Nonlinear Dynamics and Chaos I.
Problem set 4

1. Consider the discrete dynamical system

$$\begin{aligned} x_{n+1} &= Ax_n + f(x_n, y_n), \\ y_{n+1} &= By_n + g(x_n, y_n), \end{aligned}$$

where $x_n \in \mathbb{R}^c$, $y_n \in \mathbb{R}^d$, $A \in \mathbb{R}^{c\times c}, B \in \mathbb{R}^{d\times d}$; f and g are C^r functions with no linear terms. Assume that all eigenvalues of A have modulus one, and none of the eigenvalues of B have modulus one. Then the linearized system at the origin admits a center subspace E^c aligned with the x coordinate plane.

(a) Derive a general algebraic equation for the center manifold W^c, which is known to exists by a theorem analogous to the center manifold theorem for continuous dynamical systems.

(b) Find a cubic order approximation for the center manifold of the discrete system

$$\begin{aligned} x_{n+1} &= x_n + x_n y_n, \\ y_{n+1} &= \lambda y_n - x_n^2, \end{aligned}$$

where $\lambda \in (0, 1)$.

(c) Reduce the dynamics to the center manifold and determine the stability of the origin. Verify your results by a numerical simulation of a few initial conditions near the origin.

2. Consider the quadratic *Duffing equation*

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= \beta u - u^2 - \delta v, \end{aligned}$$

where $\delta > 0$, and $0 \le |\beta| \ll 1$.

(a) Construct a β–dependent center manifold up to quadratic order near the origin for small β values.

(b) Construct a stability diagram for the reduced system on the center manifold using β as a bifurcation parameter.

3. Construct a cubic-order local approximation for the unstable manifold of the hyperbolic fixed point of the pendulum equation

$$\ddot{x} + \sin x = 0.$$