Part 2

* Mathematical Theory :-

* Second order autonomous systems :-

→ $\frac{d^2x}{dt^2} = \ddot{x} = f(x)$, $f(x) = -\psi'(x)$

→ $\frac{1}{2}\dot{x}^2 + \psi(x) = \text{constant} \Rightarrow \ddot{x} = f(x)$ is conservative

⇒ Coupled 2^{nd} order system :-

→ $\dot{x} = y$, $\dot{y} = f(x)$ or (reversible system)

→ $\dot{x} = f(x,y)$ & $\dot{y} = g(x,y)$

i) $f(x,-y) = -f(x,y) \Rightarrow f$ is odd in y

ii) $g(x,-y) = g(x,y) \Rightarrow g$ is even in y.

(reversible if $t \to -t \Rightarrow y \to -y$)

→ $\vec{\nabla} \times (\vec{\nabla}\psi) = \vec{0}$

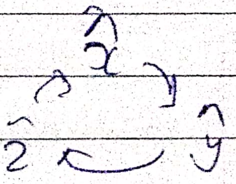
→ $\vec{\nabla} \times \vec{F} = 0 \Rightarrow \vec{F} = \vec{\nabla}\psi$

* Line & Surface elements :-

i) Cartesian coordinates :-

→ $d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$

→ $d\vec{a} = dx\,dy\,(\hat{x} \times \hat{y})$



ii) Polar coordinates :-

$\rightarrow$ $\vec{dl} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$

$\vec{da} = r\,dr\,d\theta\,(\hat{r}\times\hat{\theta})$

$\hat{r}$ $\hat{z}$ $\hat{\theta}$

* Vector Integral Th^m :

$\rightarrow$ $\displaystyle\int_{\vec{a}}^{\vec{b}} (\vec{\nabla}\psi)\cdot\vec{dl} = \psi(\vec{b}) - \psi(\vec{a})$ (Gradient th^m)

$\rightarrow$ $\displaystyle\int_S (\vec{\nabla}\times\vec{A})\cdot\vec{da} = \oint_P \vec{A}\cdot\vec{dl}$ (Strokes)

$\rightarrow$ $\displaystyle\int_V (\vec{\nabla}\cdot\vec{A})\,d\tau = \oint \vec{A}\cdot\vec{da}$ (Gauss)

