Nonlinear Dynamics and Chaos I
Solution guide for Problem Set 5

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1. Consider a planar Hamiltonian system

$$\dot{x} = \frac{\partial H(x,y)}{\partial y} + f_1(x,y),$$

$$\dot{y} = -\frac{\partial H(x,y)}{\partial x} + f_2(x,y),$$

where the twice continuously differentiable function $H(x,y)$ is the Hamiltonian associated with the system (say, the energy in classical mechanics, or the stream function in fluid mechanics), and the continuously differentiable $\mathbf{f} = (f_1, f_2)$ is a dissipative term (say, damping in classical mechanics, or compressible terms in fluid mechanics). Assume that $\nabla \cdot \mathbf{f} \neq 0$ for all $(x,y) \in \mathbb{R}^2$. (Linear damping, for instance has this property.)

Show that the above system can have no limit cycles.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underline{F}(x,y) := \begin{bmatrix} \frac{\partial H}{\partial y}(x,y) + f_1(x,y) \\ -\frac{\partial H}{\partial x}(x,y) + f_2(x,y) \end{bmatrix} \qquad -(1)$$

$$\text{div}(\underline{F}) = \frac{\partial^2 H}{\partial x \partial y}(x,y) + \frac{\partial f_1}{\partial x}(x,y) - \frac{\partial^2 H}{\partial y \partial x}(x,y) + \frac{\partial f_2}{\partial y}$$

$$= \text{div}(\underline{f}) \qquad (H \in C^2)$$

$$\neq 0 \quad \forall \ (x,y) \in \mathbb{R}^2.$$

Thus, by the Benedixois criterion, (1) does not have a periodic solution in $\mathbb{R}^2$.

2. (*Accuracy of averaging*) Show that on times scales of $\mathcal{O}(1/\epsilon)$, a solution $\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ of the dynamical system

$$\dot{\mathbf{x}} = \epsilon \mathbf{f}(\mathbf{x}, t, \epsilon), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

(ϵ is a small parameter and $\mathbf{f}$ is a smooth function that is T-periodic in time) remains $\mathcal{O}(\epsilon)$-close to any solution $\mathbf{y}(t)$ with $\mathbf{y}(0) = \mathbf{x}_0 + \mathcal{O}(\epsilon)$ of the averaged system

$$\dot{\mathbf{y}} = \epsilon \bar{\mathbf{f}}_0(\mathbf{y}), \qquad \mathbf{y} \in \mathbb{R}^n, \tag{2}$$

where $\bar{\mathbf{f}}_0(\mathbf{y}) = \frac{1}{T} \int_0^T \mathbf{f}(y, t, 0)\, dt$.

Hint: Subtract (2) from (1) and integrate to obtain an expression for $|\mathbf{x}(t) - \mathbf{y}(t)|$. Estimate $|\mathbf{x}(t) - \mathbf{y}(t)|$ from above using the facts that $\bar{\mathbf{f}}$ is Lipschitz and $\left|\hat{f} - \bar{f}\right|/\epsilon$ is uniformly bounded, where $\hat{f}$ is the right-hand side of the system into wich (1) is transformed by the averaging transformation $\mathbf{x} = \mathbf{y} + \epsilon \mathbf{w}(\mathbf{y}, t)$. Then use the following generalized Gronwall inequality:

If $u(t)$, $v(t)$, $c(t)$ are nonnegative functions, $c(t)$ is differentiable, and

$$v(t) \le c(t) + \int_0^t u(s)\, v(s)\, ds,$$

then

$$v(t) \le c(0) e^{\int_0^t u(s)\, ds} + \int_0^t c'(s)\, e^{\int_s^t u(\tau)\, d\tau}\, ds.$$

Remember from the lecture on averaging that $\dot{x} = \epsilon f(x, t, \epsilon)$ can be transformed to the differential equation

$$\dot{\tilde{x}} = \epsilon \bar{f}_0(\tilde{x}) + \epsilon^2 f_1(\tilde{x}, t, \epsilon) \qquad (1)$$

through the near-identity transformation $x = \tilde{x} + \epsilon w(\tilde{x}, t)$.

