151-0530-00L, Spring, 2020

Nonlinear Dynamics and Chaos II

Homework Assignment 1

Due: Wednesday, March 25;
please submit by email to Dr. Shobhit Jain <shjain@ethz.ch>

1. Derive the Hamiltonian equations of motion for a the coupled pendulums shown in Fig. 1. (The two point masses m are placed at the tips of two massless rods of length L. Both joints are frictionless; the constant of gravity is g.)

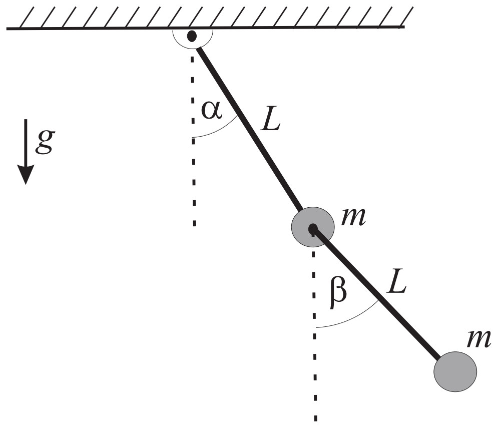

Figure 1: Coupled system of two pendulums

2. Consider the Lotka–Volterra model

$$\begin{aligned} \dot{h} &= a_1 h\,(1 - bp)\,, \\ \dot{p} &= -a_2 p\,(1 - ch)\,, \end{aligned} \tag{1}$$

for the interaction of a predator and a prey population. Here $h(t)$ and $p(t)$ denote the predator and prey populations, respectively, as a function of time; a_1, a_2, b, and c are positive parameters.

(a) Show that system (1) is Hamiltonian for $h, p > 0$ after an appropriate rescaling of time. Find the Hamiltonian. (Hint: Rewrite (1) as $\dot{h} = A(h,p)C(p)$, $\dot{p} = A(h,p)D(h)$ by defining the functions A, C and D appropriately.)

(b) Using the Hamiltonian, argue that the two species can exhibit stable coexistence, i.e., the system admits a stable fixed point. (Hint: establish *full nonlinear stability* for the fixed point).

3. Consider a two-dimensional steady *compressible* fluid flow with velocity field $\mathbf{v}(\mathbf{x}) = (u(x,y), v(x,y))$, where $\mathbf{x} = (x,y)$. Assume that the flow conserves mass, i.e., its density function $\rho(\mathbf{x}) > 0$ satisfies the equation of continuity. The latter equation, in its general form for unsteady flows, reads

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0,$$

valid or general, unsteady flow. Show that the equation of fluid particle motion becomes a canonical Hamiltonian system after a rescaling of time.

4. Consider a dynamical system defined on the two-dimensional torus $\mathbb{T}^2 = S^1 \times S^1$. Such systems admit the general form

$$\begin{aligned} \dot{\phi}_1 &= a(\phi_1, \phi_2), \\ \dot{\phi}_2 &= b(\phi_1, \phi_2), \end{aligned} \tag{2}$$

where $\phi_i \in S^1$.

(a) Show that a physical example of system (2) is found in the motion of two uncoupled linear undamped oscillators. Specifically, show that orbits of

$$\begin{aligned} \ddot{x} + x &= 0, \\ \ddot{y} + y &= 0, \end{aligned}$$

are confined to two-dimensional invariant tori of the phase space.

(b) Assume that system (2) has no fixed point (which is the case in the oscillator example). Argue that (2) then *cannot* be Hamiltonian, even after a rescaling of time. (*Hint:* Use the fact that a continuous function defined on a compact set must have a minimum and a maximum).

5. Show that for any dynamical system $\dot{q} = f(q,t)$, $q \in \mathbb{R}^n$, one can select a canonically conjugate variable $p \in \mathbb{R}^n$, such that the evolution of $(q(t), p(t))$ is governed by a Hamiltonian system. (Thus any type of dynamics can be viewed as a projection from a higher-dimensional Hamiltonian dynamical system.)

