151-0530-00L, Spring, 2020

Nonlinear Dynamics and Chaos II

Homework Assignment 5

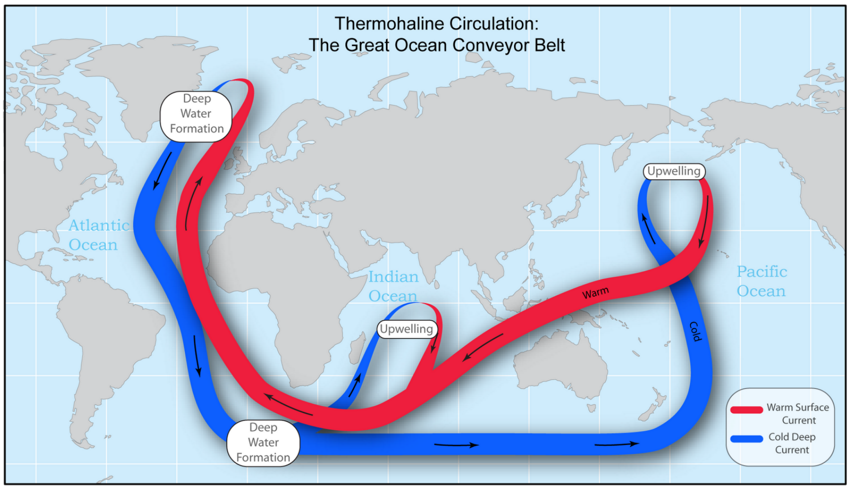
Due: Wednesday, May 13;
Please submit by email to Dr. Shobhit Jain <shjain@ethz.ch>

1. Consider a slowly forced pendulum with viscous damping:

$$\ddot{\varphi} + k\dot{\varphi} + \sin\varphi = F_0 \sin \epsilon t,$$

where $k > 0$ is the damping coefficient, $0 < F_0 < 1$ is the forcing amplitude, and $0 < \epsilon \ll 1$ is the forcing frequency. Give a complete qualitative description of the geometry of this system in the extended phase space for small enough ϵ. (*Hint:* Make the system autonomous by introducing the phase variable $\psi = \epsilon t$.)

2. Thermohaline circulation (THC) is a part of the large-scale ocean circulation that is driven by global density gradients created by surface heat and freshwater fluxes. Stommel's box model (1961) is a qualitative description of the trends and equilibria in THC. This model couples the two fundamental drivers of TLC, temperature (*thermo*) and salt concentration (*-haline*), in a nonlinear fashion.



The non-dimensional **variables** of Stommel's model are:

$x(t)$**:** temperature difference between the tropics (lower latitudes) and the North-Atlantic (higher latitudes)

$y(t)$**:** salinity (i..e, salt concentration) difference between the above two regions of the ocean

The non-dimensional **parameters** of the model are:

τ_x**:** relaxation time to a constant temperature difference between northern and southern latitudes in the absence of coupling

τ_y**:** relaxation time to zero salinity difference between higher and lower latitudes in the absence of coupling. In practice, $\tau_x/\tau_y = \epsilon \ll 1$.

μ**:** measure of freshwater flux through clouds moving from lower to higher latitudes

η : nonlinear coupling parameter between temperature and salinity evolution

With this notation, Stommel's model can be written as

$$\begin{aligned}
\dot{x} &= -\frac{1}{\tau_x}(x-1) + \frac{1}{\tau_y}x\left[1+\eta^2(x-y)^2\right], \\
\dot{y} &= \frac{\mu}{\tau_y} - \frac{1}{\tau_y}y\left[1+\eta^2(x-y)^2\right].
\end{aligned}$$

(a) Show that Stommel's model has a globally attracting slow manifold that governs the asymptotic behavior of THC. Find a leading order approximation to this manifold. (*Hint*: rescale time by letting $s = t/\tau_y$.)

(b) Compute the leading-order reduced flow on the slow manifold. Determine qualitatively the possible dynamical behaviors on the slow manifold as the parameters μ and η are varied.

Nonlinear Dynamics and Chaos II.
Homework 5

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Exercise 1

Consider the slowly forced pendulum with viscous damping

$$\ddot{\varphi} + k\dot{\varphi} + \sin\varphi = F_0 \sin(\varepsilon t)$$

$k > 0$ is the damping coefficient, $F_0 < 1$ and $0 < \varepsilon \ll 1$.

