Nonlinear Dynamics and Chaos I
Solution guide for Problem Set 4

Shobhit Jain

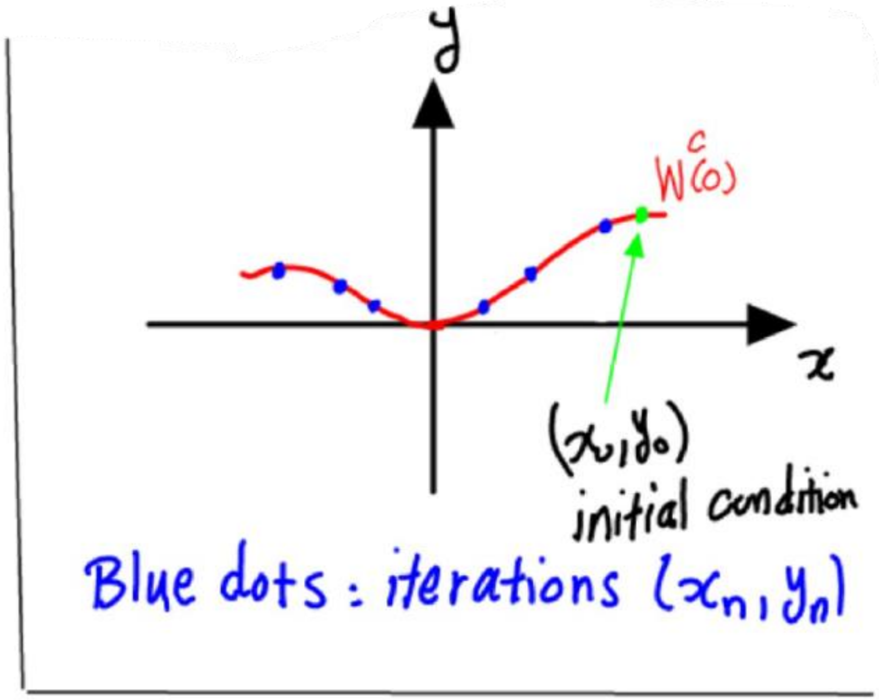
1. Consider the discrete dynamical system

$$\begin{aligned} x_{n+1} &= Ax_n + f(x_n, y_n), \\ y_{n+1} &= By_n + g(x_n, y_n), \end{aligned}$$

where $x_n \in \mathbb{R}^c$, $y_n \in \mathbb{R}^d$, $A \in \mathbb{R}^{c\times c}$, $B \in \mathbb{R}^{d\times d}$; f and g are C^r functions with no linear terms. Assume that all eigenvalues of A have modulus one, and none of the eigenvalues of B have modulus one. Then the linearized system at the origin admits a center subspace E^c aligned with the x coordinate plane.

(a) Derive a general algebraic equation for the center manifold W^c, which is known to exists by a theorem analogous to the center manifold theorem for continuous dynamical systems.

(a) Let the graph of the center manifold near the origin be given by $y = h(x)$, $h: \mathbb{R}^c \to \mathbb{R}^d$



By the invariance of the center manifold we have $y_n = h(x_n)$ for all n.

therefore, $y_{n+1} = h(x_{n+1})$

But
$$\begin{aligned} y_{n+1} &= By_n + g(x_n, y_n) \\ &= Bh(x_n) + g(x_n, h(x_n)) \end{aligned}$$

and
$$\begin{aligned} x_{n+1} &= Ax_n + f(x_n, y_n) \\ &= Ax_n + f(x_n, h(x_n)) \end{aligned}$$

Hence,
$$Bh(x_n) + g(x_n, h(x_n)) = h\big(Ax_n + f(x_n, h(x_n))\big)$$

therefore, the function $h : \mathbb{R}^c \to \mathbb{R}^d$ satisfies

$$\boxed{Bh(x) + g(x, h(x)) = h\big(Ax + f(x, h(x))\big)} \quad (1)$$

(b) Find a cubic order approximation for the center manifold of the discrete system

$$\begin{aligned} x_{n+1} &= x_n + x_n y_n, \\ y_{n+1} &= \lambda y_n - x_n^2, \end{aligned}$$

where $\lambda \in (0,1)$.

