Summary of the course/ End Sem. Exam

1. <u>Differential Calculus: Numerical ODE solvers</u>

<u>2 Elementary Mechanics</u>
2.1 Newtonian Mechanics: Single Particle. Force, Energy, Potential.
2.2 Basic Kinematic Quantities. Conservation Theorems
2.3 Rotational Motion.
2.4 Configuration and phase space
2.5 Problem Solving
3 <u>Oscillations and Motion</u>
3.1 Simple harmonic oscillator
3.2 Nonlinear Oscillations
3.4 Applications and Problem Solving
4 <u>Lagrangian and Hamiltonian Dynamics</u>
4.1 Lagrangian approach to Mechanics
4.2 Variational calculas , Eulers Equations
4.3 Hamiltonian Dynamics
4.4 Examples and Problem Solving

5.1 Central Force Motion.
5.2 Two body problem. Orbits, Gravitation.
5.3 Scattering Problem

Scattering in a Central force field.

Central Force Field Scattering

- Application of Central Forces outside of astronomy: **Scattering of particles**.
- Atomic scale scattering.

Description of scattering processes:

1 body formulation = Scattering of particles by a

Center of Force.

- *Original 2 body problem = Scattering of "particle" with the reduced mass* $\boldsymbol{\mu}$ *from a center of force*

Assumptions

1. Consider a **uniform beam** of particles (of any kind) of equal mass and energy incident on a center of force (Central force **f(r)**).
   - ***Assume*** *that **f(r)** falls off to zero at large **r**.*

2. Incident beam is characterized by an intensity (flux density)

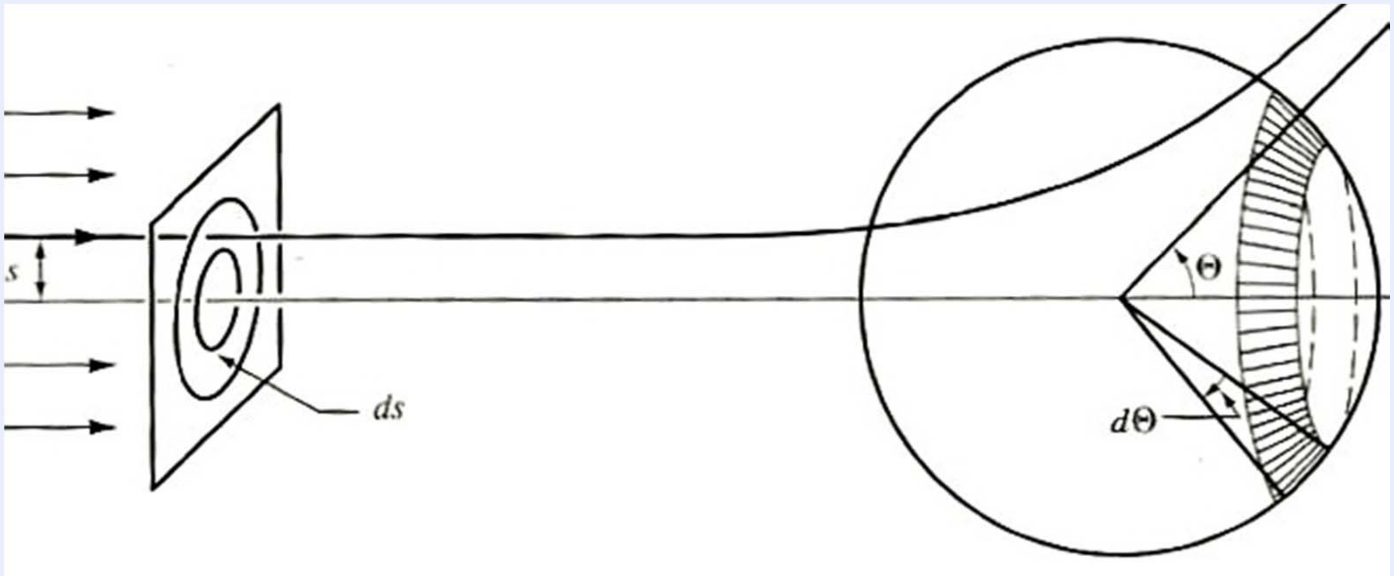
I ≡ # particles crossing a unit area (⊥ beam) per unit time .

