Examen Sisteme dinamice

Exercițiul 2

(a) $x^2y' = xy + y^2$

$\Leftrightarrow y' = \frac{xy+y^2}{x^2} = \frac{xy}{x^2} + \frac{y^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2$

Facem substituția $z = \frac{y}{x}$, unde $y' = z'x + z$

$z'x + z = z + z^2 \Leftrightarrow z' = z^2 \cdot \frac{1}{x} \Leftrightarrow \frac{dz}{dx} = z^2 \cdot \frac{1}{x} \Leftrightarrow \frac{1}{z^2} dz = \frac{1}{x} dx \Big| \int$

$\Leftrightarrow -\frac{1}{z} = \ln|x| + c_0,\ c_0 \in \mathbb{R}$

$\Leftrightarrow +\frac{1}{z} = -\ln(x) + c,\ -c_0 := c,\ c \in \mathbb{R}$

$\Leftrightarrow z = \frac{1}{c - \ln x}$

$\Rightarrow y = \frac{x}{c-\ln x},\ c \in \mathbb{R}$

(b) $y'' - 4y' + 8y = (25x-5)e^x$

I ec. omogenă:

$y'' - 4y' + 8y = 0$

Scriem ecuația caracteristică atașată:

$r^2 - 4r + 8 = 0$

$\Delta = 16 - 4\cdot 8 = -16$

$r_{1/2} = \frac{4 \pm \sqrt{16i^2}}{2}$ $\begin{cases} r_1 = \frac{4+4i}{2} = 2+2i \\ r_2 = 2-2i \end{cases}$

$y_1(x) = e^{2x}\cos(2x)$

$y_2(x) = e^{2x}\sin(2x)$

$\Rightarrow y_0 = c_1 e^{2x}\cos 2x + c_2 e^{2x}\sin(2x),\ c_1, c_2 \in \mathbb{R}.$

II ec. particulară:

$f(x) = (25x-5)e^x$

$\Rightarrow y_p = e^x(ax+b)$

$y_p' = e^x(ax+b) + e^x \cdot a$

$y_p'' = e^x(ax+b) + e^x \cdot a + e^x \cdot a$

Înlocuim în ec inițială:

$$e^x(ax+b)+2e^xa-4\left(e^x(ax+b)+e^x\cdot a\right)+8\left(e^x(ax+b)\right)=(25x-5)e^x$$

$$\Leftrightarrow e^x(ax+b)+e^x 2a-4e^x(ax+b)-4e^xa+8e^x(ax+b)=(25x-5)e^x$$

$$\Leftrightarrow e^x(ax+b+2a-4ax-4b-4a+8ax+8b)=(25x-5)e^x$$

$$\Leftrightarrow e^x(5ax+5b-2a)=(25x-5)e^x$$

$$\Leftrightarrow \begin{cases} 5a=25 & \Rightarrow a=5 \\ 5b-2a=-5 & \Rightarrow 5b-10=-5 \Leftrightarrow 5b=5 \Leftrightarrow b=1 \end{cases}$$

$$\Rightarrow y_p=e^x(5x+1)$$

$$\Rightarrow y=y_0+y_p$$

$$\Leftrightarrow y=c_1e^{2x}\cos 2x+c_2e^{2x}\sin(2x)+e^x(5x+1),\ c_1,c_2\in\mathbb{R}.$$

Exercițiul 3

$$\begin{cases} xy''+y'=4x \\ y(1)=1 \\ y'(1)=4 \end{cases}$$

$xy''+y'=4x$, facem substituția $z=y'\Rightarrow z'=y''$

$$xz'+z=4x$$

I ec. omogenă:

$$xz'+z=0$$

$$\Leftrightarrow xz'=-z \Leftrightarrow z'=-z\cdot\frac{1}{x} \Leftrightarrow \frac{dz}{dx}=-z\cdot\frac{1}{x}$$

$$\Leftrightarrow \frac{1}{z}dz=-\frac{1}{x}dx \Leftrightarrow \cancel{\ln|y|}\ \ln|z|=-\ln|x|+c_0,\ c_0\in\mathbb{R}$$

$$\Leftrightarrow \cancel{\ln|y|}\ \ln|z|=\ln(x)^{-1}+\ln c,\quad c_0=\ln c,\ c\in\mathbb{R}$$

$$\Leftrightarrow \cancel{y=\frac{1}{x}\cdot c} \Leftrightarrow z_0=\frac{1}{x}\cdot c$$

