Nonlinear Dynamics and Chaos I
Problem Session 1

Shobhit Jain

(1) Many important properties of nonlinear dynamical systems follow from Gronwall's inequality. Assume that two positive, continuous scalar functions $u(t)$ and $v(t)$ satisfy the condition

$$u(t) \leq C + \int_{t_0}^{t} u(\tau)v(\tau)\, d\tau$$

for some constant $C \geq 0$ and for all $t \geq t_0$. Then Gronwall's inequality asserts that

$$u(t) \leq Ce^{\int_{t_0}^{t} v(\tau)\, d\tau}$$

for all $t \geq t_0$.

Define $h(t) = C + \int_{t_0}^{t} u(\tau)v(\tau)\,d\tau.$ (1)

$\Rightarrow \dot{h}(t) = u(t)v(t) \leq v(t)h(t)$

From the definition of $h(t)$ and because $u, v > 0$ we have $u(t) \leq h(t)$

$$\Rightarrow \frac{\dot{h}(t)}{h(t)} \leq v(t) \Rightarrow \frac{d}{dt}\log[h(t)] \leq v(t) \quad (2)$$

Integrate both sides of (2) to get $\log \frac{h(t)}{h(t_0)} \le \int_{t_0}^{t} v(\tau)\,d\tau$

$$\Rightarrow \quad h(t) \le h(t_0) \exp\left(\int_{t_0}^{t} v(\tau)\,d\tau\right)$$

From (1) we have $h(t_0) = c$; hence $u(t) \le h(t) \le c \exp\left(\int_{t_0}^{t} v(\tau)\,d\tau\right)$

The significance of this result is that it gives a $u(t)$-independent upper bound on the growth of $u(t)$. Using Gronwall's inequality, give an upper bound on how fast the solutions of a nonlinear ODE can separate from each other in time. In particular, show that for an ODE of the form

$$\dot{x} = f(x, t), \qquad x \in \mathbb{R}^n,$$

and for two solutions starting from the initial conditions x_0 and $\hat{x}_0$ at time t_0, we have

$$|x(t, x_0) - x(t, \hat{x}_0)| \le |x_0 - \hat{x}_0|\, e^{L(t-t_0)},$$

where L is a Lipschitz constant for the function f over a domain containing the trajectories of the system over the time interval $[t_0, t]$.

The solutions $x(t, x_0)$ and $x(t, \hat{x}_0)$ of the IVP satisfy the integral equations

$$x(t, x_0) = x_0 + \int_{t_0}^{t} f(x(s, x_0), s)\, ds$$

$$x(t, \hat{x}_0) = \hat{x}_0 + \int_{t_0}^{t} f(x(s, \hat{x}_0), s)\, ds$$

Substracting these two from each other and taking the absolute values we get

$$|x(t, x_0) - x(t, \hat{x}_0)| = \left| x_0 - \hat{x}_0 + \int_{t_0}^{t} \left[f(x(s, x_0), s) - f(x(s, \hat{x}_0), s) \right] ds \right|$$

Using triangle and Jensen's inequalities we get

$$|x(t, x_0) - x(t, \hat{x}_0)| \leq |x_0 - \hat{x}_0| + \int_{t_0}^{t} \left| f(x(s, x_0), s) - f(x(s, \hat{x}_0), s) \right| ds \tag{1}$$

By Lipschitz continuity of f we have

$$\left| f(x(s,x_0),s) - f(x(s,\hat{x}_0),s) \right| \leq L \left| x(s,x_0) - x(s,\hat{x}_0) \right|$$

this together with (1) gives

$$|x(t,x_0) - x(t,\hat{x}_0)| \leq |x_0 - \hat{x}_0| + \int_{t_0}^{t} L |x(s,x_0) - x(s,\hat{x}_0)| \, ds$$

