

Nonlinear Dynamics and Chaos

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Chapter 0

Introduction

First we shall introduce the most important characters in our following exploration. The ideas and definitions here will be recurring regularly as we examine them from different perspectives and using different tools.

Definition 0.1 (Dynamical System (DS)). A triple (P, E, \mathcal{F}) , with

- P : the phase space for the dynamical variable $x \in P$,
- E : base space of the evolutionary variable (e.g. time) $t \in E$,
- \mathcal{F} : the evolution rule (deterministic) which defines the transition from one state to the next.

The two main types of evolutionary variable spaces are

- (i) Discrete dynamical systems (DDS) $t \in E = \mathbb{Z}$ with trajectory $\{x_0, x_1, \dots\}$,
- (ii) Continuous dynamical systems (CDS) $t \in E = \mathbb{R}$ with trajectory $\{x_t\}_{t \in \mathbb{R}}$.

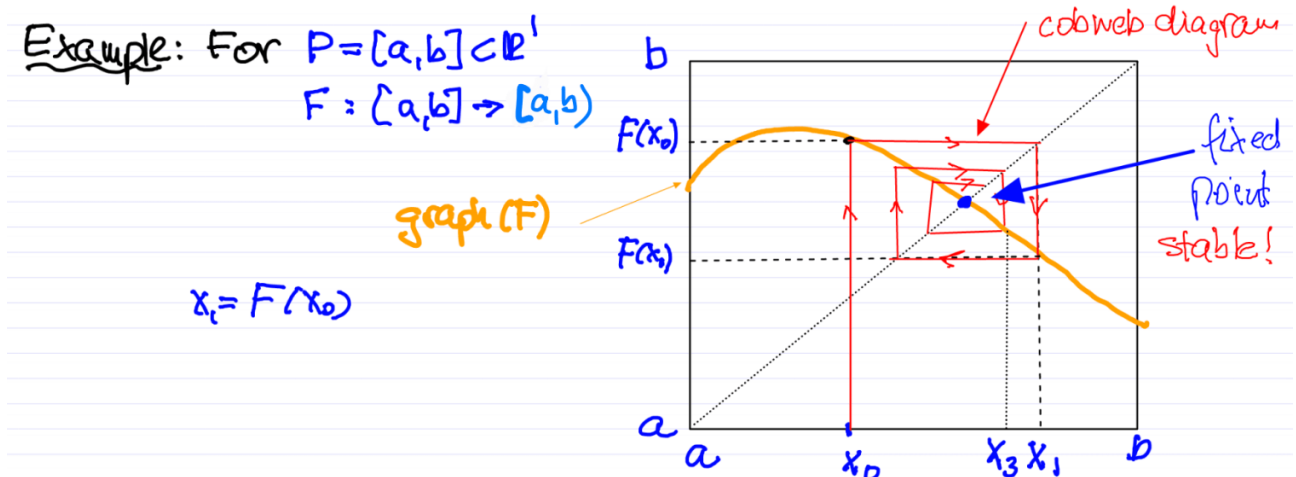
Corresponding to these there are various types of evolution rules

- (i) In a DDS we have iterated mappings

$$x_{n+1} = F(x_n, n).$$

If there is no explicit dependence on n , i.e. $\frac{\partial F}{\partial n} = 0$, then

$$x_{n+1}F(x_n) = F(F(x_{n-1})) = \underbrace{F \circ \dots \circ F}_{n+1 \text{ times}}(x_0) = F^{n+1}(x_0).$$



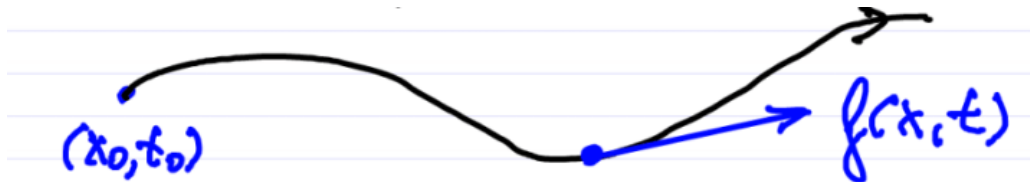
Example 0.1.

