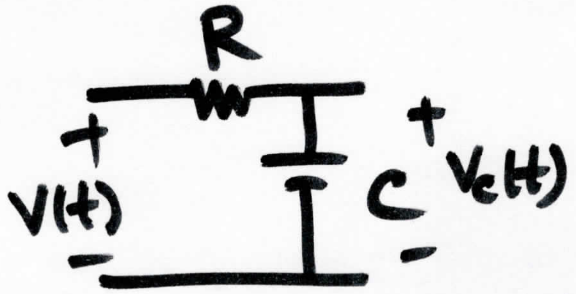
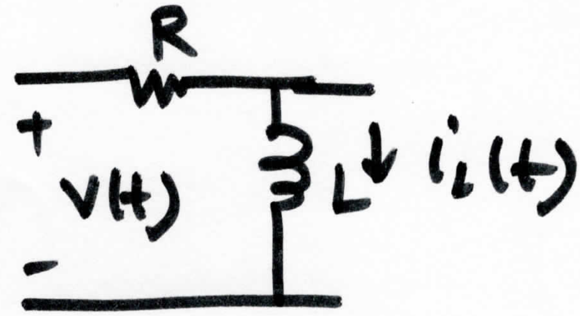
Exponential input to RC or RL Circuits





$$V(t) = RC\frac{dv_c}{dt} + V_c(t)$$

$$V(t) = R\,i_L(t) + L\frac{di_L}{dt}$$

$$\Rightarrow \quad \frac{dv_c}{dt} = -\frac{1}{RC}V_c(t) + \frac{1}{RC}V(t)$$

$$\Rightarrow \quad \dot{V}_c(t) = -\frac{1}{RC}V_c(t) + \frac{1}{RC}V(t)$$

at $t=0^-$; $\quad V_c(0-) = V_i$

$\Rightarrow$ at $t=0+$, $V_c(0+) = V_i = V_c(0-)$

$\Rightarrow \dot{V}_c(0+) = ?$

Now, $\quad \dot{V}_c(0+) = -\frac{1}{RC}V_c(0+) + \frac{1}{RC}V(0+)$

$\ddot{V}_c(0+) = ?$

$$\Rightarrow \ddot{V}_c(t) = -\frac{1}{RC}\dot{V}_c(t) + 0$$

(for DC input)

$$\ddot{V}_c(t) = -\frac{1}{RC}\dot{V}_c(t) \quad \text{for } \underline{\underline{t>0}}.$$

$$\Rightarrow \ddot{V}_c(0+) = -\frac{1}{RC}\dot{V}_c(0+) \checkmark$$

$$\Rightarrow \dddot{V}_c(t) = -\frac{1}{RC}\ddot{V}_c(t)$$

$$\Rightarrow \dddot{V}_c(0+) = -\frac{1}{RC}\ddot{V}_c(0+)$$

For RC System

$$\dot{V}_c(t) = -\frac{1}{RC}V_c(t) + \frac{1}{RC}V(t)$$

Consider, $V(t) = A e^{st}$

$$\Rightarrow V_c(t) = \underset{\substack{\downarrow \\ \text{Forced response}}}{V_F(t)} + \underset{\substack{\searrow \\ \text{Natural response}}}{V_N(t)}$$

$$V_F(t) = B e^{st} \Rightarrow \underline{\underline{B = ?}}$$

$$\Rightarrow \dot{V}_F(t) = B s e^{st}$$

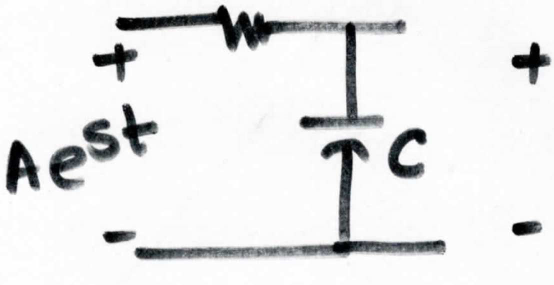
$$\Rightarrow \dot{V}_F(t) = -\frac{1}{Rc} V_F(t) + \frac{1}{Rc} A e^{st}$$

$$\Rightarrow B s e^{st} = -\frac{1}{Rc} B e^{st} + \frac{1}{Rc} A e^{st}$$

$$\Rightarrow \left[B s + \frac{1}{Rc} B \right] \cancel{e^{st}} = \frac{1}{Rc} A \cancel{e^{st}}$$

