151-0530-00L, Spring, 2020

Nonlinear Dynamics and Chaos II

Homework Assignment 4

Due: Wednesday, April 29;
Please submit by email to Dr. Shobhit Jain <shjain@ethz.ch>

1. Compute the Lyapunov-type numbers $\nu(p)$ and $\sigma(p)$ in the example

$$\dot{x} = -x(1-x^2),$$
$$\dot{y} = -by,$$

for all points $p \in M_0$, with the parameter $b \in \mathbb{R}^+$ and with overflowing-invariant manifold M_0 defined as

$$M_0 = \left\{(x,y) \in \mathbb{R}^2 : \; y = 0, \; x \in [-3/2, 3/2]\right\}.$$

(*Hint*: Use the operators $A_t(p)$ and $B_t(p)$ defined in class).

2. The stable and unstable manifolds of a normally hyperbolic invariant manifold M turn out to admit a delicate internal structure, an *invariant foliation,* which is useful in determining the exact asymptotic behavior of trajectories in $W^u(M)$ and $W^s(M)$.

More specifically, if $M \subset \mathbb{R}^n$ is a compact, C^r smooth, k-dimensional, r-normally hyperbolic invariant manifold with boundary, and $\dim[W^s(M)] = k+s$, then $W^s(M)$ has the following properties (some of which are sketched in Fig. 1.):

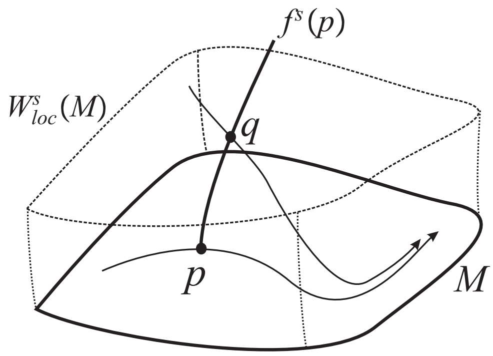

Figure 1: The geometry of stable fibers

(i) Near M, the stable manifold $W^s(M)$ can be written as

$$W^s_{\text{loc}}(M) = \cup_{p\in M} \; f^s(p),$$

where $f^s(p)$ is a C^r smooth, s-dimensional submanifold of $W^s_{\text{loc}}(M)$ for which $f^s(p) \cap M = p$. We refer to the point p on M as the base point of the *stable fiber* $f^s(p)$.

(ii) The stable fiber $f^s(p)$ is tangent to $N^s_p M$, the local section of the stable subbundle $N^s M$.

(iii) The stable fibers form a positively invariant family, i.e., $F^t(f^s(p)) \subset f^s(F^t(p))$. In words, stable fibers are mapped into stable fibers by the flow map, although individual stable fibers are not invariant under the flow.

(iv) There exist positive constants C_s and λ_s, such that for any $q \in f^s(p)$, we have $|F^t(q) - F^t(p)| < C_s e^{-\lambda_s t}$. In other words, trajectories intersecting a stable fiber will exponentially converge to the trajectory on M that passes through the base point of that stable fiber.

(v) For any $q \in f^s(p)$ and $\hat{q} \in f^s(\hat{p})$, we have

$$\frac{\|F^t(q) - F^t(p)\|}{\|F^t(\hat{q}) - F^t(p)\|} \to 0,$$

as $t \to \infty$, unless $p = \hat{p}$. In other words, out of all the trajectories that may converge to the positive half-trajectory

$$\gamma(p) = \left\{F^t(p)\right\}_{t\geq 0},$$

the trajectories starting from the stable fiber $f^s(p)$ converges at the fastest rate. One therefore obtains a local stable manifold

$$W^{ss}_{\text{loc}}(\gamma(p)) = \cup_{\tilde{p}\in\gamma(p)}\, f^s(\tilde{p})$$

for any trajectory $\gamma(p)$ on the manifold M. (The full stable manifold $W^s_{\text{loc}}(\gamma(p))$ may be larger than $W^{ss}_{\text{loc}}(\gamma(p))$, because $\gamma(p)$ may also attract trajectories within M.)

(vi) $f^s(p) \cap f^s(\hat{p}) = \emptyset$, unless $p = \hat{p}$. In other words, stable fibers with different base points do not intersect.

(vii) A stable fiber $f^s(p)$ is a C^{r-1} smooth function of its base point p.

(viii) Stable fibers C^r-smoothly persist under small C^1 perturbations of the dynamical system.