- *As a particle approaches the center of force, it is either attracted or repelled & thus it's orbit will be changed (deviate from the initial straight line path).*

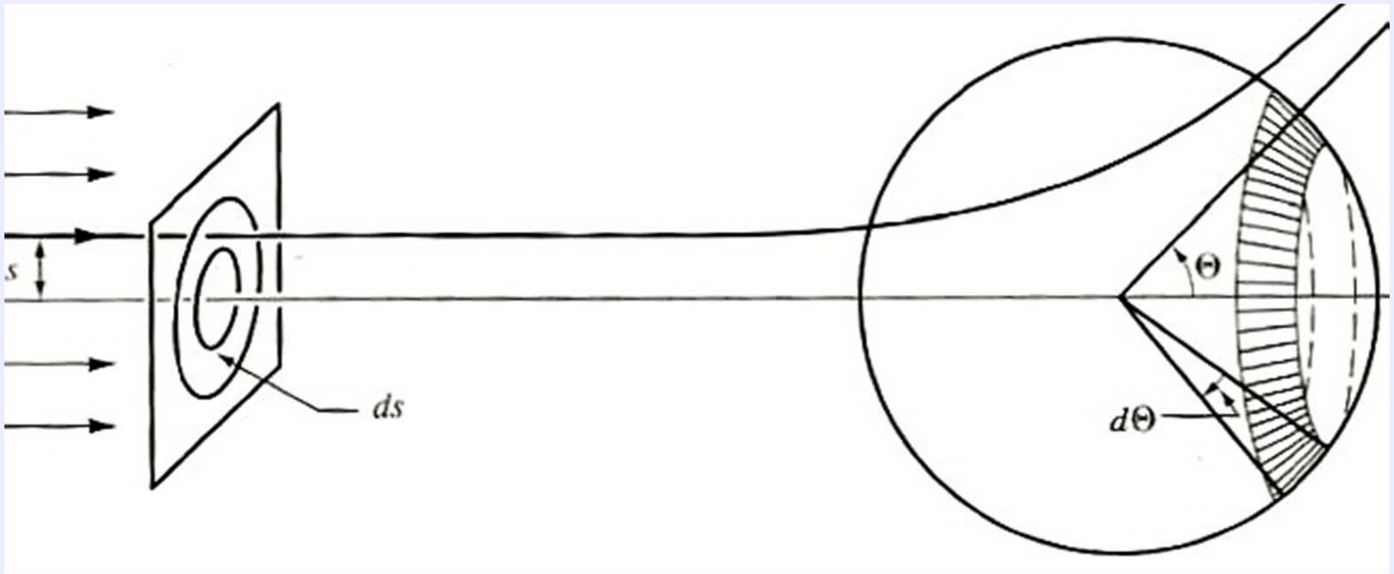
Direction of final motion is not the same as incident motion. ⇒ ≡ ***Particle is Scattered***

Interested in distribution of scattering angles that result from collisions with various impact parameters.

Impact parameter → distance of closest approach.



➢ **repulsive scattering** (as shown in the figure):



➢ ***Define:*** **Cross Section for Scattering** in a given direction (into a given solid angle $\mathbf{d\Omega}$):

$$\boldsymbol{\sigma(\Omega)d\Omega \equiv (N_s/I).}$$

With **I =** incident intensity

$\mathbf{N_s}$ **=** # particles/time scattered into solid angle $\mathbf{d\Omega}$