(b) Here, $A=[1]$, $B=[\lambda]$, $f(x_n, y_n) = x_n y_n$ and $g(x_n, y_n) = -x_n^2$.

Since h passes through the origin and it is tangent to the x axis, we have $h(0)=0$ and $h'(0)=0$. (Note that here $c=d=1 \Rightarrow h: \mathbb{R} \to \mathbb{R}$)

Therefore, the Taylor expansion of h around $x=0$ has the form

$$h(x) = ax^2 + bx^3 + O(x^4) \quad (2)$$

Eq. (1) for the current system is: $\lambda h(x) - x^2 = h(x + xh(x))$.

Substituting (2) in this equation we get

$$\lambda\left(ax^2+bx^3+O(x^4)\right)-x^2 = h\left(x+ax^3+O(x^4)\right)$$

$$= a\left(x+ax^3+O(x^4)\right)^2+b\left(x+ax^3+O(x^4)\right)^3+\dots$$

$$\Rightarrow (\lambda a-1)x^2+\lambda b x^3+O(x^4) = ax^2+bx^3+O(x^4)$$

Matching the exponents from both sides we obtain:

$$\left.\begin{array}{l} \lambda a-1 = a \Rightarrow a = 1/\lambda-1 \\ \lambda b = b \Rightarrow b = 0 \end{array}\right\} \Rightarrow \boxed{h(x) = \frac{1}{\lambda-1}x^2+O(x^4)}$$

(c) Reduce the dynamics to the center manifold and determine the stability of the origin. Verify your results by a numerical simulation of a few initial conditions near the origin.

(c) The center manifold near the origin satisfies $h(x) \simeq \frac{1}{\lambda-1} x^2$.
Hence, the dynamics on the center manifold satisfy

$$x_{n+1} = x_n + \frac{1}{\lambda-1} x_n^3$$

$$\Rightarrow x_{n+1} = x_n \left(1 + \frac{1}{\lambda-1} x_n^2\right) \quad (3)$$

In the following, we show that the fixed point $x=0$ of (3) is asymptotically stable.

First let $x_0 \in \mathbb{R}$ with $|x_0|>0$ small enough be an initial condition. Then if

$$\left| 1+\frac{1}{\lambda-1}x_0^2 \right| < 1 \qquad (4)$$

we have

$$|x_1| \leq \left| x_0\left(1+\frac{1}{\lambda-1}x_0^2\right) \right| < |x_0|$$

Inequality (4), holds iff $|x_0| < \sqrt{2(1-\lambda)}$.

that is for any x_0 with $|x_0| < \sqrt{2(1-\lambda)}$ we have $|x_1| < |x_0| < \sqrt{2(1-\lambda)}$.

For such initial conditions, we have (by induction):

$$\cdots < |x_{n+1}| < |x_n| < \cdots < |x_1| < |x_0| < \sqrt{2(1-\lambda)} \qquad (5)$$

this proves the stability of the fixed point $x=0$. To prove asymptotic stability, we need $\lim_{n\to\infty} x_n = 0$.

Note that the sequence $\{|x_n|\}$ is a decreasing (due to (5)) sequence that is bounded from below ($|x_n| \geq 0$). Therefore, it must have a limit: $\lim_{n\to\infty} |x_n| = \alpha$.

