Sequence and Series –:

A map from $\mathbb{N}$ to $\mathbb{R}$ i.e., $f : \mathbb{N} \longrightarrow \mathbb{R}$ defines a sequence of real number when image is arranged in natural order of natural number

$$\langle a_n \rangle = \langle f(n) \rangle = \langle f(1), f(2), f(3), f(4), \cdots \rangle$$

Example –:

(i) $\langle a_n \rangle = \langle n \rangle = \{1, 2, 3, 4, 5, 6, \cdots\}$

(ii) $\langle a_n \rangle = (-1)^n n = \{-1, 2, -3, 4, -5, \cdots\}$

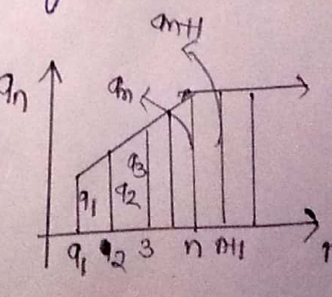
(iii) $\langle a_n \rangle = (-1)^n = \{-1, 1, -1, 1, -1, 1, \cdots\}$

Monotonic sequence –:

(i) Let $\langle a_n \rangle$ be a sequence of real number

$\langle a_n \rangle$ is monotonic increasing or non-decreasing if

$$a_n \leq a_{n+1} \quad \forall\, n \in \mathbb{N}$$



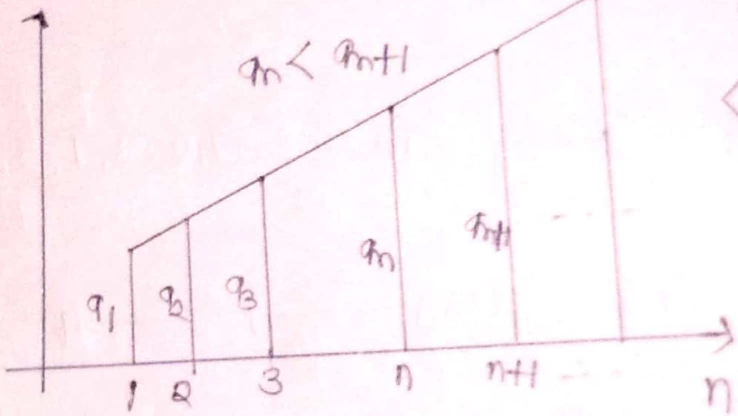
Example –: (i) $\langle a_n \rangle = \langle n^2 \rangle$

(ii) $a_{n+1} = \sqrt{2 + a_n} \quad \forall\, n \in \mathbb{N}$ is monotonically increasing sequence.

(ii) The sequence $\langle a_n \rangle$ is monotonically decreasing or non-increasing if
$a_n \geq a_{n+1} \quad \forall\, n \in \mathbb{N}$.

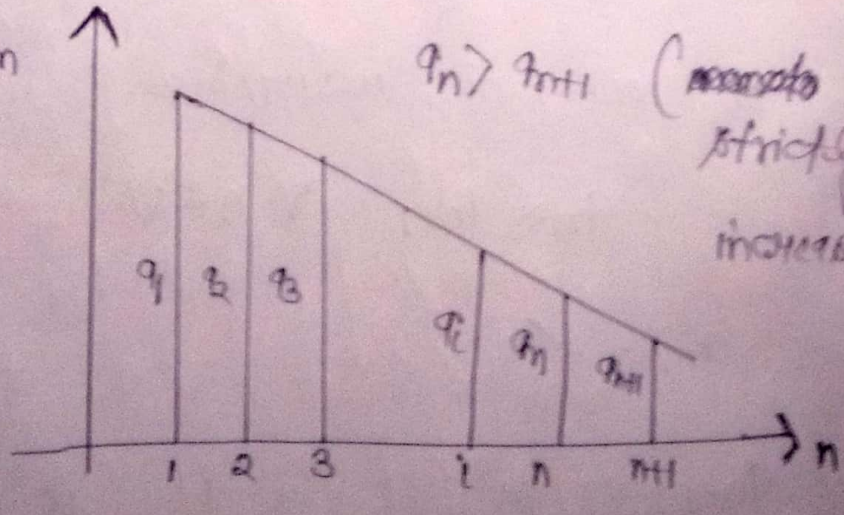
(iii) If $a_n < a_{n+1} \quad \forall\, n \in \mathbb{N}$, then $\langle a_n \rangle$ is called strictly increasing or increasing sequence.