The local unstable manifold $W^u_{\text{loc}}(M)$ has a similar invariant foliation

$$W^u_{\text{loc}}(M) = \cup_{p\in M}\, f^u(p),$$

with appropriate properties in backward time. (For more information, see S. Wiggins, *Normally Hyperbolic Invariant Manifolds in Dynamical Systems*, Springer 1994)

Consider now the three-dimensional nonlinear dynamical system

$$\begin{aligned}
\dot{x} &= -\varepsilon\left(x + y^2\right),\\
\dot{y} &= -y,\\
\dot{z} &= z,
\end{aligned}$$

with the small parameter $\varepsilon \geq 0$.

(a) Show that the set $M_0 = \{y = z = 0,\, x \in [-1, 1]\}$ is a normally hyperbolic invariant manifold for $\varepsilon = 0$.

(b) Find the manifold M_ε into which M_0 perturbs for small $\varepsilon > 0$.

(c) Using the property (v), show that for any base point $p \in W^s_{\text{loc}}(M)$, the corresponding stable fiber is the nonlinear surface.

$$f^s(p) = \left\{(x, y, z) \ \mid x = x_p + \frac{\varepsilon}{2-\varepsilon}y^2,\, z = 0\right\}.$$

(d) Find a similar expression for the unstable fibers $f^u(p)$.

(e) Verify explicitly the properties of the stable fibers listed in (i)-(vii) in this example.

(e) For any trajectory γ in M_ε, find explicit expressions for $W^{ss}_{\text{loc}}(\gamma)$, $W^{uu}_{\text{loc}}(\gamma)$, $W^s_{\text{loc}}(\gamma)$, and $W^u_{\text{loc}}(\gamma)$.

Nonlinear Dynamics and Chaos II.
Homework 4

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Exercise 1

Computing the Lyapunov-type numbers $\nu(p)$ and $\sigma(p)$ in the example

$$\dot{x} = -x(1-x^2) \tag{1}$$
$$\dot{y} = -by, \qquad b > 0, \tag{2}$$

on the overflowing invariant manifold $M = \{(x,y) \in \mathbb{R}^2 : y = 0, x \in [-3/2, 3/2]\}$.

Solution

To define the Lyapunov-type numbers at a point p on an overflowing invariant manifold M, let $w_0 \in NM_p$ be a vector in the normal space of M at p and $v_0 \in TM_p$ be a vector in the tangent space at p. Their images under the linearized backward flow are

$$v_{-t} = DF_p^{-t}(v_0) \text{ and}$$
$$w_{-t} = \Pi_{F^{-t}(p)} \circ DF_p^{-t}(w_0).$$

Here, DF_p^{-t} is the differential of the (backward) flow-map at the point p (which is a linear map on the tangent space of the full phase-space at p) and Π_q is the orthogonal projection onto the normal space of M at q.

With this notation, the Lyapunov-type numbers, for a point p on an overflowing invariant manifold M, are defined to be

$$\nu(p) = \inf\left\{a > 0 : \frac{||w_0||}{||w_{-t}||a^t} \to 0 \text{ as } t \to \infty, \forall w_0 \in NM_p\right\} \tag{3}$$

$$\sigma(p) = \inf\left\{b \in \mathbb{R} : \frac{||w_0||/||w_{-t}||^b}{||v_0||/||v_{-t}||} \to 0 \text{ as } t \to \infty, \forall w_0 \in NM_p, v_0 \in TM_p\right\}. \tag{4}$$

To calculate $\nu(p)$ and $\sigma(p)$, we first define the two operators

$$A_t(p) := DF_p^{-t}|_{TM_p} \tag{5}$$

$$B_t(p) := \Pi_p \circ DF^t_{F^{-t}(p)}|_{NM_{F^{-t}(p)}}. \tag{6}$$

Then, a theorem by Fenichel states that

$$\nu(p) = \limsup_{t\to\infty} ||B_t||^{1/t}. \tag{7}$$

If in addition, $\nu(p) < 1$ then

$$\sigma(p) = \lim\,\sup_{t\to\infty} \frac{\log ||A_t(p)||}{-\log ||B_t(p)||}. \tag{8}$$

To compute the operators $A_t(p)$ and $B_t(p)$, we calculate the flow-map of system (1)-(2) explicitly. Equations (1) and (2) are decoupled, so we can find the flow map by integrating the equations separately. The equation for y is a homogeneous, linear, scalar equation, which has the solution, assuming the initial condition is given as $y(t_0) := y_0$,

$$y(t; y_0, t_0) = y_0 e^{-b(t-t_0)}.$$

The system is autonomous, which means we can also set $t_0 = 0$ without loss of generality:

$$y(t; y_0) = y_0 e^{-bt}. \tag{9}$$

We can solve (1) by separation of variables:

$$\begin{aligned} \dot{x} &= -x(1-x^2) \\ t &= \int_{x_0}^{x} -\frac{dx'}{x'(1-x'^2)} dx'. \end{aligned}$$

