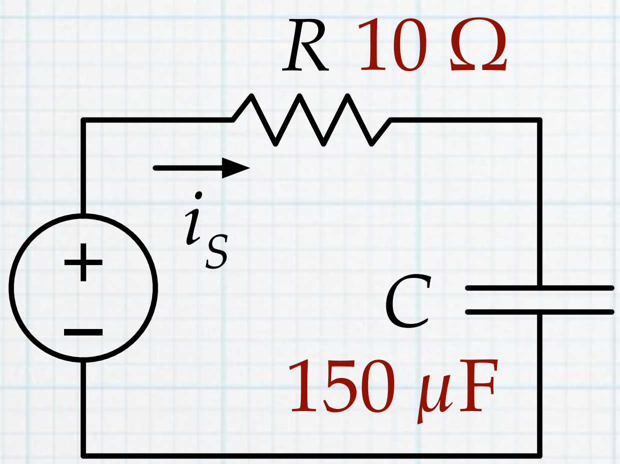
AC power

Consider a simple RC circuit. We might like to know how much power is being supplied by the source. We probably need to find the current.

$V_S = V_m \cos\omega t$

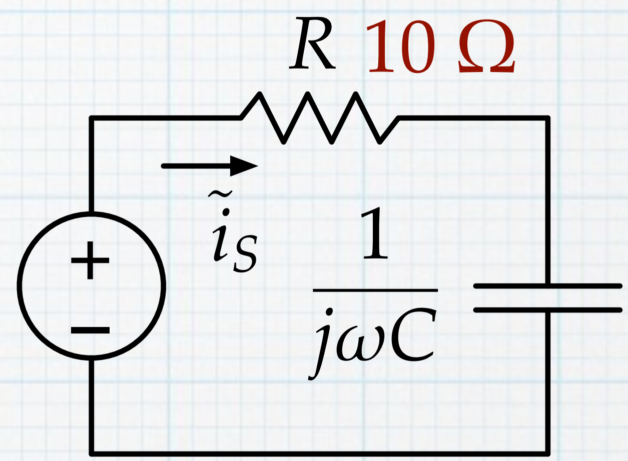
$V_m = 10$ V

$f = 60$ Hz



$\tilde{V}_s = V_m$

10 V



$-j17.7\ \Omega$

$$Z = R + \frac{1}{j\omega C} = 10\ \Omega - j17.7\ \Omega$$

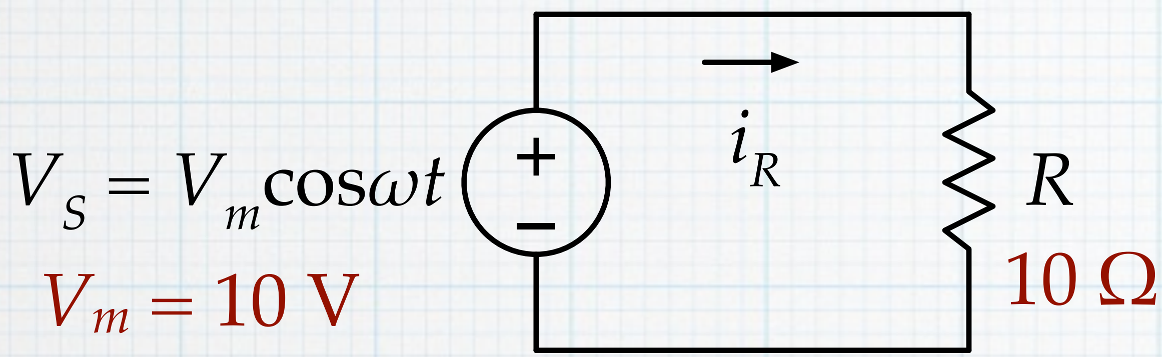
$$\tilde{i}_S = \frac{\tilde{V}_S}{Z} = \frac{V_m}{R + \frac{1}{j\omega C}} = \frac{10\text{V}}{10 - j17.7\Omega} = \frac{10\text{V}}{(20.3\Omega)\, e^{-j60.5^\circ}} = (0.49\text{A})\, e^{+j60.5^\circ}$$

Easy enough, but what about power ? We could multiply the source voltage by the above current expression, but we'll get a complex number. Does that mean anything?

$$\tilde{P}_S = \tilde{V}_S \cdot \tilde{i}_S = (10\text{V}) \left[(0.49\text{A})\, e^{j60.5^\circ}\right] = (4.9\text{W})\, e^{j60.5^\circ} = 2.413\text{W} + j4.265\text{W}$$

?!?!?!?!?!?

