SC 216 Calculus with Complex Variables

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$\boxed{\text{Solutions of } y'' + P(x)y' + Q(x) = 0 \text{ near the regular singular point } x=0}$ ---(1)

METHOD OF FROBENIUS

— We seek a solution of the form

Frobenius Series

$$y = x^m(a_0 + a_1x + a_2x^2 + \cdots) \quad \text{---(2)}$$

$m \to$ may be −ve integer, a fraction or irrational real no.

$a_0 \neq 0$ & m is a number we need to find

— Why we seek the solution of the form of Frob. series?

□ Consider the Euler's Equation

$$x^2y'' + pxy' + qy = 0 \quad \text{----(3)}$$

$$\Rightarrow \quad y'' + \frac{p}{x}y' + \frac{qy}{x^2} = 0 \quad \text{----(4)}$$

$\Rightarrow P(x) = \frac{p}{x}$ & $Q(x) = \frac{q}{x^2}$ Thus $x=0$ is a regular singular point whenever const. p & q are not both zero.

Change the independent variable from x to $z = \ln x$

$$\Rightarrow \quad y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x}$$

$$\& \quad y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy}{dz}\left(-\frac{1}{x^2}\right) + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$\Rightarrow y'' = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$y'' = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$$

Substituting y' & y'' in (4) we get

$$\boxed{\frac{d^2y}{dz^2} + (p-1)\frac{dy}{dz} + qy = 0} \quad --- (5)$$

A O.D.E. with constt. coefficients.

$\Rightarrow$ A.E. is $m^2 + (p-1)m + q = 0$

If roots are m_1 & m_2 then (5) has following independent solutions

$$\left.\begin{array}{l} e^{m_1 z} \ \& \ e^{m_2 z} \quad (\text{if } m_1 \neq m_2) \\ e^{m_1 z} \ \& \ z e^{m_1 z} \quad (\text{if } m_1 = m_2) \end{array}\right\} \text{---(6).}$$

But $e^z = x$ thus the corresponding solutions for eqn (3) will be

$$\left.\begin{array}{l} x^{m_1} \ \& \ x^{m_2} \quad (\text{if } m_1 \neq m_2) \\ x^{m_1} \ \& \ x^{m_1} \ln x \quad (\text{if } m_1 = m_2) \end{array}\right\} \text{---(7)}$$

— Equation (3) in most general form will be

$$y'' + \left(\frac{p_0 + p_1 x + p_2 x^2 + \cdots}{x}\right) y' + \left(\frac{q_0 + q_1 x + q_2 x^2 + \cdots}{x^2}\right) y = 0$$

(the most general D.E. with a regular singular pt. at $x = 0$) --- (8)

Now since we can go from (3) to (8) by replacing constt. by power series it is natural to guess ~~that~~ the solutions from (7) to the solutions of (8) can be obtained by replacing x^m by Frob. series (2)

$\therefore$ (8) will have independent solutions as eqn (2)
and $y = x^m \ln x \,(a_0 + a_1 x + a_2 x^2 + \cdots)$ $x > 0$

Why $a_0 \neq 0$?

Example $2x^2y'' + x(2x+1)y' - y = 0$ ---(9)

$\Rightarrow \quad y'' + \frac{(1/2 + x)}{x} y' + \frac{-1/2}{x^2} y = 0$

$\Rightarrow \quad xP(x) = \frac{1}{2} + x \quad \& \quad x^2Q(x) = -\frac{1}{2}$

$\Rightarrow x = 0$ is a regular singular pt.

$\Rightarrow$ We seek a solution of the form of Frobenius series

$$y = x^m(a_0 + a_1x + a_2x^2 + \dots)$$

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots$$

$\Rightarrow y' = a_0mx^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \dots$

$\& \; y'' = a_0m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + a_2(m+2)(m+1)x^m + \dots$

Now substitute these in (9) & the method is similar except we also need to find the value of m.