(ii) In a CDS we have a first order system of ordinary differential equations (ODE)

$$\dot{x} = f(x, t)$$

for $x \in P$ and $t \in E$. This yields the initial value problem (IVP):

$$\begin{cases} \dot{x} = f(x, t) \\ x(t_0) = x_0 \end{cases}$$



Assuming there exists a unique solution $\varphi(t; t_0, x_0)$ with $\dot{\varphi} = f(\varphi, t)$ and $\varphi(t_0) = x_0$, then the following flow map is well defined

$$F_{t_0}^t(x_0) := \varphi(t; t_0, x_0).$$

Such an $F_{t_0}^t$ has nice properties

- $F_{t_0}^t$ is as smooth as $f(x, t)$,
- $F_{t_0}^{t_0} = I$ and $F_{t_0}^{t_2} = F_{t_1}^{t_2} \circ F_{t_0}^{t_1}$ (group property),

- $(F_{t_0}^t)^{-1} = F_t^{t_0}$ exists and is smooth.

A special case of this is the autonomous system

$$\dot{x} = f(x).$$

The autonomy of a system implies

$$x(s, t_0, x_0) = x(\underbrace{s - t_0}_t, 0, x_0) \stackrel{!}{=} x(t, x_0).$$

And the induced flow map in this case is the one-parameter family of maps

$$F^t = F_0^t : x_0 \mapsto x(t, x_0).$$

Example 0.2 (Logistic Equation). For a resource-limited population, we have the following dynamic system for $a > 0$, $b > 0$, and the population $x \in \mathbb{R}_+ \cup \{0\}$

$$\dot{x} = ax(b - x).$$

In this case we have $E = \mathbb{R}$ and $\mathcal{F} = \{F^t\}_{t=-\infty}^{+\infty}$. This system has globally existing unique solutions (see later).

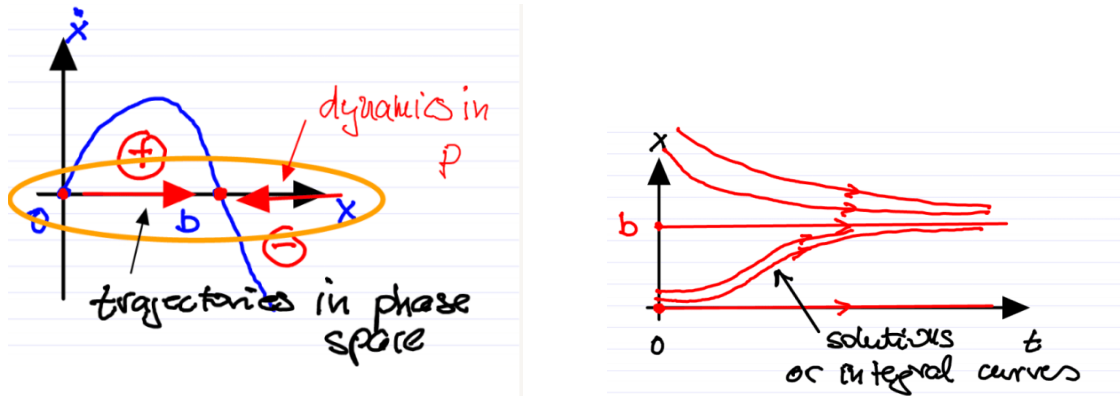


Figure 1: Left: Analysis of the right hand side. Right: Evolution in the extended phase space $P \times \mathbb{R}$.

Example 0.3 (Pendulum). Given the equation of motion

$$ml^2\ddot{\varphi} = -mgl \sin(\varphi).$$

We let $x_1 = \varphi$ and $x_2 = \dot{\varphi}$ to transform into the first-order ODE form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l} \sin(x_1). \end{cases}$$

Thus we have

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad f(x) = \begin{pmatrix} x_2 \\ -\frac{g}{l} \sin(x_1) \end{pmatrix}.$$

Qualitatively analysis gives the following facts

- $(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (\pi, 0)$ are zeros of f ,
- Energy is conserved, hence both small and large amplitude oscillations are expected,
- We have the symmetries $(x_1, x_2, t) \mapsto (x_1, -x_2, -t)$ and $(x_1, x_2, t) \mapsto (-x_1, x_2, -t)$.