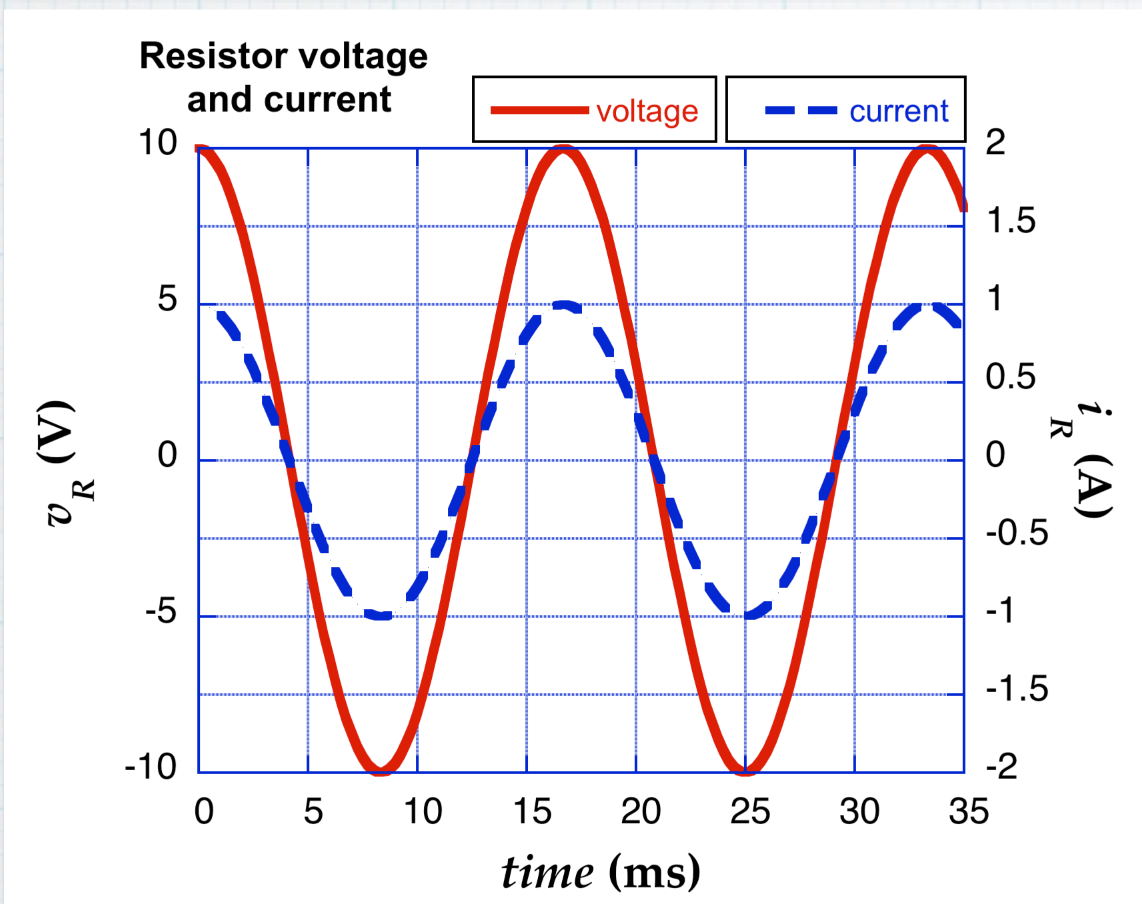
Quick review: Look at just the resistor.

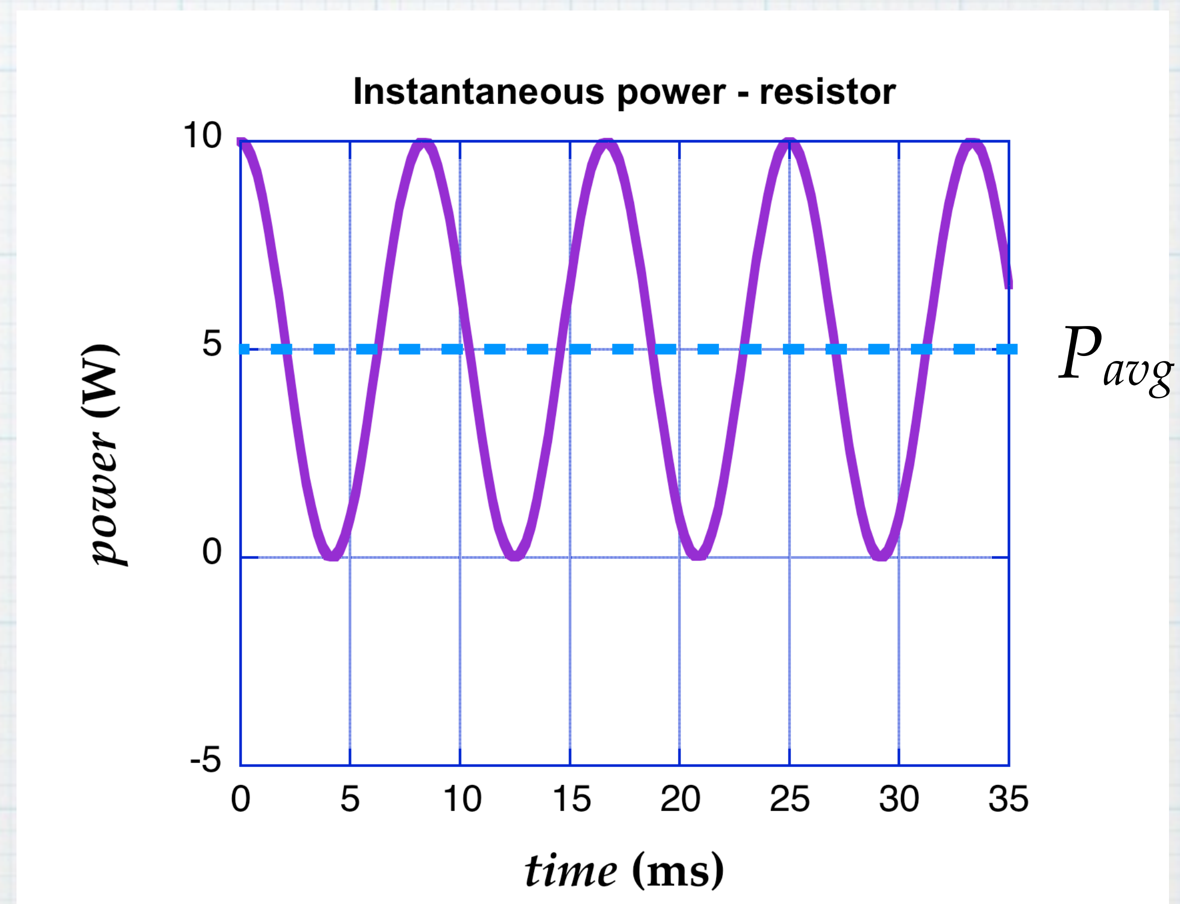


$$i_R = \frac{V_S}{R} = \frac{V_m}{R}\cos\omega t = I_m \cos\omega t$$

$$P(t) = V_S \cdot i_R = V_m I_m \cos^2 \omega t$$

$$= \frac{V_m I_m}{2}\left(1 - \cos 2\omega t\right)$$





$$P_{avg} = \frac{1}{T}\int_0^T P(t)\,dt = \frac{1}{T}\int_0^T \frac{V_m I_m}{2}\left(1 + \cos 2\omega t\right)dt = \frac{V_m I_m}{2}$$

Now consider just the capacitor.

