Nonlinear Dynamics and Chaos

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2022

Contents

Chapter 0

Introduction

First we shall introduce the most important characters in our following exploration. The ideas and definitions here will be recurring regularly as we examine them from different perspectives and using different tools. The content covered by this course can be found in the following books. For further details on some of the results, we recommend consulting these.

- J. Guckenheimer & P. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields
- F. Verhulst, Nonlinear Differential Equations and Dynamical Systems
- V. I. Arnold, Ordinary Differential Equations
- S. Strogatz, Nonlinear Dynamics and Chaos

Definition 0.1 (Dynamical System (DS))**.** A triple $(P, E, \mathcal{F})$, with

- P : the phase space for the dynamical variable $x \in P$,
- E : base space of the evolutionary variable (e.g. time) $t \in E$,
- $\mathcal{F}$: the evolution rule (deterministic) which defines the transition from one state to the next.

The two main types of evolutionary variable spaces are

(i) Discrete dynamical systems (DDS) $t \in E = \mathbb{Z}$ with trajectory $\{x_0, x_1, \ldots\}$,

(ii) Continuous dynamical systems (CDS) $t \in E = \mathbb{R}$ with trajectory $\{x_t\}_{t\in\mathbb{R}}$.

Corresponding to these there are various types of evolution rules

(i) In a DDS we have iterated mappings

$$\boxed{x_{n+1} = F(x_n, n).}$$

If there is no explicit dependence on n, i.e. $\frac{\partial F}{\partial n} = 0$, then

$$\boxed{x_{n+1} = F(x_n) = F(F(x_{n-1})) = \underbrace{F \circ \ldots \circ F}_{n+1 \text{ times}}(x_0) = F^{n+1}(x_0).}$$

Example 0.1 (Cobweb diagram of a one-dimensional DDS)*.* In such cases and in one-dimensional problems, a simple way to analyze the behavior of the system is the so-called *cobweb* diagram. We may plot x_{n+1} as a function of x_n, as demonstrated in Fig. 1. The image of an initial condition x_0 lies on the graph at $x_{n+1} = F(x_0)$. We can also compute the next iterate by horizontally projecting the point $(x_0, F(x_0))$ to the diagonal line defined by $x_{n+1} = x_n$. Following the projection of this point to the horizontal axis (x_n) we find the intersection with the graph at the point $(x_1, F(x_1))$. It follows, that fixed points on the cobweb diagram, correspond to the intersections of the graph of F with the diagonal line $x_{n+1} = x_n$.

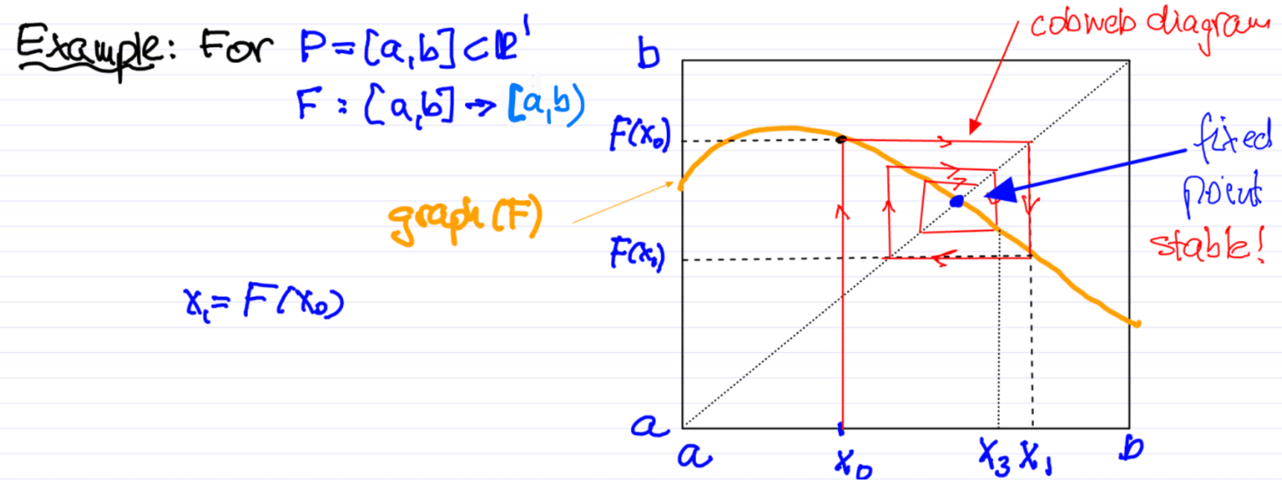

Figure 1: Analysis of a one-dimensional system defined on the interval $x \in [a, b]$ using the cobweb diagram.

