 (Cardan's Method)

The Roots of a Cubic Equation

A general cubic equation is (with all a_i real)

$$\boxed{a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0}$$

in which $a_3 \neq 0$. Dividing throughout by a_3 we get $\boxed{x^3 + b_2 x^2 + b_1 x + b_0 = 0}$

in which $\boxed{b_i = a_i / a_3}$, $i = 0, 1, 2$. (all b_i are real)

Now we transform the variable $\boxed{X = x + h}$

So that we have $(x+h)^3 + b_2 (x+h)^2 + b_1 (x+h) + b_0 = 0$

$$\Rightarrow x^3 + 3x^2 h + 3h^2 x + h^3 + b_2 (x^2 + 2xh + h^2)$$
$$+ b_1 x + b_1 h + b_0 = 0$$

Gathering all terms of the same power,

$$x^3 + x^2 (3h + b_2) + x(3h^2 + 2hb_2 + b_1)$$
$$+ (h^3 + b_2 h^2 + b_1 h + b_0) = 0$$

We choose $\boxed{h = -b_2/3}$ to get the standard form of a cubic equation going as

$$\boxed{x^3 + Px + q = 0}$$ in which,

the <u>quadratic term vanishes</u> and

$\boxed{p = 3h^2 + 2b_2 h + b_1}$, $\boxed{q = h^3 + b_2 h^2 + b_1 h + b_0}$.

<u>(p, q are real).</u>

In the equation $\boxed{x^3 + px + q = 0}$, we substitute $\boxed{x = y + z}$ to get,

$$y^3 + z^3 + 3y^2 z + 3z^2 y + p(y+z) + q = 0$$

$$\Rightarrow y^3 + z^3 + 3yz(y+z) + p(y+z) + q = 0$$

$$\Rightarrow \boxed{y^3 + z^3 + (3yz + p)(y+z) + q = 0}.$$

<u>y and z can have any value</u> under the constraint that <u>their sum is a root of $x^3 + px + q = 0$</u>. Accordingly we choose $\boxed{yz = -p/3}$, so that

$\boxed{y^3 + z^3 = -q}$ and $\boxed{y^3 z^3 = -\frac{p^3}{27}}$.

Consider an auxiliary ~~cubic~~ quadratic equation,

$\boxed{at^2 + bt + c = 0}$ whose roots are

$\boxed{\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$. Obviously $\boxed{\alpha + \beta = -\frac{b}{a}}$

and $\boxed{\alpha\beta = \frac{c}{a}}$.

Setting an equivalence between y^3, z^3 and α, β we see that y^3 and z^3 are the roots of ~~[illegible]~~ a quadratic equation, with $\boxed{-q \equiv -b/a}$ and $\boxed{\frac{c}{a} \equiv -\frac{p^3}{27}}$.

$\therefore$ $\boxed{t^2 + qt - \frac{p^3}{27} = 0}$ gives the roots as

$$t = \frac{-q \pm \sqrt{q^2 + 4p^3/27}}{2}$$

$$\Rightarrow \boxed{t = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

y^3 and z^3 are the two roots of this equation.

$\therefore$ $\boxed{y^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$ and

$\boxed{z^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$. Now since $\boxed{x = y + z}$,

$$\boxed{x = \left[-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{1/3} + \left[-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{1/3}}.$$

This is the solution of the standard form of the cubic equation by Cardan's method. Writing the discriminant $\boxed{D = \frac{q^2}{4} + \frac{p^3}{27}}$, $\boxed{x = \left(-\frac{q}{2} + \sqrt{D}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{D}\right)^{1/3}}$.

Now, $\boxed{y = \sqrt[3]{y^3}}$ and $\boxed{z = \sqrt[3]{z^3}}$ are the arithmetic roots. We have seen that any quantity has three cube roots, and so the <u>three cube roots of y^3</u> are $\boxed{y, y\omega, y\omega^2}$. Likewise the <u>three cube roots of z^3</u> are $\boxed{z, z\omega, z\omega^2}$. It appears that $\boxed{x^3+px+q=0}$ has <u>nine roots</u>, with all the combinations.

