Nonlinear Dynamics and Chaos

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Chapter 0

Introduction

First we shall introduce the most important characters in our following exploration. The ideas and definitions here will be recurring regularly as we examine them from different perspectives and using different tools.

Definition 0.1 (Dynamical System (DS))**.** A triple $(P, E, \mathcal{F})$, with

- P : the phase space for the dynamical variable $x \in P$,
- E : base space of the evolutionary variable (e.g. time) $t \in E$,
- $\mathcal{F}$: the evolution rule (deterministic) which defines the transition from one state to the next.

The two main types of evolutionary variable spaces are

(i) Discrete dynamical systems (DDS) $t \in E = \mathbb{Z}$ with trajectory $\{x_0, x_1, \ldots\}$,

(ii) Continuous dynamical systems (CDS) $t \in E = \mathbb{R}$ with trajectory $\{x_t\}_{t \in \mathbb{R}}$.

Corresponding to these there are various types of evolution rules

(i) In a DDS we have iterated mappings

$$\boxed{x_{n+1} = F(x_n, n).}$$

If there is no explicit dependence on n, i.e. $\frac{\partial F}{\partial n} = 0$, then

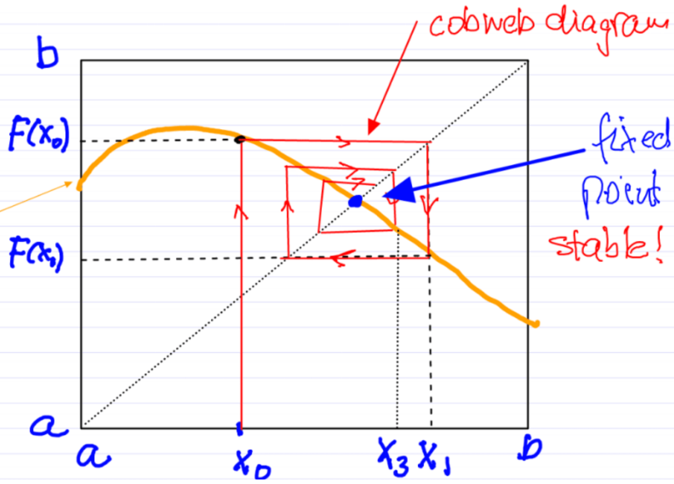
$$\boxed{x_{n+1} F(x_n) = F(F(x_{n-1})) = \underbrace{F \circ \ldots \circ F}_{n+1 \text{ times}}(x_0) = F^{n+1}(x_0).}$$

Example: For $P=[a,b]\subset\mathbb{R}^1$

$F:[a,b]\to[a,b]$

graph (F)

$x_1 = F(x_0)$



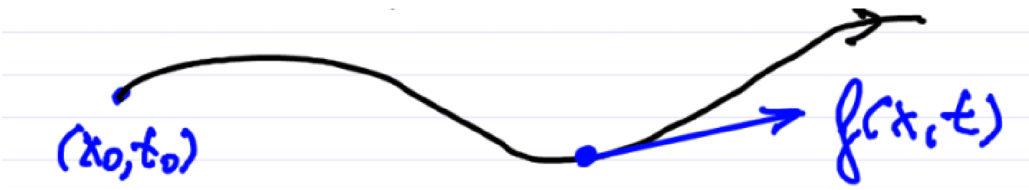
Example 0.1.

(ii) In a CDS we have a first order system of ordinary differential equations (ODE)

$$\boxed{\dot{x} = f(x,t)}$$

for $x \in P$ and $t \in E$. This yields the initial value problem (IVP):

$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0 \end{cases}$$



Assuming there exists a unique solution $\varphi(t;t_0,x_0)$ with $\dot{\varphi} = f(\phi,t)$ and $\varphi(t_0) = x_0$, then the following flow map is well defined

$$\boxed{F_{t_0}^t(x_0) := \varphi(t;t_0,x_0).}$$

Such an $F_{t_0}^t$ has nice properties

- $F_{t_0}^t$ is as smooth as $f(x,t)$,
- $F_{t_0}^{t_0} = I$ and $F_{t_0}^{t_2} = F_{t_1}^{t_2} \circ F_{t_0}^{t_1}$ (group property),

- $\left(F_{t_0}^t\right)^{-1} = F_t^{t_0}$ exists and is smooth.

A special case of this is the autonomous system

$$\boxed{\dot{x} = f(x).}$$

The autonomy of a system implies

$$x(s, t_0, x_0) = x(\underbrace{s - t_0}_{t}, 0, x_0) \stackrel{!}{=} x(t, x_0).$$

And the induced flow map in this case is the one-parameter family of maps

$$\boxed{F^t = F_0^t : x_0 \mapsto x(t, x_0).}$$

Example 0.2 (Logisitic Equation). For a resource-limited population, we have the following dynamic system for $a > 0$, $b > 0$, and the population $x \in \mathbb{R}_+ \cup \{0\}$

$$\dot{x} = ax(b - x).$$

In this case we have $E = \mathbb{R}$ and $\mathcal{F} = \{F^t\}_{t=-\infty}^{+\infty}$. This system has globally existing unique solutions (see later).

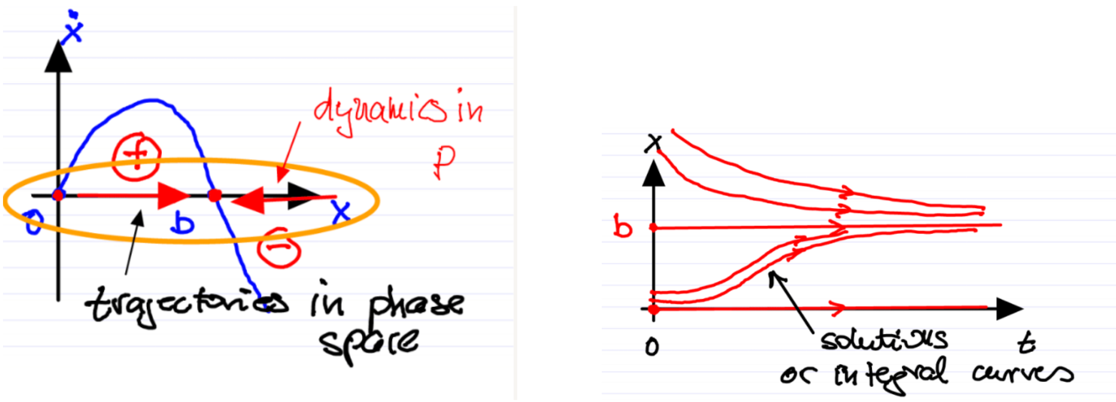


Figure 1: Left: Analysis of the right hand side. Right: Evolution in the extended phase space $P \times \mathbb{R}$.

Example 0.3 (Pendulum). Given the equation of motion

$$ml^2\ddot{\varphi} = -mgl\sin(\varphi).$$

We let $x_1 = \varphi$ and $x_2 = \dot{\varphi}$ to transform into the first-order ODE form

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\dfrac{g}{l}\sin(x_1). \end{cases}$$

Thus we have

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad f(x) = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) \end{pmatrix}.$$

Qualitatively analysis gives the following facts

- $x_1, x_2) = (0,0)$ and $(x_1, x_2) = (\pi, 0)$ are zeros of f,
- Energy is conserved, hence both small and large amplitude oscillations are expected,
- We have the symmetries $(x_1, x_2, t) \mapsto (x_1, -x_2, -t)$ and $(x_1, x_2, t) \mapsto (-x_1, x_2, -t)$.

