Sheet 1

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(i) First note that there are $kN(k)$ elements with orbit length exactly (minimal) k, for all k. Next, we want to know how many elements have (not necessarily exactly) orbits of length k. This means if $k = 4$, then we count 16 elements, as we include elements with orbit length exactly 2 and/or 1 (2 and 1 each divide 4). This entails counting how many ways we can construct a block of length k. We consider one construction C_1 to be equal to another C_2, if there exists n such that $\sigma^n(C_1) = C_2$, where σ acts cyclically. Equivalently, we could say $C_1 \cong C_2$ if

$$\exists n \geq 1 : \quad \sigma^n \left(\overline{C}_1.\overline{C_1}\right) = \overline{C}_2.\overline{C}_2.$$

Constructing a unique block in this case just means choosing a number i, $0 \leq i \leq k$, for the amount of symbols of one type, and choosing a constellation for placing these symbols. For a given i there are $\binom{k}{i}$ (k choose i) ways to place the i elements in k places. Next, we have to sum over all possible i

$$\sum_{i=0}^{k} \binom{k}{i} = 2^k.$$

In order to get the amount of elements with orbit length exactly k we have to subtract out the elements which have orbit length exactly i for $i|k$ (i divides k). There are $i \cdot N(i)$ of these elements, thus

$$kN(k) = 2^k - \sum_{i|k} iN(i).$$

Now dividing by k yields the desired result.

(ii) (a) The matrix A tells us if it is possible to transition from symbol i to symbol j in a single step if $A_{ij} = 1$, and otherwise 0. Say $A_{ii} = 1$, then $\overline{s}_i.\overline{s}_i$ is an admissible sequence and is a fixed point of σ ($\sigma\left(\overline{s}_i.\overline{s}_i\right) = \overline{s}_i.\overline{s}_i$). Noting that all fixed points of σ are in fact sequences consisting of a single symbol it is clear that the fixed points of σ on Σ_A^N are the sequences $\overline{s}_j.\overline{s}_j$ with $A_{jj} = 1$.

In conclusion, there are as many fixed points as 1s on the diagonal, and A is binary (only consists of 0s and 1s), this is equal to $\mathrm{tr}(A)$. (This holds as $\overline{s}_j.\overline{s}_j$ is fixed and the type of fixed point, further it is only in Σ_A^N if $A_{jj} = 1$.)

(b) A_{ii}^k encodes the amount of unique admissible paths from $i \to i$ in k steps. Any fixed point of σ^k has the for $\overline{C}.\overline{C}$ with C being a sequence of k symbols. We will call C admissible if for all i $C_iC_{i+1} =: s_m s_n$ and $A_{mn} = 1$. Thus we want to know how many admissible C exist, as each of these correspond to a fixed point. Furthermore no other fixed points exist as all fixed points must be of this form. Hence we will have identified all fixed points of σ^k (k-periodic orbits).

Since we repeat C infinitely, there are A_{jj}^k admissible C which start with s_j, i.e. $C_1 = s_j$. If we sum over all s_j to get the total amount of admissible C we find $\sum_{i=1}^N A_{ii}^k = \mathrm{tr}(A)$.

(iii)

(iv) First note that the orbit s^* visits $B(s, \delta)$ (ball of radius δ around s) infinitely often for all $s \notin \mathrm{Orbit}(s^*) = \mathcal{O}(s^*)$, as if there were finite visits, there there would exist $0 < \delta = 0.5 \min_k(d(\sigma^k(s^*), s))$, and there would not exist $N > 0$ with $d(\sigma^N(s^*), s) < \delta'$. Define $B' = B \setminus \mathcal{O}(s^*)$, nonempty (as the orbit is countable and B is uncountable (B is open)). For any $a \in A$ there exists $\delta > 0$ with $B(a, \delta) \subset A$, there also exists N such that $d(\sigma^N(s^*), a) < \delta$, call $\sigma^N(s^*) = s_a$. Choose any $b \in B'$, there exists $\epsilon > 0$ such that $B(b, \epsilon) \subset B$ and $M > N$ with $d(\sigma^M(s^*), b) < \epsilon$, such an M exists due to the infinite visiting property. Call $\sigma^M(s^*) = s_b$, we know that $s_a \in A$ and that $\sigma^{M-N}(s_a) = \sigma^M(s^*) = s_b$. Thus we have that

$$\sigma^{M-N}(A) \cap B \neq \emptyset,$$

and the N in question corresponds to the $M - N$ here.

