power (to line 1) is 100 mW. (*a*) Determine the total loss of the combination in dB. (*b*) Determine the power transmitted to the output end of line 2.

Solution. (*a*) The dB loss of the joint is

$$L_j(\text{dB}) = 10\log_{10}\left(\frac{1}{1-|\Gamma|^2}\right) = 10\log_{10}\left(\frac{1}{1-0.09}\right) = 0.41\ \text{dB}$$

The total loss of the link in dB is now

$$L_t(\text{dB}) = (0.20)(10) + 0.41 + (0.10)(15) = 3.91\ \text{dB}$$

(*b*) The output power will be $P_{\text{out}} = 100 \times 10^{-0.391} = 41$ mW.

10.10 VOLTAGE STANDING WAVE RATIO

In many instances, characteristics of transmission line performance are amenable to measurement. Included in these are measurements of unknown load impedances, or input impedances of lines that are terminated by known or unknown load impedances. Such techniques rely on the ability to measure voltage amplitudes that occur as functions of position within a line, usually designed for this purpose. A typical apparatus consists of a *slotted line,* which is a lossless coaxial transmission line having a longitudinal gap in the outer conductor along its entire length. The line is positioned between the sinusoidal voltage source and the impedance that is to be measured. Through the gap in the slotted line, a voltage probe may be inserted to measure the voltage amplitude between the inner and outer conductors. As the probe is moved along the length of the line, the maximum and minimum voltage amplitudes are noted, and their ratio, known as the *voltage standing wave ratio,* or VSWR, is determined. The significance of this measurement and its utility form the subject of this section.

To understand the meaning of the voltage measurements, we consider a few special cases. First, if the slotted line is terminated by a matched impedance, then no reflected wave occurs; the probe will indicate the same voltage amplitude at every point. Of course, the instantaneous voltages that the probe samples will differ in phase by $\beta(z_2 - z_1)$ rad as the probe is moved from $z = z_1$ to $z = z_2$, but the system is insensitive to the phase of the field. The equal-amplitude voltages are characteristic of an unattenuated traveling wave.

Second, if the slotted line is terminated by an open or short circuit (or in general a purely imaginary load impedance), the total voltage in the line is a standing wave and, as was shown in Example 10.1, the voltage probe provides no output when it is located at the nodes; these occur periodically with half-wavelength spacing. As the probe position is changed, its output varies as $|\cos(\beta z + \phi)|$, where z is the distance from the load, and where the phase, ϕ, depends on the load impedance. For example, if the load is a short circuit, the requirement of zero voltage at the short leads to a null occurring there, and so the voltage in the line will vary as $|\sin(\beta z)|$ (where $\phi = \pm\pi/2$).

A more complicated situation arises when the reflected voltage is neither 0 nor 100 percent of the incident voltage. Some energy is absorbed by the load and some is reflected. The slotted line, therefore, supports a voltage that is composed of both a traveling wave and a standing wave. It is customary to describe this voltage as a standing wave, even though a traveling wave is also present. We will see that the voltage does not have zero amplitude at any point for all time, and the degree to which the voltage is divided between a traveling wave and a true standing wave is expressed by the ratio of the maximum amplitude found by the probe to the minimum amplitude (VSWR). This information, along with the positions of the voltage minima or maxima with respect to that of the load, enable one to determine the load impedance. The VSWR also provides a measure of the quality of the termination. Specifically, a perfectly matched load yields a VSWR of exactly 1. A totally reflecting load produces an infinite VSWR.

To derive the specific form of the total voltage, we begin with the forward and backward-propagating waves that occur within the slotted line. The load is positioned at $z = 0$, and so all positions within the slotted line occur at negative values of z. Taking the input wave amplitude as V_0, the total phasor voltage is

$$V_{sT}(z) = V_0 e^{-j\beta z} + \Gamma V_0 e^{j\beta z} \tag{79}$$

The line, being lossless, has real characteristic impedance, Z_0. The load impedance, Z_L, is in general complex, which leads to a complex reflection coefficient:

$$\boxed{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\phi}} \tag{80}$$

If the load is a short circuit ($Z_L = 0$), ϕ is equal to π; if Z_L is real and less than Z_0, ϕ is also equal to π; and if Z_L is real and greater than Z_0, ϕ is zero. Using (80), we may rewrite (79) in the form:

$$V_{sT}(z) = V_0 \left(e^{-j\beta z} + |\Gamma| e^{j(\beta z + \phi)}\right) = V_0 e^{j\phi/2} \left(e^{-j\beta z} e^{-j\phi/2} + |\Gamma| e^{j\beta z} e^{j\phi/2}\right) \tag{81}$$

To express (81) in a more useful form, we can apply the algebraic trick of adding and subtracting the term $V_0(1 - |\Gamma|)e^{-j\beta z}$:

$$V_{sT}(z) = V_0(1 - |\Gamma|)e^{-j\beta z} + V_0 |\Gamma| e^{j\phi/2} \left(e^{-j\beta z} e^{-j\phi/2} + e^{j\beta z} e^{j\phi/2}\right) \tag{82}$$

The last term in parentheses in (82) becomes a cosine, and we write

$$\boxed{V_{sT}(z) = V_0(1 - |\Gamma|)e^{-j\beta z} + 2V_0 |\Gamma| e^{j\phi/2} \cos(\beta z + \phi/2)} \tag{83}$$

The important characteristics of this result are most easily seen by converting it to real instantaneous form:

$$\boxed{\begin{aligned}\mathcal{V}(z,t) = \text{Re}[V_{sT}(z)e^{j\omega t}] = {} & \underbrace{V_0(1-|\Gamma|)\cos(\omega t - \beta z)}_{\textit{traveling wave}} \\ & + \underbrace{2|\Gamma|V_0\cos(\beta z + \phi/2)\cos(\omega t + \phi/2)}_{\textit{standing wave}}\end{aligned}} \tag{84}$$

Equation (84) is recognized as the sum of a traveling wave of amplitude $(1-|\Gamma|)V_0$ and a standing wave having amplitude $2|\Gamma|V_0$. We can visualize events as follows: The portion of the incident wave that reflects and back-propagates in the slotted line interferes with an equivalent portion of the incident wave to form a standing wave. The rest of the incident wave (which does not interfere) is the traveling wave part of (84). The maximum amplitude observed in the line is found where the amplitudes of the two terms in (84) add directly to give $(1+|\Gamma|)V_0$. The minimum amplitude is found where the standing wave achieves a null, leaving only the traveling wave amplitude of $(1-|\Gamma|)V_0$. The fact that the two terms in (84) combine in this way with the proper phasing is not immediately apparent, but the following arguments will show that this does occur.

To obtain the minimum and maximum voltage amplitudes, we may revisit the first part of Eq. (81):

$$V_{sT}(z) = V_0\left(e^{-j\beta z} + |\Gamma|e^{j(\beta z+\phi)}\right) \tag{85}$$

First, the minimum voltage amplitude is obtained when the two terms in (85) subtract directly (having a phase difference of π). This occurs at locations

$$\boxed{z_{\min} = -\frac{1}{2\beta}(\phi + (2m+1)\pi) \quad (m = 0, 1, 2, \ldots)} \tag{86}$$

Note again that all positions within the slotted line occur at negative values of z. Substituting (86) into (85) leads to the minimum amplitude:

$$V_{sT}(z_{\min}) = V_0(1-|\Gamma|) \tag{87}$$

The same result is obtained by substituting (86) into the real voltage, (84). This produces a null in the standing wave part, and we obtain

$$\mathcal{V}(z_{\min}, t) = \pm V_0(1-|\Gamma|)\sin(\omega t + \phi/2) \tag{88}$$

The voltage oscillates (through zero) in time, with amplitude $V_0(1-|\Gamma|)$. The plus and minus signs in (88) apply to even and odd values of m in (86), respectively.

Next, the maximum voltage amplitude is obtained when the two terms in (85) add in-phase. This will occur at locations given by

$$\boxed{z_{\max} = -\frac{1}{2\beta}(\phi + 2m\pi) \quad (m = 0, 1, 2, \ldots)} \tag{89}$$

On substituting (89) into (85), we obtain

$$V_{sT}(z_{\max}) = V_0(1 + |\Gamma|) \tag{90}$$

As before, we may substitute (89) into the real instantaneous voltage (84). The effect is to produce a maximum in the standing wave part, which then adds in-phase to the running wave. The result is

$$\mathcal{V}(z_{\max}, t) = \pm V_0(1 + |\Gamma|)\cos(\omega t + \phi/2) \tag{91}$$

where the plus and minus signs apply to positive and negative values of m in (89), respectively. Again, the voltage oscillates through zero in time, with amplitude $V_0(1 + |\Gamma|)$.

Note that a voltage maximum is located at the load ($z = 0$) if $\phi = 0$; moreover, $\phi = 0$ when Γ is real and positive. This occurs for real Z_L when $Z_L > Z_0$. Thus there is a voltage maximum at the load when the load impedance is greater than Z_0 and both impedances are real. With $\phi = 0$, maxima also occur at $z_{\max} = -m\pi/\beta = -m\lambda/2$. For a zero-load impedance, $\phi = \pi$, and the maxima are found at $z_{\max} = -\pi/(2\beta)$, $-3\pi/(2\beta)$, or $z_{\max} = -\lambda/4$, $-3\lambda/4$, and so forth.

The minima are separated by multiples of one half-wavelength (as are the maxima), and for a zero load impedance, the first minimum occurs when $-\beta z = 0$, or at the load. In general, a voltage minimum is found at $z = 0$ whenever $\phi = \pi$; this occurs if $Z_L < Z_0$ where Z_L is real. The general results are illustrated in Figure 10.6.

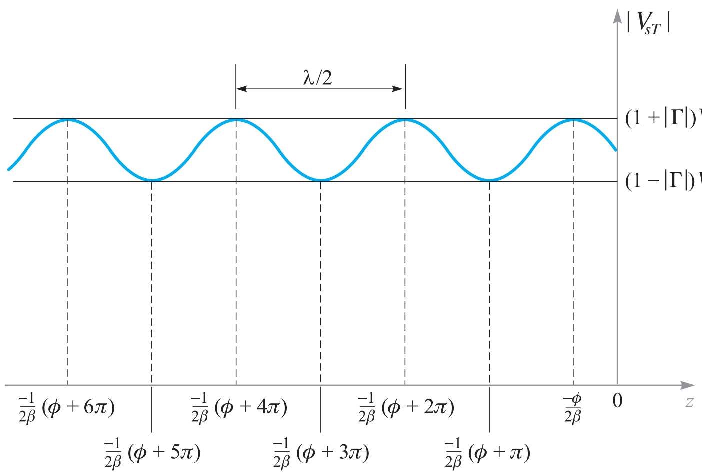


Figure 10.6 Plot of the magnitude of V_{sT} as found from Eq. (85) as a function of position, z, in front of the load (at $z = 0$). The reflection coefficient phase is ϕ, which leads to the indicated locations of maximum and minimum voltage amplitude, as found from Eqs. (86) and (89).

Finally, the voltage standing wave ratio is defined as:

$$\boxed{s \equiv \frac{V_{sT}(z_{\max})}{V_{sT}(z_{\min})} = \frac{1+|\Gamma|}{1-|\Gamma|}} \tag{92}$$

Since the absolute voltage amplitudes have divided out, our measured VSWR permits the immediate evaluation of $|\Gamma|$. The phase of Γ is then found by measuring the location of the first maximum or minimum with respect to the load, and then using (86) or (89) as appropriate. Once Γ is known, the load impedance can be found, assuming Z_0 is known.

WW Illustrations

D10.3. What voltage standing wave ratio results when $\Gamma = \pm 1/2$?

Ans. 3

EXAMPLE 10.7

Slotted line measurements yield a VSWR of 5, a 15-cm spacing between successive voltage maxima, and the first maximum at a distance of 7.5 cm in front of the load. Determine the load impedance, assuming a 50-Ω impedance for the slotted line.

Solution. The 15-cm spacing between maxima is $\lambda/2$, implying a wavelength of 30 cm. Because the slotted line is air-filled, the frequency is $f = c/\lambda = 1$ GHz. The first maximum at 7.5 cm is thus at a distance of $\lambda/4$ from the load, which means that a voltage minimum occurs at the load. Thus Γ will be real and negative. We use (92) to write

$$|\Gamma| = \frac{s-1}{s+1} = \frac{5-1}{5+1} = \frac{2}{3}$$

So

$$\Gamma = -\frac{2}{3} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

which we solve for Z_L to obtain

$$Z_L = \frac{1}{5} Z_0 = \frac{50}{5} = 10\ \Omega$$

10.11 TRANSMISSION LINES OF FINITE LENGTH

A new type of problem emerges when considering the propagation of sinusoidal voltages on finite-length lines that have loads that are not impedance matched. In such cases, numerous reflections occur at the load and at the generator, setting up a multiwave bidirectional voltage distribution in the line. As always, the objective is to determine the net power transferred to the load in steady state, but we must now include the effect of the numerous forward- and backward-reflected waves.

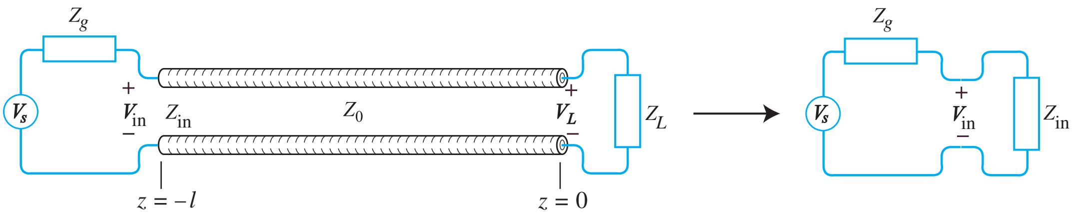


Figure 10.7 Finite-length transmission line configuration and its equivalent circuit.

Figure 10.7 shows the basic problem. The line, assumed to be lossless, has characteristic impedance Z_0 and is of length l. The sinusoidal voltage source at frequency ω provides phasor voltage V_s. Associated with the souce is a complex internal impedance, Z_g, as shown. The load impedance, Z_L, is also assumed to be complex and is located at $z = 0$. The line thus exists along the negative z axis. The easiest method of approaching the problem is not to attempt to analyze every reflection individually, but rather to recognize that in steady state, there will exist one net forward wave and one net backward wave, representing the superposition of all waves that are incident on the load and all waves that are reflected from it. We may thus write the total voltage in the line as

$$V_{sT}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \tag{93}$$

in which V_0^+ and V_0^- are complex amplitudes, composed respectively of the sum of all individual forward and backward wave amplitudes and phases. In a similar way, we may write the total current in the line:

$$I_{sT}(z) = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z} \tag{94}$$

We now define the *wave impedance,* $Z_w(z)$, as the ratio of the total phasor voltage to the total phasor current. Using (93) and (94), this becomes:

$$\boxed{Z_w(z) \equiv \frac{V_{sT}(z)}{I_{sT}(z)} = \frac{V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}}{I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z}}} \tag{95}$$

We next use the relations $V_0^- = \Gamma V_0^+$, $I_0^+ = V_0^+/Z_0$, and $I_0^- = -V_0^-/Z_0$. Eq. (95) simplifies to

$$Z_w(z) = Z_0 \left[\frac{e^{-j\beta z} + \Gamma e^{j\beta z}}{e^{-j\beta z} - \Gamma e^{j\beta z}} \right] \tag{96}$$

Now, using the Euler identity, (32), and substituting $\Gamma = (Z_L - Z_0)/(Z_L + Z_0)$, Eq. (96) becomes

$$Z_w(z) = Z_0 \left[\frac{Z_L \cos(\beta z) - jZ_0 \sin(\beta z)}{Z_0 \cos(\beta z) - jZ_L \sin(\beta z)} \right] \tag{97}$$

The wave impedance at the line input is now found by evaluating (97) at $z = -l$, obtaining

$$\boxed{Z_{\text{in}} = Z_0 \left[\frac{Z_L \cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + jZ_L \sin(\beta l)} \right]} \tag{98}$$

This is the quantity that we need in order to create the equivalent circuit in Figure 10.7.

One special case is that in which the line length is a half-wavelength, or an integer multiple thereof. In that case,

$$\beta l = \frac{2\pi}{\lambda} \frac{m\lambda}{2} = m\pi \quad (m = 0, 1, 2, \ldots)$$

Using this result in (98), we find

$$\boxed{Z_{\text{in}}(l = m\lambda/2) = Z_L} \tag{99}$$

For a half-wave line, the equivalent circuit can be constructed simply by removing the line completely and placing the load impedance at the input. This simplification works, of course, provided the line length is indeed an integer multiple of a half-wavelength. Once the frequency begins to vary, the condition is no longer satisfied, and (98) must be used in its general form to find Z_{in}.

Another important special case is that in which the line length is an odd multiple of a quarter wavelength:

$$\beta l = \frac{2\pi}{\lambda}(2m+1)\frac{\lambda}{4} = (2m+1)\frac{\pi}{2} \quad (m = 0, 1, 2, \ldots)$$

Using this result in (98) leads to

$$\boxed{Z_{\text{in}}(l = \lambda/4) = \frac{Z_0^2}{Z_L}} \tag{100}$$

An immediate application of (100) is to the problem of joining two lines having different characteristic impedances. Suppose the impedances are (from left to right) Z_{01} and Z_{03}. At the joint, we may insert an additional line whose characteristic impedance is Z_{02} and whose length is $\lambda/4$. We thus have a sequence of joined lines whose impedances progress as Z_{01}, Z_{02}, and Z_{03}, in that order. A voltage wave is now incident from line 1 onto the joint between Z_{01} and Z_{02}. Now the effective load at the far end of line 2 is Z_{03}. The input impedance to line 2 at any frequency is now

$$Z_{\text{in}} = Z_{02} \frac{Z_{03} \cos \beta_2 l + jZ_{02} \sin \beta_2 l}{Z_{02} \cos \beta_2 l + jZ_{03} \sin \beta_2 l} \tag{101}$$

Then, since the length of line 2 is $\lambda/4$,

$$Z_{\text{in}}(\text{line } 2) = \frac{Z_{02}^2}{Z_{03}} \tag{102}$$

Reflections at the Z_{01}–Z_{02} interface will not occur if $Z_{\text{in}} = Z_{01}$. Therefore, we can match the junction (allowing complete transmission through the three-line sequence) if Z_{02} is chosen so that

$$\boxed{Z_{02} = \sqrt{Z_{01} Z_{03}}} \tag{103}$$

This technique is called *quarter-wave matching* and again is limited to the frequency (or narrow band of frequencies) such that $l \doteq (2m+1)\lambda/4$. We will encounter more examples of these techniques when we explore electromagnetic wave reflection in Chapter 12. Meanwhile, further examples that involve the use of the input impedance and the VSWR are presented in Section 10.12.

10.12 SOME TRANSMISSION LINE EXAMPLES

In this section, we apply many of the results that we obtained in the previous sections to several typical transmission line problems. We simplify our work by restricting our attention to the lossless line.

Let us begin by assuming a two-wire 300 Ω line ($Z_0 = 300$ Ω), such as the lead-in wire from the antenna to a television or FM receiver. The circuit is shown in Figure 10.8. The line is 2 m long, and the values of L and C are such that the velocity on the line is 2.5×10^8 m/s. We will terminate the line with a receiver having an input resistance of 300 Ω and represent the antenna by its Thevenin equivalent $Z = 300$ Ω in series with $V_s = 60$ V at 100 MHz. This antenna voltage is larger by a factor of about 10^5 than it would be in a practical case, but it also provides simpler values to work with; in order to think practical thoughts, divide currents or voltages by 10^5, divide powers by 10^{10}, and leave impedances alone.

Because the load impedance is equal to the characteristic impedance, the line is matched; the reflection coefficient is zero, and the standing wave ratio is unity. For the given velocity and frequency, the wavelength on the line is $v/f = 2.5$ m, and the phase constant is $2\pi/\lambda = 0.8\pi$ rad/m; the attenuation constant is zero. The electrical length of the line is $\beta l = (0.8\pi)2$, or 1.6π rad. This length may also be expressed as 288°, or 0.8 wavelength.

