Nonlinear Dynamics and Chaos I
Solution guide for Problem Set 3

1. The first three modes of a convecting fluid motion in a two-dimensional layer heated from below are described by the famous *Lorenz equations*

$$
\begin{aligned}
\dot{x} &= a(y-x), \\
\dot{y} &= bx - y - xz, \\
\dot{z} &= xy - cz.
\end{aligned}
$$

Here $a > 0$ denotes the Prandtl number, $b > 0$ is the Rayleigh number, and $c > 0$ is the aspect ratio. Lorenz's original assumption is that $a > 1 + c$.

The above equations describe the evolution of the amplitudes of one velocity mode and two temperature modes. As a paradigm of chaotic dynamics, this system has inspired much of the development of the modern geometric theory of dynamical systems.

(a) Complicated dynamics arise when all possible equilibria of the system become unstable, and hence solutions cannot settle down to any simple steady state. Show that this is the case when $b > a(3+a+c)/(a-c-1)$. (*Note:* A negative sign for one of the Routh-Hurwitz determinants actually implies instability, not just the lack of asymptotic stability.)

The three fixed points of $\underline{\dot{x}} = f(\underline{x})$ are

$P_1 : x_0 = y_0 = z_0 = 0$

$P_2 : x_0 = y_0 = \sqrt{c(b-1)}, \quad z_0 = b-1$

$P_3 : x_0 = y_0 = -\sqrt{c(b-1)}, \quad z_0 = b-1$

For the system to have these three fixed points we must have $\boxed{b>1}$.

Let A denote $Df(x_0, y_0, z_0)$. Then

$$A = \begin{pmatrix} -a & a & 0 \\ b-z_0 & -1 & -x_0 \\ y_0 & x_0 & -c \end{pmatrix}$$

The characteristic polynomial of A is

$$\lambda^3 + (a+c+1)\lambda^2 + [ac + a + c + x_0^2 + a(z_0 - b)]\lambda + ac(z_0 - b + 1) + x_0^2 a + a x_0 y_0 = 0$$

Stability of P_1 :

$$\lambda^3 + (a+c+1)\lambda^2 + (ac + a + c - ab)\lambda - ac(b-1) = 0$$

A necessary condition for all roots of the above polynomial to be negative is that all its coefficients have the same sign. But here $-ac(b-1) < 0$ while λ^3 has a positive coefficient (i.e., $+1$). $\Rightarrow$ A has a positive eig. value

$\Rightarrow P_1$ is unstable.

Stability of P_2, P_3:

The Routh-Hurwitz determinants are:

$d_1 = 2ac(b-1) > 0$ $\qquad$ $d_2 = (a+b)c > 0$

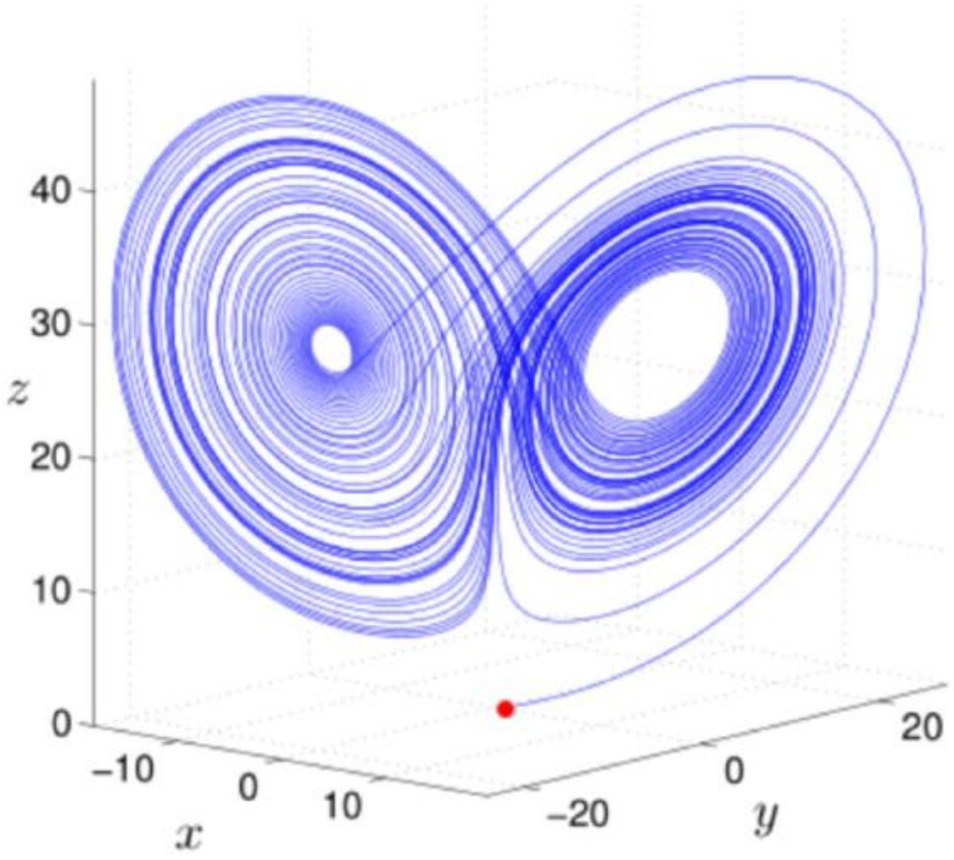
$$d_3 = \begin{vmatrix} (a+b)c & 2ac(b-1) \\ 1 & a+c+1 \end{vmatrix} = (a+b)(a+c+1)c - 2ac(b-1)$$

