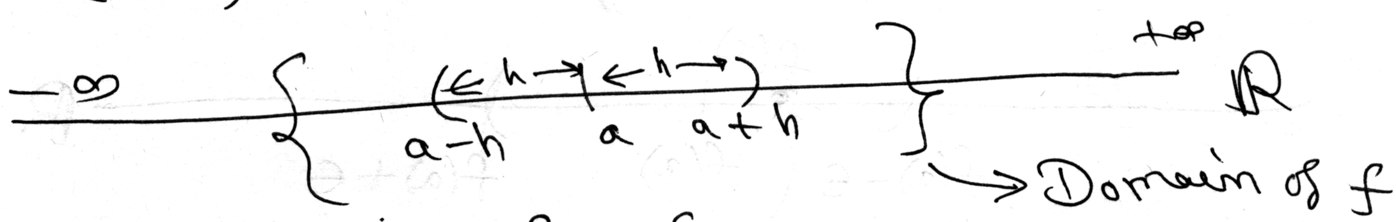
CCV-3 SC 216 Calculus with Complex Variables — Manish K. Gupta Jan 10, 2008 www.mankg.com

ANALOGY

Limit of A seq.	Limit of a function
seq $S = \{s_n\}_{n=1}^{\infty}$	f function
Domain $\mathbb{N} = \{n\}$	$\mathbb{R} = \{x\}_{x \in \mathbb{R}}$
s_n	$f(x)$
L	L
∞	a
ϵ	ϵ
N	δ
$n \geq N$ (for sufficiently large values of n; n is suff. close to ∞ ($\neq \infty$))	$0 < \lvert x-a \rvert < \delta$; x is suff. close to a but not equal to a

$\boxed{\text{Function continuous at a pt. } x_0 \in \mathbb{R} \text{ (Real line)}}$

— f is a real valued fn. whose domain containing all points of some open interval $(a-h, a+h)$ $h>0$



— We say that fn. f is cont.s at $a \in \mathbb{R}$ if

1. $\lim_{x \to a} f(x)$ exist
2. $f(a)$ is defined
3. $\lim_{x \to a} f(x) = f(a)$

$\boxed{\text{Example 1}}$ $\quad f(x) = \dfrac{\sin x}{x} \quad (x \in \mathbb{R}, x \neq 0)$

□ f is not defined at $x=0$ hence it is not cont^s at $x=0$ even though the limit

