Stable simulation of a lumped element rod

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Abstract

1 Introduction

2 Lumped element model

We consider a one dimensional rod with N elements of identical mass m and coordinate θ, coupled with spring-damper forces

$$\tau^{(i,i+1)} = -K(\theta^{(i)} - \theta^{(i+1)}) - \gamma(\omega^{(i)} - \omega^{(i+1)}), \text{ and } \tau^{(i+1,i)} = -\tau^{(i,i+1)}. \quad (1)$$

We are particularly interested in the case where K is arbitrarily large, even including $K = \infty$, which justifies the choice of implicit integration below.

The rod can also be forced externally with $\tau^{(1)}$ and $\tau^{(N)}$. In addition to this, it can be subjected to force-velocity coupling with external dynamical systems, assuming these are physical. This is done by adding input spring-dampers at either end which produce forces according to

$$\tau^{(c,i)} = -K^{(c)}(\theta^{(i)} - \bar{\theta}^{(c,i)}) - \gamma^{(c)}(\omega^{(i)} - \bar{\omega}^{(c,i)}) \quad (2)$$

where $(\bar{\cdot})$ are either last known quantities, or extrapolations. We also consider the case where only the velocity $\bar{\omega}^{(c,i)}$ is known in which case we have

$$\begin{aligned} \tau^{(c,i)} &= -K^{(c)}\delta\theta^{(i)} - \gamma^{(c)}(\omega^{(i)} - \bar{\omega}^{(c,i)}) \text{ with} \\ \dot{\delta\theta}^{(i)} &= \omega^{(i)} - \bar{\omega}^{(e,i)}. \end{aligned} \tag{3}$$

This coupling force is reported to the coupled element of course. Given the input and output signals here, this rod can be coupled to other subsystems either kinematically, or via force-velocity or even force-displacements couplings.

Given stiffness requirements, we use a stable integrator, namely, Spook. To get maximum stability, we first introduce the matrix $\bar{G}$ which contains internal, pairwise interactions as

$$\bar{G} = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ \vdots & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & 1 & -1 \end{bmatrix}. \tag{4}$$

Given its form, the implication here is that $G\theta = 0$ is the unique rest configuration, where

$$\theta = \left[\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}\right]^T. \tag{5}$$

Henceforth, quantities without superscripts are understood to be vectors or matrices. We can therefore associate a potential energy

$$U = \frac{1}{2}\theta^T \bar{G}^T K \bar{G}\theta. \tag{6}$$

If we include the external forces due drivers, again, anticipating very high stiffness, we need two additional rows so that

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ \vdots & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & 1 & -1 \\ \vdots & \vdots & \vdots & 0 & 1 \end{bmatrix}. \tag{7}$$

Under infinite stiffness, these additional rows impose the constraint

$$\omega^{(i)} = \bar{\omega}^{(e,i)}. \tag{8}$$

FiXme Note: The form of the matrix in Eqn. (7) isn't implemented as an option yet meaning that it isn't possible to drive the rod with a hard kinematic driver. The reason being that we would need a special flag to

FiXme Note: no nonholonomic coupling now

choose which form the system matrix takes. This can also be modified to include the case of displacement-displacement coupling as described further below, as well as the case of high but finite stiffness for coupling.

After a Legendre transform and discretization with fixed step h and discrete time k we arrive at

$$\begin{aligned} \begin{bmatrix} \mathcal{J} & -G^T \\ G & \tilde{K}^{-1} \end{bmatrix} \begin{bmatrix} \omega_{k+1} \\ \lambda \end{bmatrix} &= \begin{bmatrix} \mathcal{J}\omega_k + h\tau \\ -\frac{4}{h}\gamma G\theta_k + \gamma G\omega_k \end{bmatrix} \\ \theta_{k+1} &= \theta_k + h\omega_k \end{aligned} \tag{9}$$

where $\mathcal{J}$ is the (diagonal) inertia matrix and

$$\begin{aligned} \gamma &= \frac{1}{1 + 4\rho/h} \text{ and } \rho \text{ is a relaxation rate} \\ \tilde{K} &= \frac{h^2 K^{(i)}}{4\gamma} \end{aligned} \tag{10}$$

The ρ parameter has units of inverse time and describes a relaxation rate, i.e., damping, and ρ/h is the number of steps it takes to loose half the excitation amplitude locally. When this is set to $\rho/h = 2$, the energy dissipates rapidly, and this provides unconditional stability via numerical damping. However, as soon as we reach $\rho/h < 1/10$ or so, we have natural damping.

