Nonlinear Dynamics & Chaos I

Exercice Set 5 Solutions

Question 1

Consider the quadratic *Duffing equation*

$$\begin{aligned}
\dot{u} &= v, \\
\dot{v} &= \beta u - u^2 - \delta v,
\end{aligned}$$

where $\delta > 0$, and $0 \leq |\beta| \ll 1$.

(a) Construct a β-dependent center manifold up to quadratic order near the origin for small β values.

(b) Construct a stability diagram for the reduced system on the center manifold using β as a bifurcation parameter.

Solution 1

(a) Linearized dynamics around fixed point $(0, 0)$

$$\dot{\eta} = A\eta \ , \ A = \begin{bmatrix} 0 & 1 \\ \beta & -\delta \end{bmatrix} \ , \ \mathrm{eig}(A) = \lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \beta}$$

Note that $\lambda_1 = 0$, $\lambda_2 = -2\delta$ for $\beta = 0$. Thus, by the center manifold theorem, we have a 1-dimensional center manifold passing through the origin and a unique 1-dimensional stable manifold.

- Consider the extended system

$$\begin{aligned}
&\dot{\beta} = 0 \\
&\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\delta \end{bmatrix}}_{B} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \beta u - u^2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&\text{Eigenvalues of } B : \lambda_1 = 0 \ , \ \lambda_2 = -\delta \\
&\text{Eigenvectors of } B : e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \ e_2 = \begin{bmatrix} \frac{1}{\delta} \\ -1 \end{bmatrix}
\end{aligned}$$

From the eigenvalues and eigenvectors, we can perform a change of coordinates

$$\begin{bmatrix} u \\ v \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix} \ , \ T = [e_1 | e_2] = \begin{bmatrix} 1 & \frac{1}{\delta} \\ 0 & -1 \end{bmatrix} \ , \ T^{-1} = \begin{bmatrix} 1 & \frac{1}{\delta} \\ 0 & -1 \end{bmatrix} = T$$

$$\implies u = x + \frac{y}{\delta} \ , \ v = -y$$

$$
\begin{aligned}
\implies \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= T^{-1}BT \begin{bmatrix} x \\ y \end{bmatrix} + T^{-1} \begin{bmatrix} 0 \\ \beta u - u^2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & -\delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{\delta}\left(\beta\left(x + \frac{y}{\delta}\right) - \left(x + \frac{y}{\delta}\right)^2\right) \\ -\beta\left(x + \frac{y}{\delta}\right) + \left(x + \frac{y}{\delta}\right)^2 \end{bmatrix}
\end{aligned} \tag{1}
$$

Seek center manifold as a graph over center subspace locally as

$$
\begin{aligned}
y &= h(x, \beta) = a_1 x^2 + a_2 x\beta + \cancel{a_3\beta^2} + \mathcal{O}(3) \\
\dot{y} &= \frac{\partial h}{\partial x}\dot{x} + \frac{\partial h}{\partial \beta}\cancelto{0}{\dot{\beta}}
\end{aligned} \tag{2}
$$

Note: We cancel the term $a_3\beta^2$ to respect the existence of the fixed point.

Use invariance in (2):

$$
\implies \dot{y} = (2a_1 x + a_2\beta)\left[\frac{1}{\delta}\left(\beta\left(x + \frac{h(x,\beta)}{\delta}\right) - \left(x + \frac{h(x,\beta)}{\delta}\right)^2\right)\right] \tag{3}
$$

$$
\text{But also } \dot{y} = -\delta h(x,\beta) - \beta\left(x + \frac{h(x,\beta)}{\delta}\right) + \left(x + \frac{h(x,\beta)}{\delta}\right)^2 \tag{4}
$$

Comparing $\mathcal{O}(2)$ terms in (3) & (4), we get:

$$
\begin{aligned}
x^2 : &\qquad -\delta a_1 + 1 = 0 \implies a_1 = \frac{1}{\delta} \\
x\beta : &\qquad -\delta a_2 - 1 = 0 \implies -a_2 = \frac{1}{\delta}
\end{aligned}
$$

Thus, the β-dependent center manifold is given by

$$
h(x,\beta) = \frac{x^2}{\delta} - \frac{x\beta}{\delta} + \mathcal{O}(3) \tag{5}
$$

Substitute (5) into first equation in (1) to obtain reduced dynamics on the center manifold: $W^C_\beta(0)$ up to quadratic order.

$$
\begin{aligned}
\dot{x} &= \frac{1}{\delta}\left[\beta\left(x + \frac{h(x,\beta)}{\delta}\right) - \left(x + \frac{h(x,\beta)}{\delta}\right)^2\right] \\
&= \frac{1}{\delta}[\beta x - x^2] + \mathcal{O}(3)
\end{aligned}
$$

(b)

$$
\dot{x} = \frac{1}{\delta}[\beta x - x^2]
$$

Fixed points:

$$
\begin{aligned}
x &= 0, \\
\beta &= x
\end{aligned}
$$

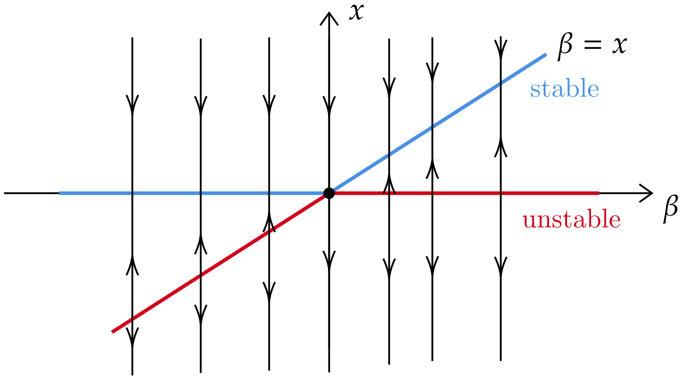


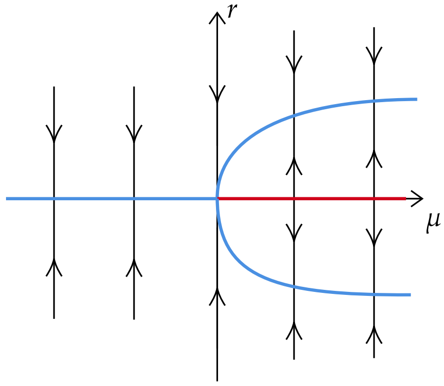
Figure 1: Transcritical bifurcation

Solution 2

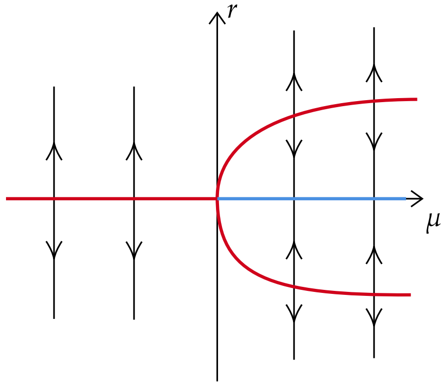
Assume that a dynamical system, depending on a parameter μ, undergoes a subcritical Hopf bifurcation at $\mu = 0$. Let

$$\begin{cases} \dot{r} = r(d_0\mu + a_0 r^2) \\ \dot{\theta} = \omega + e_0 r^2 + b_0\mu \end{cases}$$

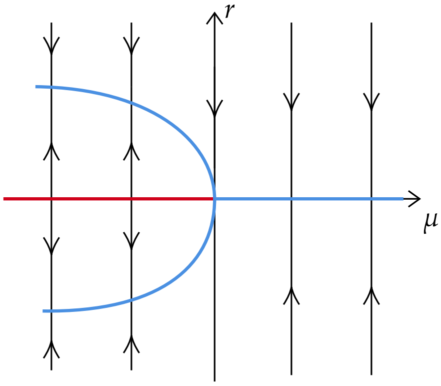
be the truncated normal form on the center manifold W^c_μ in polar coordinates. Which figure represents the correct bifurcation diagram for this system?



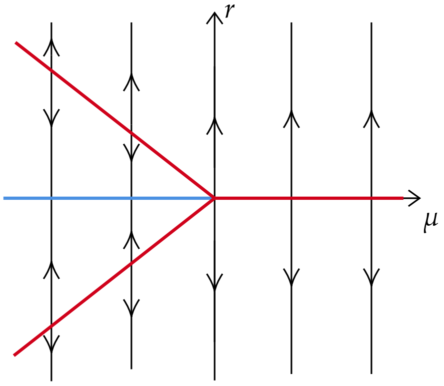
(a)



(b)



(c)

(d) 

Solution 3

Assume that the dynamical system $\dot{\mathbf{x}} = f(\mathbf{x}, \mu)$, $(\mathbf{x} \in \mathbb{R},\ \mu \in \mathbb{R})$ undergoes a codimension 1 bifurcation at $y = 0$. If $f(-x, \mu) = -f(x, \mu)$, what type of bifurcation is possible at $\mu = 0$?

(a) Saddle-node

(b) Transcritical

(c) Pitchfork

(d) None

Solution 4

Consider the dynamical system

$$\dot{x} = A(\mu)x + f(x; \mu)$$

where $x \in \mathbb{R},\ f(x, 0) = -f(-x, 0),\ \forall x \in \mathbb{R},\ \mu \in \mathbb{R},\ f \in C^1$. Which of the following statements are true?

(a) This system cannot have a saddle-node bifurcation at $\mu = 0$.

(b) This system will have either a Hopf bifurcation or a transcritical bifurcation at $\mu = 0$.

(c) This system has a hyperbolic fixed point at $x = 0$, and hence cannot have a bifurcation at $\mu = 0$.

(d) None of the above

The normal form for saddle-node bifurcation is $\dot{x} = \mu - x^2$, which the above scalar system cannot be transformed into since the right-hand-side is an odd function.

Solution 5

Consider a dynamical system

$$\dot{x} = A(\mu^2)x + f(x, \mu)$$

where $x \in \mathbb{R}^2,\ A \in \mathbb{R}^{2\times 2},\ \mu \in \mathbb{R},\ f(x, \mu) = \mathcal{O}(|x|^2),\ \nabla \cdot f(x) < 0$ for $|x| \ll 1$ where the 2×2 matrix depends on μ^2. Assume that A(0) has a purely imaginary pair of eigenvalues.
Which of the following statements are true?

(a) This system has a subcritical Hopf bifurcation at $\mu = 0$.

(b) This system has a supercritical Hopf bifurcation at $\mu = 0$.

(c) The $x = 0$ fixed point does not undergo a Hopf bifurcation.

(d) The $x = 0$ fixed point undergoes a Hopf bifurcation, but its type cannot be determined from the information given.

Refer to the Hopf-Bifurcation Theorem on Page 94 of the lecture notes. We must have $d_0 \neq 0$.

$$d_0 = \frac{d}{d\mu}\mathrm{Re}[\lambda_\mu]\big|_{\mu=0}.$$

Here,

$$\begin{aligned} \lambda_\mu &= \lambda(\mu^2) \\ \implies d_0 &= \frac{d}{d\mu}[\mathrm{Re}(\lambda(\mu^2))]\big|_{\mu=0} \\ &= 0. \end{aligned}$$