(iv) If



e.g. $\langle a_n \rangle = \langle n^2 \rangle$

(v) If $a_n > a_{n+1} \quad \forall\, n \in \mathbb{N}$, then $\langle a_n \rangle$ is called strictly decreasing or decreasing sequence.



$a_n > a_{n+1}$ (strictly increasing).

Limit of the sequence —:

Let $\langle u_n \rangle$ be a sequence of real number, if sequence tends to a limit l, then we write

$$\lim_{n\to\infty} (u_n) = l.$$

Convergence and divergence of a sequence —:

(i) Let $\langle u_n \rangle$ be a sequence of real number, the sequence $\langle u_n \rangle$ is said to be convergent if $\lim_{n\to\infty} u_n =$ finite.

i.e., if the limit of sequence is finite, the sequence is convergent.

(ii) The sequence $\langle u_n \rangle$ is said to be divergent if $\lim_{n\to\infty} u_n = \infty$ or $-\infty$.

i.e., if the limit of sequence does not tends to a finite number, the sequence is said to be divergent.

Oscillatory sequence :-

If sequence $\{\alpha_n\}_{n=1}^{\infty}$ neither converge to a finite number α nor diverge to ∞ or $-\infty$ then we say the sequence $\{\alpha_n\}_{n=1}^{\infty}$ oscillates.

Eg. $\{\alpha_n\}_{n=1}^{\infty} = \{(-1)^{n-1}\}_{n=1}^{\infty} = \{1,-1,1,-1,1,-1,\ldots\}$

oscillatory sequence

$\{\alpha_n\}_{n=1}^{\infty} = \{\frac{1}{n}\}_{n=1}^{\infty} = \{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\}$

Convergent sequence.

$\{\alpha_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty} = \{1,2,3,4,5,6,\ldots\}$

divergent sequence

Question :- check the convergence and divergence of the sequence.

(i) $\{\alpha_n\}_{n=1}^{\infty} = \{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\ldots\}$

(ii) $\{\alpha_n\}_{n=1}^{\infty} = \{\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\ldots\}$

(iii) $\{\alpha_n\}_{n=1}^{\infty} = \{1,-1,1,-1,\ldots\}$

Soln ① we have given

$$\{\alpha_n\}_{n=1}^{\infty} = \{\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3},\frac{1}{2^4},\ldots\}$$

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The nth term of the sequence

$$x_n = \frac{1}{2n}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{1}{2n}\right) = 0 \text{ finite}$$

$\Rightarrow$ The sequence $\{x_n\}_{n=1}^{\infty} = \{\frac{1}{2n}\}_{n=1}^{\infty}$ is Convergent.

(ii) We have given

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\right\}$$

The nth term of the sequence

$$x_n = \left(\frac{n}{n+1}\right)$$

$$\lim_{n\to\infty}(x_n) = \lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right) = 1$$

$\Rightarrow$ The sequence $\{x_n\}_{n=1}^{\infty} = \{\frac{n}{n+1}\}_{n=1}^{\infty}$ is Convergent.

(iii) We have given

$$\{x_n\}_{n=1}^{\infty} = \{(-1)^{n-1}\} = \{1, -1, 1, -1, 1, -1, \cdots\}$$

$$= \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ even} \end{cases}$$

$\Rightarrow$ $\{x_n\}_{n=1}^{\infty} = \{(-1)^{n-1}\}_{n=1}^{\infty}$ is an oscillatory sequence.

Note that -:

(i) $\lim_{n\to\infty} x^n = 0$ if $x<1$

and $\lim_{n\to\infty} x^n = \infty$ if $x>1$.