The integral on the right side can be evaluated using the expression (partial fraction decomposition)

$$-\frac{1}{x(1-x^2)} = -\frac{1}{x} - \frac{1}{2(1-x)} + \frac{1}{2(1+x)}.$$

Then, using Newton-Leibniz, we get

$$\begin{aligned} t &= -(\log x - \log x_0) + \frac{1}{2}\left[\log(1-x) - \log(1-x_0)\right] + \frac{1}{2}\left[\log(1+x) - \log(1+x_0)\right] \\ e^t &= \frac{x_0}{x}\sqrt{\frac{1-x^2}{1-x_0^2}} \end{aligned}$$

This expression only makes sense if x and x_0 have the same sign. In this case, we can square it to get

$$\frac{x^2 e^{2t}}{x_0^2} = \frac{1-x^2}{1-x_0^2}$$

Rearranging to solve for x^2, we have

$$x^2 = \frac{x_0^2}{x_0^2 + (1-x_0^2)e^{2t}}.$$

In the relevant case, when x and x_0 have the same sign, we can take the (positive) square root

$$x(t; x_0) = \frac{x_0}{\sqrt{x_0^2 + (1-x_0^2)e^{2t}}}. \tag{10}$$

The components of the flow-map are then given by the functions (9) and (10).

$$F^t(x_0, y_0) := \begin{bmatrix} x(t; x_0) \\ y(t; y_0) \end{bmatrix} = \begin{bmatrix} \frac{x_0}{\sqrt{x_0^2 + (1-x_0^2)e^{2t}}} \\ y_0 e^{-bt} \end{bmatrix} \tag{11}$$

The linearized backward flow-map is the matrix

$$DF^{-t}_{(x_0,y_0)} := \begin{bmatrix} \frac{\partial x(-t;x_0)}{\partial x_0} & 0 \\ 0 & \frac{\partial y(-t;y_0)}{\partial y_0} \end{bmatrix}. \tag{12}$$

To compute the first entry, I use the short-hand $a := x_0^2 + (1-x_0^2)e^{-2t}$.

$$\frac{\partial x(t;x_0)}{\partial x_0} = \frac{1}{a}\left[\sqrt{a} - \frac{x_0^2(1-e^{-2t})}{\sqrt{a}}\right] = \frac{a - x_0^2(1-e^{-2t})}{a^{3/2}} = \frac{e^{-2t}}{a^{3/2}}$$

$$DF^{-t}_{x_0} := \begin{bmatrix} \frac{e^{-2t}}{a^{3/2}} & 0 \\ 0 & e^{bt} \end{bmatrix}$$

Restricting this matrix onto the tangent space of M at $p = x_0$, means that it acts on vectors of the ambient space, that have the form

$$\mathbf{v_0} = \begin{bmatrix} v_0 \\ 0 \end{bmatrix},$$

as

$$DF^{-t}_{x_0}|_{TM_{x_0}}\mathbf{v_0} = \begin{bmatrix} \frac{e^{-2t}}{a^{3/2}} v_0 \\ 0 \end{bmatrix} \in TM_{F^{-t}(x_0)}$$

Which shows that the operator $A_t(x_0)$ is simply the scalar multiplication by

$$A_t(x_0) = \frac{e^{-2t}}{(x_0^2 + (1-x_0^2)e^{-2t})^{3/2}}. \tag{13}$$

Similarly, with the notation $x_{-t} = F^{-t}(x_0)$,

$$DF^{t}_{F^{-t}(x_0)} = \begin{bmatrix} \frac{e^{2t}}{(x_{-t}^2 + (1-x_{-t}^2)e^{2t})^{3/2}} & 0 \\ 0 & e^{-bt} \end{bmatrix}$$

Restricting it to the normal space at x_{-t} (vectors parallel to the y axis), and then projecting to the normal space at x_0 (again, vectors parallel to the y axis) we have

$$B_t(x_0) = e^{-bt}. \tag{14}$$

Substituting the expressions (13) and (14) into the theorem (7) and (8) gives

$$\nu(x_0) = \limsup_{t\to\infty} ||B_t||^{1/t} = \limsup_{t\to\infty} e^{-b} = e^{-b}.$$

$$\sigma(x_0) = \limsup_{t\to\infty} \frac{\log ||A_t(p)||}{-\log ||B_t(p)||} = \limsup_{t\to\infty} \frac{-2t - \frac{3}{2}\log(x_0^2 + (1-x_0^2)e^{-2t})}{bt} =$$

$$= -\frac{2}{b} - \limsup_{t\to\infty} \frac{3}{2b}\frac{\log(x_0^2 + (1-x_0^2)e^{-2t}))}{t}$$