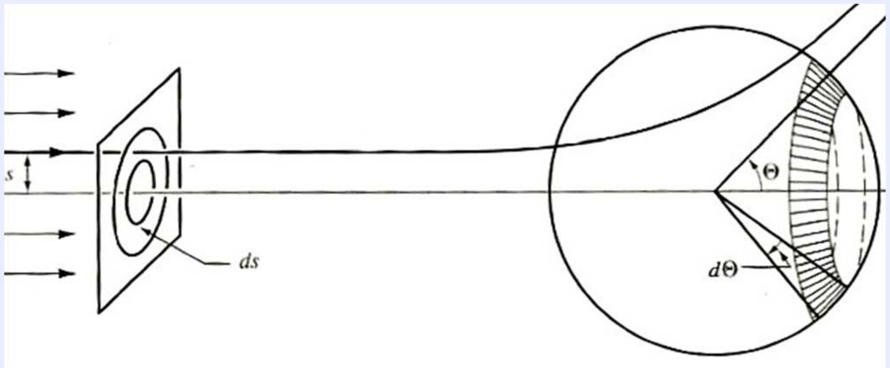
Differential Scattering Cross Section:

$\sigma(\Omega)d\Omega \equiv (N_s/I)$

I = incident intensity

N_s = # particles/time

scattered into angle $d\Omega$



- In general, the solid angle Ω depends on the spherical angles Θ, Φ. However, for central forces, there must be **symmetry** about the axis of the incident beam

$\Rightarrow \sigma(\Omega)$ ($\equiv \sigma(\Theta)$) is independent of azimuthal angle Φ

$\Rightarrow d\Omega \equiv 2\pi \sin\Theta d\Theta$, $\sigma(\Omega)d\Omega \equiv 2\pi \sin\Theta d\Theta$,

$\Theta \equiv$ Angle between incident & scattered beams, as in the figure.

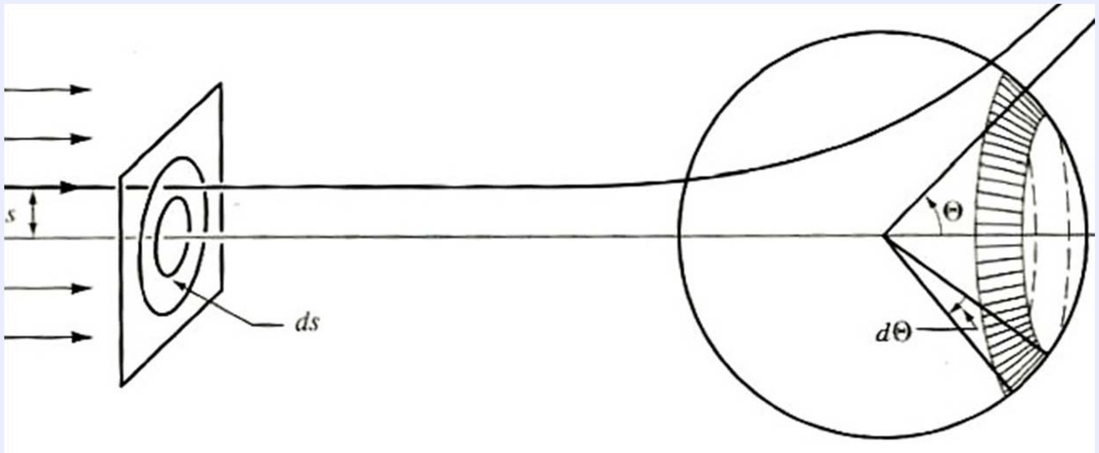
$\sigma \equiv$ "***cross section***". It has units of area

Also called the differential cross section.

In all Central Force problems, for a given particle, the orbit, & thus the amount of scattering, is determined by ??