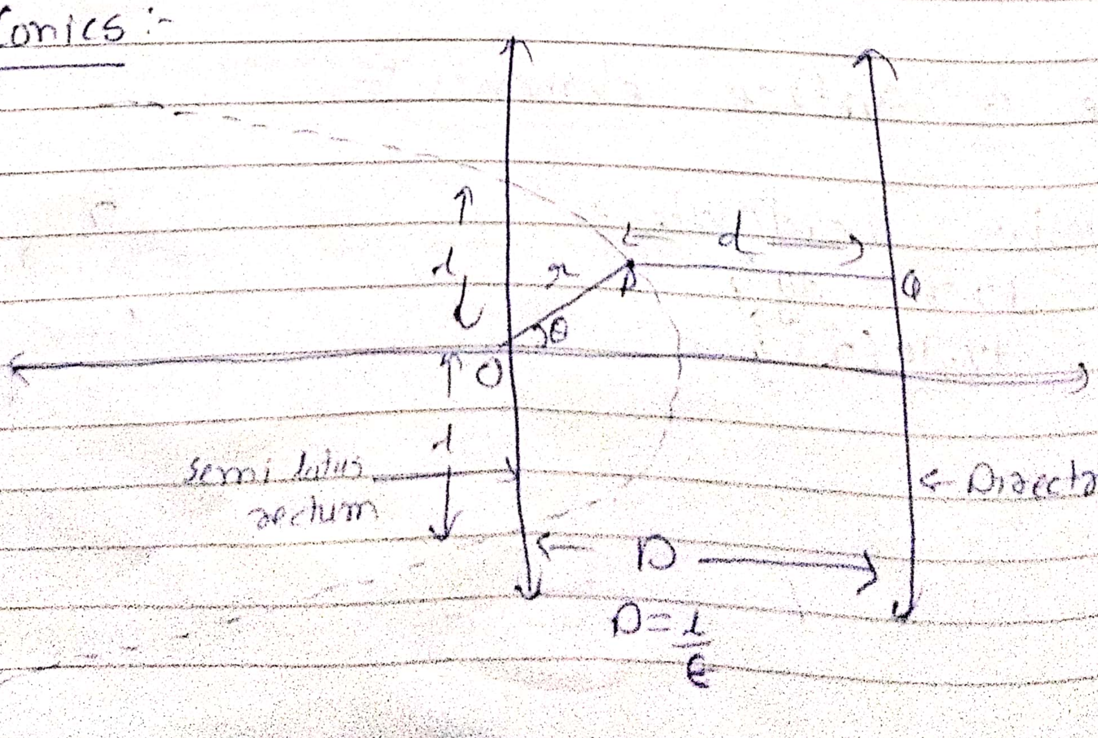
$\Rightarrow$ Solenoidal field :-

$\rightarrow$ $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \oint \vec{A}\cdot\vec{da} = 0$

$\rightarrow$ $\vec{\nabla}\times\vec{A} = 0 \Rightarrow \oint \vec{A}\cdot\vec{dl} = 0$ (Irrotational field)

$\rightarrow$ $\vec{A} = -\vec{\nabla}\psi \Rightarrow \displaystyle\int_{\vec{a}}^{\vec{b}} (\vec{\nabla}\psi)\cdot\vec{dl} = \psi(\vec{b}) - \psi(\vec{a})$ (Conservative field)

* Conics :-



→ Eccentricity $\epsilon = \frac{r}{d}$

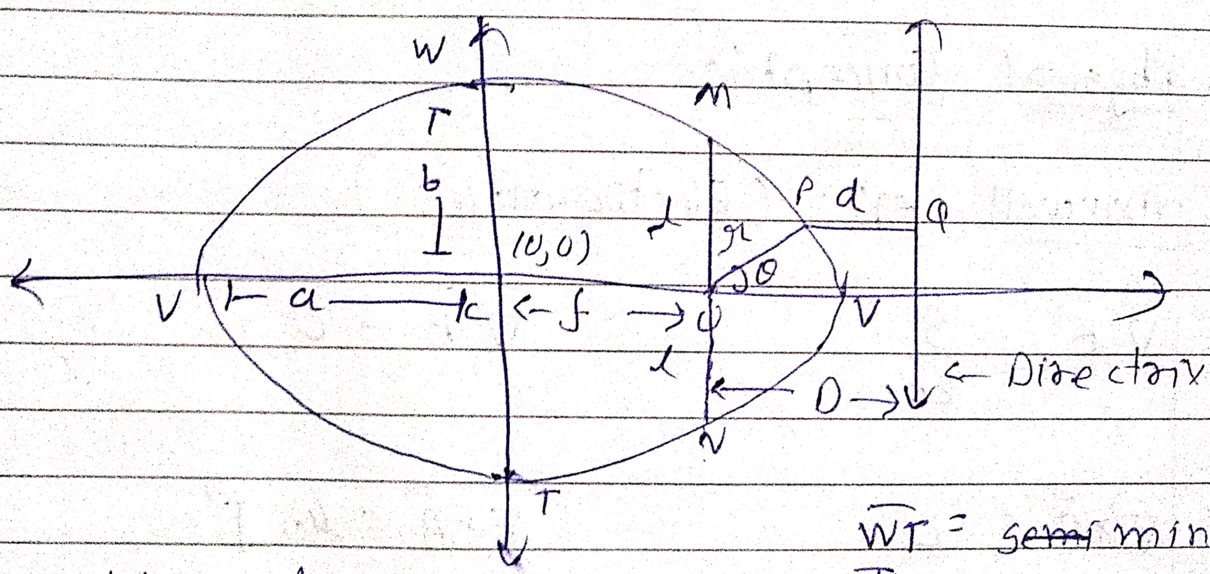
→ Polar equation : $r = \frac{l}{1+\epsilon\cos\theta}$

