EXAMEN SISTEME DINAMICE

DANCIU ALEXANDRU-MIHAI, 212

2) a) $x y' = y + 2y(\ln y - \ln x)$

$$y' = \frac{y}{x} + 2\frac{y}{x} \cdot \ln \frac{y}{x}$$

folosim substituția $z = \frac{y(x)}{x} \Rightarrow y = z \cdot x$

$\Rightarrow y' = z'x + z$

$z'x + z = z + 2z\ln z$

$z'x = 2z\ln z \,|:x \qquad z' = \frac{dz}{dx}$

$$\frac{dz}{dx} = \frac{1}{x} \cdot 2z\ln z$$

$$\frac{dz}{z\ln z} = \frac{2}{x} \cdot dx \,\Big|\int$$

$$\int \frac{1}{z\ln z}dz = \int \frac{1}{\ln z} \cdot \frac{1}{z}dz = \int \frac{1}{t} \cdot dt = \ln t + \mathcal{C}$$

$t = \ln z \qquad \int \frac{1}{z\ln z}dz = \ln(\ln z) + \mathcal{C}$

$dt = \frac{1}{z}dz$

$$\int \frac{dz}{z\ln z} = \int \frac{2}{x}dx$$

$\ln(\ln z) = 2\ln x + \ln c, \; C \in \mathbb{R}$

$\ln(\ln z) = \ln(c \cdot x^2)$

$\ln z = c \cdot x^2 \Rightarrow z = e^{c \cdot x^2}, \; c \in \mathbb{R}$

$y = z \cdot x = e^{c \cdot x^2} \cdot x, \; c \in \mathbb{R}$

$y(x) = x \cdot e^{c \cdot x^2}$, ~~$c \in \mathbb{R}$~~ $c \in \mathbb{R}$

2) b) $y'' - 6y' + 13y = 13x - 6$

$y = y_0 + y_p$ unde y_0 - sol. gen. a ec. omogene

y_p - sol. particulară

$y'' - 6y' + 13y = 0$

$r^2 - 6r + 13 = 0$

$\Delta = 36 - 4 \cdot 13 = 36 - 52 = -16 = (4i)^2$

$r_{1,2} = \dfrac{6 \pm 4i}{2} = 3 \pm 2i \Rightarrow \alpha = 3, \beta = 2$

$y_1 = e^{\alpha x} \cos \beta x \quad y_2 = e^{\alpha x} \sin \beta x$

$y_0 = c_1 \cdot y_1 + c_2 \cdot y_2, \; c_1, c_2 \in \mathbb{R}$

$y_0 = c_1 \cdot e^{3x} \cos 2x + c_2 \cdot e^{3x} \sin 2x, \; c_1, c_2 \in \mathbb{R}$

$y_p = ax + b$ - sol. particulară $\Rightarrow y_p' = a \Rightarrow y_p'' = 0$

$y_p'' - 6y_p' + 13y_p = 13x - 6$

$0 - 6 \cdot a + 13(ax + b) = 13x - 6$

$13ax + 13b - 6a = 13x - 6$

$\Rightarrow \begin{cases} 13a = 13 \Rightarrow a = 1 \\ 13b - 6a = -6 \Leftrightarrow 13b - 6 = -6 \Rightarrow b = 0 \end{cases}$

$y_p = 1 \cdot x + 0 = x.$

$y(x) = y_0 + y_p$

$y(x) = c_1 \cdot e^{3x} \cos 2x + c_2 \cdot e^{3x} \sin 2x + x, \; c_1, c_2 \in \mathbb{R}$

