151-0530-00L, Spring, 2020

Nonlinear Dynamics and Chaos II

Homework Assignment 3

Due: Wednesday, April 8;
Please submit by email to Dr. Shobhit Jain <shjain@ethz.ch>

1. Show that if λ is an eigenvalue of a linearized Hamiltonian system, then so is $-\lambda$. (*Hint:* For any symmetric matrix A and any nonsingular matrix B, the eigenvalues of BA and $B^{-1}(BA)B$ coincide.)

2. A *gradient system* is a dynamical system of the form

$$\dot{x} = -DV(x), \qquad x \in \mathbb{R}^n,$$

with some smooth function V. Apart from the absence of the symplectic matrix J, gradient systems appear similar to Hamiltonian systems. In fact, they are rather different, as the following exercises show.

(a) Show that the eigenvalues of a linearized gradient system are always real.

(b) Find a condition for V under which a fixed point of a gradient system is asymptotically stable.

(c) Show that gradient systems cannot have periodic orbits.

(d) Given a smooth function $f(x)$, propose a numerical method for finding the local minima and maxima of f.

3. Let $f : X \to Y$ be a smooth function, where $X \subset \mathbb{R}^n$ is a manifold. Prove that the graph of f,

$$\text{graph}(f) = \{(x, y) \in X \times Y : \ y = f(x)\}$$

is always a manifold.

4. Show that the tangent space of a manifold at any point is independent of the local parametrization used in its construction (i.e., another local parametrization would give the same tangent space).

5. Prove that the tangent bundle of a manifold is a manifold by constructing an explicit local parametrization.

Nonlinear Dynamics and Chaos II.
Homework 3

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Exercise 1

Show that if λ is an eigenvalue of a linearized Hamiltonian system, then so is $-\lambda$. (Hint: For any symmetric matrix A and any nonsingular matrix B, the eigenvalues of BA and $B^{-1}(BA)B$ coincide.)

Solution

Given a Hamiltonian system $\dot{x} = JDH(x)$, for $x \in \mathbb{R}^{2n}$ its linearization is

$$\dot{y} = (JD^2H)y,$$

with the matrix

$$J = \begin{bmatrix} 0 & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & 0 \end{bmatrix}.$$

From the hint, BA and $B^{-1}(BA)B$ have the same eigenvalues. This is a similarity transformation. I will call $BA = G$, and the similar matrix $B^{-1}GB = F$. The inverse transform is $G = BFB^{-1}$.

Assume λ is an eigenvalue of G with eigenvector v. Then, for F we have that

$$BFB^{-1}v = \lambda v$$

and

$$F(B^{-1}v) = \lambda(B^{-1}v), \tag{1}$$

Showing that λ is an eigenvalue of F with eigenvector $B^{-1}v$.

Also note, that for the linearized Hamiltonian, the matrix D^2HJ has the property (written in index notation)

$$(D^2H)_{ij}J_{jk} = -J_{kj}(D^2H)_{ji},$$

because of $J_{ij} = -J_{ji}$ and $D^2H_{ij} = D^2H_{ji}$.

$$D^2HJ = -(JD^2H)^T \tag{2}$$

Substituting $B = J$ and $A = D^2H$ into (1) gives

$$J^{-1}J(D^2H)J(J^{-1}v) = \lambda J^{-1}v$$
$$(D^2H)J(J^{-1}v) = \lambda J^{-1}v.$$

Using (2) we get

$$(JD^2H)^T(J^{-1}v) = -\lambda J^{-1}v.$$

This means that $-\lambda$ is an eigenvalue of $(JD^2H)^T$ (with eigenvector $J^{-1}v$), and since the eigenvalues of a matrix and its transpose coincide, it proves the statement.

Exercise 2

A gradient system is a dynamical system of the form

$$\dot{x} = -DV(x), \qquad x \in \mathbb{R}^n,$$

with some smooth function V.

(a) Show that the eigenvalues of a linearized gradient system are always real.

(b) Find a condition for V under which a fixed point of a gradient system is asymptotically stable.

(c) Show that gradient systems cannot have periodic orbits.

(d) Given a smooth function $f(x)$, propose a numerical method for finding the local minima and maxima of f.

Solution

(a): The linearized system for the gradient system is

$$\dot{y} = -D^2Vy.$$

V is assumed to be smooth, so the Hessian D^2V is a symmetric matrix, because of the symmetry of second partial derivatives.

Since symmetric matrices have real eigenvalues, the linearized gradient system too, has real eigenvalues.

(b): The fixed points satisfy $DV(x_0) = 0$. That is, x_0 is a fixed point if and only if it is a critical point of V.

Claim: If x_0 is a local minimum of V, then x_0 is asymptotically stable.