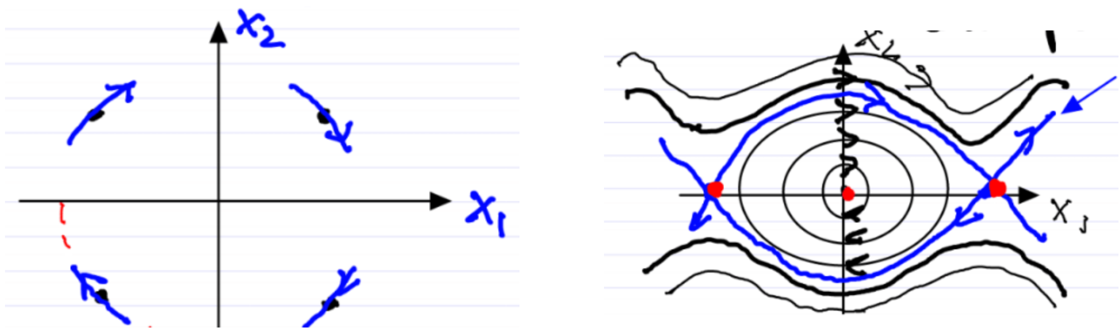


Figure 2: Left: The symmetries of the dynamic system. Right: Phase portrait of the pendulum. The blue trajectories are separatrix.

Definition 0.2. A separatrix connects fixed points, is unobservable by itself, and separates regions of similar behavior.

Example 0.4 (Exploit geometry of phase space for analysis). Two bikes *can* make it from A to B on different routes without exceeding distance D. Assume two trucks are trying to make it between A and B, on different roads in the opposite direction, carrying load of width D. Can the trucks make it without hitting each other?

The two trajectories must intersect by continuity, thus at that point the trucks must be at the same positions as the bikes, implying they are within distance D. Therefore the trucks must crash!

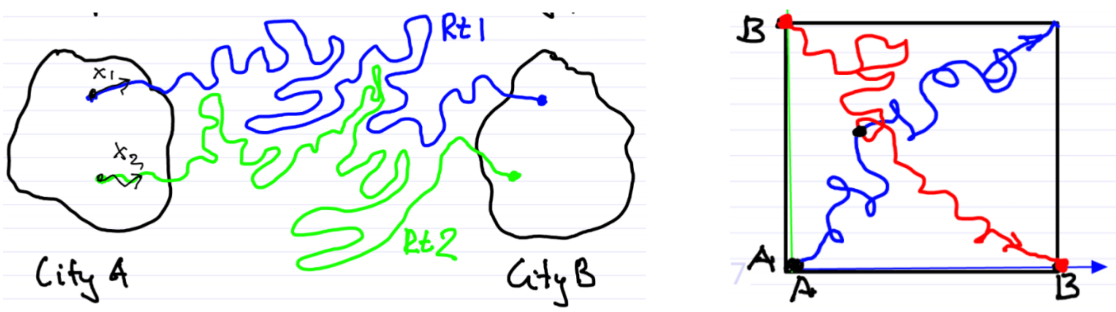


Figure 3: Left: An example of the two bike routes. Right: Blue represents the phase trajectory of the two biker, red represents the phase trajectory of the two trucks.

Chapter 1

Fundamentals

In this chapter, we first review some fundamental properties of continuous dynamical systems that will be used heavily in later chapters. As we will see, these technical results are interesting in their own right. They can help in interpreting or cross-checking numerical results or physical models for self-consistency or accuracy.

1.1 Existence and uniqueness of solutions

Consider

$$\begin{cases} \dot{x} = f(x,t); \quad x \in \mathbb{R}^n \\ x(t_0) = x_0 \end{cases}.$$

Does this initial value problem have a unique solution? We have the following theorems to help us answer that question.

Theorem 1.1 (Peano)**.** *If $f \in C^0$ near (x_0, t_0), then there exists a local solution $\varphi(t)$, i.e.,*

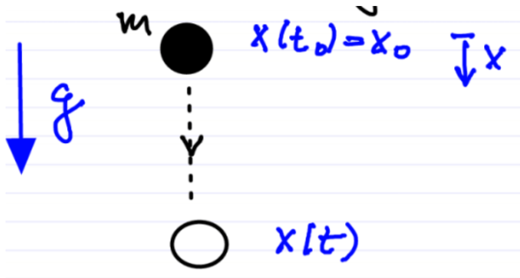
$$\dot{\varphi}(t) = f(\varphi(t), t), \varphi(t_0) = x_0;\ \forall t \in (t_0 - \epsilon, t_0 + \epsilon);\ 0 < \epsilon \ll 1.$$

Example 1.1 (Free falling mass). We have that the total energy is conserved

$$\frac{1}{2} m\dot{x}^2 = mg(x - x_0).$$

This implies that

$$\begin{cases} \dot{x} = \sqrt{2g(x - x_0)} \\ x(0) = x_0 \end{cases}$$

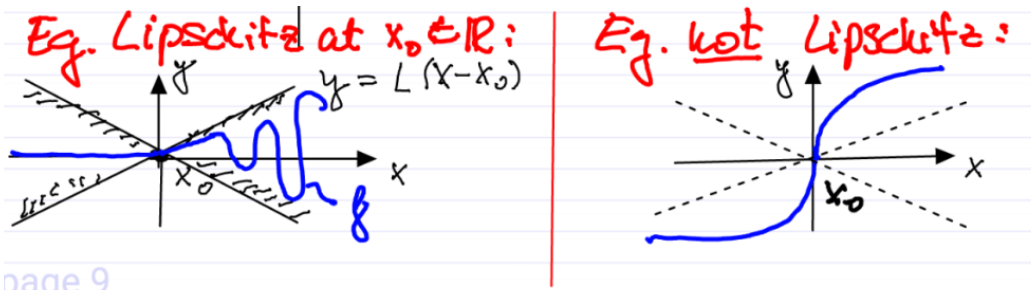


on the set $P = \{x \in \mathbb{R} : \ x \geq x_0\}$. Therefore we have that $f \in C^0$ in phase space, so by Peano's theorem (cf. 1.1), there exists a local solution. The solution is actually $x(t) = x_0 + \frac{g}{2}(t - t_0)^2$, however $x(t) = x_0$ is also a solution to the IVP, therefore we do not have a unique solution. Physically there exists a solution, but this IVP was derived from a heuristic energy-principle, not from Newton's laws, which are not equivalent.

Definition 1.1. A function is called locally Lipschitz around x_0 if there exists an open set U_{x_0} and $L > 0$ such that for all $x, y \in U_{x_0}$

$$\boxed{|f(y,t) - f(x,t)| \leq L|y - x|.}$$

Example 1.2 (Lipschitz functions)*.* Here we have an example of a Lipschitz and a non-Lipschitz function around x_0.



Theorem 1.2 (Picard)**.** *Assume*

(i) $f \in C^0$ in t near (t_0, x_0),

(ii) f is locally Lipschitz in x near (t_0, x_0).

Then there exists a unique local solution to the IVP. The proof from Arnold's ODE.

Note f is $C^1 \implies f$ is Lipschitz $\implies f$ is C^0.

Example 1.3 (Free falling mass revisted). We check if f is Lipschitz.

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} = \frac{\sqrt{2g}}{\sqrt{|x - x_0|}} \not< L|x - x_0|.$$

Thus f is not Lipschitz near x_0.

1.2 Geometric consequences of uniqueness

If the solution is unique, we have a few facts that can be derived from the geometric point of view.

(i) The trajectories of autonomous systems cannot intersect. Note that fixed points do not violate this (e.g. pendulum equations).

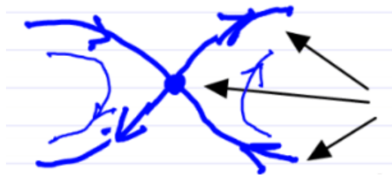


Figure 1.1: The phase portrait of the pendulum. Trajectories do not intersect since each arrow is pointing at separate trajectories.