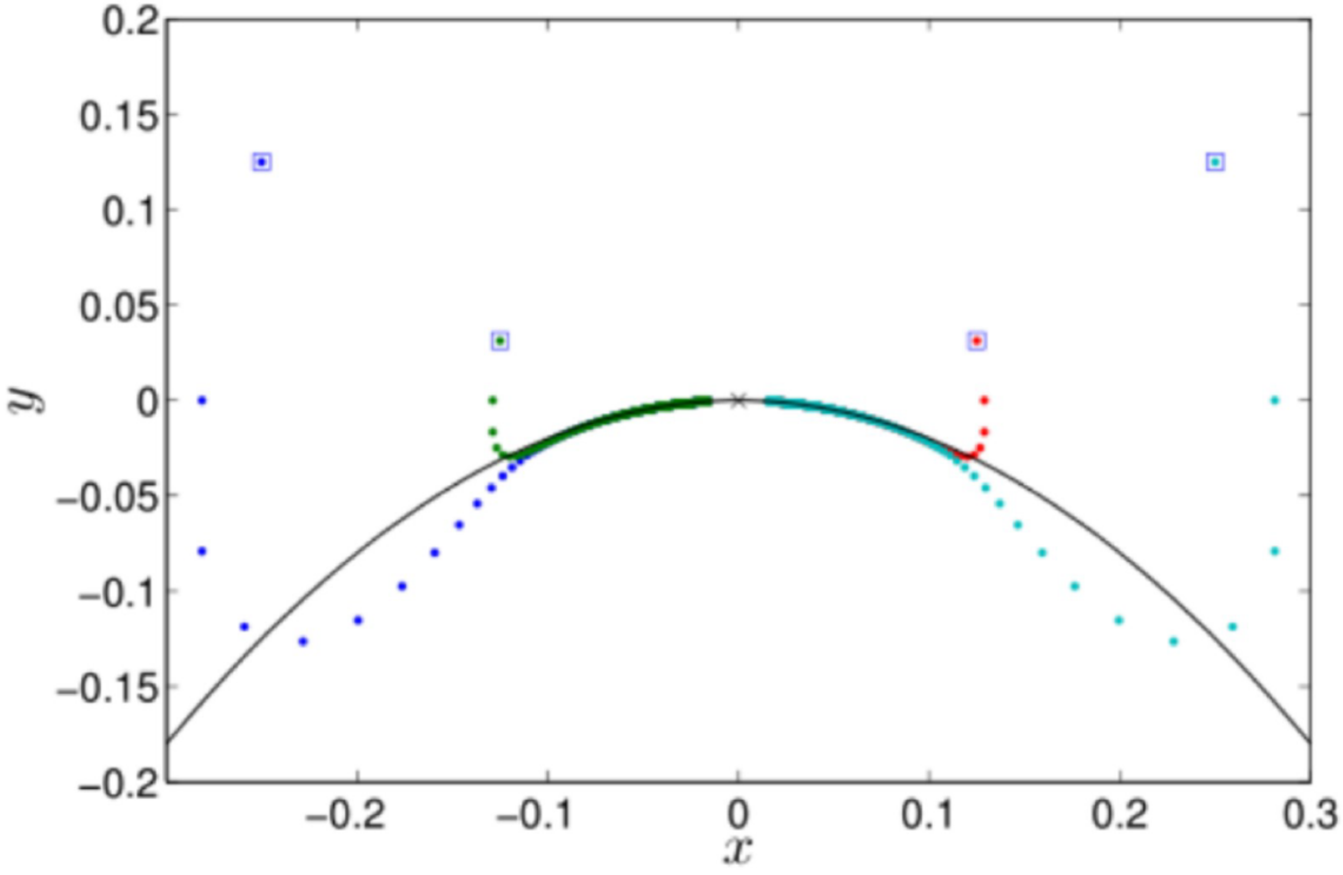
This limit, in general, doesn't have to be zero. But taking the limit $n\to\infty$ in eq. (3) we get

$$\lim_{n\to\infty} |x_{n+1}| = \lim_{n\to\infty} |x_n| \left(1 + \frac{1}{\lambda-1} \lim_{n\to\infty} |x_n|^2\right)$$

$$\Rightarrow \quad \alpha = \alpha\left(1 + \frac{1}{\lambda-1} \alpha^2\right) \Rightarrow \alpha = 0 \Rightarrow \lim_{n\to\infty} x_n = 0$$

Therefore the fixed point is asymptotically stable.

