Lecture 18:
Calculus of Variations

Finding extremum

- The calculus of variations involves finding an extremum (maximum or minimum) of a **quantity** that is expressible as an integral.

- examples:
  - *The shortest path between two points*
  - *Fermat's principle (light follows a path that is an extremum)*

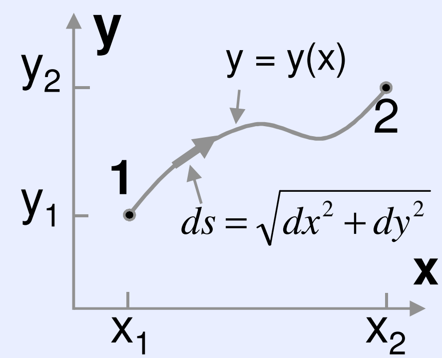
- Shortest path between two points in a plane?

Shortest Path Between 2 Points

- Shortest path between two points in a plane—a straight line. proof of this!!!—the calculus of variations provides the proof.

- Consider two points in the x-y plane (figure.)
- An arbitrary path joining the points follows the general curve $y = y(x)$, and an element of length along the path is

$$ds = \sqrt{dx^2 + dy^2}.$$



- We can rewrite this as $$ds = \sqrt{1 + y'(x)^2}\,dx,$$

because $$dy = \frac{dy}{dx}dx = y'(x)dx.$$

Thus, the length is

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\,dx.$$

Shortest Path Between 2 Points

- From an integral along a path, to an integral over x:

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx.$$

- Simplified the problem!!!, but we need to find the path for which L is an extremum (a minimum in this case).

Fermat's Principle:

Fermat's Principle:

- Finding the path light will take through a medium that has some index of refraction $n \neq 1$.
- Light travels more slowly through such a medium, and we define the index of refraction as $n = c/v$. where c is the speed of light in vacuum, and v is the speed of light in the medium. The total travel time is then

$$\tau = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \frac{1}{c}\int_1^2 n\, ds = \frac{1}{c}\int_{x_1}^{x_2} n(x,y)\sqrt{1+y'(x)^2}\, dx.$$

- Allowing the index of refraction to vary arbitrarily vs. x and y.

Variational Principles

- Usual minimizing or maximizing of a function $f(x)$, → take the derivative and find its zeroes (i.e. the values of x for which the slope of the function is zero).

- These points of zero slope may be minima, maxima, or points of inflection, but in each case we can say that the function is **stationary** at those points, meaning for values of x near such a point, the value of the function does not change (due to the zero slope).

- Similarly, we should be able to find solutions to these integrals that are stationary for infinitesimal variations in the path. This is called **calculus of variations**.

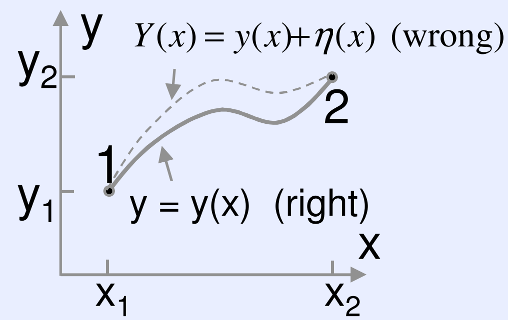
- The methods are called **variational methods**, and a principle like Fermat's Principle are called **variational principles**.

Euler-Lagrange Equation

- Variational method due to Euler and Lagrange, to find an extremum (let's consider this a minimum) for an as yet unknown curve joining two points x_1 and x_2, satisfying the integral relation

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx.$$

- The function f is a function of three variables, but because the path of integration is $y = y(x)$, the integrand can be reduced to a function of just one variable, x.
- Consider two curves joining points 1 and 2, the curve $y(x)$, and a curve $Y(x)$ that is a small displacement from the "right" curve (figure).