(ii) For non-autonomous systems, intersections in phase space are possible. In which case we can extend the phase space in order to get an autonomous system where there cannot be any intersections.

$$X = \begin{pmatrix} x \\ t \end{pmatrix}, \; F(X) = \begin{pmatrix} f(x,t) \\ 1 \end{pmatrix}; \; \dot{X} = F(X).$$

1.3 Local vs global existence

Example 1.4 (Exploding solution).

$$\begin{cases} \dot{x} = x^2 \\ x(t_0) = 1. \end{cases}$$

Integrating yields the solution $x(t) = \frac{1}{1-(t-t_0)}$. This solution blows up at $t_\infty = t_0 + 1$, therefore the solution is only local.

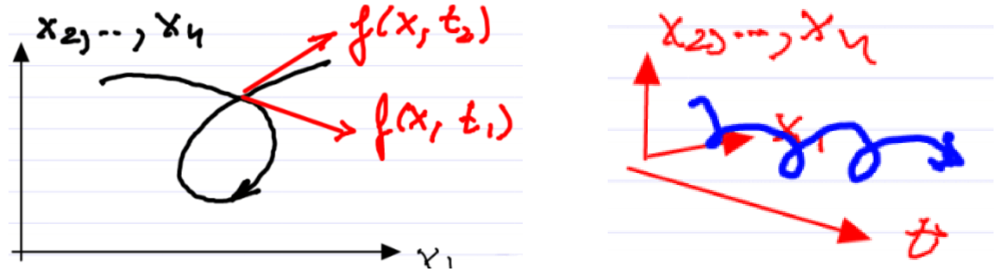
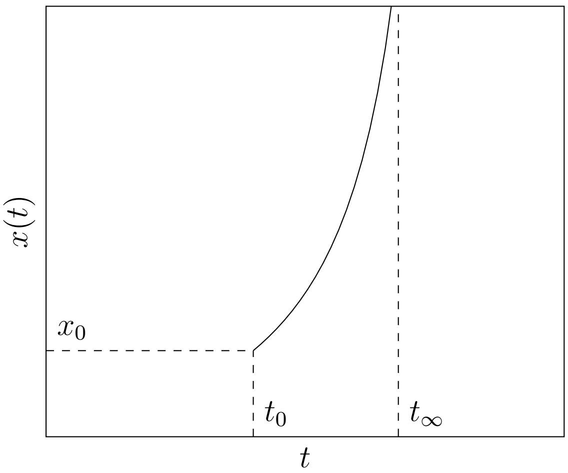


Figure 1.2: Left: Intersecting trajectories in phase space for a non-autonomous system. Right: The same trajectory in the extended phase space, without intersections.



To address this problem of local solutions not being able to be continued into global solution, we have the following theorem.

Theorem 1.3 (Continuation of solution)**.** *If a local solutions cannot be continued to to a time T, then we must have*

$$\boxed{\lim_{t\to T} |x(t)| = \infty.}$$

The proof from Arnold's ODE.

Example 1.5 (Coupled Pendulum System)*.* We set $x_1 = \varphi_1,\ x_2 = \dot{\varphi}_1,\ x_3 = \varphi_2,\ x_4 = \dot{\varphi}_2$ and get the following equation of motion

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dots \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \dots \end{cases}$$

The RHS is smooth, therefore there exists a unique local solution to any IVP. The phase space

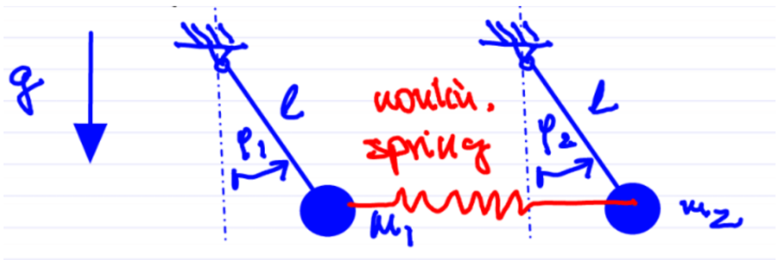


Figure 1.3: Physical setup of the coupled pendulum with a nonlinear spring.

is given by

$$P = \{x : \ x_1 \in S^1, \ x_2 \in \mathbb{R}, \ x_3 \in S^1, \ x_4 \in \mathbb{R}\} = S^1 \times \mathbb{R} \times S^1 \times \mathbb{R}.$$

Where S^1 is the 1 dimensional sphere (i.e. a circle). With this space we know that $|x_1|$ and $|x_3|$ are bounded. Due to energy being conserved we have

$$\begin{aligned} E &= T + V = \frac{1}{2} m_1 l_1 x_2^2 + \frac{1}{2} m_2 l_2 x_4^2 + \underbrace{V(x_1, x_3)}_{\geq 0} \\ E &= E_0 = \text{constant} \geq 0. \end{aligned}$$

Hence $|x_2|$ and $|x_4|$ are also bounded, therefore all solutions exist globally.

Definition 1.2. A linear system is one such that for $x \in \mathbb{R}^n, \ A(t) \in \mathbb{R}^{n\times n}$ and $A \in C^0$

$$\boxed{\dot{x} = A(t)x.}$$

Remark 1.4. Note that $S = \frac{1}{2}(A + A^T)$ is symmetric (i.e. $S = S^T$) and $\Omega = \frac{1}{2}(A - A^T)$ is skew symmetric (i.e. $\Omega = -\Omega^T$). Furthermore the eigenvalues of S, λ_i, are all real and their respective eigenvectors, e_i, are orthogonal.

Example 1.6 (Global existence in linear systems).

$$\begin{aligned} \langle x, \dot{x} \rangle = \frac{1}{2}\frac{d}{dt}|x(t)|^2 &= \langle x, A(t)x \rangle = \langle x, (S(t) + \Omega(t))x \rangle \\ &= \langle x, S(t)x \rangle + \underbrace{\langle x, \Omega(t)x \rangle}_{=0} \overset{(*)}{=} \sum_{i=1}^{n} \lambda_i(t) x_i^2 \\ &\leq \lambda_{\max}(t) \sum_{i=1}^{n} x_i^2 = \lambda_{\max}(t)|x(t)|^2. \end{aligned}$$

Where in $(*)$ we used that $x = \sum_{i=1}^{n} x_i e_i$ with $|e_i| = 1$ and $e_i \perp e_j$ for all $i \neq j$. Thus we get

$$\frac{\frac{1}{2}\frac{d}{dt}|x(t)|^2}{|x(t)|^2} \leq \lambda_{\max}(t) \implies \int_{t_0}^{t} \log\left(\frac{|x(s)|^2}{|x(t_0)|^2}\right) ds \leq \lambda_{\max}(s) ds.$$

By exponentiating both sides, we obtain

$$\boxed{|x(t)| \leq |x(t_0)| \exp\left(\int_{t_0}^{t} \lambda_{\max}(s) ds\right).}$$

Therefore, by the continuation theorem, global solutions exist as long as $\int_{t_0}^{t} \lambda_{\max}(s) ds < \infty$.

1.4 Dependence on initial conditions

Given the IVP

$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0. \end{cases}$$

With $x \in \mathbb{R}^n$ and $f \in C^r$ for some $r \geq 1$, we have the solution $x(t; t_0, x_0)$.

Question How does the solution depend on initial data? But first, why do we care about this? Because we robust solutions with respect to errors and uncertainties in the initial data.

