The phasor voltage at the load is now the sum of the incident and reflected voltage phasors, evaluated at $z = 0$:

$$V_L = V_{0i} + V_{0r} \tag{71}$$

Additionally, the current through the load is the sum of the incident and reflected currents, also at $z = 0$:

$$I_L = I_{0i} + I_{0r} = \frac{1}{Z_0}[V_{0i} - V_{0r}] = \frac{V_L}{Z_L} = \frac{1}{Z_L}[V_{0i} + V_{0r}] \tag{72}$$

We can now solve for the ratio of the reflected voltage amplitude to the incident voltage amplitude, defined as the *reflection coefficient*, $\Gamma$:

$$\boxed{\Gamma \equiv \frac{V_{0r}}{V_{0i}} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi_r}} \tag{73}$$

where we emphasize the complex nature of $\Gamma$—meaning that in general, a reflected wave will experience a reduction in amplitude and a phase shift, relative to the incident wave.

Now, using (71) with (73), we may write

$$V_L = V_{0i} + \Gamma V_{0i} \tag{74}$$

from which we find the *transmission coefficient,* defined as the ratio of the load voltage amplitude to the incident voltage amplitude:

$$\boxed{\tau \equiv \frac{V_L}{V_{0i}} = 1 + \Gamma = \frac{2Z_L}{Z_0 + Z_L} = |\tau| e^{j\phi_t}} \tag{75}$$

A point that may at first cause some alarm is that if $\Gamma$ is a positive real number, then $\tau > 1$; the voltage amplitude at the load is thus greater than the incident voltage. Although this would seem counterintuitive, it is not a problem because the load current will be lower than that in the incident wave. We will find that this always results in an average *power* at the load that is less than or equal to that in the incident wave. An additional point concerns the possibility of loss in the line. The incident wave amplitude that is used in (73) and (75) is always the amplitude that occurs *at the load*—after loss has occurred in propagating from the input.

Usually, the main objective in transmitting power to a load is to configure the line/load combination such that there is no reflection. The load therefore receives all the transmitted power. The condition for this is $\Gamma = 0$, which means that the load impedance must be equal to the line impedance. In such cases the load is said to be *matched* to the line (or vice versa). Various impedance-matching methods exist, many of which will be explored later in this chapter.

Finally, the fractions of the incident wave *power* that are reflected and dissipated by the load need to be determined. The incident power is found from (64), where this time we position the load at $z = L$, with the line input at $z = 0$.

$$\langle \mathcal{P}_i \rangle = \frac{1}{2}\mathrm{Re}\left\{ \frac{V_0 V_0^*}{|Z_0|} e^{-2\alpha L} e^{j\theta} \right\} = \frac{1}{2}\frac{|V_0|^2}{|Z_0|} e^{-2\alpha L} \cos\theta \tag{76a}$$

The reflected power is then found by substituting the reflected wave voltage into (76$a$), where the latter is obtained by multiplying the incident voltage by $\Gamma$:

$$\langle \mathcal{P}_r \rangle = \frac{1}{2}\mathrm{Re}\left\{ \frac{(\Gamma V_0)(\Gamma^* V_0^*)}{|Z_0|} e^{-2\alpha L} e^{j\theta} \right\} = \frac{1}{2}\frac{|\Gamma|^2 |V_0|^2}{|Z_0|} e^{-2\alpha L} \cos\theta \tag{76b}$$

The reflected power fraction at the load is now determined by the ratio of (76$b$) to (76$a$):

$$\boxed{\frac{\langle \mathcal{P}_r \rangle}{\langle \mathcal{P}_i \rangle} = \Gamma\Gamma^* = |\Gamma|^2} \tag{77a}$$

The fraction of the incident power that is transmitted into the load (or dissipated by it) is therefore

$$\boxed{\frac{\langle \mathcal{P}_t \rangle}{\langle \mathcal{P}_i \rangle} = 1 - |\Gamma|^2} \tag{77b}$$

The reader should be aware that the transmitted power fraction is *not* $|\tau|^2$, as one might be tempted to conclude.

Illustrations

In situations involving the connection of two semi-infinite transmission lines having different characteristic impedances, reflections will occur at the junction, with the second line being treated as the load. For a wave incident from line 1 ($Z_{01}$) to line 2 ($Z_{02}$), we find

$$\boxed{\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}} \tag{78}$$

The fraction of the power that propagates into the second line is then $1 - |\Gamma|^2$.

