Nonlinear Dynamics & Chaos I

Exercice Set 2 Solutions

Question 1

Consider the nonlinear oscillator

$$\ddot{x} + \omega_0^2 x = \varepsilon M x^2,$$

where $\varepsilon M x^2$ represents a small nonlinear forcing term ($0 \leq \varepsilon \ll 1, M > 0$).
Using Lindstedt's method, find an $\mathcal{O}(\varepsilon)$ approximation for nonlinear motions as a function of their initial position, with zero initial velocity.

Solution 1

$$\begin{aligned}
\ddot{x} + \omega_0^2 x &= \varepsilon M x^2, \quad 0 < \varepsilon \ll 1, \quad M > 0, \quad \omega_0 \neq 0 \\
x(0) &= a_0 \\
\dot{x}(0) &= 0
\end{aligned}$$

- Seek solutions of the form:

$$\begin{aligned}
x_\varepsilon(t) &= \varphi_0(t;\varepsilon) + \varepsilon\varphi_1(t;\varepsilon) + \mathcal{O}(\varepsilon^2) \\
\varphi_i(t,\varepsilon) &= \varphi_i(t + T_\varepsilon;\varepsilon)
\end{aligned}$$

- rewrite period as

$$T_\varepsilon = \frac{2\pi}{\omega(\varepsilon)}, \quad \omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + \mathcal{O}(\varepsilon^2)$$

- Rescale time:

$$\tau = \omega(\varepsilon)t \Longrightarrow \boxed{\frac{d}{d\tau} = \frac{1}{\omega(\varepsilon)}\frac{d}{dt}} \Longrightarrow \boxed{(\omega(\varepsilon))^2 x'' + \omega_0^2 x = \varepsilon M x^2} \tag{1}$$

- Plug in the new Ansatz into the rescaled equation

$$[\omega_0^2 + 2\varepsilon\omega_0\omega_1 + \mathcal{O}(\varepsilon^2)][\varphi_0'' + \varepsilon\varphi_1'' + \mathcal{O}(\varepsilon^2)] + \omega_0^2[\varphi_0 + \varepsilon\varphi_1 + \mathcal{O}(\varepsilon^2)] = \varepsilon M[\varphi_0^2 + \mathcal{O}(\varepsilon)]$$

- Collect terms of equal power of ε

  $\mathcal{O}(1)$:

$$\begin{aligned}
\omega_0^2\varphi_0'' + \omega_0^2\varphi_0 = 0, &\quad \varphi_0(0) = a_0, \quad \dot{\varphi}_0(0) = 0 \\
\Longrightarrow \varphi_0 &= a_0\cos(\tau)
\end{aligned}$$

  $\mathcal{O}(2)$:

$$\begin{aligned}
\omega_0^2\varphi_1'' + \omega_0^2\varphi_1 &= M\varphi_0^2 - 2\omega_0\omega_1\varphi_0'' = M a_0^2\cos^2(\tau) + 2a_0\omega_0\omega_1\cos(\tau) \\
&= M\frac{a_0^2}{2}[1 + \cos(2\tau)] + \underbrace{2a_0\omega_0\omega_1\cos(\tau)}_{\text{resonance}}
\end{aligned}$$

  Select $\omega_1 = 0$ to eliminate resonance terms and obtain periodic solution.
  Solve for φ_1:

$$\varphi_1'' + \varphi_1 = \frac{M a_0^2}{2\omega_0^2}[1 + \cos(2\tau)], \quad \varphi_1(0) = 0, \quad \dot{\varphi}_1(0) = 0 \tag{2}$$

- Pick solution Ansatz:

$$\varphi_1(\tau) = A\cos(\tau) + B\sin(\tau) + C\cos(2\tau) + D\sin(2\tau) + E$$

- Substituting in (2):

$$\begin{aligned}-A\cos(\tau) - B\sin(\tau) - 4C\cos(2\tau) - 4D\sin(2\tau) + A\cos(\tau) + B\sin(\tau) + C\cos(2\tau) + D\sin(2\tau) + E \\ = \frac{Ma_0^2}{2\omega_0^2}\cos(2\tau) + \frac{Ma_0^2}{2\omega_0^2}\end{aligned}$$

- Comparing coefficients:

$$\implies E = \frac{Ma_0^2}{2\omega_0^2}, \quad C = -\frac{Ma_0^2}{6\omega_0^2}, \quad D = 0$$

$$\begin{aligned}\varphi_1(0) = 0 &\implies A + C + E = 0 \implies A = -\frac{Ma_0^2}{3\omega_0^2} \\ \varphi_1'(0) = 0 &\implies B + 2D = 0 \implies B = 0\end{aligned}$$

$$\implies \boxed{\varphi_1(\tau) = -\frac{Ma_0^2}{3\omega_0^2}\cos(\tau) - \frac{Ma_0^2}{6\omega_0^2}\cos(2\tau) + \frac{Ma_0^2}{2\omega_0^2}}$$

