frac{R_3}{R_1} V_{S1} + V_{S2} + R_3 I_S = v_b \left(1 + \frac{R_3}{R_1} + \frac{R_3}{R_2}\right)$$

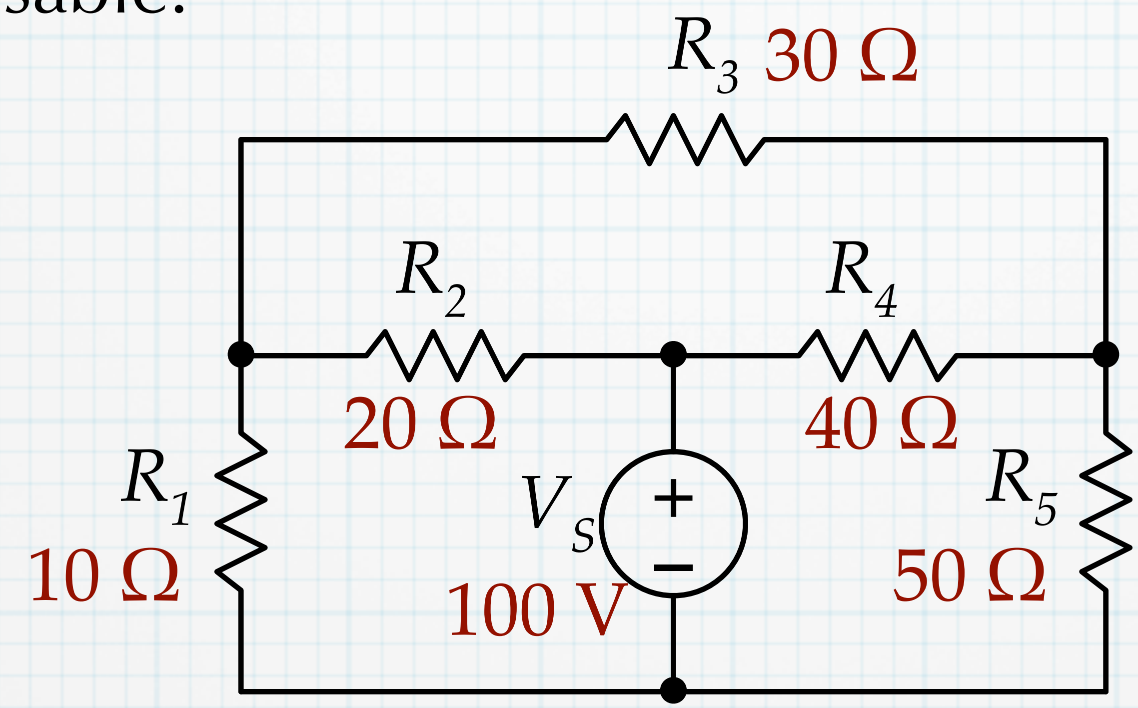
$$\frac{5\text{k}\Omega}{1\text{k}\Omega}(10\text{V}) + (20\text{V}) + (5\text{k}\Omega)(5\text{mA}) = v_b \left(1 + \frac{5\text{k}\Omega}{1\text{k}\Omega} + \frac{5\text{k}\Omega}{3\text{k}\Omega}\right)$$

