Nonlinear Dynamics & Chaos I

Exercice Set 1 Solutions

Question 1

Many important properties of nonlinear dynamical systems follow from Gronwall's inequality. Assume that two positive, continuous scalar functions $u(t)$ and $v(t)$ satisfy the condition

$$u(t) \leq C + \int_{t_0}^{t} u(\tau)v(\tau) \, \mathrm{d}\tau$$

for some constant $C \geq 0$ and for all $t \geq t_0$. Then Gronwall's inequality asserts that

$$u(t) \leq Ce^{\int_{t_0}^{t} v(\tau) \, \mathrm{d}\tau}$$

for all $t \geq t_0$. The significance of this result is that it gives a $u(t)$-independent upper bound on the growth of $u(t)$. Using Gronwall's inequality, give an upper bound on how fast the solutions of a nonlinear ODE can separate from each other in time. In particular, show that for an ODE of the form

$$\dot{x} = f(x,t), \qquad x \in \mathbb{R}^n,$$

and for two solutions starting from the initial conditions x_0 and $\hat{x}_0$ at time t_0, we have

$$|x(t,x_0) - x(t,\hat{x}_0)| \leq |x_0 - \hat{x}_0|e^{L(t-t_0)},$$

where L is a Lipschitz constant for the function f over a domain containing the trajectories of the system over the time interval $[t_0, t]$.

Hint: Substitute both solutions into the ODE, integrate the resulting two equations from t_0 to t, and estimate their normed difference.

Solution 1

(1) Define

$$h(t) = C + \int_{t_0}^{t} u(\tau)v(\tau) \, \mathrm{d}\tau \tag{1}$$

$$\implies \dot{h}(t) = u(t)v(t) \leq^{(*)} v(t)h(t)$$

$(*)$ From the definition of $h(t)$ and because $u, v > 0$ we have $u(t) \leq h(t)$

$$\implies \frac{\dot{h}(t)}{h(t)} \leq v(t) \implies \frac{d}{dt} \log[h(t)] \leq v(t) \tag{2}$$

Integrate both sides of (2) to get:

$$\log\left[\frac{h(t)}{h(t_0)}\right] \leq \int_{t_0}^{t} v(\tau) \, \mathrm{d}\tau$$

$$\implies h(t) \leq h(t_0) \exp\left(\int_{t_0}^{t} v(\tau) \, \mathrm{d}\tau\right)$$

From (1) we have $h(t_0) = C$; hence:

$$u(t) \leq h(t) \leq C \exp\left(\int_{t_0}^{t} v(\tau) \, \mathrm{d}\tau\right)$$

□

(2) The solutions $x(t, x_0)$ and $x(t, \tilde{x}_0)$ of the IVP satisfy the integral equations

$$x(t, x_0) = x_0 + \int_{t_0}^t f(x(s, x_0), s)\, \mathrm{d}s$$

$$x(t, \tilde{x}_0) = x_0 + \int_{t_0}^t f(x(s, \tilde{x}_0), s)\, \mathrm{d}s$$

Subtracting these two from each other and taking the absolute values we get

$$|x(t, x_0) - x(t, \tilde{x}_0)| = \left| x_0 - \tilde{x}_0 + \int_{t_0}^t [f(x(s, x_0), s) - f(x(s, \tilde{x}_0), s)]\, \mathrm{d}s \right|$$

Using triangular and Jensen's inequalities we get

$$|x(t, x_0) - x(t, \tilde{x}_0)| \leq |x_0 - \tilde{x}_0| + \int_{t_0}^t |f(x(s, x_0), s) - f(x(s, \tilde{x}_0), s)|\, \mathrm{d}s \tag{3}$$

By Lipschitz continuity of f we have

$$|f(x(s, x_0), s) - f(x(s, \tilde{x}_0), s)| \leq L\, |x(t, x_0) - x(t, \tilde{x}_0)|$$

This together with (3) gives

$$|x(t, x_0) - x(t, \tilde{x}_0)| \leq |x_0 - \tilde{x}_0| + \int_{t_0}^t L\, |x(t, x_0) - x(t, \tilde{x}_0)|\, \mathrm{d}s$$

Using Gronwall's inequality from Exercice 1 with $u(t) = |x(t, x_0) - x(t, \tilde{x}_0)|$, $C = |x_0 - \tilde{x}_0|$ and $v(t) = L$ (constant), we get

$$|x(t, x_0) - x(t, \tilde{x}_0)| \leq |x_0 - \tilde{x}_0| \exp\left(\int_{t_0}^t L\, \mathrm{d}s \right) = |x_0 - \tilde{x}_0|\, e^{L(t - t_0)}$$

□

Question 2

Consider a pendulum that strikes an inclined wall repeatedly, as shown in Fig. 1 below. Using the phase portrait of the pendulum discussed in class, sketch the trajectories in the phase space of this impact dynamical system for positive and negative values of the angle α, when

  (i) there us no loss of energy at impact

  (ii) the coefficient of restitution is 0.5.

