11.12 SOME TRANSMISSION LINE EXAMPLES

In this section we shall apply many of the results that we obtained in the previous sections to several typical transmission line problems. We shall simplify our work by restricting our attention to the lossless line.

Let us begin by assuming a two-wire 300 Ω line ($Z_0 = 300\ \Omega$), such as the lead-in wire from the antenna to a television or FM receiver. The circuit is shown in Figure 11.10. The line is 2 m long, and the values of L and C are such that the velocity on the line is 2.5×10^8 m/s. We shall terminate the line with a receiver having an input resistance of 300 Ω and represent the antenna by its Thevenin equivalent $Z = 300\,\Omega$ in series with $V_s = 60$ V at 100 MHz. This antenna voltage is larger by a factor of about 10^5 than it would be in a practical case, but it also provides simpler values to work with; in order to think practical thoughts, divide currents or voltages by 10^5, divide powers by 10^{10}, and leave impedances alone.

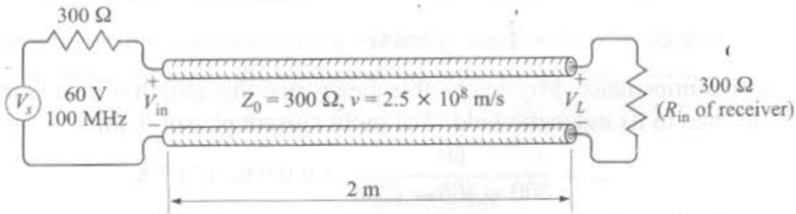

Fig. 11.10 A transmission line that is matched at both ends produces no reflections, and thus delivers maximum power to the load.

Since the load impedance is equal to the characteristic impedance, the line is matched; the reflection coefficient is zero, and the standing wave ratio is unity. For the given velocity and frequency, the wavelength on the line is $v/f = 2.5$ m, and the phase constant is $2\pi/\lambda = 0.8\pi$ rad/m; the attenuation constant is zero. The electrical length of the line is $\beta l = (0.8\pi)2$, or 1.6π rad. This length may also be expressed as 288°, or 0.8 wavelength.

The input impedance offered to the voltage source is 300 Ω, and since the internal impedance of the source is 300 Ω, the voltage at the input to the line is half of 60 V, or 30 V. The source is matched to the line and delivers the maximum available power to the line. Since there is no reflection and no attenuation, the voltage at the load is 30 V, but it is delayed in phase by 1.6π rad. Thus

$$V_{in} = 30\cos(2\pi 10^8 t)\ \text{V}$$

whereas

$$V_L = 30\cos(2\pi 10^8 t - 1.6\pi)\ \text{V}$$

The input current is

$$I_{in} = \frac{V_{in}}{300} = 0.1\cos(2\pi 10^8 t)\ \text{A}$$

while the load current is

$$I_L = 0.1\cos(2\pi 10^8 t - 1.6\pi)\ \text{A}$$

The average power delivered to the input of the line by the source must all be delivered to the load by the line,

$$P_{\text{in}} = P_L = \frac{1}{2} \times 30 \times 0.1 = 1.5 \text{ W}$$

Now let us connect a second receiver, also having an input resistance of 300 Ω, across the line in parallel with the first receiver. The load impedance is now 150 Ω, the reflection coefficient is

$$\Gamma = \frac{150 - 300}{150 + 300} = -\frac{1}{3}$$

and the standing wave ratio on the line is

$$s = \frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} = 2$$

The input impedance is no longer 300 Ω, but is now

$$Z_{\text{in}} = Z_0 \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} = 300 \frac{150 \cos 288^\circ + j300 \sin 288^\circ}{300 \cos 288^\circ + j150 \sin 288^\circ}$$

$$= 510\angle{-23.8^\circ} = 466 - j206 \ \Omega$$

which is a capacitive impedance. Physically, this means that this length of line stores more energy in its electric field than in its magnetic field. The input current phasor is thus

$$I_{s,\text{in}} = \frac{60}{300 + 466 - j206} = 0.0756\angle 15.0^\circ \text{ A}$$

and the power supplied to the line by the source is

$$P_{\text{in}} = \frac{1}{2} \times (0.0756)^2 \times 466 = 1.333 \text{ W}$$

Since there are no losses in the line, 1.333 W must also be delivered to the load. Note that this is less than the 1.50 W which we were able to deliver to a matched load; moreover, this power must divide equally between two receivers, and thus each receiver now receives only 0.667 W. Since the input impedance of each receiver is 300 Ω, the voltage across the receiver is easily found as

$$0.667 = \frac{1}{2}\frac{|V_{s,L}|^2}{300}$$

$$|V_{s,L}| = 20 \text{ V}$$

in comparison with the 30 V obtained across the single load.

