DA-IICT, Gandhinagar
MID Term Examination
Subject: Computational and Numerical Methods (CS374)

Date: 28/10/2020 | **Start Time: 02:30 PM**
Time Duration: 90 minutes | **Max. Marks: 60**

Instructions:

1. Scientific Calculator is allowed.
2. Negative Marking -1.25.
3. All questions carry equal marks.

1. Consider the Taylor's series for $\ln x$ about e. Assume that $\mid x - e \mid < 1$ and accuracy $\frac{1}{2} \times 10^{-1}$. What is the minimum number of terms required to achieve this accuracy?

   (a) 4. (b) 3. (c) 5. (d) 2.

2. If the Bisection method is used starting with the interval $[2, 3]$ for the equation $f(x) = 0$, how many steps must be taken to compute a root with relative accuracy $< 10^{-3}$?

   (a) At least 7. (b) At least 8. (c) At least 9. (d) Depends on f.

3. Consider a variation of Newton's method in which only one derivative is needed: that is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}, \qquad f'(x_0) \neq 0.$$

   If r is a simple root of f, then, the rate of convergence (p) and asymptomatic error constant (M) will be given by (**Hint**: Find M and p such that $e_{n+1} \approx M e_n^p$.)

   (a) $p = 1, M = 1 - \frac{f'(r)}{f'(x_0)}$.
   (c) $p = 1, M = \frac{f'(r)}{f'(x_0)}$.
   (b) $p = 2, M = 1 - \frac{f'(r)}{f'(x_0)}$.
   (d) $p = 2, M = \frac{f'(r)}{f'(x_0)}$.

4. If the method of functional iteration (fixed-point method) is used on $F(x) = \frac{1}{2(1+x^2)}$ starting at $x_0 = 7$, then the resulting sequence (x_n)

   (a) converges and limit lies in $\left[0, \frac{1}{2}\right]$.
   (b) does not converge.
   (c) converges and limit lies in $\left[0, \frac{1}{4}\right]$.
   (d) converges and limit lies in $\left[0, \frac{1}{6}\right]$.

5. For what values of a, the matrix $\begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix}$ is positive definite.

(a) $a > 0$. (b) $-1 < a < 1$. (c) $\frac{1}{2} < a < 1$. (d) $-1 < a < \frac{1}{2}$.

6. Which of the followings are **NOT** true:

   (a) Every matrix of the form $A = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ has an $LU-$factorization.

   (b) For Cholesky factorization of a matrix A, it is necessary that A must be symmetric.

   (c) For ϵ small enough, the linear system $\begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ can be solved by using Gauss Elimination without pivoting.

   (d) The condition number for the matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$, using $1-$norm, is 1

7. Let $L = [l_{ij}]$ and $U = [u_{ij}]$ denotes lower-triangular matrix and upper-triangular matrix of order 3, respectively. Further, let the diagonal elements $l_{ii} = 2$ for $i = 1, 2, 3$. Then, the values of l_{32} and u_{23} in the $LU-$factorization of the matrix $A = \begin{bmatrix} 6 & 10 & 0 \\ 12 & 26 & 4 \\ 0 & 9 & 12 \end{bmatrix}$ equal to

   (a) $l_{32} = u_{23} = 2$.

   (b) $l_{32} = 3$ and $u_{23} = 2$.

   (c) $l_{32} = u_{23} = 3$.

   (d) $l_{32} = 2$ and $u_{23} = 3$.

8. Assume that $0 < \epsilon < 2^{-22}$. If the Gaussian algorithm without pivoting is used to solve the following system on the Marc-32, what will be the solution vector (x_1, x_2)?

$$\begin{bmatrix} \epsilon & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

   (a) $(0, 2)$. (b) $(2, 0)$. (c) $(1, 2)$. (d) $(2, 1)$.

9. Consider the system of equations $\begin{bmatrix} 1 & 2 \\ 1 & 2.01 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. Let the constant vector $b = (4, 4)^T$ is changed to $\tilde{b} = (3, 5)^T$, then find the relative change in the solution vector x with respect to the $\infty-$norm.

   (a) 0.5. (b) 0.3. (c) 404. (d) 200.

10. If the matrix $A = [a_{ij}]$ has the property:

$$a_{ii} = 1 > \sum_{j=1, j \neq i}^{n} | a_{ij} |, \quad i = 1, 2, ..., n,$$

then which of the following iterative method(s) converges for solving the system of equations $Ax = b$.

   (a) Richardson Method only.

   (b) Gauss-Siedel Method only.

(c) Both Gauss-Siedel and Richardson Methods.

(d) All iterative methods converges.

11. Using the interpolation technique to fit the following data in the interpolating polynomial, the predicted value of y at 1.5 will be:

x	1	-2	0
y	3	-3	-7

(a) 15.5. (b) 11. (c) -3.75. (d) 3.5.

12. A table of values for $f(x) = \cos x$ in $\left[0, \frac{\pi}{4}\right]$ with equally spaced step size is to be constructed. The maximum step size, say h, that can be used if the error in linear interpolation is less than 5×10^{-6}.

