Nonlinear Dynamics & Chaos I

Exercice Set 7 Solutions

Question 1

Consider a planar Hamiltonian system

$$\dot{x} = \frac{\partial H(x,y)}{\partial y} + f_1(x,y),$$
$$\dot{y} = -\frac{\partial H(x,y)}{\partial x} + f_2(x,y),$$

where the twice continuously differentiable function $H(x,y)$ is the Hamiltonian associated with the system (say, the energy in classical mechanics, or the stream function in fluid mechanics), and the continuously differentiable $\mathbf{f} = (f_1, f_2)$ is a dissipative term (say, damping in classical mechanics, or compressible terms in fluid mechanics). Assume that $\nabla \cdot \mathbf{f} \neq 0$ for all $(x,y) \in \mathbb{R}^2$. (Linear damping, for instance has this property.)
Show that the above system can have no limit cycles.

Solution 1

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underline{F}(x,y) := \begin{bmatrix} \frac{\partial H}{\partial y}(x,y) + f_1(x,y) \\ -\frac{\partial H}{\partial x}(x,y) + f_2(x,y) \end{bmatrix} \tag{1}$$

$$\begin{aligned} \operatorname{div}(\underline{F}) &= \frac{\partial^2 H}{\partial x \partial y}(x,y) + \frac{\partial f_1}{\partial x}(x,y) - \frac{\partial^2 H}{\partial y \partial x}(x,y) + \frac{\partial f_2}{\partial y} \\ &= \operatorname{div}(\underline{f}) \qquad (H \in C^2) \\ &\neq 0 \quad \forall (x,y) \in \mathbb{R}^2 \end{aligned}$$

Thus, by the Bendixson's criterion, (1) does not have a periodic solution in $\mathbb{R}^2$.

Solution 2

Consider a planar dynamical system with the following phase portrait:

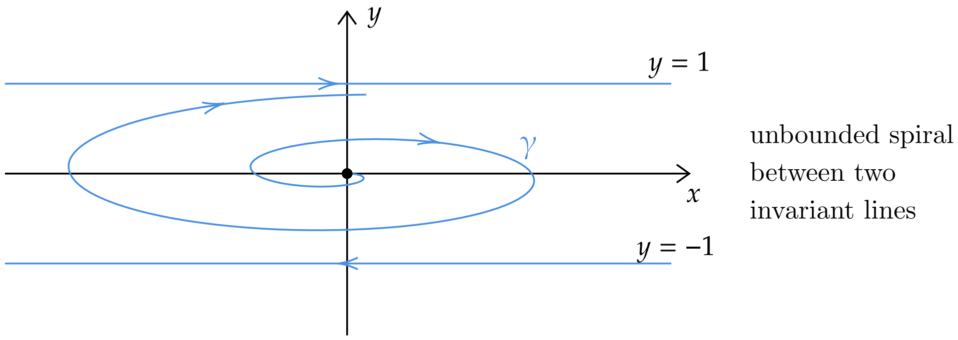

Figure 1: Phase portrait of the planar dynamical system

Which of the following statement is true?

(a) The ω-limit set of γ is empty.

(b) By the Poincaré-Bendixson theorem, the ω-limit set of γ is composed of the lines $y = 1$ and $y = -1$.

(c) The Poincaré-Bendixson theorem does not apply to γ.

(d) None of the above

Question 3 - Accuracy of averaging

Show that on time scales of $\mathcal{O}(1/\varepsilon)$, a solution $\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ of the dynamical system

$$\dot{\mathbf{x}} = \varepsilon \mathbf{f}(\mathbf{x}, t, \varepsilon), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{2}$$

(ε is a small parameter and $\mathbf{f}$ is a smooth function that is T-periodic in time) remains $\mathcal{O}(\varepsilon)$-close to any solution $\mathbf{y}(t)$ with $\mathbf{y}(0) = \mathbf{x}_0 + \mathcal{O}(\varepsilon)$ of the averaged system

$$\dot{\mathbf{y}} = \varepsilon \bar{\mathbf{f}}_0(\mathbf{y}), \qquad \mathbf{y} \in \mathbb{R}^n, \tag{3}$$

where

$$\bar{\mathbf{f}}_0(\mathbf{y}) = \frac{1}{T}\int_0^T f(y, t, 0)\,\mathrm{d}t.$$

