

# Nonlinear Dynamics & Chaos I

## Exercise Set 3 Solutions

### Question 1

Consider a ball of mass  $m$  that slides on a rotating hoop (see Fig. 1).

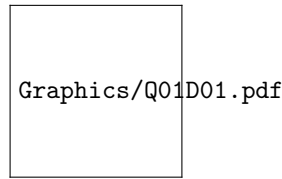


Figure 1: Mass on a hoop

The angular velocity of the hoop is  $\Omega$ , the viscous friction coefficient between the hoop and the ball is  $b$ , and the constant of gravity is  $g$ . The equation of motion for the sliding ball is given by

$$mR^2\ddot{\alpha} + bR^2\dot{\alpha} + mR^2(g/R - \Omega^2 \cos(\alpha)) \sin(\alpha) = 0.$$

- (a) Plot the location of equilibria of the ball as a function of the non-dimensionalized rotation parameter  $\nu = R\Omega^2/g$ .
- (b) Using linearization, determine the stability type of the different equilibrium branches on the plot. Identify the critical angular velocity at which a bifurcation of equilibria occurs.

### Solution 1

- (a) The equation of motion for the sliding ball is given by

$$mR^2\ddot{\alpha} + bR^2\dot{\alpha} + mR^2(g/R - \Omega^2 \cos(\alpha)) \sin(\alpha) = 0$$

Where we'll define  $\nu = R\Omega^2/g$

We can define  $x_1 = \alpha$ ,  $x_2 = \dot{\alpha}$  in order to write the ODE as  $\dot{x} = f(x)$  where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad f(x) = \begin{pmatrix} x_2 \\ -\frac{g}{R}[1 - \nu \cos(x_1)] \sin(x_1) - \frac{b}{m}x_2 \end{pmatrix}$$

In other words

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{R}[1 - \nu \cos(x_1)] \sin(x_1) - \frac{b}{m}x_2 \end{bmatrix}$$

Fixed points are found when  $f(x) = 0$ . This implies that  $x_2 = 0$  and  $[1 - \nu \cos(x_1)] \sin(x_1) = 0$

**Case 1:**  $\nu < 1$ :

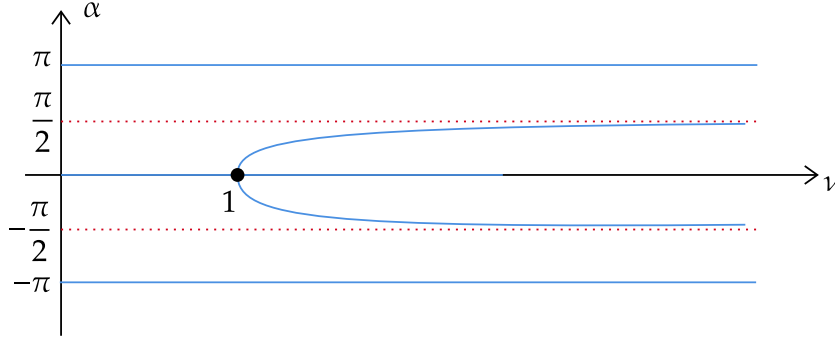
Only two fixed points exist:  $(0, 0)$  and  $(\pi, 0)$  [Note: the fixed point  $(-\pi, 0)$  is physically identical to the fixed point  $(\pi, 0)$ . Therefore, we only discuss  $(\pi, 0)$ ]

**Case 2:  $\nu > 1$ :**

Two additional fixed points emerge that correspond to the solutions of  $\cos(x_1) = \frac{1}{\nu}$ .

Let  $\alpha_0 \in (0, \pi)$  be the positive solution:  $\cos(\alpha_0) = \frac{1}{\nu}$ . Then the fixed points in this case are:

$(0, 0)$ ,  $(\pi, 0)$ ,  $(\alpha_0, 0)$  and  $(-\alpha_0, 0)$



The blue curves mark the location of the fixed points.

(b) First we compute  $\nabla f(x_1, x_2)$ :

$$\nabla f(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ \frac{g}{R}[2\nu \cos^2(x_1) - \cos(x_1) - \nu] & -\frac{b}{m} \end{pmatrix}$$

whose eigenvalues are given by

$$\lambda_{\pm} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 + \frac{g}{R}[2\nu \cos^2(x_1) - \cos(x_1) - \nu]}$$

Now we investigate the linear stability of each fixed point:

**Fixed point  $(0, 0)$ :**

$$\lambda_{\pm} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 + \frac{g}{R}(\nu - 1)}$$

- $\nu < 1 \implies \text{Re}(\lambda_+) < 0$  and  $\text{Re}(\lambda_-) < 0$ . Therefore  $(0, 0)$  is asymptotically stable.
- $\nu > 1 \implies \text{Re}(\lambda_+) > 0$  and  $\text{Re}(\lambda_-) < 0$ . Therefore  $(0, 0)$  is unstable.

