Nonlinear Dynamics & Chaos I

Exercice Set 2 Solutions

Question 1

Consider the nonlinear oscillator

$$\ddot{x} + \omega_0^2 x = \varepsilon M x^2,$$

where $\varepsilon M x^2$ represents a small nonlinear forcing term $(0 \leq \varepsilon \ll 1, M > 0)$.
Using Lindstedt's method, find an $\mathcal{O}(\varepsilon)$ approximation for nonlinear motions as a function of their initial position, with zero initial velocity.

Solution 1

$$\begin{aligned}\ddot{x} + \omega_0^2 x &= \varepsilon M x^2, \quad 0 < \varepsilon \ll 1, \quad M > 0, \quad \omega_0 \neq 0\\ x(0) &= a_0\\ \dot{x}(0) &= 0\end{aligned}$$

- Seek solutions of the form:

$$\begin{aligned}x_\varepsilon(t) &= \varphi_0(t;\varepsilon) + \varepsilon\varphi_1(t;\varepsilon) + \mathcal{O}(\varepsilon^2)\\ \varphi_i(t,\varepsilon) &= \varphi_i(t + T_\varepsilon;\varepsilon)\end{aligned}$$

- rewrite period as

$$T_\varepsilon = \frac{2\pi}{\omega(\varepsilon)}, \quad \omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + \mathcal{O}(\varepsilon^2)$$

- Rescale time:

$$\tau = \omega(\varepsilon)t \Longrightarrow \boxed{\frac{d}{d\tau} = \frac{1}{\omega(\varepsilon)}\frac{d}{dt}} \Longrightarrow \boxed{(\omega(\varepsilon))^2 x'' + \omega_0^2 x = \varepsilon M x^2} \tag{1}$$

- Plug in the new Ansatz into the rescaled equation

$$[\omega_0^2 + 2\varepsilon\omega_0\omega_1 + \mathcal{O}(\varepsilon^2)][\varphi_0'' + \varepsilon\varphi_1'' + \mathcal{O}(\varepsilon^2)] + \omega_0^2[\varphi_0 + \varepsilon\varphi_1 + \mathcal{O}(\varepsilon^2)] = \varepsilon M[\varphi_0^2 + \mathcal{O}(\varepsilon)]$$

- Collect terms of equal power of ε

  $\mathcal{O}(1)$:

$$\begin{aligned}\omega_0^2\varphi_0'' + \omega_0^2\varphi_0 = 0, \quad &\varphi_0(0) = a_0, \quad \dot{\varphi}_0(0) = 0\\ \Longrightarrow \varphi_0 &= a_0\cos(\tau)\end{aligned}$$

  $\mathcal{O}(2)$:

$$\begin{aligned}\omega_0^2\varphi_1'' + \omega_0^2\varphi_1 &= M\varphi_0^2 - 2\omega_0\omega_1\varphi_0'' = M a_0^2\cos^2(\tau) + 2a_0\omega_0\omega_1\cos(\tau)\\ &= M\frac{a_0^2}{2}[1 + \cos(2\tau)] + \underbrace{2a_0\omega_0\omega_1\cos(\tau)}_{\text{resonance}}\end{aligned}$$

  Select $\omega_1 = 0$ to eliminate resonance terms and obtain periodic solution.

  Solve for φ_1:

$$\varphi_1'' + \varphi_1 = \frac{M a_0^2}{2\omega_0^2}[1 + \cos(2\tau)], \quad \varphi_1(0) = 0, \quad \dot{\varphi}_1(0) = 0 \tag{2}$$

- Pick solution Ansatz:

$$\varphi_1(\tau) = A\cos(\tau) + B\sin(\tau) + C\cos(2\tau) + D\sin(2\tau) + E$$

- Substituting in (2):

$$-A\cos(\tau) - B\sin(\tau) - 4C\cos(2\tau) - 4D\sin(2\tau) + A\cos(\tau) + B\sin(\tau) + C\cos(2\tau) + D\sin(2\tau) + E$$
$$= \frac{Ma_0^2}{2\omega_0^2}\cos(2\tau) + \frac{Ma_0^2}{2\omega_0^2}$$

- Comparing coefficients:

$$\implies E = \frac{Ma_0^2}{2\omega_0^2}, \quad C = -\frac{Ma_0^2}{6\omega_0^2}, \quad D = 0$$

$$\varphi_1(0) = 0 \implies A + C + E = 0 \implies A = -\frac{Ma_0^2}{3\omega_0^2}$$
$$\varphi_1'(0) = 0 \implies B + 2D = 0 \implies B = 0$$

$$\implies \boxed{\varphi_1(\tau) = -\frac{Ma_0^2}{3\omega_0^2}\cos(\tau) - \frac{Ma_0^2}{6\omega_0^2}\cos(2\tau) + \frac{Ma_0^2}{2\omega_0^2}}$$

- In original time:

$$x_\varepsilon(t) = a_0\cos(\omega t) + \varepsilon\frac{Ma_0^2}{\omega_0^2}\left[-\frac{1}{3}\cos(\omega t) - \frac{1}{6}\cos(2\omega t) + \frac{1}{2}\right] + \mathcal{O}(\varepsilon^2)$$

where

$$\omega = \omega_0 + \mathcal{O}(\varepsilon^2)$$

Question 2

Consider the forced *van der Pol equation*

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = F\cos(\omega t),$$

which arises in models of self-excited oscillation, such as those of a valve generator with a cubic valve characteristic. Here $F, \omega > 0$ are parameters, and $0 \le \varepsilon \ll 1$.