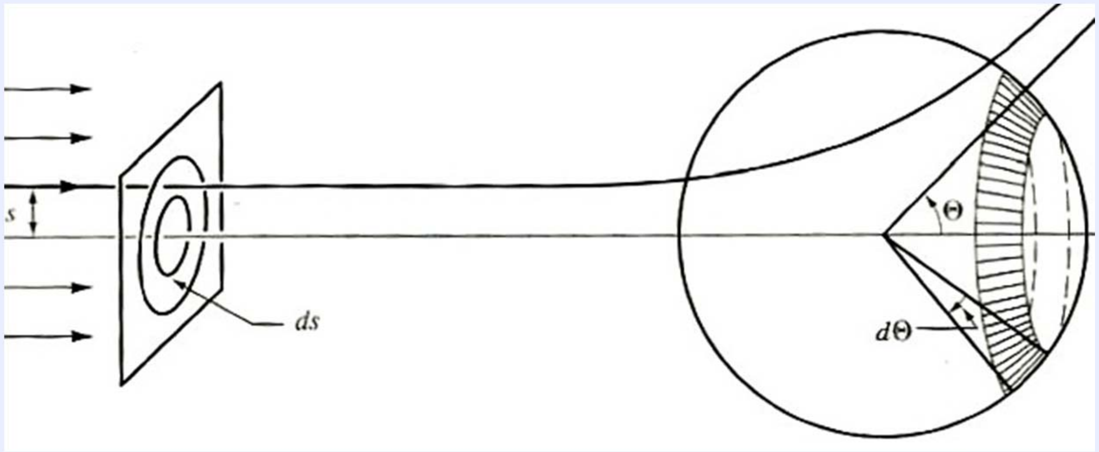
- As in all Central Force problems, for a given particle, the orbit, & thus the amount of scattering, is determined by the energy **E** & the angular momentum ℓ
- ***Define***: **Impact parameter, s**, & express the angular momentum ℓ in terms of **E** & **s**.
- Impact parameter **s** ≡ the ⊥ distance between the center of force & the incident beam velocity (fig).
- ***GOAL:*** **Given** the energy **E**, the impact parameter **s**, & the force **f(r)**, what is the cross section **σ(Θ)**?

- **Beam**, intensity **I**. Particles, mass **m**, incident speed (at $\mathbf{r} \to \infty$) = $\mathbf{v_0}$.



- **find $\mathbf{v_0}$** and ℓ **(angular momentum of the particle about force center or m2)**

- **Beam**, intensity **I**. Particles, mass **m**, incident speed (at $\mathbf{r} \to \infty$) = $\mathbf{v_0}$.



- **Energy conservation:**

$$E = T + V = (\tfrac{1}{2})mv^2 + V(r) = (\tfrac{1}{2})m(v_0)^2 + V(r \to \infty)$$

- Assume $V(r \to \infty) = 0 \;\Rightarrow\; E = (\tfrac{1}{2})m(v_0)^2$

$$\Rightarrow\; v_0 = (2E/m)^{\frac{1}{2}}$$

- **Angular momentum:** $\ell \equiv mv_0 s \equiv s(2mE)^{\frac{1}{2}}$

If E and s are fixed, **Θ can be determined.**

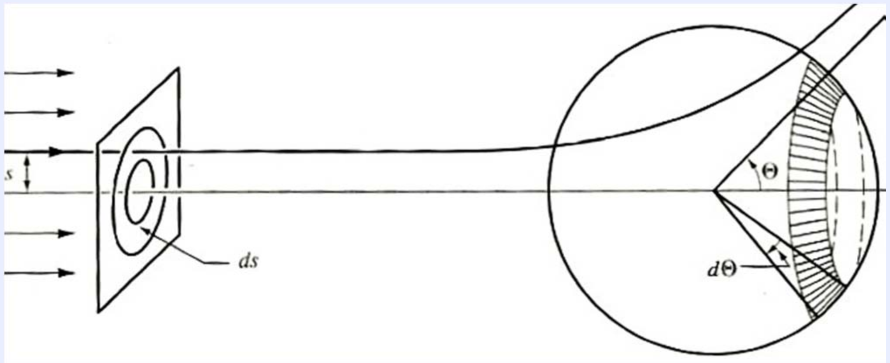
Assume→ The number of particles scattered into a solid angle $d\Omega$ lying between Θ and $\Theta+d\Theta$ must be equal to the number of the incident particles with impact parameter lying between the s and s+ds.

- **Angular momentum** $\ell \equiv \mathbf{mv_0s} \equiv \mathbf{s(2mE)^{1/2}}$

  Incident speed $\mathbf{v_0}$.

- $\mathbf{N_s} \equiv$ # particles scattered into solid angle $\mathbf{d\Omega}$ between

  $\mathbf{\Theta}$ & $\mathbf{\Theta + d\Theta}$.



