Tute-3 Soln

Date ______
Page ______

Soln 1 There are n types of coupons.

p_i = Probability that randomly chosen coupon is of type i.

→ Here $\forall i$, p_i is constant. Means you have to assume that there are infinitely many coupons from which probability for coupon with type i to be choosen is p_i and that is constant, even ~~at~~ after many trials. ⇒ Or " Assume the case of "with Replacement" or "Independent Events".

→ Given that $\boxed{\sum_{i=1}^{n} p_i = 1}$

k coupons Are tube collected.

A_i : At least one coupon of type "i" is collected among those k coupons.

& $i \neq j$.

(A) (a) $P(A_i) = ?$

A_i^c = None of the coupons are of type i.

$\therefore P(A_i^c) = (1-p_i)^k$

$\therefore P(A_i) = 1 - P(A_i^c)$

$$\boxed{= 1 - (1-p_i)^k}$$

* Long method to understand.

A_i can be seen as there are 1 or 2 or 3 ... k i types of coupons.

~~$\therefore P(A_i) = \underbrace{\binom{k}{1} p_i (1-p_i)^{k-1}}_{\text{1 'i' type}} + \underbrace{\binom{k}{2} p_i^2 (1-p_i)^{k-2}}_{\text{2 'i' type}} + \dots + \underbrace{\binom{k}{k} p_i^k}_{\text{all k 'i' type}}$~~ X

Date ________
Page ________

$$\therefore P(A_i) = \underbrace{\binom{k}{1} p_i (1-p_i)^{k-1}}_{\text{for 1 only}} + \underbrace{\binom{k}{2} p_i^2 (1-p_i)^{k-2}}_{\text{for 2 only}} + \dots + \underbrace{\binom{k}{k} p_i^k (1-p_i)^0}_{\text{for all}}$$

$$= \binom{k}{0}(1-p_i)^k + \binom{k}{1} p_i (1-p_i)^{k-1} + \dots + \binom{k}{k} p_i^k (1-p_i)^0 - \binom{k}{0}(1-p_i)^k$$

$$= (p_i + 1 - p_i)^k - 1\cdot(1-p_i)^k$$

(Using binomial theorem)

$$\boxed{= 1 - (1-p_i)^k}$$

(B) $P(A_i \cup A_j) = ?$

here., $(A_i \cup A_j)^c = A_i^c \cap A_j^c$

No coupons is of either type 'i' or type 'j'!

$$\therefore P(A_i \cup A_j)^c = (1 - p_i - p_j)^k$$

$$\boxed{\therefore P(A_i \cup A_j) = 1 - (1 - p_i - p_j)^k}$$

(C) $P(A_i | A_j) = ?$

Using formula.

$$P(A_i | A_j) = \frac{P(A_i \cap A_j)}{P(A_j)}$$

$$= \frac{-P(A_i \cup A_j) + P(A_i) + P(A_j)}{P(A_j)}$$

$$= \frac{-(1 - (1-p_i-p_j)^k) + 1 - (1-p_i)^k + 1 - (1-p_j)^k}{1-(1-p_j)^k}$$

$$\boxed{= \frac{1 - (1-p_i)^k - (1-p_j)^k + (1-p_i-p_j)^k}{1-(1-p_j)^k}}$$

Date ________
Page ________

Sol^n 2] Independent trials.
P(Success) = p
P(Failure) = 1-p.
What is the probability that n successes occur before m failures?

→ Here it is not specified that how many trials will be take.

⇒ the question can be seen as a " What is the probability that at least n successes occur before m failures.

⇒ There must be at least ~~n+m-1~~ n successes in first n+m-1 trials. Here game can be over before n+m-1 trials! And in that case we will assume that additional trials are taken to complete n+m-1 trials.

Now the probability for k successes in l trials
↳ exactly

is $= \boxed{\binom{l}{k} p^k (1-p)^{l-k}}$ p: Probability of success

here $k = n, n+1, \ldots, n+m-1$
$l = n+m-1$

∴ Probability of n successes before m failures is
$$\boxed{P_{n,m} = \sum_{i=n}^{n+m-1} \binom{n+m-1}{i} p^i (1-p)^{n+m-1-i}}$$

→ The game discussed in the question is just to give you a brief idea about the question.

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Sol^n 3] Let's learn some biology !!!

Every living things have a gene pair in the DNA and that ~~is~~ can be seen as (x, y).

now x & y are determine by its mother and father. $\rightarrow$ Mother (x_1, y_1) Father (x_2, y_2)

~~Chi~~

Child (x, y)

x from mother, y from father

$$\boxed{P(x=x_1) = P(x=y_1) = \frac{1}{2} \qquad P(y=x_2) = P(y=y_2) = \frac{1}{2}}$$

- Now Let's move ~~to~~ on to the question.

A female chimp has given birth to a child.

M_1: Male no. 1 is the father.

$\boxed{P(M_1) = p}$

M_2: Male no. 2 is the father.

$\boxed{P(M_2) = (1-p)}$

	Pair of gene.
Female	(A, A)
Child/Baby.	(A, a)
Male 1	(a, a)
Male 2	(A, a)

~~BGA~~ $\equiv$ $B(A,a)$ $\rightarrow$ Baby has gene pair (A, a)

Date ____
Page ____

→ What is the probability that Male 1 is a father?

