Corrections to Numerical Integration and Differentiation Notes

1/. Pages 13 and 14: All numerical second derivatives are to be represented as $\boxed{\mathcal{D}_h^{(2)} f(t)}$.

2/. Page 19: In the bottom part of the page, $\boxed{C = 1}$ is to be read as $\boxed{C_1 = 1}$ & $\boxed{C_2 = 2 - C_1 = 1}$.

Additional Discussions

1/. Error Consistency in the Method of Undetermined Coefficients.

i.) First Derivative: The forward difference formula is $\boxed{\mathcal{D}_h f(t) \simeq f'(t) + \frac{f''(t)\,h}{2!}}$, which we use in

$$f'(t) + \frac{f''(t)h}{2!} = (A+B) f(t) + A f'(t) h + A \frac{f''(t) h^2}{2}$$

$\therefore \boxed{(A+B) = 0}$ and $\boxed{Ah = 1}$. The second order terms with $f''(t)$ on both the L.H.S. and the R.H.S. are now $\boxed{\frac{f''(t)h}{2!} = A h^2 \frac{f''(t)}{2}}$, consistent because $\boxed{Ah = 1}$.

ii.) Second Derivative: $\boxed{\mathcal{D}_h^{(2)} f(t) \simeq f''(t) + \frac{h^2}{12} f^{(4)}(t)}$

Using which we get,

$$f''(t) + \frac{h^2}{12} f^{(4)}(t) = (A+B+C) f(t) + h(A-C) f'(t) + (A+C)\frac{h^2}{2} f''(t) + (A-C)\frac{f'''(t)h^3}{6} + \frac{(A+C)}{4!} h^4 f^{(4)}(t).$$

(P. T. O.)

Comparing L.H.S. and R.H.S. we get,

$\boxed{A+B+C=0}$, $\boxed{h(A-C)=0}$ $\Rightarrow \boxed{A=C}$.

$\boxed{(A+C)\frac{h^2}{2}=1}$ $\Rightarrow \boxed{A=1/h^2}$ Now considering the fourth-orders,

we get, $\boxed{\frac{h^2}{12}f^{(4)}(t)=\frac{(A+C)}{4!}h^4 f^{(4)}(t)}$.

Since $\boxed{\frac{(A+C)^2}{2}=1}$, we have consistency on both sides above.

2/. Gaussian Quadrature with Multiple Nodes

Given, $\boxed{I=\int_a^b f(x)\,dx=\sum_{i=1}^{n} c_i f(x_i)}$,

we note the following:

i.) The number of unknowns, weight factors c_i and nodes x_i, is $2n$, where n is number of nodes.

ii.) for odd n, $\boxed{x=0}$ is always a node.

iii.) Other nodes, whether or not n is odd or even, show a symmetric distribution of nodes about $x=0$.

iv.) The weight factors reduce in value as n increases. More nodes cover greater area.

3/. Γ functions and the Gaussian Integral

$\boxed{I=\int_{-\infty}^{\infty} e^{-x^2}\,dx=\sqrt{\pi}}$ We solve this integral by the use of Γ functions.

Define $\boxed{u = x^2} \Rightarrow du = 2x\,dx \Rightarrow \boxed{dx = \frac{du}{2\sqrt{u}}}$

Now, $I = \int_{-\infty}^{\infty} e^{-x^2}dx = 2\int_0^{\infty} e^{-x^2}dx$ (even function)

$\Rightarrow I = \cancel{2}\int_0^{\infty} e^{-u}\frac{1}{\cancel{2}}u^{-1/2}du = \int_0^{\infty} u^{1/2-1}e^{-u}du$

Now $\boxed{\Gamma(n) = \int_0^{\infty} x^{n-1}e^{-x}dx} \Rightarrow \boxed{I = \Gamma(1/2)}$ and $\boxed{\Gamma(1) = 1}$

Also $\boxed{\Gamma(n+1) = n\Gamma(n)}$, $\boxed{\Gamma(n) = (n-1)!}$ $\boxed{\Gamma(1/2) = \sqrt{\pi}}$.

Hence, $\boxed{I = \int_{-\infty}^{\infty} e^{-x^2}dx = \Gamma(1/2) = \sqrt{\pi} \simeq 1.772}$ (exact result).

4/. Points of Inflection of the Gaussian Function

$\boxed{f(x) = Ce^{-x^2/2\sigma^2}} \Rightarrow \boxed{\frac{df}{dx} = Ce^{\frac{-x^2}{2\sigma^2}} \times -\frac{\cancel{2}x}{\cancel{2}\sigma^2}}$

$\therefore \boxed{\frac{df}{dx} = -\frac{xf(x)}{\sigma^2}} \Rightarrow \frac{d^2f}{dx^2} = -\frac{1}{\sigma^2}\left[f(x) + x\frac{df}{dx}\right]$

$\Rightarrow \boxed{\frac{d^2f}{dx^2} = -\frac{f(x)}{\sigma^2}\left[1 - \frac{x^2}{\sigma^2}\right]}$ $\boxed{\frac{df}{dx} = 0}$, when $x = 0$ or $x \to \pm\infty$.

The point of inflection is for $\boxed{\frac{d^2f}{dx^2} = 0}$, when $\boxed{x = \pm\sigma}$

For $I = \int_{-\infty}^{\infty} e^{-x^2}dx$, $\boxed{2\sigma^2 = 1}$.

$\Rightarrow \boxed{\sigma \simeq \pm 0.707}$

Quadrature

Hence solving by $(|\sigma| < 1)$

$I \simeq \int_{-1}^{1} e^{-x^2}dx$ covers most of the area under the curve.