Nonlinear Dynamics & Chaos I

Exercice Set 5 Solutions

Question 1

Consider the quadratic *Duffing equation*

$$\dot{u} = v,$$
$$\dot{v} = \beta u - u^2 - \delta v,$$

where $\delta > 0$, and $0 \leq |\beta| \ll 1$.

(a) Construct a β-dependent center manifold up to quadratic order near the origin for small β values.

(b) Construct a stability diagram for the reduced system on the center manifold using β as a bifurcation parameter.

Solution 1

(a) Linearized dynamics around fixed point $(0, 0)$

$$\dot{\eta} = A\eta \ , \ A = \begin{bmatrix} 0 & 1 \\ \beta & -\delta \end{bmatrix} \ , \ \text{eig}(A) = \lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \beta}$$

Note that $\lambda_1 = 0$, $\lambda_2 = -\delta$ for $\beta = 0$. Thus, by the center manifold theorem, we have a 1-dimensional center manifold passing through the origin and a unique 1-dimensional stable manifold.

- Consider the extended system

$$\dot{\beta} = 0$$
$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\delta \end{bmatrix}}_{B} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \beta u - u^2 \end{bmatrix}$$

$$\text{Eigenvalues of } B : \lambda_1 = 0 \ , \ \lambda_2 = -\delta$$
$$\text{Eigenvectors of } B : e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \ e_2 = \begin{bmatrix} \frac{1}{\delta} \\ -1 \end{bmatrix}$$

From the eigenvalues and eigenvectors, we can perform a change of coordinates

$$\begin{bmatrix} u \\ v \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix} \ , \ T = [e_1|e_2] = \begin{bmatrix} 1 & \frac{1}{\delta} \\ 0 & -1 \end{bmatrix} \ , \ T^{-1} = \begin{bmatrix} 1 & \frac{1}{\delta} \\ 0 & -1 \end{bmatrix} = T$$

$$\implies u = x + \frac{y}{\delta} \ , \ v = -y$$

$$\begin{aligned}
\implies \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} &= T^{-1}BT\begin{bmatrix}x\\ y\end{bmatrix} + T^{-1}\begin{bmatrix}0\\ \beta u - u^2\end{bmatrix}\\
&= \begin{bmatrix}0 & 0\\ 0 & -\delta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}\frac{1}{\delta}\left(\beta\left(x+\frac{y}{\delta}\right) - \left(x+\frac{y}{\delta}\right)^2\right)\\ -\beta\left(x+\frac{y}{\delta}\right) + \left(x+\frac{y}{\delta}\right)^2\end{bmatrix}
\end{aligned} \tag{1}$$

Seek center manifold as a graph over center subspace locally as

$$\begin{aligned}
y &= h(x,\beta) = a_1x^2 + a_2x\beta + \cancel{a_3\beta^2} + \mathcal{O}(3)\\
\dot{y} &= \frac{\partial h}{\partial x}\dot{x} + \frac{\partial h}{\partial \beta}\cancelto{0}{\dot{\beta}}
\end{aligned} \tag{2}$$

Note: We cancel the term $a_3\beta^2$ to respect the existence of the fixed point.

Use invariance in (2):

$$\implies \dot{y} = (2a_1x + a_2\beta)\left[\frac{1}{\delta}\left(\beta\left(x+\frac{h(x,\beta)}{\delta}\right) - \left(x+\frac{h(x,\beta)}{\delta}\right)^2\right)\right] \tag{3}$$

$$\text{But also } \dot{y} = -\delta h(x,\beta) - \beta\left(x+\frac{h(x,\beta)}{\delta}\right) + \left(x+\frac{h(x,\beta)}{\delta}\right)^2 \tag{4}$$

Comparing $\mathcal{O}(2)$ terms in (3) & (4), we get:

$$\begin{aligned}
x^2: &\qquad -\delta a_1 + 1 = 0 \implies a_1 = \frac{1}{\delta}\\
x\beta: &\qquad -\delta a_2 - 1 = 0 \implies -a_2 = \frac{1}{\delta}
\end{aligned}$$