Using Gronwall's inequality from Exercise 1 with

$u(t) = |x(t,x_0) - x(t,\hat{x}_0)|$, $C = |x_0 - \hat{x}_0|$ and $v(t) = L$ (constant)

we get $$|x(t,x_0) - x(t,\hat{x}_0)| \leq |x_0 - \hat{x}_0| \exp\left(\int_{t_0}^{t} L \, ds\right) = |x_0 - \hat{x}_0| e^{L(t-t_0)}$$

(2) Consider a pendulum that strikes an inclined wall repeatedly, as shown in Fig. 1 below. Using the phase portrait of the pendulum discussed in class, sketch the trajectories in the phase space of this impact dynamical system for positive and negative values of the angle α, when (i) there is no loss of energy at impact (ii) the coefficient of restitution is 0.5. Identify the asymptotic behavior of the pendulum in each case

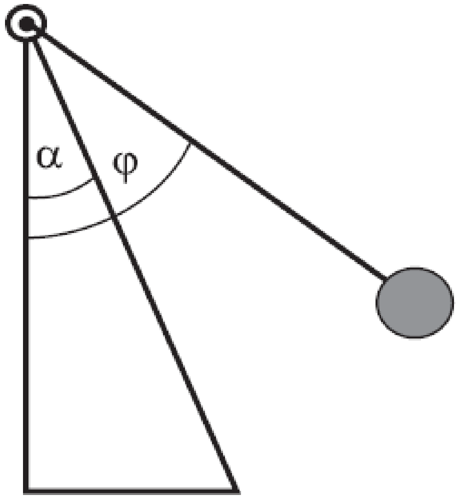
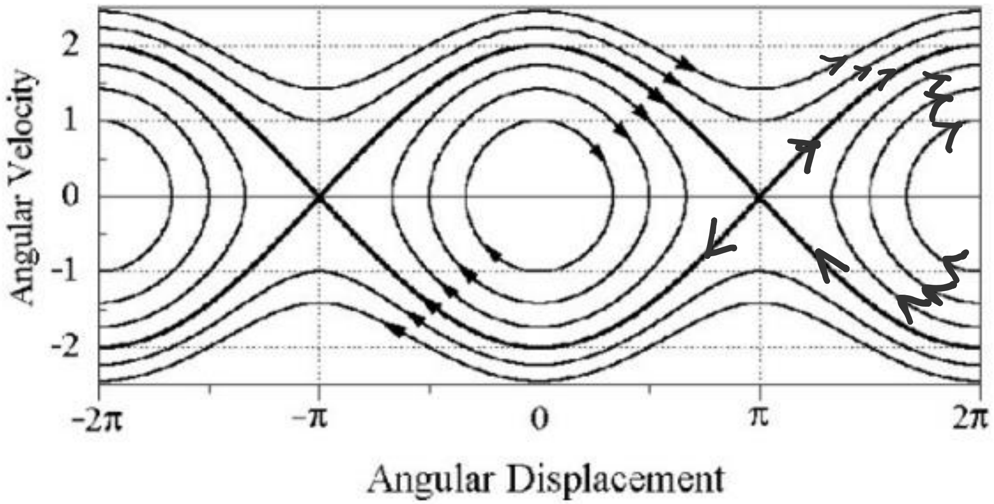


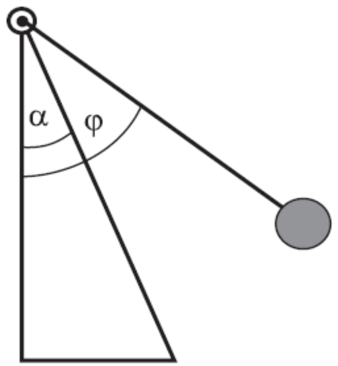
Figure 1:

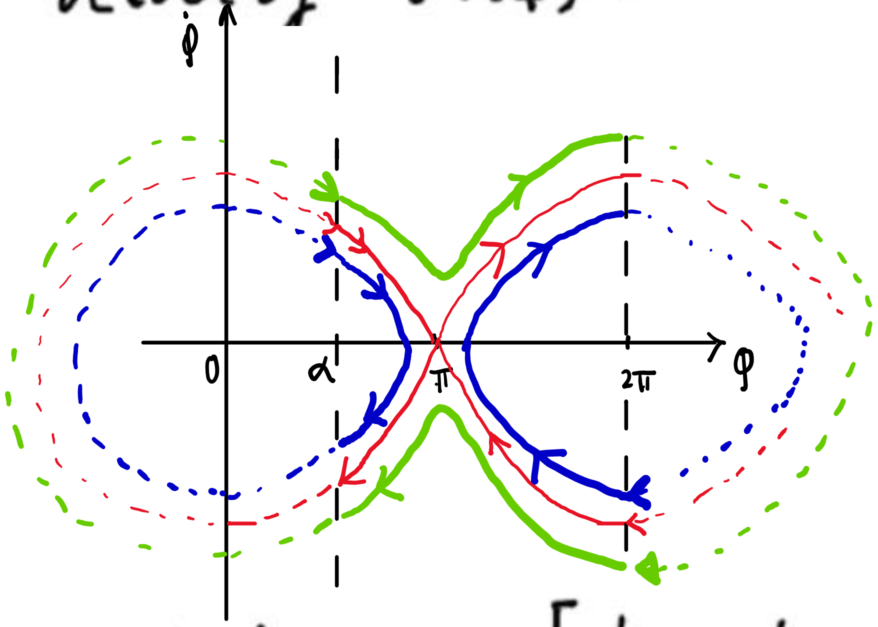
Without wall



2. Case $\alpha > 0$

(i) Post-impact velocity $v(t_+) = -v(t_-)$ Pre-impact velocity





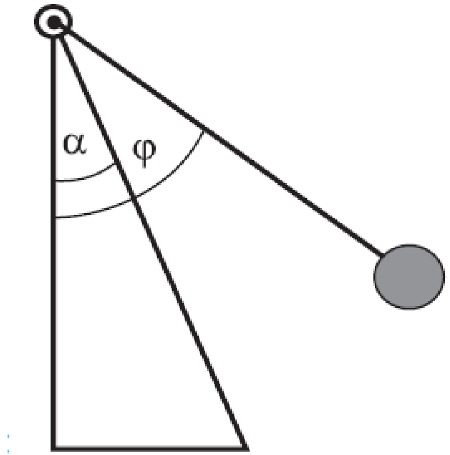
The dashed lines represent the istance of the impact.

Possible asymptotic behavior: [depending on initial conditions $(\varphi(0), \dot{\varphi}(0))$]

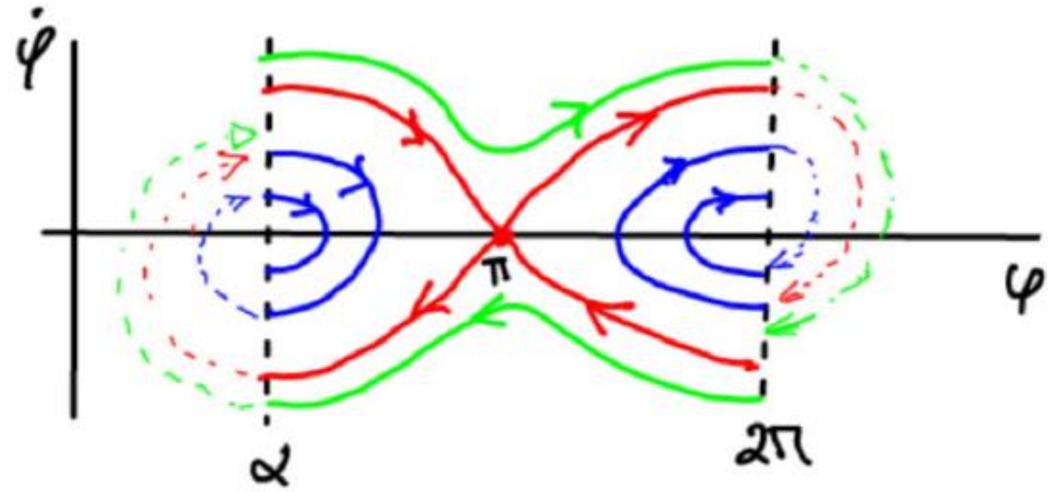
Bouncing against the inclined wall or the vertical wall

