

《数字图像处理》

第5讲 傅里叶变换

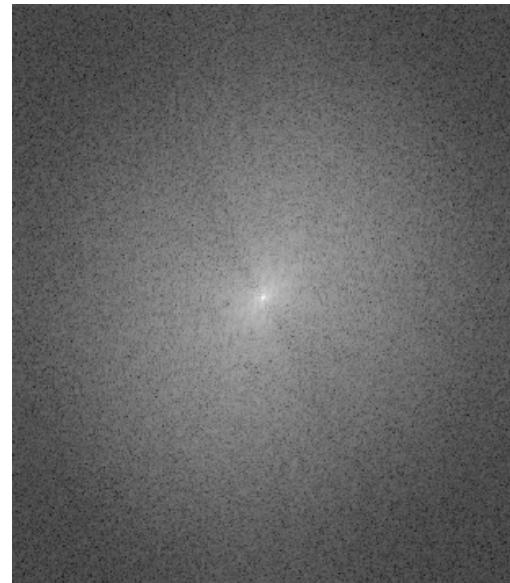
冯建江

清华大学 自动化系

2017.10.26

- 上节课介绍了空域图像增强
- 图像增强可在空域 (spatial domain) 或频域 (frequency domain) 进行
- 空域图像增强：直接修改像素值。
- 频域图像增强：先对输入图像做傅里叶变换（Fourier transform），然后修改频谱，最后做傅里叶反变换，得到增强的图像。
- 今天讲傅里叶变换，下节课讲频域图像增强

月球图像



傅里叶变换的
幅度谱
(magnitude
spectrum)

内 容

- 前言
- 一维傅里叶变换
- 二维傅里叶变换

傅里叶变换简史



- 法国数学家约瑟夫 · 傅里叶 (Joseph Fourier, 1768–1830) 1805年提出任一周期函数都可以展成三角函数的无穷级数，而非周期函数可表示为三角函数的积分
- 早期的主要用途是微分方程求解
- 1960年代，计算机的发展和快速傅里叶变换的发现促成了信号处理领域的大发展

傅里叶变换在图像处理中的主要应用

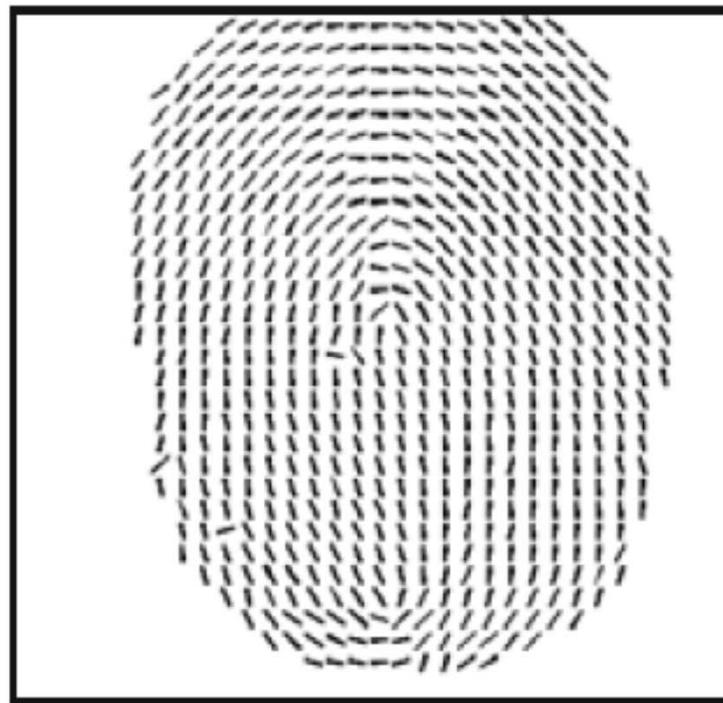
- 图像分析
- 图像滤波
- 图像压缩

应用1：图像分析

估计指纹方向场是指纹识别系统的一个关键模块。如果能得到准确的方向场，可对指纹进行有效增强，去除噪声，使特征提取更为准确，最终提高识别率。



指纹图像



方向场

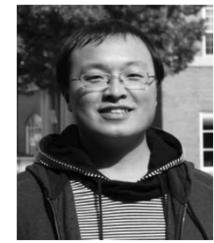

[Login](#)
Public area
[Home](#)
[Background](#)
[Benchmarks](#)
[Register](#)
[Published Results](#)
[Statistics](#)
Participant area
[Login](#)
[Upload](#)
[Pending Algorithms](#)
[Tested Algorithms](#)
[Download](#)
Past editions
[FVC2006](#)
[FVC2004](#)
[FVC2002](#)
FVC2000

Published Results

Select Benchmark Area:

Select Benchmark: All benchmarks ▾

Hide any previous version of the same algorithm

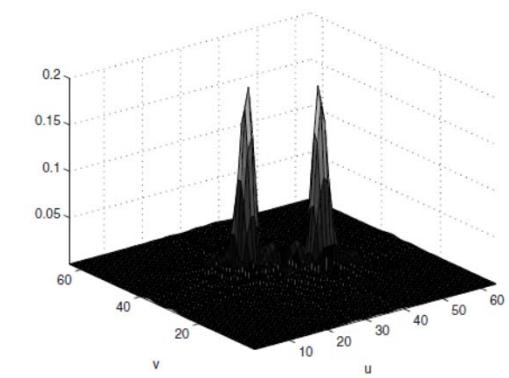
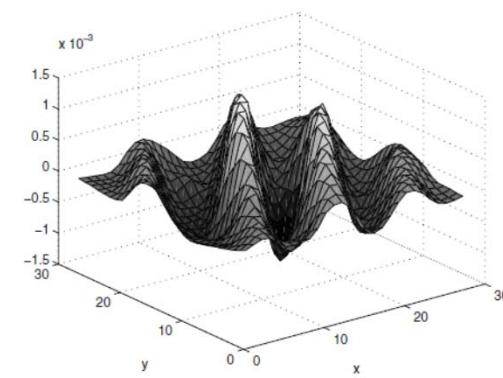
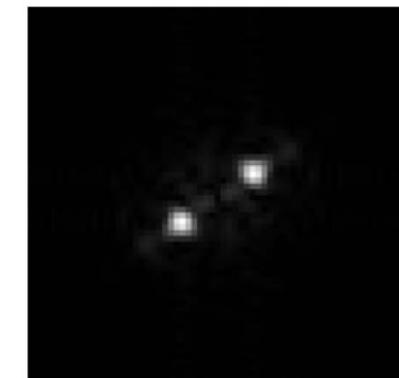
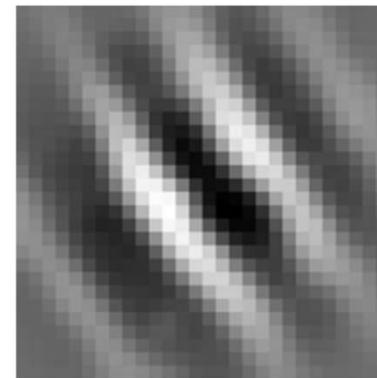
 010110
110011
101000
0001


Fingerprint Orientation Extraction

Published on	Benchmark	Participant	Type	Algorithm	Version	AvgErr _{GQ}	AvgErr _{BQ}	Show details
09/05/2013	FOE-STD-1.0	Department of Automation, Tsinghua University	Academic Research Group	LocalDict	0.1	6,08°	9,66°	
08/04/2012	FOE-STD-1.0	Institute of Automation, Chinese Academy of Sciences	Academic Research Group	ROF	1.1	5,24°	11,20°	
18/11/2011	FOE-STD-1.0	Zengbo Xu	Independent Developer	MXR	1.0.5	5,59°	11,36°	
08/11/2011	FOE-STD-1.0	Biometric System Laboratory	Academic Research Group	Adaptive-3 (Baseline)	v0.2	5,93°	13,27°	
22/11/2011	FOE-STD-1.0	Antheus Technology, Inc.	Company	AntheusOriEx	1.1.4	5,46°	17,06°	
22/11/2010	FOE-STD-1.0	School of Engineering and Information Technology, UNSW@ADFA	Academic Research Group	FOMFE	1.0	6,70°	21,44°	
19/07/2010	FOE-STD-1.0	Biometric System Laboratory	Academic Research Group	Gradient (baseline)	1.0	5,86°	21,83°	

估计指纹方向场

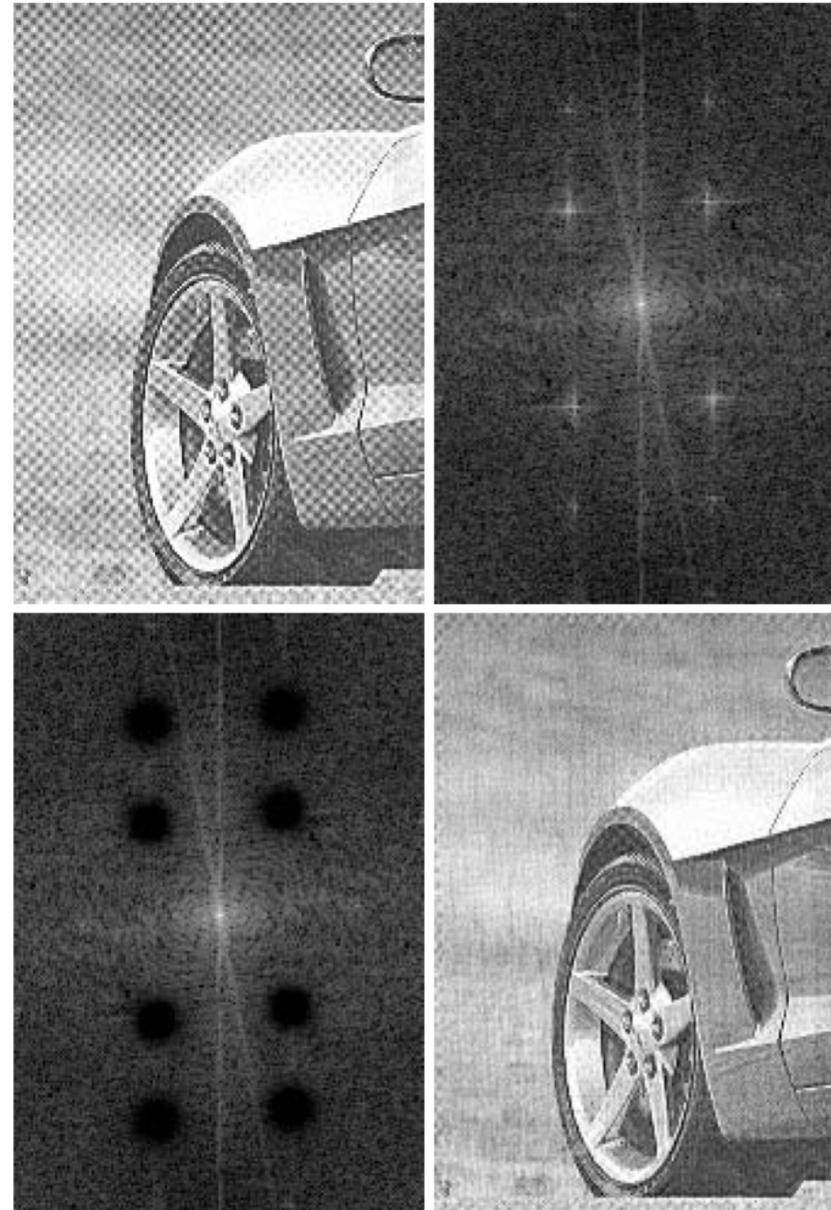
将图像分块，对每个块做离散傅里叶变换，检测幅度谱中最大值，估计出局部纹线方向。



应用2：图像增强

1. 将图像进行二维离散傅里叶变换；
2. 利用图像在该变换空间的特有性质，对变换域图像进行某种处理；
3. 反变换回图像空间，得到所需的处理效果。

陷波滤波器
(Notch
reject filter)



a b
c d

FIGURE 4.64
(a) Sampled newspaper image showing a moiré pattern.
(b) Spectrum.
(c) Butterworth notch reject filter multiplied by the Fourier transform.
(d) Filtered image.

应用3：图像压缩

JPEG图像压缩：

1. 将图像分为许多 8×8 块；
2. 对每个图像块做离散余弦变换(DCT，与DFT类似)；
3. 丢弃高频成分（因为通常高频成分的权重低）；

重建：反变换得到各图像块，然后拼接得到全图。

原图

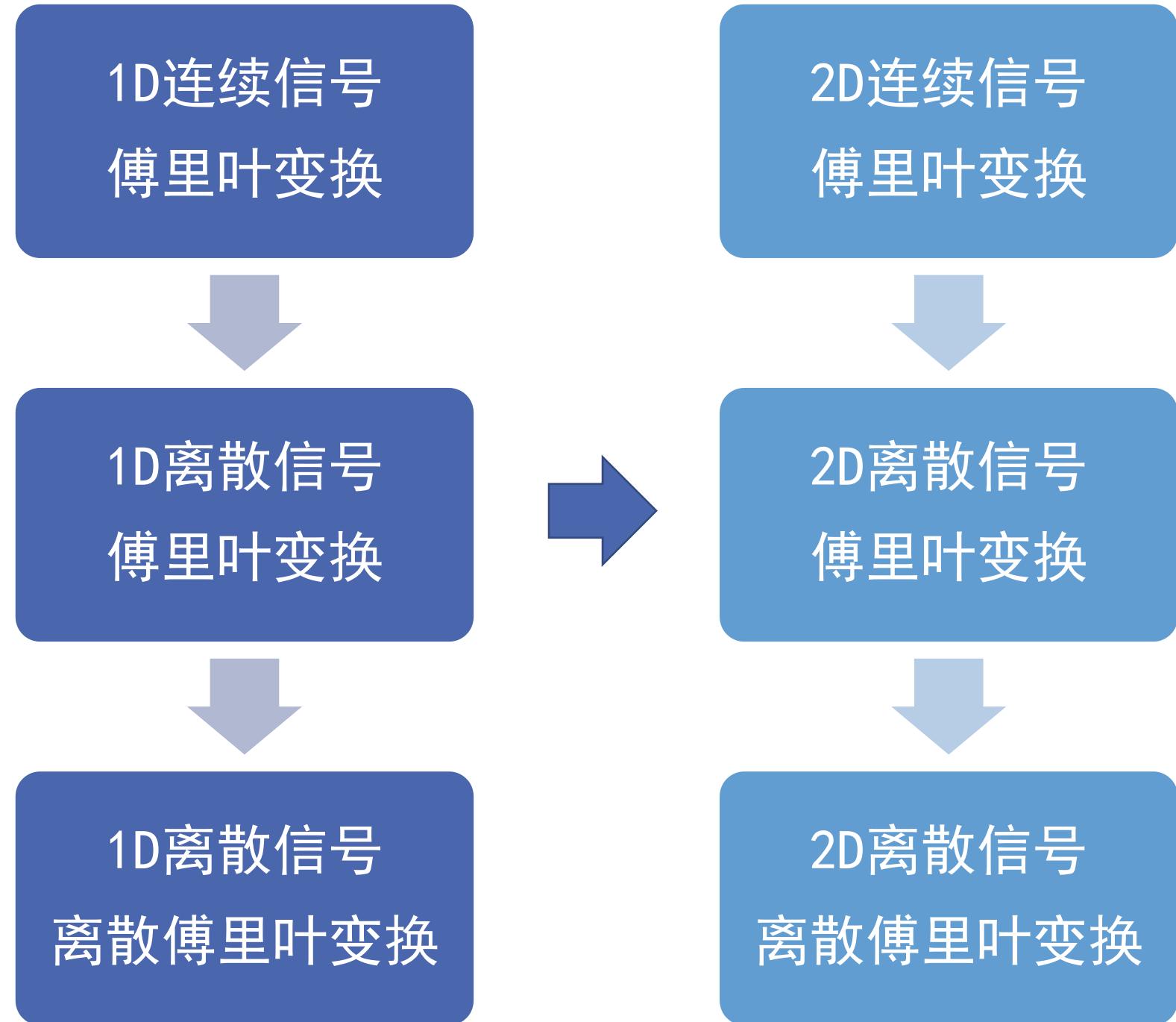


重建图



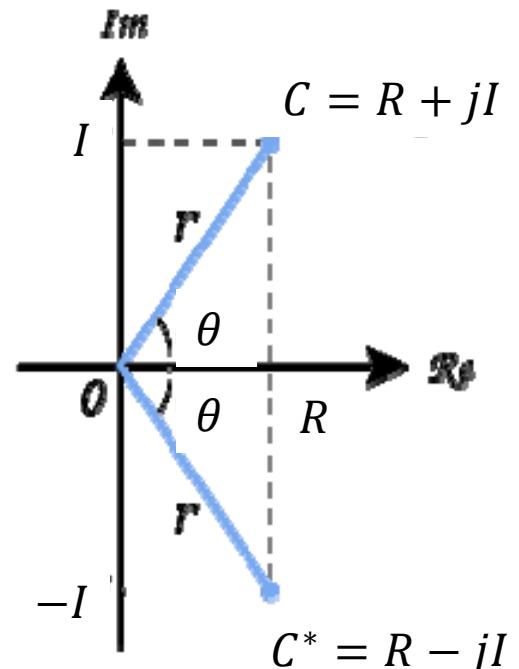
内 容

- 前言
- 一维傅里叶变换
- 二维傅里叶变换



复数 (complex number)