However, it is to be noted that $\boxed{yz = -\frac{p}{3}}$ is the equation that we <u>cubed</u> to get, $\boxed{y^3 z^3 = \frac{-p^3}{27}}$. But this cubic condition could also have been obtained from $\boxed{yz = -\frac{\omega p}{3}}$ and $\boxed{yz = -\frac{\omega^2 p}{3}}$, which are <u>NOT</u> the equations we started with.

Hence in looking for combinations that are the <u>correct solutions</u> of $\boxed{x^3 + px + q = 0}$, we should look only for those that satisfy $\boxed{yz = -\frac{p}{3}}$. Since $\boxed{\omega^3 = 1}$, such combinations are $\boxed{x_1 = y + z}$, $\boxed{x_2 = y\omega + z\omega^2}$ and $\boxed{x_3 = y\omega^2 + z\omega}$.

The <u>other six combinations</u> that we left out are solutions of either

(P.T.O.)

$\boxed{x^3 + \omega P x + q = 0}$ or $\boxed{x^3 + \omega^2 P x + q = 0}$.

Nature of the roots from $\boxed{D = \frac{q^2}{4} + \frac{p^3}{27}}$ (discriminant)

Now $\boxed{x = y + z}$, where $\boxed{y = \left(-\frac{q}{2} + \sqrt{D}\right)^{1/3}}$ and $\boxed{z = \left(-\frac{q}{2} - \sqrt{D}\right)^{1/3}}$. Obviously the solutions of y, z are the arithmetical cube roots.

Case I: When $\boxed{D > 0}$, y and z are real.

$\therefore$ $\boxed{x_1 = y + z}$ is a real root of $\boxed{x^3 + Px + q = 0}$ but $\boxed{x_2 = y\omega + z\omega^2}$ and $\boxed{x_3 = y\omega^2 + z\omega}$ are imaginary. Hence, for $D > 0$, one real root and two imaginary roots exist.

Case II: When $\boxed{D = 0}$, $\boxed{y = z = (-q/2)^{1/3}}$.

Hence with both y, z real $\boxed{x_1 = 2y = 2(-q/2)^{1/3}}$ is a real root. The other two roots are $\boxed{x_2 = y\omega + z\omega^2 = y(\omega + \omega^2)}$ and $\boxed{x_3 = y\omega^2 + z\omega = y(\omega^2 + \omega)}$.

Since $\boxed{1 + \omega + \omega^2 = 0}$ $\Rightarrow$ $\boxed{\omega + \omega^2 = -1}$. Using this condition we get $\boxed{x_2 = -y = x_3}$. Since, $y = (-q/2)^{1/3}$, $\boxed{x_2 = x_3 = -(-q/2)^{1/3}}$. Hence, for $\boxed{D = 0}$, all roots are real and two roots are equal.

Case III: When $\boxed{D<0}$, then y^3 and z^3 are in the form $\boxed{y^3 = (A+iB)}$ and $\boxed{z^3 = A-iB}$.

We write $\boxed{y = (A+iB)^{1/3} = M+iN}$ and $\boxed{z = (A-iB)^{1/3} = M-iN}$. Hence $\boxed{x_1 = (M+\cancel{iN}) + (M-\cancel{iN})}$

$\Rightarrow$ $\boxed{x_1 = 2M}$ which is a real root (with M and N both being real). Now $\boxed{x_2 = (M+iN)\omega + (M-iN)\omega^2}$

and similarly $\boxed{x_3 = (M+iN)\omega^2 + (M-iN)\omega}$. Noting $\boxed{\omega = \frac{-1+\sqrt{3}i}{2}}$ and $\boxed{\omega^2 = \frac{-1-\sqrt{3}i}{2}}$, we get.

$$x_2 = (M+iN)\left(\frac{-1+\sqrt{3}i}{2}\right) + (M-iN)\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$\Rightarrow x_2 = -\frac{M}{2} - \cancel{\frac{iN}{2}} + \cancel{\frac{M}{2}\sqrt{3}i} - \frac{N\sqrt{3}}{2} - \frac{M}{2} + \cancel{\frac{iN}{2}} - \cancel{\frac{M}{2}\sqrt{3}i} - \frac{\sqrt{3}N}{2}$$

$\Rightarrow$ $\boxed{x_2 = -M - N\sqrt{3}}$. The second root is also real.