$$\lim_{x \to 0} \frac{\sin x}{x} = (\text{exist}) = 1 \quad ▨$$

$\boxed{\text{Example ②}}$ $\quad g(x) = \dfrac{\sin x}{x} \quad (x \neq 0)$

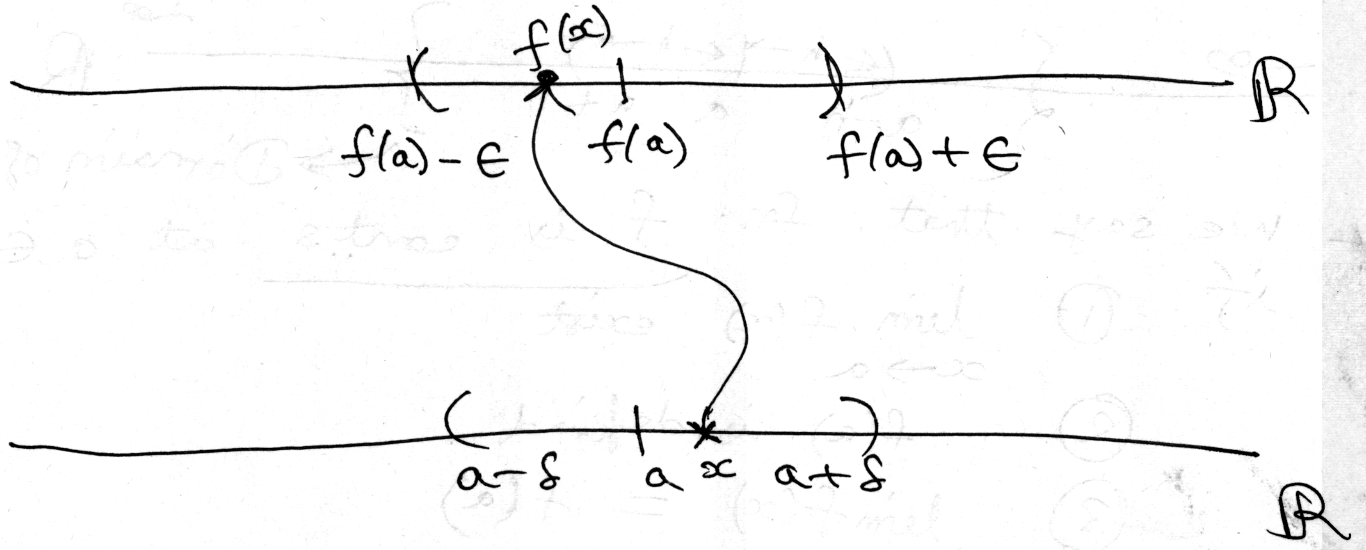
$$g(0) = 1$$

g is cont^s at $x=0$ since

$$\lim_{x \to 0} g(x) = g(0)$$

$\boxed{\text{MORE FORMAL } \epsilon - \delta \text{ def!}}$

— The real valued fn. f is cont^s at $a \in \mathbb{R}$ iff given $\epsilon > 0$ $\exists$ $\delta > 0$ s.t. $|f(x) - f(a)| < \epsilon$ $\quad (|x-a| < \delta)$



$\boxed{\text{THEOREM}}$ The real-valued fn. f is cont.s at $a \in \mathbb{R}$ iff whenever $\{x_n\}_{n=1}^{\infty}$ seq. of real nos. converges to a

then the seq.

$$\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(a)$$

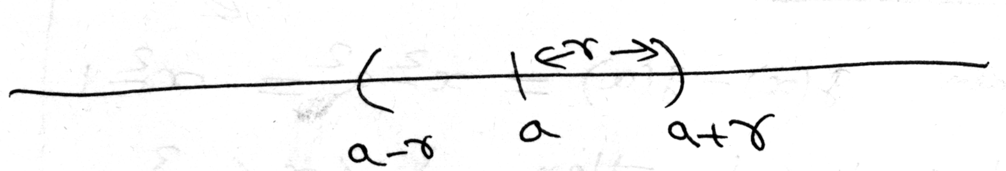
– f is cont.s at a iff

$$\lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = f(a)$$

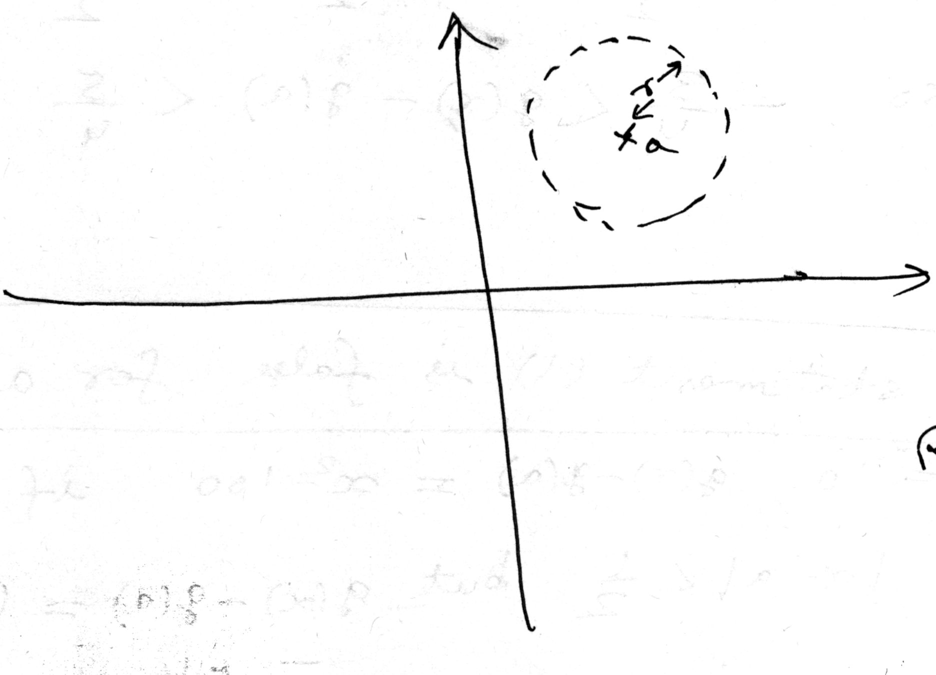
__Def:__ If $a \in \mathbb{R}$ & $r > 0$ define

$$B[a; r] = \{x \in \mathbb{R} \mid |x-a| < r\}$$

open ball of radius r about a



$\mathbb{R}^1$



$\mathbb{R}^2$

Real plane

? $\mathbb{R}^3$

Uniform Continuity

– We know that a real-valued fn. f is conts at $a \in \mathbb{R}$ if given $\epsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|f(x) - f(a)| < \epsilon \quad (|x-a| < \delta)$$

In general δ depends not only on ϵ but also on pt. a under consideration.