- In original time:

$$x_\varepsilon(t) = a_0\cos(\omega t) + \varepsilon\frac{Ma_0^2}{\omega_0^2}\left[-\frac{1}{3}\cos(\omega t) - \frac{1}{6}\cos(2\omega t) + \frac{1}{2}\right] + \mathcal{O}(\varepsilon^2)$$

where

$$\omega = \omega_0 + \mathcal{O}(\varepsilon^2)$$

Question 2

Consider the forced *van der Pol equation*

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = F\cos(\omega t),$$

which arises in models of self-excited oscillation, such as those of a valve generator with a cubic valve characteristic. Here $F, \omega > 0$ are parameters, and $0 \le \varepsilon \ll 1$.

(i) For small values of ε, find an approximation for an **exactly** $2\pi/\omega$-periodic solution of the equation. The error of your approximation should be $\mathcal{O}(\varepsilon)$.

(ii) For $\varepsilon = 0.1,\ \omega = 2$, and $F = 1$, verify your prediction numerically by solving the equation numerically. Plot your numerical solution along with your analytic prediction computed in (i).

Note: For chaotic dynamics in the forced van der Pol equation, see Section 2.1 of *Guckenheimer & Holmes*.

Solution 2

(i) Seek solutions of the form:

$$x_\varepsilon(t) = \varphi_0(t) + \varepsilon\varphi_1(t) + \mathcal{O}(\varepsilon^2)$$

Substituting this solution in the ODE $\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = F\cos(\omega t)$ we get:

$$\ddot{\varphi}_0 + \varphi_0 + \varepsilon(\ddot{\varphi}_1 + \varphi_1 + \varphi_0^2\dot{\varphi}_0 - \dot{\varphi}_0) + \mathcal{O}(\varepsilon^2) = F\cos(\omega t)$$

$$\Longrightarrow \mathcal{O}(1) : \ddot{\varphi}_0 + \varphi_0 = F\cos(\omega t) \tag{3}$$

$$\Longrightarrow \mathcal{O}(2) : \ddot{\varphi}_1 + \varphi_1 = \dot{\varphi}_0(1 - \varphi_0^2) \tag{4}$$

Since we seek solutions with period $\frac{2\pi}{\omega}$ for any $0 \le \varepsilon \ll 1$, the period of each φ_i must be $\frac{2\pi}{\omega}$. Because the initial condition for the system was not specified we will use those to enforce the periodicity condition. That is, we assume that the initial conditions also depend on ε.

$$x_\varepsilon(0) = a + b\varepsilon + O(\varepsilon^2)$$
$$\dot{x}_\varepsilon(0) = c + d\varepsilon + O(\varepsilon^2)$$

The solution that is $O(\varepsilon)$ accurate, can be obtained as a solution to equation (3). We take the following trial function

$$\varphi_0(t) = A\sin(t) + B\cos(t) + C\cos(\omega t). \tag{5}$$

The constant C can be determined by substituting into (3) as

$$-C\omega^2\cos(\omega t) + C\cos(\omega t) = F\cos(\omega t).$$

Since this has to hold for all t, the coefficients of $\cos(\omega t)$ must balance. This leads to $C = F/(1-\omega^2)$. Therefore, the solution $\varphi_0(t)$ reads as

$$\varphi_0(t) = A\sin(t) + B\cos(t) + \frac{F\cos(\omega t)}{1-\omega^2}. \tag{6}$$

To have a period $\frac{2\pi}{\omega}$, we must select $A = B = 0$. This condition can be enforced by choosing appropriate initial conditions for the ODE (3):

$$A = B = 0 \Longrightarrow a = C = \frac{F}{1-\omega^2} \quad d = 0. \tag{7}$$

$$\boxed{\varphi_0(t) = \frac{F\cos(\omega t)}{1-\omega^2}, \quad \varphi_0(0) = \frac{F}{1-\omega^2}, \quad \dot{\varphi}_0(0) = 0} \tag{8}$$

$$x_\varepsilon(t) = \varphi_0(t) + \underbrace{\mathcal{O}(\varepsilon)}_{\text{error term}}$$

Bonus: $O(\varepsilon^2)$ *accurate calculation*

We can proceed by solving the $O(\varepsilon)$ equation (4). With the solution (8) the forcing term on the right-hand side can be written as

$$\begin{aligned}
\dot{\varphi}_0(1-\varphi_0^2) &= -C\omega\sin(\omega t)\left(1-C^2\cos^2(\omega t)\right) && (9)\\
&= -C\omega\sin(\omega t)\left(1-C^2+C^2\sin^2(\omega t)\right) && (10)\\
&= -C\omega\sin(\omega t)(1-C^2)-C^3\omega\sin^3(\omega t) && (11)\\
&= -C\omega(1-C^2)\sin(\omega t)-C^3\omega\frac{3\sin(\omega t)-\sin(3\omega t)}{4} && (12)\\
&= C\omega\left(\frac{C^2}{4}-1\right)\sin(\omega t)+\frac{C^3\omega}{4}\sin(3\omega t). && (13)
\end{aligned}$$

