CCV-8

SC216 Calculus with Complex Variables

Feb 18, 2008

FUNCTIONS OF COMPLEX VARIABLES

$\mathbb{R}$ $x^2+1=0$ has no solution

If i is a root $\Rightarrow$ $i^2+1=0$ or

$i=\sqrt{-1}$

$\downarrow$

$\mathbb{C} = \{a+ib \mid a,b \in \mathbb{R}\}$ Field of complex numbers.

- If $z \in \mathbb{C} \Rightarrow z = x+iy$, $x = Re(z)$ & $y = Im(z)$

$z = (x,y)$

$$\Rightarrow \mathbb{C} = \left\{ (a,b) \mid a,b \in \mathbb{R} \text{ and } \begin{array}{l} (a,b)+(a',b') = (a+a', b+b') \\ (a,b)(a',b') = (aa'-bb', ba'+ab') \end{array} \right\}$$

Why?

Here $i = (0,1)$

$z = (x,0) + (0,1)(y,0) = x+iy \in \mathbb{C}$

$\mathbb{C}$ is a field.

$\Rightarrow$ Given $z = (x,y) \in \mathbb{C}$

$\exists\, z^{-1} \in \mathbb{C}$ s.t. $zz^{-1} = 1$

If $z^{-1} = (u,v)$

$\Rightarrow (x,y)(u,v) = 1 = (1,0)$

$\Rightarrow xu - yv = 1$ & $yu + xv = 0$

$\Rightarrow u = \dfrac{x}{x^2+y^2}$, $v = \dfrac{-y}{x^2+y^2}$

Thus $\boxed{z^{-1} = \left(\dfrac{x}{x^2+y^2}, \dfrac{-y}{x^2+y^2}\right)}$ $(z \neq 0)$

Exercise (a) $Re(iz) = -Im(z)$

(b) $Im(iz) = Re(z)$

Exercise Solve for $z^2 + z + 1 = 0$ for $z = (x, y)$

Ans. $z = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

Moduli $z = x + iy$

$|z| = \sqrt{x^2 + y^2}$ = distance between (x, y) and $(0, 0)$

$\Rightarrow \quad |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Thus $|z - z_0| = R$ represents the equation of a circle of radius R with center at z_0 and generic pt. z.

Note $|z|^2 = (Re(z))^2 + (Im(z))^2$

Thus
$$\boxed{\begin{array}{c} Re(z) \le |Re(z)| \le |z| \\ \text{and} \\ Im(z) \le |Im(z)| \le |z| \end{array}}$$

Inequalities

(1) $|z_1 + z_2| \le |z_1| + |z_2|$

(Triangle $\le$)

How to prove it?

$\boxed{\text{Theorem}}$ $|z_1 \pm z_2| \leq |z_1| + |z_2|$ — (1)

and $|z_1 \pm z_2| \geq \big||z_1| - |z_2|\big|$ — (2)

□ We can prove (2) using (1). For example

$$|z_1| = |(z_1+z_2) + (-z_2)| \leq |z_1+z_2| + |-z_2|$$

$$\Rightarrow \quad |z_1+z_2| \geq |z_1| - |z_2| \quad \text{— (3)} \quad \text{if } |z_1| \geq |z_2|$$

In the other case i.e. if $|z_1| < |z_2|$ interchange z_1 & z_2 in (3) to get

$$|z_1+z_2| \geq -(|z_1| - |z_2|) \quad \text{— (4)}$$

(3) & (4) gives $|z_1+z_2| \geq \big||z_1| - |z_2|\big|$ ▨

Exercise | Prove the negative part in (2).

$\boxed{\text{Verify that}}$ $\sqrt{2}\,|z| \geq |Re(z)| + |Im(z)|$

$\boxed{\text{Exercise}}$ sketch the set of points

(a) $|z-1+i| = 1$ (b) $|z+i| \leq 3$ (c) $|z-4i| \geq 4$

(d) $|z-4i| + |z+4i| = 10$ (e) $|z-1| = |z+i|$

$\boxed{\text{Conjugates}}$

If $z = x+iy$ $\bar{z} = x - iy$ i.e. $\bar{z} = (x, -y)$

① $\bar{\bar{z}} = z$ ② $|\bar{z}| = |z|$ ③ $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

④ $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ ⑤ $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

⑥ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ $(z_2 \neq 0)$ ⑦ $Re(z) = \frac{z + \bar{z}}{2}$