Theorem 1.5. *If $f \in C^r$ for $r \geq 1$ then $x(t; t_0, x_0)$ is C^r in (t_0, x_0). Proof in Arnold's ODE.*

The geometric meaning of this is that for $U \subset P \subset \mathbb{R}^n$ we have that $F_{t_0}^t(U)$ is a smooth deformation of U. It turns out $\left(F_{t_0}^t\right)^{-1} = F_t^{t_0}$ is also C^r, hence we have that $F_{t_0}^t$ is a diffeomor-

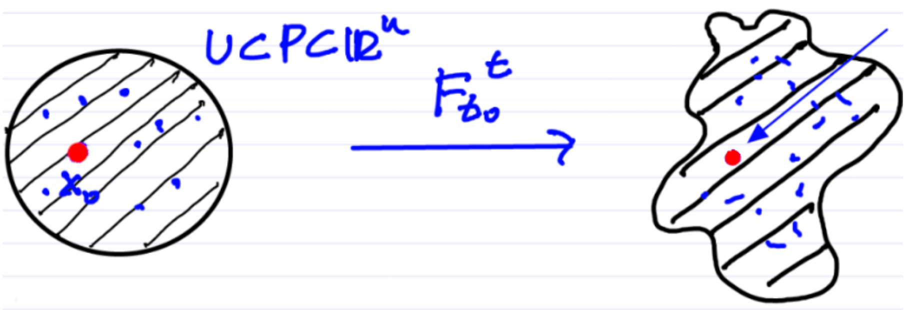


Figure 1.4: The smooth transformation of U. The red point on the right it $F_{t_0}^t(x_0)$, i.e. the image of x_0 through the evolution operator.

phism.

Now, how can we compute the Jacobian of the flow map $\frac{\partial x(t;t_0,x_0)}{\partial x_0} = DF_{t_0}^t(x_0)$? We will use the IVP.

$$\frac{d}{dt}\frac{\partial x}{\partial x_0} = D_x f(x(t;t_0,x_0),t)\frac{\partial x}{\partial x_0}.$$

The flow gradient satisfies the IVP

$$\begin{aligned}\frac{d}{dt}\left[DF_{t_0}^t(x_0)\right] &= D_x f(F_{t_0}^t(x_0),t)DF_{t_0}^t(x_0)\\ DF_{t_0}^{t_0}(x_0) &= I.\end{aligned}$$

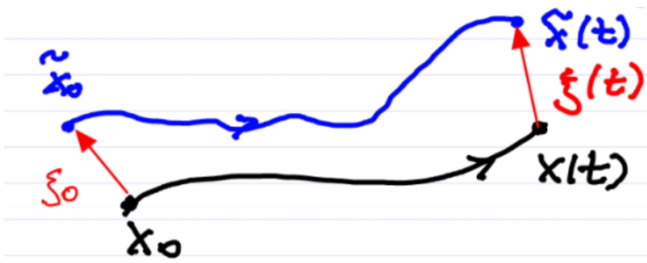
This gives us the equation of variations (linear, non-autonomous)

$$\begin{cases}\dot{M} = D_x f(x(t;t_0,x_0))M\\ M(t_0) = I.\end{cases}$$

Example 1.7 (Locations of extreme deformation in phase space). We define

$$\begin{aligned}\xi(t) &:= \tilde{x}(t) - x(t) = x(t;t_0,\tilde{x_0}) - x(t;t_0,x_0)\\ &= x(t;t_0,x_0) + \frac{\partial x}{\partial x_0}(t;t_0,x_0)\xi_0 + \mathcal{O}(|\xi_0|^2) - x(t;t_0,x_0)\\ &= DF_{t_0}^t(x_0)\xi_0 + \mathcal{O}(|\xi_0|^2).\end{aligned}$$

Where we used the Taylor expansion and assume the perturbation to x_0 is small, i.e. $|\xi_0| \ll 1$. Therefore we have



$$\begin{aligned}|\xi(t)|^2 &= \langle DF_{t_0}^t(x_0)\xi_0, DF_{t_0}^t(x_0)\xi_0\rangle + \mathcal{O}(|\xi_0|^3)\\ &= \langle \xi_0, \underbrace{\left[DF_{t_0}^t(x_0)\right]^T DF_{t_0}^t(x_0)}_{=:C_{t_0}^t(x_0)}\xi_0\rangle + \mathcal{O}(|\xi_0|^3).\end{aligned}$$

$C_{t_0}^t(x_0)$ is known as the Cauchy-Green strain tensor (field of $n \times n$ symmetric matrices). Therefore the largest possible deformation is

$$\max_{x_0,xi_0} \frac{|\xi(t)|^2}{|\xi_0|^2} = \max_{x_0,\xi_0} \frac{\langle \xi_0, C_{t_0}^t(x_0)\xi_0 \rangle}{|\xi_0|^2} = \max_{x_0} \lambda_n(x_0).$$

Where we used that $C_{t_0}^t$ is positive definite in the last equality, and that $\lambda_n(x_0)$ is the largest eigenvalue of $C_{t_0}^t(x_0)$. We typically have exponential growth.

Definition 1.3. The finite-time Lyapunov exponent is defined as

$$\boxed{\mathrm{FTLE}_{t_0}^t(x_0) := \frac{1}{2(t-t_0)} \log(\lambda_n(x_0)).}$$

The FTLE is a diagnostic quantity for Lagrangian Coherent Structure (LCS), i.e. influential surfaces governing the evolution in P.

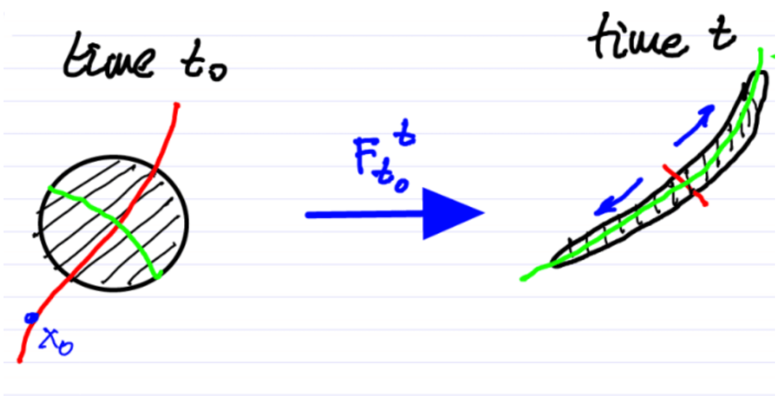


Figure 1.5: On the left the red ridge represents large values of $\mathrm{FTLE}_{t_0}^t$, on the right the green ridge the high values of $\mathrm{FTLE}_t^{t_0}$.

The ridges of $\mathrm{FTLE}_{t_0}^t$ are the repelling LCS, meanwhile the ridges of $\mathrm{FTLE}_t^{t_0}$ are the attracting LCS. Now we are left with the problem of computing $F_{t_0}^t(x_0)$. Recall that analytically we start with $F_{t_0}^t(x_0)$ and use this to calculate $DF_{t_0}^t(x_0)$. From here we can find $C_{t_0}^t(x_0)$, giving us $\lambda_n(x_0)$ and thereby the FTLE. We know approximate this process numerically.

(i) Define an initial $M \times N$ grid of initial data $x_0(i,j) \in \mathbb{R}^2$.

(ii) Launch trajectories numerically from grid points to obtain a discrete approximation of $F_{t_0}^t(x_0)$ as $F_{t_0}^t(x_0(i,j))$.

(iii) Use finite differencing to approximate $DF_{t_0}^t(x_0(i,j))$.

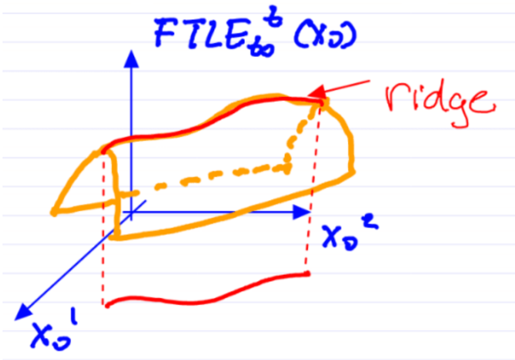


Figure 1.6: The projection of the FTLE ridge onto the initial value space.

Example 1.8 (Double gyre model using FTLE). We have the stream function

$$\Psi(x, y) = -\sin(\pi x)\sin(\pi y).$$

This gives the fluid velocity field

$$V = \begin{cases} \dot{x} = \dfrac{\partial \Psi}{\partial y} \\ \dot{y} = -\dfrac{\partial \Psi}{\partial x}. \end{cases}$$

Remark 1.6. This is an example of a Hamiltonian system of Ψ being the Hamiltonian H.