The input impedance offered to the voltage source is 300 Ω, and since the internal impedance of the source is 300 Ω, the voltage at the input to the line is half of 60 V, or 30 V. The source is matched to the line and delivers the maximum available power

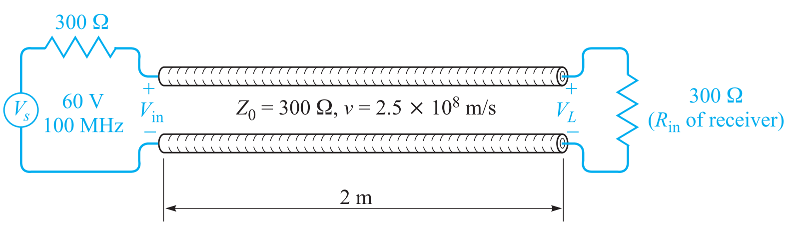


Figure 10.8 A transmission line that is matched at both ends produces no reflections and thus delivers maximum power to the load.

to the line. Because there is no reflection and no attenuation, the voltage at the load is 30 V, but it is delayed in phase by 1.6π rad. Thus,

$$V_{\text{in}} = 30\cos(2\pi 10^8 t) \text{ V}$$

whereas

$$V_L = 30\cos(2\pi 10^8 t - 1.6\pi) \text{ V}$$

The input current is

$$I_{\text{in}} = \frac{V_{\text{in}}}{300} = 0.1\cos(2\pi 10^8 t) \text{ A}$$

while the load current is

$$I_L = 0.1\cos(2\pi 10^8 t - 1.6\pi) \text{ A}$$

The average power delivered to the input of the line by the source must all be delivered to the load by the line,

$$P_{\text{in}} = P_L = \frac{1}{2} \times 30 \times 0.1 = 1.5 \text{ W}$$

Now let us connect a second receiver, also having an input resistance of 300 Ω, across the line in parallel with the first receiver. The load impedance is now 150 Ω, the reflection coefficient is

$$\Gamma = \frac{150 - 300}{150 + 300} = -\frac{1}{3}$$

and the standing wave ratio on the line is

$$s = \frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} = 2$$

The input impedance is no longer 300 Ω, but is now

$$\begin{aligned} Z_{\text{in}} &= Z_0 \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} = 300\,\frac{150\cos 288^\circ + j300\sin 288^\circ}{300\cos 288^\circ + j150\sin 288^\circ} \\ &= 510\angle{-23.8^\circ} = 466 - j206\ \Omega \end{aligned}$$

which is a capacitive impedance. Physically, this means that this length of line stores more energy in its electric field than in its magnetic field. The input current phasor is thus

$$I_{s,\text{in}} = \frac{60}{300 + 466 - j206} = 0.0756\angle 15.0^\circ \text{ A}$$

and the power supplied to the line by the source is

$$P_{\text{in}} = \frac{1}{2} \times (0.0756)^2 \times 466 = 1.333 \text{ W}$$

Since there are no losses in the line, 1.333 W must also be delivered to the load. Note that this is less than the 1.50 W which we were able to deliver to a matched load; moreover, this power must divide equally between two receivers, and thus each receiver now receives only 0.667 W. Because the input impedance of each receiver is 300 Ω, the voltage across the receiver is easily found as

$$0.667 = \frac{1}{2}\frac{|V_{s,L}|^2}{300}$$
$$|V_{s,L}| = 20 \text{ V}$$

in comparison with the 30 V obtained across the single load.

Before we leave this example, let us ask ourselves several questions about the voltages on the transmission line. Where is the voltage a maximum and a minimum, and what are these values? Does the phase of the load voltage still differ from the input voltage by 288°? Presumably, if we can answer these questions for the voltage, we could do the same for the current.

Equation (89) serves to locate the voltage maxima at

$$z_{\max} = -\frac{1}{2\beta}(\phi + 2m\pi) \quad (m = 0, 1, 2, \ldots)$$

where $\Gamma = |\Gamma|e^{j\phi}$. Thus, with $\beta = 0.8\pi$ and $\phi = \pi$, we find

$$z_{\max} = -0.625 \quad \text{and} \quad -1.875 \text{ m}$$

while the minima are $\lambda/4$ distant from the maxima;

$$z_{\min} = 0 \quad \text{and} \quad -1.25 \text{ m}$$

and we find that the load voltage (at $z = 0$) is a voltage minimum. This, of course, verifies the general conclusion we reached earlier: a voltage minimum occurs at the load if $Z_L < Z_0$, and a voltage maximum occurs if $Z_L > Z_0$, where both impedances are pure resistances.

The minimum voltage on the line is thus the load voltage, 20 V; the maximum voltage must be 40 V, since the standing wave ratio is 2. The voltage at the input end of the line is

$$V_{s,\text{in}} = I_{s,\text{in}}\, Z_{\text{in}} = (0.0756\angle 15.0^\circ)(510\angle{-23.8^\circ}) = 38.5\angle{-8.8^\circ}$$

The input voltage is almost as large as the maximum voltage anywhere on the line because the line is about three-quarters of a wavelength long, a length which would place the voltage maximum at the input when $Z_L < Z_0$.

Finally, it is of interest to determine the load voltage in magnitude *and phase*. We begin with the total voltage in the line, using (93).

$$V_{sT} = \left(e^{-j\beta z} + \Gamma e^{j\beta z}\right) V_0^+ \tag{104}$$

We may use this expression to determine the voltage at any point on the line in terms of the voltage at any other point. Because we know the voltage at the input to the line, we let $z = -l$,

$$V_{s,\text{in}} = \left(e^{j\beta l} + \Gamma e^{-j\beta l}\right) V_0^+ \tag{105}$$

and solve for V_0^+,

$$V_0^+ = \frac{V_{s,\text{in}}}{e^{j\beta l} + \Gamma e^{-j\beta l}} = \frac{38.5\angle{-8.8^\circ}}{e^{j1.6\pi} - \frac{1}{3}e^{-j1.6\pi}} = 30.0\angle 72.0^\circ \text{ V}$$

We may now let $z = 0$ in (104) to find the load voltage,

$$V_{s,L} = (1 + \Gamma)V_0^+ = 20\angle 72^\circ = 20\angle{-288^\circ}$$

The amplitude agrees with our previous value. The presence of the reflected wave causes $V_{s,\text{in}}$ and $V_{s,L}$ to differ in phase by about -279° instead of -288°.

EXAMPLE 10.8

In order to provide a slightly more complicated example, let us now place a purely capacitive impedance of $-j300\,\Omega$ in parallel with the two $300\ \Omega$ receivers. We are to find the input impedance and the power delivered to each receiver.

Solution. The load impedance is now $150\ \Omega$ in parallel with $-j300\,\Omega$, or

$$Z_L = \frac{150(-j300)}{150 - j300} = \frac{-j300}{1 - j2} = 120 - j60\ \Omega$$

We first calculate the reflection coefficient and the VSWR:

$$\Gamma = \frac{120 - j60 - 300}{120 - j60 + 300} = \frac{-180 - j60}{420 - j60} = 0.447\angle{-153.4^\circ}$$

$$s = \frac{1 + 0.447}{1 - 0.447} = 2.62$$

Thus, the VSWR is higher and the mismatch is therefore worse. Let us next calculate the input impedance. The electrical length of the line is still 288°, so that

$$Z_{\text{in}} = 300\,\frac{(120 - j60)\cos 288^\circ + j300\sin 288^\circ}{300\cos 288^\circ + j(120 - j60)\sin 288^\circ} = 755 - j138.5\ \Omega$$

This leads to a source current of

$$I_{s,\text{in}} = \frac{V_{Th}}{Z_{Th} + Z_{\text{in}}} = \frac{60}{300 + 755 - j138.5} = 0.0564\angle 7.47^\circ\ \text{A}$$

Therefore, the average power delivered to the input of the line is $P_{\text{in}} = \frac{1}{2}(0.0564)^2(755) = 1.200$ W. Since the line is lossless, it follows that $P_L = 1.200$ W, and each receiver gets only 0.6 W.

EXAMPLE 10.9

As a final example, let us terminate our line with a purely capacitive impedance, $Z_L = -j300\,\Omega$. We seek the reflection coefficient, the VSWR, and the power delivered to the load.

Solution. Obviously, we cannot deliver any average power to the load since it is a pure reactance. As a consequence, the reflection coefficient is

$$\Gamma = \frac{-j300 - 300}{-j300 + 300} = -j1 = 1\angle{-90^\circ}$$

and the reflected wave is equal in amplitude to the incident wave. Hence, it should not surprise us to see that the VSWR is

$$s = \frac{1 + |-j1|}{1 - |-j1|} = \infty$$

and the input impedance is a pure reactance,

$$Z_{\text{in}} = 300\,\frac{-j300\cos 288^\circ + j300\sin 288^\circ}{300\cos 288^\circ + j(-j300)\sin 288^\circ} = j589$$

Thus, no average power can be delivered to the input impedance by the source, and therefore no average power can be delivered to the load.

Although we could continue to find numerous other facts and figures for these examples, much of the work may be done more easily for problems of this type by using graphical techniques. We encounter these in Section 10.13.

D10.4. A 50 W lossless line has a length of 0.4λ. The operating frequency is 300 MHz. A load $Z_L = 40 + j30\ \Omega$ is connected at $z = 0$, and the Thevenin-equivalent source at $z = -l$ is $12\angle 0^\circ$ V in series with $Z_{Th} = 50 + j0\ \Omega$. Find: (a) Γ; (b) s; (c) Z_{in}.

Ans. $0.333\angle 90^\circ$; 2.00; $25.5 + j5.90\ \Omega$

D10.5. For the transmission line of Problem D10.4, also find: (a) the phasor voltage at $z = -l$; (b) the phasor voltage at $z = 0$; (c) the average power delivered to Z_L.

Ans. $4.14\angle 8.58^\circ$ V; $6.32\angle -125.6^\circ$ V; 0.320 W

10.13 GRAPHICAL METHODS: THE SMITH CHART

Transmission line problems often involve manipulations with complex numbers, making the time and effort required for a solution several times greater than are needed for a similar sequence of operations on real numbers. One means of reducing the labor without seriously affecting the accuracy is by using transmission-line charts. Probably the most widely used one is the Smith chart.[3]

Basically, this diagram shows curves of constant resistance and constant reactance; these may represent either an input impedance or a load impedance. The latter, of course, is the input impedance of a zero-length line. An indication of location along the line is also provided, usually in terms of the fraction of a wavelength from a voltage maximum or minimum. Although they are not specifically shown on the chart, the standing-wave ratio and the magnitude and angle of the reflection coefficient are very

[3] P. H. Smith, "Transmission Line Calculator," *Electronics,* vol. 12, pp. 29–31, January 1939.

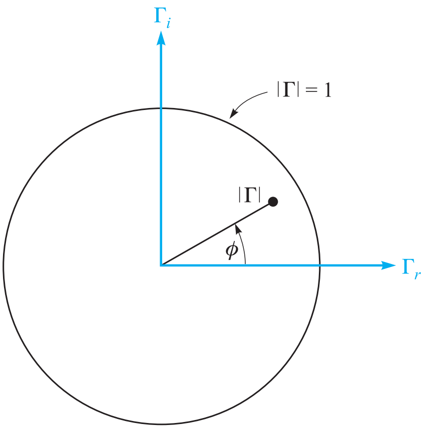


Figure 10.9 The polar coordinates of the Smith chart are the magnitude and phase angle of the reflection coefficient; the rectangular coordinates are the real and imaginary parts of the reflection coefficient. The entire chart lies within the circle $|\Gamma| = 1$.

quickly determined. As a matter of fact, the diagram is constructed within a circle of unit radius, using polar coordinates, with radius variable $|\Gamma|$ and counterclockwise angle variable ϕ, where $\Gamma = |\Gamma|e^{j\phi}$. Figure 10.9 shows this circle. Since $|\Gamma| < 1$, all our information must lie on or within the unit circle. Peculiarly enough, the reflection coefficient itself will not be plotted on the final chart, for these additional contours would make the chart very difficult to read.

The basic relationship upon which the chart is constructed is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{106}$$

The impedances that we plot on the chart will be *normalized* with respect to the characteristic impedance. Let us identify the normalized load impedance as z_L,

$$\boxed{z_L = r + jx = \frac{Z_L}{Z_0} = \frac{R_L + jX_L}{Z_0}}$$

and thus

$$\boxed{\Gamma = \frac{z_L - 1}{z_L + 1}}$$

or

$$\boxed{z_L = \frac{1 + \Gamma}{1 - \Gamma}} \tag{107}$$

In polar form, we have used $|\Gamma|$ and ϕ as the magnitude and angle of Γ. With Γ_r and Γ_i as the real and imaginary parts of Γ, we write

$$\Gamma = \Gamma_r + j\Gamma_i \tag{108}$$

Thus

$$r + jx = \frac{1 + \Gamma_r + j\Gamma_i}{1 - \Gamma_r - j\Gamma_i} \tag{109}$$

The real and imaginary parts of this equation are

$$\boxed{r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2}} \tag{110}$$

$$\boxed{x = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}} \tag{111}$$

After several lines of elementary algebra, we may write (110) and (111) in forms which readily display the nature of the curves on Γ_r, Γ_i axes,

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2 \tag{112}$$

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 \tag{113}$$

The first equation describes a family of circles, where each circle is associated with a specific value of resistance r. For example, if $r = 0$, the radius of this zero-resistance circle is seen to be unity, and it is centered at the origin ($\Gamma_r = 0$, $\Gamma_i = 0$). This checks, for a pure reactance termination leads to a reflection coefficient of unity magnitude. On the other hand, if $r = \infty$, then $z_L = \infty$ and we have $\Gamma = 1 + j0$. The circle described by (112) is centered at $\Gamma_r = 1$, $\Gamma_i = 0$ and has zero radius. It is therefore the point $\Gamma = 1 + j0$, as we decided it should be. As another example, the circle for $r = 1$ is centered at $\Gamma_r = 0.5$, $\Gamma_i = 0$ and has a radius of 0.5. This circle is shown in Figure 10.10, along with circles for $r = 0.5$ and $r = 2$. All circles are centered on the Γ_r axis and pass through the point $\Gamma = 1 + j0$.

Equation (113) also represents a family of circles, but each of these circles is defined by a particular value of x, rather than r. If $x = \infty$, then $z_L = \infty$, and $\Gamma = 1 + j0$ again. The circle described by (113) is centered at $\Gamma = 1 + j0$ and has zero radius; it is therefore the point $\Gamma = 1 + j0$. If $x = +1$, then the circle is centered at $\Gamma = 1 + j1$ and has unit radius. Only one-quarter of this circle lies within the boundary curve $|\Gamma| = 1$, as shown in Figure 10.11. A similar quarter-circle appears below the Γ_r axis for $x = -1$. The portions of other circles for $x = 0.5$, -0.5, 2, and -2 are also shown. The "circle" representing $x = 0$ is the Γ_r axis; this is also labeled in Figure 10.11.

The two families of circles both appear on the Smith chart, as shown in Figure 10.12. It is now evident that if we are given Z_L, we may divide by Z_0 to

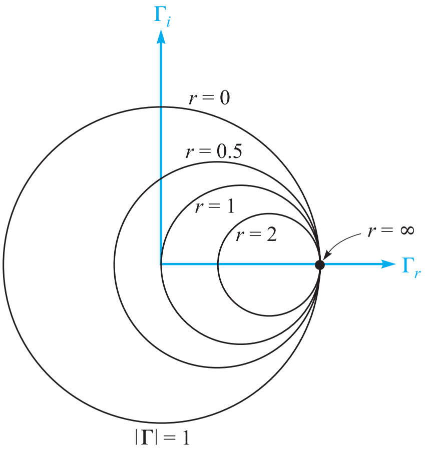


Figure 10.10 Constant-r circles are shown on the Γ_r, Γ_i plane. The radius of any circle is $1/(1 + r)$.

obtain z_L, locate the appropriate r and x circles (interpolating as necessary), and determine Γ by the intersection of the two circles. Because the chart does not have concentric circles showing the values of $|\Gamma|$, it is necessary to measure the radial distance from the origin to the intersection with dividers or a compass and use an auxiliary scale to find $|\Gamma|$. The graduated line segment below the chart in Figure 10.12 serves this purpose. The angle of Γ is ϕ, and it is the counterclockwise angle from the Γ_r axis. Again, radial lines showing the angle would clutter up the chart

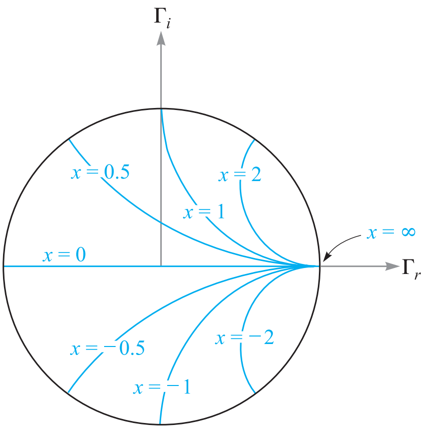


Figure 10.11 The portions of the circles of constant x lying within $|\Gamma| = 1$ are shown on the Γ_r, Γ_i axes. The radius of a given circle is $1/|x|$.

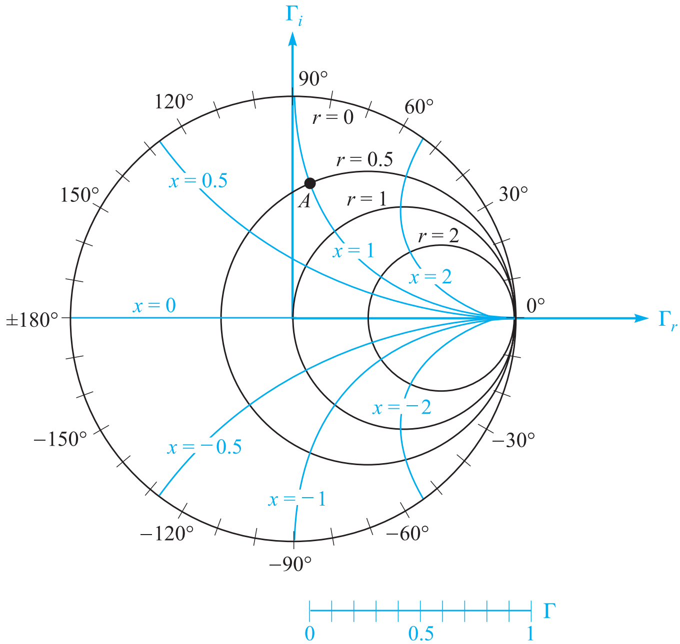


Figure 10.12 The Smith chart contains the constant-r circles and constant-x circles, an auxiliary radial scale to determine $|\Gamma|$, and an angular scale on the circumference for measuring ϕ.

badly, so the angle is indicated on the circumference of the circle. A straight line from the origin through the intersection may be extended to the perimeter of the chart. As an example, if $Z_L = 25 + j50\ \Omega$ on a 50 Ω line, $z_L = 0.5 + j1$, and point A on Figure 10.12 shows the intersection of the $r = 0.5$ and $x = 1$ circles. The reflection coefficient is approximately 0.62 at an angle ϕ of 83°.

The Smith chart is completed by adding a second scale on the circumference by which distance along the line may be computed. This scale is in wavelength units, but the values placed on it are not obvious. To obtain them, we first divide the voltage at any point along the line,

$$V_s = V_0^+(e^{-j\beta z} + \Gamma e^{j\beta z})$$

by the current

$$I_s = \frac{V_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z})$$

obtaining the normalized input impedance

$$z_{\text{in}} = \frac{V_s}{Z_0 I_s} = \frac{e^{-j\beta z} + \Gamma e^{j\beta z}}{e^{-j\beta z} - \Gamma e^{j\beta z}}$$

Replacing z with $-l$ and dividing numerator and denominator by $e^{j\beta l}$, we have the general equation relating normalized input impedance, reflection coefficient, and line length,

$$z_{\text{in}} = \frac{1 + \Gamma e^{-j2\beta l}}{1 - \Gamma e^{-j2\beta l}} = \frac{1 + |\Gamma| e^{j(\phi - 2\beta l)}}{1 - |\Gamma| e^{j(\phi - 2\beta l)}} \tag{114}$$

Note that when $l = 0$, we are located at the load, and $z_{\text{in}} = (1 + \Gamma)/(l - \Gamma) = z_L$, as shown by (107).

Equation (114) shows that the input impedance at any point $z = -l$ can be obtained by replacing Γ, the reflection coefficient of the load, by $\Gamma e^{-j2\beta l}$. That is, we decrease the angle of Γ by $2\beta l$ radians as we move from the load to the line input. Only the angle of Γ is changed; the magnitude remains constant.

Thus, as we proceed from the load z_L to the input impedance z_{in}, we move *toward* the generator a distance l on the transmission line, but we move through a *clockwise* angle of $2\beta l$ on the Smith chart. Since the magnitude of Γ stays constant, the movement toward the source is made along a constant-radius circle. One lap around the chart is accomplished whenever βl changes by π rad, or when l changes by one-half wavelength. This agrees with our earlier discovery that the input impedance of a half-wavelength lossless line is equal to the load impedance.

The Smith chart is thus completed by the addition of a scale showing a change of 0.5λ for one circumnavigation of the unit circle. For convenience, two scales are usually given, one showing an increase in distance for clockwise movement and the other an increase for counterclockwise travel. These two scales are shown in Figure 10.13. Note that the one marked "wavelengths toward generator" (wtg) shows increasing values of l/λ for clockwise travel, as described previously. The zero point of the wtg scale is rather arbitrarily located to the left. This corresponds to input impedances having phase angles of 0° and $R_L < Z_0$. We have also seen that voltage minima are always located here.

EXAMPLE 10.10

The use of the transmission line chart is best shown by example. Let us again consider a load impedance, $Z_L = 25 + j50\ \Omega$, terminating a 50-Ω line. The line length is 60 cm and the operating frequency is such that the wavelength on the line is 2 m. We desire the input impedance.

Solution. We have $z_L = 0.5 + j1$, which is marked as A on Figure 10.14, and we read $\Gamma = 0.62\angle 82°$. By drawing a straight line from the origin through A to the circumference, we note a reading of 0.135 on the wtg scale. We have $l/\lambda = 0.6/2 = 0.3$, and it is, therefore, 0.3λ from the load to the input. We therefore find z_{in} on the $|\Gamma| = 0.62$ circle opposite a wtg reading of $0.135 + 0.300 = 0.435$. This construction is shown in Figure 10.14, and the point locating the input impedance is marked B. The normalized input impedance is read as $0.28 - j0.40$, and thus $Z_{\text{in}} = 14 - j20$. A more accurate analytical calculation gives $Z_{\text{in}} = 13.7 - j20.2$.