Likewise, $x_3 = (M+iN)\left(\frac{-1-\sqrt{3}i}{2}\right) + (M-iN)\left(\frac{-1+\sqrt{3}i}{2}\right)$

$$\Rightarrow x_3 = -\frac{M}{2} - \cancel{\frac{iN}{2}} - \cancel{\frac{M\sqrt{3}i}{2}} + \frac{N\sqrt{3}}{2} - \frac{M}{2} + \cancel{\frac{iN}{2}} + \cancel{\frac{M\sqrt{3}i}{2}} + \frac{N\sqrt{3}}{2}$$

$\Rightarrow$ $\boxed{x_3 = -M + N\sqrt{3}}$. The third root is real.

Hence, for $\boxed{D<0}$, all the three roots are real, but M and N are not known from A and B. This is known as the irreducible case, because the cube root of a complex number is not generally given.

To know the roots we have to write $\boxed{A = r\cos\theta}$, $\boxed{B = r\sin\theta}$ and $\boxed{\tan\theta = B/A}$.

Now,

$$x_1 = y + z = (A+iB)^{1/3} + (A-iB)^{1/3}$$

$$\Rightarrow \quad x_1 = (r\cos\theta + i\, r\sin\theta)^{1/3} + (r\cos\theta - i\, r\sin\theta)^{1/3}$$

$$\Rightarrow x_1 = r^{1/3}(\cos\theta + i\sin\theta)^{1/3} + r^{1/3}(\cos\theta - i\sin\theta)^{1/3}$$

By Euler's formula $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$,

we can write $\boxed{e^{in\theta} = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

$$\Rightarrow x_1 = r^{1/3}\left[\cos\frac{\theta}{3} + \cancel{i\sin\frac{\theta}{3}} + \cos\frac{\theta}{3} - \cancel{i\sin\frac{\theta}{3}}\right]$$

$$\Rightarrow \boxed{x_1 = r^{1/3}\left(\cos\frac{\theta}{3}\right)\times 2 = 2r^{1/3}\cos\frac{\theta}{3}}$$

$\boxed{x_2 = y\omega + z\omega^2}$ and $\boxed{x_3 = y\omega^2 + z\omega}$.

We know that $\boxed{\omega = e^{i2\pi/3}}$ and $\boxed{\omega^2 = e^{i4\pi/3}}$

Further $\boxed{\omega^2 = 1/\omega = e^{-i2\pi/3}}$. Using these,

$$x_2 = y e^{i2\pi/3} + z e^{-i2\pi/3} = r^{1/3}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)e^{i2\pi/3} + r^{1/3}\left(\cos\frac{\theta}{3} - i\sin\frac{\theta}{3}\right)e^{-i2\pi/3}$$

$$\Rightarrow x_2 = r^{1/3} e^{i\theta/3} e^{i2\pi/3} + r^{1/3} e^{-i\theta/3} e^{-i2\pi/3}$$

$$\Rightarrow x_2 = r^{1/3} e^{i(\theta+2\pi)/3} + r^{1/3} e^{-i(\theta+2\pi)/3}$$

$$\Rightarrow x_2 = r^{1/3}\left[\cos\left(\frac{\theta+2\pi}{3}\right) + \cancel{i\sin\left(\frac{\theta+2\pi}{3}\right)} + \cos\left(\frac{\theta+2\pi}{3}\right) - \cancel{i\sin\left(\frac{\theta+2\pi}{3}\right)}\right]$$

$$\Rightarrow \boxed{x_2 = r^{1/3}\cos\left(\frac{\theta+2\pi}{3}\right)\times 2 = 2r^{1/3}\cos\left(\frac{\theta+2\pi}{3}\right)}$$

$$x_3 = y\omega^2 + z\omega = y e^{i4\pi/3} + z e^{i2\pi/3}$$

But $\boxed{\omega = 1/\omega^2 = e^{-i4\pi/3}}$. Using this,

$$x_3 = r^{1/3}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)e^{i4\pi/3} + r^{1/3}\left(\cos\frac{\theta}{3} - i\sin\frac{\theta}{3}\right) \times e^{-i4\pi/3}$$

