Nonlinear Dynamics Homework 1 Solutions

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Q. 1

A periodic orbit is characterized by a string of k symbols. There is a total of 2^k possible permutations of the symbols 1 and 2 over k positions. However if this string consists of k/i repetitions of a string of length i, that periodic orbit is already an i-orbit, and so will have to be excluded from the set of k-orbits. The number of i-orbits is $N(i)$, and there will be i occurrences of k-strings consisting of repetitions of some shift of each i-orbit. Let $< i, k >$ denote the set of $i \in \mathbb{N}$ such that k/i is an integer. Then

$$2^k - \sum_{i \in <i,k>} iN(i) \tag{1}$$

is the number of permutations of two symbols that are not a repetition of some string. Furthermore, since shifting a string does not change its represented periodic orbit, there will be k shifted versions of each string, which we handle by dividing with k:

$$N(k) = \frac{1}{k}\left(2^k - \sum_{i \in <i,k>} iN(i)\right) \tag{2}$$

Q. 2

a

The space of admissible sequences is

$$\Sigma_A^N = \{s \in \Sigma^N : A_{s_i, s_{i+1}} \neq 0 \quad \forall i \in \mathbb{Z}\}. \tag{3}$$

For a fixed point $\bar{a}.\bar{a}$, $a \in \{1, \ldots, N\}$, we must therefore have $A_{a,a} \neq 0$. So the ath element on the diagonal of A must be non-zero for $\bar{a}.\bar{a}$ to be in the space of admissible sequences. Furthermore since Σ^N contains all permutations and therefore all possible fixed points, the condition $A_{a,a} \neq 0$ is both sufficient and necessary for $\bar{a}.\bar{a}$ to be an admissible fixed point. Therefore the number of fixed points is equal to the number of non-zero diagonal elements of A, which, since $A_{i,j} \in \{0, 1\}$, is equal to $\mathrm{tr}(A)$.

b

Apply the reasoning from a) to the map σ^k, i.e. the shift map repeated k times. The matrix of admissible sequences is then A repeated k times, i.e. A^k. Again if $A^k_{a,a} = 0$, there is no admissible sequence going from a back to a in k steps. Note that e.g. $A^3_{i,j} = \sum_{k,l} A_{i,k} A_{k,l} A_{l,j}$. Each of the terms represents an admissible sequence in 3 steps from i to j. The sum is the total number of such admissible sequences. As in a), all possible sequences are in Σ^N, so $A^k_{a,a}$ is the number of ways to map a into itself in k steps. Again the total number of such admissible k-periodic points is the sum of $A^k_{i,i}$, i.e. $\mathrm{tr}(A^k)$.

Q. 3

Given any two periodic orbits $\bar{s}.\bar{s}$ and $\bar{z}.\bar{z}$, any sequence s^* defines a heteroclinic orbit between them by $\bar{s}s^*.\bar{z}$. To show this is a heteroclinic orbit, observe that

$$\begin{aligned} \lim_{N\to\infty} d(\sigma^N(\bar{s}s^*.\bar{z}), \bar{z}.\bar{z}) = 0 \\ \lim_{N\to-\infty} d(\sigma^N(\bar{s}s^*.\bar{z}), \bar{s}.\bar{s}) = 0. \end{aligned} \tag{4}$$

That is, any such trajectory approaches $\bar{z}.\bar{z}$ in forward iterations and $\bar{s}.\bar{s}$ in backward iterations of the shift map. Since there are infinitely many sequences s^*, and the periodic orbits were arbitrary, it follows that there are infinitely many heteroclinic orbits connecting any two periodic orbits.

Q. 4

Fix "points" $a \in A$ and $b \in B$. Since A and B are open sets, we can choose a neighborhood U around a and V around b such that all points closer to a than δ_U lie in U, and all points closer to b than δ_V lie in V. Pick $\delta = \min(\delta_U, \delta_V)$. The existence of a dense orbit for σ in Σ implies that there are integers $N(a)$, $N(b)$, and a sequence $s^* \in \Sigma$, such that $d(\sigma^{N(a)}(s^*), a) < \delta$ and $d(\sigma^{N(b)}(s^*), b) < \delta$. Since this sequence comes closer than δ to a and b, it passes through both U and V. Set $N = N(b) - N(a)$ and $z = \sigma^{N(a)}(s^*)$. Then $z \in A$ and $\sigma^N(z) = \sigma^{N(b)}(s^*) \in B$. The claim follows.

Q. 5

Given δ and $s \neq z$, by the definition of distance in Σ there must be some finite integer $N \in \mathbb{Z}$ such that the Nth elements $s_N \neq z_N$. Now iterate both sequences N times and compute the distance:

$$
\begin{aligned}
d(\sigma^N(s), \sigma^N(z)) &= \sum_i \frac{|\sigma^N(s)_i - \sigma^N(z)_i|}{2^{|i|}} = \\
&= \sum_i \frac{|s_{i+N} - z_{i+N}|}{2^{|i|}} \geq \frac{|s_N - z_N|}{2^{|0|}} = 1.
\end{aligned} \tag{5}
$$

Thus the claim follows from setting e.g. $\Delta = \frac{1}{2}$.