Information concerning the location of the voltage maxima and minima is also readily obtained on the Smith chart. We already know that a maximum or minimum

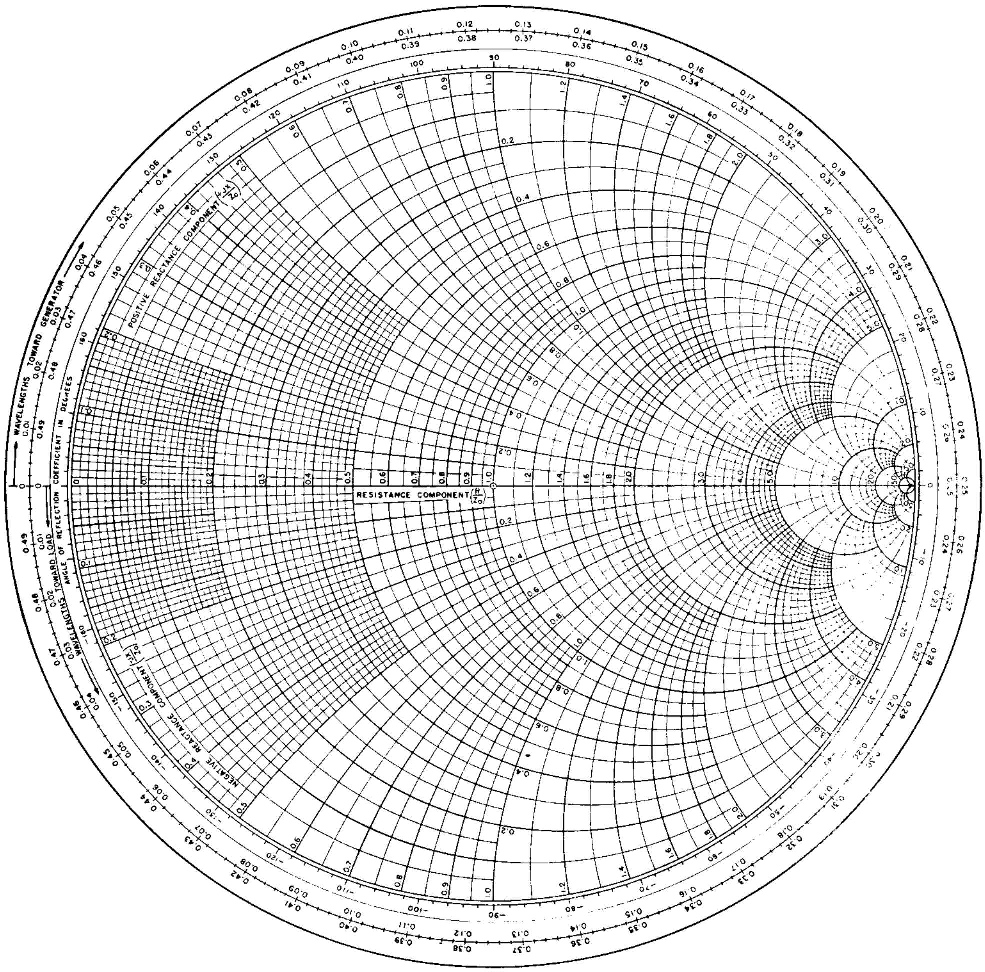


Figure 10.13 A photographic reduction of one version of a useful Smith chart (*courtesy of the Emeloid Company, Hillside, NJ*). For accurate work, larger charts are available wherever fine technical books are sold.

must occur at the load when Z_L is a pure resistance; if $R_L > Z_0$ there is a maximum at the load, and if $R_L < Z_0$ there is a minimum. We may extend this result now by noting that we could cut off the load end of a transmission line at a point where the input impedance is a pure resistance and replace that section with a resistance R_{in}; there would be no changes on the generator portion of the line. It follows, then, that the location of voltage maxima and minima must be at those points where Z_{in} is a pure resistance. Purely resistive input impedances must occur on the $x = 0$ line (the Γ_r axis) of the Smith chart. Voltage maxima or current minima occur when $r > 1$, or at wtg $= 0.25$, and voltage minima or current maxima occur when $r < 1$,

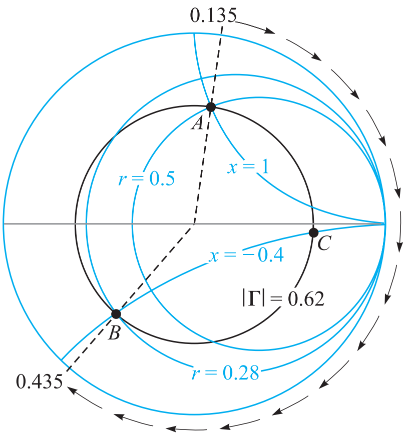


Figure 10.14 Normalized input impedance produced by a normalized load impedance $z_L = 0.5 + j1$ on a line 0.3λ long is $z_{\text{in}} = 0.28 - j0.40$.

or at wtg $= 0$. In Example 10.10, then, the maximum at wtg $= 0.250$ must occur $0.250 - 0.135 = 0.115$ wavelengths toward the generator from the load. This is a distance of 0.115×200, or 23 cm from the load.

We should also note that because the standing wave ratio produced by a resistive load R_L is either R_L/R_0 or R_0/R_L, whichever is greater than unity, the value of s may be read directly as the value of r at the intersection of the $|\Gamma|$ circle and the r axis, $r > 1$. In our example, this intersection is marked point C, and $r = 4.2$; thus, $s = 4.2$.

Transmission line charts may also be used for normalized admittances, although there are several slight differences in such use. We let $y_L = Y_L/Y_0 = g + jb$ and use the r circles as g circles and the x circles as b circles. The two differences are, first, the line segment where $g > 1$ and $b = 0$ corresponds to a voltage minimum; and second, 180° must be added to the angle of Γ as read from the perimeter of the chart. We shall use the Smith chart in this way in Section 10.14.

Special charts are also available for non-normalized lines, particularly 50 Ω charts and 20 mS charts.

D10.6. A load $Z_L = 80 - j100\ \Omega$ is located at $z = 0$ on a lossless 50-Ω line. The operating frequency is 200 MHz and the wavelength on the line is 2 m. (*a*) If the line is 0.8 m in length, use the Smith chart to find the input impedance. (*b*) What is s? (*c*) What is the distance from the load to the nearest voltage maximum? (*d*) What is the distance from the input to the nearest point at which the remainder of the line could be replaced by a pure resistance?

Ans. $79 + j99\ \Omega$: 4.50; 0.0397 m; 0.760 m

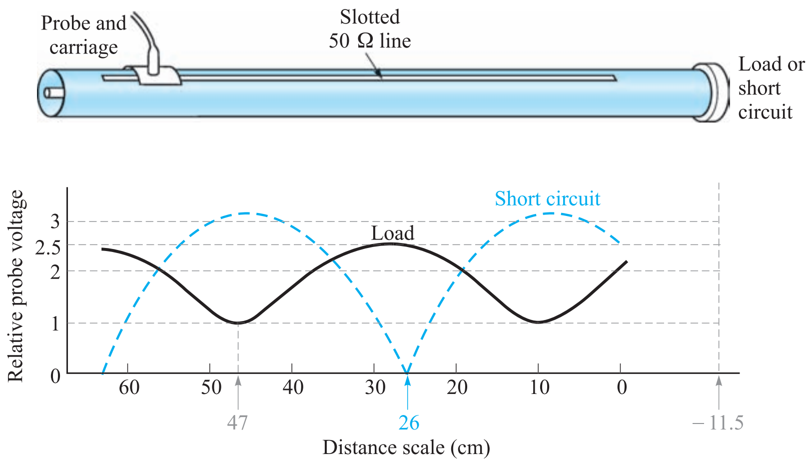


Figure 10.15 A sketch of a coaxial slotted line. The distance scale is on the slotted line. With the load in place, $s = 2.5$, and the minimum occurs at a scale reading of 47 cm. For a short circuit, the minimum is located at a scale reading of 26 cm. The wavelength is 75 cm.

We next consider two examples of practical transmission line problems. The first is the determination of load impedance from experimental data, and the second is the design of a single-stub matching network.

Let us assume that we have made experimental measurements on a 50 Ω slotted line that show there is a voltage standing wave ratio of 2.5. This has been determined by moving a sliding carriage back and forth along the line to determine maximum and minimum voltage readings. A scale provided on the track along which the carriage moves indicates that a *minimum* occurs at a scale reading of 47.0 cm, as shown in Figure 10.15. The zero point of the scale is arbitrary and does not correspond to the location of the load. The location of the minimum is usually specified instead of the maximum because it can be determined more accurately than that of the maximum; think of the sharper minima on a rectified sine wave. The frequency of operation is 400 MHz, so the wavelength is 75 cm. In order to pinpoint the location of the load, we remove it and replace it with a short circuit; the position of the minimum is then determined as 26.0 cm.

We know that the short circuit must be located an integral number of half-wavelengths from the minimum; let us arbitrarily locate it one half-wavelength away at $26.0 - 37.5 = -11.5$ cm on the scale. Since the short circuit has replaced the load, the load is also located at -11.5 cm. Our data thus show that the minimum is $47.0 - (-11.5) = 58.5$ cm from the load, or subtracting one-half wavelength, a minimum is 21.0 cm from the load. The voltage *maximum* is thus $21.0 - (37.5/2) = 2.25$ cm from the load, or $2.25/75 = 0.030$ wavelength from the load.

With this information, we can now turn to the Smith chart. At a voltage maximum, the input impedance is a pure resistance equal to sR_0; on a normalized basis, $z_{in} = 2.5$.

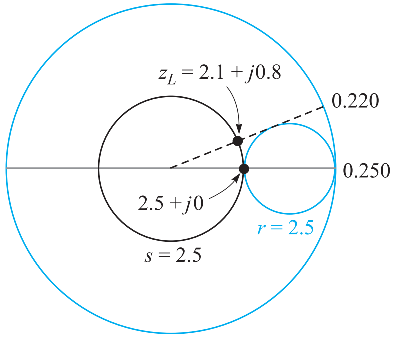


Figure 10.16 If $z_{\text{in}} = 2.5 + j0$ on a line 0.3 wavelengths long, then $z_L = 2.1 + j0.8$.

We therefore enter the chart at $z_{\text{in}} = 2.5$ and read 0.250 on the wtg scale. Subtracting 0.030 wavelength to reach the load, we find that the intersection of the $s = 2.5$ (or $|\Gamma| = 0.429$) circle and the radial line to 0.220 wavelength is at $z_L = 2.1 + j0.8$. The construction is sketched on the Smith chart of Figure 10.16. Thus $Z_L = 105 + j40\,\Omega$, a value that assumes its location at a scale reading of -11.5 cm, or an integral number of half-wavelengths from that position. Of course, we may select the "location" of our load at will by placing the short circuit at the point that we wish to consider the load location. Since load locations are not well defined, it is important to specify the point (or plane) at which the load impedance is determined.

As a final example, let us try to match this load to the 50 Ω line by placing a short-circuited stub of length d_1 a distance d from the load (see Figure 10.17). The stub line has the same characteristic impedance as the main line. The lengths d and d_1 are to be determined.

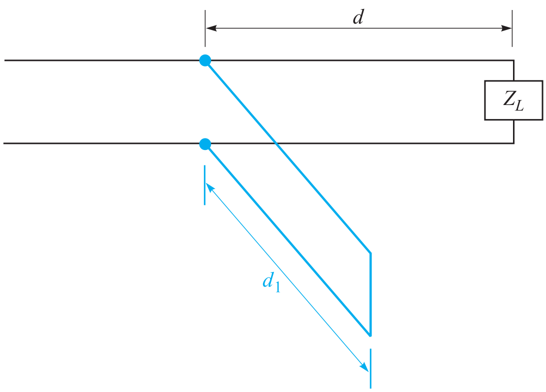


Figure 10.17 A short-circuited stub of length d_1, located at a distance d from a load Z_L, is used to provide a matched load to the left of the stub.

The input impedance to the stub is a pure reactance; when combined in parallel with the input impedance of the length d containing the load, the resultant input impedance must be $1 + j0$. Because it is much easier to combine admittances in parallel than impedances, let us rephrase our goal in admittance language: the input admittance of the length d containing the load must be $1 + jb_{\text{in}}$ for the addition of the input admittance of the stub jb_{stub} to produce a total admittance of $1 + j0$. Hence the stub admittance is $-jb_{\text{in}}$. We will therefore use the Smith chart as an admittance chart instead of an impedance chart.

The impedance of the load is $2.1 + j0.8$, and its location is at -11.5 cm. The admittance of the load is therefore $1/(2.1 + j0.8)$, and this value may be determined by adding one-quarter wavelength on the Smith chart, as Z_{in} for a quarter-wavelength line is R_0^2/Z_L, or $z_{\text{in}} = 1/z_L$, or $y_{\text{in}} = z_L$. Entering the chart (Figure 10.18) at $z_L = 2.1 + j0.8$, we read 0.220 on the wtg scale; we add (or subtract) 0.250 and find the admittance $0.41 - j0.16$ corresponding to this impedance. This point is still located on the $s = 2.5$ circle. Now, at what point or points on this circle is the real part of the admittance equal to unity? There are two answers, $1 + j0.95$ at wtg $= 0.16$, and $1 - j0.95$ at wtg $= 0.34$, as shown in Figure 10.18. We select the former value since this leads to the shorter stub. Hence $y_{\text{stub}} = -j0.95$, and the stub location corresponds to wtg $= 0.16$. Because the load admittance was found at wtg $=$ 0.470, then we must move $(0.5 - 0.47) + 0.16 = 0.19$ wavelength to get to the stub location.

Finally, we may use the chart to determine the necessary length of the short-circuited stub. The input conductance is zero for any length of short-circuited stub, so we are restricted to the perimeter of the chart. At the short circuit, $y = \infty$ and wtg $= 0.250$. We find that $b_{\text{in}} = -0.95$ is achieved at wtg $= 0.379$, as shown in Figure 10.18. The stub is therefore $0.379 - 0.250 = 0.129$ wavelength, or 9.67 cm long.

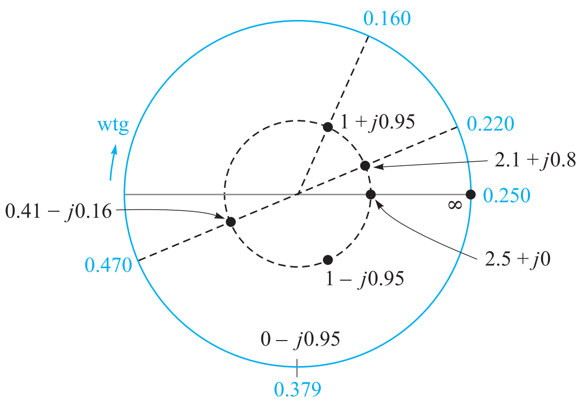


Figure 10.18 A normalized load, $z_L = 2.1 + j0.8$, is matched by placing a 0.129-wavelength short-circuited stub 0.19 wavelengths from the load.

D10.7. Standing wave measurements on a lossless 75-Ω line show maxima of 18 V and minima of 5 V. One minimum is located at a scale reading of 30 cm. With the load replaced by a short circuit, two adjacent minima are found at scale readings of 17 and 37 cm. Find: (*a*) s; (*b*) λ; (*c*) f; (*d*) Γ_L; (*e*) Z_L.

Ans. 3.60; 0.400 m; 750 MHz; $0.704\angle{-33.0}$; $77.9 + j104.7$ Ω

D10.8. A normalized load, $z_L = 2 - j1$, is located at $z = 0$ on a lossless 50-Ω line. Let the wavelength be 100 cm. (*a*) A short-circuited stub is to be located at $z = -d$. What is the shortest suitable value for d? (*b*) What is the shortest possible length of the stub? Find s: (*c*) on the main line for $z < -d$; (*d*) on the main line for $-d < z < 0$; (*e*) on the stub.

Ans. 12.5 cm; 12.5 cm; 1.00; 2.62; ∞

10.14 TRANSIENT ANALYSIS

Throughout most of this chapter, we have considered the operation of transmission lines under steady-state conditions, in which voltage and current were sinusoidal and at a single frequency. In this section we move away from the simple time-harmonic case and consider transmission line responses to voltage step functions and pulses, grouped under the general heading of *transients*. These situations were briefly considered in Section 10.2 with regard to switched voltages and currents. Line operation in transient mode is important to study because it allows us to understand how lines can be used to store and release energy (in pulse-forming applications, for example). Pulse propagation is important in general since digital signals, composed of sequences of pulses, are widely used.

We will confine our discussion to the propagation of transients in lines that are lossless and have no dispersion, so that the basic behavior and analysis methods may be learned. We must remember, however, that transient signals are necessarily composed of numerous frequencies, as Fourier analysis will show. Consequently, the question of dispersion in the line arises, since, as we have found, line propagation constants and reflection coefficients at complex loads will be frequency-dependent. So, in general, pulses are likely to broaden with propagation distance, and pulse shapes may change when reflecting from a complex load. These issues will not be considered in detail here, but they are readily addressed when the precise frequency dependences of β and Γ are known. In particular, $\beta(\omega)$ can be found by evaluating the imaginary part of γ, as given in Eq. (41), which would in general include the frequency dependences of R, C, G, and L arising from various mechanisms. For example, the skin effect (which affects both the conductor resistance and the internal inductance) will result in frequency-dependent R and L. Once $\beta(\omega)$ is known, pulse broadening can be evaluated using the methods to be presented in Chapter 12.

We begin our basic discussion of transients by considering a lossless transmission line of length l terminated by a matched load, $R_L = Z_0$, as shown in Figure 10.19*a*.

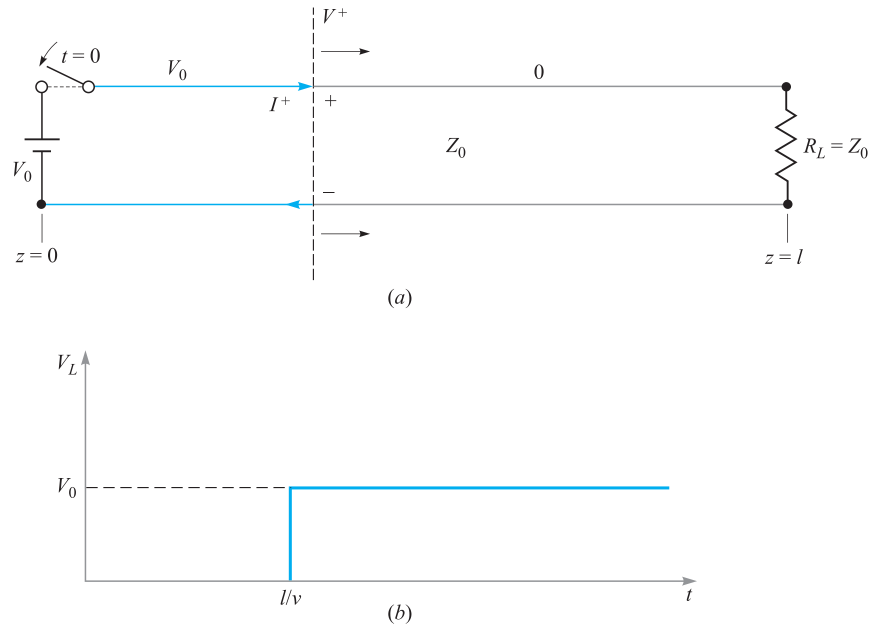


Figure 10.19 (*a*) Closing the switch at time $t = 0$ initiates voltage and current waves V^+ and I^+. The leading edge of both waves is indicated by the dashed line, which propagates in the lossless line toward the load at velocity ν. In this case, $V^+ = V_0$; the line voltage is V^+ everywhere to the left of the leading edge, where current is $I^+ = V^+/Z_0$. To the right of the leading edge, voltage and current are both zero. Clockwise current, indicated here, is treated as positive and will occur when V^+ is positive. (*b*) Voltage across the load resistor as a function of time, showing the one-way transit time delay, l/ν.

At the front end of the line is a battery of voltage V_0, which is connected to the line by closing a switch. At time $t = 0$, the switch is closed, and the line voltage at $z = 0$ becomes equal to the battery voltage. This voltage, however, does not appear across the load until adequate time has elapsed for the propagation delay. Specifically, at $t = 0$, a voltage wave is initiated in the line at the battery end, which then propagates toward the load. The leading edge of the wave, labeled V^+ in Figure 10.19, is of value $V^+ = V_0$. It can be thought of as a propagating step function, because at all points to the left of V^+, the line voltage is V_0; at all points to the right (not yet reached by the leading edge), the line voltage is zero. The wave propagates at velocity ν, which in general is the group velocity in the line.[4] The wave reaches the load at time $t = l/\nu$

[4] Because we have a step function (composed of many frequencies) as opposed to a sinusoid at a single frequency, the wave will propagate at the group velocity. In a lossless line with no dispersion as considered in this section, $\beta = \omega\sqrt{LC}$, where L and C are constant with frequency. In this case, we would find that the group and phase velocities are equal; that is, $d\omega/d\beta = \omega/\beta = \nu = 1/\sqrt{LC}$. We will thus write the velocity as ν, knowing it to be both ν_p and ν_g.

and then does not reflect, as the load is matched. The transient phase is thus over, and the load voltage is equal to the battery voltage. A plot of load voltage as a function of time is shown in Figure 10.19*b*, indicating the propagation delay of $t = l/\nu$.

