Markowitz Model

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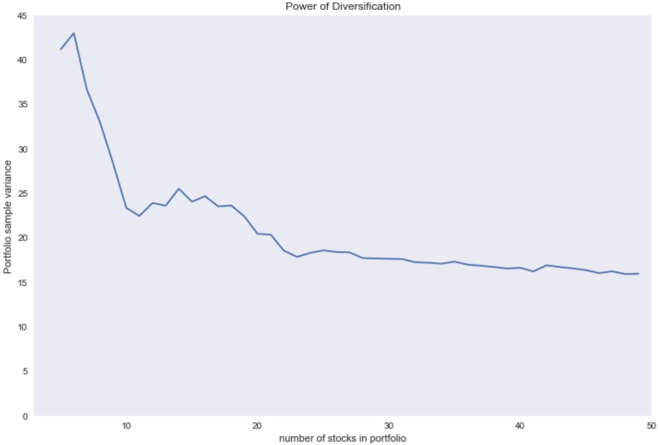
Portfolio and Risk Diversification



- Sample variance of an equal-weighted portfolio as a function of the number of stocks.

Decomposing Portfolio Variance

- Recall the portfolio variance

$$Var(r_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov\left(r_i, r_j\right).$$

- We consider the equal-weighted portfolio (i.e., $w_i = \frac{1}{n}$)

$$Var(r_{eq}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{n^2} Cov\left(r_i, r_j\right).$$

Decomposing Portfolio Variance

$$
\begin{aligned}
Var(r_{eq}) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{n^2} Cov\left(r_i, r_j\right) \\
&= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{n} \sigma_i^2 + \frac{n-1}{n} \sum_{i=1}^{n} \sum_{j \neq i} \frac{1}{n(n-1)} Cov(r_i, r_j).
\end{aligned}
$$

- Let $\bar{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} \sigma_i^2$ and $\bar{Cov} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} Cov(r_i, r_j)$, then

$$
Var(r_{eq}) = \frac{1}{n} \bar{\sigma^2} + \frac{n-1}{n} \bar{Cov}
$$

- Thus, portfolio variance is composed of two components: average variance ($\bar{\sigma^2}$) and average covariance ($\bar{Cov}$).

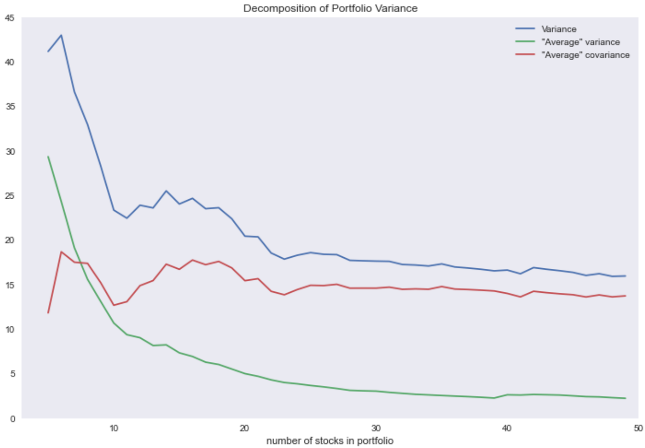
Decomposing Portfolio Variance

- What happens when more and more stocks are included?

$$\lim_{n\to\infty} Var(r_{eq}) = \lim_{n\to\infty} \frac{1}{n}\bar{\sigma^2} + \lim_{n\to\infty} \frac{n-1}{n}\bar{Cov}$$
$$= \bar{Cov}$$

- Average variance ($\bar{\sigma^2}$) can be eliminated by diversification. Average covariance ($\bar{Cov}$) cannot.
- Risk that can be eliminated by diversification $\rightarrow$ nonsystematic risk, idiosyncratic risk, firm-specific risk
- Risk that remains after diversification $\rightarrow$ systematic risk, market risk

Data?



Decomposition of Portfolio Variance
Variance
"Average" variance
"Average" covariance
0
5
10
15
20
25
30
35
40
45
10
20
30
40
50
number of stocks in portfolio

Summary

- Portfolio risk is increasing in the correlation between asset prices.
- How can we interpret the correlation between asset prices? $\rightarrow$ systematic risk
- The contribution of an asset depends on the covariance with other assets.
- In other words, how much it is exposed to systematic risk.

