1. Consider the nonlinear oscillator

$$\ddot{x} + \omega_0^2 x = \varepsilon M x^2,$$

where $\varepsilon M x^2$ represents a small nonlinear forcing term ($0 \leq \varepsilon \ll 1$, $M > 0$)

Using Lindstedt's method, find an $\mathcal{O}(\varepsilon)$ approximation for nonlinear periodic motions as a function of their initial position, with zero initial velocity.

$$\ddot{x} + \omega_0^2 x = \varepsilon M x^2 \quad , 0 < \varepsilon \ll 1 \, , \quad M > 0, \omega_0 \neq 0$$

$$x(0) = a_0, \quad \dot{x}(0) = 0$$

- Seek solutions of the form $x_\varepsilon(t) = \varphi_0(t;\varepsilon) + \varepsilon \varphi_1(t;\varepsilon) + \mathcal{O}(\varepsilon^2)$

  $\varphi_i(t,\varepsilon) = \varphi_i(t + T_\varepsilon; \varepsilon)$

- Rewrite period as $T_\varepsilon = \dfrac{2\pi}{\omega(\varepsilon)}$, $\omega(\varepsilon) = \omega_0 + \varepsilon \omega_1 + \mathcal{O}(\varepsilon^2)$

- Rescale time : $\tau = \omega(\varepsilon) t \Rightarrow \boxed{\begin{aligned} \frac{d}{d\tau} &= \frac{1}{\omega(\varepsilon)} \frac{d}{dt} \\ (\cdot)' &= \frac{1}{\omega(\varepsilon)} \dot{(\cdot)} \end{aligned}} \Rightarrow \boxed{(\omega(\varepsilon))^2 x'' + \omega_0^2 x = \varepsilon M x^2}$

- Plug in new Ansatz into rescaled eq.

$$\left[\omega_0^2 + 2\varepsilon\omega_0\omega_1 + O(\varepsilon^2)\right]\left[\varphi_0'' + \varepsilon\varphi_1'' + O(\varepsilon^2)\right] + \omega_0^2\left[\varphi_0 + \varepsilon\varphi_1 + O(\varepsilon^2)\right] = \varepsilon M\left[\varphi_0^2 + O(\varepsilon)\right]$$

- Collect terms of equal power in ε

$O(1)$

$$\omega_0^2\varphi_0'' + \omega_0^2\varphi_0 = 0, \quad \varphi_0(0) = a_0, \quad \mathring{\varphi}_0(0) = 0$$

$$\Rightarrow \quad \varphi_0 = a_0\cos\tau$$

$O(\varepsilon)$

$$\omega_0^2\varphi_1'' + \omega_0^2\varphi_1 = M\varphi_0^2 - 2\omega_0\omega_1\varphi_0'' = Ma_0^2\cos^2\tau + 2a_0\omega_0\omega_1\cos\tau$$

$$= \frac{Ma_0^2}{2}\left[1 + \cos 2\tau\right] + \underbrace{2a_0\omega_0\omega_1\cos\tau}_{\text{resonance}}$$

Select $\omega_1 = 0$ to eliminate resonance terms and obtain periodic solution.

Solve for φ_1:

$$\varphi_1'' + \varphi_1 = \frac{Ma_0^2}{2\omega_0^2}\left[\cos 2\tau + 1\right], \quad \varphi_1(0) = 0, \quad \mathring{\varphi}_1(0) = 0. \qquad \text{—(1)}$$

Pick solution Ansatz

$$\varphi_1(t) = A\cos\tau + B\sin\tau + C\cos 2\tau + D\sin 2\tau + E$$

Substituting in (1):

$$-A\cos\tau - B\sin\tau - 4C\cos 2\tau - 4D\sin 2\tau + A\cos\tau + B\sin\tau + C\cos 2\tau + D\sin 2\tau + E = \frac{Ma_0^2}{2\omega_0^2}\cos 2\tau + \frac{Ma_0^2}{2\omega_0^2}$$