Associated with the voltage wave V^+ is a current wave whose leading edge is of value I^+. This wave is a propagating step function as well, whose value at all points to the left of V^+ is $I^+ = V^+/Z_0$; at all points to the right, current is zero. A plot of current through the load as a function of time will thus be identical in form to the voltage plot of Figure 10.19*b*, except that the load current at $t = l/\nu$ will be $I_L = V^+/Z_0 = V_0/R_L$.

We next consider a more general case, in which the load of Figure 10.19*a* is again a resistor but is *not matched* to the line ($R_L \neq Z_0$). Reflections will thus occur at the load, complicating the problem. At $t = 0$, the switch is closed as before and a voltage wave, $V_1^+ = V_0$, propagates to the right. Upon reaching the load, however, the wave will now reflect, producing a back-propagating wave, V_1^-. The relation between V_1^- and V_1^+ is through the reflection coefficient at the load:

$$\frac{V_1^-}{V_1^+} = \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \tag{115}$$

As V_1^- propagates back toward the battery, it leaves behind its leading edge a total voltage of $V_1^+ + V_1^-$. Voltage V_1^+ exists everywhere ahead of the V_1^- wave until it reaches the battery, whereupon the entire line now is charged to voltage $V_1^+ + V_1^-$. At the battery, the V_1^- wave reflects to produce a new forward wave, V_2^+. The ratio of V_2^+ and V_1^- is found through the reflection coefficient at the battery:

$$\frac{V_2^+}{V_1^-} = \Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1 \tag{116}$$

where the impedance at the generator end, Z_g, is that of the battery, or zero.

V_2^+ (equal to $-V_1^-$) now propagates to the load, where it reflects to produce backward wave $V_2^- = \Gamma_L V_2^+$. This wave then returns to the battery, where it reflects with $\Gamma_g = -1$, and the process repeats. Note that with each round trip the wave voltage is reduced in magnitude because $|\Gamma_L| < 1$. Because of this the propagating wave voltages will eventually approach zero, and steady state is reached.

The voltage across the load resistor can be found at any given time by summing the voltage waves that have reached the load and have reflected from it up to that time. After many round trips, the load voltage will be, in general,

$$\begin{aligned} V_L &= V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \cdots \\ &= V_1^+ \left(1 + \Gamma_L + \Gamma_g \Gamma_L + \Gamma_g \Gamma_L^2 + \Gamma_g^2 \Gamma_L^2 + \Gamma_g^2 \Gamma_L^3 + \cdots\right) \end{aligned}$$

With a simple factoring operation, the preceding equation becomes

$$V_L = V_1^+ (1 + \Gamma_L) \left(1 + \Gamma_g \Gamma_L + \Gamma_g^2 \Gamma_L^2 + \cdots\right) \tag{117}$$

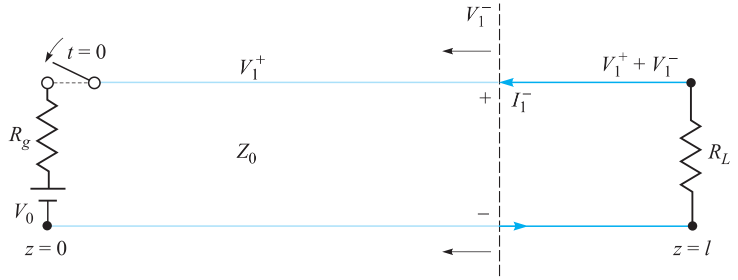


Figure 10.20 With series resistance at the battery location, voltage division occurs when the switch is closed, such that $V_0 = V_{rg} + V_1^+$. Shown is the first reflected wave, which leaves voltage $V_1^+ + V_1^-$ behind its leading edge. Associated with the wave is current I_1^-, which is $-V_1^-/Z_0$. Counterclockwise current is treated as negative and will occur when V_1^- is positive.

Allowing time to approach infinity, the second term in parentheses in (117) becomes the power series expansion for the expression $1/(1 - \Gamma_g \Gamma_L)$. Thus, in steady state we obtain

$$V_L = V_1^+ \left(\frac{1 + \Gamma_L}{1 - \Gamma_g \Gamma_L} \right) \tag{118}$$

In our present example, $V_1^+ = V_0$ and $\Gamma_g = -1$. Substituting these into (118), we find the expected result in steady state: $V_L = V_0$.

A more general situation would involve a nonzero impedance at the battery location, as shown in Figure 10.20. In this case, a resistor of value R_g is positioned in series with the battery. When the switch is closed, the battery voltage appears across the series combination of R_g and the line characteristic impedance, Z_0. The value of the initial voltage wave, V_1^+, is thus found through simple voltage division, or

$$\boxed{V_1^+ = \frac{V_0 Z_0}{R_g + Z_0}} \tag{119}$$

With this initial value, the sequence of reflections and the development of the voltage across the load occurs in the same manner as determined by (117), with the steady-state value determined by (118). The value of the reflection coefficient at the generator end, determined by (116), is $\Gamma_g = (R_g - Z_0)/(R_g + Z_0)$.

A useful way of keeping track of the voltage at any point in the line is through a *voltage reflection diagram*. Such a diagram for the line of Figure 10.20 is shown in Figure 10.21*a*. It is a two-dimensional plot in which position on the line, z, is shown on the horizontal axis. Time is plotted on the vertical axis and is conveniently expressed as it relates to position and velocity through $t = z/\nu$. A vertical line, located at $z = l$, is drawn, which, together with the ordinate, defines the z axis boundaries of the transmission line. With the switch located at the battery position, the initial voltage wave, V_1^+, starts at the origin, or lower-left corner of the diagram ($z = t = 0$). The location of the leading edge of V_1^+ as a function of time is shown as the diagonal line

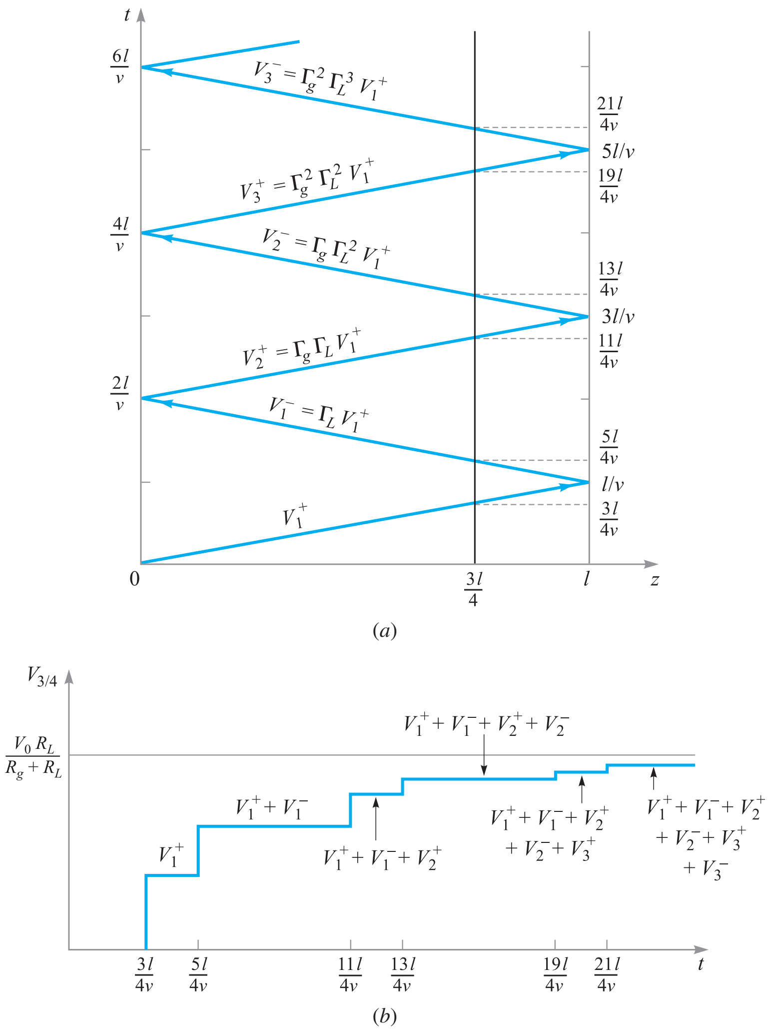


Figure 10.21 (a) Voltage reflection diagram for the line of Figure 10.20. A reference line, drawn at $z = 3l/4$, is used to evaluate the voltage at that position as a function of time. (b) The line voltage at $z = 3l/4$ as determined from the reflection diagram of (a). Note that the voltage approaches the expected $V_0 R_L/(R_g + R_L)$ as time approaches infinity.

that joins the origin to the point along the right-hand vertical line that corresponds to time $t = l/v$ (the one-way transit time). From there (the load location), the position of the leading edge of the reflected wave, V_1^-, is shown as a "reflected" line that joins the $t = l/v$ point on the right boundary to the $t = 2l/v$ point on the ordinate. From there (at the battery location), the wave reflects again, forming V_2^+, shown as a line parallel to that for V_1^+. Subsequent reflected waves are shown, and their values are labeled.

The voltage as a function of time at a given position in the line can now be determined by adding the voltages in the waves as they intersect a vertical line drawn at the desired location. This addition is performed starting at the bottom of the diagram ($t = 0$) and progressing upward (in time). Whenever a voltage wave crosses the vertical line, its value is added to the total at that time. For example, the voltage at a location three-fourths the distance from the battery to the load is plotted in Figure 10.21*b*. To obtain this plot, the line $z = (3/4)l$ is drawn on the diagram. Whenever a wave crosses this line, the voltage in the wave is added to the voltage that has accumulated at $z = (3/4)l$ over all earlier times. This general procedure enables one to easily determine the voltage at any specific time and location. In doing so, the terms in (117) that have occurred up to the chosen time are being added, but with information on the time at which each term appears.

Line current can be found in a similar way through a *current reflection diagram*. It is easiest to construct the current diagram directly from the voltage diagram by determining a value for current that is associated with each voltage wave. In dealing with current, it is important to keep track of the *sign* of the current because it relates to the voltage waves and their polarities. Referring to Figures 10.19*a* and 10.20, we use the convention in which current associated with a *forward-z* traveling voltage wave of positive polarity is positive. This would result in current that flows in the clockwise direction, as shown in Figure 10.19*a*. Current associated with a *backward-z* traveling voltage wave of positive polarity (thus flowing counterclockwise) is negative. Such a case is illustrated in Figure 10.20. In our two-dimensional transmission-line drawings, we assign positive polarity to voltage waves propagating in *either* direction if the upper conductor carries a positive charge and the lower conductor a negative charge. In Figures 10.19*a* and 10.20, both voltage waves are of positive polarity, so their associated currents will be net positive for the forward wave and net negative for the backward wave. In general, we write

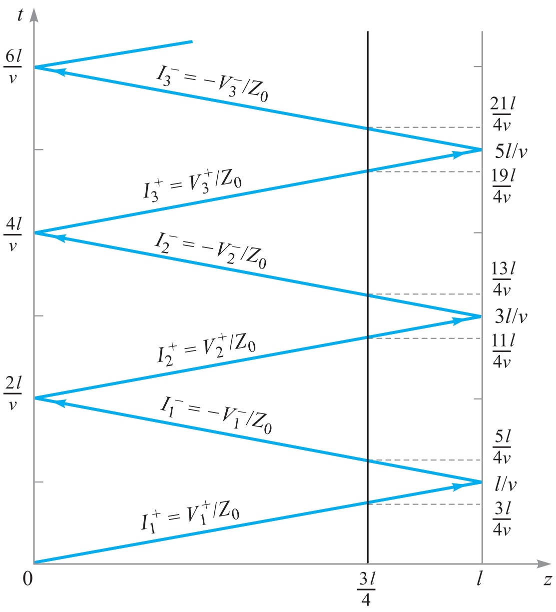
$$\boxed{I^+ = \frac{V^+}{Z_0}} \tag{120}$$

and

$$\boxed{I^- = -\frac{V^-}{Z_0}} \tag{121}$$

Finding the current associated with a backward-propagating voltage wave immediately requires a minus sign, as (121) indicates.

Figure 10.22*a* shows the current reflection diagram that is derived from the voltage diagram of Figure 10.21*a*. Note that the current values are labeled in terms of the voltage values, with the appropriate sign added as per (120) and (121). Once the current diagram is constructed, current at a given location and time can be found in exactly the same manner as voltage is found using the voltage diagram. Figure 10.22*b* shows the current as a function of time at the $z = (3/4)l$ position, determined by summing the current wave values as they cross the vertical line drawn at that location.



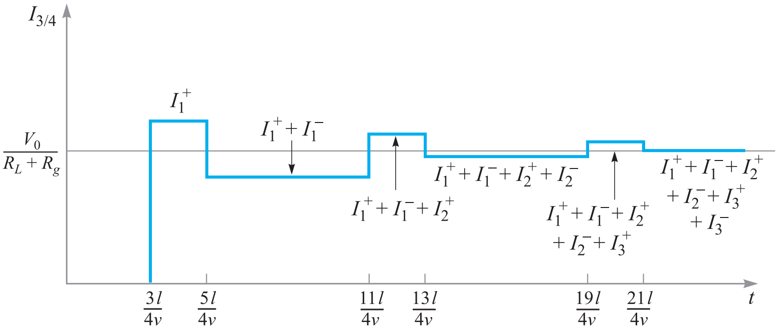
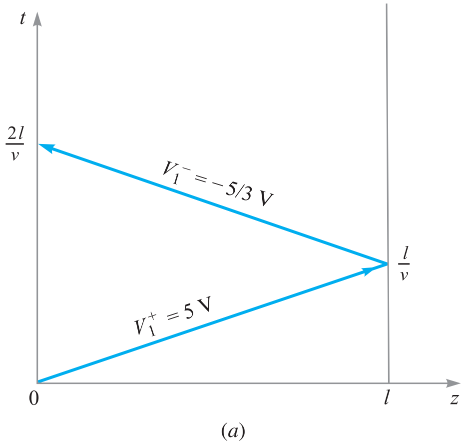


Figure 10.22 (*a*) Current reflection diagram for the line of Figure 10.20 as obtained from the voltage diagram of Figure 10.21*a*. (*b*) Current at the $z = 3l/4$ position as determined from the current reflection diagram, showing the expected steady-state value of $V_0/(R_L + R_g)$.

EXAMPLE 10.11

In Figure 10.20, $R_g = Z_0 = 50\,\Omega$, $R_L = 25\,\Omega$, and the battery voltage is $V_0 = 10$ V. The switch is closed at time $t = 0$. Determine the voltage at the load resistor and the current in the battery as functions of time.

Solution. Voltage and current reflection diagrams are shown in Figure 10.23*a* and *b*. At the moment the switch is closed, half the battery voltage appears across the



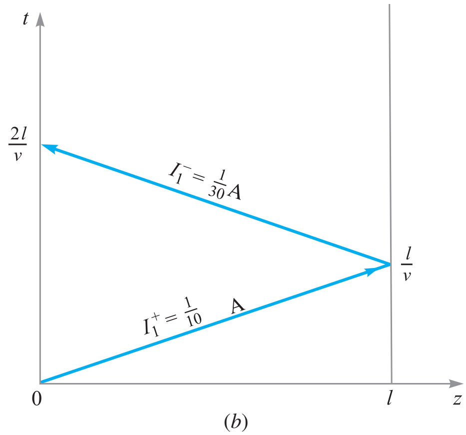


Figure 10.23 Voltage (*a*) and current (*b*) reflection diagrams for Example 10.11.

50-Ω resistor, with the other half comprising the initial voltage wave. Thus $V_1^+ = (1/2)V_0 = 5$ V. The wave reaches the 25-Ω load, where it reflects with reflection coefficient

$$\Gamma_L = \frac{25 - 50}{25 + 50} = -\frac{1}{3}$$

So $V_1^- = -(1/3)V_1^+ = -5/3$ V. This wave returns to the battery, where it encounters reflection coefficient $\Gamma_g = 0$. Thus, no further waves appear; steady state is reached.

Once the voltage wave values are known, the current reflection diagram can be constructed. The values for the two current waves are

$$I_1^+ = \frac{V_1^+}{Z_0} = \frac{5}{50} = \frac{1}{10} \text{ A}$$

and

$$I_1^- = -\frac{V_1^-}{Z_0} = -\left(-\frac{5}{3}\right)\left(\frac{1}{50}\right) = \frac{1}{30} \text{ A}$$

Note that no attempt is made here to derive I_1^- from I_1^+. They are both obtained independently from their respective voltages.

The voltage at the load as a function of time is now found by summing the voltages along the vertical line at the load position. The resulting plot is shown in Figure 10.24*a*. Current in the battery is found by summing the currents along the vertical axis, with the resulting plot shown as Figure 10.24*b*. Note that in steady state, we treat the circuit as lumped, with the battery in series with the 50- and 25-Ω resistors. Therefore, we expect to see a steady-state current through the battery (and everywhere else) of

$$I_B(\text{steady state}) = \frac{10}{50+25} = \frac{1}{7.5} \text{ A}$$

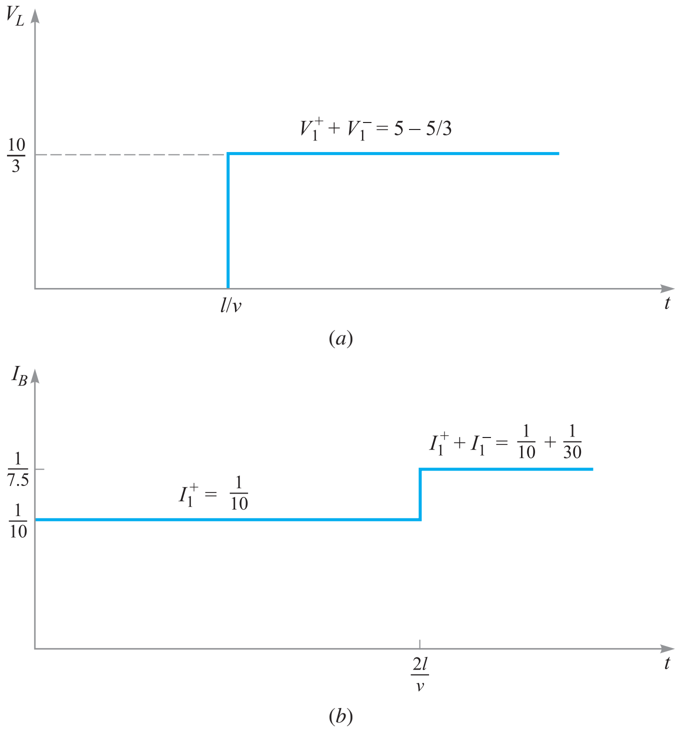


Figure 10.24 Voltage across the load (*a*) and current in the battery (*b*) as determined from the reflection diagrams of Figure 10.23 (Example 10.11).

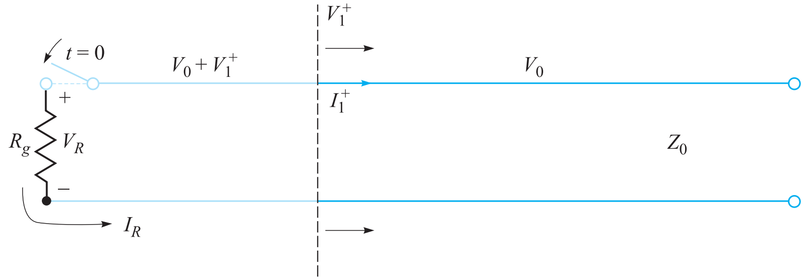


Figure 10.25 In an initially charged line, closing the switch as shown initiates a voltage wave of opposite polarity to that of the initial voltage. The wave thus depletes the line voltage and will fully discharge the line in one round trip if $R_g = Z_0$.

This value is also found from the current reflection diagram for $t > 2l/\nu$. Similarly, the steady-state load voltage should be

$$V_L(\text{steady state}) = V_0 \frac{R_L}{R_g + R_L} = \frac{(10)(25)}{50 + 25} = \frac{10}{3}\ \text{V}$$

which is found also from the voltage reflection diagram for $t > l/\nu$.

Another type of transient problem involves lines that are *initially charged*. In these cases, the manner in which the line discharges through a load is of interest. Consider the situation shown in Figure 10.25, in which a charged line of characteristic impedance Z_0 is discharged through a resistor of value R_g when a switch at the resistor location is closed.[5] We consider the resistor at the $z = 0$ location; the other end of the line is open (as would be necessary) and is located at $z = l$.

When the switch is closed, current I_R begins to flow through the resistor, and the line discharge process begins. This current does not immediately flow everywhere in the transmission line but begins at the resistor and establishes its presence at more distant parts of the line as time progresses. By analogy, consider a long line of automobiles at a red light. When the light turns green, the cars at the front move through the intersection first, followed successively by those further toward the rear. The point that divides cars in motion and those standing still is, in fact, a wave that propagates toward the back of the line. In the transmission line, the flow of charge progresses in a similar way. A voltage wave, V_1^+, is initiated and propagates to the right. To the left of its leading edge, charge is in motion; to the right of the leading edge, charge is stationary and carries its original density. Accompanying the charge in motion to the left of V_1^+ is a drop in the charge density as the discharge process occurs, and so the line voltage to the left of V_1^+ is partially reduced. This voltage will be given by the sum of the initial voltage, V_0, and V_1^+, which means that V_1^+ must

[5] Even though this is a load resistor, we will call it R_g because it is located at the front (generator) end of the line.

in fact be negative (or of opposite sign to V_0). The line discharge process is analyzed by keeping track of V_1^+ as it propagates and undergoes multiple reflections at the two ends. Voltage and current reflection diagrams are used for this purpose in much the same way as before.