The difference between these curves as some function $\eta(x)$.

$$Y(x) = y(x) + \eta(x); \qquad \eta(x_1) = \eta(x_2) = 0.$$

Euler-Lagrange Equation

- Infinitely many functions $\eta(x)$ → each will be longer that the "right" path.
- To quantify how close the "wrong" path can be to the "right" one, let's write $Y = y + \alpha\eta$, so that

$$S(\alpha) = \int_{x_1}^{x_2} f[Y, Y'(x), x]dx$$
$$= \int_{x_1}^{x_2} f[y + \alpha\eta, y' + \alpha\eta', x]dx.$$

- The shortest path is the one for which the derivative $dS/d\alpha = 0$ when $\alpha = 0$. To differentiate the above equation with respect to α, we need to evaluate the partial derivative $\partial S / \partial \alpha$ via the chain rule

$$\frac{\partial f(y + \alpha\eta, y' + \alpha\eta', x)}{\partial \alpha} = \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'},$$

so $dS/d\alpha = 0$ gives

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha} dx = \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right) dx = 0$$

Euler-Lagrange Equation

The second term by integration by parts:

$$\int_{x_1}^{x_2} \eta' \frac{\partial f}{\partial y'} dx = \left[\eta(x) \frac{\partial f}{\partial y'} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) dx,$$

but the first term above (the end-point term) is zero because $\eta(x)$ is zero at the endpoints.

- Our modified equation is then

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) dx = 0.$$

- This leads us to the Euler-Lagrange equation

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0.}$$

- We come to this conclusion because the modified equation has to be zero for any $\eta(x)$.

Euler-Lagrange Equation

- Our modified equation is then

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) dx = 0.$$

- Integral is independent of "alpha" α, but y and dy/dx are still functions of α.
- But $\eta(x)$ is an arbitrary function. So the second term has to be zero.
- This leads us to the Euler-Lagrange equation

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0.}$$

Euler-Lagrange Equation

- Summary → We can find a minimum (more generally a stationary point) for the path S if we can find a function for the path that satisfies
- $$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.}$$
- The procedure is to set up the problem so that the quantity whose stationary path we seek is expressed as

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx,$$

where $f[y(x), y'(x), x]$ is the function appropriate to our problem.

Write down the Euler-Lagrange equation, and solve for the function $y(x)$ that defines the required stationary path.

Shortest Path Between Two Points

- The problem of the shortest path between two points can be expressed as

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx.$$

- The integrand contains our function $f(y, y', x) = \sqrt{1 + y'(x)^2}.$

- The two partial derivatives in the Euler-Lagrange equation are:

$$\frac{\partial f}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}.$$

- Thus, the Euler-Lagrange equation gives us

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} = 0.$$

- This says that $\dfrac{y'}{\sqrt{1 + y'^2}} = C, \quad \text{or} \quad y'^2 = C^2(1 + y'^2).$

- The final result: $y'^2 = \text{constant}$ (call it m^2), so $y(x) = mx + b$. In other words, a straight line is the shortest path.

Examples

Minimizing, Maximizing, and Finding Stationary Points
(often dependant upon physical properties and geometry of problem)

Geodesics

A locally length-minimizing curve on a surface

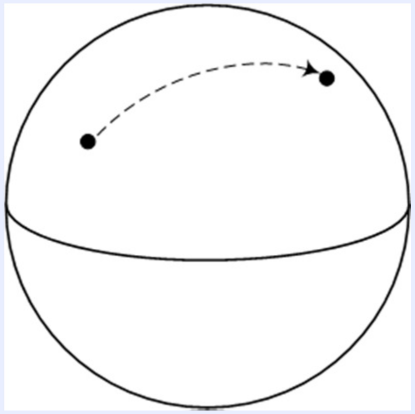
Find the equation $y = y(x)$ of a curve joining points (x_1, y_1) and (x_2, y_2) in order to minimize the arc length

$$ds = \sqrt{dx^2 + dy^2} \quad \text{and} \quad dy = \frac{dy}{dx}dx = y'(x)\,dx$$

