CT111 Introduction to Communication Systems
Lecture 4: Complex Numbers; Fourier Transform

Yash M. Vasavada

Associate Professor, DA-IICT, Gandhinagar

10th January 2018

Overview of Today's Talk

1. Signal Model
2. Complex Numbers
   - Square Root of Negative Numbers
   - Properties of Numbers
   - Complex Numbers
   - n^{th} Roots of Unity
   - Complex Powers

Overview of Today's Talk

1. Signal Model
2. Complex Numbers
   - Square Root of Negative Numbers
   - Properties of Numbers
   - Complex Numbers
   - n^{th} Roots of Unity
   - Complex Powers

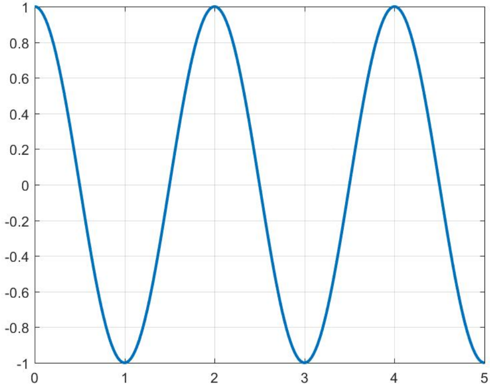
A Standard Form of Communication Signal

Electromagnetic Radiation

- We have talked about how the communication system in our brains is more powerful than any technological communication system devised so far
- However, there is one area where the technology has now outpaced the humans: ability to generate the communication signals that travel far away
- The key signal that makes the long distance communication possible is given by Electromagnetics, and it has a wavy shape that you see when you drop a pebble in a pond

A Standard Form of Communication Signal
Electromagnetic Radiation

- The basic ingredient of a communication signal is a periodic sinusoidal waveform given as: $s_r(t) = A\cos\left(\Theta(t)\right) = A\cos\left(2\pi ft + \theta\right)$



A Standard Form of Communication Signal

Sinusoidal Electromagnetic Radiation

- The basic ingredient of a communication signal is a periodic sinusoidal waveform, with a cycle duration of T_{cycle} given as: $s_r(t) = A\cos(\Theta(t)) = A\cos(2\pi ft + \theta)$
- A : is called the amplitude of the signal
- f : is the frequency of the signal, and it is the inverse of the cycle duration T_{cycle}. This is measured in Hertz
- $\Theta(t)$: is the phase angle of the signal, measured in radians
- θ : is the initial phase angle (at time $t = 0$)

A Standard Form of Communication Signal

Complex Exponential

- Sinusoidal waveform can be thought of as a component of a more general, complex-valued, waveform:

$$s(t) = A\exp\left(i\Theta(t)\right) = A\exp\left(i(2\pi ft + \theta)\right)$$
$$= s_r(t) + is_i(t)$$

  $\rightarrow$ $s(t)$: is called the complex *phasor*
  $\rightarrow$ $s_r(t)$: is the real part of this phasor
  $\rightarrow$ $s_i(t)$: is the imaginary part of this phasor
  $\rightarrow$ i : is the imaginary number $\sqrt{-1}$

- Several new concepts: complex phasor, its real and imaginary part, and $i = \sqrt{-1}$

A Real Tale of Imaginary Numbers
Taking Square Roots

- What is square-root of -1, i.e., $\sqrt{-1}$?
- Ancient mathematicians had to grapple with this, and they concluded that such a number cannot be a real number. Why?
  - $\rightarrow$ Square 1, and you get 1. Square -1, and you also get 1.
  - $\rightarrow$ Alternatively, there is not a way to take any real number, square it and get -1 as the result.
  - $\rightarrow$ This surely means conclusively that no such number as $\sqrt{-1}$ can really exist.
  - $\rightarrow$ OR does it? Let's make sure.

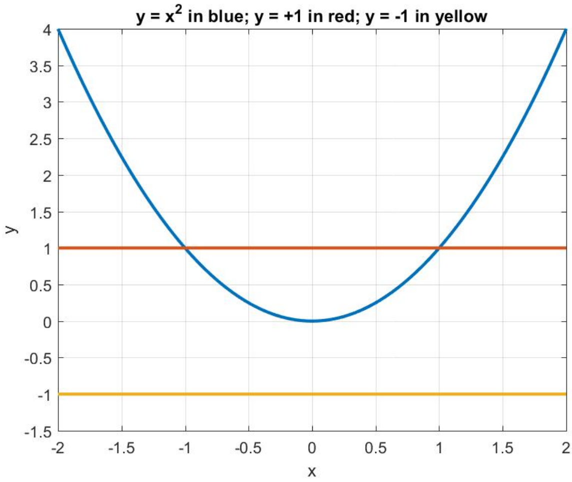
A Real Tale of Imaginary Numbers

An Algebraic View

- $y = x^2$ defines a parabola.
- This parabola intersects $y = 1$ line two times at ± 1, however...
- it never meets $y = -1$ line!

A Real Tale of Imaginary Numbers

An Algebraic/Graphical View



A Tale of Imaginary Numbers

An Algebraic View

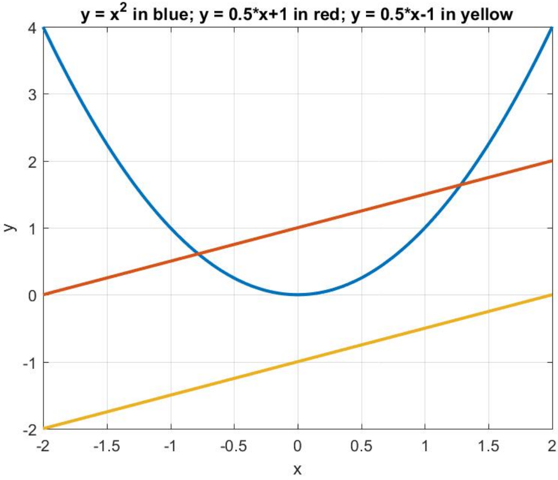
- Equation for the non-horizontal lines: $y = bx + c$.
- The solutions to quadratic equation $x^2 - bx - c = 0$ are given by the point(s) of intersection of parabola $y = x^2$ with the line $y = bx + c$.
- We know that the solution to this quadratic equation is given as

$$x = -\frac{1}{2}\left(b \pm \sqrt{b^2 - 4ab}\right)$$

When $b^2 - 4ab < 0$, we again have the problem of taking square-root of a negative number. This happens because the line never meets the parabola.

A Real Tale of Imaginary Numbers

An Algebraic/Graphical View



A Real Tale of Imaginary Numbers
An Algebraic View

- Okay, all of these settles the issue.
- Absurd to talk/think about $\sqrt{-1}$.

A Real Tale of Imaginary Numbers
Third Order Polynomials

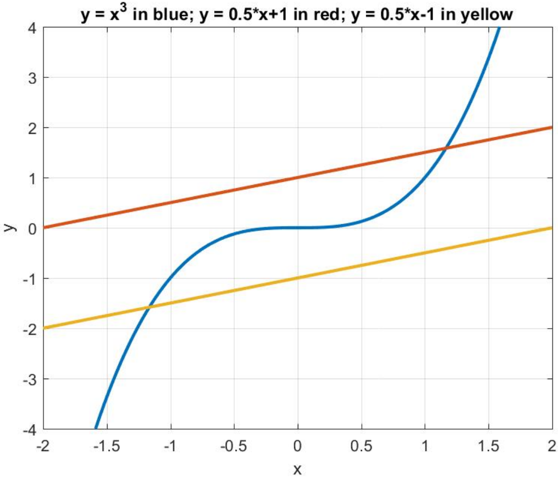
- Let's see the situation for the third-order polynomials:

$$x^3 = bx + c$$

- The line $y = bx + c$ always meets the cubic x^3

A Real Tale of Imaginary Numbers

Third Order Polynomials



A Real Tale of Imaginary Numbers
Third Order Polynomials

- Okay, looks like for the cubic polynomials $x^3 = bx + c$, we should not have to worry about situations in which square-root of a negative number is required.
- In the year 1545, a mathematician in Italy, Gerolamo Cardano, provided a general formula for solving the cubic polynomials $x^3 = bx + c$

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}},$$

where $q = c/2$ and $p = b/3$

A Real Tale of Imaginary Numbers

Bombelli's Problem

- Cardano's formula in the year 1545 for solving the cubic polynomials $x^3 = bx + c$:

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}},$$

where $q = c/2$ and $p = b/3$

- If we don't use Cardano's formula, the solution is clear, and as expected, it is a real number $x = 4$.
- However, if we *do* use Cardano's formula, we obtain

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}},$$

→ The square-root of -1 has reared its ugly head here

→ This was first observed by another mathematician Rafeal Bombelli in about 1575

A Real Tale of Imaginary Numbers

Bombelli's Problem

- Cardano's formula in the year 1545 for solving the cubic polynomials $x^3 = bx + c$:

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}},$$

  where $q = c/2$ and $p = b/3$
- If we don't use Cardano's formula, the solution is clear, and as expected, it is a real number $x = 4$.
- However, if we *do* use Cardano's formula, we obtain

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}},$$

  → The square-root of -1 has reared its ugly head here
  → This was first observed by another mathematician Rafeal Bombelli in about 1575

A Real Tale of Imaginary Numbers
Bombelli's Problem

- Cardano's formula in the year 1545 for solving the cubic polynomials $x^3 = bx + c$:

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}},$$

where $q = c/2$ and $p = b/3$
- If we don't use Cardano's formula, the solution is clear, and as expected, it is a real number $x = 4$.
- However, if we *do* use Cardano's formula, we obtain

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}},$$

→ The square-root of -1 has reared its ugly head here
→ This was first observed by another mathematician Rafeal Bombelli in about 1575

A Real Tale of Imaginary Numbers

Bombelli's Problem

- Cardano's formula in the year 1545 for solving the cubic polynomials $x^3 = bx + c$:

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}},$$

  where $q = c/2$ and $p = b/3$
- If we don't use Cardano's formula, the solution is clear, and as expected, it is a real number $x = 4$.
- However, if we *do* use Cardano's formula, we obtain

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}},$$

$\rightarrow$ The square-root of -1 has reared its ugly head here

$\rightarrow$ This was first observed by another mathematician Rafeal Bombelli in about 1575

A Real Tale of Imaginary Numbers

Bombelli's Problem

- Cardano's formula in the year 1545 for solving the cubic polynomials $x^3 = bx + c$:

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}},$$

  where $q = c/2$ and $p = b/3$
- If we don't use Cardano's formula, the solution is clear, and as expected, it is a real number $x = 4$.
- However, if we *do* use Cardano's formula, we obtain

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}},$$

  → The square-root of -1 has reared its ugly head here
  → This was first observed by another mathematician Rafeal Bombelli in about 1575

A Real Tale of Imaginary Numbers

Bombelli's Problem

How Bombelli would have likely reacted:

- First reaction: blame Cardano, he messed up!
- Second reaction (when it turned out that there was nothing wrong in Cardano's formula): hit the head against a wall
- Final solution (after cooling down and thinking through the problem): discovery of imaginary and complex numbers

A Real Tale of Imaginary Numbers
Bombelli's Problem

How Bombelli would have likely reacted:

- First reaction: blame Cardano, he messed up!
- Second reaction (when it turned out that there was nothing wrong in Cardano's formula): hit the head against a wall
- Final solution (after cooling down and thinking through the problem): discovery of imaginary and complex numbers

A Real Tale of Imaginary Numbers

Bombelli's Problem

How Bombelli would have likely reacted:

- First reaction: blame Cardano, he messed up!
- Second reaction (when it turned out that there was nothing wrong in Cardano's formula): hit the head against a wall
- Final solution (after cooling down and thinking through the problem): discovery of imaginary and complex numbers

A Real Tale of Imaginary Numbers

Bombelli's Problem

How Bombelli would have likely reacted:

- First reaction: blame Cardano, he messed up!
- Second reaction (when it turned out that there was nothing wrong in Cardano's formula): hit the head against a wall
- Final solution (after cooling down and thinking through the problem): discovery of imaginary and complex numbers