To show this, we can use V as a Lyapunov function:

If $V(x_0)$ is a minimum, that means there is an open subset U, such that $\forall x \in U$, $V(x) > V(x_0)$.

Then, define $L(x) := V(x) - V(x_0)$. This function has the property that $L(x) > 0 \; \forall x \in U \setminus \{x_0\}$. Its time derivative is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = DV(x) \cdot \dot{x} = -DV(x) \cdot DV(x) = -||DV(x)||^2 < 0.$$

The norm of the gradient is always positive if $x \neq x_0$. This shows that $L(x)$ is positive definite and $\dot{L}$ is negative definite on $U \setminus \{x_0\}$, which, by Lyapnunov's theorem, guarantees asymptotic stability for x_0.

(c): The statement is, if x_0 is not a fixed point, there is no $T > 0$ such that $x(T) = x_0$. Assume the converse:

$\exists T > 0$ such that $x(T) = x(0) = x_0$. Consider the net change in V along this trajectory, using Newton-Leibniz:

$$V(x(T)) - V(x(0)) = \int_0^T \frac{\mathrm{d}V(t)}{\mathrm{d}t}\mathrm{d}t. \tag{3}$$

From the previous exercise,

$$\int_0^T \frac{\mathrm{d}V(t)}{\mathrm{d}t}\mathrm{d}t = -\int_0^T ||DV(x(t))||^2\mathrm{d}t < 0. \tag{4}$$

This is strictly negative, if x_0 is not a fixed point. By assumption, the left hand side of (3) is 0, which is in contradiction with (4). This shows that there is no periodic orbit in a gradient system.

(d) Based on this result, we can numerically find the minima of a smooth function $f(x)$ by considering the gradient system

$$\dot{x} = -Df(x). \tag{5}$$

If $\hat{x}$ is a local minimum of $f(x)$, then there is an open subset U, such that $\forall x \in U$, $f(x) > f(\hat{x})$. By the result in (b), $\hat{x}$ is an asymptotically stable fixed point of (5).

To numerically find $\hat{x}$, take an initial guess x_0 in U. Then, we solve the initial value problem $x(0) = x_0$, with $x(t)$ satisfying (5). The solution will converge to x_0. To be specific, simply using the Euler-scheme

$$x_{n+1} = x_n - Df(x_n),$$

is the gradient descent method. For finding maxima, put $F(x) := -f(x)$, and repeat the process for the gradient system $\dot{x} = -DF(x)$.

Exercise 3

Let $f : X \to Y$ be a smooth function, where $X \subset \mathbb{R}^n$ is a (k-dimensional) manifold. Prove that the graph of f

$$\mathrm{graph}(f) = \{(x, y) \in X \times Y : y = f(x)\}$$

is always a manifold.

Solution

$X \subset \mathbb{R}^n$ is a manifold, which means $\forall x \in X$ there is an open subset $V \subset \mathbb{R}^k$ around $\xi \in \mathbb{R}^k$ and a local parametrization $\varphi : V \to \mathbb{R}^n$, $\varphi(\xi) = x$. I also assume that $Y \subset \mathbb{R}^l$ for some l. The product $X \times Y$ is a subset of the ambient space $\mathbb{R}^n \times \mathbb{R}^l$.

Let us call the graph of f, M. The goal is to show that for each $(x, y) \in M$, there is an open subset U of R^k, with $\xi' \in U$ and a local parametrization $F : U \to \mathbb{R}^n \times \mathbb{R}^l$ with $F(\xi') = (x, y) \in M$, which is a diffeomorphism from U to $M \cap W$, for an open set $W \subset \mathbb{R}^n \times \mathbb{R}^l$ [$W \cap M$ is relatively open].

Consider the map

$$F : V \subset \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^l$$

given by

$$F(\xi) = (\varphi(\xi), f \circ \varphi(\xi)).$$

This map is smooth, because its components are smooth. It is also a bijection on $M \cap W$:

- It is injective, since if $F(\xi_1) = F(\xi_2)$, we must have $\varphi(\xi_1) = \varphi(\xi_2)$ because of the first component. φ is a bijection, so $\xi_1 = \xi_2$.
- It is surjective, because if $(x, y) \in M \cap W$, then we have $f(x) = y$ and $x = \varphi(\xi)$ for some $\xi \in V$, since φ is a local parametrization for X. This means we can write $(x, y) = F(\xi) = (\varphi(\xi), f \circ \varphi(\xi)$.