- 傅里叶变换的基函数（变换核）是复函数；信号/图像的傅里叶变换往往是复函数
- 复数 $C = R + jI$. R 是实部 (real part) , I 是虚部 (imaginary part) , R 和 I 都是实数 (real number) , 虚数单位 $j = \sqrt{-1}$
- 复数 C 的共轭 (conjugate) $C^* = R - jI$
- 复数 C 可视为复平面 (直角坐标系) 的点；横坐标为实轴，纵坐标为虚轴。
- 复数的极坐标表示: $C = |C|(\cos \theta + j \sin \theta)$, 模 $|C| = \sqrt{R^2 + I^2}$, 辐角 $\theta \in [-\pi, \pi]$
- 欧拉公式: $e^{j\theta} = \cos \theta + j \sin \theta$
- 复数的极坐标表示 (指数形式) : $C = |C|e^{j\theta}$
- 复函数 $F(u) = R(u) + jI(u)$

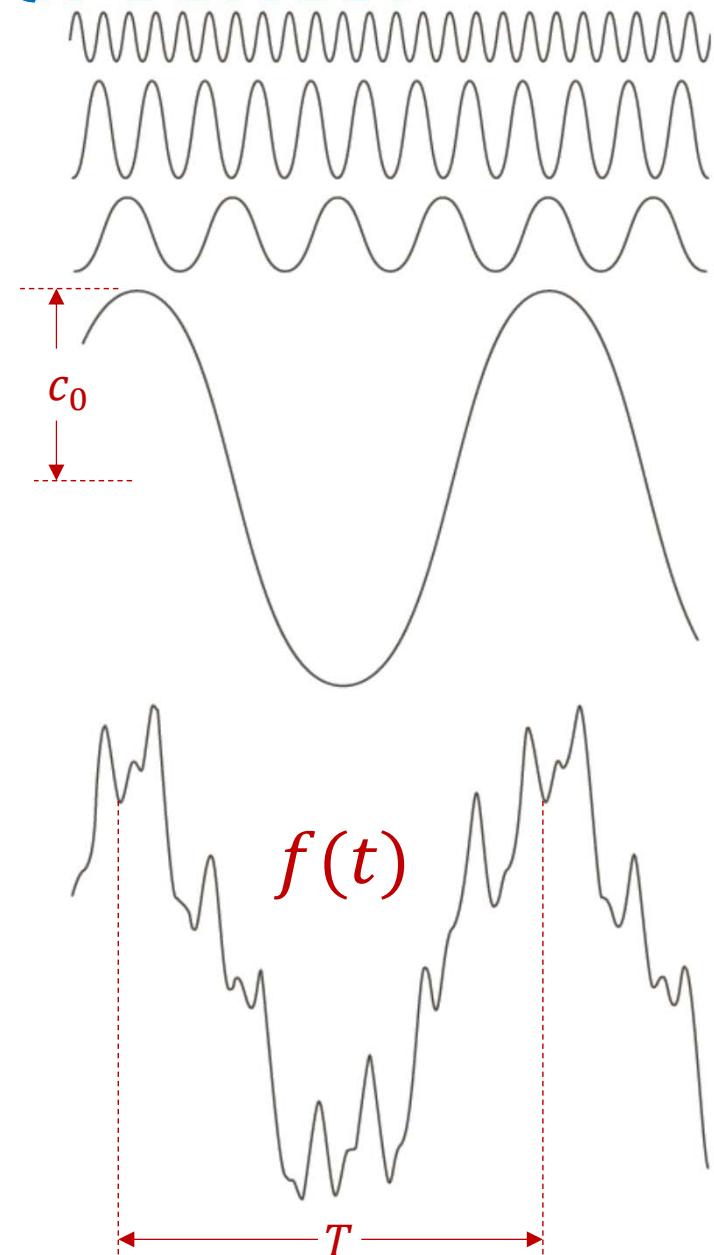


傅里叶级数 (Fourier series)

周期为 T 的连续函数 $f(t)$ 可用正弦函数和余弦函数构成的无穷级数来表示

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j \frac{2\pi n}{T} t}$$
$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j \frac{2\pi n}{T} t} dt$$
$$n = 0, \pm 1, \pm 2, \dots$$

我们研究的信号/图像是有限时长/空间范围，傅里叶变换的应用远比傅里叶级数多



连续冲激函数 (impulse function)

- 在信号与系统、傅里叶变换中，冲激函数是非常重要的概念
- 讲傅里叶变换的顺序：**先连续，后离散**
- 因为连续是基础，而且连续的数学简单
- 连续变量 t 的冲激函数 $\delta(t)$ ，也叫脉冲函数

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

- 冲激函数的采样特性：

$$\int_{-\infty}^{\infty} f(t)\delta(t) dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0)$$

离散冲激函数

- 离散变量 x 的冲激函数 $\delta(x)$

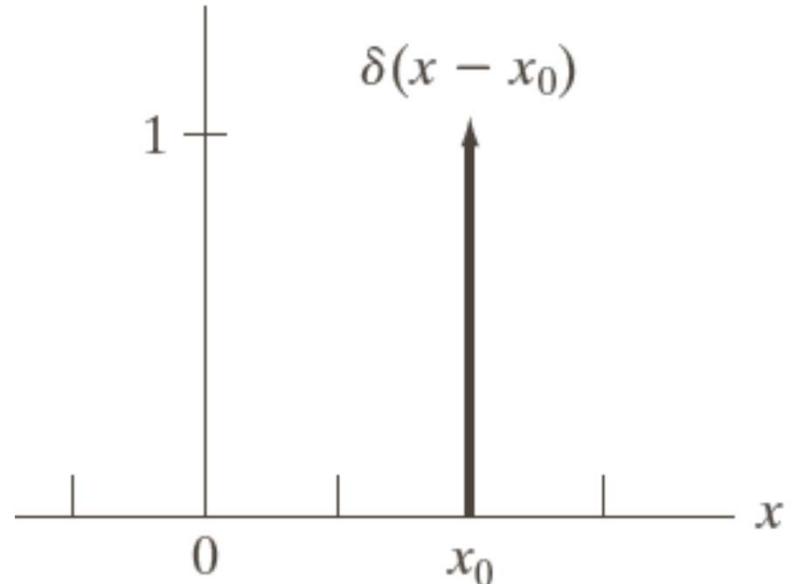
$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$$\sum_{-\infty}^{\infty} \delta(x) = 1$$

- 冲激函数的采样特性

$$\sum_{-\infty}^{\infty} f(x)\delta(x) = f(0)$$

$$\sum_{-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

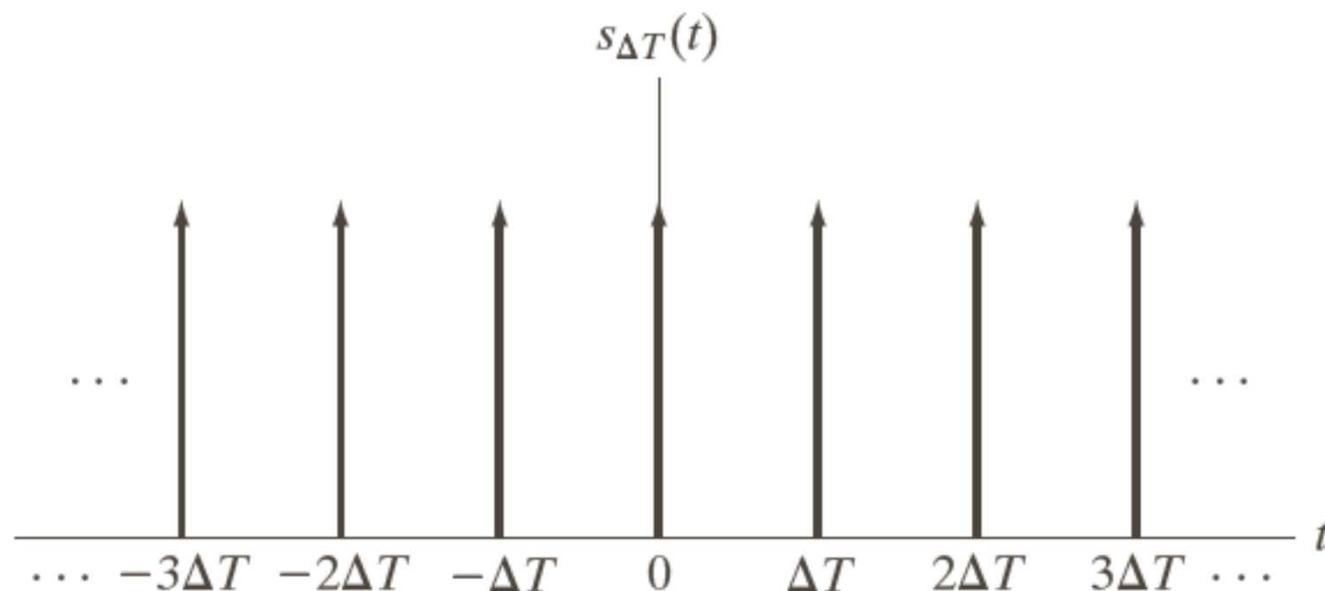


冲激串 (Impulse train)

冲激串 $S_{\Delta T}(t)$, 也叫脉冲火车, 是无数个间隔 ΔT 的冲激之和:

$$S_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

这是离散和连续通用的表达式。



1D连续信号
傅里叶变换



1D离散信号
傅里叶变换



1D离散信号
离散傅里叶变换

一元连续函数的傅里叶变换 (Fourier transform)

一元连续变量 t 的连续函数 $f(t)$ 的傅里叶变换 $F(\mu)$:

$$F(\mu) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

已知 $F(\mu)$, 通过傅里叶反变换得到 $f(t)$:

$$f(t) = \mathcal{F}^{-1}\{F(\mu)\} = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi t\mu} d\mu$$

$f(t)$ 和 $F(\mu)$ 构成傅里叶变换对: $f(t) \Leftrightarrow F(\mu)$

如果 $f(t)$ 是实函数, $F(\mu)$ 通常是复函数。

t 通常表示时间变量, $f(t)$ 的域叫时域。

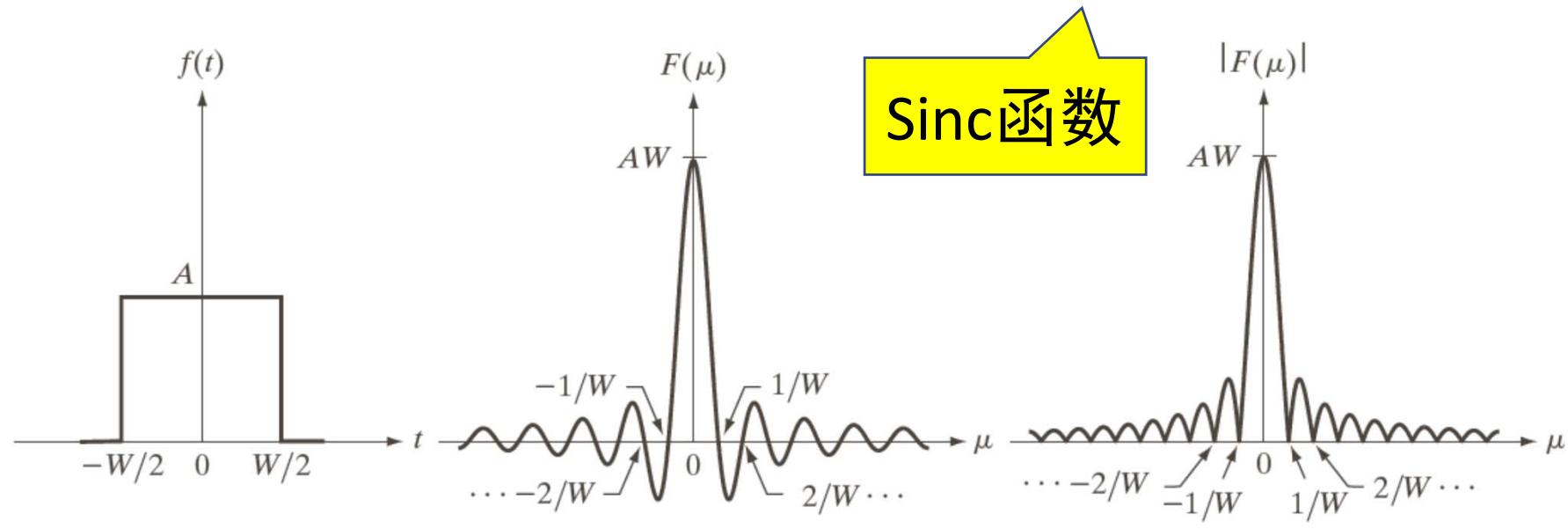
μ 是频率变量, $F(\mu)$ 的域叫频域 (frequency domain)。

傅里叶变换：例1

- 冲激函数 $\delta(t)$ 的傅里叶变换 $F(\mu) = 1$
- $\delta(t - t_0)$ 的傅里叶变换 $F(\mu) = e^{-j2\pi\mu t_0}$,
 $|F(\mu)| = 1$

傅里叶变换：例2

矩形函数的傅里叶变换 $F(\mu) = AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}$, 为实函数

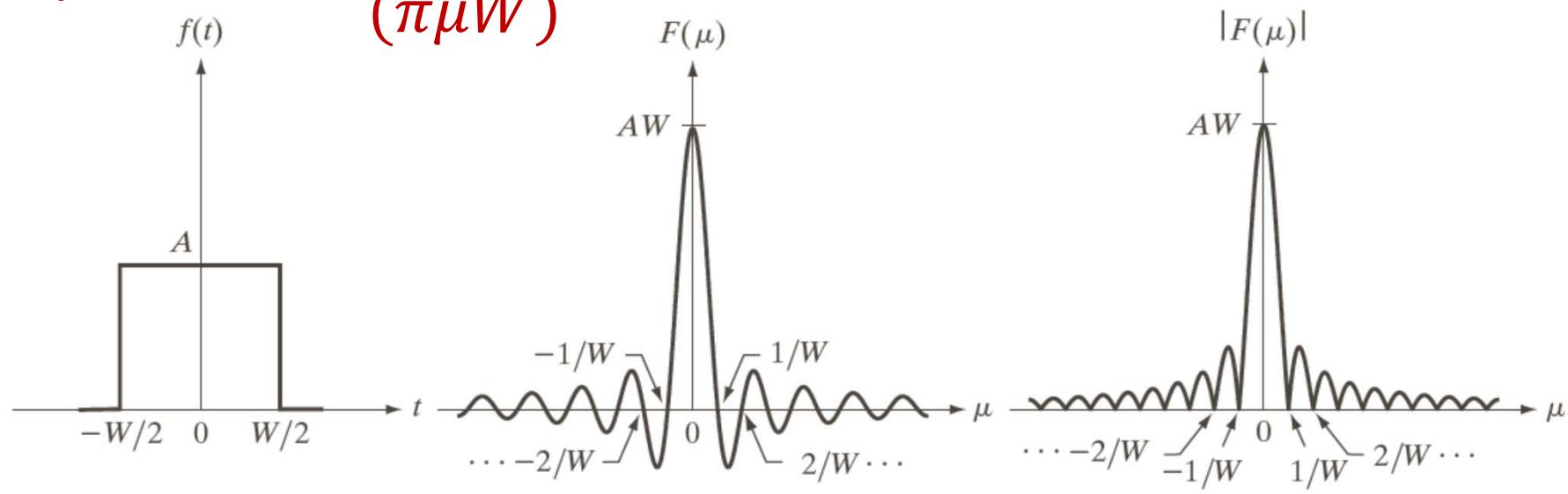


a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

由于傅里叶变换一般为复函数，为了显示方便，通常显示其幅度谱（也叫频谱） $|F(\mu)|$

$$F(\mu) = AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}$$



a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

矩形函数的幅度谱 $|F(\mu)|$ 的特点：

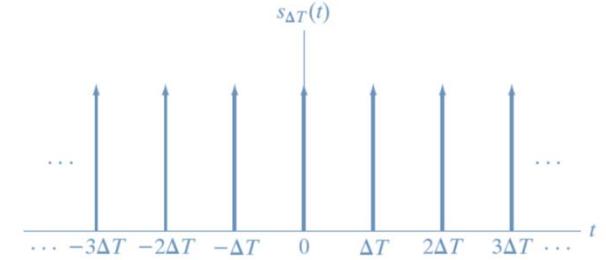
1. 零点位置与 W 成反比
2. 峰值由原点向两边衰减（高频系数小）
3. 函数延伸到正负无穷远

傅里叶变换：例3

冲激串 $s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$ 的傅里叶变换 $S(\mu)$

- $s_{\Delta T}(t)$ 可展开为傅里叶级数

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j \frac{2\pi n}{\Delta T} t}$$



$$\text{系数 } c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j \frac{2\pi n}{\Delta T} t} dt = \frac{1}{\Delta T}$$

$$s_{\Delta T}(t) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j \frac{2\pi n}{\Delta T} t}$$

- 利用傅里叶变换的对称性: $f(t) \Leftrightarrow F(\mu)$, $F(t) \Leftrightarrow f(-\mu)$

$$\mathfrak{F}\left\{e^{j \frac{2\pi n}{\Delta T} t}\right\} = \delta\left(\mu - \frac{n}{\Delta T}\right)$$

$$S(\mu) = \mathfrak{F}\{s_{\Delta T}(t)\} = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

周期为 $\frac{1}{\Delta T}$ 的
冲激串

卷积定理

- 连续变量函数 $f(t)$ 和 $h(t)$ 的卷积（信号 f 、滤波器 h ）

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau)h(t - \tau) d\tau$$

- 两个函数卷积的傅里叶变换是各自傅里叶变换的乘积，

$$\mathcal{F}\{f(t) \star h(t)\} = F(\mu)H(\mu)$$

- 两个函数乘积的傅里叶变换是各自傅里叶变换的卷积，

$$\mathcal{F}\{f(t)h(t)\} = F(\mu) \star H(\mu)$$

- 卷积定理将频域滤波和时域（空域）滤波关联了起来

卷积定理的证明

$$\begin{aligned}\mathfrak{F}\{f(t) \star h(t)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau)h(t - \tau) d\tau \right] e^{-j2\pi\mu t} dt \\&= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} h(t - \tau)e^{-j2\pi\mu t} dt \right] d\tau \quad \text{变换积分顺序} \\&= \int_{-\infty}^{\infty} f(\tau) [H(\mu)e^{-j2\pi\mu\tau}] d\tau = H(\mu) \int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau} d\tau \\&= H(\mu)F(\mu)\end{aligned}$$