$$\Rightarrow x_3 = r^{1/3} e^{i\theta/3} e^{i4\pi/3} + r^{1/3} e^{-i\theta/3} e^{-i4\pi/3}$$

$$\Rightarrow x_3 = r^{1/3} e^{i\left(\frac{\theta+4\pi}{3}\right)} + r^{1/3} e^{-i(\theta+4\pi)/3}$$

$$\Rightarrow x_3 = r^{1/3}\left[\cos\left(\frac{\theta+4\pi}{3}\right) + \cancel{i\sin\left(\frac{\theta+4\pi}{3}\right)} + \cos\left(\frac{\theta+4\pi}{3}\right) - \cancel{i\sin\left(\frac{\theta+4\pi}{3}\right)}\right]$$

$$\Rightarrow \boxed{x_3 = r^{1/3}.2\cos\left(\frac{\theta+4\pi}{3}\right) = 2r^{1/3}\cos\left(\frac{\theta+4\pi}{3}\right)}.$$

<u>Example:</u> $\boxed{x^3 - 15x - 126 = 0}$ $\quad P = -15$, $Q = -126$.

$$\therefore D = \frac{Q^2}{4} + \frac{P^3}{27} = 3969 - 125 = 3844 > 0.$$

Since $D > 0$, we have 1 real root and 2 complex conjugates.

$$y = \left(-\frac{Q}{2} + \sqrt{D}\right)^{1/3} = \left(\frac{126}{2} + \sqrt{3844}\right)^{1/3} = (63+62)^{1/3} = 5$$

$$z = \left(-\frac{Q}{2} - \sqrt{D}\right)^{1/3} = \left(\frac{126}{2} - \sqrt{3844}\right)^{1/3} = (63-62)^{1/3} = 1$$

$\therefore \boxed{y + z = x_1 = 5 + 1 = 6}$. The other other two roots are

$$\boxed{y\omega + z\omega^2 = x_2 = 5\left(\frac{-1+\sqrt{3}i}{2}\right) + 1\left(\frac{-1-\sqrt{3}i}{2}\right) = -3 + 2\sqrt{3}i}$$

and

$$\boxed{y\omega^2 + z\omega = x_3 = 5\left(\frac{-1-\sqrt{3}i}{2}\right) + 5\left(\frac{-1+\sqrt{3}i}{2}\right) = -3 - 2\sqrt{3}i}.$$

Vanishing Discriminants and Coinciding Roots.

Quadratic: $\boxed{f(x) = ax^2 + bx + c = 0}$.

The two roots are $\boxed{x_{1,2} = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}}$.

The discriminant $\boxed{D = b^2 - 4ac}$.

Consider $\boxed{f'(x) = 2ax + b = 0}$. $\Rightarrow \boxed{x = -b/2a}$

Using this $x = -b/2a$ in $\boxed{f(x)=0}$ we get,

$$\cancel{a}\frac{b^2}{4a^{\cancel{2}}a} + b\left(-\frac{b}{2a}\right) + c = 0 \Rightarrow b^2 - 2b^2 + 4ac = 0$$

$$\Rightarrow -b^2 + 4ac = 0.$$

$\Rightarrow \boxed{b^2 - 4ac = D = 0}$ The discriminant vanishes and two roots are equal when both $f(x) = f'(x) = 0$

Cubic: $\boxed{x^3 + px + q = 0}$ $f(x) = x^3 + px + q$

$\Rightarrow \boxed{f'(x) = 3x^2 + p = 0}$.

$\Rightarrow \boxed{x = (-p/3)^{1/2}}$. Use $\boxed{x = \sqrt{-\frac{p}{3}}}$ in $\boxed{f(x) = x^3 + px + q = 0}$

to get, $-\frac{p}{3}\sqrt{-\frac{p}{3}} + p\sqrt{-\frac{p}{3}} + q = 0$

$$\Rightarrow \frac{2p}{3}\sqrt{-\frac{p}{3}} = -q \Rightarrow \frac{4p^2}{9}\left(-\frac{p}{3}\right) = q^2.$$

$\Rightarrow \boxed{\frac{q^2}{4} + \frac{p^3}{27} = 0}$. Once again $\boxed{D = 0}$ when $\boxed{f(x) = f'(x) = 0}$.

