§ *6.2 Permutations and Combinations*

An Example

Example

In how many ways can we select three students from a group of five students to stand in line for a picture?

An Example

Example

In how many ways can we select three students from a group of five students to stand in line for a picture?

How many ways to select the first student?

An Example

Example

In how many ways can we select three students from a group of five students to stand in line for a picture?

How many ways to select the first student? The second?

An Example

Example

In how many ways can we select three students from a group of five students to stand in line for a picture?

How many ways to select the first student? The second? The third?

An Example

Example

In how many ways can we select three students from a group of five students to stand in line for a picture?

How many ways to select the first student? The second? The third? So how many way are there?

An Example

Example

In how many ways can we select three students from a group of five students to stand in line for a picture?

How many ways to select the first student? The second? The third? So how many way are there? $5 \cdot 4 \cdot 3 = 60$.

An Example

Example

In how many ways can we select three students from a group of five students to stand in line for a picture?

How many ways to select the first student? The second? The third? So how many way are there? $5 \cdot 4 \cdot 3 = 60$.

Example

How many ways can we arrange all five of the students for picture?

An Example

Example

In how many ways can we select three students from a group of five students to stand in line for a picture?

How many ways to select the first student? The second? The third? So how many way are there? $5 \cdot 4 \cdot 3 = 60$.

Example

How many ways can we arrange all five of the students for picture?

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$

Permutations

Theorem

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n,r) =_n P_r = n(n-1)(n-2)\dots(n-r+1)$$

r-permutations of a set with n distinct elements.

Permutations

Theorem

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) =_n P_r = n(n-1)(n-2)\ldots(n-r+1)$$

r-permutations of a set with n distinct elements.

Corollary

If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \frac{n!}{(n-r)!}$

Permutations

Theorem

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) =_n P_r = n(n-1)(n-2)\dots(n-r+1)$$

r-permutations of a set with n distinct elements.

Corollary

If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \frac{n!}{(n-r)!}$

Note: By convention, $0! = 1$

Back to Examples

Example

How many ways are there to choose the first prize winner, second prize winner and third prize winner from a contest with 100 different people?

Back to Examples

Example

How many ways are there to choose the first prize winner, second prize winner and third prize winner from a contest with 100 different people?

$$P(100,3) = \frac{100!}{(100-3)!} = \frac{100!}{97!} = 100 \cdot 99 \cdot 98 = 970,200$$

Back to Examples

Example

How many ways are there to choose the first prize winner, second prize winner and third prize winner from a contest with 100 different people?

$$P(100, 3) = \frac{100!}{(100-3)!} = \frac{100!}{97!} = 100 \cdot 99 \cdot 98 = 970,200$$

Example

Suppose a saleswoman has to visit 8 different cities. She must begin her trip in a specified city, but then she can visit the other seven cities in whatever order she wishes. In how many different orders can she visit these cities?

Back to Examples

Example

How many ways are there to choose the first prize winner, second prize winner and third prize winner from a contest with 100 different people?

$$P(100,3) = \frac{100!}{(100-3)!} = \frac{100!}{97!} = 100 \cdot 99 \cdot 98 = 970,200$$

Example

Suppose a saleswoman has to visit 8 different cities. She must begin her trip in a specified city, but then she can visit the other seven cities in whatever order she wishes. In how many different orders can she visit these cities?

$7! = 5040$

A Different Kind of Problem

Example

How many different three student committees can be formed from a group of four students?

A Different Kind of Problem

Example

How many different three student committees can be formed from a group of four students?

What is different?

A Different Kind of Problem

Example

How many different three student committees can be formed from a group of four students?

What is different? The order in which we select the students does not matter.

A Different Kind of Problem

Example

How many different three student committees can be formed from a group of four students?

What is different? The order in which we select the students does not matter.

We can select a three student committee by excluding one of the students, and there are four ways to do so. Therefore, there are 4 committees possible.

Combinations

Theorem

The number of r-combinations of a set with n elements, where $\in \mathcal{Z}^+$ and $0 \leq r \leq n$, equals

$$C(n, r) =_n C_r = \frac{n!}{r!(n-r)!}$$

Combinations

Theorem

The number of r-combinations of a set with n elements, where $\in \mathcal{Z}^+$ *and* $0 \leq r \leq n$*, equals*

$$C(n, r) =_n C_r = \frac{n!}{r!(n-r)!}$$

How does this formula relate to that for r-permutations?

Combinations

Theorem

The number of r-combinations of a set with n elements, where $\in \mathcal{Z}^+$ *and* $0 \leq r \leq n$*, equals*

$$C(n, r) =_n C_r = \frac{n!}{r!(n-r)!}$$

How does this formula relate to that for r-permutations?