Definition of Numbers

- What are the different (types of) numbers? One answer:
  1. $\mathcal{N} = \{0, 1, 2, 3, \ldots\}$
  2. $\mathcal{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
  3. $\mathcal{Q} =$ insert all the fractions
  4. $\mathcal{R} =$ complete the real number line
- What are the different (types of) numbers? Another answer:
  - → No one has seen a number 1 or number 5. This is because numbers are not like the objects of the physical world. If there was indeed such a thing as number 1 that was tangible or visible as the rocks and table-chairs are, it would be in a musueum!
  - → Numbers live inside our mental world, and are bound by certain rules

Definition of Numbers

- What are the different (types of) numbers? One answer:
  1. $\mathcal{N} = \{0, 1, 2, 3, \ldots\}$
  2. $\mathcal{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
  3. $\mathcal{Q} =$ insert all the fractions
  4. $\mathcal{R} =$ complete the real number line
- What are the different (types of) numbers? Another answer:
  - → No one has seen a number 1 or number 5. This is because numbers are not like the objects of the physical world. If there was indeed such a thing as number 1 that was tangible or visible as the rocks and table-chairs are, it would be in a musueum!
  - → Numbers live inside our mental world, and are bound by certain rules

Definition of Numbers

- What are the different (types of) numbers? One answer:
  1. $\mathcal{N} = \{0, 1, 2, 3, \ldots\}$
  2. $\mathcal{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
  3. $\mathcal{Q} =$ insert all the fractions
  4. $\mathcal{R} =$ complete the real number line
- What are the different (types of) numbers? Another answer:
  - → No one has seen a number 1 or number 5. This is because numbers are not like the objects of the physical world. If there was indeed such a thing as number 1 that was tangible or visible as the rocks and table-chairs are, it would be in a musueum!
  - → Numbers live inside our mental world, and are bound by certain rules

Definition of Numbers

- What are the different (types of) numbers? One answer:
  1. $\mathcal{N} = \{0, 1, 2, 3, \ldots\}$
  2. $\mathcal{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
  3. $\mathcal{Q} =$ insert all the fractions
  4. $\mathcal{R} =$ complete the real number line
- What are the different (types of) numbers? Another answer:
  - → No one has seen a number 1 or number 5. This is because numbers are not like the objects of the physical world. If there was indeed such a thing as number 1 that was tangible or visible as the rocks and table-chairs are, it would be in a musueum!
  - → Numbers live inside our mental world, and are bound by certain rules

Definition of Numbers

- Numbers live inside our mental world, and they follow certain rules:
  1. Numbers collectively form a set
  2. There is a zero (additive identity)
  3. There is a one (multiplicative identity)
  4. Addition and multiplication are defined and are *closed* under the set
     - $\rightarrow$ Pick any two numbers, their sum and product are also ensured to be in the same set
  5. Addition and multiplication follow several desirable properties:
     - $\rightarrow$ Associative, distributive, commutative
  6. Several other properties are optional, not all the number sets have them:
     - $\rightarrow$ Existence of additive inverse, multiplicative inverse, etc.

Definition of Numbers

- Nothing in these properties say that the numbers cannot be paired
  - → In fact, the fractional number in the set $\mathcal{Q}$ is actually a pair of numbers $(m, n) \stackrel{\text{def}}{=} \frac{m}{n}$, where the individual numbers are drawn from $\mathcal{Z}$, and where the order matters, i.e., the fractional number (m, n) is different from (n, m)
- Consider a new *type* of number x, which is also a pair of numbers $x \stackrel{\text{def}}{=} (x_r, x_i)$
  - → Individual numbers are drawn from $\mathcal{R}$; and the order matters here as well

Definition of Complex Numbers

- Consider a new type of number, which is a pair of numbers $x \stackrel{\text{def}}{=} (x_r, x_i)$
  - → Individual numbers are drawn from $\mathcal{R}$; the order matters
- Important differences compared to $\mathcal{Q}$: there is a geometrical aspect to this number
  - → x_r and x_i are from two real number lines that are perpendicular to each other
  - → We put x_r on the line drawn horizontally, this is our standard real ($\mathcal{R}$) number line
  - → We put x_i on the line drawn vertically, this is the standard real number line but it is to be thought of as getting multiplied by $i = \sqrt{-1}$ everywhere!

Definition of Complex Numbers

Interpretation of x_r, x_i

- Why did we use the subscripts r and i to denote the Cartesian coordinates?
  - → x_r: to be thought of as a number on the *real* number line
  - → x_i: to be thought of as along the *imaginary* number line
  - x: to be thought of as a *sum* of real and imaginary numbers

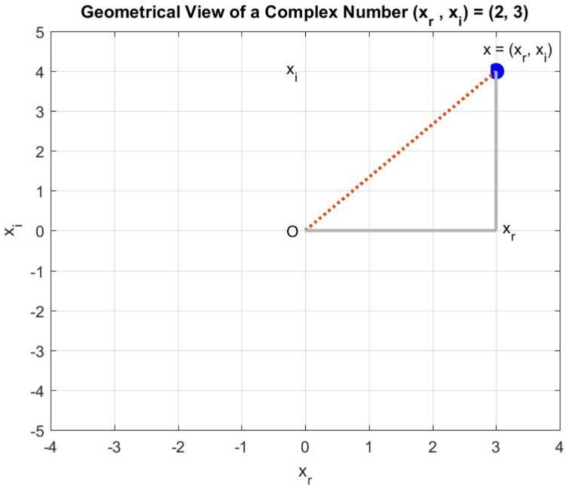
$$x = x_r + \sqrt{-1}\, x_i$$

- $\sqrt{-1}$ is cumbersome to write: denote it by i (imaginary)

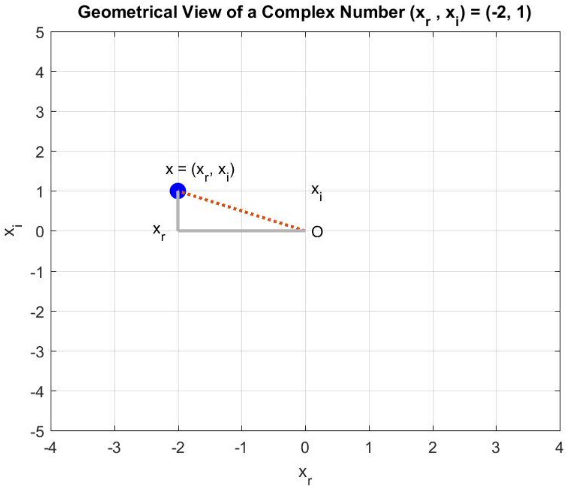
$$x = x_r + i\, x_i$$

(Matlab note: to avoid the confusion between a program variable i and the imaginary number $i = \sqrt{-1}$, Matlab uses a specific notation $1i$ instead of i)

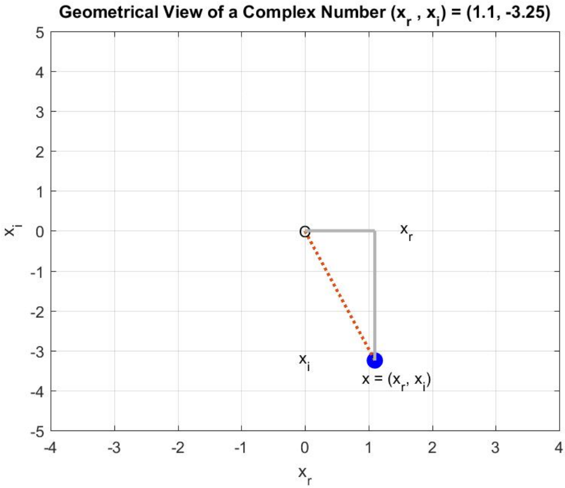
Definition of Complex Numbers
Example 1



Definition of Complex Numbers
Example 2

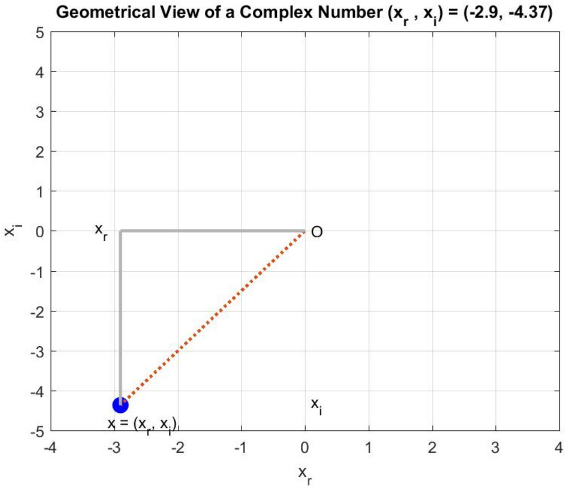


Definition of Complex Numbers
Example 3



Definition of Complex Numbers

Example 4



Definition of Complex Numbers

- This new number is like a point on a plane
  - → This *generalizes* all the other numbers, which are like a point on a line
- $x \stackrel{\text{def}}{=} (x_r, x_i)$: coordinates of this point
  - → Called the Cartesian Coordinates
- A point can be specified either by its Cartesian or Polar coordinates
- In the Polar coordinates, x is represented as $x \stackrel{\text{def}}{=} (|x|, \theta_x)$
  - → $|x|$ is called the *magnitude (or amplitude)* of x (denotes the *length of the line* connecting x to the origin of the plane)
  - → θ_x is called the *phase angle* of x
- Both these specifications are *completely* equivalent; either can be used
- Any one specification can be derived given the other

Definition of Complex Numbers
Going from Cartesian to Polar

- If $x = (x_r, x_i)$ is specified in the Cartesian coordinates
  $\rightarrow$ its Polar coordinates (a, θ) can be determined as follows:

$$a = \sqrt{x_r^2 + x_i^2}$$
$$\theta = \tan^{-1}\left(\frac{x_i}{x_r}\right)$$

- If, instead, $x = (a, \theta)$ is specified in its Polar coordinates
  $\rightarrow$ the Cartesian coordinates (x_r, x_i) can be determined as follows:

$$x_r = a\cos\theta$$
$$x_i = a\sin\theta$$

Definition of Complex Numbers
Additions and Multiplications

- Addition of two numbers a and b:
  - $\rightarrow$ To obtain $c = a + b$, add the Cartesian coordinates of a and b
  - $\rightarrow$ $c_r = a_r + b_r, \quad c_i = a_i + b_i$
- Multiplication of two numbers a and b:
  - $\rightarrow$ To obtain $d = a \times b$, multiply the magnitudes of a and b and add the phase
  - $\rightarrow$ $|c| = |a| \times |b|, \quad \theta_c = \theta_a + \theta_b$

Definition of Complex Numbers
Subtractions and Divisions

- Subtration of b from a:
  - $\rightarrow$ To obtain $c = a - b$, subtract the Cartesian coordinates of b from a
  - $\rightarrow$ $c_r = a_r - b_r, \quad c_i = a_i - b_i$
- Division of a by b:
  - $\rightarrow$ To obtain $d = a/b$, divide the magnitudes of a and b and subtract the phase
  - $\rightarrow$ $|c| = |a|/|b|, \quad \theta_c = \theta_a - \theta_b$

A Real Tale of Imaginary Numbers
Bombelli's Solution

- Bombelli applied Cardano's formula to solve $x^3 = 15x + 4$:

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}},$$

- Bombelli's ingenious approach was to show:

$$2 + 11\sqrt{-1} = \left(2 + \sqrt{-1}\right)^3$$
$$2 - 11\sqrt{-1} = \left(2 - \sqrt{-1}\right)^3$$

- Therefore,

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}} = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$$

- Problem solved, and the imaginary number got its birth

A Real Tale of Imaginary Numbers

Bombelli's Solution

- Bombelli applied Cardano's formula to solve $x^3 = 15x + 4$:

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}},$$

- Bombelli's ingenious approach was to show:

$$2 + 11\sqrt{-1} = \left(2 + \sqrt{-1}\right)^3$$
$$2 - 11\sqrt{-1} = \left(2 - \sqrt{-1}\right)^3$$