Its inverse is given by

$$F^{-1} : M \cap W \to V$$

$$F^{-1} = \varphi^{-1} \circ \pi,$$

where $\pi : X \times Y \to X$, $\pi(x, y) = x$ is the projection onto the first factor, so $F^{-1}(x, y) = \varphi^{-1}(x)$. This function is smooth, because it has a smooth extension to $W \subset \mathbb{R}^n \times \mathbb{R}^l$: π is already smooth on W, and because φ was a local parametrization, φ^{-1} has to have a smooth extension.

This shows that that around a point $(x, y) \in M$, the map F is a local parametrization from an open subset $U = V \subset \mathbb{R}^k$, with $x \in \varphi(V)$, for some local parametrization of X around x.

Exercise 4

Show that the tangent space of a manifold at any point is independent of the local parametrization used in its construction (i.e., another local parametrization would give the same tangent space).

Solution

Given an m-dimensional manifold $M \subset \mathbb{R}^n$, the tangent space at $p \in M$ is defined by

$$df_x(\mathbb{R}^m),$$

for some local parametrization around p, $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ and $f(x) = p$. Assume that there is a different local parametrization, $g : V \subset \mathbb{R}^m \to \mathbb{R}^n$, $p \in f(U) \cap g(V)$, with $f(x) = p = g(y)$.

The differentials are the linear maps

$$df_x : \mathbb{R}^m \to \mathbb{R}^n \qquad dg_y : \mathbb{R}^m \to \mathbb{R}^n.$$

Since both f and g are local parametrizations, df_x and dg_y are injective and they (as linear maps) have Rank f = Rank $g = m$.

Rank $f = \dim(\text{Image } f) = \dim(\text{Image } g) = m$. This shows that the tangent spaces we obtain using f and g are both m dimensional subspaces of $\mathbb{R}^n$. That is, the vector spaces $df_x(\mathbb{R}^m)$ and $dg_y(\mathbb{R}^m)$ are isomorphic, because they are finite dimensional, with the same dimension.

Exercise 5

Prove that the tangent bundle of a manifold is a manifold by constructing an explicit local parametrization.

Solution

Let $M \subset \mathbb{R}^n$ be an m dimensional manifold. Its tangent bundle is

$$TM = \bigcup_{p \in M} \{p\} \times TM_p.$$

We show, that TM is a $2m$ dimensional manifold. Consider a point $(p, v) \in TM$, such that $v \in TM_p$. Since M is a manifold, there is a local parametrization $\varphi : \tilde{U} \subset \mathbb{R}^m \to \mathbb{R}^n$, with $x \in \tilde{U}, \varphi(x) = p$.

Let $U = \tilde{U} \times \mathbb{R}^m$ be an open set, and define a function $\Phi : U \to \mathbb{R}^n \times \mathbb{R}^n$, by

$$\Phi(x, \eta) = (\varphi(x), d\varphi_x(\eta)).$$

Since both φ and $d\varphi_x$ are smooth functions, Φ is also smooth. We still need to show that Φ is a diffeomorphism onto $TM \cap W$ for an open set $W \subset \mathbb{R}^n \times \mathbb{R}^n$.

Φ is a bijection, because:

If $\Phi(x, \eta) = \Phi(x', \eta')$, for $(x, \eta), (x', \eta') \in TM \cap W$ then $\varphi(x) = \varphi(x')$ from the first component, which means $x = x'$. This is because φ is a local parametrization. From the second component: $d\varphi_x(\eta) = d\varphi_x(\eta')$ means $\eta = \eta'$, since the linear map $d\varphi_x$ is injective. This shows Φ is injective on the relatively open set $TM \cap W$.

For surjectivity, for a point $(p, v) \in TM \cap W$ we can find $x = \varphi^{-1}(p)$, since the local parametrization is already bijective. We also know that its inverse is smooth, so with $\eta := d(\varphi^{-1})_p(v)$, we see that

$$d\varphi_p(\eta) = d\varphi_p[d(\varphi^{-1})_p(v)] = d\varphi_{\varphi^{-1}(p)} \circ d(\varphi^{-1})_p(v) = d(\varphi^{-1} \circ \varphi)_p(v) = v,$$

by the chain rule. So, we have found a point (x, η) for each $(p, v) \in TM \cap W$, with $\Phi(x, \eta) = (p, v)$.

This also gives the inverse function on this relatively open set:

$$\Phi^{-1} : TM \cap W \to U$$

$$\Phi^{-1}(p, v) = \left[\varphi^{-1}(p), d(\varphi^{-1})_p(v)\right].$$

We know that φ^{-1} can be extended to a smooth function, and the same is true for its differential. Then, we have that $\Phi : U \to \mathbb{R}^n \times \mathbb{R}^n$ is a diffeomorphism, and is a suitable local parametrization for TM, around a point $(p, v) = \Phi(x, \eta)$,