Solution

To write the system in first order, autonomous form, let $v = \dot{\varphi}$ and $\psi = \varepsilon t$.

$$\begin{aligned}
\dot{\varphi} &= v \\
\dot{v} &= -kv - \sin\varphi + F_0 \sin\psi \\
\dot{\psi} &= \varepsilon.
\end{aligned}$$

This is a problem that has separated timescales, and can be viewed as a singularly perturbed problem, on the fast timescale. Here, φ and v are the fast variables, while ψ is the slow variable. Setting $\varepsilon = 0$, we get the fast subsystem:

$$\begin{aligned}
\dot{\varphi} &= v \\
\dot{v} &= -kv - \sin\varphi + F_0 \sin\psi \\
\dot{\psi} &= 0.
\end{aligned}$$

To obtain the critical manifold C_0, we look for fixed points of this system. Because of the first equation, we must have $v = 0$. Substituting it into the second equation, we get

$$0 = -\sin\varphi + F_0 \sin\psi.$$

Since $F_0 < 1$, this has has two solutions for all ψ.

$$\varphi_1(\psi) = \arcsin(F_0 \sin\psi) \tag{1}$$

or

$$\varphi_2(\psi) = -\arcsin(F_0 \sin\psi) + \pi. \tag{2}$$

The extended phase space is spanned by the variables (φ, v, ψ). But since φ and ψ are cyclic, the phase space can be viewed as the manifold $S^1 \times S^1 \times \mathbb{R} \sim T^2 \times \mathbb{R}$. Because we were able to write the fixed points as a graph over ψ, the critical manifold is

$$C_0 = \{(\varphi, \psi, v) \in S^1 \times S^1 \times \mathbb{R} : v = 0, \varphi = \arcsin(F_0 \sin\psi), \varphi = -\arcsin(F_0 \sin\psi) + \pi\}.$$

This is a union of two disjoint curves. The normal hyperbolicity of the manifold depends on the hyperbolicity of the fixed points, in the fast subsystem.

$$\frac{d}{dt}\begin{bmatrix}\varphi \\ v\end{bmatrix} = \begin{bmatrix} v \\ -kv - \sin\varphi + F_0 \sin\psi\end{bmatrix} := f(\varphi, v)$$
$$\dot{\psi} = 0$$

The Jacobian matrix $D_{\varphi,v} f(\varphi, v)$ is

$$D_{\varphi,v} f(\varphi, v) = \begin{bmatrix} 0 & 1 \\ -\cos\varphi & -k\end{bmatrix},$$

which we have to evaluate at the points (1) and (2) and look for its eigenvalues. The characteristic polynomial is

$$\lambda^2 + k\lambda + \cos\varphi_{1,2}(\psi) = 0,$$

with roots

$$\lambda_\pm = -\frac{k}{2} \pm \sqrt{\frac{k^2}{4} - \cos\varphi_{1,2}(\psi)}. \tag{3}$$

Here, by substituting (1) and (2), we get

$$\cos\varphi_1 = \cos(\arcsin(F_0 \sin\psi)) = \sqrt{1 - F_0^2 \sin^2\psi} > 0$$

and

$$\cos\varphi_2 = \cos(-\arcsin(F_0 \sin\psi) + \pi) = -\cos(\arcsin(F_0 \sin\psi)) = -\sqrt{1 - F_0^2 \sin^2\psi} < 0,$$

by the properties $\cos(\pi - \theta) = -\cos\theta$ and $\cos(\arcsin(x)) = \sqrt{1 - x^2}$. Evaluating (3) at these points, we see that in case of the first root, φ_1, the square root is strictly smaller than $k/2$, both eigenvalues are negative. At the root φ_2, the square root is strictly larger than $k/2$, which means λ_+ becomes positive.

Both fixed-curves $\varphi_{1,2}(\psi)$ of the fast subsystem are hyperbolic, φ_1 is asymptotically stable and φ_2 is unstable. This means that the critical manifold C_0 is a normally hyperbolic invariant manifold.

The full system is an ε-size perturbation to the fast subsystem, and we can conclude that there exists a normally hyperbolic invariant manifold C_ε, ε close to C_0, which is the slow manifold.

The full extended phase space $S^1 \times S^1 \times \mathbb{R}$, can be viewed as the product of a 2-torus and the real line. The critical manifold (and ε close to it, the slow manifold) then, is an union of two curves, that run on the 2-torus, corresponding to the section $v = 0$.

We can represent this torus as an embedded submanifold of $\mathbb{R}^3$, parametrized by (φ, ψ).