For any autonomous Hamiltonian system we have that H is constant along trajectories, we check

$$\frac{d}{dt}\Psi(x(t), y(t)) = \frac{\partial \Psi}{\partial x}\dot{x} + \frac{\partial \Psi}{\partial y}\dot{y} = 0.$$

So we have that trajectories are level curves of $\Psi(x, y)$. We can then derive the phase portrait from the level curves of Ψ. Further, we have that $\dot{x} = \frac{\partial \Psi}{\partial y} = -\pi \sin(\pi x)\cos(\pi y)$ which yields that $\text{sign}(\dot{x}) = -\text{sign}(\sin(\pi x))\text{sign}(\cos(\pi y))$. Putting these together we can construct the contour plot with arrows.

Figures here were taken from Shawn Shadden of UC Berkeley.

Example 1.9 (ABC flow). Let our dynamic system be defined as follows with $A, B, C \in \mathbb{R}$

$$\begin{cases} \dot{x} = A\sin(z) + C\cos(y) \\ \dot{y} = B\sin(x) + A\cos(z) \\ \dot{z} = C\sin(y) + B\cos(x). \end{cases}$$

We are looking for an exact solution of Euler's equation of inviscid fluids. We have an autonomous velocity field, which is known to generate chaotic fluid trajectories.

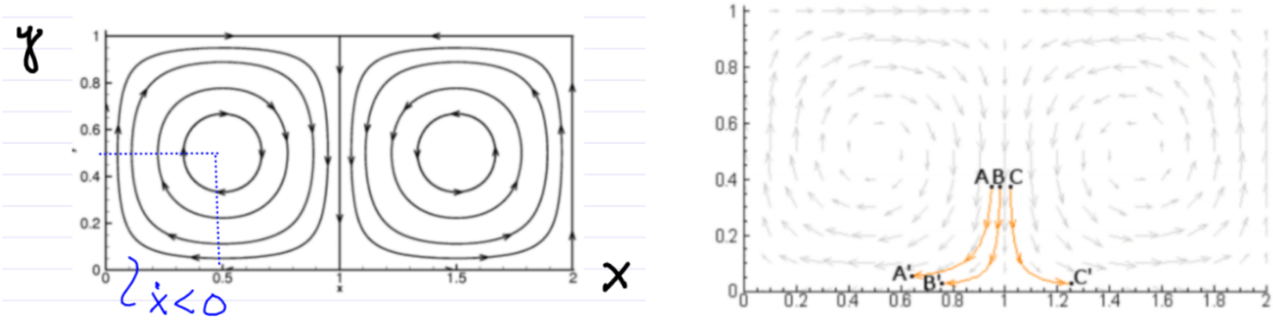


Figure 1.7:

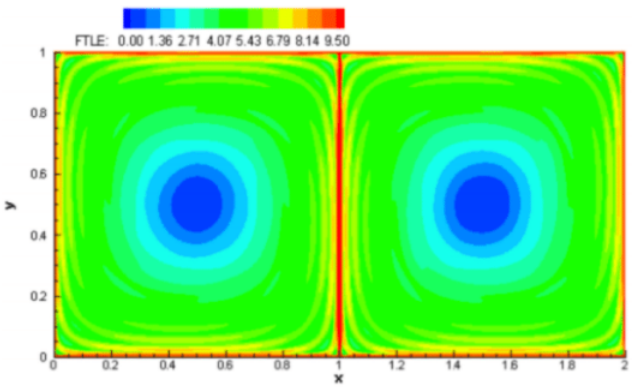


Figure 1.8: Top left: The analytic phase plot. Top right: The exploration done to calculate FTLE. Bottom: The FTLE plot.

1.5 Dependence on parameters

We now have the IVP

$$\begin{cases} \dot{x} = f(x,t,\mu) \\ x(t_0) = x_0. \end{cases}$$

With $x \in \mathbb{R}^n,\ f \in C^r,\ r \geq 1$, therefore we have a solution $x(t;t_0,x_0,\mu) \in C^r_{x_0}$.

Question How does the solution depend on μ?

Why Care? We would like robustness of solutions with respect to parameter changes or uncertainties in the model.

Example 1.10 (Perturbation Theory). Given a weakly nonlinear oscillator

$$m\ddot{x} + c\dot{x} + kx = \epsilon f(x,\dot{x},t),\ 0 \leq \epsilon \ll 1,\ x \in \mathbb{R}.$$

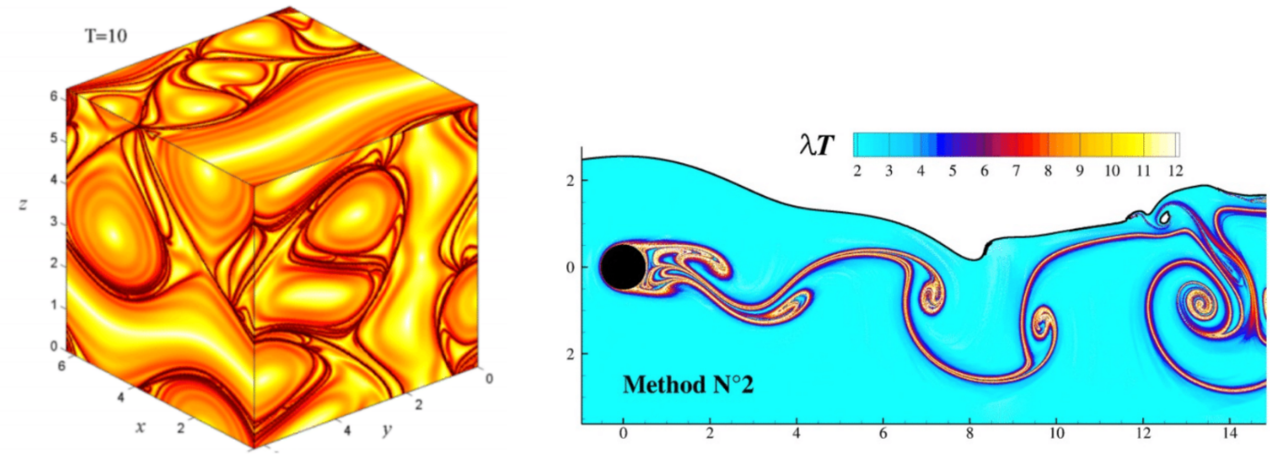


Figure 1.9: Left: numerical results of dynamic system (Guckenheimer-Holmes Physica D, 2001). Right: vortex shedding behind a cylinder under a free surface (Sun et. al, 2016).

The usual approach is to seek solutions by expanding from known solution of the linear limit, i.e.

$$x_\epsilon(t) = \varphi_0(t) + \epsilon\varphi_1(t) + \epsilon^2\varphi_2(t) + \ldots + \mathcal{O}(\epsilon^r).$$

If $x_\epsilon(t)$ is in C^r_ϵ, we have $\varphi_1(t) = \left.\frac{\partial x_\epsilon(t)}{\partial \epsilon}\right|_{\epsilon=0}$ and $\varphi_2(t) = \left.\frac{\partial^2 x_\epsilon(t)}{\partial \epsilon^2}\right|_{\epsilon=0}$

Answer Regularity with respect to μ actually follows from regularity with respect to x_0. Use the trick of extending the IVP with a dummy variable μ

$$\begin{cases} \dot{x} = f(x,t,u) \\ \dot{\mu} = 0 \\ x(t_0) = x_0 \\ \mu(t_0) = \mu_0. \end{cases}$$

Thus with $X = \begin{pmatrix} x \\ \mu \end{pmatrix} \in \mathbb{R}^{n+p}$ and $F(X_0) = \begin{pmatrix} f \\ 0 \end{pmatrix}$; $X_0 = \begin{pmatrix} x_0 \\ \mu_0 \end{pmatrix}$. Therefore we have

$$\begin{cases} \dot{X} = F(X) \\ X(t_0) = X_0 \end{cases} \tag{1.1}$$

Applying the previous result on regularity with respect to x_0 to (1.1), we have that $f \in C^r_{x,\mu}$ implies that $X(t) \in C^r_{X_0}$ in turn implying that $x(t; t_0, x_0, \cdot) \in C^r_\mu$. The solution is as smooth in parameters as the RHS of the dynamic system.