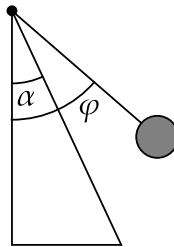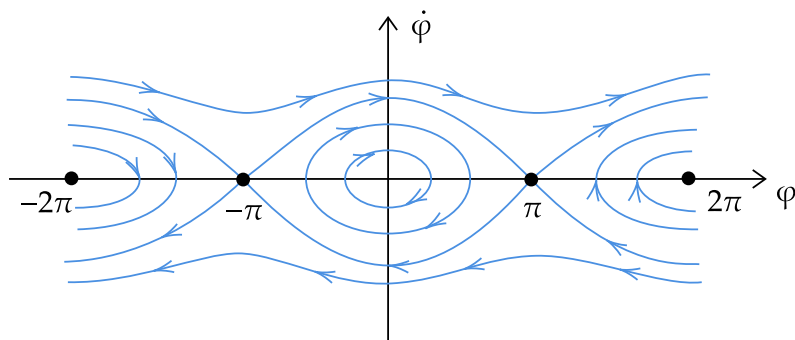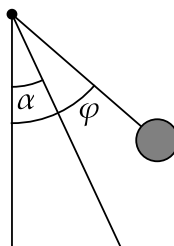Identify the asymptotic behavior of the pendulum in each case.

Figure 1: Pendulum attached to the inclined wall.

Hint: During impact, the pre-impact velocity of the pendulum is assumed to jump instantaneously to its post-impact value, while the pendulum's position remains the same.

Solution 2

Without any walls:

(1) Case $\alpha > 0$:

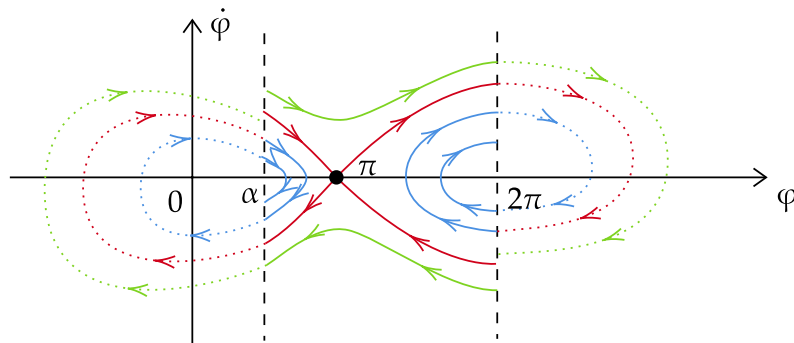(i) Post impact velocity $v(t_+) = -v(t_-)$ pre-impact velocity
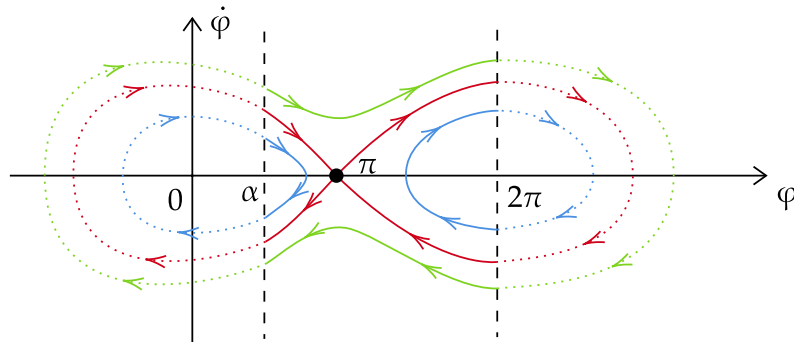
The dashed lines represent the instance of impact.