1D连续信号
傅里叶变换



1D离散信号
傅里叶变换

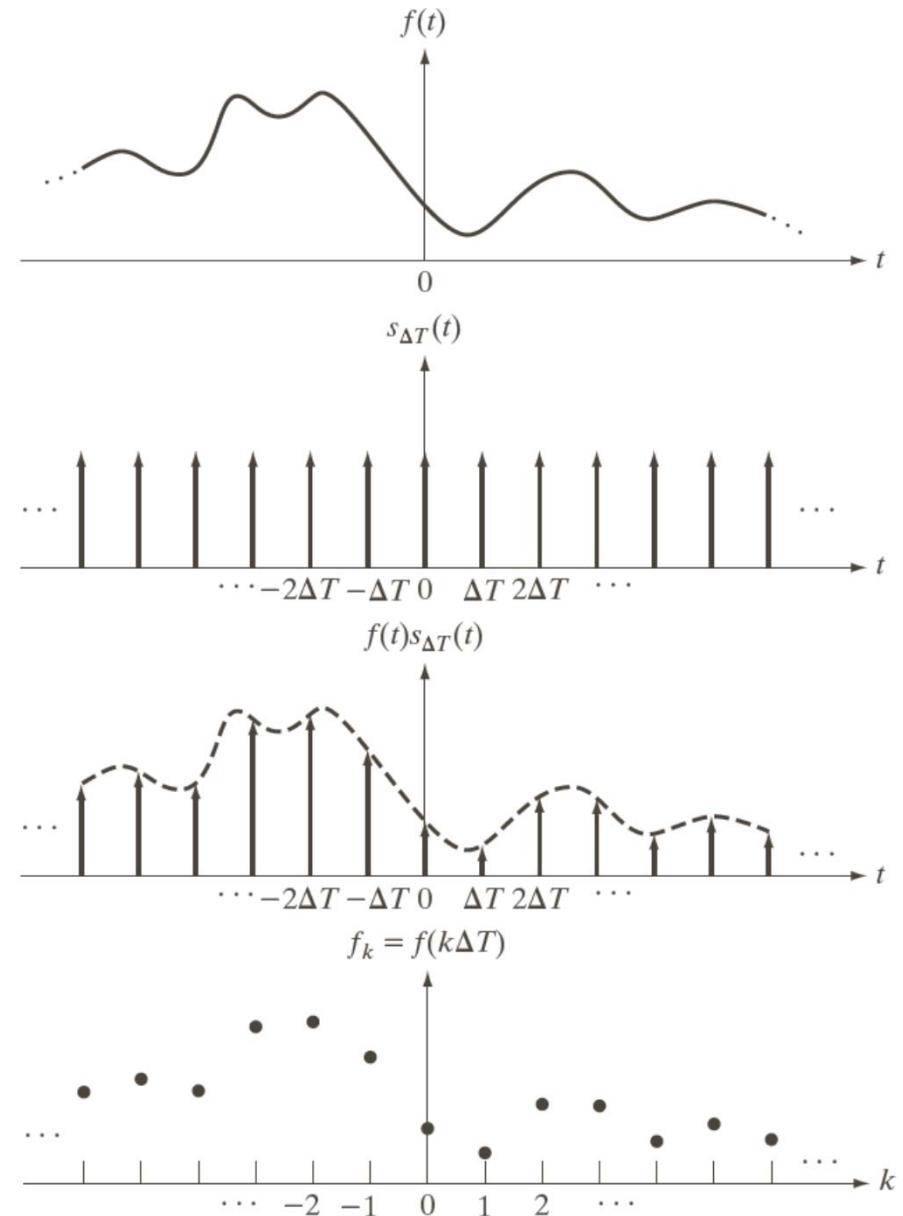


1D离散信号
离散傅里叶变换

采样

- 连续函数需要采样和量化，转化为一串离散值，才方便计算机处理
- 先只考虑采样
- 使用冲激串函数与 $f(t)$ 相乘，得到采样函数
(Sampled function)

$$\tilde{f}(t) = f(t)s_{\Delta T}(t)$$



采样函数的傅里叶变换

采样函数的傅里叶变换 $\tilde{F}(\mu) = \mathcal{F}\{f(t)s_{\Delta T}(t)\} = F(\mu) \star S(\mu)$

冲激串的傅里叶变换 $S(\mu) = \mathcal{F}\{s_{\Delta T}(t)\} = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$

$$\tilde{F}(\mu) = F(\mu) \star S(\mu) = \int_{-\infty}^{\infty} F(\tau)S(\mu - \tau) d\tau$$

$$= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau$$

$$= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau) \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

$\tilde{F}(\mu)$ 是无穷个 $F(\mu)$ 的周期性复制之和，周期为 $\frac{1}{\Delta T}$

注意：采样函数的傅里叶变换为连续函数。

采样函数的傅里叶变换：不同采样率

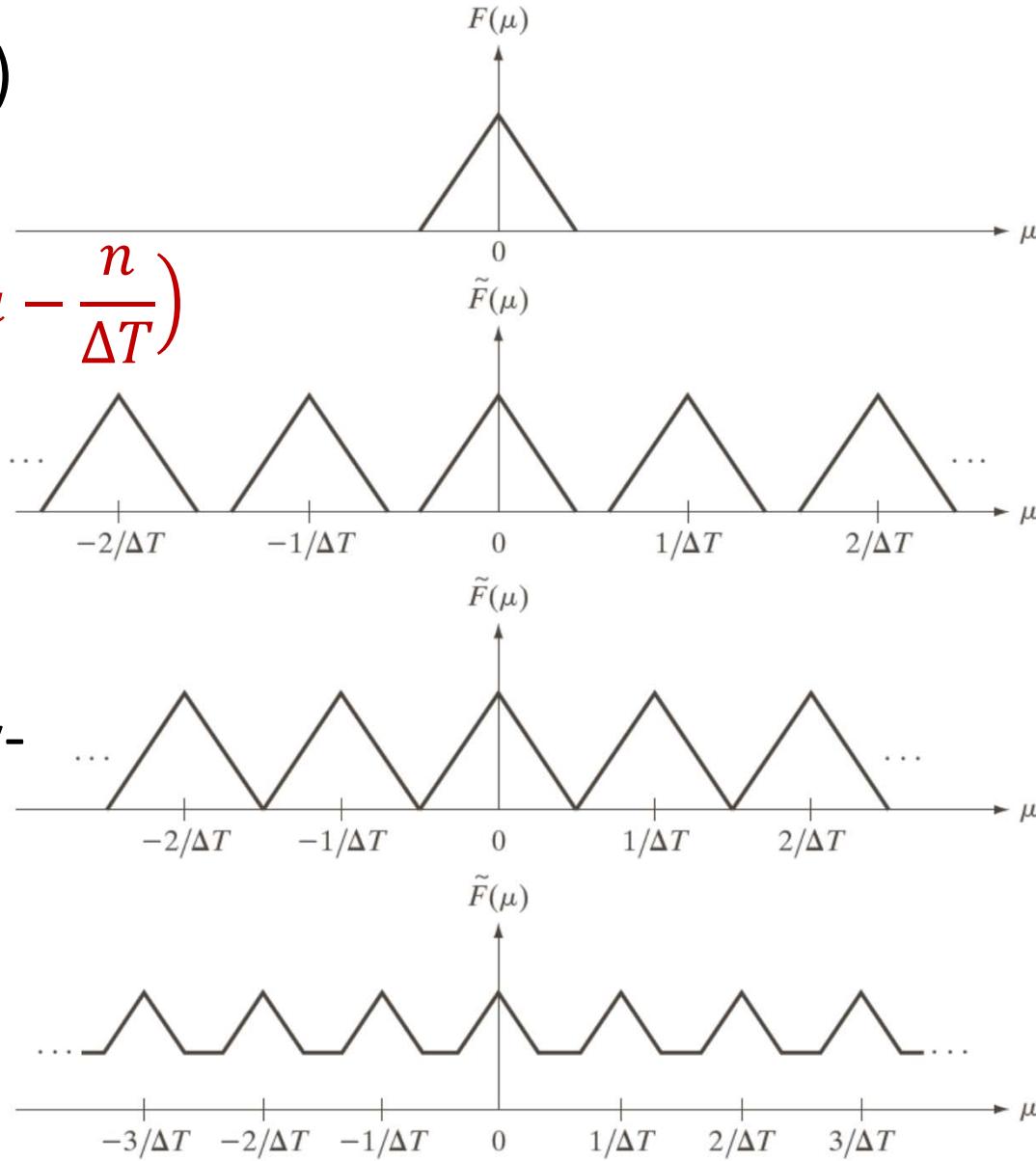
带限(band-limited)
函数 $f(t)$ 的FT

$$\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

过采样(over-sampling)的FT

临界采样(critically-sampling)的FT

欠采样(under-sampling)的FT

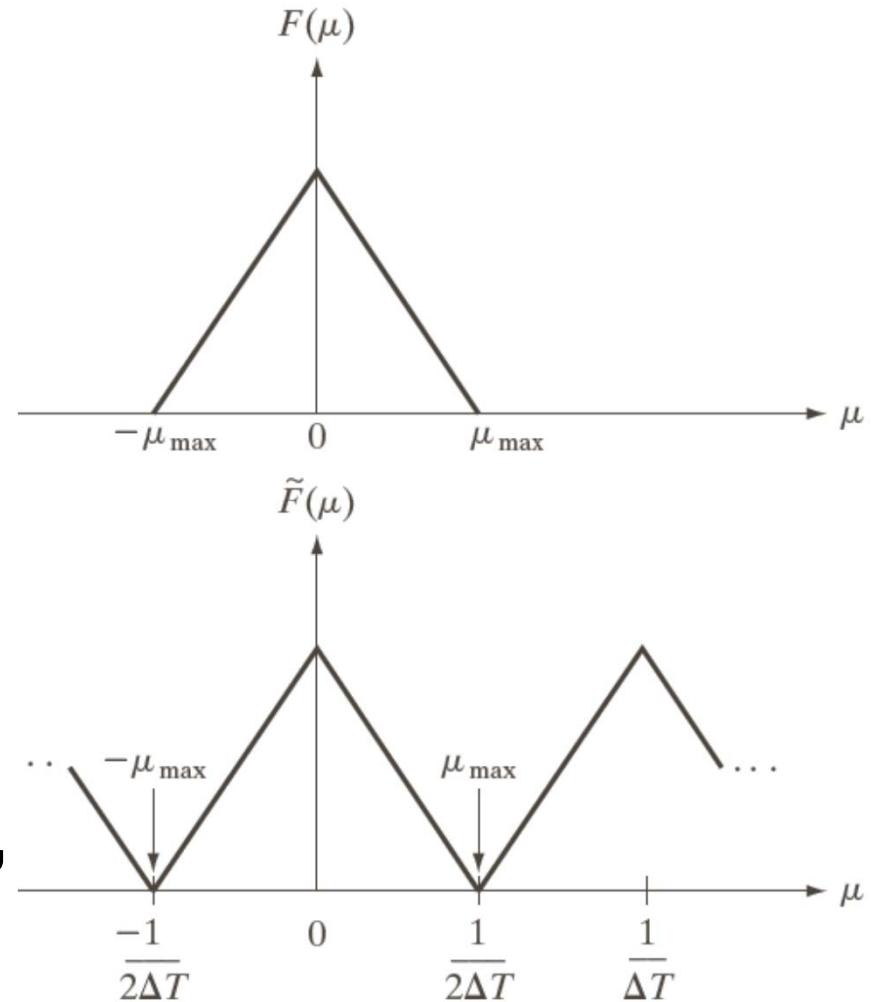


采样定理

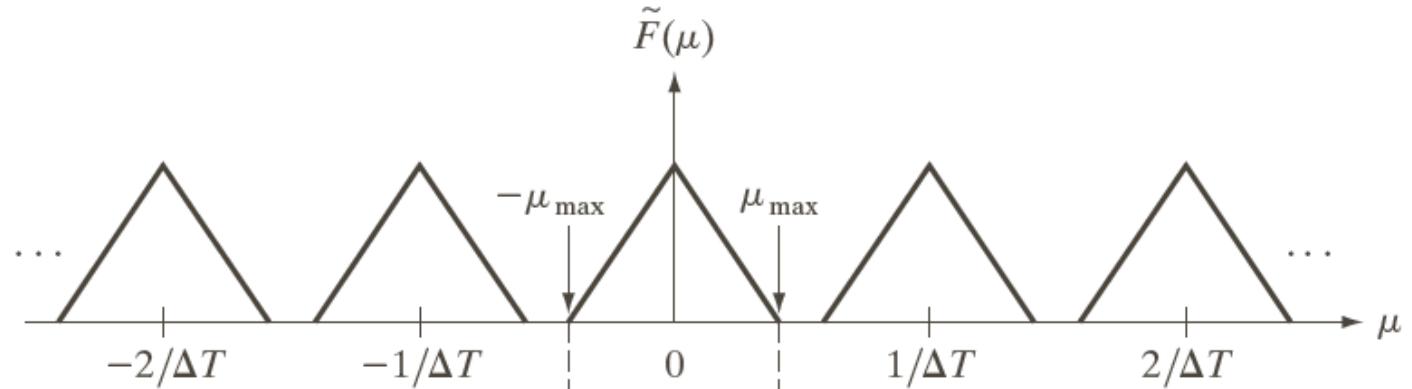
如果函数 $f(t)$ 的傅里叶变换符合下面条件，则称该函数的**带宽有限** (band-limited)

$$F(\mu) = 0 \quad \text{for } |\mu| \geq \mu_{\max}$$

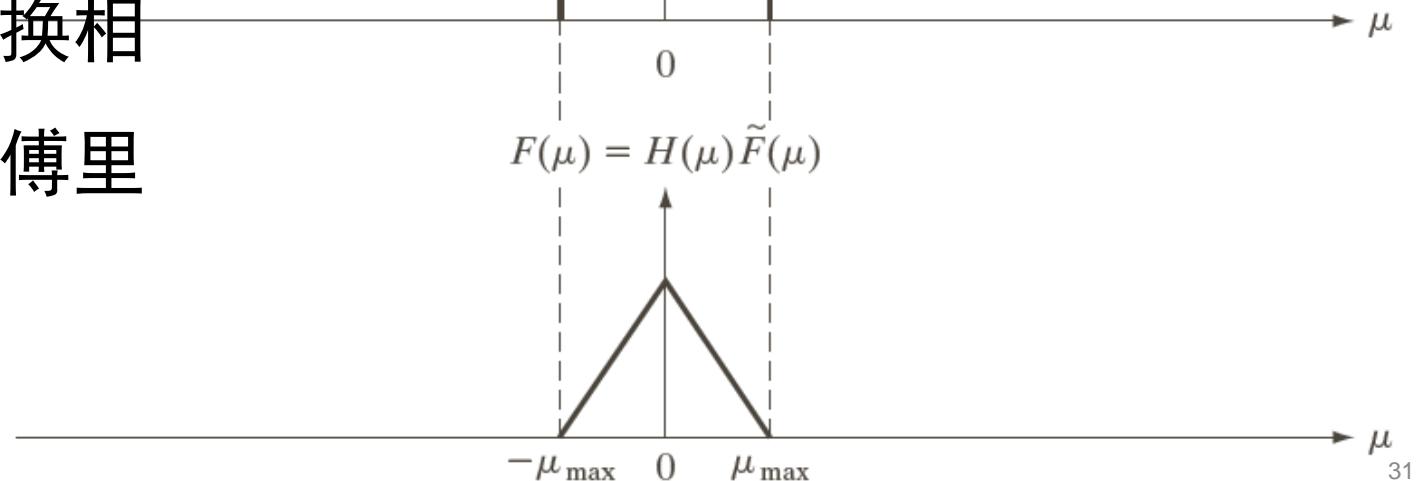
奈奎斯特采样定理 (Nyquist's Sampling Theorem)：当采样间隔 $\Delta T < \frac{1}{2\mu_{\max}}$ 时（即采样率大于奈奎斯特采样率 $2\mu_{\max}$ ），带限函数 $f(t)$ 可以从采样点完美重建（傅里叶变换可逆）。



