FUNDAMENTAL OF STAT. LEARN
& PATTERN RECOGNITION
CSE 569 - HOMEWORK #1

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Q1.
Let P_K be the event of prize being in hat $k = 1, 2, 3$

Hence, $P(P_K) = 1/3$

S_{ij} : The stranger opens hat 'j' after we select 'i' hat.

So, the probability for stranger to choose a hat given P_K has prize

$$P(S_{ij} \mid P_K) = \begin{cases} 0 & i = j \ (\because i \text{ can't be } j) \\ 0 & j = k \ (\because j \text{ can't have prize}) \\ 1/2 & i = k \ (\text{if 'i' had prize he has 2 choices}) \\ 1 & i \neq k,\ j \neq k \ (\text{if 'i' did not have prize he should choose the one other that doesn't hav prize}) \end{cases}$$

Using the above inference, solving for a state,

Say $i = 1$ and $j = 3$ (Stranger's hat)

Probability of 'i' having prize $P(P_1 | S_{13})$

Using Bayes theorem,

$$P(P_1 | S_{13}) = \frac{P(S_{13} | P_1)\, P(P_1)}{P(S_{13})}$$

Substituting values

$$= \frac{1/2 \times 1/3}{P(S_{13})} = \frac{1/6}{P(S_{13})} \longrightarrow (1)$$

$$P(S_{13}) = P(S_{13}\ P_1) + P(S_{13}\ P_2) + \underbrace{P(S_{13}\ P_3)}_{'0'}$$

as stranger confirms no prize

$$= P(S_{13} | P_1)\, P(P_1) + P(S_{13} | P_2)\, P(P_2)$$

$$= 1/2 \times 1/3 + 1 \times \frac{1}{3}$$

$$= 1/2$$

using $P(S_{13}) = \frac{1}{2}$ in 1

$$P(P_1 | S_{13}) = \frac{1/6}{1/2} = 2/6 = 1/3$$

if P_2 has the prize

$$P(P_2 / S_{13}) = 1 - P(p_1 | S_{13})$$ Since $j = 3$ is confirmed by stranger

$$= 1 - 1/3$$

$$= 2/3$$

$$\because P(P_2 | S_{13}) > P(P_1 / S_{13})$$

it is advised to switch hat after the stranger selects one. Since it has the higher probability for getting the prize

Question # 1 – Part B – Stranger Hat Problem Simulation in Python

```
#!/usr/bin/python3
import random

iteration = 1000

hat = ["empty","empty","prize"]

win = 0
lost = 0

# iteration for not switching
for i in range(1,iteration):
    random.shuffle(hat)
    # I pick a hat
    pick = random.randrange(3)
    # stranger picks a hat
    strangerPick = list(range(3))
    strangerPick.remove(pick)
    if hat.index("prize") in strangerPick:
        strangerPick.remove(hat.index("prize"))
        strangerPick = strangerPick[0]
    else:
        strangerPick = strangerPick[random.randrange(1)]
    if hat[pick] == "prize":
        win+=1
    else:
        lost += 1
```

```
print("Running 1000 iterations of hat selection and not switching
after stranger picks his...")

print("Did not switch hat and the percentage of success is:
{}".format(win/iteration))

print("percentage lost: {}".format(lost/iteration))


win = 0

lost = 0


# iteration for not switching

for i in range(1,iteration):

    random.shuffle(hat)

    # I pick a hat

    pick = random.randrange(3)

    # stranger picks a hat

    strangerPick = list(range(3))

    strangerPick.remove(pick)

    if hat.index("prize") in strangerPick:

        strangerPick.remove(hat.index("prize"))

        strangerPick = strangerPick[0]

    else:

        strangerPick = strangerPick[random.randrange(2)]

    prevPick = pick

    pick = list(range(3))

    pick.remove(prevPick)

    pick.remove(strangerPick)

    pick = pick[0]

    if hat[pick] == "prize":

        win+=1
```

```
    else:
        lost += 1

print("Running 1000 iterations of hat selection and switching after stranger picks his...")
print("Did not switch hat and the percentage of success is: {}".format(win/iteration))
print("percentage lost: {}".format(lost/iteration))
```