Since the difference between permutations and combinations is order, we destroy the order of a permutation by dividing by the number of ways to permute the r selected elements ...

Back to Examples

Example

How many 5 card poker hands are there if we are using a standard deck?

Back to Examples

Example

How many 5 card poker hands are there if we are using a standard deck?

$$_{52}C_5 = \frac{52!}{5! \cdot 47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960$$

Back to Examples

Example

How many 5 card poker hands are there if we are using a standard deck?

$$_{52}C_5 = \frac{52!}{5! \cdot 47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960$$

Example

Suppose we wanted to select 47 cards from that standard deck. In how many ways can we do this?

Back to Examples

Example

How many 5 card poker hands are there if we are using a standard deck?

$$_{52}C_5 = \frac{52!}{5! \cdot 47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960$$

Example

Suppose we wanted to select 47 cards from that standard deck. In how many ways can we do this?

$$_{52}C_{47} = \frac{52!}{47! \cdot 5!}$$

Back to Examples

Example

How many 5 card poker hands are there if we are using a standard deck?

$$_{52}C_5 = \frac{52!}{5! \cdot 47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960$$

Example

Suppose we wanted to select 47 cards from that standard deck. In how many ways can we do this?

$$_{52}C_{47} = \frac{52!}{47! \cdot 5!}$$

Why are these answers the same?

And More Examples

Example

How many ways are there to select a 5 student contingent from the school's 10 person tennis team to make up the travel team for a tournament?

And More Examples

Example

How many ways are there to select a 5 student contingent from the school's 10 person tennis team to make up the travel team for a tournament?

$${}_{10}C_5 = \frac{10!}{5! \cdot 5!} = 252$$

And More Examples

Example

How many ways are there to select a 5 student contingent from the school's 10 person tennis team to make up the travel team for a tournament?

$${}_{10}C_5 = \frac{10!}{5! \cdot 5!} = 252$$

Example

A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select six people to go on the mission, assuming they can all do all required tasks?

And More Examples

Example

How many ways are there to select a 5 student contingent from the school's 10 person tennis team to make up the travel team for a tournament?

$${}_{10}C_5 = \frac{10!}{5! \cdot 5!} = 252$$

Example

A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select six people to go on the mission, assuming they can all do all required tasks?

$${}_{30}C_6 = \frac{30!}{6! \cdot 24!} = 593,775$$

When They Become More Interesting

Example

There are 9 faculty members in mathematics and 11 in computer science. How many ways are there to select a committee to develop a discrete mathematics course if the committee must consist of three faculty members from each department?

When They Become More Interesting

Example

There are 9 faculty members in mathematics and 11 in computer science. How many ways are there to select a committee to develop a discrete mathematics course if the committee must consist of three faculty members from each department?

Number of ways to select the math faculty

When They Become More Interesting

Example

There are 9 faculty members in mathematics and 11 in computer science. How many ways are there to select a committee to develop a discrete mathematics course if the committee must consist of three faculty members from each department?

Number of ways to select the math faculty ${}_9C_3 = \frac{9!}{3! \cdot 6!}$

When They Become More Interesting

Example

There are 9 faculty members in mathematics and 11 in computer science. How many ways are there to select a committee to develop a discrete mathematics course if the committee must consist of three faculty members from each department?

Number of ways to select the math faculty ${}_9C_3 = \frac{9!}{3! \cdot 6!}$
Number of ways to select the CS faculty

When They Become More Interesting

Example

There are 9 faculty members in mathematics and 11 in computer science. How many ways are there to select a committee to develop a discrete mathematics course if the committee must consist of three faculty members from each department?

Number of ways to select the math faculty ${}_9C_3 = \frac{9!}{3! \cdot 6!}$
Number of ways to select the CS faculty ${}_{11}C_3 = \frac{11!}{3! \cdot 8!}$

When They Become More Interesting

Example

There are 9 faculty members in mathematics and 11 in computer science. How many ways are there to select a committee to develop a discrete mathematics course if the committee must consist of three faculty members from each department?

Number of ways to select the math faculty ${}_9C_3 = \frac{9!}{3! \cdot 6!}$
Number of ways to select the CS faculty ${}_{11}C_3 = \frac{11!}{3! \cdot 8!}$
How do we combine these?

When They Become More Interesting

Example

There are 9 faculty members in mathematics and 11 in computer science. How many ways are there to select a committee to develop a discrete mathematics course if the committee must consist of three faculty members from each department?

Number of ways to select the math faculty ${}_9C_3 = \frac{9!}{3! \cdot 6!}$
Number of ways to select the CS faculty ${}_{11}C_3 = \frac{11!}{3! \cdot 8!}$
How do we combine these?

$${}_9C_3 = \frac{9!}{3! \cdot 6!} \cdot {}_{11}C_3 = \frac{11!}{3! \cdot 8!} = 84 \cdot 330 = 27,720$$

More Examples

Example

How many ways are there to form a joint congressional committee with 3 senators and 5 representatives?

