SC 216 Calculus with Complex Variables

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USE OF UNKNOWN SOLUTION TO FIND ANOTHER

$$y'' + P(x)\, y' + Q(x)\, y = 0 \quad ----(1)$$

If we know two l.i. solns y_1 & y_2 of above eqn we can write general soln. But how to find y_1 & y_2?

— If we can find one soln y_1 somehow then we can find y_2.

— Let y_1 is a nonzero soln. of (1)

Assume $y_2 = v y_1$ is a soln. of (1) (Why so?)

where v is a fn. of x. We can now find y.

$$y_2' = v y_1' + v' y_1 \quad \& \quad y_2'' = v y_1'' + 2 v' y_1' + v'' y_1$$

$$\Rightarrow \quad v(y_1'' + P y_1' + Q y_1) + v'' y_1 + v'(2y_1' + P y_1) = 0$$

$\because y_1$ is a soln. $\Rightarrow \quad v'' y_1 + v'(2y_1' + P y_1) = 0$

$$\Rightarrow \quad \frac{v''}{v'} = -2\frac{y_1'}{y_1} - P$$

$$\Rightarrow \quad \ln v' = -2 \ln y_1 - \int P\, dx$$

$$\Rightarrow \quad v' = \frac{1}{y_1^2} e^{-\int P\, dx}$$

$$\Rightarrow \quad \boxed{v = \int \frac{1}{y_1^2} e^{-\int P\, dx}\, dx}$$

We need to show that y_1 & $y_2 = v y_1$ are l.i.

(Verify!)

Example $y_1 = x$ is a soln. of $x^2y'' + xy' - y = 0$
Find G.S.

□ $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$ $\quad \because P(x) = \frac{1}{x}$

$y_2 = vy_1$ where $v = \int \frac{1}{x^2} e^{-\int \frac{1}{x} dx} dx$

$\Rightarrow y_2 = \frac{x^{-2}}{-2} = -\frac{1}{2}x^{-1}$

$\because y_1$ & y_2 are l.i.

$\Rightarrow$ G.S. is $y = c_1 x + c_2 x^{-1}$

HOMO. EQ^N WITH CONSTANT COEFFICIENTS

$$y'' + P(x)y' + Q(x)y = 0$$

when $P(x) = p$ (constt.) $Q(x) = q$ (constt.)

$$y'' + py' + qy = 0$$

Possible soln. $y = e^{mx}$ for some m

$\Rightarrow (m^2 + pm + q)e^{mx} = 0$ but $e^{mx} \neq 0$

$\Rightarrow m^2 + pm + q = 0$ (A.E.)

$$m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

① If roots m_1 & m_2 are distinct & real ($p^2 - 4q > 0$)

soln. are $e^{m_1 x}$ & $e^{m_2 x}$

$\because \frac{e^{m_1 x}}{e^{m_2 x}} \neq$ constt. these are l.i.

$\Rightarrow \boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

(2) If m_1 & m_2 are distinct complex roots. $(p^2-4q<0)$

then $m_{1,2} = a \pm ib$

$\Rightarrow e^{m_1 x} = e^{(a+ib)x} = e^{ax}(\cos bx + i \sin bx)$

& $e^{m_2 x} = e^{ax}(\cos bx - i \sin bx)$

$\Rightarrow$ Soln. will be

$$y = A e^{m_1 x} + B e^{m_2 x} = A e^{ax}(\cos bx + i \sin bx) + B e^{ax}(\cos bx - i \sin bx).$$

$$\Rightarrow \boxed{y = e^{ax}\left[c_1 \cos bx + c_2 \sin bx\right]}$$

$c_1 = A+B$

$c_2 = i(A-B)$

$\Rightarrow$ two l.i. solns are

$y_1 = e^{ax} \cos bx$ (Verify!)