After the common factor x^{m-2} is canceled we get

$$a_0m(m-1) + a_1(m+1)mx + a_2(m+2)(m+1)x^2 + \dots + \left(\frac{1}{2} + x\right)\left[a_0m + a_1(m+1)x + a_2(m+2)x^2 + \dots\right] - \frac{1}{2}(a_0 + a_1x + a_2x^2 + \dots) = 0$$

$\Rightarrow$

$$\left.\begin{aligned} a_0\left[m(m-1) + \frac{1}{2}m - \frac{1}{2}\right] &= 0 \\ a_1\left[(m+1)m + \frac{1}{2}(m+1) - \frac{1}{2}\right] + a_0m &= 0 \\ a_2\left[(m+2)(m+1) + \frac{1}{2}(m+2) - \frac{1}{2}\right] + a_1(m+1) &= 0 \end{aligned}\right\} \quad \text{---(10)}$$

$\dots$

$\because a_0 \neq 0$

$$\boxed{m(m-1) + \frac{1}{2}m - \frac{1}{2} = 0} \quad \text{----- (11)}$$

indicial eq^n

roots are $m_1 = 1$ & $m_2 = -\frac{1}{2}$

For $m_1 = 1$

$$a_1 = -\frac{a_0}{2\cdot 1 + \frac{1}{2}\cdot 2 - \frac{1}{2}} = -\frac{2}{5}a_0$$

$$a_2 = -\frac{2a_1}{3\cdot 2 + \frac{1}{2}\cdot 3 - \frac{1}{2}} = -\frac{2}{7}a_1$$
$$= \frac{4}{35}a_0$$

.....

& for $m_2 = -\frac{1}{2}$ we get

$$a_1 = \frac{1/2\, a_0}{\frac{1}{2}(-\frac{1}{2}) + \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}} = -a_0$$

$$a_2 = \frac{1/2\, a_1}{\frac{3}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{3}{2} - \frac{1}{2}} = -\frac{1}{2}a_1 = \frac{1}{2}a_0$$

$\Rightarrow$ we get two Frobenius series solns in each case we have put $a_0 = 1$

$$y_1 = x\left(1 - \frac{2}{5}x + \frac{4}{35}x^2 + \cdots\right) \text{---(12)}$$

$$y_2 = x^{-1/2}\left(1 - x + \frac{1}{2}x^2 + \cdots\right) \text{---(13)}$$

Both y_1 & y_2 are l.i. for $x > 0$, (Verify!)

So a general soln. is

$$y = c_1 x\left(1 - \frac{2}{5}x + \frac{4}{35}x^2 + \cdots\right)$$

$$+ c_2 x^{-1/2}\left(1 - x + \frac{x^2}{2} + \cdots\right)$$

In more general case indicial eqn will be $m(m-1) + m p_0 + q_0 = 0$ ---(14)

The more general result is given below.

THEOREM

Assume that $x = 0$ is a regular singular pt. of the D.E. $y'' + P(x)y' + Q(x)y = 0$ and ---(15) the power series ~~soln~~ expansions

$$xP(x) = \sum_{n=0}^{\infty} p_n x^n \;\&\; x^2 Q(x) = \sum_{n=0}^{\infty} q_n x^n$$

are valid on an interval $|x| < R$, $R > 0$

Let the indicial eqn

$$m(m-1) + m p_0 + q_0 = 0$$

have real roots m_1 & m_2 with $m_2 \le m_1$. Then the eqn (15) has at least one soln.

$$y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n \quad (a_0 \neq 0)$$

on the interval $0 < x < R$, where the a_n are determined in terms of a_0 by the recursion formula

$$a_n\left[(m+n)(m+n-1) + (m+n)p_0 + q_0\right] + \sum_{k=0}^{n-1} a_k\left[(m+k)p_{n-k} + q_{n-k}\right] = 0 \quad ---(16)$$

with m is replaced by m_1 and the series

$\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$.

Further if $m_1 - m_2 \neq 0$ or a +ve integer then eqn (15) has a second independent soln.

$$y_2 = x^{m_2} \sum_{n=0}^{\infty} a_n x^n \quad (a_0 \neq 0)$$

on the same interval, where in this case the a_n are determined by formula (16) with m replaced by m_2 and again the series $\sum a_n x^n$ conv. for $|x| < R$.

$\boxed{\text{Remark}}$ How to find second soln. when $m_1 - m_2 = 0$ or a +ve integer.

$\boxed{\text{CASE A}}$ If $m_1 = m_2$ there can not exist a second Frobenius series soln.