- Therefore,

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}} = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$$

- Problem solved, and the imaginary number got its birth

A Real Tale of Imaginary Numbers

Bombelli's Solution

- Bombelli applied Cardano's formula to solve $x^3 = 15x + 4$:

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}},$$

- Bombelli's ingenious approach was to show:

$$2 + 11\sqrt{-1} = \left(2 + \sqrt{-1}\right)^3$$
$$2 - 11\sqrt{-1} = \left(2 - \sqrt{-1}\right)^3$$

- Therefore,

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}} = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$$

- Problem solved, and the imaginary number got its birth

A Real Tale of Imaginary Numbers

Bombelli's Solution

- Bombelli applied Cardano's formula to solve $x^3 = 15x + 4$:

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}},$$

- Bombelli's ingenious approach was to show:

$$2 + 11\sqrt{-1} = \left(2 + \sqrt{-1}\right)^3$$
$$2 - 11\sqrt{-1} = \left(2 - \sqrt{-1}\right)^3$$

- Therefore,

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}} = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$$

- Problem solved, and the imaginary number got its birth

Definition of Complex Numbers

Interpretation of x_r, x_i

- Why did we use the subscripts r and i to denote the Cartesian coordinates?
  - → x_r: to be thought of as a number on the *real* number line
  - → x_i: to be thought of as along the *imaginary* number line
  - x: to be thought of as a *sum* of real and imaginary numbers

$$x = x_r + \sqrt{-1}\, x_i$$

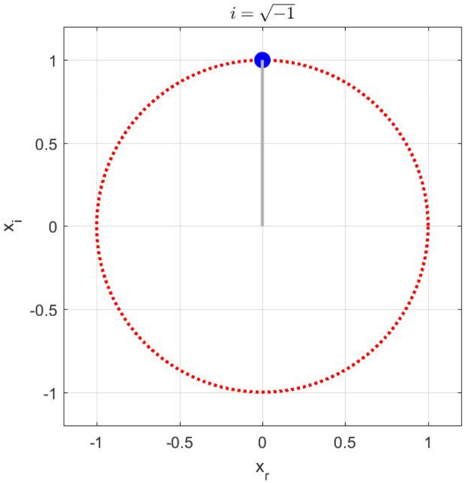
- $\sqrt{-1}$ is cumbersome to write: denote it by i (imaginary)

$$x = x_r + i\, x_i$$

(Matlab note: to avoid the confusion between a program variable i and the imaginary number $i = \sqrt{-1}$, Matlab uses a specific notation $1i$ instead of i)

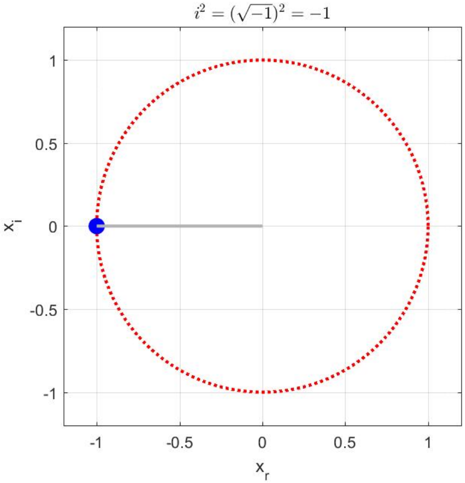
Definition of Complex Numbers

Interpretation of $i = \sqrt{-1}$



Definition of Complex Numbers

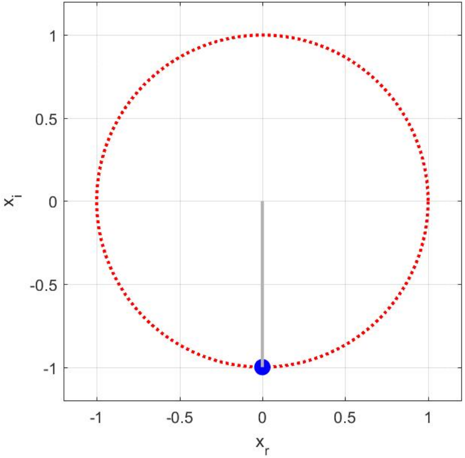
Interpretation of $i = \sqrt{-1}$



Definition of Complex Numbers

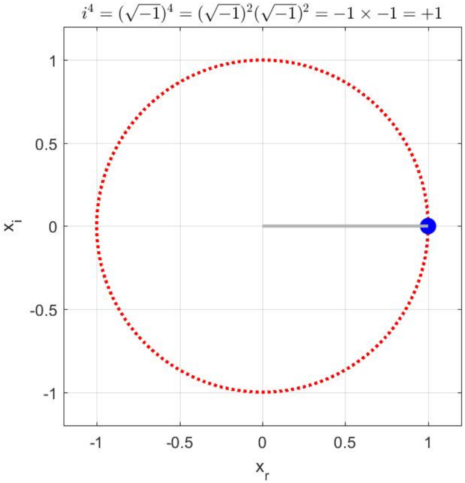
Interpretation of $i = \sqrt{-1}$

$$i^3 = (\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} = -1 \times i = -i$$



Definition of Complex Numbers

Interpretation of $i = \sqrt{-1}$



Solving n^{th}-order Equation

- We've seen that:
  - → i is a number for which $i^4 = 1$.
  - → Equivalently, i is the solution of an equation $x^4 = 1$
  - → Equivalently, i is the *root* of a polynomial $p_4(x) : x^4 - 1 = 0$.
- An n^{th}-order polynomial has a total of n roots. Where are the other three roots of $p_4(x)$.
  - → These are $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$.
  - → The 4 roots of $p_4(x)$ are along the unit circle in the complex-number plane and they are at an angle of $n \times 90^o$, where $n = 0, 1, 2$ and 3, from the positive direction of the real number line

Solving n^{th}-order Equation

- We've seen that:
  - → i is a number for which $i^4 = 1$.
  - → Equivalently, i is the solution of an equation $x^4 = 1$
  - → Equivalently, i is the *root* of a polynomial $p_4(x) : x^4 - 1 = 0$.
- An n^{th}-order polynomial has a total of n roots. Where are the other three roots of $p_4(x)$.
  - → These are $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$.
  - → The 4 roots of $p_4(x)$ are along the unit circle in the complex-number plane and they are at an angle of $n \times 90^o$, where $n = 0, 1, 2$ and 3, from the positive direction of the real number line

Solving n^{th}-order Equation

- We've seen that:
  - $\rightarrow$ i is a number for which $i^4 = 1$.
  - $\rightarrow$ Equivalently, i is the solution of an equation $x^4 = 1$
  - $\rightarrow$ Equivalently, i is the *root* of a polynomial $p_4(x) : x^4 - 1 = 0$.
- An n^{th}-order polynomial has a total of n roots. Where are the other three roots of $p_4(x)$.
  - $\rightarrow$ These are $i^2 = -1, i^3 = -i$ and $i^4 = 1$.
  - $\rightarrow$ The 4 roots of $p_4(x)$ are along the unit circle in the complex-number plane and they are at an angle of $n \times 90^o$, where $n = 0, 1, 2$ and 3, from the positive direction of the real number line

Solving n^{th}-order Equation

- What are k roots of $p_k(x) : x^k - 1 = 0$.
  → The k roots of $p_k(x)$ are along the unit circle in the complex-number plane and they are at an angle of $n \times 360^o/k$, where $n = 0, 1, 2, \ldots, k-1$, from the positive direction of the real number line

Solving n^{th}-order Equation

- What are k roots of $p_k(x) : x^k - 1 = 0$.
  - $\rightarrow$ The k roots of $p_k(x)$ are along the unit circle in the complex-number plane and they are at an angle of $n \times 360^o / k$, where $n = 0, 1, 2, \ldots, k-1$, from the positive direction of the real number line

Complex Exponentials

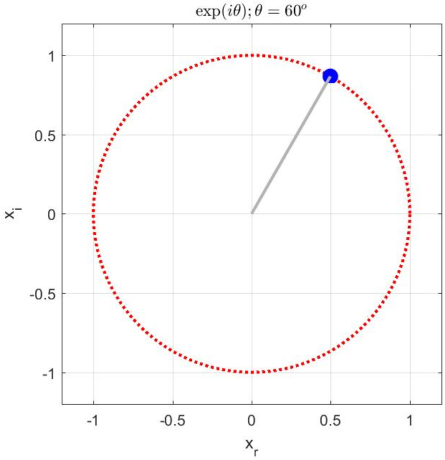
- How to make sense of $y = \exp(i\theta)$?
- It can be shown (see, for example, `http://math2.org/math/oddsends/complexity/e%5Eitheta.htm`) that

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta)$$

- This is known as Euler's Formula (often called the most beautiful equation of the math!
  - → Leads to Euler's Identity: $e^{i\pi} = -1$.

Complex Exponentials

- $\exp(i\theta) = \cos(theta) + i\sin(\theta)$, when $\theta = 60^o$



A Standard Form of Communication Signal

Complex Exponential

- Sinusoidal waveform can be thought of as a component of a more general, complex-valued, waveform:

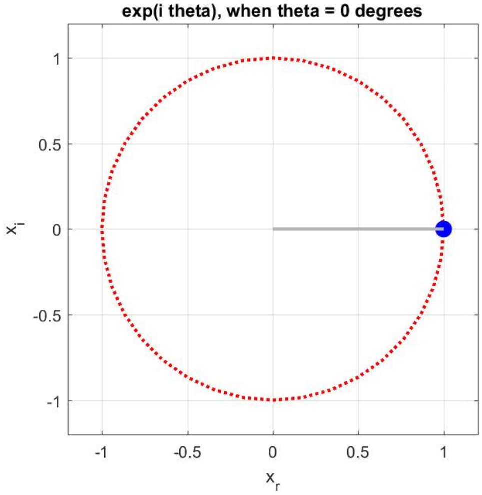
$$
\begin{aligned}
s(t) &= A\exp\left(i\Theta(t)\right) = A\exp\left(i(2\pi f t + \theta)\right) \\
&= s_r(t) + i s_i(t)
\end{aligned}
$$

  - $\rightarrow$ $s(t)$: is called the complex *phasor*
  - $\rightarrow$ $s_r(t)$: is the real part of this phasor
  - $\rightarrow$ $s_i(t)$: is the imaginary part of this phasor
  - $\rightarrow$ i : is the imaginary number $\sqrt{-1}$
- Several new concepts: complex phasor, its real and imaginary part, and $i = \sqrt{-1}$

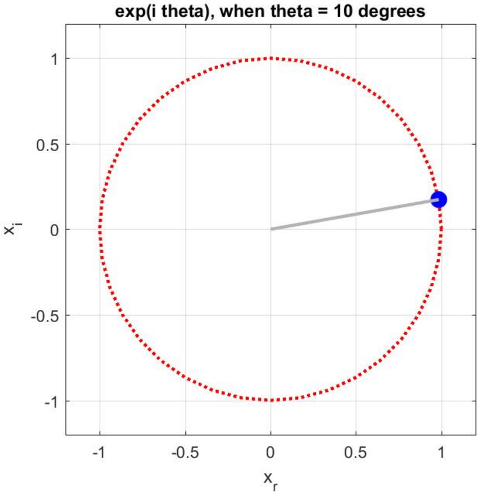
A Standard Form of Communication Signal
Complex Exponential

Let us watch a movie of the complex phasor $s(t)$ when f is positive valued.