(ii) In a CDS we have a first order system of ordinary differential equations (ODE)

$$\boxed{\dot{x} = f(x, t)}$$

for $x \in P$ and $t \in E$. This yields the initial value problem (IVP):

$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0 \end{cases}$$

Assuming there exists a unique solution $\varphi(t; t_0, x_0)$ with $\dot{\varphi} = f(\phi, t)$ and $\varphi(t_0) = x_0$, then the following flow map is well defined

$$\boxed{F_{t_0}^t(x_0) := \varphi(t; t_0, x_0).}$$

Geometrically, this solution can be viewed as a trajectory in phase space (cf. Fig. 2).

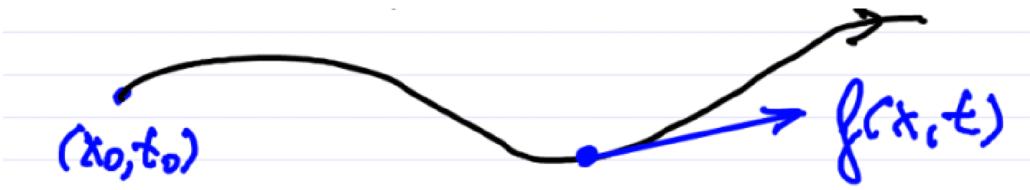

Figure 2: Trajectory of a continuous dynamical system. The right-hand-side is given by $f(x,t)$, which is the tangent vector to this curve at the point x at time t.

Such an $F_{t_0}^t$ has the properties

(i) $F_{t_0}^t$ is as smooth as $f(x,t)$,

(ii) $F_{t_0}^{t_0} = I$ and $F_{t_0}^{t_2} = F_{t_1}^{t_2} \circ F_{t_0}^{t_1}$,

(iii) $\left(F_{t_0}^t\right)^{-1} = F_t^{t_0}$ exists and is smooth.

Properties (ii) and (iii) together are called the group property. A special case of continuous dynamical systems is the autonomous system

$$\boxed{\dot{x} = f(x).}$$

The autonomy of a system implies

$$x(s, t_0, x_0) = x(\underbrace{s - t_0}_{t}, 0, x_0) \stackrel{!}{=} x(t, x_0).$$

And the induced flow map in this case is the one-parameter family of maps

$$\boxed{F^t = F_0^t : x_0 \mapsto x(t, x_0).}$$

Example 0.2 (The logisitic Equation)*.* For a resource-limited population, we have the following dynamical system for $a > 0$, $b > 0$, and the population $x \in \mathbb{R}_+ \cup \{0\}$

$$\dot{x} = ax(b - x).$$

In this case we have $E = \mathbb{R}$ and $\mathcal{F} = \{F^t\}_{t=-\infty}^{+\infty}$. This system has globally existing unique solutions (see later). We may analyze the behavior of this system by plotting the $\dot{x}$ as a function of x, analogously to the cobweb diagram. This is demonstrated in Fig. 3. At x values, where $\dot{x}$ is positive $x(t)$ is growing, while at negative values it is decreasing. This means, that fixed points, at which $x(t) = \text{const.}$ correspond to intersections of the graph with the horizontal axis.