Before we leave this example, let us ask ourselves several questions about the voltages on the transmission line. Where is the voltage a maximum and a minimum, and what are these values? Does the phase of the load voltage still differ from the input voltage by 288°? Presumably, if we can answer these questions for the voltage, we could do the same for the current.

Equation (89) serves to locate the voltage maxima at

$$z_{\max} = -\frac{1}{2\beta}(\phi + 2m\pi) \quad (m = 0, 1, 2, \ldots)$$

where $\Gamma = |\Gamma|e^{j\phi}$. Thus, with $\beta = 0.8\pi$ and $\phi = \pi$, we find

$$z_{\max} = -0.625 \quad \text{and} \quad -1.875 \text{ m}$$

while the minima are $\lambda/4$ distant from the maxima;

$$z_{\min} = 0 \quad \text{and} \quad -1.25 \text{ m}$$

and we find that the load voltage (at $z = 0$) is a voltage minimum. This, of course, verifies the general conclusion we reached earlier: a voltage minimum occurs at the load if $Z_L < Z_0$, and a voltage maximum occurs if $Z_L > Z_0$, where both impedances are pure resistances.

The minimum voltage on the line is thus the load voltage, 20 V; the maximum voltage must be 40 V, since the standing wave ratio is 2. The voltage at the input end of the line is

$$V_{s,\text{in}} = I_{s,\text{in}}\, Z_{\text{in}} = (0.0756\angle 15.0^\circ)(510\angle -23.8^\circ) = 38.5\angle -8.8^\circ$$

The input voltage is almost as large as the maximum voltage anywhere on the line because the line is about three-quarters of a wavelength long, a length which would place the voltage maximum at the input when $Z_L < Z_0$.

Finally, it is of interest to determine the load voltage in magnitude *and phase*. We begin with the total voltage in the line, using (93).

$$V_{sT} = \left(e^{-j\beta z} + \Gamma e^{j\beta z}\right) V_0^+ \tag{104}$$

We may use this expression to determine the voltage at any point on the line in terms of the voltage at any other point. Since we know the voltage at the input to the line, we let $z = -l$,

$$V_{s,\text{in}} = \left(e^{j\beta l} + \Gamma e^{-j\beta l}\right) V_0^+ \tag{105}$$

and solve for V_0^+,

$$V_0^+ = \frac{V_{s,\text{in}}}{e^{j\beta l} + \Gamma e^{-j\beta l}} = \frac{38.5\angle -8.8^\circ}{e^{j1.6\pi} - \frac{1}{3}e^{-j1.6\pi}} = 30.0\angle 72.0^\circ \text{ V}$$

We may now let $z = 0$ in (104) to find the load voltage,

$$V_{s,L} = (1+\Gamma)V_0^+ = 20\angle 72^\circ = 20\angle -288^\circ$$

The amplitude agrees with our previous value. The presence of the reflected wave causes $V_{s,\text{in}}$ and $V_{s,L}$ to differ in phase by about -279° instead of -288°.

EXAMPLE 11.11

In order to provide a slightly more complicated example, let us now place a purely capacitive impedance of $-j300\,\Omega$ in parallel with the two 300 Ω receivers. We are to find the input impedance and the power delivered to each receiver.

Solution. The load impedance is now 150 Ω in parallel with $-j300\,\Omega$, or

$$Z_L = \frac{150(-j300)}{150 - j300} = \frac{-j300}{1 - j2} = 120 - j60\,\Omega$$

We first calculate the reflection coefficient and the VSWR:

$$\Gamma = \frac{120 - j60 - 300}{120 - j60 + 300} = \frac{-180 - j60}{420 - j60} = 0.447\angle -153.4^\circ$$

$$s = \frac{1 + 0.447}{1 - 0.447} = 2.62$$

Thus, the VSWR is higher and the mismatch is therefore worse. Let us next calculate the input impedance. The electrical length of the line is still 288°, so that

$$Z_{\text{in}} = 300\,\frac{(120 - j60)\cos 288^\circ + j300\sin 288^\circ}{300\cos 288^\circ + j(120 - j60)\sin 288^\circ} = 755 - j138.5\,\Omega$$