The following figure shows the iterations of the map for four initial conditions marked by square symbols. The higher iterations are marked by dots. The black curve marks the approximate center manifold $y = \frac{1}{\lambda - 1} x^2$. In this example we chose $\lambda = 1/2$.



2. Consider the quadratic *Duffing equation*

$$\dot{u} = v,$$
$$\dot{v} = \beta u - u^2 - \delta v,$$

where $\delta > 0$, and $0 \leq |\beta| \ll 1$.

(a) Construct a β–dependent center manifold up to quadratic order near the origin for small β values.

Linearized dynamics around fixed point $(0,0)$

$\dot{\eta} = A\eta$, $A = \begin{bmatrix} 0 & 1 \\ \beta & -\delta \end{bmatrix}$, $\mathrm{eig}(A) = \lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \beta}$

Note that $\lambda_1 = 0, \lambda_2 = -2\delta$ for $\beta = 0$. Thus, by the center manifold theorem, we have a 1-dimensional center manifold passing through the origin and a unique 1-dimensional stable manifold

- Consider the extended system

$$\dot{\beta} = 0$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\delta \end{bmatrix}}_{B} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \beta u - u^2 \end{bmatrix}$$

Eigenvalues of B : $\lambda_1 = 0$, $\lambda_2 = -\delta$,

Eigenvectors of B : $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 1/\delta \\ -1 \end{bmatrix}$

Use transformation $\begin{bmatrix} u \\ v \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$,

$T = [e_1 \mid e_2] = \begin{bmatrix} 1 & 1/\delta \\ 0 & -1 \end{bmatrix}$, $T^{-1} = \begin{bmatrix} 1 & 1/\delta \\ 0 & -1 \end{bmatrix} = T$

$\Rightarrow u = x + \dfrac{y}{\delta}$, $v = -y$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = T^{-1} B T \begin{bmatrix} x \\ y \end{bmatrix} + T^{-1} \begin{bmatrix} 0 \\ \beta u - u^2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{\delta}\left(\beta\left(x + \frac{y}{\delta}\right) - \left(x + \frac{y}{\delta}\right)^2\right) \\ -\beta\left(x + \frac{y}{\delta}\right) + \left(x + \frac{y}{\delta}\right)^2 \end{bmatrix} \quad \text{--- (1)}$$

Seek center manifold as a graph over center subspace locally as

$$y = h(x,\beta) = a_1 x^2 + a_2 x\beta + \cancel{a_3 \beta^2} + O(3)$$

To respect the existence of the fixed point.

$$\dot{y} = \partial_x h \, \dot{x} + \partial_\beta h \, \cancel{\dot{\beta}}^{\,0} \qquad (2)$$

Use invariance in (2):

$$\Rightarrow \quad \dot{y} = (2a_1 x + a_2\beta)\left[\frac{1}{\delta}\left(\beta\left(x + \frac{h(x,\beta)}{\delta}\right) - \left(x + \frac{h(x,\beta)}{\delta}\right)^2\right)\right] \qquad (3)$$

But also

$$\dot{y} = -\delta\, h(x,\beta) - \beta\left(x + \frac{h(x,\beta)}{\delta}\right) + \left(x + \frac{h(x,\beta)}{\delta}\right)^2 \qquad (4)$$

Comparing $O(2)$ terms in (3) & (4), we get.

x^2: $\quad -\delta a_1 + 1 = 0 \Rightarrow a_1 = \frac{1}{\delta}$

$x\beta$: $\quad -\delta a_2 - 1 = 0 \Rightarrow -a_2 = \frac{1}{\delta}$.