**Fixed points  $(\pm\pi, 0)$ :**

$$\lambda_{\pm} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 + \frac{g}{R}(\nu + 1)}$$

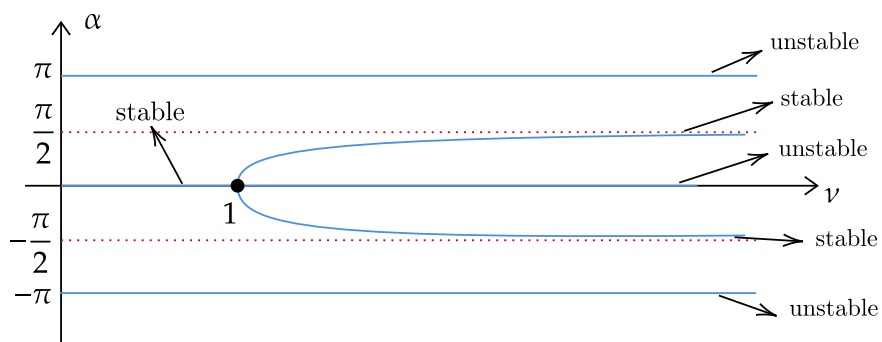
For any  $\nu \geq 0$ ,  $\text{Re}(\lambda_+) > 0 \implies (\pm\pi, 0)$  is unstable for any  $\nu \geq 0$ .

**Fixed points  $(\pm\alpha_0, 0)$**

Remember that these fixed points only exist when  $\nu > 1$ . Also  $\cos(\pm\alpha_0) = \frac{1}{\nu}$

$$\lambda_{\pm} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 + \frac{g}{R}\left(\frac{1 - \nu^2}{\nu}\right)}$$

For any  $\nu > 1$ ,  $\text{Re}(\lambda_+) < 0$  and  $\text{Re}(\lambda_-) < 0$ . Therefore the fixed points  $(\pm\alpha_0, 0)$  are asymptotically stable.



The bifurcation of equilibria occurs at  $\nu = 1 \implies \Omega^2 = \frac{g}{R} \implies \Omega = \pm \sqrt{\frac{g}{R}}$

## Question 2

Consider a discrete dynamical system given by the iterated mapping

$$x_{k+1} = f(x_k), \quad f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad x_k \in \mathbb{R}^n.$$

Assume that  $x = p$  is a fixed point for the mapping, i.e.,  $p = f(p)$ .

- (a) Derive a linearized mapping of the form

$$y_{k+1} = Ay_k \tag{1}$$

to describe the discrete dynamics in the vicinity of the fixed point.

- (b) Assume that  $A$  has eigenvalues  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  with corresponding  $n$  linearly independent eigenvectors  $s_1, \dots, s_n \in \mathbb{C}^n$ . Show that the general solution of (1) is of the form

$$y_k = c_1 \varphi_1(k) + \dots + c_n \varphi_n(k), \quad \varphi_i(k) = \lambda_i^k s_i. \tag{2}$$

- (c) Formulate a definition of stability, asymptotic stability, and instability for the  $y = 0$  fixed point of (1).  
 (d) Using (2), find a sufficient and necessary condition for the asymptotic stability you have defined in (c).

## Solution 2

- (a) Let  $x_k$  be near the fixed point  $P$  and define  $y_k = x_k - P$ . Then

$$\begin{aligned} x_{k+1} &= f(x_k) = f(P + y_k) = f(P) + Df(P)y_k + \mathcal{O}(\|y_k\|^2) \\ &= P + Df(P)y_k + \mathcal{O}(\|y_k\|^2) \end{aligned}$$

$$\implies y_{k+1} = x_{k+1} - P = Df(P)y_k + \mathcal{O}(\|y_k\|^2)$$

Now for  $\|y_k\|$  small enough the linear approximation of the map  $x_{k+1} = f(x_k)$  is  $y_{k+1} = Ay_k$  with  $A = Df(P)$ .