Bouncing back and forth between the two walls

Convergence to the upright position $\varphi = \pi$, $\dot{\varphi} = 0$



(ii) Post impact velocity $v(t_+) = -\frac{1}{2}\, v(t_-)$ Pre-impact velocity

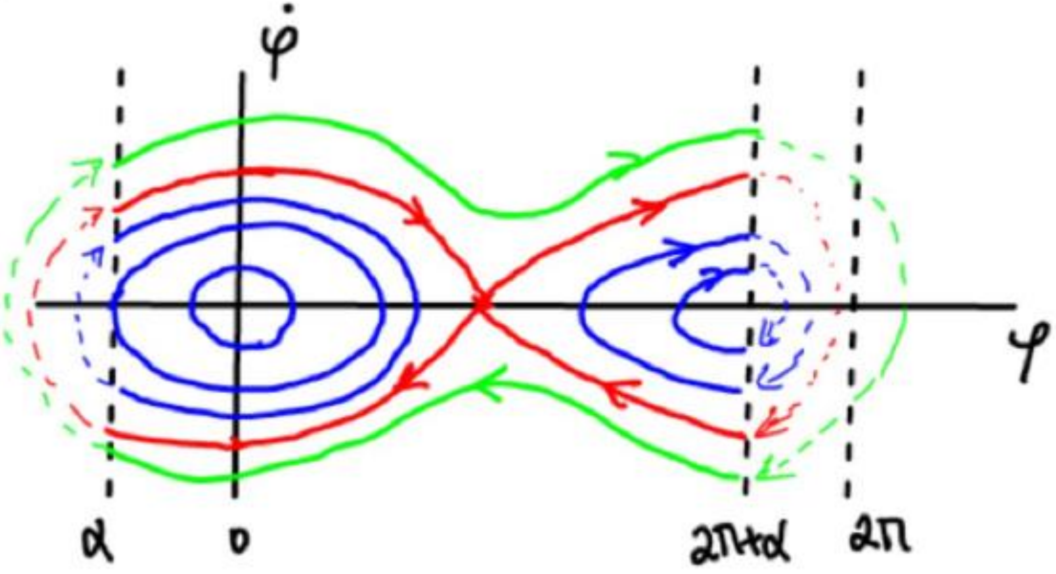


Possible asymptotics:

- convergence to the upright position
- lying against the inclined wall
- lying against the vertical wall