Comparing coefficients:

$$\Rightarrow E = \frac{Ma_0^2}{2\omega_0^2}, \quad C = -\frac{Ma_0^2}{6\omega_0^2}, \quad D = 0$$

$$\varphi_1(0) = 0 \Rightarrow A + C + E = 0 \Rightarrow A = -\frac{Ma_0^2}{3\omega_0^2}$$

$$\varphi_1'(0) = 0 \Rightarrow B + 2D = 0 \Rightarrow B = 0.$$

$$\Rightarrow \boxed{\varphi_1(\tau) = -\frac{Ma_0^2}{3\omega_0^2}\cos\tau - \frac{Ma_0^2}{6\omega_0^2}\cos 2\tau + \frac{Ma_0^2}{2\omega_0^2}}$$

In original time:

$$x_\varepsilon(t) = a_0\cos\omega t + \varepsilon\frac{Ma_0^2}{\omega_0^2}\left[-\frac{1}{3}\cos\omega t - \frac{1}{6}\cos 2\omega t + \frac{1}{2}\right] + O(\varepsilon^2)$$

where

$$\omega = \omega_0 + O(\varepsilon^2)$$

2. Consider the forced *van der Pol equation*

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = F\cos\omega t,$$

which arises in models of self-excited oscillation, such as those of a valve generator with a cubic valve characteristic. Here $F, \omega > 0$ are parameters, and $0 \le \varepsilon \ll 1$.

(i) For small values of ε, find an approximation for an **exactly** $2\pi/\omega$–periodic solution of the equation. The error of your approximation should be $\mathcal{O}(\varepsilon)$.

(i) Seek solutions of the form: $x_\epsilon(t) = \varphi_0(t) + \epsilon\varphi_1(t) + O(\epsilon^2)$

Substituting this solution in the ODE $\ddot{x} + \epsilon(x^2-1)\dot{x} + x = F\cos\omega t$ we get

$$\ddot{\varphi}_0 + \varphi_0 + \epsilon\left(\ddot{\varphi}_1 + \varphi_1 + \varphi_0^2\dot{\varphi}_0 - \dot{\varphi}_0\right) + O(\epsilon^2) = F\cos\omega t$$

$$\Rightarrow \begin{cases} O(1): & \ddot{\varphi}_0 + \varphi_0 = F\cos\omega t & (1) \\ O(\epsilon): & \ddot{\varphi}_1 + \varphi_1 = \dot{\varphi}_0(1-\varphi_0^2) & (2) \end{cases}$$

$$\text{Eq. (1)} \Rightarrow \varphi_0(t) = A\sin t + B\cos t + \frac{F\cos\omega t}{1-\omega^2} \qquad (3)$$

Since we seek solutions with period $\frac{2\pi}{\omega}$ for any $0 \leq \epsilon << 1$, the period of each φ_i must be $\frac{2\pi}{\omega}$. This holds in particular for φ_0. Therefore, we must enforce $A = B = 0$ in Eq. (3).

This condition can be enforced by choosing appropriate initial conditions for the ODE (1):

$$A = B = 0 \Rightarrow \boxed{\varphi_0(t) = \frac{F\cos\omega t}{1-\omega^2}, \quad \varphi_0(0) = \frac{F}{1-\omega^2}, \quad \dot{\varphi}_0(0) = 0} \qquad (4)$$

