Gaussian Quadrature

1/. Uses ~~a~~ an exact integration of polynomials of increasing degree.

2/. The integrating interval is not subdivided.

$$I = \int_a^b f(x)\,dx = \sum_{i=1}^{n} c_i f(x_i)$$

Example: $\boxed{f(x) = x}$ $\int_a^b f(x)\,dx$

The integral is $\int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2 - a^2}{2}$

$$\therefore \int_a^b f(x)\,dx = (b-a)\left(\frac{b+a}{2}\right) = (b-a) f\left(\frac{a+b}{2}\right)$$

OR $\int_a^b f(x)\,dx = \frac{b^2}{2} - \frac{a^2}{2} = \frac{b}{2} f(b) - \frac{a}{2} f(a)$

The integral is represented as a series of the function values at a few points multiplied by weight factors, c_i

Scaling the Limits of the Integral

Given $\boxed{\int_a^b f(x)\,dx}$ first define $\boxed{u = x - c}$. $\boxed{dx = du}$

$\boxed{c = (a+b)/2}$

i.) When $x = b$, $u = b - \frac{a}{2} - \frac{b}{2} = \frac{b-a}{2}$.

ii) when $x = a$, $u = a - \frac{a}{2} - \frac{b}{2} = -\frac{b-a}{2}$.

$$\Rightarrow \int_a^b f(x)\,dx = \int_{-(b-a)/2}^{(b-a)/2} F(u)\,du$$

$\because$ $\boxed{dx = du}$ $\boxed{F(u) = f(u+c)}$

Now Scale $\boxed{z = \frac{u}{(b-a)/2}}$ $\boxed{dz = \frac{du}{(b-a)/2}}$

i.) When $u = (b-a)/2$, $z = 1$.

ii) When $u = -(b-a)/2$, $z = -1$.

$$\Rightarrow \int_{-(b-a)/2}^{(b-a)/2} F(u)\,du = \int_{-1}^{1} F\left(z\left[\frac{b-a}{2}\right]\right)\left(\frac{b-a}{2}\right)dz$$

$\boxed{G(z) = F\left(z\left[\frac{b-a}{2}\right]\right)\left(\frac{b-a}{2}\right)} \Rightarrow \int_{-1}^{1} G(z)\,dz$ rescaled The integral

Hence integrals of the form $\int_{-1}^{1} f(x)\,dx$ are regularly used.

Approximation with one Node

$$\boxed{\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} c_i f(x_i)}$$

$x_i \rightarrow$ Node
$c_i \rightarrow$ Weight (factor).

With just one node, the approximation is

$$\boxed{\int_{-1}^{1} f(x)\,dx \simeq c_1 f(x_1)}$$

Two unknown quantities c_1 and x_1, require two conditions.

i.) First take $\boxed{f(x) = 1}$, which is the lowest order in a polynomial.

$$\Rightarrow \int_{-1}^{1} (1)\,dx = c_1 \cdot 1 \Rightarrow x\Big|_{-1}^{1} = 2 = c_1$$

$$\Rightarrow \boxed{c_1 = 2}.$$

ii.) Then take $\boxed{f(x) = x}$, which is the linear order in a polynomial.

$$\Rightarrow \int_{-1}^{1} x\,dx = c_1 x_1 \Rightarrow \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1-1}{2} = 0 = c_1 x_1 = 2x_1$$

$$\Rightarrow \boxed{x_1 = 0}$$

$$\Rightarrow \boxed{\int_{-1}^{1} f(x)\,dx \simeq 2f(0)}$$

(approximation)

Example $\int_{-1}^{1} e^x\,dx = e^x\Big|_{-1}^{1}$

$= e - \frac{1}{e}$

$= 2.3504024$

If $f(x) = e^x \Rightarrow$ Integral is $2e^0 = 2$

Approximation with Two Nodes

$$\boxed{\int_{-1}^{1} f(x)\,dx \simeq c_1 f(x_1) + c_2 f(x_2)}$$

Four unknown quantities c_1, c_2, x_1, x_2.

i.) $\boxed{f(x) = 1}$, the lowest order polynomial.
(zero order)

$$\Rightarrow \int_{-1}^{1} 1\,dx = x\Big|_{-1}^{1} = 2 \Rightarrow \boxed{c_1 + c_2 = 2}$$

ii.) $\boxed{f(x) = x}$, the linear order polynomial.

$$\Rightarrow \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0 \Rightarrow \boxed{c_1 x_1 + c_2 x_2 = 0}$$

iii.) $\boxed{f(x) = x^2}$, the quadratic order polynomial.

$$\Rightarrow \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3} \Rightarrow \boxed{c_1 x_1^2 + c_2 x_2^2 = \frac{2}{3}}$$

iv.) $\boxed{f(x) = x^3}$, the cubic order polynomial.

$$\Rightarrow \int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0 \Rightarrow \boxed{c_1 x_1^3 + c_2 x_2^3 = 0}$$

Four equations for four unknown quantities.