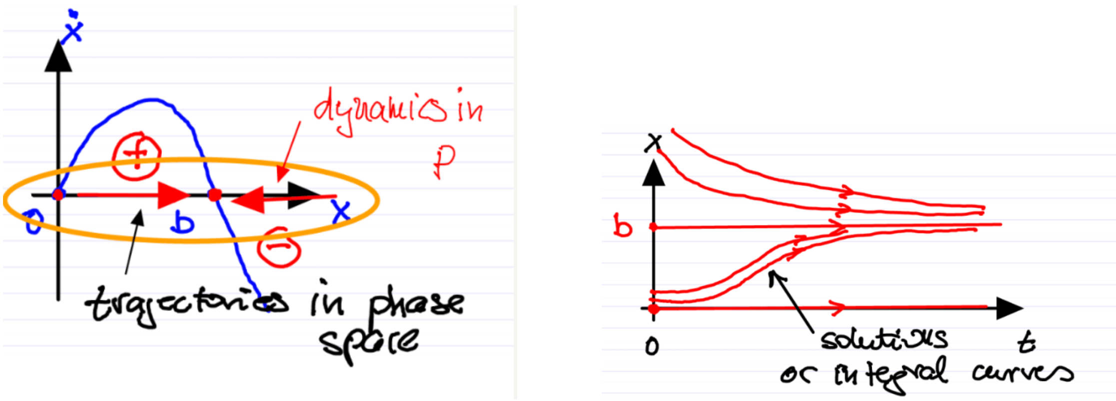

Figure 3: Left: Analysis of the right hand side. Right: Evolution in the extended phase space $P \times \mathbb{R}$.

Example 0.3 (Pendulum)*.* Given the equation of motion

$$ml^2\ddot{\varphi} = -mgl\sin(\varphi).$$

We let $x_1 = \varphi$ and $x_2 = \dot{\varphi}$ to transform into the first-order ODE form

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\dfrac{g}{l}\sin(x_1). \end{cases}$$

Thus we have

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad f(x) = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) \end{pmatrix}.$$

Qualitative analysis gives the following facts

- $(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (\pi, 0)$ are zeros of f.
- Energy is conserved, hence both small and large amplitude oscillations are expected.
- The function $f(x)$ has symmetries: it is invariant under the transformations $(x_1, x_2, t) \mapsto (x_1, -x_2, -t)$ and $(x_1, x_2, t) \mapsto (-x_1, x_2, -t)$. See the left panel of Fig. 4.

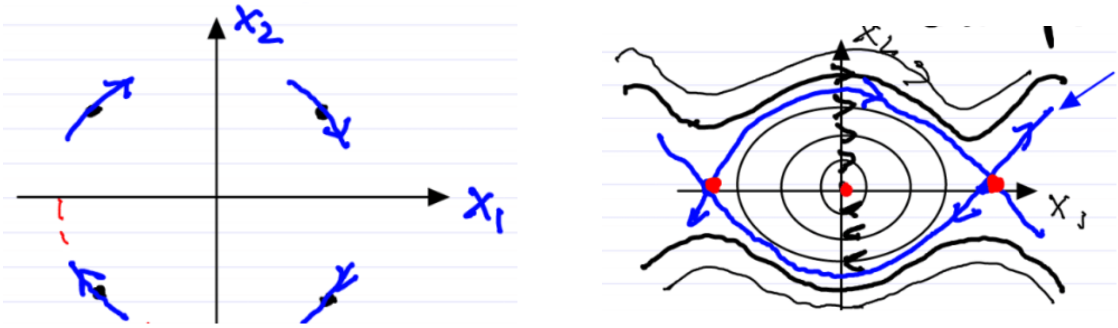

Figure 4: Left: The symmetries of the dynamic system. Right: Phase portrait of the pendulum. Red dots show the fixed points, while the blue trajectories make up the separatrix.

Definition 0.2 (Separatrix). A separatrix is a boundary (i.e., a codimension-1 surface) in phase space which separates regions of qualitatively different behaviors. In practice, it is unobservable by itself and connects different fixed points. The separatrix of the pendulum is shown in the right panel of Fig. 4.

Example 0.4 (Exploit geometry of phase space for analysis). Consider two cities, A and B. The two cities are connected by two roads, denoted by the blue and green curves of the left panel of Fig. 5. We assume that traveling on the two roads, it is possible for two bikes to make it from A to B without ever being further away from each other than a distance $d < D$.

Given this information, can two trucks make it between A and B, on different roads in the opposite direction, carrying load of width D? The two trajectories must intersect by continuity, thus at that point the trucks must be at the same positions as the bikes, implying they are within distance D. Therefore the trucks must crash! Assume two trucks are trying to make it between A and B, on different roads in the opposite direction, carrying load of width D. Can the trucks make it without hitting each other? We can view this problem as a continuous dynamical system with two coordinates, x_1 and x_2 which parametrize the two routes between A and B. This dynamical system is, in general, nonautonomous.