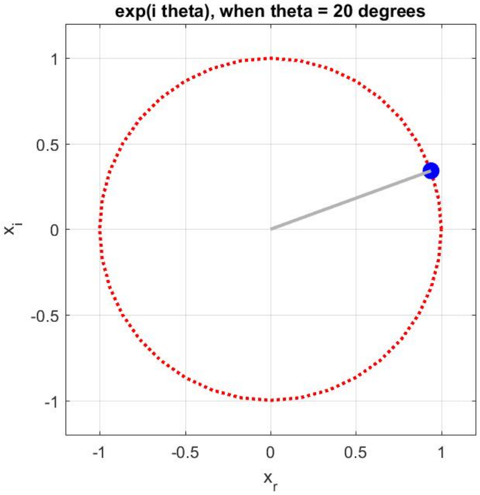
Complex Exponentials



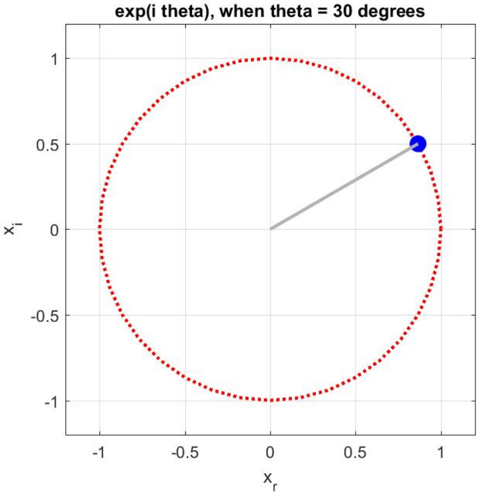
Complex Exponentials



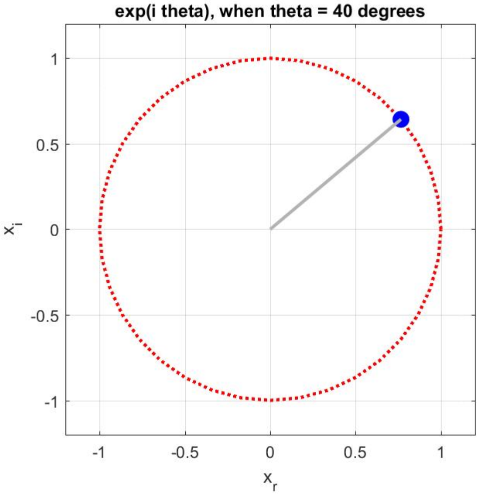
Complex Exponentials



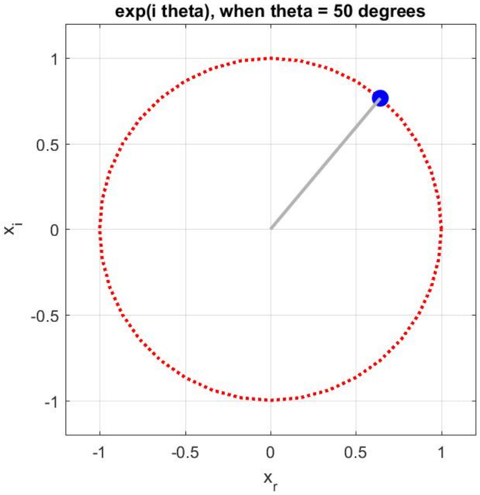
Complex Exponentials



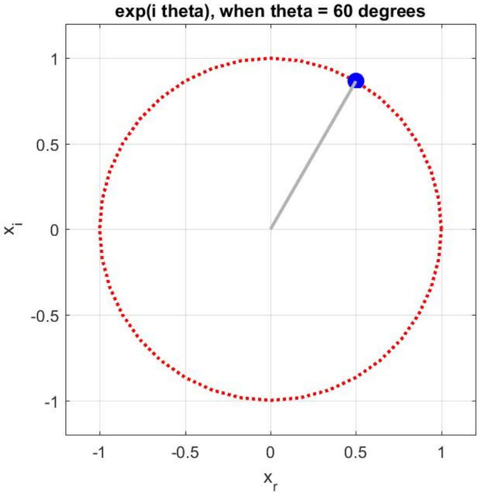
Complex Exponentials



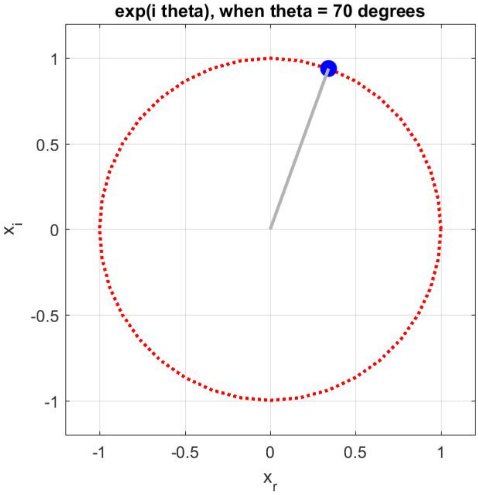
Complex Exponentials



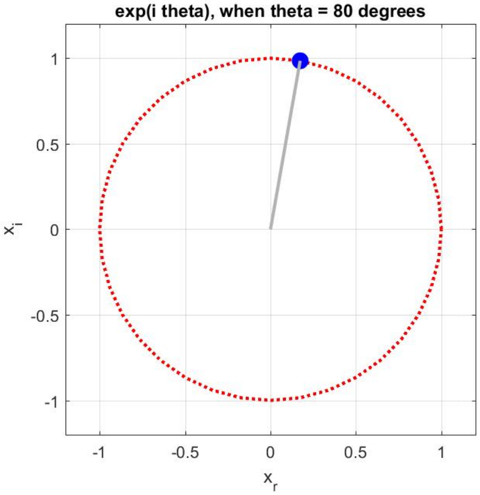
Complex Exponentials



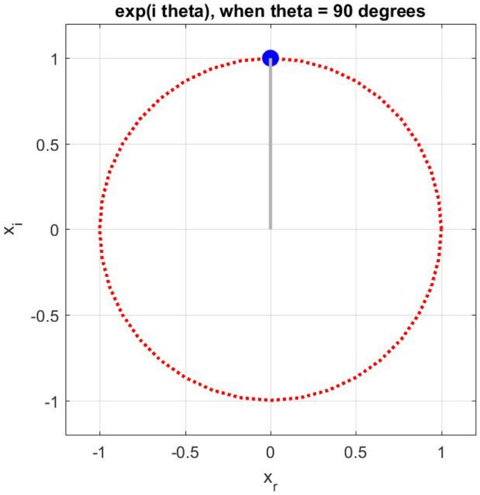
Complex Exponentials



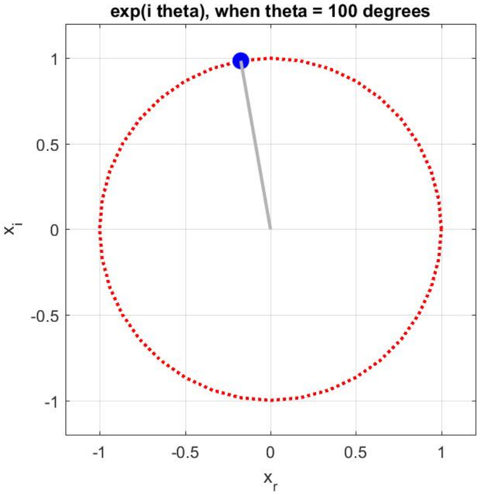
Complex Exponentials



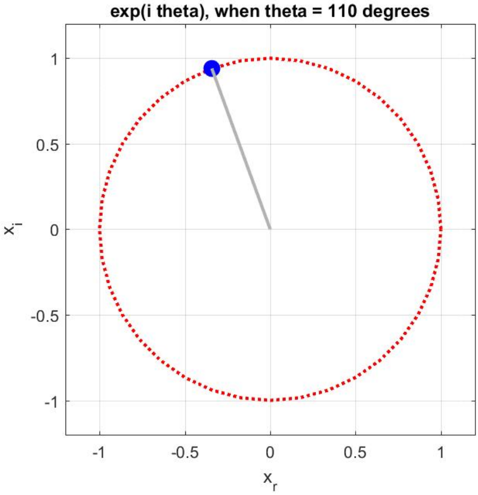
Complex Exponentials



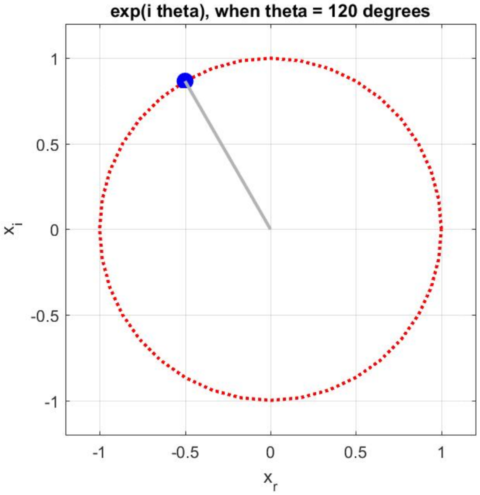
Complex Exponentials



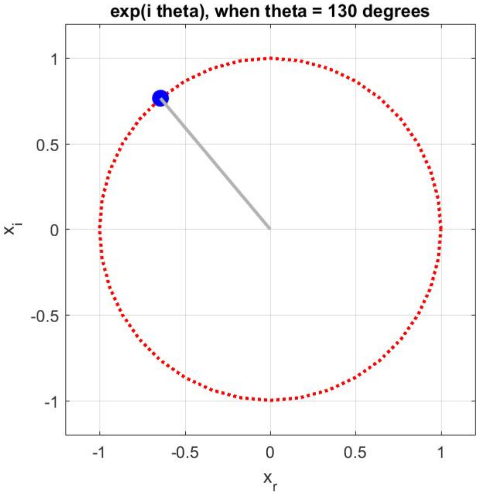
Complex Exponentials



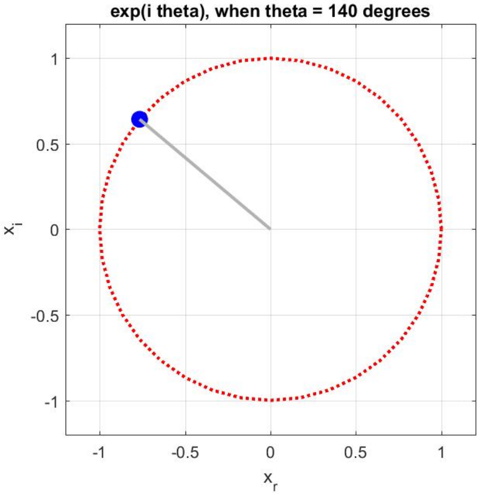
Complex Exponentials