Hint: Subtract (3) from (2) and integrate to obtain an expression for $|\mathbf{x}(t) - \mathbf{y}(t)|$. Estimate $|\mathbf{x}(t) - \mathbf{y}(t)|$ from above using the facts that $\bar{\mathbf{f}}$ is Lipschitz and $|\hat{f} - \bar{f}|/\varepsilon$ is uniformly bounded, where $\hat{f}$ is the right-hand-side of the system into which (2) is transformed by the averaging transformation $\mathbf{x} = \mathbf{y} + \varepsilon \mathbf{w}(\mathbf{y}, t)$. Then use the following generalized Gronwall inequality:
If $u(t), v(t), c(t)$ are non-negative functions, $c(t)$ is differentiable, and

$$v(t) \leq c(t) + \int_0^t u(s)v(s)\,\mathrm{d}s,$$

then

$$v(t) \leq c(0)e^{\int_0^t u(s)\,\mathrm{d}s} + \int_0^t c'(s)e^{\int_s^t u(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s.$$

Solution 3

Remember from the lecture on averaging that $\dot{x} = \varepsilon f(x, t, \varepsilon)$ can be transformed to the differential equation

$$\dot{\tilde{x}} = \varepsilon \bar{f}_0(\tilde{x}) + \varepsilon^2 f_1(\tilde{x}, t, \varepsilon) \tag{4}$$

Through the near-identity transformation $x = \tilde{x} + \varepsilon w(\tilde{x}, t)$.
Moreover, f_1 is globally bounded, i.e., there exists $L_1 > 0$ such that

$$|f_1(\tilde{x}, t, \varepsilon)| < L_1 \qquad \forall t > 0 \text{ and } \forall \tilde{x} \in \mathbb{R}^n$$

Now by construction $|x(t) - \tilde{x}(t)| = \varepsilon |w(\tilde{x}(t), t)| = \mathcal{O}(\varepsilon)$ Therefore, it suffices to show that solutions of the averaged equation

$$\dot{y} = \varepsilon \bar{f}_0(y) \tag{5}$$

remain $\mathcal{O}(\varepsilon)$ close to the solutions of (4).
Subtracting (5) from (4), integrating and dropping the tilde ($\tilde{\ }$) sign, we get

$$x(t) - y(t) = x_0 - y_0 + \varepsilon \int_0^t \left(\bar{f}_0(x(s)) - \bar{f}_0(y(s))\right)\,\mathrm{d}s + \varepsilon^2 \int_0^t f_1(x(s), s, \varepsilon)\,\mathrm{d}s$$

$$\implies |x(t) - y(t)| \leq |x_0 - y_0| + \varepsilon \int_0^t L_2 |x(s) - y(s)|\,\mathrm{d}s + \varepsilon^2 \int_0^t L_1\,\mathrm{d}s$$

where we used boundedness of f_1 and Lipschitz continuity of $\bar{f}_0$:

$$|f_0(x) - f_0(y)| \leq L_2 |x - y|$$

Therefore,

$$|x(t) - y(t)| \leq |x_0 - y_0| + \varepsilon^2 L_1 t + \int_0^t \varepsilon L_2 |x(s) - y(s)|\,\mathrm{d}s \tag{6}$$

Now apply Gronwall's inequality with $v(t) = |x(t) - y(t)|$, $u(t) = \varepsilon L_2$ and $c(t) = |x_0 - y_0| + \varepsilon^2 L_1 t$ to get

$$\begin{aligned} |x(t) - y(t)| &\leq |x_0 - y_0| e^{\varepsilon L_2 t} + \int_0^t \varepsilon L_1 e^{\varepsilon L_2 (t-s)}\,\mathrm{d}s \\ &= |x_0 - y_0| e^{\varepsilon L_2 t} + \varepsilon \frac{L_1}{L_2}\left(e^{\varepsilon L_2 t} - 1\right) \leq \left[|x_0 - y_0| + \varepsilon \frac{L_1}{L_2}\right] e^{\varepsilon L_2 t} \end{aligned}$$

Since $|x_0 - y_0| = \mathcal{O}(\varepsilon)$, we conclude that $|x(t) - y(t)| = \mathcal{O}(\varepsilon)$ as long as $t \in \left[0, \frac{1}{\varepsilon L_2}\right)$, i.e., time scales of $\mathcal{O}(1/\varepsilon)$.