Referring to Figure 10.25, we see that for positive V_0 the current flowing through the resistor will be counterclockwise and hence negative. We also know that continuity requires that the resistor current be equal to the current associated with the voltage wave, or

$$I_R = I_1^+ = \frac{V_1^+}{Z_0}$$

Now the resistor voltage will be

$$V_R = V_0 + V_1^+ = -I_R R_g = -I_1^+ R_g = -\frac{V_1^+}{Z_0} R_g$$

where the minus signs arise from the fact that V_R (having positive polarity) is produced by the negative current, I_R. We solve for V_1^+ to obtain

$$\boxed{V_1^+ = \frac{-V_0 Z_0}{Z_0 + R_g}} \qquad (122)$$

Having found V_1^+, we can set up the voltage and current reflection diagrams. The diagram for voltage is shown in Figure 10.26. Note that the initial condition of voltage V_0 everywhere on the line is accounted for by assigning voltage V_0 to the horizontal axis of the voltage diagram. The diagram is otherwise drawn as before, but with $\Gamma_L = 1$ (at the open-circuited load end). Variations in how the line discharges thus depend on the resistor value at the switch end, R_g, which determines the reflection coefficient, Γ_g, at that location. The current reflection diagram is derived from the voltage diagram in the usual way. There is no initial current to consider.

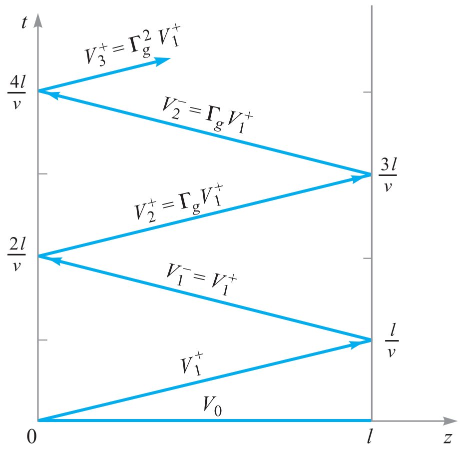


Figure 10.26 Voltage reflection diagram for the charged line of Figure 10.25, showing the initial condition of V_0 everywhere on the line at $t = 0$.

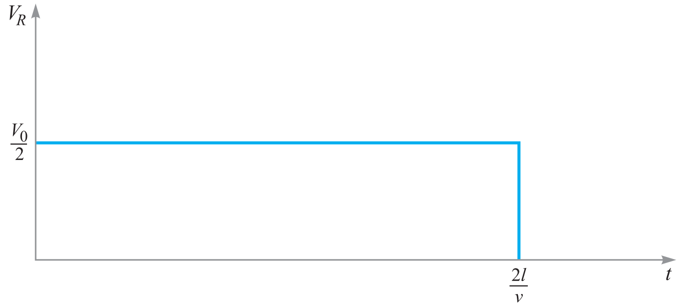


Figure 10.27 Voltage across the resistor as a function of time, as determined from the reflection diagram of Figure 10.26, in which $R_g = Z_0$ ($\Gamma = 0$).

A special case of practical importance is that in which the resistor is matched to the line, or $R_g = Z_0$. In this case, Eq. (122) gives $V_1^+ = -V_0/2$. The line fully discharges in one round trip of V_1^+ and produces a voltage across the resistor of value $V_R = V_0/2$, which persists for time $T = 2l/v$. The resistor voltage as a function of time is shown in Figure 10.27. The transmission line in this application is known as a *pulse-forming line*; pulses that are generated in this way are well formed and of low noise, provided the switch is sufficiently fast. Commercial units are available that are capable of generating high-voltage pulses of widths on the order of a few nanoseconds, using thyratron-based switches.

When the resistor is not matched to the line, full discharge still occurs, but does so over several reflections, leading to a complicated pulse shape.

EXAMPLE 10.12

In the charged line of Figure 10.25, the characteristic impedance is $Z_0 = 100\ \Omega$, and $R_g = 100/3\ \Omega$. The line is charged to an initial voltage, $V_0 = 160$ V, and the switch is closed at time $t = 0$. Determine and plot the voltage and current through the resistor for time $0 < t < 8l/v$ (four round trips).

Solution. With the given values of R_g and Z_0, Eq. (47) gives $\Gamma_g = -1/2$. Then, with $\Gamma_L = 1$, and using (122), we find

$$
\begin{aligned}
V_1^+ &= V_1^- = -3/4V_0 = -120\ \mathrm{V}\\
V_2^+ &= V_2^- = \Gamma_g V_1^- = +60\ \mathrm{V}\\
V_3^+ &= V_3^- = \Gamma_g V_2^- = -30\ \mathrm{V}\\
V_4^+ &= V_4^- = \Gamma_g V_3^- = +15\ \mathrm{V}
\end{aligned}
$$

Using these values on the voltage reflection diagram, we evaluate the voltage in time at the resistor location by moving up the left-hand vertical axis, adding voltages as we progress, and beginning with $V_0 + V_1^+$ at $t = 0$. Note that when we add voltages along the vertical axis, we are encountering the intersection points between incident and reflected waves, which occur (in time) at each integer multiple of $2l/v$. So, when moving up the axis, we add the voltages of *both* waves to our total at each occurrence. The voltage within each time interval is thus:

$$
\begin{aligned}
V_R &= V_0 + V_1^+ = 40\text{ V} && (0 < t < 2l/v)\\
&= V_0 + V_1^+ + V_1^- + V_2^+ = -20\text{ V} && (2l/v < t < 4l/v)\\
&= V_0 + V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ = 10\text{ V} && (4l/v < t < 6l/v)\\
&= V_0 + V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + V_4^+ = -5\text{ V} && (6l/v < t < 8l/v)
\end{aligned}
$$

The resulting voltage plot over the desired time range is shown in Figure 10.28*a*.

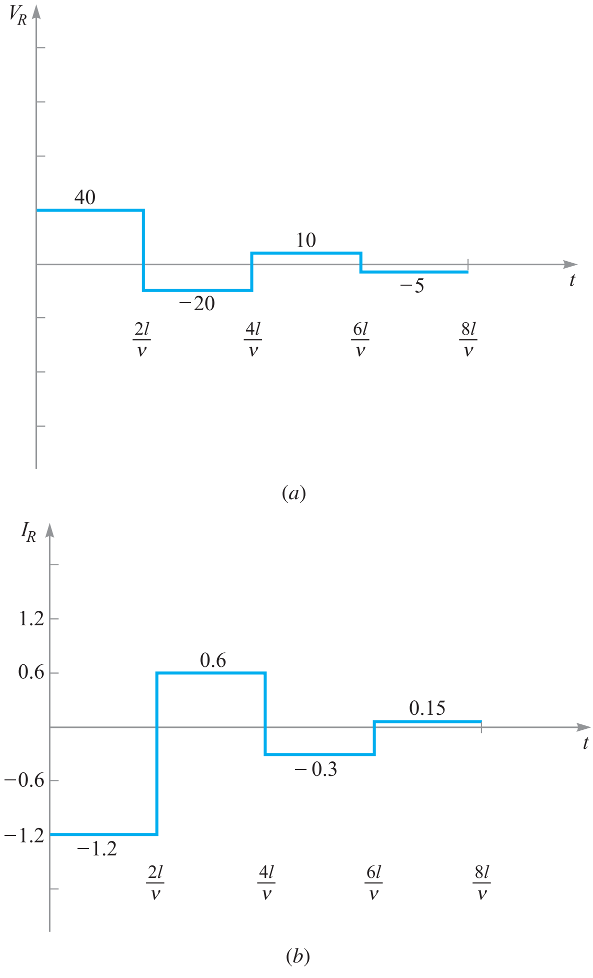


Figure 10.28 Resistor voltage (*a*) and current (*b*) as functions of time for the line of Figure 10.25, with values as specified in Example 10.12.

The current through the resistor is most easily obtained by dividing the voltages in Figure 10.28*a* by $-R_g$. As a demonstration, we can also use the current diagram of Figure 10.22*a* to obtain this result. Using (120) and (121), we evaluate the current waves as follows:

$$I_1^+ = V_1^+/Z_0 = -1.2\,\text{A}$$
$$I_1^- = -V_1^-/Z_0 = +1.2\,\text{A}$$
$$I_2^+ = -I_2^- = V_2^+/Z_0 = +0.6\,\text{A}$$
$$I_3^+ = -I_3^- = V_3^+/Z_0 = -0.30\,\text{A}$$
$$I_4^+ = -I_4^- = V_4^+/Z_0 = +0.15\,\text{A}$$

Using these values on the current reflection diagram, Figure 10.22*a*, we add up currents in the resistor in time by moving up the left-hand axis, as we did with the voltage diagram. The result is shown in Figure 10.28*b*. As a further check to the correctness of our diagram construction, we note that current at the open end of the line ($Z = l$) must always be zero. Therefore, summing currents up the right-hand axis must give a zero result for all time. The reader is encouraged to verify this.

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CHAPTER 10 PROBLEMS

WW Quizzes

10.1 The parameters of a certain transmission line operating at $\omega = 6 \times 10^8$ rad/s are $L = 0.35\ \mu$H/m, $C = 40$ pF/m, $G = 75\ \mu$S/m, and $R = 17\ \Omega$/m. Find γ, α, β, λ, and Z_0.

10.2 A sinusoidal wave on a transmission line is specified by voltage and current in phasor form:

$$V_s(z) = V_0\, e^{\alpha z}\, e^{j\beta z} \quad \text{and} \quad I_s(z) = I_0\, e^{\alpha z}\, e^{j\beta z}\, e^{j\phi}$$

where V_0 and I_0 are both real. (*a*) In which direction does this wave propagate and why? (*b*) It is found that $\alpha = 0$, $Z_0 = 50\,\Omega$, and the wave velocity is $v_p = 2.5 \times 10^8$ m/s, with $\omega = 10^8\ \text{s}^{-1}$. Evaluate R, G, L, C, λ, and ϕ.

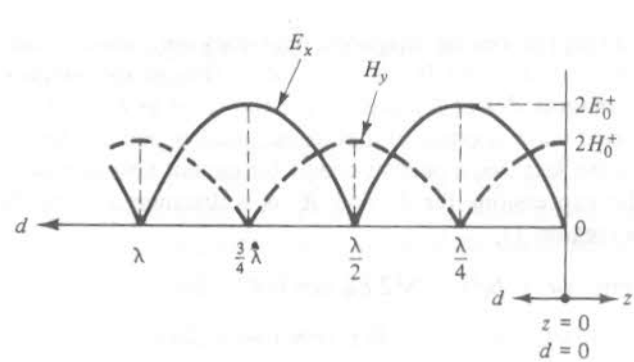

Figure 2–24 The rms values of $\mathscr{E}_x$ and $\mathscr{H}_y$ as a function of position for the case of a conducting surface located at $d = 0$.

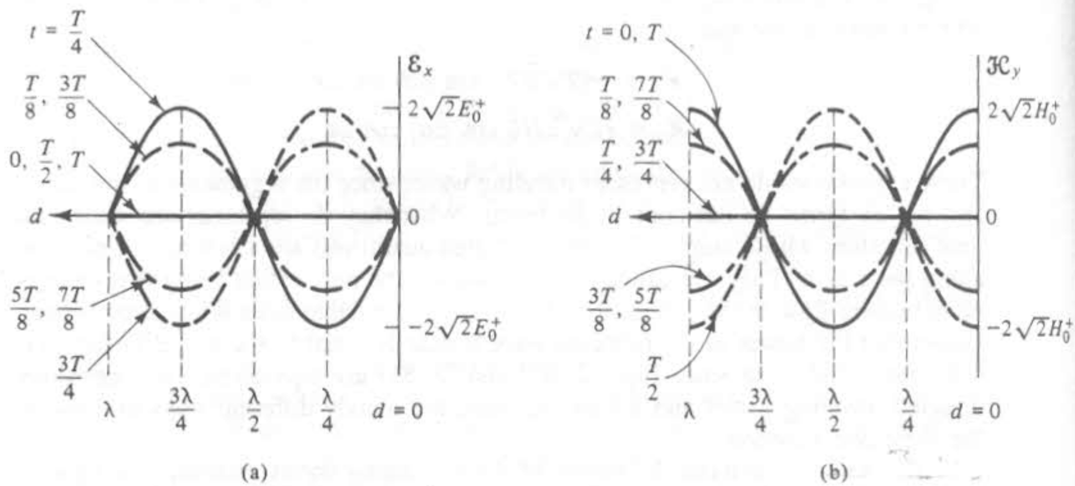

Figure 2–25 A plot of the instantaneous values of $\mathscr{E}_x$ and $\mathscr{H}_y$ as a function of position and time. The conducting surface is located at $d = 0$.

analogous to the boundary condition $\mathscr{E}_x = 0$ at the conducting surface. For any fixed value of d, both $\mathscr{E}_x$ and $\mathscr{H}_y$ are sinusoidal time functions having a period T. Observe that between a successive pair of nulls, all the sinusoidal time functions of the standing wave are in phase. That is, they all reach their maximum (and minimum) values at the same time. This contrasts with a pure traveling wave ($\alpha = 0$) wherein phase is a function of position but amplitude is not.

The analysis described here could have been carried out using phasor notation. The case of reflections at a dielectric boundary is analyzed using this approach.

Oblique incidence. Let us briefly consider the case in which the wave impinges upon the conducting surface at some angle θ_i. This angle, known as the *angle of incidence*, represents the angular distance between the propagation axis of the incident wave and a line normal to the reflecting surface. As in the case of optics, the angle of reflection θ_r equals the angle of incidence θ_i. This is true for any type polarization. The power density of the reflected wave equals that of the incident wave since a perfect conductor cannot absorb power. The proof of these results is given in Ref. 2–8.

Solution:

(a) $\bar{Z}_L = \dfrac{150 + j90}{75} = 2 + j1.2$ and from Eq. (3–45),

$$\Gamma_L = \frac{1 + j1.2}{3 + j1.2} = 0.48\underline{/28.4°}$$

At 600 MHz, $\lambda_0 = 50$ cm and since both μ_R and ϵ_R are unity, $\lambda = 50$ cm. Thus,

$$\beta l = (2\pi/50)15 = 0.6\pi \text{ rad or } 108°$$

and from Eq. (3–43) with $\alpha = 0$,

$$\Gamma_{\text{in}} = 0.48\underline{/28.4 - 2(108)} = 0.48\underline{/-187.6°}$$

From Eq. (3–52),

$$\text{SWR} = \frac{1 + 0.48}{1 - 0.48} = 2.85$$

(b) At the input ($d = l$), Eq. (3–38) reduces to $\mathbf{V}_{\text{in}} = \mathbf{V}_0^+(1 + \Gamma_{\text{in}})$. With $Z_G = 0$, $\mathbf{V}_{\text{in}} = \mathbf{V}_G$, $\mathbf{V}_0^+ = \mathbf{V}_G/(1 + \Gamma_{\text{in}})$ and therefore

$$\mathbf{V}_0^+ = \frac{10\underline{/0}}{0.524 + j0.06} = 19\underline{/-6.5°} \quad \text{V}$$

The maximum rms voltage is given by

$$V_{\max} = \{1 + |\Gamma|\}V^+$$

where for a lossless line $V^+ = V_0^+$ and $|\Gamma| = |\Gamma_{\text{in}}| = |\Gamma_L|$. Therefore,

$$V_{\max} = (1.48)(19) = 28.12 \text{ V}$$

(c) For $\alpha = 2.0$ dB/m $= 0.23$ Np/m and $l = \lambda = 0.5$ m, $\alpha l = 0.115$ Np. From Eq. (3–43),

$$|\Gamma_{\text{in}}| = |\Gamma_L|e^{-2\alpha l} = 0.48e^{-0.23} = 0.381$$

In this example, $\mathbf{V}_G = \mathbf{V}_{\text{in}}$ since $Z_G = 0$. Generally, $Z_G \neq 0$ which makes solving for $\mathbf{V}_{\text{in}}$ and $\mathbf{V}_L$ a little more involved. This case is treated in part *c* of this section.

3–4b Some Special Cases of Terminated Lines

Let us now study four special cases of load impedances terminating a lossless transmission line, namely, $Z_L = 0$ (short circuit), $Z_L = \infty$ (open circuit), $Z_L = jX$ (pure reactance) and $Z_L = R_L$ (pure resistance).

Line terminated in a short circuit. In this case $Z_L = 0$ and therefore $\Gamma_L = -1$ and the SWR is infinite. The standing wave patterns for rms voltage and current are the same as those in Fig. 2–24, where V replaced E_x and I replaces H_y. At $d = 0$ (the load end), $V = 0$, as it must be at a perfect short. Other voltage nulls occur at multiples of a half wavelength. Current, on the other hand, is a maximum at the load, its rms value being twice that of the incident current I_0^+. Because the voltage pattern is displaced with respect to the current pattern, the ratio of V to I is a function of d. For instance, at $d = \lambda/4$, V is a maximum and I is zero, which means that the impedance at that point is infinite. Thus, a quarter wavelength of line transforms the short circuit into an open circuit! This impedance transforming property of a transmission line is discussed in Secs. 3–5 and 3–6.

Line terminated in an open circuit. For an open-circuited line, $Y_L = 0$ and therefore $\Gamma_L = +1$ and the SWR is infinite. The voltage and current standing wave patterns are shown in Fig. 3–11. At the open circuit, $I = 0$, by definition, while the rms voltage is a maximum and equal to $2V_0^+$. Note that a quarter wavelength from the open, $V = 0$ and therefore the impedance at that point is zero, a short circuit. As always on a lossless line, a particular rms value of voltage or current reoccurs every half wavelength.

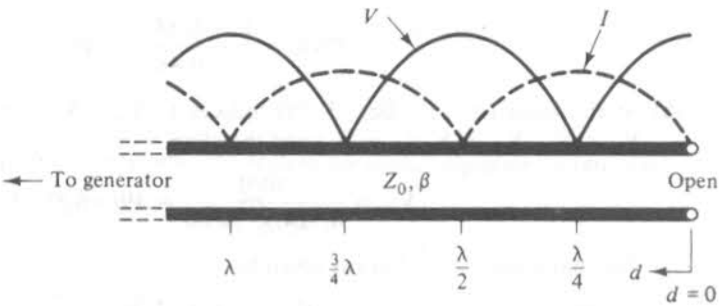

Figure 3–11 Standing-wave patterns on an open-circuited transmission line. ($\alpha = 0$.)

Reactively terminated lines. For a transmission line terminated in a purely reactive circuit, $Z_L = jX$ and therefore

$$\Gamma_L = \frac{j\bar{X} - 1}{j\bar{X} + 1} = 1\underline{/\pi - 2 \arctan \bar{X}} \tag{3–54}$$

where $\bar{X} \equiv X/Z_0$ is the normalized load reactance. For example, if $\bar{X} = +1$, $\Gamma_L = 1\underline{/90^\circ}$, while for $\bar{X} = -1$, $\Gamma_L = 1\underline{/-90^\circ}$.

Note that for a pure reactance, $|\Gamma_L| = 1$ and is independent of the value of $\bar{X}$. Thus the SWR is infinite. The physical interpretation is that in the steady state, a pure reactance cannot absorb power and hence all of the incident wave must be reflected. This same argument applies to the short and open-circuited cases. Consequently, $|\Gamma_L| < 1$ only when Z_L has a resistive component.

The standing wave patterns for $Z_L = jZ_0$ ($\bar{X} = +1$) are shown in Fig. 3–12. It is left to the reader to show that the first voltage null occurs at $d = 3\lambda/8$ (Prob. 3–17). This can be verified analytically or with the aid of a phasor diagram. The use of counterrotating phasors to determine the standing wave pattern was discussed in Sec. 2–7b and Fig. 2–28.

Resistively terminated lines. In many applications, the transmission line is terminated in a purely resistive network. That is, $Z_L = R_L$ and therefore $\Gamma_L = (R_L - Z_0)/(R_L + Z_0)$. Since R_L and Z_0 are real, Γ_L must be real. When $R_L = Z_0$, no

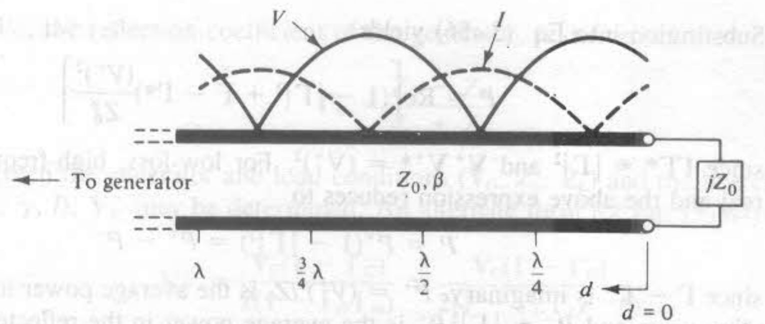

Figure 3–12 Standing-wave patterns for a reactively terminated line with $Z_L = +jZ_0$. ($\alpha = 0$.)

reflections occur. For $R_L > Z_0$, Γ_L is positive and a voltage maximum (current minimum) exists at the load. In this case,

$$\text{SWR} = \frac{R_L}{Z_0} \quad \text{since} \quad |\Gamma_L| = \frac{R_L - Z_0}{R_L + Z_0}$$

For $R_L < Z_0$, Γ_L is negative and a voltage minimum (current maximum) exists at the load. In this case,

$$\text{SWR} = \frac{Z_0}{R_L} \quad \text{since} \quad |\Gamma_L| = \frac{Z_0 - R_L}{Z_0 + R_L}$$

Thus if Z_L is *purely resistive*,

$$\text{SWR} = \frac{R_L}{Z_0} \quad \text{or} \quad \frac{Z_0}{R_L} \quad \text{whichever is greater than unity.} \tag{3–55}$$

For any finite value of R_L, SWR is finite and $|\Gamma_L| < 1$. This means that some of the incident power is absorbed by the load.