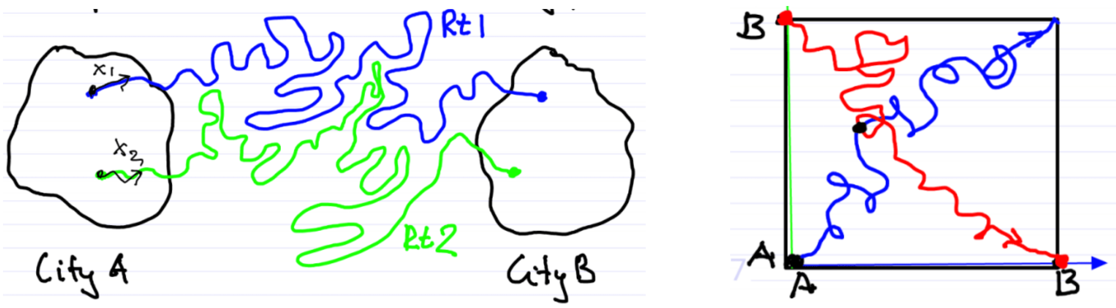

Figure 5: Left: An example of the two bike routes. Right: Blue represents the trajectory of the two bikes, red represents the trajectory of the two trucks.

The right panel of Fig. 5 shows the trajectories of the two trucks and the two bikes in phase space. The two trajectories must intersect by continuity, thus at that point the trucks must be at the same positions as the bikes, implying they are within distance D. Therefore the trucks must crash!

Chapter 1

Fundamentals

In this chapter, we first review some fundamental properties of continuous dynamical systems that will be used heavily in later chapters. As we will see, these technical results are interesting in their own right. They can help in interpreting or cross-checking numerical results or physical models for self-consistency or accuracy.

1.1 Existence and uniqueness of solutions

Consider

$$\begin{cases} \dot{x} = f(x,t); \quad x \in \mathbb{R}^n \\ x(t_0) = x_0 \end{cases}.$$

Does this initial value problem have a unique solution? We have the following theorems to help us answer that question.

Theorem 1.1 (Peano)**.** *If $f \in C^0$ near (x_0, t_0), then there exists a local solution $\varphi(t)$, i.e.,*

$$\dot{\varphi}(t) = f(\varphi(t), t), \varphi(t_0) = x_0; \ \forall t \in (t_0 - \epsilon, t_0 + \epsilon); \ 0 < \epsilon \ll 1.$$

Example 1.1 (Free falling mass). Consider a point mass of mass m at position x. The acceleration due to gravity is denoted by g. Measuring the potential energy from the reference point $x = x_0$, we have that the total energy is conserved

$$\frac{1}{2} m \dot{x}^2 = mg(x - x_0).$$

This implies that

$$\begin{cases} \dot{x} = \sqrt{2g(x - x_0)} \\ x(0) = x_0 \end{cases}$$

on the set $P = \{x \in \mathbb{R} : \ x \geq x_0\}$. Therefore we have that $f \in C^0$ in phase space, so by Peano's theorem (cf. Theorem 1.1), there exists a local solution. A schematic diagram is shown in Fig. 1.1.

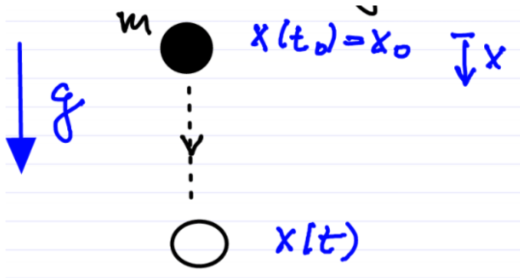

Figure 1.1: Schematic diagram of the point mass in free fall.

The solution is actually $x(t) = x_0 + \frac{g}{2}(t - t_0)^2$, however $x(t) = x_0$ is also a solution to the IVP, therefore we do not have a unique solution. Physically there exists a solution, but this IVP was derived from a heuristic energy-principle, not from Newton's laws, which are not equivalent.

Definition 1.1. A function f is called locally Lipschitz around x_0 if there exists an open set U_{x_0} and $L > 0$ such that for all $x, y \in U_{x_0}$

$$\boxed{|f(y,t) - f(x,t)| \leq L|y - x|.}$$

Example 1.2 (Lipschitz functions)*.* Fig. 1.2 shows an example of a Lipschitz and a non-Lipschitz function around x_0.