Possible asymptotic behavior: [depending on initial conditions $\varphi(0), \dot{\varphi}(0)$]

- Bouncing against the inclined wall or the vertical wall
- Bouncing back and forth between the two walls
- Convergence to the upright position $\varphi = \pi, \dot{\varphi} = 0$
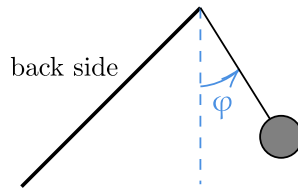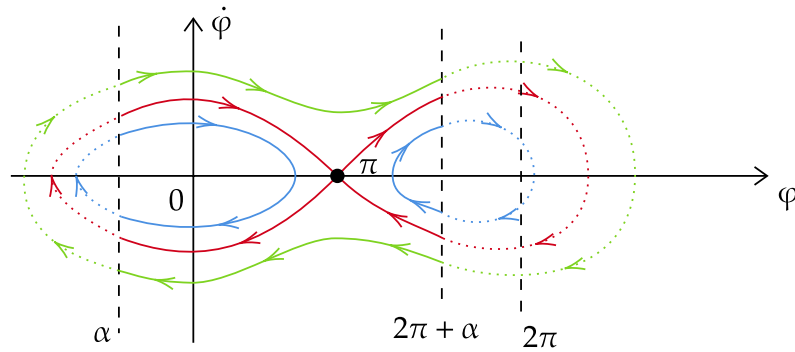
(ii) Post impact velocity $v(t_+) = -\frac{1}{2}v(t_-)$

Possible asymptotics:

- Convergence to the upright position
- Lying against the inclined wall
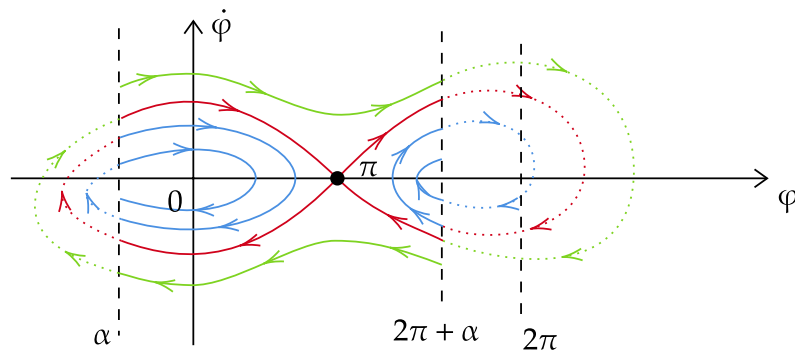- Lying against the vertical wall

(2) Case $\alpha < 0$: Solution 1

back side

φ

(i) $v(t_+) = -v(t_-)$:

$\dot{\varphi}$

π

0

φ

α $2\pi + \alpha$ 2π

Possible asymptotics:

- Convergence to upright position
- Bouncing against either side of the wall
- Bouncing back an forth between the two sides of the wall
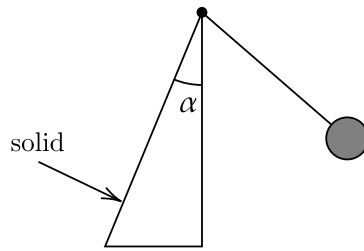- Oscillating around the vertical position $\varphi = 0$

(ii) $v(t_+) = -\frac{1}{2}v(t_-)$:

$\dot{\varphi}$

π

0

φ

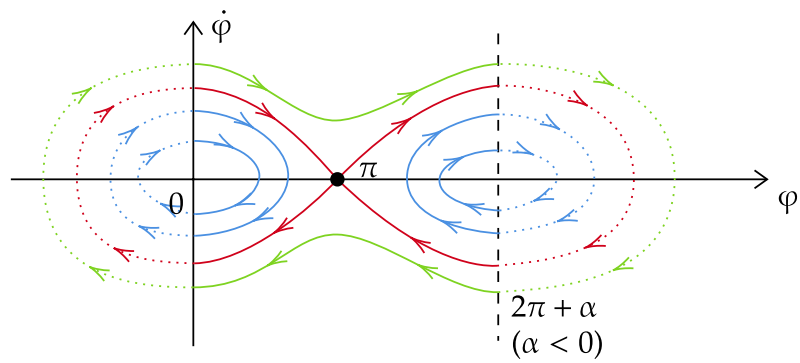α $2\pi + \alpha$ 2π

Possible asymptotics:

- Convergence to the upright position
- Oscillating around the vertical position $\varphi = 0$
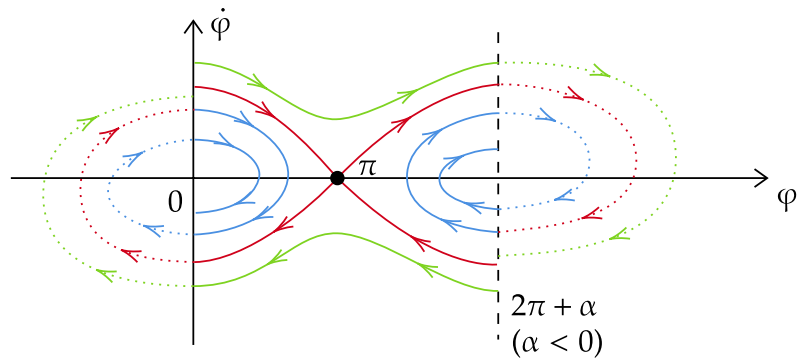- Lying on the back of the wall

(3) Case $\alpha < 0$: Solution 2

(i) $v(t_+) = -v(t_-)$:

(ii) Post impact velocity $v(t_+) = -\frac{1}{2}v(t_-)$ pre-impact velocity

Possible asymptotics:

- Convergence to the upright position
- Lying against the inclined wall
- Lying against the vertical wall

Question 3

Consider the non-dimensionalized, force-damped pendulum equation

$$\ddot{x} + k\dot{x} + \sin(x) = a\sin(t),$$

where $k \geq 0$ is the damping coefficient and $a \geq 0$ is the forcing amplitude.

(a) For vanishing damping and forcing ($a = k = 0$), compute and plot numerically the FTLE field for this system over a 100×100 grid of initial conditions, covering the square $[-\pi, \pi] \times [-\pi, \pi]$ in the phase space of the pendulum. Perform the computation for long enough times so that the FTLE plot fully reveals the separatrices of the system, as we discussed in class for the undamped pendulum.

(b) To explore the fate of these separatrices under mild damping and forcing, repeat the same FTLE computations for $a = 0.5$ and $k = 0.1$. Discuss domains of attractions and their boundaries based on the results.

Hint: Consider using MATLAB's ODE45 routine for trajectory integration. To this end, you need to assemble all initial grid points into a single long vector, and pass that vector on to ODE45 as an initial condition. You will then need to define an appropriate extended ODE, whose right-hand side will be called by ODE45 to produce an advected image of the extended initial condition vector. Subsequently, you will need to convert the elements of this advected vector back into advected grid positions. Alternatively, you could write your own 4-point Runge-Kutta solver to track trajectories simultaneously on the full grid of initial conditions (see the algorithm discussed in class).

Solution 3

Pendulum equation:
$$\ddot{x} + k\dot{x} + \sin(x) = a\sin(t)$$

where $k \geq 0$ and $a \geq 0$
The first order system representation car be written as

$$z_1 = x, \quad z_2 = \dot{x}$$
$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ a\sin(t) - kz_1 - \sin(z_1) \end{bmatrix}$$

MATLAB code for flowmap generation

```
1   %% Initialize script
2
3   close all
4   clear
5   clc
6
7   %% Grid of initial conditions
8
9   xs = linspace(-pi,pi,100);
10  ys = linspace(-pi,pi,100);
11  [X0, Y0] = meshgrid(xs,ys);
12
13  %% ODE simulation
14
15  t0 = 20;
16  tend = 0;
17
18  XF = zeros(size(X0));
19  YF = zeros(size(Y0));
20
21  for i = 1:size(X0,1)
```

```
22      for j = 1:size(X0,2)
23          x0 = X0(i,j);
24          y0 = Y0(i,j);
25
26          [t, z] = ode45(@odefun, [t0 tend], [x0,y0]);
27
28          XF(i,j) = z(end,1);
29          YF(i,j) = z(end,2);
30      end
31  end
32
33  %% Deformation Gradient
34
35  [DFxx, DFxy] = gradient(XF, xs, ys);
36  [DFyx, DFyy] = gradient(YF, xs, ys);
37
38  %% Cauchy Green Strain Tensor Components
39
40  C11 = DFxx.^2 + DFyx.^2;
41  C12 = DFxx.*DFxy + DFyx.*DFyy;
42  C21 = C12;
43  C22 = DFxy.^2 + DFyy.^2;
44
45  %% Largest eigenvalue of Cauchy Green Strain Tensor
46
47  detC = C11.*C22 - C12.*C21;
48  traceC = C11 + C22;
49
50  lambda = real(traceC./2 + sqrt((traceC./2).^2 - detC));
51
52  %% PLot FTLE
53
54  FTLE = log(lambda)/(2*abs(tend-t0));
55
56  figure(1)
57  title('FTLE${}^{20}_0$', 'interpreter', 'latex')
58  surf(X0,Y0, FTLE, 'EdgeColor','none');
59  colorbar
60
61  axis tight
62  xlabel('$x$', 'interpreter','latex')
63  ylabel('$\dot{x}$', 'interpreter','latex')
64
65
66
67  %% Function to simulate
68
69  function [rhs] = odefun(t,X)
70      a = 0.0; % or 0.5
71      k = 0.0; % or 0.1
72
73      x = X(1);
74      xd = X(2);
75
76      rhs = [xd;
77              a*sin(t) - sin(x) - k*xd];
78  end
```
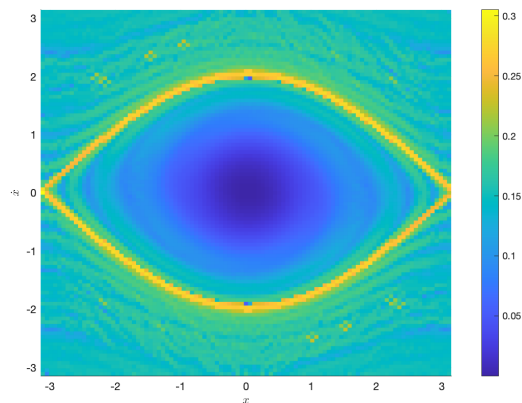
(1) $a = k = 0$