It maybe unclear why B open implies it is uncountable (ignoring $B = \emptyset$); take any $b \in B$ and there exists $0 < \epsilon < 1$ with $B(b, \epsilon) \subset B$. Now we want an injection from $(0, \epsilon)$ to B. For each $x \in (0, \epsilon)$, write the binary representation of $x = 0.b_1 b_2 \ldots$, this mapping is bijective, call it $\varphi(x)$. Now construct an element by taking $b = \ldots.s_0 s_1 s_2 \ldots$ and for every $b_i = 1$ set $s_i' = s \in \Sigma$ with $|s - s_i| = 1$, and for $b_i = 0$ $s_i' = s_i$. Set $S' = \ldots s_{-1}.s_0 s_1' s_2' s_3' \ldots$ and call this $\Psi(\varphi(x))$. Ψ is clearly injective as every binary sequence is mapped to a unique element of Σ. Next, we see that $d(\Psi(\varphi(x)), b)$ is exactly equal to x, thus for all $x \in (0, \epsilon)$ we have $\Psi(\varphi(x)) \in B(b, \epsilon) \subset B$, so we have an injection from an uncountable set into B, showing B to be uncountable.

(v) Choose $\Delta = \frac{1}{2}$. For any two symbol sequences with $d(s, s') > 0$ (non-equal), there exists a position $N \in \mathbb{Z}$ such that $s_N \neq s_N'$. Then we know that $\sigma^N(s)_0 \neq \sigma^N(s')$, therefore $d(\sigma^N(s), \sigma^N(s')) \geq 1 > \Delta$.

Sheet 2

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Problem 1 First we need to define our generalized coordinate $q = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. From here, our canonical coordinates are as follows.

$$x_1 = L \sin \alpha \qquad y_1 = L \cos \alpha$$
$$x_2 = L \sin \alpha + L \sin \beta \qquad y_2 = L \cos \alpha + L \cos \beta.$$

Next we need to find the kinetic energy

$$\begin{aligned}
T_1 &= \frac{m}{2}(\dot{x}_1^2 + \dot{y}_1^2) = \frac{m}{2} L^2 \dot{\alpha}^2 \\
T_2 &= \frac{m}{2}(L^2\dot{\alpha}^2 + L^2\dot{\beta}^2 + 2L^2 \cos(\alpha)\cos(\beta)\dot{\alpha}\dot{\beta} + 2L^2 \sin(\alpha)\sin(\beta)\dot{\alpha}\dot{\beta}) \\
&= \frac{mL^2}{2}(\dot{\alpha}^2 + \dot{\beta}^2 + 2\cos(\alpha - \beta)\dot{\alpha}\dot{\beta}).
\end{aligned}$$

In order to get the Lagrangian we also need the potential energy

$$V_1 = -mgL\cos(\alpha) \qquad V_2 = -mgL(\cos(\alpha) + \cos(\beta)).$$

So we get the Lagrangian

$$\mathcal{L} = T - V = \frac{mL^2}{2}\left(2\dot{\alpha}^2 + \dot{\beta}^2 + 2\cos(\alpha - \beta)\dot{\alpha}\dot{\beta}\right) + mgL\left(2\cos(\alpha) + \cos(\beta)\right).$$

We constructed the Lagrangian to derive the generalized momenta.

$$\begin{aligned}
p_1 &= \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = 2mL^2\dot{\alpha} + mL^2\cos(\alpha - \beta)\dot{\beta} \\
p_2 &= \frac{\partial \mathcal{L}}{\partial \dot{\beta}} = mL^2\dot{\beta} + mL^2\cos(\alpha - \beta)\dot{\alpha}.
\end{aligned}$$