Question 4 - Unsteady separation in time-periodic fluid flows

Fluid trajectories $\mathbf{x}(t) = (x(t), y(t))$ in a two-dimensional time-periodic flow satisfy the differential equations

$$\begin{aligned} \dot{x} &= u(x, y, t), \qquad u(x, y, t) = u(x, y, t + T), \\ \dot{y} &= v(x, y, t), \qquad v(x, y, t) = v(x, y, t + T), \end{aligned} \tag{7}$$

where $T > 0$ is the period, u and v are smooth velocity components satisfying the incompressibility condition $u_x + v_y \equiv 0$. Assume that the fluid is bounded by a wall at $y = 0$, on which the velocity field satisfies the no-slip boundary conditions $u(x, 0, t) = v(x, 0, t) = 0$. As a result, all boundary points are nonhyperbolic fixed points for (7).
We say that a boundary point $\mathbf{p}_0 = (x_0, 0)$ is a separation point for the flow (7) if $\mathbf{p}_0$ admits an unstable manifold $W^u(\mathbf{p}_0)$. Physically, $W^u(\mathbf{p}_0)$ is a time-dependent curve of fluid particles that shrinks to $\mathbf{p}_0$ is backward time. In forward time, $W^u(\mathbf{p}_0)$ attracts fluid particles, then ejects them from the vicinity of the wall (see Fig. 1). Show that separation points satisfy the criteria

$$\int_0^T u_y(x_0, 0, t)\,\mathrm{d}t = 0, \qquad \int_0^T v_{yy}(x_0, 0, t)\,\mathrm{d}t > 0,$$

i.e., separation takes place where the average of the wall-shear is zero and the average of v_{yy} is positive.
Hint: Use incompressibility and the boundary conditions to show that (7) can be rewritten as

$$\begin{aligned} \dot{x} &= yU(x, y, t), \\ \dot{y} &= y^2 V(x, y, t). \end{aligned}$$

To focus on the vicinity of the boundary, introduce the scaled variable $y = \varepsilon\eta$, where $0 \leq \varepsilon \ll 1$. Show that the resulting $(\dot{x}, \dot{\eta})$ equations are slowly varying and find the corresponding averaged equations. For the averaged equations, rescale time on trajectories by letting $\frac{d\tau}{dt} = \eta(t)$ in order to remove the common η factor from the right-hand-side, and look for a hyperbolic fixed point with an unstable manifold off the wall. Use the averaging theorem to relate your results to the original system (7).

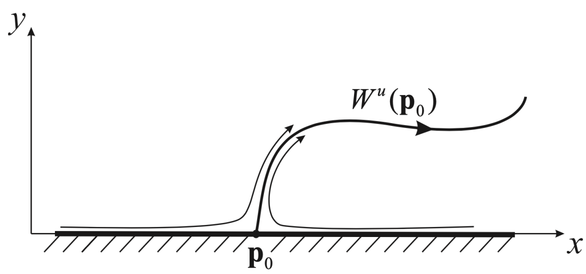

Figure 2: Unsteady separation from a no-slip wall

Solution 4

We start with the Taylor expansions of $u(x, y, t)$ and $v(x, y, t)$ in y near $y = 0$:

$$\begin{cases} u(x, y, t) = u(x, 0, t) + \dfrac{\partial u}{\partial y}(x, 0, t)y + \mathcal{O}(|y|^2) \\ v(x, y, t) = v(x, 0, t) + \dfrac{\partial v}{\partial y}(x, 0, t)y + \dfrac{1}{2}\dfrac{\partial^2 v}{\partial y^2}(x, 0, t)y^2 + \mathcal{O}(|y|^3) \end{cases} \tag{8}$$

But $u(x, 0, t) = v(x, 0, t) = 0$ for any x.
Differentiating $u(x, 0, t) = 0$ with respect to x we get $\dfrac{\partial u}{\partial x}(x, 0, t) = 0$.
By incompressibility:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$
$$\Longrightarrow \frac{\partial v}{\partial y}(x, 0, t) = -\frac{\partial u}{\partial x}(x, 0, t) = 0, \ \ \forall x$$

Hence (8) simplifies to

$$\begin{cases} u(x,y,t) = \dfrac{\partial u}{\partial y}(x,0,t)y + \mathcal{O}(|y|^2) = yU(x,y,t) \\ v(x,y,t) = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial y^2}(x,0,t)y^2 + \mathcal{O}(|y|^3) = y^2 V(x,y,t) \end{cases}$$

