(x, y, z)

$f = xyz$

Engineering Mathematics - I (BMS-01)

Unit - 4

1. Scalar point function:

A scalar point function is a function that assigns a real number (i.e. a scalar) to each point of some region of space. If to each point (x, y, z) of a region R in space there is assigned a real number f = f (x, y, z), then f is called a scalar point function. The temperature distribution within some body at a particular ~~point in~~ time. is example of scalar pt funcn

2. vector point function:

A vector point function is a function that **assigns a vector to each point of some region of space**. If to each point (x, y, z) of a region R in space there is assigned a vector u =u(x, y, z), then u is called a vector point function. Velocity at different points within a moving fluid.

$\vec{r}$

Consider the vector

$\bar{r}$

$\vec{r} = \vec{r}(x, y, z)$

$\vec{r} = \vec{r}(t)$, where 't' is scalar variable

$= \overrightarrow{f(t)}$

In cartesian

$\vec{r} = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$

$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow r = \sqrt{x^2 + y^2 + z^2}$ where $r = |\vec{r}|$

$\Rightarrow r^2 = x^2 + y^2 + z^2$

$\Rightarrow 2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \boxed{\dfrac{\partial r}{\partial x} = \dfrac{x}{r}}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

Also

$\dfrac{\partial \vec{r}}{\partial x} = \hat{i}$, $\dfrac{\partial \vec{r}}{\partial y} = \hat{j}$, $\dfrac{\partial \vec{r}}{\partial z} = \hat{k}$

<u>velocity</u>:-

$\vec{r} = f(t)$

velocity $= \dfrac{d\vec{r}}{dt} = \vec{v}$

Acceleration $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$

Note that :- $\frac{d\vec{r}}{dt}$ is a vector along the tangent to the curve at some point P.

1. velocity in the direction $\vec{b}$

$$= \vec{v}\cdot\hat{a}$$

2. Acceleration in the direction $\vec{b}$

$$= \vec{a}\cdot\hat{b}$$

Q. A particle moves along the curve $x = 2t^2,\ y = t^2 - 4t,\ z = 3t - 5$, t is time. Find the velocity and acceleration at $t=1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$

Sol:- $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t-5)\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 4t\hat{i} + (2t-4)\hat{j} + 3\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 4\hat{i} + 2\hat{j}$$

At $t=1$

$$\vec{v} = 4\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\vec{a} = 4\hat{i} + 2\hat{j}$$

Now in the direction $\vec{b} = \hat{i} - 3\hat{j} + 2\hat{k}$, velocity & acceleration are given by

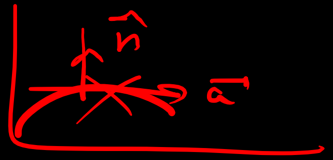
$$= \vec{v} \cdot \hat{b}$$

$$= (4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot \frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{14}} = \frac{4+6+6}{\sqrt{14}} = \frac{16}{\sqrt{14}} = \frac{8}{7}\sqrt{14}$$

Acceleration at $t=1$ in the direction of $\vec{b}$ is

$$= \vec{a} \cdot \hat{b}$$

$$= (4\hat{i} + 2\hat{j}) \cdot \frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{14}} = \frac{-2}{\sqrt{14}} = \frac{-\sqrt{14}}{7}.$$



$\bar{a} \times \bar{b} = abc\,\hat{n}$

Gradient of a scalar field

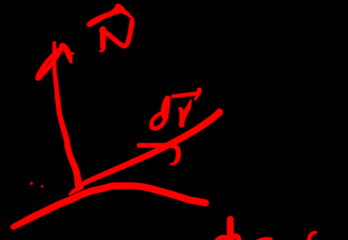
Let $\phi = \phi(x, y, z)$ be the scalar point function then gradient of the scalar field ϕ is defined as

$$\text{grad}\,\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi \quad , \quad \nabla = \text{Del} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

✓ $$\boxed{\nabla\phi = \sum \hat{i}\frac{\partial\phi}{\partial x}}$$

$\nabla\phi \cdot \delta\vec{r} = 0$

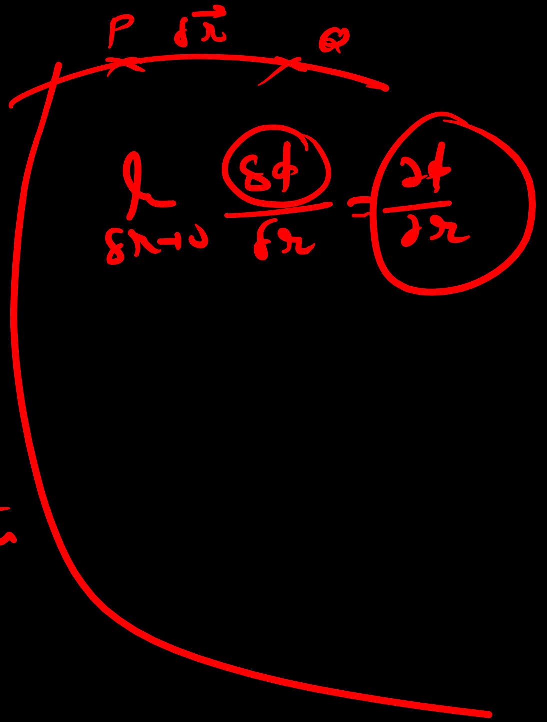


Note that :-

If ϕ is scalar point function then

$\nabla\phi$ is the normal to the surface ϕ.

Directional Derivative



$$\lim_{\delta r \to 0} \frac{\delta\phi}{\delta r} = \frac{\partial\phi}{\partial r}$$

Let $\vec{PQ} = \delta\vec{r}$ then

$\dfrac{\partial\phi}{\partial r} = \lim\limits_{\delta r \to 0} \dfrac{\delta\phi}{\delta r}$ is called the directional derivative of ϕ at P in the direction $\vec{PQ}$.

$\boxed{\nabla\phi}$

$$\left|\frac{\partial\phi}{\partial r}\right| = |\nabla\phi \cdot \hat{a}| = |\nabla\phi||\cos\theta| \le |\nabla\phi|$$

Also $\boxed{\dfrac{\partial\phi}{\partial r} = \nabla\phi \cdot \hat{a}}$ denotes the directional derivative of ϕ in the direction $\vec{a} = \vec{PQ}$.

Also :- (1) maximum value of directional derivative

$= |\nabla\phi|$

(2) maximum value of directional derivative occurs in the direction of $\nabla\phi$ itself.