## EXAMPLE 11.7

A 50 Ω lossless transmission line is terminated by a load impedance, $Z_L = 50 - j75\ \Omega$. If the incident power is 100 mW, find the power dissipated by the load.

**Solution.** The reflection coefficient is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - j75 - 50}{50 - j75 + 50} = 0.36 - j0.48 = 0.60e^{-j.93}$$

Then

$$\langle \mathcal{P}_t \rangle = (1 - |\Gamma|^2)\langle \mathcal{P}_i \rangle = [1 - (0.60)^2](100) = 64\ \mathrm{mW}$$

## EXAMPLE 11.8

Two lossy lines are to be joined end-to-end. The first line is 10 m long and has a loss rating of 0.20 dB/m. The second line is 15 m long and has a loss rating of 0.10 dB/m. The reflection coefficient at the junction (line 1 to line 2) is $\Gamma = 0.30$. The input power (to line 1) is 100 mW. ($a$) Determine the total loss of the combination in dB. ($b$) Determine the power transmitted to the output end of line 2.

**Solution.** (*a*) The dB loss of the joint is

$$L_j(\text{dB}) = 10\log_{10}\left(\frac{1}{1-|\Gamma|^2}\right) = 10\log_{10}\left(\frac{1}{1-0.09}\right) = 0.41 \text{ dB}$$

The total loss of the link in dB is now

$$L_t(\text{dB}) = (0.20)(10) + 0.41 + (0.10)(15) = 3.91 \text{ dB}$$

(*b*) The output power will be $P_{\text{out}} = 100 \times 10^{-0.391} = 41$ mW.

## 11.10 VOLTAGE STANDING WAVE RATIO

In many instances, transmission line performance characteristics are amenable to measurement. Included in these are measurements of unknown load impedances, or input impedances of lines that are terminated by known or unknown load impedances. Such techniques rely on the ability to measure voltage amplitudes that occur as functions of position within a line, usually designed for this purpose. A typical apparatus consists of a *slotted line,* which is a lossless coaxial transmission line having a longitudinal gap in the outer conductor along its entire length. The line is positioned between the sinusoidal voltage source and the impedance that is to be measured. Through the gap in the slotted line, a voltage probe may be inserted to measure the voltage amplitude between the inner and outer conductors. As the probe is moved along the length of the line, the maximum and minimum voltage amplitudes are noted, and their ratio, known as the *voltage standing wave ratio,* or VSWR, is determined. The significance of this measurement and its utility form the subject of this section.

To understand the meaning of the voltage measurements, we consider a few special cases. First, if the slotted line is terminated by a matched impedance, then no reflected wave occurs; the probe will indicate the same voltage amplitude at every point. Of course, the instantaneous voltage which the probe samples will differ in phase by $\beta(z_2 - z_1)$ rad as the probe is moved from $z = z_1$ to $z = z_2$, but the system is insensitive to the phase of the field. The equal-amplitude voltages are characteristic of an unattenuated travelling wave.

Second, if the slotted line is terminated by an open or short circuit (or in general a purely imaginary load impedance), the total voltage in the line is a standing wave and, as was shown in Example 11.1, the voltage probe provides no output when it is located at the nodes; these occur periodically with half-wavelength spacing. As the probe position is changed, its output varies as $|\cos(\beta z + \phi)|$, where $z$ is the distance from the load, and where the phase, $\phi$, depends on the load impedance. For example, if the load is a short circuit, the requirement of zero voltage at the short leads to a null occurring there, and so the voltage in the line will vary as $|\sin(\beta z)|$ (where $\phi = \pm\pi/2$).

A more complicated situation arises when the reflected voltage is neither 0 nor 100 percent of the incident voltage. Some energy is absorbed by the load and some is reflected. The slotted line therefore supports a voltage that is composed of both a travelling wave and a standing wave. It is customary to describe this voltage as a standing wave, even though a travelling wave is also present. We shall see that the voltage does not have zero amplitude at any point for all time, and the degree to which the voltage is divided between a travelling wave and a true standing wave is expressed by the ratio of the maximum amplitude found by the probe to the minimum amplitude (VSWR). This information, along with the positions of the voltage minima or maxima with respect to that of the load, enable one to determine the load impedance. The VSWR also provides a measure of the quality of the termination. Specifically, a perfectly matched load yields a VSWR of exactly 1. A totally reflecting load produces an infinite VSWR.