Conclusion: Two roots coincide when $D = 0$ due to $f(x) = f'(x) = 0$.

Roots of a General Quartic Equation: | Ferrari's method

A general quartic equation (with all a_i real) is $\boxed{a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0}$, in which $a_4 \neq 0$. Dividing throughout by a_4,

$\boxed{x^4 + 2Px^3 + Qx^2 + 2Rx + S = 0}$, in which

$\boxed{2P = a_3/a_4}$, $\boxed{Q = a_2/a_4}$, $\boxed{2R = a_1/a_4}$, $\boxed{S = \dfrac{a_0}{a_4}}$.

We now add $\boxed{(ax+b)^2}$ on both sides to get

$$x^4 + 2Px^3 + Qx^2 + 2Rx + S + a^2x^2 + 2abx + b^2 = (ax+b)^2$$

Gathering all terms of the same power,

$$x^4 + 2Px^3 + (Q+a^2)x^2 + 2(R+ab)x + (S+b^2) = (ax+b)^2.$$

Suppose the left hand side is a perfect square going as $\boxed{(x^2+Px+K)^2}$, we can write

$$x^4 + P^2x^2 + K^2 + 2Px^3 + 2PKx + 2Kx^2 = (ax+b)^2$$

Gathering all the terms ~~again~~ of the same power,

$$x^4 + 2Px^3 + (P^2+2K)x^2 + 2PKx + K^2 = (ax+b)^2.$$

On comparing we get, $\boxed{P^2 + 2K = Q + a^2}$, $\boxed{PK = R + ab}$ and $\boxed{K^2 = S + b^2}$. Three equations for a, b, K.

From these relations we get $a = \frac{Pk - R}{b}$

$\therefore\ a^2 = \frac{(Pk-R)^2}{b^2}$. But $\boxed{a^2 = P^2 + 2k - q}$ and $\boxed{b^2 = k^2 - s}$.

Hence we get $\frac{(Pk-R)^2}{(k^2-s)} = P^2 + 2k - q$

$\Rightarrow (Pk-R)^2 = (P^2+2k-q)(k^2-s)$.

$\Rightarrow \cancel{P^2k^2} - 2PkR + R^2 = \cancel{P^2k^2} + 2k^3 - qk^2 - sP^2 - 2ks + qs$

$\Rightarrow \boxed{2k^3 - qk^2 + 2(PR - s)k + (qs - sP^2 - R^2) = 0}$.

The above equation is an auxiliary Cubic equation in k, one of whose roots must be real. Using this real root in $(x^2 + Px + k)^2 = (ax + b)^2$, we get.

$x^2 + Px + k = \pm(ax + b)$

$\Rightarrow \boxed{x^2 + (P \mp a)x + (k \mp b) = 0}$, which is a pair of quadratic equations giving four roots.

We now write $\boxed{a = \pm|a|}$ and $\boxed{b = \pm|b|}$

Case 1: $\boxed{a, b > 0}$. In this case the two equations are $\boxed{x^2 + (P - |a|)x + (k - |b|) = 0}$ and

$\boxed{x^2 + (P + |a|)x + (k + |b|) = 0}$.

Case II: $\boxed{a, b < 0}$. In this case the two equations are $\boxed{x^2 + (p + |a|)x + (k + |b|) = 0}$ and $\boxed{x^2 + (p - |a|)x + (k - |b|) = 0}$.

Case III: $\boxed{a > 0, b < 0}$. The two equations are

$$\boxed{x^2 + (p - |a|)x + (k + |b|) = 0}$$ and

$$\boxed{x^2 + (p + |a|)x + (k - |b|) = 0}$$

Case IV: $\boxed{a < 0, b > 0}$. The two equations are

$$\boxed{x^2 + (p + |a|)x + (k - |b|) = 0}$$ and

$$\boxed{x^2 + (p - |a|)x + (k + |b|) = 0}$$

The alternating signs of a and b simply exchange the two equations.