$\sum \hat{i} x$

$= \hat{i}x + \hat{j}y + \hat{k}z$

$= \vec{r}$

Q (1) show that $\nabla r^n = n r^{n-2} \vec{r}$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Sol :- $\left\{ \because \nabla\phi = \sum \hat{i} \frac{\partial \phi}{\partial x} \right\}$

$$\left\{ \nabla r^n = \sum \hat{i} \frac{\partial}{\partial x} (r^n) \right.$$

$$= \sum \hat{i}\, n r^{n-1} \frac{\partial r}{\partial x}$$

$$= \sum \hat{i}\, n r^{n-1} \frac{x}{r} = n r^{n-2} \sum \hat{i} x = n r^{n-2} \vec{r} \Big\}$$

If $n=-1$ then

$$\nabla \vec{r}^{-1} = \nabla\left(\frac{1}{r}\right) = \sum \hat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$= \sum \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)$$

$$= \sum \hat{i}\left(-\frac{1}{r^2}\frac{x}{r}\right)$$

$$= -\frac{1}{r^3}\sum \hat{i}x = \frac{-\vec{r}}{r^3}$$

$\vec{N}_1$, $\vec{N}_2 =$

Q. Find a unit vector normal to the surface $x^3+y^3-3xyz=3$ at the point $(1,2,-1)$.

Sol:

$\phi = x^3 + y^3 + 3xyz - 3$

Normal to the surface $= \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^3 + y^3 + 3xyz - 3\right)$$

$$= \hat{i}(3x^2 + 3yz) + \hat{j}(3y^2 + 3xz) + \hat{k}(3xy)$$

At point $(1, 2, -1)$

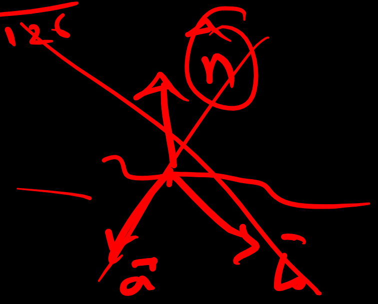
$$\vec{N} = (\nabla\phi)_{(1,2,-1)}$$

$$= -3\hat{i} + 9\hat{j} + 6\hat{k}$$

Unit normal to the surface

$$\hat{N} = \frac{\vec{N}}{|\vec{N}|} = \frac{-3\hat{i}+9\hat{j}+6\hat{k}}{\sqrt{126}}$$

$$\begin{array}{r} 81 \\ 9 \\ 36 \\ \hline 126 \end{array}$$



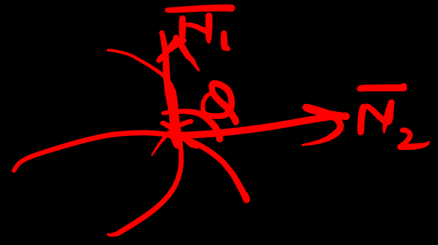
__Note that__ :- Greatest rate of ϕ

$$= |\nabla\phi|$$

$$= |\vec{N}|$$

$$= \sqrt{126}$$

Direction of greatest rate $= \nabla\phi = -3\hat{i}+9\hat{j}+6\hat{k}$



Q. Find the angle between two surfaces $x^2+y^2+z^2=9$ & $z=x^2+y^2-3$ at $(2,-1,2)$

Sol:- Let $\phi_1 = x^2+y^2+z^2-9$, $\phi_2 = z-x^2-y^2+3$

$$\nabla\phi_1 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9), \quad \nabla\phi_2 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(z-x^2-y^2+3)$$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}, \quad \nabla\phi_2 = -2x\hat{i}-2y\hat{j}+\hat{k}$$

$$\vec{N}_1 = (\nabla\phi_1)_{(2,-1,2)} = 4\hat{i}-2\hat{j}+4\hat{k}$$

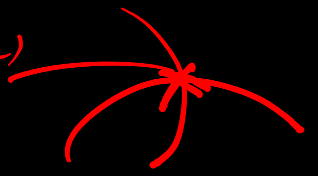
$$\vec{N}_2 = (\nabla\phi_2)_{(2,-1,2)} = -4\hat{i}+2\hat{j}+\hat{k}$$

$$\therefore \vec{N}_1\cdot\vec{N}_2 = |\vec{N}_1|\,|\vec{N}_2|\cos\theta$$

$$\cos\theta = \frac{\vec{N}_1\cdot\vec{N}_2}{|\vec{N}_1|\,|\vec{N}_2|} = \frac{-16-4+4}{6\times\sqrt{21}} = \frac{-16}{6\times\sqrt{21}} = \frac{-8}{3\sqrt{21}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-8}{3\sqrt{21}}\right)$$

(1,-1,2)



Q. Find λ & μ such that

$$\lambda x^2 - \mu yz = (\lambda+2)x \quad \text{and}$$

$$4x^2y + z^3 = 4$$ intersect orthogonally at $(1,-1,2)$.

Sol: $\phi_1 = \lambda x^2 - \mu yz - (\lambda+2)x$

$\phi_2 = 4x^2y + z^3 - 4$

$$\vec{N_1} = \nabla\phi_1 = \hat{i}(2\lambda x - (\lambda+2)) + \hat{j}(-\mu z) + \hat{k}(-\mu y)$$

$$\vec{N_2} = \nabla\phi_2 = \hat{i}(8xy) + \hat{j}(4x^2) + \hat{k}(3z^2)$$

$$\vec{N_1} = (\nabla\phi_1)_{(1,-1,2)} = \hat{i}(\lambda-2) - 2\mu\hat{j} + \mu\hat{k}$$

$$\vec{N_2} = (\nabla\phi_2)_{(1,-1,2)} = \hat{i}(-8) + 4\hat{j} + 12\hat{k}$$

$\because$ ϕ_1 & ϕ_2 are orthogonal, hence

$$\vec{N_1}\cdot\vec{N_2} = 0$$

$$-8(\lambda-2) - 8\mu + 12\mu = 0$$

$$\Rightarrow -8(\lambda-2) + 4\mu = 0$$

$$\Rightarrow \boxed{\mu = 2(\lambda-2)} \quad \text{--- (1)}$$

The point of intersection will satisfies both surfaces hence

$$\therefore \lambda x^2 - \mu yz = (\lambda+2)x$$

$$\Rightarrow \cancel{\lambda} + 2\mu = \cancel{\lambda} + 2$$

$$\Rightarrow 2\mu = 2$$

$$\boxed{\mu = 1}$$

Then by (1), $\mu = 2(\lambda - 2)$

$$\Rightarrow 1 = 2(\lambda - 2)$$

$$\Rightarrow \lambda - 2 = \frac{1}{2}$$

$$\Rightarrow \lambda = 2 + \frac{1}{2} = \frac{5}{2}$$

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① Find the directional derivative of $\frac{1}{r}$ in the direction of $\vec{r}$.

