151-0530-00L, Spring, 2020

Nonlinear Dynamics and Chaos II

Homework Assignment 4

Due: Wednesday, April 29;
Please submit by email to Dr. Shobhit Jain <shjain@ethz.ch>

1. Compute the Lyapunov-type numbers $\nu(p)$ and $\sigma(p)$ in the example

$$\dot{x} = -x(1-x^2),$$
$$\dot{y} = -by,$$

for all points $p \in M_0$, with the parameter $b \in \mathbb{R}^+$ and with overflowing-invariant manifold M_0 defined as

$$M_0 = \left\{(x,y) \in \mathbb{R}^2 : \;\; y = 0, \; x \in [-3/2, 3/2]\right\}.$$

(*Hint*: Use the operators $A_t(p)$ and $B_t(p)$ defined in class).

2. The stable and unstable manifolds of a normally hyperbolic invariant manifold M turn out to admit a delicate internal structure, an *invariant foliation,* which is useful in determining the exact asymptotic behavior of trajectories in $W^u(M)$ and $W^s(M)$.

More specifically, if $M \subset \mathbb{R}^n$ is a compact, C^r smooth, k-dimensional, r-normally hyperbolic invariant manifold with boundary, and $\dim\left[W^s(M)\right] = k+s$, then $W^s(M)$ has the following properties (some of which are sketched in Fig. 1.):

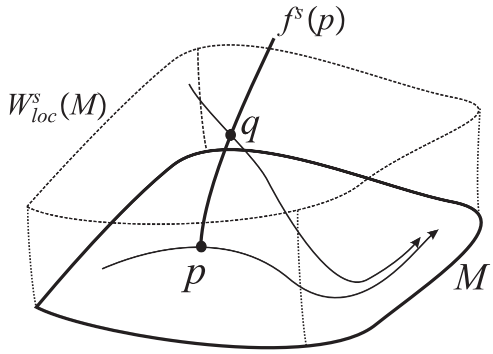

Figure 1: The geometry of stable fibers

(i) Near M, the stable manifold $W^s(M)$ can be written as

$$W^s_{\mathrm{loc}}(M) = \cup_{p \in M} \; f^s(p),$$

where $f^s(p)$ is a C^r smooth, s-dimensional submanifold of $W^s_{\mathrm{loc}}(M)$ for which $f^s(p) \cap M = p$. We refer to the point p on M as the base point of the *stable fiber* $f^s(p)$.

(ii) The stable fiber $f^s(p)$ is tangent to $N^s_p M$, the local section of the stable subbundle $N^s M$.

(iii) The stable fibers form a positively invariant family, i.e., $F^t(f^s(p)) \subset f^s(F^t(p))$. In words, stable fibers are mapped into stable fibers by the flow map, although individual stable fibers are not invariant under the flow.

(iv) There exist positive constants C_s and λ_s, such that for any $q \in f^s(p)$, we have $|F^t(q) - F^t(p)| < C_s e^{-\lambda_s t}$. In other words, trajectories intersecting a stable fiber will exponentially converge to the trajectory on M that passes through the base point of that stable fiber.

(v) For any $q \in f^s(p)$ and $\hat{q} \in f^s(\hat{p})$, we have

$$\frac{\|F^t(q) - F^t(p)\|}{\|F^t(\hat{q}) - F^t(p)\|} \to 0,$$

as $t \to \infty$, unless $p = \hat{p}$. In other words, out of all the trajectories that may converge to the positive half-trajectory

$$\gamma(p) = \left\{F^t(p)\right\}_{t \geq 0},$$

the trajectories starting from the stable fiber $f^s(p)$ converges at the fastest rate. One therefore obtains a local stable manifold

$$W^{ss}_{\text{loc}}(\gamma(p)) = \cup_{\tilde{p} \in \gamma(p)} f^s(\tilde{p})$$

for any trajectory $\gamma(p)$ on the manifold M. (The full stable manifold $W^s_{\text{loc}}(\gamma(p))$ may be larger than $W^{ss}_{\text{loc}}(\gamma(p))$, because $\gamma(p)$ may also attract trajectories within M.)

(vi) $f^s(p) \cap f^s(\hat{p}) = \emptyset$, unless $p = \hat{p}$. In other words, stable fibers with different base points do not intersect.

(vii) A stable fiber $f^s(p)$ is a C^{r-1} smooth function of its base point p.

(viii) Stable fibers C^r-smoothly persist under small C^1 perturbations of the dynamical system.

The local unstable manifold $W^u_{\text{loc}}(M)$ has a similar invariant foliation

$$W^u_{\text{loc}}(M) = \cup_{p \in M} \ f^u(p),$$

with appropriate properties in backward time. (For more information, see S. Wiggins, *Normally Hyperbolic Invariant Manifolds in Dynamical Systems*, Springer 1994)

Consider now the three-dimensional nonlinear dynamical system

$$\begin{aligned}
\dot{x} &= -\varepsilon\left(x + y^2\right), \\
\dot{y} &= -y, \\
\dot{z} &= z,
\end{aligned}$$

with the small parameter $\varepsilon \geq 0$.

(a) Show that the set $M_0 = \{y = z = 0, \, x \in [-1, 1]\}$ is a normally hyperbolic invariant manifold for $\varepsilon = 0$.

(b) Find the manifold M_ε into which M_0 perturbs for small $\varepsilon > 0$.

(c) Using the property (v), show that for any base point $p \in W^s_{\text{loc}}(M)$, the corresponding stable fiber is the nonlinear surface.

$$f^s(p) = \left\{(x, y, z) \ \mid x = x_p + \frac{\varepsilon}{2 - \varepsilon} y^2, \, z = 0\right\}.$$

(d) Find a similar expression for the unstable fibers $f^u(p)$.

(e) Verify explicitly the properties of the stable fibers listed in (i)-(vii) in this example.

(e) For any trajectory γ in M_ε, find explicit expressions for $W^{ss}_{\text{loc}}(\gamma)$, $W^{uu}_{\text{loc}}(\gamma)$, $W^s_{\text{loc}}(\gamma)$, and $W^u_{\text{loc}}(\gamma)$.