Financial Econometrics

Chapter 6: Capital Asset Pricing Model

In Choi

Sogang University

References

- Chapter 5 of Campbell, Lo and MacKinlay (1997)
- Chapters 5 and 20 of Cochrane (2001)
- Chapters 5 and 8 of Cuthbertson and Nitzsche (2004)
- Fama E.F. and K.R. French (2004): The Capital Asset Pricing Model: Theory and Evidence, *Journal of Economic Perspectives*, 18, pp. 25–46.

What does CAPM imply?

- The expected return of an asset must be linearly related to the covariance of its return with the return of the market portfolio.
- CAPM quantifies risk and the reward for bearing it.

What is CAPM?

- CAPM was developed by Markowitz (1959), Sharpe (1964; Journal of Finance) and Lintner (1965; Review of Economics and Statistics).
- CAPM is a single-period model.
- The Sharpe-Linter CAPM

$$\begin{aligned} E(R_i) &= R_f + \beta_{im}(E(R_m) - R_f) \\ \beta_{im} &= \frac{Cov(R_i, R_m)}{Var(R_m)} \end{aligned} \tag{1}$$

R_i : return on asset i
R_m : return on the market portfolio
R_f : return on risk-free asset

What is CAPM?

- Let $Z_i = R_i - R_f$ and $Z_m = R_m - R_f$. Then, if the risk-free rate is nonstochastic, (1) becomes

$$\begin{aligned} E(Z_i) &= \beta_{im} E(Z_m) \\ \beta_{im} &= \frac{Cov(Z_i, Z_m)}{Var(Z_m)}. \end{aligned}$$

In order to test for CAPM, regress Z_i on Z_m and see if
(i) the intercept is zero
(ii) β completely captures the cross-sectional variation of expected excess returns
(iii) the market risk premium is positive.

What is CAPM?

- Betas for individual stocks are calculated by using the time series regression model
$$Z_{it} = \alpha_{im} + \beta_{im} Z_{mt} + \epsilon_{it}.$$
Z_{it} : realized excess return for stock i at time t.
Z_{mt} : realized return for the market portfolio at time t (e.g., S&P 500 index, KOSPI index, Hangseng Index, etc.)

What is CAPM?

- Betas for individual stocks are typically reported in Web pages for stocks. Higher betas implies higher level of sensitivivity of stocks to market movements.
- The CAPM can be used to compute expected stock returns. This works only when the CAPM provides a good description of the data.

Efficient-Set Mathematics

- There are N risky asset returns denoted by an $N \times 1$ vector R.
- $E(R) = \boldsymbol{\mu}_R$ and $Var(R) = E(R - \boldsymbol{\mu}_R)(R - \boldsymbol{\mu}_R)' = \Omega_R$ (Ω_R is called the variance-covariance of the random vector R).

Efficient-Set Mathematics

- Notation
  - ω_a : $N \times 1$ vector of portfolio weights for an arbitrary portfolio a with weights summing to one
  - Mean of the portfolio with weights ω_a : $\mu_a = \omega_a' E(R) = \omega_a' \boldsymbol{\mu}_R$
  - Variance of the portfolio with weights ω_a : $\sigma_a^2 = \omega_a' \Omega_R \omega_a$
  - Covariance between any two portfolios a and b : $\omega_a' \Omega_R \omega_b$

- Portfolio p is the *minimum-variance portfolio* of all portfolios with mean return μ_p if its portfolio weight vector (ω_p) is the solution to the following constrained optimization:

$$\min_{\omega} \omega' \Omega_R \omega$$

subject to

$$\begin{aligned} \omega' \boldsymbol{\mu}_R &= \mu_p; \\ \omega' \mathbf{i} &= 1 \ (\mathbf{i} = [1, ..., 1]'). \end{aligned}$$

- The solution of this problem[1] is

$$\omega_p = g + h\mu_p$$

where g and h are $N \times 1$ vectors

$$\begin{aligned} g &= \frac{1}{D}\left[B(\Omega_R^{-1}\mathbf{i}) - A(\Omega_R^{-1}\boldsymbol{\mu}_R)\right] \\ h &= \frac{1}{D}\left[C(\Omega_R^{-1}\mathbf{i}) - A(\Omega_R^{-1}\boldsymbol{\mu}_R)\right] \end{aligned}$$

and $A = \mathbf{i}'\Omega_R^{-1}\boldsymbol{\mu}_R$, $B = \boldsymbol{\mu}_R'\Omega_R^{-1}\mathbf{i}$, $C = \mathbf{i}'\Omega_R^{-1}\mathbf{i}$ and $D = BC - A^2$.

[1] This can be solved by the Lagrangian approach.

- For a given μ_p, letting $R_p = \omega_p' R$ and $\sigma_p^2 = Var(R_p)$,

$$\begin{aligned} \sigma_p^2 &= E(\omega_p' R - \omega_p' \mu_R)(\omega_p' R - \omega_p' \mu_R)' \\ &= \omega_p' \Omega_R \omega_p \\ &= \frac{C\mu_p^2 - 2A\mu_p + B}{BC - A^2}. \end{aligned} \qquad (2)$$

  Plotting this in the space of (σ, μ), we obtain mean-variance frontier. The MV frontier is the locus of the minimum variance for a given expected return.