$\epsilon = 0 \Rightarrow$ circle

$0<\epsilon<1 \Rightarrow$ ellipse

$\epsilon = 1 \Rightarrow$ parabola

$\epsilon > 1 \Rightarrow$ hyperbola



$\overline{MN}$ = latus ractom = $2l$

$\overline{WT}$ = ~~semi~~ minor axis

$\overline{VV}$ = ~~semi~~ major axis

$\overline{CV} = \overline{CV} = a$

$CW = CT = b$

→ Area = $\pi ab = \pi a^2 \sqrt{1-\epsilon^2}$

→ $l = a(1-\epsilon^2)$

→ $f = a\epsilon = \sqrt{a^2-b^2}$

→ $b = a\sqrt{1-\epsilon^2}$

→ $\epsilon = \sqrt{\frac{a^2-b^2}{a^2}} = \frac{\sqrt{a^2-b^2}}{a}$

→ $D = \frac{l}{\epsilon}$

* Partial differential equation:

→ $Au_{xx} + Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$
(simple 2nd order partial differential eqn)

→ $\Delta = B^2 - 4AC$

$\Delta < 0 \Rightarrow$ elliptic
$\Delta = 0 \Rightarrow$ parabola
$\Delta > 0 \Rightarrow$ hyperbola

☆ Physical Concept:-

* Maxwell eqn: (Electrostatics)

→ $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ $\quad$ $\vec{\nabla} \cdot \vec{B} = 0$

$\vec{\nabla} \times \vec{E} = \vec{0}$ $\quad$ $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

→ $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$ $\quad$ $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

→ $\oint_S \vec{B} \cdot d\vec{a} = 0$

→ $\vec{B} = \vec{\nabla} \times \vec{A}$ $\quad$ $\vec{E} = -\vec{\nabla} V$, $\vec{A}$ = Magnetic vector
V = Electrostatic

* **Maxwell's eqⁿ of electrodynamics :-**

→ $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$ $\qquad$ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$

→ $\vec{J_D} = \varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$

→ Freespace $\Rightarrow \rho = 0, \vec{J} = \vec{0}$

* **Wave particle Duality :-**

→ $E = mc^2$ or $E = E_K + m_0 c^2$

→ $E^2 = (pc)^2 + (m_0 c^2)^2$

→ $E = h\nu = \dfrac{hc}{\lambda}$

→ $p = \dfrac{h}{\lambda}$, $\lambda = \dfrac{h}{mv}$

→ $n\lambda = 2\pi r_n$, n int

→ $L = mvr_n = n\hbar$

→ $\Delta x\cdot\Delta k_x \geq \dfrac{1}{2}$ $\qquad$ $\Delta E\cdot\Delta t \geq \dfrac{\hbar}{2}$ $\qquad$ $\boxed{k = \dfrac{2\pi}{\lambda}}$

$\Delta p_x\cdot\Delta x \geq \dfrac{\hbar}{2}$ $\qquad$ $\Delta p_x = \hbar\,\Delta k_x$

$\Delta E = \hbar\,\Delta\omega$

$\Delta\omega\,\Delta t \geq \dfrac{1}{2}$

* Physical system:

* The pendulum:

→ $\ddot{\theta} = -\omega^2\theta$, $\omega = \sqrt{\frac{g}{L}}$

⇒ Euler's Method:

→ $\dot{\theta} = \Omega$ & $\dot{\Omega} = -\frac{g\theta}{L} = -\omega^2\theta$

$\theta_{i+1} = \theta_i + \Omega \Delta t$
$\Omega_{i+1} = \Omega_i - \frac{g}{L}\theta_i \Delta t$

* Damped oscillator:

→ $mL\ddot{\theta} = -mg\sin\theta + (F - D\dot{\theta})$

$\therefore \frac{t_{os}^2}{T^2}\frac{d^2\theta}{d\tau^2} + \frac{t_d}{T}\frac{d\theta}{d\tau} = f - \sin\theta$

where, $t_{os}^2 = \frac{L}{g}$, $f = \frac{F}{mg}$

$t_d = \frac{D}{mg}$

→ $t_{os} << t_d$ ⇒ no longer oscillation

→ Fermi function, $f(E) = \frac{1}{1 + e^{(E-E_f)/K_BT}}$

* Josephson effect :-

→ DC effect : $I = I_c \sin\phi_0$

→ AC effect : $I = I_c \sin\left(\phi_0 + \frac{2eV t}{\hbar}\right)$

voltage $V = \frac{\hbar\dot{\phi}}{2e}$, $\dot{\phi} = \omega$

→ $\cancel{\frac{1}{T^2}\left(\frac{\hbar C}{2eI_c}\right)\frac{d^2\theta}{d\tau^2} + \left(\frac{\rho}{mg}\right)\frac{1}{T}\frac{d\theta}{d\tau} = f - \sin\theta}$

→ $\frac{1}{T^2}\left(\frac{\hbar C}{2eI_c}\right)\frac{d^2\phi}{d\tau^2} + \left(\frac{\hbar}{2eRI_c}\right)\frac{1}{T}\frac{d\phi}{dt} = \frac{I}{I_c} - \sin\phi$

* linear Oscillator :-

→ $\frac{d^2x}{dt^2} = \ddot{x} = -\frac{K}{m}x = -\omega^2 x$, $\omega = \sqrt{\frac{K}{m}}$

→ $\frac{v^2}{2c} + \frac{x^2}{(2c/\omega^2)} = 1$, c = constant, $\dot{x} = v$

⇒ With damping :-

→ $m\frac{d^2x}{dt^2} = -Kx - D\frac{dx}{dt}$

→ $\ddot{x} + 2b\dot{x} + \omega^2 x = 0$, $2b = \frac{D}{m}$, $\omega = \sqrt{\frac{K}{m}}$

→ For L-R-C circuit, $2b = \frac{R}{L}$ & $\omega^2 = \frac{1}{LC}$

Sol: $x \sim e^{\lambda t}$

→ $\lambda_{1,2} = -b \pm \sqrt{b^2 - \omega^2}$

i) Overdamped : $b^2 > \omega^2$

$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$; $\lambda_1, \lambda_2 < 0$

ii) Underdamped

→ $b^2 < \omega^2$

→ $x = e^{-bt}[(\alpha_1 + \beta_1)\cos(\sqrt{\omega^2 - b^2}\, t) - (\alpha_2 - \beta_2)\sin\sqrt{\omega^2 - b^2}\, t)$

→ $b = 0 \Rightarrow$ no damping

iii) Critical damping :-

→ $\omega^2 = b^2$

→ $\lambda_1 = \lambda_2 = -b$

→ $x = (A + Bt)e^{-bt}$

* Resonance :-

→ $m\frac{d^2x}{dt^2} + kx = F_0\cos(\Omega t)$

$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = f\cos(\Omega t)$, $\omega^2 = k/m$, $f = F_0/m$

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$\Rightarrow x_i = \int \frac{f_0}{\omega}$

$\rightarrow x = A\cos\omega t + B\sin\omega t + \frac{f_0}{\omega^2 - \Omega^2}\cos(\Omega t)$

$\rightarrow \omega = \Omega \Rightarrow$ resonance

$\Rightarrow$ Euler method :- (For damped oscillator)

$\rightarrow \ddot{x} + 2b\dot{x} + \omega^2 x = 0.$

$\dot{x} = v \quad \& \quad \dot{v} = -2bv - \omega^2 x.$

$\rightarrow v_{i+1} = v_i + (-2bv_i - \omega^2 x_i)\Delta t$

$x_{i+1} = x_i + v_i \Delta t$

* Particle Motion on plane :-

$\rightarrow x = r\cos\theta \qquad r^2 = x^2 + y^2$

$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$

$\rightarrow \hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \quad , \quad \hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$

$\rightarrow \vec{v} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}$

$\dot{r} = 0 \Rightarrow$ circular motion

$\dot{\theta} = 0 \Rightarrow$ linear motion

$\rightarrow \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

* Kepler's law:

i)