3–4c Power Flow Along Terminated Lines

The general problem of power flow along a terminated transmission line will now be discussed. Figure 3–10 shows a line of length l driven by an ac source and terminated in a load Z_L. $\mathbf{V}_G$ is the open-circuit voltage of the generator and Z_G is its internal impedance. From ac theory, the average power flow into an impedance Z is given by

$$P = \text{Re}\,(\mathbf{VI}^*) = VI\cos\theta_{\text{pf}} \tag{3–56}$$

where V and I are rms values, θ_{pf} is the power-factor angle and * denotes the complex conjugate. This equation also applies to the average power flow at any point along a transmission line, the direction of flow being from the generator toward the load. The voltage and current on the line are given by Eqs. (3–14) and (3–16). With $\mathbf{V}^+ = \mathbf{I}^+ Z_0$ and $\mathbf{V}^- = \Gamma\mathbf{V}^+$, they may be rewritten as

$$\mathbf{V} = \mathbf{V}^+(1 + \Gamma) \quad \text{and} \quad \mathbf{I} = \frac{\mathbf{V}^+}{Z_0}(1 - \Gamma) \tag{3–57}$$

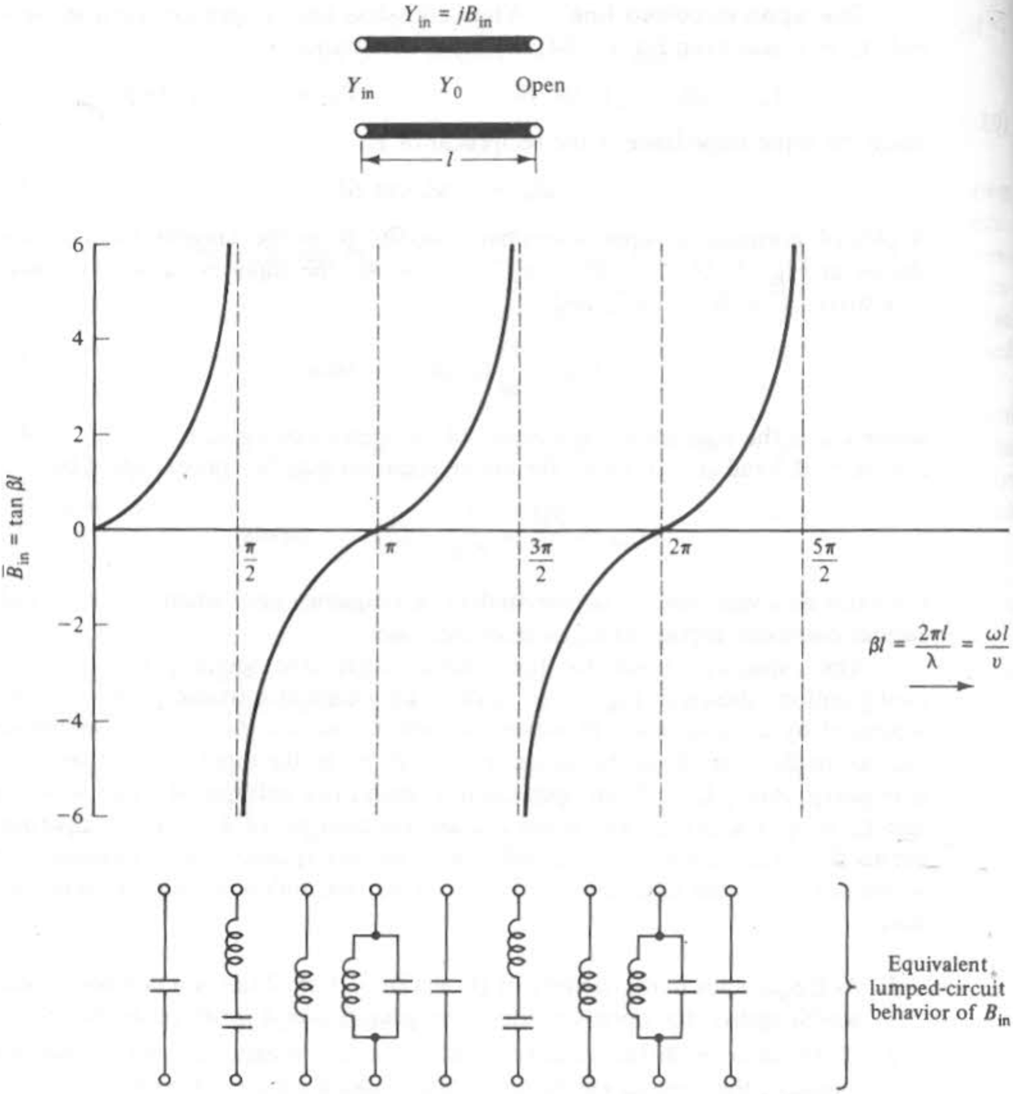

Figure 3–16 Normalized input susceptance versus βl for an open-circuited, lossless transmission line.

load end open circuited (denoted by Z_{oc}). Z_{sc} and Z_{oc} are given by Eqs. (3–87) and (3–91). Solving for Z_0 and βl yields

$$Z_0 = \sqrt{Z_{sc}\, Z_{oc}} \qquad \text{and} \qquad \beta l = \arctan \sqrt{-\frac{Z_{sc}}{Z_{oc}}} \tag{3–94}$$

For a lossless line Z_{sc} and Z_{oc} are reactive and of opposite sign. Since impedance repeats every half wavelength there are an infinite number of solutions for βl. In the absence of additional information, the primary solution ($0 \leq \beta l \leq \pi$) should be

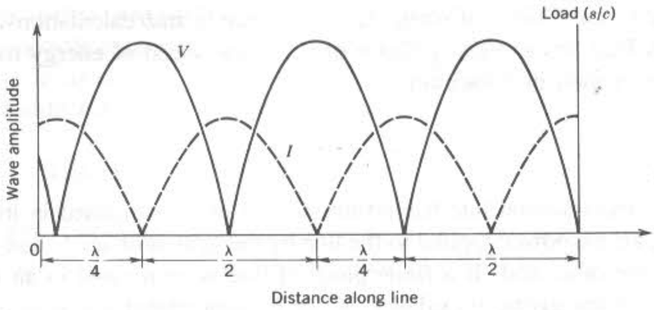

FIGURE 7-6 Lossless line terminated in a short circuit.

Consider only the forward traveling voltage and current waves for the moment. At the load, the voltage will be zero and the current a maximum because the load is a short circuit. Note that the current has a finite value since the line has an impedance. At that instant of time, the same conditions also apply at a point exactly one wavelength on the generator side of the load, and so on. The current at the load is always a maximum, although the size of this maximum varies periodically with time, since the applied wave is sinusoidal.

The reflection that takes place at the short circuit affects both voltage and current. The current now starts traveling back to the generator, unchanged in phase (series circuit theory), but *the voltage is reflected with a* 180° *phase reversal.* At a point exactly a quarter-wavelength from the load, the current is *permanently* zero (as shown in Figure 7-6). This is because the forward and reflected current waves are exactly 180° out of phase, as the reflected wave has had to travel a distance of $\lambda/4 + \lambda/4 = \lambda/2$ farther than the forward wave. The two cancel, and a current node is established. The voltage wave has also had to travel an extra distance of $\lambda/2$, but since it underwent a 180° phase reversal on reflection, its total phase change is 360°. Reinforcement will take place, resulting in a voltage antinode at precisely the same point as the current node.

A half-wavelength from the load is a point at which there will be a voltage zero and a current maximum. This arises because the forward and reverse current waves are now in phase (current has had to travel a total distance of one wavelength to return to this point). Simultaneously the voltage waves will cancel, because the 180° phase reversal on reflection must be added to the extra distance the reflected wave has to travel. All these conditions will repeat at half-wavelength distances, as shown in Figure 7-6. Every time a point is considered that is $\lambda/2$ farther from the load than some previously considered point, the reflected wave has had to travel one whole wavelength farther. Therefore it has the same relation to the forward wave as it had at the first point.

It must be emphasized that this situation is permanent for any given load and is determined by it; such waves are truly *standing* waves. All the nodes are permanently fixed, and the positions of all antinodes are constant. Many of the same conditions apply if the load is an open circuit, except that the first current minimum (and voltage

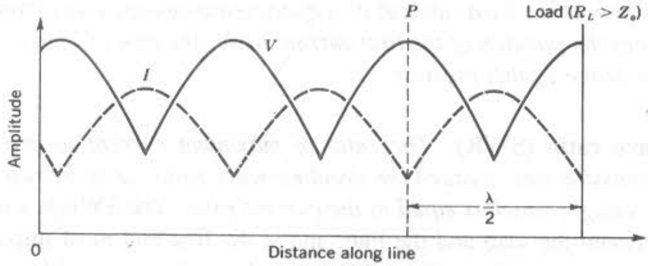

FIGURE 7-7 Lossless line terminated in a pure resistance greater than Z_0 (note that voltage SWR equals current SWR).

Consider a pure resistance connected to a transmission line, such that $R_L \neq Z_0$. Since the voltage and current vary along the line, as shown in Figure 7-7, so will the resistance or impedance. However, conditions do repeat every half-wavelength, as already outlined. The impedance at P will be equal to that of the load, if P is a half-wavelength away from the load and the line is lossless.

7-1.5 Quarter- and Half-Wavelength Lines

Sections of transmission lines that are exactly a quarter-wavelength or a half-wavelength long have important impedance-transforming properties, and are often used for this purpose at radio frequencies. Such lines will now be discussed.

Impedance inversion by quarter-wavelength lines Consider Figure 7-8, which shows a load of impedance Z_L connected to a piece of transmission line of length s and having Z_0 as its characteristic impedance. When the length s is exactly a quarter-wavelength line (or an odd number of quarter-wavelengths) and the line is lossless, then the impedance Z_s, seen when looking toward the load, is given by

$$Z_s = \frac{Z_0^2}{Z_L} \tag{7-10}$$

This relationship is sometimes called *reflective impedance;* i.e., the quarter-wavelength reflects the opposite of its load impedance. Equation (7-10) represents a very important and fundamental relation, which is somewhat too complex to derive here, but whose truth may be indicated as follows. Unless a load is resistive and equal to the characteristic impedance of the line to which it is connected, standing waves of voltage and current are set up along the line, with a node (and antinode) repetition rate

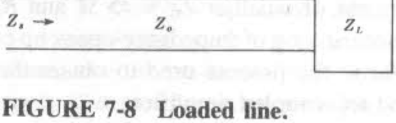

FIGURE 7-8 Loaded line.

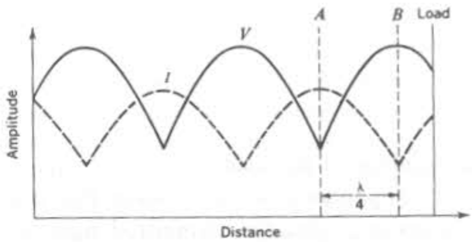

FIGURE 7-9 Standing waves along a mismatched transmission line; impedance inversion.

of $\lambda/2$. This has already been shown and is indicated again in Figure 7-9. Note that here the voltage and current minima are not zero; the load is not a short circuit, and therefore the standing-wave ratio is not infinite. Note also that the current nodes are separated from the voltage nodes by a distance of $\lambda/4$, as before.

It is obvious that at the point A (voltage node, current antinode) the line impedance is low, and at the point B (voltage antinode, current node) it is the reverse, i.e., high. In order to change the impedance at A, it would be necessary to change the SWR on the line. If the SWR were increased, the voltage minimum at A would be lower, and so would be the impedance at A. The size of the voltage maximum at B would be increased, and so would the impedance at B. Thus an increase in Z_B is accompanied by a decrease in Z_A (if A and B are $\lambda/4$ apart). This amounts to saying that *the impedance at A is inversely proportional to the impedance at B*. Equation (7-10) states this relation mathematically and also supplies the proportionality constant; this happens to be the square of the characteristic impedance of the transmission line. The relation holds just as well when the two points are not voltage nodes and antinodes, and a glance at Figure 7-9 shows that it also applies when the distance separating the points is three, five, seven and so on, quarter-wavelengths.

Another interesting property of the quarter-wave line is seen if, in Equation (7-10), the impedances are normalized with respect to Z_0. Dividing both sides by Z_0, we have

$$\frac{Z_s}{Z_0} = \frac{Z_0}{Z_L} \tag{7-11}$$

but

$$\frac{Z_s}{Z_0} = z_s$$

and

$$\frac{Z_L}{Z_0} = z_L$$

whence $Z_0/Z_L = 1/z_L$.

Substituting these results into Equation (8-11) gives

SOLUTION

Just as $z = Z/Z_0$, so $y = Y/Y_0$; this may be very simply checked.
Therefore

$$y = \frac{0.004 - j0.002}{0.0033} = 1.21 - j0.61$$

Hence the normalized susceptance required to cancel the load's normalized susceptance is $+j0.61$. From the chart, the length of line required to give a normalized input admittance of 0.61 when the line is short-circuited is given by

$$\text{Length} = 0.250 + 0.087 = 0.337\lambda$$

Since the line has air as its dielectric, the velocity factor is 1.
Therefore

$$v_c = f\lambda$$

$$\lambda = \frac{v_c}{f} = \frac{300 \times 10^6}{150 \times 10^6} = 2 \text{ m}$$

$$\text{Length} = 0.337\,\lambda = 0.337 \times 200 = 67.4 \text{ cm}$$

7-2.2 Problem Solution

In most cases, the best method of explaining problem solution with the Smith chart is to show how an actual problem of a given type is solved. In other cases, a procedure may be established without prior reference to a specific problem. Both methods of approach will be used here.

Matching of load to line with a quarter-wave transformer

EXAMPLE 7-7 Refer to Figure 7-13. A load $Z_L = (100 - j50)\ \Omega$ is connected to a line whose $Z_0 = 75\ \Omega$. Calculate

(a) The point, nearest to the load, at which a quarter-wave transformer may be inserted to provide correct matching
(b) The Z_0' of the transmission line to be used for the transformer

SOLUTION

(a) Normalize the load impedance with respect to the line; thus $(100 - j50)/75 = 1.33 - j0.67$. Plot this point *(A)* on the Smith chart. Draw a circle whose center lies at the center of the chart, passing through the plotted point. As a check, note that this circle should correspond to an SWR of just under 1.9. Moving toward the generator, i.e., clockwise, find the nearest point at which the line impedance is purely resistive (this is the intersection of the drawn circle with the only straight line on the chart). Around the rim of the chart, measure the distance from the load to this point *(B)*; this distance $= 0.500 - 0.316 = 0.184\ \lambda$. Read off the normalized resistance at *B*, here $r = 0.53$, and convert this normalized resistance into an actual resistance by multiplying by the Z_0 of the line. Here $R = 0.53 \times 75 = 39.8\ \Omega$.

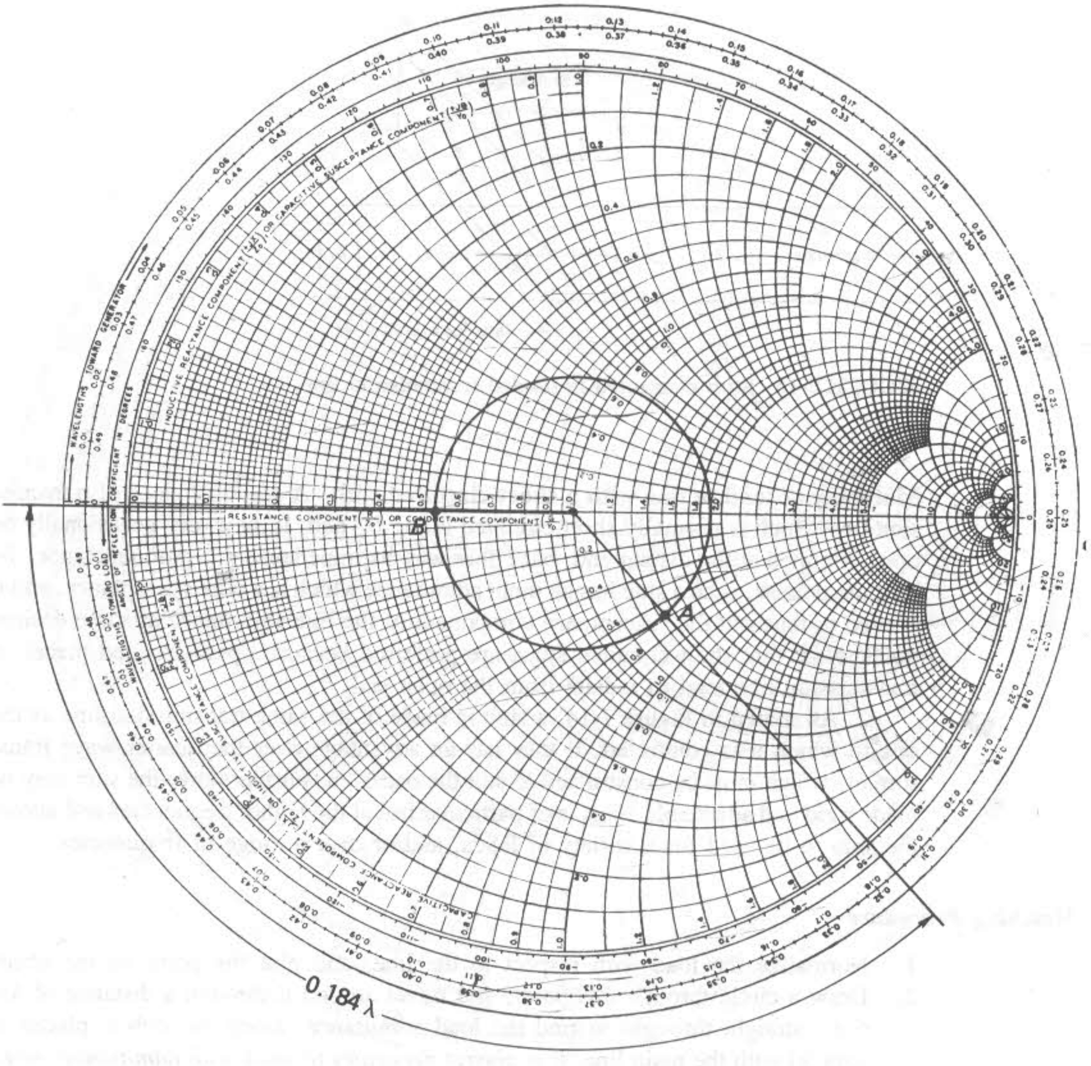

FIGURE 7-13 Smith chart solution of Example 7-7, matching with a quarter-wave transformer.

(b) 39.8 Ω is the resistance which the λ/4 transformer will have to match to the 75-Ω line, and from this point the procedure is as in Example 7-4. Therefore

$$Z_0' = \sqrt{Z_0 Z_R} = \sqrt{75 \times 39.8} = 54.5\ \Omega$$

Students at this point are urged to follow the same procedure to solve an example with identical requirements, but now $Z_L = (250 + j450)\ \Omega$ and $Z_0 = 300\ \Omega$. The answers are distance = 0.080 λ and $Z_0' = 656\ \Omega$.

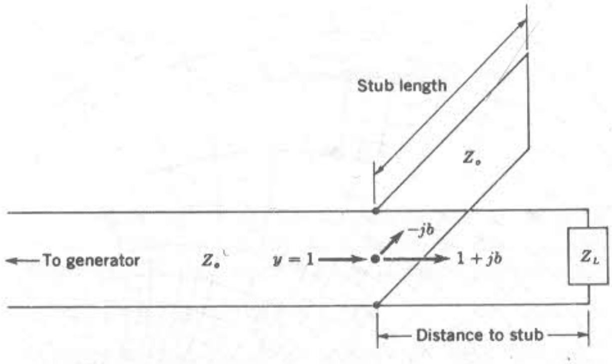

FIGURE 7-14 Stub connected to loaded transmission line.

Matching of load to line with a short-circuited stub A *stub* is a piece of transmission line which is normally short-circuited at the far end. It may very occasionally be open-circuited at the distant end, but either way its impedance is a pure reactance. To be quite precise, such a stub has an input admittance which is a pure susceptance, and it is used to tune out the susceptance component of the line admittance at some desired point. Note that short-circuited stubs are preferred because open-circuited pieces of transmission line tend to radiate from the open end.

As shown in Figure 7-14, a stub is made of the same transmission line as the one to which it is connected. It thus has an advantage over the quarter-wave transformer, which must be constructed to suit the occasion. Furthermore, the stub may be made rigid and adjustable. This is of particular use at the higher frequencies and allows the stub to be used for a variety of loads, and/or over a range of frequencies.