(ii) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x

(iii) $\lim_{n\to\infty} \left(\frac{\log n}{n}\right) = 0$

(iv) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

(v) $\lim_{n\to\infty} (n)^{1/n} = 1$

(vi) $\lim_{n\to\infty} \left[\frac{(n!)}{n}\right]^{1/n} = \frac{1}{e}$

(viii) $\lim_{n\to\infty} n x^n = 0$ if $x<1$

(ix) $\lim_{n\to\infty} (n)^h = \infty$ $(h>0)$

(x) $\lim_{n\to\infty} \left(\frac{1}{n^h}\right) = 0$ (xi) $\lim_{x\to\infty} \left[\frac{a^x-1}{x}\right] = \log a$

(xii) $\lim_{n\to\infty} \left(\frac{a^{1/n}-1}{1/n}\right) = \log a$

(xiii) $\lim_{x\to 0} \left(\frac{\sin x}{x}\right) = 1$ (xiv) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right) = 1$

Series -: Let $\{u_n\}_{n=1}^{\infty} = u_1, u_2, u_3, u_4, u_5, \ldots$

be a sequence of real number.

Then, $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + u_4 + \ldots$

is an infinite series.

Write,

$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$
$$S_3 = u_1 + u_2 + u_3$$
$$\vdots$$
$$S_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum_{i=1}^{n} u_i$$

Called nth partial sum of $\sum_{n=1}^{\infty} u_n$

$$\lim_{n\to\infty} (u_1 + u_2 + u_3 + \cdots + u_n) = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

or

$$\lim_{n\to\infty} (s_n) = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

(i) If $\lim_{n\to\infty} (s_n) =$ a finite number s (say), then we say that $\sum_{n=1}^{\infty} u_n$ is convergent and its sum is s.

(ii) If $\lim_{n\to\infty} (s_n) = +\infty$ (or $-\infty$) then we say that $\sum_{n=1}^{\infty} u_n$ diverges to $+\infty$ (or $-\infty$).

(iii) If $\{s_n\}_{n=1}^{\infty}$ oscillates, then $\sum_{n=1}^{\infty} u_n$ is oscilates.

i.e., (i) If s_n tends to a finite number as $n \to \infty$, then series $\sum u_n$ is said to be convergent.

(ii) If $s_n = \sum_{i=1}^{n} u_i$ tends to infinity as $n \to \infty$, the series $\sum_{n=1}^{\infty} u_n$ is said to be divergent.

(iii) If $\{s_n\} = \sum_{i=1}^{n} u_i$ does not tend to a unique limit, finite or infinite, the series $\sum u_n$ is called oscillatory.

Properties of series :

(i) The nature of an infinite series does not change

(i) by multiplication of all terms by a constant k

(ii) by addition or deletion of finite number of terms.

(2) If $\sum u_n$ and $\sum v_n$ are convergent, $\sum (u_n + v_n)$ is also convergent.

Question : Test the convergence of the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \infty$$

Soln The nth partial sum

$$s_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$= \frac{1\left(1 - \frac{1}{2^n}\right)}{\left(1 - \frac{1}{2}\right)} = 2\left(1 - \frac{1}{2^n}\right)$$

$$\Rightarrow \lim_{n\to\infty}(S_n) = 2 \text{ finite}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \text{ is } \underline{\text{Convergent}}.$$

Question:- Prove that the following series;

$$\frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \cdots \text{ is Convergent}$$

and find its sum.

Soln We have given

$$\sum_{n=1}^{\infty} u_n = \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \cdots$$

Here $\quad u_n = \frac{(n+1)}{(n+2)!} = \frac{(n+2)-1}{(n+2)!} = \frac{(n+2)}{(n+2)!} - \frac{1}{(n+2)!}$

$$u_n = \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$$

Now nth partial sum

$$S_n = \left(\frac{1}{2!} - \cancel{\frac{1}{3!}}\right) + \left(\cancel{\frac{1}{3!}} - \cancel{\frac{1}{4!}}\right) + \left(\cancel{\frac{1}{4!}} - \cancel{\frac{1}{5!}}\right) \setminus - \setminus + \left(\cancel{\frac{1}{(n+1)!}} - \frac{1}{(n+2)!}\right)$$

$$S_n = \frac{1}{2!} - \frac{1}{(n+2)!}$$

$$\Rightarrow \lim_{n\to\infty}(S_n) = \frac{1}{2}$$

$\therefore \sum u_n$ is Converges and its limit is $\frac{1}{2}$ $\underline{\underline{\text{Ans}}}$

Question :- Test the nature of the following series

(i) $1+\frac{5}{4}+\frac{6}{4}+\frac{7}{4}+ \cdots \infty$

(ii) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$

(iii) $\log 3+\log\left(\frac{4}{3}\right)+\log\left(\frac{5}{4}\right)+\cdots+\infty$

