Lecture - 14

Recap:

Hypergeometric random variable, estimating the size of a population.

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expected value of a sum = sum of expected values.

Cumulative distribution function

Continuous random variable.
crv

X is a crv if $\exists$ a nonnegative function f, defined over R, s.t.

$$P(X \in B) = \int_B f(x)\,dx$$

e.g.

$$f(x) = \begin{cases} C(4x-2x^2) & \text{if } 0<x<2 \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\underline{\underline{P(X>1)}}$

$$\parallel$$

$$B = (1, \infty)$$

$$\int_B f(x)\,dx = \int_1^\infty f(x)\,dx$$

$$= \int_1^2 + \int_2^\infty = 0$$

i) Compute C using $\int_{-\infty}^{\infty} f(x)\,dx = 1$

ii) Compute $\int_{1}^{\infty} f(x)\,dx$

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e.g. X = no. of hours a computer works before breaking down.

$$f(x) = \begin{cases} \lambda e^{-x/100}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

What is the probability that the computer will work between 50 & 150 hours?

i) find the value of λ

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\lambda \int_{0}^{\infty} e^{-x/100}\, dx = 1$$

$$= \lambda(-100)\, e^{-x/100}\Big]_{0}^{\infty}$$

$$= -100\lambda\,[0 - 1] = 100\lambda = 1$$

$$\therefore \lambda = \frac{1}{100}$$

ii) Compute $P(50 < x < 150)$ 5

$$\parallel$$

$$\frac{1}{100}\int_{50}^{150} e^{-x/100}\,dx = e^{-\frac{1}{2}} - e^{-\frac{3}{2}}$$

$$P(X = 50) = 0$$

$$P(X = 150) = 0$$

$$P(50 \leq X \leq 50)$$

$$\parallel$$

$$\int_{50}^{50} f(x)\,dx = 0$$

e.g.: lifetime of a mobile in hours $= X$

$$f(x) = \begin{cases} 0, & x \le 100 \\ \dfrac{100}{x^2}, & x > 100 \end{cases}$$

What is the probability that exactly 2 out of 5 such mobiles will need to be replaced within the 1st 150 hours of use.?

Step 1: $P(0 < X < 150)$

$$= \int_0^{150} f(x)\,dx = \frac{1}{3}$$

Y = no. of mobiles that need to be replaced.

$$P(Y = 2) = \binom{5}{2} p^2 (1-p)^3$$

Y is a binomial r.v., $p = \frac{1}{3}$

$= 80/243$

Cumulative distribution function

$$\underline{F(a)} = P(X < a) = P(X \leq a)$$

$$= \int_{-\infty}^{a} f(x)\,dx$$

$$\frac{dF(a)}{da} = \frac{da}{da} f(a) - \frac{d(-\infty)}{da} f(-\infty) + \int_{-\infty}^{a} \frac{\partial}{\partial a}(f(x))\,dx$$

$\searrow 0$

$= f(a)$

e.g.

X is a continuous r.v., f_X

F_X is its cumulative distribution fn.

→ $Y = 2X$

f_Y : density function of Y. in terms of density fn of X

→ Compute F_Y

→ differentiate to get f_Y

$$F_Y(a) = P(Y \le a)$$
$$= P(2X \le a)$$
$$= P(X \le a/2) = F_X(a/2)$$

$$F_Y(a) = F_{X.}(a/2)$$

$$f_Y(a) = \frac{d(a/2)}{da} f_X(a/2)$$

$$\boxed{f_Y(a) = \frac{1}{2} f_X(a/2)}$$

Expectation, Variance for crv

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$Var(X) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \left[\int_{-\infty}^{\infty} x f(x)\, dx\right]^2$$

$\parallel$

$E(X^2) -$

$[E(X)]^2$

e.g.

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute $E[X]$ and $Var(X)$

$2/3$ $1/18$

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e.g.

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute $E[e^X]$.

$Y = e^X$

$E[Y]$

i) Compute $F_X(a) \to a$

ii) Compute $F_Y(a) = \log a$

iii) Compute f_Y

iv) $E(Y)$

$$F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx = \int_{-\infty}^{0} 0 \; +$$

$$\int_0^a \underbrace{f_X(x)}\,dx = \int_0^a dx = x\Big]_0^a = a$$

$$F_Y(a) = P(Y \le a) = P(e^X \le a)$$

$$= P(X \le \log a) = F_X(\log a) = \log a$$

$$F_Y(a) = \log a$$

iii) $f_Y(a) = \frac{d}{da}(F_Y(a))$

$= \frac{d}{da}(\log a) = \frac{1}{a}$

$f_Y(a) = 1/a$ | $X: 0 \to 1$
$Y: e^0 \to e^1$

iv) $E(Y) = \int_{e^0}^{e^1} \frac{1}{a} \cdot a \cdot d(a)$

$= \int_1^e da = a \Big]_1^e$

$= \boxed{e - 1}$