- Minimizing (2) with respect to μ_p gives $\mu_p = A/C$. This means that the global minimum variance is attained when

$$\mu = A/C = \frac{\mathbf{i}'\Omega_R^{-1}\boldsymbol{\mu}_R}{C} = \left(\frac{1}{C}\Omega_R^{-1}\mathbf{i}\right)'\boldsymbol{\mu}_R.$$

This implies the weights of the global minimum variance portfolio is

$$\omega_g = \frac{1}{C}\Omega_R^{-1}\mathbf{i}$$

and its variance is

$$\sigma_g^2 = \omega_g'\Omega_R\omega_g = \frac{1}{C^2}\mathbf{i}'\Omega_R^{-1}\mathbf{i} = \frac{1}{C},$$

because $\mathbf{i}'\Omega_R^{-1}\mathbf{i}' = C$ that is derivable from $\mathbf{i}'\omega_g = 1$.

- For each minimum-variance portfolio p, except the global minimum-variance portfolio g, there exists a unique minimum-variance portfolio that has zero covariance with p (called zero-beta portfolio with respect to p). This can be proven using the fact p and op correspond to the two solutions of equation (2).

Now introduce a risk-free asset.

- Given a risk-free asset with return R_f, consider the problem of choosing the minimum-variance portfolio with expected return μ_p. That is, we consider the contrained optimization problem

$$\min_{\omega} \omega' \Omega_R \omega$$

subject to

$$\omega' \boldsymbol{\mu}_R + (1 - \omega' \mathbf{i}) R_f = \mu_p; \;\; (\mathbf{i} = [1, ..., 1]').$$

- The resulting solution is the optimal portfolio for risky assets. The weight for the risk-free asset is obtained from this.

Efficient-Set Mathematics

With a risk-free asset

- The solution is

$$\begin{aligned}\omega_p &= \frac{\mu_p - R_f}{(\boldsymbol{\mu}_R - R_f\mathbf{i})'\Omega_R^{-1}(\boldsymbol{\mu}_R - R_f\mathbf{i})} \times \Omega_R^{-1}(\boldsymbol{\mu}_R - R_f\mathbf{i}) \\ &= c_p \times \bar{\omega}.\end{aligned}$$

- This shows that weights for the risky assets are proportional to the vector $\bar{\omega}$. (Note: c_p is a constant.)

- If only risky assets are owned, we should have $\mathbf{i}'\omega_q = 1$. Thus, the risky-asset-only portfolio should be

$$\omega_q = \frac{\Omega_R^{-1}(\boldsymbol{\mu}_R - R_f\mathbf{i})}{\mathbf{i}'\Omega_R^{-1}(\boldsymbol{\mu}_R - R_f\mathbf{i})}.$$

  This is called the tangency portfolio. We have $\omega_p = \omega_q$ iff $\mu_p = \boldsymbol{\mu}_{Ri}$ for $i = 1, ..., N$ (i.e., $\mu_R = \mu_p \times \mathbf{i}$).
- If $\mu_p = R_f$, no risky assets will be owned.

- Suppsoe that an investor allocates his fund between risk-free and risky portfolio ω_q. Given the optimal portfolio ω_q for risky assets and the risk-free asset with return R_f, the asset return of the combined portfolio is

$$R_p = (1-x)R_f + xR_q$$

which gives

$$\mu_p = (1-x)R_f + x\mu_q \tag{3}$$

and

$$Var(R_p) = x^2 Var(R_q). \tag{4}$$

- Equation (4) gives

$$x = \frac{\sigma_p}{\sigma_q}.$$

Plugging this into equation (3), we obtain

$$\mu_p = R_f + \left(\frac{\mu_q - R_f}{\sigma_q} \right) \sigma_p.$$

- All efficient portfolios lie along the line from the risk-free asset through portfolio q.
- Portfolio q is a fixed bundle of risky assets held by all investors regardless of their preferences. Hence, it is called the market portfolio.
- Straight line connecting R_f and q and going beyond is called the capital market line (CML).

- Investor preferences determine where they will reside along the CML. Risk-averse investors will locate themselves in between R_f and q. Risk-loving investors will go beyond q.
- An extreme investor who does not want any risk will reside at point R_f.
- For any portfolio a, $\frac{\mu_a - R_f}{\sigma_a}$ is called the Sharpe ratio. It measures the expected excess return per unit risk or it is the price of risk.

- $\frac{\mu_q - R_f}{\sigma_q}$ is called the market price of risk.
- Portfolio q has the maximum Sharpe ratio of all portfolios of risky assets. This mean that no one will stay along the line $R_f \longrightarrow a$.