Nonlinear Dynamics and Chaos II.
Homework 2

Kaszás Bálint

January 24, 2023

Exercise 1

Derive the Hamiltonian equations of motion for the coupled pendulums

Solution

Fix the origin at the suspension point. Then, the Cartesian coordinates of the two point masses are

$$x_1 = L\sin\alpha; \quad x_2 = L\sin\beta + x_1 = L(\sin\alpha + \sin\beta)$$
$$y_1 = -L\cos\alpha; \quad y_2 = -L(\cos\alpha + \cos\beta).$$

The Lagrangian for the double pendulum in Cartesian coordinates is

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2) - mg(y_1 + y_2).$$

Calculating the velocities, in terms of the generalized coordinates α, β, we get

$$\dot{x}_1^2 + \dot{y}_1^2 = L^2\dot{\alpha}^2$$

$$\dot{x}_2 = L\cos\alpha\dot{\alpha} + L\cos\beta\dot{\beta}$$
$$\dot{y}_2 = L\sin\alpha\dot{\alpha} + L\sin\beta\dot{\beta}$$
$$\dot{x}_2^2 + \dot{y}_2^2 = L^2[\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta}(\cos\alpha\cos\beta + \sin\alpha\sin\beta)] = L^2[\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta}\cos(\alpha - \beta)].$$

The Lagrangian is

$$L(\alpha, \beta, \dot{\alpha}, \dot{\beta}) = \frac{mL^2}{2}(2\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta}\cos(\alpha - \beta)) + mgL(2\cos\alpha + \cos\beta).$$

The generalized momenta are

$$p_\alpha = \frac{\partial L}{\partial\dot{\alpha}} = mL^2(2\dot{\alpha} + \dot{\beta}\cos(\alpha - \beta))$$
$$p_\beta = \frac{\partial L}{\partial\dot{\beta}} = mL^2(\dot{\beta} + \dot{\alpha}\cos(\alpha - \beta)).$$

To invert this relation, express $\dot{\alpha}$ and $\dot{\beta}$ with p_α and p_β. From the second equation, we get

$$\dot{\beta} = \frac{p_\beta}{mL^2} - \dot{\alpha}\cos(\alpha - \beta).$$

Substituting this into the first,

$$\dot{\alpha} = \frac{p_\alpha}{2mL^2} - \frac{\dot{\beta}\cos(\alpha-\beta)}{2} = \frac{p_\alpha}{2mL^2} - \frac{1}{2}\cos(\alpha-\beta)\left(\frac{p_\beta}{mL^2} - \dot{\alpha}\cos(\alpha-\beta)\right)$$

$$\dot{\alpha}\left(1 - \frac{1}{2}\cos^2(\alpha-\beta)\right) = \frac{p_\alpha - p_\beta\cos(\alpha-\beta)}{2mL^2}$$

$$\dot{\alpha} = \frac{p_\alpha - p_\beta\cos(\alpha-\beta)}{mL^2(2-\cos^2(\alpha-\beta))}.$$

Then, using $\dot{\alpha}$ in the first equation,

$$\dot{\beta} = \frac{p_\beta}{mL^2} - \frac{p_\alpha - p_\beta\cos(\alpha-\beta)}{mL^2(2-\cos^2(\alpha-\beta))}\cos(\alpha-\beta) = \frac{2p_\beta - p_\alpha\cos(\alpha-\beta)}{mL^2(1+\sin^2(\alpha-\beta))}.$$

The Hamiltonian is obtained by a Legendre transform

$$H(\alpha, p_\alpha, \beta, p_\beta) = \dot{\alpha}p_\alpha + \dot{\beta}p_\beta - L = \frac{p_\alpha^2 + 2p_\beta^2 - 2p_\beta p_\alpha\cos(\alpha-\beta)}{mL^2(1+\sin^2(\alpha-\beta))} -$$
$$-\frac{mL^2}{2}(2\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta}\cos(\alpha-\beta)) - mgL(2\cos\alpha + \cos\beta)$$

The second term is

$$\frac{mL^2}{2}(2\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta}\cos(\alpha-\beta)) = \frac{mL^2}{2}[2\left(\frac{p_\alpha - p_\beta\cos(\alpha-\beta)}{mL^2(1+\sin^2(\alpha-\beta))}\right)^2 +$$
$$+\left(\frac{2p_\beta - p_\alpha\cos(\alpha-\beta)}{mL^2(1+\sin^2(\alpha-\beta))}\right)^2 + 2\frac{2p_\beta - p_\alpha\cos(\alpha-\beta)}{mL^2(1+\sin^2(\alpha-\beta))}\frac{p_\alpha - p_\beta\cos(\alpha-\beta)}{mL^2(1+\sin^2(\alpha-\beta))}\cos(\alpha-\beta)] =$$
$$= \frac{2p_\alpha^2 + 2p_\beta^2\cos^2(\alpha-\beta) - 4p_\alpha p_\beta\cos(\alpha-\beta) + 4p_\beta^2 + p_\alpha^2\cos^2(\alpha-\beta) - 4p_\alpha p_\beta\cos(\alpha-\beta)}{2mL^2(1+\sin^2(\alpha-\beta))^2} +$$
$$+\frac{4p_\beta p_\alpha\cos(\alpha-\beta) - 4p_\beta^2\cos^2(\alpha-\beta) - 2p_\alpha^2\cos^2(\alpha-\beta) + 2p_\beta p_\alpha\cos^3(\alpha-\beta)}{2mL^2(1+\sin^2(\alpha-\beta))^2} =$$
$$= \frac{p_\alpha^2(2-\cos^2(\alpha-\beta)) + 2p_\beta^2(2-\cos^2(\alpha-\beta)) - 2p_\beta p_\alpha\cos(\alpha-\beta)(2-\cos^2(\alpha-\beta))}{2mL^2(1+\sin^2(\alpha-\beta))^2} =$$
$$= \frac{p_\alpha^2 + 2p_\beta^2 - 2p_\beta p_\alpha\cos(\alpha-\beta)}{2mL^2(1+\sin^2(\alpha-\beta))}.$$