Thus, the β-dependent center manifold is given by

$$h(x,\beta) = \frac{x^2}{\delta} - \frac{x\beta}{\delta} + \mathcal{O}(3) \tag{5}$$

Substitute (5) into first equation in (1) to obtain reduced dynamics on the center manifold: $W^C_\beta(0)$ up to quadratic order.

$$\begin{aligned}
\dot{x} &= \frac{1}{\delta}\left[\beta\left(x+\frac{h(x,\beta)}{\delta}\right) - \left(x+\frac{h(x,\beta)}{\delta}\right)^2\right]\\
&= \frac{1}{\delta}[\beta x - x^2] + \mathcal{O}(3)
\end{aligned}$$

(b)

$$\dot{x} = \frac{1}{\delta}[\beta x - x^2]$$

Fixed points:

$$\begin{aligned}
x &= 0,\\
\beta &= x
\end{aligned}$$

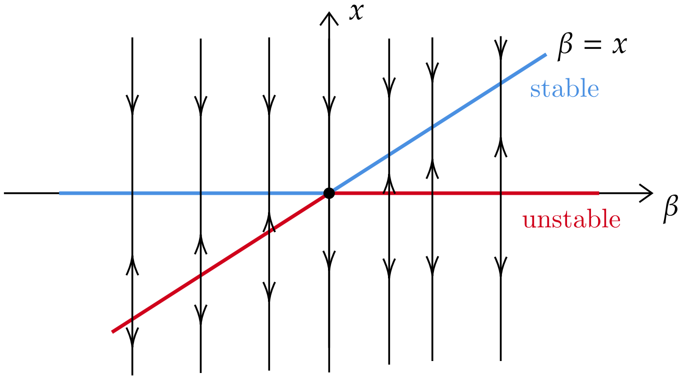

Figure 1: Transcritical bifurcation

Question 2

Consider a dynamical system that has a pair of purely imaginary eigenvalues at its fixed point for the parameter value $\mu = 0$. As we have seen, a linear change of coordinates and a center manifold reduction gives the two-dimensional reduced dynamical system

$$\dot{x} = \delta(\mu)x - \omega(\mu)y + f(x, y, \mu), \tag{6}$$
$$\dot{y} = \delta(\mu)y + \omega(\mu)x + g(x, y, \mu), \tag{7}$$

where $\delta(\mu) = \text{Re } \lambda(\mu)$, $\omega(\mu) = \text{Im } \lambda(\mu)$. (Here $\lambda(\mu)$ and $\bar{\lambda(\mu)}$ is the pair of complex eigenvalues that crosses the imaginary axis at $\mu = 0$.)
Recall that in polar coordinates, the truncated normal form of (6) can be written as

$$\dot{r} = r(d_0\mu + a_0 r^2),$$
$$\dot{\theta} = \omega_0 + b_0\mu + c_0 r^2,$$

where

$$\begin{aligned} d_0 &= \delta'(0), \quad \omega_0 = \omega(0) \\ a_0 &= \frac{1}{16}\left[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right]_{x=y=0,\mu=0} \\ &+ \frac{1}{16\omega_0}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right]_{x=y=0,\mu=0}. \end{aligned}$$

These classic formulae are used in all applications where Hopf bifurcations are analyzed.
As an application of these results, consider now the stick-slip oscillator

$$m\ddot{x} + c\dot{x} + kx = F_f, \qquad F_f = mg\mu_0\left(1 + e^{-\beta|v_0 - \dot{x}|}\right)\text{sign } (v_0 - \dot{x}),$$

where m is the mass of the oscillator, g is the constant of gravity, $\beta > 0$ is a constant, μ_0 is the Coulomb (static) friction coefficient, v_0 is the speed of the belt, x is the position of the mass on the belt, c is the coefficient of viscous damping, and k is the spring coefficient (see Fig. 2).