Moreover, f_1 is globally bounded, i.e., there exists $L_1 > 0$ s.t.

$$|f_1(\tilde{x}, t, \epsilon)| < L_1 \qquad \forall\, t > 0 \text{ and } \forall\, \tilde{x} \in \mathbb{R}^n$$

Now by construction $|x(t) - \tilde{x}(t)| = \epsilon|w(\tilde{x}(t), t)| = O(\epsilon)$.

Therefore, it suffices to show that solutions of the averaged eq.

$$\dot{y} = \epsilon \bar{f}_0(y) \qquad (2)$$

remain $O(\epsilon)$ close to the solutions of (1).

Subtracting (2) from (1), integrating and dropping the tilde ($\sim$) sign, we get

$$x(t) - y(t) = x_0 - y_0 + \epsilon \int_0^t \left(\bar{f}_0(x(s)) - \bar{f}_0(y(s)) \right) ds + \epsilon^2 \int_0^t f_1(x(s), s, \epsilon) ds$$

$$\Rightarrow |x(t) - y(t)| \leq |x_0 - y_0| + \epsilon \int_0^t L_2 |x(s) - y(s)| ds + \epsilon^2 \int_0^t L_1 ds$$

where we used boundedness of f_1 and Lipschitz continuity of $\bar{f}_0$:

$$|f_0(x) - f_0(y)| \leq L_2 |x - y|$$

Therefore,

$$|x(t) - y(t)| \leq |x_0 - y_0| + \epsilon^2 L_1 t + \int_0^t \epsilon L_2 |x(s) - y(s)| ds \qquad (3)$$

Now apply Gronwall's inequality with $v(t) = |x(t) - y(t)|$, $u(t) = \epsilon L_2$ and $c(t) = |x_0 - y_0| + \epsilon^2 L_1 t$ to get

$$|x(t) - y(t)| \leq |x_0 - y_0| e^{\epsilon L_2 t} + \int_0^t \epsilon L_1 e^{\epsilon L_2 (t-s)} ds$$

$$= |x_0 - y_0| e^{\epsilon L_2 t} + \epsilon \frac{L_1}{L_2} \left(e^{\epsilon L_2 t} - 1 \right) \leq \left[|x_0 - y_0| + \epsilon \frac{L_1}{L_2} \right] e^{\epsilon L_2 t}$$

Since $|x_0 - y_0| = O(\epsilon)$, we conclude that $|x(t) - y(t)| = O(\epsilon)$ as long as $t \in \left[0, \frac{1}{\epsilon L_2} \right)$, i.e., time scales of $O(1/\epsilon)$.

3. (*Unsteady separation in time-periodic fluid flows*) Fluid trajectories $\mathbf{x}(t) = (x(t), y(t))$ in a two-dimensional time-periodic flow satisfy the differential equations

$$
\begin{aligned}
\dot{x} &= u(x, y, t), & u(x, y, t) = u(x, y, t + T), \\
\dot{y} &= v(x, y, t), & v(x, y, t) = v(x, y, t + T),
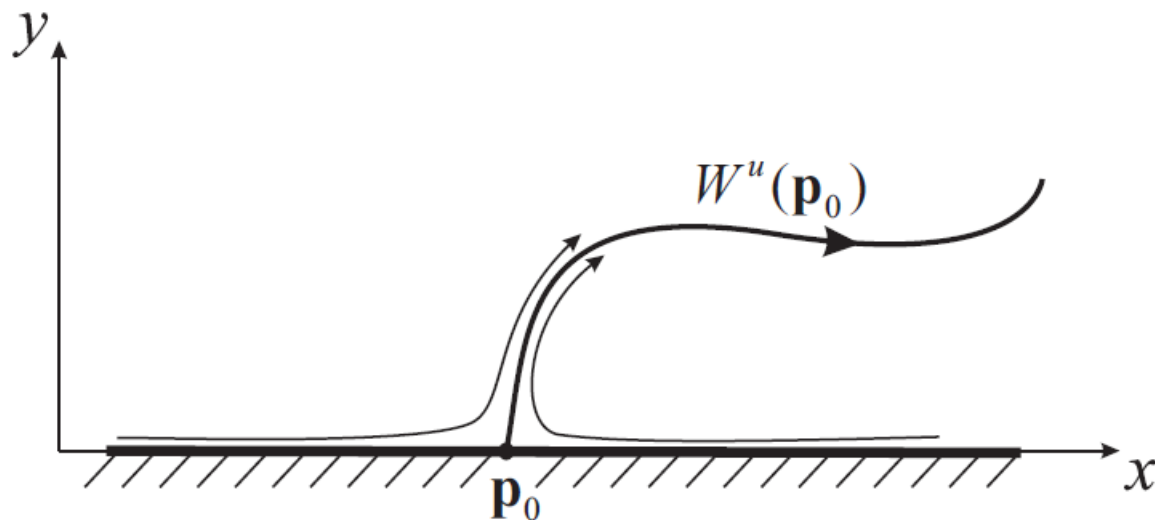\end{aligned}
\tag{3}
$$