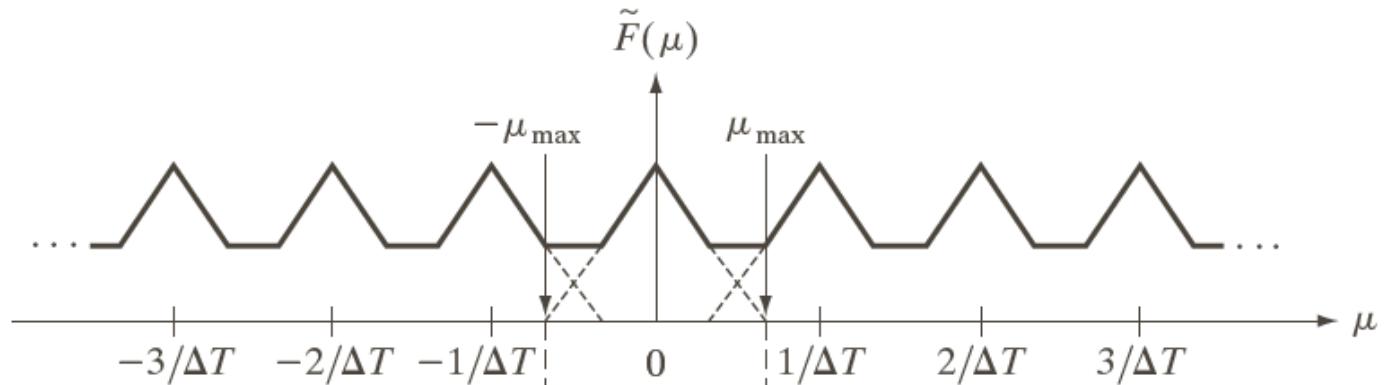
从采样信号重建原信号



重建方法：将**理想低通滤波器**与采样信号傅里叶变换相乘，然后做反傅里叶变换。

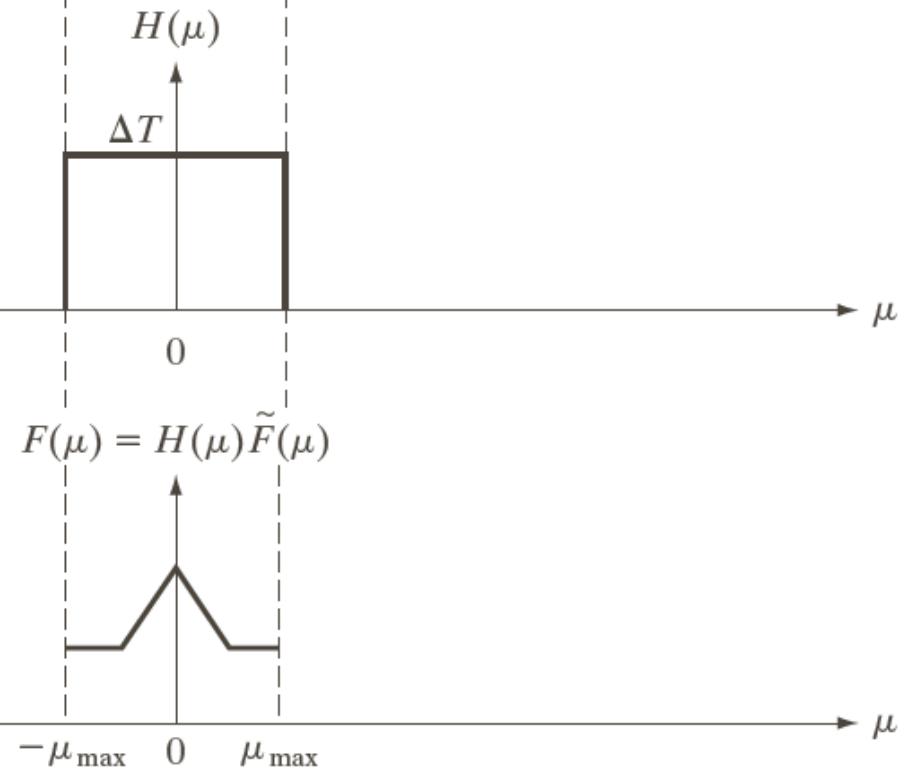


混叠现象

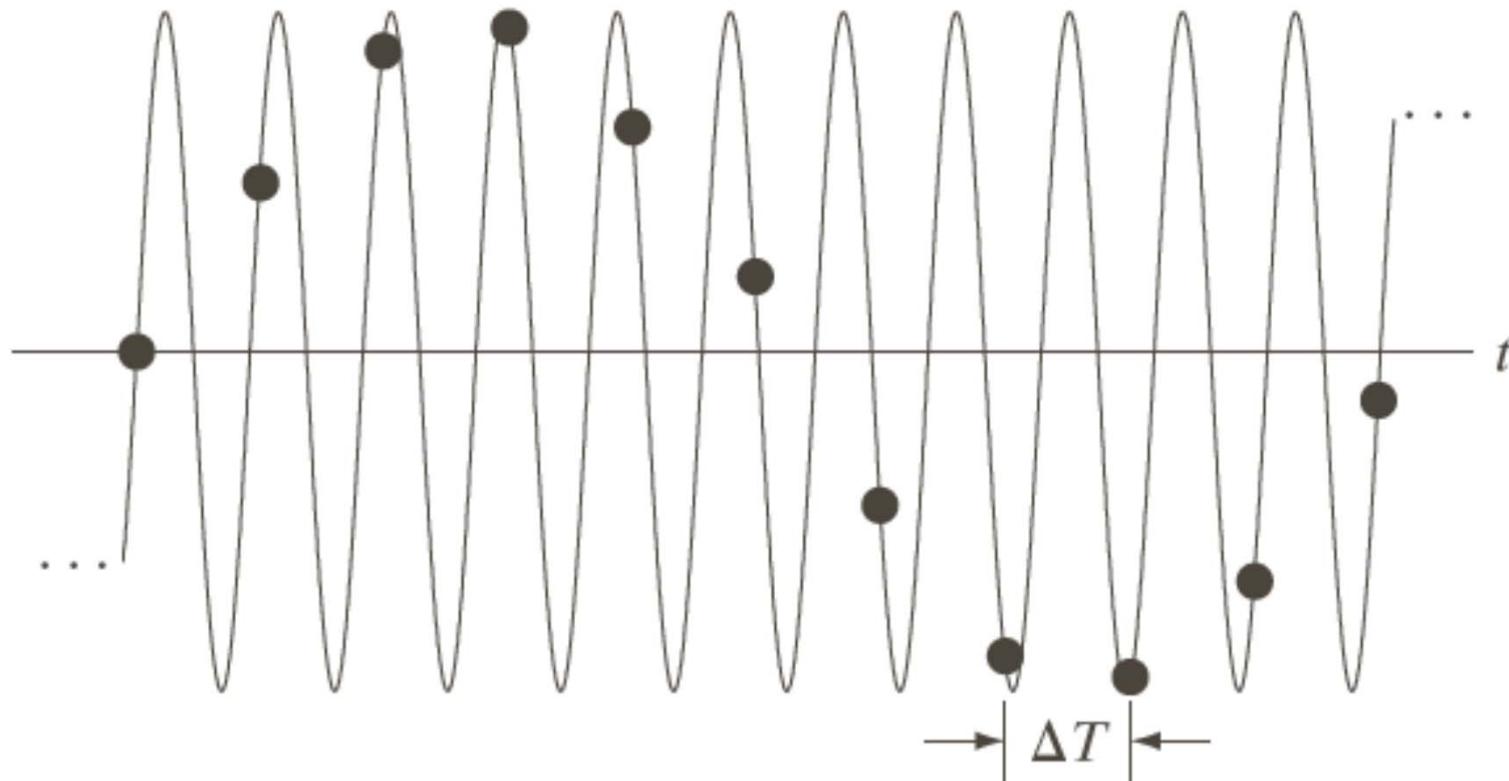


如果采样率低于奈奎斯
特采样率，采样信号的
频谱就会重叠，无法恢
复原始信号。

这种频谱重叠导致的失
真称为**混叠**（aliasing）。



混叠现象：例子



信号为正弦波 $\sin(\pi t)$, 周期为2秒;
采样率超过1样本/秒, 可避免混叠;
采样率小于1样本/秒, 发生混叠;
黑色采样点 (采样率约0.45样本/秒) 貌似低频正弦波;

1D连续信号
傅里叶变换



1D离散信号
傅里叶变换



1D离散信号
离散傅里叶变换

一维离散傅里叶变换 (DFT)

- 前面利用卷积定理证明了：采样函数 $\tilde{f}(t)$ 的傅里叶变换 $\tilde{F}(\mu)$ ，是原函数 $f(t)$ 的傅里叶变换 $F(\mu)$ 的周期性复制之和

- 下面推导从 $\tilde{f}(t)$ 计算 $\tilde{F}(\mu)$ 的公式

$$\begin{aligned}\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt = \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t) \delta(t - n\Delta T) e^{-j2\pi\mu t} dt = \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \delta(t - n\Delta T) e^{-j2\pi\mu t} dt = \\ &= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T} \quad (\text{f_n为$f(t)$的采样值})\end{aligned}$$

为了在计算机上对信号进行处理，先要得到采样信号；对于傅里叶变换，遇到了同样的问题

- $\tilde{F}(\mu)$ 是连续函数，无法在计算机上计算和处理
- 在计算机上，我们只能计算和处理 $\tilde{F}(\mu)$ 的采样版

一维离散傅里叶变换

$$\tilde{F}(\mu) = \sum_{-\infty}^{\infty} f_n e^{-j2\pi\mu n \Delta T}$$

$\tilde{F}(\mu)$ 是周期为 $1/\Delta T$ 的函数，只需对一个周期采样（没必要、不可能存储所有周期）。对其在区间 $[0, 1/\Delta T]$ 内，按照等间隔采样，

$$\mu = \frac{m}{M\Delta T}, \quad m = 0, 1, 2, \dots, M - 1$$

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi m n / M}$$

给定一组 $\{f_n\}$, $f(t)$ 的 M 个样本，上式得到一组 $\{F_m\}$ 。

一维离散傅里叶变换和逆变换

离散时间信号 $f(x)$ 的离散傅里叶变换 $F(u)$

为和2D DFT保持一致，时域变量用 x ，频域变量用 u

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \quad u = 0, 1, 2, \dots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \quad x = 0, 1, 2, \dots, M-1$$

$F(u)$ 和 $f(x)$ 均为周期为 M 的无限周期函数

$$F(u + kM) = F(u)$$

$$f(x + kM) = f(x)$$

采样间隔与频率间隔

如果 $f(x)$ 是 $f(t)$ 的 M 个样本，采样间隔为 ΔT ，序列 $\{f(x)\}, x = 0, 1, 2, \dots, M - 1$ 的时长为

$$T = M\Delta T$$

频域的采样间隔 Δu 是

$$\Delta u = \frac{1}{M\Delta T} = \frac{1}{T}$$

DFT 频域范围为

$$\Omega = M\Delta u = \frac{1}{\Delta T}$$

内 容

- 前言
- 一维傅里叶变换
- 二维傅里叶变换

二元连续冲激函数

- 连续变量 t 和 z 的冲激函数 (impulse function)

$$\delta(t, z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t, z) dt dz = 1$$

- 二元冲激函数的采样特性

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) \delta(t, z) dt dz = f(0, 0)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) \delta(t - t_0, z - z_0) dt dz = f(t_0, z_0)$$

二元离散冲激函数

- 离散变量 x 和 y 的冲激函数

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

- 二元离散冲激函数的采样特性

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y) \delta(x, y) = f(0, 0)$$

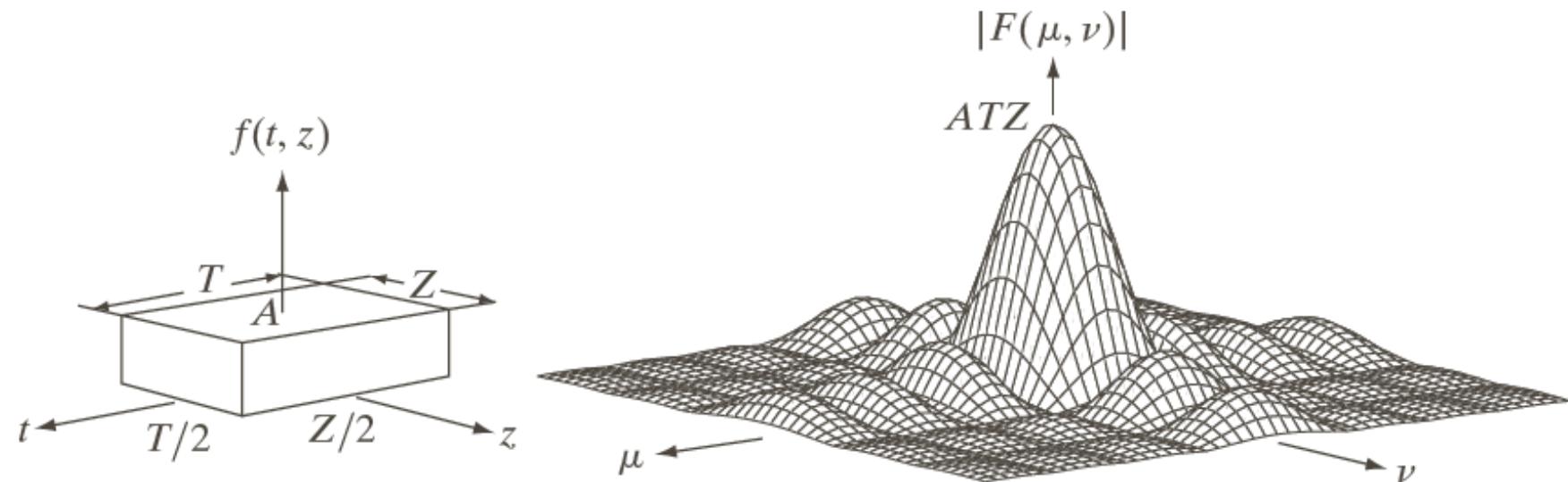
$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y) \delta(x - x_0, y - y_0) = f(x_0, y_0)$$

二维连续傅里叶变换

$$F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + \nu z)} dt dz$$
$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu) e^{j2\pi(\mu t + \nu z)} d\mu d\nu$$

二维连续傅里叶变换：例子

$$F(\mu, \nu) = ATZ \frac{\sin(\pi\mu T)}{\pi\mu T} \frac{\sin(\pi\nu Z)}{\pi\nu Z}$$



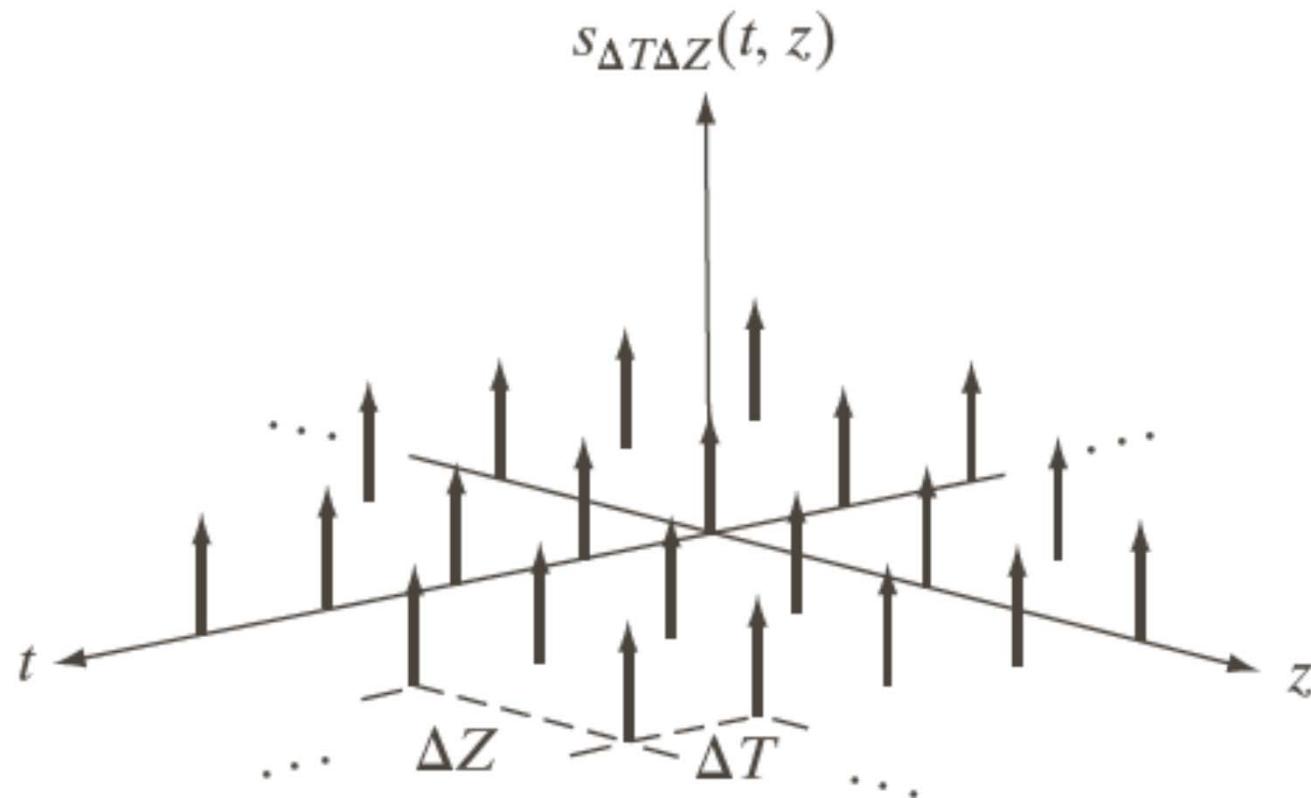
a b

FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t -axis, so the spectrum is more “contracted” along the μ -axis. Compare with Fig. 4.4.

