CT 203: Signals and Systems
Tutorial on Fourier Transform and Hilbert Transform

(Week of October 02, 2011)

1. Let $g(t) = Ae^{-bt}u(t)$. What are the range of frequencies that would contain $x\%$ of the total energy of $g(t)$?

Solution: We can first compute $E_g = \frac{A^2}{2b}$. To compute W that would contain $x\%$ of E_g we proceed as follows:

$$
\begin{aligned}
0.01xE_g &= \int_{-W}^{+W} |G(f)|^2 df \text{ (by Rayleigh's energy theorem)} \\
0.01x\frac{A^2}{2b} &= \int_{-W}^{+W} \frac{A^2}{(b^2+4\pi^2 f^2)} df \ (\because G(f) = \frac{A}{(b+j2\pi f)}) \qquad (1) \\
\frac{0.01x}{2b} &= \frac{1}{2\pi b}\int_{-W}^{+W} \frac{(b/2\pi)}{\left(f^2+\left(\frac{b}{2\pi}\right)^2\right)} df \qquad (2) \\
\frac{0.01x}{2b} &= \frac{1}{2\pi b} \times \left[\mathrm{Tan}^{-1}\left(\frac{2\pi f}{b}\right)\right]_{-W}^{+W} \\
\frac{0.01x}{2} &= \frac{1}{\pi}\mathrm{Tan}^{-1}\left(\frac{2\pi W}{b}\right) (\because \mathrm{Tan}^{-1}(-x) = -\mathrm{Tan}^{-1}(x)) \qquad (3)
\end{aligned}
$$

For $x = 50$, W can be found to be equal to $\frac{b}{2\pi}$. For $x = 99$, W can be found to be equal to $\frac{66.298b}{2\pi}$.

2. Using the duality theorem compute the FT of $z(t) = A\mathrm{Sinc}(2Wt)$.

Solution: Recall that

$$F\left[A\Pi\left(\frac{t}{\tau}\right)\right] \leftrightarrow A\tau\mathrm{Sinc}(f\tau)\ , \qquad (4)$$

where $\Pi\left(\frac{t}{\tau}\right)$ denotes the rectangular pulse of duration τ seconds centered around zero. To apply duality theorem we arrange the given $z(t)$ as follows:

$$z(t) = A^{'}(2W)\mathrm{Sinc}(2Wt) \text{ where } A^{'} = \frac{A}{2W} \qquad (5)$$

The RHS of (5) is similar to the RHS of (4) with the following associations: $A^{'} = A$ and $2W = \tau$.

Therefore, the FT of given $z(t)$ by duality theorem is $A^{'}\Pi\left(\frac{-f}{2W}\right)$ which is equal to $A^{'}\Pi\left(\frac{f}{2W}\right)$ (since the rectangle function $\Pi(.)$ has even symmetry).

3. Denote $x(t)$ and $\hat{x}(t)$ as the signal and its Hilbert transform (HT). With this notation prove the following properties of HT: (a) $x(t)$ and $\hat{x}(t)$ have the same amplitude spectrum, (b) $-x(t)$ is the Hilbert transform of $\hat{x}(t)$ and (c) $x(t)$ and $\hat{x}(t)$ are orthogonal to each other.

Solution: (a) By definition

$$\begin{aligned}
\hat{X}(f) &\triangleq F\left[\hat{x}(t)\right] = F\left[\frac{1}{\pi t} * x(t)\right] = -j\text{sgn}(f)X(f) \\
|\hat{X}(f)| &= |-j\text{sgn}(f)|X(f) \\
&= |-j\text{sgn}(f)||X(f)| \\
&= |X(f)| \ (\because |-j\text{sgn}(f)| = 1 \forall f)
\end{aligned} \tag{6}$$

which completes the proof. An upshot of property 1 is that

$$E_x = \int_{-\infty}^{+\infty} |X(f)|^2 = \int_{-\infty}^{+\infty} |\hat{X}(f)|^2 = E_{\hat{x}}$$

(b) By definition

$$\begin{aligned}
H\left[\hat{x}(t)\right] &= \frac{1}{\pi t} * \hat{x}(t) \\
F\left[H\left[\hat{x}(t)\right]\right] &= -j\text{sgn}(f) \quad \underbrace{\hat{X}(f)}_{=-j\text{sgn}(f)X(f)} \\
&= -X(f) \ , (\because \ (\text{sgn}(f))^2 = 1)
\end{aligned}$$

finally taking F^{-1} on both sides of the above equation we have the desired result.

