Nonlinear Dynamics & Chaos I

Exercice Set 6 Solutions

Solution 1

Which quantity is conserved along the trajectories of $\ddot{x} - x + x^3 = 0$?

(a) $H(x, \dot{x}) = \dot{x}^2 - x^2 + \frac{1}{4}x^4$

(b) $H(x, \dot{x}) = \dot{x}^2 - x^2 + \frac{1}{8}x^4$

(c) $H(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \frac{1}{4}x^3$

(d) $H(x, \dot{x}) = \dot{x}^2 - x^2 + \frac{1}{2}x^4$

Compute the derivative with respect to time of H.

$$\frac{dH}{dt} = 2\ddot{x}\dot{x} - 2\dot{x}x + 2x^3\dot{x} = 2\dot{x}(\ddot{x} - x + x^3),$$

which is zero along the trajectories of the dynamical system and hence is conserved.

Solution 2

Consider the dynamical system

$$\begin{cases} \dot{x} = y + f(x, y) \\ \dot{y} = x + g(x, y) \end{cases}$$

Where $x, y \in \mathbb{R}$. Which condition is sufficient for this dynamical system not to have a limit cycle?

(a) $f(x, y)g(x, y) < 0, \quad \forall x, y$

(b) $\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} < 0, \quad \forall x, y$

(c) $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} < 0, \quad \forall x, y$

(d) None of the above

Let

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} y + f(x, y) \\ x + g(x, y) \end{bmatrix}$$

$$\implies \nabla \cdot \mathbf{f} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \neq 0$$

Solution 3

Consider the system

$$\begin{cases} \dot{r} = r(1-r) \\ \dot{\theta} = \sin^2\left(\frac{\theta}{2}\right) \end{cases}$$

written in polar coordinates (r, θ). With the phase portrait depicted in the figure below, which statement is correct ?

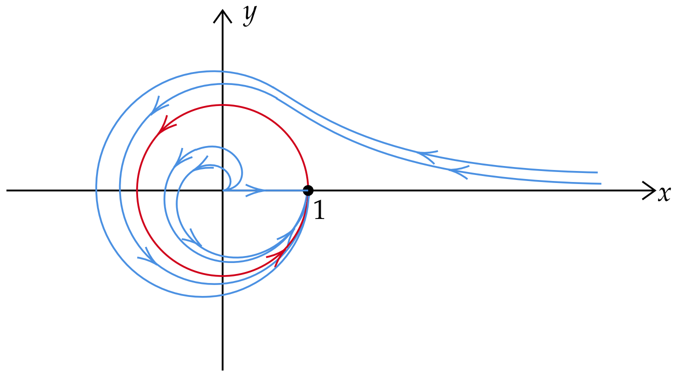


Figure 1: Phase portrait of the system.

(a) The fixed point $(x = 0, y = 0)$ is the α-limit point for any (x_0, y_0).

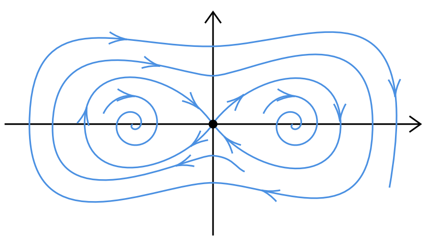
(b) The fixed point $(x = 1, y = 0)$ is asymptotically stable.

(c) The fixed point $(x = 1, y = 0)$ is the ω-limit point for any $(x_0, y_0) \neq (0, 0)$.

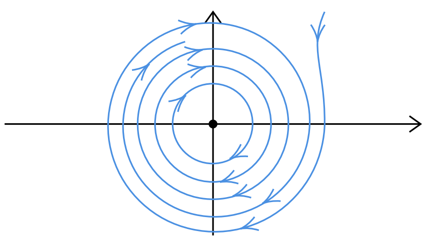
(d) The invariant curve $r = 1$ is the ω-limit point for any (x_0, y_0) with $x^2 + y^2 = 1$.

Solution 4

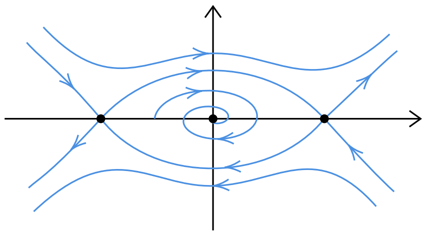
The phase portrait of four dynamical systems are depicted below. Which phase portrait is robust under small enough perturbations?