Output:

```
sdj@sdj:~/CourseAssignments/CSE569/HomeWork/HW1$ ./strangerHat.py
Running 1000 iterations of hat selection and not switching after stranger picks his...
Did not switch hat and the percentage of success is: 0.322
percentage lost: 0.677
Running 1000 iterations of hat selection and switching after stranger picks his...
Did not switch hat and the percentage of success is: 0.627
percentage lost: 0.372
```

Q2. Multivariate Normally distributed classes,

$$p(x|\omega_i) = \frac{1}{(2\pi)^{d/2} |\Sigma_i|^{1/2}} \exp\left[-\frac{1}{2}(x-\mu_i)^t \Sigma_i^{-1} (x-\mu_i)\right]$$

for $i = 1, 2$

given: $\Sigma_1 = \Sigma_2 = \Sigma$

and $\lambda_{12} = 2 \quad \lambda_{21} = 1 \quad \lambda_{11} = \lambda_{22} = 0$

According to conditional risk eqn:

$$(\lambda_{21} - \lambda_{11})\, p(x|\omega_1)\, P(\omega_1) > (\lambda_{12} - \lambda_{22})\, p(x|\omega_2)\, P(\omega_2)$$

so, $g_1(x) = (\lambda_{21} - \lambda_{11})\, p(x/\omega_1)\, P(\omega_1)$

$g_2(x) = (\lambda_{12} - \lambda_{22})\, p(x/\omega_2)\, P(\omega_2)$

substituting values,

$g_1(x) = p(x/\omega_1)\, P(\omega_1)$

$g_2(x) = 2\, p(x/\omega_2)\, P(\omega_2)$

Since in our case we have $\Sigma_1 = \Sigma_2 = \Sigma$

we have $g_i(x)$ in the form

$$g_i(x) = -\frac{1}{2}(x-\mu_i)^T \Sigma^{-1}(x-\mu_i) + \ln P(\omega_i)$$

$$\Rightarrow g_i(x) = (\Sigma^{-1}\mu_i)^T x - \frac{1}{2}\mu_i^T \Sigma^{-1}\mu_i + \ln P(\omega_i)$$

Rewriting this equation for ω_1 & ω_2 with loss factors,

$$g_1(x) = (\Sigma^{-1}\mu_1)^T x - \frac{1}{2}\mu_1^T \Sigma^{-1}\mu_1 + \ln P(\omega_1)$$

$$g_2(x) = 2(\Sigma^{-1}\mu_2)^T x - \mu_2^T \Sigma^{-1}\mu_2 + 2\ln P(\omega_2)$$

To derive eqn of decision boundary

$$g_1(x) - g_2(x) = 0$$

$$(\Sigma^{-1}\mu_1)^T x - \frac{1}{2}\mu_1^T \Sigma^{-1}\mu_1 + \ln P(\omega_1)$$
$$- 2(\Sigma^{-1}\mu_2)^T x + \mu_2^T \Sigma^{-1}\mu_2 - \ln[P(\omega_2)]^2 = 0$$

$$[(\Sigma^{-1}\mu_1)^T - 2(\Sigma^{-1}\mu_2)^T]x$$
$$- \frac{1}{2}\mu_1^T \Sigma^{-1}\mu_1 + \mu_2^T \Sigma^{-1}\mu_2 + \ln\left(\frac{P(\omega_1)}{[P(\omega_2)]^2}\right) = 0$$

$$[\{\Sigma^{-1}(\mu_1 - 2\mu_2)\}^T]x$$
$$- \frac{1}{2}\mu_1^T \Sigma^{-1}\mu_1 + \mu_2^T \Sigma^{-1}\mu_2 + \ln\left(\frac{P(\omega_1)}{[P(\omega_2)]^2}\right) = 0$$

This is the equation of the decision boundary which is a hyperplane and linear to x.

