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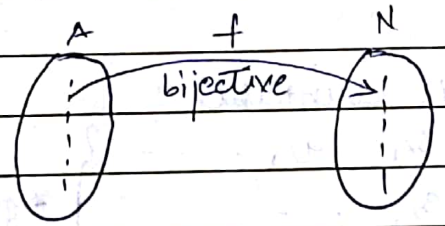
SETS!

A) COUNTABLE SET:

- A set is defined to be countable if:
  i) it is a finite set
  ii) There exists a one-one corresspondance (ie, bijectiveness) between the given function and the set of natural numbers.



- Example:
  show that Z is a countable set.

Solution:

let f be a function from N to Z

$\therefore f: N \to Z$

where $Z = \{0, \pm 1, \pm 2 \ldots\}$

for proving that Z is countable, $g: Z \to N$ should be bijective.

$\therefore$ If $f: N \to Z$ is bijective, then $g = f^{-1}$ should be bijective

$\therefore$ let us assume $f = \begin{cases} \frac{(n-1)}{2}, & n \in \text{odd} \\ -n/2, & n \in \text{even} \end{cases}$

$\therefore$ f is one-one & onto

$\therefore$ f = bijective $\quad \therefore$ g = bijetive

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Q. Show that rational numbers' set is countable.

$\mathbb{Q}$ = set of rational numbers
N = set of natural numbers

① $f: N \times N \to N$
the bijections:
a) $f(m,n) = \frac{1}{2}(m+n-1)(m+n-2) + m$

b) $g(m,n) = 2^{m-1}(2n-1)$

$\therefore$ $N \times N$ is countable.

⇒ Prove that $\mathbb{Q}^+$ is countable:

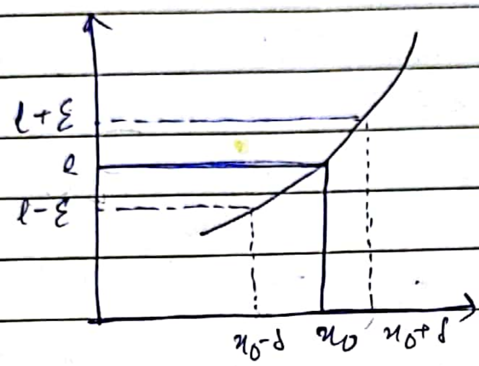
$$\left\{ \begin{array}{llll} 1/1, & 2/1, & 3/1, & 4/1, ---- \\ 1/2, & 2/2, & 3/2 ---- & \\ 1/3, & 2/3, & 3/3 --- & \\ \vdots & \vdots & \vdots & \vdots \end{array} \right\} = \mathbb{Q}^+$$

$f: N \times N \to \mathbb{Q}^+$
$f(m,n) = \frac{m}{n}$

⇒ Union of countable sets is also countable.
⇒ Subset of a countable set is also countable.
⇒ The following are equivalent:
a) S is countable (where S is a set).
b) There exists a onto map from N to S.
c) There exists a one-one map from S to N.

LIMITS:

DEFINATION: if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - l| < \varepsilon$ where $0 < |x - x_0| < \delta$



Examples:

① $f(x) = x^2$ $\lim_{x \to 0} f(x) = l$ $\therefore \lim_{x \to 0} x^2 = 0$

Solⁿ $l = 0$ $f(x) = x^2$ $x_0 = 0$

$\therefore |f(x) - l| < \varepsilon$

$|x^2 - 0| < \varepsilon$

$0 < x^2 < \varepsilon$ $\quad -\sqrt{\varepsilon} < x < \sqrt{\varepsilon}$ — ①

$|x - x_0| < \delta$ $\therefore x < \delta$

but $x < \sqrt{\varepsilon}$ (from ①)

$\therefore \underline{\sqrt{\varepsilon} = \delta}$

② $\lim_{x \to 1} (5x - 3) = 2$

Solⁿ: $l = 2$, $f(x) = 5x - 3$, $x_0 = 1$

$|f(x) - l| < \varepsilon$

$|5x - 3 - 2| < \varepsilon$

$|x - 1| < \dfrac{\varepsilon}{5}$ — ①

$|x - x_0| < \delta$ $\therefore |x - 1| < \delta$

but $|x - 1| < \varepsilon/5$ — from ①

$\therefore \delta = \varepsilon/5$

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now, $|x-x_0|<\delta$

$|x-1|<\delta$

but $\delta=\varepsilon/5$

$\therefore |x-1|<\varepsilon/5$

$|5x-5|<\varepsilon$

$|(5x-3)-2|<\varepsilon$

$\therefore |f(x)-l|<\varepsilon$

Hence proved.

(3) $\lim_{x\to 1} \frac{x^2-1}{x-1}=2$

Soln: $f(x)=\frac{x^2-1}{x-1}$, $x_0=1$, $l=2$

$|f(x)-l|<\varepsilon \quad \therefore \left|\frac{(x+1)(x-1)}{(x-1)}-2\right|<\varepsilon$

$\therefore |x+1-2|<\varepsilon \qquad \therefore |x-1|<\varepsilon$

but $|x-x_0|<\delta \qquad \therefore |x-1|<\delta$

$\therefore \delta=\varepsilon$

now, $|x-x_0|<\delta=\varepsilon$

$\therefore |x-1|<\varepsilon \quad \therefore |x+1-2|<\varepsilon$

$|f(x)-l|<\varepsilon$

Hence proved.

(4) $\lim_{x\to 5}\sqrt{x-1}=2$

Soln, $f(x)=\sqrt{x-1}$, $x_0=5$, $l=2$

$|f(x)-l|<\varepsilon \quad |\sqrt{x-1}-5|<\varepsilon$

$5-\varepsilon<\sqrt{x-1}<\varepsilon+5$

$(5-\varepsilon)^2<x-1<(\varepsilon+5)^2 \qquad x\in((5-\varepsilon)^2+1,(5+\varepsilon)^2+1)$

$\delta=\min\{\cancel{1}-(5-\varepsilon)^2,(5+\varepsilon)^2-1\}$

⑤ $\lim_{x \to 3} (3x-7) = 2$

Sol^n. $f(x) = 3x-7$, $l = 2$; $x_0 = 3$

$\therefore |f(x) - l| < \varepsilon$

$|3x-7-2| < \varepsilon$

$-\varepsilon < 3x-9 < \varepsilon$

$\frac{9-\varepsilon}{3} < x < \frac{\varepsilon+9}{3}$ $\qquad x \in \left(\frac{9-\varepsilon}{3}, \frac{\varepsilon+9}{3}\right)$

$|x-3| < \delta$

$3-\delta < x < \delta+3$

$\therefore \delta = \min\left\{3 - \frac{9-\varepsilon}{3}, \frac{\varepsilon+9}{3} - 3\right\}$

⑥ $\lim_{x \to 0} \sqrt{4-x} = 2$

Sol^n: $f(x) = \sqrt{4-x}$, $x_0 = 0$, $l = 2$

$|f(x) - l| < \varepsilon$

$|\sqrt{4-x} - 2| < \varepsilon$

$2 - \varepsilon < \sqrt{4-x} < \varepsilon + 2$

$(2-\varepsilon)^2 < 4-x < (\varepsilon+2)^2$

$4-(\varepsilon+2)^2 < x < 4-(2-\varepsilon)^2$ $\qquad \therefore x \in \left(4-(\varepsilon+2)^2, 4-(2-\varepsilon)^2\right)$

$x_0 = 0$

$|x - x_0| < \delta$ $\qquad \therefore -\delta < x < \delta$

$\therefore \delta = \min\left\{\cancel{40}\ (\varepsilon+2)^2 - 4, 4-(2-\varepsilon)^2\right\}$

$\delta = \min\{\varepsilon^2 + 4\varepsilon, -\varepsilon^2 + 4\varepsilon\}$

⑦ $f(x) = \begin{cases} x^2 , & x \neq 1 \\ 2 , & x = 1 \end{cases}$ $\qquad \lim_{x \to 1} x^2 = 1$ $\qquad l = 1$

Sol^n: $|x^2 - 1| < \varepsilon$ $\qquad 1-\varepsilon < x^2 < \varepsilon+1$ $\qquad \sqrt{1-\varepsilon} < x < \sqrt{\varepsilon+1}$

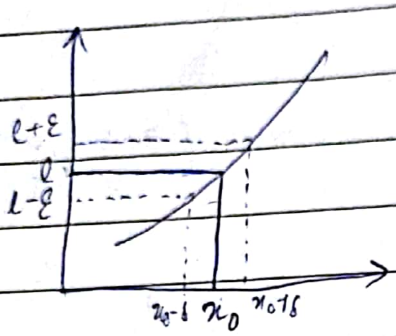
$|x-1| < \delta$ $\qquad \therefore 1-\delta < x < \delta+1$

$\therefore \delta = \min\{1-\sqrt{1-\varepsilon}, \sqrt{\varepsilon+1} - 1\}$

$\Rightarrow$ L.H.L: $\lim_{x \to x_0^-} f(x) = l$

For each $\varepsilon > 0$ there exists $\delta > 0$ such that

$|f(x) - l| < \varepsilon$ whenever $x_0 - \delta < x < x_0$



$\Rightarrow$ RHL: $\lim_{x \to x_0^+} f(x) = l$

For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - l| < \varepsilon$ for all $x \in (x_0, x_0+\delta)$

$\Rightarrow$ when LHL = RHL, $\lim_{x \to x_0} f(x) = l$

Examples:

① $\lim_{x \to 0^+} \sqrt{x} = 0$ find LHL.

$|\sqrt{x} - 0| < \varepsilon$

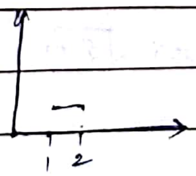
$|\sqrt{x}| < \varepsilon$

$x < \varepsilon^2$

$x \in (x_0, x_0+\delta)$ $\therefore x \in (0, \delta)$

$\therefore \delta = \varepsilon^2$

$\Rightarrow$ $\lim_{x \to 1} [x] =$ doesn't exist



PROPERTIES OF LIMITS:

$$\lim_{x\to x_0}\left(f(x)\pm g(x)\right) = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x)$$

$$\lim_{x\to x_0}\left(f(x)\cdot g(x)\right) = \lim_{x\to x_0} g(x) \cdot \lim_{x\to x_0} f(x)$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}, \text{ given } \lim_{x\to x_0} g(x) \neq 0$$

$$\lim_{x\to x_0} (f(x))^n = \left(\lim_{x\to x_0} f(x)\right)^n$$

$$\lim_{x\to x_0} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to x_0} f(x)} \quad (n = +ve)$$

SANDWITCH THEOREM:

(1) If $f(x) \leq g(x) \leq h(x)$ for all $x \in I$

(2) $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = l$ (3) $\therefore \lim_{x\to x_0} g(x) = l$

Example: $-|x| \leq \sin x \leq |x|$

$$\lim_{x\to 0} -|x| = 0 = \lim_{x\to 0} |x|$$

$$\therefore \lim_{x\to x_0} \sin x = 0$$

CONTINUITY:

Definition: $f(x)$ is said to be continuous at x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$|f(x) - l| < \varepsilon$ where $|x - x_0| < \delta$, $x = x_0$ also valid.

$\therefore$ LHL = RHL = $f(x_0)$ ← function is said to be continuous at x_0.

(1) $\lim_{x\to 3} (x^2+2x) = 15$

Prove using ε, δ defination.

$f(x) = x^2+2x \quad l = 15 \quad x_0 = 3$

$|f(x) - l| < \varepsilon \qquad |x - x_0| < \delta$

$|x^2+2x-15| < \varepsilon \qquad |x-3| < \delta$

$-\varepsilon < (x+5)(x-3) < \varepsilon \qquad -\delta < (x-3) < \delta$

$\therefore \delta + 8 < x+5 < \delta + 8 \qquad \therefore |(x+5)(x-3)| < \varepsilon = \delta \cdot (\delta+8)$

$\therefore \qquad \therefore \delta^2 + 8\delta - \varepsilon = 0$

$\delta = \min\left\{ \frac{-8+\sqrt{64+4\varepsilon}}{2}, \frac{-8-\sqrt{64+4\varepsilon}}{2} \right\}$

(2) $f(x) = x+2 \quad \lim_{x\to 3} (x+2) = 5$

$|f(x) - l| < \varepsilon$

$|x+2-5| < \varepsilon \qquad \varepsilon = 0.5$

$\therefore |x-3| < 0.5 \qquad |x-3| < \delta$

$\therefore \delta = 0.5$

(3) $\lim_{x\to 2} (x-4) = 3$ prove that limit doesn't exist.

$|f(x) - l| < \varepsilon \quad \therefore |x-4-3| < \varepsilon \qquad |x - x_0| < \delta \quad \therefore |x-2| < \delta$

$|x-7| < \varepsilon \qquad |x-2| < \delta$

$-\varepsilon < x-7 < \varepsilon \qquad -\delta < x-2 < \delta$

$-(\delta+5) < x-7 < (\delta-5)$

$\therefore \varepsilon = \delta+5 \quad , \text{or} \quad \varepsilon = \delta - 5$

$\therefore \delta = \varepsilon - 5 \qquad \delta = \varepsilon + 5$

$\delta = \min\{\varepsilon-5, \varepsilon+5\}$

$= \varepsilon - 5$

$\therefore$ for $\varepsilon = 1$, $\delta = -4$,

$\therefore$ limit doesn't exist

$$\lim_{x\to 3} (x^2+2x) = 15$$

$\lim_{x\to 2} x^2+4x$ $(x+8)$

$|f(x) - l| < \varepsilon$ $\qquad$ $|x-x_0| < \delta$

$\therefore |x^2+2x-15| < \varepsilon$ $\qquad$ $|x-3| < \delta$

$|(x+5)(x-3)| < \varepsilon$

assume $\delta = 1$. — (1)

$-1 < x-3 < 1$

$2 < x < 4$

$7 < x+5 < 9$

$|x+5| < 9$

$\therefore |(x+5)(x-3)| < \delta \cdot 9 \le \varepsilon$

$\therefore \delta = \frac{\varepsilon}{9}$ — (2)

$\therefore \delta = \min\left\{1, \frac{\varepsilon}{9}\right\}$ from (1) and (2)

CONTINUITY:

⇒ f is said to be right continuous at x_0 if $\lim_{x\to x_0^+} f(x) = f(x_0)$

Example: $f(x) = [x]$

$\lim_{x\to n^+} [x] = n$ $\qquad$ $\lim_{x\to n^-} [x] = n-1$

$f(n) = [n] = n$ $\qquad$ where $n \in I$

$\lim_{x\to x_0^+} f(x) = f(x_0)$

$\therefore$ right continuous.

⇒ f is said to be left continuous at x_0 if $\lim_{x\to x_0^-} f(x) = f(x_0)$

Example: $f(x) = (x)$ $\quad$ ((x) = ceiling function)

$f(x) = \sqrt{4-x^2} \qquad x \in [-2, 2]$

f is continuous at -2, if

$$\lim_{x\to -2^+} f(x) = f(-2)$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \sqrt{4-x^2} = 0 = f(-2)$$

$\therefore$ conti. at -2.

f is continuous at $x=2$ if

$$\lim_{x\to 2^-} f(x) = f(2)$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \sqrt{4-x^2} = 0 = f(2)$$

TYPES OF DISCONTINUITY:

Removable discontinuity:

x_0 is said to be a removable continuity of $f(x)$ if

$$\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) \neq f(x_0)$$

$$\therefore \lim_{x\to x_0} f(x) \neq f(x_0)$$

Example: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ but $f(0) =$ undefined $\left(f(x) = \frac{\sin x}{x}\right)$

$\therefore$ this is discontinuous at $x=0$

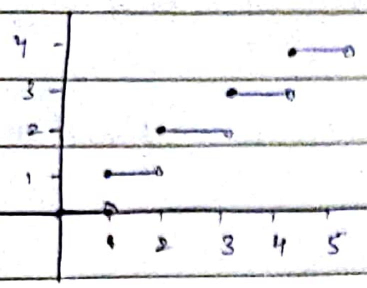
this can be removed by assigning a particular value to function at $x=0$.

$$g(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$\therefore g(x) =$ continuous.

2. Jump discontinuity:

Example: $f(x) = \lfloor x \rfloor$ (floor function)



here each $x \in I$ is jump discontinuity.

LHL $\neq$ RHL at all $x \in I$

(discontinuous from left side as LHL $\neq$ RHL $= f(x_0)$)

Example: ceiling function, $f(x) = \lceil x \rceil$

(this is discontinuous from right side as RHL $\neq$ LHL $= f(x_0)$)

3. Infinite discontinuity:

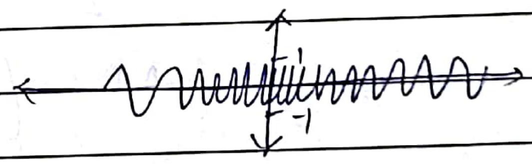
when value of $f(x)$ becomes infinite at a ~~c~~ certain point.

Example: $f(x) = \frac{1}{x^2}$

$f(0) = \infty$ $\lim_{x \to 0} \frac{1}{x^2}$ doesn't exist

4. Oscillatory discontinuity:

Example: $f(x) = \sin \frac{1}{x}$



(at $x=0$)

the functions oscillates so much so that it is impossible to find LHL or RHL.

PROPERTIES OF CONTINUOUS FUNCTIONS:

if f and g are continuous functions $x = x_0$, then:

① $f+g$, $f \cdot g$, $f-g$, kf are all continuous (k = scalar)

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(2) $\frac{f}{g}$ is continuous if $g(x_0) \neq 0$

(3) f^n is ~~continuous~~ continuous. ~~(exists)~~ $n = +ve$

(4) $\sqrt[n]{f}$ is continuous if it is defined on the domain and $n = +ve$

(5) Any polynomial function is continuous.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

(6) if $P(x)$ and $Q(x)$ are 2 polynomial functions, $\frac{P(x)}{Q(x)}$ is continuous. $(Q(x) \neq 0)$

(7) if g = continuous at $x = b$ and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g\left(\lim_{x \to c} f(x)\right) = g(b)$$

Ex: $\lim_{x \to \pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)$

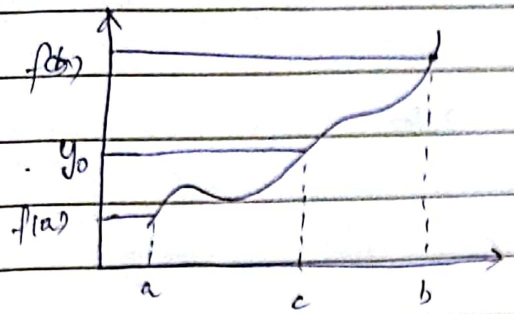
$$= \cos\left(\lim_{x \to \pi/2}\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)\right)$$

$$= \cos\left(\pi + \sin\left(\lim_{x \to \pi/2}\left(\frac{3\pi}{2} + x\right)\right)\right)$$

$$= \cos\left(\pi + \sin\left(\frac{3\pi}{2} + \frac{\pi}{2}\right)\right) = -1$$

(8) Intermediate value theorem:

If $f(x)$ is continuous on $[a, b]$, if y_0 is any value on $f(a)$ and $f(b)$, then there exists a $c \in (a, b)$ such that $f(c) = y_0$



here $y_0 = f(c)$

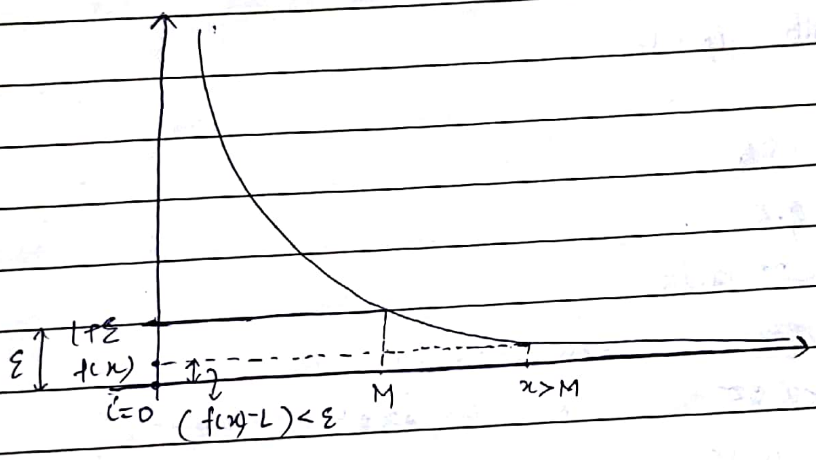
LIMITS INVOLVING INFINITY:

A) $\lim_{x \to \infty} f(x) = L$

if for every $\varepsilon > 0$, $\exists$ a real no $M > 0$, such that for all $x > M$, $|f(x) - L| < \varepsilon$

$f(x) = 1/x$

$x \to \infty$, $L = 0$



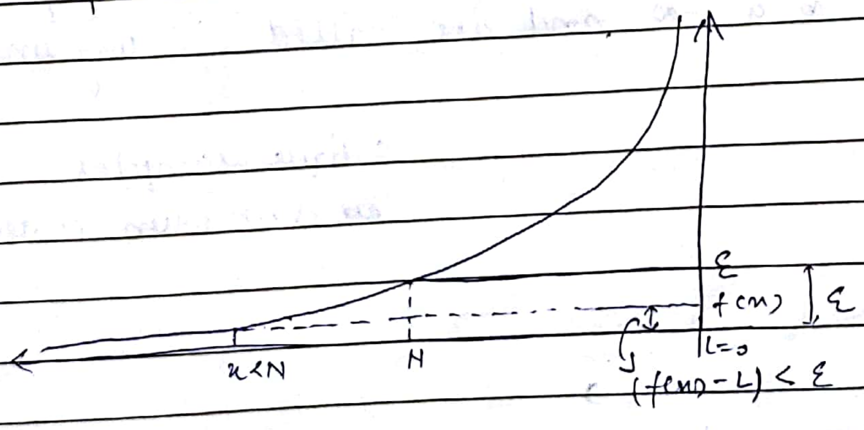
$\therefore |f(x) - L| < \varepsilon$

B) $\lim_{x \to -\infty} f(x) = L$

if for every $\varepsilon > 0$, $\exists$ a real no. N such that for all $x < N$, $|f(x) - L| < \varepsilon$

$f(x) = \frac{1}{x}$

$x \to -\infty$, $L = 0$



* THOMAS CALCULUS:

① $f(x) = 3-2x \quad x_0 = 3 \quad \varepsilon = 0.02$

$$\lim_{x\to 3} 3-2x = -3$$

$$\therefore |3-2x+3| < 0.02$$

$$-0.02 < 6-2x < 0.02$$

$$-0.01 < 3-x < 0.01$$

$$2.99 > x > -3.01$$

$$\therefore \delta = 0.01$$

② $f(x) = \sqrt{1-5x} \quad x_0 = -3 \quad \varepsilon = 0.5$

$$\lim_{x\to -3} f(x) = \sqrt{16} = 4 = L$$

$$-0.5+4 < \sqrt{1-5x} < 0.5+4$$

$$3.5 < \sqrt{1-5x} < 4.5$$

$$12.25 < 1-5x < 19.25$$

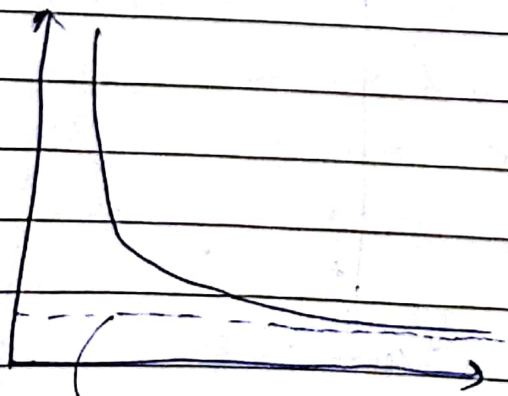
$$-3.85 < x < -2.25$$

$$\therefore \delta = \min\{-3+3.85,\ 3+-2.25\}$$

$$\therefore \delta = 0.75$$

ASYMTOTES:

asymtotes are the lines which will touch the graph of f(x) at ∞ or -∞ and are called limiting lines also.



horizontal asymtote

Oblique asymptotes are exist when power of numerator is 1 > than denominator

① $f(x) = \frac{11x+2}{2x^3-1}$ find its asymptote (horizontal):

$$\lim_{x\to\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to\infty} \frac{11/x^2 + 2/x^3}{2 - 1/x^3} = 0$$

$\therefore$ asymptote (horizontal) $\Rightarrow y = 0$

② find the oblique asymptote ~~(vertical asymptote)~~

$$f(x) = \frac{x^2-3}{2x-4}$$

$$= \left(\frac{x}{2}+1\right) + \frac{1}{2x-4}$$

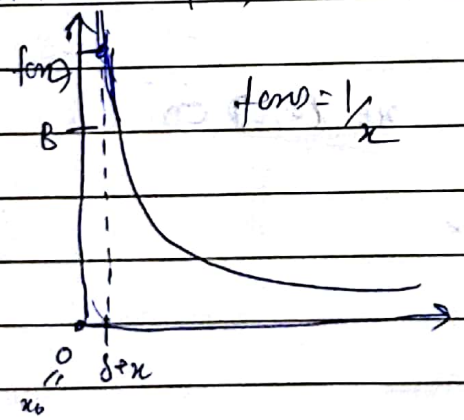
$$\lim_{x\to\infty} \frac{1}{2x-4} = 0$$

$\therefore$ oblique asymptote $= \left(\frac{x}{2}+1\right)$

③ vertical asymptote:

$$\lim_{x\to x_0} f(x) = \infty$$

For every real number $B>0$, $\exists\ \delta>0$ for $0<|x-x_0|<\delta$ such that $f(x) > B$



A line $x=a$ is a vertical asymptote of this curve if either $\lim_{x\to a^+} f(x) = \pm\infty$ ✓

or $\lim_{x\to a^-} f(x) = \pm\infty$

DIFFERENTIABILITY:

continuous $\not\Rightarrow$ differentiable

but differentiable $\Rightarrow$ continuous.