$$x_\varepsilon(t) = \varphi_0(t) + O(\varepsilon)$$

error term

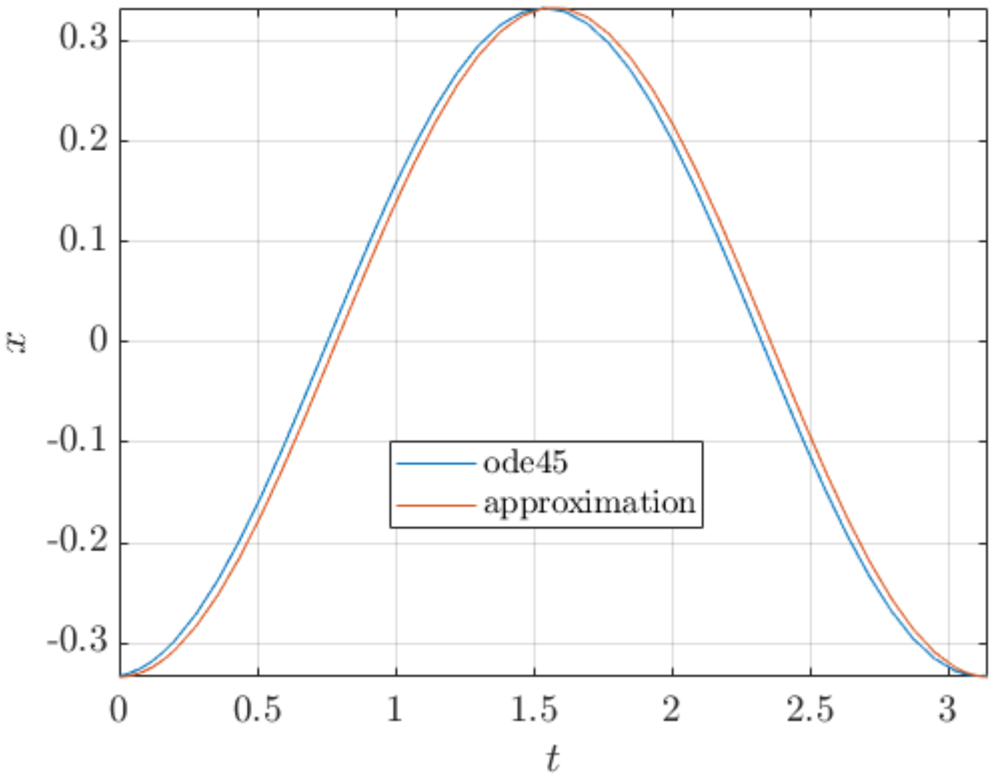
(ii) For $\varepsilon = 0.1$, $\omega = 2$, and $F = 1$, verify your prediction numerically by solving the equation numerically. Plot your numerical solution along with your analytic prediction computed in (i).

We solve the ODE numerically to obtain a solution $x(t)$ and compare this solution to the perturbed approximation $x_\epsilon(t)$ given by (4)

The I.C. for the ODE are chosen s.t. $x(0) = x_\epsilon(0)$ and $\dot{x}(0) = \dot{x}_\epsilon(0)$

Therefore, $x(0) = \varphi_0(0)$, $\dot{x}(0) = \dot{\varphi}_0(0)$ where $\varphi_0(0)$ and $\dot{\varphi}_1(0)$ are given in (4).

$$z_1 = x, \quad z_2 = \dot{x}, \qquad \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ F\cos(\omega t) - \epsilon(1 - z_1^2)z_2 - z_1 \end{bmatrix}$$



0.3
0.2
0.1
0
-0.1
-0.2
-0.3
x
0
0.5
1
1.5
2
2.5
3
t
ode45
approximation

3. Consider a ball of mass m that slides on a rotating hoop (see Fig. 1).

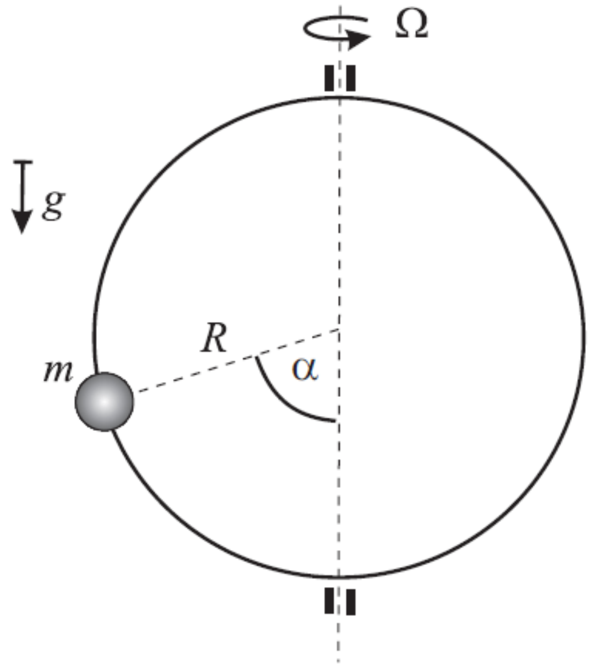


Figure 1: Mass on a loop

The angular velocity of the hoop is Ω, the viscous friction coefficient between the hoop and the ball is b, and the constant of gravity is g. The equation of motion for the sliding ball is given by

