Nonlinear Dynamics & Chaos I

Exercice Set 5 Solutions

Question 1

Consider the quadratic *Duffing equation*

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= \beta u - u^2 - \delta v,\end{aligned}$$

where $\delta > 0$, and $0 \leq |\beta| \ll 1$.

(a) Construct a β-dependent center manifold up to quadratic order near the origin for small β values.

(b) Construct a stability diagram for the reduced system on the center manifold using β as a bifurcation parameter.

Solution 1

(a) Linearized dynamics around fixed point $(0,0)$

$$\dot{\eta} = A\eta \ , \ A = \begin{bmatrix} 0 & 1 \\ \beta & -\delta \end{bmatrix} \ , \ \mathrm{eig}(A) = \lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \beta}$$

Note that $\lambda_1 = 0$, $\lambda_2 = -\delta$ for $\beta = 0$. Thus, by the center manifold theorem, we have a 1-dimensional center manifold passing through the origin and a unique 1-dimensional stable manifold.

- Consider the extended system

$$\dot{\beta} = 0$$
$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\delta \end{bmatrix}}_{B} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \beta u - u^2 \end{bmatrix}$$

$$\text{Eigenvalues of } B : \lambda_1 = 0 \ , \ \lambda_2 = -\delta$$
$$\text{Eigenvectors of } B : e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \ e_2 = \begin{bmatrix} \frac{1}{\delta} \\ -1 \end{bmatrix}$$

From the eigenvalues and eigenvectors, we can perform a change of coordinates

$$\begin{bmatrix} u \\ v \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix} \ , \ T = [e_1|e_2] = \begin{bmatrix} 1 & \frac{1}{\delta} \\ 0 & -1 \end{bmatrix} \ , \ T^{-1} = \begin{bmatrix} 1 & \frac{1}{\delta} \\ 0 & -1 \end{bmatrix} = T$$

$$\implies u = x + \frac{y}{\delta} \ , \ v = -y$$

$$
\begin{aligned}
\implies \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} &= T^{-1}BT\begin{bmatrix}x\\ y\end{bmatrix} + T^{-1}\begin{bmatrix}0\\ \beta u - u^2\end{bmatrix}\\
&= \begin{bmatrix}0 & 0\\ 0 & -\delta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}\frac{1}{\delta}\left(\beta\left(x+\frac{y}{\delta}\right) - \left(x+\frac{y}{\delta}\right)^2\right)\\ -\beta\left(x+\frac{y}{\delta}\right) + \left(x+\frac{y}{\delta}\right)^2\end{bmatrix} \qquad (1)
\end{aligned}
$$

Seek center manifold as a graph over center subspace locally as

$$
\begin{aligned}
y &= h(x,\beta) = a_1x^2 + a_2x\beta + \cancel{a_3\beta^2} + \mathcal{O}(3)\\
\dot{y} &= \frac{\partial h}{\partial x}\dot{x} + \frac{\partial h}{\partial \beta}\cancelto{0}{\dot{\beta}} \qquad (2)
\end{aligned}
$$

Note: We cancel the term $a_3\beta^2$ to respect the existence of the fixed point.

Use invariance in (2):

$$
\implies \dot{y} = (2a_1x + a_2\beta)\left[\frac{1}{\delta}\left(\beta\left(x + \frac{h(x,\beta)}{\delta}\right) - \left(x+\frac{h(x,\beta)}{\delta}\right)^2\right)\right] \qquad (3)
$$

$$
\text{But also } \dot{y} = -\delta h(x,\beta) - \beta\left(x+\frac{h(x,\beta)}{\delta}\right) + \left(x+\frac{h(x,\beta)}{\delta}\right)^2 \qquad (4)
$$

Comparing $\mathcal{O}(2)$ terms in (3) & (4), we get:

$$
\begin{aligned}
x^2: &\qquad -\delta a_1 + 1 = 0 \implies a_1 = \frac{1}{\delta}\\
x\beta: &\qquad -\delta a_2 - 1 = 0 \implies -a_2 = \frac{1}{\delta}
\end{aligned}
$$

Thus, the β-dependent center manifold is given by

$$
h(x,\beta) = \frac{x^2}{\delta} - \frac{x\beta}{\delta} + \mathcal{O}(3) \qquad (5)
$$

Substitute (5) into first equation in (1) to obtain reduced dynamics on the center manifold: $W^C_\beta(0)$ up to quadratic order.

$$
\begin{aligned}
\dot{x} &= \frac{1}{\delta}\left[\beta\left(x+\frac{h(x,\beta)}{\delta}\right) - \left(x+\frac{h(x,\beta)}{\delta}\right)^2\right]\\
&= \frac{1}{\delta}[\beta x - x^2] + \mathcal{O}(3)
\end{aligned}
$$

(b)

$$
\dot{x} = \frac{1}{\delta}[\beta x - x^2]
$$

Fixed points:

$$
\begin{aligned}
x &= 0,\\
\beta &= x
\end{aligned}
$$

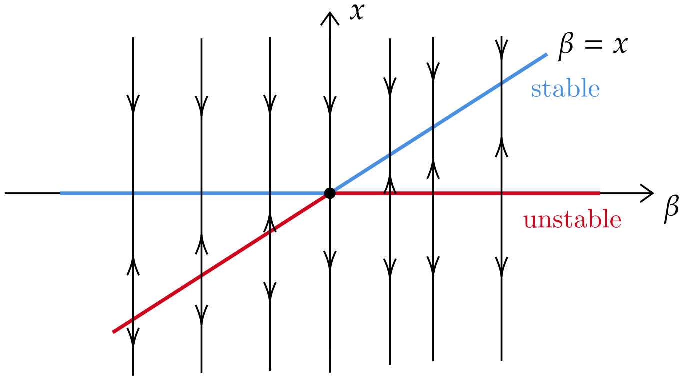

Figure 1: Transcritical bifurcation