(a) 0.04. (b) 0.006. (c) 0.06. (d) 0.4.

Answer Key:

1) Option (a).
2) Option (b).
3) Option (a).
4) Option (a).
5) Option (c).
6) Option (c) and Option (d).
7) Option (b).
8) Option (a).
9) Option (a).
10) Option (c).
11) Option (b).
12) Option (b).

Detailed Solution

(1) Error in Taylor's series of $\ln x$ about $x = e$ will be

$$R_n(x) = \frac{(x-e)^{n+1}}{(n+1)!} f^{n+1}(\xi) \quad \text{, where } \xi \text{ lies b/w } x \text{ \& } e.$$

and $$f^{n+1}(\xi) = \frac{(-1)^n n!}{\xi^{n+1}}$$

$$\Rightarrow R_n(x) = \frac{(x-e)^{n+1} (-1)^n}{(n+1)\, \xi^{n+1}}$$

Note that $|x-e|<1$ & ξ lies b/w x & e

$$\Rightarrow |\xi - e| < 1 \Rightarrow e-1 < \xi < e+1$$

$$\Rightarrow \frac{1}{\xi} < \frac{1}{e-1} \Rightarrow \frac{1}{\xi^{n+1}} < \frac{1}{(e-1)^{n+1}}$$

$$\therefore |R_n(x)| = \frac{|x-e|^{n+1}}{(n+1)\,\xi^{n+1}} < \frac{1}{(e-1)^{n+1}(n+1)} < \frac{1}{2}\times 10^{-1}$$

$$\Rightarrow n = 3$$

$\therefore$ Number of terms $= 3+1 = 4$.

② Given that Bisection method is used for solving $f(x)=0$, starting with the interval $[2,3]$.
Denote $a_0=2$, $b_0=3$. Let 'c' is ~~the~~ an exact root.

<u>I^{st} step</u> $x_0 = \dfrac{a_0+b_0}{2}$,

Then, $|x_0-c| \leq \frac{1}{2}(b_0-a_0)$

and root lies either in $[a_0, x_0]$ or in $[x_0, b_0]$ due to Bolzano's Thm.

We say next interval $[a_1, b_1]$.

$$\left([a_1, b_1] \subseteq [a_0, b_0] \quad \& \quad b_1 - a_1 = \frac{1}{2}(b_0 - a_0) \right)$$

$\underline{\text{II}^{nd} \text{ Step:}}$ Define $x_1 = \dfrac{a_1 + b_1}{2}$.

Then, $|x_1 - c| \leq \frac{1}{2}(b_1 - a_1) = \frac{1}{2^2}(b_0 - a_0)$

and so on.

Repeated application of Bolzano's Theorem implies the following

* $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}] \subseteq [a_{n-2}, b_{n-2}]$

$$\subseteq \ldots \subseteq [a_0, b_0].$$

* $$x_n = \frac{a_n + b_n}{2},$$

* $$|x_n - c| \le \frac{1}{2}(b_n - a_n) = \frac{1}{2^{n+1}}(b_0 - a_0) = \frac{1}{2^{n+1}}$$

Also, $c \geq 2$, therefore

$$\frac{|x_n - c|}{|c|} \leq \frac{1}{2^{n+2}}(b_0 - a_0) = \frac{1}{2^{n+2}} < 10^{-3}$$

$$\Rightarrow n \geq 8.$$

On the other hand, if you start sequence of approximation with $x_1, x_2, \ldots$ whose n-th term is

$$x_n = \frac{a_{n-1} + b_{n-1}}{2}.$$

Then, $|x_n - c| \leq \frac{1}{2^n}(b_0 - a_0) \Rightarrow n \geq 9.$

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}, \quad n = 0, 1, 2, 3, \ldots \quad (1)$$

Given that (i) $f'(x_0) \neq 0$

(ii) $f(\alpha) = 0$ but $f'(\alpha) \neq 0$.

Let $e_n = x_n - \alpha$.

From (1)

$$e_{n+1} = e_n - \frac{f(e_n + \alpha)}{f'(x_0)} = e_n - \frac{f(\alpha) + e_n f'(\xi)}{f'(x_0)}$$

$$\Rightarrow e_{n+1} = e_n - \frac{f'(\xi)}{f'(x_0)} e_n \;, \text{ where } \xi \text{ lies b/w } x_n \,\&\, x_{\ldots}$$

$$\simeq e_n - \frac{f'(\alpha)}{f'(x_0)} e_n$$

$$= \left[1 - \frac{f'(\alpha)}{f'(x_0)}\right] e_n$$

$$= M e_n^p$$

$$\Rightarrow M = 1 - \frac{f'(\alpha)}{f'(x_0)} \;, \quad p = \underline{1}.$$

4. $F(x) = \dfrac{1}{2(1+x^2)}$, $x_0 = 7$.

Clearly, $0 \leq F(x) \leq \dfrac{1}{2} \quad \forall x$.

Let $I = [0, \tfrac{1}{2}]$. Then $F : I \to I$.

