

# Nonlinear Dynamics & Chaos I

## Exercise Set 5 Solutions

### Question 1

Consider the quadratic *Duffing equation*

$$\begin{aligned}\dot{u} &= v, \\ \dot{v} &= \beta u - u^2 - \delta v,\end{aligned}$$

where  $\delta > 0$ , and  $0 \leq |\beta| \ll 1$ .

- (a) Construct a  $\beta$ -dependent center manifold up to quadratic order near the origin for small  $\beta$  values.
- (b) Construct a stability diagram for the reduced system on the center manifold using  $\beta$  as a bifurcation parameter.

### Solution 1

- (a) Linearized dynamics around fixed point  $(0,0)$

$$\dot{\eta} = A\eta, \quad A = \begin{bmatrix} 0 & 1 \\ \beta & -\delta \end{bmatrix}, \quad \text{eig}(A) = \lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \beta}$$

Note that  $\lambda_1 = 0$ ,  $\lambda_2 = -\delta$  for  $\beta = 0$ . Thus, by the center manifold theorem, we have a 1-dimensional center manifold passing through the origin and a unique 1-dimensional stable manifold.

- Consider the extended system

$$\begin{aligned}\dot{\beta} &= 0 \\ \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\delta \end{bmatrix}}_B \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \beta u - u^2 \end{bmatrix}\end{aligned}$$

Eigenvalues of  $B$ :  $\lambda_1 = 0$ ,  $\lambda_2 = -\delta$

Eigenvectors of  $B$ :  $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $e_2 = \begin{bmatrix} \frac{1}{\delta} \\ -1 \end{bmatrix}$

From the eigenvalues and eigenvectors, we can perform a change of coordinates

$$\begin{bmatrix} u \\ v \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}, \quad T = [e_1 | e_2] = \begin{bmatrix} 1 & \frac{1}{\delta} \\ 0 & -1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & \frac{1}{\delta} \\ 0 & -1 \end{bmatrix} = T$$

$$\implies u = x + \frac{y}{\delta}, \quad v = -y$$

$$\begin{aligned}
 \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= T^{-1} B T \begin{bmatrix} x \\ y \end{bmatrix} + T^{-1} \begin{bmatrix} 0 \\ \beta u - u^2 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & -\delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{\delta} \left( \beta \left( x + \frac{y}{\delta} \right) - \left( x + \frac{y}{\delta} \right)^2 \right) \\ -\beta \left( x + \frac{y}{\delta} \right) + \left( x + \frac{y}{\delta} \right)^2 \end{bmatrix}
 \end{aligned} \tag{1}$$

Seek center manifold as a graph over center subspace locally as

$$\begin{aligned}
 y &= h(x, \beta) = a_1 x^2 + a_2 x \beta + a_3 \beta^2 + \mathcal{O}(3) \\
 \dot{y} &= \frac{\partial h}{\partial x} \dot{x} + \frac{\partial h}{\partial \beta} \dot{\beta}
 \end{aligned} \tag{2}$$

Note: We cancel the term  $a_3 \beta^2$  to respect the existence of the fixed point.

Use invariance in (2):

$$\Rightarrow \dot{y} = (2a_1 x + a_2 \beta) \left[ \frac{1}{\delta} \left( \beta \left( x + \frac{h(x, \beta)}{\delta} \right) - \left( x + \frac{h(x, \beta)}{\delta} \right)^2 \right) \right] \tag{3}$$

$$\text{But also } \dot{y} = -\delta h(x, \beta) - \beta \left( x + \frac{h(x, \beta)}{\delta} \right) + \left( x + \frac{h(x, \beta)}{\delta} \right)^2 \tag{4}$$

Comparing  $\mathcal{O}(2)$  terms in (3) & (4), we get:

$$\begin{aligned}
 x^2 : \quad & -\delta a_1 + 1 = 0 \Rightarrow a_1 = \frac{1}{\delta} \\
 x\beta : \quad & -\delta a_2 - 1 = 0 \Rightarrow -a_2 = \frac{1}{\delta}
 \end{aligned}$$

Thus, the  $\beta$ -dependent center manifold is given by

$$h(x, \beta) = \frac{x^2}{\delta} - \frac{x\beta}{\delta} + \mathcal{O}(3) \tag{5}$$

Substitute (5) into first equation in (1) to obtain reduced dynamics on the center manifold:  $W_\beta^C(0)$  up to quadratic order.