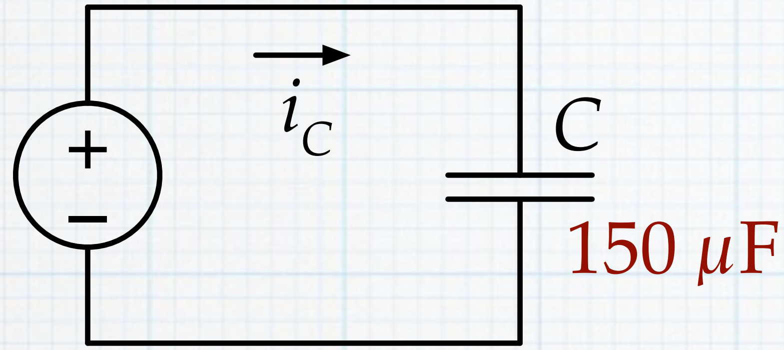
$$i_C = C\frac{dv_c}{dt} = -\omega C V_m \sin\omega t$$

$$= -I_m \sin\omega t \quad I_m = 0.566\ \text{A}$$

$V_m = 10$ V

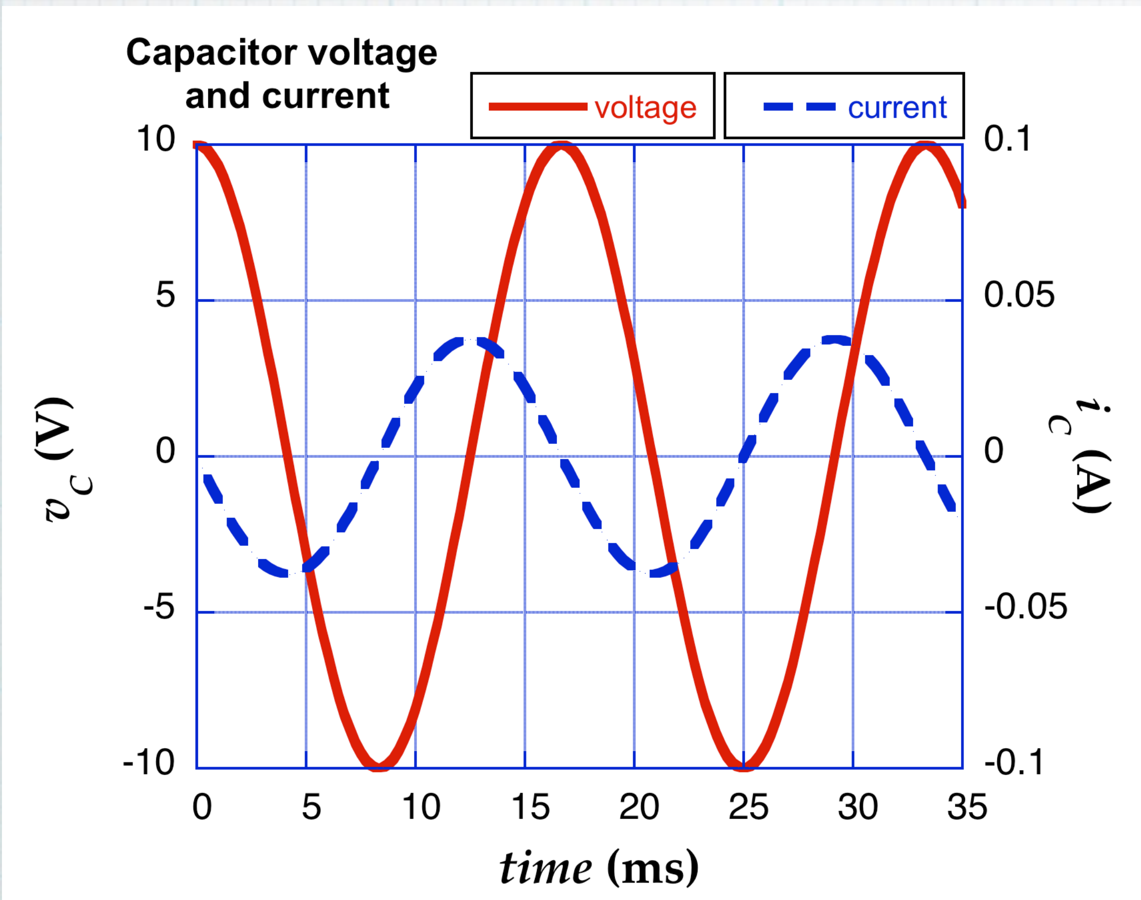
$V_S = V_m \cos\omega t$

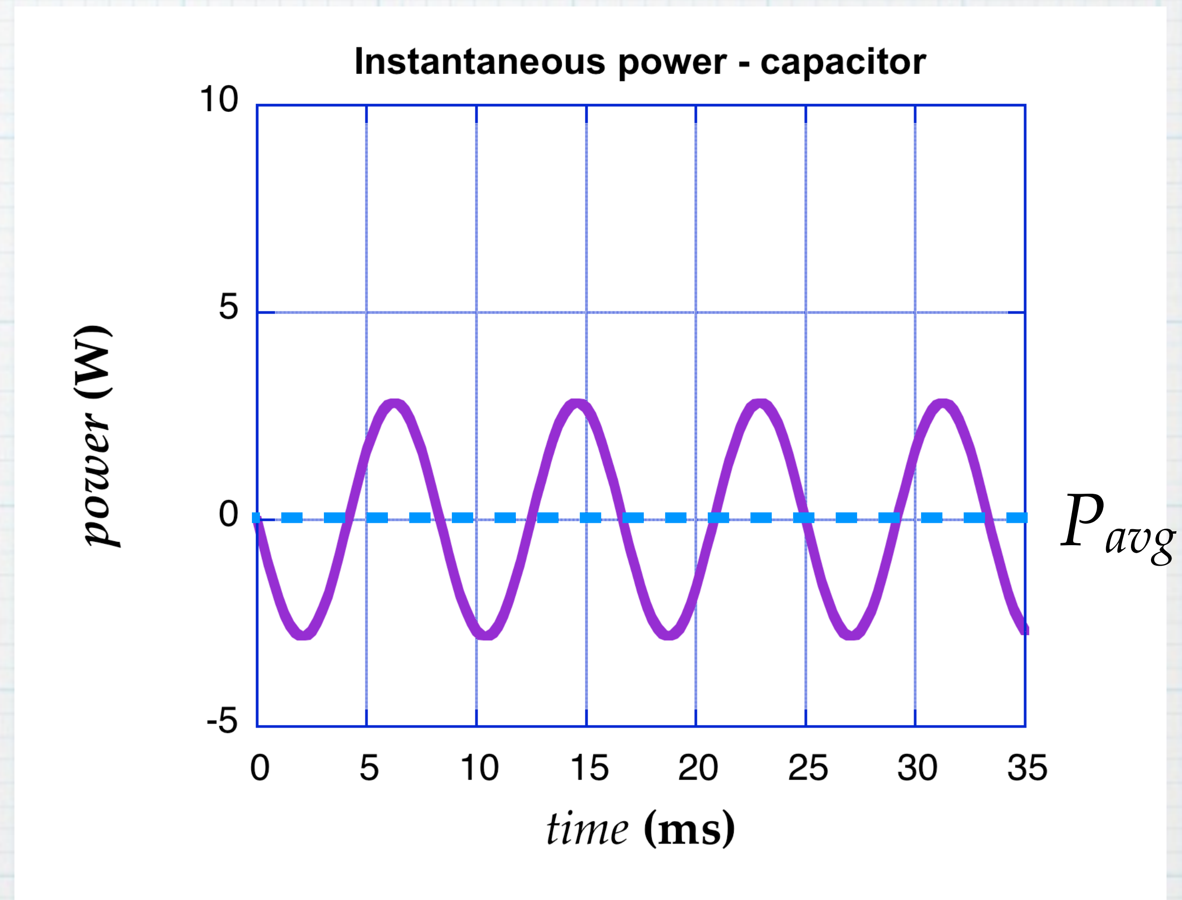
$f = 60$ Hz



$$P(t) = -V_m I_m \cos\omega t \sin\omega t$$