& $y_2 = e^{ax} \sin bx$

(3) If m_1 & m_2 are equal real roots $(p^2-4q=0)$

then $m_1 = m_2 = -\frac{p}{2} = m$ (say)

$\Rightarrow$ one soln. will be $y_1 = e^{mx}$, $m = -\frac{p}{2}$

We can find other l.i. soln. by

assuming $y_2 = v y_1$

where $v = \int \frac{1}{y_1^2} e^{-\int p\,dx}\,dx$

$$= \int \frac{1}{e^{-px}} e^{-px}\,dx = x$$

$\Rightarrow$ $y_1 = e^{-p/2 x} = e^{mx}$

& $y_2 = v y_1 = x e^{mx}$ are l.i. solns.

$\Rightarrow$ $\boxed{y = c_1 e^{mx} + c_2 x e^{mx}}$ is a G.S.

THE METOD OF UNDETERMINED COEFFICIENTS

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{——— (1)}$$

homo. eqn

$$y'' + P(x)y' + Q(x)y = R(x) \quad \text{--- (2)}$$

non homo. eqn

We know that if $y_g(x)$ is the G.S. of (1) & $y_p(x)$ is a P.S. of (2) then G.S. of (2) is given by

$$y(x) = y_g(x) + y_p(x)$$

How to find $y_p(x)$?

— Suppose $P(x) = \text{constt.} = p$ (say)
$Q(x) = \text{constt.} = q$ (say)

$$\Rightarrow \quad y'' + py' + qy = R(x) \quad \text{--- (3)}$$

If $R(x) = \begin{cases} \text{exponential fn.} \\ \text{a sine fn.} \\ \text{a cosine fn.} \\ \text{a polynomial} \\ \text{or a combination of the above} \end{cases}$

then we can apply the method of undetermined coefficients

— Consider $y'' + py' + qy = e^{ax}$ --- (4)

$\because$ differentiating an exponential gives exponentials we guess

$$y_p = A e^{ax} \text{ is a P.S. of (4)} \quad \text{--- (5)}$$

$A \rightarrow$ undetermined coefficient we want to find s.t. (5) will satisfy (4)

Substitute (5) into (4)

$$A\,(a^2+pa+q)\,e^{ax} = e^{ax}$$

$$\Rightarrow \quad A = \frac{1}{a^2+pa+q} \quad \text{---(6)}$$

with the exception when a is a root of Auxiliary Eq^n (A.E.) $m^2+pm+q=0$ --(7)

In this case we try as

$$y_p = A\,x\,e^{ax} \quad \text{----(8)}$$

$$\Rightarrow \quad \underbrace{A\,(a^2+pa+q)}_{=0}\,x\,e^{ax} + A\,(2a+p)\,e^{ax} = e^{ax}$$

$\because$ a is root $\qquad e^{ax} \neq 0$

$$\Rightarrow \quad A = \frac{1}{2a+p} \text{ is valid coff. for (8)} \quad \text{---(9)}$$

except when $a = -p/2$ and in this case we try something like

$$y_p = A\,x^2\,e^{ax} \quad \text{--(10)}$$

$$\Rightarrow \quad \underbrace{A\,(a^2+pa+q)}_{=0}\,x^2e^{ax} + \underbrace{2A\,(2a+p)}_{0}\,x\,e^{ax} + 2A\,e^{ax} = e^{ax}$$

$$\Rightarrow \quad A = \frac{1}{2} \quad \text{--(11)} \qquad e^{ax} \neq 0$$

SUMMARY

- If a is not ~~the~~ a root of A.E. (7) then $y''+py'+qy = e^{ax}$ has a P.S. $= A\,e^{ax}$
- If a is simple root of A.E. (7) then P.S. $= A\,x\,e^{ax}$
- If a is a double root of A.E. (7) then P.S. $= A\,x^2 e^{ax}$

- For $y'' + py' + qy = \begin{cases} \sin bx \\ \text{or} \\ \cos bx \end{cases}$ or combination of these —— (12)

Try for $y_p = A \sin bx + B \cos bx$ —— (13)

& find A & B

Note this will not work if (13) satisfies homo. eqn of (12). then we try

$y_p = x\,(A \sin bx + B \cos bx)$ etc. —— (14)