To derive the specific form of the total voltage, we begin with the forward and backward-propagating waves that occur within the slotted line. The load is positioned at $z = 0$, and so all positions within the slotted line occur at negative values of $z$. Taking the input wave amplitude as $V_0$, the total phasor voltage is

$$V_{sT}(z) = V_0 e^{-j\beta z} + \Gamma V_0 e^{j\beta z} \tag{79}$$

The line, being lossless, has real characteristic impedance, $Z_0$. The load impedance, $Z_L$, is in general complex, which leads to a complex reflection coefficient:

$$\boxed{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi}} \tag{80}$$

If the load is a short circuit ($Z_L = 0$), $\phi$ is equal to $\pi$; if $Z_L$ is real and less than $Z_0$, $\phi$ is also equal to $\pi$; and if $Z_L$ is real and greater than $Z_0$, $\phi$ is zero. Using (80), we may rewrite (79) in the form:

$$V_{sT}(z) = V_0 \left( e^{-j\beta z} + |\Gamma| e^{j(\beta z + \phi)} \right) = V_0 e^{j\phi/2} \left( e^{-j\beta z} e^{-j\phi/2} + |\Gamma| e^{j\beta z} e^{j\phi/2} \right) \tag{81}$$

To express (81) in a more useful form, we can apply the algebraic trick of adding and subtracting the term $V_0(1 - |\Gamma|) e^{-j\beta z}$:

$$V_{sT}(z) = V_0(1 - |\Gamma|) e^{-j\beta z} + V_0 |\Gamma| e^{j\phi/2} \left( e^{-j\beta z} e^{-j\phi/2} + e^{j\beta z} e^{j\phi/2} \right) \tag{82}$$

The last term in parentheses in (82) becomes a cosine, and we write

$$\boxed{V_{sT}(z) = V_0(1 - |\Gamma|) e^{-j\beta z} + 2V_0 |\Gamma| e^{j\phi/2} \cos(\beta z + \phi/2)} \tag{83}$$

The important characteristics of this result are most easily seen by converting it to real instantaneous form:

$$\boxed{\begin{aligned} \mathcal{V}(z,t) = \mathrm{Re}[V_{sT}(z) e^{j\omega t}] &= \underbrace{V_0(1 - |\Gamma|) \cos(\omega t - \beta z)}_{\textit{traveling wave}} \\ &+ \underbrace{2|\Gamma| V_0 \cos(\beta z + \phi/2) \cos(\omega t + \phi/2)}_{\textit{standing wave}} \end{aligned}} \tag{84}$$

Equation (84) is recognized as the sum of a traveling wave of amplitude $(1 - |\Gamma|)V_0$ and a standing wave having amplitude $2|\Gamma|V_0$. We can visualize events as follows: The portion of the incident wave that reflects and back-propagates in the slotted line interferes with an equivalent portion of the incident wave to form a standing wave. The rest of the incident wave (which does not interfere) is the traveling wave part of (84). The maximum amplitude observed in the line is found where the amplitudes of the two terms in (84) add directly to give $(1 + |\Gamma|)V_0$. The minimum amplitude is found where the standing wave achieves a null, leaving only the traveling wave amplitude of $(1 - |\Gamma|)V_0$. The fact that the two terms in (84) combine in this way with the proper phasing is not immediately apparent, but the following arguments will show that this does occur.

To obtain the minimum and maximum voltage amplitudes, we may revisit the first part of Eq. (81):

$$V_{sT}(z) = V_0\left(e^{-j\beta z} + |\Gamma|e^{j(\beta z+\phi)}\right) \tag{85}$$

First, the minimum voltage amplitude is obtained when the two terms in (85) subtract directly (having a phase difference of $\pi$). This occurs at locations

$$\boxed{z_{\min} = -\frac{1}{2\beta}(\phi + (2m+1)\pi) \quad (m = 0, 1, 2, \ldots)} \tag{86}$$

Note again that all positions within the slotted line occur at negative values of $z$. Substituting (86) into (85) leads to the minimum amplitude:

$$V_{sT}(z_{\min}) = V_0(1 - |\Gamma|) \tag{87}$$

The same result is obtained by substituting (86) into the real voltage, (84). This produces a null in the standing wave part, and we obtain

$$\mathcal{V}(z_{\min}, t) = \pm V_0(1 - |\Gamma|)\sin(\omega t + \phi/2) \tag{88}$$

The voltage oscillates (through zero) in time, with amplitude $V_0(1 - |\Gamma|)$. The plus and minus signs in (88) apply to even and odd values of $m$ in (86), respectively.