(c) Recall $x(t)$ and $\hat{x}(t)$ are orthogonal if and only if $\int_{-\infty}^{+\infty} x(t)(\hat{x}(t))^* dt = 0$. Consider

$$\begin{aligned}
&\int_{-\infty}^{+\infty} x(t)(\hat{x}(t))^* dt \\
&= \int_{-\infty}^{+\infty} X(f) \quad \underbrace{(\hat{X}(f))^*}_{=j\text{sgn}(f)X^*(f)} \quad df \\
&= j\int_{-\infty}^{+\infty} |X(f)|^2 \text{sgn}(f) df \\
&= 0 \ ,
\end{aligned}$$

where the last step follows because $|X(f)|^2\text{sgn}(f)$ is an odd function as $|X(f)|^2$ is an even function of f ($\because |X(-f)| = |X(f)|$ as $x(t)$ is real).

4. Compute the Hilbert transform of the causal rectangular pulse $x(t) = A\Pi\left(\frac{t-\frac{\tau}{2}}{\tau}\right)$ (where as before $\Pi\left(\frac{t}{\tau}\right)$ denotes the rectangular pulse of duration τ seconds centered around zero).

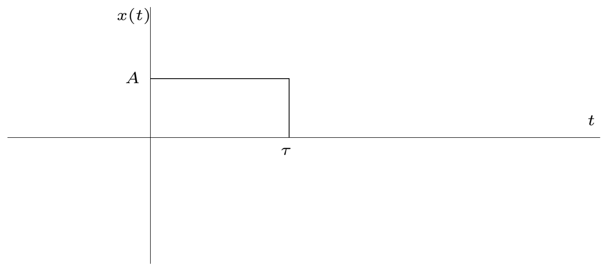

Figure 1: The Figure above shows a causal rectangular pulse of amplitude A and duration τ seconds, whose Hilbert transform needs to be computed.

Solution: Given $x(t) = A\Pi\left(\frac{t-\frac{\tau}{2}}{\tau}\right)$ (shown in Fig. 1). Therefore,

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda} x(t-\lambda) d\lambda. \tag{7}$$

Case 1: For $\hat{x}(t)$ for $t < 0$ and $t > \tau$, $\hat{x}(t)$ can be computed as follows:

$$\begin{aligned}
\hat{x}(t) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda} x(t-\lambda) d\lambda \\
&= \frac{A}{\pi}\int_{t-\tau}^{t} \frac{1}{\lambda} d\lambda \text{ (for } t > 0 \text{ and } t > \tau) \\
&= \frac{A}{\pi} \ln\left(\frac{t}{t-\tau}\right) \text{ (for } t > 0 \text{ and } t > \tau) \qquad (8)
\end{aligned}$$

(the limits of integration in the second equation above can be obtained by referring to Fig 2)

Case 2: For $0 < t < \frac{\tau}{2}$, $\hat{x}(t)$ can be computed as follows:

$$\begin{aligned}
\hat{x}(t) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda} x(t-\lambda) d\lambda \\
&= \frac{A}{\pi}\int_{t-\tau}^{-t} \frac{1}{\lambda} d\lambda \text{ (for } 0 < t < \frac{\tau}{2}) \\
&= \frac{A}{\pi} \ln\left(\frac{t}{\tau-t}\right) \text{ (for } 0 < t < \frac{\tau}{2}) \qquad (9)
\end{aligned}$$

(the limits of integration in the second equation above can be obtained by referring to Fig 3)

Case 3: For $t = \frac{\tau}{2}$, $\hat{x}(t)$ can be seen from Fig. 4 to be equal to 0.

