151-0530-00L, Spring, 2020

Nonlinear Dynamics and Chaos II

Homework Assignment 1

Due: Thursday, March 11, in class

1. Show that the map the binary expansion map $g(x) = 2x \bmod 1$ is chaotic on $[0, 1]$.

2. Let A denote the transition matrix for a sub-shift $\sigma\colon \Sigma_A^N \mapsto \Sigma_A^N$ of finite type on N symbols.

   (a) Show that the number of fixed points of σ is equal to trace(A).

   (b) Show that the total number of *admissible* $k-$periodic points (i..e, $k-$periodic points whose minimal period may be less than k) is equal to trace(A^k).

3. The idea of the Lyapunov-exponent (see part I of this course) can be extended from continuous to discrete dynamical systems. Specifically, for one-dimensional iterated maps $x_{n+1} = f(x_n)$ defined by a smooth function $f : \mathbb{R} \to \mathbb{R}$, assume that trajectories separate from each other at an asymptotic rate of $e^{\lambda(x_0)n}$ for some exponent $\lambda \in \mathbb{R}$, which we may informally call the Lyapunov exponent associated with the initial condition x_0 from which the iteration starts. Note that $\lambda(x_0)$ may also be negative, which indicates exponential convergence between the trajectories starting close to x_0. Positive values of $\lambda(x_0)$ indicate sensitive dependence on initial conditions near x_0, which is a necessary condition for chaotic behavior, as we have seen. A more precise definition of $\lambda(x_0)$ formalizes these ideas by letting

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \lim_{\delta\to 0+} \log \frac{|f^n(x_0+\delta) - f^n(x_0)|}{\delta},$$

   whenever the limits involved in this expression exist.

   (a) Show that

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \left| f'(f^i(x_0)) \right|,$$

   whenever the limit $n \to \infty$ exists.

   (b) Compute $\lambda(x_0)$ as the average of ten $\lambda(x_0^i)$ values for randomly selected initial conditions $x_0^1, \dots, x_0^{10}$ for the logistic map $f(x) = ax(1-x)$, for 50 a values taken from the parameter range $a \in [3, 4]$. Plot the resulting average Lyapunov exponents as a function of a and identify the approximate a value over which an overall sensitivity for initial conditions arises and hence chaos is possible. Include a copy of your code with the results.(*Hint:* see Olsen, L.F., and Degn, H., Chaos in biological systems, *Quart. Rev. Biophys.* **18** (1985) 165-225, Fig. 6 for the plot you should obtain from your calculations.)