Q3. 1 - Dimensional 2-class classifier

$$p(x|\omega_1) = \begin{cases} \frac{-x+1}{2} & x \in [-1,1] \\ 0 & \text{else} \end{cases}$$

$$p(x|\omega_2) = \begin{cases} \frac{x+1}{2} & x \in [-1,1] \\ 0 & \text{else} \end{cases}$$

a) $P(\omega_1) = P(\omega_2) = 0.5$

According to Bayes decision rule, decide ω_1

$P(\omega_1|x) > P(\omega_2|x)$

$\Rightarrow \quad p(x|\omega_1)\, P(\omega_1) > p(x|\omega_2)\, P(\omega_2) \rightarrow (1)$

Substituting given values.

$\Rightarrow \quad \frac{-x+1}{2}(0.5) > \frac{x+1}{2}(0.5)$

$\Rightarrow \quad -x+1 > x+1$

$\Rightarrow \quad 0 > x$

Hence $x = 0$ is the decision boundary.

and decide ω_1 if $x < 0$ else ω_2.

Computing Bayes Error:

$$P(\text{error}) = \int_{-1}^{1} P(\text{error}|x)\, p(x)\, dx \longrightarrow (2)$$

Based on decision boundary

$$= \underbrace{\int_{-1}^{0} P(\text{error}|x)\, p(x)\, dx}_{\text{deciding } \omega_1} + \underbrace{\int_{0}^{1} P(\text{error}|x)\, p(x)\, dx}_{\text{deciding } \omega_2}$$

$$= \int_{-1}^{0} P(\omega_2|x)\, p(x)\, dx + \int_{0}^{1} P(\omega_1|x)\, p(x)\, dx$$

Applying Bayes rule,

$$= \int_{-1}^{0} \frac{p(x|\omega_2)\, P(\omega_2)}{\cancel{p(x)}} \cancel{p(x)}\, dx + \int_{0}^{1} \frac{p(x|\omega_1)\, P(\omega_1)}{\cancel{p(x)}} \cancel{p(x)}\, dx$$

$$= \int_{-1}^{0} P(\omega_2) \left\{ \frac{x+1}{2} \right\} dx + \int_{0}^{1} P(\omega_1) \left\{ \frac{-x+1}{2} \right\} dx$$

$$= 0.5 \left[\frac{x^2}{4} + \frac{x}{2} \right]_{-1}^{0} + 0.5 \left[\frac{x}{2} - \frac{x^2}{4} \right]_{0}^{1} = 0.25$$

b) $P(\omega_1) = 0.7$ $P(\omega_2) = 0.3$

using eqn (1) decide ω_1 if,

$$p(x|\omega_1)\, P(\omega_1) > p(x|\omega_2)\, P(\omega_2)$$

$$0.7\left[\frac{-x+1}{2}\right] > 0.3\left[\frac{x+1}{2}\right]$$

$$\frac{7}{10}\left[\frac{-x}{2}+\frac{1}{2}\right] > \frac{3}{10}\left[\frac{x}{2}+\frac{1}{2}\right]$$

$$\frac{-7x}{20}+\frac{7}{20} > \frac{3x}{20}+\frac{3}{20}$$

$$\Rightarrow \frac{7}{20}-\frac{3}{20} > \frac{7x}{20}+\frac{3x}{20} \Rightarrow \frac{4}{\cancel{20}} > \frac{10x}{\cancel{20}}$$

$$\Rightarrow \frac{2}{5} > x$$

Hence, $x = 2/5$ is the decision boundary

and decide ω_1 if $x < 2/5$ else ω_2

Computing Bayes error:

using eqn (2)

$$P(\text{error}) = \int_{-1}^{1} P(\text{error}|x)\, p(x)\, dx$$

$$= \int_{-1}^{2/5} P(\omega_2|x)\, p(x)\, dx + \int_{2/5}^{1} P(\omega_1|x)\, p(x)\, dx$$

using Bayes rule

$$= \int_{-1}^{2/5} p(x|\omega_2)\, P(\omega_2)\, dx + \int_{2/5}^{1} p(x|\omega_1)\, P(\omega_1)\, dx$$

$$= \frac{3}{10}\left[\frac{x^2}{4} + \frac{x}{2}\right]_{-1}^{2/5} + \frac{7}{10}\left[\frac{x}{2} - \frac{x^2}{4}\right]_{2/5}^{1}$$

$$= \frac{3}{10}\left[\frac{49}{100}\right] + \frac{7}{10}\left[\frac{9}{100}\right] = 0.210$$

c) $P(\omega_1) = P(\omega_2) = 0.5$

Accounting loss as follows,

$\lambda_{21} = 1 \quad \lambda_{12} = 2 \quad \lambda_{11} = \lambda_{22} = 0$

According to conditional risk eqn

decide ω_1 if

$(\lambda_{21} - \lambda_{11})\, P(\omega_1|x) > (\lambda_{12} - \lambda_{22})\, P(\omega_2|x)$