We get $\boxed{c_2 = 2 - c_1}$ and $\boxed{x_2 = -\frac{c_1 x_1}{2 - c_1}}$.
(from the first two equations).

$\Rightarrow \frac{2}{3} = c_1 x_1^2 + (2 - c_1) \cdot \left(\frac{c_1 x_1}{2 - c_1} \right)^2$ | from the third equation.

$\Rightarrow \boxed{\frac{2}{3} = c_1 x_1^2 + \frac{(c_1 x_1)^2}{2 - c_1}}$.

From the fourth equation we get,

$$- c_1 x_1^3 = c_2 x_2^3 = (2 - c_1) \cdot \frac{-(c_1 x_1)^3}{(2 - c_1)^3}$$

$$\Rightarrow \cancel{+} c_1 x_1^3 = \cancel{+} \frac{c_1^3 x_1^3}{(2 - c_1)^2} \Rightarrow 1 = \frac{c_1^2}{(2 - c_1)^2}$$

$\Rightarrow (2 - c_1)^2 = c_1^2 \Rightarrow \boxed{2 - c_1 = \pm c_1}$, of which the only possible solution is $\boxed{2 - c_1 = c_1}$ $\Rightarrow \boxed{c_1 = 1}$ $\therefore$ With $\boxed{c = 1}$, $\boxed{c_2 = 2 - c_1 = 1}$.

$$\Rightarrow \frac{2}{3} = x_1^2 + x_1^2 \Rightarrow 2x_1^2 = \frac{2}{3} \Rightarrow \boxed{x_1 = \frac{1}{\sqrt{3}}}$$

From $c_1 x_1 = - c_2 x_2$ we get $\boxed{x_2 = -\frac{1}{\sqrt{3}}}$

(x_1 and x_2 can exchange their signs).

Hence, $\boxed{\int_{-1}^{1} f(x)\, dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)}$.

Example: $\boxed{f(x) = e^x}$. The exact integral is $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} e^x\,dx = e - \frac{1}{e} = 2.3504024$.

By Gaussian quadrature we get,

$$\int_{-1}^{1} f(x)\,dx = e^{1/\sqrt{3}} + e^{-1/\sqrt{3}} = 2.342696,$$

which is very close to 2.3504024 above.

Example: Integrate $\boxed{f(x) = \frac{1}{x}}$ between 3.1 and 3.9.

$$\Rightarrow I = \int_{3.1}^{3.9} \frac{dx}{x} = \ln\left(\frac{3.9}{3.1}\right) = 0.22957444$$

(the exact integral).

By Gaussian quadrature:

define $\boxed{u = x - c}$ where $c = \frac{3.1+3.9}{2} = 3.5$.

$\Rightarrow \boxed{du = dx}$ and $\boxed{x = u + c}$

$$\Rightarrow I = \int_{3.1}^{3.9} \frac{dx}{x} = \int_{-0.4}^{0.4} \frac{du}{u+3.5}.$$

Now scale $\boxed{z = \frac{u}{0.4}}$

Hence $\boxed{u = 0.4z}$ and $\boxed{du = 0.4\,dz}$

$$\Rightarrow I = \int_{-1}^{1} \frac{0.4\,dZ}{0.4Z + 3.5} = 0.4\int_{-1}^{1} F(z)\,dZ,$$

in which $\boxed{F(z) = \dfrac{1}{0.4Z + 3.5}}$

$$\Rightarrow I = 0.4\int_{-1}^{1} F(z)\,dZ = 0.4\left[F\left(\tfrac{1}{\sqrt{3}}\right) + F\left(-\tfrac{1}{\sqrt{3}}\right)\right]$$

$$\Rightarrow I = 0.4\left[\frac{1}{0.4(1/\sqrt{3}) + 3.5} + \frac{1}{0.4(-1/\sqrt{3}) + 3.5}\right]$$

$$\Rightarrow I = 0.4\left[0.26802896 + 0.30589834\right]$$

$\Rightarrow I = 0.22957092$ (close to the exact value of 0.22957444)

Example: The Gaussian Integral is.

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \simeq \underline{1.772}$$ (Through an exact integral)

We find $$\int_{-1}^{1} e^{-x^2}\,dx \simeq e^{-(1/\sqrt{3})^2} + e^{-(-\frac{1}{\sqrt{3}})^2}$$
$$\simeq 2e^{-1/3} \simeq \underline{1.433}$$

By the Gaussian quadrature we get a value that is comparable to the exact value of 1.772.