3) $\begin{cases} (1+x^3)\cdot y'' - 3x^2\cdot y' = 6x^2 \\ y(0) = 2 \\ y(1) = 5 \end{cases}$

reducem gradul prin substituția $y' = z \Rightarrow y'' = z'$

$(1+x^3)\cdot z' - 3x^2\cdot z = 6x^2$

$y \quad z = z_0 + z_p$

$(1+x^3)z' - 3x^2 z = 0$

$z' = \frac{3x^2}{1+x^3} z \qquad \frac{dz}{dx} = z'$

$\frac{dz}{dx} = \frac{3x^2}{1+x^3} z$

$\frac{dz}{z} = \frac{3x^2}{1+x^3} dx$

$\int \frac{1}{z} dz = \int \frac{(1+x^3)'}{1+x^3} dx$

$\ln z = \ln(1+x^3) + \ln c_1, \ c_1 \in \mathbb{R}$

$z = c_1(1+x^3), \ c_1 \in \mathbb{R}$

$y' = z = c_1(1+x^3) = c_1 x^3 + c_1$

$y = \int (c_1 x^3 + c_1)\, dx = c_1 \cdot \frac{x^4}{4} + c_1 x + C_2, \ c_1, c_2 \in \mathbb{R}$

$\left.\begin{matrix} y(0) = c_1\cdot 0 + c_1\cdot 0 + c_2 \\ y(0) = 2 \end{matrix}\right\} \Rightarrow c_2 = 2$

$\left.\begin{matrix} y(1) = c_1 \cdot \frac{1}{4} + c_1 + 2 \\ y(1) = 5 \end{matrix}\right\} \Rightarrow \frac{5}{4} c_1 + 2 = 5$

$\frac{5}{4} c_1 = 3 \Rightarrow c_1 = 3 \cdot \frac{4}{5} = \frac{12}{5}$

$y(x) = \frac{12}{5} \cdot \frac{x^4}{4} + \frac{12}{5} x + 2$

$y(x) = \frac{3x^4}{5} + \frac{12x}{5} + 2$

7) a) $\begin{cases} x'(t) = x - y \\ y'(t) = x - xy^2 \end{cases}$

$\begin{cases} f_1(x,y) = x - y \\ f_2(x,y) = x - xy^2 \end{cases}$ $\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases}$

$\Rightarrow \begin{cases} x - y = 0 & \Rightarrow x = y \\ x - xy^2 = 0 & \Rightarrow x - x \cdot x^2 = 0 \end{cases}$

$x(1 - x^2) = 0$

$c_1)$ $x = 0$
$y = 0$
$(0,0)$ pct. de echilibru

$c_2)$ $1 - x^2 = 0$
$x^2 = 1$
$x_{1,2} = \pm 1$ $\Rightarrow$ $(1,1)$ și $(-1,-1)$ pct. de echilibru
$y = x$

b) Calculăm matricea Jacobi

$$J(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1-y^2 & -2xy \end{pmatrix}$$

pt $(0,0)$

$J(0,0) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ $|\lambda I_2 - J(0,0)| = 0$

$\begin{vmatrix} \lambda - 1 & 1 \\ -1 & \lambda \end{vmatrix} = 0$

$\lambda(\lambda - 1) + 1 = 0$

$\lambda^2 - \lambda + 1 = 0$

$\Delta = 1 - 4 = -3 = (\sqrt{3}\, i)^2$

$\lambda_{1,2} = \frac{1 \pm \sqrt{3}\, i}{2}$ $\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$, $\lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

Deoarece $Re(\lambda) > 0 \Rightarrow (0,0)$ este instabil de tip focus

7) b) continuare

pt $(1, 1)$ $\quad J(1,1) = \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix}$

$|\lambda I_2 - J(1,1)| = 0 \iff \begin{vmatrix} \lambda - 1 & 1 \\ 0 & \lambda + 2 \end{vmatrix} = 0$

$(\lambda - 1)(\lambda + 2) - 0 = 0$

$\lambda_1 = 1 \qquad \lambda_2 = -2$

Deoarece $Re(\lambda_1) > 0 \Rightarrow (1, 1)$ este instabil, de tip şa

$\lambda_1 \cdot \lambda_2 = -2 < 0$

pt $(-1, -1)$ $\quad J(-1,-1) = \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix}$