Derivation of CAPM

- CAPM can be derived using the efficient set mathematics.
- Suppose that an investor decides to allocate x_i to a security i with mean return μ_i and variance σ_i^2 and $1 - x_i$ to the market portfolio q. This portfolio's expected return is

$$\begin{aligned}
\mu_p &= x_i \mu_i + (1 - x_i)\mu_q; \\
\sigma_p &= \sqrt{x_i^2 \sigma_i^2 + (1 - x_i)^2 \sigma_q^2 + 2x_i(1 - x_i)\sigma_{iq}}.
\end{aligned}$$

Derivation of CAPM

- But moving off the market portfolio is not an optimal investment strategy and thus it should be $x_i = 0$. At $x_i = 0$,

$$\frac{\partial \mu_p}{\partial \sigma_p} \big|_{x_{i=0}} = \frac{\left(\mu_i - \mu_q\right) \sigma_q}{\sigma_{iq} - \sigma_q^2}.$$[2]

But this should be equal to the slope of the CML

$$\frac{\mu_q - R_f}{\sigma_q}.$$

Thus, we obtain

$$\mu_i = R_f + \frac{\sigma_{iq}}{\sigma_q^2}(\mu_q - R_f).$$

[2] Consider $\frac{\partial \mu_p}{\partial x_i} \frac{\partial x_i}{\partial \sigma_p}$ evaluated at $x_i = 0$. Note also that $\sigma_p = \sigma_q$ at $x_i = 0$.

- Let Z_t be an $N \times 1$ vector of excess returns for N assets at time t. According to the Sharpe-Lintner CAPM,

$$Z_t = \alpha + \beta Z_{mt} + \varepsilon_t$$

where $\alpha = 0$, Z_{mt} is the time period t market portfolio excess return and ε_t is an idiosyncratic error $(\sim iid(0, \Sigma))$.

- Testing for CAPM amounts to testing the null hypothesis

$$H_0 : \alpha = 0.$$

  Of course, this is only an aspect of CAPM.
- In addition, we need to know whether $\beta > 0$ and whether there are other factors that affect the expected return.

- Let

$$
\begin{aligned}
\hat{\alpha} &= \bar{Z} - \hat{\beta}\bar{Z}_m; \\
\hat{\beta} &= \frac{\sum_{t=1}^{T}(Z_t - \bar{Z})(Z_{mt} - \bar{Z}_m)}{\sum_{t=1}^{T}(Z_{mt} - \bar{Z}_m)^2}; \\
\bar{Z} &= \frac{1}{T}\sum_{t=1}^{T} Z_t; \;\; \bar{Z}_m = \frac{1}{T}\sum_{t=1}^{T} Z_{mt}; \\
\hat{\Sigma} &= \frac{1}{T}\sum_{t=1}^{T}(Z_t - \hat{\alpha} - \hat{\beta}Z_{mt})(Z_t - \hat{\alpha} - \hat{\beta}Z_{mt})'.
\end{aligned}
$$

- Wald statistic for $H_0 : \alpha = 0$ is defined by

$$
\begin{aligned}
J &= \hat{\alpha}' [Var(\hat{\alpha})]^{-1} \hat{\alpha} \\
&= T \left[1 + \frac{\bar{Z}_m^2}{\frac{1}{T} \sum_{t=1}^{T} (Z_{mt} - \bar{Z}_m)^2} \right]^{-1} \hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha}.
\end{aligned}
$$

- When T is large, $J \sim \chi^2(N)$.

- There are other types of tests available (cf. Campbell, Lo and MacKinlay).
- Time series empirical evidence is not favorable to CAPM as shown in Table 5.3 of Campbell, Lo and MacKinlay.

- Figure 20.8 of Cochrane (2001, p.436) plots returns and betas of 10 portfolios of NYSE stocks sorted by size, and those of portfolios made up of corporate bonds and long-term government bonds.

- We find:
  - (i) Portfolios with higher average returns have higher betas as CAPM predicts.
  - (ii) The smallest firms seem to earn an average return a few percentage points too high given their betas (small-firm effect).
  - (iii) The long-term and corporate bonds have mean returns in line with their low betas, despite their standard deviations nearly as high as those of stocks.

- Suppose that a company is considering new projects that will provide cash streams. The projects will be financed by equities only.
- The question is which projects should be pursued and which should not be.
- If a projcect provides a cash stream which has positive present value, it should be pursued.
- When calculating the present value, many companies in the real world use the discount rate that is implied by CAPM.
- Why? The expected return implied by CAPM is the opportunity cost of equity.

Example

A company's expected return calculated by CAPM is 16.495%. A new project will provide cash \$14 million next year, while its cost is \$10 million. The present value of the project is

$$\frac{14}{1+0.16495} - 10 = 2.0177.$$

The calculation shows that this project is worth pursuing.

- If a company's projects are financed by debts and preferred stocks, the cost of capital is easier to calculate: it is the cost of borrowing.

Example

(Weighted average cost of capital) A company's expected return calculated by CAPM is 16.495%,and the cost of borrowing for the company is 8% per annum. A new project will provide cash \$14 million next year, while its cost is \$10 million out of which \$4 million will be financed by equities and \$6 million by debt. The average cost of capital is

$$\frac{4}{6+4} \times 0.16495 + \frac{6}{6+4} \times 0.08 = 0.11398.$$

The present value of the project is

$$\frac{14}{1+0.11398} - 10 = 2.5676.$$