Cross section definition

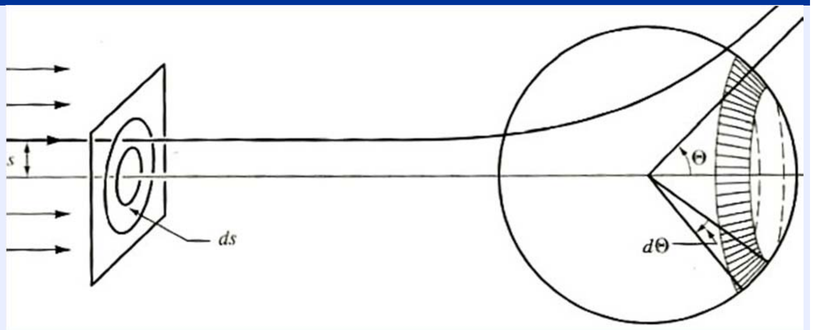
$$\Rightarrow \qquad \mathbf{N_s = 2\pi I\sigma(\Theta)\sin\Theta d\Theta}$$

- $\mathbf{N_i} \equiv$ # incident particles with impact parameter between **s** & **s + ds** . $\mathbf{N_i = 2\pi Isds}$

- **Conservation of particle number**

  $\Rightarrow \mathbf{N_s = N_i}$ or: $\mathbf{2\pi I\sigma(\Theta)\sin\Theta|d\Theta| = 2\pi Is|ds|}$

  2πI cancels out! (Use absolute values because **N**'s are always **> 0,** but **ds** & **dΘ** can have any sign.)



$\boldsymbol{\sigma(\Theta)\sin\Theta|d\Theta| = s|ds|}$ **(1)**

- **s** = a function of energy **E** & scattering angle **Θ:**

  $s = s(\Theta,E)$

(1) $\Rightarrow \sigma(\Theta) = (s/\sin\Theta)\,(|ds|/|d\Theta|)$ **(2)**

- To compute **σ(Θ)** we clearly need $s = s(\Theta,E)$
- **Alternatively,** could use $\Theta = \Theta(s,E)$ & rewrite **(2)** as:

  $\sigma(\Theta) = (s/\sin\Theta)/[(|d\Theta|/|ds|)]$ **(2´)**

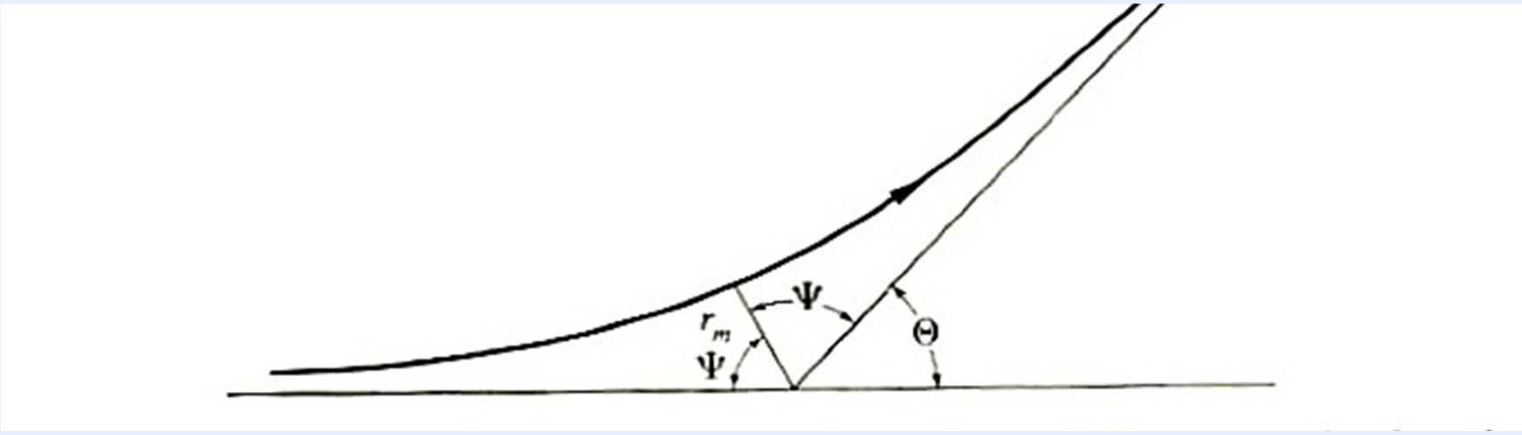
- Get $\Theta = \Theta(s,E)$ from the orbit eqtn. For general central force (**θ** is the angle which describes the orbit $r = r(\theta)$; $\theta \neq \Theta$)