② " " " " $\frac{1}{r^n}$ " " " $\vec{r}$

Sol: (2) Given that $\phi = \frac{1}{r^n} = r^{-n}$

Directional derivative $= \nabla\phi \cdot \hat{a}$

$= \nabla r^{-n} \cdot \hat{r}$

$= \sum \hat{i} \frac{\partial}{\partial x}(r^{-n}) \cdot \frac{\vec{r}}{r}$

$= \sum \hat{i} \left(-n r^{-n-1}\right) \frac{\partial r}{\partial x} \cdot \frac{\vec{r}}{r}$

$= -n \sum \hat{i}\, r^{-n-1} \frac{x}{r} \cdot \frac{\vec{r}}{r}$

$= -n r^{-n-3} \sum \hat{i} x \cdot \vec{r}$

$= -n r^{-n-3}\, \vec{r} \cdot \vec{r}$

$= -n r^{-n-3}\, r^2$

$= -n r^{-n-1}$ Ans

① If $n = 1$, $\phi = \frac{1}{r}$

then Directional deriv. $= \frac{-1}{r^2}$ Ans.

Evaluate (i) $\nabla(\log r)$ (ii) $\nabla(\log r^n)$ (iii) $\nabla(f(r))$

Sol: (ii) $\nabla(\log r^n) = \sum \hat{i}\frac{\partial}{\partial x}(\log r^n)$

$$= \sum \hat{i}\,\frac{1}{r^n}\,n r^{n-1}\frac{\partial r}{\partial x}$$

$$= \sum \hat{i}\,\frac{1}{r^n}\,n r^{n-1}\frac{x}{r}$$

$$= \frac{n}{r^2}\sum \hat{i}x = \frac{n\vec{r}}{r^2}$$

(iii) $\nabla(f(r)) = \sum \hat{i}\frac{\partial}{\partial x}(f(r)) = \sum \hat{i}\,f'(r)\frac{\partial r}{\partial x}$

$$= \sum \hat{i}\,f'(r)\frac{x}{r}$$

$$= \frac{f'(r)}{r}\sum \hat{i}x$$

$$= \frac{f'(r)}{r}\vec{r}$$

$$= f'(r)\,\nabla r$$

$\left\{\because\ \nabla r = \sum \hat{i}\frac{\partial}{\partial x}(r)\right.$

$$= \sum \hat{i}\frac{\partial r}{\partial x}$$

$$= \sum \hat{i}\frac{x}{r}$$

$$\left.= \frac{1}{r}\vec{r}\right\}$$

Q. If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at pt. $(1,1,1)$ has maximum magnitude $\boxed{15}$ in the direction parallel to the line

$$\boxed{\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}.}$$

Find values of a, b & c.

$\nabla\phi$ ✓

Sol: $\nabla\phi = \hat{i}(2axy + cz^2) + \hat{j}(ax^2 + 2byz) + \hat{k}(by^2 + 2czx)$

$$\vec{N_1} = (\nabla\phi)_{(1,1,1)} = \hat{i}(2a+c) + \hat{j}(a+2b) + \hat{k}(b+2c)$$

Given that $|\nabla\phi| = 15$

$$\Rightarrow \sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2} = 15$$

$$\Rightarrow \boxed{(2a+c)^2 + (a+2b)^2 + (b+2c)^2 = 225} \quad \text{—— } (1)$$

Also at $(1,1,1)$

$\vec{N_1} = \nabla\phi_1$

$= \hat{i}(2a+c) + \hat{j}(a+2b) + \hat{k}(b+2c)$

$\because$ maximum value of directional derivative occurs in the direction

$\vec{n} = \hat{i}(2a+c) + \hat{j}(a+2b) + \hat{k}(b+2c)$ and it parallel to $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$

i.e the vector $2\hat{i} - 2\hat{j} + \hat{k}$

Hence $$\frac{2a+c}{2} = \frac{a+2b}{-2} = \frac{b+2c}{1}$$

$-6a - 2c - 4b = 0$, $a + 4b + 4c = 0$

$\Rightarrow$ $3a + 2b + c = 0$ — (A)

$a + 4b + 4c = 0$ — (B)

$$\Rightarrow \frac{a}{4} = \frac{-b}{11} = \frac{c}{12-2} = \lambda$$

$\Rightarrow a = 4\lambda,\ b = -11\lambda,$

$c = 10\lambda$

Using ①

$$\lambda^2 = \frac{25}{81}$$

$$\Rightarrow \lambda = \pm\frac{5}{9}$$

$a = \pm\frac{20}{9},\ b = \mp\frac{55}{9}$

$c = \pm\frac{55}{9}.$

Divergence of a vector point function

Let $\vec{v}$ is the vector point function, then

$$\text{div}\,\vec{v} = \nabla\cdot\vec{v}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{v}$$

$$= \hat{i}\cdot\frac{\partial\vec{v}}{\partial x} + \hat{j}\cdot\frac{\partial\vec{v}}{\partial y} + \hat{k}\cdot\frac{\partial\vec{v}}{\partial z}$$

$$= \sum \hat{i}\cdot\frac{\partial\vec{v}}{\partial x}$$

If $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ then

$$\text{div}\,\vec{v} = \nabla\cdot\vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$\nabla\phi$

$$\boxed{\nabla\phi = \sum \hat{i}\frac{\partial\phi}{\partial x}}$$

Curl of a vector point func^n

let $\vec{V}$ is the vector point func^n then

$$\text{curl}\,\vec{V} = \nabla \times \vec{V}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{V}$$

$$= \hat{i} \times \frac{\partial \vec{V}}{\partial x} + \hat{j} \times \frac{\partial \vec{V}}{\partial y} + \hat{k} \times \frac{\partial \vec{V}}{\partial z}$$

$$= \sum \hat{i} \times \frac{\partial \vec{V}}{\partial x}$$

If $\vec{V} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ then

$$\text{curl}\,\vec{V} = \nabla \times \vec{V}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \hat{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \#$$

Note that :-

① $\operatorname{div}\vec{v}$ represents the rate of outflow per unit volume at a given point of the fluid.

② $\operatorname{curl}\vec{v} = 2\vec{\omega}$

$\Rightarrow \vec{\omega} = \frac{1}{2}\operatorname{curl}\vec{v}$

i.e. the angular velocity at any pt is equal to half of the curl of linear velocity.

Defn: ① A vector $\vec{v}$ is said to be solenoidal if $\operatorname{div}\vec{v} = 0$.

② A vector $\vec{v}$ is said to be irrotational if $\operatorname{curl}\vec{v} = 0$.

Q Evaluate $\operatorname{div}\vec{r}$, $\operatorname{curl}\vec{r}$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\operatorname{div}\vec{V} = \nabla\cdot\vec{V}$

Sol:- $\operatorname{div}\vec{r} = \sum \hat{i}\cdot\frac{\partial\vec{r}}{\partial x}$

$= \sum \hat{i}\cdot\hat{i} = \hat{i}\cdot\hat{i} + \hat{j}\cdot\hat{j} + \hat{k}\cdot\hat{k} = 3$

$\operatorname{curl}\vec{r} = \sum \hat{i}\times\frac{\partial\vec{r}}{\partial x} = \sum \hat{i}\times\hat{i} = \hat{i}\times\hat{i} + \hat{j}\times\hat{j} + \hat{k}\times\hat{k} = 0$

Q Show that $\operatorname{div}(\operatorname{grad} r^n) = n(n+1)r^{n-2}$ and hence evaluate $\operatorname{div}\left(\nabla\left(\frac{1}{r}\right)\right)$.