$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ = rate of change of function
= instantaneous change
: $f'(x)$

APPLICATIONS OF DERIVATIVES:

① maxima and minima:

a) global maximum / absolute maximum:

domain of $f = D$

if $f(x) \leq f(c)$ where $c, x \in D$

then $f(c)$ = global maximum.

b) local maximum / relative maximum:

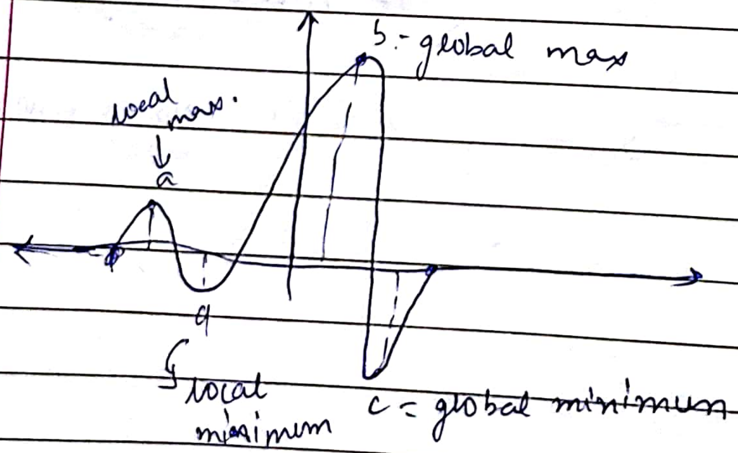
$f(x) \leq f(c)$ for all $x \in [a,b] \in D$

c) global minimum

$f(c) \leq f(x)$ for all $x \in D$

d) local minimum

$f(c) \leq f(x)$ for all $x \in [a,b] \subset D$



③ <u>Extreme value theorem:</u>

if $f(x)$ is continuous on a closed bounded interval then extreme value exists.

$f(x) = x^2$ $\quad [1, 2]$ $\quad$ (max and min exist)

$f(x) = x^2$ $\quad (-\infty, \infty)$ $\quad$ (min exist, max doesn't)

⑤ <u>1st derivative theorem:</u>

if f has a local max or min at an interior point c of its domain and f' exists at c, then

$$f'(c) = 0$$

<u>Proof:</u> let f has a local max at c.

$$f(x) \le f(c) \qquad \forall\, x \in (c-\delta, c+\delta)$$

since c is an interior point and $f'(c)$ exists

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \text{ exists}$$

and

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \text{ exists}$$

$$\lim_{x \to c^+} \frac{-ve}{+ve} \le 0 \qquad (\because f(x) \le f(c) \text{ \& } x > c)$$

$$\lim_{x \to c^-} \frac{-ve}{-ve} \ge 0 \qquad (\because f(x) \le f(c) \text{ and } x < c)$$

but $f'(c)$ exists

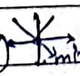
$$\therefore \lim_{x \to c^+} = \lim_{x \to c^-} = f'(c) \quad \therefore f'(c) = 0$$

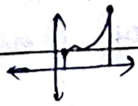
$\therefore$ Proved.

Similar argument for local minimum.

<u>NOTE:</u> fo can possibly have an extreme value where:

i) interior point where $f'(c) = 0$

ii) interior point where f' doesn't exist (Eg: $f(x) = |x|$)  min

iii) end points of domain. 

Sufficient condition:

if function is differentiable upto n times on a domain D.

$f'(c)=0$,

look for the first ~~odd~~ even non-zero derivative.

$f^{(1)}=0 \quad f^{(3)}=0 \quad \dots \quad f^{(n=odd)}=0$

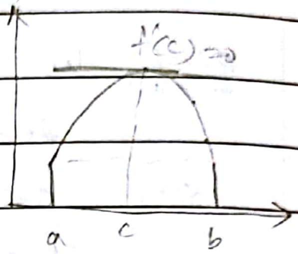
i) if $f^{(n=even)}>0$, $c=$ min (local)

ii) if $f^{(n=even)}<0$, $c=$ max (local)

iii) if $f^{(n=odd)}=0$, $c=$ neither local max/min.

ROLE'S THEOREM:

f is defined on $[a,b]$



if (i) f is continuous on $[a,b]$

(ii) f is differentiable on (a,b)

(iii) $f(a)=f(b)$

Then there exists $c\in(a,b)$ such that $f'(c)=0$

f is differentiable on all internal points! (a,b)

f is continuous on $[a,b]$.

By extreme value theorem, maximum & minimum exists/occurs.

∴ max/min occurs at interior points or end points

i) if max/min occurs at $c\in(a,b)$

$f'(c)=0$

ii) if max/min occurs at end points.

then because $f(a)=f(b)$, it will be a constant function.

∴ for all $c\in(a,b)$ $f'(c)=0$

∴

Examples:

$f(x)=x^3+3x+1$ ~~Proof~~ show that only one real solution.

$f(-1)=-3$

$f(0)=1$

as the function is continuous, the function crosses the x-axis at least once.

assume that $f(x)=$ has 2 real roots.

$\therefore f(x_1)=f(x_2)=0$

By Rolle's theorem, $\exists$ $c\in(x_1,x_2)$, such that $f'(c)=0$

$f'(x) = 3x^2+3$

$f'(x)=0 \quad \therefore \quad 3x^2+3=0 \quad \therefore x^2=-1$

↳ not possible

$\therefore$ no $c\in(x_1,x_2)$ such that $f'(c)=0$

$\therefore$ assumption is wrong

$\therefore$ only once crosses the x-axis

$\therefore$ only one real root.

MEAN VALUE THEOREM (MVT)

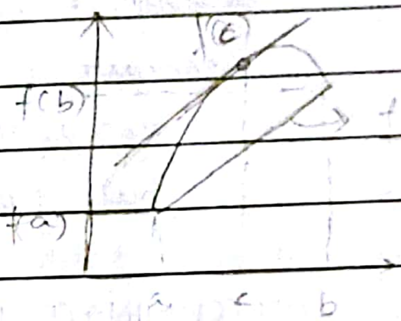
let $f(x)$ be:

i) continuous in $x\in[a,b]$

ii) differentiable on $x\in(a,b)$

Then $\exists$ $c\in(a,b)$, such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$



Proof: Eq. of line joining $(a,f(a))$ and $(b,f(b))$

$$g(x)= f(a) + \left\{\frac{f(b)-f(a)}{b-a}\right\}(x-a) \qquad \{y=mx+c\}$$

$h(x)=f(x)-g(x)$ (distance betn line and curve at any point)

$$= f(x)-f(a)-\left\{\frac{f(b)-f(a)}{b-a}\right\}(x-a)$$

$f(x)=$ continuous $\quad g(x)=$ continuous

$\therefore h(x)=f(x)-g(x)=$ continuous.

similarly, all 3 are differentiable.

$h(a)=0 \qquad h(b)=0 \qquad \therefore h(a)=h(b)$

$\therefore$ by Rolle's theorem, $\exists$ $c\in(a,b)$ such that $h'(c)=0$

$\therefore h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$

$\therefore h'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a}$

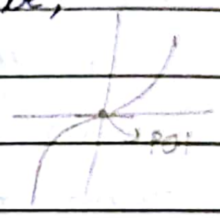
$\therefore f'(c) = \frac{f(b)-f(a)}{b-a}$

CONCAVE UPWARD AND CONCAVE DOWNWARD (CONVEX):

A differentiable function $f(x)$ is concave up on an interval I if $f''(x) > 0$ on I
concave down on I if
$f''(x) < 0$ on I.

POINT OF INFLECTION:

~~concav~~ point at which concavity changes ~~occurs~~, ie, concave to convex or convex to concave.



$f'(x) = 0$ at points of inflection
but neither max nor min.

INDETERMINATE FORMS:

$\frac{0}{0}$, $0 \cdot \infty$, $\frac{\infty}{\infty}$ etc

Suppose $f(a) = g(a) = 0$, and f, g are differentiable on open interval I containing 'a' and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

) $\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$

$= \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}} - \frac{1}{2}}{2x}$

$= \frac{1}{4} \lim_{x\to 0} -\frac{1}{2}(1+x)^{3/2} = \frac{1}{8}$

② $\lim_{x\to 0} \frac{x - \sin x}{x^3}$

$= \lim_{x\to 0} \frac{1-\cos x}{3x^2}$

$= \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6}$

2^{nd} MEAN VALUE THEOREM:

f and g are ~~continuous~~
1) continuous on $[a,b]$
2) differentiable on (a,b)

and also $g'(x) \neq 0$ on (a,b)

Then $\exists$ $c \in (a,b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

TAYLOR'S THEOREM:

$f(x)$ is n times continuously differentiable and the $(n+1)$th derivative exists on $[a,b]$

let $x_0 \in [a,b]$

For every x near x_0 (taylor's series)

$$f(x) = f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots$$

$$\cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \boxed{\frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\eta)}$$

where $\eta \in (x_0, x)$

Remainder

= error term

① $f(x) = \sin x$ $x_0 = 0$

Find the taylor's series explansion of $\sin x$ upto 4 terms and obtain an error ~~bound~~ bound.

$x \in [0,1]$

Sol^n: $f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^4(x) = \sin x \qquad f^4(0) = 0$

$f^5(x) = \cos x \qquad f^5(0) = 1$

$f^6(x) = -\sin x \qquad f^6(0) = 0$

$f^7(x) = -\cos x \qquad f^7(0) = -1$

$$\therefore f(x) = \frac{(x-0)}{1} f'(0) + f(0) + \frac{(x-0)^2}{2!} f''(0) - - -$$

$$- - - + f^7(0) \frac{(x-0)}{7!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

Remainder term !

$$R = \frac{x^8}{8!} \sin \eta$$

$$|R| = \left| \frac{x^8}{8!} \sin \eta \right| \leq \left| \frac{x^8}{8!} \max \{\sin x, x \in [0,1]\} \right|$$

$$= \frac{1^8}{8!} \sin 1 \leq \frac{\sin 1}{8!}$$

INTEGRATION:

→ Finite number of discontinuities : integration is possible

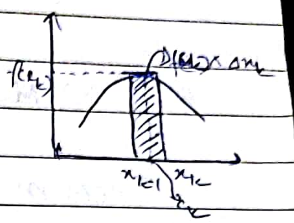
→ $\int_a^b f(x)\,dx$

$a = x_0$ .... $b = x_n$

$\Delta x_k = x_k - x_{k-1}$

c_k = any point in k^{th} interval

$\sum_{1}^{n} f(c_k) \times (\Delta x_k)$ = sum of area of rectangles



$$\therefore \lim_{\Delta x_k \to 0} \sum_{1}^{n} f(c_k)\,\Delta x_k = \int_a^b f(x)\,dx$$

$\sum_{1}^{n} f(c_i)\,\Delta x_k$ = Riemann's sum

a

$$\lim_{n \to \infty} \sum_{1}^{n} f(c_k)\,\Delta x_k = \int_a^b f(x)\,dx$$

(1) $f(x) = \begin{cases} 1 & x = \text{rational} \\ 0 & x = \text{irrational} \end{cases}$

check if $f(x)$ is integrable over $[0,1]$.

sol^n: let P be a part in $x \in [0,1]$

and $[x_{k-1}, x_k]$ be the k^{th} subinterval

(i) c_k = point that gives max value of $f(x)$ in k^{th} interval.

$$\sum_{1}^{n} f(c_k)\,\Delta x_k = \sum_{1}^{n} \Delta x_k \qquad [\because f(x_k) = 1 \text{ (max)}]$$

$$= 1$$

(ii) c_k = point giving min value of $f(x)$ in k^{th} interval

$$\therefore \sum_{1}^{n} f(c_k)\,\Delta x_k = \sum_{1}^{n} 0 \cdot \Delta x_k = 0 \qquad [\because f(x_k) = 0 \text{ (min)}]$$

as $I(i) \neq I(ii)$

$\therefore$ not integrable.

MVT FOR INTEGRATION:

→ If x is continuous on $[a,b]$, then at some $c \in [a,b]$ such that : $f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$

FUNDAMENTAL THEOREMS ON CALCULUS:

① Theorem 1:

→ if f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is

(i) continuous $[a,b]$

(ii) differentiable on (a,b)

(iii) $F'(x) = f(x)$ ie, its derivative is $f(x)$

② Theorem 2:

→ i) if f is continuous on $[a,b]$ and

ii) $\exists$ F such that $F'(x) = f(x)$

then :

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

ie, $f(x)$ is integrable

APPLICATION OF ~~DERIVATIVE~~ INTEGRALS:

① Area bounded by curve $y = f(x)$ over $[a,b]$

i) sub divide the interval $[a,b]$ at zeros of $f(x)$

ii) integrate

iii) add absolute values of integrals.

t: Let f be continuous on symmetric interval $[-a,a]$

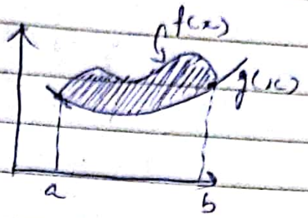
i) f = even $\therefore \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

ii) f = odd. $\therefore \int_{-a}^{a} f(x)\,dx = 0$

⇒ for integral wrt y, $I = \int_a^b f(y)\,dy$

(2) area bounded by 2 curves.

$$A = \int_a^b (f(x) - g(x))\,dx$$



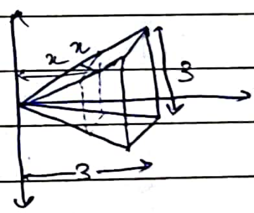
(3) Volume using cross-section:

Def^n The volume of a solid of integrable cross-sectional area $A(x)$ from $x=a$ to $x=b$ is the integral

$$V = \int_a^b A(x)\,dx$$

Q: A pyramid is 3 m high, having a square base with side 3m. The cross-sectional area of the pyramid ⊥ular to the altitude x m down is x m square. Find the volume.

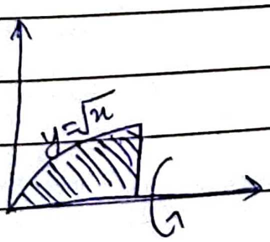
$A(x) = x^2$ $\qquad \therefore V = \int_0^3 x^2\,dx = 9\ m^3$

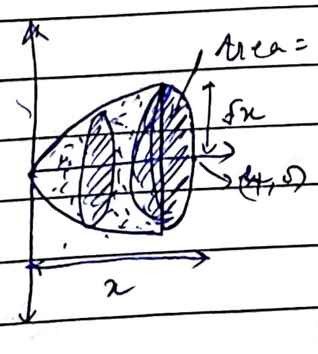


(4) solids of revolution:

The solid generated by rotating a plane region about an axis in its plane is called solid of revolution.

Q:





Area $= \pi(\sqrt{x})^2 = \pi x$

$\therefore A(x) = \pi x$

$\therefore$ volume =

$V = \int_0^4 \pi x\,dx$

$V = 8\pi\ m^3$

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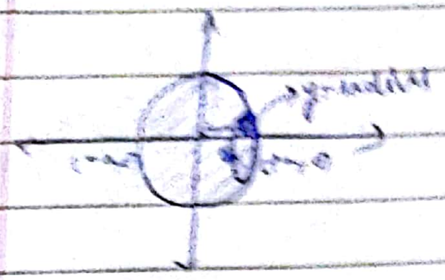
Q. The circle $x^2+y^2=a^2$ is rotated about the x-axis to generate a sphere. Find V.



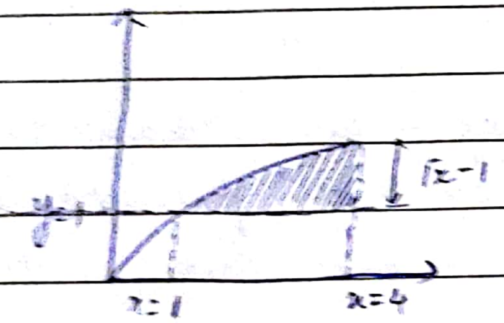
$A(x) = \pi y^2 = \pi(a^2-x^2)$

$V = \int_{-a}^{a} \pi(a^2-x^2)$

$= \frac{\pi}{3}\left[a^2\right]_{-a}^{a} - \frac{1}{3}\left[x^3\right]_{-a}^{a}$

$= \frac{4\pi a^3}{3}$

Q. Find V of solid generated by ~~y~~ revolving the region bounded by $y=\sqrt{x}$, $y=1$ and $x=4$ about $y=1$



$\therefore A(x) = \pi(\sqrt{x}-1)^2$

$\therefore V = \int_1^4 \pi(\sqrt{x}-1)^2 = \pi\int_1^4 (x-2\sqrt{x}+1)\,dx$

$= \pi\left[\frac{x^2}{2} - \frac{2}{3}x^{3/2} + x\right]_1^4$

$= \pi\left[\frac{16}{2} - \frac{32}{3} + 4 - \frac{1}{2} + \frac{4}{3} - 1\right]$

$= \pi\left(\frac{15}{2} - \frac{28}{3} + 3\right) = \frac{7\pi}{6}$

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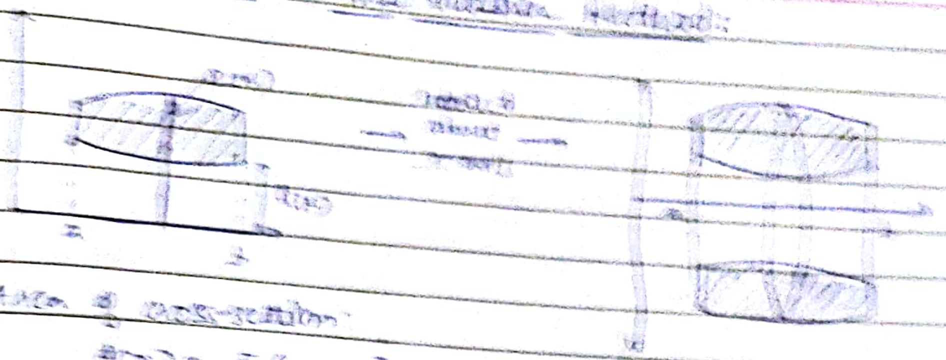
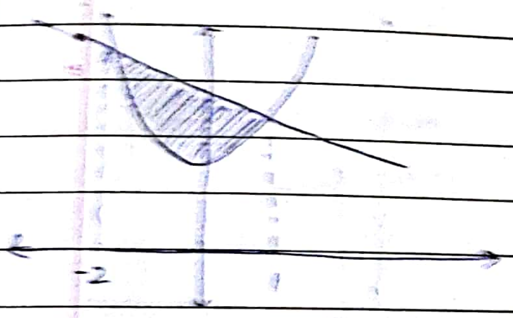
(5) Solids of Revolution: The Washer Method:



Area of cross-section:

$$A(x) = \pi \{ (R(x))^2 - \pi (r(x))^2$$

Q. The region bounded by the curve $y = x^2+1$ and $y = -x+3$ is revolved about x-axis to generate a solid. Find the volume of the solid.



$$A(x) = \pi \{ (R(x))^2 - (r(x))^2 \}$$

$$R(x) = -x+3$$

$$r(x) = x^2+1$$

$$A(x) = \pi \{ (-x+3)^2 - (x^2+1)^2 \}$$

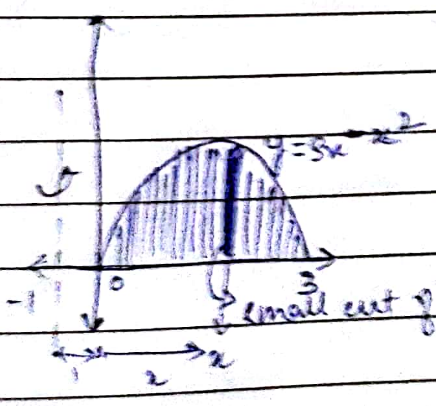
$$= \pi \{ x^2 - 6x + 9 - x^4 - 2x^2 - 1 \}$$

$$= \pi \{ -x^4 - x^2 - 6x + 8 \}$$

$$\therefore V = \int_{-1}^{2} A(x) = \frac{117\pi}{5}$$

(6) Volume of solid using with [illegible] cylindrical shells:

Slicing with cylinders (SHELL METHOD):



small cut of curve

each slice can be rotated individually to form hollow cylinders

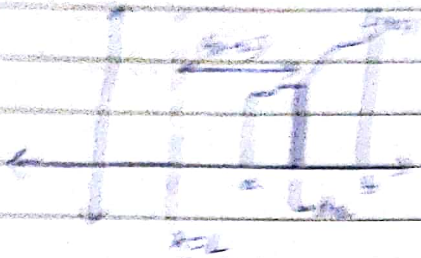
$\therefore$ volume of the cylindrical shell $= 2\pi \times r \times h \times \Delta x$

here $r = x+1$ (from line $x = -1$) $h = f(x) = y = 3x - x^2$

[illegible]

[illegible]

Special formula: [illegible]



$$\boxed{[illegible]}$$

③ **Arc length of curve:**

Def: If f' is continuous on $[a,b]$ then length of curve (arc length) $y=f(x)$ from $(a, f(a))$ to $(b, f(b))$ is:

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Q. Find length of curve:

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \qquad x \in [0,1]$$

$$\frac{dy}{dx} = \frac{2\times 4\sqrt{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{2}} \times x^{1/2}$$

$$= 2\sqrt{2}\sqrt{x}$$

$$\therefore \quad L = \int_0^1 \sqrt{1+8x}\,dx$$

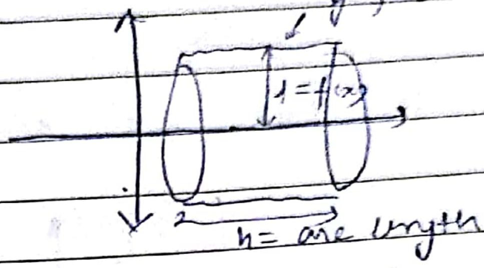
$$= \frac{1\cdot 2}{8\cdot 3}\left[(1+8x)^{3/2}\right]_0^1$$

$$= \frac{1}{12} \times \frac{3\times3\times3-1}{12} = \frac{9}{4} - \frac{1}{12} = \frac{13}{6}$$

(8) Surfaces of revolution:

If the function $f(x) \geq 0$ is continuous and differentiable on $[a,b]$, the area of the surface generated by revolving the graph $y=f(x)$ about x-axis is:

$$\boxed{S = \int_a^b 2\pi f(x) \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}$$



$\because \; S = \int 2\pi r h$

Q Find the surface area generated by revolving the curve $f(x)=2\sqrt{x}$ about x-axis. $x \in [1,2]$.

$$S = \int_1^2 2\pi \times f(x) \times \sqrt{\left(\frac{dy}{dx}\right)^2+1}\, dx$$

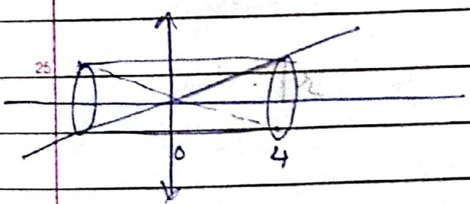
$$= \int_1^2 2\pi \times 2\sqrt{x} \times \sqrt{\frac{1}{x}+1}\, dx$$

$$= 4\pi \int_1^2 \sqrt{x+1}\, dx$$

$$= 4\pi \int_1^2$$

EXAMPLES:

(1) $y = \frac{x}{2}$ $\quad x \in [0,4]$ rotated about x-axis.



$$S = \int_0^4 2\pi \times \frac{x}{2} \times \sqrt{1+\left(\frac{1}{2}\right)^2}\, dx$$

$$= \pi \int_0^4 x \sqrt{\frac{5}{4}}\, dx$$

$$= \frac{\pi\sqrt{5}}{2} \times \frac{16}{2} = 4\sqrt{5}\pi$$

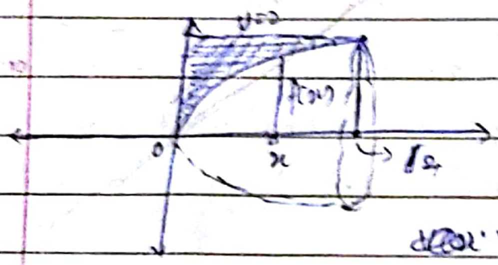
about y-axis:

$x = 2y \qquad y \in [0, 2]$

$$S = \int_0^2 2y \times 2\pi \times \sqrt{1+4}\, dy$$

$$= 8\sqrt{5}\,\pi$$

(2) $y = \sqrt{x}$, $y = 2$, $x = 0$ around x-axis. Find volume.