$$= -\frac{V_m I_m}{2}\sin 2\omega t$$

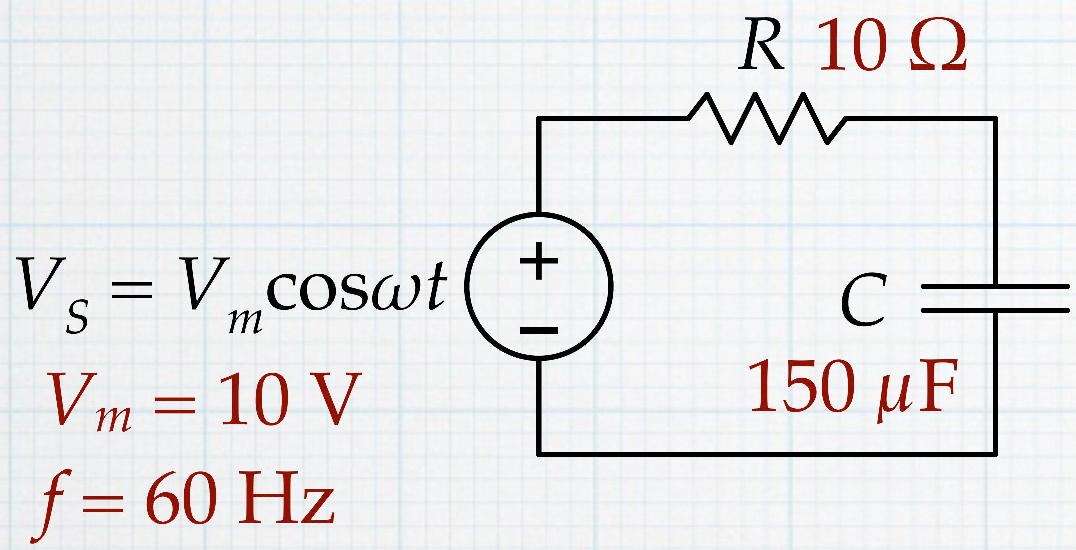




$$P_{avg} = \frac{1}{T}\int_0^T P(t)\,dt = \frac{1}{T}\int_0^T \left(-\frac{V_m I_m}{2}\right)\sin(2\omega t)\,dt = 0\ !!$$

There's a lot of energy going back and forth, but there is no net power flow.