$\operatorname{curl}(\operatorname{grad}\phi) = 0$

Sol: $\operatorname{div}(\operatorname{grad} r^n)$

$= \operatorname{div}(\nabla r^n)$

$= \nabla\cdot(\nabla r^n)$

$= \nabla^2 r^n$

$= \nabla\cdot(\nabla r^n)$

$= \nabla\cdot\left(\sum i\frac{\partial}{\partial x} r^n\right)$

$= \nabla\cdot\left(\sum i\, n r^{n-1}\frac{\partial r}{\partial x}\right)$

$= \nabla\cdot\left(\sum \hat{i}\, n r^{n-1}\frac{x}{r}\right)$

$$= n\left(\nabla\cdot\sum \hat{i}\,x\,r^{n-2}\right)$$

$$\nabla\cdot\vec{v} = \operatorname{div}\vec{v} = \sum \hat{i}\frac{\partial}{\partial x}\cdot\vec{v}$$

$$\to \operatorname{div}(u\vec{v}) = \nabla u\cdot\vec{v} + u\operatorname{div}\vec{v}$$

$$\operatorname{div}(r^{n-2}\vec{r}) = \nabla r^{n-2}\cdot\vec{r} + r^{n-2}\operatorname{div}\vec{r}$$

$$= (n-2)r^{n-4}\vec{r}\cdot\vec{r} + r^{n-2}\times 3$$

$$\nabla r^n = n r^{n-2}\vec{r}$$

$$= n\left(\nabla\cdot\vec{r}\,r^{n-2}\right)$$

$$= n\sum \hat{i}\frac{\partial}{\partial x}\cdot\left(\vec{r}\,r^{n-2}\right)$$

$$= n\sum \hat{i}\cdot\left(\frac{\partial\vec{r}}{\partial x}r^{n-2} + \vec{r}\,(n-2)r^{n-3}\frac{\partial r}{\partial x}\right)$$

$$= n\sum \hat{i}\cdot\left(\hat{i}\,r^{n-2} + \vec{r}\,(n-2)r^{n-3}\frac{x}{r}\right)$$

$$= n\sum \hat{i}\cdot\left(\hat{i}\,r^{n-2} + \vec{r}\,(n-2)r^{n-4}x\right)$$

$$= n\,r^{n-2}\sum\hat{i}\cdot\hat{i} + n(n-2)r^{n-4}\left(\sum\hat{i}\cdot\vec{r}\,x\right)$$

$$= n\,r^{n-2} + n(n-2)\,r^{n-4}\,\vec{r}\cdot\sum\hat{i}x$$

$$= 3n\,r^{n-2} + n(n-2)\,r^{n-4}\,\vec{r}\cdot\vec{r}$$

$$= 3n\,r^{n-2} + n(n-2)\,r^{n-2}$$

$$= 3n\,r^{n-2} + n(n-2)\,r^{n-2}$$

$$= n\,r^{n-2}(3+n-2)$$

$$= n(n+1)\,r^{n-2}$$

$$\operatorname{div}(\operatorname{grad} r^n)$$

$$= \operatorname{div}(n r^{n-2}\vec{r})$$

$$= n\operatorname{div}(r^{n-2}\vec{r})$$

$$= n\{\nabla r^{n-2}\cdot\vec{r} + r^{n-2}\operatorname{div}\vec{r}\}$$

$$= n\{(n-2)r^{n-4}\vec{r}\cdot\vec{r} + 3r^{n-2}\}$$

$$= n(n+1)\,r^{n-2}$$

Also $\operatorname{div}\left(\nabla \frac{1}{r}\right) = \nabla \cdot (\nabla r^{-1}) = \nabla^2 r^{-1} = 0$ #

Q Show that $\operatorname{div}(r^n \vec{r}) = (n+3) r^n$ and $r^n \vec{r}$ is solenoidal iff $n = -3$ whereas it is irrotational for all n.

Sol:- $\operatorname{div}(r^n \vec{r})$

$$= \sum \hat{i} \cdot \frac{\partial}{\partial x}(r^n \vec{r})$$

$$= \sum \hat{i} \cdot \left(n r^{n-1} \frac{\partial r}{\partial x} \vec{r} + r^n \frac{\partial \vec{r}}{\partial x}\right)$$

$$= \sum \hat{i} \cdot \left(n r^{n-1} \frac{x}{r} \vec{r} + r^n \hat{i}\right)$$

$$= \sum \hat{i} \cdot \left(\underline{n r^{n-2}} \vec{r} x + r^n \hat{i}\right)$$

$$= n r^{n-2} \sum \hat{i} x \cdot \vec{r} + r^n \sum \hat{i} \cdot \hat{i}$$

$$= n r^{n-2} \vec{r} \cdot \vec{r} + 3 r^n$$

$$= n r^{n-2} r^2 + 3 r^n$$

$$= (n+3) r^n$$

$$\Rightarrow \operatorname{div}(r^n \vec{r}) = 0 \text{ iff } n = -3$$

$[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$

Similarly, we can show that

$$\text{curl } r^n \vec{r} = 0 \quad \forall n.$$

$\Rightarrow r^n \vec{r}$ is irrotational for all n.

→ Q Show that $\dfrac{\vec{r}}{r^3}$ is solenoidal as well as irrotational.

Q Evaluate $\text{div}(\vec{a}\times\vec{r})$ & $\text{curl}(\vec{a}\times\vec{r})$

Sol:- $\text{div}(\vec{a}\times\vec{r})$

$$= \sum \hat{i}\cdot\frac{\partial}{\partial x}(\vec{a}\times\vec{r})$$

$$= \sum \hat{i}\cdot(0\times\vec{r} + \vec{a}\times\hat{i})$$

$$= \sum \hat{i}\cdot(\vec{a}\times\hat{i}) = \sum[\hat{i}\,\vec{a}\,\hat{i}]$$

$$= \sum[\vec{a}\,\hat{i}\,\hat{i}]$$

$$= \sum \vec{a}\cdot(\hat{i}\times\hat{i}) = 0$$

$\text{curl}(\vec{a}\times\vec{r})$

$$= \sum \hat{i}\times\frac{\partial}{\partial x}(\vec{a}\times\vec{r})$$

$$= \sum \hat{i}\times(\vec{a}\times\hat{i})$$

$$= \sum(\hat{i}\cdot\hat{i})\vec{a} - \sum(\hat{i}\cdot\vec{a})\hat{i}$$

$$= 3\vec{a} - \vec{a} = 2\vec{a}$$

$\because \sum(\hat{i}\cdot\vec{a})\hat{i} = \vec{a}$

↓

Pf:-

$\boxed{\cancel{F = \nabla\phi}}$

$\Big\{$ let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{a}\cdot\hat{i} = a_1, \quad \vec{a}\cdot\hat{j} = a_2, \quad \vec{a}\cdot\hat{k} = a_3$

