Lecture - 15

Recap:

Cumulative distribution function

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute $E[e^X] = e - 1$

Theorem:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx \rightarrow$$

$$= \int_{-\infty}^{\infty} e^x \, 1. \, dx = \int_0^1 e^x \, dx$$

$$= e^x \Big]_0^1 = e^1 - e^0 = e - 1$$

$g(x)$ is non negative.

<u>Lemma</u>: Y is a nonnegative random variable.

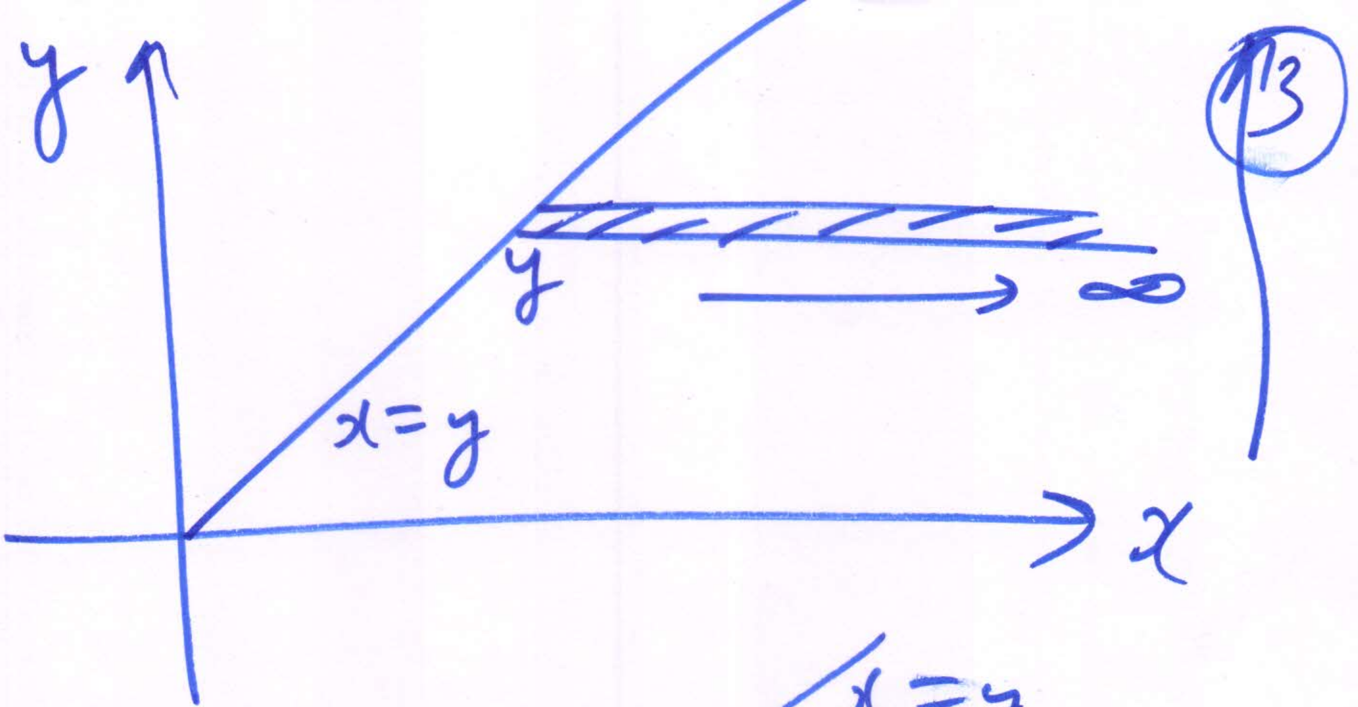
$$E[Y] = \int_0^\infty P(Y > y)\, dy$$

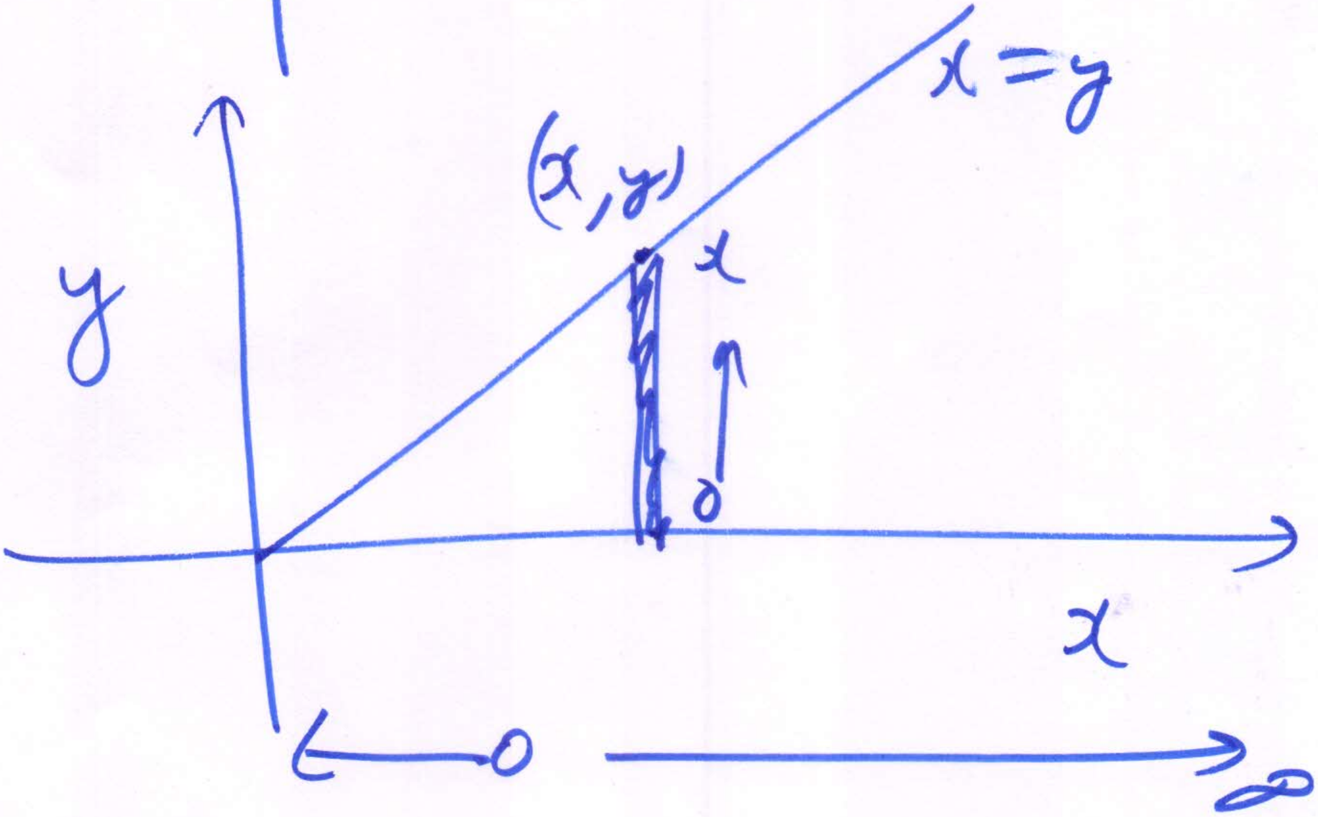
$$\parallel$$

$$\int_0^\infty y\, f(y)\, dy.$$

$$P(Y > y) = \int_y^\infty f_Y(x)\, dx$$

$$E[Y] = \int_{\underset{\text{over } y}{0}}^{\infty} \int_{\underset{\text{over } x}{y}}^{\infty} f_Y(x)\, dx\, dy = \int_{\underset{\text{over } x}{0}}^{\infty} \left[\int_{\underset{\text{over } y}{0}}^{x} dy \right] f_Y(x)\, dx$$





$$= \int_0^\infty y\Big]_0^x f_X(x)\,dx = \int_0^\infty x\, f_Y(x)\,dx$$

$$= \int_0^\infty x \cdot f_Y(x) \cdot dx = E[Y]$$

$$E[Y] = \int_0^\infty P(Y > y)\, dy.$$

$$E[g(X)] = \int_0^\infty \underline{P(g(X) > y)}\, dy$$

$$P(g(X) > y) = \int_{x:\, g(x) > y} f(x)\, dx$$

$$P(X \in B) = \int_B f(x)\, dx$$

$$= \int_0^\infty \left[\int_{\substack{x: \\ g(x) > y}} f(x)\, dx \right] dy$$

Change the order of integration

$$\int_0^\infty \left(\int_{x:\, g(x)>y} f(x)\,dx \right) dy$$

Over y over x

$$= \int_{\substack{\text{Over} \\ x: \\ g(x)>0}} \left(\int_{\substack{0 \\ \text{Over } y}}^{g(x)} 1 \, dy \right) f(x)\,dx$$

