Nonlinear Dynamics and Chaos I

Problem Set 3.5 - Questions

Dynamic mode decomposition

A popular method of data driven modeling is the *dynamic mode decomposition (DMD).* Assume that we observe a dynamical process and record n observables at $m+1$ discrete time instants, i. e. $x_0, ..., x_m \in \mathbb{R}^n$. These measurements are separated by a time interval of Δt. The method finds a linear, discrete dynamical system that best approximates the observed trajectories as

$$x_{k+1} \approx B x_k,$$

for some constant matrix $B \in \mathbb{R}^{n \times n}$. To this end, the data is assembled into two matrices

$$X = [x_0, ..., x_{m-1}] \in \mathbb{R}^{n \times m} \quad \text{and} \quad Y = [x_1, ..., x_m] \in \mathbb{R}^{n \times m}.$$

The matrix Y contains the same data as X but shifted forward in time by Δt. The matrix B solves the equation

$$Y = BX$$

in the least-squared sense and is given by

$$B = (YX^T)(XX^T)^\dagger,$$

where $A^\dagger$ denotes the *pseudoinverse* of A.

Question 1

Analyze the following scalar dynamical system ($n = 1$) with DMD that has multiple fixed points

$$\dot{x} = x - x^3.$$

The fixed points are located at $x = 0$ and $x = 1$. We disregard the fixed point at $x = -1$ by assuming $x \geq 0$.

(a) What is the stability type of the fixed points $x = 0$ and $x = 1$?

(b) Generate a trajectory numerically started from the initial condition $x_0 = 0.001$ sampled at $t = 0,\ \Delta t,\ 2\Delta t\ , ...,\ m\Delta t$ with $\Delta t = 0.1$ and $m = 50$. Fit a DMD model to this data. What is the approximate linear model that you obtain? Is it accurate in reproducing the trajectory?

(c) Consider a longer trajectory with $m = 120$. The trajectory converges to the fixed point $x = 1$. Can the DMD model predict this? If not, what do we observe instead?

Question 2

Suppose that we are given velocity measurements of a two dimensional flow past a cylinder. We see that the initially steady flow slowly develops oscillations as shown in Fig. 1. Without any knowledge of the underlying equations, our goal is to obtain a DMD model for this phenomenon.

Note: For larger datasets such as this one, it is convenient to use an equivalent formula for the matrix B. It can be shown that

$$B = YX^\dagger.$$

In addition, we may also reduce the effective dimensionality of the model. With the singular value decomposition of the data matrix $X = U\Sigma V^T$, we may choose to keep only the r most important singular values $X \approx U_r\Sigma_r V_r^T$. The truncated DMD matrix then can be defined as

$$B_r = YV_r(\Sigma_r)^\dagger U_r^T.$$

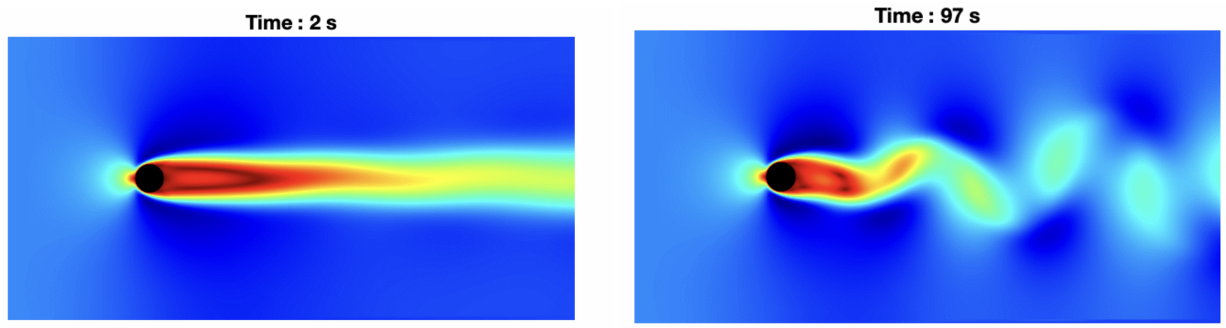

Figure 1: Velocity snapshots of a 2 dimensional flow past a cylinder.

(a) Prepare the data for the analysis. Load the provided data from

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Cylinderflow.mat
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into memory (in either python or Matlab). It contains measurements of the two velocity components (u, v) of the system at discrete spatial locations and at discrete time snapshots,

$$u(x_i, y_j, t_k) \quad v(x_i, y_j, t_k)$$

stored in a multi dimensional array of size $(N_x, N_y, 2, m)$. Flatten the first 3 dimensions to get a single matrix X of size $n \times m$, where n is the number of total observations $n = 2N_xN_y$. Similarly, generate the matrix Y by shifting X one time step into the future.

(b) Fit a DMD model to the matrices X and Y. What are the leading DMD eigenvalues that you obtain? Visualize some of the leading DMD modes by plotting the u component on the original grid. Note that you will need to reshape the eigenvector to an array of size $(N_x, N_y, 2)$.

(c) Predict the time evolution of the second fluid flow that you find in the same datafile. Make a prediction by advancing its initial vector in time using the matrix B. Visualize the prediction by plotting the time evolution of the kinetic energy

$$E_k = \frac{1}{2}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} u(x_i, y_j, t_k)^2 + v(x_i, y_j, t_k)^2.$$

This can be easily computed as $\frac{1}{2}||x_k||^2$ if x_k is the flattened data vector. Does this match the true time evolution?