II ec. particulară:

$$\cancel{y_p=\frac{1}{x}\cdot f}\quad z_p=\frac{1}{x}\cdot f$$

$$\cancel{y}\quad z_p'=-\frac{1}{x^2}\cdot f+\frac{1}{x}\cdot f'$$

Înlocuim în ec:

$x\left(-\frac{1}{x^2}\cdot f+\frac{1}{x}\cdot f'\right)+\frac{1}{x}\cdot f=4x$

$\Leftrightarrow -\frac{x}{x^2}\cdot f+\frac{x}{x}\cdot f'+\frac{1}{x}\cdot f=4x$

$\Leftrightarrow -\cancel{\frac{1}{x}f}+f'+\cancel{\frac{1}{x}f}=4x \Leftrightarrow f'=4x \Leftrightarrow f=4\frac{x^2}{2}=2x^2$

$\Rightarrow z_p=\frac{1}{\cancel{x}}\cdot 2x^{\cancel{2}}=2x$

$\Rightarrow z=z_0+z_p=\frac{1}{x}\cdot c+2x$, $c\in\mathbb{R}$

Dar $y'=z$

$\Rightarrow y'=\frac{1}{x}\cdot c+2x \Leftrightarrow y=\int\frac{1}{x}\cdot c+2x\,dx=c_1\ln|x|+\cancel{2}\frac{x^2}{\cancel{2}}+c_2$

$=c_1\ln|x|+x^2+c_2$, $c:=c_1$, $c_1,c_2\in\mathbb{R}$

$\left.\begin{array}{l} y(1)=1 \\ y(1)=c_1\ln 1+1+c_2=1+c_2 \end{array}\right\}\Rightarrow 1+c_2=1 \Leftrightarrow c_2=0.$

$y'=\frac{1}{x}\cdot c_1+2x$

$y'(1)=4$ | $\cancel{\frac{c_1}{4}+8=4 \Leftrightarrow \frac{c_1}{4}=-4 \Leftrightarrow c_1=-16}$

$\cancel{y'(1)=\frac{c_1}{4}+8}$

$\Rightarrow \cancel{y=x^2-16}$

$\left.\begin{array}{l} y'(1)=c_1+2 \\ y'(1)=4 \end{array}\right| \Rightarrow c_1+2=4 \Leftrightarrow c_1=2$

$\cancel{y=x^2+2}$

$y=2\ln|x|+x^2$

Exercițiul 5

$x' = a\cdot x^2 - x^3 - a + x \quad , a \in \mathbb{R}$

$f(x) = ax^2 - x^3 - a + x$

$f(x) = 0 \Leftrightarrow ax^2 - x^3 - a + x = 0$

$\Leftrightarrow (x-1)(x+1)(a-x) = 0$

$\Rightarrow x_1^* = -1,\ x_2^* = 1,\ x_3^* = a$ sol ecuației, $a \in \mathbb{R}$.

pct de echilibru.

$f'(x) = 2ax - 3x^2 + 1$

$f'(-1) = -2a - 2 = -2(a+1)$

dacă $a+1 > 0 \Leftrightarrow a > -1 \Rightarrow f'(-1) < 0 \Rightarrow x_1^* = -1$ asimptotic stabil

dacă $a+1 < 0 \Leftrightarrow a < -1 \Rightarrow f'(-1) > 0 \Rightarrow x_1^* = -1$ instabil.

$f'(1) = 2a - 3 + 1 = 2a - 2 = 2(a-1)$

dacă $a-1 > 0 \Leftrightarrow a > -1 \Rightarrow f'(1) > 0 \Rightarrow x_1^* = 1$ instabil

dacă $a-1 < 0 \Leftrightarrow a < -1 \Rightarrow f'(1) < 0 \Rightarrow x_1^* = 1$ asimptotic stabil

$f'(a) = 2a^2 - 3a^2 + 1 = -a^2 + 1 = 1 - a^2$

dacă $a \in (-\infty, -1) \Rightarrow f'(a) < 0 \Rightarrow x_1^* = a$ asimptotic stabil, $a \in (-\infty, -1)$

dacă $a \in (-1, 1) \Rightarrow f'(a) > 0 \Rightarrow x_1^* = a$ instabil, $a \in (-1, 1)$

dacă $a \in (1, \infty) \Rightarrow f'(a) < 0 \Rightarrow x_1^* = a$ asimptotic stabil, $a \in (1, \infty)$