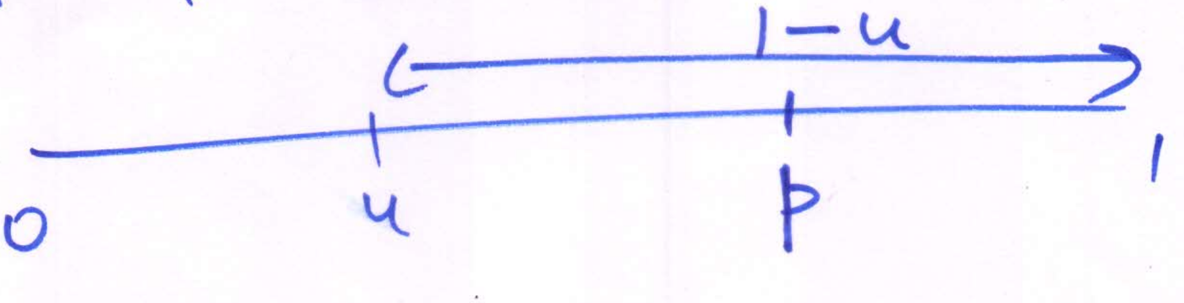
$$= \int_0^\infty g(x)\, f(x)\, dx = E[g(X)]$$

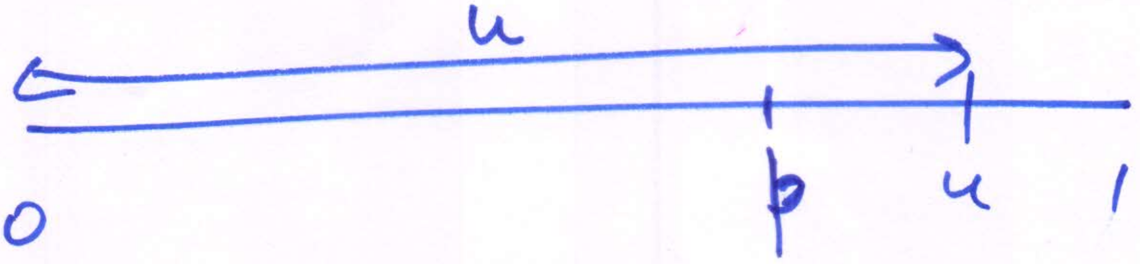
General result also holds true

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

e.g. A stick of length 1 is split at a point U, that is uniformly distributed over $(0,1)$. Determine the <u>expected length of the piece</u> that ~~coints~~ contains point p, $0 \le p \le 1$.

p is a fixed point on the stick.





$$f(u) = \begin{cases} 1 & 0 \leq u \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$f(u) = \int_0^1 a \, . \, du$$

$$= a$$

$$1 = \int_{-\infty}^{\infty} f(u) \, du$$

$$= \int_0^1 a \, . \, du = a$$

$$\Rightarrow a = 1$$

$$\int_\alpha^\beta f(u) du = 1$$

$$f(u) = \frac{1}{\beta - \alpha}$$

$L =$ length of the piece 8
that contains point p.

$$L = \begin{cases} 1-u & \text{if } \underset{0 \text{ to } p}{u < p} \\ u & \text{if } \underset{p \text{ to } 1}{u > p} \end{cases}$$

$E[g(u)]$

$$E[L] = \int_0^1 L(u) f(u)\, du$$

$$= \int_0^p (1-u)\, du + \int_p^1 u\, du$$

$$= p - p^2 + \frac{1}{2}$$

e.g. If you are 5 minutes early, the cost is 6s.
if you are 5 minutes late, the cost is 3s.
Travel time is a c.r.v. X with density fn $f(x)$.
You leave t minutes early from home. What is the best value of t to minimize the cost?
Define a new r.v. Cost C which is dependent on the travel time X.

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$$C(X) = \begin{cases} 6(t - X) & \text{if } \underset{15}{X} \le \underset{20}{t} \quad (0 \text{ to } t) \\ 3(X - t) & \text{if } \underset{30}{X} > \underset{20}{t} \quad (t \text{ to } \infty) \end{cases}$$

→ compute $E[C(X)]$

→ differentiate $E[C(X)]$ w.r.t. t to get minimum

$$E[C(X)] = \int_{-\infty}^{\infty} C(X) f(x)\, dx$$