When $m_1 - m_2 (> 0) \in \mathbb{Z}$

Put $m = m_2$ in (16) and write it as

$$a_n f(m_2 + n) = -a_0(m_2 p_n + q_n) - \cdots - a_{n-1}\left[(m_2+n-1)p_1 + q_1\right] \quad \text{--- (17)}$$

where $f(m) = m(m-1) + m p_0 + q_0$

Since $f(m_2+n) = 0$ for certain +ve integer n there is a difficulty in calculating a_n.

$\boxed{\text{CASE B}}$ If R.H.S. of eqn (17) is $\neq 0$ when $f(m_2+n) = 0$ then there is no possible way of continuing the coff. $\Rightarrow \nexists$ exist a second Frobenius series soln.

$\boxed{\text{CASE C}}$ If R.H.S. of eqn (17) is $= 0$ when $f(m_2+n) = 0$, then a_n can be assigned any value.

In particular, we can put $a_n = 0$ and continue. $\Rightarrow$ In this case there does exist ($\exists$) a second Frobenius series soln.

Question: What form second soln. takes when $m_1 - m_2 = 0$ or $(>0) \in \mathbb{Z}$

□ Define a +ve integer k by $k = m_1 - m_2 + 1$ then indicial eqn (14) can be written as

$$(m-m_1)(m-m_2) = m^2 - (m_1+m_2)m + m_1 m_2 = 0$$

$$\Rightarrow p_0 - 1 = -(m_1+m_2)$$

$$\text{or } m_2 = 1 - p_0 - m_1 \quad \Rightarrow k = 2m_1 + p_0$$

now we can find second soln. y_2 from the known soln.

$$y_1 = x^{m_1}(a_0 + a_1 x + \cdots)$$ by writing

$$y_2 = v y_1 \quad \text{where} \quad v' = \frac{1}{y_1^2} \int e^{-\int p(x)\,dx}$$

$$= \frac{1}{x^{2m_1}(a_0+a_1x+\cdots)} e^{-\int ((p_0/x) + p_1 + \cdots)\,dx}$$

$$= \frac{1}{x^{2m_1}(a_0+a_1x+\cdots)} e^{(-p_0 \ln x - p_1 x - \cdots)}$$

$$= \frac{1}{x^k(a_0+a_1x+\cdots)} e^{-(p_1 x - \cdots)} = \frac{1}{x^k} g(x) \text{ (say)}$$

— $g(x)$ is analytic at $x=0$ with $g(0) = \frac{1}{a_0^2}$ so in some interval about the pt. $x=0$, we have

$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots, \quad b_0 \neq 0$$

$$\Rightarrow \quad v' = b_0 x^{-k} + b_1 x^{-k+1} + \cdots + b_{k-1} x^{-1} + b_k + \cdots$$

$$\Rightarrow \quad v = \frac{b_0 x^{-k+1}}{-k+1} + \frac{b_1 x^{-k+2}}{-k+2} + \cdots + b_{k-1} \ln x + b_k x + \cdots$$

$$\Rightarrow \quad y_2 = y_1 v = y_1 \left(\frac{b_0 x^{-k+1}}{-k+1} + \cdots + b_{k-1} \ln x + b_k x + \cdots \right)$$

$$= b_{k-1} y_1 \ln x + x^{m_1} (a_0 + a_1 x + \cdots) \left(\frac{b_0 x^{-k+1}}{-k+1} + \cdots \right)$$

need factor x^{-k+1} & use $m_1 - k + 1 = m_2$ & multiplying the 2 power series we get

$$\boxed{y_2 = b_{k-1} y_1 \ln x + x^{m_2} \sum_{n=0}^{\infty} c_n x^n} \quad \text{— (18)}$$

as our second soln.

The general form of the second soln. is

$$\boxed{y_2 = y_1 \ln x + x^{m_2} \sum_{n=0}^{\infty} c_n x^n} \quad \text{— (19)}$$

$\boxed{\text{Example}}$ Gauss's Hypergeometric eqn

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0 \quad \text{——— (1)}$$

$\left.\begin{matrix} a \\ b \\ c \end{matrix}\right\}$ constt.

$\square$

$\Rightarrow P(x) = \dfrac{c-(a+b+1)x}{x(1-x)}$ & $Q(x) = \dfrac{-ab}{x(1-x)}$

$\Rightarrow x=0$ & $x=1$ are the only singular pt. on the x-axis.