Thus, the β-dependent center manifold is given by

$$h(x,\beta) = \frac{x^2}{8} - \frac{x\beta}{8} + O(3), \quad \text{—— (5)}$$

Substitute (5) into first equation in (1) to obtain reduced dynamics on the center manifold: $W^c_\beta(0)$ upto quadratic order.

$$\dot{x} = \frac{1}{8}\left[\beta\left(x + \frac{h(x,\beta)}{8}\right) - \left(x + \frac{h(x,\beta)}{8}\right)^2\right]$$

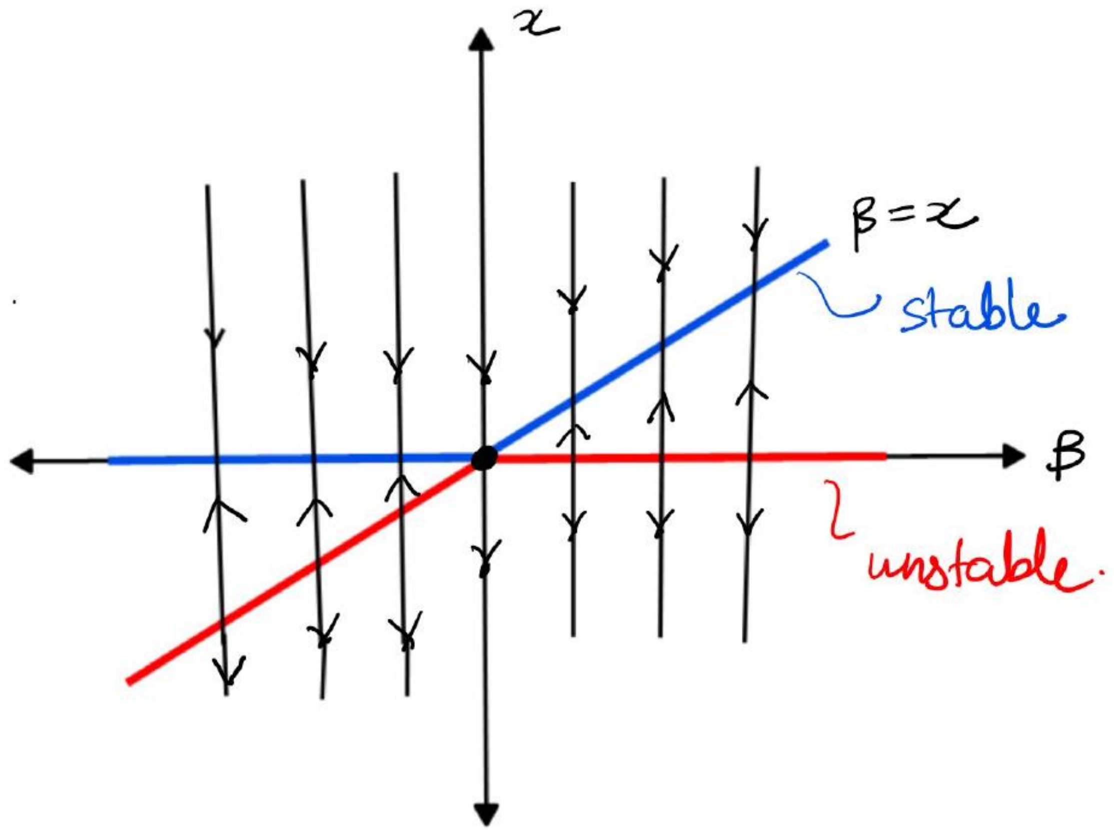
$$= \frac{1}{8}\left[\beta x - x^2\right] + O(3)$$

(b) Construct a stability diagram for the reduced system on the center manifold using β as a bifurcation parameter.

$$\dot{x} = \frac{1}{\delta}\left[\beta x - x^2\right]$$

fixed pts :