- (b) Take any  $y_0 \in \mathbb{R}^n$ . Since  $s_1, \dots, s_n \in \mathbb{C}^n$  are linearly independent there are constants  $c_1, \dots, c_n \in \mathbb{C}$  such that  $y_0 = c_1 s_1 + \dots + c_n s_n$ .

Now define

$$\begin{aligned} y_1 &= Ay_0 = c_1 As_1 + \dots + c_n As_n \\ &= c_1 \lambda s_1 + \dots + c_n \lambda s_n \\ &= c_1 \varphi_1(1) + \dots + c_n \varphi_n(1) \end{aligned}$$

Similarly, for any  $k \geq 1$ ,

$$\begin{aligned} y_k &= Ay_{k-1} = c_1 \lambda_1^{k-1} As_1 + \dots + c_n \lambda_n^{k-1} As_n \\ &= c_1 \varphi_1(k) + \dots + c_n \varphi_n(k) \end{aligned} \tag{3}$$

It's easy to check that  $y_{k+1} = Ay_k$  for any  $k \geq 0$ . Since  $y_0 \in \mathbb{R}^n$  was arbitrary,  $c_1 \varphi_1(k) + c_2 \varphi_2(k) + \dots + c_n \varphi_n(k)$  is a general solution of  $y_{k+1} = Ay_k$ .

- (c) **Definition of stability:**

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ such that } \forall y_0 \in \mathbb{R}^n \text{ with } \|y_0\| \leq \delta \text{ we have } \|y_k\| \leq \varepsilon \text{ for any } k \geq 0$$

**Definition of asymptotic stability:**

$y = 0$  is asymptotically stable if and only if:

- $y = 0$  is stable
- $\exists \delta > 0$  such that  $\forall y_0 \in \mathbb{R}^n$  with  $\|y_0\| < \delta$  we have  $\lim_{k \rightarrow \infty} \|y_k\| = 0$

**Definition of instability:**

$y = 0$  is unstable if it's not stable !

- (d) We claim that the necessary and sufficient condition for asymptotic stability of the origin is  $|\lambda_i| < 1$  for  $i = 1, 2, \dots, n$

Sufficient: From (c) any solution of  $y_{k+1} = Ay_k$  can be written as:

$$y_{k+1} = \sum_{i=1}^n c_i \lambda_i^k s_i$$

Without loss of generality, we assume that the eigenvectors  $s_i$  are normalized, i.e.,  $\|s_i\| = 1 \forall i \in \{1, 2, \dots, n\}$ . Then

$$\|y_{k+1}\| \leq \sum_{i=1}^n |c_i| |\lambda_i|^k \|s_i\| = \sum_{i=1}^n |c_i| |\lambda_i|^k$$

But since  $|\lambda_i| < 1$ , we have  $\lim_{k \rightarrow \infty} |\lambda_i|^k = 0$ . Which implies

$$\lim_{k \rightarrow \infty} \sum_{i=1}^n |c_i| |\lambda_i|^k = 0$$

Hence,

$$\lim_{k \rightarrow \infty} \|y_{k+1}\| = 0 \quad (4)$$

Also note that since  $|\lambda_i| < 1 \forall i \in \{1, \dots, n\}$ , the matrix norm  $\|A\| < 1$ .

Hence  $\|y_{k+1}\| = \|Ay_k\| < \|y_k\| \implies y = 0$  is also stable. This together with (4) implies asymptotic stability of the fixed point  $y = 0$ .

Necessity: Assume there is  $i_0 \in \{1, 2, \dots, n\}$  such that  $|\lambda_{i_0}| \geq 1$ .

It is enough to show that  $\exists y_0 \in \mathbb{R}^n$  such that  $\lim_{k \rightarrow \infty} \|A^k y_0\| \neq 0$

[This is due to the fact that  $y_k = A^k y_0$  and that one can rescale  $y_0$  as  $r y_0$  for  $0 < r \ll 1$  small enough such that  $\|r y_0\| < \delta, \forall \delta > 0$ ]

To show that such  $y_0 \in \mathbb{R}^n$  exists, note that  $\|A^k s_{i_0}\| = \|\lambda_{i_0}^k s_{i_0}\| = |\lambda_{i_0}|^k \geq 1 \forall k \geq 0$ .

This is, however, not enough since  $s_{i_0} \in \mathbb{C}^n$  while we need a vector in  $\mathbb{R}^n$ .