$$95\text{V} = v_b (7.667)$$

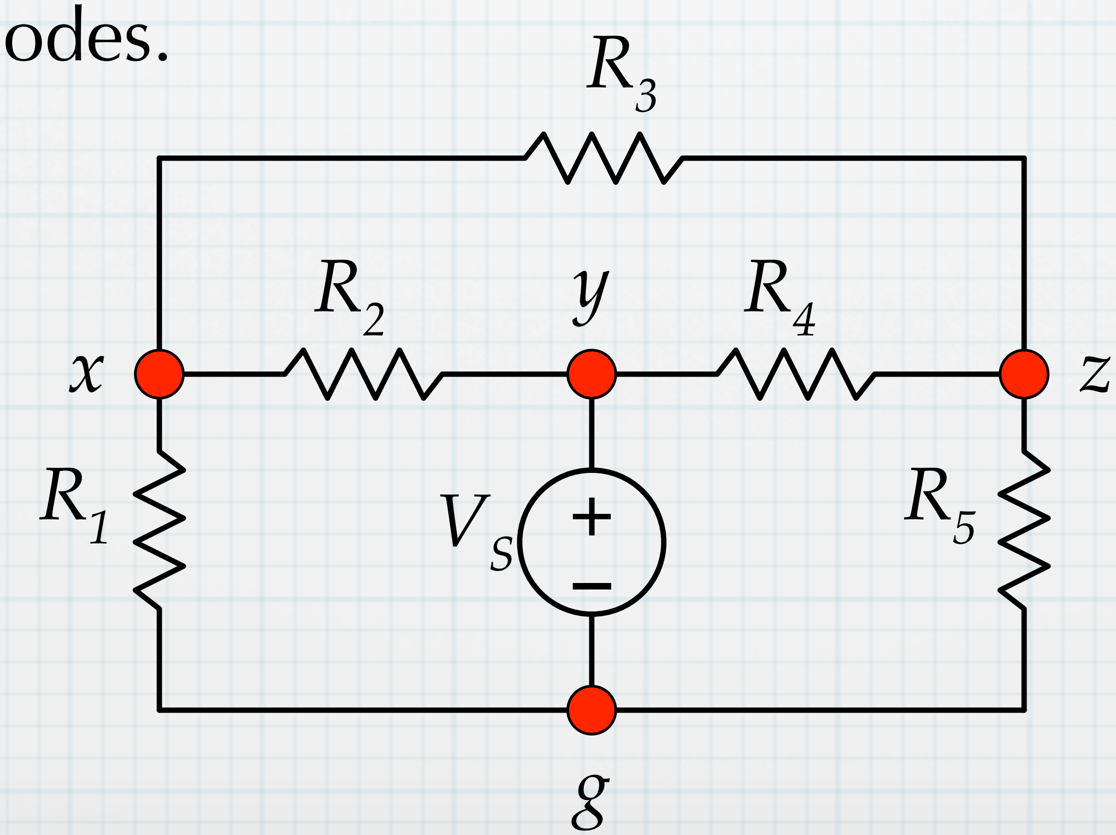
$$v_b = \boxed{12.4\text{V}}$$

Example

In the circuit below, we might like to find the currents of the resistors. The resistor that bridges across the top makes the short-cut methods unusable.