(iv) $\sum\left(\sqrt{n+1}-\sqrt{n}\right)$

Soln :- (i) we have given

$$\sum_{n=1}^{\infty} u_n = 1+\frac{5}{4}+\frac{6}{4}+\frac{7}{4}+\cdots+\infty$$

$$= \frac{4}{4}+\frac{5}{4}+\frac{6}{4}+\frac{7}{4}+\cdots+\infty$$

$$= \sum_{n=1}^{\infty}\left(\frac{4+(n-1)}{4}\right) = \sum_{n=1}^{\infty}\left(\frac{n+3}{4}\right)$$

$$u_n = \left(\frac{n+3}{4}\right) = \frac{3}{4}+\frac{1}{4}(n)$$

nth partial sum -

$$S_n = \frac{3}{4}+\frac{1}{4}(1+2+3+\cdots+n)$$

$$= \frac{3}{4}+\frac{1}{2}\,\frac{n}{2}(n+1)$$

$$\lim_{n\to\infty}(S_n) = \infty \quad (\text{divergent})$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{n+3}{4}\right) \text{ is divergent.}$$

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(ii) We have given

$$\sum_{n=1}^{\infty} u_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Here, $u_n = \frac{1}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} = \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$\therefore$ The nth partial sum

$$S_n = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right)$$

$$S_n = \left(1 - \frac{1}{n+1}\right)$$

$\Rightarrow \lim_{n\to\infty} (S_n) = 1$ (finite)

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is Convergent and its sum is 1.

Ans

(iii) We have given $= \log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \cdots$

$$\sum_{n=1}^{\infty} u_n = \log 3 + \log\left(\frac{4}{3}\right) + \log\left(\frac{5}{4}\right) + \cdots + \infty$$

$$= \sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$$

Here $u_n = \log(n+2) - \log n$

$\therefore$ The nth partial sum

$$S_n = (\cancel{\log 2} - \log 1) + (\cancel{\log 3} - \cancel{\log 2}) + (\cancel{\log 4} - \cancel{\log 3}) + \cdots + \log(n+1) - \cancel{\log n}$$

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$$S_n = \log(n+1) - \log 1 = \log(n+1)$$

$$\lim_{n\to\infty} S_n = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ is divergent.}$$ <u>Ans</u>

(4) We have given

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$$

Here $u_n = \sqrt{n+1} - \sqrt{n} = \dfrac{(n+1) - n}{\sqrt{n+1} + (\sqrt{n})}$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

The nth partial sum

$$S_n = (\cancel{\sqrt{2}} - 1) + (\cancel{\sqrt{3}} - \cancel{\sqrt{2}}) + (\cancel{\sqrt{4}} - \cancel{\sqrt{3}})$$
$$\cdots + (\sqrt{n+1} - \cancel{\sqrt{n}})$$

$$S_n = \sqrt{n+1} - 1$$

$$\lim_{n\to\infty} (S_n) = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \text{ is divergent.}$$

<u>properties of Geometric series</u> :-

The series $1 + r + r^2 + r^3 + \cdots \infty$ is

(i) convergent if $|r| < 1$

(ii) divergent if $r \geq 1$

(iii) oscillatory if $r \leq -1$.

Question-: Test the nature of the following series

(i) $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \cdots \infty$

(ii) $1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3 + \cdots \infty$

Soln-: (i) we have given

$$1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots$$

Geometric series.

The rth term $= \left(\frac{3}{4}\right) < 1$

$\Rightarrow$ Series is Convergent

The nth partial sum

$$s_n = 1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^{n-1}$$

$$= \frac{1\left(1-\left(\frac{3}{4}\right)^n\right)}{\left(1-3/4\right)} = 4\left(1-\left(\frac{3}{4}\right)^n\right)$$

$$\Rightarrow \lim_{n\to\infty}(s_n) = 4(1-0) = 4$$

$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^{n-1}$ is Convergent and Converges to 4 Ans

(ii) We have given

$$1+\left(\frac{4}{3}\right)+\left(\frac{4}{3}\right)^2+\left(\frac{4}{3}\right)^3+ \cdots +\infty$$

Geometric series

Here the

$$r^{th} \text{ term} = \left(\frac{4}{3}\right) > 1$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^{n-1} \text{ is divergent.}$$