$\boxed{\text{Example}}$ Find a P.S. of

$$y'' + y = \sin x \quad \text{—— (15)}$$

□ Homo. eqn $y'' + y = 0$

$\Rightarrow$ G.S. of homo. eqn will be

$$y = c_1 \sin x + c_2 \cos x$$

$\Rightarrow$ we can not take $y_p = A \sin x + B \cos x$

rather we try $y_p = x(A \sin x + B \cos x)$

$\Rightarrow \quad y_p' = A \sin x + B \cos x + x(A \cos x - B \sin x)$

$y_p'' = 2A \cos x - 2B \sin x + x(-A \sin x - B \cos x)$

$\Rightarrow$ ~~as~~ from (15) we get

$2A \cos x - 2B \sin x = \sin x$

$\Rightarrow A = 0 \;\&\; B = -\frac{1}{2}$

$\Rightarrow y_p = -\frac{1}{2} x \cos x$

$\Rightarrow$ G.S. $\boxed{y = c_1 \sin x + c_2 \cos x - \frac{1}{2} x \cos x}$

— For $y'' + py' + qy = a_0 + a_1 x + \dots + a_n x^n$
A polynomial

We ~~try~~ P.S. as

$y_p = A_0 + A_1 x + \dots + A_n x^n$ (Why?)

— If const. $q = 0$ then we take

$y_p = x(A_0 + A_1 x + \dots + A_n x^n)$

— If both $p = q = 0$ then solve by direct integration

$\boxed{\text{Example}}$ Find the G.S. of

$y'' - y' - 2y = 4x^2$ — (19)

□ Homo. eqn: $y'' - y' - 2y = 0$

$\Rightarrow$ A.E.: $m^2 - m - 2 = 0$

$\Rightarrow (m-2)(m+1) = 0$

$\therefore$ G.S. of homo. eqn: $y_g = c_1 e^{2x} + c_2 e^{-x}$

$\because$ RHS ~~is a~~ of (19) is a polynomial of deg 2

we try for P.S. as $y_p = A + Bx + cx^2$

$\Rightarrow$

$2c - (B + 2cx) - 2(A + Bx + cx^2) = 4x^2$

$$\Rightarrow \left.\begin{aligned} 2c - B - 2A &= 0 \\ -2c - 2B &= 0 \\ -2c &= 4 \end{aligned}\right\} \Rightarrow \begin{aligned} c &= -2 \\ B &= 2 \\ \& \ A &= -3 \end{aligned}$$

$\Rightarrow y_p = -3 + 2x - 2x^2$

$\Rightarrow$ G.S. $y = c_1 e^{2x} + c_2 e^{-x} - 3 + 2x - 2x^2$

General Method of variation of parameters

$$y'' + P(x)\,y' + Q(x)\,y = R(x) \quad \text{---(1)}$$

Non Homo.

Consider the homo. D.E.

$$y'' + P(x)\,y' + Q(x)\,y = 0 \quad \text{---(2)}$$

Suppose $y = c_1 y_1(x) + c_2 y_2(x)$ is a G.S. of (2). ---(3)

Now we ~~see~~ replace constt. c_1 by $v_1(x)$ & c_2 by $v_2(x)$ and seek a soln. of the form

$$y(x) = v_1(x)\,y_1(x) + v_2(x)\,y_2(x) \quad \text{---(4)}$$

of (1)

$$\Rightarrow y' = (v_1 y_1' + v_2 y_2') + (v_1' y_1 + v_2' y_2) \quad \text{---(5)}$$

To ~~simpl~~ make it simpler

Put $v_1' y_1 + v_2' y_2 = 0$ ---(6)

$$\Rightarrow y' = v_1 y_1' + v_2 y_2' \quad \text{---(7)}$$

$$\Rightarrow y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2' \quad \text{---(8)}$$

$\Rightarrow$ Subst. in (1) gives

$$v_1 \underbrace{(y_1'' + P y_1' + Q y_1)}_{=0} + v_2 \underbrace{(y_2'' + P y_2' + Q y_2)}_{=0} + v_1' y_1' + v_2' y_2' = R(x) \quad \text{---(9)}$$

$$\Rightarrow v_1' y_1' + v_2' y_2' = R(x) \quad \text{---(10)}$$

(6) & (10) together gives

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = R(x)$$

This can be solved for u_1' & u_2' to give

$$u_1' = \frac{-y_2 R(x)}{W(y_1, y_2)} \quad \& \quad u_2' = \frac{y_1 R(x)}{W(y_1, y_2)}$$