⑧ $Im(z) = \frac{z - \bar{z}}{2i}$ ⑨ $z\bar{z} = |z|^2$

⑪ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$ ⑩ $|z_1 z_2| = |z_1||z_2|$

Prove that $|z_1 z_2| = |z_1||z_2|$

$\square$

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 z_2)(\bar{z}_1 \bar{z}_2)$$

$$= (z_1 \bar{z}_1)(z_2 \bar{z}_2) = |z_1|^2 |z_2|^2 = (|z_1||z_2|)^2$$

$\Rightarrow$ result hold. $\blacksquare$

Note this also tells us that $|z^2| = |z|^2$ & $|z^3| = |z|^3$ & so on

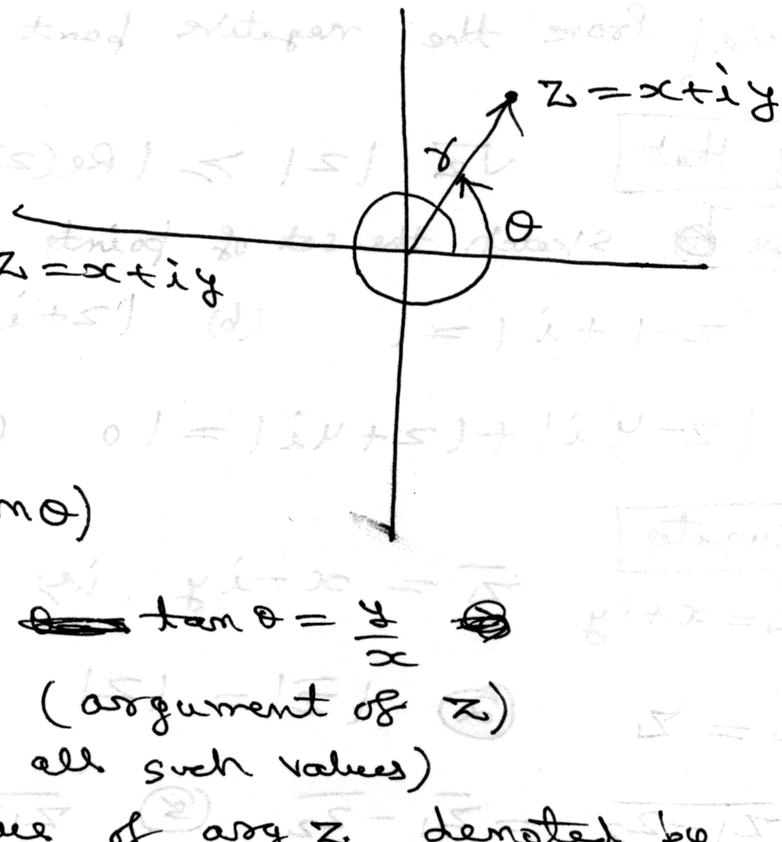
$\boxed{\text{Exercise 1}}$ Sketch the set of pts.

(a) $Re(\bar{z} - i) = 2$ (b) $|2z - i| = 4$

$\boxed{\text{Exercise}}$

(a) $\overline{\left(\dfrac{z_1}{z_2 z_3}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2 \bar{z}_3}$ (b) $\left|\dfrac{z_1}{z_2 z_3}\right| = \dfrac{|z_1|}{|z_2||z_3|}$

$\boxed{\text{Exponential form}}$



If (r, θ) are the polar coordinates of $z = x + iy$

then $x = r\cos\theta$
$y = r\sin\theta$

$$z = r(\cos\theta + i\sin\theta)$$

$r = |z|$ and ~~...~~ $\tan\theta = \dfrac{y}{x}$

$\theta = \arg z$ (argument of z)
(set of all such values)

Principle value of $\arg z$ denoted by $\operatorname{Arg} z$ is the ! Θ s.t. $-\pi < \Theta \le \pi$

Note that

$$\arg z = \operatorname{Arg} z + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

when z is a $-$ve real no. $\operatorname{Arg} z$ has value π, not $-\pi$

(4)