Finally we would like to calculate the Hamiltonian $\mathcal{H} = p \cdot \dot{q} - \mathcal{L}$. To do this, we first have to find the correct coordinate transformation, we see that $p = A\dot{q}$; therefore we are interested in finding A^{-1}.

$$A = \begin{bmatrix} 2mL^2 & mL^2\cos(\alpha-\beta) \\ mL^2\cos(\alpha-\beta) & mL^2 \end{bmatrix}$$
$$A^{-1} = \frac{1}{m^2L^4\left(2-\cos^2(\alpha-\beta)\right)} \begin{bmatrix} mL^2 & -mL^2\cos(\alpha-\beta) \\ -mL^2\cos(\alpha-\beta) & 2mL^2 \end{bmatrix}.$$

Plugging in this transformation we find

$$\begin{aligned}\mathcal{H} &= p \cdot A^{-1}p - \mathcal{L} \\ &= \frac{mL^2p_1^2 + 2mL^2p_2^2 - 2mL^2\cos(\alpha-\beta)p_1p_2}{2m^2L^4\left(2-\cos^2(\alpha-\beta)\right)} - 2mgL\cos(\alpha) - mgL\cos(\beta).\end{aligned}$$

This equality can also be seen as using $\mathcal{H} = T+V$ and $T = \frac{1}{2}p^TA^{-1}p$. Recall, $2T = \frac{\partial\mathcal{L}}{\partial\dot{q}} \cdot \dot{q} = p \cdot q$.

Problem 2 First we would like to find if we can find a quantity which is independent of time. To do this we examine the following

$$\frac{dh}{dp} = \frac{dh}{dt}\frac{dt}{dp} = \frac{\dot{h}}{\dot{p}} = -\frac{a_1h(1-bp)}{a_2p(1-ch)}.$$

Next we split up the differentials to find

$$\frac{a_2(1-ch)}{h}dh + \frac{a_1(1-bp)}{p}dp = 0.$$

We can then integrate both sides to find

$$a_2\ln(h) - a_2ch + a_1\ln(p) - a_1bp = \text{constant}.$$

Hence we have a conserved quantity, to confirm this applying $\frac{d}{dt}$ to this quantity yields 0. Call this quantity U. As in the lecture, we can see that $\dot{U}(x(t)) = 0$ implies that $DU(x(t)) \cdot \dot{x}(t) = DU \cdot f = 0$, i.e. the two terms are orthogonal. So we can write

$$f(x) = P(x)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} DU(x).$$

So we find $\dot{x} = P(x)JDU(x)$, a noncanonical Hamiltonian system. Plugging this into our current system, we find

$$\begin{pmatrix} \dot{h} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} a_1h(1-bp) \\ -a_2p(1-ch) \end{pmatrix} = P(h,p)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{a_2}{h} - a_2c \\ \frac{a_1}{p} - a_1b \end{pmatrix} = P(h,p)\begin{pmatrix} \frac{a_1}{p} - a_1b \\ -\frac{a_2}{h} + a_2c \end{pmatrix}.$$

Comparing terms reveals that $P(h,p) = hp$, because $h,p > 0$ we can rescale time

$$\tau = \int_0^t h(s)p(s)ds.$$

Framing the time derivative with respect to this scaling yields

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = P(h,p)x'.$$

Hence we find

$$P(h,p)x' = P(x)JDU(x) \quad \implies \quad x' = JDU(x).$$

Thus we have shown that the Lotka-Volterra system is Hamiltonian, for $\mathcal{H} = U$.

To test if the system admits a stable fixed point, we have to check if $\mathcal{H}$ is definite (since the Hamiltonian is conserved, if it is negative definite, we can just take $-\mathcal{H}$). The Hessian is

$$\begin{pmatrix} -\frac{a_2}{h^2} & 0 \\ 0 & -\frac{a_1}{p^2} \end{pmatrix}.$$

This is clearly negative definite, as all of the variables are positive. The fixed point is found at the point where $\dot{x} = 0$, i.e. $p = \frac{1}{b}$ and $h = \frac{1}{c}$.