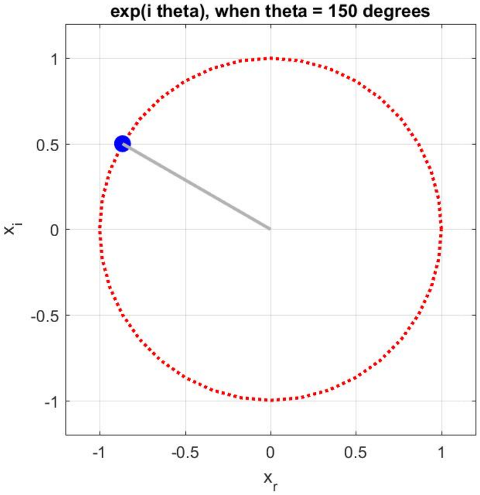
Complex Exponentials



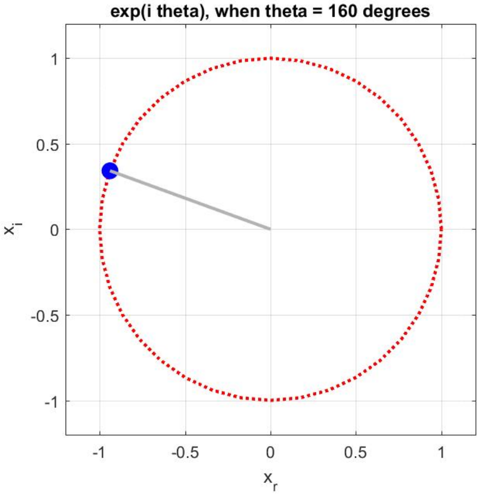
Complex Exponentials



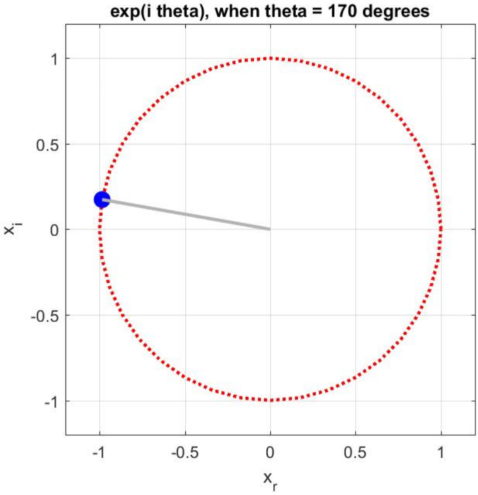
Complex Exponentials



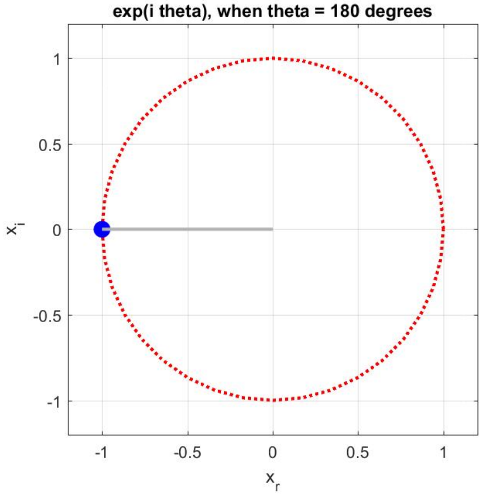
Complex Exponentials



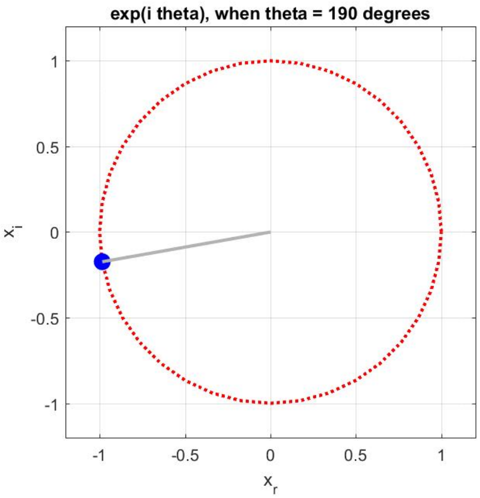
Complex Exponentials



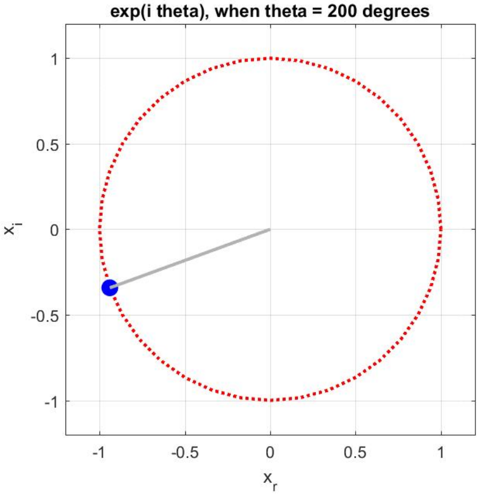
Complex Exponentials



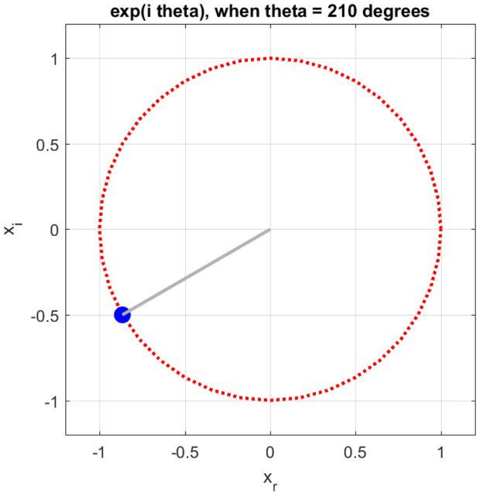
Complex Exponentials



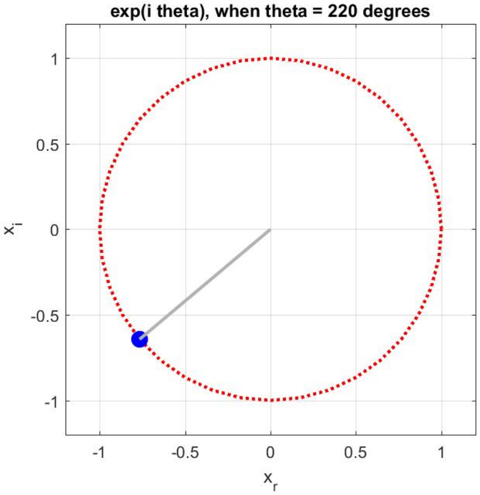
Complex Exponentials



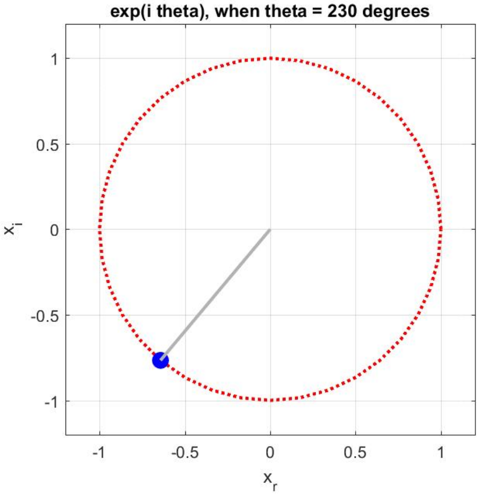
Complex Exponentials



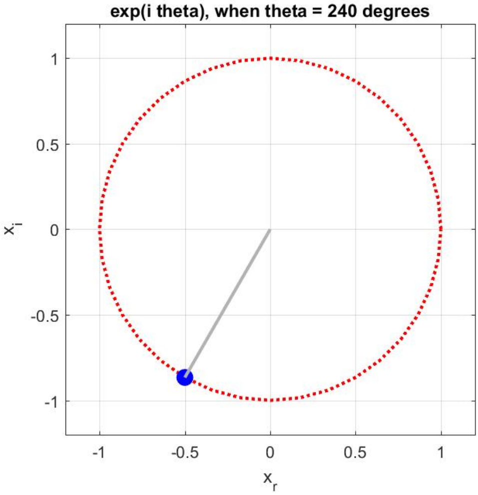
Complex Exponentials



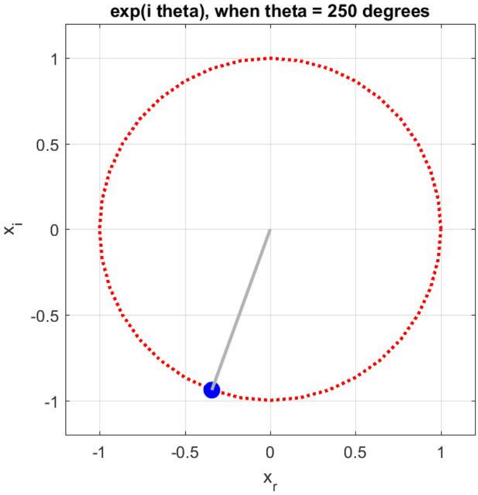
Complex Exponentials



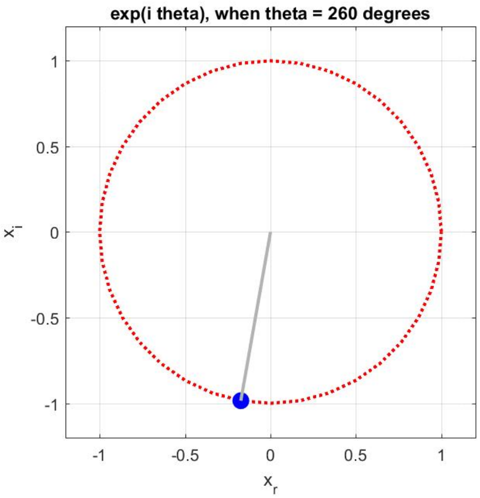
Complex Exponentials



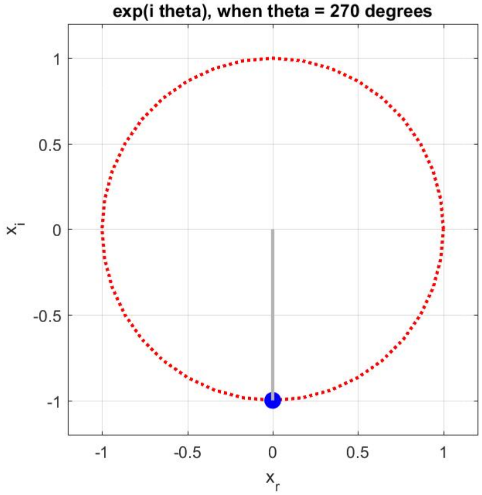
Complex Exponentials