二维采样

二维冲激串：

$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$



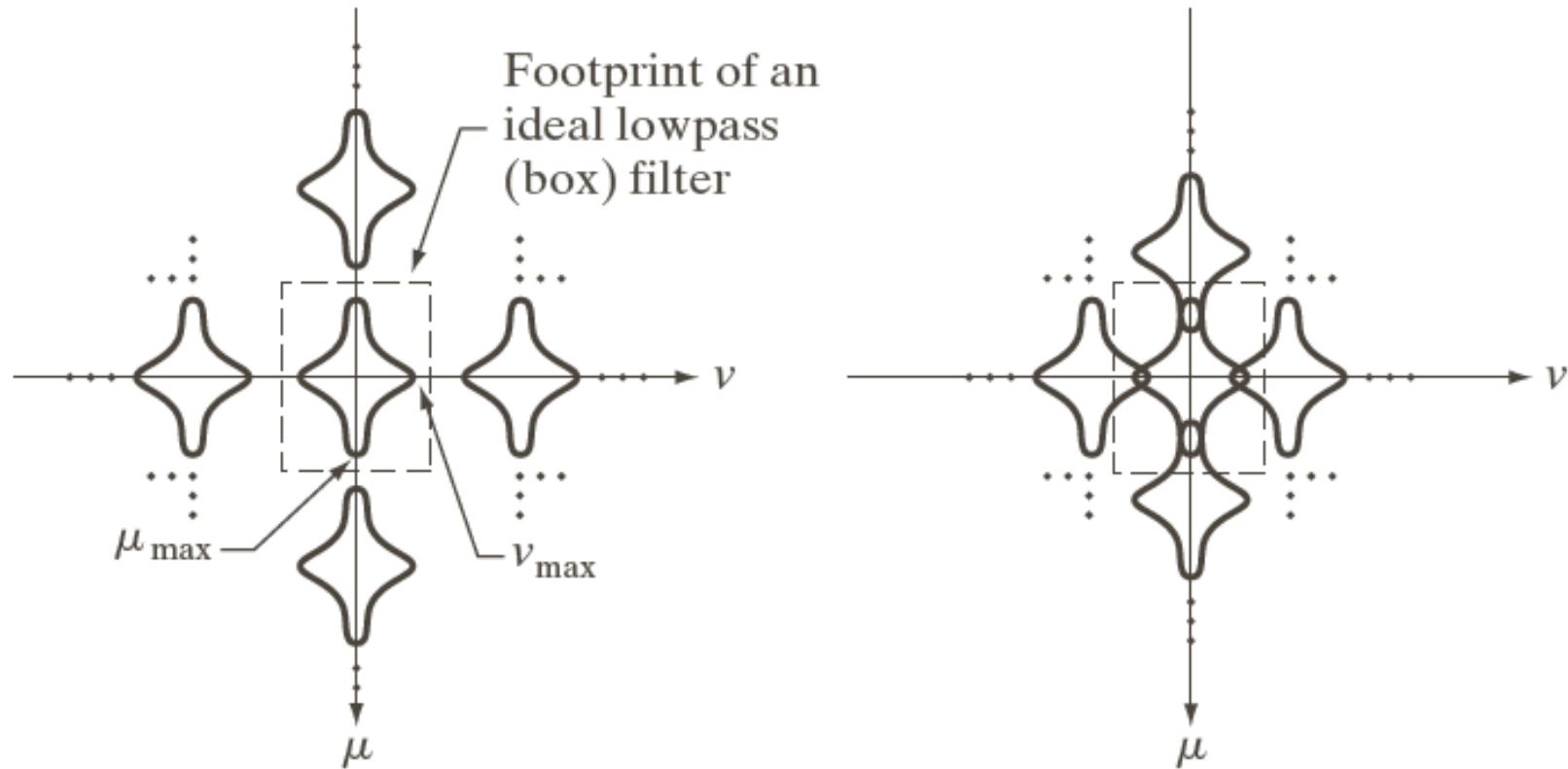
二维采样定理

如果函数 $f(t, z)$ 的傅里叶变换符合下面条件，则称该函数的带宽有限。

$$F(\mu, \nu) = 0 \quad \text{for } |\mu| \geq \mu_{\max} \text{ or } |\nu| \geq \nu_{\max}$$

二维采样定理 (2D Sampling Theorem)：当采样间隔 $\Delta T < \frac{1}{2\mu_{\max}}$ 而且 $\Delta Z < \frac{1}{2\nu_{\max}}$ 时，带限函数 $f(t, z)$ 可以从采样值完美重建。

过采样与欠采样



过采样函数的傅里叶变换

欠采样函数的傅里叶变换

混叠现象

- 类似于一维信号，二维图像只有在空域无限时，才可能是带宽有限的。
- 计算机处理的二维图像是空域有限的，因此带宽无限的。（证明方法：截断的二维图像可视为空域无限图像与矩形函数的乘积；而矩形函数的傅里叶变换是无限带宽的；卷积定理）
- 实际中，混叠不可避免。
- 图像的混叠分为空域混叠和时域混叠（例如转动的螺旋桨）。
- 本课程只讲空域混叠

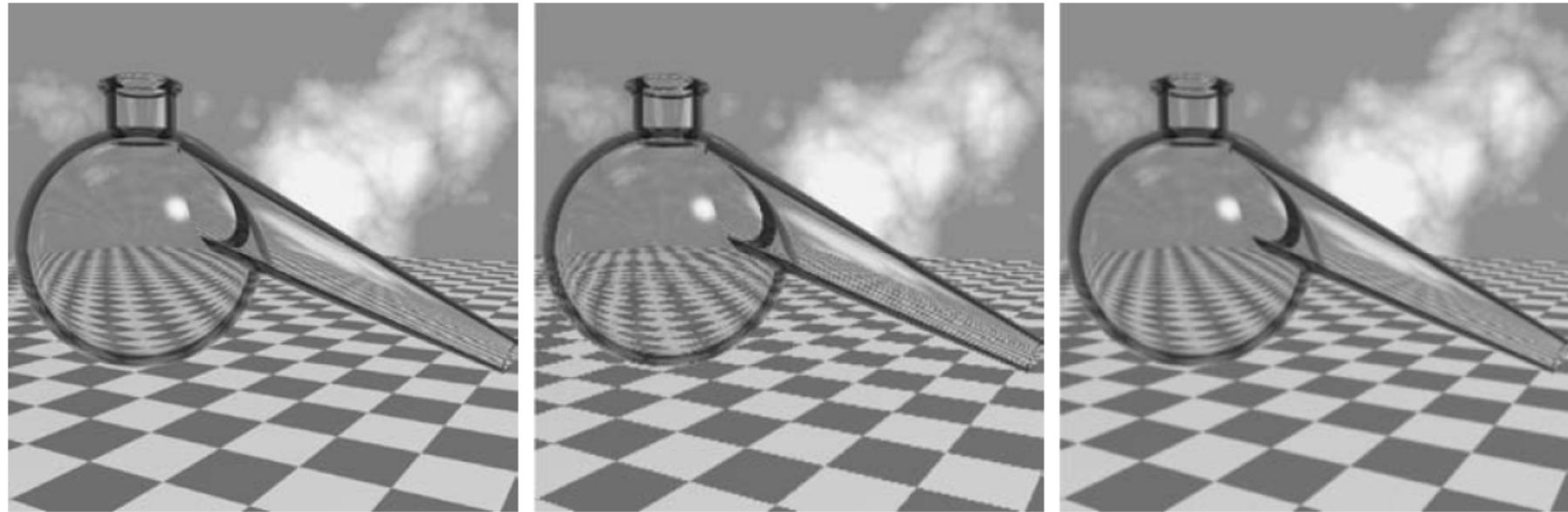
混叠现象



a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3×3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

混叠现象



a b c

FIGURE 4.18 Illustration of jaggies. (a) A 1024×1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5×5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)

二维离散傅里叶变换 (2D DFT)

$f(x, y)$ 为 $M \times N$ 数字图像

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

$$F(u, v) = R(u, v) + jI(u, v)$$

$$F(u, v) = |F(u, v)| e^{j\phi(u, v)}$$

幅度谱

相位谱

计算傅里叶变换的系数，相当于计算2个矩阵的内积

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cos[2\pi(ux/M + vy/N)]$$
$$-j \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \sin[2\pi(ux/M + vy/N)]$$

$$F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

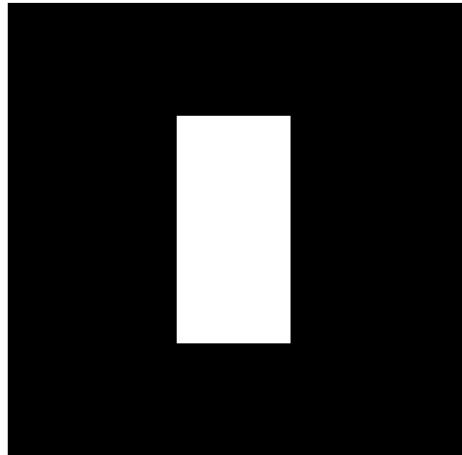
$$F(0, N/2) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cos(\pi y) - j \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \sin(\pi y)$$

$$F(M/2, N/2)$$

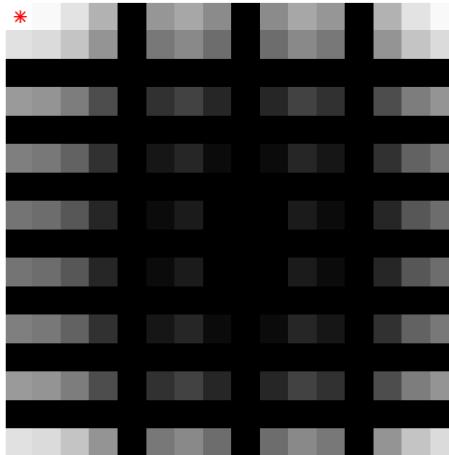
$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cos(\pi x + \pi y) - j \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \sin(\pi x + \pi y)$$

$$\text{直流成分 } F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

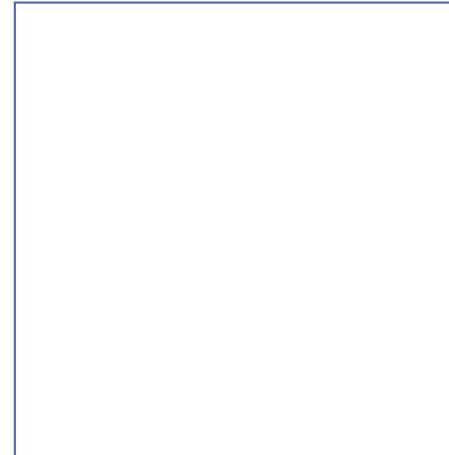
16 × 16图像



幅度谱



real part

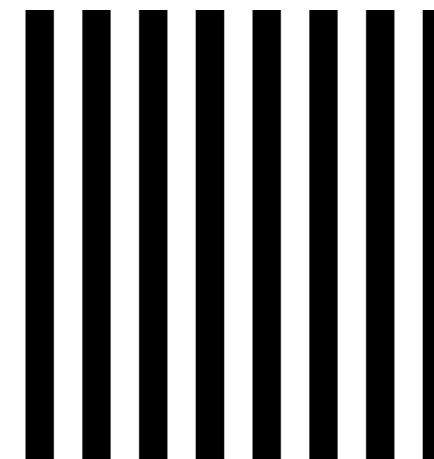
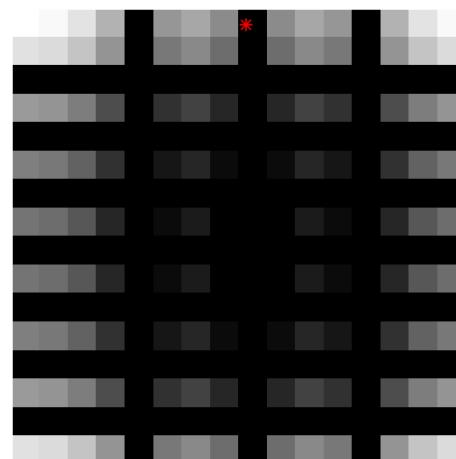


imaginary part



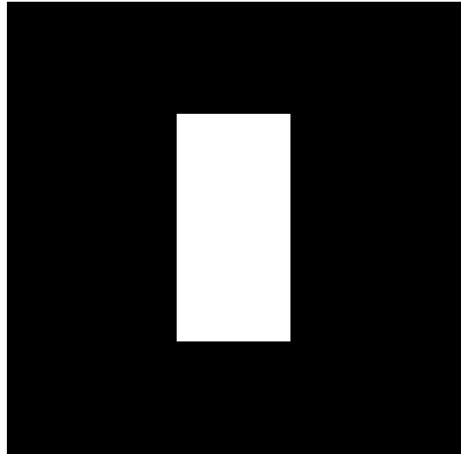
基函数图像1为白， -1为黑， 0为灰

$$F(M/2,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \cos(\pi x) - j \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \sin(\pi x)$$

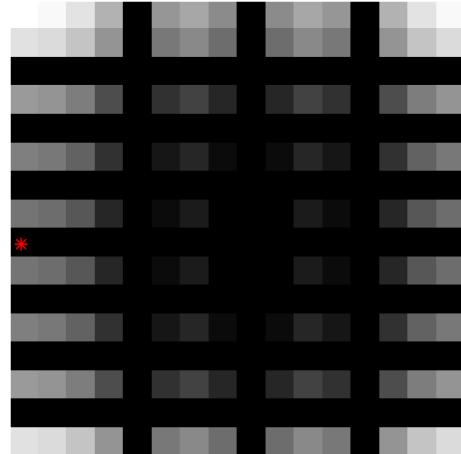


$$F(0, N/2) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cos(\pi y) - j \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \sin(\pi y)$$

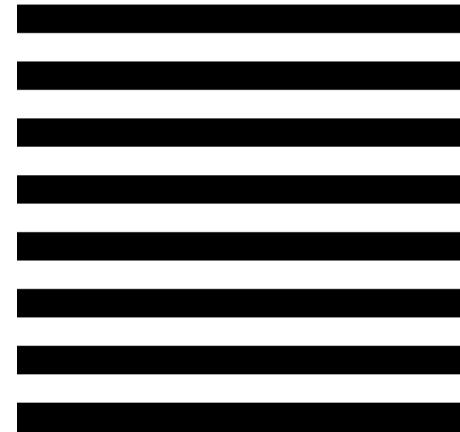
16 × 16图像



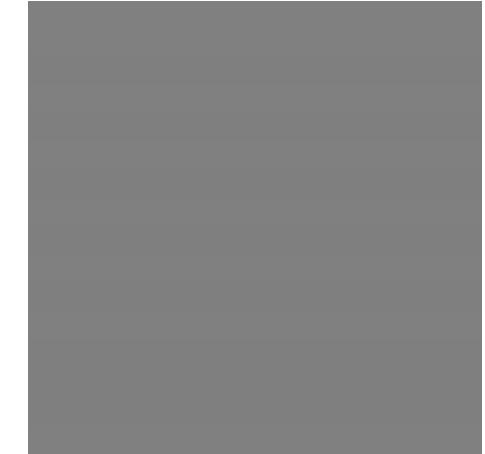
幅度谱



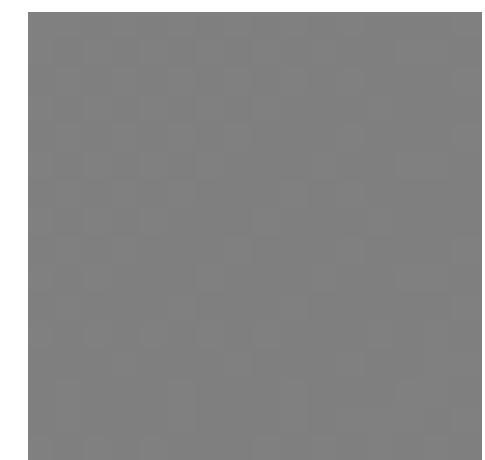
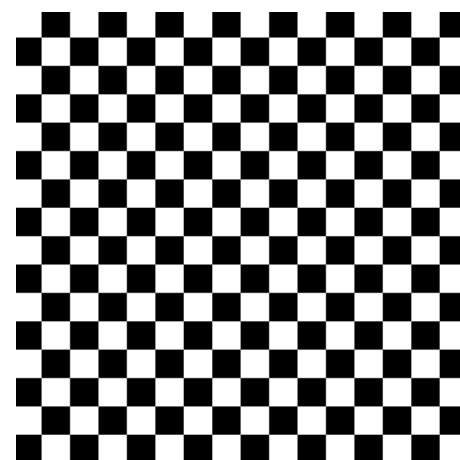
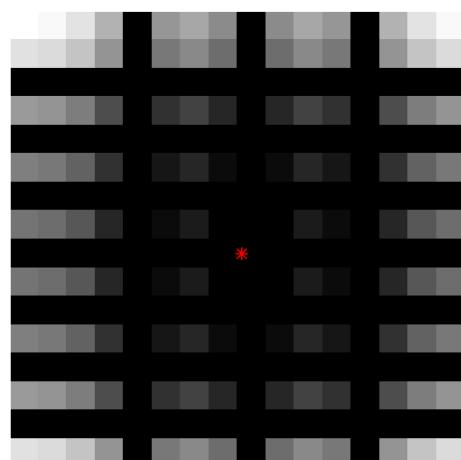
real part