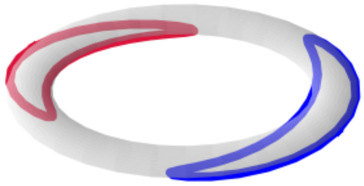

Figure 1: Slow manifold of the slowly forced pendulum. The torus shown is the $v = 0$ section of phase space. The colored curves make up the slow manifold, the blue one is $\varphi_1(\psi)$ and the red one is $\varphi_2(\psi)$.

The slow manifold is shown in Fig. 1. The red curve is the unstable part of C_ε and the blue one is the stable part. They trace out closed curves, φ_1 oscillating between $-\arcsin(F_0)$ and $\arcsin(F_0)$ and φ_2 between $\pi + \arcsin(F_0)$ and $\pi - \arcsin(F_0)$. In the extended phase space, $S^1 \times S^1 \times \mathbb{R}$ (the torus shown is just a submanifold of this), the curve φ_2 has a local stable manifold and an unstable manifold. Trajectories in the stable manifold would converge to φ_2, while all other nearby trajectories would converge to φ_1.

Exercise 2

Stommel's model for the Thermohaline circulation is

$$\dot{x} = -\frac{1}{\tau_x}(x-1) + \frac{1}{\tau_y}x[1+\eta^2(x-y)^2] \tag{4}$$

$$\dot{y} = \frac{\mu}{\tau_y} - \frac{1}{\tau_y}y[1+\eta^2(x-y)^2]. \tag{5}$$

The variables are x, y, where $x(t)$ is the temperature-gradient and $y(t)$ is the salinity gradient. The parameter μ is the freshwater flux, η is the nonlinear coupling constant. The relaxation times for the two processes are τ_x and τ_y, which satisfy $\tau_x/\tau_y = \varepsilon \ll 1$.

(a) Show that Stommel's model has a globally attracting slow manifold that governs the asymptotic behavior of THC. Find a leading order approximation to this manifold.

Solution

Introducing the new timescale $s = t\tau_y$, $d/dt = 1/\tau_y d/ds$:

$$\frac{1}{\tau_y}\frac{dx}{ds} = -\frac{1}{\tau_x}(x-1) + \frac{1}{\tau_y}x[1+\eta^2(x-y)^2]$$

$$\frac{1}{\tau_y}\frac{dy}{ds} = \frac{\mu}{\tau_y} - \frac{1}{\tau_y}y[1+\eta^2(x-y)^2].$$

Using the definition of ε, we have the singular perturbation problem, with x the fast variable and y the slow variable.

$$\varepsilon\frac{dx}{ds} = -(x-1) + \varepsilon x[1+\eta^2(x-y)^2] := f(x,y,\varepsilon) \tag{6}$$

$$\frac{dy}{ds} = \mu - y[1+\eta^2(x-y)^2] := g(x,y). \tag{7}$$

Switching timescales again, by $s = \varepsilon\tau$ and denoting the differentiation with respect to τ, by $(\cdot)'$

$$\begin{aligned} x' &= -(x-1) + \varepsilon x[1 + \eta^2(x-y)^2] \\ y' &= \varepsilon\left(\mu - y[1 + \eta^2(x-y)^2]\right). \end{aligned}$$

Setting $\varepsilon = 0$, we get the fast subsystem:

$$x' = -(x-1)$$

and $y' = 0$.

We look for the fixed point of this subsystem, which is $x = 1$, for any y, which means we can write the critical manifold as

$$C_0 = \{(x,y) : x = 1\}. \tag{8}$$

The stability of the critical manifold depends on the stability of the fixed point $x = 1$ in the fast subsystem. Since

$$\frac{d}{dx}f(x,y,0) = -1$$

this fixed point is hyperbolic and also attracting with a rate $e^{-\tau}$. Because of this, the critical manifold C_0 is a normally attracting invariant manifold for the system, for $\varepsilon = 0$.

As a result of Fenichel's theorem, we conclude that for small enough ε, there is a normally attracting invariant manifold C_ε for the system (4), which is $O(\varepsilon)$ close to C_0. This is the slow manifold, which we can represent as a graph

$$C_\varepsilon = \{(x,y) : x = \varphi(y,\varepsilon)\}.$$

Because of the smoothness properties, we can expand it in terms of ε: $x = \varphi(y,\varepsilon) = \varphi_0(y) + \varepsilon\varphi_1(y) + O(\varepsilon^2) = 1 + \varphi_1(y) + O(y^2)$.