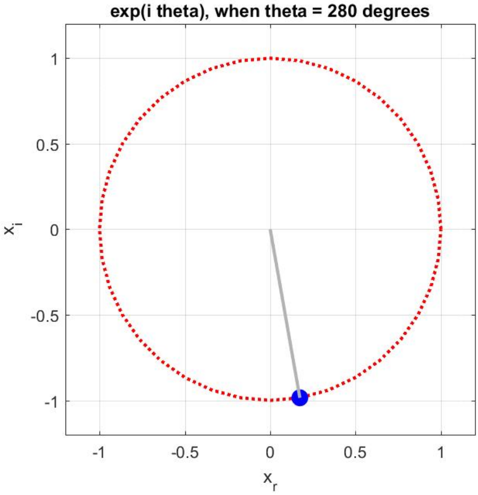
Complex Exponentials



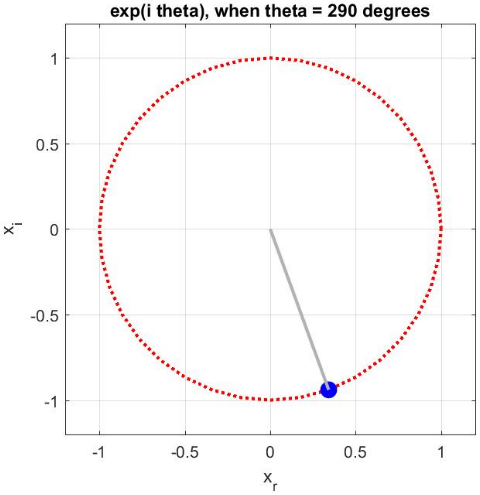
Complex Exponentials



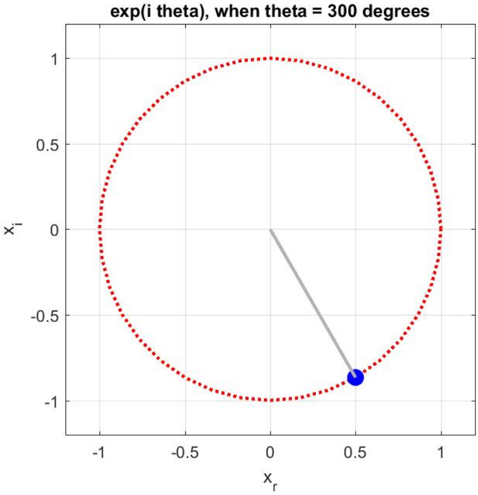
Complex Exponentials



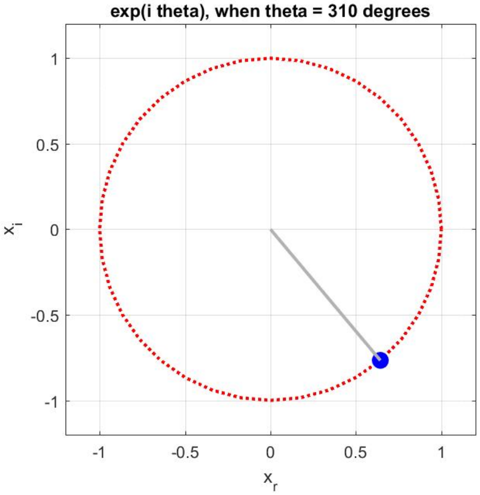
Complex Exponentials



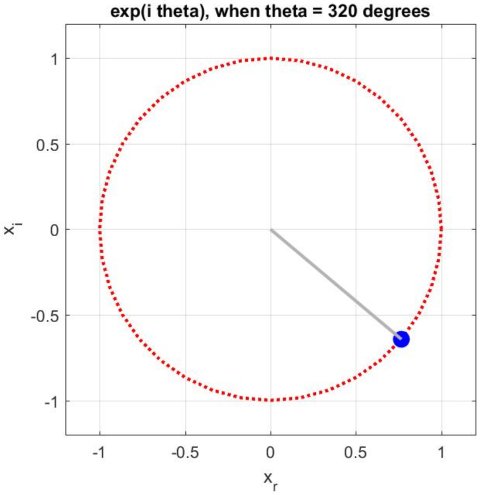
Complex Exponentials



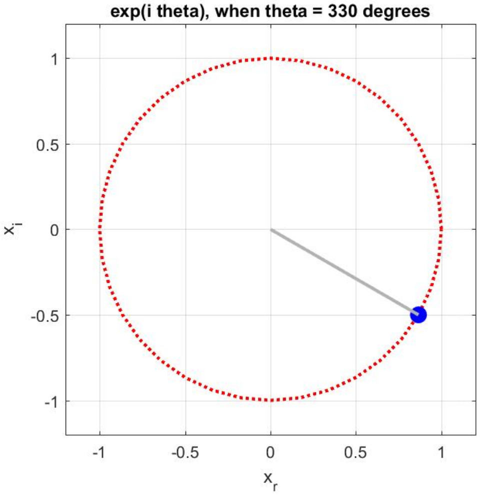
Complex Exponentials



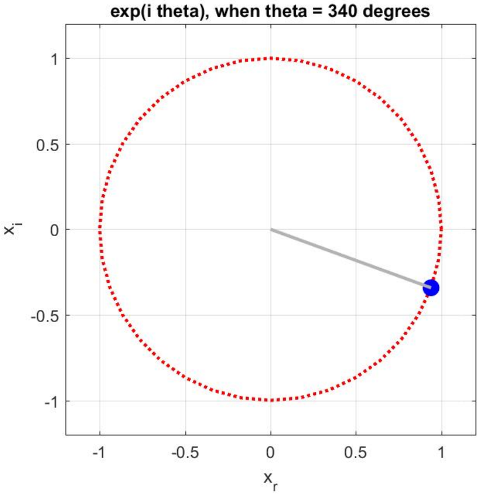
Complex Exponentials



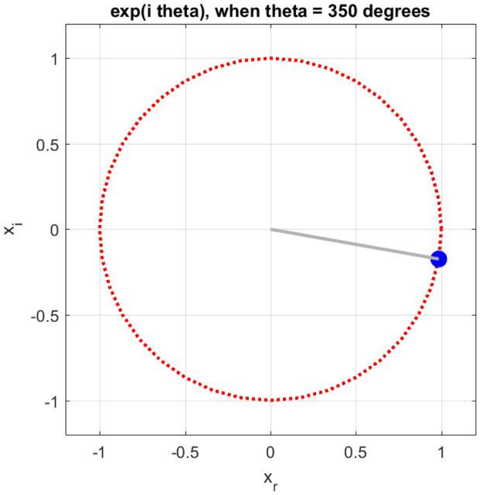
Complex Exponentials



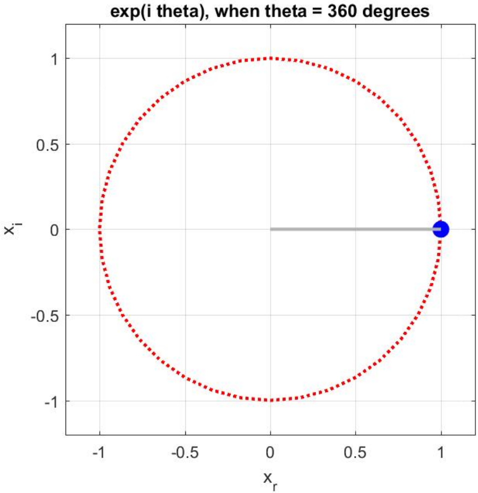
Complex Exponentials



Complex Exponentials



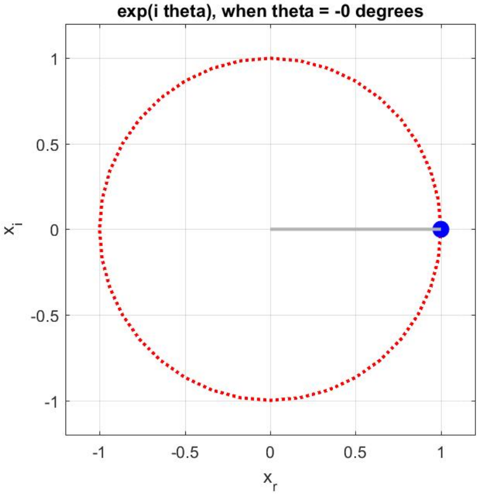
Complex Exponentials



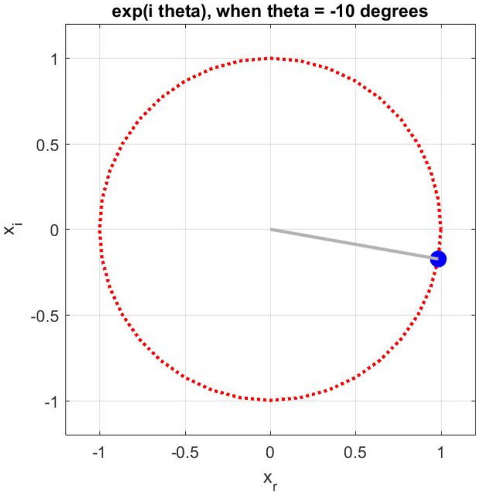
A Standard Form of Communication Signal
Complex Exponential

Next is a movie of the complex phasor $s(t)$ when f is negative valued.