Back to the RC circuit.



$$V_S(t) = V_m \cos\omega t$$

$$i_S(t) = I_m \cos(\omega t + \theta)$$

$I_m = 0.566$ A $\quad \theta = 60.5°$

$$P(t) = V_S(t) \cdot i_S(t)$$ Instantaneous power.

$$= V_m I_m [\cos\omega t][\cos(\omega t + \theta)]$$

$$= V_m I_m \left[\frac{\cos\theta}{2} + \frac{\cos(2\omega t + \theta)}{2}\right]$$ First trig identity.

$$= \frac{V_m I_m}{2}[\cos\theta + \cos\theta\cos(2\omega t) - \sin\theta\sin(2\omega t)]$$

Second trig identity.

$$\cos a \cos b = \frac{\cos(a+b)}{2} + \frac{\cos(a-b)}{2}$$

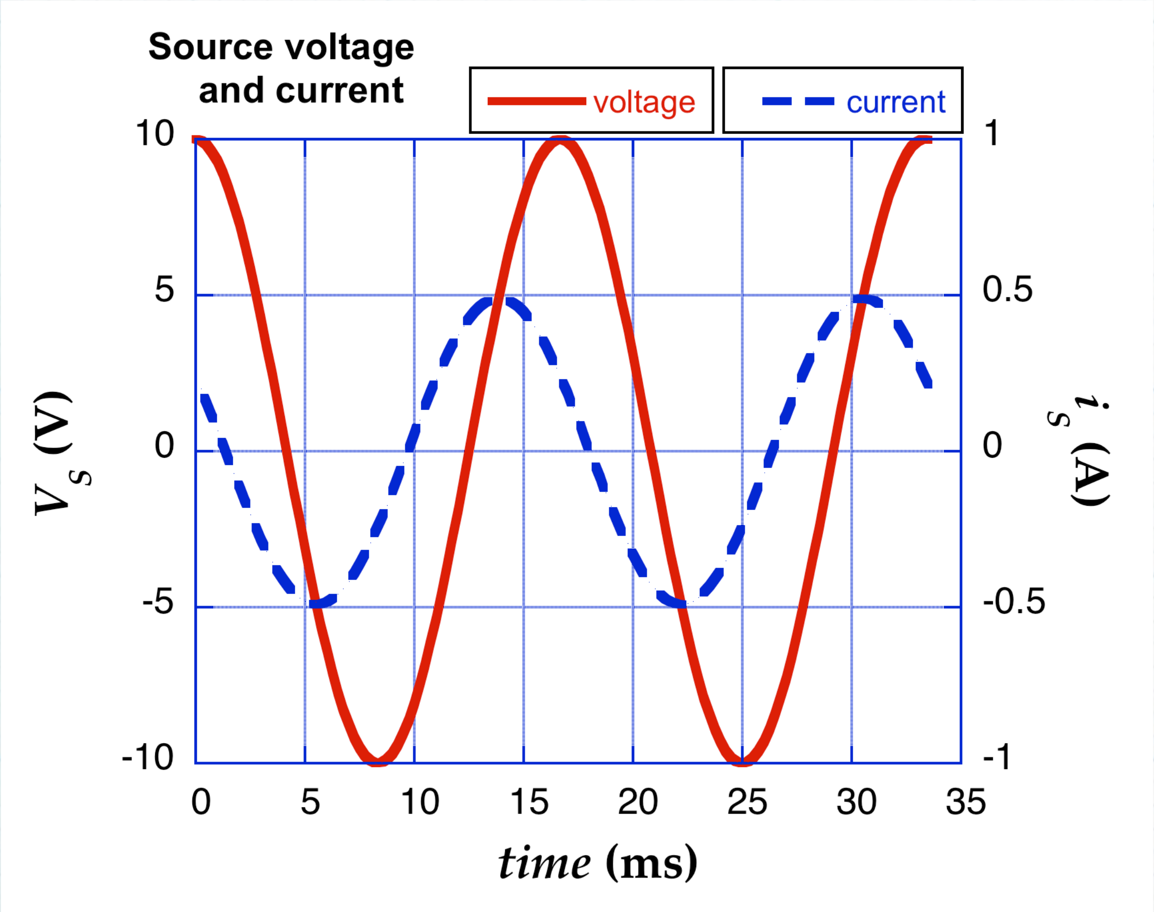
$$\cos a \cos b = \frac{\cos(a+b)}{2} + \frac{\cos(a-b)}{2}$$