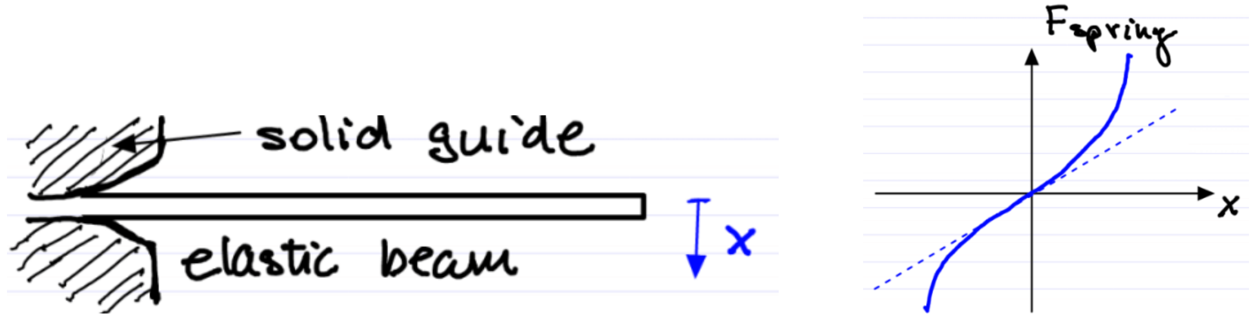


Figure 1.10: Setup for the nonlinear springboard.

Example 1.11 (Periodic Oscillations of a nonlinear springboard). By increasing x, the effective free length of the beam is shortened, thereby stiffening spring type non-linearity. This yield the following equations of motion

$$\begin{cases} \ddot{x} + x + \epsilon x^3 = 0; & 0 \leq \epsilon \ll 1 \\ x(0) = a_0; & \dot{x}(0) = 0. \end{cases}$$

So we have weak non-linearity with no known explicit solution. Rewriting this as a first order ODE ($x_1 = x;\ x_2 = \dot{x}$), and note that the RHS is $C^r_{x,\mu}$, therefore there exists a unique local solution that is also C^r_μ. Thus the expansion is justified

$$x_\epsilon(t) = \varphi_0(t) + \epsilon\varphi_1(t) + \ldots + \mathcal{O}(\epsilon^r). \tag{1.2}$$

We can see that for $\epsilon = 0$ we find that $\varphi_0(t) = a_0 \cos(t)$.

Now we look specifically for T-periodic solutions, as we would expect such a solution physically, therefore we have

$$\varphi_i(t) = \varphi_i(t + T).$$

The period T still has to be determined. Plugging this Ansatz to (1.2) into the IVP to get

$$\begin{aligned} \mathcal{O}(1): \quad & \ddot{\varphi}_0 + \varphi_0 && = 0 \\ \mathcal{O}(\epsilon): \quad & \ddot{\varphi}_1 + \underbrace{\varphi_1}_{\omega=1} && = -\varphi_0^3 = -a_0^3\cos^3(t) = -a_0^3\left[\frac{1}{4}\cos(3t) + \frac{3}{4}\underbrace{\cos(t)}_{\text{resonance}}\right]. \end{aligned} \tag{1.3}$$

We can see that (1.3) is a second order linear oscillator with resonant forcing. We have the initial conditions

$$\varphi_1(0) = 0; \quad \dot{\varphi}_1(t) = 0.$$

This holds as $\varphi_0 = a_0 \cos(t)$ already solves the IVP. We find the solution of (1.3)

$$\begin{aligned}\varphi_1(t) &= \varphi_1^{\text{hom}}(t) + \varphi_1^{\text{part}}(t) \\ &= \underbrace{A\cos(t) + B\sin(t)}_{\text{TBD from initial conditions}} + \underbrace{C\cos(3t) + Dt\cos(t) + Et\sin(t)}_{\text{TBD from (1.3)}}.\end{aligned}$$

Observe that $\varphi_1(t)$ contains $t\cos(t)$ and $t\sin(t)$, thus it cannot be periodic, so our Ansatz already fails for $i = 1$. We conclude that no solution of this type exists; our Ansatz was too restrictive and T should depend on ϵ.

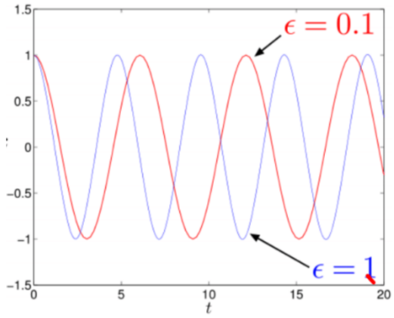


Figure 1.11: x for $a_0 = 1$.

Lindstedt's idea We should seek solution of the form

$$x_\epsilon(t) = \varphi_0(t;\epsilon) + \epsilon\varphi_1(t;\epsilon) + \epsilon^2\varphi_2(t;\epsilon) + \mathcal{O}(\epsilon^3).$$

Furthermore φ_i should be T_ϵ periodic, i.e. the period depends on the non-linearity ϵ.

$$\varphi_i(t + T_\epsilon;\epsilon) = \varphi_i(t;\epsilon).$$

Rewriting the period as

$$T_\epsilon = \frac{2\pi}{\omega(\epsilon)}; \quad \omega(\epsilon) = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \mathcal{O}(\epsilon^3).$$

We then rescale time according to $\tau = \omega(\epsilon)t$ to find

$$\frac{d}{d\tau} = \frac{1}{\omega(\epsilon)}\frac{d}{dt} \implies \boxed{[\omega(\epsilon)]^2 x'' + x + \epsilon x^3 = 0.}$$

Where we have taken x' to represent $\frac{dx}{d\tau}$. Plugging in our new Ansatz into the rescaled ODE yields

$$\left(1 + 2\epsilon\omega_1 + \mathcal{O}(\epsilon^2)\right)\left[\varphi_0'' + \epsilon\varphi'' + \mathcal{O}(\epsilon^2)\right] + \left[\varphi_0 + \epsilon\varphi_1 + \mathcal{O}(\epsilon^2)\right] + \epsilon\left[\varphi_0^3 + \mathcal{O}(\epsilon)\right] = 0.$$

Matching equal powers of ϵ yields

$$\mathcal{O}(1): \ \varphi_0'' + \varphi_0 = 0 \implies \varphi_0(\tau) = a_0 \cos(\tau); \quad \varphi_0(0) = a_0; \quad \dot{\varphi}_0(0) = 0$$

$$\mathcal{O}(\epsilon): \ \varphi_1'' + \varphi_1 = -\phi_0^3 - 2\omega_1 \varphi_0'' = \left(2\omega_1 a_0 - \frac{3}{4}a_0^3\right) \underbrace{\cos(\tau)}_{\text{resonance}} - \frac{a_0^3}{4}\cos(3\tau);$$

$$\varphi_1(0) = 0; \quad \dot{\varphi}_1(0) = 0.$$

From the first line, we can see the initial condition are fulfilled. In this step we used that $\dot{\varphi}(t=0) = 0$ if and only if $\omega(\epsilon)\varphi'(0) = 0$. We get the solution

$$\varphi_1(t) = A\cos(\tau) + B\sin(\tau) + C\cos(3\tau) + D\tau\cos(\tau) + E\tau\sin(\tau).$$

The presence of resonance again excludes periodic solutions, but now we can select ω_1 to eliminate these terms.

$$2\omega_1 a_0 - \frac{3}{4}a_0^3 = 0 \implies \boxed{\omega_1 = \frac{3}{8}a_0^2.}$$

This successfully eliminates the resonance and determines the missing frequency term at $\mathcal{O}(\epsilon)$. Thus we find

$$x_\epsilon(\tau) = a_0\cos(\tau) - \frac{\epsilon}{32}a_0^3\left(\cos(\tau) - \cos(3\tau)\right) + \mathcal{O}(\epsilon^2).$$

In the original time scaling this is

$$x_\epsilon(t) = a_0\cos(\omega t) - \frac{\epsilon}{32}a_0^3\left(\cos(\omega t) - \cos(3\omega t)\right) + \mathcal{O}(\epsilon^2); \quad \omega = 1 + \frac{3}{8}\epsilon a_0^2 + \mathcal{O}(\epsilon^2).$$

This procedure can be continued to higher order terms, where we select ω_2 so that the $\mathcal{O}(\epsilon^2)$ terms cancel.