Next, the maximum voltage amplitude is obtained when the two terms in (85) add in-phase. This will occur at locations given by

$$\boxed{z_{\max} = -\frac{1}{2\beta}(\phi + 2m\pi) \quad (m = 0, 1, 2, \ldots)} \tag{89}$$

On substitution of (89) into (85), we obtain

$$V_{sT}(z_{\max}) = V_0(1 + |\Gamma|) \tag{90}$$

As before, we may substitute (89) into the real instantaneous voltage (84). The effect is to produce a maximum in the standing wave part, which then adds in-phase to the running wave. The result is

$$\mathcal{V}(z_{\max}, t) = \pm V_0(1 + |\Gamma|)\cos(\omega t + \phi/2) \tag{91}$$

where the plus and minus signs apply to positive and negative values of $m$ in (89), respectively. Again, the voltage oscillates through zero in time, with amplitude $V_0(1 + |\Gamma|)$.

Note that a voltage maximum is located at the load ($z = 0$) if $\phi = 0$; moreover, $\phi = 0$ when $\Gamma$ is real and positive. This occurs for real $Z_L$ when $Z_L > Z_0$. Thus there is a voltage maximum at the load when the load impedance is greater than $Z_0$ and both impedances are real. With $\phi = 0$, maxima also occur at $z_{\max} = -m\pi/\beta = -m\lambda/2$. For a zero load impedance, $\phi = \pi$, and the maxima are found at $z_{\max} = -\pi/(2\beta), -3\pi/(2\beta)$, or $z_{\max} = -\lambda/4, -3\lambda/4$, and so forth.

The minima are separated by multiples of one half-wavelength (as are the maxima), and for a zero load impedance, the first minimum occurs when $-\beta z = 0$, or at the load. In general, a voltage minimum is found at $z = 0$ whenever $\phi = \pi$; this occurs if $Z_L < Z_0$ where $Z_L$ is real. The general results are illustrated in Figure 11.7.

Finally, the voltage standing wave ratio is defined as

$$\boxed{s \equiv \frac{V_{sT}(z_{\max})}{V_{sT}(z_{\min})} = \frac{1 + |\Gamma|}{1 - |\Gamma|}} \tag{92}$$

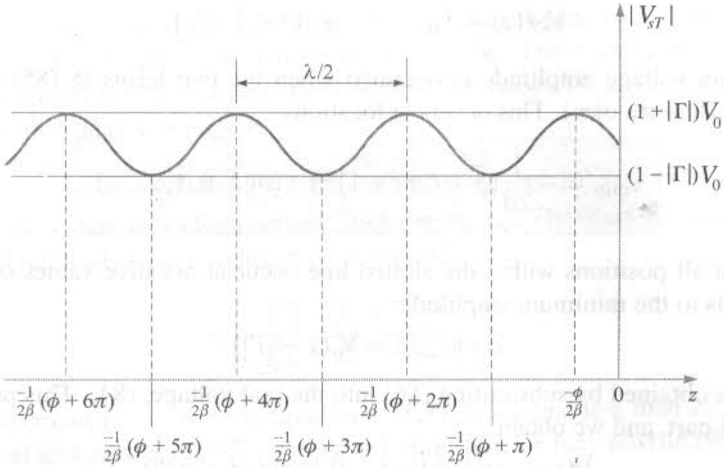


**Fig. 11.7** Plot of the magnitude of $V_{sT}$ as found from Eq. (85) as a function of position, $z$, in front of the load (at $z = 0$). The reflection coefficient phase is $\phi$, which leads to the indicated locations of maximum and minimum voltage amplitude, as found from Eqs. (86) and (89).

Illustrations

Since the absolute voltage amplitudes have divided out, our measured VSWR permits the immediate evaluation of $|\Gamma|$. The phase of $\Gamma$ is then found by measuring the location of the first maximum or minimum with respect to the load, and then using (86) or (89) as appropriate. Once $\Gamma$ is known, the load impedance can be found, assuming $Z_0$ is known.