On one hand, on the manifold we have $x = \varphi(y,\varepsilon)$, which we can differentiate with respect to τ.

$$x' = \frac{d\varphi}{dy}y' = \varepsilon^2\frac{d\varphi_1(y)}{dy}\left(\mu - y - y\eta^2(1 + \varepsilon\varphi_1(y) - y)^2\right) = O(\varepsilon^2). \tag{9}$$

On the other hand, we can use the x' equation restricted to the manifold.

$$\begin{aligned} x' = f(\varphi(y,\varepsilon),y,\varepsilon) &= -\varepsilon\varphi_1(y) + \varepsilon x[1 + \eta^2(x-y)^2] = \\ &= -(x-1) + \varepsilon x[1 + \eta^2(x^2 - 2xy + y^2)] = \\ &= -\varepsilon\varphi_1(y) + \varepsilon + \varepsilon\eta^2(1 - 2y + y^2) + O(\varepsilon^2). \end{aligned}$$

For the manifold to be invariant, these two expressions must match in all orders of ε. Since in (9), there were no order ε terms, we must get 0 for the coefficient of the $O(\varepsilon)$ term in the latter expression. This means

$$\varphi_1(y) = 1 + \eta^2(1-y)^2.$$

To leading-order, the slow manifold is described by the graph $x = 1 + \varepsilon(1 + \eta^2(1-y)^2)$.

(b) Compute the leading-order reduced flow on the slow manifold. Determine qualitatively the possible dynamical behaviors on the slow manifold as the parameters μ and η are varied.

Solution

To derive the reduced order flow, we Taylor-expand the equation governing the slow variable (on the original timescale, s):

$$g(x,y) = g(\varphi_0(y),y) + \varepsilon \left(\frac{dg}{dx}|_{(\varphi_0(y),y)} \right) \varphi_1(y) + O(\varepsilon^2)$$

From (7), $g(x,y) = \mu - y[1+\eta^2(x-y)^2]$

$$g(x,y) = \mu - y[1+\eta^2(1-y)^2] - 2\varepsilon\varphi_1(y)\eta^2 y(1-y) + O(\varepsilon^2)$$

Plugging in the expression for φ_1, we have

$$g(x,y) = \mu - y[1+\eta^2(1-y)^2] - 2\varepsilon(1+\eta^4 y(1-y)^3) + O(\varepsilon^2)$$

Keeping only the leading order term, we get the reduced dynamics on the slow manifold

$$\frac{dy}{ds} = \mu - y[1+\eta^2(1-y)^2].$$

On the slow manifold, the long time behavior of solutions is dictated by the fixed points. They are the roots of the right hand side:

$$\mu = y[1+\eta^2(1-y)^2]$$

The graph of these functions are shown in Fig. 2. The fixed points of the system are given by the intersection points. To see the effect of the parameters, note that the shape of the cubic

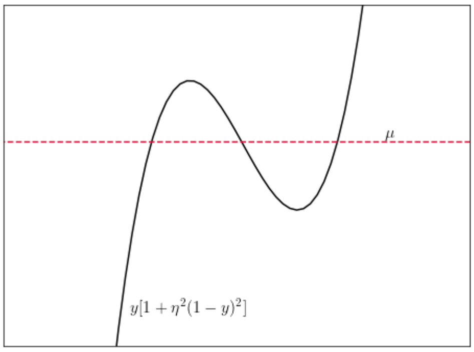

Figure 2: Graphs of the functions $f(y) = \mu$ and $f(y) = y[1+\eta^2(1-y)^2]$.

function is only governed by η. It can have at most two critical points, one minimum and one maximum. These are:

$$\frac{d}{dy} y[1+\eta^2(1-y)^2] = 0$$

$$1 + \eta^2(1-y)^2 - 2y\eta^2(1-y) = 0$$

$$3\eta^2 y^2 - 4y\eta^2 + 1 + \eta^2 = 0.$$

This equation has two roots (the minimum and maximum of the cubic function) if

$$16\eta^4 - 12\eta^2(1+\eta^2) > 0,$$

or $\eta^2 > 3$. In this case, the intersection of this graph with $y = \mu$ can give 1, 2 or 3 values. If y is smaller or larger than any of the critical values of the graph, there is only one solution. In these cases the right hand side of the system $\mu - y[1 + \eta^2(1 - y)^2]$ changes sign at the fixed points, and they are stable.

When μ reaches (one of) the critical values, a new fixed point appears in a bifurcation.

When μ is between the critical values, we have 3 fixed points, two stable and one unstable.

When the above discriminant is 0, at $\eta^2 = 3$, the cubic graph has only one critical point, and for all values of μ we only have one fixed point. This fixed point is stable.