$\Rightarrow \lambda_{21}\, p(x|\omega_1)\, P(\omega_1) > \lambda_{12}\, p(x|\omega_2)\, P(\omega_2)$

$\Rightarrow p(x|\omega_1) > 2\, p(x|\omega_2)$

$\Rightarrow \frac{-x}{2} + 1/2 > 2\left(\frac{x}{2} + \frac{1}{2}\right)$

$\Rightarrow -1 + 1/2 > x + x/2$

$\Rightarrow -1/3 > x$

Hence, $x = -1/3$ is the decision boundary

Conditional Risk's can be given by,

$$R(\alpha_1|x) = \lambda_{11} P(\omega_1|x) + \lambda_{12} P(\omega_2|x)$$

$$R(\alpha_2|x) = \lambda_{21} P(\omega_1|x) + \lambda_{22} P(\omega_2|x)$$

substituting values of "λ"

$$R(\alpha_1|x) = 2P(\omega_2|x) \quad R(\alpha_2|x) = P(\omega_1|x)$$

and the overall risk is given by

$$R = \int R(\alpha(x)|x)\, p(x)\, dx$$

$$= \int_{-1}^{-1/3} R(\alpha_1|x)\, p(x)\, dx + \int_{-1/3}^{1} R(\alpha_2|x)\, p(x)\, dx$$

$$= 2\int_{-1}^{-1/3} P(\omega_2|x)\, p(x)\, dx + \int_{-1/3}^{1} P(\omega_1|x)\, p(x)\, dx$$

$$= 2\int_{-1}^{-1/3} p(x|\omega_2)\, P(\omega_2)\, dx + \int_{-1/3}^{1} p(x|\omega_1)\, P(\omega_1)\, dx$$

$$= \left[\frac{x^2}{4} + \frac{x}{2}\right]_{-1/3}^{1} + \frac{1}{2}\left[\frac{x}{2} - \frac{x^2}{4}\right]_{-1/3}^{+1} = \frac{10}{9}$$

$$\boxed{R = 10/9}$$

Q4.

$$X \longrightarrow Y \longrightarrow Z \longrightarrow W$$

$P(x_1) = 0.60$

$P(y_1 | x_1) = 0.40$ $\quad$ $P(z_1 | y_0) = 0.60$

$P(y_1 | x_0) = 0.30$ $\quad$ $P(w_1 | z_1) = 0.45$

$P(z_1 | y_1) = 0.25$ $\quad$ $P(w_1 | z_0) = 0.30$

(a) find $P(w_0 | x_1)$

finding z occurs with either z_0 (or) z_1

when x_1 happens

so, $P(z_1 | x_1)$

$= P(z_1 | y_1) P(y_1 | x_1) + P(z_1 | y_0) P(y_0 | x_1)$

$= (0.25 \times 0.40) + (0.60 \times 0.60)$

$P(z_1 | x_1) = 0.46$

$P(z_0 | x_1) = P(z_0 | y_1) P(y_1 | x_1)$

$+ P(z_0 | y_0) P(y_0 | x_1)$ (or)