Example: $\boxed{x^4 - 2x^3 - 5x^2 + 10x - 3 = 0}$.

$\boxed{P = -1}$, $\boxed{Q = -5}$, $\boxed{R = 5}$, $\boxed{S = -3}$. The auxiliary cubic equation is

$$\boxed{2k^3 - Qk^2 + 2(PR - S)k + (QS - SP^2 - R^2) = 0}$$

$$\therefore\ 2k^3 + 5k^2 + 2(-1 \times 5 + 3)k + (15 + 3 - 25) = 0$$

$$\Rightarrow \boxed{2k^3 + 5k^2 - 4k - 7 = 0}.$$ The solution of this cubic equation is $k = -1$. We can check it from $-2 + 5 + 4 - 7 = 0$. Now $a^2 = P^2 + 2k - Q = 1 - 2 + 5 = 4$. Similarly

$b^2 = K^2 - S \Rightarrow b^2 = 1 + 3 = 4$. And finally $ab = PK - R = 1 - 5 = -4$. Hence we have $\boxed{a^2 = 4}$, $\boxed{b^2 = 4}$ and $\boxed{ab = -4}$. Clearly $\boxed{a = \pm 2}$ and $\boxed{b = \pm 2}$ but <u>a and b must be of opposite signs</u> so that $\boxed{ab = -4}$.

We choose $\boxed{a = 2}$ and $\boxed{b = -2}$. With these used in $\boxed{x^2 + (P \mp a)x + (K \mp b) = 0}$.

$\therefore$ We get $x^2 + (-1-2)x + (-1+2) = 0$ and $x^2 + (-1+2)x + (-1-2) = 0$.

$\Rightarrow \boxed{x^2 - 3x + 1 = 0}$ or $\boxed{x^2 + x - 3 = 0}$.

The roots of the first equation are

$$x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2},$$

and of the second equation are $\boxed{x = \frac{-1 \pm \sqrt{1+12}}{2}}$

$$\Rightarrow \boxed{x = \frac{-1 \pm \sqrt{13}}{2}}.$$

If we choose $\boxed{a = -2}$ and $\boxed{b = 2}$, we get $\boxed{x^2 + (-1+2)x + (-1-2) = 0}$, which gives $\boxed{x^2 + x - 3 = 0}$. The other equation is $\boxed{x^2 + (-1-2)x + (-1+2) = 0}$ giving $\boxed{x^2 - 3x + 1 = 0}$ $\Rightarrow$ When a and b exchange signs, the two equations are exchanged.

Root of a Quartic Equation without the Cubic Term

Given an equation $\boxed{x^4 + qx^2 + Rx + S = 0}$,

We write $\boxed{x^4 + qx^2 + Rx + S = (x^2 + kx + l)\times(x^2 - kx + m)}$.

The right hand side is expanded as

$$x^4 - \cancel{kx^3} + mx^2 + \cancel{kx^3} - k^2x^2 + kmx + lx^2 - klx + lm = x^4 + (m - k^2 + l)x^2 + (km - kl)x + lm$$

This we write as $\boxed{x^4 + (m - k^2 + l)x^2 + k(m - l)x + lm = 0}$

and compare with $\boxed{x^4 + qx^2 + Rx + S = 0}$ to get $\boxed{lm = S}$, $\boxed{k(m - l) = R}$, $\boxed{m + l - k^2 = q}$.

From the last two conditions we get

$\boxed{m + l = q + k^2}$ and $\boxed{m - l = R/k}$.

Hence on adding the two equations above we get $\boxed{2m = q + k^2 + \frac{R}{k}}$ and on taking the difference we get $\boxed{2l = q + k^2 - \frac{R}{k}}$.