To complete the proof, note that  $s_{i_0} = \xi + i\eta$  with  $\xi, \eta \in \mathbb{R}^n$ .

$$\implies 1 \leq \|A^k s_{i_0}\|^2 = \|A^k \xi + i A^k \eta\|^2 = \|A^k \xi\|^2 + \|A^k \eta\|^2$$

Therefore, either  $\|A^k \xi\| \geq \frac{1}{\sqrt{2}}$  or  $\|A^k \eta\| \geq \frac{1}{\sqrt{2}}$ .

Without loss of generality assume  $\|A^k \xi\| \geq \frac{1}{\sqrt{2}}$ . Now let  $y_0 = \xi$  to get

$$\underbrace{\|y_k\| = \|A^k y_0\| \geq \frac{1}{\sqrt{2}}}_{\text{true for every } k \geq 0} \implies \lim_{k \rightarrow \infty} \|y_k\| \neq 0$$

### Question 3

The first three modes of a convection fluid motion in a two-dimensional layer heated from below are described by the famous *Lorenz equations*

$$\begin{aligned}\dot{x} &= a(y - x), \\ \dot{y} &= bx - y - xz, \\ \dot{z} &= xy - cz.\end{aligned}$$

Here  $a > 0$  denotes the Prandtl number,  $b > 0$  is the Rayleigh number, and  $c > 0$  is the aspect ratio. Lorenz's original assumption is that  $a > 1 + c$ .

The above equations describe the evolution of the amplitudes of one velocity mode and two temperature modes. As a paradigm of chaotic dynamics, this system has inspired much of the development of the modern geometric theory of dynamical systems.

- (a) Complicated dynamics arise when all possible equilibria of the system become unstable, and hence solutions cannot settle down to any simple steady state. Show that this is the case when

$$b > \frac{a(3 + a + c)}{a - c - 1}$$

*Note:* Note: A negative sign for one of the Routh-Hurwitz determinants actually implies instability, not just the lack of asymptotic stability.

- (b) Solve the Lorenz equations numerically for  $a = 10$ ,  $b = 28$ , and  $c = 8/3$ , choosing an initial condition close to  $x = y = z = 0$ . Plot the trajectory in three dimensions to show that it converges to a complicated surface, a *chaotic attractor*.

### Solution 3

- (a) By setting  $f(x) = 0$ , we obtain three fixed points for  $\dot{x} = f(x)$ . This can be seen by noting that the first equation  $a(y_0 - x_0) = 0$  implies  $x_0 = y_0$ . From the second equation, we get

$$\begin{aligned}bx_0 - y_0 - x_0z_0 &= 0 \\ z_0 &= b - 1.\end{aligned}$$

Then the third equation reduces to

$$\begin{aligned}x_0y_0 - cz_0 &= 0 \\ x_0 = y_0 &= \pm\sqrt{c(b-1)}.\end{aligned}$$

The resulting three fixed points are

$$\begin{aligned}P_1 : x_0 &= y_0 = z_0 = 0 \\ P_2 : x_0 &= y_0 = \sqrt{c(b-1)}, z_0 = b - 1 \\ P_3 : x_0 &= y_0 = -\sqrt{c(b-1)}, z_0 = b - 1.\end{aligned}$$

For the system to have these three fixed points we must have  $\boxed{b > 1}$ . If  $0 < b \leq 1$ , the only fixed point is  $P_1$ .

Let  $A$  denote  $Df(x_0, y_0, z_0)$ . Then:

$$A = \begin{pmatrix} -a & a & 0 \\ b - z_0 & -1 & -x_0 \\ y_0 & x_0 & -c \end{pmatrix}$$

The eigenvalues  $\lambda$  of  $A$  are defined as the roots of the characteristic polynomial

$$\det |A - \lambda I| = 0.$$

For the matrix  $A$  this takes the form

$$A = \begin{vmatrix} -a - \lambda & a & 0 \\ b - z_0 & -1 - \lambda & -x_0 \\ y_0 & x_0 & -c - \lambda \end{vmatrix} = 0.$$

We may expand this determinant according to the first row as

$$\begin{aligned} (-a - \lambda) \begin{vmatrix} -1 - \lambda & -x_0 \\ x_0 & -c - \lambda \end{vmatrix} - a \begin{vmatrix} b - z_0 & -x_0 \\ y_0 & -c - \lambda \end{vmatrix} &= 0 \\ - (a + \lambda) [(1 + \lambda)(c + \lambda) + x_0^2] - a(b - z_0)(-c - \lambda) - ax_0y_0 &= 0 \end{aligned}$$