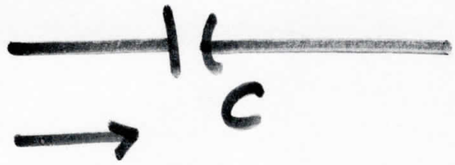
$$\Rightarrow \frac{B [sRc + 1]}{\cancel{Rc}} = \frac{A}{\cancel{Rc}}$$

$$\Rightarrow \boxed{B = \frac{A}{1 + sRc}}$$



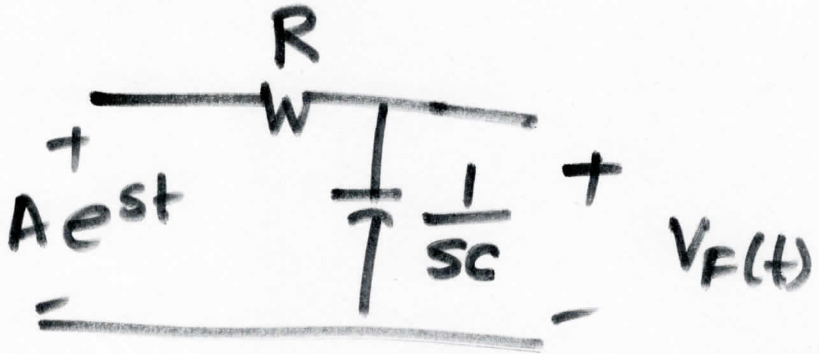
$$V_F(t) = \frac{A}{1 + sRc} e^{st}$$

$+ \; V_F(t) \; - \; = B e^{st}$



$$I_F(t) = C \frac{dV_F}{dt} = sCBe^{st}$$

$$\frac{V_F(t)}{I_F(t)} = \frac{\cancel{Be^{st}}}{sC\cancel{Be^{st}}} = \frac{1}{sC} = X(s)$$



$$= \frac{\frac{1}{sC}}{R + \frac{1}{sC}} \times Ae^{st}$$

$$= \frac{A}{1+sRC} e^{st}$$

So,

$$V_c(t) = V_F(t) + V_N(t)$$

$$= \frac{A}{1+sRC} e^{st} + K e^{-t/RC}$$

for $t > 0$

at $t = 0-$, $V_c(t) = V_i = V_c(0+)$

$$\Rightarrow \quad \underset{\downarrow \atop V_i}{V_c(0+)} = \frac{A e^0}{1+sRc} + K e^0$$

$$\Rightarrow \quad K = \left[V_i - \frac{A}{1+sRc} \right]$$

So,

$$V_c(t) = \frac{A}{1+sRc} e^{st} + \left[V_i - \frac{A}{1+sRc} \right] e^{-t/Rc}$$

$$= \frac{A}{1+sRc} \left[\frac{e^{st} - e^{-t/Rc}}{1} \right] + V_i e^{-t/Rc}$$

$\underline{\underline{s = -\frac{1}{Rc}}}$

$s \neq -\frac{1}{Rc}$

$$\lim_{s \to -\frac{1}{Rc}} \frac{e^{st} - e^{-t/Rc}}{1+sRc} = \lim_{s \to -\frac{1}{Rc}} \frac{t e^{st}}{Rc} = \frac{t}{Rc} e^{-t/Rc}$$