Also $lm = S \Rightarrow \boxed{4lm = 4S}$ Using the formulae of $2m$ and $2l$ we get,

$$\boxed{4lm = 4S = \left[(q + k^2) + \frac{R}{k}\right]\left[(q + k^2) - \frac{R}{k}\right]}.$$

$$\Rightarrow 4s = (q+k^2)^2 - \frac{R^2}{k^2}$$ | using the form $(a+b)(a-b) = a^2-b^2$

$$\Rightarrow 4sk^2 = k^2(q^2 + 2qk^2 + k^4) - R^2$$

$$\Rightarrow \boxed{k^6 + 2qk^4 + (q^2 - 4s)k^2 - R^2 = 0}$$

The above is a cubic equation in k^2, which has one root that is real and positive. (Every equation of an even degree, with a negative last term, has at least two real roots, one positive and one negative).

Here R is real. $\Rightarrow R^2 > 0$. $\therefore -R^2 < 0$ (negative).

We use the real root $\boxed{k^2 > 0}$ to solve

$\boxed{x^2 + kx + l = 0}$ and $\boxed{x^2 - kx + m = 0}$.

Now $\boxed{m = \frac{q+k^2}{2} + \frac{R}{2k}}$, $\boxed{l = \frac{q+k^2}{2} - \frac{R}{2k}}$.

If $k^2 > 0 \Rightarrow k$ is real. Write $k = \pm|k|$.

i.) When $k = +|k|$, $\boxed{m = \left(\frac{q+k^2}{2}\right) + \frac{R}{2|k|}}$

and $\boxed{l = \left(\frac{q+k^2}{2}\right) - \frac{R}{2|k|}}$.

ii.) When $k = -|k|$, $\boxed{m = \left(\frac{q+k^2}{2}\right) - \frac{R}{2|k|}}$

and $\boxed{l = \left(\frac{q+k^2}{2}\right) + \frac{R}{2|k|}}$. Obviously, m and l exchange values, when k changes sign from $+|k|$ to $-|k|$.

Example: $\boxed{x^4 - 2x^2 + 8x - 3 = 0}$.

Comparing with $\boxed{x^4 + Qx^2 + Rx + S = 0}$, we see $\boxed{Q = -2}$, $\boxed{R = 8}$, $\boxed{S = -3}$.

Factorising $\boxed{(x^2 + kx + l)(x^2 - kx + m) = 0}$, we have $\boxed{lm = S} \Rightarrow \boxed{lm = -3}$, $\boxed{k(m - l) = R}$

$\Rightarrow \boxed{k(m - l) = 8}$, $\boxed{m + l - k^2 = Q} \Rightarrow \boxed{m + l - k^2 = -2}$.

Also, $\boxed{k^6 + 2Qk^4 + (Q^2 - 4S)k^2 - R^2 = 0}$, which becomes $\boxed{k^6 - 4k^4 + 16k^2 - 64 = 0}$, using the values of Q, R and S. The solution of this equation is $\boxed{k^2 = 4}$. $\Rightarrow \boxed{k = \pm 2}$.

(Check: $\cancel{64} - \cancel{4 \times 16} + \cancel{16 \times 4} - \cancel{64} = 0$) $\rightarrow$ verified.

i.) $\boxed{k = +2}$. $\Rightarrow \boxed{m = \frac{Q + k^2}{2} + \frac{R}{2k}} \Rightarrow \boxed{m = 3}$

and $l = \boxed{\frac{Q + k^2}{2} - \frac{R}{2k}} \Rightarrow \boxed{l = -1}$ Using k, l, m we have $(x^2 + 2x - 1)(x^2 - 2x + 3) = 0$

$\Rightarrow \boxed{x^2 + 2x - 1 = 0} \Rightarrow \boxed{x = \frac{-2 \pm \sqrt{4 + 4}}{2} = -1 \pm \sqrt{2}}$.

and $\boxed{x^2 - 2x + 3 = 0} \Rightarrow \boxed{x = \frac{2 \pm \sqrt{4 - 12}}{2} = 1 \pm \sqrt{2}\, i}$

ii.) $\boxed{k = -2} \Rightarrow \boxed{m = \frac{Q + k^2}{2} - \frac{R}{2|k|}} \Rightarrow \boxed{m = -1}$ and

$\boxed{l = \frac{Q + k^2}{2} + \frac{R}{2|k|}} \Rightarrow \boxed{l = 3}$. But we now have $\boxed{x^2 - 2x + 3 = 0}$ and $\boxed{x^2 + 2x - 1 = 0}$ with the same four roots.