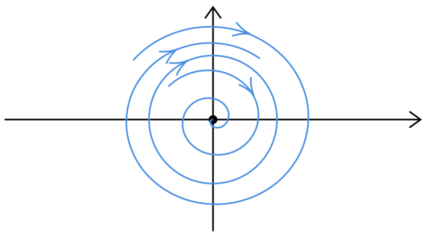
(a)



(b)



(c)



(d)

(a) has a homoclinic connection, (b) has a center-type fixed point, (c) has heteroclinic connections, all of which are structurally unstable.

Solution 5

A one-degree-of-freedom mechanical system has a first integral

$$E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + V(x)$$

where the graph of $V(x)$ is shown below. Which figure may correspond to the phase portrait of the mechanical system?

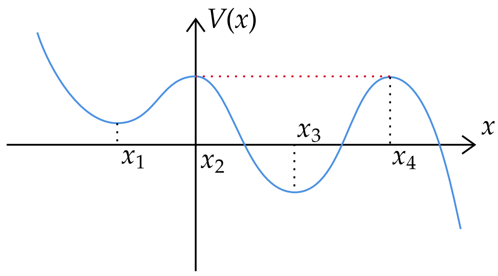
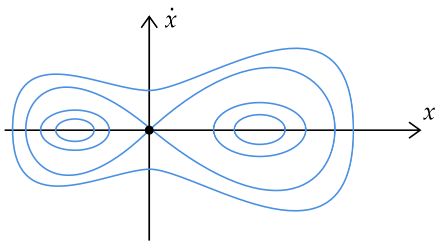
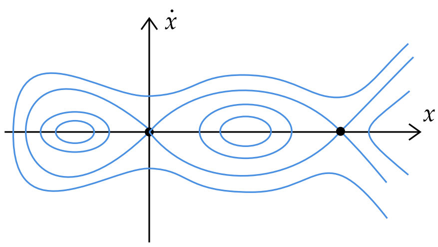


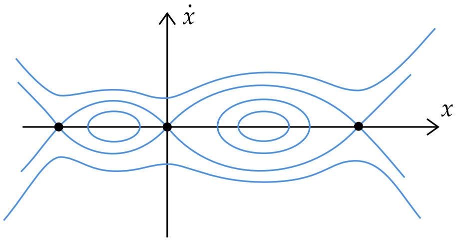
Figure 2: Potential function $V(x)$ of the mechanical system.

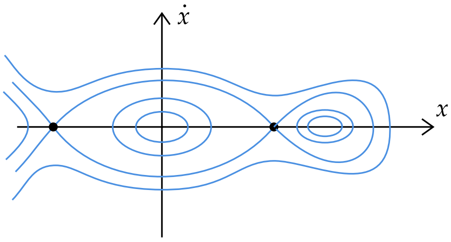


(a)



(b)

(c) 

(d) 

Solution 6

Consider a planar dynamical system with the following phase portrait:

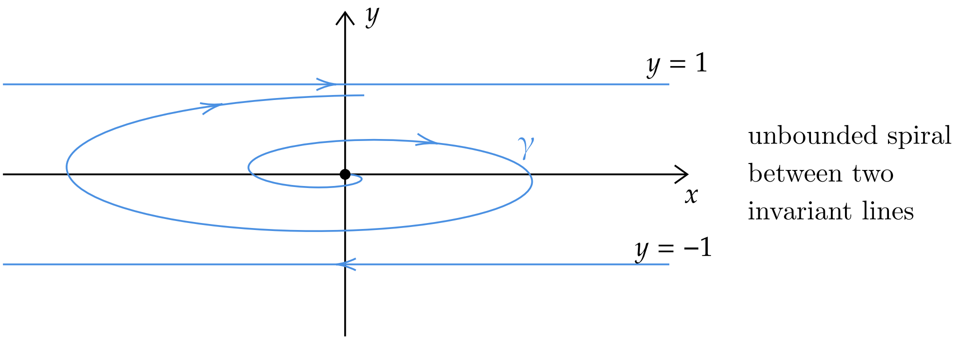


Figure 3: Phase portrait of the planar dynamical system

Which of the following statement is true?

(a) The ω-limit set of γ is empty.

(b) By the Poincaré-Bendixson theorem, the ω-limit set of γ is composed of the lines $y = 1$ and $y = -1$.

(c) The Poincaré-Bendixson theorem does not apply to γ.

(d) None of the above

Solution 7

Consider the Van der Pol equation

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0, \quad \alpha > 0, \quad x \in \mathbb{R}$$

Which of the following statements are true?