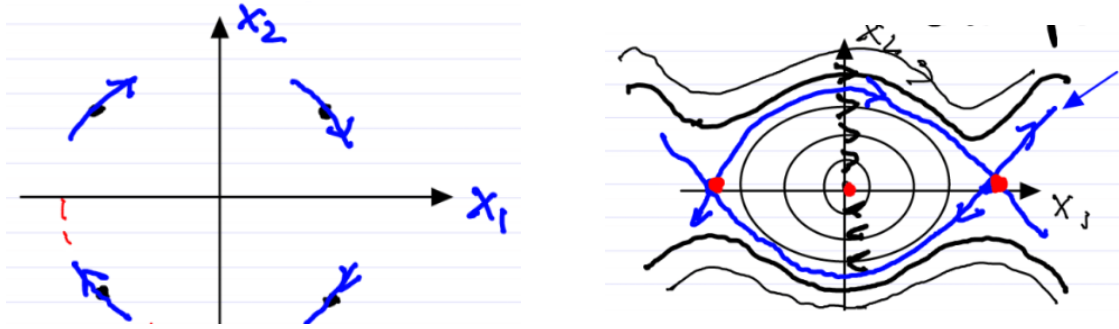


Figure 2: Left: The symmetries of the dynamic system. Right: Phase portrait of the pendulum. The blue trajectories are separatrix.

Definition 0.2. A separatrix connects fixed points, is unobservable by itself, and separates regions of similar behavior.

Example 0.4 (Exploit geometry of phase space for analysis). Two bikes *can* make it from A to B on different routes without exceeding distance D . Assume two trucks are trying to make it between A and B , on different roads in the opposite direction, carrying load of width D . Can the trucks make it without hitting each other?

The two trajectories must intersect by continuity, thus at that point the trucks must be at the same positions as the bikes, implying they are within distance D . Therefore the trucks must crash!

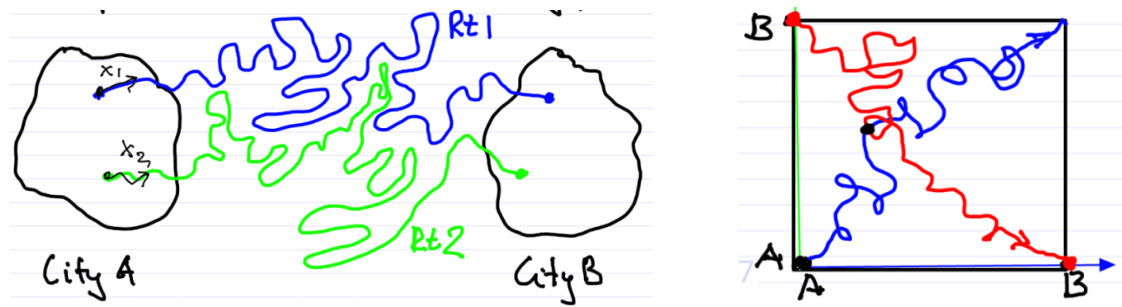


Figure 3: Left: An example of the two bike routes. Right: Blue represents the phase trajectory of the two bikers, red represents the phase trajectory of the two trucks.

Chapter 1

Fundamentals

In this chapter, we first review some fundamental properties of continuous dynamical systems that will be used heavily in later chapters. As we will see, these technical results are interesting in their own right. They can help in interpreting or cross-checking numerical results or physical models for self-consistency or accuracy.

1.1 Existence and uniqueness of solutions

Consider

$$\begin{cases} \dot{x} = f(x, t); & x \in \mathbb{R}^n \\ x(t_0) = x_0 \end{cases}.$$

Does this initial value problem have a unique solution? We have the following theorems to help us answer that question.

Theorem 1.1 (Peano). *If $f \in C^0$ near (x_0, t_0) , then there exists a local solution $\varphi(t)$, i.e.,*

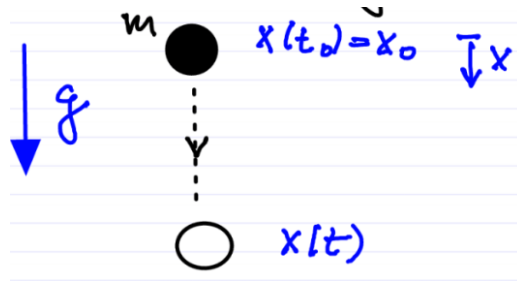
$$\dot{\varphi}(t) = f(\varphi(t), t), \varphi(t_0) = x_0; \quad \forall t \in (t_0 - \epsilon, t_0 + \epsilon); \quad 0 < \epsilon \ll 1.$$