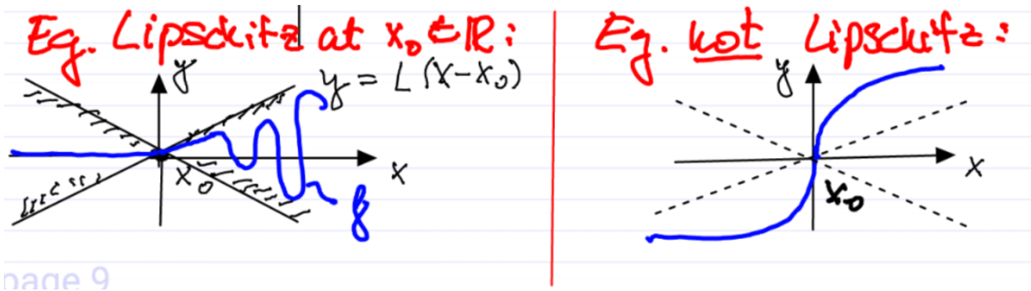

Figure 1.2: Interpretation of the Lipschitz property.

Theorem 1.2 (Picard)**.** *Assume*

(i) $f \in C^0$ *in* t *near* (t_0, x_0)*,*

(ii) f is locally Lipschitz in x near (t_0, x_0).

Then there exists a unique local solution to the IVP. The proof can be found in Arnold's book on ODEs.

Note the following relations. If f is $C^1 \implies f$ is Lipschitz $\implies f$ is C^0.

Example 1.3 (Free falling mass revisted). We check if f is Lipschitz.

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} = \frac{\sqrt{2g}}{\sqrt{|x - x_0|}} \not< L|x - x_0|.$$

Thus f is not Lipschitz near x_0.

1.2 Geometric consequences of uniqueness

If the solution is unique, we have a few facts that can be derived from the geometric point of view.

(i) The trajectories of autonomous systems cannot intersect. Note that fixed points do not violate this. See Fig. 1.3 which shows the phase portrait of the pendulum.

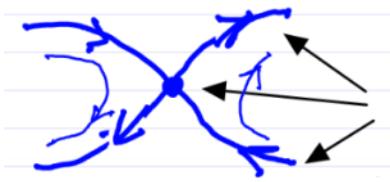

Figure 1.3: The phase portrait of the pendulum. Trajectories do not intersect since each arrow is pointing at separate trajectories.

(ii) For non-autonomous systems, intersections in phase space are possible: a trajectory may occupy the same point x at different time instants (see the left panel of Fig. 1.4). In this case we can extend the phase space in order to get an autonomous system where there cannot be any intersections.

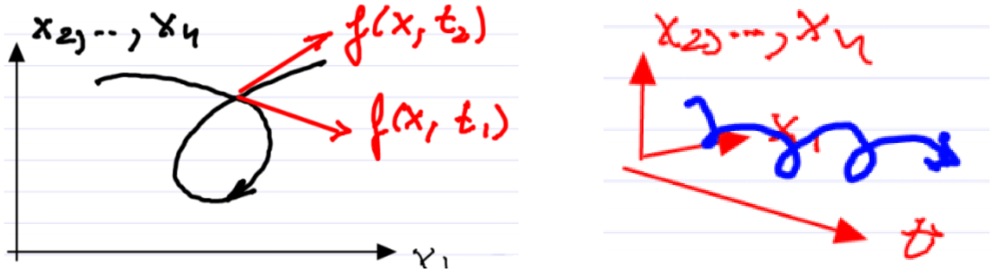

Figure 1.4: Left: Intersecting trajectories in phase space for a non-autonomous system. Right: The same trajectory in the extended phase space, without intersections.

$$X = \begin{pmatrix} x \\ t \end{pmatrix}, \ F(X) = \begin{pmatrix} f(x,t) \\ 1 \end{pmatrix}; \ \dot{X} = F(X).$$

1.3 Local vs global existence

Example 1.4 (Exploding solution).

$$\begin{cases} \dot{x} = x^2 \\ x(t_0) = 1. \end{cases}$$

Integrating yields the solution $x(t) = \frac{1}{1-(t-t_0)}$. This solution blows up at $t_\infty = t_0 + 1$, therefore the solution is only local. This is demonstrated in Fig. 1.5.

