151-0530-00L, Spring, 2020

Nonlinear Dynamics and Chaos II

Homework Assignment 1

Due: Wednesday, March 25;
please submit by email to Dr. Shobhit Jain <shjain@ethz.ch>

1. Derive the Hamiltonian equations of motion for a the coupled pendulums shown in Fig. 1. (The two point masses m are placed at the tips of two massless rods of length L. Both joints are frictionless; the constant of gravity is g.)

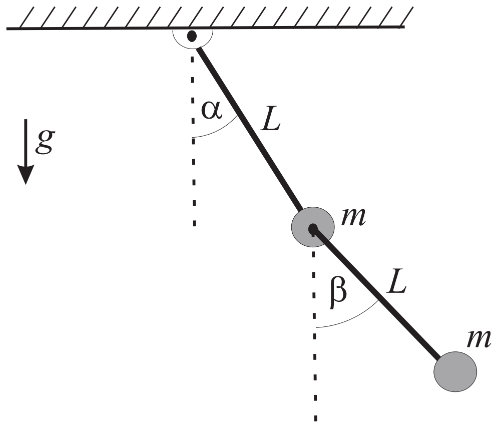

Figure 1: Coupled system of two pendulums

2. Consider the Lotka–Volterra model

$$\begin{aligned} \dot{h} &= a_1 h\,(1 - bp)\,, \\ \dot{p} &= -a_2 p\,(1 - ch)\,, \end{aligned} \tag{1}$$

for the interaction of a predator and a prey population. Here $h(t)$ and $p(t)$ denote the predator and prey populations, respectively, as a function of time; a_1, a_2, b, and c are positive parameters.

(a) Show that system (1) is Hamiltonian for $h, p > 0$ after an appropriate rescaling of time. Find the Hamiltonian. (Hint: Rewrite (1) as $\dot{h} = A(h,p)C(p)$, $\dot{p} = A(h,p)D(h)$ by defining the functions A, C and D appropriately.)

(b) Using the Hamiltonian, argue that the two species can exhibit stable coexistence, i.e., the system admits a stable fixed point. (Hint: establish *full nonlinear stability* for the fixed point).

3. Consider a two-dimensional steady *compressible* fluid flow with velocity field $\mathbf{v}(\mathbf{x}) = (u(x,y), v(x,y))$, where $\mathbf{x} = (x,y)$. Assume that the flow conserves mass, i.e., its density function $\rho\,(\mathbf{x}) > 0$ satisfies the equation of continuity. The latter equation, in its general form for unsteady flows, reads

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0,$$

valid or general, unsteady flow. Show that the equation of fluid particle motion becomes a canonical Hamiltonian system after a rescaling of time.

4. Consider a dynamical system defined on the two-dimensional torus $\mathbb{T}^2 = S^1 \times S^1$. Such systems admit the general form

$$\begin{aligned} \dot{\phi}_1 &= a(\phi_1, \phi_2), \\ \dot{\phi}_2 &= b(\phi_1, \phi_2), \end{aligned} \tag{2}$$

where $\phi_i \in S^1$.

(a) Show that a physical example of system (2) is found in the motion of two uncoupled linear undamped oscillators. Specifically, show that orbits of

$$\begin{aligned} \ddot{x} + x &= 0, \\ \ddot{y} + y &= 0, \end{aligned}$$

are confined to two-dimensional invariant tori of the phase space.

(b) Assume that system (2) has no fixed point (which is the case in the oscillator example). Argue that (2) then *cannot* be Hamiltonian, even after a rescaling of time. (*Hint:* Use the fact that a continuous function defined on a compact set must have a minimum and a maximum).

5. Show that for any dynamical system $\dot{q} = f(q, t)$, $q \in \mathbb{R}^n$, one can select a canonically conjugate variable $p \in \mathbb{R}^n$, such that the evolution of $(q(t), p(t))$ is governed by a Hamiltonian system. (Thus any type of dynamics can be viewed as a projection from a higher-dimensional Hamiltonian dynamical system.)