$= 1 - P(z_1 | x_1)$

$P(z_0 | x_1) = 0.54$

$$P(w_0 | x_1) = P(w_0 | z_1)\, P(z_1 | x_1) + P(w_0 | z_0)\, P(z_0 | x_1)$$

$$= (1 - P(w_1 | z_1)) \times 0.46 + (1 - P(w_1 | z_0)) \times 0.54$$

$$= (1 - 0.45) \times 0.46 + (1 - 0.30) \times 0.54$$

$$= 0.6255$$

$$\boxed{P(w_0 | x_1) = 0.6255}$$

(b) $P(x_0 | w_1)$, find the value

According to Bayes theorem

$$P(x_0 | w_1) = \frac{P(w_1 | x_0)\, P(x_0)}{P(w_1)} \rightarrow (A)$$

w_1 occurs based on the value of Z which has 2 states Z_0 & Z_1

Hence, generalizing to get the value

$$P(W_1) = \sum_{Z \in \{0,1\}} P(W_1, Z)$$

$$= \sum_{Z \in \{0,1\}} P(W_1 | Z)\, P(Z)$$

$$= P(W_1 | Z_0)\, P(Z_0) + P(W_1 | Z_1)\, P(Z_1)$$

$$P(W_1) = (0.30 \times P(Z_0)) + (0.55 \times P(Z_1)) \rightarrow 1$$

finding $P(Z_0)$ & $P(Z_1)$

$$P(Z_0) = \sum_{Y \in \{0,1\}} P(Z_0, Y)$$

$$= \sum_{Y \in \{0,1\}} P(Z_0 | Y)\, P(Y)$$

$$= P(Z_0 | Y_0)\, P(Y_0) + P(Z_0 | Y_1)\, P(Y_1)$$

$$= (0.4 \times P(Y_0)) + (0.75 \times P(Y_1)) \rightarrow 2$$

$$P(Z_1) = \sum_{Y \in \{0,1\}} P(Z_1, Y)$$

$$= (0.25 \times P(Y_1)) + (0.60 \times P(Y_0)) \to (3)$$

$$P(Y_0) = \sum_{X \in \{0,1\}} P(Y_0, X)$$

$$= (0.7 \times P(X_0)) + (0.6 \times P(X_1))$$

$$= (0.7 \times 0.4) + (0.6 \times 0.6)$$

$$P(Y_0) = 0.64 \to (4)$$

$$P(Y_1) = \sum_{X \in \{0,1\}} P(Y_1, X)$$

$$= (0.4 \times P(X_1)) + (0.3 \times P(X_0))$$

$$= (0.4 \times 0.6) + (0.3 \times 0.4)$$

$$P(Y_1) = 0.36 \to (5)$$

using (4) & (5) in (3) & (2)

$$P(Z_1) = 0.474 \qquad P(Z_0) = 0.526$$

substituting these values in eqn ①

$P(W_1) = 0.4185 \longrightarrow$ ⑥

finding $P(W_1 | x_0)$:

$P(Z_1 | x_0) = P(Z_1 | y_1)\, P(y_1 | x_0) + P(Z_1 | y_0)\, P(y_0 | x_0)$

$= (0.25 \times 0.30) + (0.6 \times 0.7)$

$P(Z_1 | x_0) = 0.495$

$P(Z_0 | x_0) = P(Z_0 | y_1)\, P(y_1 | x_0) + P(Z_0 | y_0)\, P(y_0 | x_0)$

$= (0.75 \times 0.30) + (0.4 \times 0.7)$

$P(Z_0 | x_0) = 0.505$

$P(W_1 | x_0) = P(W_1 | Z_0)\, P(Z_0 | x_0) + P(W_1 | Z_1)\, P(Z_1 | x_0)$

$$= (0.30 \times 0.505) + (0.45 \times 0.495)$$

$$= 0.37425 \rightarrow (7)$$

So, substituting values in eqn (A)

$$P(x_0|\omega_1) = \frac{P(w_1|x_0)\,P(x_0)}{P(w_1)}$$

$$= \frac{0.37425 \times 0.4}{0.4185}$$

$$\boxed{P(x_0|w_1) = 0.3577}$$

Q5. If the underlying class-conditionals are Gaussian densities, then decision boundary is linear (hyperplanes).

False

Explanation :

The decision boundary can take any form of hyperquadrics - hyperplanes, pair of hyperplanes, hypersphere, hyperellipsoid hyperparaboloids . And this depends on the Gaussian distribution, if the Gaussian distribution is arbitrary (each category having different co-variance matrix in multivariate Normal density) then it can lead to bayes decision boundary that are any of the above mentioned hyperquadric form .