After collecting the coefficients of the different powers of  $\lambda$ , the characteristic equation of  $A$  is:

$$\lambda^3 + (a + c + 1)\lambda^2 + [ac + a + c + x_0^2 + a(z_0 - b)]\lambda + ac(z_0 - b + 1) + x_0^2a + ax_0y_0 = 0$$

**Stability of  $P_1$ :**

$$\lambda_3 + (a + c + 1)\lambda^2 + (ac + a + c - ab)\lambda - ac(b - 1) = 0$$

A necessary condition for all roots of the above polynomial to be negative is that all its coefficients have the same sign. But here  $-ac(b - 1) < 0$  while  $\lambda^3$  has a positive coefficient (i.e.  $+1$ ).  $\implies A$  has a positive eigenvalue.

$\implies P_1$  is unstable.

**Stability of  $P_2, P_3$ :**

Note that in the characteristic equation, we only have products of  $x_0$  and  $y_0$ , i. e.  $x_0^2$  and  $x_0y_0$ . This means that the equation is invariant to changing the sign of  $x_0$  and  $y_0$  and we get the same eigenvalues at  $P_2$  and at  $P_3$ .

The Routh-Hurwitz determinants are:

$$\begin{aligned} d_1 &= 2ac(b - 1) > 0 \\ d_2 &= (a + b)c > 0 \\ d_3 &= \begin{vmatrix} (a + b)c & 2ac(b - 1) \\ 1 & a + c + 1 \end{vmatrix} = (a + b)(a + c + 1)c - 2ac(b - 1) \end{aligned}$$

For  $P_2$  and  $P_3$  to be unstable, we must have  $d_3 < 0$

$$d_3 < 0 \iff b > \frac{a(3 + a + c)}{a - (c + 1)} \quad \underbrace{\qquad}_{\text{follows from } a > c + 1} > 1$$

```
(b)      %% Initiate Script
2        close all
3        clear all
4        clc
5
6        %% Params & Initial Condition
7
8        a = 10;
9        b = 28;
10       c = 8/3;
11
12       x0 = [0.101; 0.1; 0.1];
13
14       %% Function & Simulation
15
```

```
16     f = @(t,x) [a * (x(2) - x(1));
17                 b * x(1) - x(2) - x(1) * x(3);
18                 x(1) * x(2) - c * x(3)];
19
20     [t ,x] = ode45(f, [0, 1000], x0);
21
22     %% Plot
23
24     figure(1)
25     hold on
26     plot3(x(:,1), x(:,2),x(:,3))
```



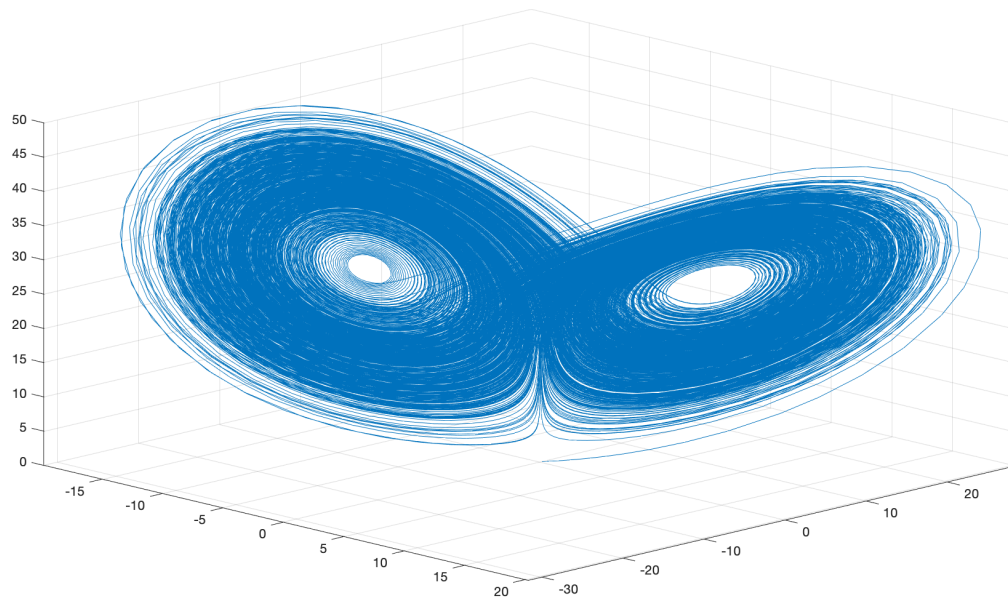


Figure 2: Simulation of the Lorenz equations.