**D11.3** What voltage standing wave ratio results when $\Gamma = \pm 1/2$?

**Ans.** 3

## EXAMPLE 11.9

Slotted line measurements yield a VSWR of 5, a 15 cm spacing between successive voltage maxima, and the first maximum at a distance of 7.5 cm in front of the load. Determine the load impedance, assuming a 50 Ω impedance for the slotted line.

**Solution.** The 15 cm spacing between maxima is $\lambda/2$, implying a wavelength of 30 cm. Since the slotted line is air-filled, the frequency is $f = c/\lambda = 1$ GHz. The first maximum at 7.5 cm is thus at a distance of $\lambda/4$ from the load, which means that a voltage minimum occurs at the load. Thus $\Gamma$ will be real and negative. We use (92) to write

$$|\Gamma| = \frac{s-1}{s+1} = \frac{5-1}{5+1} = \frac{2}{3}$$

So

$$\Gamma = -\frac{2}{3} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

which we solve for $Z_L$ to obtain

$$Z_L = \frac{1}{5}Z_0 = \frac{50}{5} = 10\,\Omega$$

## 11.11 TRANSMISSION LINES OF FINITE LENGTH

A new type of problem emerges when considering the propagation of sinusoidal voltages on finite-length lines which have loads that are not impedance-matched. In such cases, numerous reflections occur at the load and at the generator, setting up a multiwave bidirectional voltage distribution in the line. As always, the objective is to determine the net power transferred to the load in steady state, but we must now include the effect of the numerous forward and backward reflected waves.

Figure 11.8 shows the basic problem. The line, assumed to be lossless, has characteristic impedance $Z_0$ and is of length $l$. The sinusoidal voltage source at frequency $\omega$ provides phasor voltage $V_s$. Associated with the souce is a complex internal impedance, $Z_g$, as shown. The load impedance, $Z_L$, is also assumed to be complex and is located at $z = 0$. The line thus exists along the negative $z$ axis. The easiest method of approaching the problem is not to attempt to analyze every reflection individually, but rather to recognize that in steady state, there will exist one net forward wave and one net backward wave, representing the superposition of all waves that are incident on the load and all waves that are reflected from it. We may thus write the total voltage in the line as

$$V_{sT}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \tag{93}$$

in which $V_0^+$ and $V_0^-$ are complex amplitudes, composed respectively of the sum of all individual forward and backward wave amplitudes and phases. In a similar way, we may write the total current in the line:

$$I_{sT}(z) = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z} \tag{94}$$

We now define the *wave impedance*, $Z_w(z)$, as the ratio of the total phasor voltage to the total phasor current. Using (93) and (94), this becomes

$$\boxed{Z_w(z) \equiv \frac{V_{sT}(z)}{I_{sT}(z)} = \frac{V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}}{I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z}}} \tag{95}$$

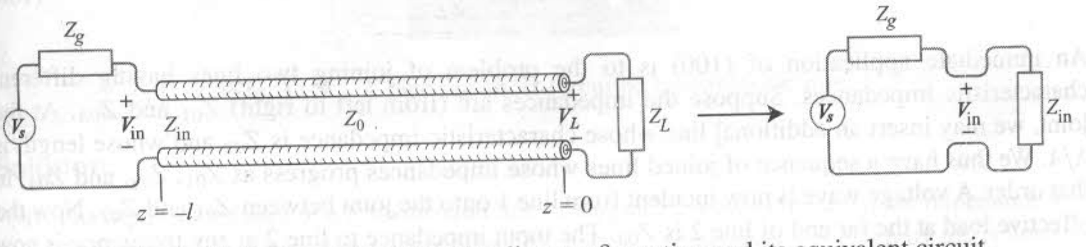


**Fig. 11.8** Finite-length transmission line configuration and its equivalent circuit.

We next use the relations $V_0^- = \Gamma V_0^+$, $I_0^+ = V_0^+/Z_0$, and $I_0^- = -V_0^-/Z_0$.

Equation (95) simplifies to

$$Z_w(z) = Z_0 \left[ \frac{e^{-j\beta z} + \Gamma e^{j\beta z}}{e^{-j\beta z} - \Gamma e^{j\beta z}} \right] \tag{96}$$

Now, using the Euler identity, (32), and substituting $\Gamma = (Z_L - Z_0)/(Z_L + Z_0)$, Eq. (96) becomes

$$Z_w(z) = Z_0 \left[ \frac{Z_L \cos(\beta z) - jZ_0 \sin(\beta z)}{Z_0 \cos(\beta z) - jZ_L \sin(\beta z)} \right] \tag{97}$$

The wave impedance at the line input is now found by evaluating (97) at $z = -l$, obtaining

$$\boxed{Z_{\text{in}} = Z_0 \left[ \frac{Z_L \cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + jZ_L \sin(\beta l)} \right]} \tag{98}$$

This is the quantity that we need in order to create the equivalent circuit in Figure 11.8.