Utility Function - Expected Utility Hypothesis

- (definition) For a random variable X with distribution function $F()$ and a support of $[0, \bar{x}]$, the expected utility function is

$$E\left[u\left(X\right)\right] = \int_0^{\bar{x}} u(x)dF(x)$$

or, for discrete variables

$$E\left[u\left(X\right)\right] = \sum p_i u(x_i)$$

- (definition) Risk premium p is the maximum amount an agent is willing to pay to avoid risk associated with X. That is,

$$u(E(X) - p) = E[u(X)]$$

- (definition) Certainty equivalent C_e is defined by the difference between the expected value and risk premium

$$u(C_e) = E[u(X)]$$

Utility Function - Risk Aversion

- Intuitively, an agent is risk averse if he/she prefers sure bet.
- An agent is risk averse if his/her utility function is concave.
  - $u^{''}(w) < 0$: risk averse
  - $u^{''}(w) = 0$: risk neutral
  - $u^{''}(w) > 0$: risk lover

Utility Function - Measures of Risk Aversion

- Arrow-Prat measure of risk aversion
  - Absolute risk aversion coeffcient (ARA): $-\frac{u^{"}(w)}{u^{'}(w)}$
  - Relative risk aversion coeffcient (RRA): $-\frac{u^{"}(w)w}{u^{'}(w)}$

Utility Function - CARA and CRRA utility function

- CARA (Constant Absolute Risk Aversion) utility function:

$$u(w) = a - e^{-cw}$$

    - $ARA(w) = c$
    - $RRA(w) = cw$

- CRRA (Constant Relative Risk Aversion) utility function:

$$u(w) = \frac{w^{1-x}}{1-x}$$

    - (Homework) Check $ARA(w)$ and $RRA(w)$ for this case.

Risk Aversion and Portfolio Choice

- We are interested in the portfolio choice of an agent with utility $u()$.
- What elements will the agent consider?
- We can specialize the problem so that the agent only considers the mean and variance of his/her consumption or wealth.

- (proposition 1) An agent only considers the mean and variance of his/her wealth if $u()$ is quadratic and the distribution of wealth is arbitrary.
  $\rightarrow$ Let $u(w) = w - \frac{a}{2}w^2$ for $w \in \left[0, a^{-1}\right]$ and $a > 0$.
  Then,

$$\begin{aligned} E\left[u\left(w\right)\right] &= E\left(w\right) - \frac{a}{2}E\left[w^2\right] \\ &= E\left(w\right) - \frac{a}{2}\left[Var\left(w\right) + E\left(w\right)^2\right] \\ &= V\left[E\left(w\right), Var\left(w\right)\right] \end{aligned}$$

- (proposition 2) An agent only considers the mean and variance of his/her wealth if $u()$ is concave and wealth is normally distributed
  $\rightarrow$ Let $W \sim N\left(\mu, \sigma^2\right)$.
  Then, for $Z \sim N(0,1)$ W can be expressed as $W = \mu + \sigma Z$

$$
\begin{aligned}
E[u(W)] &= E\left[u\left(\mu + \sigma Z\right)\right] \\
&= \int_{-\infty}^{\infty} u\left(\mu + \sigma Z\right) f(z) dz \\
&= V\left(\mu, \sigma^2\right)
\end{aligned}
$$

- For portfolio analysis, we only need to care about mean and variance of portfolio returns (much more convenient!).

Optimal Portfolio

- We have seen that only mean and variance matter when it comes to portfolio choices.
- Then, what is the optimal portfolio?
- A portfolio which maximizes expected return given a certain level of risk (varaiance), or
- A portfolio which minimizes risk (variance) given a certain level of expected return.

Optimal Portfolio

(Case 1: Two Risky Assets)

- We are interested in constructing a portfolio $r_p = w_1 r_1 + w_2 r_2$ which minimizes $Var(r_p)$ given a certain level of expected return μ.

$$\min_{w_1, w_2} Var(r_p) = w_1^2 \sigma_1^2 + 2 w_1 w_2 \sigma_{12} + w_2^2 \sigma_2^2$$

s.t

$$E(r_p) = w_1 \mu_1 + w_2 \mu_2 = \mu,$$

$$w_1 + w_2 = 1$$

Optimal Portfolio

(Case 1: Two Risky Assets)

- From the constraints,

$$w_2^* = 1 - w_1^*,$$

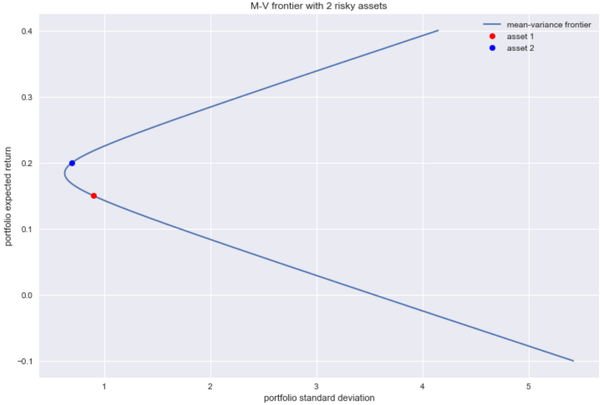
$$w_1^* = \frac{\mu - \mu_2}{\mu_1 - \mu_2},$$

- Plugging w_1^* and w_2^* to the objective function, we get

$$Var(r_p) = {w_1^*}^2 \sigma_1^2 + 2 w_1^* w_2^* \sigma_{12} + {w_2^*}^2 \sigma_2^2$$

(Case 1: Example)

$$\mu_1 = 0.15,\ \sigma_1 = 0.9,\ \sigma_{12} = 0.2,\ \mu_2 = 0.2,\ \sigma_2 = 0.7$$



Optimal Portfolio

- The mean-variance frontier is a grapth of the lowest possible variance that can be attained for a given expected portfolio return.
- It can be interpreted as the risk-return opportunities available to the investor.
- Global minimum-variance portfolio has a smaller variance than that of either of the individual assets.
- Naturally, investors will choose their portfolios on the upper half of the mean variance frontier (efficient frontier), depending on his/her degree of risk aversion.