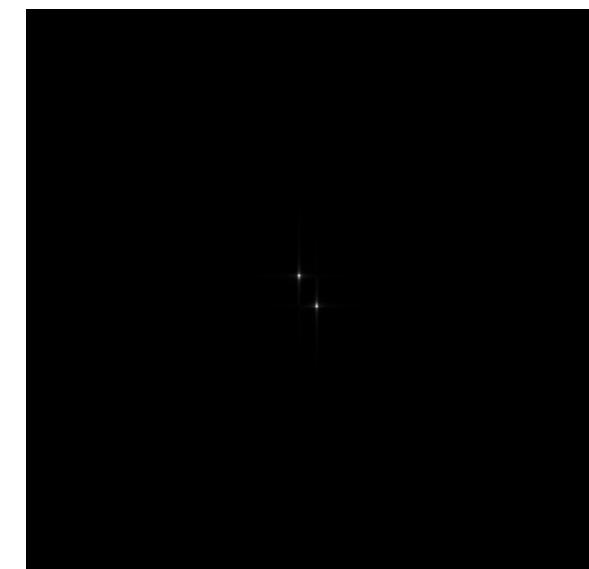
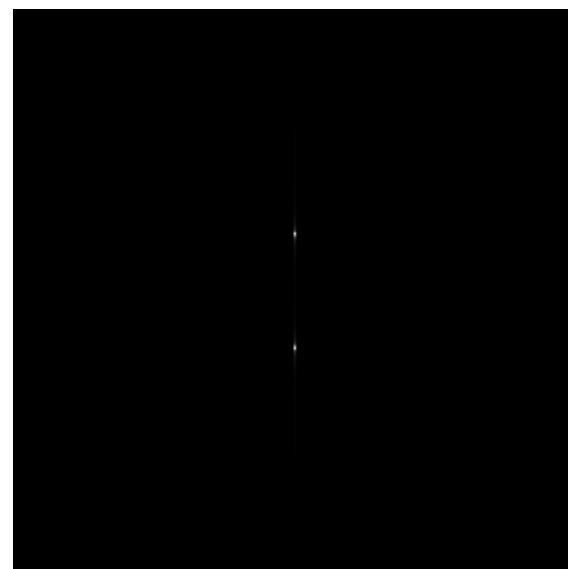
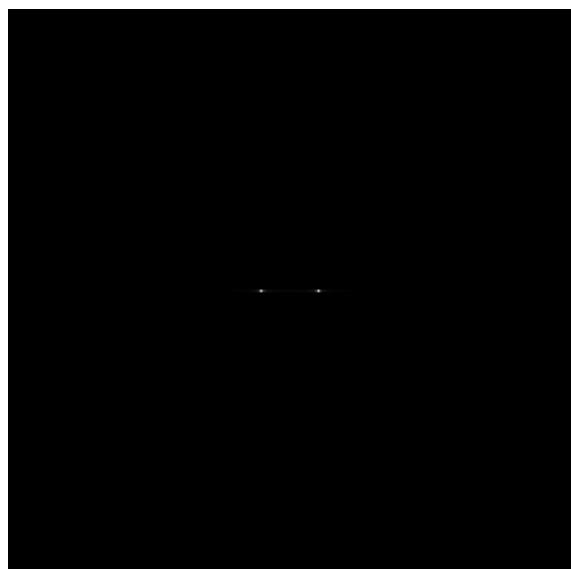
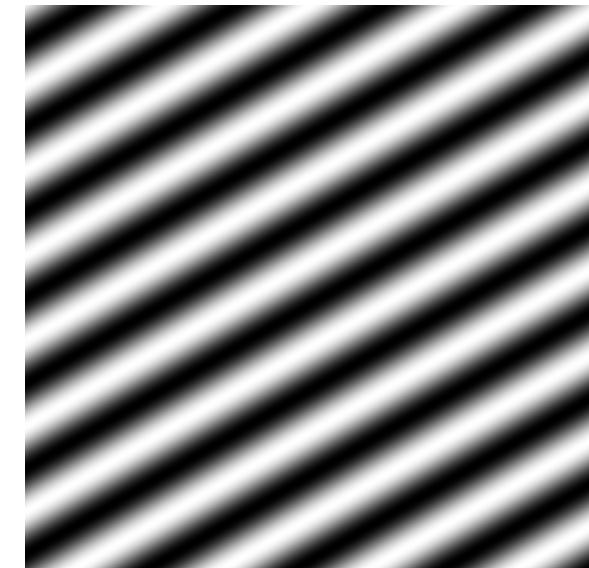
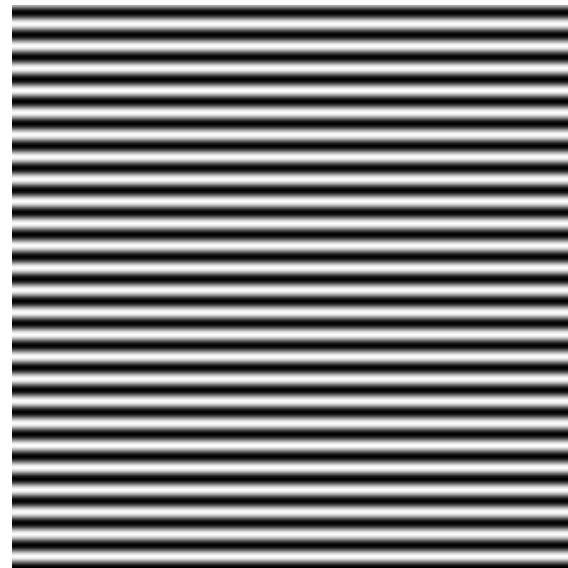
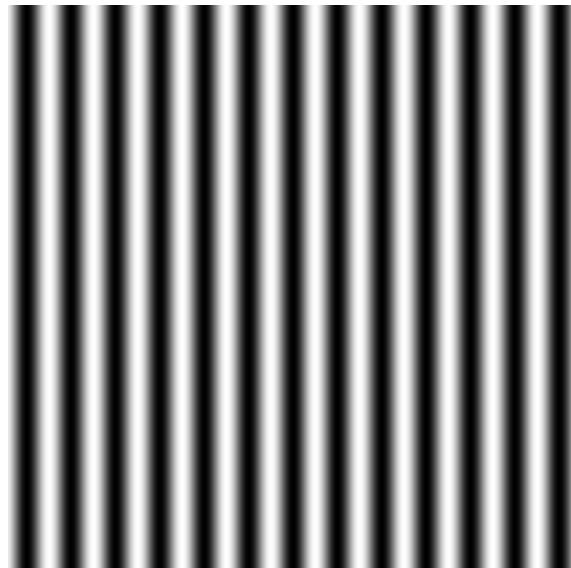
imaginary part



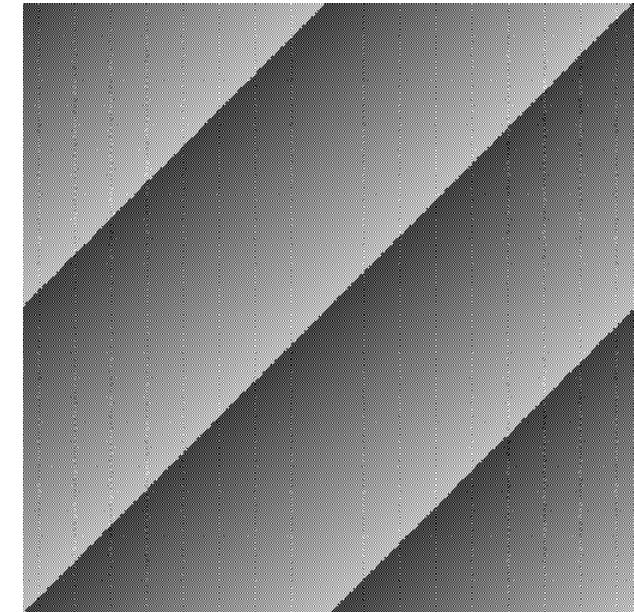
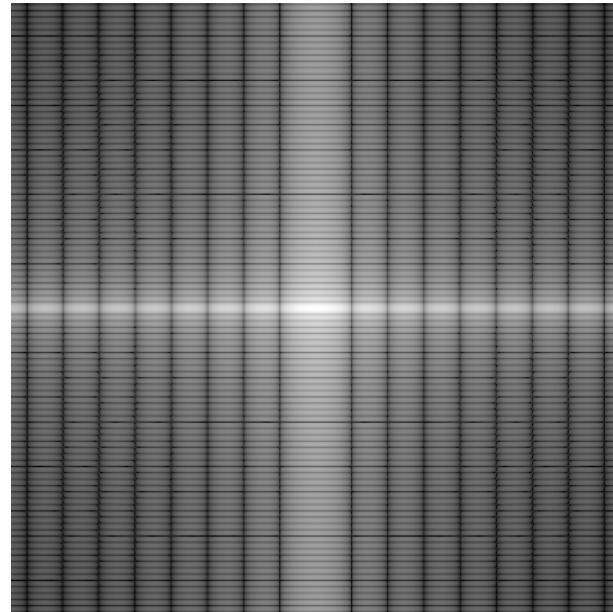
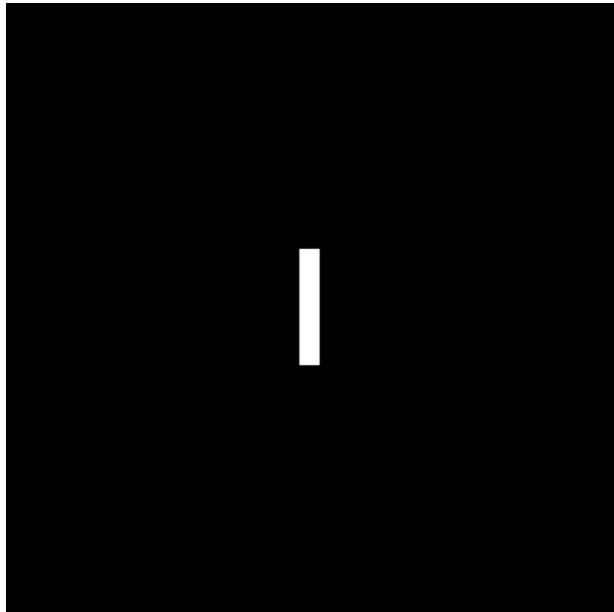
$$F(M/2, N/2) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cos(\pi x + \pi y) - j \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \sin(\pi x + \pi y)$$



不同参数正弦波的2D DFT



矩形函数的2D DFT



2D DFT的属性

- 采样间隔
- 平移和旋转
- 周期性
- 对称性
- 卷积定理

空域和频域的采样间隔之间的关系

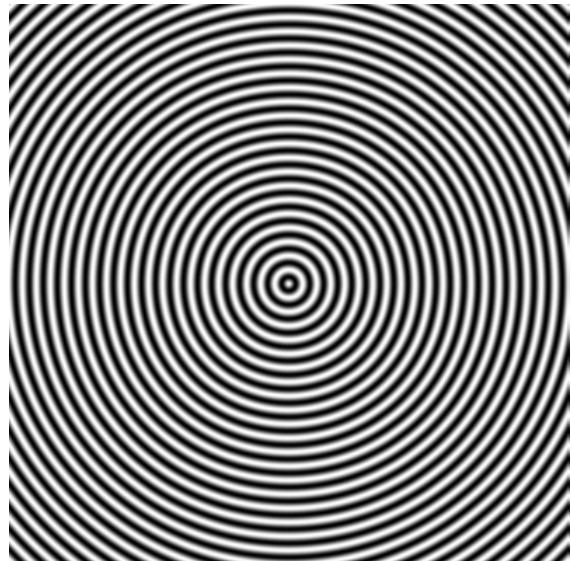
- 数字图像及其DFT都是离散函数
- 空域采样间隔 (ΔT 和 ΔZ) 与频域采样间隔 (Δu 和 Δv) 之间的关系： $\Delta u = \frac{1}{M\Delta T}$, $\Delta v = \frac{1}{N\Delta Z}$
- 通常图像空域两个维度的采样间隔 ΔT 和 ΔZ 是相同的



- 可见，频域采样间隔与图像的高和宽有关，如果图像是长方形，频域两个维度的采样间隔不同。这给频谱分析和滤波器设计带来不便
- 方便起见，可通过补零，将图像调为正方形

空域和频域的采样间隔之间的关系

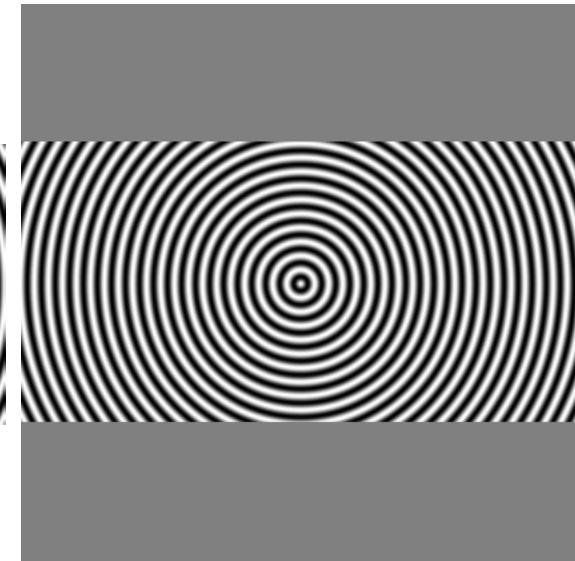
图像



400×400 像素

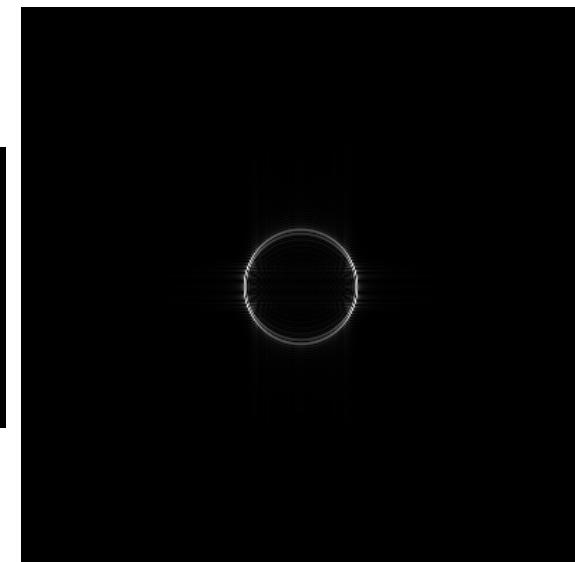
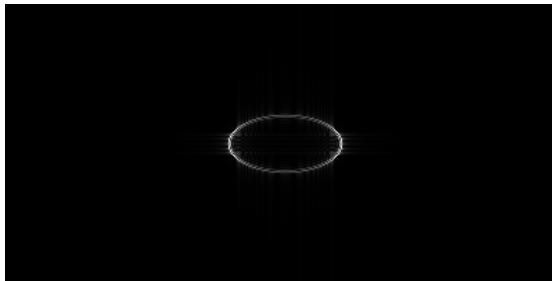
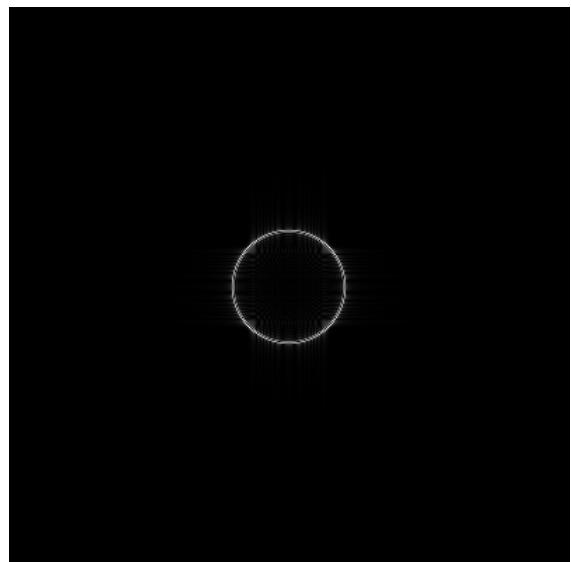


200×400 像素



补零

幅度谱



```
close all
M = 400; N = 400;
[X,Y] = meshgrid(-M/2:M/2-1,-N/2:N/2-1);
phase = 2*pi*0.1*sqrt(X.^2+Y.^2);
f = cos(phase);
figure(1),imshow(f,[ ])

F1 = fft2(f);
F2 = fftshift(F1);
figure(2),imshow(abs(F2),[ ])

f2 = f(M/4:3*M/4-1,:);
figure(3),imshow(f2,[ ])
F1 = fft2(f2);
F2 = fftshift(F1);
figure(4),imshow(abs(F2),[ ])

f3 = zeros(M,N);
f3(M/4:3*M/4-1,:) = f(M/4:3*M/4-1,:);
figure(5),imshow(f3,[ ])
F1 = fft2(f3);
F2 = fftshift(F1);
figure(6),imshow(abs(F2),[ ])
```

平移

$$F(u, v) = \sum_{\substack{x=0 \\ x=M}}^{M-1} \sum_{\substack{y=0 \\ y=N}}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^1 \sum_{v=0}^N F(u, v) e^{j2\pi(ux/M + vy/N)}$$

