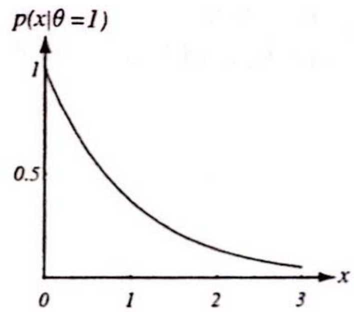
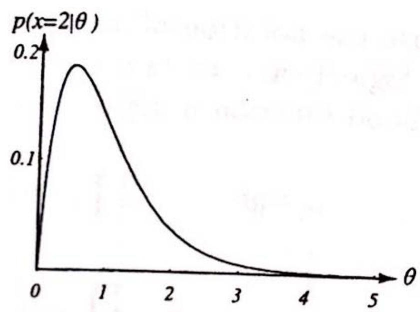
CSE 569 Homework #2: Outline for the solutions

Problem 1.

(1) The plot is roughly like



(2) The plot is roughly like



(3)

The log likelihood is

$$l(\theta) = \sum \log p(x_k \mid \theta) = n \log \theta - \theta \sum x_k$$

Differentiate w.r.t θ, and set to zero, we got the solution as

$$\hat{\theta} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i}$$

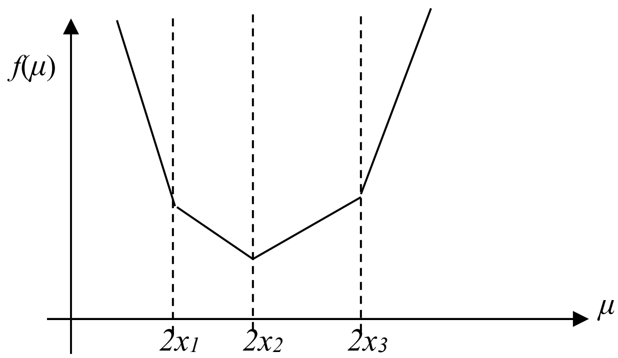
Problem 2.

(a) By setting $\int_{-\infty}^{\infty} p(x)dx = 1,$ we will find c=1. (In solving the integral, you will need to do it on two intervals, from $-\infty$ to $\mu/2$, and from $\mu/2$ to ∞, respectively, to get rid of the absolute sign in $|2x-\mu|$.)

(b) The likelihood of the data under μ is $p(D|\mu) = p(x_1)p(x_2)p(x_3)$,and the log-likelihood is

$$l(\mu) = \log(p(x_1)p(x_2)p(x_3)) = -(|2x_1 - \mu| + |2x_2 - \mu| + |2x_3 - \mu|)$$

The MLE solution requires us to find a μ such that $l(\mu)$ is maximized, or equivalently $(|2x_1-\mu|+|2x_2-\mu|+|2x_3-\mu|)$ is minimized. Note that you will not be able to take derivative due to the absolute sign ($f(\mu) = |2x_1-\mu|+|2x_2-\mu|+|2x_3-\mu|$ is not differentiable for all μ). So we will have to go to the basic definition of MLE and find a μ that minimizes $f(\mu)$. A careful look at $f(\mu)$ tells us that it is a piece-wise linear function of μ, as illustrated below (where without loss of generality we assume $x_1 < x_2 < x_3$)



Therefore, the MLE of μ is $2x_2$, or in general, 2 times the middle one in the given 3 points if we sort them first.

Problem 3.

(1) For just 1 data point, the likelihood function is just $\dfrac{\lambda^{x_1}}{x_1!}e^{-\lambda}$

Take the partial derivative, and then set to 0, we will find the MLE is just x_1.

(Further discussion on Part (1): In general, with n samples, the MLE is the sample mean. n=1 is the special case with only one observation x_1, but in this special case, if $x_1$happen to be 0, then MLE is not properly defined, since we would have 0 as an estimate but the Poisson distribution assumes a $\lambda>0$. In general, if the sample mean happen to be 0, the MLE is not properly define.)

(2) P is the Poisson distribution. $f(\lambda|D)=$

$$P(D\,|\,\lambda)f(\lambda)/\int P(D\,|\,\lambda)f(\lambda)d\lambda = \ldots$$

See more detail on the next two pages.

Problem 4. (No solution for Problem 4.)

(2) This part needs some extra efforts on getting the results right, although it is easy to write down the initial steps.

Let's first compute $f(\lambda|D)$,

$$f(\lambda|D) = \frac{P(D|\lambda)f(\lambda)}{\int P(D|\lambda) f(\lambda)\, d\lambda}, \quad \Leftarrow P(D|\lambda) = \frac{\lambda^x}{x!} e^{-\lambda}$$

(From Part (1))

So if we can get $W \triangleq \int P(D|\lambda) f(\lambda) d\lambda$, we can simply plug W, $P(D|\lambda)$, and $f(\lambda)$ into the above expression to get $f(\lambda|D)$.

With that, finding the mean of $f(\lambda|D)$ can be solved by $\int \lambda f(\lambda|D) d\lambda$.

It is conceptually easy & clear to this piont.

Now, let's do the actually computation of W and $\int \lambda f(\lambda|D) d\lambda \triangleq E(\lambda|D)$

$$W = \int P(D|\lambda) f(\lambda) d\lambda = \int_0^\infty \frac{\lambda^x}{x!} e^{-\lambda} \cdot e^{-\lambda}\, d\lambda = \int_0^\infty \frac{\lambda^x e^{-2\lambda}}{x!} d\lambda$$

$$= \int_0^\infty \frac{(2\lambda)^x e^{-2\lambda}}{2^x\, x!} d\lambda \underset{\uparrow}{=} \frac{1}{2^{x+1}} \int_0^\infty \frac{t^x e^{-t}}{x!} dt$$

Change of Variable: $t = 2\lambda$

Now, we'll need to know the integral is actually for a Gamma distribution with parameters $\{k = x+1, \theta = 1\}$, and thus the integral $= 1$.

e.g. see wikipedia.

$$\therefore W = \frac{1}{2^{x+1}}$$

$$\therefore f(\lambda|D) = 2^{x+1} \frac{\lambda^x}{x!} e^{-2\lambda}, \quad \lambda > 0.$$

Now, we need to compute $E[\lambda|D] \triangleq \int_0^\infty \lambda f(\lambda|D) d\lambda$

$$E(\lambda|D) = \int_0^\infty \lambda f(\lambda|D)\, d\lambda$$

$$= \int_0^\infty 2^{x+1} \frac{\lambda^{x+1}}{x!} e^{-2\lambda}\, d\lambda$$

This appears again hard to compute, but we will again use the same trick of Gamma distribution to help,

$$= \int_0^\infty \left(\frac{1}{2}\right)(x+1) \underbrace{\frac{\lambda^{x+1} e^{-2\lambda}}{(x+1)!\left(\frac{1}{2}\right)^{x+2}}} d\lambda$$

Looking at this part, it is a Gamma distribution with parameter $\{k = x+2 \ \& \ \theta = \frac{1}{2}\}$, and thus

$$= \underline{\frac{x+1}{2}}, \quad \text{(since the rest integrates to 1)}.$$

As you can see, now, even if we have only one data point x which happens to be 0, we will get a meaningful estimate of $\frac{0+1}{2} = \frac{1}{2}$. (This was shown not properly defined for MLE in Part (1)).