$\cancel{A(x) = \pi [f(x)]^2}$

$A(x) = \pi(4 - x)$

$$V = \int_0^4 A(x)\, dx$$

$$V = \pi \int_0^4 (4 - x)\, dx$$

$$= \pi \left[4x - \frac{x^2}{2} \right]_0^4 = \pi(16 - 8) = 8\pi$$

DIFFERENTIAL EQUATIONS:

→ function: $f\left(\frac{dy}{dx}, x, y\right) = 0$

→ n^{th} order differential equation:

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, x, y\right) = 0$$

① $\frac{dy}{dx} + x = 0$

$$\therefore \frac{dy}{dx} = -x$$

$$\int dy = -\int x\,dx$$

$$y = \frac{-x^2}{2} + c$$

Seperation of variable method:

variables can be seperated.

Above question is an example.

Homogeneous differential equation:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

- $y/x = v$ $\qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore v + x\frac{dv}{dx} = f(v)$$

② $\frac{dy}{dx}$

#

$$\frac{dy}{dx} = \frac{a_1x + a_2y + a_3}{b_1x + b_2y + b_3}$$

Case II: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Eg: $\frac{dy}{dx} = \frac{x+y+1}{2x+y+3}$

let $x = X + h$

$y = Y + l$

$\therefore dx = dX$

$dy = dY$

$\therefore \frac{dY}{dX} = \frac{X+h+Y+l+1}{2X+2h+Y+l+3}$

$= \frac{X+Y+(h+l+1)}{2X+Y+(2h+l+3)}$

let $h+l+1 = 0$

$2h+l+3 = 0$

$\therefore \underline{h = -2} \quad \underline{l = 1}$

$\therefore \frac{dY}{dX} = \frac{X+Y}{2X+Y}$

now eq is homogeneous differential eq.

$\therefore \frac{Y}{X} = V$

$\therefore \frac{dY}{dX} = V + X\frac{dV}{dX}$

$\therefore V + X\frac{dV}{dX} = \frac{1+V}{2+V}$

$X\frac{dV}{dX} = \frac{1-V-V^2}{2+V}$

$\therefore \frac{2+V}{1-V-V^2}\, dV = \frac{dX}{X}$

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$$\frac{dy}{dx} + y P(x) = Q(x)$$

$$e^{\int P dx}\left(\frac{dy}{dx} + y P(x)\right) = Q e^{\int P dx}$$

$$\frac{d}{dx}\left(e^{\int P dx} \cdot y\right) = Q e^{\int P dx}$$

integrating:

$$e^{\int P dx} \cdot y = \int Q e^{\int P dx}\, dx$$

integrating factor: $IF = e^{\int P dx}$

Example: $\frac{dy}{dx} + y x^2 = x^2$

$$IF = e^{\int x^2 dx} = e^{x^3/3}$$

$$\therefore \cancel{y \cdot e^{x^3/3}} = \cancel{\int x^3 e^{x^3/3} dx}$$

$$= \cancel{3x^3 e^{x^3/3}} - \cancel{9\int x^2 e^{x^3/3} dx}$$

$$= \cancel{3x^3 e^{x^3/3}} - \cancel{9\int e^t dt}$$

$$= \cancel{3x^3 e^{x^3/3}} - \cancel{9 e^{x^3/3}} + c$$

$$\frac{dy}{dx} + y P(x) = y^n Q(x) \qquad y^{-n+1}$$

$$\therefore \frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}} P(x) = Q(x)$$

Let $\frac{1}{y^{n-1}} = z \qquad \frac{n-1}{y^n}\frac{dy}{dx} = \frac{dz}{dx}$

LINEARLY INDEPENDENT:

Eg: $\sin x$, $\cos x$

$\downarrow$ y_1, $\downarrow$ y_2

$c_1 \sin x + c_2 \cos x = 0$ — (1)

for all $x \in R$, there is no certain (c_1, c_2) for which the expression (1) is valid always.

$\therefore$ this is possible only when $c_1 = c_2 = 0$.

$\therefore$ they are linearly independent.

↳ $\sin x$ and $\cos x$.

Eg: $y_1 = 3x^2$

$y_2 = 2x^2$

$c_1(3x^2) + c_2(2x^2) = 0$

here for $((c_1 = -2), (c_2 = 3))$ the expression is valid.

$\therefore$ y_1 and y_2 are linearly dependent.

) $\frac{d^2y}{dx^2} - 4y = 0$

here $y_1 = e^{2x}$

$y_2 = 2e^{2x}$ } solutions.

$\therefore$ sol^n is $y = c_1 e^{2x}$

also, $y = c_2 e^{(-2x)}$ is also a solution.

$\therefore$ general solution $\Rightarrow$ $\underline{y = c_1 e^{2x} + c_2 e^{-2x}}$.

- here e^{2x} and e^{-2x} are linearly independent.

) $\frac{d^2y}{dx^2} - 4y = e^{3x}$ e^{-3x} e^{3x}

$y_1 = \frac{1}{5} e^{3x}$ ← one solution.

(particular solution)

~~$y = \frac{2}{5}e^{3x}$~~ ~~is also solution.~~

linear combinations of $y = \frac{1}{5}e^{3x}$ are not sol^{n}s

~~to~~ of ~~yo~~ $\frac{d^2y}{dx^2} - 4y = e^{3x}$.

sol $y = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{5} e^{3x}$

the differential equation is satisfied.

SOLUTION:

$$\frac{d^2y}{dx^2} + f_1(x,y)\frac{dy}{dx} + f_2(x,y)\,y = f_3(x)$$

i) find general solution for:

$$\frac{d^2y}{dx^2} + f_1(x,y)\frac{dy}{dx} + f_2(x,y)\,y = 0$$

ii) find particular solution for:

$$\frac{d^2y}{dx^2} + f_1(x,y)\frac{dy}{dx} + f_2(x,y)\,y = f_3(x)$$

iii) combine the general solution + particular solⁿ.

(2) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 8e^{2x}$

i) homogeneous solution:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$$

let e^{mx} be a solⁿ

$$\therefore m^2e^{mx} + 5me^{mx} - 6e^{mx} = 0$$

$$m^2 + 5m - 6 = 0$$

$$(m-1)(m+6) = 0$$

$$\therefore m = 1 \text{ or } m = -6$$

ii) particular solution:

$$y = \frac{1}{8}e^{2x} \qquad y = Ae^{mx}$$

iii) $\therefore$ general solution $\rightarrow$

$$y = c_1e^{x} + c_2e^{-6x} + \frac{1}{8}e^{2x}$$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x$$

i) homogeneous.

$$\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = 0 \qquad y = e^{mx}$$

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

$$y = e^{x} \text{ a } e^{2x}$$

ii) particular solⁿ

$$\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = x$$

$$(ax^2 + bx + c = y$$

$$\therefore\ 2a - 6ax - 3b + 2ax^2 + 2bx + 2c = x$$

$$\therefore\ a = 0 \qquad b = \frac{1}{2} \qquad c = \frac{3}{4}$$

$$\therefore\ y = \frac{x}{2} + \frac{3}{4}$$

↳ particular solⁿ

iii) general solⁿ:

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{x}{2} + \frac{3}{4}$$

(5) $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x} + x$$

i) homogeneous:

$$y = e^{x} \text{ a } e^{2x}$$

ii) particular:

$$y = ax + b + ce^{3x}$$

$$9ce^{3x} - 3((a + 3ce^{3x}) + 2(ax + b + ce^{3x}) = e^{3x} + x$$

$$\cancel{9ce^{3x}} - \cancel{9ce^{3x}} + 2ce^{3x} - 3a + 2b + 2ax = e^{3x} + x$$

$$2ce^{3x} = e^{3x} \quad \therefore c = \frac{1}{2} \qquad 2b - 3a = 0 \qquad a = \frac{1}{2}$$

$$\therefore b = \frac{3}{4}$$

iii) $$\therefore\ y = c_1 e^{x} + c_2 e^{2x} + \frac{x}{2} + \frac{3}{4} + \frac{e^{3x}}{2}$$

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(4) $\frac{d^4y}{dx^4} - 9\frac{d^2y}{dx^2} = x^2 + e^{3x}$

i) Homogeneous:

$$\frac{d^4y}{dx^4} - 9\frac{d^2y}{dx^2} = 0$$

$y = e^{mx}$

$m^4 e^{mx} - 9m^2 e^{mx} = 0$

$m^2 e^{mx}(m^2 - 9) = 0$

$m = 0, 3, -3$

$\therefore \; y = c_1 + c_2 e^{3x} + c_3 e^{-3x}$

ii) particular: $\frac{d^4y}{dx^4} - 9\frac{d^2y}{dx^2} = x^2 + e^{3x}$ $ax^3 + bx^2 -$

$y = x^4 e^{3x}$

$y' = 3x^4 e^{3x} - 4e^{3x}x^3$

$y'' = 9e^{3x}x^4 + 12x^3 e^{3x} + 12e^{3x}x^2 + 12e^{3x}x^3$

(5) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

for $y = e^{mx}$ $\therefore m = 1$

for $y = xe^x$

$\therefore (xe^x + 2e^x) - 2(xe^x + e^x) + (xe^x) = 0$

$\therefore$ general solution:

$y = c_1 e^x + c_2 x e^x$

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② $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

$y = e^{mx}$

$m^3 - 3m^2 + 3m - 1 = 0$

$(m-1)^3 = 0$

$\therefore m = 1$

$\therefore y_1 = e^x$

$y_2 = xe^x$

$y_3 = x^2e^x$

(including complex numbers):

③ $\frac{d^2y}{dx^2} + y = 0$

$y = e^{mx}$

~~y=~~ $m^2 + 1 = 0$

$\therefore m = +i, -i$

$\therefore y = e^{ix}, e^{-ix}$

$= (\cos x + i\sin x), (\cos x - i\sin x)$

$e^{(a+ib)x}, e^{(a-ib)x}$ $(\because m = a \pm ib)$

$\therefore y = c_1e^{ax}(\cos bx + i\sin bx) + c_2e^{ax}(\cos bx - i\sin bx)$

$= e^{ax}[\underbrace{(c_1+c_2)}_{A_1}\cos bx + \underbrace{i(c_1-c_2)}_{A_2}\sin bx]$

$= e^{ax}(A_1\cos bx + A_2\sin bx)$

② FOR DIFFERENTIAL EQ OF TYPE : $a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} \cdots + a_n y$

put $y = e^{mx}$

$\therefore e^{mx}(a_0m^n + a_1m^{n-1} \cdots + a_0) = 0$

i) if all are distinct and real

let roots be $m_1, m_2 \ldots$

$\therefore y = c_1e^{m_1x} + c_2e^{m_2x} \cdots + c_ne^{m_nx}$

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ii) If $m_1 = m_2, m_3, m_4 \cdots m_n$

$y = (C_1 + x C_2) e^{m_1 x} + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$

iii) if $m_1 = m_2 = m_3 = \cdots = m_k, m_{k+1}, \cdots, m_n$

$y = (C_1 + x C_2 + x^2 C_3 + \cdots + x^{k-1} C_k) e^{m_1} + C_{k+1} e^{m_{k+1} x} + \cdots C_n e^{m_n x}$

iv) $m = a \pm ib, m_3, m_4 \cdots m_n$

$y = e^{ax} [C_1 \cos bx + C_2 \sin bx] + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$

v) $m_1 = m_2 = a + ib \quad m_3 = m_4 = a - ib, m_5 \cdots m_n$

$y = e^{ax} ((C_1 + C_2) \cos bx + (C_3 - C_4) \sin bx) + C_5 e^{m_5 x} + \cdots + C_n e^{m_n x}$

) $\frac{d^3 y}{dx^3} + 2 \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$

$y = e^{mx}$

$m^3 + 2m^2 + m = 0$

$m(m^2 + 2m + 1) = 0$

$m = 0 \quad \cancel{m = \frac{-2 \pm \sqrt{4 - 4}}{2}} \quad m = -1$

$\therefore y = C_1 e^{0x} + C_2 e^{-x} + C_3 x e^{-x}$

$\therefore y = C_1 + e^{-x} (C_2 + x C_3)$

) $\frac{d^4 y}{dx^4} + y = 0$

$y = e^{mx}$

$m^4 e^{mx} + e^{mx} = 0 \quad m^4 = -1$

$m^2 = \pm i \quad m = \pm\sqrt{i}, \pm\sqrt{-i}$

$m = \sqrt{i}, -\sqrt{i}, i\sqrt{i}, -i\sqrt{i}$

$i = e^{i\pi/2} \quad \therefore \sqrt{i} = e^{i\pi/4} = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4}$

$\therefore \sqrt{i} = 1/\sqrt{2} + i/\sqrt{2} \quad -\sqrt{i} = -1/\sqrt{2} - i/\sqrt{2}$

$\therefore m = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$$m = \pm\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$m = \pm\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

and $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

and $\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

and $\frac{+1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$$\therefore y = e^{x/\sqrt{2}}\left(c_1 e^{+i} + c_2 e^{+1-i}\right) + e^{-x/\sqrt{2}}\left(c_3 e^{-1+i} + c_4 e^{-1-i}\right)$$

$$= e^{x/\sqrt{2}}\left[\left(A_1 \cos x + A_2 i \sin x\right) + e^{-x}\left(A_3 \cos x - \ldots\right.\right.$$

(15) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{x}$

i) homogeneous:

$y = e^{mx}$

$m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$

$\therefore y = e^{x}, e^{2x}$

ii) particular:

$y = xe^{x}(A)$

$\therefore A(xe^{x} + 2e^{x}) - 3A(xe^{x} + e^{x}) + 2A\,xe^{x} = e^{x}$

$\cancel{Ax} + 2A - \cancel{3Ax} - 3A + \cancel{2Ax} = 1$

$\boxed{A = -1}$

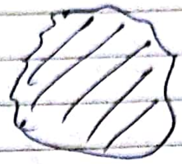
$\therefore y = -xe^{x}$

iii) general solution:

$$y = c_1 e^{x} + c_2 e^{2x} - xe^{x}$$

FUNCTIONS OF SEVERAL VARIABLES:

① Functions of two variables:


R

For ~~the region~~ R,

i) interior point:

A point (x_0, y_0) is said to be an interior point ~~for~~ of R if there exists a disc, centered ~~in~~ at (x_0, y_0) with some finite radius is ~~entirely~~ entirely contained in R.

ii) Boundary point:

A point (x_0, y_0) is called a boundry point if ~~it when~~ the smallest possible disc centred at (x_0, y_0) will contain some points not in R.

⇒ Level curve:

The ~~set~~ set of points in the plane ~~above~~ where of a function $f(x,y)$ has a constant value, ie, $f(x,y) = c$ is called a level curve of f.

⇒ level surface:

The set of points (x, y, z) in a space where a function $f(x,y,z)$ is a constant value, ie, $f(x,y,z) = c$ is called a level ~~curve~~ surface.

Eg: $f(x,y) = x^2 + y^2$

$\therefore$ level curves : $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$

LIMITS OF A FUNCTION WITH TWO VARIABLES:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for any $\varepsilon > 0$, there exists a corresponding δ, such that for all (x,y) in the domain of f.

$|f(x) - L| < \varepsilon$ when $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

Eg: $\lim_{(x,y)\to(x_0,y_0)} x = x_0$

$f(x,y) = x \qquad L = x_0$

$\varepsilon > 0$

$|f(x,y) - L| < \varepsilon$

$\therefore |x - x_0| < \varepsilon$

and $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

let $\varepsilon < \delta$

$\therefore 0 < (x-x_0)^2 + (y-y_0)^2 < \delta$

$\therefore 0 < (x-x_0)^2 < \delta^2$

$|x - x_0| < \sqrt{\delta^2} = \delta = \varepsilon$

$\therefore \varepsilon = \delta$

$\Rightarrow$ PROPERTIES:

If L, M, K are real numbers and $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$, then:

i) $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$

ii) $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$

iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) \cdot g(x,y) = ML$

iv) $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \qquad (M \neq 0)$

v) $\lim_{(x,y)\to(x_0,y_0)} k f(x,y) = kL$

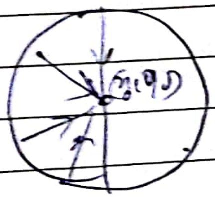
vi) $\lim_{(x,y)\to(x_0,y_0)} \left(f(x,y)\right)^{r/s} = L^{r/s}$

($r, s \in I$ and $s \neq 0$)

Eg: ① $\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = \sqrt{\lim_{(x,y)\to(3,-4)} (x^2+y^2)} = 5$

② $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \frac{x(\sqrt{x}+\sqrt{y})\cancel{(\sqrt{x}-\sqrt{y})}}{\cancel{(\sqrt{x}-\sqrt{y})}} = 0$

③ ~~...~~ using definition of limits, find $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$



Along the line $x=0$, the function has value 0, when $y \neq 0$.
$\therefore$ if limit exists, $L=0$

$\therefore f(x,y) = \frac{4xy^2}{x^2+y^2}$ $\quad L=0$

For any $\varepsilon > 0$,

$\left|\frac{4xy^2}{x^2+y^2}\right| < \varepsilon$ when $|x^2+y^2| < \delta^2$

$y^2 < (x^2+y^2)$ and $x^2 < x^2+y^2$

$\therefore \frac{4|x|y^2}{x^2+y^2} < \frac{4|x|(x^2+y^2)}{x^2+y^2} = 4|x| = 4\sqrt{x^2} < 4\sqrt{x^2+y^2} < \varepsilon$

$\therefore 4\delta = \varepsilon \therefore \delta = \frac{\varepsilon}{4}$

$\therefore$ for $\varepsilon > 0$ there exists $\delta > 0$

$\therefore$ limit exists, $L=0$

Hence proved.

⑥ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{x^2+1}$ | $f(0,0)=0$

$l = 0$

$|f(x,y) - l| < \varepsilon$

$\left|\dfrac{x+y}{x^2+1} - 0\right| < \varepsilon$

$\left|\dfrac{x+y}{x^2+1}\right| \le |x+y| \le |x|+|y|$

now, $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

$\sqrt{x^2+y^2} < \delta$

as $|x| \le \sqrt{x^2+y^2}$

$|y| \le \sqrt{x^2+y^2}$

$\therefore \quad |x| + |y| \le 2\sqrt{x^2+y^2}$

$\therefore \left|\dfrac{x+y}{x^2+1}\right| \le 2\sqrt{x^2+y^2}$

$\therefore \boxed{\delta = \dfrac{\varepsilon}{2}}$

POLAR COORDINATES:

① $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3}{x^2+y^2}$

put $x = r\cos\theta$ $y = r\sin\theta$ and $\lim\limits_{r\to 0}$

$\therefore \lim\limits_{r\to 0} \dfrac{r^3\cos^3\theta}{r^2\sin^2\theta + r^2\cos^2\theta}$

$= \lim\limits_{r\to 0} \dfrac{r\cos^3\theta}{\sin^2\theta + \cos^2\theta}$

$= \lim\limits_{r\to 0} \dfrac{r\cos^3\theta}{1}$

$= 0$

By ~~2d~~ method:

② $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

$x = r\cos\theta \quad y = r\sin\theta$

$f(r,\theta) = \frac{r^2\cos^2\theta}{r^2\sin^2\theta + r^2\cos^2\theta} = \cos^2\theta \qquad \therefore \lim_{r\to 0} f(r,\theta) = \cos^2\theta$

as $\cos\theta$ varies for θ,
lim doesn't exist

③ $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$

$x = r\cos\theta \quad y = r\sin\theta$

$f(r,\theta) = \frac{2r^2\cos^2\theta\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta}$

$= \frac{2r\sin\theta\cos^2\theta}{r^2\cos^4\theta + r\sin^2\theta} \qquad \frac{2\times 0\times\sin\theta\cos^2\theta}{0+\sin^2\theta}$

for a fix θ (non-zero):
$\lim_{r\to 0} f(r,\theta) = 0 = L_1$

Approaching through curve $y = x^2$, $\therefore r\sin\theta = r^2\cos^2\theta$

$f(r,\theta) = \frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}$

$= \frac{2\sin^2\theta}{\sin^2\theta + \sin^2\theta} = 1$

$\therefore \lim_{r\to 0} f(r,\theta) = 1 = L_2$

$\therefore$ as $L_1 \neq L_2$, limit doesn't exist

④ $\lim_{(x,y)\to(0,0)} f(x,y)$ $\quad f(x,y) = \frac{2x}{x^2 + x + y^2}$

$x = r\cos\theta \quad y = r\sin\theta$

$f(r,\theta) = \frac{2r\cos\theta}{r^2\cos^2\theta + r^2\sin^2\theta + r\cos\theta}$

$= \frac{2\cos\theta}{r + \cos\theta}$

for fix θ,

$\lim_{r\to 0} f(r,\theta) = 2$

~~along~~ for $\theta = \pi/2$

the denominator is not defined

$\therefore \lim_{r\to 0} f(r,\theta)$ = doesn't exist

$\therefore$ limit doesn't exist.

⑤ $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$

Method 1: $x = r\cos\theta \quad y = r\sin\theta$

$f(r,\theta) \lim_{r\to 0} = \frac{2\cos\theta\sin\theta}{1} = \sin 2\theta$

$\therefore \lim_{r\to 0} f(r,\theta) = \sin 2\theta$

$\therefore$ for different θ, $\sin 2\theta$ is different

$\therefore$ limit doesn't exist

Method 2: let $y = mx$

$\therefore f(x,y) = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1+m^2}$

for different m, $\lim_{(x,y)\to(0,0)} = \frac{2m}{1+m^2}$ has diff. values

$\therefore$ limit doesn't exist

③ $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$

method 2: using $y = mx^2$

CONTINUITY:

$f(x,y)$ is continuous at (x_0,y_0) if :

i) f is defined at (x_0,y_0)

ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ should exist

iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

① $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

for $(x,y) \neq (0,0)$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$ $(x = r\cos\theta,\ r\sin\theta = y)$

$= \lim\limits_{r\to 0} \dfrac{r\sin 2\theta}{1} = \sin 2\theta$

$\therefore$ doesn't exist

for all other (x,y) limit exist

$\therefore$ discontinuous.

② $\lim\limits_{P\to(1,0,-1)} \dfrac{e^{x+z}}{z^2+\cos\sqrt{xy}}$

$= \dfrac{e^0}{1+\cos 0} = \dfrac{1}{2}$

PARTIAL DERIVATIVES:

$z = f(x, y)$

$$\left.\frac{\delta f}{\delta x}\right|_{P(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$\left.\frac{\delta f}{\delta y}\right|_{(x_0, y_0)} = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

Examples:

① $f(x, y) = x \cos xy$

$$\frac{\delta f}{\delta x} = -xy \sin xy + \cos xy$$

$$\frac{\delta f}{\delta y} = -x^2 \sin xy$$

IMPLICIT:

$yz - \log z = x + y$

$$y \frac{\delta z}{\delta x} - \frac{1}{z} \frac{\delta z}{\delta x} = 1$$

$$\frac{\delta z}{\delta x}\left(y - \frac{1}{z}\right) = 1$$

$$\frac{\delta z}{\delta x} = \frac{z}{yz - 1}$$

$$y \frac{\delta z}{\delta y} + z - \frac{1}{z} \frac{\delta z}{\delta y} = 1$$

$$\frac{\delta z}{\delta y} = \frac{(1 - z) z}{yz - 1}$$

PARTIAL DERIVATIVES AND CONTINUITY:

For the existance of partial derivative the function need not be continuous.