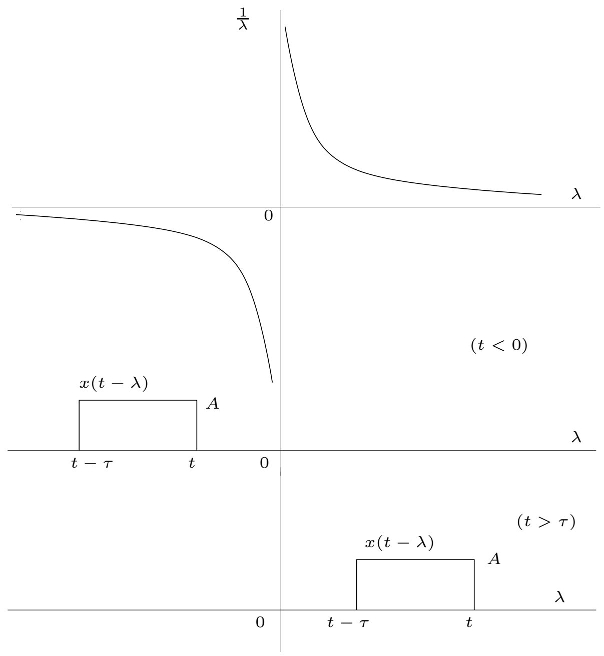

Figure 2: The Figure above shows the plots of $(1/\lambda)$ and $x(t-\lambda)$ for $t<0$ and $t>\tau$.

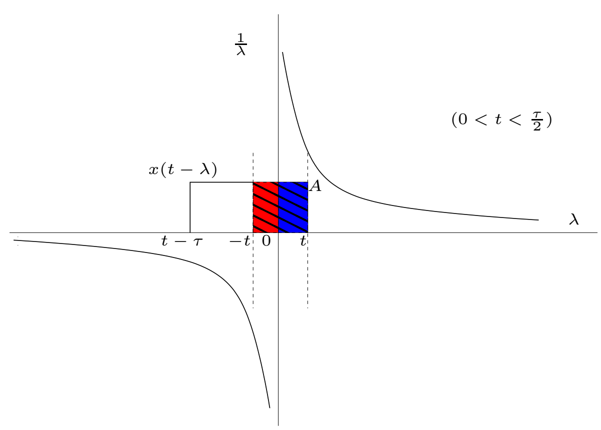

Figure 3: The Figure above shows the plots of $(1/\lambda)$ and $x(t-\lambda)$ for $0<t<\frac{\tau}{2}$. As can be seen from the figure, the area of $\frac{1}{\lambda}x(t-\lambda)$ for $\lambda \in (-t,0)$ (shown in red) and $\lambda \in (0,t)$ (shown in blue) are exactly same and therefore cancel out each other.

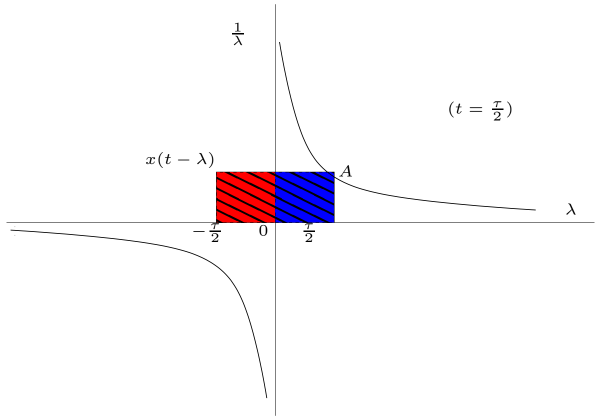

Figure 4: The Figure above shows the plots of $(1/\lambda)$ and $x(t-\lambda)$ for $t = \frac{\tau}{2}$. As can be seen from the figure, the area of $\frac{1}{\lambda}x(t-\lambda)$ for $\lambda \in (-\frac{\tau}{2}, 0)$ (shown in red) and $\lambda \in (0, +\frac{\tau}{2})$ (shown in blue) are exactly same and therefore cancel out each other.