Substituting it into the Hamiltonian,

$$H(\alpha, p_\alpha, \beta, p_\beta) = \frac{p_\alpha^2 + 2p_\beta^2 - 2p_\beta p_\alpha\cos(\alpha-\beta)}{2mL^2(1+\sin^2(\alpha-\beta))} - mgL(2\cos\alpha + \cos\beta).$$

Hamilton's Equations are

$$\dot{\alpha} = \frac{p_\alpha - p_\beta\cos(\alpha-\beta)}{mL^2(2-\cos^2(\alpha-\beta))}$$
$$\dot{\beta} = \frac{2p_\beta - p_\alpha\cos(\alpha-\beta)}{mL^2(1+\sin^2(\alpha-\beta))}$$
$$\dot{p}_\alpha = -\frac{\partial H}{\partial\alpha}$$
$$\dot{p}_\beta = -\frac{\partial H}{\partial\beta}$$

For the momentum equations, let $\lambda(\alpha,\beta) = 2mL^2(1+\sin^2(\alpha-\beta))$.

$$\dot{p}_\alpha = -\frac{\partial H}{\partial \alpha} = -\frac{1}{\lambda^2}\left(2\lambda p_\alpha p_\beta \sin(\alpha-\beta) - \frac{\partial \lambda}{\partial \alpha}(p_\alpha^2 + 2p_\beta^2 - 2p_\alpha p_\beta \cos(\alpha-\beta))\right) - 2mgL\sin\alpha$$

$$= \frac{-2p_\alpha p_\beta \sin(\alpha-\beta)}{\lambda} + \frac{2mL^2(2\sin(\alpha-\beta)\cos(\alpha-\beta))(p_\alpha^2 + 2p_\beta^2 - 2p_\alpha p_\beta\cos(\alpha-\beta))}{\lambda^2} - 2mgL\sin\alpha$$

$$\dot{p}_\alpha = -2mgL\sin\alpha - \frac{p_\alpha p_\beta \sin(\alpha-\beta)}{mL^2(1+\sin^2(\alpha-\beta))} + \frac{\sin(2(\alpha-\beta))(p_\alpha^2 + 2p_\beta^2 - 2p_\alpha p_\beta\cos(\alpha-\beta)))}{2mL^2(1+\sin^2(\alpha-\beta))}$$

$$\dot{p}_\beta = -mgL\sin\beta + \frac{p_\alpha p_\beta \sin(\alpha-\beta)}{mL^2(1+\sin^2(\alpha-\beta))} - \frac{\sin(2(\alpha-\beta))(p_\alpha^2 + 2p_\beta^2 - 2p_\alpha p_\beta\cos(\alpha-\beta)))}{2mL^2(1+\sin^2(\alpha-\beta))}$$

Exercise 2

Consider the Lotka–Volterra model

$$\begin{aligned}\dot{h} &= a_1 h(1-bp)\\ \dot{p} &= -a_2 p(1-ch)\end{aligned}$$

for the interaction of a predator and a prey population. Here $h(t)$ and $p(t)$ denote the predator and prey populations, respectively, as a function of time. $a_1, a_2, b, c > 0$.

(a) Show that the system is Hamiltonian for $h, p > 0$ for an appropriate rescaling of time.