For example, let

$$g(x) = x^2 \quad (-\infty < x < \infty)$$

then with $\epsilon = 2$ the statement

$$|g(x) - g(a)| < 2 \quad \left(|x-a| < \tfrac{1}{2}\right) \qquad \text{—— (1)}$$

is true if $a = 1$ ~~See explan~~

For then $g(x) - g(a) = x^2 - a^2 = x^2 - 1$

if $|x-a| < \frac{1}{2}$ then $\frac{1}{2} < x < \frac{3}{2}$

& so $-\frac{3}{4} < g(x) - g(a) < \frac{5}{4}$

Now statement (1) is false for $a = 10$

When $a = 10$, $g(x) - g(a) = x^2 - 100$ if $x = 10\frac{1}{4}$

then $|x-a| < \frac{1}{2}$ but $g(x) - g(a) = (10\frac{1}{4})^2 - 10^2$

$= 5\frac{1}{16} > 2$

$\Rightarrow |g(x) - g(a)| > 2$

Thus (even though g is cont̤s at the pt. $a = 10$ as well as at the pt. $a = 1$) ~~the~~ the number $\delta = \frac{1}{2}$ is usable at $a = 1$ but not at $a = 10$ as a δ corresponding to $\epsilon = 2$

$\boxed{\text{Claim}}$ There is no one $\delta > 0$ s.t. the statement

$$|g(x) - g(a)| < 2 \quad (|x-a| < \delta) \qquad \text{—— (2)}$$

is true $\forall a \in \mathbb{R}$

$\square$ For $g(x) - g(a) = (x-a)(x+a)$

Suppose there were a δ for which (2) ~~holds~~ held $\forall a$

then we would have for $a > 0$ & $x = a + \frac{\delta}{2}$

$$|g(x) - g(a)| = |x-a| \cdot |x+a| = \frac{\delta}{2} \cdot \left|2a + \frac{\delta}{2}\right| < 2 \text{ (by def)}$$

$$\Rightarrow \quad a\delta < 2 \quad \forall a > 0$$

which is false

Thus for the fn. $g = x^2$ corresponding to $\epsilon = 2$ there is no δ that will work $\forall a$ simultaneously (Nevertheless g is cont̤s at each $a \in \mathbb{R}$)

Such

— A fn. f is called ~~a~~ uniformly conts if it is conts fn. and given $\epsilon > 0$ we can always choose δ so that δ depends only on ϵ but not on a

$\boxed{\text{Example}}$ $f(x) = x^2$ is uniformly conts on $[-1, 1]$

$\boxed{\text{Example}}$ $f(x) = \sin\frac{1}{x}$ is conts & bounded on $(0, \frac{2}{\pi})$ but is not uniformly conts

$\boxed{\text{Example}}$ ~~Sine fn~~ $f(x) = \sin x$ $-\infty < x < \infty$ is uniformly conts

$\boxed{\text{Example}}$ $f(x) = \begin{cases} 1 & \text{when } x \text{ is irrational} \\ -1 & \text{when } x \text{ is rational} \end{cases}$

is disconts at every pt.

$\boxed{\text{Example}}$ $g(x) = \sqrt{x}$ $(0 \leq x < \infty)$

g is conts on $[0, \infty)$

Example

Characteristic fn. of rational numbers

$$\chi(x) = 1 \quad (x \in \mathbb{R},\ x \text{ rational}) \quad x \in \mathbb{Q}$$

$$\chi(x) = 0 \quad (x \in \mathbb{R} \text{ but } x \notin \mathbb{Q})$$

□ This $\chi(x)$ is defined for any $a \in \mathbb{R}$

but $\lim_{x \to a} \chi(x)$ does not exist for any a

Suppose $\lim_{x \to a} \chi(x) = L$ (exist) for some $L \in \mathbb{R}$

Given $\epsilon = \frac{1}{3}$ there would exist $\delta > 0$ s.t.

$$|\chi(x) - L| < \frac{1}{3} \text{ if } 0 < |x-a| < \delta$$

Now in the interval $(a, a+\delta)$ there is both rational & irrational numbers.

$\therefore$ If $x \in (a, a+\delta)$ is rational we would have $|1-L| < \frac{1}{3}$

while if $x \in (a, a+\delta)$ is irrational we would have $|0-L| < \frac{1}{3}$

$\therefore$ A contradiction follows

Thus the fn. $\chi(x)$ is not conts.

$\boxed{\text{THEOREM}}$

f is cont.s at a iff given $\epsilon > 0$

$\exists\ \delta > 0$ s.t. $f(x) \in B[f(a); \epsilon]$

if $x \in B[a; \delta]$

i.e. open ball is mapped by f into the open ball

$\boxed{\text{THEOREM}}$ If the real-valued fn. f is cont.s on the closed bounded interval $[a, b]$ then f is uniformly cont.s on $[a, b]$.