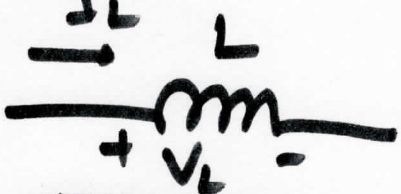
So, for $s = -\frac{1}{RC}$,

$$V_c(t) = \frac{A\, t\, e^{-t/RC}}{RC} + V_i\, e^{-t/RC}$$

$\downarrow$

Resonance

$$\boxed{X_c(s) = \frac{1}{Cs}}$$



$$\boxed{X_L(s) = Ls}$$

$$V_L = L\frac{dI_L}{dt}$$

$$I_L = I_0 e^{st}$$

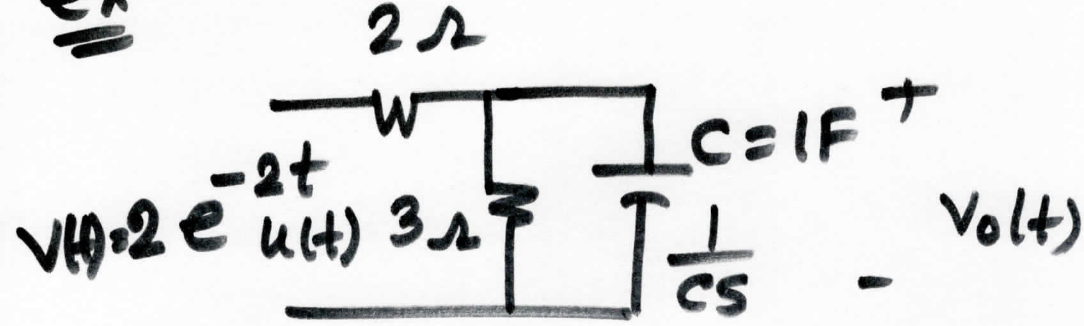
$$\Rightarrow V_L = L(I_0 s e^{st})$$

$$= I_0 L s e^{st}$$

$$X_L = \frac{V_L}{I_L} = \frac{I_0 L s e^{st}}{I_0 e^{st}}$$

$$= Ls$$

ex



$$s = -2 \qquad V(t) = A e^{st} = 2e^{-2t}$$

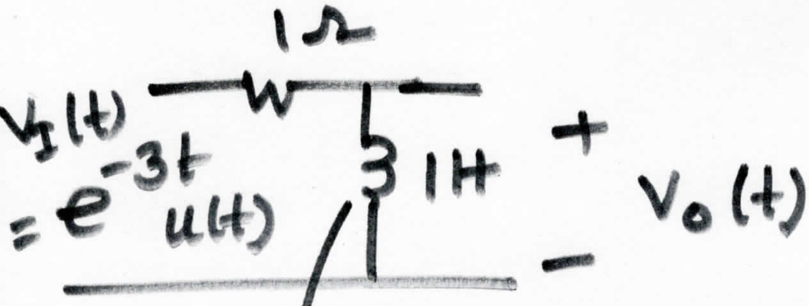
$$V_F = \frac{\left(3 \parallel \frac{1}{s}\right)}{2 + \left(3 \parallel \frac{1}{s}\right)} \times A e^{st}$$

$$= \frac{\dfrac{3 \times 1/s}{3 + 1/s}}{2 + \dfrac{3 \times 1/s}{3 + 1/s}} \times A e^{st}$$