Figure 1: Unsteady separation from a no-slip wall

where $T > 0$ is the period, u and v are smooth velocity components satisfying the incompressibility condition $u_x + v_y \equiv 0$. Assume that a the fluid is bounded by a wall at $y = 0$, on which the velocity field satisfies the no-slip boundary conditions $u(x, 0, t) = v(x, 0, t) = 0$. As a result, all boundary points are nonhyperbolic fixed points for (3).

We say that a boundary point $\mathbf{p_0} = (x_0, 0)$ is a separation point for the flow (3) if $\mathbf{p_0}$ admits an unstable manifold $W^u(\mathbf{p_0})$. Physically, $W^u(\mathbf{p_0})$ is a time-dependent curve of fluid particles that shrinks to $\mathbf{p_0}$ is backward time. In forward time, $W^u(\mathbf{p_0})$ attracts fluid particles, then ejects them from the vicinity of the wall (see Fig. 1). Show that separation points satisfy the criteria

$$\int_0^T u_y(x_0, 0, t)\, dt = 0, \qquad \int_0^T v_{yy}(x_0, 0, t)\, dt > 0,$$

i.e., separation takes place where the average of the wall-shear is zero and the average of v_{yy} is positive.

Hint: Use incompressibility and the boundary conditions to show that (3) can be rewritten as

$$\dot{x} = yU(x, y, t),$$
$$\dot{y} = y^2V(x, y, t).$$

To focus on the vicinity of the boundary, introduce the scaled variable $y = \epsilon\eta$, where $0 \le \epsilon \ll 1$. Show that the resulting $(\dot{x}, \dot{\eta})$ equations are slowly varying and find the corresponding averaged equations. For the averaged equations, rescale time on trajectories by letting $d\tau/dt = \eta(t)$ in order to remove the common η factor from the right-hand side, and look for a hyperbolic fixed point with an unstable manifold off the wall. Use the averaging theorem to relate your results to the original system (3).

We start with the Taylor expansions of $u(x,y,t)$ and $v(x,y,t)$ in y near $y=0$:

$$\begin{cases} u(x,y,t) = u(x,0,t) + \dfrac{\partial u}{\partial y}(x,0,t)\, y + O(|y|^2) \\[2mm] v(x,y,t) = v(x,0,t) + \dfrac{\partial v}{\partial y}(x,0,t)\, y + \dfrac{1}{2}\dfrac{\partial^2 v}{\partial y^2}(x,0,t)\, y^2 + O(|y|^3) \end{cases} \quad (1)$$

But $u(x,0,t) = v(x,0,t) = 0$ for any x.

Differentiating $u(x,0,t) = 0$ w.r.t. x we get $\dfrac{\partial u}{\partial x}(x,0,t) = 0$.

By incompressibility, $\dfrac{\partial u}{\partial x} = -\dfrac{\partial v}{\partial y} \implies \dfrac{\partial v}{\partial y}(x,0,t) = -\dfrac{\partial u}{\partial x}(x,0,t) = 0, \ \forall x.$

Hence (1) Simplifies to

$$\begin{cases} u(x,y,t) = \dfrac{\partial u}{\partial y}(x,0,t)\, y + O(|y|^2) = y\, U(x,y,t) \\[2mm] v(x,y,t) = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial y^2}(x,0,t)\, y^2 + O(|y|^3) = y^2\, V(x,y,t) \end{cases}$$