For P_2 and P_3 to be unstable, we must have $d_3 < 0$.

$$d_3 < 0 \iff b > \frac{a(3+a+c)}{a-(c+1)} > 1$$

Follows from $a > c+1$

(b) Solve the Lorenz equations numerically for $a = 10$, $b = 28$, and $c = 8/3$, choosing an initial condition close to $x = y = z = 0$. Plot the trajectory in three dimensions to show that it converges to a complicated surface, a *chaotic attractor*.



2. Recall from the last problem set that a ball of mass m sliding on a hoop rotating with angular velocity Ω satisfies the differential equation

$$mR^2\ddot{\alpha} + mR^2\left(g/R - \Omega^2\cos\alpha\right)\sin\alpha = 0 \tag{1}$$

if there is no friction between the hoop and the mass. Assume that the parameter values are such that the lower equilibrium position of the ball is stable in linear approximation.

(a) Show that in this case, the equilibrium is also nonlinearly stable. (*Hint*: Note that system (1) is not conservative: an external torque is required to maintain the constant rotation of the hoop. As a result, the total mechanical energy of the system is not expected to work as a Lyapunov function. To find another candidate for a Lyapunov function, find a quantity that *is* conserved along trajectories. Such a quantity can be found by multiplying equation (1) by $\dot{\alpha}$ and integrating in time.)

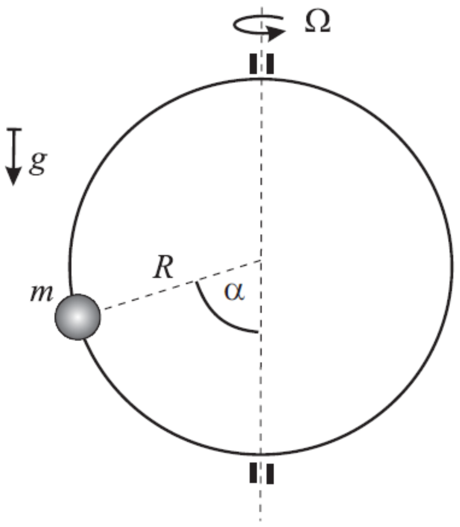


Figure 1: Mass on a loop

From the previous assignment, we know that the lower equilibrium is unstabe when $\Omega^2 > g/R$. Hence, in the following we assume

$$\boxed{\Omega^2 < g/R} \quad (1)$$

$$mR^2\ddot{\alpha} + mR^2\left(g/R - \Omega^2\cos\alpha\right)\sin\alpha = 0$$

(a) Multiplying the equation of motion by $\dot{\alpha}$, we find that

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\alpha}^2 - \frac{g}{R}\cos\alpha + \frac{1}{4}\Omega^2\cos 2\alpha\right) = 0 \quad (2)$$

Let $x_1 := \alpha$, $x_2 := \dot{\alpha}$. Eq. (2) implies that the function

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{g}{R}(1 - \cos x_1) + \frac{\Omega^2}{4}(\cos 2x_1 - 1)$$

is constant along trajectories, i.e. $\frac{d}{dt}V(x_1(t), x_2(t)) = 0$.