$\therefore \sum(\hat{i}\cdot\vec{a})\hat{i} = (\hat{i}\cdot\vec{a})\hat{i} + (\hat{j}\cdot\vec{a})\hat{j} + (\hat{k}\cdot\vec{a})\hat{k}$

$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a} \Big\}$

---

1. $\operatorname{div}(u\vec{v}) = \nabla u\cdot\vec{v} + u\operatorname{div}\vec{v}$
2. $\operatorname{curl}(u\vec{v}) = \nabla u\times\vec{v} + u\operatorname{curl}\vec{v}$
3. $\operatorname{div}(\vec{v}\times\vec{w}) = \vec{w}\cdot\operatorname{curl}\vec{v} - \vec{v}\cdot\operatorname{curl}\vec{w}$
4. $\operatorname{curl}(\vec{v}\times\vec{w}) = \vec{v}\operatorname{div}\vec{w} - \vec{w}\operatorname{div}\vec{v} + (\vec{w}\cdot\nabla)\vec{v} - (\vec{v}\cdot\nabla)\vec{w}$
5. $\operatorname{div}(\operatorname{grad}\phi) = \nabla\cdot(\nabla\phi) = \nabla^2\phi$
   $= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$
6. $\operatorname{curl}(\operatorname{grad}\phi)$
   $= \nabla\times(\nabla\phi) = 0$
7. $\operatorname{div}(\operatorname{curl}\vec{v})$
   $= \nabla\cdot(\nabla\times\vec{v}) = (\nabla\times\nabla)\cdot\vec{v}$
   $= 0$
8. $\operatorname{curl}(\operatorname{curl}\vec{v})$
   $= \operatorname{div}(\operatorname{curl}\vec{v}) - \nabla^2\vec{v}$

Proof:

① $\operatorname{div}(u\vec{v})$

$= \nabla\cdot(u\vec{v})$

$= \sum \hat{i}\cdot\frac{\partial}{\partial x}(u\vec{v})$

$= \sum \hat{i}\cdot\left(\frac{\partial u}{\partial x}\vec{v} + u\frac{\partial \vec{v}}{\partial x}\right)$

$= \sum \hat{i}\frac{\partial u}{\partial x}\cdot\vec{v} + u\sum \hat{i}\cdot\frac{\partial \vec{v}}{\partial x}$

$= \nabla u\cdot\vec{v} + u\operatorname{div}\vec{v}$

$\boxed{\begin{array}{l}\operatorname{curl}\vec{v}\\ = \sum \hat{i}\times\frac{\partial \vec{v}}{\partial x}\end{array}}$

$\nabla = \sum \hat{i}\frac{\partial}{\partial x}$, $\operatorname{div}\vec{v} = \nabla\cdot\vec{v}$
$= \sum \hat{i}\cdot\frac{\partial \vec{v}}{\partial x}$

③

$\operatorname{div}(\vec{v}\times\vec{w})$

$= \sum \hat{i}\cdot\frac{\partial}{\partial x}(\vec{v}\times\vec{w})$

$= \sum \hat{i}\cdot\left(\frac{\partial \vec{v}}{\partial x}\times\vec{w} + \vec{v}\cdot\frac{\partial \vec{w}}{\partial x}\right)$

$= \sum \hat{i}\cdot\left(\frac{\partial \vec{v}}{\partial x}\times\vec{w}\right) + \sum \hat{i}\cdot\left(\vec{v}\times\frac{\partial \vec{w}}{\partial x}\right)$

$= \left(\sum \hat{i}\times\frac{\partial \vec{v}}{\partial x}\right)\cdot\vec{w} - \sum \hat{i}\cdot\left(\frac{\partial \vec{w}}{\partial x}\times\vec{v}\right)$

$= \operatorname{curl}\vec{v}\cdot\vec{w} - \left(\sum \hat{i}\times\frac{\partial \vec{w}}{\partial x}\right)\cdot\vec{v}$

$= \vec{w}\cdot\operatorname{curl}\vec{v} - \vec{v}\cdot\operatorname{curl}\vec{w}$ #

Q. If $\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$. Show that $\vec{F}\cdot \text{curl}\,\vec{F} = 0$ and find $\text{div}\,\vec{F}$

Sol:

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= \hat{i}(-1-0) - \hat{j}(-1-0) + \hat{k}(0-1) = -\hat{i} + \hat{j} - \hat{k}$$

$$\vec{F}\cdot \text{curl}\,\vec{F} = [(x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}] \cdot [-\hat{i} + \hat{j} - \hat{k}]$$

$$= -x-y-1 + 1 + x + y$$

$$= 0$$

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(x+y+z) + \frac{\partial}{\partial y}(1) + \frac{\partial}{\partial z}(-x-y)$$

$$= 1 + 0 - 0) = 1$$

Note: $\text{curl}(\text{grad}\,\phi) = 0$

If $\text{grad}\,\phi = \vec{F}$

i.e $\vec{F} = \nabla\phi$

then $\boxed{\text{curl}\,\vec{F} = 0}$

i.e if the field conservative / Irrotational then $\vec{F} = \nabla\phi$ and $\text{curl}\,\vec{F} = 0$, then ϕ is known as scalar potential.

Q. Show that $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$ is irrotational and find scalar potential. Is the motion possible for an incompressible fluid?

$\vec{F} = \nabla\phi$ $\quad$ $\phi(x,y,z)$

$z = f(x,y)$

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

Sol: Given $\vec{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1) = 0$$

$\Rightarrow \vec{F}$ is irrotational.

$$\Rightarrow \boxed{\vec{F} = \nabla\phi}$$

Now $\vec{F}\cdot d\vec{r} = \nabla\phi \cdot dr$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

$\therefore \vec{F}\cdot d\vec{r} = d\phi$

on integrating

$$\phi = \int \vec{F}\cdot d\vec{r} + c$$

$$= \int \left((y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k} \right)\cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \right) + c$$

$$= \int \left[(y+z)dx + (z+x)dy + (x+y)dz \right] + c$$

$$= \int \left[y\,dx + x\,dy + z\,dx + x\,dz + z\,dy + y\,dz \right] + c$$

$$= \int \left[d(xy) + d(xz) + d(yz) \right] + c$$

$$= xy + xz + yz + c.$$

Again $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(z+x) + \frac{\partial}{\partial z}(x+y)$

$= 0+0+0 = 0$

$\Rightarrow \vec{F}$ is divergence free

$\Rightarrow$ motion is feasible for incompressible fluid.