This leads to a source current of

$$I_{s,\text{in}} = \frac{V_{Th}}{Z_{Th} + Z_{\text{in}}} = \frac{60}{300 + 755 - j138.5} = 0.0564\angle 7.47^\circ\ \text{A}$$

Therefore, the average power delivered to the input of the line is $P_{\text{in}}^{+} = \frac{1}{2}(0.0564)^2(755) = 1.200$ W. Since the line is lossless, it follows that $P_L = 1.200$ W, and each receiver gets only 0.6 W.

EXAMPLE 11.12

A local transmitter represented by an equivalent voltage source of 50 V in series with an internal impedance of $R_g = 50\,\Omega$ is operating at 500 MHz. The transmitter is connected to an antenna being represented by a complex load of $Z_L = 200 - j50\,\Omega$ through a 4 m long, 50 Ω Teflon filled ($\epsilon_r = 2.10$) lossless transmission line as shown in Figure 11.11.

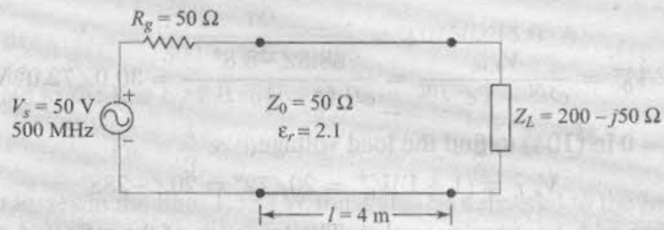

Fig. 11.11 Lossless transmission line.

Determine

(*a*) the reflection coefficient and the VSWR at the load
(*b*) the input impedance seen by the transmitter
(*c*) the time-averaged power delivered to the transmission line by the transmitter
(*d*) the time-averaged power delivered to the antenna
(*e*) the time-averaged power dissipated in the internal impedance R_g
(*f*) the total power supplied by the transmitter

Solution.

(*a*) The reflection coefficient at the load is given by

$$\Gamma = \frac{Z_L - Z_0}{Z_L - Z_0} = \frac{(200 - j50) - 50}{(200 - j50) + 50} = 0.6154 - j0.0769 \equiv 0.6202\angle - 0.1244$$

The voltage standing-wave ratio at the load is determined as

$$s \equiv \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.6202}{1 - 0.6202} = 4.27$$

(*b*) For determining the input impedance, let us first calculate the phase constant β for the transmission line shown in Figure 11.11.

$$\beta = \frac{2\pi}{\lambda} \equiv \frac{2\pi f}{v_p} \equiv \frac{2\pi \times 500 \times 10^6}{(3 \times 10^8/\sqrt{2.1})} = 15.18 \text{ rad/m}$$

The input impedance seen by the transmitter is then calculated as follows

$$Z_{\text{in}} = Z_0 \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} = 50 \frac{(200 - j50)\cos(15.18 \times 4) + j50 \sin(15.18 \times 4)}{50\cos(15.18 \times 4) + j(200 - j50)\sin(15.18 \times 4)}$$
$$= (14.664 - j24.17)\,\Omega$$

(*c*) For calculating the time-averaged power delivered to the transmission line by the source, the configuration shown in Figure 11.11 can be replaced by its equivalent circuit at the generator end as shown below in Figure 11.12

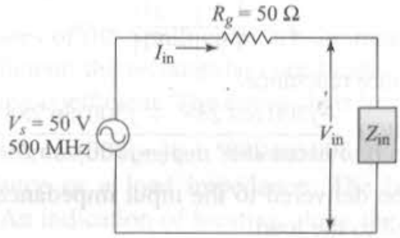

Fig. 11.12 Equivalent circuit.

The terminal voltage V_{in} in Figure 11.12 can be determined as

$$V_{\text{in}} = V_s \frac{Z_{\text{in}}}{Z_{\text{in}} + R_g} = 50 \frac{14.664 - j24.17}{14.664 - j24.17 + 50} = 20.48\angle -38.3^\circ \text{ V}$$

Similarly, the input terminal current can be calculated as

$$I_{\text{in}} = \frac{V_s}{R_g + Z_{\text{in}}} = \frac{50}{14.664 - j24.17 + 50} = 0.7243\angle 20.5^\circ \text{ A}$$

The time-averaged power delivered to the transmission line by the transmitter can finally be determined as

$$P_{\text{in}} = \frac{1}{2}(I_{\text{in}})^2\, \Re[Z_{\text{in}}] = \frac{1}{2} \times (0.7243)^2 \times 14.664 = 3.846 \text{ W}$$

(*d*) As the transmission line is lossless, hence the total power delivered to the transmission line by the source would be absorbed by the load, which represents the antenna in Figure 11.11. Hence, the time-averaged power delivered to the antenna is 3.846 W.