① $f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

at $(x, y) = (0, 0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \qquad (\text{given that } x \neq 0 \text{ or } y \neq 0)$$

$$f(0,0) = 1$$

$\therefore$ function is not continuous.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1-1}{h} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} \frac{f(0,k+0) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{1-1}{k} = 0$$

$\therefore$ Partial derivative exists at origin even though function is discontinuous at $(0,0)$.

MIXED PARTIAL DERIVATION:

$f(x,y) \rightarrow$ function.

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x} \quad \left(\text{ie; } \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \neq \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)$$

The two are equal only if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous.

DERIVATIVE:

→ If the partial derivatives of a function are continuous throughout an open region R, then f is differentiable at every point on R.

→ Here, differentiability ⇒ continuity

→ Let $w = f(x, y)$

If $f(x, y)$ is differentiable and $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then $w = f(x(t), y(t))$ is differentiable and

$$\frac{dw}{dt} = \frac{\delta f}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{\delta f}{\delta y} \cdot \frac{\delta y}{\delta t}$$

→ if $w = f(x, y, z)$

$x = g(r, s) \quad y = h(r, s) \quad z = k(r, s)$

$$\therefore \frac{dw}{dr} = \frac{\delta f}{\delta x} \cdot \frac{\delta x}{\delta r} + \frac{\delta f}{\delta y} \cdot \frac{\delta y}{\delta r} + \frac{\delta f}{\delta z} \cdot \frac{\delta z}{\delta r}$$

$$\frac{dw}{ds} = \frac{\delta f}{\delta x} \cdot \frac{\delta x}{\delta s} + \frac{\delta f}{\delta y} \cdot \frac{\delta y}{\delta s} + \frac{\delta f}{\delta z} \cdot \frac{\delta z}{\delta s}$$

DIRECTIONAL DERIVATIVES:

→ if $f(x, y)$ is differentiable, then the rate at which f changes wrt t along a differentiable curve, $x = x(t)$, $y = y(t)$ is:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

→ Defination:

The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector.

$$u = u_1\hat{i} + u_2\hat{j}$$

is the number

$$\left.\frac{df}{ds}\right|_{u, P_0} = \lim_{s \to 0} \left(\frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \right)$$

(provided the limit exists.)

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Example:

Find derivative of $f(x,y) = x^2 + xy$ at $P_0(x_0, y_0) = (1,2)$ in direction of unit vector

$$u = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$$

$$\left.\frac{df}{ds}\right|_{u,P_0} = \lim_{s \to 0} \frac{f(x_0 + u_1 s, y_0 + u_2 s) - f(x_0, y_0)}{s}$$

$$f(x_0, y_0) = 1^2 + 2 = 3$$

$$f(x_0 + u_1 s, y_0 + u_2 s) = \left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right)$$

$$\left.\frac{df}{ds}\right|_{u,P_0} = \lim_{s \to 0} \frac{(1 + s/\sqrt{2})^2 + (1 + s/\sqrt{2})(2 + s/\sqrt{2}) - 3}{s}$$

$$= \lim_{s \to 0} \frac{\cancel{1} + s^2/2 + 2s/\sqrt{2} + \cancel{2} + s^2/2 + 3s/\sqrt{2} - \cancel{3}}{s}$$

$$= \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

GRADIENT VECTOR:

at $P_0(x_0, y_0)$ & $u = u_1\hat{i} + u_2\hat{j}$

$$\left.\frac{df}{ds}\right|_{P_0} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{ds}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{ds}\right)$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right) \cdot (u_1\hat{i} + u_2\hat{j})$$

$\therefore$ gradient vector $= \vec{\nabla f} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

$$\therefore \left.\frac{df}{ds}\right|_{P_0} = \left.\vec{\nabla f}\right|_{P_0} \cdot \vec{u}$$

$$\therefore \left.\frac{df}{ds}\right|_{P_0} = |\vec{\nabla f}|\,|\vec{u}| \cos\theta$$

$$= |\nabla f| \cos\theta$$

$(\because \vec{u}$ is a unit vector$)$

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⇒ IMPORTANT RESULTS:

① The function f increases most rapidly when $\cos\theta = 1$, ie, when u is in the direction of ∇f. That is, at each point P, f increases most rapidly in the direction of gradient vector ∇f ~~and~~ at P.

② Similarly, f decreases most rapidly when $\cos\theta = -1$, ie, when u is in direction opposite to ∇f.

③ for any direction orthogonal to ∇f is the direction of zero change.

EXAMPLE:

① Find the direction $\frac{x^2}{2} + \frac{y^2}{2} = f(x,y)$

i) increase max at $(1,1)$

ii) decreases max at $(1,1)$

iii) no change at $(1,1)$

sol^n $\nabla f = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\Big|_{(1,1)} = (x\hat{i} + y\hat{j})\Big|_{(1,1)} = \hat{i} + \hat{j}$

i) in direction ($\because \cos\theta = 1$

$\therefore \vec{u} = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$

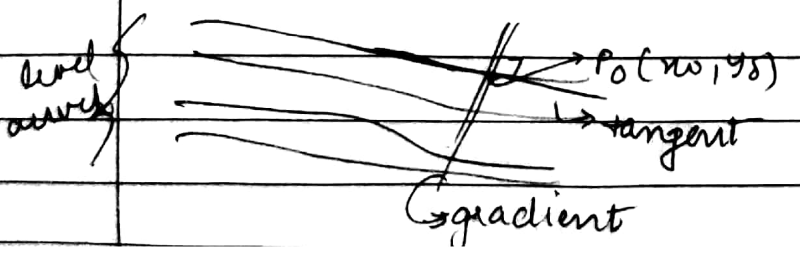
ii) opp direction ($\because \cos\theta = -1$)

$\therefore \vec{u} = -\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$

iii) no change

$\therefore \vec{u} = \frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$ or $\vec{u} = \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})$

GRADIENT AND ~~CURVES~~ TANGENTS TO LEVEL CURVES:



if a differentiable function $f(x,y)$ has a constant value c along a curve $L: x(t)\hat{i} + y(t)\hat{j}$ then
$f(x(t), y(t)) = c$ is a level curve.
then,

$$\frac{d}{dt}(f(x(t), y(t))) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = 0$$

$$\therefore \underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)}_{\text{gradient vector}} \cdot \underbrace{\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}\right)}_{\text{tangent vector}} = 0$$

$$\therefore \vec{\nabla f} \cdot \frac{d\vec{r}}{dt} = 0$$

$\therefore$ gradient vector $\perp$ tangent vector.

$\Rightarrow$ PROPERTIES:

1) $\nabla(f+g) = \nabla f + \nabla g$
2) $\nabla(f-g) = \nabla f - \nabla g$
3) $\nabla(kf) = k\nabla f$
4) $\nabla(fg) = (\nabla f)g + (\nabla g)f$
5) $\nabla\left(\frac{f}{g}\right) = \frac{(\nabla f)g - (\nabla g)f}{g^2}$ $\quad (g \neq 0)$

FUNCTIONS OF 3 VARIABLES:

directional derivative:

$$\left.\frac{df}{ds}\right|_u = D_u f = \vec{\nabla f} \cdot \vec{u}$$

$$\left(\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$

EXTREME VALUES AND SADDLE POINTS:

Defination:

let $f(x,y)$ be defined on Region R containing the point (a,b)

i) (a,b) is said to be the local maximum ~~value~~ point of f if $f(a,b) \geq f(x,y)$ $\forall$ $(x,y) \in$ open disc centred at (a,b)

ii) (a,b) is said to be local minimum point of f if $f(a,b) \leq f(x,y)$ $\forall$ $(x,y) \in$ open disc centred at (a,b)

iii) global max: $f(a,b) \geq f(x,y)$ $\forall (x,y) \in R$

iv) global min: $f(a,b) \leq f(x,y)$ $\forall (x,y) \in R$

FIRST DERIVATIVE TEST:

If $f(x,y)$ has a local maximum a local minimum value at an interiorpoint ~~of~~ (a,b) of R and if the first partial derivative exists there, then

$$f_x(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0$$

$$f_y(a,b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = 0$$

→ critical points:

an interia point of the domain of the function $f(x,y)$ where both f_x and f_y are zero or the points where one or both (f_x, f_y) do not exist. These points are called $f(x,y)$.

→ Saddle points:

The critical point which is neither max or min, is called a saddle point.

Sufficient condition:

Result:

Suppose that $f(x,y)$ and its 1^{st} and 2^{nd} partial derivatives are continuous through out a disc centred at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$

Then,

① local maximum at (a,b) if

$$f_{xx}\Big|_{(a,b)} \left(= \frac{\partial^2 f}{\partial x^2}\Big|_{(a,b)} \right) < 0$$

and $\quad f_{xx} f_{yy} - (f_{xy})^2 > 0 \quad$ at (a,b)

$$\therefore \text{ Hessian matrix}\Big|_{(a,b)} > 0$$

$$\left(\text{Hessian matrix}\Big|_{(a,b)} = \left| \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \right|_{(a,b)} \right)$$

② local minimum:

$$f_{xx} > 0 \quad \text{and} \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$$

③ saddle point:

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} < 0$$

④ Test is inclusive if : $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = 0$

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Example:

(1) Find the local extreme values:

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$$

$$f_{xx} = \frac{\partial}{\partial x}(y - 2x - 2) = -2$$

$$f_{yy} = \frac{\partial}{\partial y}(x - 2y - 2) = -2$$

$$f_{xy} = f_{yx} = \frac{\partial}{\partial x}(x - 2y - 2) = 1$$

$$\therefore \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4 - 1 = 3$$

$\therefore f_{xx} < 0$

$f_{xx} f_{yy} - (f_{xy})^2 > 0$

$\therefore$ maximum found at $(-2, -2)$.

$$\left\{ \begin{array}{lll} \because f_x = 0 & \text{and} & f_y = 0 \\ \therefore x - 2y - 2 = 0 & \text{and} & 2x - y + 2 = 0 \end{array} \right\}$$

(2) $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

$f_x = -6x + 6y = 0 \implies x = y$

$f_y = 6y - 6y^2 + 6x = 0$

$\therefore 2y - y^2 = 0$

$y = 0$ or $y = 2$

$\therefore$ critical points: $(0,0)$ and $(2,2)$

$f_{xx} = -6$

$f_{yy} = 6 - 12y$

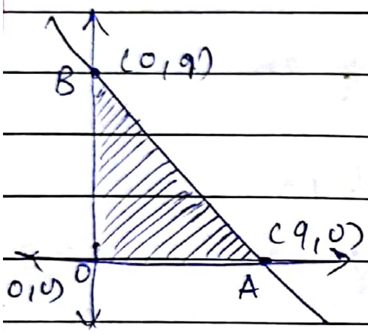
$f_{xy} = 6$

$\therefore$ for $(0,0)$: $\begin{vmatrix} -6 & 6 \\ 6 & 6 \end{vmatrix} = -2\times 36 < 0$

for $(2,2)$: $\begin{vmatrix} -6 & -6 \\ 6 & -18 \end{vmatrix} = 18\times 6 + 6\times 6 > 0$

$\therefore$ maximum at $(2,2)$
saddle point at $(0,0)$.

$f(x,y) = 2+2x+2y-x^2-y^2$
on a triangular region in 1^{st} quad bounded by the lines $x=0$, $y=0$, $y=9-x$



For interior points
$f_x = 2-2x = 0$
$f_y = 2-2y = 0$

$\therefore$ critical point : $(1,1)$

$f(1,1) = 4$

$\rightarrow f(x,0) = 2+2x-x^2$
at $(0,0)$, $f(0,0) = 2$
at $(9,0)$, $f(9,0) = -61$
$f'(x,0) = 2-2x = 0 \quad \therefore x=1$
$\therefore f(1,0) = 3$

$\rightarrow f(0,y) = 2+2y-y^2$
$f(0,9) = -61$
$f'(0,y) = 2-2y \Rightarrow y=1$
$\therefore f(0,1) = 3$

$\rightarrow f(x,9-x) = 2+\cancel{2x}+18-\cancel{2x}-x^2-(9-x)^2$
$= 20-x^2-(81-18x+x^2)$
$= -61+18x$
$f'(x,9-x) = 18$

$\therefore$ min at $(0,9)$ and $(9,0)$, max at $(0,1)$ and $(1,0)$

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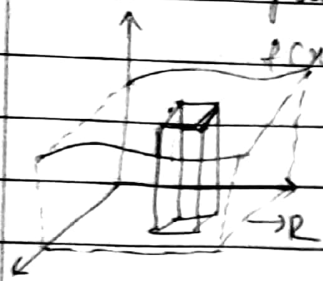
MULTIPLE INTEGRALS:

Double integrals to find volume:



$$S_n = \sum_{k=1}^{n} \Delta x_k \Delta y_k f(x_k, y_k)$$

as $\|P\| \to 0$, $n \to \infty$

$$\therefore \text{volume} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{1}^{n} \Delta x_k \Delta y_k f(x_k, y_k)$$

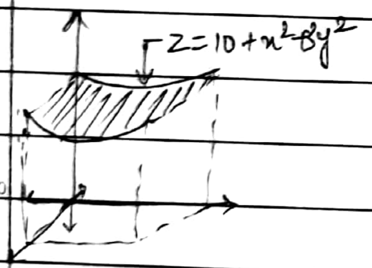
$$\therefore V = \iint_R f(x,y)\, dx\, dy$$

⇒ **Fubin's Theorem:**

if $f(x,y)$ is continuous throughout rectangular region R $a \le x \le b$, $c \le y \le d$,

$$\iint_R f(x,y)\, dA = \int\left(\int_R f(x,y)\, dx\right) dy = \int\left(\int_R f(x,y)\, dy\right) dx$$

1) Find volume bounded above by eliptical paraboloid, $z = 10 + x^2 + 3y^2$, and below by the rectange. R: $0 \le x \le 1$, $0 \le y \le 2$.



$$\therefore \text{volume} = \iint_R f(x,y)\, dx\, dy$$

$$\therefore V = \int_0^2 \left(\int_0^1 (10 + x^2 + y^2)\, dx\right) dy$$

$$= \int_0^2 \left(10 + \frac{1}{3} + 3y^2\right) dy$$

$$= \left[10y + \frac{y}{3} + y^3\right]_0^2 = \frac{86}{3}$$

2) Double Integrals over a general region:

here limits will be functions of x.

if $f(x,y)$ is +ve and continuous over R, the volume bounded by $z = f(x,y)$ and R is:

$$V = \iint_R f(x,y)\, dx\, dy$$

if R is is the region bounded by $y = g_1(x)$ and $y = g_2(x)$ and on sides by $x = a$ and $x = b$

$$V = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx$$

Fubin's Result:

① $x \in [a,b]$, $y \in [g_1(x), g_2(x)]$.

then $V = \iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$

② $y \in [c,d]$ and $x \in [h_1(y), h_2(y)]$.

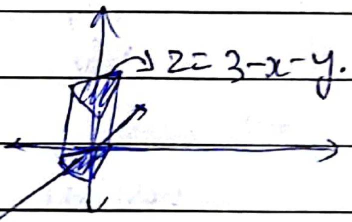
then $V = \iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$

⇒ **Example:**

Find the volume of the prism whose base is triangular in the x-y plane bounded by the x-axis and the lines $y=x$, $x=1$ whose top lines in the plane $f(x,y) = 3-x-y$

$y=0$ to $y=x$ and

$x=0$ to $x=1$



$$\therefore V = \int_0^1 \left(\int_0^x (3-x-y)\, dy \right) dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_0^x dx$$

$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2} \right) dx$$

$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3} - \frac{x^3}{6} \right]_0^1$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_0^1$$

$$= \frac{3}{2} - \frac{1}{2}$$

$$= 1$$

FINDING LIMITS OF INTEGRATION:

① using vertical cross-section

$$\iint_R f(x,y)\,dy\,dx$$

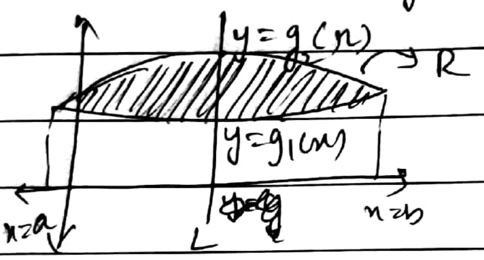
Step 1: sketch the region of integration

Step 2: finding limits of integration

→ imagine a vertical line L cutting through the region R in the increasing direction of y.

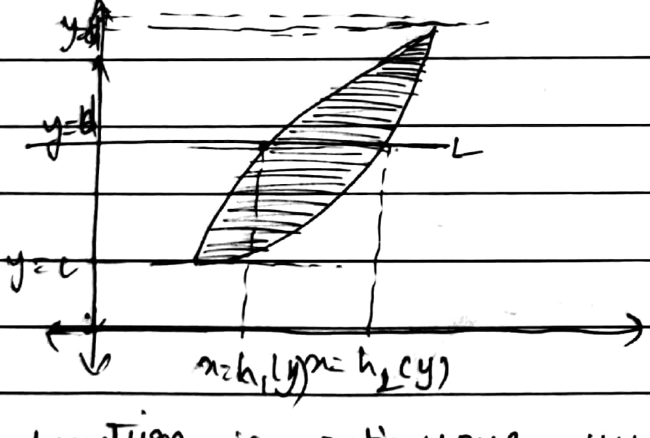
→ mark the y-values where L ~~exit~~ enters and leaves the region.

These are the y limits of integration



→ choose x limits that include all the vertical lines through R, parallel to L.

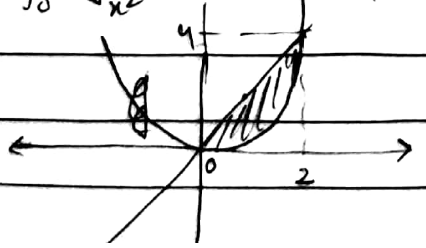
② using horizontal cross-section:



$$\iint_R f(x,y)\,dx\,dy$$

if function is continuous, then $\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx$

Example ① $\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$



$$\therefore V = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy$$

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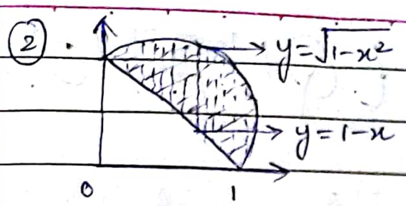
(2) 

$$V = \iint_R f(x,y)\, dy\, dx$$

$$V = \iint_R f(x,y)\, dx\, dy$$

$$V = \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x,y)\, dy\, dx$$

$$V = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x,y)\, dx\, dy$$

PROPERTIES OF DOUBLE INTEGRALS:

(1) $\iint_R c f(x,y)\, dA = c \iint_R f(x,y)\, dA$

(2) $\iint_R (f(x,y) + g(x,y))\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$

(3) domination property.

(a) $\iint_R f(x,y)\, dA \geq 0$ — if $f(x,y) \geq 0$ on R

(b) $\iint_R f(x,y)\, dA \leq 0$ if $f(x,y) \leq 0$ on R

(4) Additivity wrt R.

if $R = R_1 + R_2$; $V = \iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$

Area by double integral:

$$\iint_R dA = \iint_R dx\, dy.$$

TRIPPLE INTEGRALS:

$$V = \iiint_D dx\, dy\, dz$$

Example: Find volume $z = x^2 + 3y^2$ and $8 - x^2 - y^2$

$$V = \iiint_{z = x^2+3y^2}^{z = 8-x^2-y^2} dz\, dy\, dx$$

for y limits, $z = 0$, $y = \pm\sqrt{\frac{8-2x^2}{4}}$

for x-limit, $y = 0$; $x = \pm 2$

$$\therefore V = \int_{-2}^{2} \int_{-\sqrt{8-2x^2/4}}^{\sqrt{8-2x^2/4}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dx\, dy$$

DIFFERENTIAL EQUATION

GENERAL 1st ORDER DIFFERENTIAL EQUATION:

$\frac{dy}{dx} = f(x,y)$ → (1) first order ordinary differential equation

Solution:

y = dependent variable
x = independent variable

Any differentiable function $y(x)$ which statisfies (1) is called a solution of the differential equation.

A) Seperable equation:

$$\frac{dy}{dx} = f(x,y) \quad -(1)$$

if we can write $f(x,y) = g(x)\,h(y)$, then (1) is called seprable differentiable equation.

$$\therefore \frac{dy}{dx} = g(x)\,h(y)$$

$$\therefore \frac{1}{h(y)}\,dy = g(x)\,dx$$

B) HOMOGENOUS EQUTIONS:

$$\frac{dy}{dx} = f(x,y) \quad -(1)$$

if $f(x,y)$ is homogenous equation, then (1) is called homogenous equation.

now, $f(x,y)$ is a homogenous equation if

$$f(tx,ty) = t^n f(x,y) \qquad t \in R$$

solution:

take $y = xv$

$\therefore \frac{dy}{dx} = x\frac{dv}{dx} + v$

$\therefore$ replace and solve in variable seperable form.

LINEAR FIRST ORDER DIFFERTIABLE EQUATION:

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

(P(x) and Q(x) are continuous functions of x only.)

solution:

Integrating factor (IF) $= e^{\int P(x)dx}$

$$\therefore y\, e^{\int P(x)dx} = \int Q(x)\, e^{\int P(x)dx}\, dx$$

→ we multiply a suitable function v(x) that transforms the LHS of the derivative eqn into derivative of the product v(x)·y.

here v(x) is called integrating factor.

$$v(x)\left(\frac{dy}{dx} + P(x)\,y = Q(x)\right)$$

$$\int \left(v(x)\frac{dy}{dx} + P(x)\cdot v(x)\,y\right) = \int v(x)\,Q(x)$$

$$\therefore \int \frac{d}{dx}\left(v(x)\cdot y\right) = \int v(x)\cdot Q(x)$$

$$\therefore \boxed{y\,v(x) = \int v(x)\,Q(x)\,dx}$$

$$\frac{d}{dx}(y\cdot v(x)) = y\frac{dv}{dx} + v(x)\frac{dy}{dx} = v(x)\frac{dy}{dx} + P(x)\cdot v(x)\,y$$

$$\therefore \frac{dv}{dx} = P(x)\,v(x)$$

$$\therefore \log v(x) = \int P(x)\,dx$$

$$\therefore \boxed{v(x) = e^{\int P(x)\,dx}}$$

∴ final solution:

$$\boxed{y e^{\int P(x)\,dx} = \int Q(x)\, e^{\int P(x)\,dx}\, dx}$$

EXAMPLE:

$$x\frac{dy}{dx} - 3y = x^2 \Rightarrow \frac{dy}{dx} - \frac{3}{x}(y) = x$$

$$IF = \frac{1}{x^3}$$

$$\therefore \; y/x^3 = \int x \times \frac{1}{x^3}\, dx = \frac{-1}{x} + c$$

$$\therefore \; \boxed{y = -x^2 + cx^3}$$

Find particular solution for $3xy' - y = \log x + 1$ when $y(1) = -2$ $(x>0)$

$$\frac{dy}{dx} - \frac{1}{3x}y = \left(\frac{\log x + 1}{3x}\right)$$

$$IF = \frac{1}{x^{1/3}}$$

$$\frac{y}{x^{1/3}} = \int \left(\frac{\log x + 1}{3x}\right) \times x^{-1/3}\, dx$$

$$= \frac{1}{3}\int (\log x + 1)\, x^{-4/3}\, dx$$

$$= \frac{1}{3}\left(-3 x^{-1/3} - 3x^{-1/3}\log x + 9 x^{-1/3}\right) + c$$

$=$

D) EQUATIONS REDUCIBLE TO LINEAR EQUATION:

$$\frac{dy}{dx} + yP(x) = y^n Q(x)$$

$$\therefore \frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P(x) = Q(x) \quad - \text{(1)}$$

$$\frac{1}{y^{(n-1)}} = z$$

$$\therefore \frac{dz}{dx} = \frac{-1}{y^{n-1+1}} \times (n-1)\frac{dy}{dx}$$

$$\frac{dz}{dx} = (1-n)\left(\frac{1}{y^n}\frac{dy}{dx}\right)$$

$$\therefore \text{(1)} \equiv \frac{1}{(1-n)}\frac{dz}{dx} + zP(x) = Q(x)$$

$$\therefore \frac{dz}{dx} + (1-n)zP(x) = (1-n)Q(x)$$

Example:

(1) $x\frac{dy}{dx} + y = x^4 y^3$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{xy^2} = x^3$$

$$\frac{1}{y^2} = z \qquad \therefore \frac{dz}{dx} = \frac{-2}{y^3}\frac{dy}{dx}$$

$$\therefore \frac{-1}{2}\frac{dz}{dx} = \frac{1}{y^3}\frac{dy}{dx}$$

$$\therefore \frac{-1}{2}\frac{dz}{dx} + \frac{z}{x} = x^3 \qquad \therefore \frac{dz}{dx} - \frac{2z}{x} = -2x^3$$

$$\therefore IF = \frac{1}{x^2}$$

$$\therefore \frac{z}{x^2} = \int \frac{x^3}{x^2}\,dx = \frac{x^2}{2} + c$$

$$\therefore \frac{1}{y^2} = \frac{x^4}{2} + c$$

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② $\frac{dy}{dx}(x^2y^3+xy)=1$

$\frac{dx}{dy} = x^2y^3 + xy$

$\frac{dx}{dy} - xy = x^2y^3$

$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}y = y^3$

$\frac{1}{x} = z \quad -\frac{dz}{dy} = \frac{1}{x^2}\frac{dx}{dy}$

$-\frac{dz}{dy} - zy = y^3$

$\frac{dz}{dy} + y = -y^3$

$\therefore IF = e^{y^2/2}$

$\therefore ze^{y^2/2} = -\int e^{y^2/2}y^3\,dy$

$= -$

SECOND ORDER LINEAR DIFFERENTIAL EQUATION:

$$P(x)\, y''(x) + Q(x)\, y'(x) + R(x)\, y(x) = G(x) \quad -(1)$$

↳ 2nd order non-homogenous eq.