  $\theta(r) = \int(\ell/r^2)(2m)^{-\frac{1}{2}}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-\frac{1}{2}}dr$

- Orbit eqtn. **General central force:**

$$\theta(r) = \int (\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2} dr \qquad (3)$$

- Considering purely **repulsive scattering**. See figure:



- **Closest approach distance** ≡ $\mathbf{r_m}$. Orbit must be symmetric about $\mathbf{r_m}$ ⇒ (see figure):

Scattering angle $\Theta \equiv \boldsymbol{\pi - 2\Psi}$. Also, orbit angle

$\boldsymbol{\theta = \pi - \Psi}$ ***in the special case*** $\mathbf{r = r_m}$

$\Rightarrow$ Rearrange **(3)** as:

$$\mathbf{\Psi = \int (dr/r^2)[(2mE)/(\ell^2) - (2mV(r))/(\ell^2) - 1/(r^2)]^{-1/2} \quad (4)}$$

- Integrate from $\mathbf{r_m}$ to $\mathbf{r} \to \infty$

- Angular momentum in terms of impact parameter **s** & energy **E**: $\ell \equiv \mathbf{mv_0s} \equiv \mathbf{s(2mE)}^{1/2}$.

Put this into **(4)** & get for scattering angle **Θ**:

$\Rightarrow$ Rearrange **(3)** as:

$$\Psi = \int (dr/r^2)[(2mE)/(\ell^2) - (2mV(r))/(\ell^2) - 1/(r^2)]^{-1/2} \quad (4)$$

- Integrate from r_m to $r \to \infty$
- Angular momentum in terms of impact parameter

s & energy **E**: $\ell \equiv mv_0 s \equiv s(2mE)^{1/2}$.

Put this into **(4)** & get for scattering angle **Θ**:

$$\Theta(s) = \pi - 2\int dr(s/r)[r^2\{1 - V(r)/E\} - s^2]^{-1/2} \qquad (4')$$

Changing integration variables to **u = 1/r**:

$$\Theta(s) = \pi - 2\int s\,du\,[1 - V(r)/E - s^2u^2]^{-1/2} \qquad (4'')$$

- Integrate from $u = 0$ to $u = u_m = 1/r_m$

4- Direct numerical computation!

- ***Summary:*** Scattering by a ***general central force***:
- **Scattering angle Θ = Θ(s,E)** (**s =** impact parameter, **E =** energy):

  $$\Theta(s) = \pi - 2\int s\,du\,[1 - V(r)/E - s^2u^2]^{-1/2} \quad (4'')$$

  Integrate from **u = 0** to $u = u_m = 1/r_m$
- **Scattering cross section:**

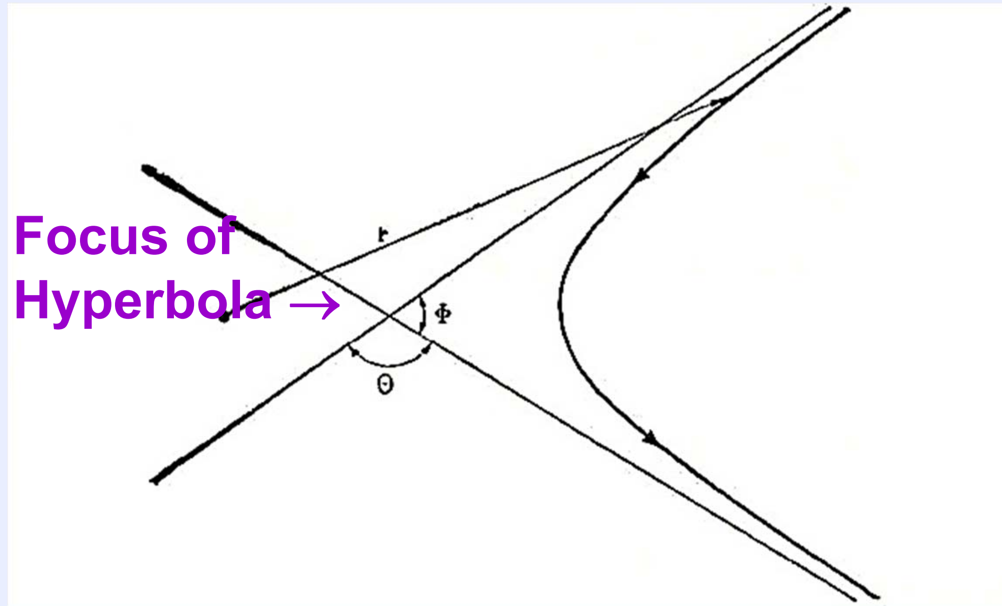
  $$\sigma(\Theta) = (s/\sin\Theta)\,(|ds|/|d\Theta|) \qquad (2)$$