(*e*) The time-averaged power dissipated in the internal impedance R_g can be determined using the equivalent circuit of Figure 11.12, which provides following expression:

$$P_g = \frac{1}{2}(I_{\text{in}})^2 R_g = \frac{1}{2} \times (0.7243)^2 \times 50 = 13.1 \text{ W}$$

(*f*) The total power supplied by the transmitter would be given by

$$P_{\text{Total}} = P_{\text{in}} + P_g = (3.846 + 13.1) = 16.946 \text{ W}$$

EXAMPLE 11.13

As a final example, let us terminate the line shown in Figure 11.10 with a purely capacitive impedance, $Z_L = -j300\,\Omega$. We seek the reflection coefficient, the VSWR, and the power delivered to the load.

Solution. Obviously, we cannot deliver any average power to the load since it is a pure reactance. As a consequence, the reflection coefficient is

$$\Gamma = \frac{-j300 - 300}{-j300 + 300} = -j1 = 1\angle{-90^\circ}$$

and the reflected wave is equal in amplitude to the incident wave. Hence it should not surprise us to see that the VSWR is

$$s = \frac{1 + |-j1|}{1 - |-j1|} = \infty$$

and the input impedance is a pure reactance,

$$Z_{\text{in}} = 300\,\frac{-j300\cos 288^\circ + j300\sin 288^\circ}{300\cos 288^\circ + j(-j300)\sin 288^\circ} = j589$$

Thus, no average power can be delivered to the input impedance by the source, and therefore no average power can be delivered to the load.

Although we could continue to find numerous other facts and figures for these examples, much of the work may be done more easily for problems of this type by using graphical techniques. We shall encounter these in Section 11.13.

D11.4 A 50 W lossless line has a length of 0.4λ. The operating frequency is 300 MHz. A load $Z_L = 40 + j30\ \Omega$ is connected at $z = 0$, and the Thevenin-equivalent source at $z = -l$ is $12\angle 0^\circ$ V in series with $Z_{Th} = 50 + j0\ \Omega$. Find: (*a*) Γ; (*b*) s; (*c*) Z_{in}.

Ans. $0.333\angle 90^\circ$; 2.00; $25.5 + j5.90\ \Omega$

D11.5 For the transmission line of Problem D11.4, also find: (*a*) the phasor voltage at $z = -l$; (*b*) the phasor voltage at $z = 0$; (*c*) the average power delivered to Z_L.

Ans. $4.14\angle 8.58^\circ$ V; $6.32\angle{-125.6^\circ}$ V; 0.320 W

11.13 GRAPHICAL METHODS

Animations

Transmission line problems often involve manipulations with complex numbers, making the time and effort required for a solution several times greater than are needed for a similar sequence of operations on real numbers. One means of reducing the labor without seriously affecting the accuracy is by using transmission-line charts. Probably the most widely used one is the Smith chart.[3]

[3] P. H. Smith, "Transmission Line Calculator," *Electronics*, vol. 12, pp. 29–31, January 1939.

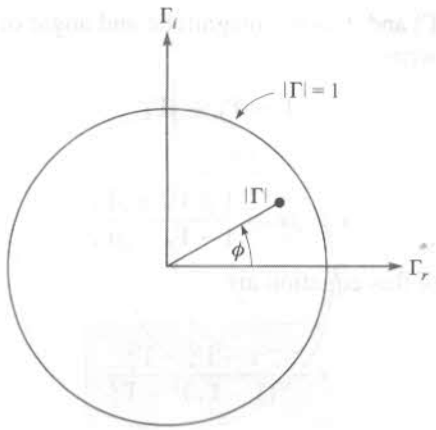

Fig. 11.13 The polar coordinates of the Smith chart are the magnitude and phase angle of the reflection coefficient; the rectangular coordinates are the real and imaginary parts of the reflection coefficient. The entire chart lies within the circle $|\Gamma| = 1$.