Since $\nu(x_0) = e^{-b} < 1$, we could use the second part of the theorem as well. In the last line, we have the lim. sup. of product of two functions, one of which is $1/t$, which goes to zero as $t \to \infty$. The limit of the numerator is

$$\lim_{t\to\infty} \frac{3}{2b}\log(x_0^2 + (1-x_0^2)e^{-2t})) = \frac{3}{b}\log x_0.$$

If this limit exists, then the whole limit is zero, since it is a product of two convergent functions. However, for $x_0 = 0$, $\log x_0$ is divergent. Then, we use the original expression and set $x_0 = 0$:

$$\limsup_{t\to\infty} \frac{3}{2b}\frac{\log(x_0^2 + (1-x_0^2)e^{-2t}))}{t} = \limsup_{t\to\infty} \frac{3}{2b}\frac{\log e^{-2t}}{t} = -\frac{3}{b}.$$

Then, the Lyapunov-type numbers on the manifold M are

$$\nu(x_0) = e^{-b}$$
$$\sigma(x_0) = \begin{cases} \frac{1}{b}, & \text{for } x_0 = 0 \\ -\frac{2}{b} & \text{otherwise.} \end{cases}$$

Exercise 2

Consider the three dimensional nonlinear dynamical system

$$\dot{x} = -\varepsilon(x + y^2) \tag{15}$$
$$\dot{y} = -y \tag{16}$$
$$\dot{z} = z, \tag{17}$$

with a small parameter $\varepsilon \geq 0$.

(a) Show that the set $M_0 = \{y = z = 0, x \in [-1, 1]\}$ is a normally hyperbolic invariant manifold for $\varepsilon = 0$.

Solution

The set M_0 is an invariant manifold, since setting $\varepsilon = 0$ and restricting to $y = z = 0$, the dynamics becomes $\dot{x} = \dot{y} = \dot{z} = 0$, there is no dynamics on this set. It is also a 1-manifold with boundary, $\partial M_0 = \{-1, 1\}$.

The system can be seen as the fast-time formulation of a singularly perturbed problem. In the limit $\varepsilon = 0$, the fast subsystem is

$$\dot{y} = -y$$
$$\dot{z} = z,$$

and the critical manifold is M_0. To establish normal hyperbolicity, we restrict the dynamics to invariant subspaces x =const., where it is given by the fast subsystem. In this case, the fast subsystem is linear for all x, and has nonzero eigenvalues, showing that the dynamics in the direction normal to M_0 is hyperbolic.

A direct application of Fenichel's definition of normal hyperbolicity is not applicable, since the manifold is not overflowing invariant. This can be fixed by a local modification of the vector field near the boundary points (Wiggins (1994) - 7.2) .

(b) Find the manifold M_ε into which M_0 perturbs for small $\varepsilon > 0$.

Solution

Restricting the perturbed system onto the set $M_\varepsilon := \{y = z = 0, x \in [-1, 1]\}$, the dynamics is $\dot{x} = -\varepsilon x$. This means that M_ε is an invariant manifold of the perturbed system, which coincides with the unperturbed manifold M_0.

(c) Show that for any base point $p \in M_\varepsilon$ the corresponding stable fiber $f^s(p) \subset W^s_{\text{loc}}(M_\varepsilon)$,

$$f^s(p) = \left\{(x, y, z) : x = x_p + \frac{\varepsilon}{2-\varepsilon}y^2, z = 0\right\}.$$

Solution

The local stable manifold is contained in the $x - y$ plane. Because of the normal hyperbolicty of M_ε, it admits an invariant foliation:

$$W^s_{\text{loc}}(M_\varepsilon) = \bigcup_{p \in M_\varepsilon} f^s(p).$$

In particular, this foliation has the property, that given a base point $p \in M_\varepsilon$, the trajectories starting in $f^s(p) \subset W^s_{\text{loc}}(M_\varepsilon)$ have the fastest rate of convergence towards the trajectory $\gamma(p) \subset M_\varepsilon$ through p. That is, given $q \in f^s(p)$ and $q' \in f^s(p')$

$$\frac{||F^t(q) - F^t(p)||}{||F^t(q') - F^t(p)||} \to 0, \text{ as } t \to \infty. \tag{18}$$