## Question 4

Recall from Question 1 that a ball of mass  $m$  sliding on a hoop rotating with angular velocity  $\Omega$  satisfies the differential equation

$$mR^2\ddot{\alpha} + bR^2\dot{\alpha} + mR^2(g/R - \Omega^2 \cos(\alpha)) \sin(\alpha) = 0 \quad (5)$$

if there is no friction between the hoop and the mass. Assume that the parameter values are such that the lower equilibrium position of the ball is stable in linear approximation.

- (a) Show that in this case, the equilibrium is also nonlinearly stable.

*Hint:* Note that system (5) is not conservative: an external torque is required to maintain the constant rotation of the hoop. As a result, the total mechanical energy of the system is not expected to work as a Lyapunov function. To find another candidate for a Lyapunov function, find a quantity that *is* conserved along trajectories. Such a quantity can be found by multiplying equation (5) by  $\dot{\alpha}$  and integrating in time.

- (b) Prove that the equilibrium *cannot* be asymptotically stable for the nonlinear system.

*Hint:* use the Lyapunov function you have found in (a).

## Solution 4

$$mR^2\ddot{\alpha} + mR^2[g/R - \Omega^2 \cos(\alpha)] \sin(\alpha) = 0$$

From the previous assignment, we know that the lower equilibrium is unstable when  $\Omega^2 > g/R$ . Hence, in the following we assume

$$\Omega^2 < \frac{g}{R} \quad (6)$$

- (a) Multiplying the equation of motion by  $\dot{\alpha}$ , we find that

$$\frac{d}{dt} \left( \frac{1}{2} \dot{\alpha}^2 - \frac{g}{R} \cos(\alpha) + \frac{1}{4} \Omega^2 \cos(2\alpha) \right) = 0 \quad (7)$$

Let  $x_1 := \alpha$ ,  $x_2 := \dot{\alpha}$ . Equation (7) implies that the function

$$V(x_1, x_2) = \frac{1}{2} x_2^2 + \frac{g}{R} (1 - \cos(x_1)) + \frac{\Omega^2}{4} (\cos(2x_1) - 1)$$

is constant along trajectories, i.e.  $\frac{d}{dt} V(x_1(t), x_2(t)) = 0$ .

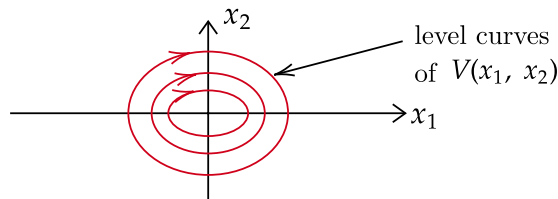
Moreover,  $V(0, 0) = 0$ . On the other hand:

$$\nabla V(0, 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \nabla^2 V(0, 0) = \begin{pmatrix} \frac{g}{R} - \Omega^2 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows from (6) that  $\nabla^2 V(0, 0)$  is positive definite.

$\implies$  By Lyapunov's direct method, the lower equilibrium is stable.

- (b) The fixed point  $(0, 0)$  cannot be asymptotically stable since the trajectories of the system coincide with level curves of  $V(x_1, x_2)$ . since  $\frac{dV}{dt} = 0$  along trajectories. But the above analysis shows that around  $(0, 0)$  the level curves of  $V$  are closed curves:



## Question 5

Consider the damped pendulum equation

$$\ddot{x} + c\dot{x} + \sin(x) = 0. \quad (8)$$

- (a) Using the energy of the pendulum as a Lyapunov function, can we conclude the nonlinear asymptotic stability of the  $x = 0$  equilibrium? Give detailed reasoning why.
- (b) A theorem due to Krasovski states the following: Assume that  $x = 0$  is a fixed point for the  $n$ -dimensional dynamical system  $\dot{x} = f(x)$ . Assume that there exists a smooth scalar function  $V(x)$  such that
- (i)  $V(x)$  is positive definite on an open neighborhood  $U$  of  $x = 0$
  - (ii)  $\dot{V}$  is negative semi-definite on the same neighborhood
  - (iii) the only trajectory lying *completely* in the set  $S = \{x \in U : \dot{V} = 0\}$  is the fixed point  $x = 0$ . Then  $x = 0$  is asymptotically stable.