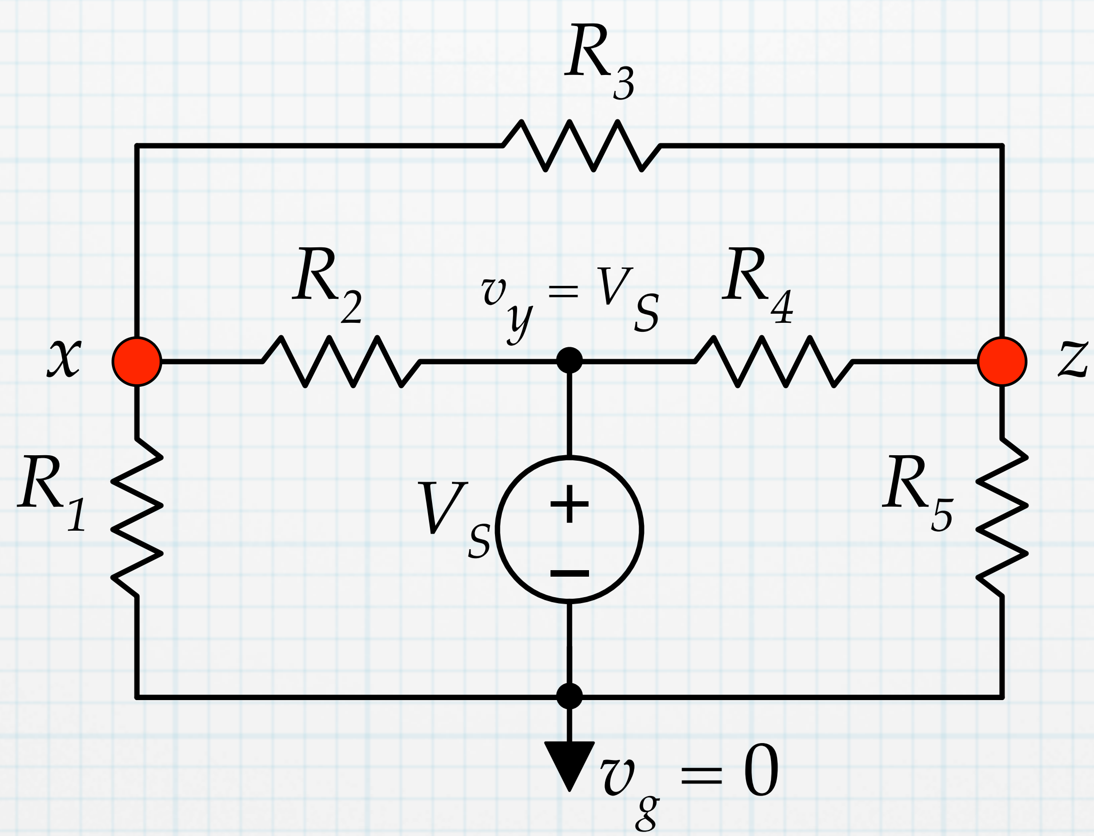


1. Identify the nodes.

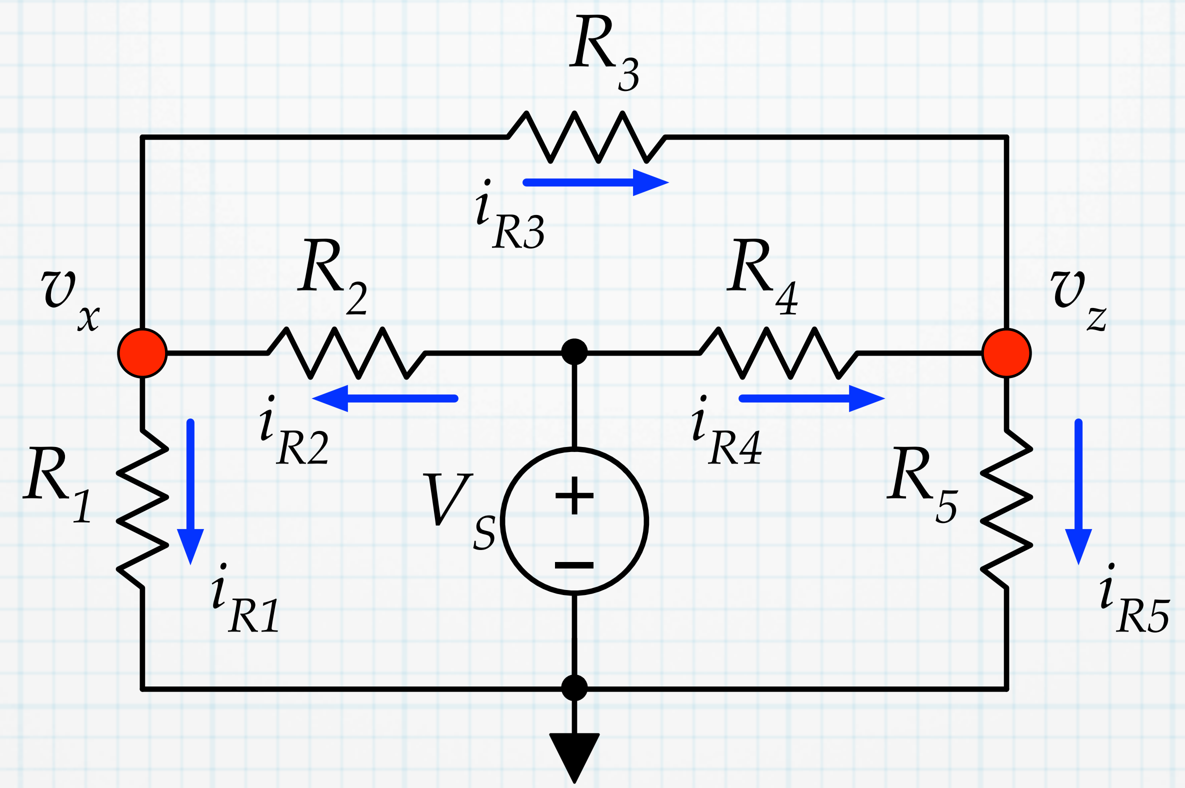


2. Choose one to be ground. Due to the single voltage source, the bottom node seems to be a likely choice.

3. Identify nodes for which the voltage is known. With g as ground, then we see that $v_y = V_S$.



5&6. Assign a variable for the unknown voltage. Assign currents for each branch connected to the node.



2 nodes this time!

7. Write KCL equations balancing the currents at the node.

x: $i_{R2} = i_{R1} + i_{R3}$ $\quad\quad$ z: $i_{R4} + i_{R3} = i_{R5}$

8. Express the resistor currents in terms of the node voltages.

$$i_{R1} = \frac{v_x}{R_1} \qquad i_{R2} = \frac{V_S - v_x}{R_2} \qquad i_{R3} = \frac{v_x - v_z}{R_3} \qquad i_{R4} = \frac{V_S - v_z}{R_4} \qquad i_{R5} = \frac{v_z}{R_5}$$

