Nonlinear Dynamics and Chaos I.
Problem set 5

1. Consider a planar Hamiltonian system

$$\begin{aligned}\dot{x} &= \frac{\partial H(x,y)}{\partial y} + f_1(x,y),\\ \dot{y} &= -\frac{\partial H(x,y)}{\partial x} + f_2(x,y),\end{aligned}$$

   where the twice continuously differentiable function $H(x,y)$ is the Hamiltonian associated with the system (say, the energy in classical mechanics, or the stream function in fluid mechanics), and the continuously differentiable $\mathbf{f} = (f_1, f_2)$ is a dissipative term (say, damping in classical mechanics, or compressible terms in fluid mechanics). Assume that $\nabla \cdot \mathbf{f} \neq 0$ for all $(x,y) \in \mathbb{R}^2$. (Linear damping, for instance has this property.)

   Show that the above system can have no limit cycles.

2. (*Accuracy of averaging*) Show that on times scales of $\mathcal{O}(1/\epsilon)$, a solution $\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ of the dynamical system

$$\dot{\mathbf{x}} = \epsilon \mathbf{f}(\mathbf{x}, t, \epsilon), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

   (ϵ is a small parameter and $\mathbf{f}$ is a smooth function that is T-periodic in time) remains $\mathcal{O}(\epsilon)$-close to any solution $\mathbf{y}(t)$ with $\mathbf{y}(0) = \mathbf{x}_0 + \mathcal{O}(\epsilon)$ of the averaged system

$$\dot{\mathbf{y}} = \epsilon \bar{\mathbf{f}}_0(\mathbf{y}), \qquad \mathbf{y} \in \mathbb{R}^n, \tag{2}$$

   where $\bar{\mathbf{f}}_0(\mathbf{y}) = \frac{1}{T}\int_0^T f(y,t,0)\,dt$.

   Hint: Subtract (2) from (1) and integrate to obtain an expression for $|\mathbf{x}(t) - \mathbf{y}(t)|$. Estimate $|\mathbf{x}(t) - \mathbf{y}(t)|$ from above using the facts that $\bar{\mathbf{f}}$ is Lipschitz and $\left|\hat{f} - \bar{f}\right|/\epsilon$ is uniformly bounded, where $\hat{f}$ is the right-hand side of the system into wich (1) is transformed by the averaging transformation $\mathbf{x} = \mathbf{y} + \epsilon \mathbf{w}(\mathbf{y}, t)$. Then use the following generalized Gronwall inequality:

   If $u(t)$, $v(t)$, $c(t)$ are nonnegative functions, $c(t)$ is differentiable, and

$$v(t) \leq c(t) + \int_0^t u(s)\,v(s)\,ds,$$

   then

$$v(t) \leq c(0)e^{\int_0^t u(s)ds} + \int_0^t c'(s)\,e^{\int_s^t u(\tau)\,d\tau}\,ds.$$

3. (*Unsteady separation in time-periodic fluid flows*) Fluid trajectories $\mathbf{x}(t) = (x(t), y(t))$ in a two-dimensional time-periodic flow satisfy the differential equations

$$\begin{aligned}\dot{x} &= u(x,y,t), \qquad u(x,y,t) = u(x,y,t+T),\\ \dot{y} &= v(x,y,t), \qquad v(x,y,t) = v(x,y,t+T),\end{aligned} \tag{3}$$

   where $T > 0$ is the period, u and v are smooth velocity components satisfying the incompressibility condition $u_x + v_y \equiv 0$. Assume that a the fluid is bounded by a wall at $y = 0$, on which the velocity

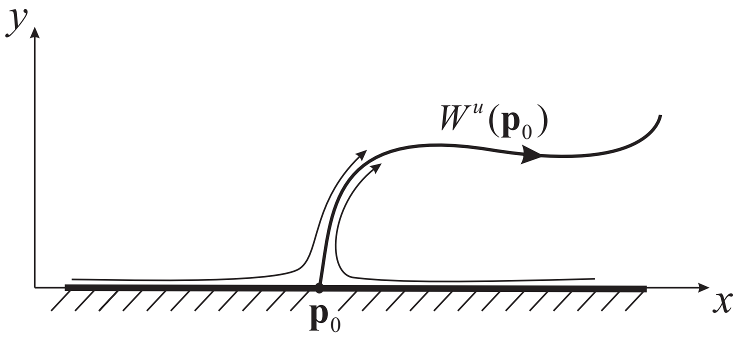

Figure 1: Unsteady separation from a no-slip wall

field satisfies the no-slip boundary conditions $u(x, 0, t) = v(x, 0, t) = 0$. As a result, all boundary points are nonhyperbolic fixed points for (3).

We say that a boundary point $\mathbf{p}_0 = (x_0, 0)$ is a separation point for the flow (3) if $\mathbf{p}_0$ admits an unstable manifold $W^u(\mathbf{p}_0)$. Physically, $W^u(\mathbf{p}_0)$ is a time-dependent curve of fluid particles that shrinks to $\mathbf{p}_0$ is backward time. In forward time, $W^u(\mathbf{p}_0)$ attracts fluid particles, then ejects them from the vicinity of the wall (see Fig. 1). Show that separation points satisfy the criteria

$$\int_0^T u_y(x_0, 0, t)\, dt = 0, \qquad \int_0^T v_{yy}(x_0, 0, t)\, dt > 0,$$

i.e., separation takes place where the average of the wall-shear is zero and the average of v_{yy} is positive.
Hint: Use incompressibility and the boundary conditions to show that (3) can be rewritten as

$$\begin{aligned} \dot{x} &= yU(x, y, t), \\ \dot{y} &= y^2 V(x, y, t). \end{aligned}$$

To focus on the vicinity of the boundary, introduce the scaled variable $y = \epsilon\eta$, where $0 \leq \epsilon \ll 1$. Show that the resulting $(\dot{x}, \dot{\eta})$ equations are slowly varying and find the corresponding averaged equations. For the averaged equations, rescale time on trajectories by letting $d\tau/dt = \eta(t)$ in order to remove the common η factor from the right-hand side, and look for a hyperbolic fixed point with an unstable manifold off the wall. Use the averaging theorem to relate your results to the original system (3).