One special case is that in which the line length is a half-wavelength, or an integer multiple thereof. In that case,

$$\beta l = \frac{2\pi}{\lambda}\frac{m\lambda}{2} = m\pi \quad (m = 0, 1, 2, \ldots)$$

Using this result in (98), we find

$$\boxed{Z_{\text{in}}(l = m\lambda/2) = Z_L} \tag{99}$$

For a half-wave line, the equivalent circuit can be constructed simply by removing the line completely and placing the load impedance at the input. This simplification works, of course, provided the line length is indeed an integer multiple of a half-wavelength. Once the frequency begins to vary, the condition is no longer satisfied, and (98) must be used in its general form to find $Z_{\text{in}}$.

Another important special case is that in which the line length is an odd multiple of a quarter wavelength:

$$\beta l = \frac{2\pi}{\lambda}(2m + 1)\frac{\lambda}{4} = (2m + 1)\frac{\pi}{2} \quad (m = 0, 1, 2, \ldots)$$

Using this result in (98) leads to

$$\boxed{Z_{\text{in}}(l = \lambda/4) = \frac{Z_0^2}{Z_L}} \tag{100}$$

An immediate application of (100) is to the problem of joining two lines having different characteristic impedances. Suppose the impedances are (from left to right) $Z_{01}$ and $Z_{03}$. At the joint, we may insert an additional line whose characteristic impedance is $Z_{02}$ and whose length is $\lambda/4$. We thus have a sequence of joined lines whose impedances progress as $Z_{01}$, $Z_{02}$, and $Z_{03}$, in that order. A voltage wave is now incident from line 1 onto the joint between $Z_{01}$ and $Z_{02}$. Now the effective load at the far end of line 2 is $Z_{03}$. The input impedance to line 2 at any frequency is now

$$Z_{\text{in}} = Z_{02} \frac{Z_{03} \cos \beta_2 l + jZ_{02} \sin \beta_2 l}{Z_{02} \cos \beta_2 l + jZ_{03} \sin \beta_2 l} \tag{101}$$

Then, since the length of line 2 is $\lambda/4$,

$$Z_{\text{in}}(\text{line } 2) = \frac{Z_{02}^2}{Z_{03}} \tag{102}$$

Reflections at the $Z_{01}$–$Z_{02}$ interface will not occur if $Z_{\text{in}} = Z_{01}$. Therefore, we can match the junction (allowing complete transmission through the three-line sequence) if $Z_{02}$ is chosen so that

$$\boxed{Z_{02} = \sqrt{Z_{01}Z_{03}}} \tag{103}$$

This technique is called *quarter-wave matching,* and again is limited to the frequency (or narrow band of frequencies) such that $l \doteq (2m+1)\lambda/4$. We will encounter more examples of these techniques when we explore electromagnetic wave reflection in Chapter 13. Meanwhile, further examples that involve the use of the input impedance and the VSWR are presented in Section 11.12.

## EXAMPLE 11.10

The transmission lines in the UHF and microwave frequency range can also be used as circuit elements showing very useful properties. The basic idea here is to consider a piece of lossless transmission line terminated in either *open* or *short* at its end. The input impedance of this special line can then exhibit either inductive or capacitive behavior depending upon the length and the operating frequency. Consider a piece of lossless transmission line of length '$l$' filled with Teflon ($\epsilon_r = 2.1$), and characteristic impedance $Z_0$ as shown in Figure 11.9. The line is shorted at its end.

(*a*) Determine the expression for the input impedance of this line starting from Eq. (98).

(*b*) Now, suppose, the shorted 50 Ω transmission line shown in Figure 11.9 is to be designed to provide a capacitive reactance with an equivalent capacitance of 5 pF at the input for a device operating at 2 GHz. Find the shortest possible length $l$ of the line.

(*c*) Explain whether there is any change in the behavior of the input impedance of the shorted line if the frequency is changed to 3 GHz.

