Subiect Tînu

1. Studiați existența derivatelor după direcție ale funcției $f(x,y)=\sqrt[3]{x+y^2}$ în pct. $(0,0)$.
Este funcția derivată parțial în acest punct? Justificați.

$$\lim_{t\to 0}\frac{f(x+t\cdot v)-f(x)}{t}$$, unde x e orice punct (de obicei 0).

$$\lim_{t\to 0}\frac{f((0,0)+t(v_1,v_2))-f(0,0)}{t}=\lim_{t\to 0}\frac{f(tv_1,tv_2)-f(0,0)}{t}$$

$$=\lim_{t\to 0}\frac{f(tv_1,tv_2)}{t}=\lim_{t\to 0}\frac{\sqrt[3]{tv_1+t^2v_2^2}}{t}=\lim_{t\to 0}\sqrt[3]{\frac{\cancel{t}(v_1+tv_2^2)}{t^{\cancel{3}2}}}$$

$\Rightarrow$ I $v_1>0 \Rightarrow \lim\limits_{t\to 0}=+\infty$

II $v_1<0 \Rightarrow \lim\limits_{t\to 0}=-\infty$

III $v_1=0 \Rightarrow \lim\limits_{t\to 0}\sqrt[3]{\frac{\cancel{t^2}v_2^2}{t^{\cancel{3}}}}\Rightarrow$ nu are limită, (lim laterali sunt diferite)

Funcția e derivabilă parțial în (x,y) dacă $\exists$ derivatele parțiale și sunt finite. Derivatele parțiale sunt cazuri particulare de derivate după direcție!

$\frac{\partial f}{\partial x}=f'_{(1,0)}(0,0)=\infty \Rightarrow$ nu e derivabilă parțial

$\frac{\partial f}{\partial y}=f'_{(0,1)}(0,0)=$ nu are limită $\Rightarrow$ nu e derivabilă parțial

2. Calculați integrala improprie $\int_3^{\infty}\frac{1}{x^2-x-2}\,dx$

$$\int_3^{\infty}\frac{1}{x^2-x-2}\,dx=\lim_{v\to\infty}\int_3^{v}\frac{1}{x^2-2\cdot\frac{1}{2}\cdot x-2}\,dx=\lim_{v\to\infty}\int_3^{v}\frac{1}{\left(x-\frac{1}{2}\right)^2-\frac{9}{4}}\,dx$$

$t=x-\frac{1}{2} \Leftrightarrow dt=dx$ pt $x=3 \Rightarrow t=\frac{5}{2}$ pt $x=v \Rightarrow t=v-\frac{1}{2}$

$$=\lim_{v\to\infty}\int_{\frac{5}{2}}^{v-\frac{1}{2}}\frac{1}{t^2-\frac{9}{4}}\,dt=\lim_{v\to\infty}\frac{2}{3}\cdot \operatorname{arctg}\frac{2t}{3}\Big|_{\frac{5}{2}}^{v-\frac{1}{2}}$$

$$=\lim_{v\to\infty}\frac{2}{3}\left(\operatorname{arctg}\frac{2v-1}{3}-\operatorname{arctg}\frac{5}{3}\right)=\frac{2}{3}\left(\frac{\pi}{2}-\operatorname{arctg}\frac{5}{3}\right)$$

3. Fie funcția $f:(0,\infty)^2 \to \mathbb{R}$, $f(x,y) = x\sqrt{y} + \frac{y}{\sqrt{x}}$. Det. $\alpha \in \mathbb{R}$ a.î. $\forall (x,y) \in (0,\infty)^2$ $\alpha \cdot \frac{x^2}{y^2} \cdot \frac{\partial^2 f}{\partial x^2}(x,y) + 2 \frac{\partial^2 f}{\partial y^2}(x+y) + \frac{x}{y} \frac{\partial^2 f}{\partial x \partial y}(x,y) = 0$

$\frac{\partial f}{\partial x} = \sqrt{y} + y \cdot (x^{-\frac{1}{2}})' = \sqrt{y} + y \cdot \left(-\frac{1}{2}\right) \cdot x^{-\frac{3}{2}}$

$x^{-\frac{1}{2}-\frac{1}{2}}$

$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{2} y \cdot \left(-\frac{3}{2}\right) \cdot x^{-\frac{5}{2}} = \frac{3}{4} y \cdot x^{-\frac{5}{2}}$

$\frac{\partial f}{\partial y} = \left(x \cdot y^{\frac{1}{2}} + \frac{1}{\sqrt{x}} \cdot y\right)' = \frac{1}{2} x \cdot y^{-\frac{1}{2}} + \frac{1}{\sqrt{x}}$

$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{4} x \cdot y^{-\frac{3}{2}}$

$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \left(\frac{1}{2} x \cdot y^{-\frac{1}{2}} + \frac{1}{\sqrt{x}}\right)'_x = \frac{1}{2} y^{-\frac{1}{2}} + \left(-\frac{1}{2}\right) \cdot x^{-\frac{3}{2}}$

