Tutorial-6

Q.1 Find all the local maxima, local minima, and saddle points of the functions

(I) $f(x,y) = e^{x^2+y^2-4x}$

(II) $f(x,y) = \ln(x+y) + x^2 y$

(III) $f(x,y) = 1 - \sqrt[3]{x^2+y^2}$

Q.3 Find two numbers a and b with $a \le b$ such that

$$\int_a^b (6-x-x^2)\,dx$$ has its largest value.

Q.2 Find the absolute maxima and minima of the function $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x=0$, $y=2$, $y=2x$ in the 1st quadrant.

Q.4 Let $f(x,y) = x^2 - xy + y^2 - y$. Find the directions u and the values of $D_u f(1,-1)$ for which

(I) $D_u f(1,-1)$ is largest

(II) $D_u f(1,-1)$ is smallest

(III) $D_u f(1,-1) = 0$

(IV) $D_u f(1,-1) = 4$

(V) $D_u f(1,-1) = -3$

Q.5 Is there a direction u in which the rate of change of $f(x,y) = x^2 - 3xy + 4y^2$ at $P(1,2)$ equals 14 ? Give reason for your answer.

Q.6 A flat circular plate has the shape of the region $x^2+y^2 \le 1$. The plate including the boundary where $x^2+y^2=1$, is heated so that the temperature at the point (x,y) is

$$T(x,y) = x^2+2y^2-x.$$

Find the temperatures at the hottest and coldest points on the plate.

Tutorial-6 Solution

Q.1 (i) $f(x,y) = e^{x^2+y^2-4x}$

$$\frac{\partial f}{\partial x} = (2x-4)\, e^{x^2+y^2-4x} = 0 \Rightarrow (2x-4) = 0$$
$$\frac{\partial f}{\partial y} = 2y\, e^{x^2+y^2-4x} = 0 \Rightarrow 2y = 0$$

$x = 2,\ y = 0$

The critical point is $(2,0)$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(2,0)} = 2e^{x^2+y^2-4x} + (2x-4)^2 e^{x^2+y^2-4x}\Big]_{(2,0)}$$
$$= 2e^{-4} = \frac{2}{e^4}$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(2,0)} = 2e^{x^2+y^2-4x} + 4y^2 e^{x^2+y^2-4x}\Big]_{(2,0)}$$
$$= 2e^{-4} = \frac{2}{e^4}$$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(2,0)} = (2x-4)\cdot 2y\, e^{x^2+y^2-4x}\Big]_{(2,0)} = 0$$

$$f_{xx}\cdot f_{yy} - f_{xy}^2\Big|_{(2,0)} = \frac{4}{e^8} > 0$$

$$f_{xx}\Big|_{(2,0)} = \frac{2}{e^4} > 0$$

So $f(x,y)$ has a local minimum at $(2,0)$

$$f(2,0) = \frac{1}{e^4}$$

(ii)

$$f(x,y) = \ln(x+y) + x^2 y$$

Sol^n

$$f_x(x,y) = 2x + \frac{1}{x+y} = 0$$

$$f_y(x,y) = \frac{1}{x+y} - 1 = 0 \Rightarrow \frac{1}{x+y} = 1$$

$$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$$

$$y = 1 - (-\tfrac{1}{2}) = \frac{3}{2}$$

$\left(-\frac{1}{2}, \frac{3}{2}\right)$ is the critical point.

$$f_{xx}\Big|_{(-\frac{1}{2},\frac{3}{2})} = 2 - \frac{1}{(x+y)^2}\Big]_{(-\frac{1}{2},\frac{3}{2})} = 2 - \frac{1}{1} = 2-1 = 1$$

$$f_{yy}\Big|_{(-\frac{1}{2},\frac{3}{2})} = \frac{-1}{(x+y)^2}\Big|_{(-\frac{1}{2},\frac{3}{2})} = -1$$

$$f_{xy}\Big|_{(-\frac{1}{2},\frac{3}{2})} = \frac{-1}{(x+y)^2}\Big|_{(-\frac{1}{2},\frac{3}{2})} = -1$$

$$f_{xx}f_{yy} - (f_{xy})^2\Big|_{(-\frac{1}{2},\frac{3}{2})} = -1-1 = -2 < 0$$

So $\left(-\frac{1}{2}, \frac{3}{2}\right)$ is a saddle point of $f(x,y)$.

(iii)

$$f(x,y) = 1 - \sqrt[3]{x^2+y^2}$$

Ans

$$\left.\begin{aligned} f_x(x,y) &= \frac{-2x}{3(x^2+y^2)^{2/3}} = 0 \\ f_y(x,y) &= \frac{-2y}{3(x^2+y^2)^{2/3}} = 0 \end{aligned}\right\} \text{No solution to the system.}$$

However we must also consider where the partial derivatives are undefined. This occurs when $x=0, y=0$.