so

$$ds = \sqrt{1 + y'(x)^2}\,dx$$

$$L = \int_C ds = \int_C \sqrt{1 + y'(x)^2}\,dx$$



Geodesics minimize path length

Fermat's Principle

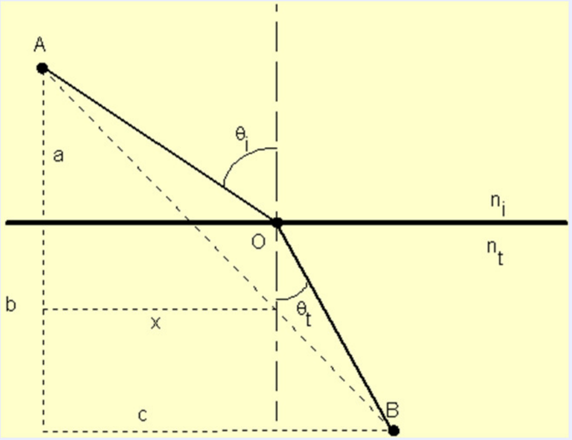
Refractive index of light in an inhomogeneous medium

$v = {}^{c}\!/_{n}$, where *v* = velocity in the medium and *n* = refractive index

Time of travel = $T = \int_C dt = \int_C \frac{ds}{v} = \frac{1}{c}\int_C n\,ds$

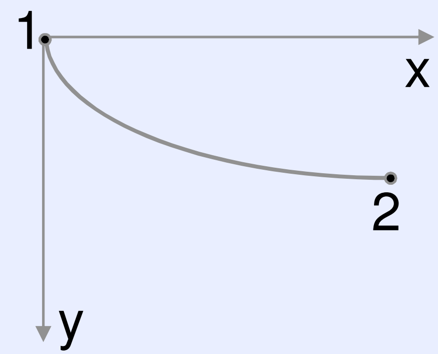
$$T = \int_C n(x, y)\sqrt{1 + y'(x)^2}\,dx$$

Fermat's principle states that the path must minimize the time of travel.



The Brachistochrone

- Statement of the problem:
  - *Given two points 1 and 2, with 1 higher above the ground, in what shape could we build a track for a frictionless roller-coaster so that a car released from point 1 would reach point 2 in the shortest possible time? See the figure, which takes point 1 as the origin, with y positive downward.*



- Force on the particle is constant, ignore friction.
- Field is conservative. Total energy is constant.
- KE=1/2mv^2; PE=-mgy

The Brachistochrone

- Solution:
  - *The time to travel from point 1 to 2 is* $\tau = \int_1^2 \frac{ds}{v},$ *where* $v = \sqrt{2gy}$ *from kinetic energy considerations.*
  - *Since this depends on* y*, we will take* y *as the independent variable, hence*

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'(y)^2 + 1}\, dy.$$

  - *Our integral now becomes:*

$$\tau = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{x'^2 + 1}}{\sqrt{y}}\, dy.$$

  - *From the Euler-Lagrange equation:*

Since we are using y as the independent variable, we swap x and y

$$\frac{\partial f}{\partial x} = \frac{d}{dy} \frac{\partial f}{\partial x'}.$$

cont'd

- *Since* $f = \dfrac{\sqrt{x'^2+1}}{\sqrt{y}}$, *clearly* $\dfrac{\partial f}{\partial x} = 0$, *and so* $\dfrac{\partial f}{\partial x'} = \text{constant}$
- *Evaluating this derivative and squaring it , we will have*

$$\frac{x'^2}{y(x'^2+1)} = \text{constant} = \frac{1}{2a}$$

where the constant is renamed $1/2a$ for future convenience.

- *Solving for x' we have:* $x' = \sqrt{\dfrac{y}{2a-y}}$. *Finally, to get x we integrate:* $x = \int \sqrt{\dfrac{y}{2a-y}}\, dy.$
- *Change of variable, by the substitution $y = a(1 - \cos\theta)$, which gives dy*
- *The two equations that give the path are then:* $x = a(\theta - \sin\theta)$ *in terms of θ.*

$$y = a(1-\cos\theta)$$

$$x = a\int (1-\cos\theta)\, d\theta = a(\theta - \sin\theta) + \text{const.}$$

cont’d

- Solution, cont’d:
  - *This curve is called a cycloid, and is a very special curve.*
  - *it is the curve traced out by a wheel rolling (upside down) along the x axis.*
  - *Constant of integration →0*
  - *Another remarkable thing is that the time it takes for a cart to travel this path from 2→3 is the same, no matter where 2 is placed, from 1 to 3! Thus, oscillations of the cart along that path are exactly isochronous (period perfectly independent of amplitude).*