Example 1.1 (Free falling mass). We have that the total energy is conserved

$$\frac{1}{2}m\dot{x}^2 = mg(x - x_0).$$

This implies that

$$\begin{cases} \dot{x} = \sqrt{2g(x - x_0)} \\ x(0) = x_0 \end{cases}$$

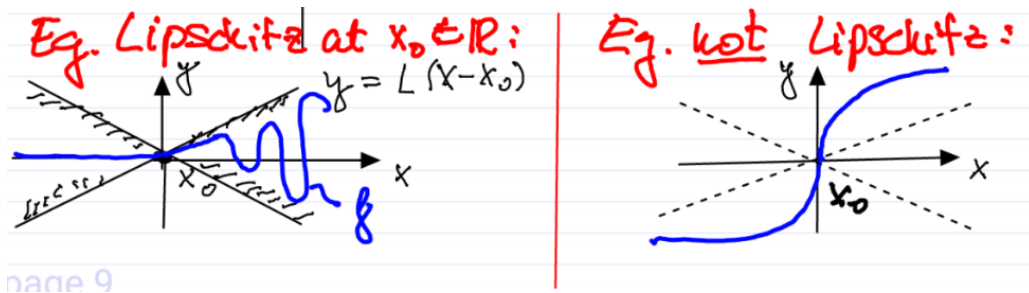


on the set $P = \{x \in \mathbb{R} : x \geq x_0\}$. Therefore we have that $f \in C^0$ in phase space, so by Peano's theorem (cf. ??), there exists a local solution. The solution is actually $x(t) = x_0 + \frac{g}{2}(t - t_0)^2$, however $x(t) = x_0$ is also a solution to the IVP, therefore we do not have a unique solution. Physically there exists a solution, but this IVP was derived from a heuristic energy-principle, not from Newton's laws, which are not equivalent.

Definition 1.1. A function is called locally Lipschitz around x_0 if there exists an open set U_{x_0} and $L > 0$ such that for all $x, y \in U_{x_0}$

$$|f(y, t) - f(x, t)| \leq L|y - x|.$$

Example 1.2 (Lipschitz functions). Here we have an example of a Lipschitz and a non-Lipschitz function around x_0 .



Theorem 1.2 (Picard). Assume

- (i) $f \in C^0$ in t near (t_0, x_0) ,
- (ii) f is locally Lipschitz in x near (t_0, x_0) .

Then there exists a unique local solution to the IVP. The proof from Arnold's ODE.

Note f is $C^1 \implies f$ is Lipschitz $\implies f$ is C^0 .

Example 1.3 (Free falling mass revisited). We check if f is Lipschitz.

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} = \frac{\sqrt{2g}}{\sqrt{|x - x_0|}} \not\leq L|x - x_0|.$$

Thus f is not Lipschitz near x_0 .

1.2 Geometric consequences of uniqueness

If the solution is unique, we have a few facts that can be derived from the geometric point of view.

- (i) The trajectories of autonomous systems cannot intersect. Note that fixed points do not violate this (e.g. pendulum equations).

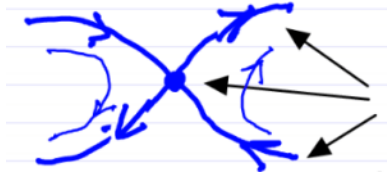


Figure 1.1: The phase portrait of the pendulum. Trajectories do not intersect since each arrow is pointing at separate trajectories.

- (ii) For non-autonomous systems, intersections in phase space are possible. In which case we can extend the phase space in order to get an autonomous system where there cannot be any intersections.

$$X = \begin{pmatrix} x \\ t \end{pmatrix}, \quad F(X) = \begin{pmatrix} f(x, t) \\ 1 \end{pmatrix}; \quad \dot{X} = F(X).$$

1.3 Local vs global existence

Example 1.4 (Exploding solution).

$$\begin{cases} \dot{x} = x^2 \\ x(t_0) = 1. \end{cases}$$

Integrating yields the solution $x(t) = \frac{1}{1-(t-t_0)}$. This solution blows up at $t_\infty = t_0 + 1$, therefore the solution is only local.