Moreover, the nth partial sum

$$S_n = 1+\left(\frac{4}{3}\right)+\left(\frac{4}{3}\right)^2+ \cdots +\left(\frac{4}{3}\right)^n$$

$$= \frac{1\left(\left(\frac{4}{3}\right)^n-1\right)}{\left(\frac{4}{3}-1\right)} = 3\left(\left(\frac{4}{3}\right)^n-1\right)$$

$$\Rightarrow \lim_{n\to\infty}(S_n) = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^{n-1} \text{ is divergent}$$

Ans

Necessary Conditions for Convergent series :

For every Convergent series $\sum u_n$

$$\lim_{n\to\infty} u_n = 0$$

~~Convergent~~ Moreover if $\lim_{n\to\infty}(u_n) \neq 0$

$$\Rightarrow \sum u_n \text{ is not Convergent.}$$

Proof :- Let $\sum_{n=1}^{\infty} u_n$ is a series of positive terms.

Let $S_n = u_1 + u_2 + u_3 + \dots + u_n$ be the nth partial sum of $\sum_{n=1}^{\infty} u_n$.

Given that $\sum_{n=1}^{\infty} u_n$ is convergent

$\Rightarrow \lim_{n\to\infty} (S_n) = k$ (a finite quantity)

Also $\lim_{n\to\infty} (S_{n-1}) = k$ (a finite quantity)

Now $S_n = S_{n-1} + u_n$

$\Rightarrow u_n = S_n - S_{n-1}$

$\Rightarrow \lim_{n\to\infty} (u_n) = \lim_{n\to\infty} [S_n - S_{n-1}] = k - k = 0$

$\Rightarrow \boxed{\lim_{n\to\infty} (u_n) = 0}$ proved

Converse of above theorem is not true.

i.e. if $\lim_{n\to\infty} (u_n) = 0$

$\not\Rightarrow \sum_{n=1}^{\infty} u_n$ is convergent.

Consider the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \dots + \frac{1}{\sqrt{n}} + \dots \infty$$

Here $u_n = \frac{1}{\sqrt{n}}$

$\Rightarrow \lim_{n\to\infty}(u_n) = 0$, but

The nth partial sum

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{n}}$$

$$> \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}$$

$$> \frac{n}{\sqrt{n}} = \sqrt{n}$$

$$\Rightarrow \lim_{n\to\infty}(S_n) \to \infty$$

Thus, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent

although $\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right) = 0$.

So $\lim_{n\to\infty}(u_n) = 0$ is a necessary condition but not a sufficient condition for convergence.

Note that :

If $\lim_{n\to\infty}(u_n) \neq 0$, then the series $\sum_{n=1}^{\infty} u_n$ must be divergent.

Question : Test the convergence of the series

$$\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \cdots + \sqrt{\frac{n}{2(n+1)}} + \cdots$$

Solution Here $u_n = \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{1}{2(1+\frac{1}{n})}}$

$$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\frac{1}{\sqrt{2(1+1/n)}} = \frac{1}{\sqrt{2}} \neq 0$$

$\Rightarrow \sum u_n$ is not convergent (divergent).

<u>Question</u>: Examine the convergence of series

(i) $\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}$ (ii) $\sum_{n=1}^{\infty}\cos\left(\frac{1}{n}\right)$

<u>Solution</u>: (i) We have given

$$\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}} = \sum_{n=1}^{\infty} u_n$$

Here, $u_n = \sqrt{\frac{n}{n+1}} \Rightarrow \lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\sqrt{\frac{1}{1+1/n}}$

$$\Rightarrow \lim_{n\to\infty}(u_n) = 1 \neq 0$$

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is divergent.

(ii) We have given

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\cos\left(\frac{1}{n}\right)$$

Here, $u_n = \cos\left(\frac{1}{n}\right)$

$$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\cos\left(\frac{1}{n}\right)\right) = \cos(0) = 1 \neq 0$$

$\Rightarrow \sum_{n=1}^{\infty}\cos\left(\frac{1}{n}\right)$ is divergent.

p-series test to check the convergence of the series -:

The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ is

(i) Convergent if $p > 1$

(ii) divergent if $p \leq 1$.

② **Comparison Test -:**

If two positive term $\sum u_n$ and $\sum v_n$ be such that

$$\lim_{n\to\infty} \left(\frac{u_n}{v_n}\right) = \text{finite quantity, non-zero.}$$

Then both series $\sum u_n$ & $\sum v_n$ are converges or diverges togenther

i.e., Convergence of $\sum v_n \iff$ Convergence of $\sum u_n$.