- **"Steps":** To solve a scattering problem:
  1. *Given force **f(r)**, compute potential **V(r)**.*
  2. *Compute **Θ(s)** using **(4´´)**.*
  3. *Compute **σ(Θ)** using **(2)**.*

assignment

- Relations between orbit angle **θ** scattering angle **Θ**, & auxillary angle **Ψ** in the scattering problem, to get **Θ = Θ(s)** & thus **the scattering cross section.**

$$\Theta = \pi - 2\Psi$$



Ψ = direction of incoming asymptote. Determined by **r**→ ∞

Example problem

Problem

Examine the scattering produced by a repulsive central force $f = kr^{-3}$. Show that the differential cross section is given by

$$\sigma(\Theta)d\Theta = \frac{k}{2E}\frac{(1-x)dx}{x^2(2-x)^2\sin\pi x}$$

where x is the ratio Θ/π and E is the energy.

- ***Summary:*** Scattering by a ***general central force***:
- **Scattering angle Θ = Θ(s,E)** (**s =** impact parameter, **E =** energy):

  $$\Theta(s) = \pi - 2\int s\,du\,[1 - V(r)/E - s^2u^2]^{-1/2} \quad (4'')$$

  Integrate from **u = 0** to $\mathbf{u = u_m = 1/r_m}$
- **Scattering cross section:**

  $$\sigma(\Theta) = (s/\sin\Theta)\,(|ds|/|d\Theta|) \qquad (2)$$

- **"Steps":** To solve a scattering problem:
  1. *Given force **f(r)**, compute potential **V(r)**.*
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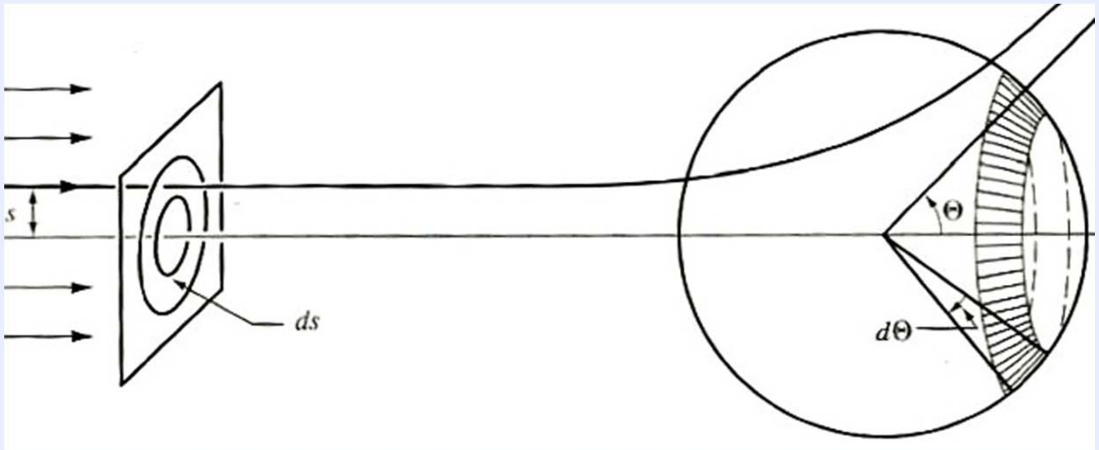