$$mR^2\ddot{\alpha} + bR^2\dot{\alpha} + mR^2\left(g/R - \Omega^2\cos\alpha\right)\sin\alpha = 0.$$

(a) Plot the location of equilibria of the ball as a function of the non-dimensionalized rotation parameter $\nu = R\Omega^2/g$.

(a) Defining $x_1 = \alpha$, $x_2 = \dot{\alpha}$, write the ODE as $\dot{x} = f(x)$ where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad f(x) = \begin{pmatrix} x_2 \\ -\frac{g}{R}(1-\nu\cos x_1)\sin x_1 - b/m\, x_2 \end{pmatrix}$$

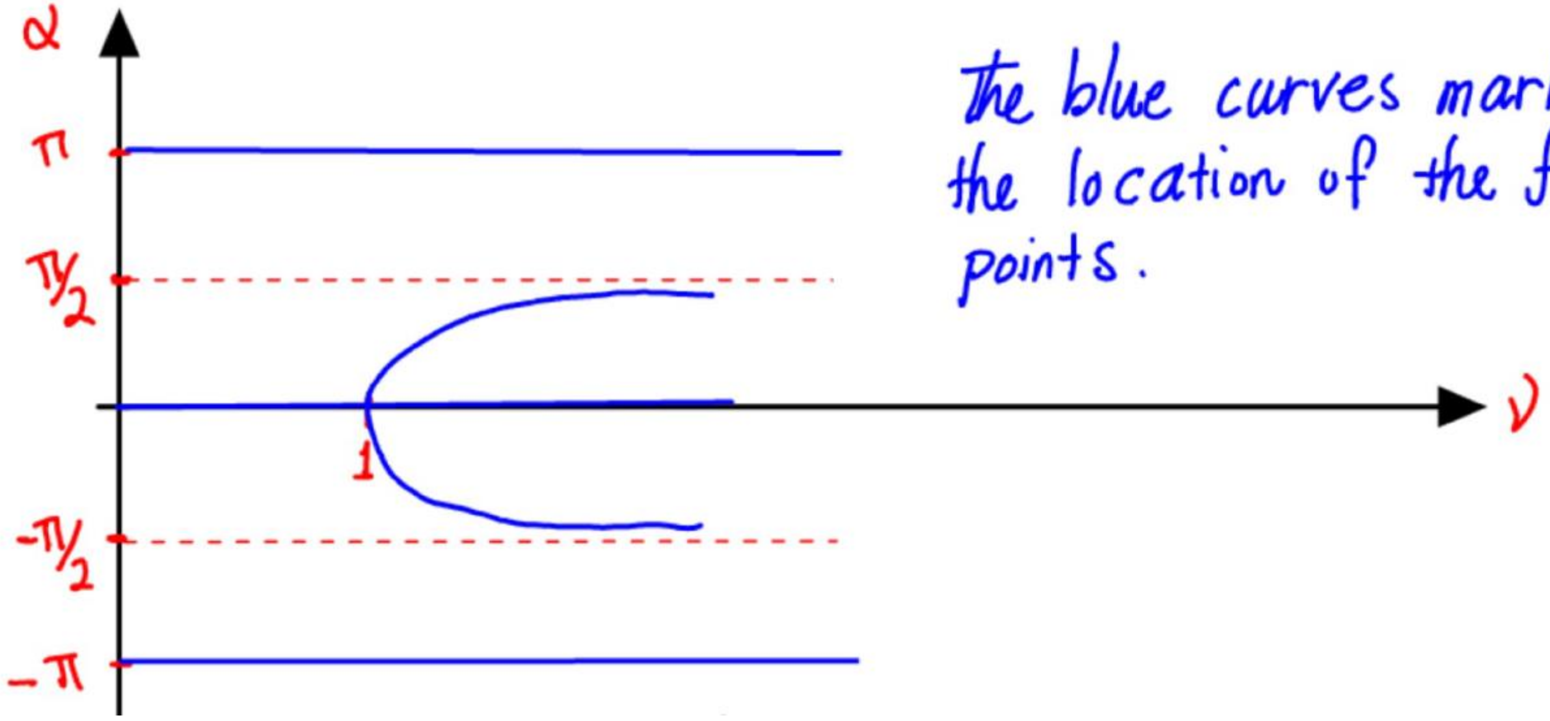
Fixed points: $\quad f(x) = 0 \quad \Rightarrow \quad x_2 = 0$ and $(1-\nu\cos x_1)\sin x_1 = 0$