So critical point is $(0,0)$.

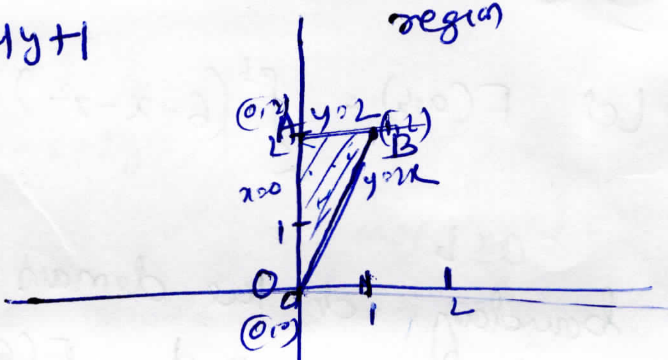
We can not use 2nd derivative test on the partial derivatives are not defined at $(0,0)$.

See $f(0,0) = 1$

$f(x,y) = 1 - \sqrt[3]{x^2+y^2} \le 1$ for all (x,y)

$\Rightarrow (0,0)$ is a local maximum.

$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$



(i) <u>On OA</u>, $f(x,y) = f(0,y) = y^2 - 4y + 1$ on $0 \le y \le 2$

$f'(0,y) = 2y - 4 = 0 \Rightarrow y = 2$

$f(0,0) = 1$, $f(0,2) = -3$

(ii) <u>On AB</u>, $f(x,y) = f(x,2) = 2x^2 - 4x - 3$ on $0 \le x \le 1$

$f'(x,2) = 4x - 4 = 0 \Rightarrow x = 1$

$f(0,2) = -3$ and $f(1,2) = -5$

(iii) <u>On OB</u>, $f(x,y) = f(x,2x) = 6x^2 - 12x + 1$ on $0 \le x \le 1$

end points $(0,0)$, $(1,2)$

$f(0,0) = 1$, $f(1,2) = -5$

(iv) <u>For interior points</u>

$$\left.\begin{aligned} f_x(x,y) &= 4x - 4 = 0 \\ f_y(x,y) &= 2y - 4 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

$(1,2)$ is not an interior point.

So from (i), (ii), (iii), (iv), absolute maximum is 1 at $(0,0)$. and absolute minimum is -5 at $(1,2)$.

~~Q.3~~ Sol

Q-3

Let $F(a,b) = \int_a^b (6-x-x^2)\,dx$

$a \le b$

The boundary of the domain of F is the line $a=b$ in the ab-plane, and $F(a,a)=0$.

So F is identically 0 on the boundary of its domain.

For interior critical points we have:

$$\frac{\partial F}{\partial a} = -(6-a-a^2)=0 \Rightarrow a=-3,2$$

$$\frac{\partial F}{\partial b} = (6-b-b^2)=0 \Rightarrow b=-3,2.$$

Since $a \le b$, there is only one interior critical point $(-3,2)$.

$$F_{aa}\Big|_{(-3,2)} = -(-1-2a)\Big|_{(-3,2)} = -(-1-2(-3)) = -5<0$$

$$F_{bb}\Big|_{(-3,2)} = (-1-2b)\Big|_{(-3,2)} = -1-2\cdot 2 = -5$$

$$F_{ab}\Big|_{(-3,2)} = 0$$

$$F_{aa}F_{bb}-(F_{ab})^2\Big|_{(-3,2)} = 25>0$$

at $F_{aa}\Big|_{(-3,2)} = -5<0$

So $(-3,2)$ is a point of local maximum.

So at $a=-3$, $b=2$ the funct $\int_a^b (6-x-x^2)\,dx$ has its largest value.

4 $\quad f(x,y) = x^2 - xy + y^2 - y$

Sol $\quad \nabla f = (2x-y)\hat{i} + (-x+2y-1)\hat{j}$

(i) $\nabla f(1,-1) = 3\hat{i} - 4\hat{j}$ $\quad$ (maximum / largest)

$|\nabla f(1,-1)| = \sqrt{3^2+4^2} = 5$

$\vec{u} = \dfrac{\nabla f(1,-1)}{|\nabla f(1,-1)|} = \dfrac{3}{5}\hat{i} - \dfrac{4}{5}\hat{j}$ $\quad$ direct

largest $\underline{D_u f(1,-1)} = \nabla f \cdot \vec{u}\Big|_{(1,-1)} = (3\hat{i} - 4\hat{j}) \cdot \left(\dfrac{3}{5}\hat{i} - \dfrac{4}{5}\hat{j}\right)$

