

Nonlinear Dynamics and Chaos I

Problem Set 3 - Questions

Question 1

Consider a ball of mass m that slides on a rotating hoop (see Fig. 1).

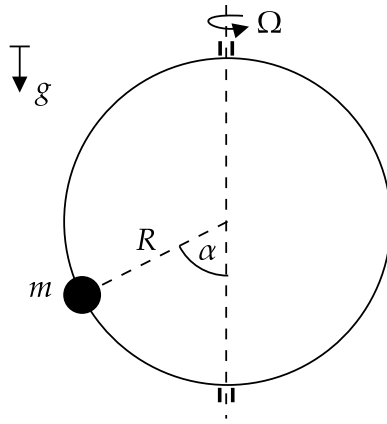


Figure 1: Mass on a hoop

The angular velocity of the hoop is Ω , the viscous friction coefficient between the hoop and the ball is b , and the constant of gravity is g . The equation of motion for the sliding ball is given by

$$mR^2\ddot{\alpha} + bR^2\dot{\alpha} + mR^2(g/R - \Omega^2 \cos(\alpha)) \sin(\alpha) = 0.$$

- Plot the location of equilibria of the ball as a function of the non-dimensionalized rotation parameter $\nu = R\Omega^2/g$.
- Using linearization, determine the stability type of the different equilibrium branches on the plot. Identify the critical angular velocity at which a bifurcation of equilibria occurs.

Question 2

Consider a discrete dynamical system given by the iterated mapping

$$x_{k+1} = f(x_k), \quad f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad x_k \in \mathbb{R}^n.$$

Assume that $x = p$ is a fixed point for the mapping, i.e., $p = f(p)$.

- Derive a linearized mapping of the form

$$y_{k+1} = Ay_k \tag{1}$$

to describe the discrete dynamics in the vicinity of the fixed point.

- Assume that A has eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ with corresponding n linearly independent eigenvectors $s_1, \dots, s_n \in \mathbb{C}^n$. Show that the general solution of (1) is of the form

$$y_k = c_1\varphi_1(k) + \dots + c_n\varphi_n(k), \quad \varphi_i(k) = \lambda_i^k s_i. \tag{2}$$

- Formulate a definition of stability, asymptotic stability, and instability for the $y = 0$ fixed point of (1).
- Using (2), find a sufficient and necessary condition for the asymptotic stability you have defined in (c).

Question 3

The first three modes of a convecing fluid motion in a two-dimensional layer heated from below are described by the famous *Lorenz equations*

$$\begin{aligned}\dot{x} &= a(y - x), \\ \dot{y} &= bx - y - xz, \\ \dot{z} &= xy - cz.\end{aligned}$$

Here $a > 0$ denotes the Prandtl number, $b > 0$ is the Rayleigh number, and $c > 0$ is the aspect ratio. Lorenz's original assumption is that $a > 1 + c$.

The above equations describe the evolution of the amplitudes of one velocity mode and two temperature modes. As a paradigm of chaotic dynamics, this system has inspired much of the development of the modern geometric theory of dynamical systems.

- (a) Complicated dynamics arise when all possible equilibria of the system become unstable, and hence solutions cannot settle down to any simple steady state. Show that this is the case when

$$b > \frac{a(3 + a + c)}{a - c - 1}$$

Note: Note: A negative sign for one of the Routh-Hurwitz determinants actually implies instability, not just the lack of asymptotic stability.

- (b) Solve the Lorenz equations numerically for $a = 10$, $b = 28$, and $c = 8/3$, choosing an initial condition close to $x = y = z = 0$. Plot the trajectory in three dimensions to show that it converges to a complicated surface, a *chaotic attractor*.

Question 4

Recall from Question 1 that a ball of mass m sliding on a hoop rotating with angular velocity Ω satisfies the differential equation

$$mR^2\ddot{\alpha} + mR^2(g/R - \Omega^2 \cos(\alpha)) \sin(\alpha) = 0 \quad (3)$$

if there is no friction between the hoop and the mass, i.e. we assume $b = 0$. Assume that the parameter values are such that the lower equilibrium position of the ball is stable in linear approximation.

- (a) Show that in this case, the equilibrium is also nonlinearly stable.

Hint: Note that system (3) is not conservative: an external torque is required to maintain the constant rotation of the hoop. As a result, the total mechanical energy of the system is not expected to work as a Lyapunov function. To find another candidate for a Lyapunov function, find a quantity that *is* conserved along trajectories. Such a quantity can be found by multiplying equation (3) by $\dot{\alpha}$ and integrating in time.

- (b) Prove that the equilibrium *cannot* be asymptotically stable for the nonlinear system.

Hint: use the Lyapunov function you have found in (a).

Question 5

Consider the damped pendulum equation

$$\ddot{x} + c\dot{x} + \sin(x) = 0. \quad (4)$$

- (a) Using the energy of the pendulum as a Lyapunov function, can we conclude the nonlinear asymptotic stability of the $x = 0$ equilibrium? Give detailed reasoning why.
- (b) A theorem due to Krasovski states the following: Assume that $x = 0$ is a fixed point for the n -dimensional dynamical system $\dot{x} = f(x)$. Assume that there exists a smooth scalar function $V(x)$ such that
- $V(x)$ is positive definite on an open neighborhood U of $x = 0$
 - \dot{V} is negative semi-definite on the same neighborhood
 - the only trajectory lying *completely* in the set $S = \{x \in U : \dot{V} = 0\}$ is the fixed point $x = 0$. Then $x = 0$ is asymptotically stable.

Use Krasovski's theorem to conclude the asymptotic stability of the origin for (4).

Question 6

Consider an n -degree-of-freedom holonomic mechanical system (i.e. one that has only position-dependent constraints) with generalized coordinates $q \in \mathbb{R}^n$ and generalized velocities $\dot{q} \in \mathbb{R}^n$. The total energy of the system is of the form

$$E(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} + V(q),$$

where $M \in \mathbb{R}^{n \times n}$ is the mass matrix (symmetric and positive definite), and $V(q)$ is the potential energy. In the absence of external forces, the associated Lagrangian equations of motion are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0,$$

where $L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - V(q)$ is the Lagrangian of the mechanical system.

Show that if $V(q)$ admits a strict local minimum at a point q_0 , then q_0 is a (nonlinearly) stable equilibrium for the mechanical system. (This result is also known as *Dirichlet's Theorem* in classical mechanics).