9. Substitute resistor currents into the KCL equations to form the node-voltage equations.

$$x: \quad \frac{V_S - v_x}{R_2} = \frac{v_x}{R_1} + \frac{v_x - v_z}{R_3}$$

$$z: \quad \frac{V_S - v_z}{R_4} + \frac{v_x - v_z}{R_3} = \frac{v_z}{R_5}$$

10. Solve the equations

$$V_S - v_x = \frac{R_2}{R_1}v_x + \frac{R_2}{R_3}v_x - \frac{R_2}{R_3}v_z$$

$$V_S - v_z + \frac{R_4}{R_3}v_x - \frac{R_4}{R_3}v_z = \frac{R_4}{R_5}v_z$$

$$\left[1 + \frac{R_2}{R_1} + \frac{R_2}{R_3}\right]v_x - \frac{R_2}{R_3}v_z = V_S$$

$$-\frac{R_4}{R_3}v_x + \left[1 + \frac{R_4}{R_3} + \frac{R_4}{R_5}\right]v_z = V_S$$

$$\left[1 + \frac{20\ \Omega}{10\ \Omega} + \frac{20\ \Omega}{30\ \Omega}\right]v_x - \frac{20\ \Omega}{30\ \Omega}v_z = 100\text{ V}$$

$$-\frac{40\ \Omega}{30\ \Omega}v_x + \left[1 + \frac{40\ \Omega}{30\ \Omega} + \frac{40\ \Omega}{50\ \Omega}\right]v_z = 100\text{ V}$$