Also note that

$$\begin{cases} U(x, 0, t) = \dfrac{\partial u}{\partial y}(x, 0, t) \\[2mm] V(x, 0, t) = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial y^2}(x, 0, t) \end{cases} \qquad (2) \qquad \left[\begin{array}{l}\text{Higher-order terms}\\ \text{are identically Zero}\end{array}\right]$$

therefore,

$$\begin{cases} \dot{x} = y\, U(x, y, t) \\[2mm] \dot{y} = y^2\, V(x, y, t) \end{cases}$$

Scaling y as $y = \epsilon \eta$, we get

$$\begin{cases} \dot{x} = \epsilon \eta\, U(x, \epsilon \eta, t) \\[2mm] \dot{\eta} = \epsilon \eta^2\, V(x, \epsilon \eta, t) \end{cases} \qquad (3)$$

Since U and V are also T-periodic, averaging theory applies to (3) with the averaged equations

$$\begin{cases} \dot{x} = \epsilon \eta\, \bar{U}(x) \\[2mm] \dot{\eta} = \epsilon \eta^2\, \bar{V}(x) \end{cases} \qquad (4)$$

where

$$\begin{cases} \bar{U}(x) = \dfrac{1}{T}\displaystyle\int_0^T U(x, 0, s)\, ds = \dfrac{1}{T}\int_0^T \dfrac{\partial u}{\partial y}(x, 0, s)\, ds \\[4mm] \bar{V}(x) = \dfrac{1}{T}\displaystyle\int_0^T V(x, 0, s)\, ds = \dfrac{1}{2T}\int_0^T \dfrac{\partial^2 v}{\partial y^2}(x, 0, s)\, ds \end{cases} \qquad (5)$$

Rescaling time as $\frac{d\tau}{dt} = \eta(t)$ and denoting the derivative w.r.t. τ by prime sign (') we get

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau} \frac{d\tau}{dt} = \eta x' \quad , \quad \dot{\eta} = \eta \eta'$$

Substitutting these expressions in (5), we get

$$\begin{cases} x' = \epsilon \overline{U}(x) \\ \eta' = \epsilon \eta \, \overline{V}(x) \end{cases} \qquad (6)$$

Equation (6) has a fixed point $(x_0, \eta = 0)$ on the wall if and only if $\overline{U}(x_0, 0) = 0$ Using (5), we have

$$\overline{U}(x_0) = 0 \quad \Longleftrightarrow \quad \boxed{\int_0^T \frac{\partial u}{\partial y}(x_0, 0, s)ds = 0} \qquad (7)$$

Now we turn to the stability of the fixed point $(x_0, 0)$ on the wall by linearizing (6) around this fixed point:

$$\underline{\xi}' = \epsilon \underbrace{\begin{pmatrix} \frac{\partial \bar{U}}{\partial x}(x_0) & 0 \\ & \\ 0 & \bar{V}(x_0) \end{pmatrix}}_{:= A} \underline{\xi}$$

The matrix A has eigenvalues $\epsilon \frac{\partial \bar{U}}{\partial x}(x_0)$ and $\epsilon \bar{V}(x_0)$ correspoinding to eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

For the unstable manifold to be off the wall, we need $\bar{V}(x_0) > 0$.

Using (5), we have

$$\bar{V}(x_0) > 0 \iff \boxed{\int_0^T \frac{\partial^2 v}{\partial y^2}(x_0, 0, s)\, ds > 0} \quad (8)$$

Exercise : Show that $\dfrac{\partial \bar{U}}{\partial x}(x_0) = -2\bar{V}(x_0)$ and hence $(x_0, 0)$ is a hyperbolic fixed point of (6) given Condition (8) holds.

The conditions (7) and (8) together imply that there exists a hyperbolic fixed point of the averaged system (6).

The theory of averaging guarantees, the existence of a fixed point $(x_0^\epsilon, 0)$ of the original time-periodic flow which is $O(\epsilon)$ close to the fixed point $(x_0, 0)$ of the averaged system.

Moreover, $(x_0^\epsilon, 0)$ has an unstable manifold W_ϵ^u $\mathcal{O}(\epsilon)$-close to W^u of the averaged system.

Remark :

A hyperbolic fixed point of the averaged system, in general, signals a nearby limit cycle of the original system. But, in the above example, it signals a fixed point of the original system since the points on the wall don't move due to the no-slip boundary condition.