Now $xP(x) = c + [c-(a+b+1)]x + \ldots$

& $x^2Q(x) = -\dfrac{abx}{1-x} = -abx(1+x+x^2+\ldots)$

$$= -abx - abx^2 \ldots\ldots$$

$\Rightarrow x=0$ (and similarly $x=1$) is a regular singular pt. (R.S.P.)

$\Rightarrow p_0 = c$ & $q_0 = 0$

so Indicial eqn is

$m(m-1) + mc = 0$ or $m[m-(1-c)] = 0$

$\Rightarrow \quad m_1 = 0$ & $m_2 = 1-c$

If $1-c$ is not a +ve integer i.e. if $c \neq 0$ or a $-$ve integer then we will have a soln. of the form m (Why?)

$$y = x^0 \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + \ldots$$

$$a_0 \neq 0 \quad \text{——— (2)}$$

$\Rightarrow$ After substitution

we get

$$a_{n+1} = \frac{(a+n)(b+n)}{(n+1)(c+n)} a_n \quad ---(3)$$

Put $a_0 = 1$ & find a_n

$$a_1 = \frac{ab}{1 \cdot c}, \quad a_2 = \frac{a(a+1)\,b(b+1)}{1 \cdot 2 \cdot c(1+c)}$$

$$a_3 = \frac{a(a+1)(a+2)\,b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)}, \ldots$$

$\Rightarrow$

$$y = 1 + \frac{ab}{1 \cdot c} x + \frac{a(a+1)\,b(b+1)}{1 \cdot 2 \cdot c \cdot (c+1)} x^2 + \ldots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\,b(b+1)\cdots(b+n-1)}{n!\,c(c+1)\cdots(c+n-1)} x^n \quad ---(4)$$

This is called <u>Hypergeometric series</u>

denoted by $F(a, b, c, x)$

- If we put $a = 1$ & $c = b$ we get

$$F(1, b, b, x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

(geometric series)

- If a or $b = 0$ or a $-$ve integer the series (4) breaks off and is a polynomial otherwise ratio test shows that it converges for $|x| < 1$

Since (3) gives

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{(a+n)(b+n)}{(n+1)(c+n)}\right| |x| \to |x| \text{ as } n \to \infty$$

$\Rightarrow$ if $c \neq 0$ or $\neq -$ve integer $F(a, b, c, x)$ is analytic on $|x| < 1$.

Note $F(a, b, c, x) = F(b, a, c, x)$

— If $1-c \neq 0$ or $\neq$ -ve integer $\Rightarrow c \neq$ +ve integer then $\exists$ a second independent soln. of (1) near $x=0$ with exponent $m_2 = 1-c$

$\Rightarrow y = x^{1-c}(a_0 + a_1 x + a_2 x^2 + \dots)$ in (1)

$\Rightarrow$ Put & solve

More instructive manner would be

put $y = x^{1-c} z$ & (1) becomes

$$x(1-x) z'' + \left[(2-c) - \left([a-c+1] + [b-c+1] + 1\right)x\right] z' - (a-c+1)(b-c+1) z = 0$$

which is hypergeometric eq^n ——— (5) with constt. a, b, c replaced by $a-c+1$, $b-c+1$ & $2-c$

we know what soln. $\because$ (5) has ~~soln~~.

we get $Z = F(a-c+1, b-c+1, 2-c, x)$

$\Rightarrow y = x^{1-c} F(a-c+1, b-c+1, 2-c, x)$

$\Rightarrow$ when c is not an integer we have

$$\boxed{y = c_1 F(a, b, c, x) + c_2 x^{1-c} F(a-c+1, b-c+1, 2-c, x)}$$

as a general soln of the hypergeometric eq^n near the singular pt. $x=0$.

Solve the eq^n (1) near the pt. $x=1$

by introducing $t = 1-x$

$\Rightarrow t(1-t) y'' + [(a+b-c+1) - (a+b+1) t] y' - aby = 0$

we get

$y = c_1 F(a, b, a+b-c+1, 1-x) + c_2 (1-x)^{c-a-b} F(c-b, c-a, c-a-b+1, 1-x)$