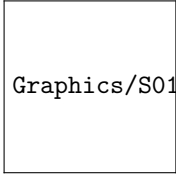
$$\begin{aligned}
 \dot{x} &= \frac{1}{\delta} \left[ \beta \left( x + \frac{h(x, \beta)}{\delta} \right) - \left( x + \frac{h(x, \beta)}{\delta} \right)^2 \right] \\
 &= \frac{1}{\delta} [\beta x - x^2] + \mathcal{O}(3)
 \end{aligned}$$

(b)

$$\dot{x} = \frac{1}{\delta} [\beta x - x^2]$$

Fixed points:

$$\begin{aligned}
 x &= 0, \\
 \beta &= x
 \end{aligned}$$



Graphics/S01D01.pdf

Figure 1: Transcritical bifurcation

## Question 2

Consider a dynamical system that has a pair of purely imaginary eigenvalues at its fixed point for the parameter value  $\mu = 0$ . As we have seen, a linear change of coordinates and a center manifold reduction gives the two-dimensional reduced dynamical system

$$\dot{x} = \delta(\mu)x - \omega(\mu)y + f(x, y, \mu), \quad (6)$$

$$\dot{y} = \delta(\mu)y + \omega(\mu)x + g(x, y, \mu), \quad (7)$$

where  $\delta(\mu) = \operatorname{Re} \lambda(\mu)$ ,  $\omega(\mu) = \operatorname{Im} \lambda(\mu)$ . (Here  $\lambda(\mu)$  and  $\bar{\lambda}(\mu)$  is the pair of complex eigenvalues that crosses the imaginary axis at  $\mu = 0$ .)

Recall that in polar coordinates, the truncated normal form of (6) can be written as

$$\begin{aligned} \dot{r} &= r(d_0\mu + a_0r^2), \\ \dot{\theta} &= \omega_0 + b_0\mu + c_0r^2, \end{aligned}$$

where

$$\begin{aligned} d_0 &= \delta'(0), \quad \omega_0 = \omega(0) \\ a_0 &= \frac{1}{16} [f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}]_{x=y=0, \mu=0} \\ &\quad + \frac{1}{16\omega_0} [f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}]_{x=y=0, \mu=0}. \end{aligned}$$

These classic formulae are used in all applications where Hopf bifurcations are analyzed.

As an application of these results, consider now the stick-slip oscillator

$$m\ddot{x} + c\dot{x} + kx = F_f, \quad F_f = mg\mu_0 \left(1 + e^{-\beta|v_0 - \dot{x}|}\right) \operatorname{sign}(v_0 - \dot{x}),$$

where  $m$  is the mass of the oscillator,  $g$  is the constant of gravity,  $\beta > 0$  is a constant,  $\mu_0$  is the Coulomb (static) friction coefficient,  $v_0$  is the speed of the belt,  $x$  is the position of the mass on the belt,  $c$  is the coefficient of viscous damping, and  $k$  is the spring coefficient (see Fig. 2).

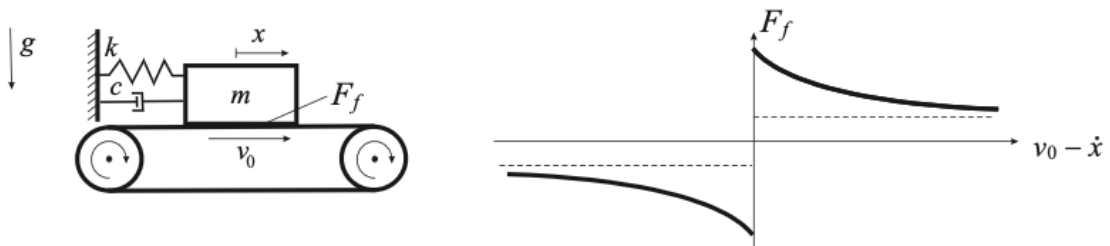


Figure 2: Stick-slip oscillator and its dry-friction force as a function of the relative velocity between the mass and the belt.