$\therefore P(M_1 | B(A,4)) = ?$

$$\therefore P(M_1 | B(A,4)) = \frac{P(B(A,4)|M_1)\,P(M_1)}{P(B(A,4)|M_1)\,P(M_1) + P(B(A,4)|M_2)\,P(M_2)}$$

$$= \frac{1\cdot P}{1\cdot P + (1-P)\frac{1}{2}} \qquad (\because M_1 \text{ has } (a,4))$$

$$\boxed{= \frac{2P}{P+1}}$$

Soln 4] n men in a party. Their hats are removed and mixed up. Now ~~to~~ match occurs if a men select his own hat.

(a) What is ~~a~~ the probability that no matches occur?

Method 1 → $P(a) = \dfrac{\text{No. of ways in which no matches occur.}}{\text{total ways of arrangement.}}$

here total ways of arrangement $= n!$

Now to find total ways in which no matches occur, we will use concept of Derangement and Inclusion - Exclusion.

~~S_i = #Ways in which i Derangements occurs.~~

~~$\therefore \delta_n = S_0 - S_1 + S_2 - S_3 + \dots + (-1)^n$~~ X

d_n = #ways in which no one gets their hats.

Now using inclusion exclusion argument

$$\boxed{d_n = S_0 - S_1 + S_2 - S_3 + \dots + (-1)^n S_n} \quad \text{—(1)}$$

$$\boxed{S_i = \binom{n}{i}(n-i)!} \quad \text{(2)}$$

→ i things go in correct place and rest of the things go "anywere"!

Scanned by CamScanner

Date ______
Page ______

to explain in better way

M_i: Men i gets his correct hat.

$\therefore d_n = N(M_0^c \cap M_1^c \cap M_2^c \cap \dots \cap M_n^c)$.

- Now ~~4nt~~ using eq^n 2 in eq^n 1.

$$\therefore d_n = \binom{n}{0} n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \dots + (-1)^n \binom{n}{n} 0!$$

$$= n! - \frac{n!\,(n-1)!}{(n-1)!\,1!} + \frac{n!\,(n-2)!}{(n-2)!\,2!} + \dots + \frac{(-1)^n\, n!\, 0!}{0!\, n!}$$

$$\boxed{\therefore d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right)}$$

now expected probability

$$p(4) = \frac{d_n}{n!}$$

$$\boxed{= \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right)}$$

→ If you don't get this, plz try to understand the inclusion & exclusion argument.

* Method-2 / Approach 2.

E: Event that no matches occur.

M: First person got his correct hat.

$P(E) = P_n$ → to undertand or to make it explicit the dependence on n.

NOW, $P_n = P(E) = P(E|M)\,P(M) + P(E|M^c)\,P(M^c)$

here $P(E|M) = 0$

Date ______
Page ______

$\therefore P_n = P(E|M^c)\,P(M^c)$

$$\therefore \boxed{P_n = P(E|M^c)\frac{(n-1)}{n}} \quad \text{— ① }$$

→ Now, $P(E|M^c)$ → The probability of no matches when $(n-1)$ men select from a set of $(n-1)$ hats, that does not contains the hat of one of these men. This can ~~be~~ happen in two ways

1: There are no matches and the extra man does not select that extra hat.

2: There are no matches and the extra man select the extra hat.

→ Here case 1 can be seen as $n-1$ man and none chooses their hats. (for our convenience that extra hat is now belonging to that extra man)
so the probability of first case is $\boxed{P_{n-1}}$

→ Now the probability of second case is

$$\underbrace{\boxed{\frac{1}{(n-1)}}}_{\text{extra man chooses extra hat.}}\ \underbrace{\boxed{P_{n-2}}}_{\text{no matches for rest of the } n-2 \text{ men.}}$$

$$\therefore P_n = \frac{(n-1)}{n}\left(P_{n-1} + \frac{1}{n-1}P_{n-2}\right)$$

$$\therefore P_n = \left(\frac{n-1}{n}\right)P_{n-1} + \frac{1}{n}P_{n-2}$$

$$\therefore \boxed{P_n - P_{n-1} = \frac{1}{n}(P_{n-1} - P_{n-2})} \quad \text{— ②}$$

now $\boxed{\begin{aligned} P_1 &= 0 \\ P_2 &= \tfrac{1}{2} \end{aligned}}$ (1 man 1 hat)

Putting this in ~~eqn~~ eq^n 2

$$P_3 - P_2 = -\frac{(P_2 - P_1)}{3} = -\frac{1}{3!} \Rightarrow P_3 = \frac{1}{2!} - \frac{1}{3!}$$

$$P_4 - P_3 = -\frac{(P_3 - P_2)}{4} = \frac{1}{4!} \Rightarrow P_4 = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$$

$$\therefore \boxed{P_n = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{(-1)^n}{n!}}$$

Which is same as answer we got earlier using method I.

B What is the probability that ~~no matches~~ exactly k matches occur?

→ there are $\binom{n}{k}$ ways to choose k people who will get their correct hat. And for them, fixed group of k men will get correct hat.

$$\therefore P(B) = \underbrace{\binom{n}{k}}_{\text{ways to choose.}} \times \underbrace{\frac{1}{n} \times \frac{1}{(n-1)} \times \dots \times \frac{1}{(n-(k-1))}}_{\text{probability that } k \text{ men will get correct hat.}} \times \underbrace{P_{n-k}}_{\text{rest of them will get wrong hat.}}$$

$$= \frac{n!}{(n-k)!\,k!} \times \frac{1}{n \times (n-1) \times (n-2) \dots \times (n-k+1)} \times P_{n-k}$$

$$= \frac{n!}{n!\,k!} \times P_{n-k}$$

Date
Page

$\therefore P(B) = \dfrac{P_{n-k}}{k!}$

$$\boxed{= \dfrac{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{(-1)^{(n-k)}}{(n-k)!}}{k!}}$$