Assume that:

i) $P(x), Q(x), R(x), G(x)$ are continuous

ii) $P(x) \neq 0$.

A) $G(x) = 0$ (homogeneous):

$$P(x)\, y''(x) + Q(x)\, y'(x) + R(x)\, y(x) = 0 \quad -(2)$$

↳ 2nd order linear homogenous eq.

i) if $P(x), Q(x), R(x)$ are all constants, then the equation (2) becomes.

$$ay'' + by' + cy = 0$$

↳ linear homogenous with constant coefficients.

by looking at the equation, we can guess that one of solution is of the type $y = e^{mx}$ ~~as constant~~ $m \in R \cup Z$

NOTE: 1. If $y_1(x)$ and $y_2(x)$ are two solutions of linear homogeneous equation (2) then for constants c_1 and c_2 the function $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is also a solution of this differential equation.

∴ This is called ~~as~~ superposition principle.

2. If P, Q, R are continuous functions $P(x)$ is never zero, then the linear homogenous equation (2) has two linearly independent solutions (y_1 and y_2).

eg: x and $2x$ are dependent

x and e^x are independent

e^x and e^{-x} are independent

3. To confirm dependence of variables:

if y_1 and y_2 are 2 solutions

then they are independent if

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$$

Eg: $y_1 = x$ $\quad y_2 = x+3$

$$\begin{vmatrix} x & x+3 \\ 1 & 1 \end{vmatrix} = x-(x+3) = -3 \neq 0$$

$\therefore$ linearly independent.

Eg: $y_1 = \sin x$ $\quad y_2 = \cos x$

$$\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$

$\therefore$ linearly independent.

$y = e^{mx}$

$\therefore y' = me^{mx} \quad y'' = m^2 e^{mx}$

$\therefore am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$

$e^{mx} \neq 0$

$\therefore \boxed{am^2 + bm + c = 0}$ (auxiliary eq.) (a characteristic equation)

$\therefore \boxed{m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$

here $D = b^2 - 4ac$.

case-1: $D > 0$

$\therefore$ we have 2 real solutions for m ($m_1 \neq m_2$)

$\therefore y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ $\quad \left(\begin{array}{c} m_1, m_2 \in \mathbb{R} \\ m_1 \neq m_2 \end{array} \right)$

Example:

$$y'' - y' - 6y = 0$$

$$y = e^{mx}$$

$$\therefore \quad m^2 - m - 6 = 0$$

$$(m-3)(m+2) = 0$$

$$m = 3, \quad m = -2$$

$$\therefore \quad y_1 = e^{3x} \qquad y_2 = e^{-2x}$$

$$\therefore \quad y = c_1 y_1 + c_2 y_2$$

$$\therefore \quad \boxed{y = c_1 e^{3x} + c_2 e^{-2x}}$$

Case-2: $D = 0$

$$m = \frac{-b}{2a}$$

$\therefore \; y_1 = e^{(-b/2a)x}$ is one solution

$\therefore \; y_2 = x e^{(-b/2a)x}$ is another solution.

$$(x e^{mx})$$

$$y_2' = mx e^{mx} + e^{mx}$$

$$y_2'' = m^2 x e^{mx} + m e^{mx} + m e^{mx}$$

$$\therefore \; ay'' + by' + cy = am^2 x e^{mx} + 2am e^{mx} + bmx e^{mx} + b e^{mx} + cx e^{mx}$$

$$= (2am + b) e^{mx} + (am^2 + bm + c) x e^{mx} = 0$$

$\therefore \; 2am + b = 0$ and $am^2 + bm + c = 0$

Hence proved that y_2 is a solution.

$\therefore$ general solution: $y = c_1 e^{mx} + x c_2 e^{mx}$

$$\therefore \quad \boxed{y = (c_1 + c_2 x) e^{mx}}$$

Example:

solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

$y = e^{mx}$

$m^2 + 4m + 4 = 0$

$m = -2, -2$

$\therefore y_1 = e^{-2x}$

$y_2 = xe^{-2x}$

$\therefore$ general solution : $y = (C_1 + C_2 x)e^{-2x}$

CASE-3 : $b^2 - 4ac < 0$

Auxiliary equation has 2 complex roots

$m_1 = \alpha + i\beta \qquad m_2 = \alpha - i\beta$

$y_1 = e^{(\alpha+i\beta)x} \qquad y_2 = e^{(\alpha-i\beta)x}$

but $e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x} \cdot e^{i\beta x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))$

$y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x} \cdot e^{-i\beta x} = e^{\alpha x}(\cos(\beta x) - i\sin(\beta x))$

we need two linearly independent real solutions

$y_3 = \frac{y_1}{2} + \frac{y_2}{2} = \frac{1}{2}(e^{\alpha x} \times 2\cos(\beta x))$

$\therefore y_3 = e^{\alpha x}\cos\beta x$

$y_4 = \frac{1}{2i}(y_1 - y_2) = \frac{1}{2i}(2e^{\alpha x} i\sin(\beta x))$

$\therefore y_4 = e^{\alpha x}\sin(\beta x)$

$\therefore y_3$ and y_4 are independent real solutions.

$\therefore$ general solution:

$$y(x) = c_1 e^{\alpha x} \cos\beta x + c_2 e^{\alpha x} \sin\beta x$$

$$\therefore \boxed{y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)}$$

Example:

$$y'' - 4y' + 5y = 0$$

$$y = e^{mx}$$

$$\therefore m^2 - 4m + 5 = 0$$

$$D < 0$$

$$\therefore m = \frac{4 \pm i\sqrt{4}}{2} = 2 \pm i$$

$$\therefore \alpha = 2 \qquad \beta = 1$$

$$\therefore y_1 = \frac{1}{2}\left(e^{(2+i)x} + e^{(2-i)x}\right) = \frac{e^{2x}}{2}(\cos x + i\sin x + \cos x - i\sin x)$$

$$y_1 = e^{2x}\cos x$$

$$y_2 = \frac{1}{2i}\left(e^{(2+i)x} - e^{(2-i)x}\right) = \frac{1}{2i}(e^{2x})(\cos x + i\sin x - \cos x + i\sin x)$$

$$\therefore y_2 = e^{2x}\sin x$$

$$\therefore y = e^{2x}(c_1\cos x + c_2 \sin x)$$

B) $G(x) \neq 0$ (non-homogeneous) ($ay'' + by' + c = G(x)$)

~~solved~~ steps:

i) homogeneous solved (y_c)

ii) particular solution solved (y_p)

iii) homogenous sol^n + particular sol^n = general sol^n ($y = y_c + y_p$)

$\therefore$ $ay'' + by' + cy = 0$ (complementary function)

$ay'' + by' + cy = G(x)$ (particular ~~function~~ integral / solution)

To find y_p:

Method of undetermined coefficients:

Examples:

$y'' - 2y' - 3y = 1 - x^2$

i) y_c:

$y = e^{mx}$

$m^2 - 2m - 3 = 0$

$(m+1)(m-3) = 0$

$m = -1 \quad m = 3$

$\therefore y_c = c_1 e^{-x} + c_2 e^{3x}$

ii) y_p:

A) $y = am^4 + bm^3 + cm^2 + d$

$y' = 4am^3 + 2bm^2 + 2cm$

$y'' = 12am^2 + 4bm + 2c$

$\therefore (12am^2 + 4bm + 2c) - 2(4am^3 + 2bm^2 + 2cm)$
$- 3(am^4 + bm^3 + cm^2 + d) = 1 - x^2$

$12am^2 + 4bm + 2c - 8am^3 - 4bm^2 - 4cm - 3am^4 - 3bm^3 - 3cm^2 - d$
$= -x^2 + 1$

$\therefore 12a - 4b - 3c = -1$ — (1)

$4b - 4c = 0 \Rightarrow b = c$

$2c - d = 0 \Rightarrow d = 2c$

$-8a - 3b = 0$

$\Rightarrow 4a = \frac{-3b}{2} \quad \therefore 12a = \frac{-9b}{2}$

$\therefore \frac{-9b}{2} - 4b - 3b = -1$

$(9 + 8 + 6)b = 1$

$b = \frac{1}{23}$

$\therefore a = \frac{-3}{8} \times \frac{1}{23}$ $\quad a = \frac{1}{23}$ $\quad d = \frac{2}{23}$

B) assume $y = ax^2 + bx + c$

$y' = 2ax + b$

$y'' = 2a$

$\therefore 2a - 2(2ax+b) - 3(ax^2+bx+c) = 1 - x^2$

$2a - 4ax - 2b - 3ax^2 - 3bx - 3c = 1 - x^2$

$4a + 3b = 0$ — (1)

$2a - 2b - 3c = 1$

$3a = 1$

$\boxed{a = \frac{1}{3}}$ $\quad \therefore \boxed{b = \frac{-4}{9}}$ $\quad \boxed{c = \frac{5}{27}}$

$$\therefore \boxed{y_p = \frac{x^2}{3} - \frac{4x}{9} + \frac{5}{27}}$$

$\therefore$ general solution: $y = y_c + y_p$

$$\therefore \boxed{y = c_1 e^{-x} + c_2 e^{3x} + \frac{x^2}{3} - \frac{4}{9}x + \frac{5}{27}}$$

(2) $y'' - 3y' + 2y = 5e^{2x}$

i) y_c:

$m^2 - 3m + 2 = 0 \qquad (m-1)(m-2) = 0$

$m = 1$ or $m = 2$

$\therefore y_c = c_1 e^{x} + c_2 e^{2x}$

~~ii) y_p:~~

$\cancel{y = Ae^{x}}$

$\cancel{\therefore Ae^{x} - 3Ae^{x} + 2Ae^{x} = 5e^{x}}$

$\cancel{\therefore A = 5}$

$\cancel{\therefore y_p = 5e^{x}}$

general solution :

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ii) y_p:

$$y = axe^x$$

$$\therefore (2ae^x + axe^x) - 3(ae^x + axe^x) + 2axe^x = 5e^x$$

$$2a + \cancel{ax} - 3a - \cancel{3ax} + \cancel{2ax} = 5$$

$$a = -5$$

$$\therefore y_p = -5xe^x$$

$$y'' - 6y' + 9y = e^{3x}$$

i) y_c:

$$m^2 - 6m + 9 = 0 \qquad m = 3, 3$$

$$\therefore y_1 = e^{3x} \qquad y_2 = xe^{3x}$$

$$\therefore y_c = c_1 e^{3x} + c_2 xe^{3x}$$

ii) y_p:

$$y = ax^2 e^{3x}$$

$$y' = 2xae^{3x} + 3x^2ae^{3x}$$

$$y'' = 2ae^{3x} + 6xae^{3x} + 6xae^{3x} + 9x^2ae^{3x}$$

$$\therefore 2ae^{3x} + \cancel{12xae^{3x}} + \cancel{9x^2ae^{3x}} - \cancel{12xae^{3x}} - \cancel{18x^2ae^{3x}} + \cancel{9ax^2e^{3x}} = e^{3x}$$

$$2a = 1 \qquad \boxed{a = \frac{1}{2}}$$

$$\therefore y_p = \frac{1}{2}x^2 e^{3x}$$

$\therefore$ general solution:

$$y = \frac{1}{2}x^2e^{3x} + c_1e^{3x} + c_2xe^{3x}$$

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$y'' - y' = 5e^{x} - \sin 2x$

i) $y = e^{mx}$

$m^2 - m = 0$

$m = 0 \quad m = 1$

$\therefore y_c = c_1 + c_2 e^{x}$

ii) $y = axe^{x} + b\sin 2x + c\cos 2x$

$y' = axe^{x} + ae^{x} + 2b\cos 2x - 2c\sin 2x$

$\therefore y'' = 2ae^{x} + axe^{x} - 4b\sin 2x - 4c\cos 2x$

$\therefore \cancel{2ae^{x}} + \cancel{axe^{x}} - 4b\sin 2x - 4c\cos 2x - \cancel{axe^{x}} - \cancel{ae^{x}} - 2b\cos 2x + 2c\sin 2x$ (with ae^{x} written above)

$= 5e^{x} - \sin 2x$

$ae^{x} - \sin 2x(4b - 2c) - \cos 2x(4c + 2b) = 5e^{x} - \sin 2x$

$\boxed{a = 5}$

$4b - 2c = 1 \qquad 4c + 2b = 0 \qquad \therefore b = -2c$

$-8c - 2c = 1 \qquad \boxed{c = \frac{-1}{10}} \qquad \therefore \boxed{b = \frac{1}{5}}$

$\therefore y_p = 5xe^{x} + \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x$

$\therefore$ general solution:

$y = c_1 + c_2 e^{x} + 5e^{x} + \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x$

NOTE: possible ~~substitutions~~ forms of particular solution.

$G(x)$	Possible y_p
i) $e^{\lambda x}$ ($\lambda \neq$ root of homo)	$y_p = Ae^{\lambda x}$
ii) $e^{\lambda x}$ ($\lambda =$ single root of homo)	$y_p : Axe^{\lambda x}$
iii) $e^{\lambda x}$ ($\lambda =$ double root of homo)	$y_p : Ax^2e^{\lambda x}$
iv) $\sin kx$ or $\cos kx$ ($ki \neq$ root of homo)	$y_p : B\cos kx + C\sin kx$
v) $px^2 + qx + r$ ($0 \neq$ root of homo)	$y_p : ax^2 + bx + c$
vi) $px^2 + qx + r$ ($0 =$ single root of homo)	$y_p : ax^3 + bx^2 + cx$
vii) $px^2 + qx + r$ ($0 =$ double root of homo)	$y_p : ax^4 + bx^3 + cx^2$

METHOD OF VARIATION OF PARAMETER:

$$ay'' + by' + cy = G(x) \quad -(1)$$

i) solve associated homogeneous ~~solu~~ equation

~~by~~ let $y_c = c_1y_1 + c_2y_2$

ii) assume that

$$y_p = c_1(x)\,y_1(x) + c_2(x)\,y_2(x)$$

$$y_p' = c_1'(x)\,y_1(x) + c_2'(x)\,y_2(x) + c_1(x)\,y_1'(x) + y_2'(x)\,c_2(x)$$

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assume $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$

$\therefore y_p' = c_1(x)y_1'(x) + c_2(x)y_2'(x)$

$y_p'' = c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + c_1(x)y_1''(x) + c_2(x)y_2''(x)$

substituting in (1)

$a(c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + c_1(x)y_1''(x) + c_2(x)y_2''(x))$
$+ b(c_1(x)y_1'(x) + c_2(x)y_2'(x)) + c(c_1(x)y_1(x) + c_2(x)y_2(x))$
$= G(x)$

$\therefore c_1(x)(ay_1'' + by_1' + cy_1) + c_2(x)(ay_2'' + by_2' + cy_2)$

$+ a(c_1'y_1' + c_2'y_2') = G(x)$

but $ay_i'' + by_i' + cy_i = 0 \quad (i = 1, 2)$

$\therefore \boxed{a(c_1'y_1' + c_2'y_2') = G(x)}$

$$\therefore \boxed{\begin{aligned} c_1'y_1 + c_2'y_2 &= 0 \\ c_1'y_1' + c_2'y_2' &= \frac{G(x)}{a} \end{aligned}}$$

$\therefore c_1' = \dfrac{w_1}{w} \qquad c_2' = \dfrac{w_2}{w}$

where $w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ $\quad w_1 = \begin{vmatrix} 0 & y_2 \\ G(x)/a & y_2' \end{vmatrix}$ $\quad w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & G(x)/a \end{vmatrix}$

$\therefore$

(1) $y'' + y = \tan x$

i) homogeneous:

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$\therefore y_1 = \cos x \qquad y_2 = \sin x$$

ii) particular:

$$y_p = c_1(x)\cos x + c_2(x)\sin x$$

$$c_1' \cos x + c_2' \sin x = 0$$

$$-c_1' \sin x + c_2' \cos x = \tan x$$

$$c_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\tan x \sin x}{\cos^2 x + \sin^2 x} = \frac{-\sin^2 x \sec x}{1}$$

$$c_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \tan x}{\cos^2 x + \sin^2 x} = \frac{\sin x}{1}$$

$$\therefore c_1 = \int c_1' \, dx = \cancel{\int \frac{\sin^2 x}{\cos^2 x} \cos x \, dx}$$

$$= \int \frac{-\sin^2 x}{\cos x} \, dx$$

$$= \int \left(\frac{1 - \sin^2 x}{\cos x} - \frac{1}{\cos x} \right) dx$$

$$= \int (\cos x - \sec x) \, dx$$

$$= \sin x - \log|\sec + \tan x|$$

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$c_2 = \int c_2' \, dx = \int \sin x = -\cos x$

$\therefore y_p = (\sin x - \log|\sec x + \tan x|) \cos x + (\sin x)(-\cos x)$

$y_p = -\cos x \log|\sec x + \tan x|$

$\Rightarrow$ For 3^{rd} ~~variables~~ order eq.

$$w = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \qquad w_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ \frac{G(x)}{a} & y_2'' & y_3'' \end{vmatrix}$$

$c_1' = \frac{w_1}{w} \quad c_2' = \frac{w_2}{w} \quad c_3' = \frac{w_3}{w}$

PARTIAL DERIVATIVES:

$z = f(x,y)$

$$\left.\frac{\delta f}{\delta x}\right|_{P(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$\left.\frac{\delta f}{\delta y}\right|_{(x_0,y_0)} = \lim_{k\to 0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k}$$

Examples:

① $f(x,y) = x\cos xy$

$$\frac{\delta f}{\delta x} = -xy\sin xy + \cos xy$$

$$\frac{\delta f}{\delta y} = -x^2\sin xy$$

IMPLICIT:

$yz - \log z = x+y$

$$y\frac{\delta z}{\delta x} - \frac{1}{z}\frac{\delta z}{\delta x} = 1 \qquad\qquad y\frac{\delta z}{\delta y} + z - \frac{1}{z}\frac{\delta z}{\delta y} = 1$$

$$\frac{\delta z}{\delta x}\left(y - \frac{1}{z}\right) = 1 \qquad\qquad \frac{\delta z}{\delta y} = \frac{(1-z)z}{yz-1}$$

$$\frac{\delta z}{\delta x} = \frac{z}{yz-1}$$

PARTIAL DERIVATIVES AND CONTINUITY:

For the existance of partial derivative the function need not be continuous.

① $f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

at $(x,y) = (0,0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \quad \text{(given that } x \neq 0 \text{ or } y \neq 0\text{)}$$

$$f(x,y) = 1$$

$\therefore$ function is not continuous.

$$\left.\frac{\delta f}{\delta x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1-1}{h} = 0$$

$$\left.\frac{\delta f}{\delta y}\right|_{(0,0)} = \lim_{k\to 0} \frac{f(0,k+0) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{1-1}{k} = 0$$

$\therefore$ Partial derivative exists at origin even though function is discontinuous at $(0,0)$.

MIXED PARTIAL DERIVATION:

$f(x,y) \to$ function.

$$\frac{\delta^2 f}{\delta x \delta y} \neq \frac{\delta^2 f}{\delta y \delta x} \quad \left(\text{ie; } \frac{\delta}{\delta x}\left(\frac{\delta f}{\delta y}\right) \neq \frac{\delta}{\delta y}\left(\frac{\delta f}{\delta x}\right)\right)$$

The two are equal only if $\frac{\delta f}{\delta x}$ and $\frac{\delta f}{\delta y}$ exist and are continuous.

DERIVATIVE:

→ If the partial derivative of a function are continuous throughout an open region R then f is differentiable at every point on R.

→ Here, differentiability ⇒ continuity

→ Let $w = f(x, y)$

If $f(x, y)$ is differentiable and $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then $w = f(x(t), y(t))$ is differentiable and

$$\frac{dw}{dt} = \frac{\delta f}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{\delta f}{\delta y} \cdot \frac{\delta y}{\delta t}$$

→ if $w = f(x, y, z)$

$x = g(r, s) \quad y = h(r, s) \quad z = k(r, s)$

$$\therefore \frac{dw}{dr} = \frac{\delta f}{\delta x} \cdot \frac{\delta x}{\delta r} + \frac{\delta f}{\delta y} \cdot \frac{\delta y}{\delta r} + \frac{\delta f}{\delta z} \cdot \frac{\delta z}{\delta r}$$

$$\frac{dw}{ds} = \frac{\delta f}{\delta x} \cdot \frac{\delta x}{\delta s} + \frac{\delta f}{\delta y} \cdot \frac{\delta y}{\delta s} + \frac{\delta f}{\delta z} \cdot \frac{\delta z}{\delta s}$$

DIRECTIONAL DERIVATIVES:

→ if $f(x, y)$ is differentiable, then the rate at which f changes wrt t along a differentiable curve, $x = x(t)$, $y = y(t)$ is:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

→ Defination:

The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector.

$$u = u_1\hat{i} + u_2\hat{j}$$

is the number

$$\left.\frac{df}{ds}\right|_{u, P_0} = \lim_{s \to 0} \left(\frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \right)$$

(provided the limit exists.)

Example:

Find derivative of $f(x,y) = x^2 + xy$ at $P_0(x_0, y_0) = (1,2)$ in direction of unit vector

$$u = \frac{1}{\sqrt{2}}(\hat{i}+\hat{j})$$

$$\left.\frac{df}{ds}\right|_{u,P_0} = \lim_{s\to 0} \frac{f(x_0+u_1 s, y_0+u_2 s) - f(x_0, y_0)}{s}$$

$$f(x_0, y_0) = 1^2 + 2 = 3$$

$$f(x_0+u_1 s, y_0+u_2 s) = \left(1+\frac{s}{\sqrt{2}}\right)^2 + \left(1+\frac{s}{\sqrt{2}}\right)\left(2+\frac{s}{\sqrt{2}}\right)$$

$$\left.\frac{df}{ds}\right|_{u,P_0} = \lim_{s\to 0} \frac{(1+s/\sqrt{2})^2 + (1+s/\sqrt{2})(2+s/\sqrt{2}) - 3}{s}$$

$$= \lim_{s\to 0} \frac{\cancel{1} + s^2/2 + 2s/\sqrt{2} + \cancel{2} + s^2/2 + 3s/\sqrt{2} - \cancel{3}}{s}$$

$$= \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

GRADIENT VECTOR:

at $P_0(x_0, y_0)$ & $u = u_1\hat{i} + u_2\hat{j}$

$$\left.\frac{df}{ds}\right|_{P_0} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{ds}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{ds}\right)$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right) \cdot (u_1\hat{i} + u_2\hat{j})$$

$\therefore$ gradient vector $= \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

$$\therefore \left.\frac{df}{ds}\right|_{P_0} = \left.\vec{\nabla} f\right|_{P_0} \cdot \vec{u}$$

$$\therefore \left.\frac{df}{ds}\right|_{P_0} = |\vec{\nabla} f|\,|\vec{u}|\cos\theta$$

$$= |\nabla f| \cos\theta$$

($\because$ $\vec{u}$ is a unit vector)

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⇒ IMPORTANT RESULTS :

1. The function f increases most rapidly when $\cos\theta = 1$, ie, when u is in the direction of ∇f. That is, at each point P, f increases most rapidly in the direction of gradient vector ∇f ~~and~~ at P.
2. Similarly, f decreases most rapidly when $\cos\theta = -1$, ie, when u is in direction opposite to ∇f.
3. for any direction orthogonal to ∇f is the direction of zero change.