- Find a condition under which the system has an asymptotically stable fixed point.
- Show that a subcritical Hopf bifurcation takes place when the above condition is violated. (Use  $v_0$  as a bifurcation parameter.)
- Calculate the approximate amplitude of the bifurcating limit cycle.

## Solution 2

(a)

Let  $x_1 = x$  and  $x_2 = \dot{x}$ . Then the system can be written as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}F_f(x_2).\end{aligned}$$

The fixed point is at

$$x_1^0 = \frac{1}{k}F_f(0) = \frac{mg\mu_0}{k} \left(1 + e^{-\beta|v_0|}\right) \text{sign}(v_0), \quad x_2^0 = 0.$$

Let us now shift the origin to the fixed point by introducing new coordinates as  $z_1 = x_1 - x_1^0$  and  $z_2 = x_2$ . Then

$$\dot{z}_1 = z_2 \tag{8}$$

$$\dot{z}_2 = -\frac{k}{m}z_1 - \frac{c}{m}z_2 + \frac{1}{m}F_f(z_2) - \frac{1}{m}F_f(0), \tag{9}$$

with the fixed point at  $z_1 = z_2 = 0$ . The linearized system is given by

$$\dot{\xi} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & \mu \end{pmatrix} \xi, \tag{10}$$

where we have introduced the parameter

$$\mu = \frac{1}{m} (F_f'(0) - c) = g\beta\mu_0 e^{-\beta|v_0|} - \frac{c}{m}. \tag{11}$$

The eigenvalues of the coefficient matrix are

$$\lambda_{1,2} = \frac{\mu}{2} \pm \frac{1}{2} \sqrt{\mu^2 - \frac{4k}{m}}. \tag{12}$$

For the remainder of the discussion, let us assume that  $|\mu|$  is not too big; specifically  $\mu^2 < \frac{4k}{m}$ . This is in line with our previous assumption to treat  $\mu$  as a bifurcation parameter. In this case, the real part of  $\lambda_{1,2}$  can simply be read off from (12) as

$$\text{Re}(\lambda_{1,2}) = \frac{\mu}{2}.$$

As a result, if  $\mu < 0$  then  $\text{Re}(\lambda_1) < 0$  and  $\text{Re}(\lambda_2) < 0$ , hence the fixed point is asymptotically stable by the Hartman-Grobman theorem. This condition translates into

$$\mu < 0 \Leftrightarrow g\beta\mu_0 e^{-\beta|v_0|} < \frac{c}{m} \Leftrightarrow |v_0| > \frac{1}{\beta} \log\left(\frac{mg\beta\mu_0}{c}\right).$$

Hence, if

$$|v_0| > \frac{1}{\beta} \log\left(\frac{mg\beta\mu_0}{c}\right), \tag{13}$$

then  $(z_1 = z_2 = 0)$  is an asymptotically stable fixed point.

### Remark

Note that if the viscous damping  $c$  is large enough such that  $c > mg\beta\mu_0$  then  $\log\left(\frac{mg\beta\mu_0}{c}\right) < 0$ . Then the condition (13) is satisfied for any  $v_0 \neq 0$  and  $(z_1 = z_2 = 0)$  is asymptotically stable for any  $v_0$ .