Note that -: v_n can be obtain from u_n by taking common the ~~lowest~~ Highest degree term from numerator and dinominator.

Question:- Test the convergence of the series

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots$$

Soln We have given

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}=\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots$$

Here $u_n=\dfrac{1}{\sqrt{n}+\sqrt{n+1}}=\dfrac{1}{\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)}$

Thus $v_n=\dfrac{1}{\sqrt{n}}$

and $\lim\limits_{n\to\infty}\left(\dfrac{u_n}{v_n}\right)=\lim\limits_{n\to\infty}\left(\dfrac{1}{1+\sqrt{1+\frac{1}{n}}}\right)=1\neq 0$

$\Rightarrow$ $\sum u_n$ and $\sum v_n$ converges or diverges together

Since $\sum v_n=\sum\frac{1}{\sqrt{n}}$ is divergent as $p=\frac{1}{2}<1$ (by p-series test)

$\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$ is ~~convergent~~ divergent.

Question:- Using Comparison test discuss the convergence of $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\sqrt{n}-1}{n^2+1}\right)$

Soln Here,

$$u_n=\left(\frac{\sqrt{n}-1}{n^2+1}\right).$$

$$u_n = \frac{\sqrt{n}\left(1-\frac{1}{\sqrt{n}}\right)}{n^2\left(1+\frac{1}{n^2}\right)} = \frac{1}{n^{3/2}}\frac{\left(1-\frac{1}{\sqrt{n}}\right)}{\left(1+\frac{1}{n^2}\right)}$$

Here, let $v_n = \frac{1}{n^{3/2}}$

Now $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{\left(1-\frac{1}{\sqrt{n}}\right)}{\left(1+\frac{1}{n^2}\right)}\right) = 1 \neq 0$

Then by comparison test

$\sum u_n$ and $\sum v_n$ converges or diverges together.

Since $\sum v_n = \sum \frac{1}{n^{3/2}}$ is of the form $\sum \frac{1}{n^p}$

(Here $p = 3/2 > 1$)

$\Rightarrow \sum v_n = \sum \frac{1}{n^{3/2}}$ is convergent

$\Rightarrow \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{\sqrt{n}-1}{(n^2+1)}$ is convergent. #

Question :- Test the convergence or divergence of the series.

$$\sum_{n=1}^{\infty}\left(\frac{2n^2+3n}{5+n5}\right)$$

Soln Here, $u_n = \left(\frac{2n^2+3n}{5+n5}\right)$.

$$u_n = \frac{n^2\left(2+\frac{3}{n}\right)}{n^5\left(1+\frac{5}{n^5}\right)} = \frac{1}{n^3}\left(\frac{2+\frac{3}{n}}{\frac{5}{n^5}+1}\right)$$

Let $v_n = \left(\frac{1}{n^3}\right)$

then $\lim\limits_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim\limits_{n\to\infty}\left(\frac{2+\frac{3}{n}}{\frac{5}{n^5}+1}\right) = 2 \neq 0$

Thus, by Comparison test $\sum u_n$ & $\sum v_n$ is

Convergent or divergent together.

Here $\sum v_n = \sum\limits_{n=1}^{\infty}\frac{1}{n^3}$ is Convergent ($p=3>1$

by p- series test)

$$\Rightarrow \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\left(\frac{2n^2+3n}{5+n^5}\right) \text{ is Convergent.}$$

#

Question-: Using Comparison test, discuss the Convergence of $\sum\limits_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{1}{n}\right)$

Soln-: Here,

$$u_n = \frac{1}{n}\sin\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}\left\{\frac{1}{n} - \frac{1}{3!}\frac{1}{n^3} + \frac{1}{5!}\frac{1}{n^5} + \cdots\right\}$$

$$= \frac{1}{n^2}\left\{1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4} + \cdots\right]$$

let $v_n = \frac{1}{n^2}$

$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left[1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4} - \cdots\right]$$

$$= 1 \neq 0 \text{ finite.}$$

Then by comparison test

$\sum u_n$ & $\sum v_n$ are converges or diverges together.