Equation (4) now takes the form

$$\ddot{\varphi}_1+\varphi_1 = C\omega\left(\frac{C^2}{4}-1\right)\sin(\omega t)+\frac{C^3\omega}{4}\sin(3\omega t); \quad \varphi_1(0)=b \quad \dot{\varphi}_1(0)=d. \tag{14}$$

The appropriate trial functions is

$$\varphi_1(t) = D\cos t + E\sin t + G\sin(\omega t) + H\sin(3\omega t). \tag{15}$$

Upon substitution we find the following equation

$$-G\omega^2\sin(\omega t)-9H\omega^2\sin(3\omega t)+G\sin(\omega t)+H\sin(3\omega t) = C\omega\left(\frac{C^2}{4}-1\right)\sin(\omega t)+\frac{C^3\omega}{4}\sin(3\omega t).$$

Balancing the coefficients of $\sin(\omega t)$ and $\sin(3\omega t)$ we end up with

$$\begin{aligned}
G &= \frac{C\omega}{(1-\omega^2)}\left(\frac{C^2}{4}-1\right)\\
H &= \frac{C^3\omega}{4(1-9\omega^2)}.
\end{aligned}$$

To enforce periodicity with period $T=\frac{2\pi}{\omega}$ to order ε we must have $D=E=0$. This can be achieved by choosing the initial conditions of the φ_1 equation appropriately. This means we must have

$$\begin{aligned}
\varphi_1(0) &= \boxed{b=0} && (16)\\
\dot{\varphi}_1(0) &= \boxed{G\omega+3H\omega=d}. && (17)
\end{aligned}$$

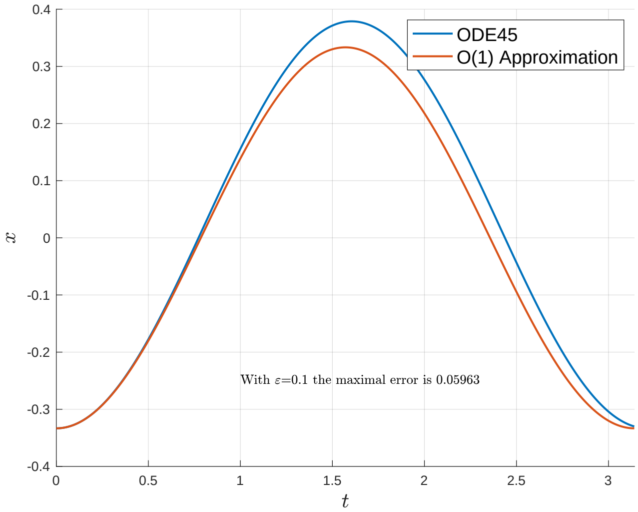
Combining the solution $\varphi_1(t)$ with (8) we finally obtain, that the T-periodic solution is given by

$$\begin{aligned}
x_\varepsilon(t) &= \frac{F}{1-\omega^2}\cos(\omega t)+\varepsilon\frac{F\omega}{(1-\omega^2)^2}\left(\frac{F^2}{4(1-\omega^2)^2}-1\right)\sin(\omega t) && (18)\\
&+\varepsilon\frac{F^3\omega}{4(1-\omega^2)^3(1-9\omega^2)}\sin(3\omega t) && (19)\\
x_\varepsilon(0) &= \frac{F}{1-\omega^2}+O(\varepsilon^2) && (20)\\
\dot{x}_\varepsilon(0) &= \varepsilon\left[\frac{F\omega^2}{(1-\omega^2)^2}\left(\frac{F^2}{4(1-\omega^2)^2}-1\right)+3\frac{F^3\omega^2}{4(1-\omega^2)^3(1-9\omega^2)}\right]+O(\varepsilon^2) && (21)
\end{aligned}$$