Further,

$$|F(x) - F(y)| = \frac{1}{2}\left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right|$$

$$= \frac{1}{2}\left|\frac{y^2 - x^2}{(1+x^2)(1+y^2)}\right| = \frac{1}{2}\left|\frac{(y-x)(y+x)}{(1+x^2)(1+y^2)}\right|$$

$$= \frac{1}{2} \frac{|x-y|\,|x+y|}{(1+x^2)(1+y^2)}$$

$$\leq \frac{1}{2} \frac{|x-y|\left(|x|+|y|\right)}{1 \cdot 1}$$

$\left[\because x, y \in \left[0, \frac{1}{2}\right]\right.$
$\left.|x|, |y| \leq \frac{1}{2}\right]$

$$\leq \frac{1}{2}|x-y|\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{1}{2}|x-y|$$

$\Rightarrow$ F is a contraction.

$\therefore$ By the Contraction Mapping Thm, F has a unique fixed point, say s, in I. Also,

$$s = \lim_{n} x_n$$

where x_n is the sequence generated by following iterative scheme

$$x_{n+1} = f(x_n).$$

5) A symmetric matrix is +ve definite iff $|A_k| > 0 \ \forall\ k = 1, 2, 3 \ldots n$.

Here A_k denotes the leading principal minor of order k.

Given $A = \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix}$

$|A_1| = |1| > 0$

$|A_2| = \begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix} = 1 - a^2 > 0 \Rightarrow -1 < a < 1$

$|A_3| = \begin{vmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{vmatrix} = 2a^3 - 3a^2 + 1 > 0 \Rightarrow \frac{1}{2} < a < 1$

6 (a) $\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$ is solvable.

$= \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) If $A = LL^T$, then $A^T = A$. (verify)

(c) Cannot be solved. (Similar steps need to follow as in lecture).

(d) $k_1(A) = \|A\|_1 \, \|A^{-1}\|_1 = 2$.

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$$A = \begin{bmatrix} 2 & 0 & 0 \\ l_{21} & 2 & 0 \\ l_{31} & l_{32} & 2 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Given

$$A = \begin{bmatrix} 6 & 10 & 0 \\ 12 & 26 & 4 \\ 0 & 9 & 12 \end{bmatrix}$$

On comparing the entries on both sides, we get

$l_{32} = 3$ & $u_{23} = 2$.

(8) We wish to solve the following system

$$\begin{bmatrix} \varepsilon & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

using Gauss-Elimination without pivoting on Marc-32.

$R_2 \to R_2 - \frac{1}{\varepsilon} R_1$, we get

$$\begin{bmatrix} \varepsilon & 2 \\ 0 & -1-\frac{2}{\varepsilon} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1-\frac{4}{\varepsilon} \end{bmatrix}$$

⇒ Applying Backsubstitution method we get:

$$x_2 = \frac{(-1-4\varepsilon^{-1})}{(-1-2\varepsilon^{-1})} \approx \frac{-4\varepsilon^{-1}}{-2\varepsilon^{-1}} = 2$$

(on Marc-32)

$$\varepsilon x_1 + 2x_2 = 4$$

$$\Rightarrow x_1 = 0$$

∴ Solution will be $(0, 2)$.

(9) On solving $\begin{bmatrix} 1 & 2 \\ 1 & 2.01 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

we get $x = (x_1, x_2)^T = (4, 0)$.

On solving $\begin{bmatrix} 1 & 2 \\ 1 & 2.01 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

we get $\tilde{x} = (x_1, x_2)^T = (-397, 400)$

We wish to find $\dfrac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} = \dfrac{401}{4} = 100.25.$

(10) Mathematically, any iterative method for solving $Ax=b$ will be of the following general form:

$$x^{(k+1)} = G x^{(k)} + C \quad \text{--- (1)}$$

where G is called the iteration matrix

Necassary and sufficient condition for converging (1) will be $\rho(G) < 1$.

(11) Divided Difference Table for the given data will be

	x	f	I^{st} D.D.	II^{nd} D.D.
x_0	1	3		
			$\frac{-3-3}{-2-1} = 2$	
x_1	-2	-3		$\frac{-2-2}{0-1} = 4$
			$\frac{-7+3}{0+2} = -2$	
x_2	0	-7		

$\therefore$ Newton's Interpolating polynomial will be

$$p(x) = f(x_0) + f[x_0, x_1](x - x_0) + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$

$\Rightarrow p(x) = 3 + 2(x-1) + 4(x-1)(x+2)$

$\Rightarrow p(1.5) = 11.$

12 For Linear interpolation, we have (x_0, f_0) & (x_1, f_1), with $h = x_1 - x_0$.

Error in Linear Interpolation (from the lectures) for equispaced data will be

$$|E_1| \leq \frac{h^2}{8} M_2 , \quad M_2 = \max_{[0, \frac{\pi}{4}]} |f''(x)| \leq 1$$

$$\Rightarrow \quad |E_1| \leq \frac{h^2}{8} < 5 \times 10^{-6}$$

$$\Rightarrow \quad h^2 < 40 \times 10^{-6}$$

$$\Rightarrow \quad h < 0.0063245$$