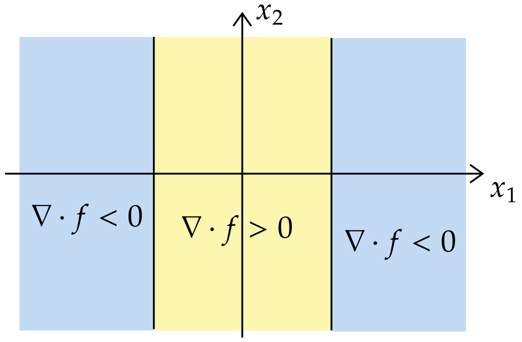
(a) This equation cannot have limit cycles.

(b) Any limit cycle of this equation must intersect at least one of the two lines $\{x = 1\}$, $\{x = -1\}$.

(c) Any limit cycles of this equation is necessarily unstable.

(d) None of the above

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2\\ -\alpha(x_1^2-1)x_2 - x_1\end{bmatrix} \quad \nabla\cdot f \qquad\qquad = -\alpha(x_1^2-1)$$



Solution 8

Consider a particle sliding frictionlessly on the following terrain:

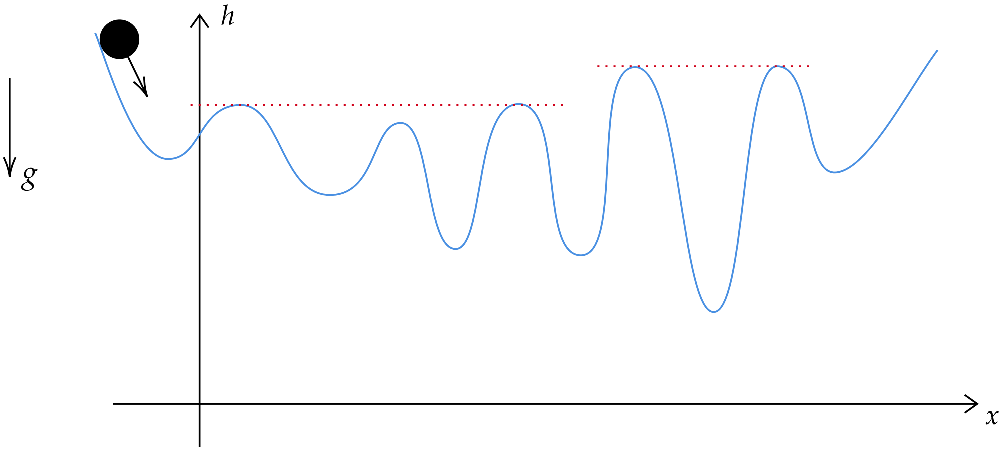
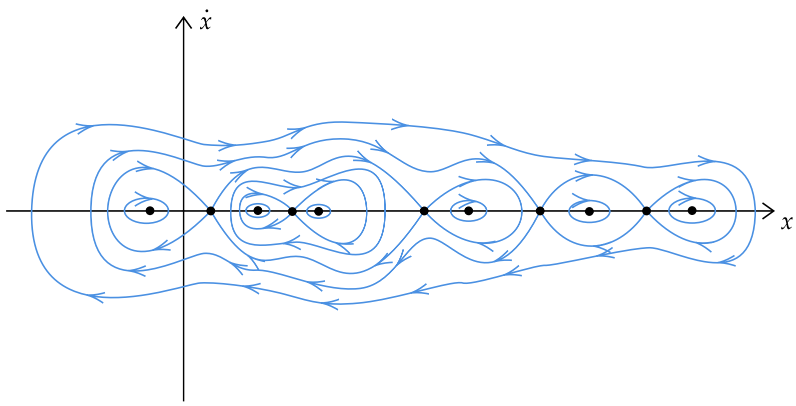


Figure 4: Terrain on which the particle slides

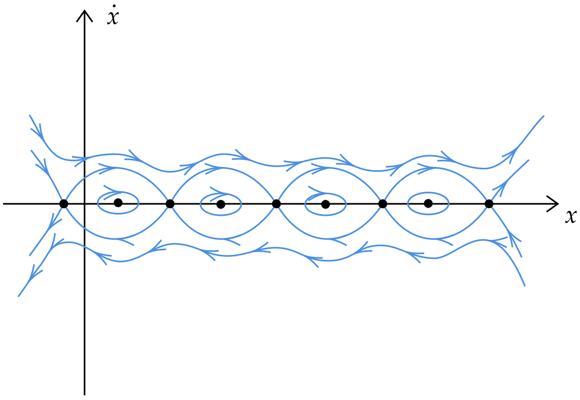
Which of the following statements are true?

(a) The phase portrait of the dynamical system describing the motion of this particle cannot be drawn, as the available information is insufficient.

(b) A qualitative sketch of the phase protait is as follows:



(c) A qualitative sketch of the phase portrait is as follows:



(d) This dynamical system must have at least one attracting limit cycle.