Problem 3 The first thing we will assume (due to steady flow) is that $\rho_t = 0$. With this in hand, if we plug it into our given dynamics we get $\nabla \cdot (\rho v) = 0$. We can rewrite this to get

$$\nabla \times \begin{pmatrix} -\rho v \\ \rho u \\ 0 \end{pmatrix} = 0.$$

If we assume that the domain is simply connected then we know (from the lecture) that there is a potential function.

$$\begin{pmatrix} -\rho v \\ \rho u \\ 0 \end{pmatrix} = \nabla \Psi = \begin{pmatrix} \partial_x \Psi \\ \partial_y \Psi \\ \partial_z \Psi \end{pmatrix}.$$

Where $\Psi = \Psi(x,y,t)$ this implies the following

$$\dot{x} = \partial_y \Psi, \quad \dot{y} = -\partial_x \Psi \quad \implies \quad \dot{\underline{x}} = \rho(\underline{x})JD\Psi(\underline{x},t).$$

Similar to before, we can now rescale time with $\tau = \int_0^t \rho(s)ds$, to find $\underline{x}' = JD\Psi(\underline{x},t)$, i.e. a canonical Hamiltonian system.

Problem 4 In the lecture, we saw that each singular oscillator has a phase space given by a cylinder. If we confine to a given energy level $E_0 > 0$, we are given a 'slice' of the cylinder, more precisely a submanifold diffeomorphic to S^1. Hence, we can choose an energy level for each oscillator, and we get that the orbit of each oscillator can be described by a respective copy of S^1. Taking the cross product of these gives us $S^1 \times S^1 = \mathbb{T}$, where equality here denotes existence of a diffeomorphism. We cannot take an energy level of 0, as this is the fixed point, and S^1 is not diffeomorphic to a point.

For general dynamic systems on the torus, we can reverse engineer this last idea. If $\mathcal{H}$ is a Hamiltonian function, it is at least $\mathcal{C}^1$, therefore we have a maximum and a minimum, these points are our candidates for fixed points. However, if this point was actually a fixed point of the system, then we would have an orbit which is a point, hence could not be diffeomorphic to the torus.

Problem 5 We will attempt to construct a Hamiltonian $\mathcal{H}$.

$$\frac{\partial \mathcal{H}}{\partial p} = \dot{q} = f(q,t) \qquad \Longrightarrow \qquad \mathcal{H} = f(q,t)p + g(q,t)$$
$$-\dot{p} = \frac{\partial \mathcal{H}}{\partial q} = f_q(q,t)p + g_q(q,t).$$

Thus we can choose $p \in \mathbb{R}^n$ with the given dynamics and $g \in \mathcal{C}^1$ such that $\dot{\mathcal{H}} = 0$. These then give us a Hamiltonian system, which when projected down to the space that q lives in, gives us the desired dynamics.

Sheet 3

Trevor Winstral

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Problem 1 Let the linearized Hamiltonian system be represented by $\dot{y} = JAy$. We have that $A = D^2H(x_0)$ fulfills

$$-J(JA) = J^T(JA) = A = A^T = A^TJ^TJ = (JA)^TJ.$$

Thus calculating the characteristic polynomial of JA gives

$$\begin{aligned}\varphi(x) &= \det(xI - JA) = \det(J)\det(xI - JA) = \det(xJ - JJA) \\ &= \det(xJ + (JA)^TJ) = \det(xI + (JA)^T)\det(J) = \det([xI + JA]^T) \\ &= \det(xI + JA) = \varphi(-x).\end{aligned}$$

Therefore if $\varphi(\lambda) = 0$ then $\varphi(-\lambda) = 0$.

