Sheet 1

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(i) First note that there are $kN(k)$ elements with orbit length exactly (minimal) k, for all k. Next, we want to know how many elements have (not necessarily exactly) orbits of length k. This means if $k = 4$, then we count 16 elements, as we include elements with orbit length exactly 2 and/or 1 (2 and 1 each divide 4). This entails counting how many ways we can construct a block of length k. We consider one construction C_1 to be equal to another C_2, if there exists n such that $\sigma^n(C_1) = C_2$, where σ acts cyclically. Equivalently, we could say $C_1 \cong C_2$ if

$$\exists n \geq 1 : \quad \sigma^n \left(\overline{C}_1.\overline{C_1}\right) = \overline{C}_2.\overline{C}_2.$$

Constructing a unique block in this case just means choosing a number i, $0 \leq i \leq k$, for the amount of symbols of one type, and choosing a constellation for placing these symbols. For a given i there are $\binom{k}{i}$ (k choose i) ways to place the i elements in k places. Next, we have to sum over all possible i

$$\sum_{i=0}^{k} \binom{k}{i} = 2^k.$$

In order to get the amount of elements with orbit length exactly k we have to subtract out the elements which have orbit length exactly i for $i|k$ (i divides k). There are $i \cdot N(i)$ of these elements, thus

$$kN(k) = 2^k - \sum_{i|k} iN(i).$$

Now dividing by k yields the desired result.

(ii) (a) The matrix A tells us if it is possible to transition from symbol i to symbol j in a single step if $A_{ij} = 1$, and otherwise 0. Say $A_{ii} = 1$, then $\overline{s}_i.\overline{s}_i$ is an admissible sequence and is a fixed point of σ ($\sigma\left(\overline{s}_i.\overline{s}_i\right) = \overline{s}_i.\overline{s}_i$). Noting that all fixed points of σ are in fact sequences consisting of a single symbol it is clear that the fixed points of σ on Σ_A^N are the sequences $\overline{s}_j.\overline{s}_j$ with $A_{jj} = 1$.

In conclusion, there are as many fixed points as 1s on the diagonal, and A is binary (only consists of 0s and 1s), this is equal to $\mathrm{tr}(A)$. (This holds as $\overline{s}_j.\overline{s}_j$ is fixed and the type of fixed point, further it is only in Σ_A^N if $A_{jj} = 1$.)

(b) A_{ii}^k encodes the amount of unique admissible paths from $i \to i$ in k steps. Any fixed point of σ^k has the for $\overline{C}.\overline{C}$ with C being a sequence of k symbols. We will call C admissible if for all i $C_i C_{i+1} =: s_m s_n$ and $A_{mn} = 1$. Thus we want to know how many admissible C exist, as each of these correspond to a fixed point. Furthermore no other fixed points exist as all fixed points must be of this form. Hence we will have identified all fixed points of σ^k (k-periodic orbits).

Since we repeat C infinitely, there are A_{jj}^k admissible C which start with s_j, i.e. $C_1 = s_j$. If we sum over all s_j to get the total amount of admissible C we find $\sum_{i=1}^N A_{ii}^k = \mathrm{tr}(A)$.

(iii)

(iv) First note that the orbit s^* visits $B(s, \delta)$ (ball of radius δ around s) infinitely often for all $s \notin \mathrm{Orbit}(s^*) = \mathcal{O}(s^*)$, as if there were finite visits, there there would exist $0 < \delta = 0.5 \min_k(d(\sigma^k(s^*), s))$, and there would not exist $N > 0$ with $d(\sigma^N(s^*), s) < \delta'$. Define $B' = B \setminus \mathcal{O}(s^*)$, nonempty (as the orbit is countable and B is uncountable (B is open)). For any $a \in A$ there exists $\delta > 0$ with $B(a, \delta) \subset A$, there also exists N such that $d(\sigma^N(s^*), a) < \delta$, call $\sigma^N(s^*) = s_a$. Choose any $b \in B'$, there exists $\epsilon > 0$ such that $B(b, \epsilon) \subset B$ and $M > N$ with $d(\sigma^M(s^*), b) < \epsilon$, such an M exists due to the infinite visiting property. Call $\sigma^M(s^*) = s_b$, we know that $s_a \in A$ and that $\sigma^{M-N}(s_a) = \sigma^M(s^*) = s_b$. Thus we have that

$$\sigma^{M-N}(A) \cap B \neq \emptyset,$$

and the N in question corresponds to the $M - N$ here.

It maybe unclear why B open implies it is uncountable (ignoring $B = \emptyset$); take any $b \in B$ and there exists $0 < \epsilon < 1$ with $B(b, \epsilon) \subset B$. Now we want an injection from $(0, \epsilon)$ to B. For each $x \in (0, \epsilon)$, write the binary representation of $x = 0.b_1 b_2 \ldots$, this mapping is bijective, call it $\varphi(x)$. Now construct an element by taking $b = \ldots.s_0 s_1 s_2 \ldots$ and for every $b_i = 1$ set $s_i' = s \in \Sigma$ with $|s - s_i| = 1$, and for $b_i = 0$ $s_i' = s_i$. Set $S' = \ldots s_{-1}.s_0 s_1' s_2' s_3' \ldots$ and call this $\Psi(\varphi(x))$. Ψ is clearly injective as every binary sequence is mapped to a unique element of Σ. Next, we see that $d(\Psi(\varphi(x)), b)$ is exactly equal to x, thus for all $x \in (0, \epsilon)$ we have $\Psi(\varphi(x)) \in B(b, \epsilon) \subset B$, so we have an injection from an uncountable set into B, showing B to be uncountable.

(v) Choose $\Delta = \frac{1}{2}$. For any two symbol sequences with $d(s, s') > 0$ (non-equal), there exists a position $N \in \mathbb{Z}$ such that $s_N \neq s_N'$. Then we know that $\sigma^N(s)_0 \neq \sigma^N(s')$, therefore $d(\sigma^N(s), \sigma^N(s')) \geq 1 > \Delta$.