Basically, this diagram shows curves of constant resistance and constant reactance; these may represent either an input impedance or a load impedance. The latter, of course, is the input impedance of a zero-length line. An indication of location along the line is also provided, usually in terms of the fraction of a wavelength from a voltage maximum or minimum. Although they are not specifically shown on the chart, the standing-wave ratio and the magnitude and angle of the reflection coefficient are very quickly determined. As a matter of fact, the diagram is constructed within a circle of unit radius, using polar coordinates, with radius variable $|\Gamma|$ and counterclockwise angle variable ϕ, where $\Gamma = |\Gamma|e^{j\phi}$. Figure 11.13 shows this circle. Since $|\Gamma| < 1$, all our information must lie on or within the unit circle. Peculiarly enough, the reflection coefficient itself will not be plotted on the final chart, for these additional contours would make the chart very difficult to read.

The basic relationship upon which the chart is constructed is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{106}$$

The impedances which we plot on the chart will be *normalized* with respect to the characteristic impedance. Let us identify the normalized load impedance as z_L,

$$\boxed{z_L = r + jx = \frac{Z_L}{Z_0} = \frac{R_L + jX_L}{Z_0}}$$

and thus

$$\boxed{\Gamma = \frac{z_L - 1}{z_L + 1}}$$

or

$$\boxed{z_L = \frac{1 + \Gamma}{1 - \Gamma}} \tag{107}$$

In polar form, we have used $|\Gamma|$ and ϕ as the magnitude and angle of Γ. With Γ_r and Γ_i as the real and imaginary parts of Γ, we write

$$\Gamma = \Gamma_r + j\Gamma_i \tag{108}$$

Thus

$$r + jx = \frac{1 + \Gamma_r + j\Gamma_i}{1 - \Gamma_r - j\Gamma_i} \tag{109}$$

The real and imaginary parts of this equation are

$$\boxed{r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2}} \tag{110}$$

$$\boxed{x = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}} \tag{111}$$

After several lines of elementary algebra, we may write (110) and (111) in forms which readily display the nature of the curves on Γ_r, Γ_i axes,

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2 \tag{112}$$

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 \tag{113}$$

The first equation describes a family of circles, where each circle is associated with a specific value of resistance r. For example, if $r = 0$ the radius of this zero-resistance circle is seen to be unity, and it is centered at the origin ($\Gamma_r = 0$, $\Gamma_i = 0$). This, for a pure reactance termination leads to a reflection coefficient of unity magnitude. On the other hand, if $r = \infty$, then $z_L = \infty$ and we have $\Gamma = 1 + j0$. The circle described by (112) is centered at $\Gamma_r = 1$, $\Gamma_i = 0$ and has zero radius. It is therefore the point $\Gamma = 1 + j0$, as we decided it should be. As another example, the circle for $r = 1$ is centered at $\Gamma_r = 0.5$, $\Gamma_i = 0$ and has a radius of 0.5. This circle is shown in Figure 11.14, along with circles for $r = 0.5$ and $r = 2$. All circles are centered on the Γ_r axis and pass through the point $\Gamma = 1 + j0$.

Equation (113) also represents a family of circles, but each of these circles is defined by a particular value of x, rather than r. If $x = \infty$, then $z_L = \infty$, and $\Gamma = 1 + j0$ again. The circle described by (113) is centered at $\Gamma = 1 + j0$ and has zero radius; it is therefore the point $\Gamma = 1 + j0$. If $x = +1$, then the circle is centered at $\Gamma = 1 + j1$ and has unit radius. Only one-quarter of this circle lies within the boundary curve $|\Gamma| = 1$, as shown in Figure 11.15. A similar quarter-circle appears below the Γ_r axis for $x = -1$. The portions of other circles for $x = 0.5$, -0.5, 2, and -2 are also shown. The "circle" representing $x = 0$ is the Γ_r axis; this is also labeled in Figure 11.15.