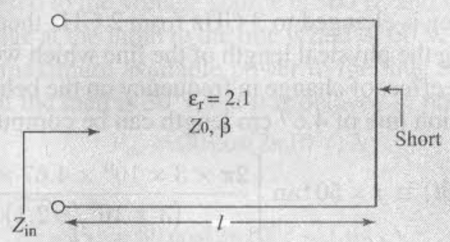


**Fig. 11.9** A piece of short-circuited transmission line.

## Solution.

(*a*) The input impedance of a transmission line of length '$l$', whose characteristic impedance and the propagation constant are defined by '$Z_0$' and '$\beta$', respectively, is given by the following expression as per Eq. (98):

$$Z_{\text{in}} = Z_0 \left[\frac{Z_L \cos(\beta l) + j\,Z_0 \sin(\beta l)}{Z_0 \cos(\beta l) + j\,Z_L \sin(\beta l)}\right] \tag{i}$$

where $Z_L$ is the terminated load impedance. Now, for the short-circuited transmission line shown in Figure 11.9, the load impedance would be zero, i.e., $Z_L = 0$, which basically modifies the basic equation for the input impedance as

$$Z_{\text{in}}^{\text{sc}} = Z_0 \left[\frac{0 \times \cos(\beta l) + j\, Z_0 \sin(\beta l)}{Z_0 \cos(\beta l) + j\, 0 \times \sin(\beta l)}\right] \equiv j\, Z_0 \tan(\beta l) \tag{ii}$$

From the above equation, it can be postulated that a piece of short-circuited transmission line can provide either inductive or capacitive reactance at the input of the line depending upon the fact that whether the sign of the tangent function is positive or negative.

(*b*) For realizing a capacitive reactance at the input of the shorted transmission line shown in Figure 11.9, the argument of tangent function in the above equation should be selected in such a way that the value of $\tan(\beta l)$ becomes negative at the operating frequency. Under this condition, the above equation can be written as

$$Z_{\text{in}}^{\text{sc}} = jZ_0 \tan(\beta l) \equiv -\frac{j}{\omega\, C_{\text{eq}}}$$

$$\Rightarrow \tan(\beta l) \equiv -\frac{1}{\omega Z_0 C_{\text{eq}}} = -\frac{1}{2 \times \pi \times 2 \times 10^9 \times 50 \times 5 \times 10^{-12}} = -0.3183$$

Now, the tangent function would be negative when its argument $\theta \equiv \beta l$ lies either in the second or the fourth quadrant. It is, however, given that the length of the line '$l$' should be minimum. Hence, the argument of the tangent function should lie in the second quadrant, i.e.,

$$\frac{\pi}{2} < \beta l < \pi$$

The value of the shortest possible length '$l_s$' can then be computed as

$$\tan(\beta l_s) = -0.3183 \equiv \tan(0.9019\,\pi)$$

$$\Rightarrow l_s = \frac{0.9019\,\pi}{\beta} \equiv \frac{0.9019\,\pi}{2\pi/\lambda} \equiv \frac{0.9019\,v_p}{2f} = \frac{0.9019\,c}{2f\sqrt{\epsilon_r}} = \frac{0.9019 \times 3 \times 10^8}{2 \times 2 \times 10^9 \times \sqrt{2.1}} = 4.67 \text{ cm}$$

(*c*) If the operating frequency is changed to 3 GHz from 2 GHz then the phase constant $\beta$ would change without changing the physical length of the line which would be kept fixed at 4.67 cm as computed above. The effect of change in frequency on the behavior of the input impedance of the shorted transmission line of 4.67 cm length can be computed as follows

$$Z_{in}^{sc} = jZ_0 \tan(\beta l) \equiv j \times 50 \tan\left[\frac{2\pi \times 3 \times 10^9 \times 4.67 \times 10^{-2}}{(3 \times 10^8/\sqrt{2.1})}\right] = j100.85$$

Hence, at 3 GHz, the input impedance of the shorted 50 Ω transmission line shown in Figure 11.9 behaves like an inductive reactance. The equivalent inductance of this line at 3 GHz can be computed using the following expression

$$Z_{\text{in}}^{sc} = j100.85 \equiv j\omega L_{\text{eq}}$$

$$\Rightarrow L_{\text{eq}} = \frac{100.85}{2 \times \pi \times 3 \times 10^9} = 5.35 \text{ nH}$$