$\boxed{\text{Example}}$ $-1-i$ has principal argument $-\frac{3\pi}{4}$ (lies in 3rd quadrant)

$$\operatorname{Arg}(-1-i) = -\frac{3\pi}{4}$$

Note $\because -\pi < \Theta \leq \pi$ it is not true that

$$\operatorname{Arg}(-1-i) = \frac{5\pi}{4}$$

$$\arg(-1-i) = -\frac{3\pi}{4} + 2n\pi \quad (n \in \mathbb{Z})$$

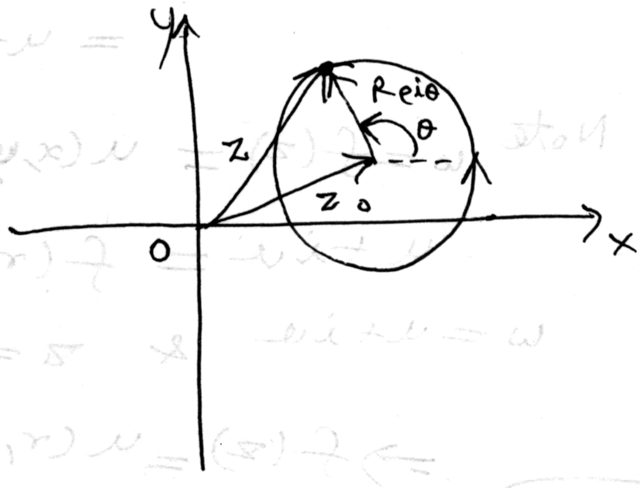
$$z = re^{i\theta} \qquad (\because e^{i\theta} = \cos\theta + i\sin\theta)$$

If $|z - z_0| = R \Rightarrow z = z_0 + Re^{i\theta}$
$(0 \leq \theta \leq 2\pi)$

$\boxed{\text{Properties}}$

① $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

② $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$



$\boxed{\text{Example}}$ $z = -\frac{2}{1+\sqrt{3}i}$

$\Rightarrow \arg(z) = \arg(-2) - \arg(1+\sqrt{3}i)$

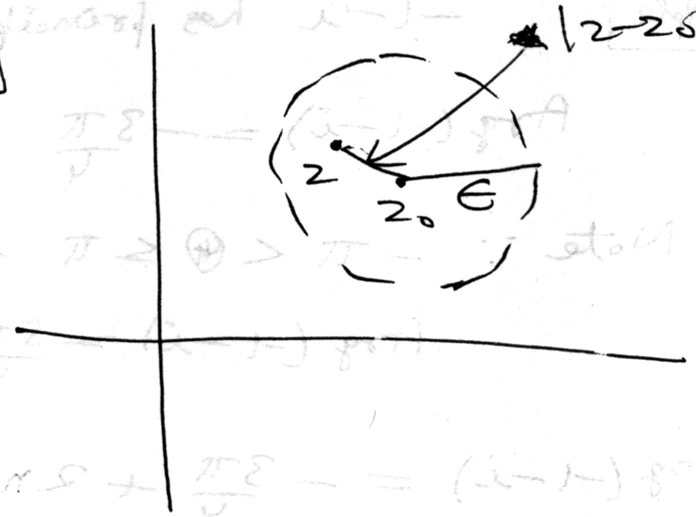
Since $\operatorname{Arg}(-2) = \pi$ & $\operatorname{Arg}(1+\sqrt{3}i) = \frac{\pi}{3}$

one value of $\arg z$ is $\frac{2\pi}{3}$ $(\because \frac{2\pi}{3} \in (-\pi, \pi))$

$\Rightarrow \operatorname{Arg} z = \frac{2\pi}{3}$

Region of complex plane

$|z-z_0| < \epsilon$



$0 < |z-z_0| < \epsilon$

deleted nbd.

$|z| = 1$ circle of radius 1

FUNCTIONS

$$f : \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{(complex valued fn.)}$$

$$w = f(z) \qquad z = x+iy \in \mathbb{C}$$

$$= u + iv \in \mathbb{C}$$

Note $w = f(z) = u(x,y) + i\, v(x,y)$

$u + iv = f(re^{i\theta})$ for polar coordinates

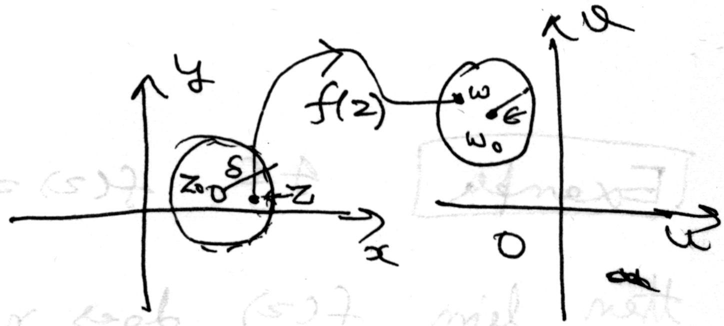
$w = u + iv$ & $z = re^{i\theta}$

$$\Rightarrow f(z) = u(r,\theta) + i\, v(r,\theta)$$

Example

$f(z) = |z|^2 = x^2 + y^2 + i\,0$

Limit of a fn.