To use this property to find the nonlinear surface $f^s(p)$, we first calculate the flow-map of system (15). The y and z coordinates decouple from x, and they can be integrated first

$$y(t; y_0) = y_0 e^{-t} \qquad z(t; z_0) = z_0 e^t.$$

Substituting this into the x equation, we get the inhomogeneous, linear scalar equation

$$\dot{x} = -\varepsilon x - \varepsilon y_0^2 e^{-2t},$$

which we can solve by variation of constants. The solution of the homogeneous part is $x_h = ce^{-\varepsilon t}$. To obtain a particular solution for the inhomogeneous equation, look for a solution of the form $x(t) = c(t)e^{-\varepsilon t}$. This leads to

$$\dot{c} = -y_0^2 \varepsilon e^{(\varepsilon - 2)t}$$

$$c = \frac{y_0^2 \varepsilon}{2 - \varepsilon} e^{(\varepsilon - 2)t}.$$

The full solution is the sum of the general solution to the homogeneous part and the particular solution

$$x(t) = c_0 e^{-\varepsilon t} + \frac{y_0^2 \varepsilon}{2 - \varepsilon} e^{-2t}.$$

Eliminating the constant of integration c_0, using the initial condition $x(0) = x_0$, we get

$$x(t; x_0) = \left(x_0 - \frac{y_0^2 \varepsilon}{2 - \varepsilon}\right) e^{-\varepsilon t} + \frac{y_0^2 \varepsilon}{2 - \varepsilon} e^{-2t}.$$

The flow-map of system (15) is

$$F^t(x_0, y_0, z_0) = \begin{bmatrix} \left(x_0 - \frac{y_0^2 \varepsilon}{2-\varepsilon}\right) e^{-\varepsilon t} + \frac{y_0^2 \varepsilon}{2-\varepsilon} e^{-2t} \\ y_0 e^{-t} \\ z_0 e^t \end{bmatrix}. \tag{19}$$

Consider the points $p = (x_p, 0, 0), p' = (x'_p, 0, 0) \in M_\varepsilon$ and $q = (x_q, y_q, 0) \in f^s(p) \subset W^u_{\text{loc}}(M_\varepsilon)$, $q' = (x'_q, y'_q, 0) \in f^s(p') \subset W^u_{\text{loc}}(M_\varepsilon)$. Then,

$$F^t(q) - F^t(p) = \begin{bmatrix} \left(x_q - \frac{y_q^2 \varepsilon}{2-\varepsilon}\right) e^{-\varepsilon t} + \frac{y_q^2 \varepsilon}{2-\varepsilon} e^{-2t} - x_p e^{-\varepsilon t} \\ y_q e^{-t} \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-\varepsilon t} \left(x_q - \frac{y_q^2 \varepsilon}{2-\varepsilon} - x_p\right) + \frac{y_q^2 \varepsilon}{2-\varepsilon} e^{-2t} \\ y_q e^{-t} \\ 0 \end{bmatrix}. \tag{20}$$

Similarly, for the denominator of (18)

$$F^t(q') - F^t(p) = \begin{bmatrix} e^{-\varepsilon t}\left(x'_q - \frac{y'^2_q\varepsilon}{2-\varepsilon} - x_p\right) + \frac{y'^2_q\varepsilon}{2-\varepsilon}e^{-2t} \\ y'_q e^{-t} \\ 0 \end{bmatrix}$$

In general, both the denominator and the numerator of (18) go to 0 as $t \to \infty$ (because they are the sums of exponentially decaying terms), the limit of the ratio is undetermined. However, since $\varepsilon \ll 1$, the slowest decaying term in the first component of the flow-map is the one which is multiplied by $e^{-\varepsilon t}$. We can set its coefficient to 0, which guarantees that the numerator goes to 0 at a faster rate than the denominator. This way, the ratio will go to 0.

This establishes a relation between x_p, the base point and the points $(x_q, y_q, 0)$ on the stable fiber connected to it. This relation is

$$x_q = x_p + \frac{y_q^2\varepsilon}{2-\varepsilon},$$

as claimed.

(d) Find a similar expression for the unstable fibers $f^u(p)$.

Solution

The local unstable manifold of M_ε, is the local stable manifold under the backward flow. Reversing time in the original system, we obtain

$$\begin{aligned} \dot{x} &= \varepsilon(x + y^2) \\ \dot{y} &= y \\ \dot{z} &= -z, \end{aligned}$$

for which the local stable manifold is contained in the $x - z$ plane. To obtain the foliation, we use the property (18) for the backward flow map.

$$F^{-t}(x_0, y_0, z_0) = \begin{bmatrix} \left(x_0 - \frac{y_0^2\varepsilon}{2-\varepsilon}\right)e^{\varepsilon t} + \frac{y_0^2\varepsilon}{2-\varepsilon}e^{2t} \\ y_0 e^t \\ z_0 e^{-t} \end{bmatrix}$$