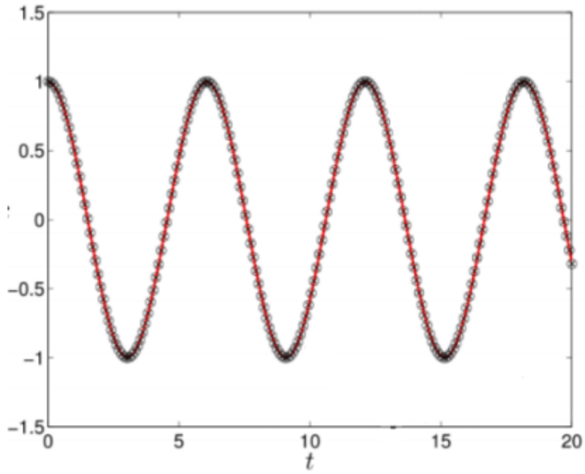


Figure 1.12: Approximation (dots) vs analytic solution (solid line).

Chapter 2

Stability of fixed points

Now we would like to begin to explore the behavious of dynamic systems around fixed points. This will allow us to find out if we should expect to observe a fixed stated, and to understand what happens if we perturb the system away from this fixed state.

2.1 Basic definitions

Consider

$$\dot{x} = f(x,t),\ x \in \mathbb{R}^n,\ f \in C^1.$$

Assume that $x = 0$ is a fixed point, i.e. $f(0,t) = 0$ for all $t \in \mathbb{R}$. If the fixed point is originally at $x + 0 \neq 0$, shift it to zero by letting $\tilde{x} := x - x_0$, therefore

$$\dot{\tilde{x}} = \dot{x} = f(x_0 + \tilde{x}, t) = \tilde{f}(\tilde{x}, t).$$

Question How does the dynamical system behave near its equilibrium state?

Definition 2.1 (Lyapupnov Stability)**.** $x = 0$ is stable if for all t_0, for all $\epsilon > 0$ small enough, there exists a $\delta = \delta(t_0, \epsilon)$, such that for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| \leq \delta$, we have

$$\|x(t; t_0, x_0)\| \leq \epsilon \quad \forall t \geq t_0.$$

Example 2.1 (Stability of lower equilibrium of the pendulum). Recall we have $\ddot{\varphi} + \sin(\phi) = 0$, and we transform this into a first order ODE by setting $x_1 = \varphi$ and $x_2 = \dot{\varphi}$ to obtain

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\sin(x_1). \end{cases}$$

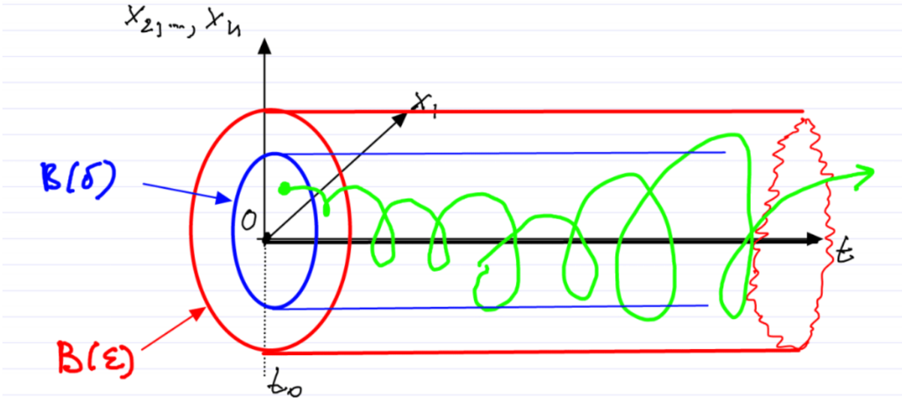


Figure 2.1: An example such a δ, $B(r)$ represents the n-dimensional ball of radius r.

For small $\epsilon > 0$, this geometric procedure gives a $\delta(\epsilon) > 0$ such that the definition of stability is satisfied for $x = 0$. Therefore $x = 0$ is (Lyapunov) stable.

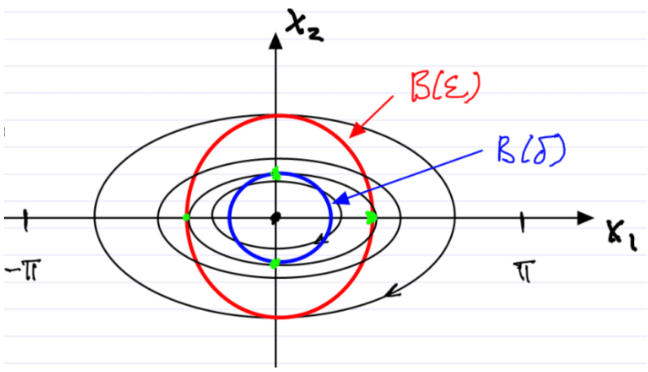


Figure 2.2: Stability of lower equilibrium for the pendulum, here $0 < \epsilon < \pi$.

Definition 2.2 (Asymptotic stability)**.** $x = 0$ is asymptotically stable if

(i) it is stable,

(ii) for all t_0, there exists $\delta_0(t_0)$ such that for every x_0 with $\|x_0\| \leq \delta_0$ we have

$$\boxed{\lim_{t \to \infty} x(t; t_0, x_0) = 0.}$$

Definition 2.3 (Domain of attraction)**.** The set of all x_0's for which

$$\boxed{\lim_{t \to \infty} x(t; t_0, x_0) = 0.}$$

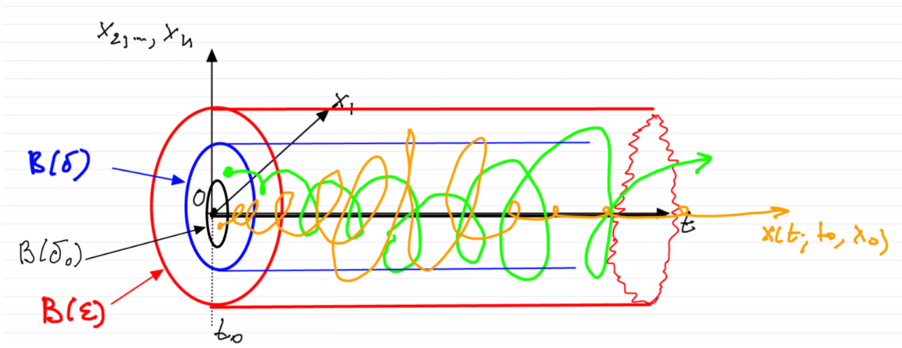


Figure 2.3: An example for an asymptotically stable fixed point (yellow trajectory).