More Examples

Example

How many ways are there to form a joint congressional committee with 3 senators and 5 representatives?

${}_{100}C_3 \cdot {}_{435}C_5$

More Examples

Example

How many ways are there to form a joint congressional committee with 3 senators and 5 representatives?

$_{100}C_3 \cdot {}_{435}\, C_5$

Example

How many different ‘words’ can we make using the letters of ‘RED SOX’ if we can use at least 3 but no more than 5 letters?

More Examples

Example

How many ways are there to form a joint congressional committee with 3 senators and 5 representatives?

$_{100}C_3 \cdot _{435}C_5$

Example

How many different 'words' can we make using the letters of 'RED SOX' if we can use at least 3 but no more than 5 letters?

$_6P_3 +_6 P_4 +_6 P_5 = 20 + 15 + 6 = 41$

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet if

- The fleet has five cars, three foreign and two domestic?

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet if

- The fleet has five cars, three foreign and two domestic?
  ${}_7C_3 \cdot {}_4C_2 = 35 \cdot 6 = 210$

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet if

- The fleet has five cars, three foreign and two domestic?
  ${}_7C_3 \cdot {}_4C_2 = 35 \cdot 6 = 210$
- The fleet can be of any size but must contain the same number of foreign and domestic cars?

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet if

- The fleet has five cars, three foreign and two domestic?
  ${}_7C_3 \cdot {}_4C_2 = 35 \cdot 6 = 210$
- The fleet can be of any size but must contain the same number of foreign and domestic cars?

$$\begin{aligned} {}_7C_1 \cdot {}_4C_1 + {}_7C_2 \cdot {}_4C_2 + {}_7C_3 \cdot {}_4C_3 + {}_7C_4 \cdot {}_4C_4 \\ = 7 \cdot 4 + 21 \cdot 6 + 35 \cdot 4 + 35 \cdot 1 \\ = 28 + 126 + 140 + 35 \\ = 329 \end{aligned}$$

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet (provided we can only have at most one of each make) if

- The fleet has four cars and one is a Chevy?

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet (provided we can only have at most one of each make) if

- The fleet has four cars and one is a Chevy?
  ${}_{10}C_3 = 120$

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet (provided we can only have at most one of each make) if

- The fleet has four cars and one is a Chevy?
  ${}_{10}C_3 = 120$
- The fleet has four cars, two of each type, and it cannot have a Chevy and a Honda?

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet (provided we can only have at most one of each make) if

- The fleet has four cars and one is a Chevy?
  ${}_{10}C_3 = 120$
- The fleet has four cars, two of each type, and it cannot have a Chevy and a Honda?
  How many ways if there is no Chevy?

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet (provided we can only have at most one of each make) if

- The fleet has four cars and one is a Chevy?
  ${}_{10}C_3 = 120$
- The fleet has four cars, two of each type, and it cannot have a Chevy and a Honda?
  How many ways if there is no Chevy?
  ${}_{7}C_2 \cdot {}_{3}C_2 = 21 \cdot 3 = 63$

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet (provided we can only have at most one of each make) if

- The fleet has four cars and one is a Chevy?
  $_{10}C_3 = 120$
- The fleet has four cars, two of each type, and it cannot have a Chevy and a Honda?
  How many ways if there is no Chevy?
  $_7C_2 \cdot_3 C_2 = 21 \cdot 3 = 63$
  How many ways if there is no Honda?

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet (provided we can only have at most one of each make) if

- The fleet has four cars and one is a Chevy?
  ${}_{10}C_3 = 120$
- The fleet has four cars, two of each type, and it cannot have a Chevy and a Honda?
  How many ways if there is no Chevy?
  ${}_7C_2 \cdot {}_3C_2 = 21 \cdot 3 = 63$
  How many ways if there is no Honda?
  ${}_6C_2 \cdot {}_4C_2 = 15 \cdot 6 = 90$

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet (provided we can only have at most one of each make) if

- The fleet has four cars and one is a Chevy?
  ${}_{10}C_3 = 120$
- The fleet has four cars, two of each type, and it cannot have a Chevy and a Honda?
  How many ways if there is no Chevy?
  ${}_7C_2 \cdot {}_3C_2 = 21 \cdot 3 = 63$
  How many ways if there is no Honda?
  ${}_6C_2 \cdot {}_4C_2 = 15 \cdot 6 = 90$
  How do we combine these?