$$= \frac{3}{2(1+3s)+3} \times A e^{st}$$

$$= \frac{3}{2(1+3\times(-2))+3} \times 2 e^{-2t}$$

$$= -\frac{6}{7} e^{-2t} u(t)$$



$X_L = LS$
$= S$

$$\frac{V_o}{V_I} = \frac{S}{S+1} = \frac{(-3)}{(-3)+1}$$
$$= 3/2$$

$$V_o(t) = V_F + V_N$$
$$= \frac{3}{2} \times e^{-3t} + K e^{-t/1}$$

$I_L(0-) = 0A = I_L(0+)$ $\quad [\tau = \frac{L}{R} = \frac{1}{1} = 1]$

$V_o(0+) = e^0 = 1V$

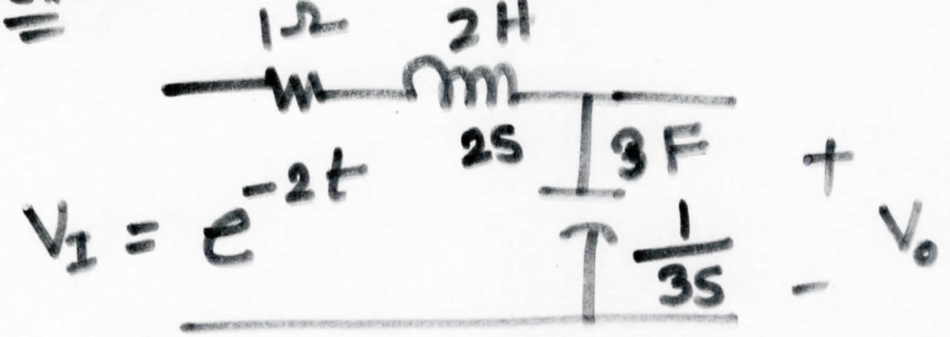
<u>at $t = 0+$</u>

$$1 = \frac{3}{2} e^0 + K e^0$$

$$\Rightarrow K = 1 - 3/2 = -1/2$$

$$\Rightarrow \boxed{V_o(t) = \frac{3}{2} e^{-3t} - \frac{1}{2} e^{-t}}$$

ex

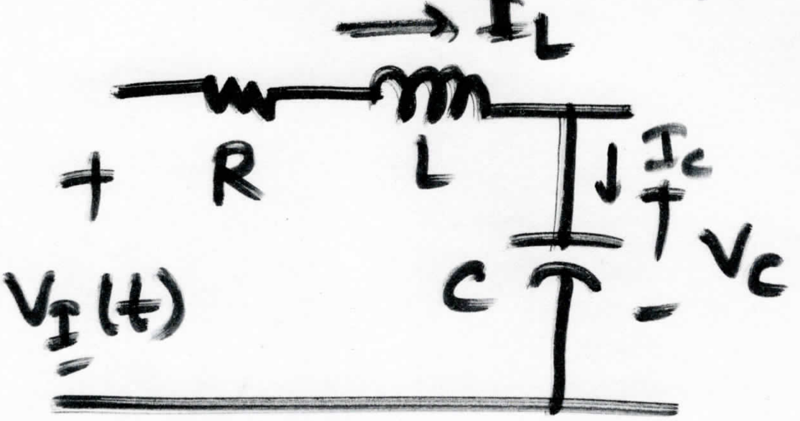


$$V_F(t) = \frac{\frac{1}{3s}}{1+2s+\frac{1}{3s}} \times e^{st}$$

$$= \frac{1}{6s^2+3s+1} e^{st}$$

$$= \frac{1}{24-6+1} e^{-2t}$$

$$= \frac{1}{19} e^{-2t}$$



KVL $$V_I(t) = I_L R + L\frac{dI_L}{dt} + V_c$$

$$I_L = C\frac{dV_c}{dt}$$

$$\boxed{V_I(t) = RC\frac{dV_c}{dt} + LC\frac{d^2V_c}{dt^2} + V_c}$$

<u>V_N</u>

$$LC\frac{d^2V_c}{dt^2} + RC\frac{dV_c}{dt} + V_c = 0$$

$$V_N = Ae^{st}$$

$$\Rightarrow LCAs^2e^{st} + RCAse^{st} + Ae^{st} = 0$$

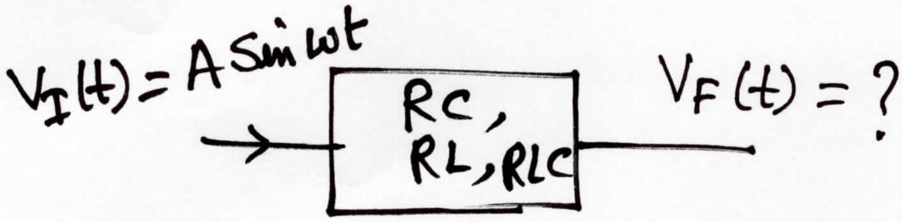
$$\Rightarrow Ae^{st}\left[LCs^2 + RCs + 1\right] = 0$$