The two families of circles both appear on the Smith chart, as shown in Figure 11.16. It is now evident that if we are given Z_L, we may divide by Z_0 to obtain z_L, locate the appropriate r and x circles (interpolating as necessary), and determine Γ by the intersection of the two circles. Since the chart does not have concentric circles showing the values of $|\Gamma|$, it is necessary to measure the radial

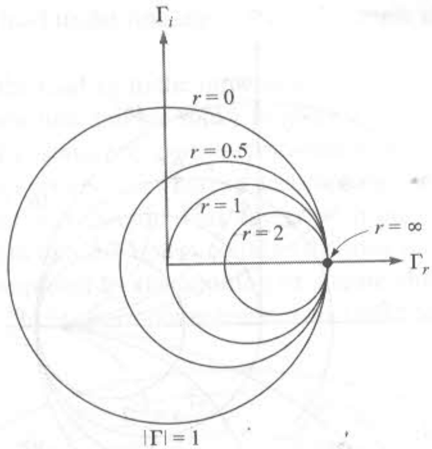

Fig. 11.14 Constant-r circles are shown on the Γ_r, Γ_i plane. The radius of any circle is $1/(1+r)$.

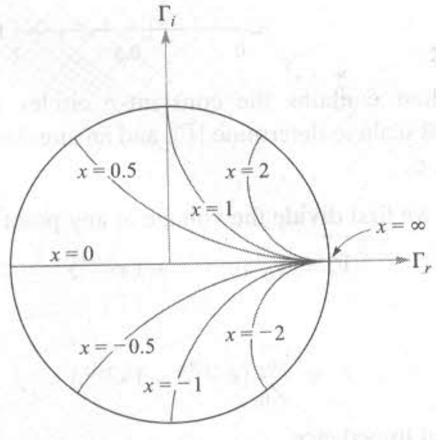

Fig. 11.15 The portions of the circles of constant x lying within $|\Gamma| = 1$ are shown on the Γ_r, Γ_i axes. The radius of a given circle is $1/|x|$.

distance from the origin to the intersection with dividers or compass and use an auxiliary scale to find $|\Gamma|$. The graduated line segment below the chart in Figure 11.16 serves this purpose. The angle of Γ is ϕ, and it is the counterclockwise angle from the Γ_r axis. Again, radial lines showing the angle would clutter up the chart badly, so the angle is indicated on the circumference of the circle. A straight line from the origin through the intersection may be extended to the perimeter of the chart. As an example, if $Z_L = 25 + j50\,\Omega$ on a 50 Ω line, $z_L = 0.5 + j1$, and point A on Figure 11.16 shows the intersection of the $r = 0.5$ and $x = 1$ circles. The reflection coefficient is approximately 0.62 at an angle ϕ of 83°.

The Smith chart is completed by adding a second scale on the circumference by which distance along the line may be computed. This scale is in wavelength units, but the values placed on it are

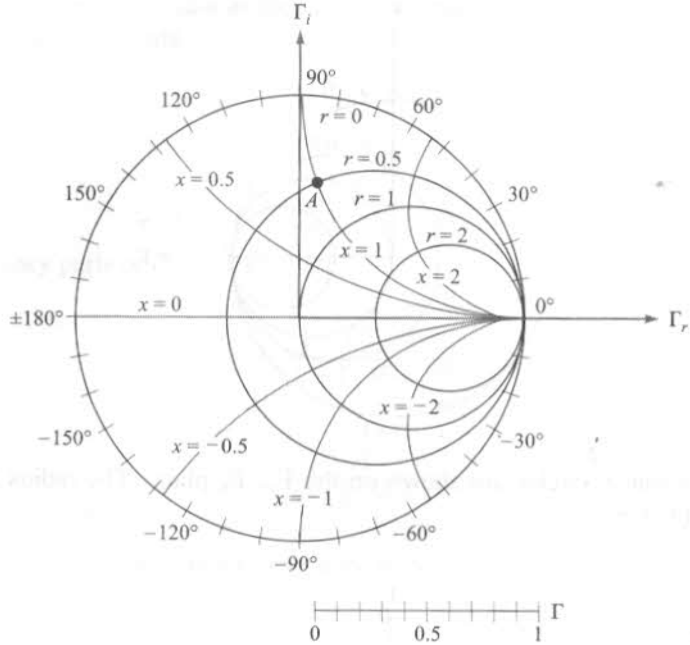

Fig. 11.16 The Smith chart contains the constant-r circles and constant-x circles, an auxiliary radial scale to determine $|\Gamma|$, and an angular scale on the circumference for measuring ϕ.