$\vec{F} = \nabla\phi$

$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$d\vec{r} = dx\hat{i}+dy\hat{j}+dz\hat{k}$

$r = (x^2+y^2+z^2)^{\frac{1}{2}}$

Q Show that $\vec{F} = \vec{r}\,r^{-3}$ is irrotational and hence find ϕ.

Sol: we have already proved that $\operatorname{curl}\vec{F} = \operatorname{curl}(\vec{r}\,r^{3}) = 0$

$\Rightarrow \vec{F}$ is irrotational.

$\therefore\ d\phi = \vec{F}\cdot d\vec{r} = \frac{\vec{r}}{r^3}\cdot d\vec{r} = \frac{(x\hat{i}+y\hat{j}+z\hat{k})\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})}{(x^2+y^2+z^2)^{3/2}}$

$d\phi = \frac{x\,dx+y\,dy+z\,dz}{(x^2+y^2+z^2)^{3/2}}$

$$\Rightarrow \quad \phi = \int \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}} + C$$

let $x^2+y^2+z^2 = t$

$$\Rightarrow x\,dx + y\,dy + z\,dz = \frac{dt}{2}$$

$$\phi = \int \frac{dt}{2\,t^{3/2}} + C$$

$$\phi = \frac{1}{2}\int t^{-3/2}\,dt + C$$

$$= \frac{1}{2}\,\frac{t^{-1/2}}{-1/2} + C$$

$$= -\frac{1}{\sqrt{t}} + C$$

$$\therefore \phi = \frac{-1}{\sqrt{x^2+y^2+z^2}} + C$$

or $\boxed{\phi = -\frac{1}{r} + C}$

Q Show that $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ is irrotational, and find ϕ. Hence evaluate the line integral from $(1,2)$ to $(2,1)$.

Sol: we can easily show that $\text{curl}\,\vec{F} = 0$

$\therefore \quad d\phi = \vec{F}\cdot d\vec{r}$

$$= \left[(x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}\right]\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= (x^2 - y^2 + x)\,dx - (2xy + y)\,dy$$

$$= (x^2 + x)\,dx - y\,dy - (y^2\,dx + 2xy\,dy)$$

on integration

$$\phi = \int \left[(x^2 + x)\,dx - y\,dy - d(xy^2)\right] + c$$

$$\therefore \quad \boxed{\phi = \frac{x^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} - xy^2 + C}$$

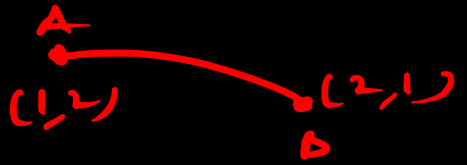
$\frac{5}{6} - 6$

$-\frac{31}{6}$

$\frac{8}{3} - \frac{1}{2}$

$\frac{13}{6} + \frac{31}{6}$

$\frac{44}{6} = \frac{22}{3}$



$$\therefore \quad \phi = \vec{F} \cdot d\vec{r}$$

on integrating:

$$[\phi]_A^B = \int_A^B \vec{F} \cdot d\vec{r}$$

$$\Rightarrow \int_A^B \vec{F} \cdot d\vec{r} = [\phi]_B - [\phi]_A$$

$$= [\phi]_{(2,1)} - [\phi]_{(1,2)}$$

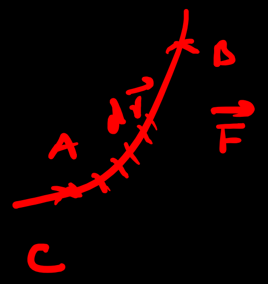
$$= \left(\frac{8}{3} + \frac{4}{2} - \frac{1}{2} - 2 + C\right) - \left(\frac{1}{3} + \frac{1}{2} - 2 - 4 + C\right)$$

$$= \frac{22}{3}.$$

Vector Integration

(i) Line integral

Any integral which is evaluated over a curve is said to be line integral.



A line integral is denoted by

$$I = \int_C \vec F \cdot d\vec r$$

If $\vec F = F_1\hat i + F_2\hat j + F_3\hat k$, $d\vec r = dx\,\hat i + dy\,\hat j + dz\,\hat k$ then

$$I = \int_C \vec F \cdot d\vec r = \int_C [F_1\,dx + F_2\,dy + F_3\,dz]$$

$$\int_0^\infty \cos bx\, e^{-ax} x^{n-1}\,dx = \frac{\Gamma(n)\cos(n\tan^{-1}\tfrac{b}{a})}{(a^2+b^2)^{n/2}}$$

$$\iiint F(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz$$

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)} \int_h^k F(u)\, u^{l+m+n-1}\,du$$

$$h \le x+y+z \le k$$

$$\iiint_K (x+y+z+1)^2\,dx\,dy\,dz \quad x^{1-1} y^{1-1} z^{1-1}$$

$$\int_a^b \frac{(x-a)^{l-1}}{(b-x)^{m+l/n}}$$

$$0 \le x+y+z \le 1$$

$$1 \le \boxed{x+y+z+1} \le 2 \quad (u)$$

$$=$$

$$= \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(3)} \int_1^2 u^2\, u^{1+1+1-1}\,du$$

#

$$\int_0^\infty \cos\left(\frac{\pi x^2}{2}\right)dx$$

6

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \int_0^\infty \cos x\; x^{\frac{1}{2}-1}\,dx$$

$$= \Gamma(1/2)\; \times$$

→ If the curve is closed then

$\int_C \vec{F}\cdot d\vec{r}$ is said to be circulation of $\vec{F}$ round the curve c.

→ If circulation is zero, then $\vec{F}$ is said to be irrotational and ~~any force~~ the force $\vec{F}$ in this case is said to be conservative force.

→ Also work done by a particle during the displacement from A to B is given by $W = \int_C \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot d\vec{r}$

We know that if $\vec{F}$ is irrotational / conservative, $\vec{F}\cdot d\vec{r} = d\phi$

hence

$$W = \int_A^B d\phi = [\phi]_B - [\phi]_A$$

Thus ϕ depends on the value of ϕ at the ends points ~~and~~ not on the path joining A and B.