Use Krasovski's theorem to conclude the asymptotic stability of the origin for (8).

## Solution 5

(a)

$$E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + (1 - \cos(x))$$

$$y = (y_1, y_2) := (x, \dot{x})$$

$$y = f(y) = \begin{bmatrix} y_2 \\ -cy_2 - \sin(y_1) \end{bmatrix} \implies E(y) = \frac{1}{2}y_2^2 + (1 - \cos(y_1))$$

(i)

$$E(0) = 0, DE(0) = 0, D^2E(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\implies$  Hessian is positive definite.

$\implies E$  is positive-definite near the origin

(ii)

$$\begin{aligned} \dot{E}(y) &= \langle DE(y), f(y) \rangle = (\sin(y_1), y_2) \cdot (y_2, -cy_2 - \sin(y_1)) \\ &= \sin(y_1)y_2 - cy_2^2 - \sin(y_1)y_2 \\ &= -cy_2^2 \leq 0 \end{aligned}$$

$E$  is positive definite around the origin and  $\dot{E}$  is negative semi-definite.

Indeed, we cannot find an open set  $U$  around the origin where

$$\dot{E}(y) < 0 \quad \forall y \in U \setminus \{0\} \quad [\dot{E}(y) = 0 \text{ for any } y = (y_1, 0) \text{ with } y_1 \neq 0]$$

Thus, theorem 2 is not applicable to conclude nonlinear asymptotic stability of the origin.

- (b) We use Krasovski's theorem with  $V = E, U \subset (-\pi, \pi) \times \mathbb{R}$  open set around the origin in  $S' \times \mathbb{R}$  such that the statements (i) & (ii) in the hypothesis of Krasovski are satisfied as shown above in part a).

$$S = \{y \in U | \dot{E}(y) = 0\} \subset \underbrace{\{(y_1, 0) | y_1 \in (-\pi, \pi)\}}_{\tilde{S}} \quad (9)$$

Indeed, the only trajectory of the system completely contained in the set  $\tilde{S}$  on the  $y_1$ -axis is the origin (cf. phase portrait).  $\implies S$  contains only the fixed point as a trajectory of the system.

Hence, the hypothesis of Krasovski's theorem is satisfied and the origin is asymptotically stable for the nonlinear damped pendulum.

## Question 6

Consider an  $n$ -degree-of-freedom holonomic mechanical system (i.e. one that has only position-dependent constraints) with generalized coordinates  $q \in \mathbb{R}^n$  and generalized velocities  $\dot{q} \in \mathbb{R}^n$ . The total energy of the system is of the form

$$E(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} + V(q),$$

where  $M \in \mathbb{R}^{n \times n}$  is the mass matrix (symmetric and positive definite), and  $V(q)$  is the potential energy. In the absence of external forces, the associated Lagrangian equations of motion are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0,$$

where  $L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} + V(q)$  is the Lagrangian of the mechanical system.

Show that if  $V(q)$  admits a strict local minimum at a point  $q_0$ , then  $q_0$  is a (nonlinearly) stable equilibrium for the mechanical system. (This result is also known as *Dirichlet's Theorem* in classical mechanics).

## Solution 6

First construct the function:

$$\begin{aligned} \bar{E}(q, \dot{q}) &= E(q, \dot{q}) - V(q_0) \\ &= \frac{1}{2} \dot{q}^T M \dot{q} + V(q) - V(q_0) \end{aligned}$$

Now, at  $(q, \dot{q}) = (q_0, 0)$  we have  $\bar{E}(q_0, 0) = 0$

Note that  $M(q)$  is positive definite for all  $q$  and  $V(q) - V(q_0)$  is positive around  $q = q_0$ . (Since  $V$  has a local minimum at  $q_0$ )

$\implies \bar{E}(q, \dot{q})$  is positive definite around  $(q_0, 0)$ .

But  $\frac{d\bar{E}}{dt} = \frac{dE}{dt}$  since  $V(q_0)$  is a constant.

We show that  $\frac{dE}{dt} = 0$ .