We can of course eliminate the Lagrange multipliers λ and solve instead

$$\left[\mathcal{J} + G^T\tilde{K}G\right]\omega_{k+1} = \mathcal{J}\omega_k + h\tau - \frac{4\gamma}{h}G^T\tilde{K}G\theta_k + \gamma G^T\tilde{K}G\omega_k. \tag{11}$$

where $\tilde{K}$ is defined as before in Eqn. (10). This requires finite K however. But we can mix the two methods and treat the external forcing terms using the relevant part of Eqn. (11) and the rest with Eqn. (9).

3 Coupling details

Coupling to other simulations comes in various forms. A pure torque coupling simply adds $h\tau^{(i,c)}$ to the RHS in Eqn. (9). A velocity or displacement coupling adds $\frac{h^2}{4\gamma}K^{(i,c)}$ to the diagonal as well as

$$h\tau^{(i,c)} = -hK^{(i,c)}(\theta^{(i)} - \bar{\theta}^{(i,c)}) + \frac{h^2}{4}(\omega^{(i)} - \bar{\omega}^{(i)}) \tag{12}$$

or

$$h\tau^{(i,c)} = -K^{(i,c)}\delta\theta^{(i)} + \frac{h^2}{4}(\omega^{(i)} - \bar{\omega}^{(i)}) \tag{13}$$

on the RHS of Eqn. (9). For the latter case, we update $\delta\theta^{(i)}$ with

$$\delta\theta^{(i)}_{k+1} = \delta\theta^{(i)}_k + h(\omega^{(i)}_{k+1} - \bar{\omega}^{(i,c)}). \tag{14}$$

Output variables.

Name	Symbol	Variable	Value ref.	Default	Causality
First angle	θ_1	`theta1`	0		
Last angle	θ_2	`theta2`	1		
First angular speed	ω_1	`omega1`	2		
Last angular speed	ω_2	`omega2`	3		
First angular acceleration	α_1	`alpha1`	4		
Last angular acceleration	α_2	`alpha2`	5		
First angle difference	$\delta\theta_1$	`dtheta1`	6		
Last angle difference	$\delta\theta_2$	`dtheta2`	7		
First output torque	$\tau^{(c,1)}$	`out_torque1`	8		
Last output torque	$\tau^{(c,N)}$	`out_torque2`	9		

Inputs.

Name	Symbol	Variable	Value ref.	Default	Causality
First input torque	$\tau^{(1)}$	`tau1`	10		
Last input torque	$\tau^{(N)}$	`tau2`	11		
First driving displacement	$\bar{\theta}^{(c,1)}$	`theta_drive1`	12		
First driving velocity	$\bar{\omega}^{(c,1)}$	`omega_drive1`	13		
Last driving displacement	$\bar{\theta}^{(c,N)}$	`theta_drive2`	14		
Last driving velocity	$\bar{\omega}^{(c,N)}$	`omega_drive2`	15		

Parameters.

Name	Symbol	Variable	Value ref.	Default	Causality
Moment of inertia	$\mathcal{J}$	`J`	16		
Compliance	K^{-1}	`compliance`	17		
Damping rate	ρ/h	`D`	18		
First driving spring	$K^{(c,1)}$	`K_drive1`	19		
First driving damping	$\gamma^{(c,1)}$	`D_drive1`	20		
Last driving spring	$K^{(c,2)}$	`K_drive2`	21		
Last driving damping	$\gamma^{(c,2)}$	`D_drive2`	22		
Time step	h	`step`	23		
Number of elements	N	`n_elements`	24		
First initial angle	$\theta_0^{(1)}$	`theta01`	25		
Last initial angle	$\theta_0^{(N)}$	`theta02`	26		
First initial angular speed	$\omega_0^{(1)}$	`omega01`	27		
Last initial angular speed	$\omega_0^{(2)}$	`omega02`	28		
Sign for the first driver	$\sigma^{(1)}$	`driver_sign1`	29		
Sign for the last driver	$\sigma^{(N)}$	`driver_sign2`	30		
Integrate angle difference 1		`integrate_dt1`	31		
Integrate angle difference N		`integrate_dt2`	32		

4 Conclusion

Acknowledgments

References