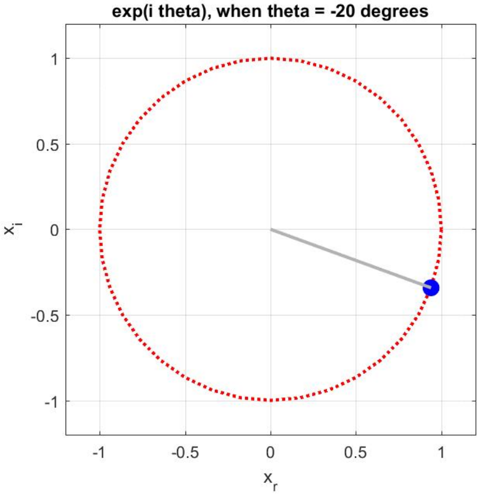
Complex Exponentials



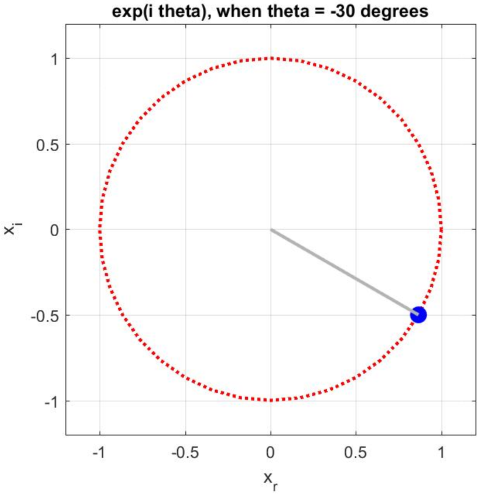
Complex Exponentials



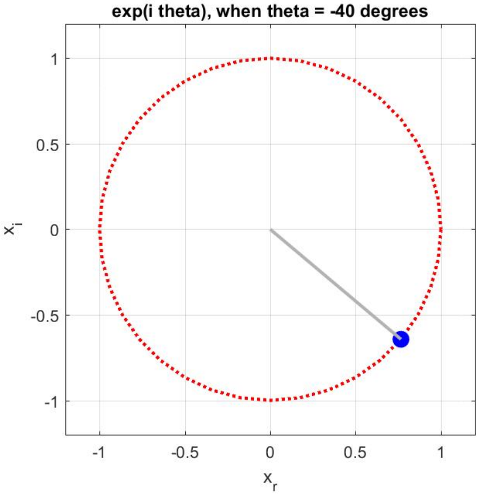
Complex Exponentials



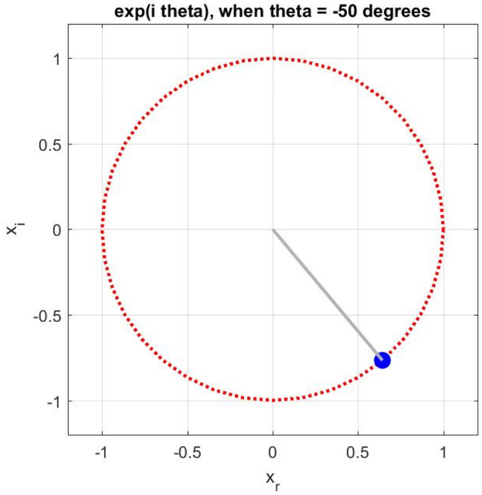
Complex Exponentials



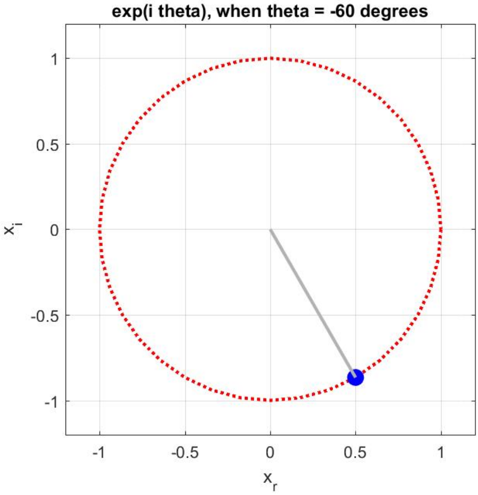
Complex Exponentials



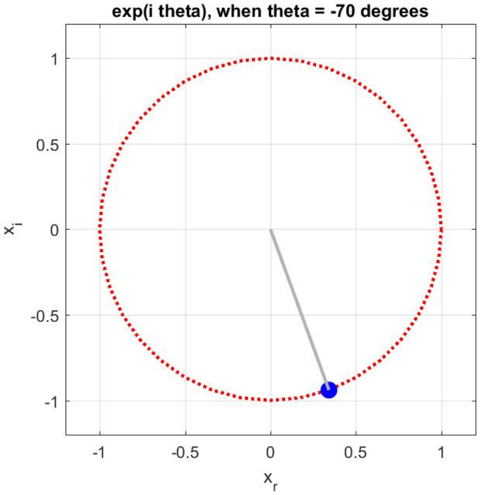
Complex Exponentials



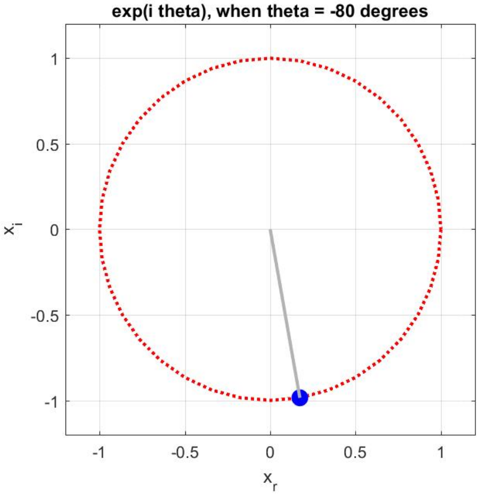
Complex Exponentials



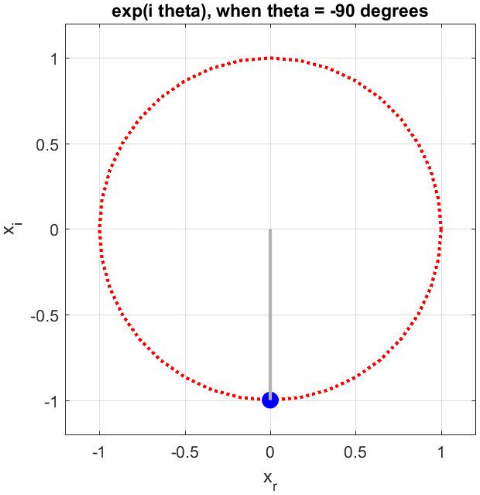
Complex Exponentials



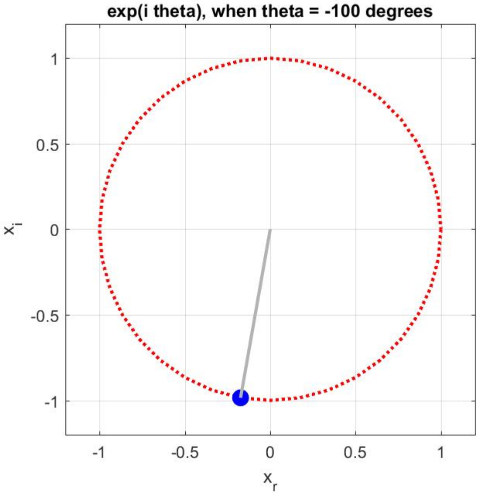
Complex Exponentials



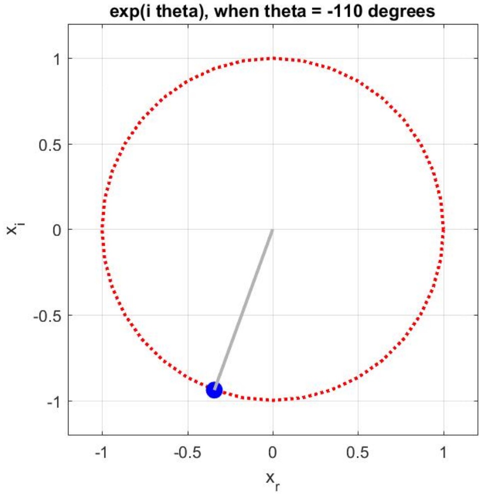
Complex Exponentials



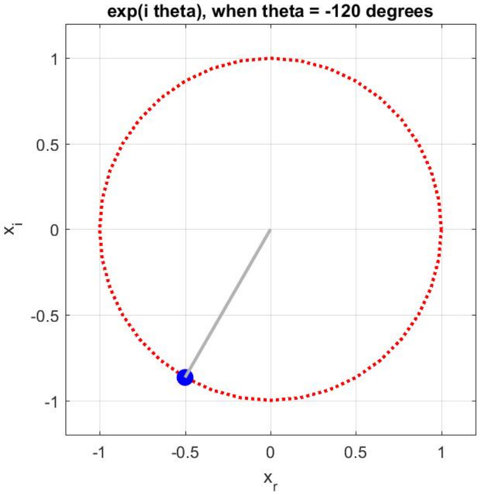
Complex Exponentials



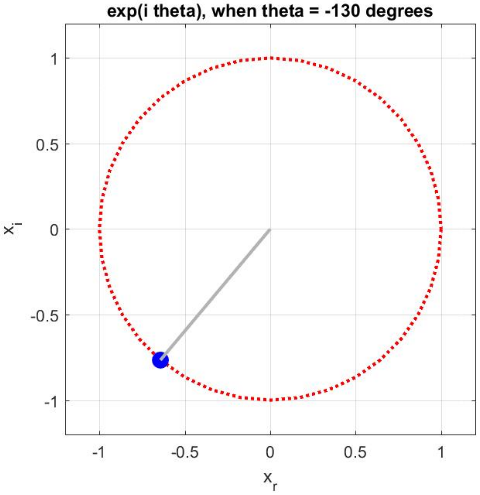
Complex Exponentials



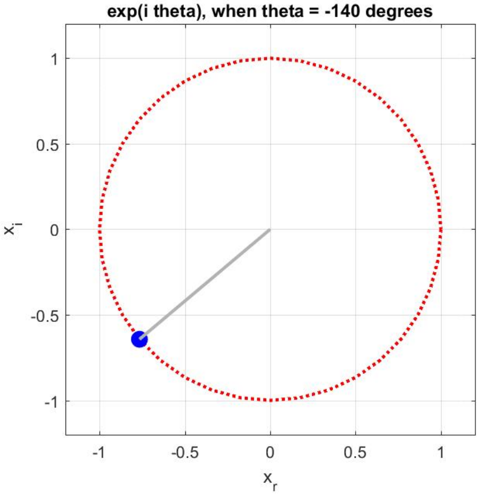
Complex Exponentials



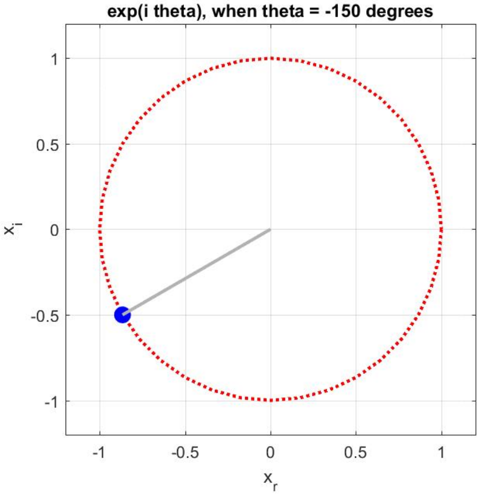
Complex Exponentials



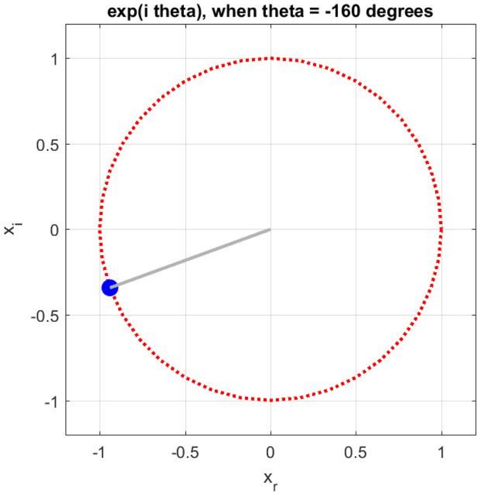
Complex Exponentials



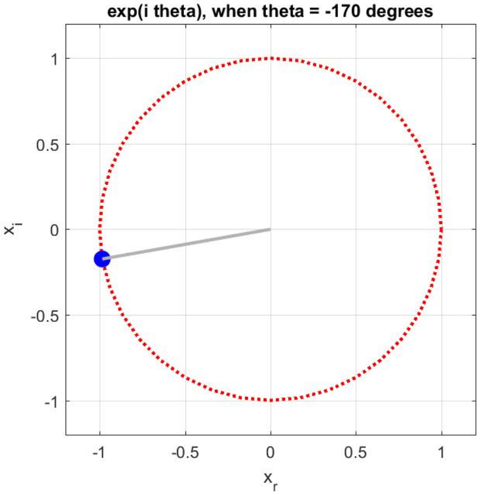
Complex Exponentials



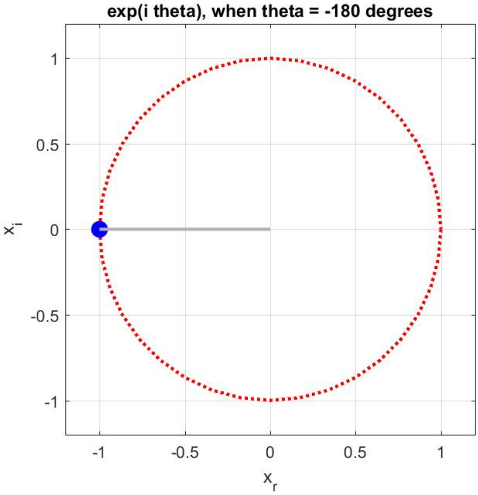
Complex Exponentials



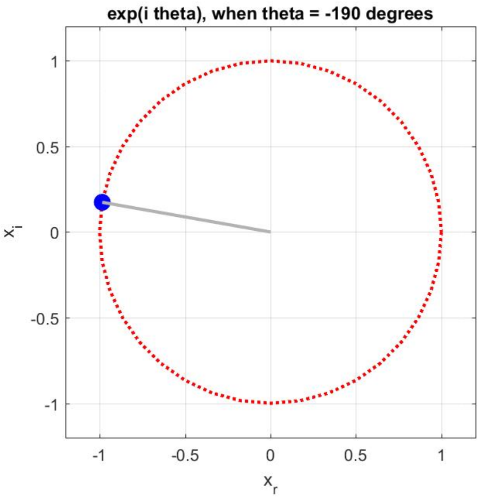
Complex Exponentials



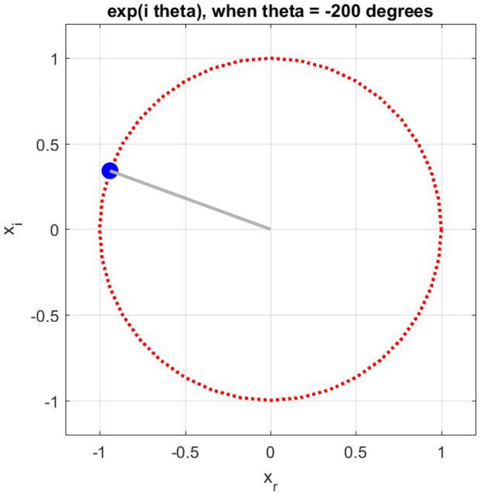
Complex Exponentials



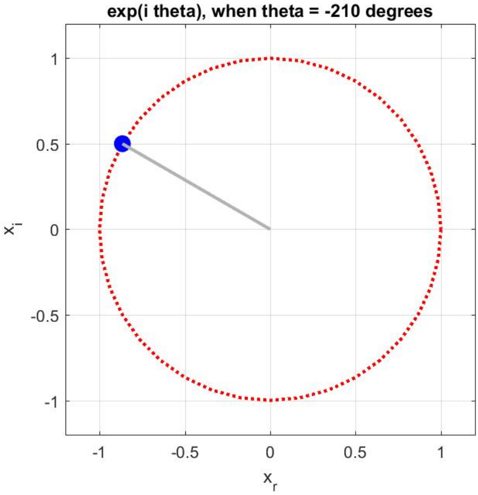
Complex Exponentials



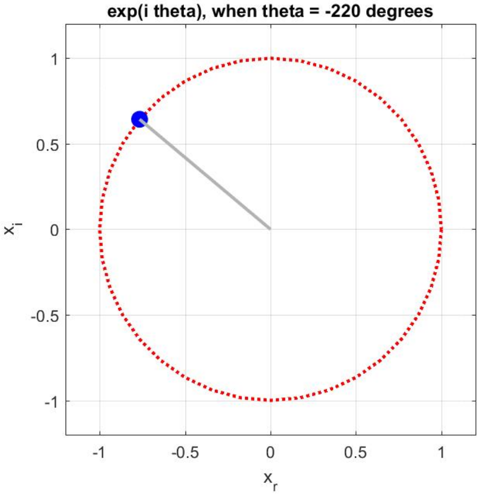
Complex Exponentials



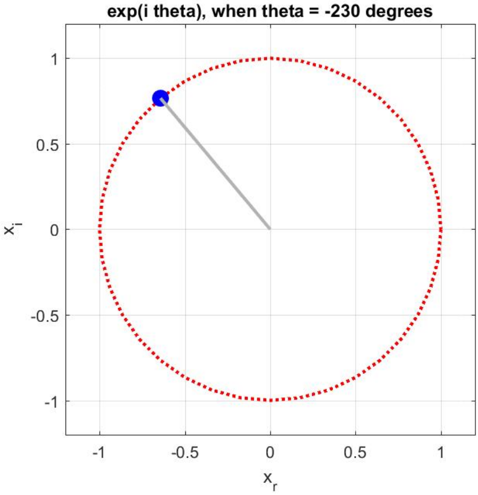
Complex Exponentials