$$\Rightarrow LCs^2 + RCs + 1 = 0$$

$$s = s_1, s_2$$

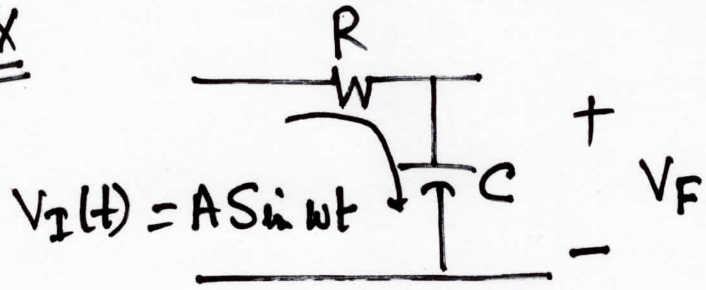
$$V_N(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

Sinusoidal Input



$$V_F(t) = B_1 \sin \omega t + B_2 \cos \omega t$$

ex



$$V_F = B_1 \sin \omega t + B_2 \cos \omega t$$

$$RC \frac{dV_F}{dt} + V_F = V_I$$

$$\Rightarrow \quad RC\left[B_1 \omega \cos \omega t + B_2 \omega (-\sin \omega t)\right] +$$

$$B_1 \sin \omega t + B_2 \cos \omega t = A \sin \omega t$$

$$\Rightarrow \quad \underbrace{\left[B_1 RC\omega + B_2\right]}_{=0} \cos \omega t + \underbrace{\left(B_1 - B_2 RC\omega - A\right)}_{=0} \sin \omega t = 0$$

1) $B_2 + B_1 WRC = 0 \Rightarrow B_2 = -B_1 WRC$

2) $B_1 - B_2 WRC - A = 0$

$$\Rightarrow B_1 = A + B_2 WRC$$

$$= A - (B_1 WRC) WRC$$

$$\Rightarrow B_1 = A - B_1 (WRC)^2$$

$$\Rightarrow B_1 \left[1 + (WRC)^2\right] = A$$

$$\Rightarrow B_1 = \frac{A}{1+(WRC)^2}$$

So, $B_2 = -B_1 WRC$

$$= -\frac{A\,WRC}{1+(WRC)^2}$$

$$\Rightarrow V_F = B_1 \sin wt + B_2 \cos wt$$

$$= \frac{A}{1+(WRC)^2} \sin wt - \frac{A\,WRC}{1+(WRC)^2} \cdot \cos wt$$

$$= \frac{A}{\sqrt{1+(WRC)^2}} \left[\frac{1}{\sqrt{1+(WRC)^2}} \sin wt - \frac{WRC}{\sqrt{1+(WRC)^2}} \cos wt\right]$$

$$= \frac{A}{\sqrt{1+(\omega RC)^2}} \cdot \sin(\omega t - \alpha)$$

where

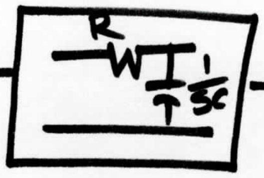
$$\cos\alpha = \frac{1}{\sqrt{1+(\omega RC)^2}}$$