(b)

For  $0 < \mu \ll 1$  we have  $\text{Re}(\lambda_{1,2}) > 0$ . At  $\mu = 0$  the eigenvalues cross the imaginary axis at

$$\lambda_{1,2}(\mu = 0) = \pm i \sqrt{\frac{4k}{m}}.$$

Now let  $-1 \ll \mu < 0$ . Then the eigenvalues are

$$\lambda_{1,2} = \frac{\mu}{2} \pm i \frac{1}{2} \sqrt{\frac{4k}{m} - \mu^2}.$$

Define

$$\delta(\mu) = \frac{\mu}{2}, \quad \omega(\mu) = \frac{1}{2} \sqrt{\frac{4k}{m} - \mu^2}. \quad (14)$$

Then, we can separate the linear and nonlinear terms from the system (8) and write it as

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} F_f(z_2) - \frac{1}{m} F'_f(0) z_2 - \frac{1}{m} F_f(0) \end{pmatrix}, \quad (15)$$

where we have denoted the linear part as

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & \mu \end{pmatrix}.$$

The eigenvalues of  $A$  are  $\delta(\mu) \pm i\omega(\mu)$ . To simplify the calculation of the eigenvectors, we note that (as a consequence of (14))

$$\mu = 2\delta \quad \frac{k}{m} = \delta^2 + \mu^2$$

and hence

$$A = \begin{pmatrix} 0 & 1 \\ -\delta^2 - \omega^2 & 2\delta \end{pmatrix}.$$

We then search for a vector  $s$  such that

$$As - (\delta + i\omega)s = \begin{pmatrix} -\delta - i\omega & 1 \\ -\delta^2 - \omega^2 & \delta - i\omega \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}.$$

For example, the non-normalized vector

$$s = \begin{pmatrix} 1 \\ \delta + i\omega \end{pmatrix}$$

is a good choice. To separate the real and imaginary parts of the eigenvector we write it as

$$s = \begin{pmatrix} 1 \\ \delta(\mu) \end{pmatrix} \pm i \begin{pmatrix} 0 \\ \omega(\mu) \end{pmatrix}.$$

Selecting the real and imaginary parts of  $s$  as basis vectors then puts  $A$  in the desired block-diagonal form, that is

$$A = VDV^{-1},$$

where

$$D = \begin{pmatrix} \delta & \omega \\ -\omega & \delta \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 \\ \delta & \omega \end{pmatrix}.$$

This means, that under the change of variables

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = V \begin{pmatrix} u \\ v \end{pmatrix}$$

we get the transformed dynamical system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \delta & \omega \\ -\omega & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \frac{1}{m\omega} \begin{pmatrix} 0 \\ F_f(\delta u + \omega v) - F'_f(0)(\delta u + \omega v) - F_f(0) \end{pmatrix}. \quad (16)$$

This is the desired form for the dynamical system defined in the problem description. Note that we may put

$$\begin{aligned} f(x, y, \mu) &= 0 \\ g(x, y, \mu) &= \frac{1}{m\omega} F_f(\delta u + \omega v) - F'_f(0)(\delta u + \omega v) - F_f(0) \end{aligned}$$

to compute the desired parameters

$$d_0 = \delta'(0) = \frac{1}{2}, \quad \omega_0 = \omega(0) = \sqrt{\frac{k}{m}}, \quad a_0 = \frac{kg\mu_0\beta^3 e^{-\beta|v_0|}}{16m},$$

where we have used that

$$g_{vvv}(0, 0, 0) = \frac{k}{m^2} F'''(0) = \frac{kg\mu_0\beta^3 e^{-\beta|v_0|}}{m}.$$

According to the Hopf-Bogdanov theorem, the radial component of the normal form of the dynamics can be written as

$$\dot{r} = r \left( \frac{\mu}{2} + \frac{kg\mu_0\beta^3 e^{-\beta|v_0|}}{16m} r^2 \right), \quad (17)$$

which has fixed points

$r = 0$  which corresponds to the stable fixed point

$$r = \pm \sqrt{\frac{-8\mu m e^{\beta|v_0|}}{kg\mu_0\beta^3}}, \text{ which corresponds to the unstable limit cycle.}$$

Expressed as a function of  $v_0$ , the bifurcation occurs at

$$v_C = \frac{1}{\beta} \log \left( \frac{mg\beta\mu_0}{c} \right).$$

(c)

For  $\mu < 0$  the amplitude of the unstable limit cycle is

$$r_0 = \sqrt{\frac{-8\mu m e^{\beta|v_0|}}{kg\mu_0\beta^3}}$$