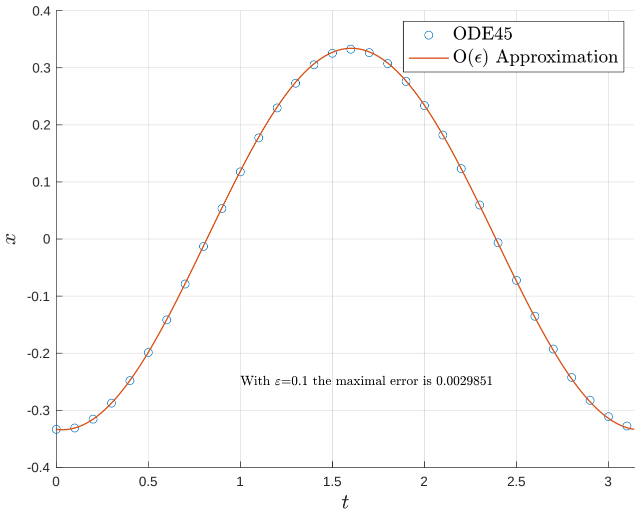
(ii) We solve the ODE numerically to obtain a solution $x(t)$ and compare this solution to the perturbed approximation $x_\varepsilon(t)$ given by (8) up to $O(\varepsilon)$.

The initial conditions for the ODE are chosen such that $x(0) = x_\varepsilon(0)$ and $\dot{x}(0) = \dot{x}_\varepsilon(0)$. Therefore, $x(0) = \varphi_0(0), \quad \dot{x}(0) = \dot{\varphi}_0(0)$ where $\varphi_0(0)$ and $\dot{\varphi}_0(0)$ are given in (7).
Equivalent first order system of differential equations:

$$z_1 = x, \quad z_2 = \dot{x}, \quad \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ F\cos(\omega t) - \varepsilon(z_1^2 - 1)z_2 - z_1 \end{bmatrix} \tag{22}$$



(iii) We now solve the same ODE (22) numerically to obtain a solution $x(t)$ started from the $O(\varepsilon^2)$ accurate initial conditions given by (16) and compare this solution to refined perturbed approximation $x_\varepsilon(t)$ given by (18) up to $O(\varepsilon^2)$.



MATLAB code

```
%% Initiate Script

close all
clear all
clc

%% define parameters

epsilon = 0.1;
omega = 2;
```

```
F = 1;

% initial condition
t0 = [F / (1 - omega.^2), 0]';

% time steps
tt_approx = 0:0.01:pi;
tt_sim = tt_approx;

%% Function and simulation

fun = @(t,x) [x(2); F*cos(omega * t) -  epsilon * (x(1).^2 - 1).*x(2) - x(1)];

opts = odeset('RelTol',1e-4,'AbsTol',1e-6);
[~ , xtrue] = ode45(fun, tt_sim, t0, opts);

%% Approximation

xApprox = F * cos(omega .* tt_approx) / (1 - omega.^2);

%% Plot results

figure(1)
hold on
plot(tt_sim, xtrue(:,1),'linewidth',1.5,'DisplayName','ODE45');
plot(tt_approx, xApprox,'linewidth',1.5,'DisplayName','O(1) Approximation');
xlabel('$t$','interpreter','Latex','FontSize',16)
ylabel('$x$','interpreter','Latex','FontSize',16)
legnd1 = legend;
legnd1.NumColumns = 1;
legnd1.FontSize = 14;
xlim([0, pi])
hold off
grid on

%% Print the error

error = max( abs( xtrue(:,1)' - xApprox));
string_to_print = ['With $\varepsilon$=', num2str(epsilon), ' the maximal error is ', num2str(error)];
text(1, -0.25, string_to_print, 'interpreter','Latex');

%%
C = F/(1-omega.^2);
H = omega*C*(C^2/4 -1)/(1-omega^2);
J = C^3*omega /(4*(1-9*omega^2));
xd01 = epsilon * (H*omega + 3*J*omega);
t1 = t0 + [0; xd01];
xApprox = F * cos(omega .* tt_approx) / (1 - omega.^2) + epsilon * H * sin(omega*tt_approx);
xApprox = xApprox + epsilon * J * sin(3*omega*tt_approx);

fun = @(t,x) [x(2); F*cos(omega * t) -  epsilon * (x(1).^2 - 1).*x(2) - x(1)];

opts = odeset('RelTol',1e-4,'AbsTol',1e-6);
[~ , xtrue] = ode45(fun, tt_sim, t1, opts);



figure(2)
```

```
hold on
plot(tt_sim(1:10:end), xtrue(1:10:end,1),'o','DisplayName','ODE45');
plot(tt_approx, xApprox,'linewidth',1.0,'DisplayName','O($\epsilon$) Approximation');
xlabel('$t$','interpreter','Latex','FontSize',16)
ylabel('$x$','interpreter','Latex','FontSize',16)
legnd1 = legend('interpreter','Latex');
legnd1.NumColumns = 1;
legnd1.FontSize = 14;
xlim([0, pi])
hold off
grid on

error = max( abs( xtrue(:,1)' - xApprox));
string_to_print = ['With $\varepsilon$=', num2str(epsilon), ' the maximal error is ', num2str(error)];
text(1, -0.25, string_to_print, 'interpreter','Latex');
```