Nonlinear Dynamics and Chaos I.
Problem set 2

1. Consider the nonlinear oscillator
$$\ddot{x} + \omega_0^2 x = \varepsilon M x^2,$$
where $\varepsilon M x^2$ represents a small nonlinear forcing term ($0 \leq \varepsilon \ll 1$, $M > 0$)

   Using Lindstedt's method, find an $\mathcal{O}(\varepsilon)$ approximation for nonlinear periodic motions as a function of their initial position, with zero initial velocity.

2. Consider the forced *van der Pol equation*
$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = F\cos\omega t,$$
which arises in models of self-excited oscillation, such as those of a valve generator with a cubic valve characteristic. Here $F, \omega > 0$ are parameters, and $0 \leq \varepsilon \ll 1$.

   (i) For small values of ε, find an approximation for an **exactly** $2\pi/\omega$–periodic solution of the equation. The error of your approximation should be $\mathcal{O}(\varepsilon)$.

   (ii) For $\varepsilon = 0.1$, $\omega = 2$, and $F = 1$, verify your prediction numerically by solving the equation numerically. Plot your numerical solution along with your analytic prediction computed in (i).

   Note: For chaotic dynamics in the forced van der Pol equation, see Section 2.1 of *Guckenheimer & Holmes.*

3. Consider a ball of mass m that slides on a rotating hoop (see Fig. 1).

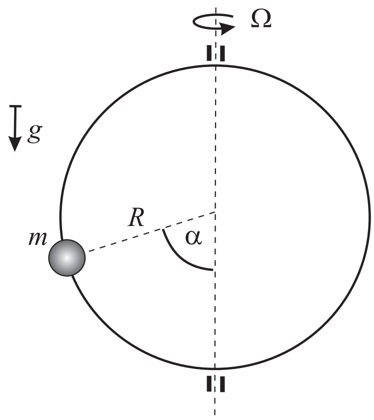

Figure 1: Mass on a loop

The angular velocity of the hoop is Ω, the viscous friction coefficient between the hoop and the ball is b, and the constant of gravity is g. The equation of motion for the sliding ball is given by
$$mR^2\ddot{\alpha} + bR^2\dot{\alpha} + mR^2\left(g/R - \Omega^2\cos\alpha\right)\sin\alpha = 0.$$

(a) Plot the location of equilibria of the ball as a function of the non-dimensionalized rotation parameter $\nu = R\Omega^2/g$.

(b) Using linearization, determine the stability type of the different equilibrium branches on the plot. Identify the critical angular velocity at which a bifurcation of equilibria occurs

4. Consider a discrete dynamical system given by the iterated mapping

$$x_{k+1} = f(x_k), \qquad f\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad x_k \in \mathbb{R}^n.$$

Assume that $x = p$ is a fixed point for the mapping, i.e., $p = f(p)$.

(a) Derive a linearized mapping of the form

$$y_{k+1} = Ay_k \tag{1}$$

to describe the discrete dynamics in the vicinity of the fixed point.

(b) Assume that A has eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with corresponding n linearly independent eigenvectors $s_1, \ldots, s_n \in \mathbb{C}^n$. Show that the general solution of (1) is of the form

$$y_k = c_1\varphi_1(k) + \ldots + c_n\varphi_n(k), \qquad \varphi_i(k) = \lambda_i^k s_i. \tag{2}$$

(c) Formulate a definition of stability, asymptotic stability, and instability for the $y = 0$ fixed point of (1).

(d) Using (2), find a sufficient and necessary condition for the asymptotic stability you have defined in (c).