not obvious. To obtain them, we first divide the voltage at any point along the line,

$$V_s = V_0^+(e^{-j\beta z} + \Gamma e^{j\beta z})$$

by the current

$$I_s = \frac{V_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z})$$

obtaining the normalized input impedance

$$z_{\text{in}} = \frac{V_s}{Z_0 I_s} = \frac{e^{-j\beta z} + \Gamma e^{j\beta z}}{e^{-j\beta z} - \Gamma e^{j\beta z}}$$

Replacing z with $-l$ and dividing numerator and denominator by $e^{j\beta l}$, we have the general equation relating normalized input impedance, reflection coefficient, and line length,

$$\boxed{z_{\text{in}} = \frac{1 + \Gamma e^{-j2\beta l}}{1 - \Gamma e^{-j2\beta l}} = \frac{1 + |\Gamma| e^{j(\phi - 2\beta l)}}{1 - |\Gamma| e^{j(\phi - 2\beta l)}}} \tag{114}$$

Note that when $l = 0$, we are located at the load, and $z_{\text{in}} = (1 + \Gamma)/(l - \Gamma) = z_L$, as shown by (107).

Equation (114) shows that the input impedance at any point $z = -l$ can be obtained by replacing Γ, the reflection coefficient of the load, by $\Gamma e^{-j2\beta l}$. That is, we decrease the angle of Γ by $2\beta l$

radians as we move from the load to the line input. Only the angle of Γ is changed; the magnitude remains constant.

Thus, as we proceed from the load z_L to the input impedance z_{in}, we move *toward* the generator a distance l on the transmission line, but we move through a *clockwise* angle of $2\beta l$ on the Smith chart. Since the magnitude of Γ stays constant, the movement toward the source is made along a constant-radius circle. One lap around the chart is accomplished whenever βl changes by π rad, or when l changes by one-half wavelength. This agrees with our earlier discovery that the input impedance of a half-wavelength lossless line is equal to the load impedance.

The Smith chart is thus completed by the addition of a scale showing a change of 0.5λ for one circumnavigation of the unit circle. For convenience, two scales are usually given, one showing

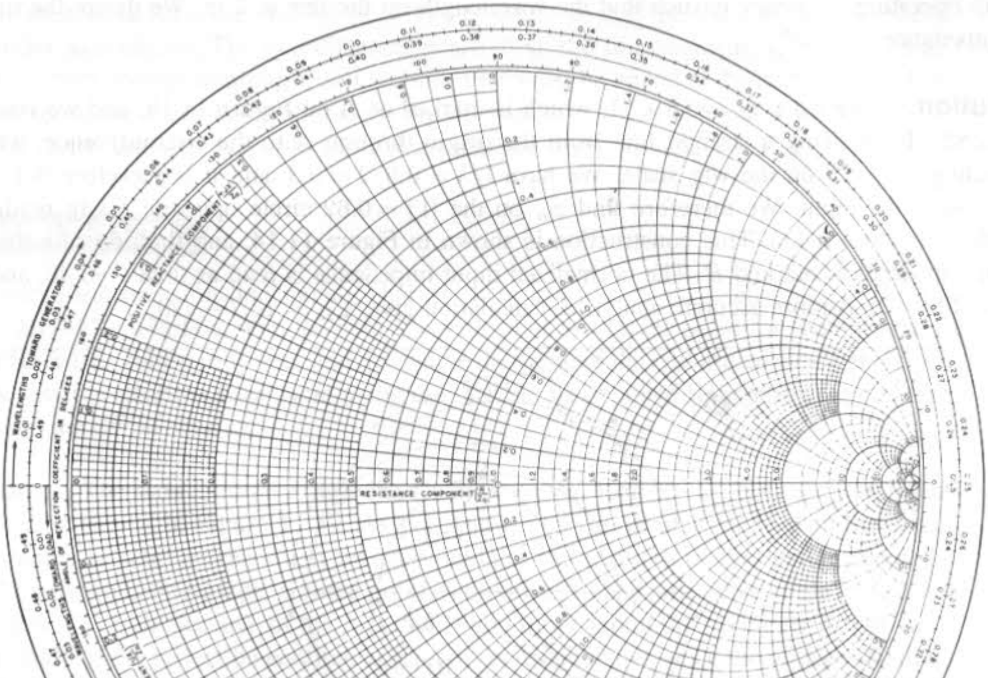

Fig. 11.17 A photographic reduction of one version of a useful Smith Chart (*courtesy of the Emeloid Company, Hillside, N.J.*). For accurate work, larger charts are available wherever fine technical books are sold.