Ques: If $\vec{F} = 3xy\,\hat{i} - y^2\hat{j}$ then evaluate $\int_C \vec{F}\cdot d\vec{r}$ for the arc of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Sol:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (3xy\,\hat{i} - y^2\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C [3xy\,dx - y^2\,dy]$$

As $y = 2x^2 \Rightarrow dy = 4x\,dx$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C [6x^3\,dx - 4x^4 \times 4x\,dx]$$

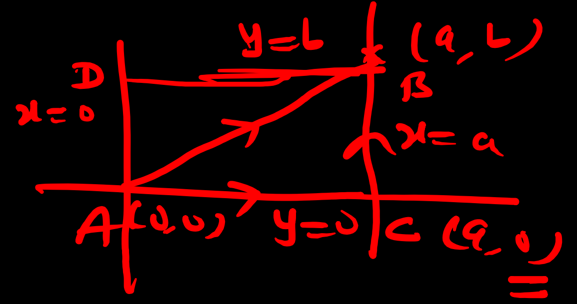
$$= \int_0^1 (6x^3 - 16x^5)\,dx = \frac{6}{4} - \frac{16}{6}$$

$$= \frac{3}{2} - \frac{8}{3} = \frac{-7}{6}.$$

Q Find the total work done by a force $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ in moving a point from $(0,0)$ to (a,b) along the rectangle bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$.

Sol:

$$\int_C \vec{F}\cdot d\vec{r} = \int_{AC} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r}$$



Along AC

$y = 0 \Rightarrow dy = 0$

$$\int_{AC} \vec{F}\cdot d\vec{r} = \int_{AC} \vec{F}\cdot d\vec{r} = \int_{AC} (x^2\hat{i})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int x^2\,dx = \int_0^a x^2\,dx = \frac{a^3}{3}$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\iint_c \vec{F}\cdot\hat{n}\,dS$$

Along CB $\quad x = a \Rightarrow dx = 0$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left[(a^2 + y^2)\hat{i} - 2ay\hat{j} \right] \cdot (dx\hat{i} + dy\hat{j})$$

$$= \int_C (-2ay\, dy)$$

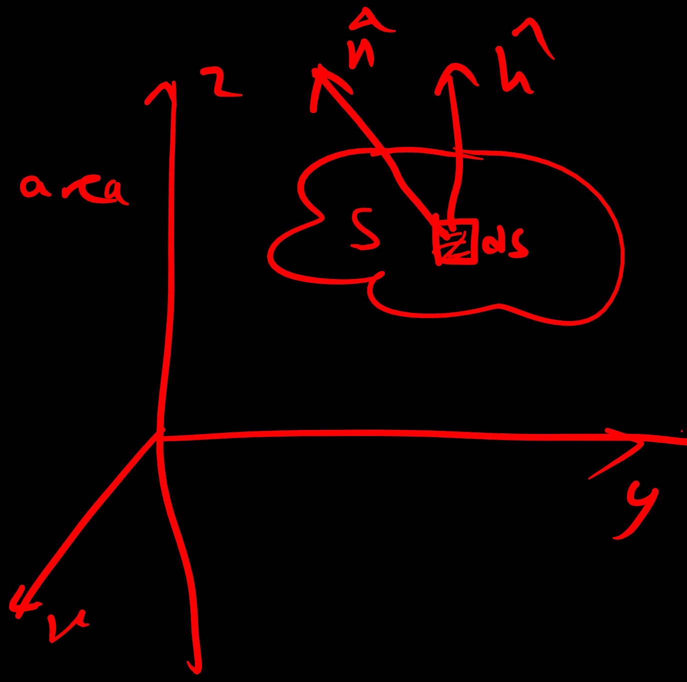
$$= -2a \int_0^b y\, dy = -2a \frac{b^2}{2} = -ab^2$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} - ab^2 \text{ ans.}$$

Surface integral

Any integral which is evaluated over a surface is known as surface integral.

If $d\vec{S}$ is small element area of the surface area S and $\hat{n}$ be the outward normal to this surface then



$$\iint_S \vec{F}\cdot\hat{n}\, ds \quad \text{or} \quad \iint_S \vec{F}\cdot d\vec{S}$$

is known surface integral, where

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

and

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

volume integral

Any integral which is evaluated over a volume is known as volume integral and it is given by

$$\iiint_V \vec{F}\,dv \quad \text{or} \quad \iiint_V \phi\,dV.$$

Gauss divergence theorem

If $\vec{F}$ is a vector point function having continuous first order partial derivative in the region V bounded by a closed surface S

then $$\boxed{\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dv}$$

where $\hat{n}$ is the outward unit normal drawn to the surface S.

$\vec{F} = x\hat{i} + y\hat{j} + \cancel{z\hat{k} = \vec{r}}$

$\operatorname{div}\vec{F} = 3$

$\iiint_V \cancel{3\, dv = 3 \times V} \; \cancel{= 3 \times \frac{4}{3}\pi \times 1^3} = 4\pi$

Stokes theorem:

If S is any open surface bounded by a closed curve C and $\vec{F}$ be the vector point function having continuous first order partial derivatives then

$$\boxed{\int_C \vec{F} \cdot \vec{dr} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds}$$

where $\hat{n}$ is the outward unit normal.

Green's theorem:

If C is a regular closed curve in the xy-plane and R is the region bounded by C, then

$$\int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Note: If $M = -y$, $N = x$ then $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$

$$\therefore \int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\Rightarrow \int_C (-y\,dx + x\,dy) = \iint_R (1+1)\,dx\,dy$$

$$\Rightarrow \boxed{\iint_R dx\,dy = \frac{1}{2}\int_C (x\,dy - y\,dx)}$$

* $\therefore$ Area bounded by a simple closed curve C is

$$= \frac{1}{2}\int_C (x\,dy - y\,dx).$$

Q Show that $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = 0$ for any closed surface S.

Sol: $\because \iint_S \vec{V}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{V}\,dv$

Let $\vec{V} = \text{curl}\,\vec{F}$

$$\Rightarrow \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,(\text{curl}\,\vec{F})\,dv$$

$$= 0 \quad \left\{ \because \text{div}(\text{curl}\,\vec{F}) = \nabla\cdot(\nabla\times\vec{F}) = 0 \right\}$$

Q Evaluate $\iint_S \vec{r}\cdot\hat{n}\,ds$ when $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

sol:- $\therefore \iint_S \vec{r}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{r}\,dv$ {Gauss div theorem}

$\therefore \text{div}\,\vec{r} = \sum \hat{i}\cdot\frac{\partial\vec{r}}{\partial x} = \sum \hat{i}\cdot\hat{i} = 3$

$\therefore \iint_S \vec{r}\cdot\hat{n}\,ds = \iiint_V 3\,dv = 3\iiint_V dv$

$= 3V$ #

Q Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$

and S is the surface of the sphere having centre at $(3,-1,2)$ and radius 3.

