151-0530-00L, Spring, 2020

Nonlinear Dynamics and Chaos II

Homework Assignment 5

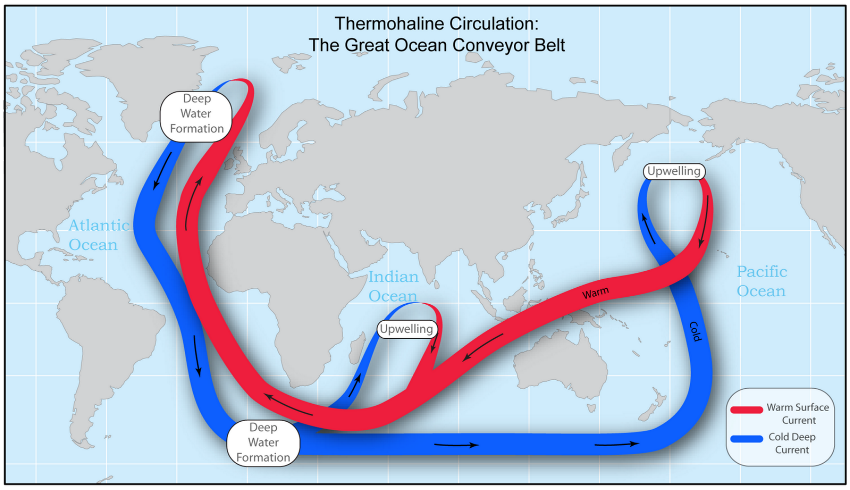
Due: Wednesday, May 13;
Please submit by email to Dr. Shobhit Jain <shjain@ethz.ch>

1. Consider a slowly forced pendulum with viscous damping:

$$\ddot{\varphi} + k\dot{\varphi} + \sin\varphi = F_0 \sin \epsilon t,$$

where $k > 0$ is the damping coefficient, $0 < F_0 < 1$ is the forcing amplitude, and $0 < \epsilon \ll 1$ is the forcing frequency. Give a complete qualitative description of the geometry of this system in the extended phase space for small enough ϵ. (*Hint:* Make the system autonomous by introducing the phase variable $\psi = \epsilon t$.)

2. Thermohaline circulation (THC) is a part of the large-scale ocean circulation that is driven by global density gradients created by surface heat and freshwater fluxes. Stommel's box model (1961) is a qualitative description of the trends and equilibria in THC. This model couples the two fundamental drivers of TLC, temperature (*thermo*) and salt concentration (*-haline*), in a nonlinear fashion.



The non-dimensional **variables** of Stommel's model are:

$x(t)$**:** temperature difference between the tropics (lower latitudes) and the North-Atlantic (higher latitudes)

$y(t)$**:** salinity (i..e, salt concentration) difference between the above two regions of the ocean

The non-dimensional **parameters** of the model are:

τ_x**:** relaxation time to a constant temperature difference between northern and southern latitudes in the absence of coupling

τ_y: relaxation time to zero salinity difference between higher and lower latitudes in the absence of coupling. In practice, $\tau_x/\tau_y = \epsilon \ll 1$.

μ: measure of freshwater flux through clouds moving from lower to higher latitudes

η : nonlinear coupling parameter between temperature and salinity evolution

With this notation, Stommel's model can be written as

$$\begin{aligned}
\dot{x} &= -\frac{1}{\tau_x}(x-1) + \frac{1}{\tau_y}x\left[1+\eta^2(x-y)^2\right], \\
\dot{y} &= \frac{\mu}{\tau_y} - \frac{1}{\tau_y}y\left[1+\eta^2(x-y)^2\right].
\end{aligned}$$

(a) Show that Stommel's model has a globally attracting slow manifold that governs the asymptotic behavior of THC. Find a leading order approximation to this manifold. (*Hint*: rescale time by letting $s = t/\tau_y$.)

(b) Compute the leading-order reduced flow on the slow manifold. Determine qualitatively the possible dynamical behaviors on the slow manifold as the parameters μ and η are varied.