$|\lambda I_2 - J(-1,-1)| = 0$

$\begin{vmatrix} \lambda - 1 & 1 \\ 0 & \lambda + 2 \end{vmatrix} = 0$

$\lambda_1 = 1 \quad \lambda_2 = -2 \Rightarrow (-1, -1)$ este instabil de tip şa

5) $x' = ax^2 - x^3 - 4a + 4x$

$f(x) = ax^2 - x^3 - 4a + 4x$

Pentru a afla punctele de echilibru se solvăm ecuația $f(x) = 0$

$-x^3 + ax^2 + 4x - 4a = 0$

$x^2(a - x) + 4(x - a) = 0$

$x^2(a - x) - 4(a - x) = 0$

$(a - x)(x^2 - 4) = 0$

$c_1)$ $a - x = 0$
$\Rightarrow -x = -a$
$x_1 = a$
punct de echilibru

$c_2)$ $x^2 = 4$
$x_{2,3} = \pm 2$
pct. de echilibru

Pentru a studia stabilitatea avem nevoie de $f'(x)$

$f'(x) = 2ax - 3x^2 + 4$

$f'(2) = 2a \cdot 2 - 3 \cdot 4 + 4$

$f'(2) = -8 + 4a$

pentru $a = 2 \Rightarrow f'(2) = 0$ nu se poate ~~sta~~ deduce stabilitatea pentru $x = 2$.

$a \in (2, +\infty) \Rightarrow f'(2) > 0$. $x = 2$ este instabil

$a \in (-\infty, 2) \Rightarrow f'(2) < 0$. $x = 2$ este local asimptotic stabil

$f'(-2) = -4a - 8$

pt $a = -2$ $f'(-2) = 0$ nu se poate deduce stabilitatea pt $x = -2$

$a \in (-2, +\infty)$ $f'(-2) < 0$ $x = -2$ este local asimptotic stabil

$a \in (-\infty, -2)$ $f'(-2) > 0$ $x = -2$ este instabil

5) continuare

$f'(a) = 2a^2 - 3a^2 + 4$

$f'(a) = -a^2 + 4$

pt $a = \pm 2 \Rightarrow f'(a) = 0$ nu se poate deduce stabilitatea punctului $x = a$

pt $a \in (-2, 2) \Rightarrow f'(a) > 0 \Rightarrow x = a$ este instabil

pt $a \in (-\infty, -2) \cup (2, +\infty) \Rightarrow f'(a) < 0 \Rightarrow x = a$ este local asimptotic stabil.

4) $\begin{cases} x'(t) = -x - y - x^3 \\ y'(t) = x - y - y^3 \end{cases}$

$f_1(x,y) = -x - y - x^3$

$f_2(x,y) = x - y - y^3$

$V(x,y) = x^2 + y^2$

$V(0,0) = 0^2 + 0^2 = 0$, $V(x,y) > 0$, $\forall (x,y) \in \mathbb{R}^2 \setminus \{0,0\}$

$\dot{V}(x,y) = \frac{\partial V}{\partial x} \cdot f_1 + \frac{\partial V}{\partial y} \cdot f_2 =$

$= 2x \cdot (-x - y - x^3) + 2y(x - y - y^3) =$

$= -2x^2 - \cancel{2xy} - 2x^4 + \cancel{2xy} - 2y^2 - 2y^4$

$= -2(x^2+y^2) - 2(x^4+y^4) = \underbrace{-2\underbrace{(x^2+y^2+x^4+y^4)}_{\geq 0}}_{\leq 0}$

$\dot{V}(x,y) \leq 0$, $\forall (x,y) \in \mathbb{R}^2$ ~~$\setminus \{(0,0)\}$~~ $\Rightarrow X^*(0,0)$ este local stabil