Sol:-

$$\iint_S \vec{F}\cdot\hat{n}\,ds$$

$$= \iiint_V \operatorname{div}\vec{F}\,dv$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\left((2x+3z)\right)+\frac{\partial}{\partial y}(-xz-y)+\frac{\partial}{\partial z}(y^2+2z)\right)dv$$

$$= \iiint_V (2-1+2)\,dv \quad = 3V$$

$$= 3\times\frac{4}{3}\pi r^3$$

$$= \cancel{3}\times\frac{4}{\cancel{3}}\pi\times 3^3$$

$$= 108\pi$$

Q Evaluate $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS$, $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$

where S is the surface of the cuboid

$0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$

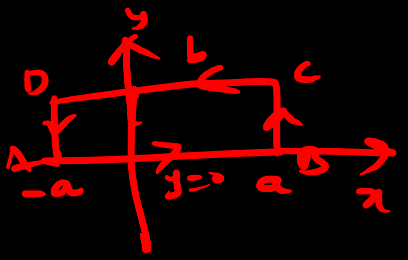
Sol:- $\iint_S \vec{F} \cdot \hat{n}\, dS = \iiint \operatorname{div} \vec{F}\, dV$

$= \int_0^a \int_0^b \int_0^c (\quad)\, dx\, dy\, dz$

$= abc\left(a + \frac{b}{2}\right).$

Q1. verify stokes theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken round the rectangle bounded by the lines

$x = \pm a$, $y = 0$, $y = b$



Q2. If $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$. show that $\iint_S \vec{F}\cdot\hat{n}\, ds = 6V$

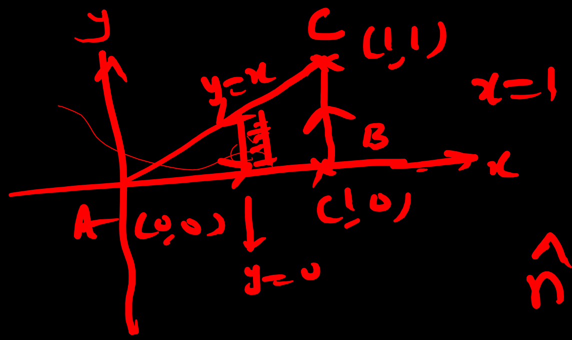
Q3. verify stokes theorem when $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ where c is the boundary of the triangle with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$.

we have to evaluate

Sol 3:- $\int_C \vec{F}\cdot d\vec{r}$

where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$

$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$



$\hat{n} = \hat{k}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CA} \vec{F}\cdot d\vec{r}$$

Now for $\int_{AB} \vec{F}\cdot d\vec{r}$:-

$y=0 \Rightarrow dy=0$, $z=0 \Rightarrow dz=0$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^1 (x^2\hat{j} - x\hat{k})\cdot(dx\,\hat{i} + 0\hat{j} + 0\hat{k})$$

$$= 0$$

$$\int_{BC} \vec{F}\cdot d\vec{r}$$

$x=1 \Rightarrow dx=0,\quad z=0 \Rightarrow dz=0$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int (y^2\hat{i} + \hat{j} - \hat{k})\cdot(0\hat{i}+dy\hat{j}+0\hat{k})$$

$$= \int_0^1 dy = 1$$

$$\int_{CA} \vec{F}\cdot d\vec{r}$$

$y=x \Rightarrow dy=dx,\quad z=0 \Rightarrow dz=0$

$$\int_{CA} \vec{F}\cdot d\vec{r} = \int_{CA} (x^2\hat{i} + x^2\hat{j} - x\hat{k})\cdot(dx\hat{i}+dx\hat{j})$$

$$= \int_1^0 (x^2+x^2)\,dx = \frac{2}{3}x - 1 = \frac{-2}{3}$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CA} \vec{F}\cdot d\vec{r}$$

$$= 0 + 1 - \frac{2}{3}$$

$$= \frac{1}{3} \text{ ns.}$$

Now we calculate $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -x-z \end{vmatrix} = \hat{j} + \hat{k}\,(2x-2y)$$

$$\iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, ds$$

$$= \iint \left(\hat{j} + (2x-2y)\hat{k} \right) \cdot \hat{k} \, ds$$

$$= \iint (2x-2y)\, ds$$

$$\left\{ \text{use } ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{1} \right\}$$

$$= \iint (2x-2y)\, dx\,dy$$

$$= \int_0^1 \int_{y=0}^{x} (2x-2y)\, dx\,dy$$

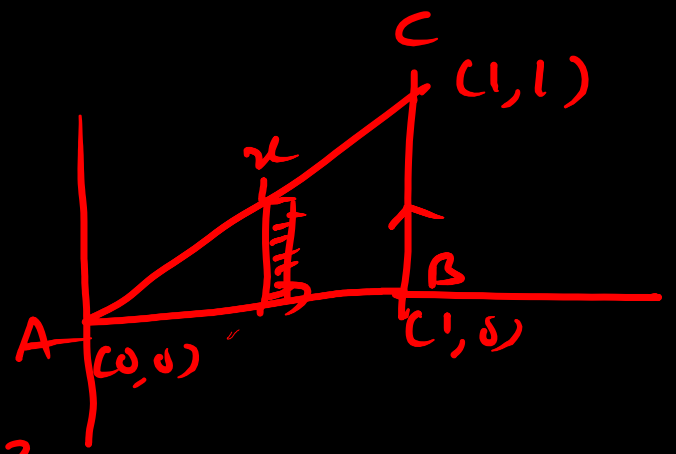
$$= \int_0^1 \left(2xy - y^2\right)_0^x dx = \int_0^1 (2x^2 - x^2)\, dx$$

$$= \int_0^1 x^2\, dx = \frac{1}{3}$$

$\therefore \boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds}$

$\therefore$ Stokes theorem verified.

Q Evaluate $\int_c (x^2 y\,dx + x^2 dy)$ where c is the boundary of the triangle $(0,0)$, $(1,0)$, $(1,1)$.



Sol:-

$$\int_c (x^2 y\,dx + x^2 dy)$$

$$= \int_c (M\,dx + N\,dy), \quad M = x^2 y, \; N = x^2$$

$$= \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$$= \iint_S (2x - x^2)\,dx\,dy = \int_0^1 \int_0^x (2x - x^2)\,dx\,dy$$

$$= \int_0^1 (2x - x^2) \times (y)_0^x \, dx$$

$$= \int_0^1 (2x - x^2) \times x \, dx$$

$$= \int_0^1 (2x^2 - x^3) \, dx = \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}.$$

Q Evaluate $\int_c [(y - \sin x)\,dx + \cos x\,dy]$ where c is the triangle formed by $y = 0$, $x = \frac{\pi}{2}$, $y = \frac{2}{\pi}x$

- L1: https://youtu.be/DSGRar3tfho

L2: https://youtu.be/qSGtnB73hvo

L3: https://youtu.be/NXMJVYheNfA

L4: https://youtu.be/TaHZ_jvHG2I

- L5: https://youtu.be/ecvZqe0YEOc
- L6: https://youtu.be/CpBPEwEMSs4
- L7: https://youtu.be/WxkyjLJ-w4s
- L8: https://youtu.be/2GurwBNuJc8
- L9: https://youtu.be/ckqotqUWSYg