Solution

The system can be written as

$$\begin{aligned}\dot{h} &= hp\left(\frac{a_1}{p} - a_1 b\right)\\ \dot{p} &= hp\left(a_2 c - \frac{a_2}{h}\right).\end{aligned}$$

We can rescale time by $A(h,p) = hp$, which is a positive function for $h, p > 0$, introducing a new time variable

$$\tau = \int_0^t h(s)p(s)ds.$$

$$\begin{bmatrix}\frac{dh}{d\tau}\\ \frac{dp}{d\tau}\end{bmatrix} = \begin{bmatrix}\frac{a_1}{p} - a_1 b\\ a_2 c - \frac{a_2}{h}\end{bmatrix} := \begin{bmatrix}C(p)\\ D(h)\end{bmatrix}$$

For this system to be Hamiltonian, C and D must be the the appropriate partial derivatives of a function $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. That is,

$$\frac{\partial H}{\partial p} = C(p)$$

and

$$\frac{\partial H}{\partial h} = -D(h).$$

We can integrate the equations to get

$$
\begin{aligned}
&\frac{a_1}{p} - a_1 b = \frac{\partial H}{\partial p} \\
&H(h,p) = a_1 \log p - a_1 b p + F(h) \\
&F'(h) = \frac{\partial H}{\partial h} = -a_2 c + \frac{a_2}{h} \\
&F(h) = -a_2 c h + a_2 \log h + K \\
&H(h,p) = a_1 \log p - a_1 b p + a_2 \log h - a_2 c h + K,
\end{aligned}
$$

where K is a constant. Taking H as the Hamiltonian, the system can be written as (in the rescaled time)

$$
\begin{aligned}
\frac{dh}{d\tau} &= \frac{\partial H}{\partial p} \\
\frac{dp}{d\tau} &= -\frac{\partial H}{\partial h}.
\end{aligned}
$$

(b) Using the Hamiltonian, argue that the two species can exhibit stable coexistence, i.e., the system admits a stable fixed point.

Solution

In the region $h, p > 0$, (where the rescaling is valid), the system has a single fixed point, defined by

$$C(h) = D(h) = 0.$$

This is satisfied by $\frac{a_1}{p_0} = a_1 b$ and $\frac{a_2}{h_0} = a_2 c$, or

$$(h_0, p_0) = \left(\frac{1}{b}, \frac{1}{c}\right).$$

To establish stability of (h_0, p_0), take H as a Lyapunov function. (h_0, p_0) is a critical point of H, which is conserved along trajectories, $\dot{H} = 0$.

The Hessian matrix of H is

$$D^2 H = \begin{bmatrix} \frac{\partial^2 H}{\partial h^2} & \frac{\partial^2 H}{\partial h \partial p} \\ \frac{\partial^2 H}{\partial h \partial p} & \frac{\partial^2 H}{\partial p^2} \end{bmatrix} = \begin{bmatrix} -\frac{a_2}{h^2} & 0 \\ 0 & -\frac{a_1}{p^2} \end{bmatrix}$$

$$D^2 H_{(h_0,p_0)} = \begin{bmatrix} -a_2 b^2 & 0 \\ 0 & -a_1 c^2 \end{bmatrix}.$$

This is negative definite at (h_0, p_0), so by taking $V = -H$ as a Lyapunov function, we can conclude nonlinear stability.

Exercise 3

Consider a two-dimensional steady compressible fluid flow with velocity field $\mathbf{v}(\mathbf{x}) = (u(x,y), v(x,y))$. Assume that the flow conserves mass, i.,e., its density function $\rho(\mathbf{x}) > 0$ satisfies the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

valid for general, unsteady flow. Show that the equation of fluid particle motion becomes a canonical Hamiltonian system after a rescaling of time.

Solution

For a steady flow, $\frac{\partial \rho}{\partial t} = 0$, which means $\rho\mathbf{v}$ is divergence free, by the continuity equation. This condition means

$$\frac{\partial(\rho u)}{\partial x} = -\frac{\partial(\rho v)}{\partial y}.$$

If we extend $\rho\mathbf{v}$ to be a 3 dimensional vector, the above relation means that there is a vector-potential $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$. In particular, with a scalar function $\Psi : \mathbb{R}^2 \to \mathbb{R}$,

$$\begin{bmatrix} \rho u \\ \rho v \\ 0 \end{bmatrix} = \text{rot } \mathbf{A} = \text{rot } \begin{bmatrix} 0 \\ 0 \\ \Psi \end{bmatrix} = \begin{bmatrix} \partial_y \Psi \\ -\partial_x \Psi \\ 0 \end{bmatrix}.$$

The (massless) fluid particles' motion obeys the differential equation

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v. \end{aligned}$$

Multiplying the equations by the density, and substituting Ψ, gives the desired form

$$\begin{aligned} \rho\dot{x} &= \rho u \\ \rho\dot{y} &= \rho v \\ \dot{x} &= \frac{1}{\rho}\frac{\partial \Psi}{\partial y} \\ \dot{y} &= -\frac{1}{\rho}\frac{\partial \Psi}{\partial x}. \end{aligned}$$