$$\lim_{z \to z_0} f(z) = w_0$$

$\forall \epsilon > 0, \exists \delta$ s.t.

$|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$

Example If $f(z) = \dfrac{iz}{2}$ in the open disk $|z| < 1$

then $\displaystyle\lim_{z \to 1} f(z) = \frac{i}{2}$

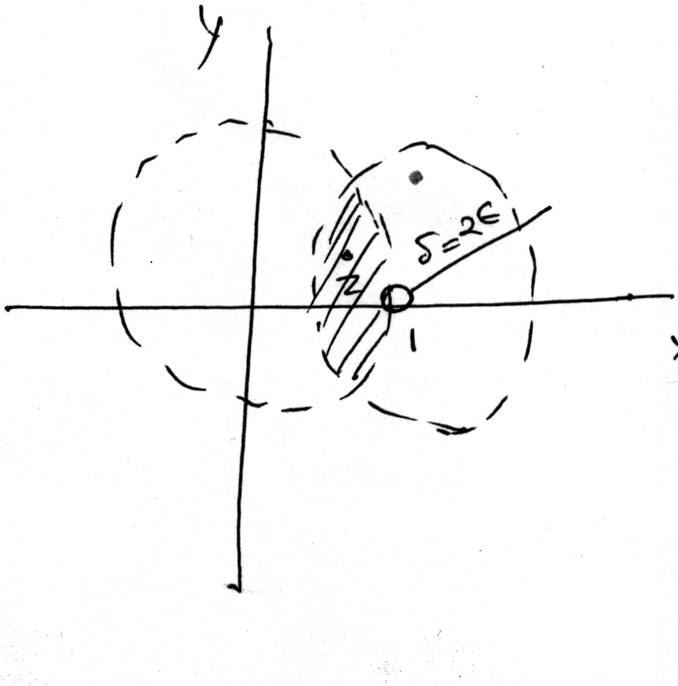
☐ Pt. 1 is on the boundary of the domain of def of f.

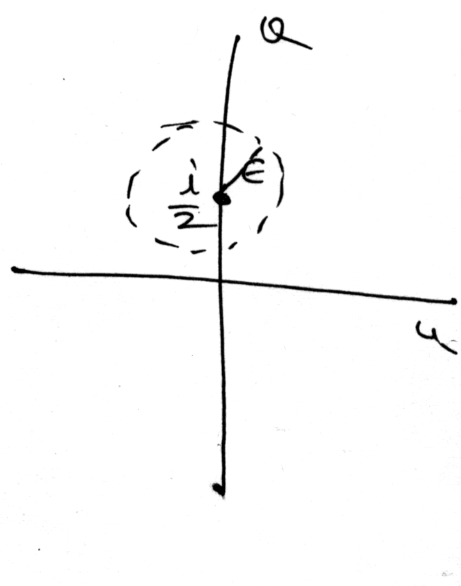
Now when $|z| < 1$

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \frac{|z-1|}{2}$$

Thus for any such z and any +ve no. ϵ

$$\left| f(z) - \frac{i}{2} \right| < \epsilon \text{ whenever } 0 < |z-1| < 2\epsilon$$





$\boxed{\text{Example}}$ If $f(z) = \dfrac{z}{\bar{z}}$

then $\lim\limits_{z \to 0} f(z)$ does not exist.

$\square$ If the limit exist then $z = (x, y) \to (0, 0)$ in any manner. ~~exist~~ Now when $z = (x, 0)$ a pt. on real axis

$$f(z) = \frac{x + i0}{x - i0} = 1$$

& when $z = (0, y)$ on imaginary axis

$$f(z) = \frac{0 + iy}{0 - iy} = -1$$

$\therefore$ limit does not exist $\blacksquare$