Tute 7 Solⁿ

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Solⁿ-1 X is a normal random variable with parameters μ & σ^2.

$E[X] = ?$ $\quad Var[X] = ?$

To make it easy let's start with standard normal random variable.

$$Z = \frac{X-\mu}{\sigma}$$

now, $E(Z) = \int_{-\infty}^{\infty} x f_Z(x)\,dx$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-x^2/2}\,dx$$

$$\boxed{= 0} \quad \left(\text{since } \frac{1}{\sqrt{2\pi}} x e^{-x^2/2} \text{ is an odd function}\right)$$

$$Var(Z) = \int_{-\infty}^{\infty} x^2 f_Z(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot x e^{-x^2/2}\,dx.$$

(Integration By Parts)

$$= \frac{1}{\sqrt{2\pi}}\left[x \int x e^{-x^2/2}\,dx - \int_{-\infty}^{\infty} 1 \int x e^{-x^2/2}\,dx\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 + \int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right]$$

$$\therefore Var[Z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx$$

$$\boxed{= 1}$$

Now, $Z = \dfrac{X - \mu}{\sigma}$

$$\therefore E[Z] = \frac{E[X] - \mu}{\sigma}$$

$$\therefore E[X] = \mu + \sigma E[Z]$$

$$= \mu + 0$$

$$\therefore \boxed{E[X] = \mu}$$

Same way $Z = \dfrac{X - \mu}{\sigma}$

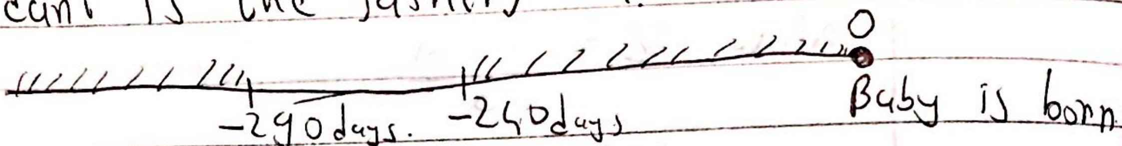
$$\therefore Var[Z] = Var\left[\frac{X}{\sigma}\right]$$

$$\therefore Var[Z] = \frac{1}{\sigma^2} Var[X]$$

$$\therefore Var[X] = \sigma^2 Var[Z]$$

$$\boxed{\therefore Var[X] = \sigma^2}$$

Sol^n.2 X : The length of the gestation.
X is normally distributed with parameters
$\mu = 270$
$\sigma^2 = 100$
P(Defendant is the father) = ?



$= P(X > 290 \wedge X < 240)$
$= P(X > 290) + P(X > 240)$
$= P\left(\frac{X-270}{100} > 2\right) + P\left(\frac{X-270}{100} < -3\right)$

$= 1 - \Phi(2) + \cancel{0} + \Phi(-3)$
$= \cancel{2u}\ 1 - \Phi(2) + 1 - \Phi(3)$
$= 1 - 0.9772 + 1 - 0.9987$
$\boxed{= 0.0241}$

Sol^{n}3 There are 100 people on diet.
at least 65 % people have lower the colestrol count ⇒ nutritionist endorses new diet.
P{Nutritionist endorses the new diet} = ?
• Here it is given that diet has no effect on colestrol count.
⇒ Probability for each person's count to be lower than it was before $= \frac{1}{2}$.

X : Number of people whose count is lowered.

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Desired probability

$$= \sum_{i=65}^{100} \binom{100}{i} \left(\frac{1}{2}\right)^{100} = P\{X \geq 64.5\}$$

here $n = 100$ $p = \frac{1}{2}$ $\mu = np$

$$\Rightarrow \mu = \frac{100}{2} = \boxed{50}$$

$$\Rightarrow \sigma^2 = np(1-p) = 100 \times \frac{1}{2} \times \frac{1}{2}$$

$$Z = \frac{X-\mu}{\sigma} \qquad \boxed{= 25}$$

$$\therefore \text{Probability} = P\left\{\frac{X-50}{\sqrt{25}} \geq \frac{64.5-50}{\sqrt{25}}\right\}$$

$$= P\{Z \geq 2.9\}$$

$$= 1 - \Phi(2.9)$$

$$= 1 - 0.9981$$

$$\boxed{= 0.0019}$$

Soln-4 X : Random variable with parameter λ.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$E[x]$, $Var[x] = ?$