$$\sin\alpha = \frac{\omega RC}{\sqrt{1+(\omega RC)^2}}$$

$$\Rightarrow \alpha = \tan^{-1}(\omega RC)$$

$$\boxed{V_F(t) = \frac{A}{\sqrt{1+(\omega RC)^2}} \sin(\omega t - \alpha)}$$

$V_I(t) = e^{st}$



$V_o(t) = \frac{1}{1+sRC} e^{st}$

$$V_I(t) = A \sin \omega t$$

$$= A \left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right]$$

$$= \frac{A}{2j} e^{j\omega t} - \frac{A}{2j} e^{-j\omega t}$$

$$e^{j\omega t} \longrightarrow \boxed{RC} \longrightarrow \frac{1}{1+j\omega RC}\,e^{j\omega t}$$

$$e^{-j\omega t} \qquad\qquad \frac{1}{1-j\omega RC}\,e^{-j\omega t}$$

$$V_0(t) = \frac{A}{2j} \times \frac{1}{1+j\omega RC}\, e^{j\omega t} - \frac{A}{2j}\,\frac{1}{1+j\omega RC}\, e^{-j\omega t}$$

$$= \frac{A}{2j} \times M e^{-j\alpha} \times e^{j\omega t} - \frac{A}{2j} M e^{j\alpha} e^{-j\omega t}$$

$$= AM \left[\frac{e^{j(\omega t-\alpha)} - e^{-j(\omega t-\alpha)}}{2j} \right]$$

$$= AM \sin(\omega t - \alpha)$$

$$\frac{1}{1+j\omega RC} = M e^{-j\alpha}$$

where $M = \dfrac{1}{\sqrt{1+(\omega RC)^2}}$ &

$$-\alpha = -\tan^{-1}(\omega RC)$$

$$\frac{a+jb}{c+jd} = \frac{\sqrt{a^2+b^2}\, e^{j\tan^{-1}\left(\frac{b}{a}\right)}}{\sqrt{c^2+d^2}\, e^{j\tan^{-1}(d/c)}} = M e^{j\phi}$$

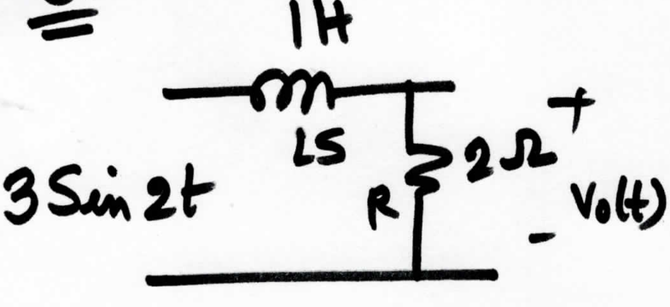
where $M = \dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$ & $\phi = \tan^{-1}\left(\frac{b}{a}\right) - \tan^{-1}\left(\frac{d}{c}\right)$

$V_I(t) = e^{st}$ $\qquad$ $V_o(t) = H(s)\, e^{st}$

$$H(s)\Big|_{s=j\omega} = H(j\omega) = M e^{j\phi}$$

$$\frac{V_I(t) = A \sin \omega t}{V_o(t)} = AM \sin[\omega t + \phi)$$

ex



$$H(s) = \frac{R}{R + LS} = \frac{2}{2 + S}$$

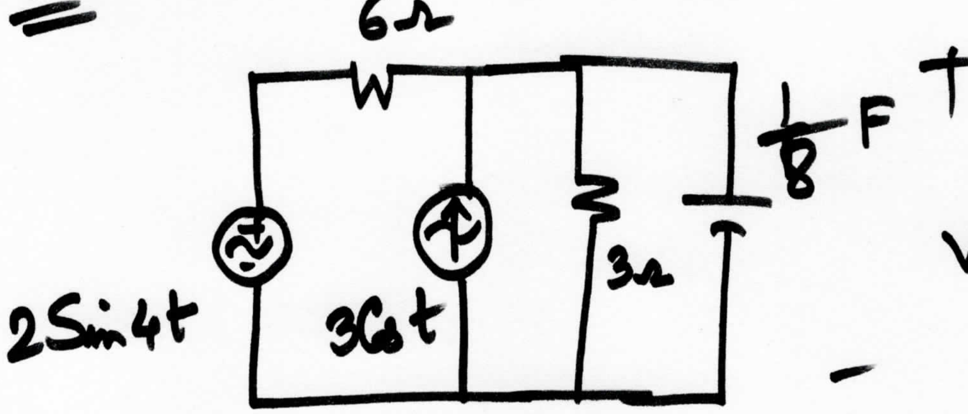
$V_I(t) = 3 \sin 2t \longrightarrow \omega = 2$ rad/sec.