Case I when $\underline{\nu < 1}$ only two fixed points exist: $(0,0)$ and $(\pi, 0)$

[Note: the fixed point $(-\pi, 0)$ is physically identical to the fixed point $(\pi, 0)$. Therefore, we only discuss $(\pi, 0)$]

Case II When $\underline{\nu > 1}$ two additional fixed points emerge that correspond to the solutions of $\cos x_1 = \frac{1}{\nu}$.

Let $\alpha_0 \in (0,\pi)$ be the positive solution : $\cos\alpha_0 = \frac{1}{\nu}$. Then the fixed points in this case are : $(0,0)$, $(\pi,0)$, $(\alpha_0,0)$, $(-\alpha_0,0)$



the blue curves mark the location of the fixed points.

(b) Using linearization, determine the stability type of the different equilibrium branches on the plot. Identify the critical angular velocity at which a bifurcation of equilibria occurs

First we compute $\nabla f(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ \frac{g}{R}\left(2\nu\cos^2 x_1 - \cos x_1 - \nu\right) & -b/m \end{pmatrix}$

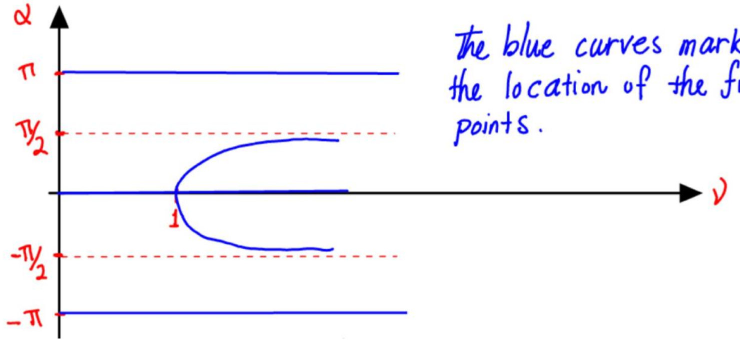
whoes eigenvalues are given by

$$\lambda_{\pm} = \frac{-b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 + g/R\left(2\nu\cos^2 x_1 - \cos x_1 - \nu\right)}$$

Now we investigate the linear stability of each fixed point:

fixed point (0,0):

$$\lambda_{\pm} = \frac{-b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 + \frac{g}{R}(\nu - 1)}$$



The blue curves mark the location of the fixed points.

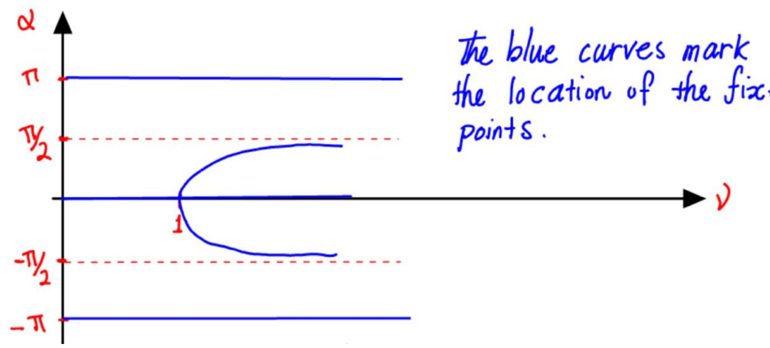
- $\nu < 1 \Rightarrow \mathrm{Re}(\lambda_+) < 0$ and $\mathrm{Re}(\lambda_-) < 0$

  $\Rightarrow$ (0,0) is asymptotically stable.