EXAMPLE :

1. Find the direction $\frac{x^2}{2} + \frac{y^2}{2} = f(x,y)$
   i) increase max at $(1,1)$
   ii) decreases max at $(1,1)$
   iii) no change at $(1,1)$

Sol^n $\nabla f = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\Big|_{(1,1)} = (x\hat{i} + y\hat{j})\Big|_{(1,1)} = \hat{i} + \hat{j}$

i) in direction ($\because \cos\theta = 1$

$\therefore \vec{u} = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$

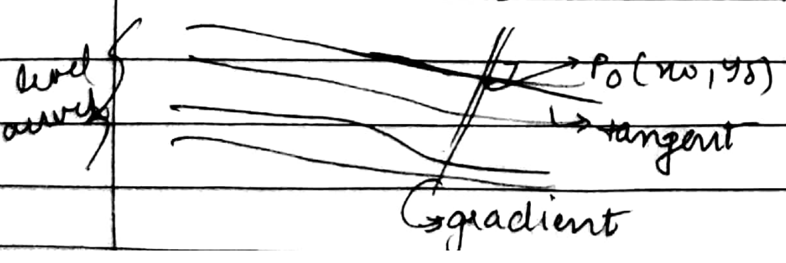
ii) opp direction ($\because \cos\theta = -1$)

$\therefore \vec{u} = -\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$

iii) no change

$\therefore \vec{u} = \frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$ or $\vec{u} = \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})$

GRADIENT AND ~~CURVES~~ TANGENTS TO LEVEL CURVES :



if a differentiable function $f(x,y)$ has a constant value c along a curve $r = x(t)\hat{i} + y(t)\hat{j}$ then $f(x(t), y(t)) = c$ is a level curve.

then,

$$\frac{d}{dt}\left(f(x(t), y(t))\right) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = 0$$

$$\therefore \underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)}_{\text{gradient vector}} \cdot \underbrace{\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}\right)}_{\text{tangent vector}} = 0$$

$$\therefore \vec{\nabla f} \cdot \frac{d\vec{r}}{dt} = 0$$

$\therefore$ gradient vector $\perp$ tangent vector.

$\Rightarrow$ PROPERTIES:

1) $\nabla(f+g) = \nabla f + \nabla g$
2) $\nabla(f-g) = \nabla f - \nabla g$
3) $\nabla(kf) = k\nabla f$
4) $\nabla(fg) = (\nabla f)g + (\nabla g)f$
5) $\nabla\left(\frac{f}{g}\right) = \frac{(\nabla f)g - (\nabla g)f}{g^2}$ $\quad (g \neq 0)$

FUNCTIONS OF 3 VARIABLES:

directional derivative:

$$\left.\frac{df}{ds}\right|_u = D_u f = \vec{\nabla f}\cdot\vec{u}$$

$$\left(\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$

EXTREME VALUES AND SADDLE POINTS:

Defination:

let $f(x,y)$ be defined on Region R containing the point (a,b)

i) (a,b) is said to be the local maximum ~~value~~ point of f if $f(a,b) \geq f(x,y)$ $\forall$ $(x,y) \in$ open disc centred at (a,b)

ii) (a,b) is said to be local minimum point of f if $f(a,b) \leq f(x,y)$ $\forall$ $(x,y) \in$ open disc centred at (a,b)

iii) global max: $f(a,b) \geq f(x,y)$ $\forall (x,y) \in R$

iv) global min: $f(a,b) \leq f(x,y)$ $\forall (x,y) \in R$

FIRST DERIVATIVE TEST:

If $f(x,y)$ has a local maximum a local minimum value at an interiorpoint of (a,b) of R and if the first partial derivative exists there, then

$$f_x(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0$$

$$f_y(a,b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = 0$$

→ critical points:

an interia point of the domain of the function $f(x,y)$ where both f_x and f_y are zero or the points where one or both (f_x, f_y) do not exist. These points are called $f(x,y)$.

→ Saddle points:

The critical point which is neither max or min, is called a saddle point.

Sufficient condition:

Result:

suppose that $f(x,y)$ and it's 1^{st} and 2^{nd} partial derivatives are continuous through out a disck centred at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$

Then,

① local maximum at (a,b) if

$$f_{xx}\Big|_{(a,b)} \left(= \frac{\partial^2 f}{\partial x^2}\Big|_{(a,b)} \right) < 0$$

and $\quad f_{xx} f_{yy} - (f_{xy})^2 > 0 \quad$ at (a,b)

$$\therefore \text{ Hessian matrix}\Big|_{(a,b)} > 0$$

$$\left(\text{Hessian matrix}\Big|_{(a,b)} = \left| \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \right|_{(a,b)} \right)$$

② local minimum:

$$f_{xx} > 0 \quad \text{and} \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$$

③ saddle point:

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} < 0$$

④ Test is inclusive if : $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = 0$

Example:

① Find the local extreme values:

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$$

$$f_{xx} = \frac{\partial}{\partial x}(y - 2x - 2) = -2$$

$$f_{yy} = \frac{\partial}{\partial y}(x - 2y - 2) = -2$$

$$f_{xy} = f_{yx} = \frac{\partial}{\partial x}(x - 2y - 2) = 1$$

$$\therefore \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4 - 1 = 3$$

$\therefore f_{xx} < 0$

$f_{xx} f_{yy} - (f_{xy})^2 > 0$

$\therefore$ maximum found at $(-2, -2)$.

$$\left\{ \begin{array}{lll} \because f_x = 0 & \text{and} & f_y = 0 \\ \therefore x - 2y - 2 = 0 & \text{and} & 2x - y + 2 = 0 \end{array} \right\}$$

② $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

$f_x = -6x + 6y = 0 \Rightarrow x = y$

$f_y = 6y - 6y^2 + 6x = 0$

$\therefore 2y - y^2 = 0$

$y = 0$ or $y = 2$

$\therefore$ critical points : $(0,0)$ and $(2,2)$

$f_{xx} = -6$

$f_{yy} = 6 - 12y$

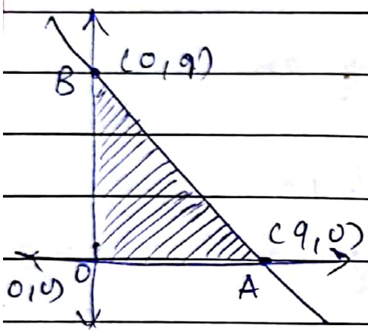
$f_{xy} = 6$

∴ for $(0,0)$: $\begin{vmatrix} -6 & 6 \\ 6 & 6 \end{vmatrix} = -2\times 36 < 0$

for $(2,2)$: $\begin{vmatrix} -6 & -6 \\ 6 & -18 \end{vmatrix} = 18\times 6 + 6\times 6 > 0$

∴ maximum at $(2,2)$
saddle point at $(0,0)$.

$f(x,y) = 2+2x+2y-x^2-y^2$
on a triangular region in 1st quad bounded by the lines $x=0$, $y=0$, $y=9-x$



For interior points
$f_x = 2-2x = 0$
$f_y = 2-2y = 0$

∴ critical point : $(1,1)$

$f(1,1) = 4$

→ $f(x,0) = 2+2x-x^2$
at $(0,0)$, $f(0,0)=2$
at $(9,0)$, $f(9,0) = -61$
$f'(x,0) = 2-2x = 0$ ∴ $x=1$
∴ $f(1,0) = 3$

→ $f(0,y) = 2+2y-y^2$
$f(0,9) = -61$
$f'(0,y) = 2-2y \Rightarrow y=1$
∴ $f(0,1) = 3$

→ $f(x,9-x) = 2+\cancel{2x}+18-\cancel{2x} - x^2 - (9-x)^2$
$= 20 - x^2 - (81 - 18x + x^2)$
$= -61 + 18x$
$f'(x,9-x) = 18$

∴ min at $(0,9)$ and $(9,0)$, max at $(0,1)$ and $(1,0)$

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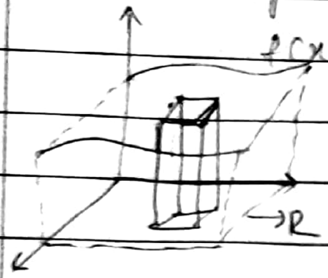
MULTIPLE INTEGRALS:

Double integrals to find volume:



$$S_n = \sum_{k=1}^{n} \Delta x_k \Delta y_k f(x_k, y_k)$$

as $\|P\| \to 0$, $n \to \infty$

$$\therefore \text{volume} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{1}^{n} \Delta x_k \Delta y_k f(x_k, y_k)$$

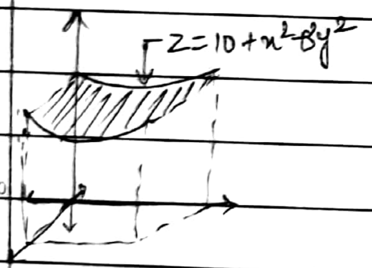
$$\therefore V = \iint_R f(x,y)\, dx\, dy$$

⇒ **Fubini's Theorem:**

if $f(x,y)$ is continuous throughout rectangular region R $a \le x \le b$, $c \le y \le d$,

$$\iint_R f(x,y)\, dA = \int\left(\int_R f(x,y)\, dx\right) dy = \int\left(\int_R f(x,y)\, dy\right) dx$$

1) Find volume bounded above by elliptical paraboloid, $z = 10 + x^2 + 3y^2$, and below by the rectange. R: $0 \le x \le 1$, $0 \le y \le 2$.



$$\therefore \text{volume} = \iint_R f(x,y)\, dx\, dy$$

$$\therefore V = \int_0^2 \left(\int_0^1 (10 + x^2 + y^2)\, dx\right) dy$$

$$= \int_0^2 \left(10 + \frac{1}{3} + 3y^2\right) dy$$

$$= \left[10y + \frac{y}{3} + y^3\right]_0^2 = \frac{86}{3}$$

2) Double Integrals over a general region:

here limits will be functions of x.

if $f(x,y)$ is +ve and continuous over R, the volume bounded by $z = f(x,y)$ and R is:

$$V = \iint_R f(x,y)\, dx\, dy$$

if R is is the region bounded by $y = g_1(x)$ and $y = g_2(x)$ and on sides by $x = a$ and $x = b$

$$V = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx$$

Fubini's Result:

① $x \in [a,b]$, $y \in [g_1(x), g_2(x)]$.

then $V = \iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$

② $y \in [c,d]$ and $x \in [h_1(y), h_2(y)]$.

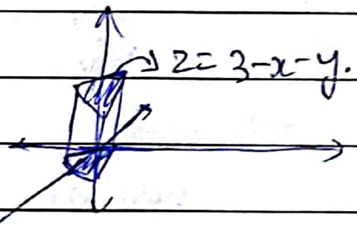
then $V = \iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$

⇒ Example:

Find the volume of the prism whose base is triangular in the x-y plane bounded by the x-axis and the lines $y = x$, $x = 1$ whose top lines in the plane $f(x,y) = 3 - x - y$

$y = 0$ to $y = x$ and

$x = 0$ to $x = 1$



$$\therefore V = \int_0^1 \left(\int_0^x (3 - x - y)\, dy \right) dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_0^x dx$$

$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2} \right) dx$$

$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3} - \frac{x^3}{6} \right]_0^1$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_0^1$$

$$= \frac{3}{2} - \frac{1}{2}$$

$$= 1$$

FINDING LIMITS OF INTEGRATION:

① using vertical cross-section

$$\iint_R f(x,y)\,dy\,dx$$

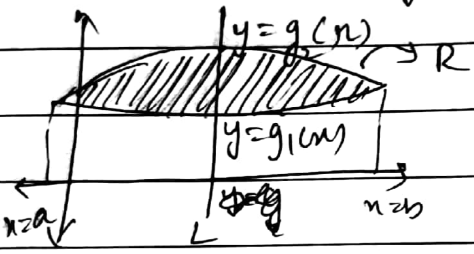
Step 1: sketch the region of integration

Step 2: finding limits of integration

→ imagine a vertical line L cutting through the region R in the increasing direction of y.

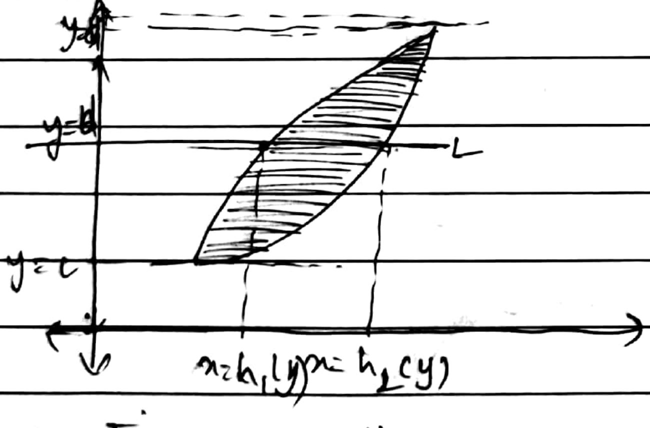
→ mark the y-values where L ~~exit~~ enters and leaves the region.

These are the y limits of integration



→ choose x limits that include all the vertical lines through R, parallel to L.

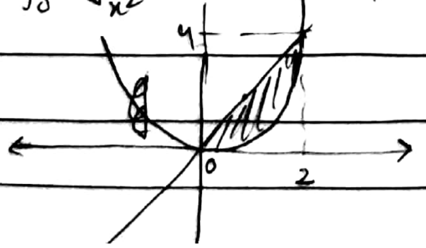
② using horizontal cross-section:



$$\iint_R f(x,y)\,dx\,dy$$

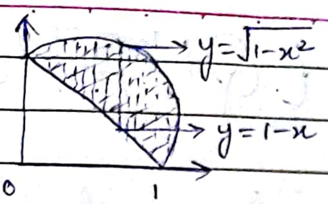
if function is continuous, then $\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx$

Example ① $\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$



$$\therefore V = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy$$

(2)



$$V=\iint_R f(x,y)\,dy\,dx$$

$$V=\iint_R f(x,y)\,dx\,dy$$

$$V=\int_0^1\int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

$$V=\int_0^1\int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

PROPERTIES OF DOUBLE INTEGRALS:

(1) $\iint_R c f(x,y)\,dA = c\iint_R f(x,y)\,dA$

(2) $\iint_R (f(x,y)+g(x,y))\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

(3) domination property.

(a) $\iint_R f(x,y)\,dA \geq 0$ — if $f(x,y) \geq 0$ on R

(b) $\iint_R f(x,y)\,dA \leq 0$ if $f(x,y) \leq 0$ on R

(4) Additivity wrt R.

if $R=R_1+R_2$; $V=\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

Area by double integral:

$$\iint_R dA = \iint_R dx\,dy.$$

TRIPPLE INTEGRALS:

$$V=\iiint_D dx\,dy\,dz$$

Example: Find volume $z=x^2+3y^2$ and $8-x^2-y^2$

$$V=\iiint_{z=x^2+3y^2}^{z=8-x^2-y^2} dz\,dy\,dx$$

for y limits, $z=0$, $y=\pm\sqrt{\frac{8-2x^2}{4}}$

for x-limit, $y=0$; $x=\pm 2$

$$\therefore V=\int_{-2}^{2}\int_{-\sqrt{8-2x^2/4}}^{\sqrt{8-2x^2/4}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$$

DIFFERENTIAL EQUATION

GENERAL 1^{st} ORDER DIFFERENTIAL EQUATION:

$\frac{dy}{dx} = f(x,y)$ → (1) first order ordinary differential equation

Solution:

y = dependent variable

x = independent variable

Any differentiable function $y(x)$ which statisfies (1) is called a solution of the differential equation.

A) Seperable equation:

$$\frac{dy}{dx} = f(x,y) \quad -(1)$$

if we can write $f(x,y) = g(x)\,h(y)$, then (1) is called seprable differentiable equation.

$$\therefore \frac{dy}{dx} = g(x)\,h(y)$$

$$\therefore \frac{1}{h(y)}\,dy = g(x)\,dx$$

3) HOMOGENOUS EQUTIONS:

$$\frac{dy}{dx} = f(x,y) \quad -(1)$$

if $f(x,y)$ is homogenous equation, then (1) is called homogenous equation.

now, $f(x,y)$ is a homogenous equation if

$$f(tx,ty) = t^n f(x,y) \qquad t \in R$$

Solution :

take $y = xv$

$\therefore \frac{dy}{dx} = x\frac{dv}{dx} + v$

$\therefore$ replace and solve in variable seperable form.

LINEAR FIRST ORDER DIFFERTIABLE EQUATION :

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

($P(x)$ and $Q(x)$ are continuous functions of x only.)

Solution :

Integrating factor (IF) $= e^{\int P(x)\,dx}$

$\therefore y\, e^{\int P(x)dx} = \int Q(x)\, e^{\int P(x)dx}\, dx$

→ we multiply a suitable function $v(x)$ that transforms the LHS of the derivative eqn ~~of~~ into derivative of the product $v(x) \cdot y$.

here $v(x)$ is called integrating factor.

$$v(x)\left(\frac{dy}{dx} + P(x)\,y = Q(x)\right)$$

$$\int\left(v(x)\frac{dy}{dx} + P(x)\cdot v(x)\,y\right) = \int v(x)\,Q(x)$$

$$\therefore \int \frac{d}{dx}\left(v(x) \cdot y\right) = \int v(x) \cdot Q(x)$$

$$\therefore \boxed{y\,v(x) = \int v(x)\,Q(x)\,dx}$$

$$\frac{d}{dx}(y \cdot v(x)) = y\frac{dv}{dx} + v(x)\frac{dy}{dx} = v(x)\frac{dy}{dx} + P(x) \cdot v(x)\,y$$

$$\therefore \frac{dv}{dx} = P(x)\,v(x)$$

$$\therefore \log v(x) = \int P(x)\,dx$$

$$\therefore \boxed{v(x) = e^{\int P(x)\,dx}}$$

∴ final solution:

$$\boxed{y e^{\int P(x)\,dx} = \int Q(x)\, e^{\int P(x)\,dx}\, dx}$$

EXAMPLE:

$$x\frac{dy}{dx} - 3y = x^2 \Rightarrow \frac{dy}{dx} - \frac{3}{x}(y) = x$$

$$IF = \frac{1}{x^3}$$

$$\therefore \; y/_{x^3} = \int x \times \frac{1}{x^3}\, dx = \frac{-1}{x} + c$$

$$\therefore \; \boxed{y = -x^2 + cx^3}$$

Find particular solution for $3xy' - y = \log x + 1$ when $y(1) = -2$ $(x>0)$

$$\frac{dy}{dx} - \frac{1}{3x}y = \left(\frac{\log x + 1}{3x}\right)$$

$$IF = \frac{1}{x^{1/3}}$$

$$\frac{y}{x^{1/3}} = \int \left(\frac{\log x + 1}{3x}\right) \times x^{-1/3}\, dx$$

$$= \frac{1}{3}\int (\log x + 1)\, x^{-4/3}\, dx$$

$$= \frac{1}{3}\left(-3x^{-1/3} - 3x^{-1/3}\log x + 9x^{-1/3}\right) + c$$

$=$

D) EQUATIONS REDUCIBLE TO LINEAR EQUATION:

$$\frac{dy}{dx} + yP(x) = y^n Q(x)$$

$$\therefore \frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P(x) = Q(x) \quad - (1)$$

$$\frac{1}{y^{(n-1)}} = z$$

$$\therefore \frac{dz}{dx} = \frac{-1}{y^{n-1+1}} \times (n-1)\frac{dy}{dx}$$

$$\frac{dz}{dx} = (1-n)\left(\frac{1}{y^n}\frac{dy}{dx}\right)$$

$$\therefore (1) \equiv \frac{1}{(1-n)}\frac{dz}{dx} + zP(x) = Q(x)$$

$$\therefore \frac{dz}{dx} + (1-n)zP(x) = (1-n)Q(x)$$

Example:

(1) $x\frac{dy}{dx} + y = x^4y^3$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{xy^2} = x^3$$

$$\frac{1}{y^2} = z \qquad \therefore \frac{dz}{dx} = \frac{-2}{y^3}\frac{dy}{dx}$$

$$\therefore \frac{-1}{2}\frac{dz}{dx} = \frac{1}{y^3}\frac{dy}{dx}$$

$$\therefore \frac{-1}{2}\frac{dz}{dx} + \frac{z}{x} = x^3 \qquad \therefore \frac{dz}{dx} - \frac{2z}{x} = -2x^3$$

$$\therefore IF = \frac{1}{x^2}$$

$$\therefore \frac{z}{x^2} = \int \frac{x^3}{x^2}dx = \frac{x^2}{2} + c$$

$$\therefore \frac{1}{y^2} = \frac{x^4}{2} + c$$

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(2) $\frac{dy}{dx}(x^2y^3+xy)=1$

$$\frac{dx}{dy} = x^2y^3 + xy$$

$$\frac{dx}{dy} - xy = x^2y^3$$

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}y = y^3$$

$$\frac{1}{x} = z \qquad -\frac{dz}{dy} = \frac{1}{x^2}\frac{dx}{dy}$$

$$-\frac{dz}{dy} - zy = y^3$$

$$\frac{dz}{dy} + y = -y^3$$

$$\therefore IF = e^{y^2/2}$$

$$\therefore z e^{y^2/2} = -\int e^{y^2/2} y^3 \, dy$$

$$= -$$

SECOND ORDER LINEAR DIFFERENTIAL EQUATION:

$$P(x)\,y''(x) + Q(x)\,y'(x) + R(x)\,y(x) = G(x) \quad \text{—(1)}$$

↳ 2nd order non-homogenous eq.

Assume that:

i) $P(x), Q(x), R(x), G(x)$ are continuous

ii) $P(x) \neq 0$.

A) $G(x) = 0$ (homogeneous):

$$P(x)\,y''(x) + Q(x)\,y'(x) + R(x)\,y(x) = 0 \quad \text{—(2)}$$

↳ 2nd order liner homogenous eq.

i) if $P(x), Q(x), R(x)$ are all constants, then the equation (2) becomes.

$$ay'' + by' + cy = 0$$

↳ linear homogenous with constant coefficients.

by looking at the equation, we can guess that one of solution is of the type $y = e^{mx}$ ~~m is constant~~ $m \in \mathbb{R} \cup \mathbb{Z}$

NOTE: 1. If $y_1(x)$ and $y_2(x)$ are two solutions of linear homogeneous equation (2) then for constants c_1 and c_2 the function

$y(x) = c_1 y_1(x) + c_2 y_2(x)$ is also a solution of this differential equation.

∴ This is called ~~as~~ superposition principle.

2. If P, Q, R are continuous functions $P(x)$ is never zero, then the linear homogenous equation (2) has two linearly independent solutions (y_1 and y_2).

Eg: x and $2x$ are dependent

x and e^x are independent

e^x and e^{-x} are independent

3. To confirm dependence of variables:

if y_1 and y_2 are 2 solutions

then they are independent if

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$$

Eg: $y_1 = x \qquad y_2 = x+3$

$$\begin{vmatrix} x & x+3 \\ 1 & 1 \end{vmatrix} = x-(x+3) = -3 \neq 0$$

$\therefore$ linearly independent.

Eg: $y_1 = \sin x \qquad y_2 = \cos x$

$$\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$

$\therefore$ linearly independent.

$y = e^{mx}$

$\therefore y' = me^{mx} \qquad y'' = m^2 e^{mx}$

$\therefore am^2e^{mx} + bme^{mx} + ce^{mx} = 0$

$e^{mx} \neq 0$

$\therefore \boxed{am^2 + bm + c = 0}$ (auxiliary eq.) (a characteristic equation)

$\therefore \boxed{m = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}}$

here $D = b^2 - 4ac$.

case-1: $D > 0$

$\therefore$ we have 2 real solutions for m ($m_1 \neq m_2$)

$\therefore y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ $\left(\begin{matrix} m_1, m_2 \in \mathbb{R} \\ m_1 \neq m_2 \end{matrix}\right)$

Example:

$$y'' - y' - 6y = 0$$

$$y = e^{mx}$$

$$\therefore \quad m^2 - m - 6 = 0$$

$$(m-3)(m+2) = 0$$

$$m = 3, \quad m = -2$$

$$\therefore \quad y_1 = e^{3x} \qquad y_2 = e^{-2x}$$

$$\therefore \quad y = c_1 y_1 + c_2 y_2$$

$$\therefore \quad \boxed{y = c_1 e^{3x} + c_2 e^{-2x}}$$

Case-2: $D = 0$

$$m = \frac{-b}{2a}$$

$\therefore \quad y_1 = e^{(-b/2a)x}$ is one solution

$\therefore \quad y_2 = x e^{(-b/2a)x}$ is another solution.

$$(x e^{mx})$$

$$y_2' = mx e^{mx} + e^{mx}$$

$$y_2'' = m^2 x e^{mx} + m e^{mx} + m e^{mx}$$

$$\therefore \quad ay'' + by' + cy = am^2 x e^{mx} + 2am e^{mx} + bmx e^{mx} + b e^{mx} + cx e^{mx}$$

$$= (2am + b) e^{mx} + (am^2 + bm + c) x e^{mx} = 0$$

$$\therefore \quad 2am + b = 0 \quad \text{and} \quad am^2 + bm + c = 0$$

Hence proved that y_2 is a solution.