Matching Procedure

1. Normalize the load with respect to the line, and plot the point on the chart.
2. Draw a circle through this point, and travel around it through a distance of $\lambda/4$ (i.e., straight through) to find the load admittance. Since the stub is placed in parallel with the main line, *it is always necessary to work with admittances when making stub calculations*.
3. *Starting from this new point (now using the Smith chart as an admittance chart)*, find the point nearest to the load at which the normalized admittance is $1 \pm jb$. *This point is the intersection of the drawn circle with the $r = 1$ circle,* which is the only circle through the center of the chart. This is the point at which a stub designed to tune out the $\pm jb$ component will be placed. Read off the distance traveled around the circumference of the chart; this is the distance to the stub.
4. To find the length of the short-circuited stub, start from the point $\infty, j\,\infty$ on the right-hand rim of the chart, since that is the admittance of a short circuit.
5. Traveling clockwise around the circumference of the chart, find the point at which the susceptance tunes out the $\pm jb$ susceptance of the line at the point at which the stub is to be connected. For example, if the line admittance is $1 + j0.43$, the

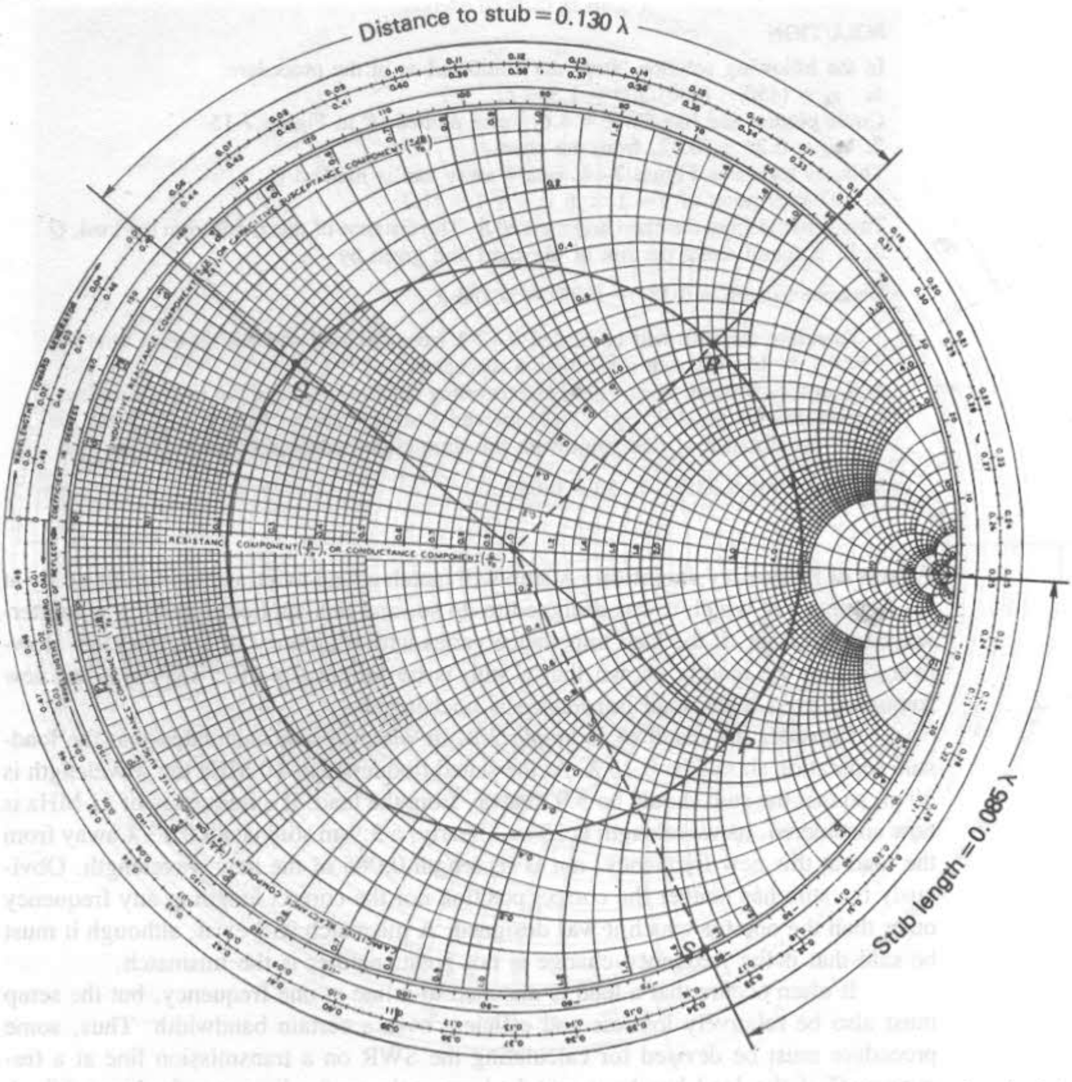

FIGURE 7-15 Smith chart solution of Example 7-8, matching with a short-circuited stub.

required susceptance is $-j0.43$. Ensure that the correct polarity of susceptance has been obtained; this is always marked on the chart on the left-hand rim.

6. Read off the distance in wavelengths from the starting point $\infty, j\infty$ to the new point, (e.g., $b = -0.43$ as above). This is the required length of the stub.

EXAMPLE 7-8 (Refer to Figure 7-15.) A series *RC* combination, having an impedance $Z_L = (450 - j600)\ \Omega$ at 10 MHz, is connected to a 300-Ω line. Calculate the position and length of a short-circuited stub designed to match this load to the line.

SOLUTION

In the following solution, steps are numbered as in the procedure:

1. $z_L = (450 - j600)/300 = 1.5 - j2.$

Circle plotted and has SWR = 4.6. Point plotted, P in Figure 7-15.

2. $y_L = 0.24 + j0.32$, from the chart.

This, as shown in Figure 7-14, is $\lambda/4$ away and is marked Q.

3. Nearest point of $y = 1 \pm jb$ is $y = 1 + j1.7$.

This is found from the chart and marked R. The distance of this point from the load, Q to R, is found along the rim of the chart and given by

Distance to stub = $0.181 - 0.051 = 0.130\ \lambda$

Therefore the stub will be placed $0.13\ \lambda$ from the load and will have to tune out $b = +1.7$; thus the stub must have a susceptance of -1.7.

4, 5, and 6. Starting from $\infty, j\infty$, and traveling clockwise around the rim of the chart, one reaches the point $0, -j1.7$; it is marked S on the chart of Figure 7-15. From the chart, the distance of this point from the short-circuit admittance point is

Stub length = $0.335 - 0.250 = 0.085\ \lambda$

Effects of frequency variation A stub will match a load to a transmission line only at the frequency at which it was designed to do so, and this applies equally to a quarter-wave transformer. If the load impedance varies with frequency, this is obvious. However, it may be readily shown that a stub is no longer a perfect match at the new frequency even if the load impedance is unchanged.

Consider the result of Example 7-8, in which it was calculated that the load-stub separation should be $0.13\ \lambda$. At the stated frequency of 10 MHz the wavelength is 30 m, so that the stub should be 3.9 m away from the load. If a frequency of 12 MHz is now considered, its wavelength is 25 m. Clearly, a 3.9-m stub is not $0.13\ \lambda$ away from the load at this new frequency, nor is its length 0.085 of the new wavelength. Obviously the stub has neither the correct position nor the correct length at any frequency other than the one for which it was designed. A mismatch will exist, although it must be said that if the frequency change is not great, neither is the mismatch.

It often occurs that a load is matched to a line at one frequency, but the setup must also be relatively lossless and efficient over a certain bandwidth. Thus, some procedure must be devised for calculating the SWR on a transmission line at a frequency f'' if the load has been matched correctly to the line at a frequency f'. A procedure will now be given for a line and load matched by means of a short-circuited stub; the quarter-wave transformer situation is analogous.

EXAMPLE 7-9 (Refer to Figure 7-16.) Calculate the SWR at 12 MHz for the problem of Example 7-8.

SOLUTION

For the purpose of the procedure, it is assumed that the calculation involving the position and length of a stub has been made at a frequency f', and it is now necessary to calculate the SWR on the main line at f''. Matter referring specifically to the example will be shown.

in (ii), and its value is greater than unity, (ii) can also be written as

$$|\Gamma| \equiv \frac{r-1}{r+1} \tag{iii}$$

Finally, the value of $|\Gamma|$ can be substituted from (iii) into (i) to obtain the following expression for the voltage standing wave ratio s:

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} \equiv \frac{1+\frac{r-1}{r+1}}{1-\frac{r-1}{r+1}} \equiv r \tag{iv}$$

From the above equation, we can say that the value of the normalized impedance at the point M, where the constant $|\Gamma|$ circle cuts the real axis, is purely real, and is equal to the standing-wave ratio.

In the second step, let us consider the point m on the left-hand side of the Smith chart intersecting the constant r circle. It can be observed from the Smith chart that at any point on the left-hand side of the origin O, the value of the normalized resistance would be less than unity, i.e., $r < 1$. Hence, at the point m, the normalized impedance is purely resistive, and the value of the normalized resistance r is less than unity, which then modifies Eq. (iii) as follows:

$$|\Gamma| \equiv \frac{1-r}{1+r} \tag{v}$$

The expression given by Eq. (5) is based on the simple fact that the magnitude of the reflection coefficient, i.e., the value of $|\Gamma|$ should always be a positive number which is not possible using the form given by (iii) if the value of 'r' is assumed to be less than unity. Once the value of $|\Gamma|$ is defined using (v), the voltage standing-wave ratio 's' can be computed using (i), i.e.,

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} \equiv \frac{1+\frac{1-r}{1+r}}{1-\frac{1-r}{1+r}} \equiv \frac{1}{r} \tag{vi}$$

Hence, the value of the normalized impedance at the point 'm', where the constant $|\Gamma|$ circle cuts the real axis on the left-hand side of the origin 'O', is purely real, and is equal to the inverse of the standing wave ratio.

From Eq. (iv), it can also be postulated that the standing-wave ratio along the line can be obtained by plotting the constant $|\Gamma|$ circle on the Smith chart, and noting the point of intersection of this circle with the positive real axis represented by the point M in Figure 11.19. The value of the normalized resistance r circle, which passes through the point M, then provides the value of the voltage standing wave ratio s on the line.

EXAMPLE 11.16

What are the limitations of a quarter-wave transformer? Explain whether it is possible to match a reactive load having $Z_L = (100 + j150)\,\Omega$ with a lossless transmission line having $Z_0 = 50\,\Omega$ using a quarter-wave transformer. If yes, find out the characteristic impedance of the transformer and other relevant parameters required to achieve this matching.

Solution. The quarter-wave transformer is a piece of transmission line, which is used either for matching two different impedances or for matching a load with a transmission line of different characteristic impedance. It has two major limitations when used for matching purposes. The first limitation is that it can only be used to match the *real* impedances. The second limitation is that the matching is achieved in the very narrow frequency band with the condition that the ideal matching is achieved only at those frequencies where the length of the line is an odd multiple of quarter wavelength, i.e., $l \equiv (2m+1)\lambda/4$.

Now, if one wants to match a reactive load with a transmission line having a *real* value of the characteristic impedance then it can only be achieved if the reactive load is first converted into a *resistive* load. In other words, a piece of transmission line of appropriate length should be connected at the load so that the input impedance becomes *resistive* at the new position. Hence, you have to locate the position on the line shown in Figure 11.20 such that if a transmission line of length l is connected at the load, then the impedance Z_B seen from the new position towards the load becomes purely resistive. Now, you know from the Smith chart theory that the impedance can be purely resistive either at the position of the voltage maximum or at the voltage minimum. In principle, the quarter-wave section can be inserted at any of these two points. However, from the practical point of view, one tries to find a position which is closer to the load in order to minimize the length l of the extra line. The whole procedure is explained below with the help of the Smith chart shown in Figure 11.21.

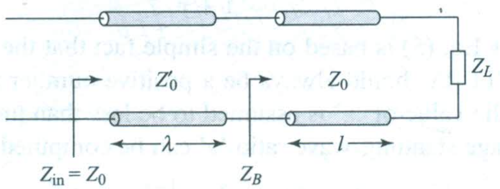

Fig. 11.20 Load line.

Major steps

1. First of all, connect a line of length l shown in Figure 11.20 so that the impedance seen from this point towards the load $Z_L = (100 + j150)\ \Omega$ becomes purely resistive. The characteristic impedance of this line is assumed to be same as that of the actual transmission line, i.e., $Z_0 = 50\ \Omega$.
2. Normalize the load impedance $Z_L = (100 + j150)\ \Omega$ with respect to the characteristic impedance of the line, i.e., $Z_L = \frac{(100+j150)}{50} = (2 + j3)$.
3. Plot $Z_L = (2 + j3)$ point on the Smith chart shown by L in Figure 11.21. Draw a constant $|\Gamma|$ circle on the Smith chart towards the generator (anti-clockwise direction) from the point L with the origin as O and the radius as OL as shown in this figure. This circle intersects the real axis on the right-hand side of the chart at the point M, where the voltage is maximum and the impedance is purely real.
4. The value of the normalized resistance at the point M may be read as $r = 7$. The distance between the load position L and the voltage maximum position M may be determined from the WTG scale provided on the Smith chart, which is given by $l = (0.25 - 0.214)\lambda \equiv 0.036\lambda$. It means that if you move a distance of 0.036 λ from the load towards the generator, then you would arrive at a position where the impedance becomes purely resistive, and this impedance can now be matched with the quarter-wave transformer.

5. The actual value of the resistive impedance at a distance $l = .036\lambda$ from the load in Figure 11.20 is then given by $Z_B = 50 \times 7 = 350\ \Omega$.
6. This impedance can now be matched with a lossless transmission line having characteristic impedance of $Z_0 = 50\ \Omega$ with the help of a quarter-wave transformer. The characteristic impedance of the quarter-wave section in this case would be given by

$$Z_0' = \sqrt{Z_B \times Z_0} = \sqrt{350 \times 50} = 132.29\ \Omega.$$

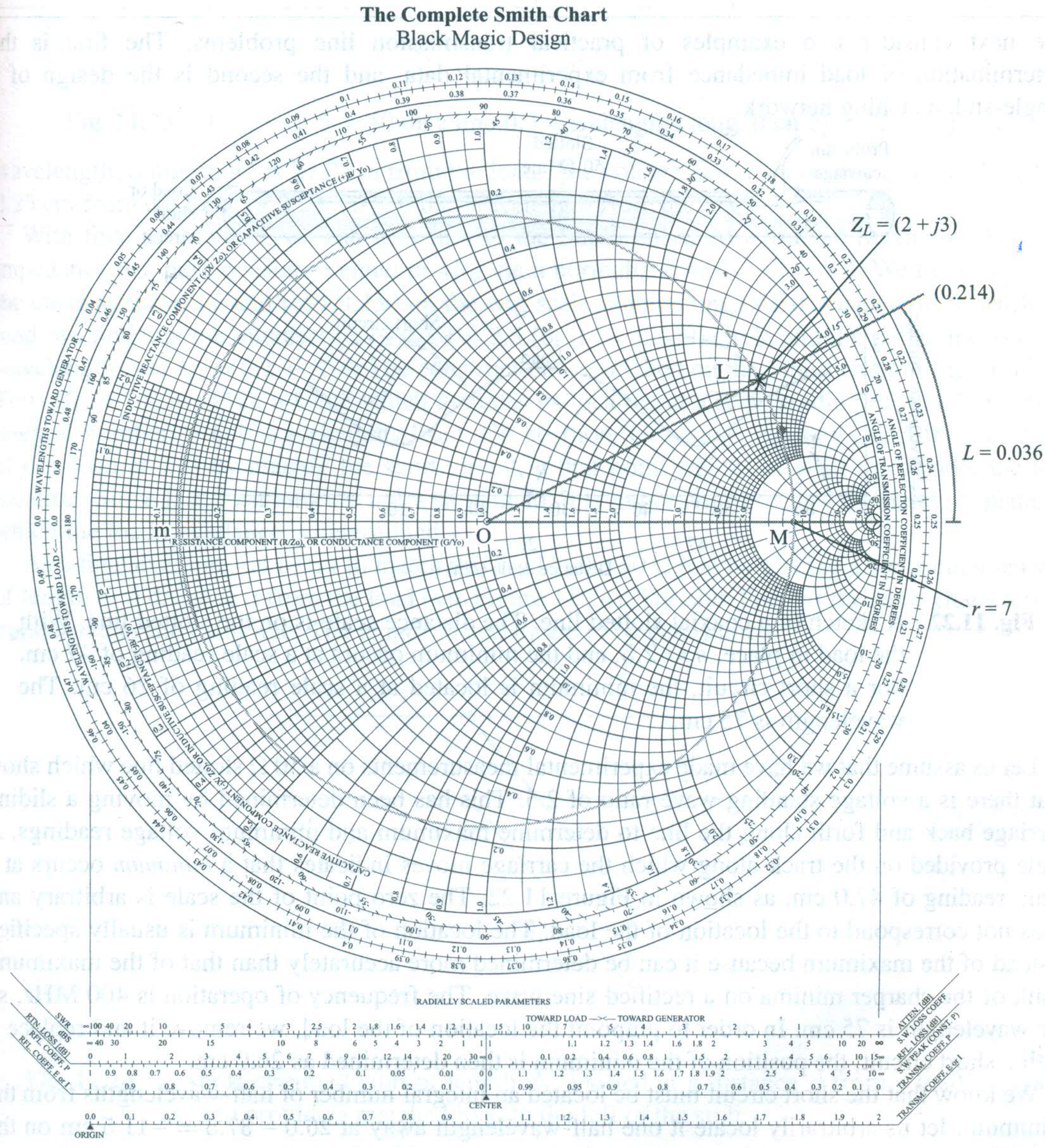

Fig. 11.21 Smith chart.

Since it is much easier to combine admittances in parallel than impedances, let us rephrase our goal in admittance language: the input admittance of the length d containing the load must be $1 + jb_{\text{in}}$ for the addition of the input admittance of the stub jb_{stub} to produce a total admittance of $1 + j0$. Hence the stub admittance is $-jb_{\text{in}}$. We shall therefore use the Smith chart as an admittance chart instead of an impedance chart.

The impedance of the load is $2.1 + j0.8$, and its location is at -11.5 cm. The admittance of the load is therefore $1/(2.1 + j0.8)$, and this value may be determined by adding one-quarter wavelength on the Smith chart, since Z_{in} for a quarter-wavelength line is R_0^2/Z_L, or $z_{\text{in}} = 1/z_L$, or $y_{\text{in}} = z_L$. Entering the chart (Figure 11.25) at $z_L = 2.1 + j0.8$, we read 0.220 on the wtg scale; we add (or subtract) 0.250 and find the admittance $0.41 - j0.16$ corresponding to this impedance. This point is still located on the $s = 2.5$ circle. Now, at what point or points on this circle is the real part of the admittance equal to unity? There are two answers, $1 + j0.95$ at wtg $= 0.16$, and $1 - j0.95$ at wtg $= 0.34$, as shown in Figure 11.25. Let us select the former value since this leads to the shorter stub. Hence $y_{\text{stub}} = -j0.95$, and the stub location corresponds to wtg $= 0.16$. Since the load admittance was found at wtg $= 0.470$, then we must move $(0.5 - 0.47) + 0.16 = 0.19$ wavelength to get to the stub location.

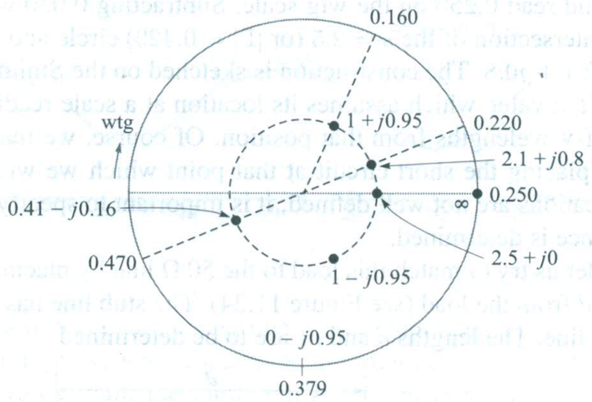

Fig. 11.25 A normalized load, $z_L = 2.1 + j0.8$, is matched by placing a 0.129-wavelength short-circuited stub 0.19 wavelengths from the load.

Finally, we may use the chart to determine the necessary length of the short-circuited stub. The input conductance is zero for any length of short-circuited stub, so we are restricted to the perimeter of the chart. At the short circuit, $y = \infty$ and wtg $= 0.250$. We find that $b_{\text{in}} = -0.95$ is achieved at wtg $= 0.379$, as shown in Figure 11.25. The stub is therefore $0.379 - 0.250 = 0.129$ wavelength, or 9.67 cm long.

EXAMPLE 11.17

It is desired to measure the impedance of an unknown load using the 50 Ω coaxial slotted line set-up shown in Figure 11.22. The experiment is performed by terminating the slotted line in the unknown impedance. The voltage standing-wave ratio measured using this set-up is found to be 3.0, and the voltage minimum is recorded at the scale reading of 50 cm.

Afterwards, the load is removed and a short is placed at the load position. The two consecutive voltage minima are now observed at scale readings of 20 cm and 45 cm, respectively. Using the Smith chart, determine (a) the wavelength, (b) the load impedance, and (c) the reflection coefficient of the load. (d) What would be the location of the first voltage minimum if the actual load is replaced by an open circuit?

Solution.

