Lecture - 28

Recap:

$$E[g(X,Y)] = \iint g(x,y) f(x,y)\,dx\,dy$$

$$E[X+Y] = E[X] + E[Y]$$

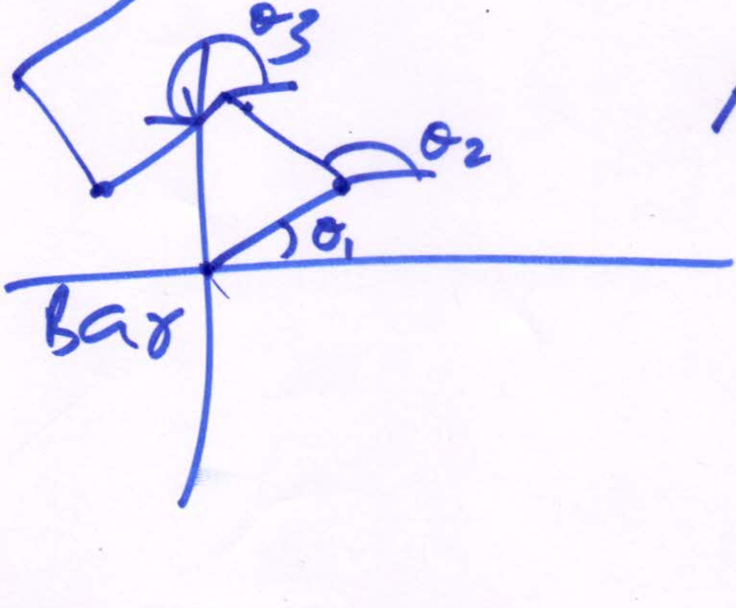
Doesn't assume independence

No. of runs of 1's

Coupon collecting problem

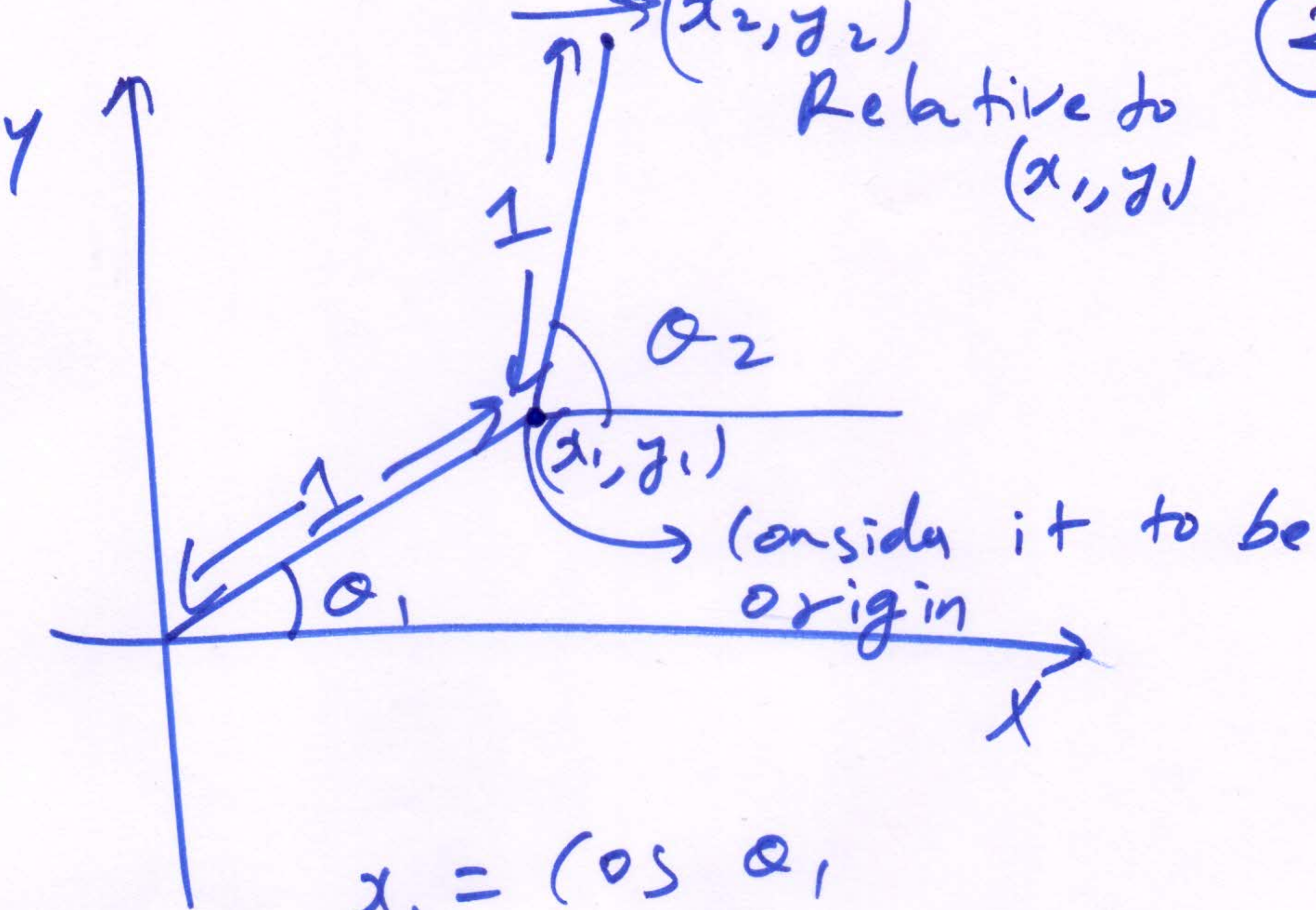
Drunkard's walk in a plane

step length = 1 unit



After n steps, he is D ~~units~~ distance away.

$E[D^2]$.



$$x_1 = \cos\theta_1$$

$$y_1 = \sin\theta_1$$

$$x_2 = \cos\theta_2$$

$$y_2 = \sin\theta_2$$

$$\vdots$$

$$x_n = \cos\theta_n$$

$$y_n = \sin\theta_n$$

$$D = \sqrt{(x_1 + x_2 + \cdots + x_n)^2 + (y_1 + y_2 + \cdots + y_n)^2}$$

3

$$D^2 = (\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_n)^2 + (\sin\theta_1 + \sin\theta_2 + \cdots + \sin\theta_n)^2$$

$$= n + \sum_{i \neq j}\sum \cos\theta_i \cos\theta_j + \sum_{i \neq j}\sum \sin\theta_i \sin\theta_j$$

independent

$$E[D^2] = n + \sum_{i \neq j}\sum E[\cos\theta_i] E[\cos\theta_j] + \sum_{i \neq j}\sum E[\sin\theta_i] E[\sin\theta_j]$$

$$\int_0^{2\pi} \cos\theta_i \, d\theta_i = 0$$

Time Complexity Analysis of Quick Sort Algorithm

3	6	1	7	8	9	5	2	10	4
5	9	3	10	11	14	8	4	17	6

$\{5, 9, 3, 10, 11, 14, 8, 4, 17, 6\}.$

$I_{3,7} = 1$ $\quad I_{1,2} = 1$ $\quad I_{4,8} = 0$