Optimal Portfolio

(Case 2: One Risky Asset and One Riskfree Asset)

- No we have a riskfree asset f and a risky asset 1.
- Then, $Var\left(r_f\right) = 0$ and $Cov\left(r_f, r_1\right)$=0
- The optimization problem becomes

$$\min_{w_1, w_2} Var\left(r_p\right) = w_1^2 \sigma_1^2$$

s.t

$$E\left(r_p\right) = w_o r_f + w_1 \mu_1 = \mu,$$

$$w_0 + w_1 = 1$$

Optimal Portfolio

(Case 2: One Risky Asset and One Riskfree Asset)

- From the constraint, we get

$$w_1^* = \frac{\mu - r_f}{\mu_1 - r_f}$$

- Then

$$Var(r_p) = \sigma_p^2 = \left(\frac{\mu - r_f}{\mu_1 - r_f}\right)^2 \sigma_1^2,$$

$$\sigma_p = \left|\frac{\mu - r_f}{\mu_1 - r_f}\right| \sigma_1.$$

- This can be written as

$$\mu = r_f \pm s_1 \sigma_p$$

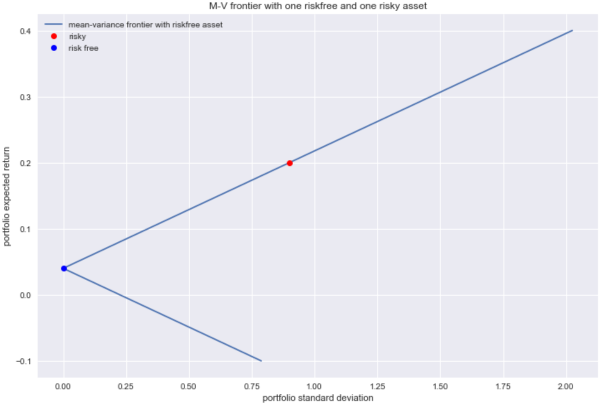
$$\text{where } s_1 \equiv \frac{\mu_1 - r_f}{\sigma_1}$$

- s_1 is the Sharpe ratio.

Optimal Portfolio

(Case 2: Example)

$$\mu_1 = 0.2,\ \sigma_1 = 0.9,\ r_f = 0.7$$



Optimal Portfolio

(Case 3: N Risky Asset and One Riskfree Asset)

- Now, its time for generalization.
- With N risky assetas and one riskfree asset,
- Then, the optimization problem becomes

$$\min_{w} w^{'} \sum w$$

s.t

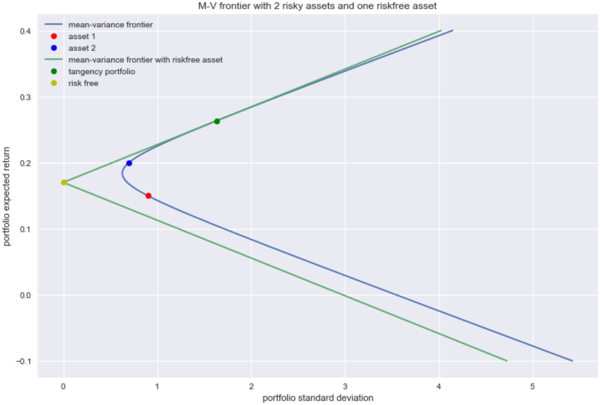
$$w^{'} \left(E - r_f \mathbb{I}\right) = \mu - r_f$$

(Case 3: N Risky Asset and One Riskfree Asset)

- A bit too complicated? Let's do it a bit differently.
- We have an alternative way of solving the problem: maximizing Sharpe ratio given the mean-variance frontier with N risky assets only.
- Note that we can treat portfolios as if they are single assets.
- Then, we can choose any portfolio between the riskfree asset and any points on the mean-varince frontier of risky assets only.
- Then, which portfolio is the best? $\rightarrow$ The efficient frontier is obtained by maximizing the slope (i.e., Sharpe ratio).

(Case 3: Example)

$$\mu_1 = 0.15,\ \sigma_1 = 0.9,\ \sigma_{12} = 0.2,\ \mu_2 = 0.2,\ \sigma_2 = 0.7,\ r_f = 0.17$$



Optimal Portfolio

(Case 3: N Risky Asset and One Riskfree Asset)

- Which point gives the highest Sharpe ratio? All the points on the efficient frontier.
- The efficient frontier is also called as Capital Market Line(CML).
- Note that it is the highest Sharpe ratio which can be obtained from the efficient frontier with risky assets only.