Note $W(y_1, y_2) \neq 0$ $\because$ y_1 & y_2 are l. i.

$$\Rightarrow u_1 = \int -\frac{y_2 R(x)}{W(y_1, y_2)}\,dx$$

$$\& \quad u_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)}\,dx$$

$\Rightarrow$ p.s. of (1) will be

$$y = y_1 \int \frac{-y_2 R(x)}{W(y_1, y_2)}\,dx + y_2 \int \frac{y_1 R(x)}{W(y_1, y_2)}\,dx$$

$\boxed{\text{Example}}$ Find p.s. of $y'' + y = \operatorname{cosec} x$

□ Homo. eqⁿ $y'' + y = 0$

$\Rightarrow y(x) = c_1 \sin x + c_2 \cos x$ is a G.S. of homo eqⁿ

$\Rightarrow y_1 = \sin x$ & $y_2 = \cos x$ & $W(y_1, y_2) = -1 \neq 0$

$$u_1 = \int \frac{-\cos x \csc x}{-1}\,dx = \ln(\sin x)$$

$$u_2 = \int \frac{\sin x \csc x}{-1}\,dx = -x$$

$\Rightarrow y = \sin x \ln(\sin x) - x \cos x$ is p.s.

HIGHER ORDER LINEAR EQ'S

ALL THE PREVIOUS METODS CAN BE GENERALIZED TO HIGHER ORDERS.

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = f(x) \quad (1)$$

$f(x)$ is cont's on $[a, b]$

G.S. of (1) is

$$y(x) = y_g(x) + y_p(x)$$

$y_p(x)$ is P.S. of (1) & $y_g(x)$ is G.S. of homo. eq^n $y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$

A.E. put $y = e^{rx}$

$$r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = 0$$

— If roots are distinct

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$$

is a sdn.

— If $r_1 = r_2 = \dots = r_k$ is a real root of multiplicity k of the A.E. then the first k terms can be replaced by

$$(c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1}) e^{r_1 x}$$

— If roots are complex & suppose $a + ib$ & $a - ib$ are roots of multiplicity k then

$$e^{ax} \left[(A_1 + A_2 x + \dots + A_k x^{k-1}) \cos bx + (B_1 + B_2 x + \dots + B_k x^{k-1}) \sin bx \right]$$

is a part of the G.S.

$\boxed{\text{Example}}$ $y^{(4)} - 5y'' + 4y = 0$

has A.E.

$$r^4 - 5r^2 + 4 = (r-1)(r+1)(r-2)(r+2) = 0$$

$\therefore$ G.S. $y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$

$\boxed{\text{Example}}$

$$y^{(4)} - 8y'' + 16y = 0$$

A.E.

$$r^4 - 8r^2 + 16 = (r-2)^2 (r+2)^2 = 0$$

$\therefore$ G.S.

$$y = (c_1 + c_2 x) e^{2x} + (c_3 + c_4 x) e^{-2x}$$

$\boxed{\text{Example}}$

$$y^{(4)} - 2y''' + 2y'' - 2y' + y = 0$$

A.E. $r^4 - 2r^3 + 2r^2 - 2r + 1 = 0$

$$\Rightarrow (r-1)^2 (r^2+1) = 0$$

$\therefore$ The G.S. is

$$y = (c_1 + c_2 x) e^{x} + c_3 \cos x + c_4 \sin x$$

Heaviside's OPERATOR METHOD

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_n y = f(x) \quad \text{— (1)}$$

Put $Dy = \frac{dy}{dx}$, $D^2 y = \frac{d^2 y}{dx^2}$, ..., $D^n y = \frac{d^n y}{dx^n}$

$\Rightarrow$ (1) becomes

$$D^n y + a_1 D^{n-1} y + \dots + a_{n-1} D y + a_n y = f(x) \quad (2)$$

or $$(D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n)\, y = f(x)$$

or $$p(D)\, y = f(x) \quad \text{---- (3)}$$

$p(D)$ is the A. Polynomial $p(r)$ with r replaced with D.