$\{5, 9, 3, 8, 4, 6\}\ 10\ \{11, 14, 17\}$

$\{5, 3, 4\}\ 6\ \{9, 8\}\ 10\ \{11\}\ 14\ \{17\}$

$3, 4, 5, 6, 8, 9, 10, 11, 14, 17$

$X =$ no. of comparisons. $= \cancel{\sum_{i \neq j} I_{ij}}$

$$I_{ij} = \begin{cases} 1 & \text{if } i \,\&\, j \text{ are compared} \\ 0 & \text{otherwise.} \end{cases}$$

index

$I_{4,9} = 0 \qquad I_{5,7} = 1$

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} I_{i,j}$$

$$E[X] = E\left[\sum \sum I_{i,j} \right]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[I_{i,j}]$$

$$I_{i,j} = \begin{cases} 1 \\ 0 \end{cases}$$

Probability that the i^{th} smallest & the j^{th} smallest numbers get compared with each other.

In the starting

$\{i, i+1, \cdots, j-1, j\}$

are in the same bracket (are together).

when will i & j get compared?

$$\frac{2}{j-i+1} = E[I_{i,j}]$$

$\{5, 9, 3, 10, 11, 14, 8, 4, 17, 6\}$

$I_{4,8}$

less than 4 $\{1, 2, 3\}$ $\{3, 4, 5\}$

$\{14, 17\}$

$$\frac{2}{5}$$

7

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$\sum_{j=i+1}^{n} \frac{2}{j-i+1} \simeq \int_{i+1}^{n} \frac{2}{x-i+1}\,dx$$

$$= 2 \log(x-i+1) \Big]_{i+1}^{n}$$

$$= 2\left[\log(n-i+1) - \log 2\right]$$

$$\sum_{i=1}^{n-1} 2\log(n-i+1) = 2\int_{1}^{n-1} \log(n-y+1)\,dy$$

$\simeq n \log n$