Moreover, $V(0,0) = 0$. On the other hand:

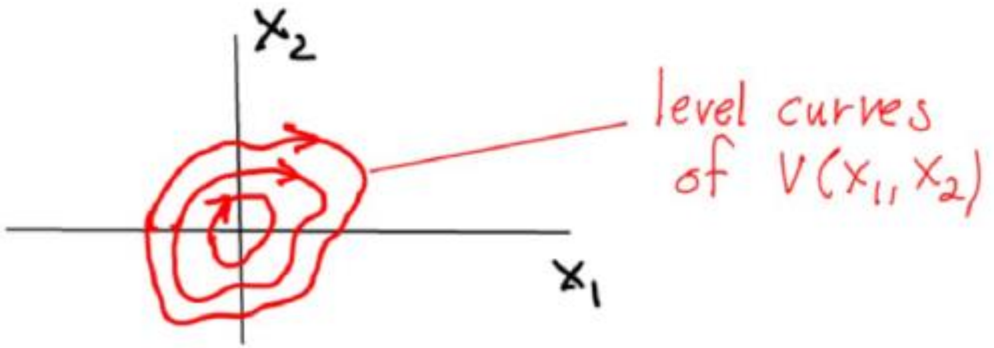
$$\nabla V(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \nabla^2 V(0,0) = \begin{pmatrix} g/R - \Omega^2 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows from (1) that $\nabla^2 V(0,0)$ is positive definite.

$\Rightarrow$ By Lyapunov's direct method, the lower equilibrium is stable.

(b) Prove that the equilibrium *cannot* be asymptotically stable for the nonlinear system (*Hint*: use the Lyapunov function you have found in (a))

(b) the fixed point $(0,0)$ cannot be asymptotically stable since the trajectories of the system coincide with level curves of $V(x_1,x_2)$, Since $\frac{dV}{dt}=0$ along trajectories. But the above analysis shows that around $(0,0)$ the level curves of V are closed curves.



3. Consider the damped pendulum equation

$$\ddot{x} + c\dot{x} + \sin x = 0. \tag{2}$$

(a) Using the energy of the pendulum as a Lyapunov function, can we conclude the nonlinear asymptotic stability of the $x = 0$ equilibrium? (Give detailed reasoning why.)

$$E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + (1 - \cos x)$$

$$y = (y_1, y_2) := (x, \dot{x})$$

$$\dot{y} = f(y) = \begin{bmatrix} y_2 \\ -c y_2 - \sin y_1 \end{bmatrix} \Rightarrow E(y) = \frac{1}{2} y_2^2 + (1 - \cos y_1)$$

(i) $E(0) = 0$, $DE(0) = 0$, $D^2E(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ Hessian is positive definite.

$\Rightarrow$ E is positive-definite near the origin

(ii) $\dot{E}(y) = \langle DE(y), f(y) \rangle = (\sin y_1, y_2) \cdot (y_2, -cy_2 - \sin y_1)$

$= \sin y_1 \, y_2 - cy_2^2 - \sin y_1 \, y_2$

$= -c y_2^2 \leq 0$

E is positive definite around the origin and $\dot{E}$ is negative semi-definite.

Indeed, we cannot find an open set $\mathcal{U}$ around the origin where $\dot{E}(y) < 0 \;\; \forall \; y \in \mathcal{U} \setminus \{0\}$ $\left(\dot{E}(y) = 0 \text{ for any } y = (y_1, 0) \text{ with } y_1 \neq 0 \right)$.

Thus, theorem 2 is not applicable to conclude nonlinear asymptotic stability of the origin.

(b) A theorem due to Krasovski states the following: Assume that $x = 0$ is a fixed point for the n-dimensional dynamical system $\dot{x} = f(x)$. Assume that there exists a smooth scalar function $V(x)$ such that (i) $V(x)$ is positive definitive on an open neighborhood U of $x = 0$ (ii) $\dot{V}$ is negative semi-definite on the same neighborhood (ii) the only trajectory lying *completely* in the set $S = \left\{x \in U : \dot{V} = 0\right\}$ is the fixed point $x = 0$. Then $x = 0$ is asymptotically stable.

Use Krasovski's theorem to conclude the asymptotic stability of the origin for (2).

b) We use Krasovski's theorem with $V = E$, $U \subset (-\pi, \pi) \times \mathbb{R}$ open set around the origin in $S^1 \times \mathbb{R}$ s.t. the statements (i) & (ii) in the hypothesis of Krasovski are satisfied as shown above in part a).