Figure 2: $FTLE_0^{20}$

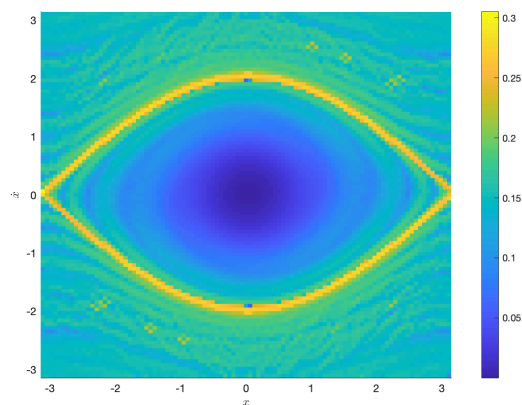The thick yellow lines are the separatrices.
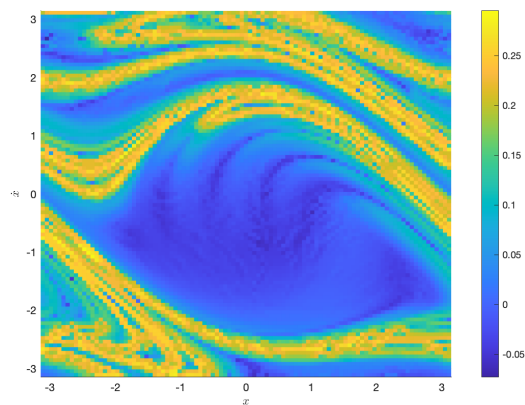

Figure 3: $FTLE_{20}^0$
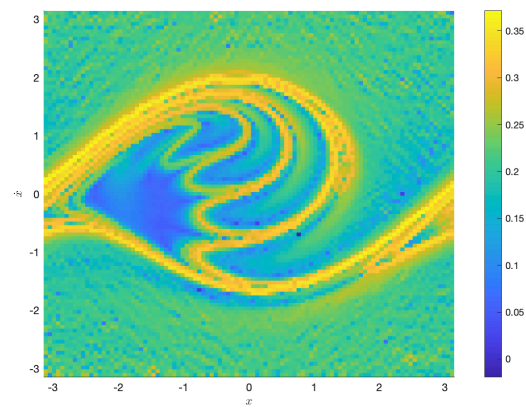
(2) $a = 0.5$ and $k = 0.1$
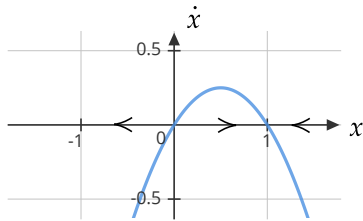