Problem 2

(i) Say x_0 is a fixed point, then set $y = x - x_0$. We have $\dot{y} = -DV(y + x_0) \approx -D^2V(x_0)y$. Because $D^2V(x_0)$ is symmetric (it is the Hessian), all of the eigenvalues are real.

(ii) If the Hessian is positive definite, then for y small, the Hessian dominates $\dot{y}$. Hence $y \to 0$, i.e. x_0 is asymptotically stable. Note that $V(x(t))' = \nabla V(x(t)) \cdot x'(t) = \nabla V(x(t)) \cdot -\nabla V(x(t)) = -\|\nabla V(x(t))\| \leq 0$.

(iii) Except for at fixed points, we can see (previous equation), is strictly decreasing along orbits. If we had a periodic orbit, then $x(t) = x(t + T)$, however $V(x(t)) > V(x(t + T))$, a contradiction.

(iv) Numerically integrating $\dot{x}$ will lead to the local minimum of V, as x is always going in the direction of the negative gradient, i.e. 'downhill the steepest way'. Analogously, integrating in backwards time will converge to the local maximum.

Note From here on, to preserve notation a chart/local diffeomorphism will be denoted without mentioning the base point. Since we work with the arbitrary base points (we always need to prove something works for every point $x \in X$), the proof holds for the entire manifold.

Problem 3 First, we note that we need a local diffeomorphism for every point in X. We will call this diffeomorphism $\varphi : U \to V$, where U is a local neighborhood of x, and V an open set in $\mathbb{R}^n$. We will tentatively define $\varphi : (x, y) \mapsto x$ (the projection), with inverse $\varphi^{-1} : x \mapsto (x, f(x))$. These are clearly smooth (due to smoothness of f) thus φ_x is a diffeomorphism. This shows that the graph of f is a manifold of $\mathbb{R}^n$.

Problem 4 Let φ and ψ ($X \to \mathbb{R}^n$) be two different charts parameterizing the manifold X, then there exists a diffeomorphism $\phi : X \to X$, such that $\psi = \varphi \circ \phi$. This is due to all manifolds being diffeomorphic to themselves. ϕ allows us to take the set of all smooth curves on $\mathbb{R}^n$ and map it bijectively to itself, hence for any curve γ used to generate to range of $d\varphi(0)$ corresponds to another curve $\phi \circ \gamma =: \tilde{\gamma}$. We have that $\psi \circ \gamma = \varphi \circ \phi \circ \gamma = \varphi \circ \tilde{\gamma}$. Hence $\text{Range}(d\psi(0)) \subset \text{Range}(d\varphi(0))$. The same argument holds with the roles of φ and ψ swapped, hence we have equality, and thus the tangent space is independent of parameterization.

Problem 5 We are looking for a chart $\varphi : \text{TM} \to \mathbb{R}^{2n}$. Let ϕ be a chart for M, this is a map between manifolds, namely $M \to \mathbb{R}^n$, inducing a map from $T_xM \to T_{\phi(x)}\mathbb{R}^n \cong \mathbb{R}^n$. Call the map $T_xM \to \mathbb{R}^n$ (where we used the implied existence of a map via the $\cong$ in the previous equation) f. Then we will propose $\varphi(x, v) = (\phi(x), f(v))$, this is clearly a diffeomorphism, and we can see it is smooth with smooth inverse coordinate wise.

Sheet 4

Trevor Winstral

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Problem 1 Since there is no interaction between x and y, we can solve each equation independently.

$$x(t) = \frac{1}{\sqrt{c_1 e^{2t} + 1}}; \quad y(t) = c_2 e^{-bt}.$$

Setting $t = 0$ we can solve for the initial conditions $p = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and find

$$x(t) = \frac{1}{\sqrt{({x_0}^{-2} - 1)e^{2t} + 1}}; \quad y(t) = y_0 e^{-bt}.$$

Thus we have the operator F^t for the initial point p. Next, we have to calculate $A_t(p)$ and $B_t(p)$.