- **Energy conservation:**

$$E = T + V = (½)mv^2 + V(r) = (½)m(v_0)^2 + V(r \to \infty)$$

- Assume $V(r \to \infty) = 0 \Rightarrow E = (½)m(v_0)^2$

$$\Rightarrow v_0 = (2E/m)^{½}$$

- **Angular momentum:** $\ell \equiv mv_0 s \equiv s(2mE)^{½}$

Equation of the Orbit

$$\dot{r} = \frac{d}{dt}(r) = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\frac{1}{u} = -\frac{\ell}{\mu}\frac{du}{d\phi}$$

$$\mu\ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}.$$

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\left(-\frac{\ell}{\mu}\frac{du}{d\phi}\right) = -\frac{\ell^2 u^2}{\mu^2}\frac{d^2 u}{d\phi^2}.$$

$$-\mu\frac{\ell^2 u^2}{\mu^2}\frac{\partial^2 u}{\partial\phi^2} = F(r) + \frac{\ell^2 u^3}{\mu}$$ or

$$\boxed{u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2}F(r).}$$

we substituted $u = 1/r$.

solution

$$U = k/2r^2 = ku^2/2$$ ------------- 1

$$l = mv_0 s = (2mE)^{1/2} s$$ -------------- 2

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{dU}{du} = -\frac{mk}{l^2}u$$

$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{mk}{l^2}\right)u = 0$$

Solution →

$$u = A\cos\gamma\theta + B\sin\gamma\theta$$

$$\gamma = \sqrt{1 + \frac{mk}{l^2}}.$$

solution

Initial conditions :

initially the particle is at angle $\theta = \pi$ and a great distance from the force cer

$$u(\theta = \pi) = 0 \quad \longrightarrow \quad A\cos\gamma\pi + B\sin\gamma\pi = 0$$

$$\longrightarrow \quad A = -B\tan\gamma\pi.$$

particle head off to $r = \infty$ at angle $\theta = \theta_s$

$$A\cos\gamma\theta_s + B\sin\gamma\theta_s = 0.$$

Manipulate and Find γ in terms of θ

solution

$$-\cos\gamma\theta_s \tan\gamma\pi + \sin\gamma\theta_s = 0$$

$$\begin{aligned} -\cos\gamma\theta_s \sin\gamma\pi + \sin\gamma\theta_s \cos\gamma\pi &= 0 \\ \longrightarrow \qquad \sin\gamma(\theta_s - \pi) &= 0 \\ \longrightarrow \qquad \gamma(\theta_s - \pi) &= \pi \end{aligned}$$

$$\gamma = \frac{1}{x-1}.$$

$$\gamma = \sqrt{1 + \frac{mk}{l^2}}.$$

→

$$1 + \frac{mk}{l^2} = \frac{1}{(x-1)^2}.$$

Get an equation for s in terms of E

Soln.

$$1 + \frac{k}{2Es^2} = \frac{1}{(x-1)^2}$$

→

$$s^2 = -\frac{k}{2E}\left[\frac{(x-1)^2}{x(x-2)}\right].$$

Differential cross section is:

$$\sigma(\theta)d\Omega = \frac{|\, s\, ds\,|}{\sin\theta}.$$

$$
\begin{aligned}
2s\,ds &= -\frac{k}{2E}\left[\frac{2(x-1)}{x(x-2)} - \frac{(x-1)^2}{x^2(x-2)} - \frac{(x-1)^2}{x(x-2)^2}\right]dx \\
&= -\frac{k}{2E}\left[\frac{2x(x-1)(x-2) - (x-1)^2(x-2) - x(x-1)^2}{x^2(x-2)^2}\right] \\
&= -\frac{k}{2E}\left[\frac{2(1-x)}{x^2(x-2)^2}\right].
\end{aligned}
$$

$$
\sigma(\theta)d\Omega = \frac{k}{2E}\left[\frac{(1-x)}{x^2(x-2)^2\sin\theta}\right]dx
$$