- $\nu > 1 \Rightarrow \mathrm{Re}(\lambda_+) > 0$ and $\mathrm{Re}(\lambda_-) < 0$

  $\Rightarrow$ (0,0) is unstable.

fixed points $(\pm\pi, 0)$



$$\lambda_\pm = \frac{-b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 + \frac{g}{R}(\nu+1)}$$

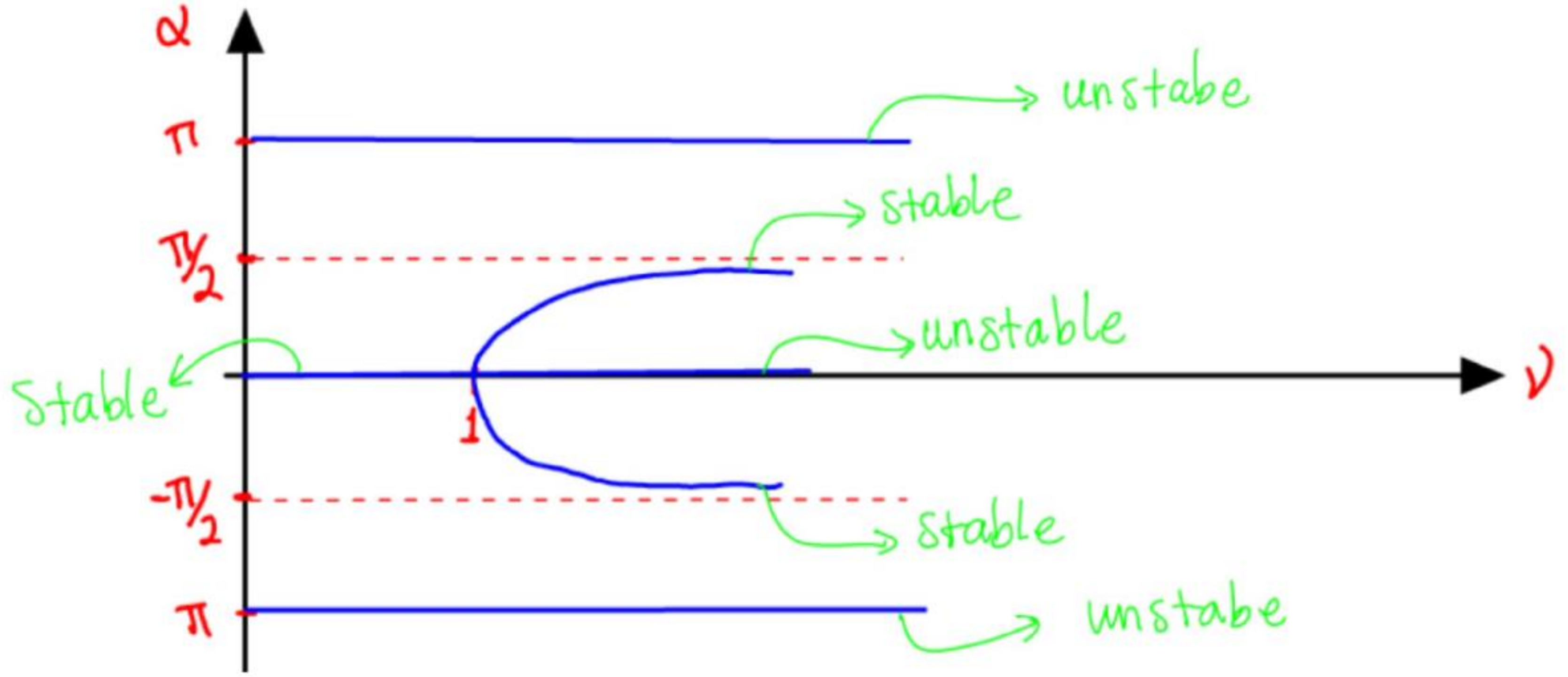
For any $\nu \geq 0$, $\operatorname{Re}(\lambda_+) > 0$ $\Rightarrow$ $(\pm\pi, 0)$ is unstable for any $\nu \geq 0$