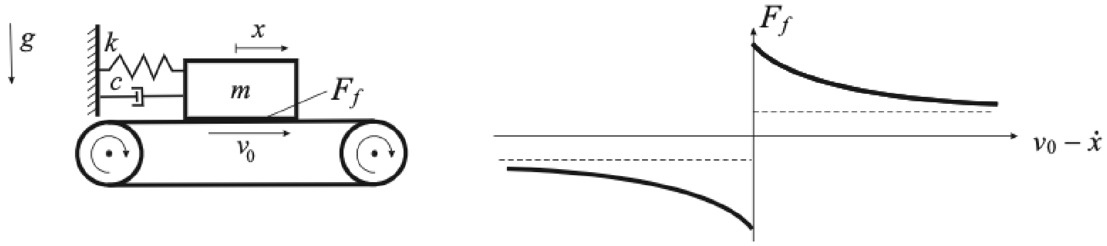

Figure 2: Stick-slip oscillator and its dry-friction force as a function of the relative velocity between the mass and the belt.

(a) Find a condition under which the system has an asymptotically stable fixed point.

(b) Show that a subcritical Hopf bifurcation takes place when the above condition is violated. (Use v_0 as a bifurcation parameter.)

(c) Calculate the approximate amplitude of the bifurcating limit cycle.

Solution 2

(a)

Let $x_1 = x$ and $x_2 = \dot{x}$. Then the system can be written as

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}F_f(x_2).\end{aligned}$$

The fixed point is at

$$x_1^0 = \frac{1}{k}F_f(0) = \frac{mg\mu_0}{k}\left(1 + e^{-\beta|v_0|}\right)\text{sign }(v_0), \quad x_2^0 = 0.$$

Let us now shift the origin to the fixed point by introducing new coordinates as $z_1 = x_1 - x_1^0$ and $z_2 = x_2$. Then

$$\dot{z}_1 = z_2 \tag{8}$$
$$\dot{z}_2 = -\frac{k}{m}z_1 - \frac{c}{m}z_2 + \frac{1}{m}F_f(z_2) - \frac{1}{m}F_f(0), \tag{9}$$

with the fixed point at $z_1 = z_2 = 0$. The linearized system is given by

$$\dot{\xi} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & \mu \end{pmatrix}\xi, \tag{10}$$

where we have introduced the parameter

$$\boxed{\mu = \frac{1}{m}\left(F_f'(0) - c\right) = g\beta\mu_0 e^{-\beta|v_0|} - \frac{c}{m}}. \tag{11}$$

The eigenvalues of the coefficient matrix are

$$\lambda_{1,2} = \frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - \frac{4k}{m}}. \tag{12}$$

For the remainder of the discussion, let us assume that $|\mu|$ is not too big; specifically $\mu^2 < \frac{4k}{m}$. This is in line with our previous assumption to treat μ as a bifurcation parameter. In this case, the real part of $\lambda_{1,2}$ can simply be read off from (12) as

$$\text{Re }(\lambda_{1,2}) = \frac{\mu}{2}.$$

As a result, if $\mu < 0$ then Re $(\lambda_1) < 0$ and Re $(\lambda_2) < 0$, hence the fixed point is asymptotically stable by the Hartman-Grobman theorem. This condition translates into

$$\mu < 0 \Leftrightarrow g\beta\mu_0 e^{-\beta|v_0|} < \frac{c}{m} \Leftrightarrow |v_0| > \frac{1}{\beta}\log\left(\frac{mg\beta\mu_0}{c}\right).$$

Hence, if

$$|v_0| > \frac{1}{\beta}\log\left(\frac{mg\beta\mu_0}{c}\right), \tag{13}$$

then $(z_1 = z_2 = 0)$ is an asymptotically stable fixed point.

Remark

Note that if the viscous damping c is large enough such that $c > mg\beta\mu_0$ then $\log\left(\frac{mg\beta\mu_0}{c}\right) < 0$. Then the condition (13) is satisfied for any $v_0 \neq 0$ and $(z_1 = z_2 = 0)$ is asymptotically stable for any v_0.

