

Stable simulation of a lumped element rod

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Abstract

1 Introduction

2 Lumped element model

We consider a one dimensional rod with N elements of identical mass m and coordinate θ , coupled with spring-damper forces

$$\tau^{(i,i+1)} = -K(\theta^{(i)} - \theta^{(i+1)}) - \gamma(\omega^{(i)} - \omega^{(i+1)}), \text{ and } \tau^{(i+1,i)} = \tau^{(i,i+1)}. \quad (1)$$

We are particularly interested in the case where K is arbitrarily large, even including $K = \infty$, which justifies the choice of implicit integration below.

The rod can also be forced externally with $\tau^{(1)}$ and $\tau^{(N)}$. In addition to this, it can be subjected to force-velocity coupling with external dynamical systems, assuming these are physical. This is done by adding input spring-dampers at either end which produce forces according to

$$\tau^{(c,i)} = -K^{(c)}(\theta^{(i)} - \bar{\omega}^{(c,i)}) - \gamma^{(c)}(\omega^{(i)} - \bar{\omega}^{(c,i)}) \quad (2)$$

where $(\bar{\cdot})$ are either last known quantities, or extrapolations. We also consider the case where only the velocity $\bar{\omega}^{(c,i)}$ is known in which case we have

$$\begin{aligned}\tau^{(c,i)} &= -K^{(c)}\delta\theta^{(i)} - \gamma^{(c)}(\omega^{(i)} - \bar{\omega}^{(c,i)}) \text{ with} \\ \delta\theta^{(i)} &= \omega^{(i)} - \bar{\omega}^{(e,i)}.\end{aligned}\tag{3}$$

This coupling force is reported to the coupled element of course. Given the input and output signals here, this rod can be coupled to other subsystems either kinematically, or via force-velocity or even force-displacements couplings.

Given stiffness requirements, we use a stable integrator, namely, SPOOK. To get maximum stability, we first introduce the matrix \bar{G} which contains internal, pairwise interactions as

$$\bar{G} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ -1 & 1 & 0 & \dots \\ 0 & -1 & 1 & \vdots \\ \vdots & \dots & \ddots & \ddots \\ 0 & \dots & \dots & -1 & 1 \end{bmatrix}.\tag{4}$$

Given its form, the implication here is that $G\theta = 0$ is the unique rest configuration. Henceforth, quantities without superscripts are understood to be vectors or matrices. We can therefore associate a potential energy

$$U = \frac{K}{2}\theta\bar{G}^T\bar{G}\theta.\tag{5}$$

If we include the external forces due drivers, again, anticipating very high stiffness, we need two additional rows so that

$$G = \begin{bmatrix} 1 & 0 & 0 & \dots \\ -1 & 1 & 0 & \dots \\ 0 & -1 & 1 & \vdots \\ \vdots & \dots & \ddots & \ddots \\ 0 & \dots & \dots & -1 & 1 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}\tag{6}$$

Under infinite stiffness, these additional rows impose the constraint

$$\omega^{(i)} = \bar{\omega}^{(e,i)}.\tag{7}$$

This can also be modified to include the case of displacement-displacement coupling as described further below, as well as the case of high but finite stiffness for coupling.

After a Legendre transform and discretization with fixed step h and discrete time k we arrive at

$$\begin{aligned}\begin{bmatrix} M & -G^T \\ G & \tilde{K}^{-1} \end{bmatrix} \begin{bmatrix} \omega_{k+1} \\ \lambda \end{bmatrix} &= \begin{bmatrix} M\omega_k + hf \\ -\frac{4}{h}\gamma Gx_k + \gamma Gv_k \end{bmatrix} \\ x_{k+1} &= x_k + hv_k\end{aligned}\tag{8}$$

where M is the (diagonal) mass matrix and

$$\rho = \frac{1}{1 + 4\tau/h} \text{ and } \tau \text{ is a relaxation rate} \quad (9)$$

$$\tilde{K} = \frac{h^2 K^{(i)}}{4\rho}$$

The τ parameter has units of inverse time and describes a relaxation rate, i.e., damping, and τ/h is the number of steps it takes to loose half the excitation amplitude locally. When this is set to $\tau/h = 2$, the energy dissipates rapidly, and this provides unconditional stability via numerical damping. However, as soon as we reach $\tau/h < 1/10$ or so, we have natural damping.

We can of course eliminate the Lagrange multipliers λ and solve instead

$$\left[M + G^T \tilde{K} G \right] v_{k+1} = M v_k + h f - \frac{4\gamma}{h} G^T \tilde{K} G x_k + \gamma G^T \tilde{K} G v_k. \quad (10)$$

where \tilde{K} is defined as before in Eqn. (9). This requires finite K however. But we can mix the two methods and treat the external forcing terms using the relevant part of Eqn. (10) and the rest with Eqn. (8).

Output variables.

Name	Symbol	Variable	Value ref.	Default	Causality
First angle	θ_1	theta1	0		
Last angle	θ_2	theta2	1		
First angular speed	ω_1	omega1	2		
Last angular speed	ω_2	omega2	3		
First angular acceleration	α_1	alpha1	4		
Last angular acceleration	α_2	alpha2	5		
First angle difference	$\delta\theta_1$	dtheta1	6		
Last angle difference	$\delta\theta_2$	dtheta2	7		
First output torque	$\tau^{(c,1)}$	out_torque1	8		
Last output torque	$\tau^{(c,N)}$	out_torque2	9		

Inputs.

Name	Symbol	Variable	Value ref.	Default	Causality
First input torque	$\tau^{(1)}$	<code>tau1</code>	10		
Last input torque	$\tau^{(N)}$	<code>tau2</code>	11		
First driving displacement	$\bar{\theta}^{(c,1)}$	<code>theta_drive1</code>	12		
First driving velocity	$\bar{\omega}^{(c,1)}$	<code>omega_drive1</code>	13		
Last driving displacement	$\bar{\theta}^{(c,N)}$	<code>theta_drive2</code>	14		
Last driving velocity	$\bar{\omega}^{(c,N)}$	<code>omega_drive2</code>	15		

Parameters.

Name	Symbol	Variable	Value ref.	Default	Causality
Moment of inertia	\mathcal{J}	J	16		
Compliance	K^{-1}	compliance	17		
Damping rate	τ/h	D	18		
First driving spring	$K^{(c,1)}$	K_drive1	19		
First driving damping	$\gamma^{(c,1)}$	D_drive1	20		
Last driving spring	$K^{(c,2)}$	K_drive2	21		
Last driving damping	$\gamma^{(c,2)}$	D_drive2	22		
Time step	h	step	23		
Number of elements	N	n_elements	24		
First initial angle	$\theta_0^{(1)}$	theta01	25		
Last initial angle	$\theta_0^{(N)}$	theta02	26		
First initial angular speed	$\omega_0^{(1)}$	omega01	27		
Last initial angular speed	$\omega_0^{(2)}$	omega02	28		
Sign for the first driver	$\sigma^{(1)}$	driver_sign1	29		
Sign for the last driver	$\sigma^{(N)}$	driver_sign2	30		
Integrate angle difference 1		integrate_dt1	31		
Integrate angle difference N		integrate_dt2	32		

3 Conclusion

Acknowledgments

References