fixed points $(\pm\alpha_0, 0)$

Remember that these fixed points only exist when $\nu > 1$. Also $\cos(\pm\alpha_0) = \frac{1}{\nu}$

$$\lambda_\pm = \frac{-b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 + \frac{g}{R}\left(\frac{1-\nu^2}{\nu}\right)}$$

For any $\nu > 1$, $\operatorname{Re}(\lambda_+) < 0$ and $\operatorname{Re}(\lambda_-) < 0$ $\Rightarrow$ fixed points $(\pm\alpha_0, 0)$ are asymptotically stable.



The bifurcation of equilibria occurs at $\nu = 1 \Rightarrow \Omega^2 = \dfrac{g}{R} \Rightarrow \Omega = \pm\sqrt{\dfrac{g}{R}}$

4. Consider a discrete dynamical system given by the iterated mapping

$$x_{k+1} = f(x_k), \qquad f\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad x_k \in \mathbb{R}^n.$$

Assume that $x = p$ is a fixed point for the mapping, i.e., $p = f(p)$.

(a) Derive a linearized mapping of the form

$$y_{k+1} = Ay_k \tag{1}$$

to describe the discrete dynamics in the vicinity of the fixed point.

(a) Let x_k be near the fixed point p and define $y_k := x_k - p$

then

$$x_{k+1} = f(x_k) = f(p + y_k) = f(p) + Df(p)\, y_k + O(\|y_k\|^2)$$

$$= p + Df(p)\, y_k + O(\|y_k\|^2)$$

$$\Rightarrow \quad y_{k+1} = x_{k+1} - p = Df(p)\, y_k + O(\|y_k\|^2)$$

Now for $\|y_k\|$ small enough the linear approximation of the map $x_{k+1} = f(x_k)$ is $y_{k+1} = Ay_k$ with $A = Df(p)$.

(b) Assume that A has eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with corresponding n linearly independent eigenvectors $s_1, \ldots, s_n \in \mathbb{C}^n$. Show that the general solution of (1) is of the form

$$y_k = c_1\varphi_1(k) + \ldots + c_n\varphi_n(k), \qquad \varphi_i(k) = \lambda_i^k s_i. \tag{2}$$

(b) Take any $y_0 \in \mathbb{R}^n$. Since $s_1, \ldots, s_n \in \mathbb{C}^n$ are linearly independent, there are constants $c_1, \ldots, c_n \in \mathbb{C}$ s.t. $y_0 = c_1 s_1 + \cdots + c_n s_n$.

Now define

$$y_1 = Ay_0 = c_1 A s_1 + \cdots + c_n A s_n$$
$$= c_1 \lambda_1 s_1 + \cdots + c_n \lambda_n s_n$$
$$= c_1 \varphi_1(1) + \cdots + c_n \varphi_n(1)$$

Similarly, for any $k \geq 1$,

$$y_k = A y_{k-1} = c_1 \lambda_1^{k-1} A s_1 + \cdots + c_n \lambda_n^{k-1} A s_n$$
$$= c_1 \varphi_1(k) + \cdots + c_n \varphi_n(k) \qquad (1)$$

It's easy to check that $y_{k+1} = A y_k$ for any $k \geq 0$. Since $y_0 \in \mathbb{R}^n$ was arbitrary, $c_1 \varphi_1(k) + c_2 \varphi_2(k) + \cdots + c_n \varphi_n(k)$ is a general solution of $y_{k+1} = A y_k$.