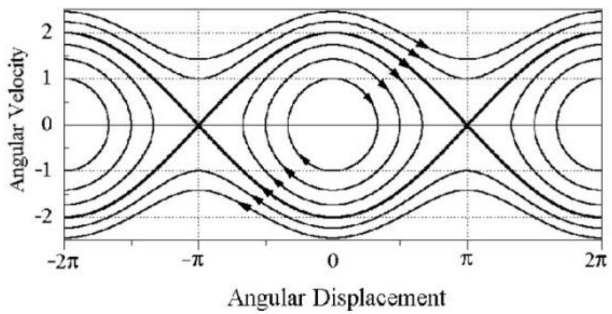
$x = 0$,
$\beta = x$



Transcritical Bifurcation

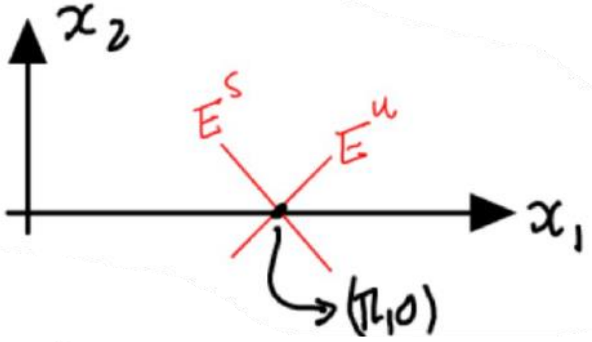
3. Construct a cubic-order local approximation for the unstable manifold of the hyperbolic fixed point of the pendulum equation

$$\ddot{x} + \sin x = 0.$$



Let $x_1 = x$ and $x_2 = \dot{x}$. Then $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 \end{cases}$

By linearization, one can show that the fixed point $(\pi, 0)$ is an unstable hyperbolic fixed point with stable and unstable linear subspaces spanned by $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$, respectively.

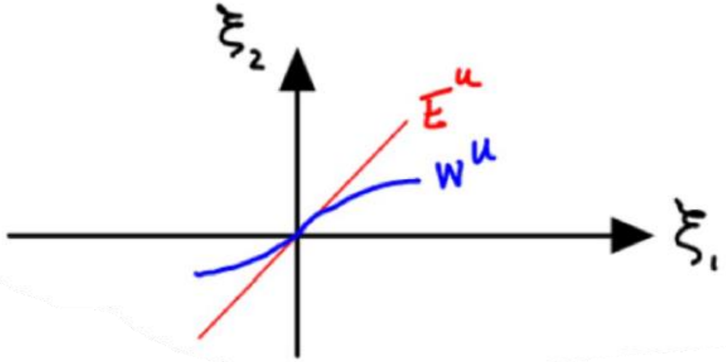


For convenience, we shift the origin by the transformation $\xi_1 = x_1 - \pi$ and $\xi_2 = x_2$ such that in (ξ_1, ξ_2) the origin is the hyperbolic fixed point.

In this coordinate system, the DS becomes:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \sin\xi_1 \end{cases} \qquad (\text{since } -\sin x_1 = -\sin(\xi_1+\pi) = \sin\xi_1$$

the unstable manifold passing through the origin is a graph over ξ_1 and tangent to E^u.



If this graph is given by $\xi_2 = h(\xi_1)$,

the Taylor expansion of h looks like: $h(\xi_1) = \underbrace{0}_{h(0)=0} + \underbrace{\xi_1}_{h'(0)=1} + a\xi_1^2 + b\xi_1^3 + O(\xi_1^4)$

By invariance of the unstable manifold we have $\dot{\xi}_2 = h'(\xi_1)\,\dot{\xi}_1$

Therefore, $\sin\xi_1 = \left(1 + 2a\xi_1 + 3b\xi_1^2 + O(3)\right)\left(\xi_1 + a\xi_1^2 + b\xi_1^3 + O(4)\right)$

the Taylor expansion of $\sin\xi_1$ around $\xi_1 = 0$ reads

$$\sin\xi_1 = \xi_1 - \frac{1}{6}\xi_1^3 + O(\xi_1^5)$$

Matching exponents we get $a = 0$ and $b = -\frac{1}{24}$.

therefore, the graph of unstable manifold satisfies:

$$\xi_2 = \xi_1 - \frac{1}{24}\xi_1^3 + O(\xi_1^4) \quad \text{or} \quad \boxed{x_2 = x_1 - \pi - \frac{1}{24}(x_1 - \pi)^3 + O(|x_1 - \pi|^4)}$$