Case $\alpha<0$: Solution 1

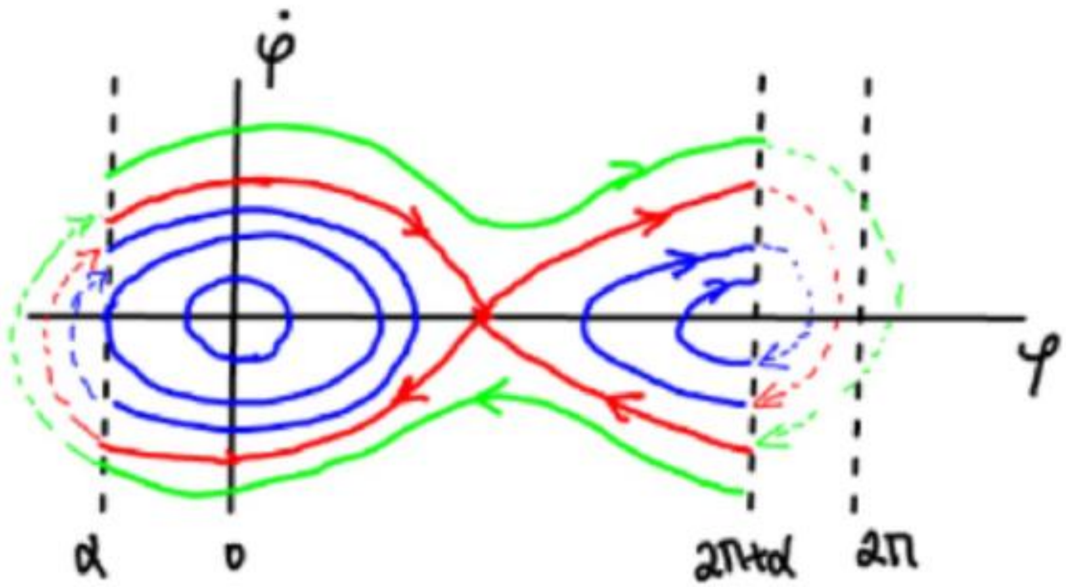
i) $v(t_+) = - v(t_-)$



Possible asymptotics:

- Convergence to upright Position
- Bouncing against either side of the wall.
- Bouncing back and forth between the two sides of the wall
- Oscillating around the vertical position $\varphi=0$

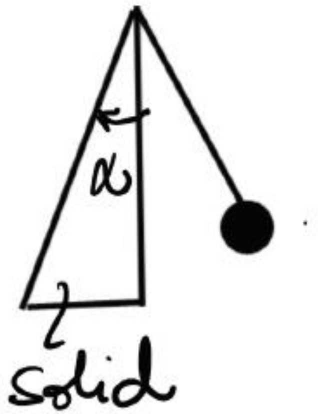
(ii) $v(t_+) = -\frac{1}{2} v(t_-)$



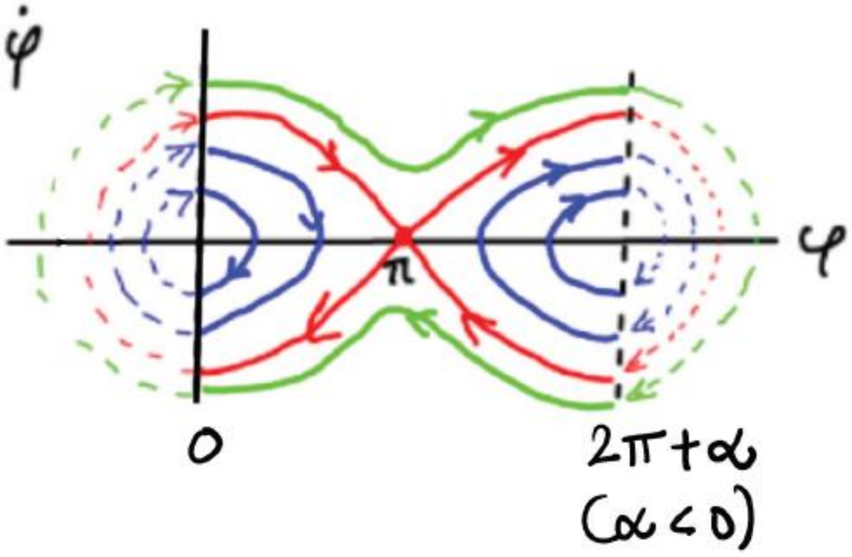
Possible asymptotics:

- Convergence to the upright Position
- Oscillating arround the vertical position $\varphi = 0$
- lying on the back of the wall