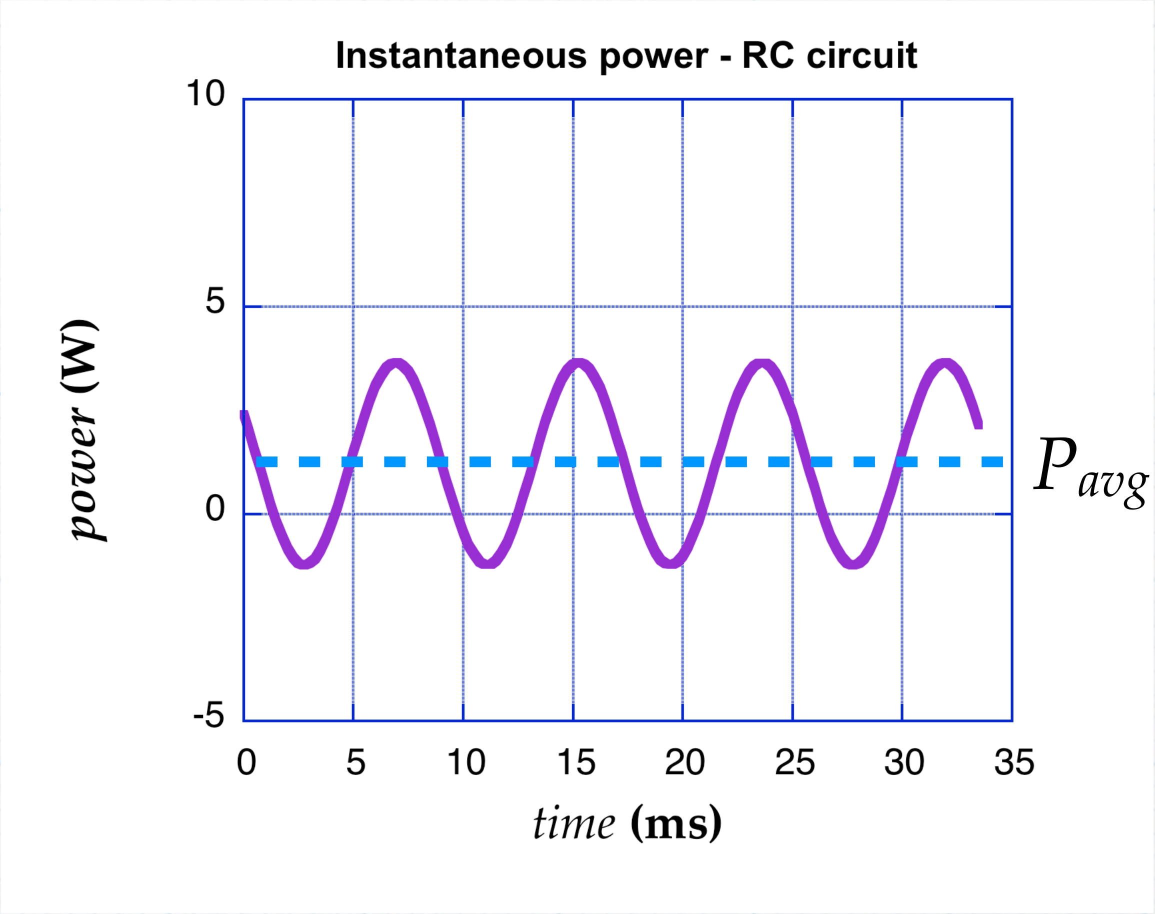
$$P_{inst}(t) = \underbrace{\frac{V_m I_m}{2} \cos\theta \left[1 + \cos(2\omega t)\right]}_{\text{resistor}} - \underbrace{\frac{V_m I_m}{2} \sin\theta \sin(2\omega t)}_{\text{capacitor}}$$

In comparing these terms with the expressions found for the resistor and capacitor individually, we can identify the first term as the power dissipated in the resistor and the second term as the power traded back and forth with the capacitor.

The source *appears* to be sending out power in the amount of $V_m I_m$. However, the reactive element captures some of that and sends it back later. This reduces the amount of power available to be dissipated in the resistor.

The reduction in the amount of power delivered to the resistor (the power that is converted to other forms of energy) is represented by the factor $\cos\theta$





Average power delivered

$$P_{avg}(t) = \frac{1}{T}\int_0^T P_{inst}(t)\,dt = \frac{V_m I_m}{2}\cos\theta = V_{RMS} I_{RMS}\cos\theta$$

Definitions

Power systems experts have developed a terminology and a set of units to describe these relationships.

Apparent power → $S = \frac{V_m I_m}{2} = V_{RMS} I_{RMS}$

Represents power that the source *thinks* is being sent out. The units for apparent power are volt-amps (VA).

Real power → $P = \frac{V_m I_m}{2} \cos\theta = V_{RMS} I_{RMS} \cos\theta$

The power being dissipated in the resistor (i.e. power that is converted to another form). The units are watts (W), just like our earlier discussion of power. This has the same value as the average power.

Reactive power → $Q = -\frac{V_m I_m}{2} \sin\theta = V_{RMS} I_{RMS} \sin\theta$

Represents of the power being exchanged back and forth between the source and reactive elements. The units are *volt-amps reactive* (VAR). Note that reactive power can be positive or negative, reflecting the nature of the reactance in the circuit. Also known as *imaginary power.*

Also

$\theta \rightarrow$ power angle

$\cos\theta \rightarrow$ power factor

With these definitions, we can write the expression for the instantaneous power in a couple of different ways.

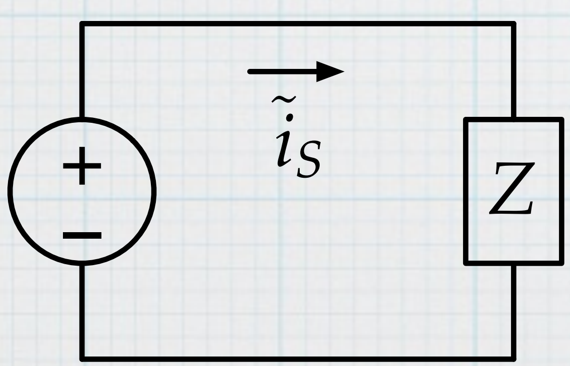
$$P_{isnt} = S\cos\theta\left[1+\cos\left(2\omega t\right)\right] - S\sin\theta\sin\left(2\omega t\right)$$

$$P_{inst}\left(t\right) = P\left[1+\cos\left(2\omega t\right)\right] + Q\sin\left(2\omega t\right)$$

Note that everything about power depends on the impedance. The power angle is simply the negative of the angle of the impedance.

$$V_S = V_m \cos\omega t$$

$$\tilde{V}_s = V_m$$



$$Z = R + jX$$

$$= |Z|e^{j\theta_z}$$

$$\tilde{i}_S = \frac{\tilde{V}_S}{Z} = \frac{V_m}{|Z|e^{j\theta_z}} = I_m e^{-j\theta_z}$$

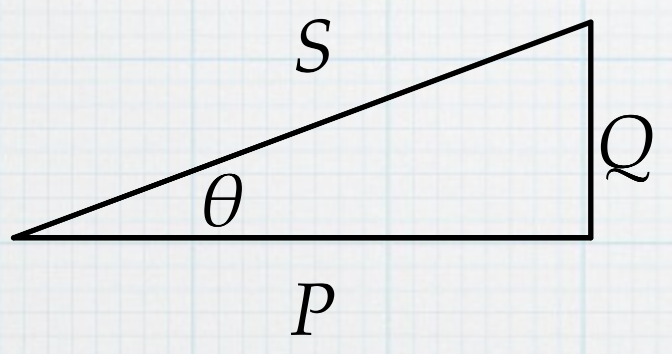
$$i_S\left(t\right) = I_m\cos\left(\omega t - \theta_z\right)$$

Power triangle

From the definitions for apparent, real, and reactive power, we see that they can be related by an equation for a right triangle:

$$S^2 = P^2 + Q^2$$

This suggests a visual aid for understanding the relationships.