Taking points $p = (x_p, 0, 0), p' = (x'_p, 0, 0) \in M_\varepsilon$ and $q = (x_q, 0, z_q) \in f^u(p) \subset W^u_{\text{loc}}(M_\varepsilon)$, $q' = (x'_q, 0, z'_q) \in f^u(p') \subset W^u_{\text{loc}}(M_\varepsilon)$, and evaluating the expressions $F^{-t}(p) - F^{-t}(q)$ and $F^{-t}(q') - F^{-t}(p)$ gives

$$F^{-t}(q) - F^{-t}(p) = \begin{bmatrix} e^{\varepsilon t}(x_q - x_p) \\ 0 \\ z_q e^{-t} \end{bmatrix}; \qquad F^{-t}(q') - F^{-t}(p) = \begin{bmatrix} e^{\varepsilon t}\left(x'_q - x_p\right) \\ 0 \\ z'_q e^{-t} \end{bmatrix}. \tag{21}$$

Following a similar argument as we made with the stable fibers, the ratio (18) converges to zero, if the numerator goes to zero much faster than the denominator. In this case, the numerator and the denominator do not go to zero term-by-term, since the exponent of the first term is positive. The ratio is

$$\frac{||F^{-t}(q) - F^{-t}(p)||}{||F^{-t}(q') - F^{-t}(p)||} = \frac{e^{2\varepsilon t}(x_q - x_p)^2 + z_q^2 e^{-2t}}{e^{2\varepsilon t}(x'_q - x_p)^2 + z'^2_q e^{-2t}}.$$

To ensure a faster convergence in the denominator, we can set $x_q = x_p$ and see that the ratio goes to zero

$$\frac{||F^{-t}(q) - F^{-t}(p)||}{||F^{-t}(q') - F^{-t}(p)||} = \frac{z_q^2 e^{-2t}}{e^{2\varepsilon t}(x_q' - x_p)^2 + z_q'^2 e^{-2t}} = \frac{z_q^2}{(x_q' - x_p)^2 e^{2t(\varepsilon+1)} + z_q'^2} \to 0.$$

The unstable fibers are the straight lines parallel to the z axis, given by

$$f^u(p) = \{(x, y, z) : x = x_p, y = 0\}.$$

(e) Verify explicitly the properties of the stable fibers in this example.

- $f^s(p)$ is a C^r smooth (given by a smooth graph), 1 dimensional submanifold of the 2 dimensional manifold $W^s_{\text{loc}}(M_\varepsilon)$. Its intersection with the manifold is $f^s(p) \cap M_\varepsilon = \{(x, y, z) : x = x_p, y = 0, z = 0\} = p$, the base point.
- The local section of the stable subbundle $N^s_p M_\varepsilon$ is the stable subspace of the normal space at p: $N^s_p M_\varepsilon$. In this example, the normal space is the plane $N_p M_\varepsilon = \{(x, y, z) : x = x_p\}$, the stable subspace is the line $N^s_p M_\varepsilon = \{(x, y, z) : x = x_p, z = 0\}$. The local tangent vector to $f^s(p)$ is $(0, 1, 0)$, which is in this subspace.
- The stable fibers form a positively invariant family. The stable fibers of the images of the basepoints are

$$f^s(F^t(p)) = f^s(F^t(x_p, 0, 0)) = \left\{(x, y, z) : x = x_p e^{-\varepsilon t} + \frac{\varepsilon}{2-\varepsilon} y^2, z = 0\right\}.$$

While the images of the stable fibers are:

$$\begin{aligned}
&F^t(f^s(p)) = F^t\left(\left\{(x, y, z) : x = x_p + \frac{\varepsilon}{2-\varepsilon} y^2, z = 0\right\}\right) = \\
&= \left\{F^t(\overline{x}, \overline{y}, \overline{z}) : \overline{x} = x_p + \frac{\varepsilon}{2-\varepsilon} \overline{y}^2, \overline{z} = 0\right\} = \\
&= \{(x, y, z) : x = x(t; \overline{x}), y = y(t; \overline{y}); z = z(t; \overline{z})\} = \\
&= \left\{(x, y, z) : x = \left(x_p + \frac{\varepsilon}{2-\varepsilon} \overline{y}^2 - \frac{\varepsilon}{2-\varepsilon} \overline{y}^2\right) e^{-\varepsilon t} + \frac{\varepsilon}{2-\varepsilon} \overline{y}^2 e^{-2t}, y = \overline{y} e^{-t}; z = 0\right\} = \\
&= \left\{(x, y, z) : x = x_p e^{-\varepsilon t} + \frac{\varepsilon}{2-\varepsilon} \overline{y}^2 e^{-2t}, y = \overline{y} e^{-t}; z = 0\right\}
\end{aligned}$$

Since we can eliminate $\overline{y}$ using the second coordinate $\overline{y} = y e^t$, this is the same set as

$$\left\{(x, y, z) : x = x_p e^{-\varepsilon t} + \frac{\varepsilon}{2-\varepsilon} y^2; z = 0\right\},$$

which coincides with $f^s(F^t(p))$.