now, $E[X^n] = \int_0^\infty x^n f(x)\, dx \quad (n > 0)$

$$= \int_0^\infty x^n \lambda e^{-\lambda x}\, dx$$

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$$\therefore E[x^n] = x^n \int \lambda e^{-\lambda x} - \int_0^\infty \left(\frac{d}{dx} x^n \int \lambda e^{-\lambda x}\, dx\right) dx$$

$$= \left[-x^n e^{-\lambda x}\right]_0^\infty + \int_0^\infty n x^{n-1} e^{-\lambda x}\, dx$$

$$= 0 + \int_0^\infty \frac{n}{\lambda} \cdot \lambda e^{-\lambda x} x^{n-1}\, dx$$

$$\therefore \boxed{E[x^n] = \frac{n}{\lambda} E[X^{n-1}]}$$

$$\therefore \boxed{E[X] = \frac{1}{\lambda}} \qquad (n=1)$$

$$\therefore E[X^2] = \frac{2}{\lambda}\left(\frac{1}{\lambda}\right) = \boxed{\frac{2}{\lambda^2}} \qquad (n=2)$$

$$\therefore Var[X] = E[X^2] - (E[X])^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\therefore \boxed{Var[X] = \frac{1}{\lambda^2}}$$

Sol^n-5 $X \to$ Random variable with probability density function.

$$f(x) = \begin{cases} c(1-x^2), & -1 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) $c = ?$

we know that $\int_{-\infty}^{\infty} f(x)\, dx = 1$

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$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_{-1}^{1} c(1-x^2) = 1$$

$$\therefore \left[cx - \frac{cx^3}{3}\right]_{-1}^{1} = 1$$

$$\therefore \left[c - \frac{c}{3}\right] - \left[-c + \frac{c}{3}\right] = 1$$

$$\therefore 2c - \frac{2c}{3} = 1$$

$$\therefore \frac{4c}{3} = 1$$

$$\boxed{\therefore c = \frac{3}{4}}$$

(b) Find cumulative distribution function of X.

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

case 1 $\cancel{x \in (-1, 1)}$ $x > (-1)$

$$\Rightarrow F(x) = \int_{-1}^{x} f(x)\,dx$$

$$= \int_{-1}^{x} \frac{3}{4}(1-x^2)\,dx$$

$$= \left[\frac{3}{4}x - \frac{3}{4}\frac{x^3}{\times 3}\right]_{-1}^{x}$$

$$= \left[\frac{3}{4}x - \frac{x^3}{4}\right] - \left[\frac{-3}{4} + \frac{1}{4}\right]$$

$$= \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right)$$

Case 2 $\cancel{x \notin}$ $x \le (-1)$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{x} 0\,dx$$

$$\therefore \boxed{F(x) = 0}$$

$$\therefore F(x) = \begin{cases} \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right) & x > (-1) \\ 0 & x \le (-1) \end{cases}$$

Soln - 6] X is a normal random variable with parameters $\mu = 10$, $\sigma^2 = 36$

a] $P\{X > 5\} = P\left\{\frac{X-\mu}{\sigma} > \frac{5-\mu}{\sigma}\right\}$

$$= P\left\{Z > \frac{5-10}{6}\right\}$$

$$= P\left\{Z > -\frac{5}{6}\right\}$$

$$= 1 - \Phi(-5/6)$$

$$= \cancel{1} - 0.\cancel{2}033$$

$$\boxed{= 0.7967}$$

$$= 1 - 0.2023$$

$$\boxed{= 0.7977}$$

b] $P\{4 < X < 16\} = P\left\{\frac{4-10}{6} < Z < \frac{16-10}{6}\right\}$

$$= P\{-1 < Z < 1\}$$

$$= \Phi(1) - \Phi(-1)$$

$= 0.841 - 0.159$

$\boxed{= 0.683}$

c] $P\{X<8\} = P\left\{Z < \frac{8-10}{6}\right\}$

$= P\left\{Z < -\frac{1}{3}\right\}$

$= \Phi\left(-\frac{1}{3}\right)$

$\boxed{= 0.369}$

d] $P\{X<20\} = P\left\{Z < \frac{20-10}{6}\right\}$

$= P\left\{Z < \frac{5}{3}\right\}$

$= \Phi\left\{\frac{5}{3}\right\}$

$\boxed{= 0.952}$

e] $P\{X>16\} = P\left\{Z > \frac{16-10}{6}\right\}$

$= P\{Z > 1\}$

$= 1 - \Phi(1)$

$\boxed{= 0.159}$