Nonlinear Dynamics & Chaos I

Exercice Set 5 Questions

Question 1

Consider the quadratic *Duffing equation*

$$\dot{u} = v,$$
$$\dot{v} = \beta u - u^2 - \delta v,$$

where $\delta > 0$, and $0 \leq |\beta| \ll 1$.

(a) Construct a β-dependent center manifold up to quadratic order near the origin for small β values.

(b) Construct a stability diagram for the reduced system on the center manifold using β as a bifurcation parameter.

Question 2

Consider a dynamical system that has a pair of purely imaginary eigenvalues at its fixed point for the parameter value $\mu = 0$. As we have seen, a linear change of coordinates and a center manifold reduction gives the two-dimensional reduced dynamical system

$$\dot{x} = \delta(\mu)x - \omega(\mu)y + f(x, y, \mu), \tag{1}$$
$$\dot{y} = \delta(\mu)y + \omega(\mu)x + g(x, y, \mu), \tag{2}$$

where $\delta(\mu) = \text{Re } \lambda(\mu)$, $\omega(\mu) = \text{Im } \lambda(\mu)$. (Here $\lambda(\mu)$ and $\lambda(\bar{\mu})$ is the pair of complex eigenvalues that crosses the imaginary axis at $\mu = 0$.)
Recall that in polar coordinates, the truncated normal form of (1) can be written as

$$\dot{r} = r(d_0\mu + a_0 r^2),$$
$$\dot{\theta} = \omega_0 + b_0\mu + c_0 r^2,$$

where

$$\begin{aligned} d_0 &= \delta'(0), \quad \omega_0 = \omega(0) \\ a_0 &= \frac{1}{16}\left[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right]_{x=y=0,\mu=0} \\ &+ \frac{1}{16\omega_0}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right]_{x=y=0,\mu=0}. \end{aligned}$$

These classic formulae are used in all applications where Hopf bifurcations are analyzed.
As an application of these results, consider now the stick-slip oscillator

$$m\ddot{x} + c\dot{x} + kx = F_f, \qquad F_f = mg\mu_0\left(1 + e^{-\beta|v_0 - \dot{x}|}\right) \text{sign } (v_0 - \dot{x}),$$

where m is the mass of the oscillator, g is the constant of gravity, $\beta > 0$ is a constant, μ_0 is the Coulomb (static) friction coefficient, v_0 is the speed of the belt, x is the position of the mass on the belt, c is the coefficient of viscous damping, and k is the spring coefficient (see Fig. 1).

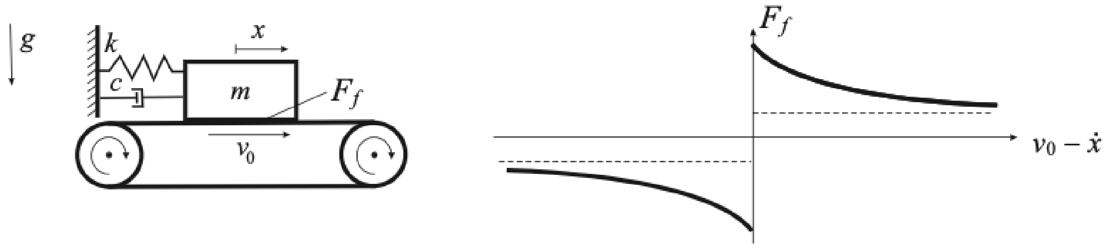

Figure 1: Stick-slip oscillator and its dry-friction force as a function of the relative velocity between the mass and the belt.

(a) Find a condition under which the system has an asymptotically stable fixed point.

(b) Show that a subcritical Hopf bifurcation takes place when the above condition is violated. (Use v_0 as a bifurcation parameter.)

(c) Calculate the approximate amplitude of the bifurcating limit cycle.