- There exist positive constants C_s and λ_s, such that for any $q \in f^s(p)$ we have $|F^t(q) - F^t(p)| < C_s e^{-\lambda_s t}$.

For a point $q = (x_q, y_q, 0) \in f^s(p)$, we can use the expression describing the fiber to get

$$F^t(q) - F^t(p) = \begin{bmatrix} \left(x_q - \frac{y_q^2 \varepsilon}{2-\varepsilon}\right) e^{-\varepsilon t} + \frac{y_q^2 \varepsilon}{2-\varepsilon} e^{-2t} - x_p e^{-\varepsilon t} \\ y_q e^{-t} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{y_q^2 \varepsilon}{2-\varepsilon} e^{-2t} \\ y_q e^{-t} \\ 0 \end{bmatrix}.$$

$$|F^t(q) - F^t(p)|^2 = \frac{y_q^2\varepsilon^2}{(2-\varepsilon)^2}e^{-4t} + y_q^2 e^{-2t} = y_q^2 e^{-2t}\left(\frac{\varepsilon^2}{(2-\varepsilon)^2}e^{-2t} + 1\right)$$

$$|F^t(q) - F^t(p)| = |y_q|e^{-t}\sqrt{\frac{\varepsilon^2}{(2-\varepsilon)^2}e^{-2t} + 1}$$

The square root is bounded by the constant $L = \sqrt{\frac{\varepsilon^2}{(2-\varepsilon)^2} + 1}$, so we can bound the distance between the trajectory through p and the trajectory through q as

$$|F^t(q) - F^t(p)| \leq |y_q|Le^{-t}.$$

The local stable manifold is compact set, that contains M_ε intersected with the $x-y$ plane, so we can choose a $C > 0$, such that $C > |y_q|$ for any $(x_q, y_q, z_q) \in W^s_{\mathrm{loc}}(M_\varepsilon)$. Then, we can set $C_s = CL$ and $\lambda_s = 1$.

- Given a base point $p \in M_\varepsilon$, the trajectories starting in $f^s(p) \subset W^s_{\mathrm{loc}}(M_\varepsilon)$ have the fastest rate of convergence towards the trajectory $\gamma(p) \subset M_\varepsilon$ through p. That is, given $q \in f^s(p)$ and $q' \in f^s(p')$
$$\frac{||F^t(q) - F^t(p)||}{||F^t(q') - F^t(p)||} \to 0, \text{ as } t \to \infty.$$
We already used this property, to arrive at the expression for the stable fibers.

- $f^s(p) \cap f^s(p') = \{\}$ if $p \neq p'$.
$$f^s(p)\cap f^s(p') = \left\{(x,y,z) : x = x_p + \frac{\varepsilon}{2-\varepsilon}y^2, z = 0\right\}\cap\left\{(x',y',z') : x' = x'_p + \frac{\varepsilon}{2-\varepsilon}y'^2, z' = 0\right\}$$
For this to be nonempty, we need $x = x'$, $y = y'$, $z = z'$, but this means
$$x_p + \frac{\varepsilon}{2-\varepsilon}y^2 = x'_p + \frac{\varepsilon}{2-\varepsilon}y^2,$$
which cannot be equal if $x_p \neq x'_p$.

- The stable fiber $f^s(p)$ is a C^{r-1} smooth function of its base point p. The set $f^s(p) = f^s(x_p)$, which is given by a graph of a C^∞ function of x_p.

- The stable fibers C^r-smoothly persist under small C^1 perturbations of the dynamical system.

  First, consider that the perturbation is only in ε: $\varepsilon := \varepsilon_0 + \Delta\varepsilon$. The stable fiber is well defined for any $\varepsilon \neq 2$ and it is also differentiable in ε.

  For a general C^1-small perturbation to (15), (say, order δ) the flow-map is also perturbed by an order δ term, since we can Taylor-expand it to first order
$$F^t(p;\delta) = F^t(p;0) + \delta\frac{\partial F^t}{\partial\delta} + O(\delta^2).$$
  Then, it can probably (?) be argued that this is only an order δ perturbation to the equation that prescribes the maximal convergence rate for points in the fiber (exercise (c)).