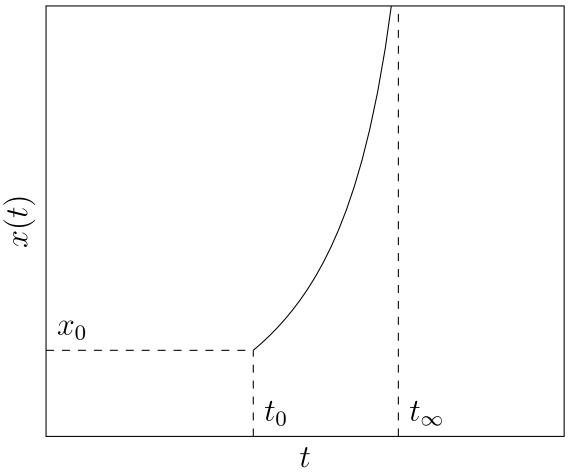

Figure 1.5: Solution to the ODE $\dot{x} = x^2$ started from $x(t_0) = 1$.

To address this problem of local solutions not being able to be continued into global solution, we have the following theorem.

Theorem 1.3 (Continuation of solution)**.** *If a local solutions cannot be continued to a time $t = T$, then we must have*

$$\boxed{\lim_{t\to T} |x(t)| = \infty.}$$

The proof can be found in Arnold's book on ODEs.

Example 1.5 (Coupled Pendulum System)*.* Consider two pendulums of masses m_1 and m_2. They both have length l. The angles of these pendula are denoted by φ_1 and φ_2. Let us assume that they are coupled by a nonlinear spring, which can be described by a potential $V(\varphi_1, \varphi_2)$. This setup is illustrated in Fig. 1.6. We set $x_1 = \varphi_1,\ x_2 = \dot{\varphi}_1,\ x_3 = \varphi_2,\ x_4 = \dot{\varphi}_2$ and get the following equation of motion

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dots \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \dots \end{cases}$$

The RHS is smooth, therefore there exists a unique local solution to any IVP. The phase space

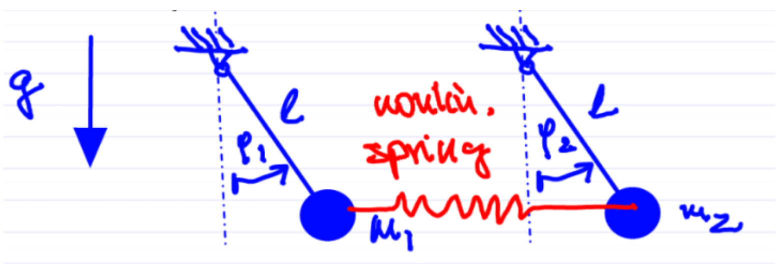

Figure 1.6: Physical setup of the coupled pendulum with a nonlinear spring.

is given by

$$P = \{x:\ x_1 \in S^1,\ x_2 \in \mathbb{R},\ x_3 \in S^1,\ x_4 \in \mathbb{R}\} = S^1 \times \mathbb{R} \times S^1 \times \mathbb{R}.$$

Where S^1 is the 1 dimensional sphere (i.e. a circle). With this space we know that $|x_1|$ and $|x_3|$ are bounded. Due to energy being conserved we have

$$E = T + V = \frac{1}{2} m_1 l_1 x_2^2 + \frac{1}{2} m_2 l_2 x_4^2 + \underbrace{V(x_1, x_3)}_{\geq 0}$$

$$E = E_0 = \text{constant} \geq 0.$$

Hence $|x_2|$ and $|x_4|$ are also bounded, therefore all solutions exist globally.

Definition 1.2. A linear system is one such that for $x \in \mathbb{R}^n,\ A(t) \in \mathbb{R}^{n\times n}$ and $A \in C^0$

$$\boxed{\dot{x} = A(t)x.}$$

Remark 1.4. Note that $S = \frac{1}{2}(A + A^T)$ is symmetric (i.e. $S = S^T$) and $\Omega = \frac{1}{2}(A - A^T)$ is skew symmetric (i.e. $\Omega = -\Omega^T$). Furthermore the eigenvalues of S, λ_i, are all real and their respective eigenvectors, e_i, are orthogonal.