We can bring it to the canonical form, by introducing a rescaling of time, $\tau = \int_0^t \frac{1}{\rho(x(s),y(s))} ds$.

$$\frac{dx}{d\tau} = \frac{\partial \Psi}{\partial y} \tag{1}$$

$$\frac{dy}{d\tau} = -\frac{\partial \Psi}{\partial x} \tag{2}$$

Exercise 4

Consider a dynamical system defined on the two-dimensional torus, $\mathbb{T}^2 = S^1 \times S^2$. Such systems admit the general form

$$\begin{aligned} \dot{\phi}_1 &= a(\phi_1, \phi_2) \\ \dot{\phi}_2 &= b(\phi_1, \phi_2), \end{aligned}$$

where $\phi_i \in S^1$.

(a) Show that a physical example is found in the motion of two uncoupled linear undamped oscillators. Specifically, show that orbits of

$$\begin{aligned} \ddot{x} + x &= 0 \\ \ddot{y} + y &= 0 \end{aligned}$$

are confined to two-dimensional tori of the phase space.

Solution

The orbits of the linear equation are described in the space space, spanned by the variables $x, \dot{x} = v_x, y, \dot{y} = v_y$.

The differential equations are satisfied by

$$x(t) = r_x \cos(t + \delta_x) \qquad y(t) = r_y \cos(t + \delta_y).$$

This can be verified by direct substitution,

$$\ddot{x} = -r_x \cos(t + \delta_x) = -x(t) \qquad \ddot{y} = -r_y \cos(t + \delta_y) = -y(t).$$

The velocity variables are

$$v_x = \dot{x} = -r_x \sin(t + \delta_x) \qquad v_y = \dot{y} = -r_y \sin(t + \delta_y).$$

The trajectory is given by the parametrization, using the notation $\phi_1 = t + \delta_x$, $\phi_2 = t + \delta_y$

$$\begin{bmatrix} x \\ v_x \\ y \\ v_y \end{bmatrix} = \begin{bmatrix} r_x \cos(\phi_1) \\ -r_x \sin(\phi_1) \\ r_y \cos(\phi_2) \\ -r_y \sin(\phi_2) \end{bmatrix}$$

Which describes a 2-torus, embedded in $\mathbb{R}^4$.

(b) Assume that the system has no fixed point (which is the case in the oscillator example). Argue that the system then cannot be Hamiltonian, even after a rescaling of time.

Solution

Assume the converse, that the system is Hamiltonian, in the generalized sense. That is, there is a smooth function $H : \mathbb{T}^2 \to \mathbb{R}$ and $F : \mathbb{T}^2 \to \mathbb{R}$, $F(\phi_1, \phi_2) \neq 0$

$$\dot{\phi}_1 = F(\phi_1, \phi_2) \frac{\partial H}{\partial \phi_2} \tag{3}$$

$$\dot{\phi}_2 = -F(\phi_1, \phi_2) \frac{\partial H}{\partial \phi_1}. \tag{4}$$

Because F cannot be zero, all possible fixed points must correspond to critical points of H. We also know that H is a continuous function, defined on a compact domain ($\mathbb{T}^2$ is compact). Then, by a theorem from analysis, H must have a minimum and a maximum value on $\mathbb{T}^2$.

Since the 2-torus is a manifold without a boundary, these extremum points must correspond to critical points of H, where $DH = 0$.

We conclude that if the original system, $[\dot{\phi}_1, \dot{\phi}_2]$, is Hamiltonian, then it must have a fixed point.

If we assume that $[\dot{\phi}_1, \dot{\phi}_2]$ does not have a fixed point, then it cannot be Hamiltonian.

Exercise 5

Show that for any dynamical system $\dot{q} = f(q, t)$, $q \in \mathbb{R}^n$, one can select a canonically conjugate variable $p \in \mathbb{R}^n$, such that the evolution of $(q(t), p(t))$ is governed by the Hamiltonian system. (Thus any type of dynamics can be viewed as a projection from a higher-dimensional Hamiltonian dynamical system.)

Solution

Consider the function $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, defined by

$$H(x, p, t) = \mathbf{f}(x, t) \cdot \mathbf{p}.$$

This function defines a Hamiltonian dynamical system on $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$. The evolution equations are

$$\dot{x} = f(x, t) \tag{5}$$
$$\dot{p} = -\nabla_x H(x, p, t) = -\nabla_x \mathbf{f}(x, t) \cdot \mathbf{p} \tag{6}$$

Projecting this system to any $\mathbf{p} =$ constant subspace gives the original dynamics, defined by the ODEs

$$\dot{q} = f(q, t).$$