$\alpha \cdot \frac{x^2}{y^2} \cdot \frac{3}{4} y \cdot x^{-2} \cdot x^{-\frac{1}{2}} + 2 \cdot \left(-\frac{1}{4}\right) \cdot x \cdot y^{-1} \cdot y^{-\frac{1}{2}} + \frac{x}{y} \cdot \frac{1}{2} y^{-\frac{1}{2}} + \left(-\frac{1}{2}\right) \cdot x^{-\frac{3}{2}} = 0$

$\frac{3}{4}\alpha \cdot \frac{1}{y\sqrt{x}} - \cancel{\frac{1}{2} \cdot \frac{x}{y} \cdot \frac{1}{\sqrt{y}}} + \cancel{\frac{1}{2} \cdot \frac{x}{y} \cdot \frac{1}{\sqrt{y}}} - \frac{1}{2} \cdot \frac{1}{y\sqrt{x}} = 0.$

$\alpha \cdot \frac{3}{4} \cdot \frac{1}{y\sqrt{x}} - \frac{1}{2} \cdot \frac{1}{y\sqrt{x}} = 0.$

$\alpha = \frac{1}{2y\sqrt{x}} \cdot \frac{4y\sqrt{x}}{3} \Rightarrow \alpha = \frac{2}{3}.$

4. a) $\lim_{n\to\infty} \frac{\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}}{\ln n} \overset{S\text{-}C}{=} \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$

$\lim_{n\to\infty} \frac{\cancel{1} + \cancel{\frac{1}{2}} + \ldots + \frac{1}{n+1} - \cancel{1} - \cancel{\frac{1}{2}} - \ldots - \frac{1}{n}}{\ln(n+1) - \ln n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\ln(n+1) - \ln n}$

$= \lim_{n\to\infty} \frac{1}{\ln\left(\frac{n+1}{n}\right)^{n+1}} = \frac{1}{\ln \lim_{n\to\infty} \left[\left(1 + \frac{1}{n}\right)^n\right]^{\frac{n+1}{n}}} = \frac{1}{\ln e^{\lim_{n\to\infty} \frac{n+1}{n}}} = \frac{1}{\ln e} = 1$

b) Studiați convergența s.t.p: $\sum_{k=1}^{\infty} \underbrace{\left(\frac{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}}{n}\right)^a}_{x_n}$ în funcție de variabila $a > 0$

Când avem $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ îl putem scrie $/ \ln n$ ca să studiem mai ușor convergența.

luăm $y_n = \left(\frac{\ln n}{n}\right)^a$

C.C.
$\Rightarrow \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \left(\frac{1 + \frac{1}{2} + \ldots + \frac{1}{n}}{\cancel{n}} \cdot \frac{\cancel{n}}{\ln n}\right)^a = \lim_{n\to\infty} \left(\underbrace{\frac{1 + \frac{1}{2} + \ldots + \frac{1}{n}}{\ln n}}_{=1}\right)^a = 1^a = 1.$

nr' ℓ

$$\Rightarrow \sum a_n \sim \sum b_n$$

$$\sum_{n \geq 1}^{\infty} \left(\frac{\ln n}{n} \right)^a$$

I D'Alembert.

$$\lim_{n \to \infty} \left(\frac{\ln n}{n} \cdot \frac{n+1}{\ln(n+1)} \right)^a = 1, \text{ nu decide}$$

II Raabe-Duhamel

$$\lim_{n \to \infty} n \left[\left(\frac{n+1}{n} \cdot \frac{\ln n}{\ln(n+1)} \right)^a - 1 \right] = \lim_{n \to \infty} n \underbrace{\frac{\left(\frac{(n+1) \ln n}{n \ln(n+1)} \right)^a - 1}{\frac{(n+1) \ln n}{n \ln(n+1)} - 1}}_{\to a} \left(\frac{(n+1) \ln n}{n \ln(n+1)} - 1 \right)$$

$$a \lim_{n \to \infty} \frac{(n+1) \ln n}{\ln(n+1)} - 1 = a \lim_{n \to \infty} \frac{(n+1) \ln n - n \ln(n+1)}{\ln(n+1)}$$

$$= a \lim_{n \to \infty} \frac{n \ln n + \ln n - n \ln(n+1)}{\ln(n+1)} = a \lim_{n \to \infty} \underbrace{\frac{\ln n}{\ln(n+1)}}_{1} + a \lim_{n \to \infty} \frac{n \ln \frac{n}{n+1}}{\ln(n+1)}$$

$$= a + a \cdot \lim_{n \to \infty} \frac{\ln \left(1 + \frac{-1}{n+1} \right)^n}{\ln(n+1)} = a + a \lim_{n \to \infty} \frac{e^{-1}}{\ln(n+1)} \overset{\to 0}{=} a + a \cdot 0 = a.$$

I $a < 0 \Rightarrow \sum b_n$ - divergentă $\Rightarrow \sum x_n$ - divergentă

II $a > 0 \Rightarrow \sum b_n$ - convergentă $\Rightarrow \sum x_n$ - convergentă.

III $a = 0 \Rightarrow \sum \frac{\ln n}{n} > \sum \frac{1}{n}$ - divergentă $\Rightarrow \sum b_n$ - divergentă.