(f) For any trajectory $\gamma \subset M_\varepsilon$, find explicit expressions for $W^{ss}_{\mathrm{loc}}(\gamma)$, $W^{uu}_{\mathrm{loc}}(\gamma)$, $W^s_{\mathrm{loc}}(\gamma)$, $W^u_{\mathrm{loc}}(\gamma)$.

From the definition,

$$W^{ss}_{\mathrm{loc}}(\gamma) = \bigcup_{\overline{p}\in\gamma(p)} f^s(\overline{p}).$$

On the manifold, the trajectory is given by $\gamma(p) = \gamma(x_p) = x_p e^{-\varepsilon t}$. After substituting, we obtain

$$W^{ss}_{\text{loc}}(\gamma) = \bigcup_{x\in\gamma(x_p)} \left\{(\overline{x},\overline{y},\overline{z}) : \overline{x} = x + \frac{\varepsilon}{2-\varepsilon}\overline{y}^2, \overline{z} = 0\right\} = \bigcup_{t\geq 0} \left\{(\overline{x},\overline{y},\overline{z}) : \overline{x} = x_p e^{-\varepsilon t} + \frac{\varepsilon}{2-\varepsilon}\overline{y}^2, \overline{z} = 0\right\},$$

which is the same set as (if $x_p > 0$) $W^{ss}_{\text{loc}}(\gamma) = \{(x,y,z) : \frac{\varepsilon}{2-\varepsilon}y^2 \leq x \leq x_p + \frac{\varepsilon}{2-\varepsilon}y^2, z = 0\}$. And in case $x_p < 0$ $W^{ss}_{\text{loc}}(\gamma) = \{(x,y,z) : \frac{\varepsilon}{2-\varepsilon}y^2 + x_p \leq x \leq \frac{\varepsilon}{2-\varepsilon}y^2, z = 0\}$.

For the unstable fibers, we have (if $x_p > 0$)

$$W^{uu}_{\text{loc}}(\gamma) = \bigcup_{\overline{p}\in\gamma(p)} f^u(\overline{p}) = \bigcup_{t\geq 0} \{x_p e^{-\varepsilon t}, y = 0\} = \{(x,y,z) : x \leq x_p, y = 0\}$$

and if $x_p < 0$

$$W^{uu}_{\text{loc}}(\gamma) = \{(x,y,z) : x \geq x_p, y = 0\}$$

The local stable manifold of trajectory $\gamma(p)$ is the set of initial conditions, through which trajectories converge to $\gamma(p)$.

$$W^s_{\text{loc}}(\gamma) = \{(x,y,z) : ||F^t(x,y,z) - \gamma(p)|| \to 0 \text{ as } t \to \infty\}.$$

Using (20), we can write the difference between the trajectories as

$$F^t(x,y,z) - \gamma(x_p) = F^t(x,y,z) - F^t(x_p,0,0) = \begin{bmatrix} e^{-\varepsilon t}\left(x - \frac{y^2\varepsilon}{2-\varepsilon} - x_p\right) + \frac{y^2\varepsilon}{2-\varepsilon}e^{-2t} \\ ye^{-t} \\ ze^{t} \end{bmatrix}.$$

For convergence, we need to set $z = 0$ and then we have

$$||F^t(x,y,0) - F^t(x_p,0,0)|| = \left[e^{-\varepsilon t}\left(x - \frac{y^2\varepsilon}{2-\varepsilon} - x_p\right) + \frac{y^2\varepsilon}{2-\varepsilon}e^{-2t}\right]^2 + y^2 e^{-2t},$$

which converges for all x, y. $W^s_{\text{loc}}(\gamma) = \{(x,y,z) : z = 0\}$.

The local unstable manifold of trajectory $\gamma(p)$ is the set of initial conditions, through which trajectories converge to $\gamma(p)$ in backward time.

Using the expression (21), we have

$$F^{-t}(x,y,z) - F^{-t}(x_p,0,0) = \begin{bmatrix} e^{\varepsilon t}\left(x - \frac{y^2\varepsilon}{2-\varepsilon} - x_p\right) + \frac{y^2\varepsilon}{2-\varepsilon}e^{2t} \\ ye^{t} \\ ze^{-t} \end{bmatrix}.$$

Setting $y = 0$, we get

$$||F^{-t}(x,0,z) - F^{-t}(x_p,0,0)|| = e^{2\varepsilon t}(x - x_p)^2 + z^2 e^{-2t},$$

which converges if $x = x_p$. $W^u_{\text{loc}}(\gamma) = \{(x,y,z) : x = x_p, y = 0\}$.