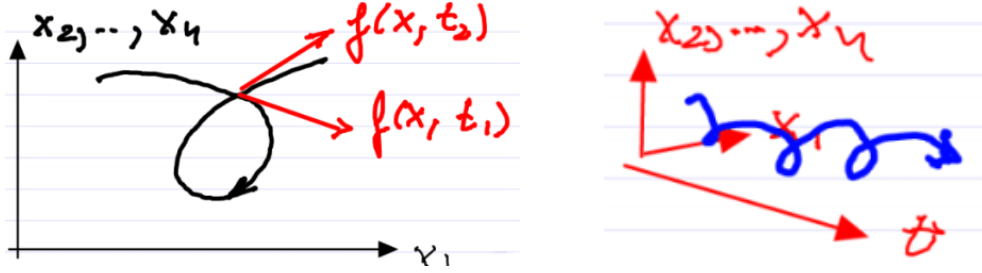
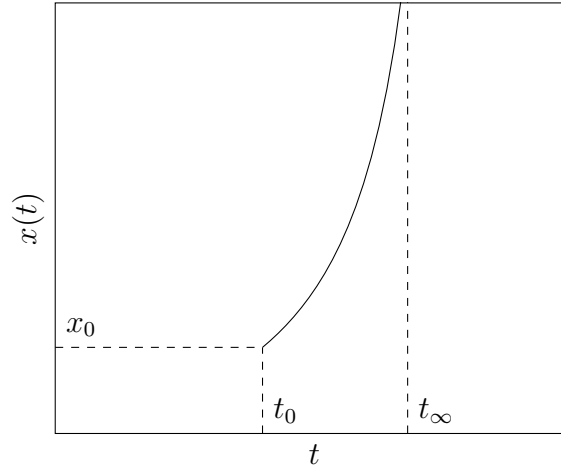


Figure 1.2: Left: Intersecting trajectories in phase space for a non-autonomous system. Right: The same trajectory in the extended phase space, without intersections.



To address this problem of local solutions not being able to be continued into global solution, we have the following theorem.

Theorem 1.3 (Continuation of solution). *If a local solutions cannot be continued to to a time T , then we must have*

$$\lim_{t \rightarrow T} |x(t)| = \infty.$$

The proof from Arnold's ODE.

Example 1.5 (Coupled Pendulum System). We set $x_1 = \varphi_1$, $x_2 = \dot{\varphi}_1$, $x_3 = \varphi_2$, $x_4 = \dot{\varphi}_2$ and get the following equation of motion

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dots \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \dots \end{cases}$$

The RHS is smooth, therefore there exists a unique local solution to any IVP. The phase space

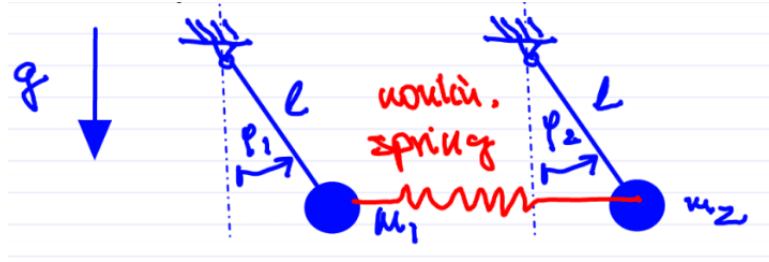


Figure 1.3: Physical setup of the coupled pendulum with a nonlinear spring.

is given by

$$P = \{x : x_1 \in S^1, x_2 \in \mathbb{R}, x_3 \in S^1, x_4 \in \mathbb{R}\} = S^1 \times \mathbb{R} \times S^1 \times \mathbb{R}.$$

Where S^1 is the 1 dimensional sphere (i.e. a circle). With this space we know that $|x_1|$ and $|x_3|$ are bounded. Due to energy being conserved we have

$$E = T + V = \frac{1}{2}m_1 l_1^2 \dot{x}_2^2 + \frac{1}{2}m_2 l_2^2 \dot{x}_4^2 + \underbrace{V(x_1, x_3)}_{\geq 0}$$

$$E = E_0 = \text{constant} \geq 0.$$

Hence $|x_2|$ and $|x_4|$ are also bounded, therefore all solutions exist globally.

Definition 1.2. A linear system is one such that for $x \in \mathbb{R}^n$, $A(t) \in \mathbb{R}^{n \times n}$ and $A \in C^0$

$$\dot{x} = A(t)x.$$

Remark 1.4. Note that $S = \frac{1}{2}(A + A^T)$ is symmetric (i.e. $S = S^T$) and $\Omega = \frac{1}{2}(A - A^T)$ is skew symmetric (i.e. $\Omega = -\Omega^T$). Furthermore the eigenvalues of S , λ_i , are all real and their respective eigenvectors, e_i , are orthogonal.