First note that, in general, the Lagrangian equation of motion is a system of  $n$  coupled equations with each equation given by:

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{q}_k} \right] - \frac{\partial L}{\partial q_k} = 0 \quad k = 1, 2, \dots, n$$

Multiply each equation by  $\dot{q}_k$  and sum over  $k$  to get:

(We'll use Einstein's notation: sum over repeated indices)

$$\dot{q}_k \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} \right] = 0 \quad (10)$$

Since  $L = \frac{1}{2} M_{ij} \dot{q}_i \dot{q}_j - V$  we have:

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}_k} &= M_{ik} \dot{q}_i, \quad \frac{\partial L}{\partial q_k} = \frac{1}{2} \frac{\partial M_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial V}{\partial q_k} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} &= M_{ik} \ddot{q}_i + \frac{\partial M_{ik}}{\partial q_j} \dot{q}_j \dot{q}_i \end{aligned} \quad (11)$$

Substituting (11) into (10), we get

$$M_{ik} \ddot{q}_i \dot{q}_k + \frac{\partial M_{ik}}{\partial q_j} \dot{q}_j \dot{q}_i \dot{q}_k - \frac{1}{2} \frac{\partial M_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j \dot{q}_k + \frac{\partial V}{\partial q_k} \dot{q}_k = 0$$

Since there is a sum over repeated indices we have:

$$M_{ik} \ddot{q}_i \dot{q}_k \equiv M_{ij} \ddot{q}_i \dot{q}_j \quad \frac{\partial M_{ik}}{\partial q_j} \dot{q}_j \dot{q}_i \dot{q}_k \equiv \frac{\partial M_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j \dot{q}_k$$

$$\begin{aligned} \implies \underbrace{M_{ij}\ddot{q}_i\dot{q}_j + \frac{1}{2}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j\dot{q}_k + \frac{\partial V}{\partial q_k}\dot{q}_k}_{= \frac{d}{dt} \left[ \frac{1}{2}M_{ij}\dot{q}_i\dot{q}_j + V(q) \right]} &= 0 \\ \implies \left[ \frac{1}{2}\dot{q}^T M(q)\dot{q} + V(q) \right] &= 0 \implies \frac{dE}{dt} = 0 \implies \frac{d\bar{E}}{dt} = 0 \end{aligned} \tag{12}$$

Using  $\bar{E}$  as the Lyapunov function, we conclude that  $(q_0, 0)$  is a stable equilibrium point.

## Solution 7

For what values of  $a$  and  $b$  does the dynamical system  $\dot{x} = Ax$  have a center fixed point at the origin ?

$$A = \begin{pmatrix} a & -b \\ b & 2 \end{pmatrix}$$

- (a)  $a = -2, b = 2$
- (b)  $a = \frac{1}{2}, b = 2$
- (c)  $a = -2, b = -4$
- (d)  $a = \frac{1}{2}, b = -4$

Compute the eigenvalues of  $A$  and check for which values of  $a, b$  we have  $\operatorname{Re}(\lambda_i) = 0$ .

## Solution 8

Consider the dynamical system

$$\ddot{\varphi} + a\dot{\varphi} + (b - \omega_0^2 \cos(\varphi)) \sin(\varphi) = 0$$

where  $a, b, \omega_0$  are positive constants. Which of the following is not correct?

- (a) If  $\omega_0^2 < b$ , then  $(0, 0)$  is an asymptotically stable fixed point.
- (b) If  $\omega_0^2 > b$ , then  $(0, 0)$  is a (Lyapunov) unstable fixed point.
- (c) For any  $a, b, \omega_0$ ,  $(\pi, 0)$  is an unstable fixed point.
- (d) For any  $a, b, \omega_0$ ,  $(0, 0)$  is a (Lyapunov) stable fixed point.

$a, b, c$  are correct by linearization. This implies that (d) is incorrect.

## Solution 9

Consider a dynamical system of the form

$$\ddot{y} + p(t)\dot{y} + q(t)y = 0 \tag{13}$$

with  $C^1$  functions  $p(t)$  and  $q(t)$ . The transformation

$$x(t) = y(t)e^{-\frac{1}{2} \int_{t_0}^t p(s) ds}$$

puts this system in the form

$$\ddot{x} + \omega(t)x = 0 \tag{14}$$

For an appropriate  $C^1$  function  $\omega(t)$ . Does the Lyapunov stability of  $x = 0$  in (14) imply the Lyapunov stability of  $y = 0$  in (13)?

- (a) Yes, the two systems are topologically equivalent.
- (b) Only for a large enough  $t$ .
- (c) Yes, when  $p(t) < 0$  for all  $t \in \mathbb{R}$ .
- (d) None of the above