$$\begin{aligned}
A_t(p) = DF^{-t}\big|_{T_pM_0} &= \left.\begin{pmatrix} \frac{e^{-2t}}{x_0^3\left(({x_0}^{-2}-1)e^{-2t}+1\right)^{\frac{3}{2}}} & 0 \\ 0 & e^{bt} \end{pmatrix}\right|_{T_pM_0} \\
B_t(p) = \Pi_p\, DF^t(F^{-t}(p))\big|_{N_{F^{-t}(p)}M_0} &= \Pi_p \left.\begin{pmatrix} \frac{e^{2t}}{F^{-t}(p)_1^3\left((F^{-t}(p)_1^{-2}-1)e^{2t}+1\right)^{\frac{3}{2}}} & 0 \\ k0 & e^{-bt} \end{pmatrix}\right|_{N_{F^{-t}(p)M_0}} \\
&= \Pi_p \left.\begin{pmatrix} \frac{e^{2t}\left[({x_0}^{-2}-1)e^{-2t}+1\right]^{\frac{3}{2}}}{\left([(x_0^2-1)e^{-2t}+1]-1)e^{2t}+1\right)^{\frac{3}{2}}} & 0 \\ 0 & e^{-bt} \end{pmatrix}\right|_{N_{F^{-t}(p)M_0}} \\
&= \Pi_p \left.\begin{pmatrix} \frac{e^{2t}\left[({x_0}^{-2}-1)e^{-2t}+1\right]^{\frac{3}{2}}}{x_0^3} & 0 \\ 0 & e^{-bt} \end{pmatrix}\right|_{N_{F^{-t}(p)M_0}}.
\end{aligned}$$

Now we need to define the tangent and normal spaces for $p \in M_0$. Geometrically, these can be easily classified (with the ambient space being $\mathbb{R}^2$, with tangent vectors from point p

pointing in the positive/negative x direction, and normal vectors from point p pointing in the positive/negative y direction. The projection becomes redundant as the map maps into the normal space at p. Thus we find

$$\nu(p) = \lim_{t\to\infty} \|B_t(p)\|^{\frac{1}{t}} = \lim_{t\to\infty} \left(e^{-bt}\right)^{\frac{1}{t}} = e^{-b} < 1.$$

Therefore, we can calculate σ as follows

$$\sigma(p) = \limsup_{t\to\infty} \frac{\log\|A_t(p)\|}{-\log\|B_t(p)\|} = \frac{-2t - 3\log(x_0) - \frac{3}{2}\log\left((x_0^{-2}-1)e^{-2t}+1\right)}{bt} = \frac{-2}{b}.$$

This covers all trajectories, except those from the fixed point at $(0,0)$ which is a singularity of this equation. There we get $\|A_t(p)\| = e^t$ (the linearization is then $D\dot{x}|_{x=0} = -1$, yielding the linearized flow of $F^t(x_0) = x_0e^{-t}$ (which we then plug in for $A_t(p)$)), therefore $\sigma(p) = \frac{1}{b}$.

Problem 2

(i) First we must show that the manifold is invariant. The derivative of all points on M_0 for $\epsilon = 0$ is 0 (by plugging in). Therefore, the manifold is clearly invariant as it is full of fixed points. Next we need to compute ν and σ, to do this we must once again compute B_t, we do not need A_t as there is no tangential compression as $\dot{x} = 0$. We begin by calculating the flow map.

$$F^t(p) = \begin{pmatrix} x_0 \\ y_0e^{-t} \\ z_0e^t \end{pmatrix}$$

Next we can calculate B_t

$$B_t(p) = \Pi_p DF^t(F^{-t}(p))\Big|_{N_{F^{-t}(p)}M_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^t \end{pmatrix}\Bigg|_{N_{F^{-t}(p)}M_0}.$$

The normal space of M_0 with base point p comprises of vectors going in the y and z directions. Thus the operator norm of B_t is e^t, by taking the normal vector pointing purely in the z direction.

$$\lim_{t\to\infty} \left(e^t\right)^{\frac{1}{t}} = e \not< 1.$$

Thus M_0 is not a NHIM since $\nu > 1$. If however, we restrict $B_t(p)$ further to only the stable directions (this excludes the normal vectors with z components), we find that

$$\nu = \lim_{t\to\infty} \left(e^{-t}\right)^{\frac{1}{t}} = e^{-1} < 1.$$

With this restriction M_0 is a NHIM with $\sigma = 0$ and $\nu = e^{-1}$.