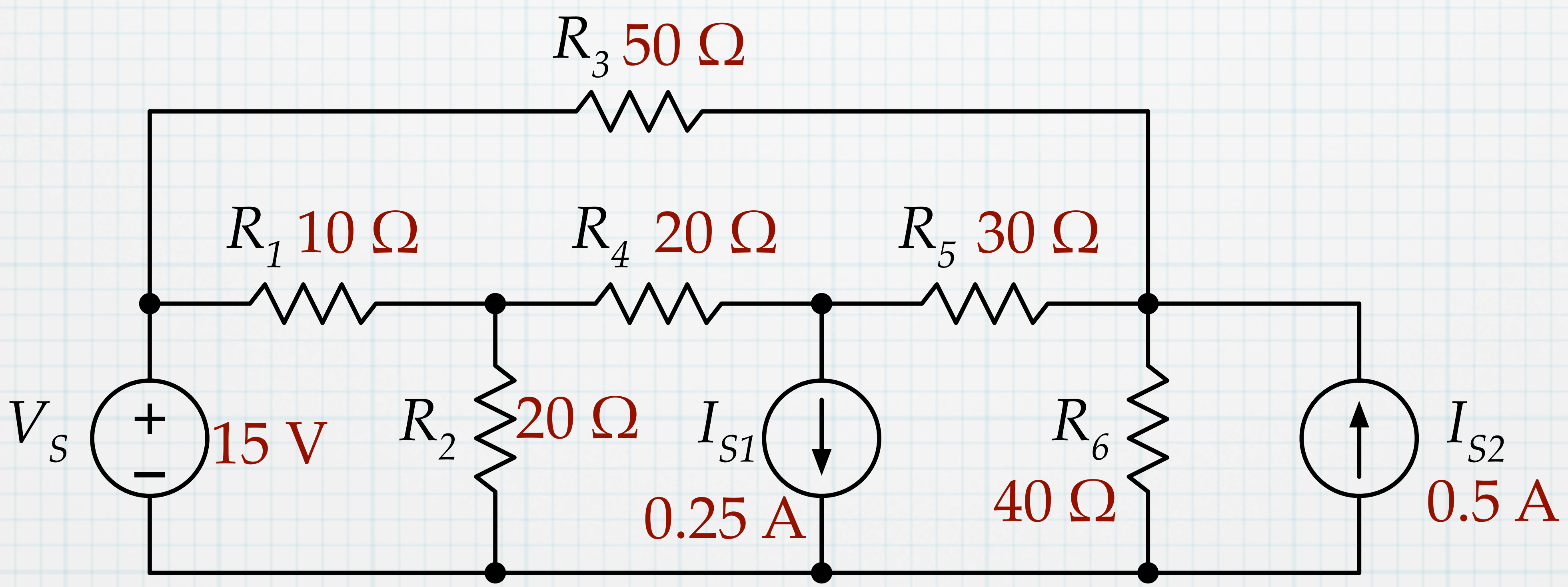
$$3.667v_x - 0.667v_z = 100\text{ V}$$

$$-1.333v_x + 3.133v_z = 100\text{ V}$$

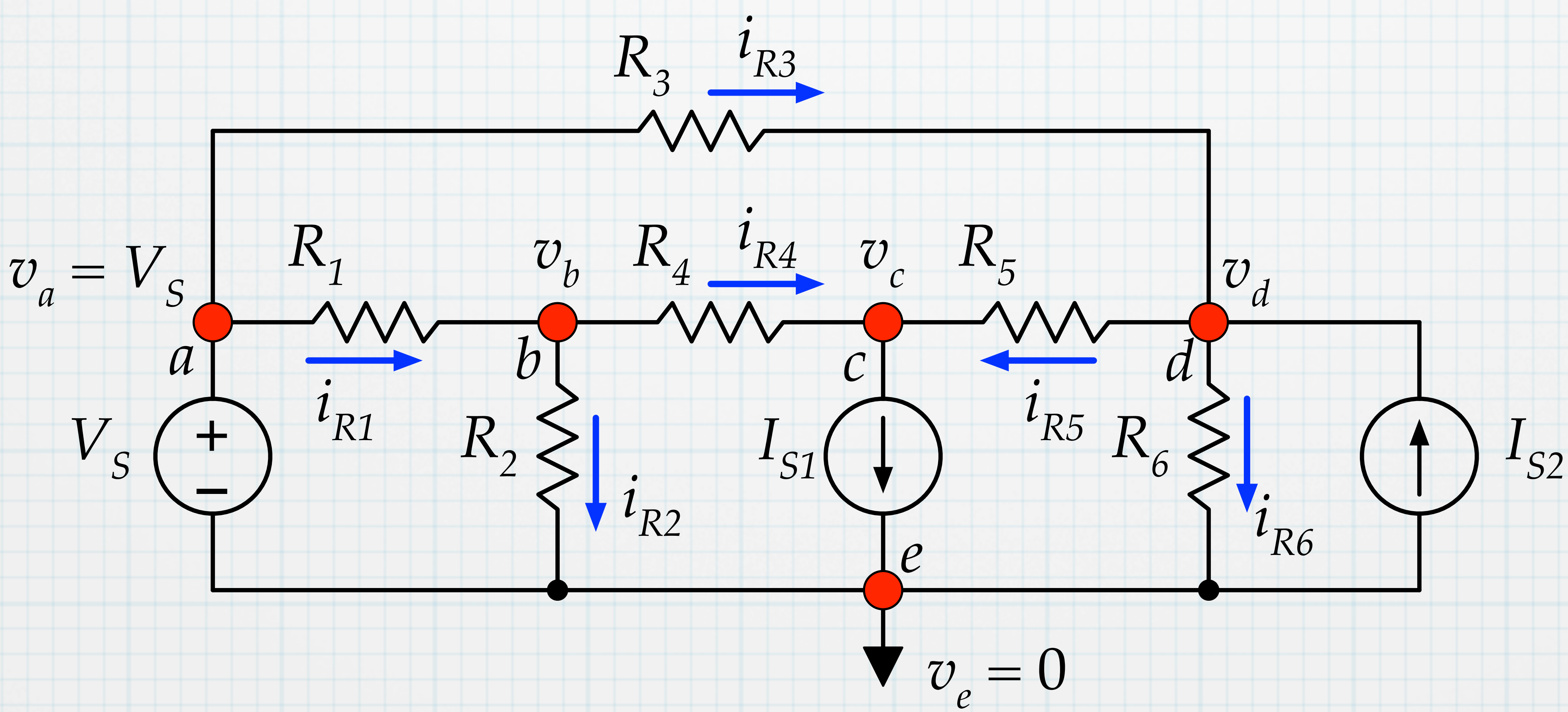
$$\boxed{v_x = 35.85\text{ V} \quad \text{and} \quad v_z = 47.175\text{ V}}$$