If the power angle is zero, the circuit consists of purely resistive elements and the all of the power from the source will be dissipated in resistors (real power = apparent power). As the impedance becomes more reactive, the power angle increase increase, and less of power is delivered to the resistive portion.

Complex power

Now back to the original question: What can we learn about the power distribution from the complex voltage and current? Taking the product of the voltage and current phasers:

$$\tilde{V}_S \tilde{i}_S = V_m \left(I_m e^{j\theta} \right) = V_m I_m \cos\theta + j V_m I_m \sin\theta$$

Comparing this to the result on slide 5 or the definitions on slide 7, we see that this is slightly off. First, we need to divide by 2 to get the correct average values. (Or use RMS quantities.) Secondly, the sign of the reactive power is wrong, which is say that sign of the angle is wrong. We should use the complex conjugate of the current to get the right sign when calculating complex power.

$$\tilde{P} = \frac{\tilde{V}_S \tilde{i}_Z^*}{2} = \frac{V_m I_m}{2} \cos\theta - j \frac{V_m I_m}{2} \sin\theta$$

$$= P + jQ$$

Now we see why reactive power can be also called imaginary power.

Be careful with possible confusion. If the voltage and current are specified in RMS quantities, then we do not divide by 2.

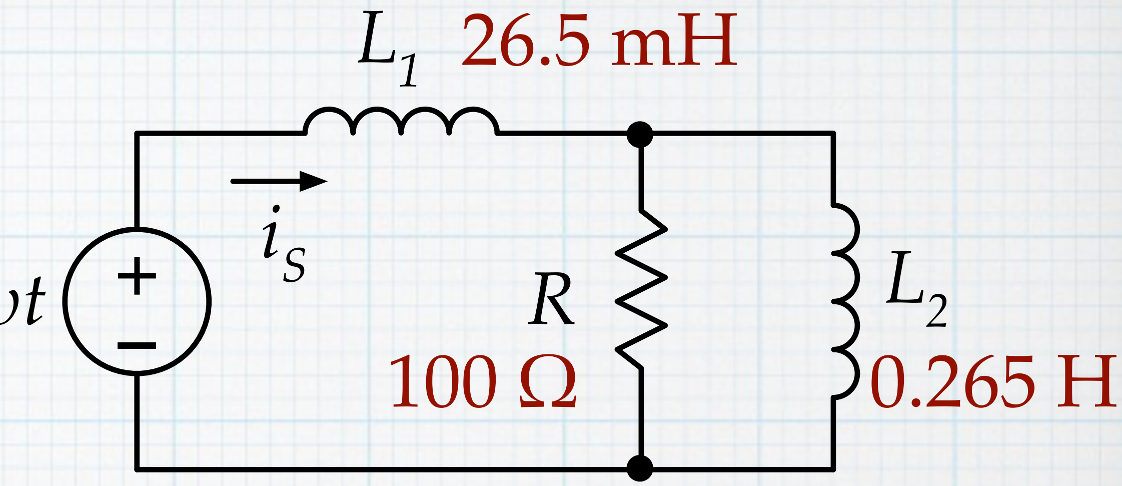
Example

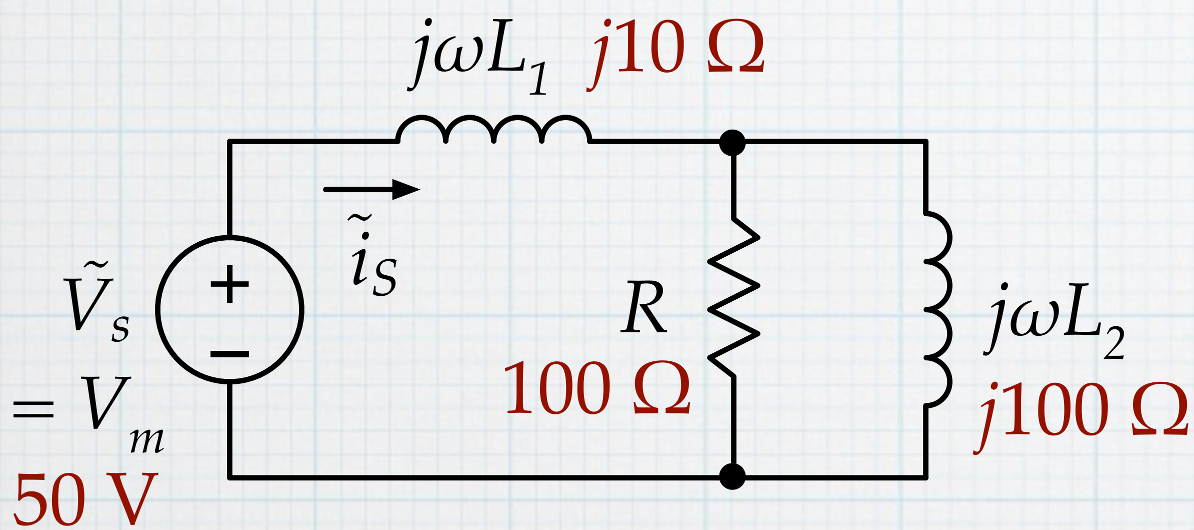
Find the apparent power, real power, and reactive power for the inductive circuit at right.