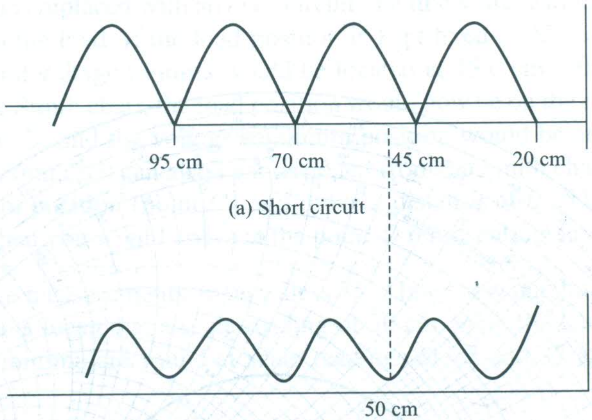

Fig. 11.26 Coaxial slotted line measurement.

The positions of the minimum under the load and the short conditions is shown in Figure 11.26. The detailed procedure is explained below.

(*a*) The consecutive voltage minima under the short conditions are at observed at 45 cm and 20 cm as seen in Figure 11.26(a). As these minima are spaced half wavelength apart, hence the wavelength may be calculated as

$$\lambda = (45 - 20) \times 2 = 50 \text{ cm}$$

(*b*) For finding the load impedance, first draw the constant VSWR circle with $s = 3$ as shown on the Smith chart. Mark the point A, where the $s = 3$ intersects the left-hand side of the real axis. This point A basically represents the voltage minimum point corresponding to the load. In other words, you can consider the point $z = 50$ cm in Figure 11.26 (b) mapped to the point A on the Smith chart.

Now, you can assume the load to be located at any of the minima point corresponding to the short, i.e., the load position can be either at $z = 20$ cm or 45 cm as shown in Figure 11.26 (a). It is convenient to consider the point which is nearest to the load minimum. Hence, the load can be assumed to be positioned at $z = 45$ cm.

It basically means that if you start from the load minima position 'A' on the Smith chart ($z = 50$ cm), and move a distance of $d = 5$ cm $= 0.1\ \lambda$ towards load then you would arrive at the load position ($z = 45$ cm).

On the Smith chart, move on the constant $s = 3$ circle from the point 'A' towards the load until you reach 0.1λ marked on the 'Wavelength towards Load' scale. Draw a line from the centre of the chart to this point intersecting the constant $s = 3$ circle at the point B shown in the chart. The constant r and constant x circles passing through the point 'B' provide the load point, which may be read as

$$Z_L = (0.5 - j0.6)$$

The actual value of the load impedance may then be calculated as

$$Z_L = 50 \times (0.5 - j0.6) = (25 - j30)\ \Omega$$

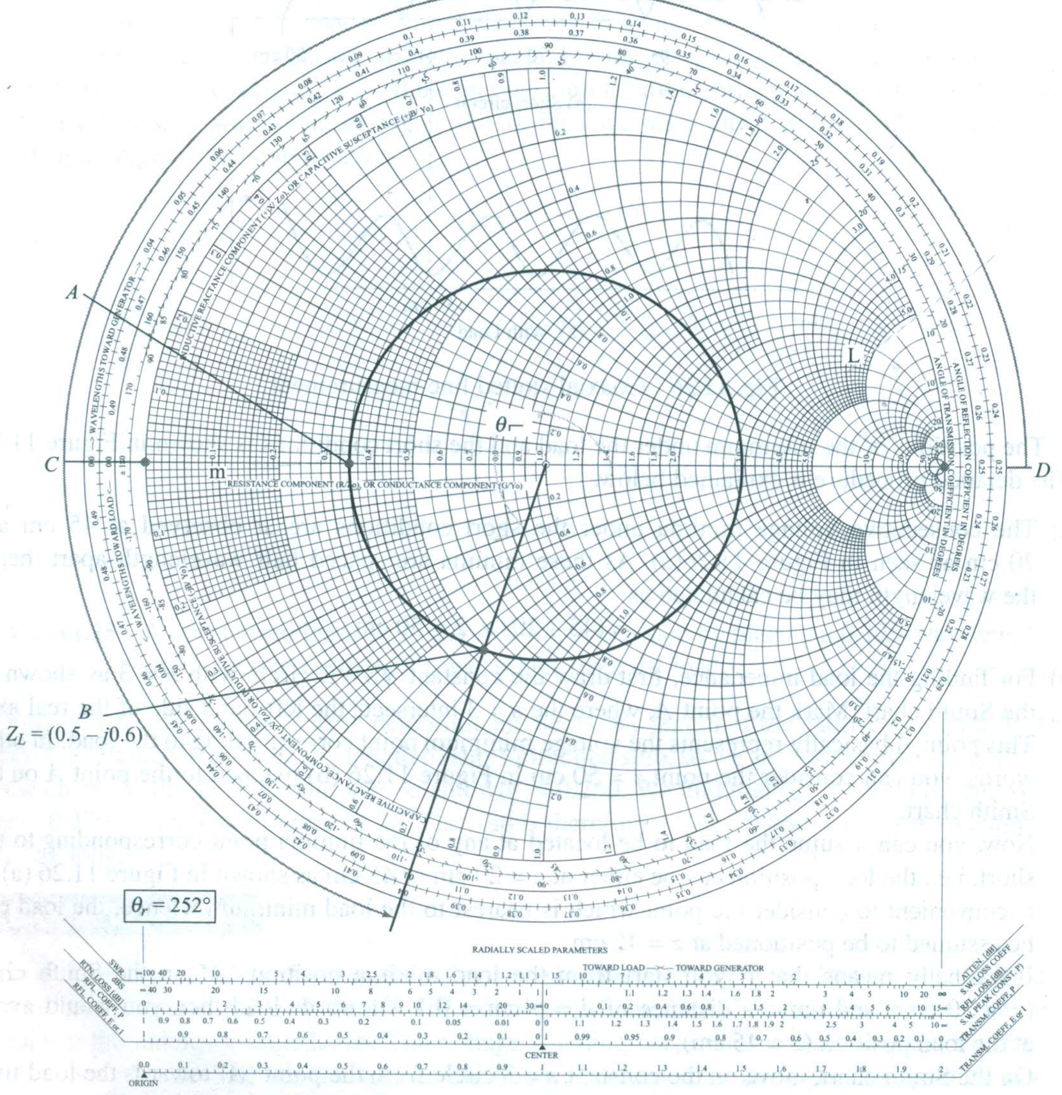

Fig. 11.27 Smith chart.

(*c*) The reflection coefficient of the load can be found out by measuring $|\Gamma|$ and the angle θ_Γ from the Smith chart, which are given as

$$|\Gamma| = 0.5 \text{ and } \theta_\Gamma = 252°$$

$$\text{Hence, } \Gamma = |\Gamma|\, e^{j\theta_r} = 0.5e^{j252°} = 0.5e^{j4.398} = -0.1545 - j0.4755$$

$|\Gamma|$ can also be found out analytically using $|\Gamma| = \dfrac{\rho - 1}{\rho + 1} = \dfrac{3-1}{3+1} = 0.5$

(*d*) When the actual load is replaced with an *open* circuit, the first voltage minimum would be shifted by 0.25λ away from the load. If the load position is kept fixed at 45.0 cm as shown in Figure 11.26 (a) then the first voltage minima would be located at 45.0 cm + 0.25λ = 57.5 cm on the scale reading. On the Smith chart, the load position would now be on the extreme right-hand side denoted by the point D, and the voltage minimum position would be at the extreme left-hand side denoted by the point C. It can also be interpreted from the Smith chart that if you start from the voltage minimum position (point C) and move a distance of 0.25λ on the outer circle of chart towards load then you would arrive at the point D representing an *open* load.

D11.7 Standing wave measurements on a lossless 75 Ω line show maxima of 18 V and minima of 5 V. One minimum is located at a scale reading of 30 cm. With the load replaced by a short circuit, two adjacent minima are found at scale readings of 17 and 37 cm. Find: (*a*) s; (*b*) λ; (*c*) f; (*d*) Γ_L; (*e*) Z_L.

Ans. 3.60; 0.400 m; 750 MHz; $0.704\angle{-33.0}$; $77.9 + j104.7\,\Omega$

EXAMPLE 11.18

A load impedance given by $Z_L = 100 + j100\,\Omega$ is to be matched to a transmission line having characteristic impedance of 150 Ω. Design a matching network consisting of a short-circuited shunt stub having characteristic impedance of 50 Ω, which is connected at some distance d_{stub} away from the load. The stub should be located at shortest possible distance from the load, and its length should also be minimum. Please use the Smith chart and explain clearly all the steps.

Solution. First, we have to find out the location of the stub d_{stub} with reference to the load which is based on the criterion that the real part of the impedance at this point should be equal to the characteristic impedance of the transmission line. A short-circuited stub is then to be connected in parallel at this point, and the length of this stub L_{stub} should be determined such that it cancels the imaginary part of the load impedance. The overall arrangement of the shunt stub with the load and the transmission line is shown in Figure 11.28. It is also to be noted that for shunt stubs, it becomes more convenient to work with admittances, rather than impedances.

The overall procedure of matching the given load with the transmission line using the Smith chart is given below:

(*a*) Calculate the normalized load impedance and determine the corresponding location marked by the point 'L' on the Smith chart.

$$Z_L = \frac{100 + j100}{150} = (0.667 + j0.667)$$

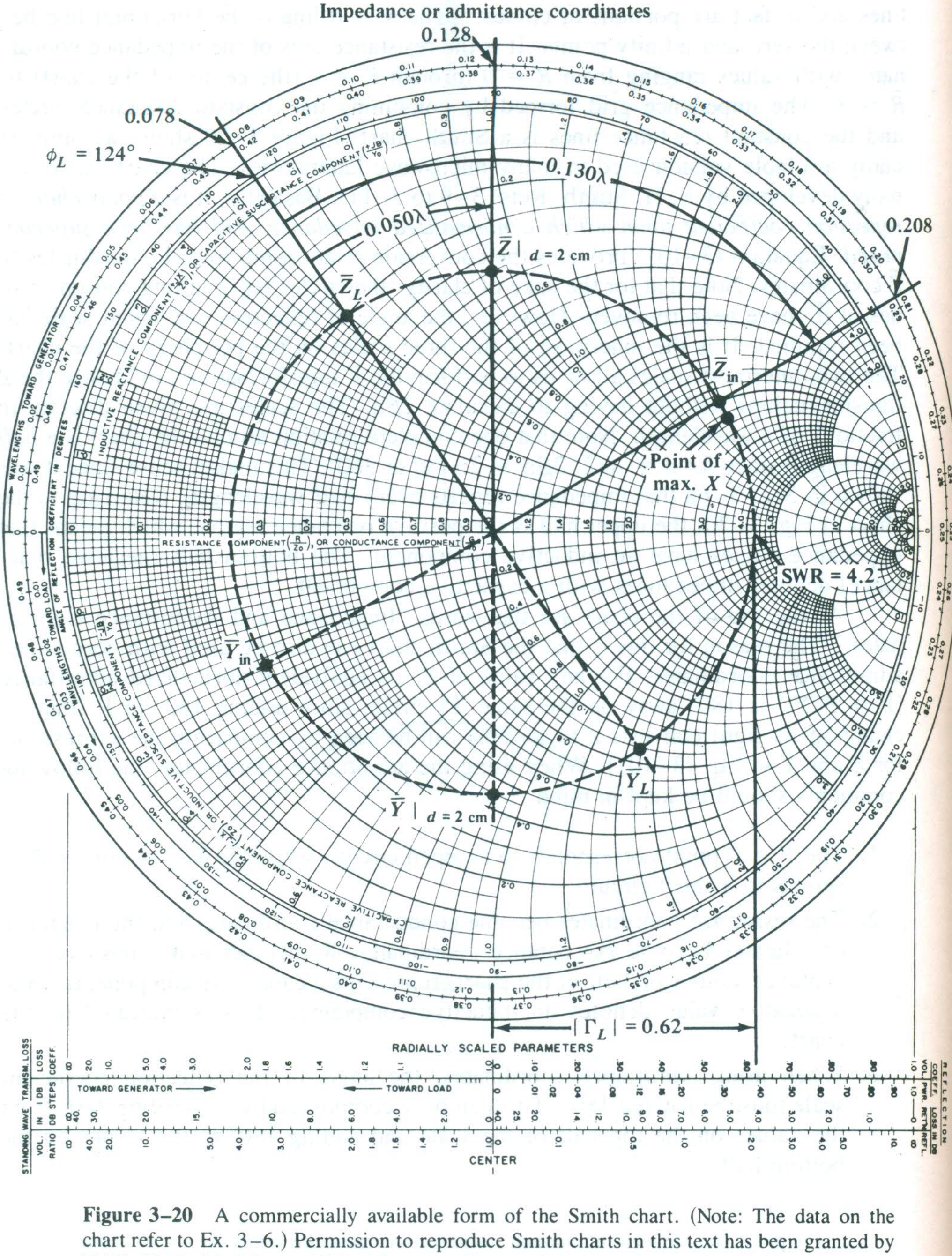

Figure 3–20 A commercially available form of the Smith chart. (Note: The data on the chart refer to Ex. 3–6.) Permission to reproduce Smith charts in this text has been granted by Phillip H. Smith, Murray Hill, N.J., under his renewal copyright issued in 1976.

Note that there are two wavelength scales on the periphery of the chart. One is labeled *Wavelengths toward Generator* and the other *Wavelengths toward Load*. The first one is used when determining the impedance at a point *nearer the input* than the known impedance. This clockwise rotation is referred to as *moving toward the generator*, the assumption being that a generator is connected to the input. The second scale is used in determining the impedance at a point *nearer the load* than the known impedance. This counterclockwise rotation is referred to as *moving toward the load*.

The following example illustrates the graphical procedure for determining the impedance and admittance transformation due to a length of transmission, line as well as other useful properties of the Smith chart.

Example 3–6:

A 5.2 cm length of lossless 100 ohm line is terminated in a load impedance $Z_L = 30 + j50$ ohms.

(a) Calculate $|\Gamma_L|$, ϕ_L, and the SWR along the line.

(b) Determine the impedance and admittance at the input and at a point 2.0 cm from the load end. The signal frequency is 750 MHz and $\lambda = \lambda_0$.

Solution:

(a) Plot the normalized load impedance $\bar{Z} \equiv Z_L/Z_0 = 0.30 + j0.50$ on the Smith chart in Fig. 3–20. This is done by starting from the 0.30 point on the resistance axis and moving up 0.50 reactance units along the constant resistance circle. Next, draw a circle with its center at $\bar{Z} = 1$ (the center of the chart) and a radius equal to the distance between $\bar{Z} = 1$ and $\bar{Z}_L$. This is shown in the figure and will hereafter be referred to as the SWR circle. It is, in fact, the constant $|\Gamma|$ circle for the given value of load impedance. The $\bar{Z}_L$ point on the Smith chart corresponds to its Γ_L value on the polar chart. Since the angle of the reflection coefficient scale has been retained, the value of ϕ_L (124° in this case) can be obtained from the chart as shown in the figure. The value of $|\Gamma_L|$ is obtained by measuring the radius of the SWR circle on the Reflection Coefficient-Vol. scale located at the bottom of the chart. In this example problem, $|\Gamma_L| = 0.62$.

One reason that the constant $|\Gamma|$ circle is called the *SWR circle* is that its intersection with the right half of the resistance axis (that is, between 1 and ∞) yields the SWR due to $\bar{Z}_L$. For this case, the SWR is 4.2. Since SWR is so easily obtained, many engineers prefer to calculate $|\Gamma|$ from Eq. (3–53) rather than using the Reflection Coefficient-Vol. scale. The reason that the resistance axis between 1 and ∞ serves as a SWR scale is that it corresponds to positive real values of Γ. As such it represents the magnitude of Γ for all normalized impedances on the SWR circle. Equation (3–101) transforms these values of Γ into $\bar{R} = (1 + |\Gamma|)/(1 - |\Gamma|)$, which is exactly the equation for SWR (Eq. 3–52). The unity SWR circle is simply a point at the center of the chart, while the infinite SWR circle is the periphery of the chart and is equivalent to $|\Gamma| = 1.00$.

(b) A graphical method of obtaining Γ_{in} when Γ_L and βl are known has been described. For a lossless line, it consists of rotating clockwise $2\beta l$ on the constant $|\Gamma|$ circle. The procedure for obtaining $\bar{Z}_{in}$ from $\bar{Z}_L$ is *exactly the same* except that the impedance coordinates of the Smith chart are used. The steps are as follows:

1. Plot $\bar{Z}_L$ (0.30 + j0.50 in this case) and draw its SWR circle (4.20 in this case).
2. Draw a radial line from the center of the chart through $\bar{Z}_L$ to the periphery. Read the value on the *Wavelengths toward Generator* scale (0.078 in this case).

This value in itself has no physical meaning. It is merely the starting point of the clockwise rotation in the next step.

3. Since $\lambda_0 = 40$ cm at 750 MHz and the input is 5.2 cm from the load, rotate clockwise *from* 0.078 a distance $l/\lambda = 5.2/40 = 0.130$. Draw a radial line from the center of the chart through the 0.208 point on the outer scale as shown in Fig. 3–20. The intersection of the radial line with the SWR circle represents $\bar{Z}_{in}$ since it corresponds to the Γ_{in} point on the polar reflection coefficient chart. In this case, $\bar{Z}_{in} = 2 + j2$ or $Z_{in} = 200 + j200$ ohms.

To obtain the impedance at $d = 2$ cm, start from $\bar{Z}_L$ and rotate clockwise $2/40 = 0.050$ and draw a radial line through the 0.128 point as shown. Its intersection with the SWR circle yields $\bar{Z} = 0.47 + j0.93$ or $Z = 47 + j93$ ohms.

This simple graphical method of determining the impedance transformation due to a length of lossless line is the most useful characteristic of the Smith chart. The procedure for determining the admittance transformation is *exactly the same* since all admittance points are directly opposite their corresponding impedance points on the SWR circle. Thus from the figure, $\bar{Y}_L = 0.88 - j1.47$, $\bar{Y}_{in} = 0.25 - j0.25$, and at $d = 2.0$ cm, $\bar{Y} = 0.43 - j0.87$. Multiplying these values by $Y_0 = 0.01$ mho gives the unnormalized admittance values.

Part *a* of this example problem illustrates how to determine $|\Gamma_L|$, ϕ_L, and SWR when Z_L and Z_0 are known. Since the line is lossless, the SWR and $|\Gamma|$ are the same at all other points on the line. The angle of the reflection coefficient, however, is a function of position and can be read on the periphery of the chart. In this example, $\phi_{in} = 30°$ while at $d = 2$ cm, ϕ is equal to 88°.

Part *b* describes the graphical solution to the impedance/admittance transformation equation for lossless lines.[16] Given $\bar{Z}_L$, the normalized impedance at any other point on the line is obtained by rotating clockwise on a fixed SWR circle the appropriate distance d/λ. Thus for a given load impedance, the SWR circle represents the locus of all possible impedance and admittance values available on the lossless line. Stated another way, given $\bar{Z}_L$, it is the locus of all possible values of $\bar{Z}_{in}$ and $\bar{Y}_{in}$ obtainable by varying the line length l. This is an example of how a good graphical procedure can show the effect of a variable (l) on the desired result (Z_{in}). For instance, it is obvious from Fig. 3–20 that if $\bar{Z}_L = 0.30 + j0.50$, varying the line length will never result in $\bar{Z}_{in} = 0.70 + j0.40$ since it is not on the SWR circle containing $\bar{Z}_L$. If the reader is ambitious, try proving this analytically. To further emphasize the chart's usefulness, consider the ease with which the following problem is solved. Given the impedance values in Ex. 3–6, what value of line length maximizes the reactive portion of the input impedance? With $\bar{Z}_L = 0.30 + j0.50$ plotted in Fig. 3–20, the SWR circle represents all possible values of $\bar{Z}_{in}$ that can be obtained by varying l. A brief look at the chart shows that the reactive portion is maximized at the point where the SWR circle is tangent to the reactance lines. A positive reactive maximum occurs when $\bar{Z}_{in} \approx 2.3 + j2.0$. A radial line through this point intersects the *Wavelengths toward Generator* scale at 0.216. Therefore, the 100 ohm line must be $(0.216 - 0.078)$ or 0.138λ long.

[16] For a lossy line, a second SWR circle is required. It is obtained by multiplying $|\Gamma_L|$ by $e^{-2\alpha d}$ and converting the resulting $|\Gamma|$ to SWR. The intersection of the radial line with the circle defined by the new SWR value yields $\bar{Z}$ at d units from the load.

Reflection Coefficients and VSWR of selected loads

Reflection Coefficient (Γ) = $\frac{(Z_L - Z_0)}{(Z_L + Z_0)}$ = $|\Gamma|\ e^{j\phi}$

Type of Load	Γ	$\|\Gamma\|$	ϕ(degrees)	VSWR = (1+$\|\Gamma\|$) / (1- $\|\Gamma\|$)	
Open Circuit $Z_L = \infty$	1	1	0	∞	True Standing Wave
Short Circuit $Z_L = 0$	-1	1	180	∞	True Standing Wave
Pure Reactance Load $Z_L = jX_L$		1		∞	True Standing Wave
Perfectly Matched Load $Z_L = Z_0$	0	0		1	Travelling Wave
$Z_L = R_L$ and $R_L > Z_0$	+ve		0	R_L / Z_0	Standing Wave
$Z_L = R_L$ and $R_L < Z_0$	-ve		180	Z_0 / R_L	Standing Wave
Z_L = R_L + j X_L					Standing Wave