Case 4: For $\frac{\tau}{2} < t < \tau$, $\hat{x}(t)$ can be computed as follows:

$$\begin{aligned}
\hat{x}(t) &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{\lambda}x(t-\lambda)d\lambda \\
&= \frac{A}{\pi}\int_{-t+\tau}^{t}\frac{1}{\lambda}d\lambda \text{ (for } \frac{\tau}{2} < t < \tau) \\
&= \frac{A}{\pi}\ln\left(\frac{t}{\tau - t}\right) \text{ (for } \frac{\tau}{2} < t < \tau) \qquad (10)
\end{aligned}$$

(the limits of integration in the second equation above can be obtained by referring to Fig 5)

Therefore combining all cases

$$\hat{x}(t) = \begin{cases} \frac{A}{\pi}\ln\left(\frac{t}{t-\tau}\right) & t < 0 \\ \frac{A}{\pi}\ln\left(\frac{t}{\tau-t}\right) & 0 < t < \tau \\ \frac{A}{\pi}\ln\left(\frac{t}{t-\tau}\right) & t > \tau \end{cases} \qquad (11)$$

The (11) can be succinctly represented as $\hat{x}(t) = \frac{A}{\pi}\ln\left|\frac{t}{t-\tau}\right|$. The Hilbert transform of the causal rectangular pulse is shown in Fig. 6

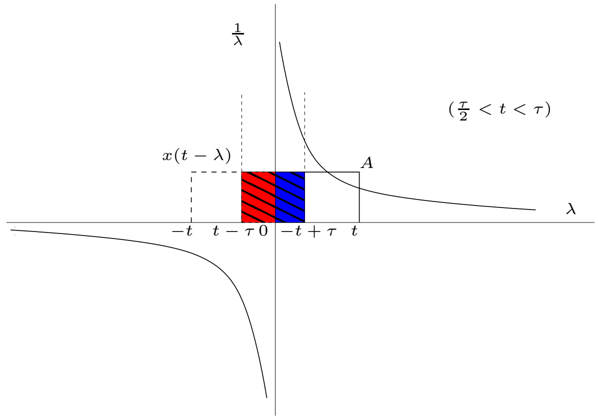

Figure 5: The Figure above shows the plots of $(1/\lambda)$ and $x(t-\lambda)$ for $\frac{\tau}{2} < t < \tau$. As can be seen from the figure, the area of $\frac{1}{\lambda}x(t-\lambda)$ for $\lambda \in (t-\tau, 0)$ (shown in red) and $\lambda \in (0, -t+\tau)$ (shown in blue) are exactly same and therefore cancel out each other.

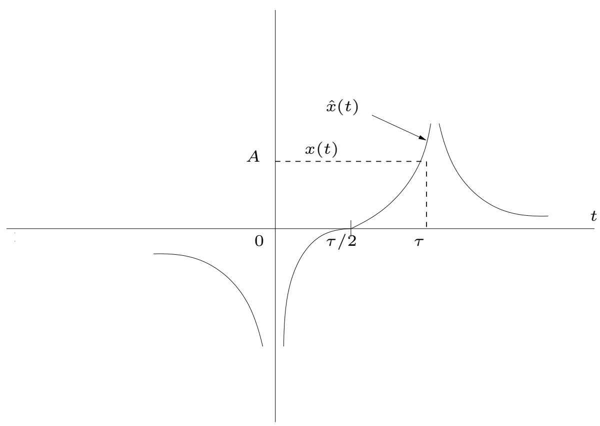

Figure 6: The Figure above shows the Hilbert transform of the given causal rectangular pulse of amplitude A and duration τ .