(b)

For $0 < \mu \ll 1$ we have Re $(\lambda_{1,2}) > 0$. At $\mu = 0$ the eigenvalues cross the imaginary axis at

$$\lambda_{1,2}(\mu = 0) = \pm i\sqrt{\frac{4k}{m}}.$$

Now let $-1 \ll \mu < 0$. Then the eigenvalues are

$$\lambda_{1,2} = \frac{\mu}{2} \pm i\frac{1}{2}\sqrt{\frac{4k}{m} - \mu^2}.$$

Define

$$\boxed{\delta(\mu) = \frac{\mu}{2}, \quad \omega(\mu) = \frac{1}{2}\sqrt{\frac{4k}{m} - \mu^2}.} \tag{14}$$

Then, we can separate the linear and nonlinear terms from the system (8) and write it as

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m}F_f(z_2) - \frac{1}{m}F_f'(0)z_2 - \frac{1}{m}F_f(0) \end{pmatrix}, \tag{15}$$

where we have denoted the linear part as

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & \mu \end{pmatrix}.$$

The eigenvalues of A are $\delta(\mu) \pm i\omega(\mu)$. To simplify the calculation of the eigenvectors, we note that (as a consequence of (14))

$$\mu = 2\delta \quad \frac{k}{m} = \delta^2 + \mu^2$$

and hence

$$A = \begin{pmatrix} 0 & 1 \\ -\delta^2 - \omega^2 & 2\delta \end{pmatrix}.$$

We then search for a vector s such that

$$As - (\delta + i\omega)s = \begin{pmatrix} -\delta - i\omega & 1 \\ -\delta^2 - \omega^2 & \delta - i\omega \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}.$$

For example, the non-normalized vector

$$s = \begin{pmatrix} 1 \\ \delta + i\omega \end{pmatrix}$$

is a good choice. To separate the real and imaginary parts of the eigenvector we write it as

$$s = \begin{pmatrix} 1 \\ \delta(\mu) \end{pmatrix} \pm i \begin{pmatrix} 0 \\ \omega(\mu) \end{pmatrix}.$$

Selecting the real and imaginary parts of s as basis vectors then puts A in the desired block-diagonal form, that is

$$A = VDV^{-1},$$

where

$$D = \begin{pmatrix} \delta & \omega \\ -\omega & \delta \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & 0 \\ \delta & \omega \end{pmatrix}.$$

This means, that under the change of variables

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = V \begin{pmatrix} u \\ v \end{pmatrix}$$

we get the transformed dynamical system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \delta & \omega \\ -\omega & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \frac{1}{m\omega} \begin{pmatrix} 0 \\ F_f(\delta u + \omega v) - F_f'(0)(\delta u + \omega v) - F_f(0) \end{pmatrix}. \tag{16}$$

This is the desired form for the dynamical system defined in the problem description. Note that we may put

$$f(x, y, \mu) = 0$$
$$g(x, y, \mu) = \frac{1}{m\omega} F_f(\delta u + \omega v) - F_f'(0)(\delta u + \omega v) - F_f(0)$$

to compute the desired parameters

$$d_0 = \delta'(0) = \frac{1}{2}, \quad \omega_0 = \omega(0) = \sqrt{\frac{k}{m}}, \quad a_0 = \frac{kg\mu_0\beta^3 e^{-\beta|v_0|}}{16m},$$

where we have used that

$$g_{vvv}(0, 0, 0) = \frac{k}{m^2} F'''(0) = \frac{kg\mu_0\beta^3 e^{-\beta|v_0|}}{m}.$$

According to the Hopf-Bogdanov theorem, the radial component of the normal form of the dynamics can be written as

$$\boxed{\dot{r} = r\left(\frac{\mu}{2} + \frac{kg\mu_0\beta^3 e^{-\beta|v_0|}}{16m} r^2\right)}, \tag{17}$$

which has fixed points

$$r = 0 \text{ which corresponds to the stable fixed point}$$
$$r = \pm\sqrt{\frac{-8\mu m e^{\beta|v_0|}}{kg\mu_0\beta^3}}, \text{ which corresponds to the unstable limit cycle.}$$

Expressed as a function of v_0, the bifurcation occurs at

$$v_C = \frac{1}{\beta} \log\left(\frac{mg\beta\mu_0}{c}\right).$$

(c)

For $\mu < 0$ the amplitude of the unstable limit cycle is

$$r_0 = \sqrt{\frac{-8\mu m e^{\beta|v_0|}}{kg\mu_0\beta^3}}$$