Car Fleet Example

Example

A fleet is to be chosen from a set of 7 different make foreign cars and 4 different make domestic cars. How many ways can we choose a fleet (provided we can only have at most one of each make) if

- The fleet has four cars and one is a Chevy?
  ${}_{10}C_3 = 120$
- The fleet has four cars, two of each type, and it cannot have a Chevy and a Honda?
  How many ways if there is no Chevy?
  ${}_7C_2 \cdot {}_3C_2 = 21 \cdot 3 = 63$
  How many ways if there is no Honda?
  ${}_6C_2 \cdot {}_4C_2 = 15 \cdot 6 = 90$
  How do we combine these? $63 + 90 = 153$

Car Fleet Example

Example

So there are 153 ways to have a fleet of 4 cars with no Chevy and no Honda. Right?

Car Fleet Example

Example

So there are 153 ways to have a fleet of 4 cars with no Chevy and no Honda. Right?

What are we neglecting to consider?

Car Fleet Example

Example

So there are 153 ways to have a fleet of 4 cars with no Chevy and no Honda. Right?
What are we neglecting to consider?
${}_6C_2 \cdot {}_3C_2 = 15 \cdot 3 = 45$

Car Fleet Example

Example

So there are 153 ways to have a fleet of 4 cars with no Chevy and no Honda. Right?
What are we neglecting to consider?
${}_6C_2 \cdot {}_3C_2 = 15 \cdot 3 = 45$
So the total is $153 - 45 = 108$ ways.

Car Fleet Example

An alternate way to consider this?

Car Fleet Example

An alternate way to consider this?

How many different fleets could we have if there are no make restrictions but we want two foreign and two domestic cars?

Car Fleet Example

An alternate way to consider this?

How many different fleets could we have if there are no make restrictions but we want two foreign and two domestic cars?

${}_7C_2 \cdot {}_4C_2 = 21 \cdot 6 = 126$

Car Fleet Example

An alternate way to consider this?

How many different fleets could we have if there are no make restrictions but we want two foreign and two domestic cars?

$_7C_2 \cdot {}_4C_2 = 21 \cdot 6 = 126$

How many have Hondas and Chevy's?

Car Fleet Example

An alternate way to consider this?

How many different fleets could we have if there are no make restrictions but we want two foreign and two domestic cars?

${}_7C_2 \cdot {}_4C_2 = 21 \cdot 6 = 126$

How many have Hondas and Chevy's?

${}_6C_1 \cdot {}_3C_1 = 6 \cdot 3 = 18$

Car Fleet Example

An alternate way to consider this?

How many different fleets could we have if there are no make restrictions but we want two foreign and two domestic cars?

${}_7C_2 \cdot {}_4C_2 = 21 \cdot 6 = 126$

How many have Hondas and Chevy's?

${}_6C_1 \cdot {}_3C_1 = 6 \cdot 3 = 18$

So, we have $126 - 18 = 108$ ways.

Poker Example

Example

How many ways are there to get a full house with a standard deck?

Poker Example

Example

How many ways are there to get a full house with a standard deck?

Suit for the set?

Poker Example

Example

How many ways are there to get a full house with a standard deck?

Suit for the set? ${}_{13}C_1 = 13$

Poker Example

Example

How many ways are there to get a full house with a standard deck?

Suit for the set? ${}_{13}C_1 = 13$ And which cards?

Poker Example

Example

How many ways are there to get a full house with a standard deck?

Suit for the set? ${}_{13}C_1 = 13$ And which cards? ${}_{4}C_3 = 4$.

Poker Example

Example

How many ways are there to get a full house with a standard deck?

Suit for the set? ${}_{13}C_1 = 13$ And which cards? ${}_{4}C_3 = 4$.

Suit for the pair?

Poker Example

Example

How many ways are there to get a full house with a standard deck?

Suit for the set? ${}_{13}C_1 = 13$ And which cards? ${}_{4}C_3 = 4$.

Suit for the pair? ${}_{12}C_1 = 12$

Poker Example

Example

How many ways are there to get a full house with a standard deck?

Suit for the set? ${}_{13}C_1 = 13$ And which cards? ${}_4C_3 = 4$.

Suit for the pair? ${}_{12}C_1 = 12$ and which cards?

Poker Example

Example

How many ways are there to get a full house with a standard deck?

Suit for the set? ${}_{13}C_1 = 13$ And which cards? ${}_4C_3 = 4$.

Suit for the pair? ${}_{12}C_1 = 12$ and which cards? ${}_4C_2 = 6$.

Poker Example

Example

How many ways are there to get a full house with a standard deck?

Suit for the set? ${}_{13}C_1 = 13$ And which cards? ${}_4C_3 = 4$.

Suit for the pair? ${}_{12}C_1 = 12$ and which cards? ${}_4C_2 = 6$.

So, the number of full houses would therefore be $13 \cdot 12 \cdot 4 \cdot 6 = 3744$.