Exercițiul 7

$$\begin{cases} x'(t) = -4y(t) \\ y'(t) = x(t) \end{cases}$$

(a) $\begin{cases} x' = -4y \\ y' = x \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$

$\begin{cases} x' = -4y \\ y' = x \end{cases}$

$x'' = -4y' = -4x$

$\Leftrightarrow x'' + 4x = 0$

ec. caracteristică atașată:

$r^2 + 4 = 0 \Rightarrow r_{1/2} = \pm 2i$

$\Rightarrow x_1 = \cos 2t,\ x_2 = \sin 2t$

$\Rightarrow x = c_1 \cos 2t + c_2 \sin 2t,\ c_1, c_2 \in \mathbb{R}$.

$y' = x \Leftrightarrow y = \int c_1 \cos 2t + c_2 \sin 2t \, dx$

$\Leftrightarrow y = c_1 \frac{\sin 2t}{2} + c_2\left(-\frac{\cos 2t}{2}\right)$

$x = c_1 \cos 2t + c_2 \sin 2t$

$\left.\begin{array}{l} x(0) = \eta_1 \\ x(0) = c_1 \end{array}\right\} \Rightarrow c_1 = \eta_1$

$y = \frac{c_1}{2} \sin 2t + \left(-\frac{c_2}{2}\right) \cos 2t$

$\left.\begin{array}{l} y(0) = \eta_2 \\ y(0) = -\frac{c_2}{2} \end{array}\right| \Leftrightarrow -\frac{c_2}{2} = \eta_2 \Leftrightarrow -c_2 = 2\eta_2 \Leftrightarrow c_2 = -2\eta_2$

sol pb Cauchy $\begin{cases} x = \eta_1 \cos 2t - 2\eta_2 \sin 2t \\ y = \frac{\eta_1}{2} \sin 2t + \eta_2 \cos 2t \end{cases}$

$x, y : I_\eta \to \mathbb{R}$

$f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

Fluxul $f(t, \eta_1, \eta_2) = \left(\eta_1 \cos 2t - 2\eta_2 \sin 2t,\ \frac{\eta_1}{2} \sin 2t + \eta_2 \cos 2t\right)$

(b) $(0,0)$ pct de echilibru

$f_1(x,y) = -4y$

$f_2(x,y) = x$

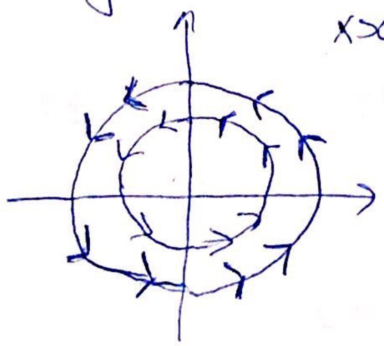
$A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$, $\det(\lambda I_2 - A) = 0 \Leftrightarrow \cancel{\begin{vmatrix} 0 & 4 \\ -1 & 0 \end{vmatrix} = 0}$

~~$\Rightarrow Re\,\lambda_1/\lambda_2 = 0.$~~

$\Leftrightarrow \begin{vmatrix} \lambda & +4 \\ -1 & \lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 4 = 0 \Leftrightarrow \lambda^2 = -4 \Rightarrow \lambda_{1,2} = \pm 2i$

Avem valori proprii simple $\Rightarrow$ $(0,0)$ pct de echilibru local stabil de tip centru.

Portret fazic:



$x > 0, y > 0$

$x' < 0$

$y' > 0$

Exercițiul 6

$$\begin{cases} y' = 3x^2 + xy^2 \\ y(0) = 1 \end{cases}$$

$x_0 = 0$
$y^0 = 1$

$f(x,y) = 3x^2 + xy^2$, $f: \mathbb{R}^2 \to \mathbb{R}^2$

$\left|\frac{\partial f}{\partial y}(x,y)\right| = 2|xy| \xrightarrow[y\to\infty]{} \infty \Rightarrow f$ nu este lipschitz pe $[a,b] \times \mathbb{R}$ în raport cu y.