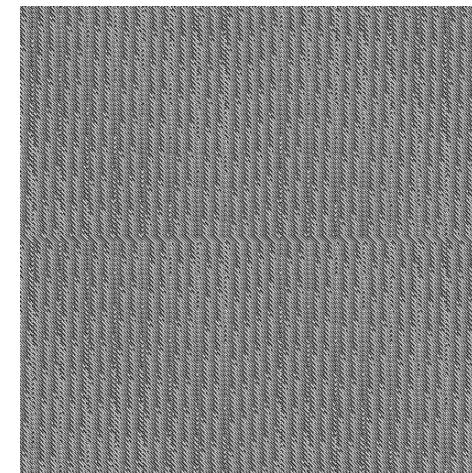
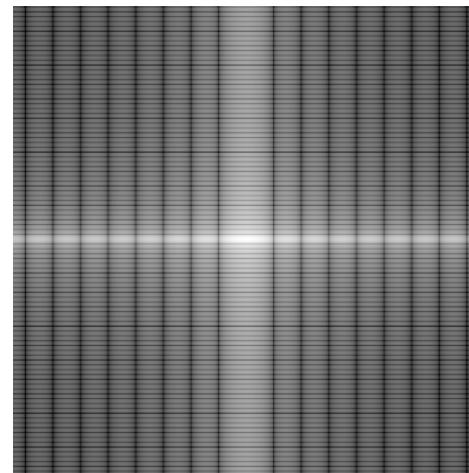
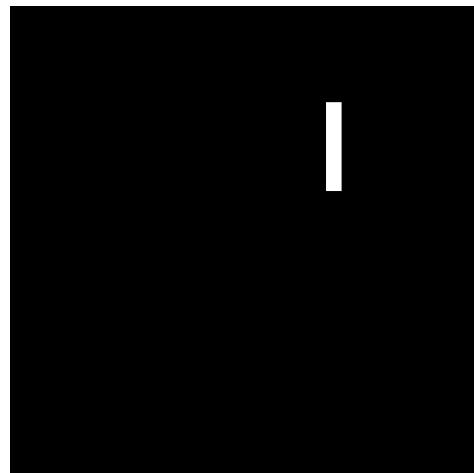
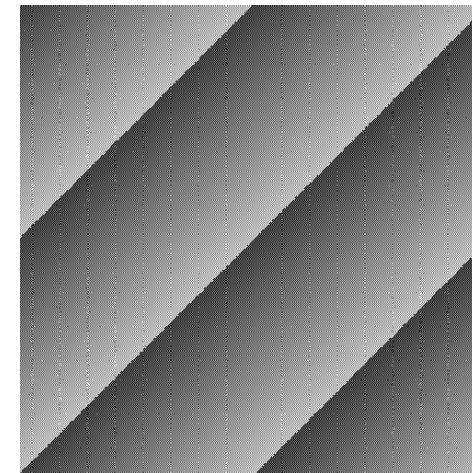
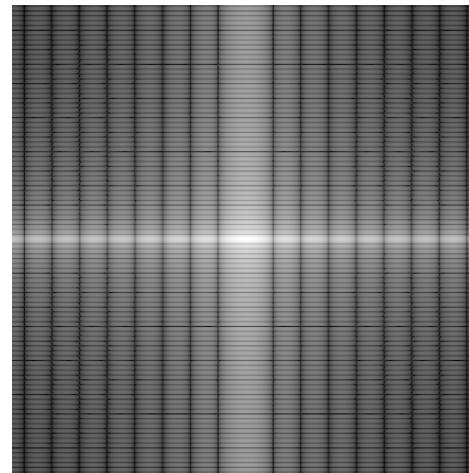
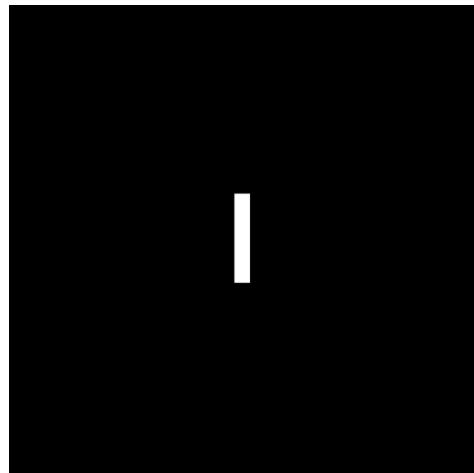
$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v) e^{-j2\pi(x_0 u/M + y_0 v/N)}$$

$$F(u - u_0, v - v_0) \Leftrightarrow f(x, y) e^{j2\pi(u_0 x/M + v_0 y/N)}$$

平移

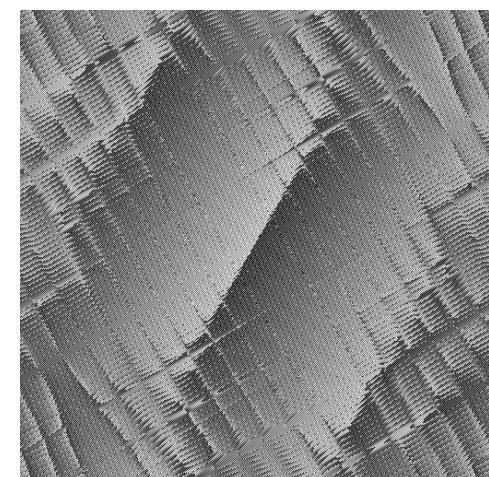
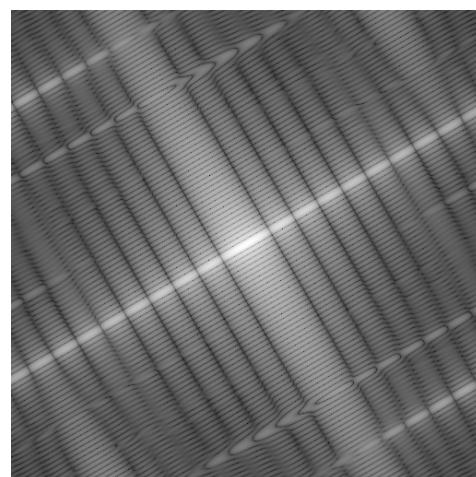
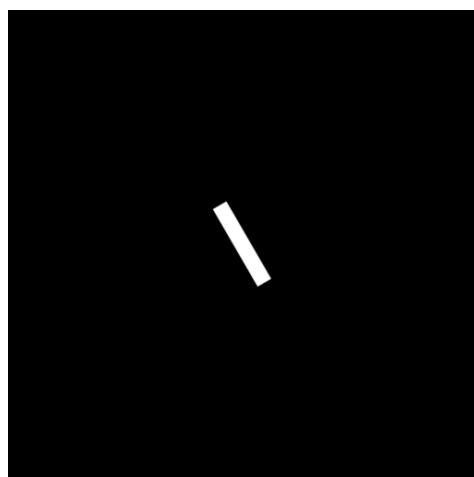
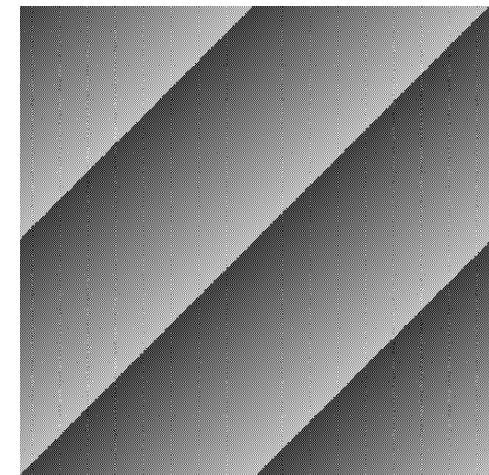
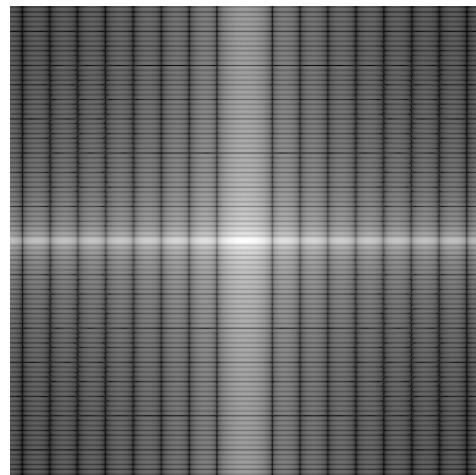
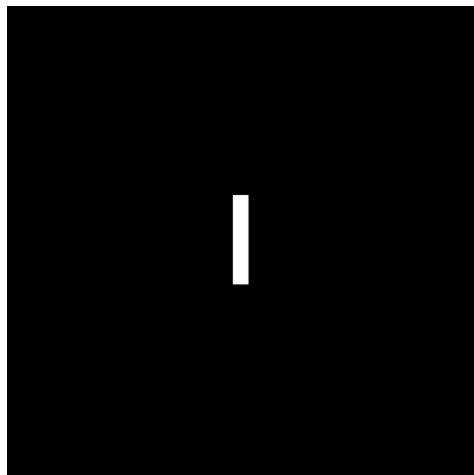
$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(x_0 u/M + y_0 v/N)}$$

图像平移，幅度谱不变，相位谱改变



旋转

$$f(r, \theta + \theta_0) \Leftrightarrow F(w, \varphi + \theta_0)$$



```
%ex0413_RectDFT
f = imread('..\data\Fig0424(a)(rectangle).tif');
f = imresize(f,0.5);
F = fftshift(fft2(f));
figure(1),clf,subplot(2,3,1),imshow(f,[])
subplot(2,3,2),imshow(log(1+abs(F)),[])
subplot(2,3,3),imshow(angle(F),[])

f2 = imread('..\data\Fig0425(a)(translated_rectangle).tif');
F2 = fftshift(fft2(f2));
figure(1),subplot(2,3,4),imshow(f2,[])
subplot(2,3,5),imshow(log(1+abs(F2)),[])
subplot(2,3,6),imshow(angle(F2),[])

f3 = imrotate(f,30,'bilinear','crop');
F3 = fftshift(fft2(f3));
figure(2),clf,subplot(2,3,1),imshow(f,[])
subplot(2,3,2),imshow(log(1+abs(F)),[])
subplot(2,3,3),imshow(angle(F),[])
figure(2),subplot(2,3,4),imshow(f3,[])
subplot(2,3,5),imshow(log(1+abs(F3)),[])
subplot(2,3,6),imshow(angle(F3),[])
```

周期性

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M+vy/N)}$$
$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M+vy/N)}$$

从定义可以得出， $F(u, v)$ 和 $f(x, y)$ 均为周期函数。

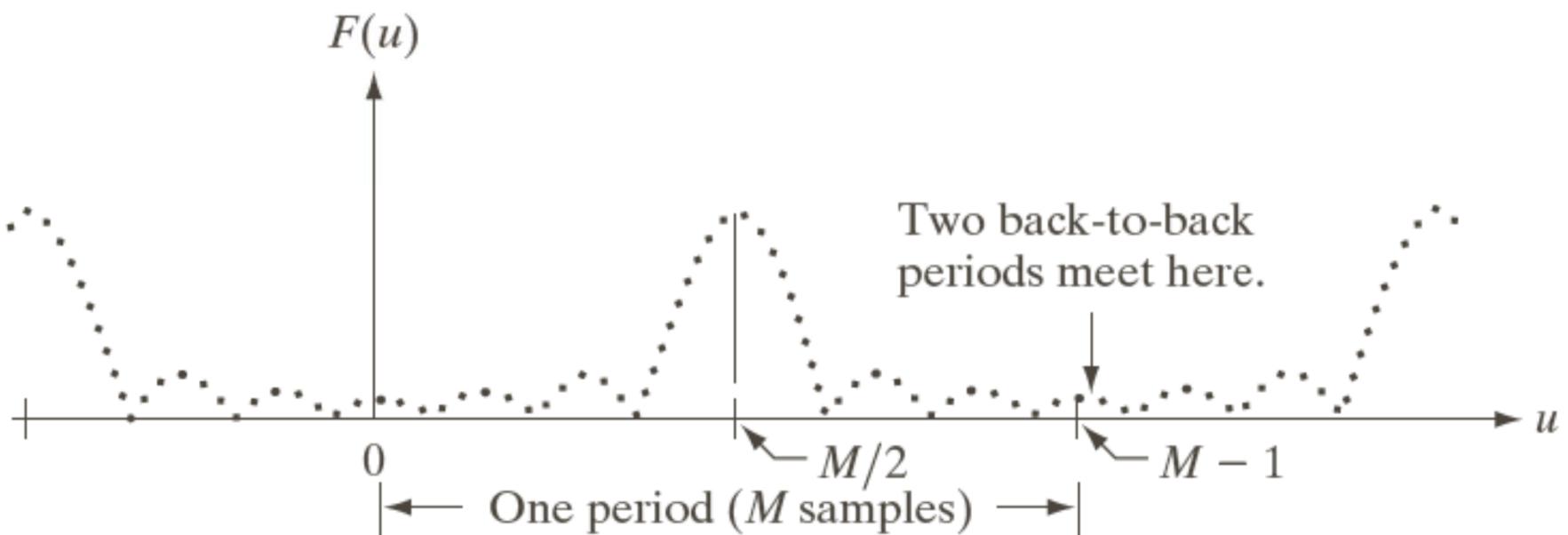
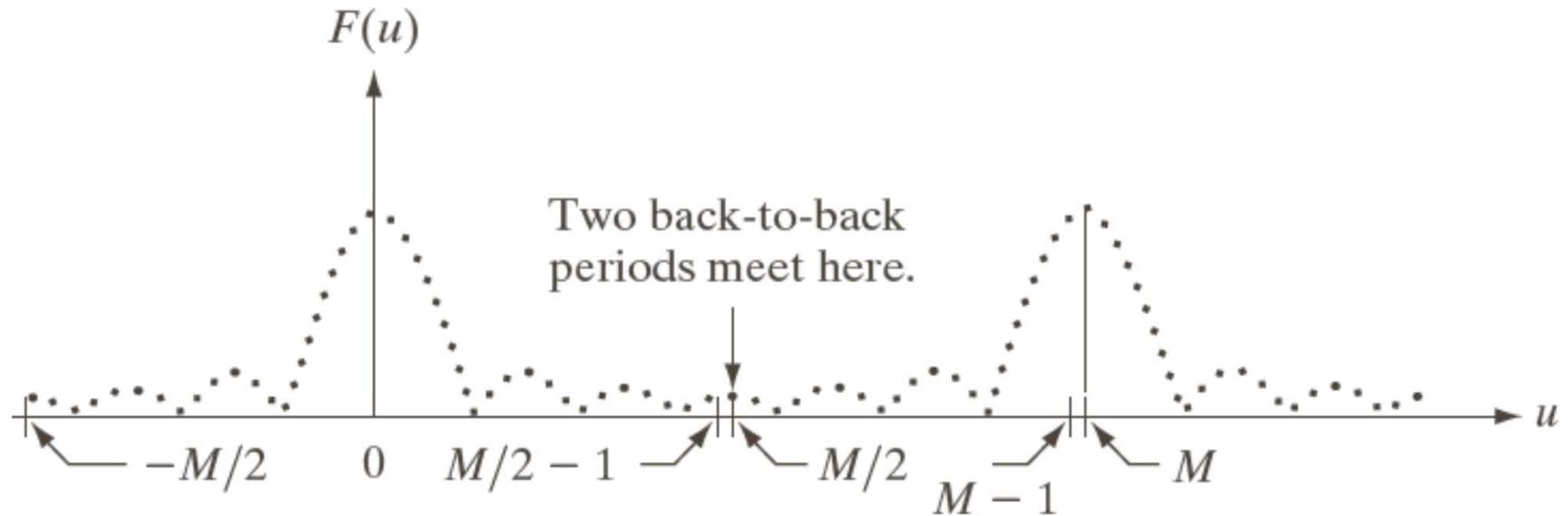
$$F(u, v) = F(u + k_1 M, v + k_2 N)$$
$$f(x, y) = f(x + k_1 M, y + k_2 N)$$

k_1 和 k_2 为整数。

如何理解 $f(x, y)$ 为周期函数？

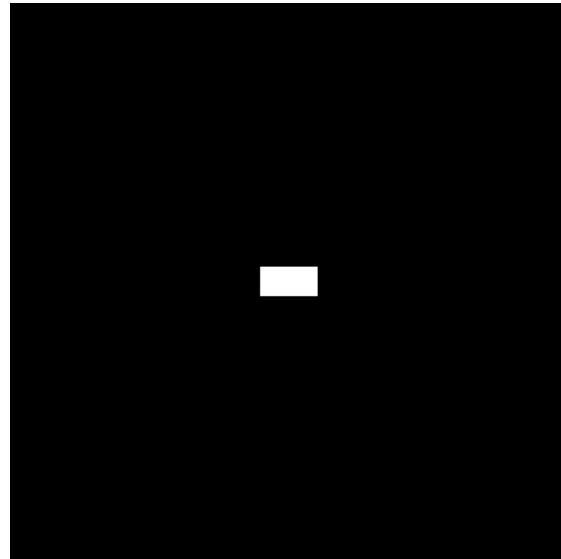
输入图像本不是周期函数。但是如果对其进行频域滤波，那么它就被当做周期函数了。

周期性

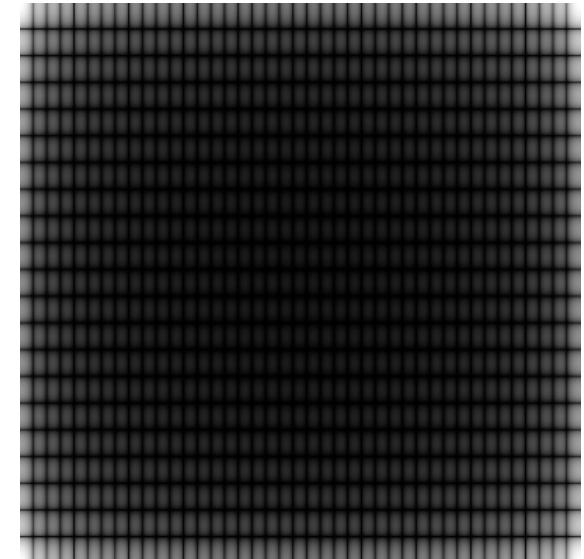


居中显示

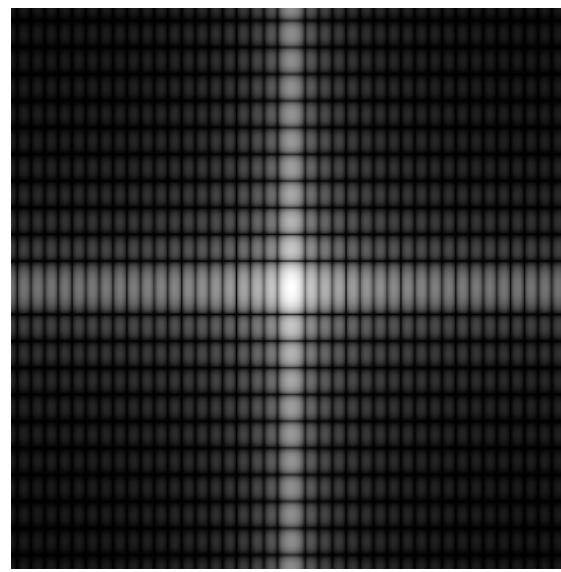
- 原始图像 $f(x, y)$ 的 DFT $F(u, v)$ 的零频（直流成分）位于左上角
- 为简化滤波器 $H(u, v)$ 的设计，需将 $F(u, v)$ 零频居中