(c) Formulate a definition of stability, asymptotic stability, and instability for the $y = 0$ fixed point of (1).

(c) Def. of stability : $\forall \epsilon > 0$, $\exists\, \delta(\epsilon) > 0$ s.t. $\forall\, y_0 \in \mathbb{R}^n$ with $\|y_0\| \leq \delta$ we have $\|y_k\| \leq \epsilon$ for any $k \geq 0$.

Def. of asymptotic Stability :

$y = 0$ is asymptotically stable iff :

(i) $y = 0$ is stable

(ii) $\exists\, \delta > 0$ s.t. $\forall y_0 \in \mathbb{R}^n$ with $\|y_0\| < \delta$ we have $\lim_{k \to \infty} \|y_k\| = 0$

Def. of instability : $y = 0$ is unstable if it's not stable!

(d) Using (2), find a sufficient and necessary condition for the asymptotic stability you have defined in (c).

We claim that the necessary and sufficient condition for asymptotic stability of the origin is $|\lambda_i| < 1$ for $i = 1, 2, \dots, n$

Sufficient: From (c) any solution of $y_{k+1} = A y_k$ can be written as

$$y_{k+1} = \sum_{i=1}^{n} c_i \lambda_i^k s_i$$

Without loss of generality, we assume that the eigenvectors s_i are normalized, i.e., $\|s_i\| = 1 \quad \forall\, i \in \{1, 2, \dots, n\}$.

Then $$\|y_{k+1}\| \le \sum_{i=1}^{n} |c_i| |\lambda_i|^k \|s_i\| = \sum_{i=1}^{n} |c_i| |\lambda_i|^k$$

But since $|\lambda_i|<1$, we have $\lim_{k\to\infty} |\lambda_i|^k = 0$. Which implies

$$\lim_{k\to\infty} \sum_{i=1}^{n} |c_i|\,|\lambda_i|^k = 0$$

Hence, $\lim_{k\to\infty} \|y_{k+1}\| = 0$. (2)

Also note that since $|\lambda_i|<1$ $\forall i\in\{1,\dots,n\}$, the matrix norm $\|A\|<1$. Hence, $\|y_{k+1}\| = \|Ay_k\| < \|y_k\|$. $\Rightarrow$ $y=0$ is also stable. This together with (2) implies asy. stability of the fixed point $y=0$.

<u>Necessity</u>: Assume there is $i_0 \in \{1, 2, \cdots, n\}$ s.t. $|\lambda_{i_0}| \geq 1$.

It is enough to show that $\exists\, y_0 \in \mathbb{R}^n$ s.t. $\lim_{k \to \infty} \|A^k y_0\| \neq 0$

[this is due to the fact that $y_k = A^k y_0$ and that one can rescale y_0 as $r y_0$ for $0 < r \ll 1$ small enough s.t. $\|r y_0\| < \delta$, $\forall \delta > 0$]

To show that such $y_0 \in \mathbb{R}^n$ exists, note that $\|A^k s_{i_0}\| = \|\lambda_{i_0}^k s_{i_0}\| = |\lambda_{i_0}|^k \geq 1$ $\forall k \geq 0$

This is, however, not enough since $s_{i_0} \in \mathbb{C}^n$ while we need a vector in $\mathbb{R}^n$.

To complete the proof, note that $s_{i_0} = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}^n$.

$$\Rightarrow 1 \leq \|A^k s_{i_0}\|^2 = \|A^k \xi + i A^k \eta\|^2 = \|A^k \xi\|^2 + \|A^k \eta\|^2$$

therefore, either $\|A^k \xi\| \geq \frac{1}{\sqrt{2}}$ or $\|A^k \eta\| \geq 1/\sqrt{2}$.

Without loss of generality assume $\|A^k \xi\| \geq 1/\sqrt{2}$. Now let $y_0 = \xi$ to get

$$\|y_k\| = \|A^k y_0\| \geq \frac{1}{\sqrt{2}} \Rightarrow \lim_{k\to\infty} \|y_k\| \neq 0$$

true for every $k \geq 0$