(iii) $S = \{y \in \mathcal{U} \mid \dot{E}(y) = 0\} \subset \underbrace{\{(y_1, 0) \mid y_1 \in (-\pi, \pi)\}}_{\tilde{S}}$

Indeed, the only trajectory of the system completely contained in the set $\tilde{S}$ on the y_1-axis is the origin (cf. phase portrait). $\Rightarrow$ S contains only the fixed point as a trajectory of the system.

Hence, the hypothesis of Krasovski's theorem is satisfied and the origin is asymptotically stable for the nonlinear damped pendulum.

4. Consider an n-degree-of-freedom holonomic mechanical system (i.e., one that has only position-dependent constraints) with generalized coordinates $q \in \mathbb{R}^n$ and generalized velocities $\dot{q} \in \mathbb{R}^n$. The total energy of the system is of the form

$$E(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + V(q),$$

where $M \in \mathbb{R}^{n\times n}$ is the mass matrix (symmetric and positive definite), and $V(q)$ is the potential energy. In the absence of external forces, the associated Lagrangian equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0,$$

where $L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q)$ is the Lagrangian of the mechanical system.

Show that if $V(q)$ admits a strict local minimum at a point q_0, then q_0 is a (nonlinearly) stable equilibrium for the mechanical system. (This result is also known as *Dirichlet's Theorem* in classical mechanics).

4. First construct the function:

$$\bar{E}(q,\dot{q}) = E(q,\dot{q}) - V(q_0)$$
$$= \frac{1}{2}\dot{q}^T M \dot{q} + V(q) - V(q_0)$$

Now, at $(q,\dot{q}) = (q_0, 0)$ we have $\bar{E}(q_0, 0) = 0$

Note that $M(q)$ is positive definite for all q and $V(q) - V(q_0)$ is positive around $q = q_0$. (since V has a local minimum at q_0)

$\Rightarrow$ $\bar{E}(q,\dot{q})$ is positive definite around $(q_0, 0)$.

But $\frac{d\bar{E}}{dt} = \frac{dE}{dt}$ since $V(q_0)$ is a constant.

we show that $\frac{dE}{dt} = 0$.

First note that, in general, the Lagrangian eq. of motion is a system of n coupled equations with each eq. given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \qquad k = 1, 2, \ldots, n$$

Multiply each equation by $\dot{q}_k$ and sum over k to get

$$(1) \quad \dot{q}_k \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} \right] = 0$$

{we use Einstein's notation: Sum over repeated indices}

Since $\quad L = \frac{1}{2} M_{ij}\, \dot{q}_i \dot{q}_j - V \quad$ we have:

$$\left.\begin{aligned} &\frac{\partial L}{\partial \dot{q}_k} = M_{ik}\, \dot{q}_i \quad , \quad \frac{\partial L}{\partial q_k} = \frac{1}{2} \frac{\partial M_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial V}{\partial q_k} \\ &\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = M_{ik}\, \ddot{q}_i + \frac{\partial M_{ik}}{\partial q_j} \dot{q}_j \dot{q}_i \end{aligned}\right\} \quad (2)$$

Substituting (2) into (1), we get

$$M_{ik}\ddot{q}_i\dot{q}_k + \frac{\partial M_{ik}}{\partial q_j}\dot{q}_j\dot{q}_i\dot{q}_k - \frac{1}{2}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j\dot{q}_k + \frac{\partial V}{\partial q_k}\dot{q}_k = 0$$

Since there is a sum over repeated indices we have

$$M_{ik}\ddot{q}_i\dot{q}_k \equiv M_{ij}\ddot{q}_i\dot{q}_j \qquad \frac{\partial M_{ik}}{\partial q_j}\dot{q}_j\dot{q}_i\dot{q}_k \equiv \frac{\partial M_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j\dot{q}_k$$

$$\Rightarrow \underbrace{M_{ij}\ddot{q}_i\dot{q}_j + \frac{1}{2}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j\dot{q}_k + \frac{\partial V}{\partial q_k}\dot{q}_k}_{= \frac{d}{dt}\left[\frac{1}{2}M_{ij}\dot{q}_i\dot{q}_j + V(q)\right]} = 0$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T M(q)\dot{q} + V(q)\right] = 0 \Rightarrow \frac{dE}{dt} = 0 \Rightarrow \frac{d\bar{E}}{dt} = 0$$

Using $\bar{E}$ as the Lyapunov function, we conclude that $(q_0, 0)$ is a <u>stable</u> equilibrium point.