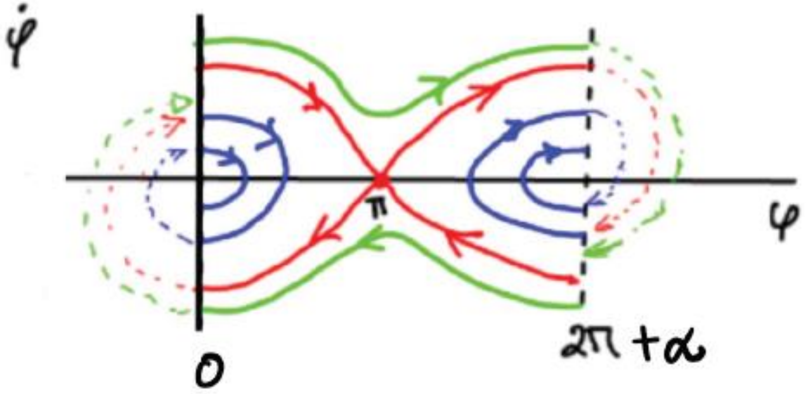
Case : $\alpha<0$: Solution 2



(i) $v(t^+) = -v(t^-)$



(ii) Post impact velocity $v(t_+) = -\frac{1}{2} v(t_-)$ Pre-impact velocity



Possible asymptotics:

- convergence to the upright position
- lying against the inclined wall
- lying against the vertical wall

(3) Consider the non-dimensionalized, forced-damped pendulum equation

$$\ddot{x} + k\dot{x} + \sin x = a \sin t,$$

where $k \geq 0$ is the damping coefficient and $a \geq$) is the forcing amplitude.

(a) For vanishing damping and forcing ($a = k = 0$), compute and plot numerically the FTLE field for this system over a 100×100 grid of initial conditions, covering the square $[-\pi, \pi] \times [-\pi, \pi]$ in the phase space of the pendulum. Perform the computation for long enough times so that the FTLE plot fully reveals the separatrices of the system, as we discussed in class for the undamped pendulum.

$$z_1 = x, \quad z_2 = \dot{x},$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ a\sin(t) - kz_1 - \sin(z_1) \end{bmatrix}$$

MATLAB code for flowmap generation

```matlab
xs = linspace(-pi,pi,100);
ys = linspace(-pi,pi,100);

[X0, Y0] = meshgrid(xs,ys);

% simulation of ODE
t0 = 20;
tend = 0;

XF = zeros(size(X0));
YF = zeros(size(Y0));

for i = 1:size(X0,1)
    for j = 1:size(X0,2)
        x0 = X0(i,j);
        y0 = Y0(i,j);

        [t, z] = ode45(@odefun, [t0 tend], [x0; y0]);

        XF(i,j) = z(end,1);
        YF(i,j) = z(end,2);
    end
end
```

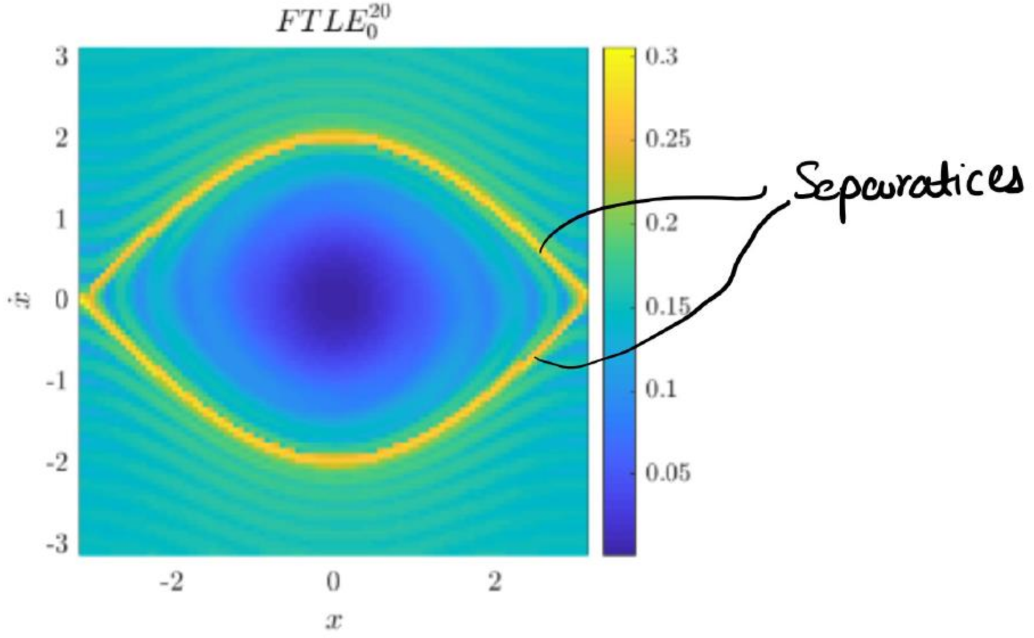
```matlab
function [rhs] = odefun(t,X)
% right hand side of the ode in first order form
a = 0; % 0.5
k = 0;% 0.1

x = X(1);
xd = X(2);

rhs = [xd;
       a*sin(t) - sin(x) - k * xd];
end
```