→ $h = r^2\dot{\theta}$, $r = \frac{1}{u}$, $\dot{\theta} = hu^2$, $K = GMm$

→ $2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

→ $F(1/u) = F(r) = \frac{-K}{r^2}$, $K = \frac{mh^2}{l}$

→ Orbit eq^n : $r = \frac{l}{1 + \epsilon\cos\theta}$

where,

$l = \frac{mh^2}{K}$, $K = GMm$

$\epsilon = \frac{Cmh^2}{K} = \sqrt{1 + \frac{2Eh^2}{G^2M^2m}}$

→ $E > 0 \Rightarrow$ hyperbolic orbit

$E = 0 \Rightarrow$ Parabolic orbit

$-\frac{G^2M^2m}{2h^2} < E < 0 \Rightarrow$ Elliptical orbit

→ Potential Energy : $V = -Ku = -\frac{GMm}{r}$

ii)

→ $\frac{d\vec{A}}{dt} = \dot{\vec{A}} = \frac{h}{2}\hat{z}$ = constant

iii)

→ $T = 2\pi\sqrt{\frac{m}{K}}\, a^{3/2} \Rightarrow T^2 = \frac{4\pi^2}{GM} a^3$

$\boxed{T^2 \propto a^3}$ $\left(\sqrt{1-\epsilon^2} = \sqrt{\frac{mh^2}{Ka}}\right)$

* Numerical scheme for planetary motion :-

→ $\vec{F} = \vec{F_x} + \vec{F_y} = -\frac{G M_0 m_E}{r^2}\hat{r} = -\frac{G M_0 m_E}{r^2}\left(\frac{x}{r}\hat{x} + \frac{y}{r}\hat{y}\right)$

→ $\dot{x} = v_x$, $\dot{y} = v_y$, $\dot{v_x} = -\frac{G M_0 x}{r^3}$, $\dot{v_y} = -\frac{G M_0 y}{r^3}$

⇒ Euler method :-

→ $x_{i+1} = x_i + v_{x,i} \times \Delta t$ $\quad y_{i+1} = y_i + v_{y,i}(\Delta t)$

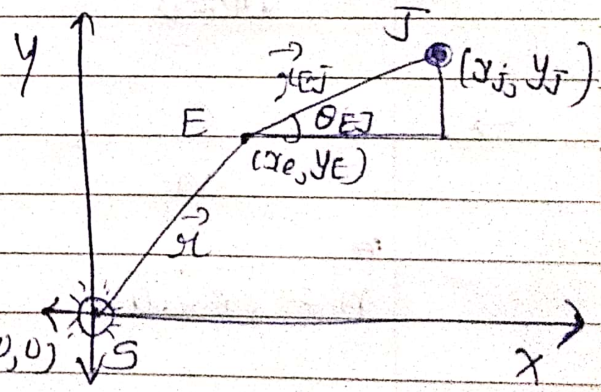
$r_i = \sqrt{x_i^2 + y_i^2}$

→ $v_{x,i+1} = v_{x,i} - \frac{G M_0 x_i (\Delta t)}{r_i^3}$ $\quad v_{y,i+1} = v_{y,i} - \frac{G M_0 y_i (\Delta t)}{r_i^3}$

⇒ Three body problem :-

$\vec{F}_{EJ,x} = -\frac{G m_E m_J (x_E - x_J)}{r_{EJ}^3}\hat{x}$

$\vec{F}_{EJ,y} = -\frac{G m_E m_J (y_E - y_J)}{r_{EJ}^3}\hat{y}$



→ $\frac{dv_{x,E}}{dt} = -\frac{G M_0 x_e}{r^3} - \frac{G m_J (x_E - x_J)}{r_{EJ}^3}$

$\frac{dv_{y,E}}{dt} = -\frac{G M_0 y_E}{r^3} - \frac{G m_J (y_E - y_J)}{r_{EJ}^3}$

→ Poisson's eq^n : $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

→ Laplace's eq^n : $\nabla^2 V = 0$ {in charge free region}

$\Rightarrow \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

(Elliptical 2nd order differential eq^n)

* <u>EM- Wave :-</u>

→ $\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} = \nabla^2 \vec{E}$, $\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = \nabla^2 \vec{B}$

→ $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

→ $v = \frac{c}{n}$, n = Reflactive index

$n = \sqrt{\varepsilon_r \mu_r}$