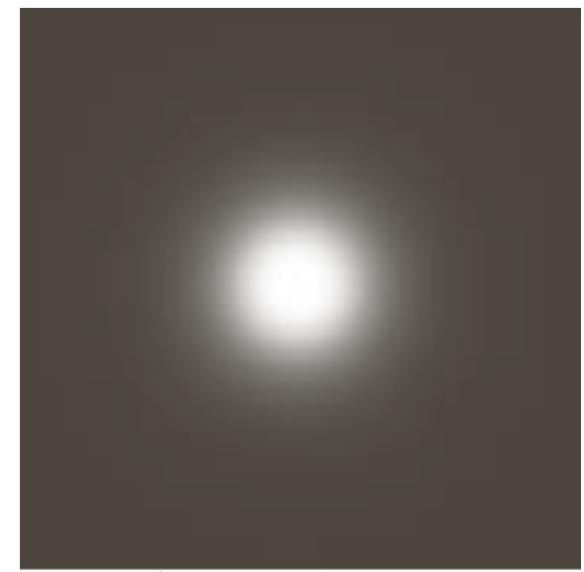
原始图像



原始图像的DFT

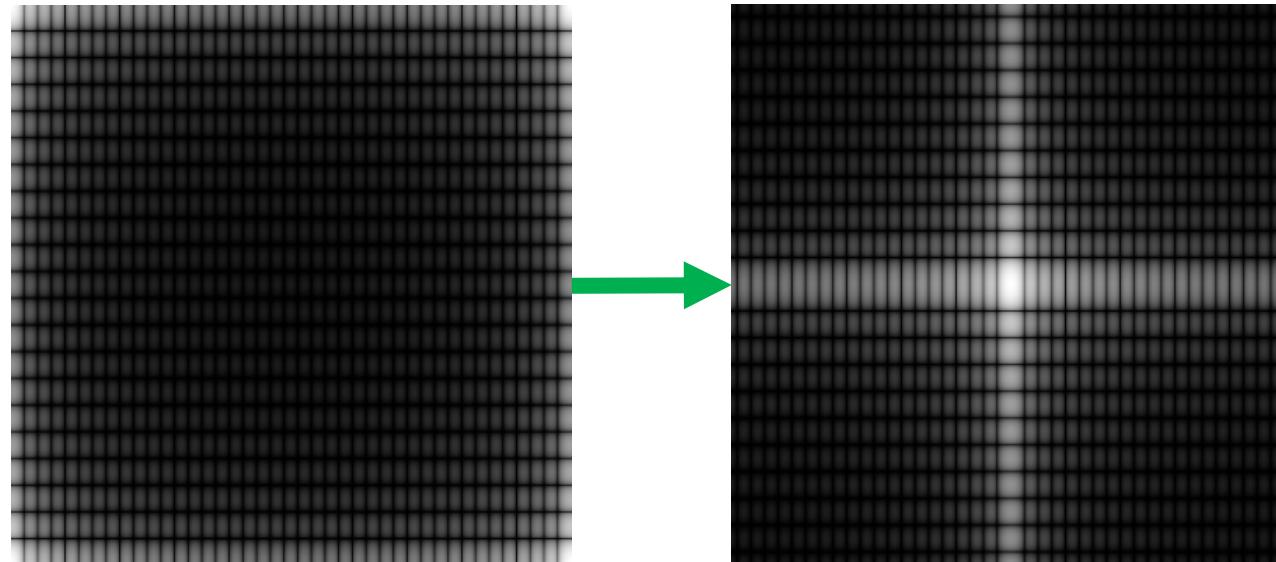


零频居中的DFT



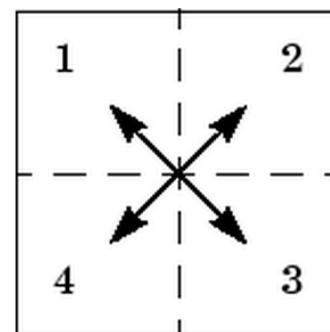
高斯低通滤波器 $H(u, v)$

居中显示的2种方法



方法1：
计算 $f(x, y)(-1)^{x+y}$ 的DFT
原理是DFT的平移属性

方法2：
MATLAB的fftshift



```
close all
f = zeros(400,400);
[M,N] = size(f);
f(M/2-10:M/2+10,N/2-20:N/2+20) = 1;

F1 = fft2(f);
figure(1),imshow(f),title('Original image')
figure(2),imshow(log(1+abs(F1)),[]),title('DFT')

[X,Y] = meshgrid(0:M-1,0:N-1);
F2 = fft2(f .* ((-ones(M,N)).^(X+Y)));
figure(3),imshow(log(1+abs(F2)),[]),title('DFT shifted by (-1)^(x+y)')

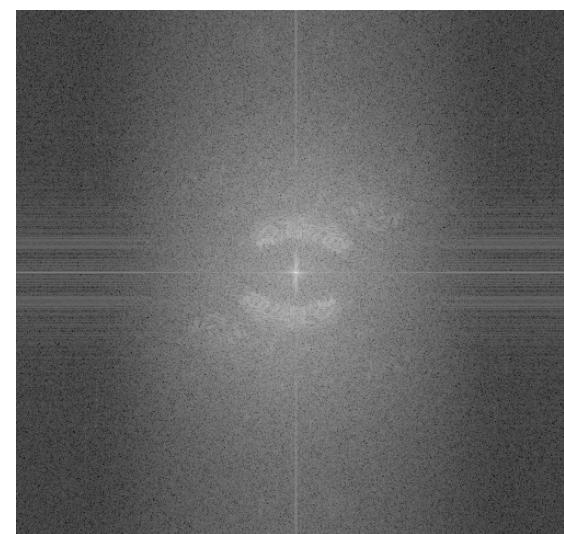
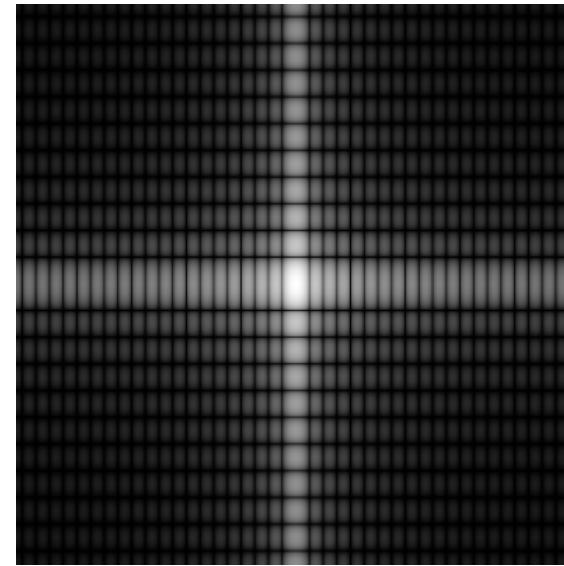
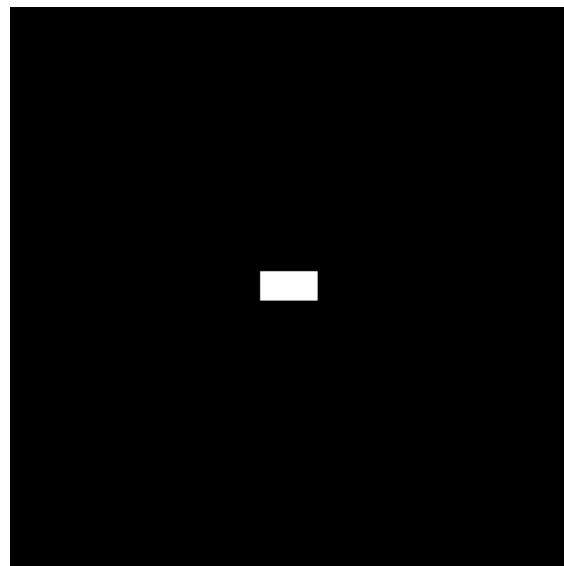
F3 = fftshift(F1);
figure(4),imshow(log(1+abs(F3)),[]),title('DFT shifted by fftshift function')
```

对称性

$$f(x, y) \Leftrightarrow F(u, v)$$

- 实函数 $f(x, y), F(u, v)$ 共轭对称 (conjugate symmetric) , 即 $F^*(u, v) = F(-u, -v)$
- 实函数 $f(x, y), F(u, v)$ 的实部 $R(u, v)$ 为偶函数, 虚部 $I(u, v)$ 为奇函数
- 实偶函数 $f(x, y), F(u, v)$ 也是实偶函数
- 实奇函数 $f(x, y), F(u, v)$ 是虚奇函数

对称性



卷积定理

$$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$$

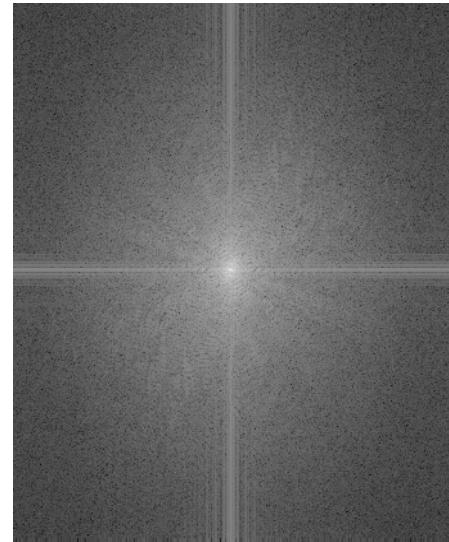
卷积定理将频域滤波和空域滤波关联了起来

傅立叶变换：振幅谱和相位谱

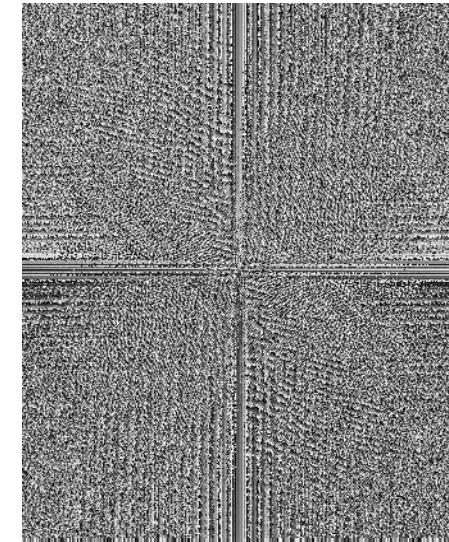
Face image



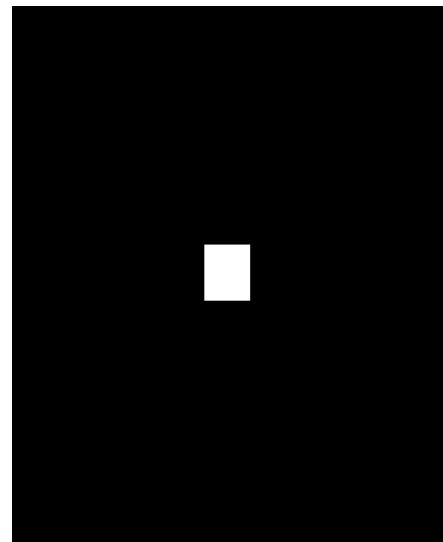
magnitude



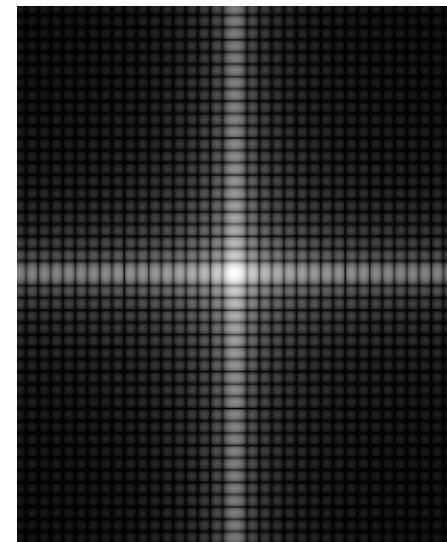
phase



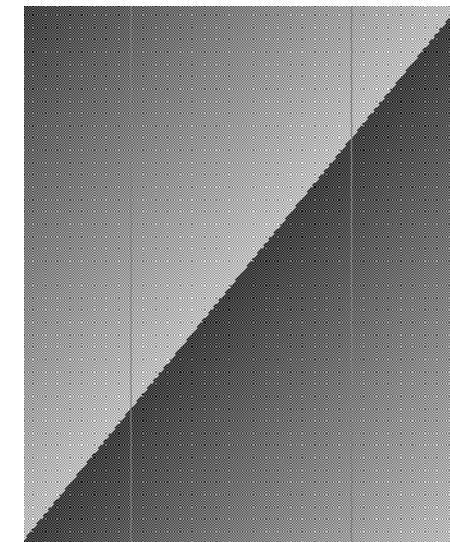
Rectangle image



magnitude



phase

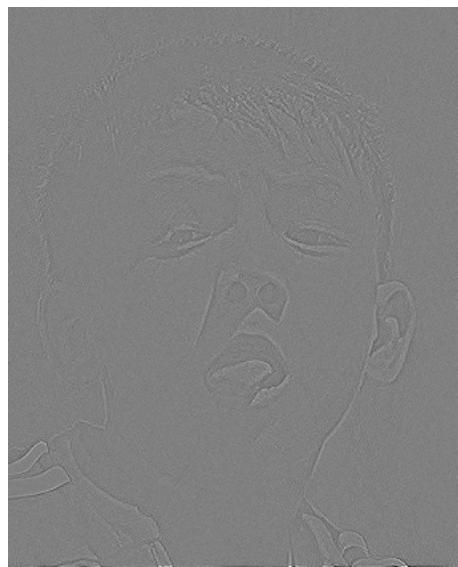


重建图像（相位更重要）

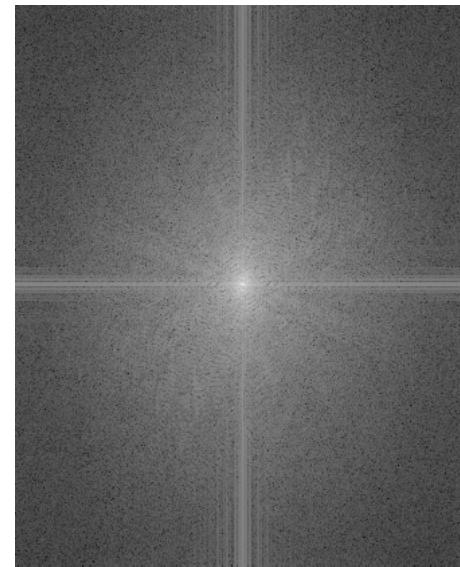
Face image



phase of face



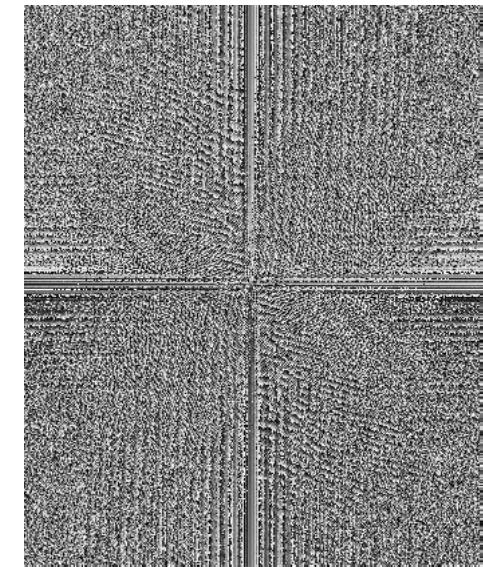
magnitude



magnitude of face



phase



phase of face +
magnitude of rectangle



重建图像的code

```
close all
f1 = rgb2gray(imread('..\data\bobby.bmp'));
F1 = fftshift(fft2(f1));
P1 = angle(F1);
M1 = abs(F1);
figure(1), subplot(1,3,1), imshow(f1), title('image')
subplot(1,3,2), imshow(log(M1+1),[]), title('magnitude')
subplot(1,3,3), imshow(P1,[]), title('phase')

% rectangle
[h, w] = size(f1);
f2 = zeros(h, w);
side_len = 0.1;
r1 = ceil((0.5-side_len/2)*h);
r2 = ceil((0.5+side_len/2)*h);
c1 = ceil((0.5-side_len/2)*w);
c2 = ceil((0.5+side_len/2)*w);
f2(r1:r2,c1:c2) = 1;
F2 = fftshift(fft2(f2));
P2 = angle(F2);
M2 = abs(F2);
figure(2), subplot(1,3,1), imshow(f2), title('image')
subplot(1,3,2), imshow(log(M2+1),[]), title('magnitude')
subplot(1,3,3), imshow(P2,[]), title('phase')

% use phase of face to reconstruct
F1_phase = cos(P1) + li*sin(P1);
recon1_phase = (ifft2(ifftshift(F1_phase)));
figure(3), subplot(1,3,1), imshow(real(recon1_phase), []), title('phase of face')

% use magnitude of face to reconstruct
recon1_magnitude = (ifft2(ifftshift(M1)));
figure(3), subplot(1,3,2), imshow(real(recon1_magnitude), []), title('magnitude of face')

% use phase of face and magnitude of rectangle to reconstruct
F1_phasel_mag2 = F1_phase .* M2;
recon1_phasel_mag2 = (ifft2(ifftshift(F1_phasel_mag2)));
figure(3), subplot(1,3,3), imshow(real(recon1_phasel_mag2), []), title('phase of face + magnitude of rectangle')
```