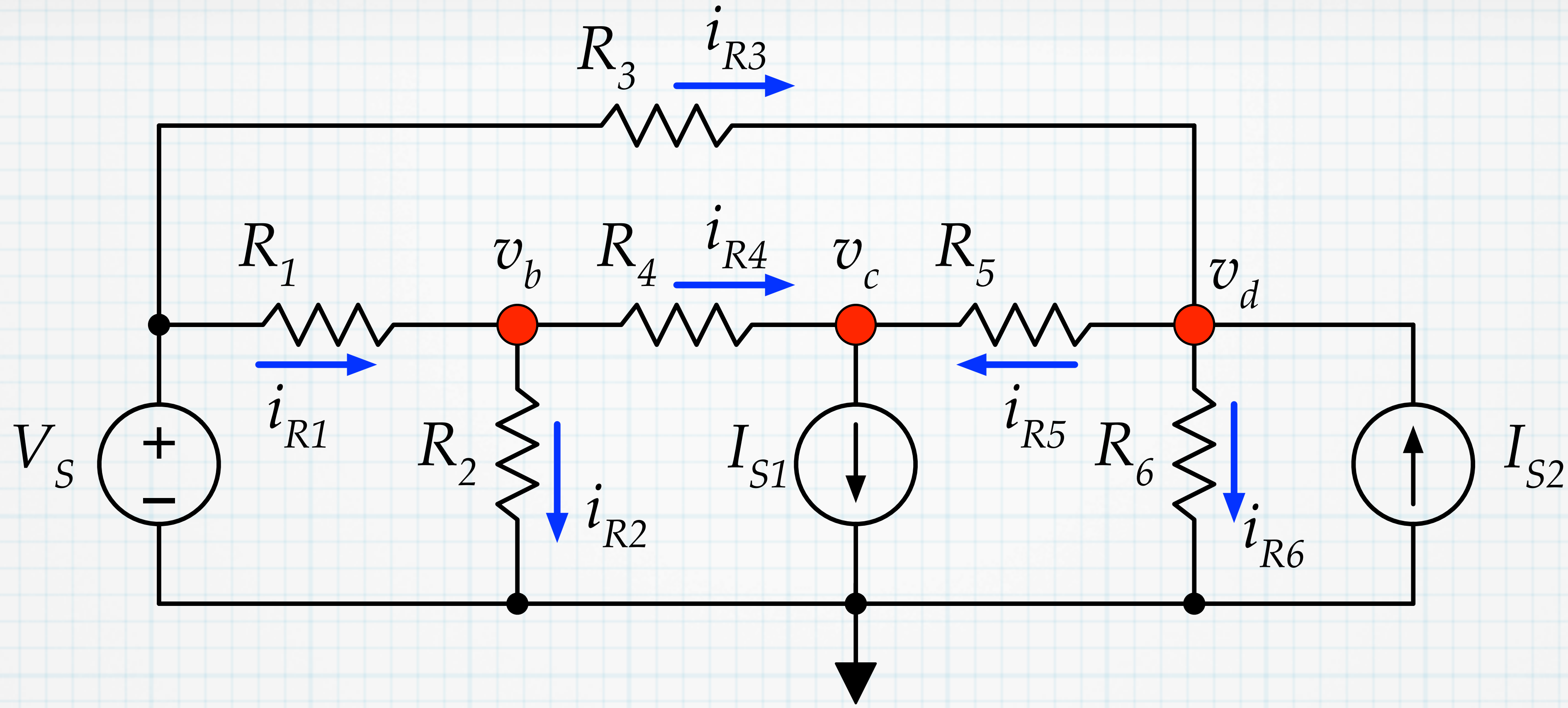
Final example (a big one)