Also note that

$$\begin{cases} U(x,y,t) = \dfrac{\partial u}{\partial y}(x,0,t) \\ V(x,y,t) = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial y^2}(x,0,t) \end{cases} \tag{9}$$

Higher-order terms are identically zero

Therefore,

$$\begin{cases} \dot{x} = yU(x,y,t) \\ \dot{y} = y^2 V(x,y,t) \end{cases}$$

Scaling y as $y = \varepsilon\eta$, we get:

$$\begin{cases} \dot{x} = \varepsilon\eta U(x,\varepsilon\eta,t) \\ \dot{\eta} = \varepsilon\eta^2 V(x,\varepsilon\eta,t) \end{cases} \tag{10}$$

Since U and V are also T-periodic, averaging theory applies to (10) with the averaged equations

$$\begin{cases} \dot{x} = \varepsilon\eta\bar{U}(x) \\ \dot{\eta} = \varepsilon\eta^2\bar{V}(x) \end{cases} \tag{11}$$

where

$$\begin{cases} \bar{U}(x) = \dfrac{1}{T}\displaystyle\int_0^T U(x,0,s)\,\mathrm{d}s = \dfrac{1}{T}\int_0^T \dfrac{\partial u}{\partial y}(x,0,s)\,\mathrm{d}s \\ \bar{V}(x) = \dfrac{1}{T}\displaystyle\int_0^T V(x,0,s)\,\mathrm{d}s = \dfrac{1}{2T}\int_0^T \dfrac{\partial^2 v}{\partial y^2}(x,0,s)\,\mathrm{d}s \end{cases} \tag{12}$$

Rescaling time as $\dfrac{d\tau}{dt} = \eta(t)$ and denoting the derivative with respect to τ by prime sign ($'$) we get

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \eta x' \quad , \quad \dot{\eta} = \eta\eta'$$

Substituting these expressions in (12), we get

$$\begin{cases} x' = \varepsilon\bar{U}(x) \\ \eta' = \varepsilon\eta\bar{V}(x) \end{cases} \tag{13}$$

Equation (13) has a fixed point $(x_0, \eta = 0)$ on the wall if and only if $\bar{U}(x_0, 0) = 0$. Using (12), we have

$$\bar{U}(x_0) = 0 \iff \boxed{\int_0^T \frac{\partial u}{\partial y}(x_0,0,s)\,\mathrm{d}s = 0} \tag{14}$$

Now we turn to the stability of the fixed point $(x_0, 0)$ on the wall by linearising (13) around this fixed point:

$$\underset{\sim}{\xi}' = \varepsilon \underbrace{\begin{pmatrix} \dfrac{\partial \bar{U}}{\partial x}(x_0) & 0 \\ 0 & \bar{V}(x_0) \end{pmatrix}}_{:=A} \underset{\sim}{\xi}$$

The matrix A has eigenvalues $\varepsilon\dfrac{\partial\bar{U}}{\partial x}(x_0)$ and $\varepsilon\bar{V}(x_0)$ corresponding to eigenvectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ respectively.
For the unstable manifold to be off the wall, we need $\bar{V}(x_0) > 0$.
Using (12), we have

$$\bar{V}(x_0) > 0 \iff \boxed{\int_0^T \frac{\partial^2 v}{\partial y^2}(x_0,0,s)\,\mathrm{d}s > 0} \tag{15}$$

Exercice:

Show that

$$\frac{\partial \bar{U}}{\partial x}(x_0) = -2\bar{V}(x_0)$$

and hence $(x_0, 0)$ is a hyperbolic fixed point of (13) given condition (15) holds.
The conditions (14) and (15) together imply that there exists a hyperbolic fixed point of the averaged system (13).
The theory of averaging guarantees, the existence of a fixed point $(x_0^\varepsilon, 0)$ of the original time-periodic flow which is $\mathcal{O}(\varepsilon)$ close to the fixed point $(x_0^\varepsilon, 0)$ of the averaged system.
Moreover, $(x_0^\varepsilon, 0)$ has an unstable manifold $W_\varepsilon^{\mathrm{u}}$ $\mathcal{O}(\varepsilon)$-close to W^{u} of the averaged system.

Remark:

A hyperbolic fixed point of the averaged system, in general, signals a nearby limit cycle of the original system. But, in the above example, it signals a fixed point of the original system since the points on the wall don't move due to the no-slip boundary condition.