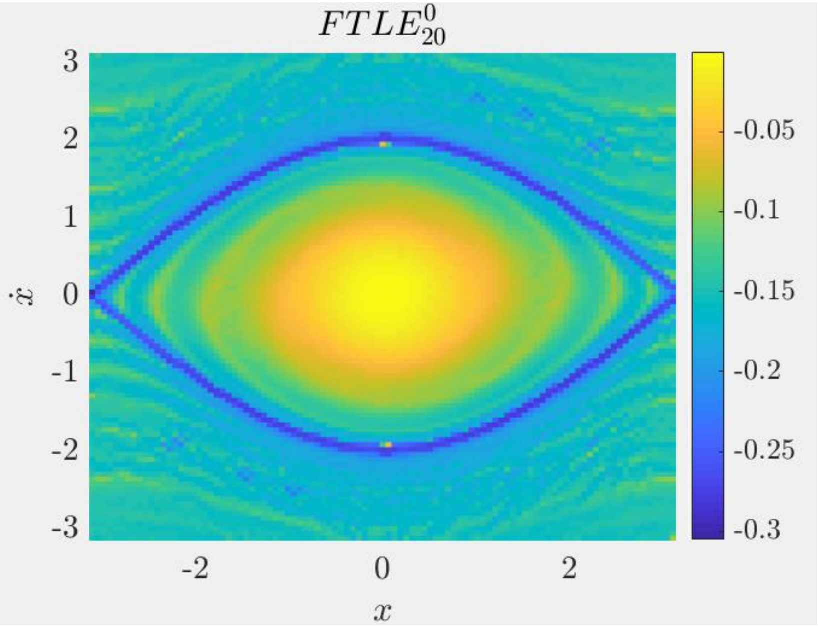
MATLAB code for computing FTLE

```matlab
%% Deformation gradient

[DFxx, DFxy] = gradient(XF, xs, ys);
[DFyx, DFyy] = gradient(YF, xs, ys);



%% Cauchy Green Strain
C11 = DFxx.^2 + DFyx.^2;
C12 = DFxx.*DFxy + DFyx.*DFyy;
C21 = C12;
C22 = DFxy.^2 + DFyy.^2;

%% Largest eigenvalue of Cauchy Green Strain tensor
detC = C11.*C22 - C12.*C21;
traceC = C11 + C22;

lambda = real(traceC/2 + sqrt((traceC./2).^2 - detC));

%% plot
FTLE = log(lambda)/(2*(tend - t0));

figure
surf(X0,Y0, FTLE,'EdgeColor', 'none')

axis tight
xlabel('$$x$$')
ylabel('$$\dot{x}$$')
```

a.)

$a = k = 0$


$FTLE_0^{20}$
Separatices
$\dot{x}$
x
3
2
1
0
-1
-2
-3
-2
0
2
0.3
0.25
0.2
0.15
0.1
0.05


$FTLE_{20}^0$
$\dot{x}$
x
3
2
1
0
-1
-2
-3
-2
0
2
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3

(b) To explore the fate of theses separatrices under mild damping and forcing, repeat the same FTLE computations for $a = 0.5$ and $k = 0.1$. Discuss domains of attractions and their boundaries based on the results.