$f: [a,b] \times \mathbb{R} \to \mathbb{R}$

$\Delta = [-a, a] \times [1-b, 1+b]$

$f: \Delta \to \mathbb{R}$, $\left|\frac{\partial f}{\partial y}(x,y)\right| = 2|x \cdot y| \leq 2(1+b) \Rightarrow f$ este lipschitz în rap cu y pe Δ

TE! în lucrare $\Rightarrow \exists!\, y^* \in C([-h,h], [1-b, 1+b])$ unde

$h = \min\{a, \frac{b}{M_f}\}$

$M_f = \max|f(x,y)| = 3a^2 + ab^2$

ec. integrală Volterra echiv cu pb Cauchy

$$y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\, ds$$

$$y(x) = 1 + \int_0^x 3s^2 + s \cdot (y(s))^2 ds$$

$$y(x) = 1 + 3\int_0^x s^2 + \int_0^x s \cdot (y(s))^2 ds$$

$$y(x) = 1 + \cancel{3}x^3 + \int_0^x s \cdot (y(s))^2 ds$$

Șirul aprox succesive:

$y_0 \in C([-h,h], [1-b, 1+b])$

$$y_{m+1}(x) = 1 + x^3 + \int_0^x s \cdot (y(s))^2 ds$$

$y_0(x) \equiv 1 \Rightarrow y_0 \in C([-h,h], [1-b, 1+b])$

$$y_1(x) = 1 + \frac{x^3}{\cancel{3}} + \int_0^x s\, ds = 1 + \frac{x^3}{\cancel{3}} + \frac{s^2}{2}\Big|_0^x = 1 + \frac{x^3}{\cancel{3}} + \frac{x^2}{2} = 1 + x^3 + \frac{x^2}{2}$$

$$y_2(x) = 1 + x^3 + \int_0^x s \cdot (y_1(s))^2 ds = 1 + x^3 + \int_0^x s\left(1 + s^3 + \frac{s^2}{2}\right)^2 ds$$

$$\cancel{= 1 + x^3 + \int_0^x s + s^4 + \frac{s^3}{2}\, ds = 1 + x^3 + \frac{x^2}{2} + \frac{x^5}{5} + \frac{x^4}{8}}$$

Ex 6 continuare

$$y_2(x) = 1+x^3+\int_0^x s\left(1+s^3+\frac{s^2}{2}\right)^2 ds$$

$$y_2(x) = 1+x^3+\int_0^x s\left(s^6+s^5+\frac{s^4}{4}+2s^3+s^2+1\right) ds$$

$$= 1+x^3+\int_0^x s^7+s^6+\frac{s^5}{4}+2s^4+s^3+s\, ds$$

$$= 1+x^3+\frac{x^8}{8}+\frac{x^7}{7}+\frac{x^6}{24}+\frac{2x^5}{5}+\frac{x^4}{4}+\frac{x^2}{2}$$

Exercitiul 4

$$\begin{cases} x' = -x+e^x y \\ y' = -e^x x - y \end{cases}$$

$V(x,y) = x^2+y^2$, $D = \mathbb{R}^2$

$V(0,0)=0$, $V(x,y)>0$, $\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\dot{V}(x,y) = \frac{\partial V}{\partial x}(x,y)\cdot f_1(x,y) + \frac{\partial V}{\partial y}(x,y)\cdot f_2(x,y)$$

$$= 2x(-x+e^x y) + 2x(-e^x x - y) = -2x^2+2xe^x y - 2x^2e^x - 2xy$$

$$= -2x^2(\underbrace{1+e^x}_{>0}) + 2xy(e^x-1)$$

Exercitiul 1

$$\begin{cases} x'(t) = r x \\ x(0) = x_0 \end{cases}$$

$x_0 = 1000 \xrightarrow{10\ ani} x_1 = 50000$

Soluția modelului este $x(t) = x_0 \cdot e^{r \cdot t}$

$x(0) = x_0 \cdot e^{r \cdot t}$, $1000 = 1000 \cdot e^{r}$

$x(10) = x_0 \cdot e^{r \cdot t}$, $50\,000 = 1000 \cdot e^{r \cdot 10}$

$$\Leftrightarrow e^{r \cdot 10} = \frac{50\,000}{1000} \Leftrightarrow e^{r \cdot 10} = 50$$