Example 1.6 (Global existence in linear systems).

$$\begin{aligned} \langle x, \dot{x} \rangle &= \frac{1}{2} \frac{d}{dt} |x(t)|^2 = \langle x, A(t)x \rangle = \langle x, (S(t) + \Omega(t))x \rangle \\ &= \langle x, S(t)x \rangle + \underbrace{\langle x, \Omega(t)x \rangle}_{=0} \stackrel{(*)}{=} \sum_{i=1}^n \lambda_i(t) x_i^2 \\ &\leq \lambda_{\max}(t) \sum_{i=1}^n x_i^2 = \lambda_{\max}(t) |x(t)|^2. \end{aligned}$$

Where in (*) we used that $x = \sum_{i=1}^n x_i e_i$ with $|e_i| = 1$ and $e_i \perp e_j$ for all $i \neq j$. Thus we get

$$\frac{\frac{1}{2} \frac{d}{dt} |x(t)|^2}{|x(t)|^2} \leq \lambda_{\max}(t) \implies \int_{t_0}^t \log \left(\frac{|x(s)|^2}{|x(t_0)|^2} \right) ds \leq \lambda_{\max}(s) ds.$$

By exponentiating both sides, we obtain

$$|x(t)| \leq |x(t_0)| \exp \left(\int_{t_0}^t \lambda_{\max}(s) ds \right).$$

Therefore, by the continuation theorem, global solutions exist as long as $\int_{t_0}^t \lambda_{\max}(s) ds < \infty$.

1.4 Dependence on initial conditions

Given the IVP

$$\begin{cases} \dot{x} = f(x, t) \\ x(t_0) = x_0. \end{cases}$$

With $x \in \mathbb{R}^n$ and $f \in C^r$ for some $r \geq 1$, we have the solution $x(t; t_0, x_0)$.

Question How does the solution depend on initial data? But first, why do we care about this? Because we robust solutions with respect to errors and uncertainties in the initial data.

Theorem 1.5. *If $f \in C^r$ for $r \geq 1$ then $x(t; t_0, x_0)$ is C^r in (t_0, x_0) . Proof in Arnold's ODE.*

The geometric meaning of this is that for $U \subset P \subset \mathbb{R}^n$ we have that $F_{t_0}^t(U)$ is a smooth deformation of U . It turns out $(F_{t_0}^t)^{-1} = F_t^{t_0}$ is also C^r , hence we have that $F_{t_0}^t$ is a diffeomor-

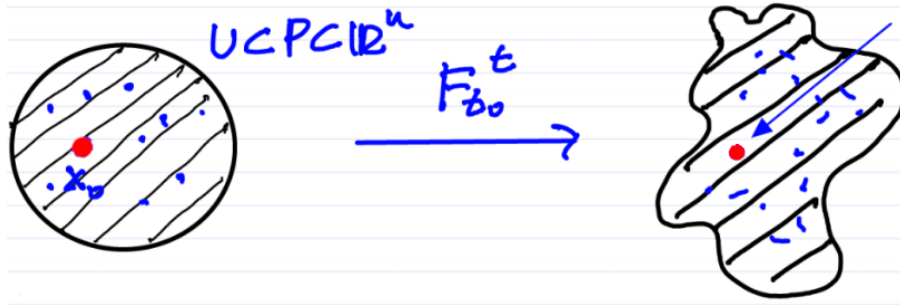


Figure 1.4: The smooth transformation of U . The red point on the right is $F_{t_0}^t(x_0)$, i.e. the image of x_0 through the evolution operator.

phism.

Now, how can we compute the Jacobian of the flow map $\frac{\partial x(t; t_0, x_0)}{\partial x_0} = DF_{t_0}^t(x_0)$? We will use the IVP.

$$\frac{d}{dt} \frac{\partial x}{\partial x_0} = D_x f(x(t; t_0, x_0), t) \frac{\partial x}{\partial x_0}.$$

The flow gradient satisfies the IVP

$$\begin{aligned} \frac{d}{dt} [DF_{t_0}^t(x_0)] &= D_x f(F_{t_0}^t(x_0), t) DF_{t_0}^t(x_0) \\ DF_{t_0}^{t_0}(x_0) &= I. \end{aligned}$$

This gives us the equation of variations (linear, non-autonomous)

$$\begin{cases} \dot{M} = D_x f(x(t; t_0, x_0))M \\ M(t_0) = I. \end{cases}$$