Nonlinear Dynamics and Chaos I.
Problem set 3

1. The first three modes of a convecting fluid motion in a two-dimensional layer heated from below are described by the famous *Lorenz equations*

$$\begin{aligned}\dot{x} &= a(y-x),\\ \dot{y} &= bx - y - xz,\\ \dot{z} &= xy - cz.\end{aligned}$$

Here $a > 0$ denotes the Prandtl number, $b > 0$ is the Rayleigh number, and $c > 0$ is the aspect ratio. Lorenz's original assumption is that $a > 1 + c$.

The above equations describe the evolution of the amplitudes of one velocity mode and two temperature modes. As a paradigm of chaotic dynamics, this system has inspired much of the development of the modern geometric theory of dynamical systems.

(a) Complicated dynamics arise when all possible equilibria of the system become unstable, and hence solutions cannot settle down to any simple steady state. Show that this is the case when $b > a(3+a+c)/(a-c-1)$. (*Note:* A negative sign for one of the Routh-Hurwitz determinants actually implies instability, not just the lack of asymptotic stability.)

(b) Solve the Lorenz equations numerically for $a = 10$, $b = 28$, and $c = 8/3$, choosing an initial condition close to $x = y = z = 0$. Plot the trajectory in three dimensions to show that it converges to a complicated surface, a *chaotic attractor.*

2. Recall from the last problem set that a ball of mass m sliding on a hoop rotating with angular velocity Ω satisfies the differential equation

$$mR^2\ddot{\alpha} + mR^2\left(g/R - \Omega^2\cos\alpha\right)\sin\alpha = 0 \tag{1}$$

if there is no friction between the hoop and the mass. Assume that the parameter values are such that the lower equilibrium position of the ball is stable in linear approximation.

(a) Show that in this case, the equilibrium is also nonlinearly stable. (*Hint*: Note that system (1) is not conservative: an external torque is required to maintain the constant rotation of the hoop. As a result, the total mechanical energy of the system is not expected to work as a Lyapunov function. To find another candidate for a Lyapunov function, find a quantity that *is* conserved along trajectories. Such a quantity can be found by multiplying equation (1) by $\dot{\alpha}$ and integrating in time.)

(b) Prove that the equilibrium *cannot* be asymptotically stable for the nonlinear system (*Hint*: use the Lyapunov function you have found in (a))

3. Consider the damped pendulum equation

$$\ddot{x} + c\dot{x} + \sin x = 0. \tag{2}$$

(a) Using the energy of the pendulum as a Lyapunov function, can we conclude the nonlinear asymptotic stability of the $x = 0$ equilibrium? (Give detailed reasoning why.)

(b) A theorem due to Krasovski states the following: Assume that $x = 0$ is a fixed point for the n-dimensional dynamical system $\dot{x} = f(x)$. Assume that there exists a smooth scalar function $V(x)$ such that (i) $V(x)$ is positive definitive on an open neighborhood U of $x = 0$ (ii) $\dot{V}$ is negative semi-definite on the same neighborhood (ii) the only trajectory lying *completely* in the set $S = \Big\{ x \in U \,:\, \dot{V} = 0 \Big\}$ is the fixed point $x = 0$. Then $x = 0$ is asymptotically stable.

Use Krasovski's theorem to conclude the asymptotic stability of the origin for (2).

4. Consider an n-degree-of-freedom holonomic mechanical system (i.e., one that has only position-dependent constraints) with generalized coordinates $q \in \mathbb{R}^n$ and generalized velocities $\dot{q} \in \mathbb{R}^n$. The total energy of the system is of the form

$$E(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + V(q),$$

where $M \in \mathbb{R}^{n\times n}$ is the mass matrix (symmetric and positive definite), and $V(q)$ is the potential energy. In the absence of external forces, the associated Lagrangian equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0,$$

where $L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q)$ is the Lagrangian of the mechanical system.

Show that if $V(q)$ admits a strict local minimum at a point q_0, then q_0 is a (nonlinearly) stable equilibrium for the mechanical system. (This result is also known as *Dirichlet's Theorem* in classical mechanics).