$\Rightarrow$ solving for y will give

$$y = \frac{1}{p(D)} f(x) \quad \text{— (4)}$$

$1/p(D)$ is an operation on $f(x)$ to yield y.

SIMPLE CASE $\quad Dy = f(x)$

$$\Rightarrow y = \frac{1}{D} f(x) = \int f(x)\, dx$$

$\Rightarrow$ $\frac{1}{D^2}$ means integrate twice

Consider $$(D - r)\, y = f(x) \quad \text{— (5)}$$

$$\Rightarrow \quad y = \frac{1}{D - r} f(x)$$

(5) is $$\frac{dy}{dx} - ry = f(x)$$

Soln is

$$y = e^{rx}\int e^{-rx} f(x)\,dx$$

$$\Rightarrow \boxed{\frac{1}{D-r} f(x) = e^{rx}\int e^{-rx} f(x)\,dx} \quad (6)$$

METHOD - I

$$y = \frac{1}{p(D)} f(x) = \frac{1}{(D-r_1)(D-r_2)\cdots(D-r_n)} f(x)$$

$$= \frac{1}{D-r_1}\cdot\frac{1}{D-r_2}\cdots\frac{1}{D-r_n} f(x)$$

we apply n inverse operation & use (6) to get the result.

Example Find P.S. of $y'' - 3y' + 2y = xe^{x}$

D We have $(D^2 - 3D + 2)y = xe^{x}$

$$\Rightarrow (D-1)(D-2)y = xe^{x}$$

$$\Rightarrow y = \frac{1}{D-1}\,\frac{1}{D-2}\, xe^{x}$$

But $\dfrac{1}{D-2} xe^{x} = e^{2x}\int e^{-2x} x e^{x}\,dx$

$$= -(1+x)e^{x}$$

$$\Rightarrow y = \frac{1}{D-1}\left[-(1+x)e^{x}\right]$$

$$= -e^{x}\int e^{-x}(1+x)e^{x}\,dx$$

$$= -\frac{1}{2}(1+x)^2 e^{x}$$

Example Find a P.S. of

$$y'' - y = e^{-x}$$

☐ We have $(D^2 - 1)y = e^{-x}$

$$\Rightarrow (D-1)(D+1)y = e^{-x}$$

$$\Rightarrow y = \frac{1}{D-1}\frac{1}{D+1}e^{-x}$$

But $\frac{1}{D+1}e^{-x} = e^{-x}\int e^{x}e^{-x}\,dx = xe^{-x}$

$$\Rightarrow y = \frac{1}{D-1}\{xe^{-x}\} = e^{x}\int e^{-x}xe^{-x}\,dx$$

$$= \left(-\frac{1}{2}x - \frac{1}{4}\right)e^{-x}$$

METHOD 2: PARTIAL FRACTIONS

$$y = \frac{1}{b(D)}f(x) = \frac{1}{(D-r_1)(D-r_2)\cdots(D-r_n)}f(x)$$

$$y = \frac{1}{b(D)}f(x) = \left[\frac{A_1}{D-r_1} + \frac{A_2}{D-r_2} + \cdots + \frac{A_n}{D-r_n}\right]f(x)$$

Old Example

$$y'' - 3y' + 2y = xe^{x}$$

$$y = \frac{1}{(D-1)(D-2)}xe^{x} = \left[\frac{1}{D-2} - \frac{1}{D-1}\right]xe^{x}$$

$$= \frac{1}{D-2}xe^{x} - \frac{1}{D-1}xe^{x}$$

$$= e^{2x}\int e^{-2x}xe^{x}\,dx - e^{x}\int e^{-x}xe^{x}\,dx$$

$$= -1(1+x)e^{x} - \frac{1}{2}x^2e^{x}$$

$$= -\left(1+x+\frac{1}{2}x^2\right)e^{x}$$

$\boxed{\text{OLD EXAMPLE}}$ $y'' - y = e^{-x}$

$$\Rightarrow y = \frac{1}{(D-1)(D+1)} e^{-x} = \frac{1}{2}\left[\frac{1}{D-1} - \frac{1}{D+1}\right] e^{-x}$$

