Nonlinear Dynamics & Chaos I

Exercice Set 4 Questions

Question 1

Consider the discrete dynamical system

$$x_{n+1} = Ax_n + f(x_n, y_n),$$
$$y_{n+1} = By_n + g(x_n, y_n),$$

where $x_n \in \mathbb{R}^c$, $y_n \in \mathbb{R}^d$, $A \in \mathbb{R}^{c\times c}$, $B \in \mathbb{R}^{d\times d}$; f and g are C^r functions with no linear terms. Assume that all eigenvalues of A have modulus one, and none of the eigenvalues of B have modulus one. Then the linearized system at the origin admits a center subspace E^c aligned with the x-coordinate plane.

(a) Derive a general algebraic equation for the center manifold W^c, which is known to exist by a theorem analogous to the center manifold theorem for continuous dynamical systems.

(b) Find a cubic order approximation for the center manifold of the discrete system

$$x_{n+1} = x_n + x_n y_n,$$
$$y_{n+1} = \lambda y_n - x_n^2,$$

where $\lambda \in (0, 1)$.

(c) Reduce the dynamics to the center manifold and determine the stability of the origin. Verify your results by a numerical simulation of a few initial conditions near the origin.

Question 2

Construct a cubic-order local approximation for the unstable manifold of the hyperbolic fixed point of the pendulum equation

$$\ddot{x} + \sin(x) = 0.$$

Question 3

Consider the discrete dynamical system

$$\begin{cases} x_{n+1} = x_n + x_n y_n \\ y_{n+1} = \frac{1}{2} y_n - x_n^2 \end{cases}$$

Let $h : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ be the local graph of the center manifold around $(0, 0)$. $(0 < \varepsilon \ll 1)$. Find the expression that h satisfies.

(a) $h(x + h(x)) - \dfrac{1}{2}h(x) = x^2$

(b) $h(x + h(x)) - \dfrac{1}{2}h(x) = -x^2$

(c) $h(x + xh(x)) - \dfrac{1}{2}h(x) = x^2$

(d) $h(x + xh(x)) - \dfrac{1}{2}h(x) = -x^2$

Question 4

Consider the following dynamical system

$$\begin{cases} \dot{x} = xy \\ \dot{y} = -y + x^2 \end{cases}$$

Which expression describes the reduced dynamics on the center manifold?

(a) $\dot{x} = x^3(1 - 2x^2) + \mathcal{O}(x^5)$

(b) $\dot{y} = y^3(1 - 2y^2) + \mathcal{O}(y^5)$

(c) $\dot{x} = x^3(1 + 2x^2) + \mathcal{O}(x^5)$

(d) $\dot{y} = y^3(1 + 2y^2) + \mathcal{O}(y^5)$

Question 5

Consider the following dynamical system

$$\begin{cases} \dot{x} = -x^3 \\ \dot{y} = -y \end{cases}$$

Let $y = h(x)$ be the graph of the center manifold of $(0, 0)$. Which of the following expressions is accurate ?
Hint: $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{y}{x^3}$

(a) The dynamical system has a unique center manifold with $h(x) = e^{-\frac{1}{2x^2}}$

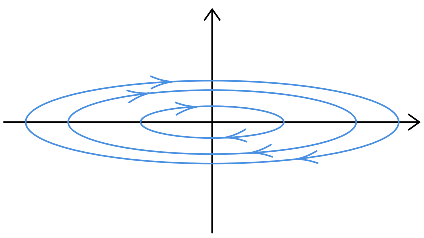
(b) The dynamical system has a unique center manifold with $h(x) = e^{-\frac{1}{x^2}}$

(c) The dynamical system has infinitely many center manifolds with $h(x) = \begin{cases} ae^{-\frac{1}{2x^2}} & x < 0 \\ 0 & x = 0 \\ be^{-\frac{1}{2x^2}} & x > 0 \end{cases} \quad \forall a, b \in \mathbb{R}$

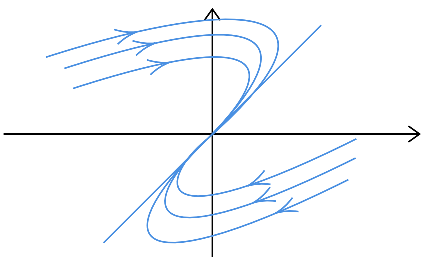
(d) The dynamical system has infinitely many center manifolds with $h(x) = \begin{cases} ae^{-\frac{1}{x^2}} & x < 0 \\ 0 & x = 0 \\ be^{-\frac{1}{x^2}} & x > 0 \end{cases} \quad \forall a, b \in \mathbb{R}$

Question 6

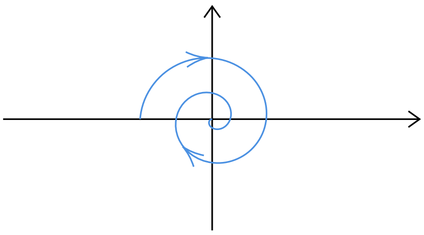
The phase portrait of four planar dynamical systems are shown below. In which case is the origin not Lyapunov stable?



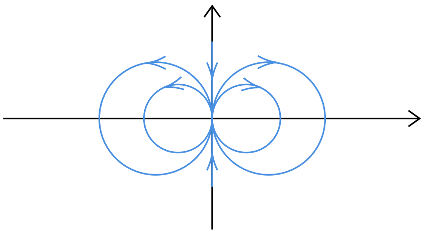
(a)



(b)



(c)



(d)

Question 7

Consider the dynamical system below

$$\dot{x} = |x|^2(Ax + f(x))$$

where , $x \in \mathbb{R}^n,\ f \in C^1,\ A \in \mathbb{R}^{n\times n},\ f = \mathcal{O}(|x|^2)$ and the matrix A has precisely one pair of purely imaginary eigenvalues, and $(n-2)$ eigenvalues with negative real parts.
Which of the following statements are true?

(a) The origin $x = 0$ is unstable.

(b) $\dim(W^c(0)) = 2$

(c) $\dim(W^c(0)) = n$

(d) None of the above