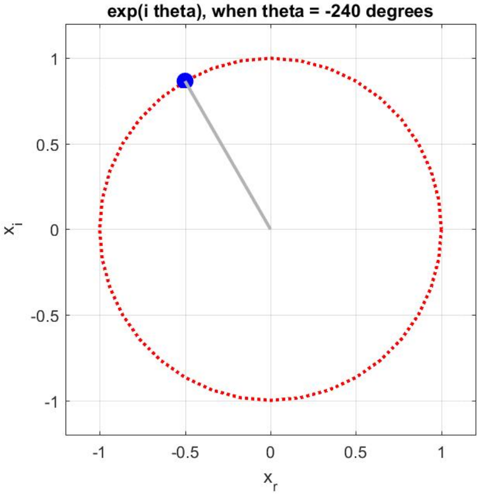
Complex Exponentials



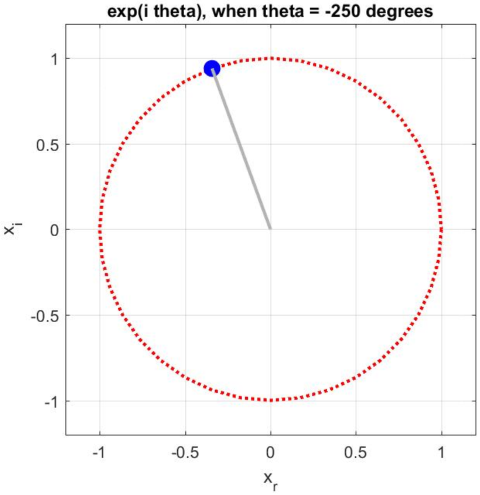
Complex Exponentials



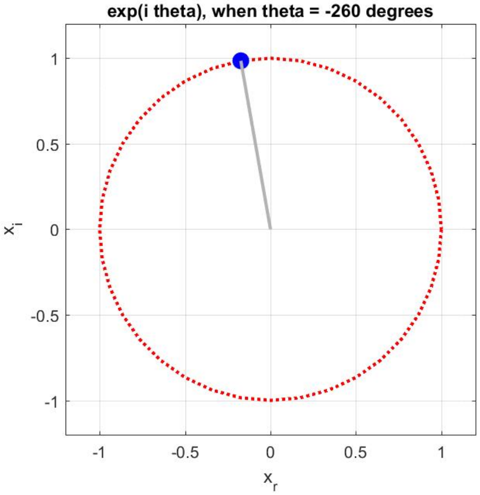
Complex Exponentials



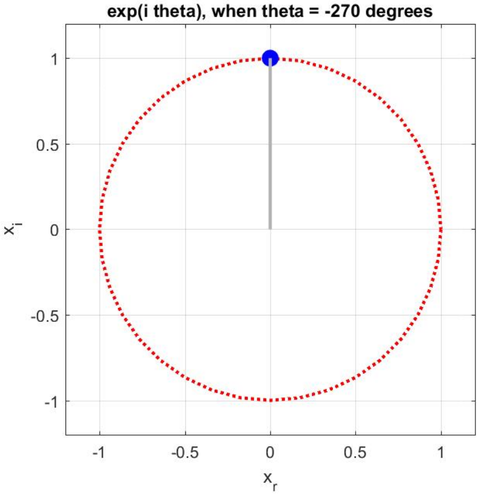
Complex Exponentials



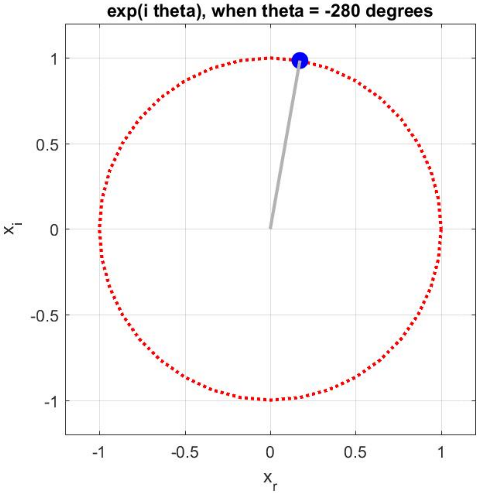
Complex Exponentials



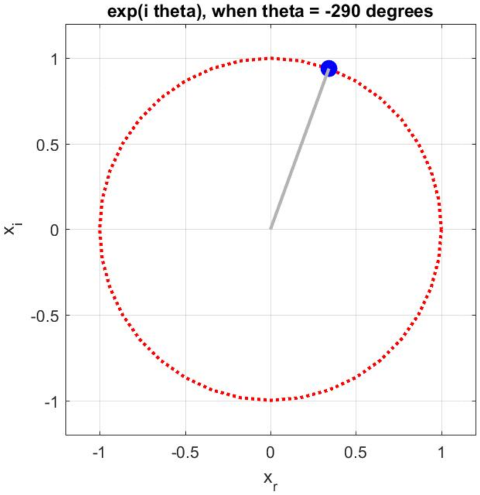
Complex Exponentials



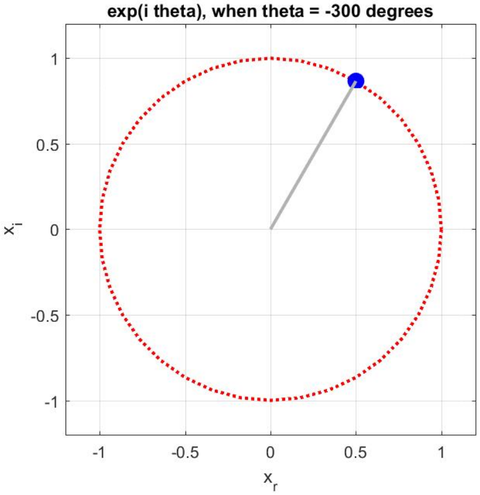
Complex Exponentials



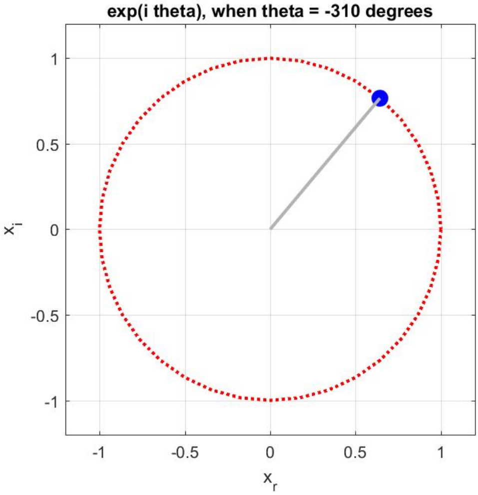
Complex Exponentials



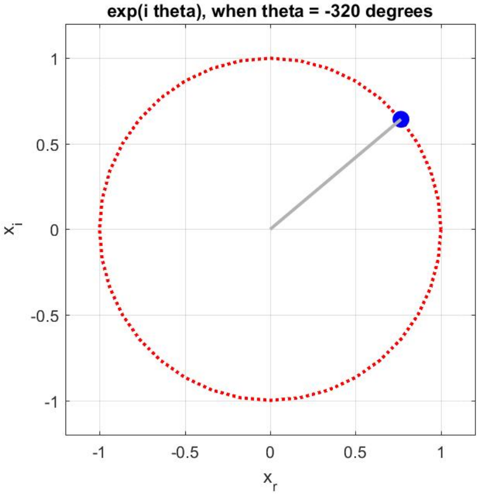
Complex Exponentials



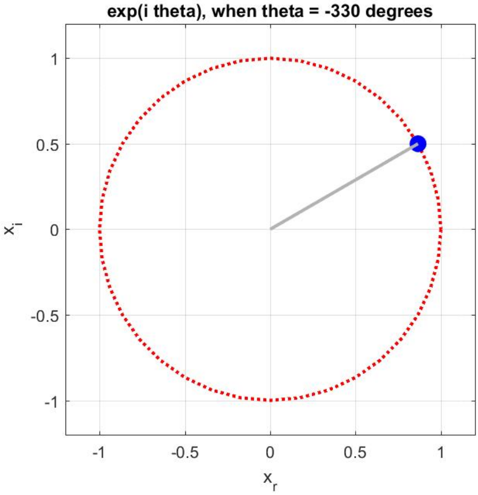
Complex Exponentials



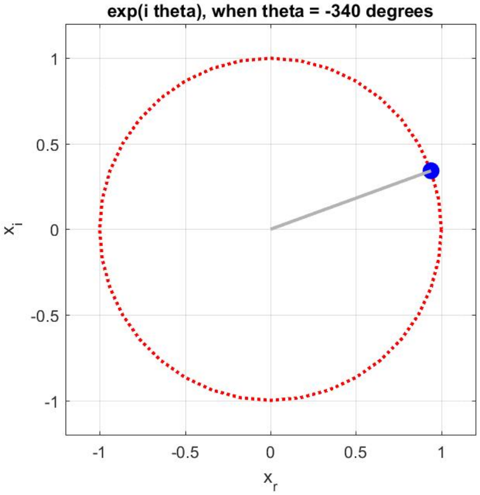
Complex Exponentials



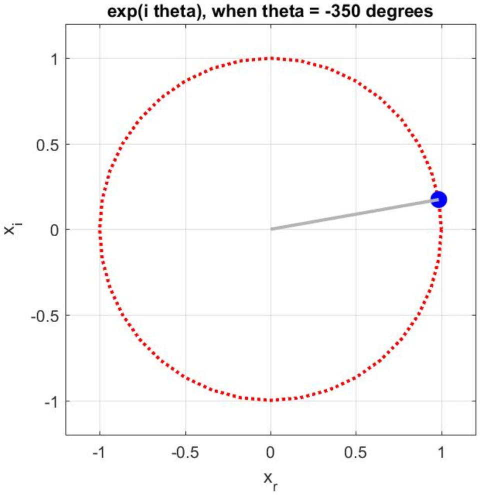
Complex Exponentials



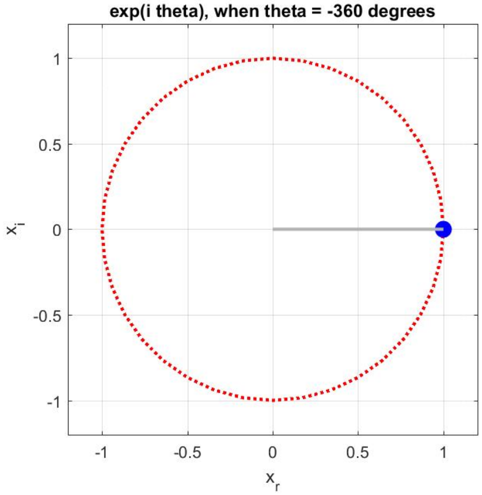
Complex Exponentials



Complex Exponentials



Complex Exponentials



Cosine Waveform

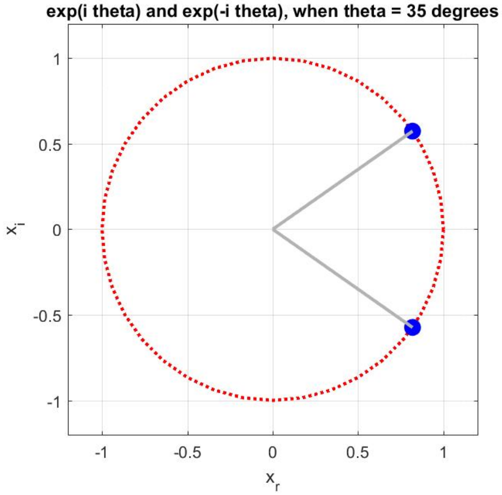
- Cosine waveform can be thought of comprised of two complex phasors, one with frequency f and another with frequency $-f$

$$\exp\left(i\,2\pi f t\right) + \exp\left(i\,2\pi(-f)t\right) = 2\cos\left(2\pi f t\right)$$

- Similarly, Sine waveform can also be thought of as comprised of the same two phasors

$$\exp\left(i\,2\pi f t\right) - \exp\left(i\,2\pi(-f)t\right) = i \times 2\sin\left(2\pi f t\right)$$

Cosine and Sine Waveforms



Cosine and Sine Waveforms