$$= \frac{1}{2} e^{x} \int e^{-x} e^{-x}\, dx - \frac{1}{2} e^{-x} \int e^{x} e^{-x}\, dx$$

$$y = -\frac{1}{4} e^{-x} - \frac{1}{2} x e^{-x}$$

- If some of the factors are repeated then it has terms $\frac{A_1}{(D-r_1)} + \frac{A_2}{(D-r_1)^2} + \dots + \frac{A_k}{(D-r_1)^k}$.

$\boxed{\text{METHOD 3}}$ $\boxed{\text{SERIES EXPANSIONS}}$

- When $f(x)$ is a polynomial then expand $\frac{1}{p(D)}$ in a power series in D

$$y = \frac{1}{p(D)} f(x) = (1 + b_1 D + b_2 D^2 + \dots) f(x)$$

Note $D^k x^n = 0$ if $k > n$

$\boxed{\text{Example}}$ $y''' - 2y'' + y = x^4 + 2x + 5$

$\square$ $(D^3 - 2D^2 + 1) y = x^4 + 2x + 5$

$$\Rightarrow y = \frac{1}{1 - 2D^2 + D^3} (x^4 + 2x + 5)$$

But $\frac{1}{1-2D^2+D^3} = 1 + 2D^2 - D^3 + 4D^4 - 4D^5 + \dots$

$$\Rightarrow y = (1 + 2D^2 - D^3 + 4D^4 - 4D^5 + \dots)(x^4 + 2x + 5)$$

$$= (x^4 + 2x + 5) + 2(12x^2) - 24x + 4(24)$$

$$y = x^4 + 24x^2 - 22x + 101$$

Remember

$$\frac{1}{1-r} = 1+r+r^2+r^3+\cdots$$

& $$\frac{1}{1+r} = 1-r+r^2-r^3+\cdots$$

Example $y''' + y'' + y' + y = x^5 - 2x^2 + x$

□ We have $(D^3+D^2+D+1)\,y = x^5-2x^2+x$

$$\Rightarrow y = \frac{1}{1+D+D^2+D^3}(x^5-2x^2+x)$$

$$= \frac{1}{1-D^4}(1-D)\,(x^5-2x^2+x)$$

$$= \frac{1}{1-D^4}\left[(x^5-2x^2+x) - (5x^4-4x+1)\right]$$

$$= (1+D^4+D^8+\cdots)(x^5-5x^4-2x^2+5x-1)$$

$$= (x^5-5x^4-2x^2+5x-1)+(120x-120)$$

$$= x^5-5x^4-2x^2+125x-121$$

METHOD - IV The Exponential shift rule

If $f(x) = e^{kx}g(x)$

then $(D-r)f(x) = (D-r)\,e^{kx}g(x)$

$$= e^{kx}D\,g(x) + k e^{kx}g(x) - r e^{kx}g(x)$$

$$(D-r)f(x) = e^{kx}(D+k-r)\,g(x)$$

$\Rightarrow$ $$\boxed{p(D)\,e^{kx}g(x) = e^{kx}p(D+k)\,g(x)}$$

Similarly $$\boxed{\frac{1}{p(D)}e^{kx}g(x) = e^{kx}\frac{1}{p(D+k)}g(x)}$$

$\boxed{\text{Example}}$ - $y'' - 3y' + 2y = xe^x$

$\Rightarrow$ $(D^2 - 3D + 2)\, y = xe^x$

$\Rightarrow y = \dfrac{1}{(D^2 - 3D + 2)}\, xe^x$

$= e^x \dfrac{1}{(D+1)^2 - 3(D+1) + 2}\, x$

$= e^x \dfrac{1}{D^2 - D}\, x = -e^x \dfrac{1}{D} \dfrac{1}{D-1}\, x$

$= -e^x \left(\dfrac{1}{D} + 1 + D + D^2 + \dots \right) x$

$= -e^x \left(\frac{1}{2} x^2 + x + 1 \right)$ ▨