$$H(j\omega)\Big|_{\omega=2} = \frac{2}{2 + j\omega} = \frac{2}{2 + j2} = \frac{1}{1 + j}$$

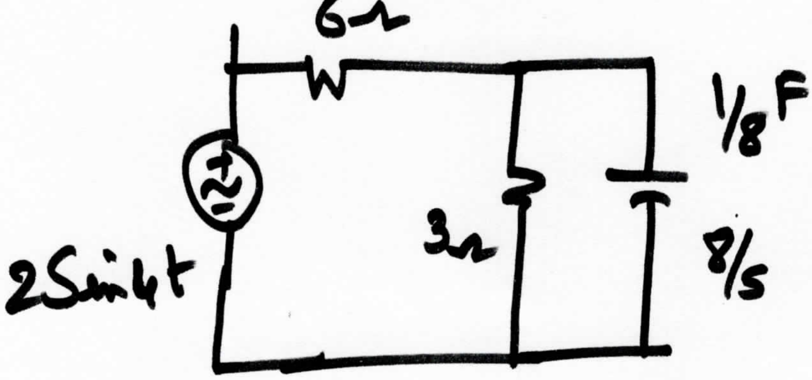
$$= \frac{1}{\sqrt{2}} \angle -45^\circ$$

$$\Rightarrow V_o(t) = \frac{3}{\sqrt{2}} \sin(2t - 45^\circ)$$

ex



$V_o(t) = ?$



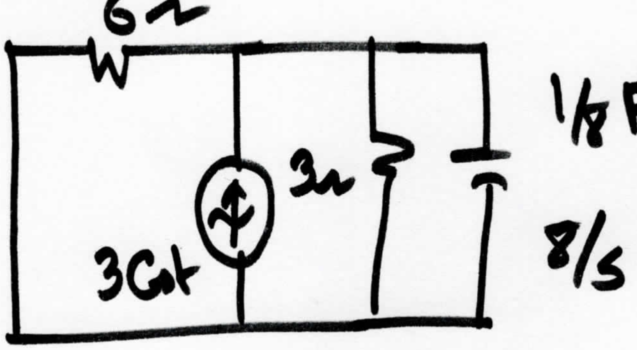
$$V_o'(t) = 2 \times \frac{1}{3\sqrt{2}} \sin(4t - 45°)$$

$$H(s) = \frac{\frac{3 \times 8/s}{3 + 8/s}}{6 + \frac{3 \times 8/s}{3 + 8/s}}$$

$$= \frac{24}{6(3s+8) + 24}$$

$$= \frac{4}{3s + 8 + 4} = \frac{4}{3s+12}$$

$$H(j4) = \frac{4}{12 + j12} = \frac{\cancel{4}\,1}{3(1+j)} = \frac{1}{3\sqrt{2}} \angle -45°$$



$$V_o''(t) = \frac{3\times 8}{\sqrt{17}}\cos(t-\tan^{-1}1/4)$$

$$3 \| 6 = 2$$

$$H(s) = \frac{2\times 8/s}{2+8/s} = \frac{16}{2s+8}$$

$$= \frac{8}{s+4}$$

$$H(j1) = \frac{8}{j1+4} = \frac{8}{\sqrt{17}}\angle -\tan^{-1}(1/4)$$

So,

$$V_o(t) = \frac{2}{3\sqrt{2}}\sin(4t-45^\circ) + \frac{24}{\sqrt{17}}\cos\left(t-\tan^{-1}\frac{1}{4}\right)$$