$V_s = V_m \cos\omega t$

$V_m = 50\ \text{V}$

$f = 60\ \text{Hz}$





Total impedance seen by the source:

$$Z = j\omega L_1 + R \,\Big|\Big|\, (j\omega L_2)$$

$$= j10\Omega + (100\Omega) \,\Big|\Big|\, (j100\Omega)$$

$$= 50\Omega + j60\Omega = (78.1\Omega)\, e^{j50.2^\circ}$$

$$\tilde{i}_S = \frac{\tilde{V}_S}{Z} = (0.64\text{A})\, e^{-j50.2^\circ}$$

$$\tilde{P}_S = \frac{\tilde{V}_S \cdot \tilde{i}_S^*}{2} = \frac{(50\text{V}) \cdot (0.64\text{A})\, e^{+j50.2^\circ}}{2}$$

$$= (16.0\text{VA})\, e^{+j50.2^\circ}$$

$$= 10.24\text{W} + j12.29\text{VAR}$$

$S = 16.0\ \text{VA}$

$P = 10.24\ \text{W}$

$Q = 12.29\ \text{VAR}$

Example

An AC source is providing power to a load that contains some reactance. Measurements show that the real power being delivered is 88 W and the reactive power is –47.5 VAR. What are the apparent power, the power angle, and the power factor for the system?

$P = 88\text{ W}$ and $Q = -47.5\text{ VAR}$

$$S = \sqrt{P^2 + Q^2}$$

$$= \sqrt{(88\text{W})^2 + (47.5\text{VAR})^2}$$

$$= 100\text{VA}$$

$$\sin\theta = \frac{Q}{S} = \frac{-47.5}{100} = -0.475$$

$$\theta = -28.4^\circ$$

Example

An AC source running at 120 V_{RMS} is delivering power to a load with impedance of 50 Ω – j 28.9 Ω. Find the real and reactive power being delivered.

(Note the use of RMS units, which means that we do not divide by 2 when calculating the powers. Or else we can change the voltage to its peak value and then divide everything by 2.)

$$Z = 50\Omega - j28.9\Omega = (57.75\Omega)\, e^{-j30^\circ}$$

$$\tilde{i}_S = \frac{\tilde{V}_S}{Z} = (2.08\mathrm{A}_{\mathrm{RMS}})\, e^{+j30^\circ}$$

$$\tilde{P}_S = \tilde{V}_S \cdot \tilde{i}_S^* = (249.6\mathrm{VA})\, e^{-j30^\circ}$$

$$= 216.2\mathrm{W} + j124.8\mathrm{VAR}$$

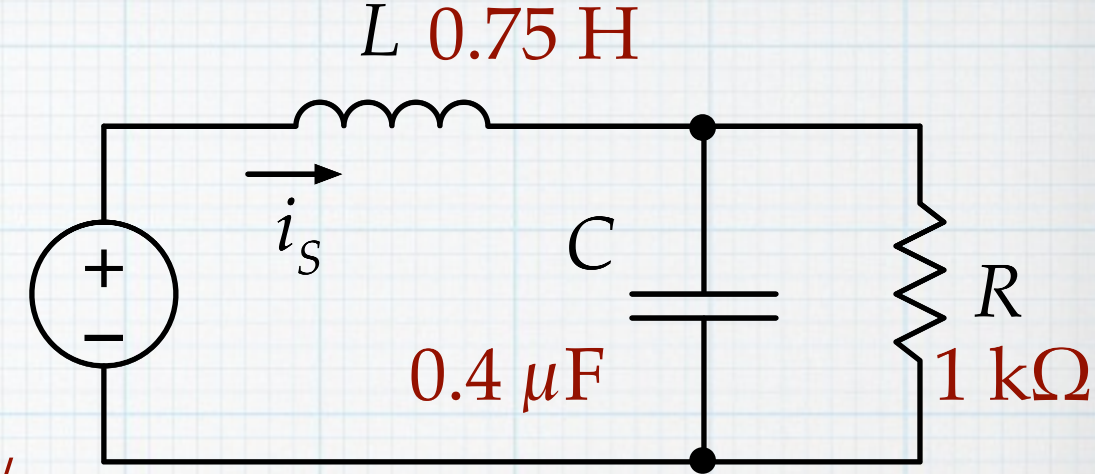
$P = 216.2$ W

$Q = 124.8$ VAR

Example

Find the apparent power, real power, and reactive power for the inductive circuit at right.

Then add a component in series with the inductor that will cause the power angle to be 0°.



$V_S = V_m \cos\omega t$

$V_m = 25$ V

$\omega = 1000$ rad/s

Total impedance seen by the source:

$$Z = j\omega L + R \left|\left| \left(\frac{1}{j\omega C}\right)\right.\right.$$

$$= j750\Omega + (1000\Omega) \left|\left| (-j2500\Omega)\right.\right.$$

$$= 861.6\Omega + j405.4\Omega = (952.2\Omega)\, e^{j25.2^\circ}$$

$$\tilde{i}_S = \frac{\tilde{V}_S}{Z} = (26.26\text{mA})\, e^{-j25.2^\circ}$$

$$\tilde{P}_S = \frac{\tilde{V}_S \cdot \tilde{i}_S^*}{2} = \frac{(25\text{V}) \cdot (0.0263\text{A})\, e^{+j25.2^\circ}}{2}$$

$$\tilde{P}_S = (0.329\text{VA})\, e^{+j25.2^\circ}$$

$$= 0.297\text{W} + j0.140\text{VAR}$$

The total impedance appears inductive (positive reactance). To offset that, add some capacitance in series.

$$\frac{1}{j\omega C} = -j405.4\Omega$$

$$C = 2.47\mu F$$

Then Q = 0 and

$$P = \frac{(25\text{V})^2}{2\,(861.6\Omega)} = 0.363\text{W}$$