Example 2.2 (Damped pendulum). We have the equation of motion

$$\ddot{\varphi} + c\dot{\varphi} + \sin(\varphi) = 0, \quad c > 0.$$

Transforming into a first-order ODE with $x_1 = \varphi$ and $x_2 = \dot{\varphi}$ gives

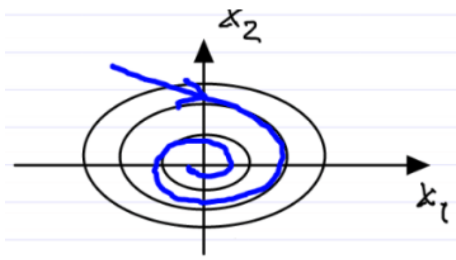
$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -cx_2 - \sin(x_1). \end{cases}$$

The total energy is given by

$$E = \frac{1}{2}x_2^2 + (1 - \cos(x_1)) .$$

Further we have the rate of energy change

$$\frac{d}{dt}E(x_1(t), x_2(t)) = x_2 \left(\dot{x_2} + \sin(x_1)\right) = -cx_2^2.$$



Therefore, along trajectories energy decreases monotonically. By the C^0 dependence on initial conditions, the trajectories remain close to the undamped oscillations for small $c > 0$. We conclude that trajectories are inward spirals for $c > 0$ small. $x = 0$ is still Lyapunov stable, but asymptotic stability does not yet follow (is the limit of $x(t)$ equal to 0?).

Remark 2.1 (Lasalle's invariance principle). This conclusion follows rigorously from Lasalle's invariance principle, namely if we assume that $\dot{x} = f(x)$, $f \in C^1$, and that there exists a $V \in C^1$ with

$$\dot{V} = \frac{dV(x(t))}{dt} \leq 0.$$

Then the set of accumulation points for any trajectory is contained in the set of trajectories that stay within the set $I = \{x \in \mathbb{R}^n : \ \dot{V}(x) = 0\}$.

Example 2.3. We are given the following dynamic system in polar coordinates

$$\begin{cases} \dot{r} = r(1-r) \\ \dot{\theta} = \sin^2\left(\dfrac{\theta}{2}\right). \end{cases}$$

Note that $r = 0$ is a fixed point, the set $r = 1$ is an invariant circle, and the set $\theta = 0$ is an invariant set. An invariant set is a set in which if the dynamic system is 'started' on the set, it remains in the set for all time. Examining the radial evolution reveals that the that the equation of motion decouples. We see that $\dot{\theta} \geq 0$, so rotation is either positive or null.

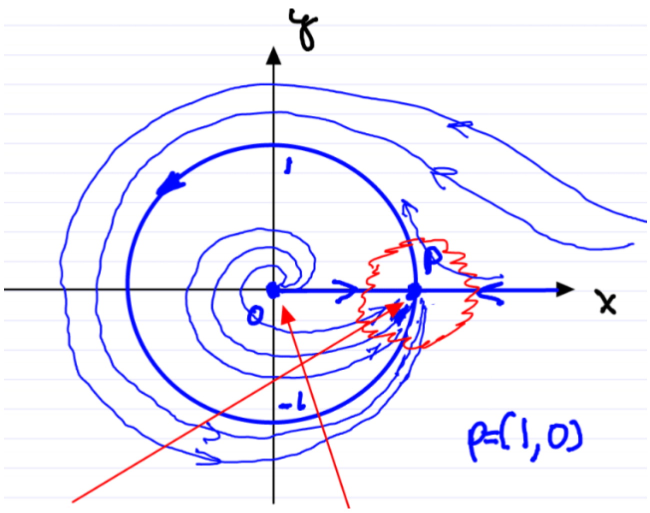


Figure 2.4: Phase portrait of the dynamic system, with arrows pointing to the two unstable equilibria.

However, we have that $p = (1, 0)$ is an example of an attractor: a set with an open neighborhood of points that all approach the set as $t \to \infty$.

Definition 2.4 (Invariant set)**.** $S \subset P$ is an invariant set for the flow map $F^t : P \to P$ if $F^t(S) = S$ for all $t \in \mathbb{R}$.

Definition 2.5 (Unstable point)**.** A fixed point $x = 0$ is unstable if it is not stable.

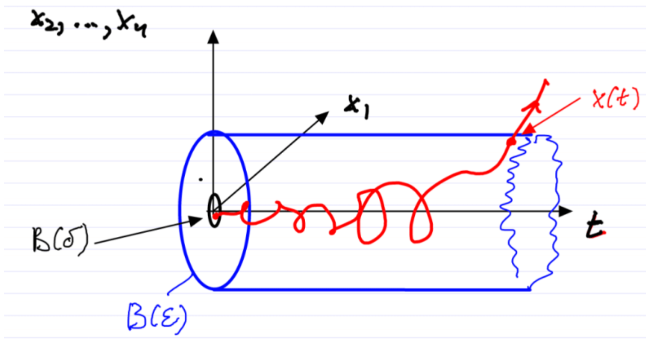
Remark 2.2. We can negate a mathematical statement by using the reverse relational operators outside the statements involving these operators i.e. $\exists \to \forall$ and $\forall \to \exists$. For example we have for continuity $\forall \epsilon\ \exists \delta :\ |f(x) - f(y)| < \epsilon$ if $|x - y| < \delta$, meanwhile for discontinuity we have $\exists \epsilon :\ \forall \delta :\ |f(x) - f(y)| \not< \epsilon$ for $|x - y| < \delta$.

In our case for stability we have

$$\forall \epsilon, t_0 : \quad \exists \delta > 0 : \quad \forall x_0 \text{ with } |x_0| < \delta : \quad |x(t)| \leq \epsilon \quad \forall t \geq t_0.$$

Meanwhile for unstability

$$\exists \epsilon, t_0 : \qquad \underbrace{\forall \delta > 0}_{\text{"for arbitrarily small"}} \quad : \quad \exists x_0 \text{ with } |x_0| < \delta : \quad |x(t)| > \epsilon \quad \underbrace{\exists t \geq t_0}_{\text{"for some"}} .$$



Remark 2.3. By C^0 dependence on initial conditions, if $x(t; t_0, x_0)$ leaves $B(\epsilon)$, then for $\tilde{x}_0$ close enough to x_0, $x(t; t_0, \tilde{x}_0)$ also leaves $B(\epsilon)$. Therefore if the measure of such trajectories in nonzero, the instability is observable!

Example 2.4 (Unstable fixed point of pendulum)*.* We have that infinitely many trajectories converge to the fixed point, yet it is still unstable. In fact, the converging trajectories form a measure-zero set, thus the stability near the unstable equilibrium is unobservable.

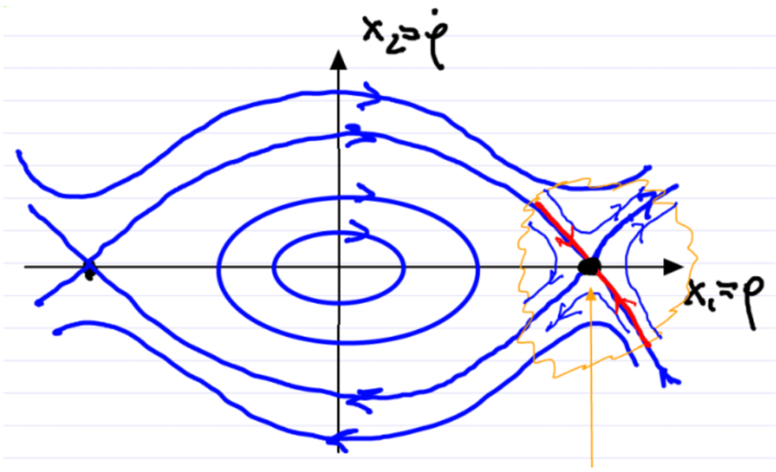


Figure 2.5: The phase portrait around the unstable fixed point of the pendulum, with the stable trajectories (red).

2.2 Stability based on linearization

We would like to derive a more general method to analyze the stability of fixed points, thus we try to simplify our system around the fixed point and discover what this can tell us about the full (unsimplified) system. In the following section we shall always assume that our system is autonomous. We will have the following setup

$$\dot{x} = f(x), \quad f \in C^1, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \quad p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \in \mathbb{R}^n.$$