$= \dfrac{9}{5} + \dfrac{16}{5} = \dfrac{25}{5} = 5$

(ii) $\quad -\nabla f(1,-1) = -3\hat{i} + 4\hat{j}$

$\vec{u} = \dfrac{-\nabla f(1,-1)}{|\nabla f(1,-1)|} = -\dfrac{3}{5}\hat{i} + \dfrac{4}{5}\hat{j}$

smallest $D_u f(1,-1) = \nabla f \cdot \vec{u}$

$= (3\hat{i} - 4\hat{j}) \cdot \left(-\dfrac{3}{5}\hat{i} + \dfrac{4}{5}\hat{j}\right)$

$= -\dfrac{9}{5} - \dfrac{16}{5} = -5$

(iii) $\quad \underline{D_u f(1,-1) = 0}$ in the direct $\quad$ of $u = \dfrac{4}{5}\hat{i} + \dfrac{3}{5}\hat{j}$

or $u = -\dfrac{4}{5}\hat{i} - \dfrac{3}{5}\hat{j}$

(iv) Let $u = u_1\hat{i} + u_2\hat{j}$ $\quad$ unit vect

$\underline{|u|\ \sqrt{u_1^2+u_2^2} = 1}$

$D_u f(1,-1) = \underline{\nabla f(1,-1) \cdot u} = (3\hat{i} - 4\hat{j}) \cdot (u_1\hat{i} + u_2\hat{j})$

$= 3u_1 - 4u_2 = 4 \quad \Rightarrow \quad u_2 = -\left(\dfrac{4 - 3u_1}{4}\right) = \dfrac{3}{4}u_1 - 1$

But $\sqrt{u_1^2+u_2^2} = 1 \quad \Rightarrow u_1^2 + u_2^2 = 1 \quad \Rightarrow u_1^2 + \left(\dfrac{3}{4}u_1 - 1\right)^2 = 1$

$\Rightarrow u_1^2 + \dfrac{9}{16}u_1^2 + 1 - \dfrac{6}{4}u_1 = 1$

$\Rightarrow \dfrac{25}{16}u_1^2 - \dfrac{6}{4}u_1 = 0 \quad \Rightarrow u_1\left(\dfrac{25}{16}u_1 - \dfrac{6}{4}\right) = 0$

$\Rightarrow u_1 = 0$ or $u_1 = \dfrac{6}{4} \times \dfrac{16}{25} = \dfrac{24}{25}$

If $u_1 = 0$, $u_2 = -1$

So one vector $\boxed{u = -\hat{j}}$

If $u_1 = \frac{24}{25}$, $u_2 = \frac{3}{4} \times \frac{24}{25} - 1 = \frac{18}{25} - 1 = \frac{-7}{25}$

Another vector $\boxed{u = \frac{24}{25}\hat{i} - \frac{7}{25}\hat{j}}$

For Both the direction vector u, $D_u f(1,-1) = 4$.

(v) Similarly as in (iv).

Ans $\boxed{u = -\hat{i}}$

$\boxed{u = \frac{7}{25}\hat{i} + \frac{24}{25}\hat{j}}$

Q.5

$f(x,y) = x^2 - 3xy + 4y^2$

$P(1,2)$

$\nabla f = (2x-3y)\hat{i} + (-3x+8y)\hat{j}$

$\nabla f\big|_{(1,2)} = -4\hat{i} + 13\hat{j}$

$|\nabla f(1,2)| = \sqrt{(-4)^2 + (13)^2} = \sqrt{16+169} = \sqrt{185} < 14$.

So $\sqrt{185}$ is the maximum rate of change.

So there is no direction u in which the rate of change of $f(x,y)$ at $P(1,2)$ equals 14.

Solution

$T(x,y) = x^2+2y^2-x$

$$\left.\begin{array}{l} T_x = 2x-1 = 0 \\ T_y = 4y = 0 \end{array}\right\} \Rightarrow y=0,\ x=\tfrac{1}{2}$$

$(\tfrac{1}{2},0)$ is interior point.

$T(\tfrac{1}{2},0) = -\tfrac{1}{4}$

On the boundary $x^2+y^2=1$

$T(x,y) = x^2+2(1-x^2)-x = -x^2-x+2$ for $-1\le x\le 1$

$\Rightarrow T'(x,y) = -2x-1=0 \Rightarrow x=-\tfrac{1}{2},\ y=\pm\tfrac{\sqrt{3}}{2}$

$T(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2}) = \tfrac{9}{4}$

$T(-\tfrac{1}{2},-\tfrac{\sqrt{3}}{2}) = \tfrac{9}{4}$

$x=-1,\ y=0$
And $T(-1,0) = 1-(-1) = 2$

$x=1,\ y=0,\quad T(1,0) = 1-1=0$

So the hottest is $\tfrac{9}{4}^{\circ}$ at $(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2})$ and at $(-\tfrac{1}{2},-\tfrac{\sqrt{3}}{2})$

the coldest is $-\tfrac{1}{4}^{\circ}$ at $(\tfrac{1}{2},0)$.