Again, we might like to know how much power is being generated and dissipated in the various elements for the circuit below. Our short-cut methods are useless here, but the node-voltage works in exactly the same manner as the previous examples. The math is a bit more tedious for a bigger circuit – there will be 3 equations in 3 unknowns in this case – but the method is still straight forward.



1. Identify the nodes.
2. Choose one to be ground. (The bottom one is good, again.)
3. Identify known node voltages. (The left-most node is obviously at V_S.)
4. Assign variable names to the remaining, unknown nodes.
5. Assign resistor currents in each branch.





7. Write KCL equations balancing the currents at the node.

$$i_{R1} = i_{R2} + i_{R4} \qquad i_{R4} + i_{R5} = I_{S1} \qquad i_{R3} + I_{S1} = i_{R5} + i_{R6}$$

8. Express the resistor currents in terms of the node voltages.

9. Substitute resistor currents into the KCL equations.

$$\frac{V_S - v_b}{R_1} = \frac{v_b}{R_2} + \frac{v_b - v_c}{R_4} \qquad \frac{v_b - v_c}{R_4} + \frac{v_d - v_c}{R_5} = I_{S1}$$

$$\frac{V_S - v_d}{R_3} + I_{S2} = \frac{v_d - v_c}{R_5} + \frac{v_d}{R_6}$$

10. Solve the equations

$$\frac{V_S - v_b}{R_1} = \frac{v_b}{R_2} + \frac{v_b - v_c}{R_4}$$

$$\frac{v_b - v_c}{R_4} + \frac{v_d - v_c}{R_5} = I_{S1}$$

$$\frac{V_S - v_d}{R_3} + I_{S2} = \frac{v_d - v_c}{R_5} + \frac{v_d}{R_6}$$

$$\left[1 + \frac{R_1}{R_2} + \frac{R_1}{R_4}\right] v_b - \frac{R_1}{R_4} v_c = V_S$$

$$v_b - \left[1 + \frac{R_4}{R_5}\right] v_c + \frac{R_4}{R_5} v_d = R_4 I_{S1}$$

$$-\frac{R_3}{R_5} v_c + \left[1 + \frac{R_3}{R_5} + \frac{R_3}{R_6}\right] v_d = V_S + R_3 I_{S2}$$

$$2v_b - 0.5v_c = 15\text{V}$$

$$v_b - 1.667v_c + 0.667v_d = 5\text{V}$$

$$-1.667v_c + 3.917v_d = 40\text{V}$$

$$\boxed{\begin{aligned} v_b &= 9.554\ \text{V} \\ v_c &= 8.217\ \text{V} \\ v_d &= 13.709\ \text{V} \end{aligned}}$$

Two web-site solvers: http://math.bd.psu.edu/~jpp4/finitemath/3x3solver.html

http://www.1728.org/unknwn3.htm