Example 1.6 (Global existence in linear systems).

$$\begin{aligned}\langle x, \dot{x}\rangle &= \frac{1}{2}\frac{d}{dt}|x(t)|^2 = \langle x, A(t)x\rangle = \langle x, (S(t) + \Omega(t))x\rangle \\ &= \langle x, S(t)x\rangle + \underbrace{\langle x, \Omega(t)x\rangle}_{=0} \overset{(*)}{=} \sum_{i=1}^{n} \lambda_i(t)x_i^2 \\ &\leq \lambda_{\max}(t)\sum_{i=1}^{n} x_i^2 = \lambda_{\max}(t)|x(t)|^2.\end{aligned}$$

Where in $(*)$ we used that $x = \sum_{i=1}^n x_i e_i$ with $|e_i| = 1$ and $e_i \perp e_j$ for all $i \neq j$. Thus we get

$$\frac{\frac{1}{2}\frac{d}{dt}|x(t)|^2}{|x(t)|^2} \leq \lambda_{\max}(t) \implies \int_{t_0}^{t} \log\left(\frac{|x(s)|^2}{|x(t_0)|^2}\right) ds \leq \lambda_{\max}(s)ds.$$

By exponentiating both sides, we obtain

$$\boxed{|x(t)| \leq |x(t_0)| \exp\left(\int_{t_0}^{t} \lambda_{\max}(s)ds\right).}$$

Therefore, by the continuation theorem, global solutions exist as long as $\int_{t_0}^{t} \lambda_{\max}(s)ds < \infty$.

1.4 Dependence on initial conditions

Given the IVP

$$\begin{cases}\dot{x} = f(x,t) \\ x(t_0) = x_0.\end{cases}$$

With $x \in \mathbb{R}^n$ and $f \in C^r$ for some $r \geq 1$, we have the solution $x(t; t_0, x_0)$.

Question How does the solution depend on initial data? But first, why do we care about this? Because we expect solution that are robust with respect to errors and uncertainties in the initial data.

Theorem 1.5. *If $f \in C^r$ for $r \geq 1$ then $x(t; t_0, x_0)$ is C^r in (t_0, x_0). Proof in Arnold's ODE.*

The geometric meaning of this is that for $U \subset P \subset \mathbb{R}^n$ we have that $F_{t_0}^t(U)$ is a smooth deformation of U (see Fig. 1.7). It turns out $\left(F_{t_0}^t\right)^{-1} = F_t^{t_0}$ is also C^r, hence we have that $F_{t_0}^t$

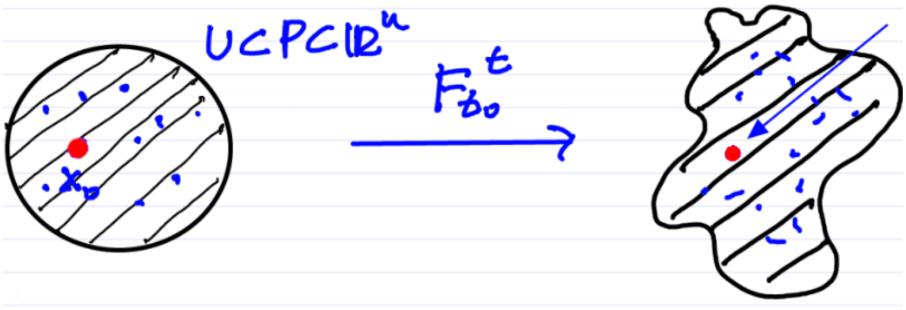

Figure 1.7: The smooth transformation of U. The red point on the right is $F_{t_0}^t(x_0)$, i.e., the image of x_0 under the evolution operator.

is a diffeomorphism.

Now, how can we compute the Jacobian of the flow map $\frac{\partial x(t;t_0,x_0)}{\partial x_0} = DF_{t_0}^t(x_0)$? We start from the IVP and take the gradient (with respect to x_0) of both sides. On the left hand side we can exchange order of the time derivative and the gradient and on the right hand side we use the chain rule. We end up with the equation

$$\frac{d}{dt}\frac{\partial x}{\partial x_0} = D_x f(x(t; t_0, x_0), t)\frac{\partial x}{\partial x_0}.$$

This means, that the flow map gradient satisfies the IVP

$$\begin{aligned} \frac{d}{dt}\left[DF_{t_0}^t(x_0)\right] &= D_x f(F_{t_0}^t(x_0), t) DF_{t_0}^t(x_0) \\ DF_{t_0}^{t_0}(x_0) &= I. \end{aligned}$$

This is called the equation of variations, which is a linear, non-autonomous ODE for the matrix $M = DF_{t_0}^t(x_0)$

$$\begin{cases} \dot{M} = D_x f(x(t; t_0, x_0))M \\ M(t_0) = I. \end{cases}$$