(i) For small values of ε, find an approximation for an **exactly** $2\pi/\omega$-periodic solution of the equation. The error of your approximation should be $\mathcal{O}(\varepsilon)$.

(ii) For $\varepsilon = 0.1,\ \omega = 2$, and $F = 1$, verify your prediction numerically by solving the equation numerically. Plot your numerical solution along with your analytic prediction computed in (i).

Note: For chaotic dynamics in the forced van der Pol equation, see Section 2.1 of *Guckenheimer & Holmes*.

Solution 2

(i) Seek solutions of the form:

$$x_\varepsilon(t) = \varphi_0(t) + \varepsilon\varphi_1(t) + \mathcal{O}(\varepsilon^2)$$

Substituting this solution in the ODE $\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = F\cos(\omega t)$ we get:

$$\ddot{\varphi}_0 + \varphi_0 + \varepsilon(\ddot{\varphi}_1 + \varphi_1 + \varphi_0^2\dot{\varphi}_0 - \dot{\varphi}_0) + \mathcal{O}(\varepsilon^2) = F\cos(\omega t)$$

$$\Longrightarrow \mathcal{O}(1) : \ddot{\varphi}_0 + \varphi_0 = F\cos(\omega t) \tag{3}$$
$$\Longrightarrow \mathcal{O}(2) : \ddot{\varphi}_1 + \varphi_1 = \dot{\varphi}_0(1 - \varphi_0^2) \tag{4}$$

From equation (3):

$$\varphi_0(t) = A\sin(t) + B\cos(t) + \frac{F\cos(\omega t)}{1 - \omega^2} \tag{5}$$

Since we seek solutions with period $\frac{2\pi}{\omega}$ for any $0 \le \varepsilon \ll 1$, the period of each φ_i must be $\frac{2\pi}{\omega}$. This holds in particular for φ_0. Therefore, we must enforce $A = B = 0$ in equation (5).

This condition can be enforced by choosing appropriate initial conditions for the ODE (3):

$$A = B = 0 \Longrightarrow \boxed{\varphi_0(t) = \frac{F\cos(\omega t)}{1 - \omega^2}, \quad \varphi_0(0) = \frac{F}{1 - \omega^2}, \quad \dot{\varphi}_0(0) = 0} \tag{6}$$

$$x_\varepsilon(t) = \varphi_0(t) + \underbrace{\mathcal{O}(\varepsilon)}_{\text{error term}}$$

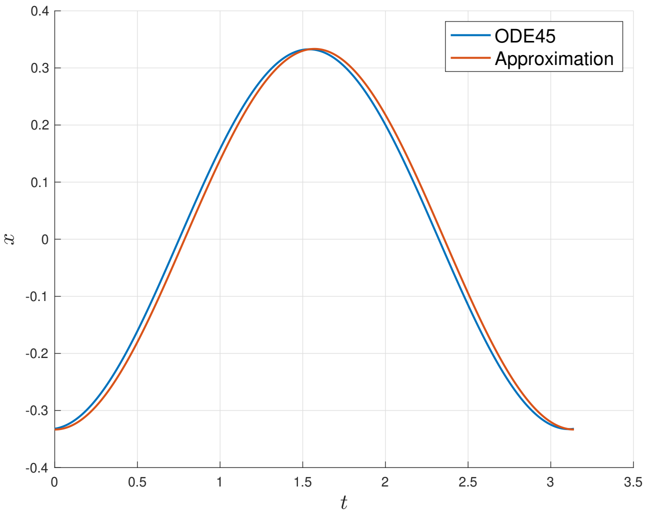
(ii) We solve the ODE numerically to obtain a solution $x(t)$ and compare this solution to the perturbed approximation $x_\varepsilon(t)$ given by (6).

The initial conditions for the ODE are chosen such that $x(0) = x_\varepsilon(0)$ and $\dot{x}(0) = \dot{x}_\varepsilon(0)$.

Therefore, $x(0) = \varphi_0(0), \quad \dot{x}(0) = \dot{\varphi}_0(0)$ where $\varphi_0(0)$ and $\dot{\varphi}_0(0)$ are given in (6).

Equivalent first order system of differential equations:

$$z_1 = x, \quad z_2 = \dot{x}, \quad \begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix} z_2 \\ F\cos(\omega t) + \varepsilon(z_1^2 - 1)z_2 - z_1\end{bmatrix} \tag{7}$$



MATLAB code

```
%% Initiate Script

close all
clear all
clc

%% define parameters

epsilon = 0.1;
omega = 2;
F = 1;

% initial condition
t0 = [F / (1 - omega.^2), 0]';

% time steps
tt_approx = 0:0.01:pi;
tt_sim = 0:0.01: 101 * pi;

%% Function and simulation

fun = @(t,x) [x(2); F*cos(omega * t) + epsilon * (x(1).^2 - 1).*x(2) - x(1)];

opts = odeset('RelTol',1e-4,'AbsTol',1e-6);
[~ , xtrue] = ode45(fun, tt_sim, t0, opts);

%% Approximation

xApprox = F * cos(omega .* tt_approx) / (1 - omega.^2);

%% Plot results

xtrue_steady_state = xtrue(end - length(xApprox) + 1: end, 1); % only take the last 315 values

figure(1)
hold on
plot(tt_approx, xtrue_steady_state,'linewidth',1.5,'DisplayName','ODE45');
plot(tt_approx, xApprox,'linewidth',1.5,'DisplayName','Approximation');
```

```
xlabel('$t$','interpreter','Latex','FontSize',16)
ylabel('$x$','interpreter','Latex','FontSize',16)
legnd1 = legend;
legnd1.NumColumns = 1;
legnd1.FontSize = 14;
hold off
grid on
```