$\therefore$ general solution: $y = c_1 e^{mx} + x c_2 e^{mx}$

$$\therefore \quad \boxed{y = (c_1 + c_2 x) e^{mx}}$$

Example:

solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

$y = e^{mx}$

$m^2 + 4m + 4 = 0$

$m = -2, -2$

$\therefore y_1 = e^{-2x}$

$y_2 = xe^{-2x}$

$\therefore$ general solution : $y = (C_1 + C_2 x)e^{-2x}$

CASE-3 : $b^2 - 4ac < 0$

Auxiliary equation has 2 complex roots

$m_1 = \alpha + i\beta \qquad m_2 = \alpha - i\beta$

$y_1 = e^{(\alpha+i\beta)x} \qquad y_2 = e^{(\alpha-i\beta)x}$

but $e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}\cdot e^{i\beta x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))$

$y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}\cdot e^{-i\beta x} = e^{\alpha x}(\cos(\beta x) - i\sin(\beta x))$

we need two linearly independent real solutions

$y_3 = \frac{y_1}{2} + \frac{y_2}{2} = \frac{1}{2}(e^{\alpha x} \times 2\cos(\beta x))$

$\therefore y_3 = e^{\alpha x}\cos\beta x$

$y_4 = \frac{1}{2i}(y_1 - y_2) = \frac{1}{2i}(2e^{\alpha x} i\sin(\beta x))$

$\therefore y_4 = e^{\alpha x}\sin(\beta x)$

$\therefore y_3$ and y_4 are independent real solutions.

$\therefore$ general solution:

$$y(x) = c_1 e^{\alpha x} \cos\beta x + c_2 e^{\alpha x} \sin\beta x$$

$$\therefore \boxed{y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)}$$

Example:

$$y'' - 4y' + 5y = 0$$

$$y = e^{mx}$$

$$\therefore m^2 - 4m + 5 = 0$$

$$D < 0$$

$$\therefore m = \frac{4 \pm i\sqrt{4}}{2} = 2 \pm i$$

$$\therefore \alpha = 2 \qquad \beta = 1$$

$$\therefore y_1 = \frac{1}{2}\left(e^{(2+i)x} + e^{(2-i)x}\right) = \frac{e^{2x}}{2}(\cos x + i\sin x + \cos x - i\sin x)$$

$$y_1 = e^{2x}\cos x$$

$$y_2 = \frac{1}{2i}\left(e^{(2+i)x} - e^{(2-i)x}\right) = \frac{1}{2i}(e^{2x})(\cos x + i\sin x - \cos x + i\sin x)$$

$$\therefore y_2 = e^{2x}\sin x$$

$$\therefore y = e^{2x}(c_1\cos x + c_2\sin x)$$

B) $G(x) \neq 0$ (non-homogeneous) ($ay'' + by' + c = G(x)$)

~~solved~~ steps:

i) homogeneous solved (y_c)

ii) particular solution solved (y_p)

iii) homogenous sol^n + particular sol^n = general sol^n
($y = y_c + y_p$)

$\therefore$ $ay'' + by' + cy = 0$ (complementary function)

$ay'' + by' + cy = G(x)$ (particular ~~function~~ integral / solution)

To find y_p:

Method of undetermined coefficients:

Examples:

$y'' - 2y' - 3y = 1 - x^2$

i) y_c:

$y = e^{mx}$

$m^2 - 2m - 3 = 0$

$(m+1)(m-3) = 0$

$m = -1 \quad m = 3$

$\therefore y_c = c_1 e^{-x} + c_2 e^{3x}$

ii) y_p:

A) $y = am^4 + bm^3 + cm^2 + d$

$y' = 4am^3 + 2bm^2 + 2cm$

$y'' = 12am^2 + 4bm + 2c$

$\therefore (12am^2 + 4bm + 2c) - 2(4am^3 + 2bm^2 + 2cm)$
$- 3(am^4 + bm^3 + cm^2 + d) = 1 - x^2$

$12am^2 + 4bm + 2c - 8am^3 - 4bm^2 - 4cm - 3am^4 - 3bm^3 - 3cm^2 - d$
$= -x^2 + 1$

$\therefore 12a - 4b - 3c = -1 \quad (1)$

$4b - 4c = 0 \Rightarrow b = c$

$2c - d = 0 \Rightarrow d = 2c$

$-8a - 3b = 0$

$\Rightarrow 4a = \frac{-3b}{2} \quad \therefore 12a = \frac{-9b}{2}$

$\therefore \frac{-9b}{2} - 4b - 3b = -1$

$(9 + 8 + 6)b = 1$

$b = \frac{1}{23}$

$\therefore a = \frac{-3}{8} \times \frac{1}{23}$ $\quad a = \frac{1}{23}$ $\quad d = \frac{2}{23}$

B) assume $y = ax^2 + bx + c$

$y' = 2ax + b$

$y'' = 2a$

$\therefore \quad 2a - 2(2ax + b) - 3(ax^2 + bx + c) = 1 - x^2$

$2a - 4ax - 2b - 3ax^2 - 3bx - 3c = 1 - x^2$

$4a\cancel{x} + 3b = 0 \quad - (1)$

$2a - 2b - 3c = 1$

$3a = 1$

$\boxed{a = \frac{1}{3}} \qquad \therefore \boxed{b = \frac{-4}{9}} \qquad \boxed{c = \frac{5}{27}}$

$\therefore \boxed{y_p = \frac{x^2}{3} - \frac{4x}{9} + \frac{5}{27}}$

$\therefore$ general solution: $y = y_c + y_p$

$\therefore \boxed{y = c_1 e^{-x} + c_2 e^{3x} + \frac{x^2}{3} - \frac{4}{9}x + \frac{5}{27}}$

(2) $y'' - 3y' + 2y = 5e^{2x}$

i) y_c:

$m^2 - 3m + 2 = 0 \qquad (m-1)(m-2) = 0$

$m = 1 \quad$ or $m = 2$

$\therefore y_c = c_1 e^{x} + c_2 e^{2x}$

~~ii) y_p:~~

~~$y = Ae^{x}$~~

~~$\therefore Ae^{x} - 3Ae^{x} + 2Ae^{x} = 5e^{x}$~~

~~$\therefore A = 5$~~

~~$\therefore y_p = 5e^{x}$~~

~~general solution:~~

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ii) y_p:

$y = axe^x$

$\therefore (2ae^x + axe^x) - 3(ae^x + axe^x) + 2axe^x = 5e^x$

$2a + \cancel{ax} - 3a - \cancel{3ax} + \cancel{2ax} = 5$

$a = -5$

$\therefore y_p = -5xe^x$

(4)

$y'' - 6y' + 9y = e^{3x}$

i) y_c:

$m^2 - 6m + 9 = 0 \qquad m = 3, 3$

$\therefore y_1 = e^{3x} \qquad y_2 = xe^{3x}$

$\therefore y_c = c_1 e^{3x} + c_2 xe^{3x}$

ii) y_p:

$y = ax^2 e^{3x}$

$y' = 2xae^{3x} + 3x^2 ae^{3x}$

$y'' = 2ae^{3x} + 6xae^{3x} + 6xae^{3x} + 9x^2 ae^{3x}$

$\therefore 2ae^{3x} + \cancel{12xae^{3x}} + \cancel{9x^2ae^{3x}} - \cancel{12xae^{3x}} - \cancel{18x^2ae^{3x}}$
$+ \cancel{9ax^2e^{3x}} = e^{3x}$

$2a = 1 \qquad \boxed{a = \frac{1}{2}}$

$\therefore y_p = \frac{1}{2}x^2 e^{3x}$

$\therefore$ general solution:

$y = \frac{1}{2}x^2 e^{3x} + c_1 e^{3x} + c_2 xe^{3x}$

$y'' - y' = 5e^{x} - \sin 2x$

i) $y = e^{mx}$

$m^2 - m = 0$

$m = 0 \quad m = 1$

$\therefore y_c = c_1 + c_2 e^{x}$

ii) $y = axe^{x} + b\sin 2x + c\cos 2x$

$y' = axe^{x} + ae^{x} + 2b\cos 2x - 2c\sin 2x$

$\therefore y'' = 2ae^{x} + axe^{x} - 4b\sin 2x - 4c\cos 2x$

$\therefore \cancel{2ae^{x}}\ ae^{x} + \cancel{axe^{x}} - 4b\sin 2x - 4c\cos 2x - \cancel{axe^{x}} - \cancel{ae^{x}} - 2b\cos 2x + 2c\sin 2x$

$= 5e^{x} - \sin 2x$

$ae^{x} - \sin 2x(4b - 2c) \mp \cos 2x(4c + 2b) = 5e^{x} - \sin 2x$

$\boxed{a = 5}$

$4b - 2c = 1 \qquad 4c + 2b = 0 \qquad \therefore \cancel{2}b = -\overset{2}{\cancel{4}}c$

$-8c - 2c = 1 \qquad \boxed{c = \frac{-1}{10}} \qquad \therefore \boxed{b = \frac{1}{5}}$

$\therefore y_p = 5xe^{x} + \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x$

$\therefore$ general solution:

$y = c_1 + c_2 e^{x} + 5xe^{x} + \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x$

NOTE: possible ~~substitutions~~ forms of particular solution.

$G(x)$	Possible y_p
i) $e^{\lambda x}$ ($\lambda \neq$ root of homo)	$y_p = Ae^{\lambda x}$
ii) $e^{\lambda x}$ ($\lambda =$ single root of homo)	$y_p : Axe^{\lambda x}$
iii) $e^{\lambda x}$ ($\lambda =$ double root of homo)	$y_p : Ax^2e^{\lambda x}$
iv) $\sin kx$ or $\cos kx$ ($ki \neq$ root of homo)	$y_p : B\cos kx + C\sin kx$
v) $px^2 + qx + r$ ($0 \neq$ root of homo)	$y_p : ax^2 + bx + c$
vi) $px^2 + qx + r$ ($0 =$ single root of homo)	$y_p : ax^3 + bx^2 + cx$
vii) $px^2 + qx + r$ ($0 =$ double root of homo)	$y_p : ax^4 + bx^3 + cx^2$

METHOD OF VARIATION OF PARAMETER:

$$ay'' + by' + cy = G(x) \quad -(1)$$

i) solve associated homogeneous ~~solu~~ equation

~~by~~ let $y_c = c_1y_1 + c_2y_2$

ii) assume that

$$y_p = c_1(x)\,y_1(x) + c_2(x)\,y_2(x)$$

$$y_p' = c_1'(x)\,y_1(x) + c_2'(x)\,y_2(x) + c_1(x)\,y_1'(x) + y_2'(x)\,c_2(x)$$

Assume $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$

$\therefore y_p' = c_1(x)y_1'(x) + c_2(x)y_2'(x)$

$y_p'' = c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + c_1(x)y_1''(x) + c_2(x)y_2''(x)$

Substituting in ①

$a(c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + c_1(x)y_1''(x) + c_2(x)y_2''(x))$
$+ b(c_1(x)y_1'(x) + c_2(x)y_2'(x)) + c(c_1(x)y_1(x) + c_2(x)y_2(x))$
$= G(x)$

$\therefore c_1(x)(ay_1'' + by_1' + cy_1) + c_2(x)(ay_2'' + by_2' + cy_2)$

$+ a(c_1'y_1' + c_2'y_2') = G(x)$

but $ay_i'' + by_i' + cy_i = 0 \quad (i = 1, 2)$

$\therefore \boxed{a(c_1'y_1' + c_2'y_2') = G(x)}$

$$\therefore \boxed{\begin{aligned} c_1'y_1 + c_2'y_2 &= 0 \\ c_1'y_1' + c_2'y_2' &= \frac{G(x)}{a} \end{aligned}}$$

$\therefore c_1' = \frac{W_1}{W} \qquad c_2' = \frac{W_2}{W}$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ $\quad W_1 = \begin{vmatrix} 0 & y_2 \\ G(x)/a & y_2' \end{vmatrix}$ $\quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & G(x) \end{vmatrix}$

$\therefore$

① $y'' + y = \tan x$

i) homogeneous:

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$\therefore y_1 = \cos x \qquad y_2 = \sin x$

ii) particular:

$$y_p = c_1(x) \cos x + c_2(x) \sin x$$

$$c_1' \cos x + c_2' \sin x = 0$$

$$-c_1' \sin x + c_2' \cos x = \tan x$$

$$c_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\tan x \sin x}{\cos^2 x + \sin^2 x} = \frac{-\sin^2 x \sec x}{1}$$

$$c_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \tan x}{\cos^2 x + \sin^2 x} = \frac{\sin x}{1}$$

$$\therefore c_1 = \int c_1' \, dx = \cancel{\int \left(\frac{\sin^2 x}{\cos^2 x}\right) \cos x \, dx}$$

$$= \int \frac{-\sin^2 x}{\cos x} \, dx$$

$$= \int \left(\frac{1 - \sin^2 x}{\cos x} - \frac{1}{\cos x} \right) dx$$

$$= \int (\cos x - \sec x) \, dx$$

$$= \sin x - \log|\sec + \tan x|$$

$$c_2 = \int c_2' \, dx = \int \sin x = -\cos x$$

$$\therefore y_p = (\sin x - \log|\sec x + \tan x|)\cos x + (\sin x)(-\cos x)$$

$$y_p = -\cos x \log|\sec x + \tan x|$$

$\Rightarrow$ FOR 3^{rd} ~~variables~~ order eq.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ \frac{G(x)}{a} & y_2'' & y_3'' \end{vmatrix}$$

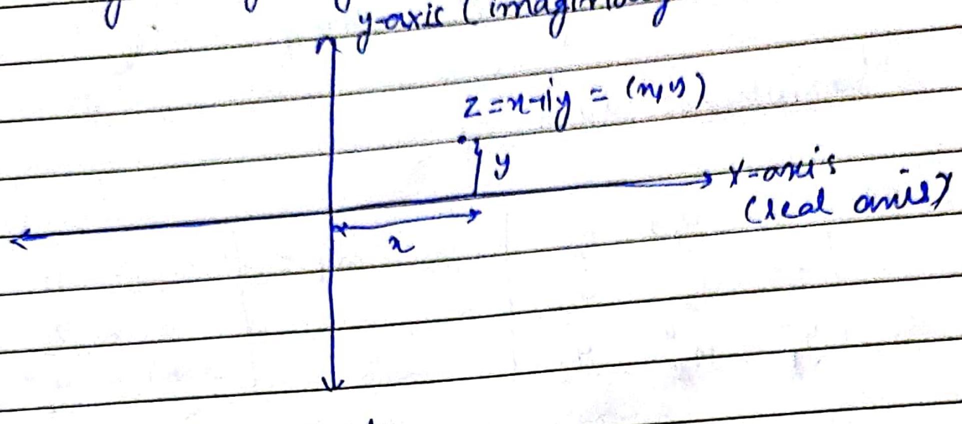
$$c_1' = \frac{W_1}{W} \qquad c_2' = \frac{W_2}{W} \qquad c_3' = \frac{W_3}{W}$$

COMPLEX NUMBER:

$z = x + iy$

x = real part

y = imaginary part



$z_1 = x_1 + iy_1$ $\quad$ $z_2 = x_2 + iy_2$

① complex addition:

$$z = z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\, i$$

$$\therefore \; Re(z_1) + Re(z_2) = Re(z)$$

$$Im(z_1) + Im(z_2) = Im(z)$$

② complex multiplication:

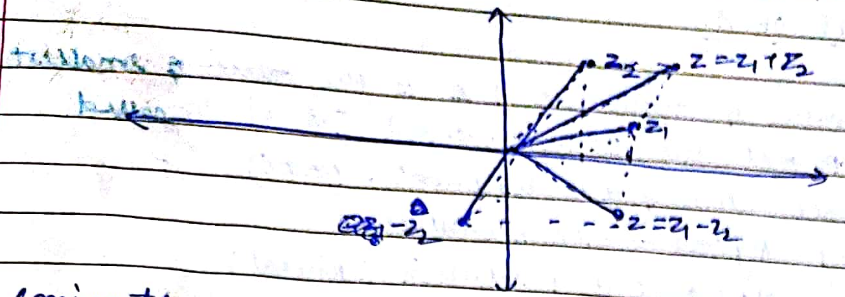
$$\boxed{z = z_1 z_2 = (x_1x_2 - y_1y_2) + (x_1y_2 + y_1x_2)}$$

③ complex division:

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(\overline{x_2 + iy_2})}{(x_2 + iy_2)(\overline{x_2 + iy_2})}$$

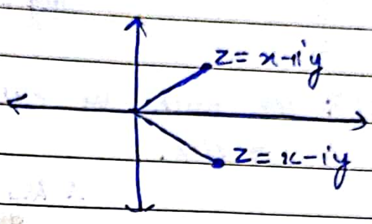
$$\boxed{z = \frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}}$$



→ conjugate of a complex number:

$z = x+iy$

$\bar{z} = x-iy$



→ properties of conjugate:

i) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

ii) $\overline{z_1-z_2} = \bar{z}_1 - \bar{z}_2$

iii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

iv) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

v) $z\bar{z} = |z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$

⇒ POLAR FORM:

$z = x+iy$

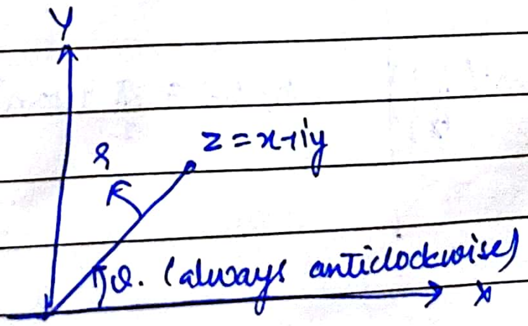
$x = r\cos\theta \qquad y = r\sin\theta$

$\therefore z = r(\cos\theta + i\sin\theta)$

where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(m)$

(m=slope)

$$\boxed{z = r(\cos\theta + i\sin\theta) = re^{i\theta} = x+iy}$$



ARGUMENT OF z:

→ Principle value of argument of z is the ~~value of~~ smallest value of θ that lies in the interval $[-\pi, \pi]$ is called the principle value of the argument. $(z \neq 0)$

→ defined by $Arg(z) \rightarrow$ principle value
$arg(z) \rightarrow$ any solution (general)

NOTE: We must be careful about the quadrant in which z lies.

$$\therefore Arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore Arg(i) = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}\infty = \frac{\pi}{2}$$

$$\therefore Arg(1) = \tan^{-1}\left(\frac{0}{1}\right) = \tan^{-1}(0) = 0$$

$$\therefore Arg(1-i) = \tan^{-1}\left(\frac{-1}{1}\right) = \frac{3\pi}{4}$$

$$\therefore Arg(1+i) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

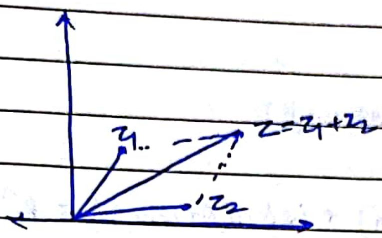
$$\therefore Arg(-1-i) = \tan^{-1}\left(\frac{-1}{-1}\right) = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

$$Arg(z) = \frac{\pi}{2} + \tan^{-1}\left(\frac{|y|}{|x|}\right) \qquad Arg(z) = \tan^{-1}\left(\cancel{\frac{y}{x}}\ \frac{|y|}{|x|}\right)$$

$$Arg(z) = \pi + \tan^{-1}\left(\frac{|y|}{|x|}\right) \qquad Arg(z) = \frac{3\pi}{2} + \tan^{-1}\left(\frac{|y|}{|x|}\right)$$

TRINGLE INEQUALITY :

$$|z_1 + z_2| \leq |z_1| + |z_2|$$



for n complex numbers:

$$|z_1 + z_2 + z_3 + \cdots + z_n| \leq |z_1| + |z_2| + |z_3| + \cdots + |z_n|$$

⇒ If $|z| = r$ and $\arg(z) = \theta$

$z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ (polar coordinates)

i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

$$\therefore\ r(\cos(\theta+2k\pi)+i\sin(\theta+2\pi k)) = R^n(\cos\phi+i\sin\phi)$$

$$\therefore\ r = R^n$$

$$\boxed{R = r^{1/n}}$$

$$\boxed{\phi = \frac{\theta+2k\pi}{n}}$$

$$\therefore\ z^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$$

n^{th} ROOT OF UNITY!

$$z = 1 = \cos\theta + i\sin\theta$$

$$= \cos(2k\pi) + i\sin(2\pi k)$$

$$\therefore\ z^{1/n} = 1^{1/n} = 1^{1/n}\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)$$

$z^{1/3}$ = cube root of unity = i) $\cos\theta + i\sin\theta$

ii) $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

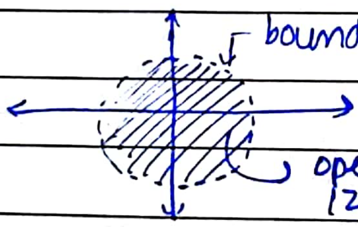
iii) $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$\sqrt[n]{1} = 1 + \omega + \omega^2 + \dots + \omega^{n-1}$

Properties! $\omega^n = 1$

$1 + \omega + \omega^2 = 0$

→ $|z| < 1$ $\quad \sqrt{x^2+y^2} < 1$ $\quad \therefore\ x^2+y^2 < 1$



boundary points not included

open disc
$|z| < 1$, centre $(0,0)$
$r = 1$

→ $|z| \leq 1$: closed disc with boundary points included

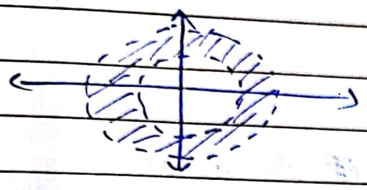
→ $|z| = r$ → circle with centre $(0,0)$ and radius r

→ $|z-a| = r$ → circle with centre a and radius r

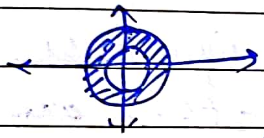
→ $|z-a| < r$ → ~~circle~~ ~~with~~ disc with centre a and radius r

→ $|z-a| \le r$ → closed disc with centre a and radius r

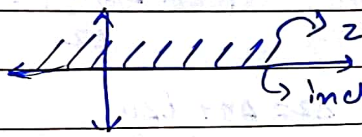
→ $r_1 < |z-a| < r_2$ → open ring with inner radius r_1 and outer radius r_2, centre a.



→ $r_1 \le |z-a| \le r_2$ → closed ring with inner radius r_1 and outer radius r_2, centre a.



→ upper half plane:



similarly, lower half plane,
right half plane
left half plane

DOMAIN:

→ open connected subset of C is called domain.