6) $\begin{cases} y' = x + 2y \\ y(0) = 1 \end{cases}$

metoda lui Euler cu pas $h = 0,1$ pe $[0,1]$

$h = 0,1 \Rightarrow N = 10$

$f(x,y) = x + 2y$

$x_0 = 0, \; y_0 = 1$

$y_{n+1} = y_n + f(x_n, y_n) \cdot h, \; n = 0, \dots, 9$

$x_1 = 0,1$

$y_{n+1} = y_n + (x_n + 2y_n) \cdot h$

$x_2 = 0,2$

$x_3 = 0,3$

$\vdots$

$x_{10} = 1$

$y_1 = y_0 + (x_0 + 2y_0) \cdot h$

$y_1 = 1 + (0 + 2 \cdot 1) \cdot 0,1$

$y_1 = 1 + 2 \cdot 0,1 = 1,2$

$y_2 = y_1 + (x_1 + 2y_1) \cdot h$

$y_2 = 1,2 + (0,1 + 2 \cdot 1,2) \cdot 0,1$

$y_2 = 1,2 + (2,5) \cdot 0,1 = 1,2 + 0,25 = 1,45$

$y_3 = y_2 + (x_2 + 2 \cdot y_2) \cdot h$

$y_3 = 1,45 + (0,2 + 2 \cdot 1,45) \cdot 0,1$

$y_3 = 1,45 + 3,1 \cdot 0,1 = 1,45 + 0,31$

$y_3 = 1,76$

1) $\begin{cases} x'(t) = 2x - xy \\ y'(t) = -4y + 2xy \end{cases}$

$$\frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{2x - xy}{-4y + 2xy}$$

$$\frac{dx}{dy} = \frac{2x - xy}{-4y + 2xy} \Leftrightarrow \frac{dx}{dy} = \frac{x(2-y)}{y(2x-4)} \Leftrightarrow$$

$\Leftrightarrow$ $dx\, y(2x-4) = dy\,(2-y)x$

$$\frac{2x-4}{x}\,dx = \frac{2-y}{y}\,dy$$

$$\left(2 - \frac{4}{x}\right)dx = \left(\frac{2}{y} - 1\right)dy \;\Big|\int$$

$x - 4\ln|x| = 2\ln|y| - y$ – ec. dif a orbitelor

1) $\begin{cases} x'(t) = 2x - xy \\ y'(t) = -4y + 2xy \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases} \Rightarrow y = \dfrac{y'}{2x-4}$

$x' = 2x - xy \quad /'$

$x'' = x'(2-y) \cancel{+} x \cdot y'$

$x'' = x'\left(2 - \dfrac{y'}{2x-4}\right) - xy'$

$x'' = 2x' - \dfrac{x'y'}{2x-4} - xy'$

$x'' = 2x' - \left(\dfrac{x'}{2x-4} + x\right)y'$

$y' = \dfrac{x'' - 2x'}{-\left(\frac{x'}{2x-4} + x\right)}$

$\cancel{\dfrac{x'' - 2x'}{-\left(\frac{x'}{2x-4} + x\right)} = (2x-4)} \cdot$

$x' = x\left(2 - \dfrac{\frac{x''-2x'}{-\left(\frac{x'}{2x-4}+x\right)}}{2x-4}\right)$

$\dfrac{x'(2x-4)}{x} = 2(2x-4) - \dfrac{x''-2x'}{+\frac{-x' - x(2x-4)}{2x-4}}$

$\dfrac{x'\cancel{(2x-4)}}{x} = 2\cancel{(2x-4)} - \dfrac{(x''-2x')\cancel{(2x-4)}}{-x' - 2x^2 + 4x}$

$\dfrac{x'}{x} = 2 - \dfrac{x''-2x'}{-x'-2x^2+4x}$

$\dfrac{x'}{x} = \dfrac{\cancel{-2x'} - 4x^2 + 8x - x'' \cancel{+2x'}}{-x' - 2x^2 + 4x}$

$-4x^3 + 8x^2 - x\,x'' = -(x')^2 - 2x^2x' + 4x\,x'$

$\cancel{-x\,x^4}$