(ii) The invariant manifold $M_\epsilon = M_0$.

(iii) Using property (v), we need for the trajectory starting from $q \in f^s(x_0)$ to converge the fastest, therefore the difference between the trajectories must have the maximal exponential decay. This immediately gives us that the stable fibre cannot contain nonzero values in the z coordinate, as this direction is unstable. In the y direction, this is trivially given by $y = y_q e^{-t}$. For the x direction we will use the ansatz $x(t) = C_1 e^{-\epsilon t} + C_2 e^{-2t}$. Calculating we find

$$\dot{x}(t) = -\epsilon C_1 e^{-\epsilon t} - 2C_2 e^{-2t} \stackrel{!}{=} -\epsilon(x + y^2) = -\epsilon(C_1 e^{-\epsilon t} + C_2 e^{-2t} + y_q^2 e^{-2t}).$$

Hence $C_2 = \frac{\epsilon}{2-\epsilon} y_0^2$, and we have to find C_1 such that $x(0) = x_q$. This condition then requires $C_1 = x_q - \frac{\epsilon}{2-\epsilon} y_q^2$. Therefore we have

$$x(t) = (x_q - \frac{\epsilon}{2-\epsilon} y_q^2) e^{-\epsilon t} + \frac{\epsilon}{2-\epsilon} y_q^2 e^{-2t}$$

The decay between the trajectories $F^t(x_0)$ and $F^t(x_q)$ to be maximized is then

$$(x_q - \frac{\epsilon}{2-\epsilon} y_q^2) e^{-\epsilon t} + \frac{\epsilon}{2-\epsilon} y_q e^{-2t} - x_0 e^{-\epsilon t} = (x_q - \frac{\epsilon}{2-\epsilon} y_q^2 - x_0) e^{-\epsilon t} + \frac{\epsilon}{2-\epsilon} y_q e^{-2t}.$$

Thus we set the first factor to 0 and get $x_q = x_0 + \frac{\epsilon}{2-\epsilon} y_q$. In conclusion the stable fibre from a point p is $f^s(p) = \big\{(x, y, z) : \ x = x_p + \frac{\epsilon}{2-\epsilon} y^2, \ z = 0\big\}$.

(iv) For the unstable fibres, we will use the same process in backwards time. We know that any component in the y direction will decay towards M_ϵ in forwards time, thus in backwards time, it is unstable. This allows us to set $y = 0$. As above we set the first term equal to 0, to eliminate the $e^{\epsilon t}$ term (note the direction of time has changed), and we find $x = x_0$. All together this gives $f^u(p) = \{(x, y, z) : \ x = x_p, \ y = 0\}$.

(v)

(vi) Using fact (v) we find

$$W^{ss}_{\text{loc}}(\gamma(p)) = \cup_{p' \in \gamma(p)} f^s(p') = \left\{(x, y, z) : \ x = x_0 + \frac{\epsilon}{2-\epsilon} y^2, \ z = 0, \ x_0 \in \gamma(p), \ y \in \mathbb{R}\right\},$$

$$\begin{aligned} W^{uu}_{\text{loc}}(\gamma(p)) = \cup_{p' \in \gamma(p)} f^u(p') &= \{(x, y, z) : \ y = 0, \ x \in \gamma(p), \ z \in \mathbb{R}\}, \\ W^s_{\text{loc}}(\gamma(p)) &= \{(x, y, z) : \ x \in M_\epsilon, \ z = 0\}, \\ W^u_{\text{loc}}(\gamma(p)) &= \{(x, y, z) : \ x \in M_\epsilon, \ y = 0\}. \end{aligned}$$