→ open set: all points interior points

→ connected - no ~~sep~~ seperation

COMPLEX FUNC:

$$D \subset C$$

$$f : D \to C$$

$f(z) = w$ $\quad z =$ complex variable $\quad z \in D$

$$\therefore f(z) = u(x,y) + iv(x,y)$$

Eg: $f(z) = 3z^2 - z$

LIMITS:

A function is said to be have limit l as z approaches z_0.

$$\lim_{z \to z_0} f(z) = l$$

if $|z - z_0| < \delta$

for $|f(z) - l| < \varepsilon$, $\varepsilon > 0$

CONTINUITY:

$$f : D \to C$$

f is said to be continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

DIFFERENTIABILITY:

f is said to be differentiable at $z = z_0$ $(z \neq 0)$

if $\lim\limits_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists.

$$\Delta z = \Delta x + i\Delta y$$

$$\therefore f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

ALSO, $f'(z_0) = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$

PROPERTIES:

i) $(kf)' = kf'$ $\quad k =$ constant

ii) $(f \pm g)' = f' \pm g'$

iii) $(fg)' = f'g + g'f$

iv) $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - g'f}{g^2}$

1) Check if $f(z) = \bar{z}$ is differentiable.

f is differentiable if $\lim\limits_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists

$$\therefore \lim_{\Delta z \to 0} \frac{\overline{(z + \Delta z)} - (\bar{z})}{\Delta z} = \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$\Delta z \to 0 \Rightarrow \Delta x \to 0$ & $\Delta y \to 0$

$$\therefore \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{(\Delta x + i\Delta y)}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

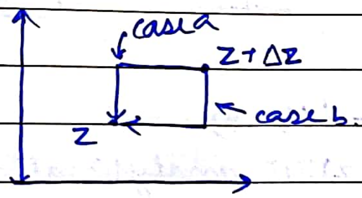
case a: $\Delta x \to 0$ before $\Delta y \to 0$

$$\therefore \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

case b: $\Delta y \to 0$ before $\Delta x \to 0$

$$\therefore \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

As limits in both the cases is not equal, the function is not differentiable.



ANALYTIC FUNCTION:

→ Let D be a domain.

A function $f(z)$ is said to be analytic on D if $f(z)$ is defined and differentiable at all points in D.

→ A function is analytic at a point z_0 if $f(z)$ is differentiable in a neighbourhood of z_0.

1. All polynomials are analytic on whole complex plane.

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

2. $\sin z, \cos z, e^z$ → analytic on complex plane.
3. if f = analytic & g = analytic

i) $(f+g)$ = analytic

ii) $(f-g)$ = analytic

iii) (fg) = analytic

iv) $\frac{f}{g}$ may or mayn't be analytic depending on g, i.e.,

not analytic where $g(z)=0$.

~~Let f(z)=u(x,y)+iv(x,y) be~~

COUCHY-RIEMAN EQUATION:

Let $f(z)=u(x,y)+iv(x,y)$ be defined and continuous at $z=x+iy$, and analytic at the point z. Then at point z the first order ~~derivative~~ partial derivative of u and v exist and satisfy the Cauchy-Rieman equation,

$$\boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}}$$

$f(z)=u(x,y)+iv(x,y)$

given that $f(z)$ is analytic at z.

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} \text{ exists.}$$

$$z=x+iy$$

$$\Delta z=\Delta x+i\Delta y$$

$$\therefore f'(z)=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{[u(x+\Delta x,y+\Delta y)+iv(x+\Delta x,y+\Delta y)]-[u(x,y)+iv(x,y)]}{\Delta x+i\Delta y}.$$

if we assume $\Delta y\to 0$ first,

$$f'(z)=\lim_{\Delta x\to 0}\frac{[u(x+\Delta x,y)+iv(x+\Delta x,y)]-[u(x,y)+iv(x,y)]}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}+i\,\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad -(1)$$

If we assume $\Delta x \to 0$ first

$$f'(z) = \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) + iv(x, y+\Delta y)] - (u(x,y) + iv(x,y))}{i\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad -(2)$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

as $f'(z)$ remains same ($\because$ analytic)

(1) = (2)

$$\therefore \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

comparing:

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \text{ and } \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

$\therefore$ → Analytic if Cochy-Rieman satisfied + 1st partial derivative conti

→ if Cochy-Rieman staisfied & 1st partial derivative conti, function is analytic

EXPONENTIAL FUNCTION:

$$e^z = e^{x+iy} = e^x \cdot e^{iy}$$

$$\boxed{e^z = e^x(\cos y + i\sin y)}$$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \qquad \cosh z = \frac{1}{2}(e^z + e^{-z})$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \qquad \sinh z = \frac{1}{2}(e^z - e^{-z})$$

$$\ln(z) = \ln(re^{i\theta}) = \ln r + i\theta \qquad |z| > 0, \; \theta = \arg(z)$$

if $\theta = Arg(z)$

$\therefore Ln\, z = \ln|z| + i\, Arg(z)$

$\therefore \boxed{\ln(z) = Ln(z) \pm 2\pi n i}$

Eg: $\ln(1+i) = \ln(\sqrt{2}\, e^{i\pi/4}) = \ln(\sqrt{2}) + i\frac{\pi}{4}$

COMPLEX INTEGRALS:

A) LINE INTEGRALS:

$I = \int_c f(z)\, dz$

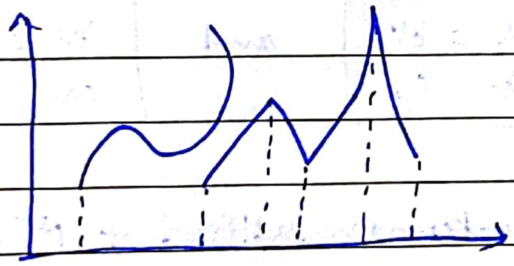
c is called the path of Integral

Necessary / sufficient condition:

→ $f(z)$ = continuous

→ c is piecewise smooth curve

Eg:



smooth on individual intervals.

we can represent the curve c by a ~~parabola form~~ parametric form

$z(t) = x(t) + iy(t)$

Eg: $c: |z| = 1$

$\therefore c: z = \cos t + i\sin t \qquad t \in [0, 2\pi]$

→ a ... b ← real line

$t_0\ t_1\ t_2\ \ldots\ t_n$ ← on the curve

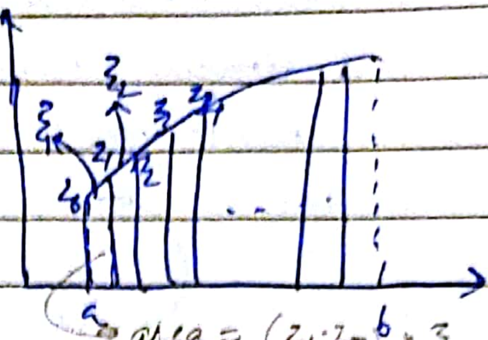
$z_0\ z_1\ z_2\ \ldots\ z_n$

let z_1 be a point on curve betn (z_0, z_1)

ξ_2 be a point in (z_1, z_2)

⋮

$S_n = \sum_{i=1}^{n} f(\xi_i)\Delta z_i$



area $= (z_1 - z_0) \times \xi_1$
$= \Delta z_1 \times \xi_1$

As $n \to \infty$, $S_n \to I$

$\therefore I = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta z_i$

If c is a simple closed curve, the integral is called contour's integral.

$I = \oint_c f(z)\,dz$

Result:

Let c be a ~~ ~~ piecewise smooth curve represented by $z = \gamma(t)$: ~~ ~~ $a \le t \le b$

Let $f(z)$ be a continuous function

Then,

$$\int_c f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,d\theta$$

$z = \gamma(t)$ $\qquad \frac{dz}{dt} = \gamma'(t)$

$$\therefore \boxed{\int_c f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt}$$

$\int_c z\,dz$, $C : |z| = 1$

$\gamma(t) = \cos t + i\sin t \qquad 0 \le t \le 2\pi$

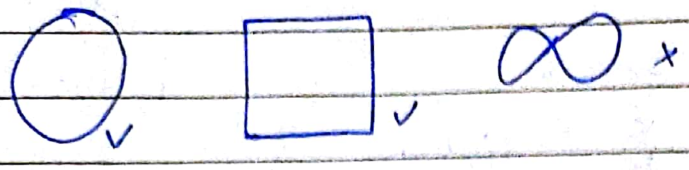
$= \int_0^{2\pi} (\cos t + i\sin t)\,dt$

$= \int_0^{2\pi} e^{it} \times ie^{it} \times dt$

⇒ **SIMPLE CLOSED CURVE:**

A simple closed path/curve is a closed curve that doesn't touch a intersect itself.

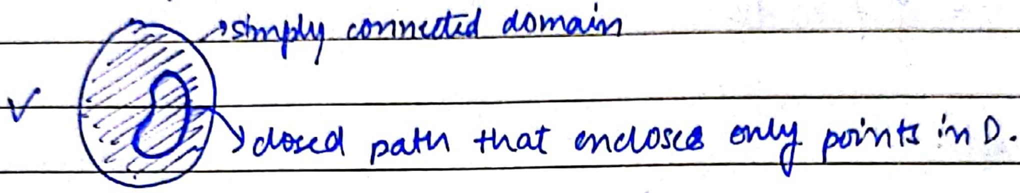
Eg:

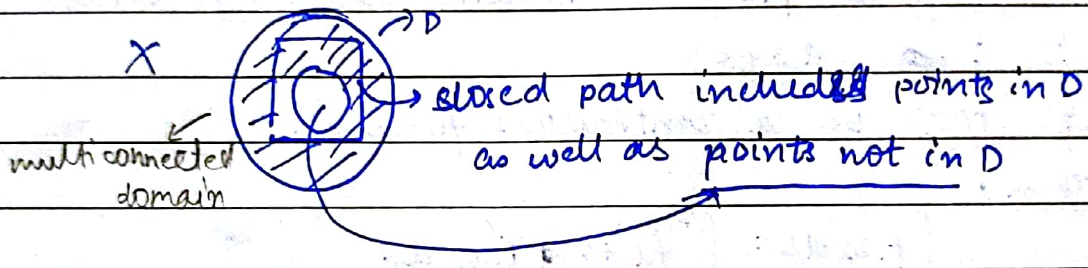


⇒ **SIMPLY CONNECTED DOMAIN:**

A domain D such that every simple closed path in D encloses only points in D.

Eg.





CAUCHY'S INTEGRAL THEOREM:

→ If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\, dz = 0$$

→ polynomials of z are analytic functions.

Eg: $\int \cos z \, dz = 0$

$|z| = 1$ $\qquad z = \gamma(t) = e^{it}$

$I = \int_0^{2\pi} \cos(e^{it}) \cdot i e^{it} \, dt$ $\qquad e^{it} = p \quad dp = ie^{it}$

$t = 0 \to p = 1$

$= \int_1^1 \cos p \, dp$ $\qquad t = 2\pi \to p = 1$

$= 0$

Eg: $\int_{|z|=1} \bar{z}\, dz$ $\qquad \gamma(t) = e^{it} = \cos t + i \sin t$

$= \int_0^{2\pi} f(\gamma(t))\, \gamma'(t)\, dt$

$= \int_0^{2\pi} e^{-it} \cdot i e^{it}\, dt$

$= i \int_0^{2\pi} dt = 2\pi i$

→ Analyticity is sufficient condition but not necessary for Cauchy's theorem.

Eg! $\int_{|z|=1} \frac{1}{z^2}\, dz$

$\gamma(t) = e^{it} \qquad t \in [0, 2\pi]$

$\gamma'(t) = i e^{it}$

$\therefore I = \int_0^{2\pi} \frac{1}{e^{2it}}\, i e^{it}\, dt$

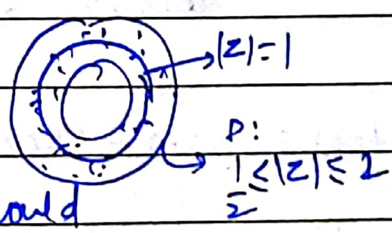
$= i \int_0^{2\pi} e^{-it}\, dt$

$= 0$

→ Simply connectedness is essential:

Eg: We have seen

$\oint_{|z|=1} dz/z = 2\pi i$



By Cauchy's theorem, $\oint_{|z|=1} \frac{dz}{z}$ should be zero

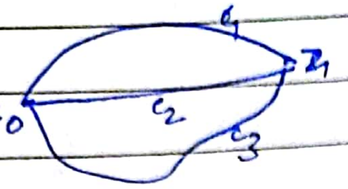
but it is not true.

$\therefore$ it is necessary for the function to have a simply connected domain; whereas, analyticity is not essential.

⇒ INDEPENDENT OF PATH:

If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

Eg:



$$I = \int_{z_0 \, c_1}^{z_1} f(z)\,dz = \int_{z_0 \, c_2}^{z_1} f(z)\,dz = \int_{z_0 \, c_3}^{z_1} f(z)\,dz$$

⇒ CAUCHY INTEGRAL FORMULA:

Statement: If $f(z)$ is analytic in a simply connected domain D, then for any $z_0 \in D$ and any ~~simply con~~ simple closed path c in D that encloses z_0,

$$\boxed{\oint_c \left(\frac{f(z)}{z-z_0}\,dz \right) = 2\pi i \, f(z_0)}$$

Here, $f(z)$ is analytic but $\frac{f(z)}{z-z_0}$ is not analytic at z_0

$\therefore \oint$ Integral is not zero because Cauchy's theorem doesn't apply here.

Proof: $f(z) = f(z_0) + (f(z) - f(z_0))$

$$\underbrace{\oint_c \frac{f(z)}{z-z_0}\,dz}_{I} = \underbrace{\oint_c \frac{f(z_0)}{z-z_0}\,dz}_{I_1} + \underbrace{\oint_c \frac{f(z)-f(z_0)}{z-z_0}\,dz}_{I_2}$$

$\therefore I = I_1 + I_2$ [can be solved using $z(t) = z_0 + Re^{it}$

$$I_1 = \oint_c \frac{f(z_0)}{z-z_0}\,dz = f(z_0) \oint_c \frac{1}{z-z_0}\,dz$$

$$= 2\pi i \, f(z_0)$$

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$$I_2 = \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

since $f(z)$ is analytic. ∴ it is differentiable and continuous.

$$\therefore |f(z)-f(z_0)| < \varepsilon \quad \text{where} \quad |z-z_0| < \delta$$

$$\therefore \left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \frac{|f(z)-f(z_0)|}{|z-z_0|} < \frac{\varepsilon}{\rho}$$

where $\rho < \delta$

* ⇒ ML-inequality :

$$\left|\int_C f(z)\,dz\right| \le ML$$

L = length of curve C

M = constant such that $|f(\xi)| \le M$

$$|S_n| = \left|\sum_{i=1}^{n} f(\xi_i)\,\Delta z_i\right|$$

$$\le \sum_{i=1}^{n} |f(\xi)|\,|\Delta z_i|$$

$$\le M \sum_{i=1}^{n} |\Delta z_i|$$

$$\le ML$$

$$\therefore \boxed{\left|\int_C f(z)\,dz\right| \le ML}$$

$$\therefore I_2 = \left|\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \le \underbrace{\frac{\varepsilon}{\rho}}_{M} \times \underbrace{2\pi\rho}_{L} = 2\pi\varepsilon$$

as $\varepsilon \to 0$, $I_2 = 0$

$$\therefore I = I_1 + I_2 = 2\pi i f(z_0) + 0$$

$$\therefore \boxed{I = 2\pi i f(z_0)}$$

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$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

$f^{(n)}(a)$ = n^{th} derivative of the function at point

Eg: $\int_{|z|=1} \frac{1}{z}\,dz = 2\pi i$

$f(z)=1 \quad a=0,\ n=0$

$\therefore \int_{|z|=1} \frac{1}{z}\,dz = \frac{2\pi i}{0!} f^{(0)}(0)$

$= \frac{2\pi i}{1} \times 1 = 2\pi i$

Eg: $\int_{|z|=1} \frac{1}{z^2}\,dz$

$n=1 \quad f(z)=1,\ a=0$

$\therefore \int_{|z|=1} \frac{1}{z^2}\,dz = \frac{2\pi i}{1!} \times f'(0)$

$= \frac{2\pi i}{1} \times 0 = 0$

Eg: $\int_{|z|=5} \frac{\cos z}{z^2}\,dz \qquad f(z)=\cos z,\ n=1,\ a=0$

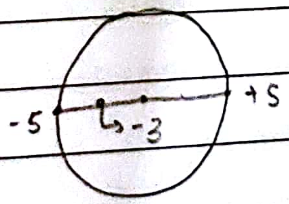
$= \frac{2\pi i}{1!} \times f'(0)$

$= 2\pi i \times (-\sin 0) = 0$

Eg: $\int_{|z|=5} \frac{\sin z}{z^2}\,dz = \frac{2\pi i}{1!} f'(0)$

$= \frac{2\pi i}{1} \times \cos 0 = 2\pi i$

Eg: $\int_{|z|=5} \frac{z^2+3z}{(z+3)^3}\,dz \qquad f(z)=z^2+3z,\ a=-3,\ n=2$

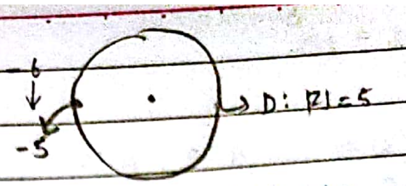


$\therefore I = \int_{|z|=5} \frac{z^2+3z}{(z+3)^3}\,dz$

$= \frac{2\pi i}{2!} f''(-3)$

$= \frac{2\pi i}{2} \times 2 = 2\pi i$

Eg: $\displaystyle\int_{|z|=5} \frac{z^2+3z}{(z+6)^3}\,dz$



the point at which the function is not analytic is outside the curve

∴ the function is analytic on and inside the curve

∴ integral is zero (by Cauchy's theorem).

$$I=\int_{|z|=5}\frac{z^2+3z}{(z+6)^3}=0$$

Eg: $\displaystyle\oint_{|z-1|=2} \frac{z^2+1}{z^2-1}\,dz = \oint_{|z-1|=2}\frac{z^2+1}{(z-1)(z+1)}\,dz$

(∵ 2 points of non-analyticity)

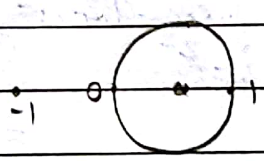
$$\frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{(z-1)}-\frac{1}{(z+1)}\right)$$

$$\therefore I = \frac{1}{2}\left(\oint_{|z-1|=2}\frac{z^2+1}{z-1}\,dz - \oint_{|z-1|=2}\frac{z^2+1}{z+1}\,dz\right)$$

$$=\frac{1}{2}\left(\frac{2\pi i}{0!}f(1) - \frac{2\pi i}{0!}f(-1)\right)$$

$$=\frac{1}{2}\left(\frac{2\pi i}{1}\times 2 - \frac{2\pi i}{0!}\times 2\right)=0$$

Eg: $\displaystyle\oint_{|z-1|=1}\frac{z^2+1}{z^2-1}\,dz$



$$=\frac{1}{2}\left(\oint_{|z-1|=1}\frac{z^2+1}{z-1}\,dz - \oint_{|z-1|=1}\frac{z^2+1}{z+1}\,dz\right)$$

↳ this is analytic

$$=\frac{1}{2}\oint_{|z-1|=1}\frac{z^2+1}{z-1}\,dz$$

$$=\frac{1}{2}\times\frac{2\pi i}{0!}f(1)= 2\pi i \quad 2\pi i$$

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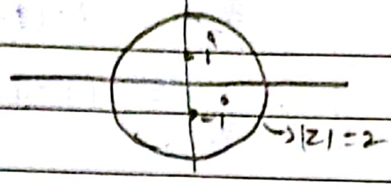
... -1 and +1
$\sin z$ = unbounded

Eg: $\displaystyle\int_{|z|=2} \frac{\sin z}{(z^2+1)^2}\,dz$ not analytic at i and $-i$

$$= \frac{1}{2}\int_{|z|=2} \frac{\sin z}{(z-i)^2}\,dz - \frac{1}{2}\int_{|z|=2} \frac{\sin z}{(z+i)^2}\,dz$$



$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{2}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

$$\therefore \left(\frac{1}{z^2+1}\right)^2 = \frac{1}{4}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)^2$$

⇒ ENTIRE FUNCTION:

A function

ENTIRE FUNCTION:

A function which is analytic in the whole complex plane is called an entire function.

Eg: $\sin z$, $\cos z$, e^z

LIOUVILLE'S THEOREM:

If an entire function $f(z)$ is bounded in absolute value, then $f(z)$ must be constant.

MORERA'S THEOREM: (corollary of Cauchy's theorem)

If $f(z)$ is continuous on a simply connected domain D, and if $\oint_C f(z)\,dz = 0$ for every closed path in D, then $f(z)$ is analytic in D.

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POWER SERIES:

→ $\frac{1}{z} = \frac{1}{1-(1-z)} = \frac{1}{(1-(1-z))}$

$\therefore \frac{1}{z} = (1-(1-z))^{-1}$

$\frac{1}{z} = 1 + (1-z)^{1} + (1-z)^2 + (1-z)^3 + \cdots$

this series is valid only if $|1-z|<1$ (power series about the point 1)
(this power series is convergent)

$\left[(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \quad \text{when } |x|<1 \right]$

→ every analytic function can be represented by using a power series about ~~you~~ some point.

→ $\frac{1}{z} = \frac{1}{2-(2-z)} = (2-(2-z))^{-1}$

$= \frac{1}{2}\left(1-\left(1-\frac{z}{2}\right)\right)^{-1}$

$= \frac{1}{2}\left[1 + \left(1-\frac{z}{2}\right)^2 + \left(1-\frac{z}{2}\right)^3 + \cdots\right]$

↳ power series about the point 2

(valid when $\left|\frac{2-z}{2}\right|<1$)
|||
convergent

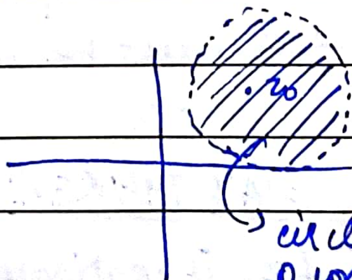
→ general power series about the point z_0

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

a_n ↳ coefficient

centre $= z_0$

radius $= 1 =$ radius of convergence



↳ circle of convergence of the given power series

→ formula to find radius of convergence.

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \text{(ratio test)}$$

$$R = \lim_{n\to\infty} |a_n|^{1/n}$$

→ Examples:

(1) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} (z-3i)^n$

centre $= 3i = z_0$

M1: $a_n = \dfrac{(2n)!}{(n!)^2}$ $\quad a_{n+1} = \dfrac{(2(n+1))!}{((n+1)!)^2}$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left(\frac{(2n)! / (n!)^2}{(2n+2)! / ((n+1)!)^2} \right)$$

$$= \lim_{n\to\infty} \left(\frac{\cancel{(n+1)}(n+1)}{2\cancel{(n+1)}(2n+1)} \right)$$

$$= \lim_{n\to\infty} \frac{1}{2} \left(\frac{1 + 1/n}{2 + 1/n} \right)$$

$$= \frac{1}{4}$$

∴ circle of convergence will have centre $3i$ and radius $\frac{1}{4}$.

(2) $\displaystyle\sum_{n=1}^{\infty} n (z + i\sqrt{2})^n$

$z_0 = -i\sqrt{2}$ $\quad a_n = n$ $\quad a_{n+1} = n+1$

$$\therefore R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

* (3) TAYLOR's SERIES:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

$$\therefore f(x) = f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots$$

(4) $\sum_{n=0}^{\infty} \frac{2^{20n}}{n!} (2-3)^n$

LAURENT SERIES:

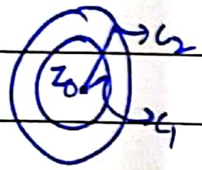
→ If $f(z)$ is analytic inside annulus formed by two concentric circles c_1 and c_2 with centre z_0, then $f(z)$ can be represented by a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

→ Here, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ → analytic part

$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ ⟶ principal part



→ Examples:

① $z^2 e^{1/z}$

at $z=0$, the function $f(z) = z^2 e^{1/z}$ is not analytic.

$$e^{1/z} = 1 + \left(\frac{1}{z}\right) + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots$$

$\therefore$ $f(z) = z^2 e^{1/z}$

$$= z^2 + z + \frac{1}{2!} + \frac{1/z}{3!} + \cdots$$

$\therefore$ Laurent series : $f(z) = z^2 + z + \frac{1}{2!} + \frac{1}{z\,3!} + \frac{1}{z^2 4!} + \cdots$

$\therefore$ analytic part $= z^2 + z + \frac{1}{2!}$

principal part $= \frac{1}{z\,3!} + \frac{1}{z^2 4!} + \cdots$

(4) $f(z) = \frac{1}{z^3 - z^4}$

$= \frac{1}{z^3(1-z)}$

∴ not analytic at $z=0$ and $z=1$

∴ Laurent series bet^n the $z=0$ circle and $z=1$ circle.

∴ $f(z) = \frac{1}{z^3}\left(\frac{1}{1-z}\right)$

$= \frac{1}{z^3}\left(1 + z^0 + z^2 + z^3 + \cdots\right)$

$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z^0 + z^2 + \cdots$

↳ laurent series about $z=0$

∴ analytic part $= 1 + z + z^2 + z^3 + \cdots + \frac{1}{z}$

principal part $= \frac{1}{z^3} + \frac{1}{z^2}$

→ about the point $z=1$