Figure 4: $FTLE_0^{20}$

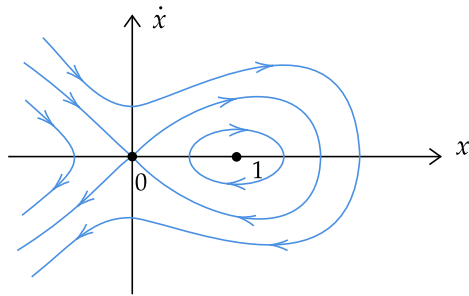Figure 5: FTLE_{20}^{0}

Solution 4

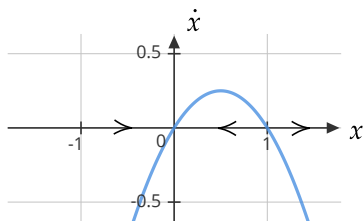Which figure describes the phase portrait of the following dynamical system ?
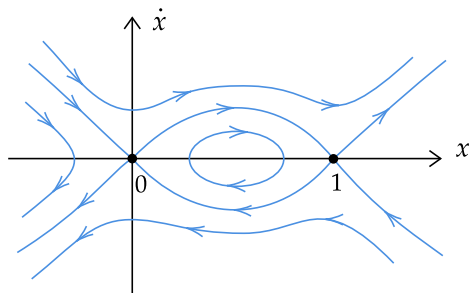
$$\ddot{x} = x(1-x)$$

(a)

(b)

(c)

(d)

Solution 5

Consider the following dynamical system with the solutions $x(t; t_0, x_0)$. For which systems does the following estimate fail for any constant $C \geq 0$.

$$||x(t; t_0, x_0) - x(t; t_0, \tilde{x}_0)|| \leq ||x - \tilde{x}_0|| \, e^{C(t-t_0)}, \quad \forall t \geq t_0, \quad x_0, \tilde{x}_0 \geq 0$$

(a) $\dot{x} = \sin(x)$

(b) $\dot{x} = x^2$
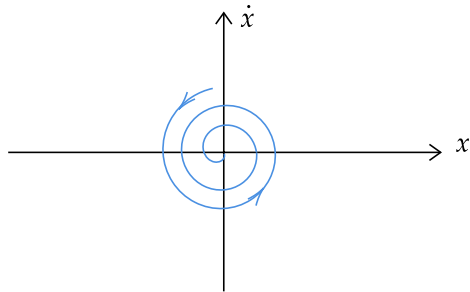
(c) $\dot{x} = 100x$

(d) $\dot{x} = e^{-x}$

This question asks which solution fails the Lipschitz continuity of the right-hand side.
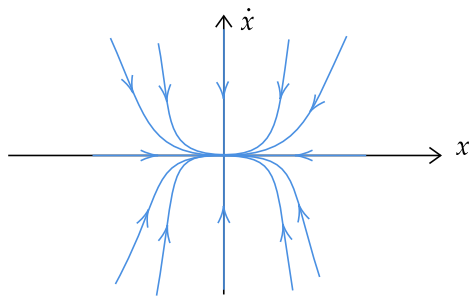
Solution 6

Which figure describes the phase portrait of the dynamical system
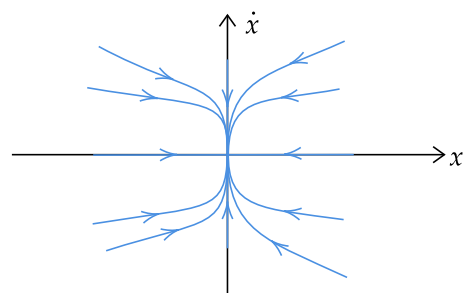
$$\ddot{x} + 2\dot{x} + \sin(x) = 0$$
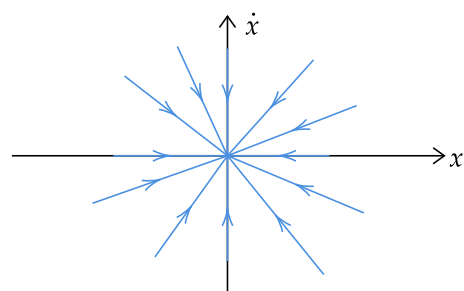
near the equilibrium point $x = 0,\ \dot{x} = 0$?

(a)

(b)

(c)

(d)

$$\lambda_{1,2} = -1$$

Solution 7

Consider the 1D map

$$x_{n+1} = x_n^2(1 - x_n^2)$$

Which relation is the correct linearization around the fixed point $x = 0$?

(a) $y_{n+1} = y_n(1 - y_n)$

(b) $y_{n+1} = y_n(1 - y_n^2)$

(c) $y_{n+1} = 0$

(d) $y_{n+1} = y_n$