Lecture-26

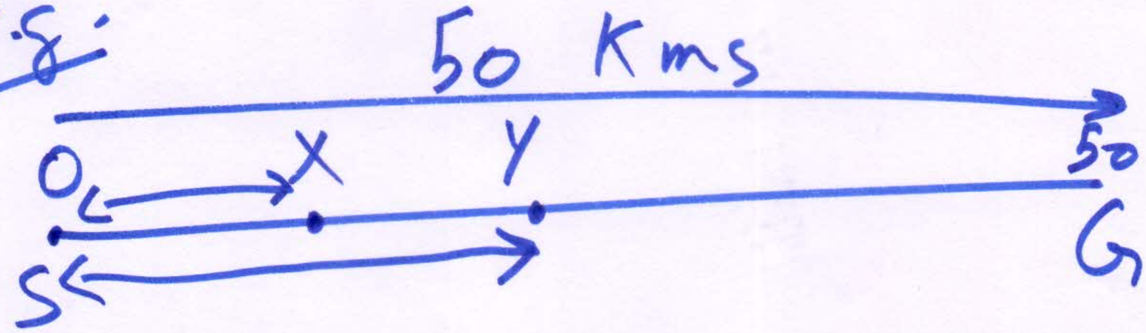
Recap: Conditional distributions.

→ Discrete, Poisson

→ Continuous random variables

Properties of Expectation.

e.g.



S. G. Highway

X: Accident

Y: ambulance.

$E[|X-Y|] = ?$

On an average, how far away is the ambulance from the accident site?

$E[g(X,Y)] =$ This is result does not need independence of X & Y.

$$\iint g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$E[g(X)] = \int g(x)\, f_X(x)\, dx$$

Discrete: $E[g(X,Y)] = \sum_y \sum_x g(x,y)\, p(x,y)$

$$E[|X-Y|] = \int_0^{50}\!\!\int |X-Y|\, f_{X,Y}(x,y)\, dx\, dy$$

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

(independent)

$$\int_0^x (x-y)\, dy \qquad = \frac{1}{50}\cdot\frac{1}{50} \qquad \int_x^{50} (y-x)\, dy$$

0 ← y → x ← y → 50

e.g.

$$g(X,Y) = X+Y$$

Not assuming independence.

$$E[g(X,Y)] = \iint g(X,Y)\, f(X,Y)\, dx\, dy$$

$$E[X+Y] = \iint (x+y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \iint x\, \underset{\uparrow\ \text{simplify}}{f(x,y)}\, dx\, dy + \iint y\, f(x,y)\, dx\, dy$$

$$\int y \left[\int f(x,y)\, dx \right] dy$$

$$= \int x \left[\int f(x,y)\, dy \right] dx$$

$$= \underbrace{\int x\, f_X(x)\, dx}_{E[X]} + \underbrace{\int y\, f_Y(y)\, dy}_{E[Y]}$$

$E[X+Y] = E[X] + E[Y]$ 4

very important result

profound implications.

$$E[X_1 + \dots + X_n] = \sum_{i=1}^{n} E[X_i]$$

Not assuming independence

e.g. Expected value of a binomial r.v. $= np$.

no. of attempts

success probability.

$$X_1, X_2, \vdots, X_n \begin{cases} 1 & \text{if } i^{th} \text{ trial is a success} \\ 0 & \text{if } i^{th} \text{ trial is a failure} \end{cases}$$

X = no. of successes = 4

H	H	T	T	H	T	H	T	T	T
1	1	0	0	1	0	1	0	0	0
X_1	X_2	X_3	X_4	X_5	6	7	8	9	10

$$X = \sum_{i=1}^{n} X_i \rightarrow \text{indicator r.v.}$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right]$$

Thm. $\longrightarrow$

$$= \sum_{i=1}^{n} \left[E[X_i]\right] = np$$

$$E[X_i] = 1(p) + 0(1-p) = p$$

eg:- Hypergeometric random variable. $E[X] = \frac{mn}{N}$

Total of N balls.

m white

$N-m$ black.

Choose $\underline{\underline{n}}$ balls randomly.

$X =$ no. of white balls in the sample of n balls.

$\{0, 1, 2, \cdots, n\}$

Expectation of Sum = Sum of Expectations.

$$X_1, X_2, \vdots, X_n = \begin{cases} 1 & \text{if } i^{th} \text{ ball is white} \\ 0 & \text{if } i^{th} \text{ ball is Black} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

$$E[X] = \sum_{i=1}^{n} E[X_i] = \frac{nm}{N}$$

$$E[X_i] = 1 \cdot \frac{m}{N} + 0\left(1 - \frac{m}{N}\right)$$

$$\begin{tikzcd}[column sep=small] & 1 \quad m/N \\ X_i \arrow[ur] \arrow[dr] & \\ & 0 \quad \left(1 - \frac{m}{N}\right) \end{tikzcd}$$

e.g.: 10 people in a room.

Each has a hat. They throw it in a box. Randomly pick one.

$X =$ no. of people who get their own hat back

$$E[X] = \sum_{i=1}^{10} E[X_i] = \frac{1}{10} \cdot 10 = 1$$

$$X_i = \begin{cases} 1 & \frac{1}{10} \quad i^{th} \text{ person gets his own hat} \\ 0 & \frac{9}{10} \quad \text{not} \end{cases}$$

X_{10}

$X = X_1 + \cdots + X_{10}$

Boole's inequality.

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

$$P(A \cup B) \leq P(A) + P(B)$$

x_1 1 if A_i takes place

x_2 =

$\vdots$ 0 if A_i doesn't happen

x_n

Total no. of events that happen.

$$X = \sum_{i=1}^{n} x_i \in \{0, 1, \cdots, n\}$$

$$Y = \begin{cases} 0 & \text{if } X = 0 \\ 1 & \text{if } X \geq 1 \end{cases}$$

$$P(Y=1) = P(\cup A_i)$$

$$P(Y=0) = 1 - P(\cup A_i)$$

$$E[X] = E[\Sigma X_i] = \Sigma E[X_i]$$

$$= \sum_{i=1}^{n} P(A_i)$$

$$E[Y] = P(\cup A_i)$$

$$X \geq Y$$

$$E[X] \geq E[Y]$$

$$X_i \begin{cases} \nearrow 1 & P(A_i) \\ \searrow 0 & (1-P(A_i) \end{cases}$$

$$\Sigma P(A_i) \geq P(\cup A_i)$$

Boole's inequality.