Here $\sum v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ $(p = 2 > 1)$ is convergent by p-series test

$\Rightarrow \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{1}{n}\right)$ is convergent by comparison test. #

Question-: Determine convergence or divergence of series $\sum_{n=1}^{\infty} \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$

Soln Here $u_n = \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$

$$u_n = \frac{n^{2/3}\left(2 - 1/n^2\right)^{1/3}}{n^{3/4}\left(3 + 2/n^2 + 5/n^3\right)^{1/4}}$$

$$u_n = \frac{1}{n^{3/4 - 2/3}}\left(\frac{\left(2 - 1/n^2\right)^{1/3}}{\left(3 + 2/n^2 + 5/n^3\right)^{1/4}}\right)$$

$$u_n = \frac{1}{n^{1/12}} \frac{\left(2-\frac{1}{n^2}\right)^{1/3}}{\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{1/4}}$$

Let $v_n = \frac{1}{n^{1/12}}$

$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{\left(2-\frac{1}{n^2}\right)^{1/3}}{\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{1/4}}\right)$$

$$= \frac{2^{1/3}}{3^{1/4}} \neq 0 \text{ finite.}$$

Then by Comparison test $\sum u_n$ & $\sum v_n$ Converges or diverges together.

Since $\sum v_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/12}}$ is divergent ($p = \frac{1}{12} < 1$ by p-series test.

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is divergent by Comparison test.

#

D'Alembert's Ratio Test :

If $\sum_{n=1}^{\infty} u_n$ is a series of positive term such that $\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = k$, then

(i) The series is Convergent if $k<1$

(ii) The series is divergent if $k>1$.

(iii) The test fail if $k=1$ (go and check by comparison test.

Question:- Prove that

$$1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\cdots$$

Converges and find its sum.

Solution:- We have given

$$\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = 1+\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots$$

Here, $u_n = \left(\frac{2}{3}\right)^{n-1}$;

$$u_{n+1} = \left(\frac{2}{3}\right)^{n}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left[\left(\frac{2}{3}\right)^n\left(\frac{3}{2}\right)^{n-1}\right]$$

$$= \left(\frac{2}{3}\right) < 1$$

Then By D'alembert's ratio test

$\sum u_n$ is Convergent.

The nth partial sum

$$S_n = 1+\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2+\cdots+\left(\frac{2}{3}\right)^{n-1}$$

$$S_n = \frac{1\left(1-\left(\frac{2}{3}\right)^n\right)}{\left(1-\frac{2}{3}\right)} = 3\left(1-\left(\frac{2}{3}\right)^n\right)$$

$$\Rightarrow \lim_{n\to\infty}(S_n) = 3$$

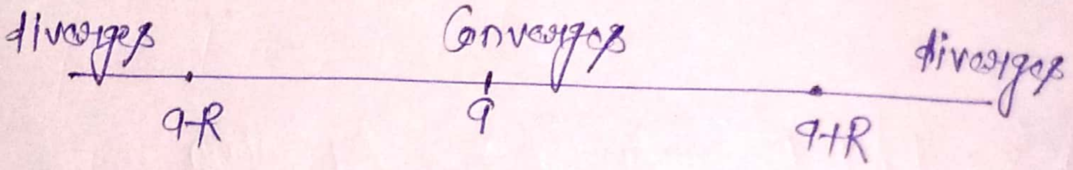
Thus $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$ is Convergent and its sum is 3 Ans

Radius of Convergence :

The radius of convergence is defined as

$$\frac{1}{R} = \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)$$

The series converges for all x in $(a-R, a+R)$ and diverges if $x < a-R$ or $x > a+R$



Converges for ~~[illegible]~~ $|x-a| < R$

diverges for $x > a-R$ or $x > a+R$.

Here, the interval is $(a-R, a+R)$

Radius of convergence for the series in R and a is the centre.

Radius of convergence is the radius of biggest circle in which series converges.

Question :- Find the radius of convergence for the series $\sum_{n=1}^{\infty} \frac{x^n}{n+2}$.

Soln Here, $u_n = \frac{x^n}{n+2}$

$u_{n+1} = \left(\frac{x^{n+1}}{n+3}\right)$

$$\therefore \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{x^{n+1}}{n+3}\cdot\frac{(n+2)}{x^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1+2/n}{1+3/n}\right)(x)$$

$$= x$$

$\therefore$ By ratio test, the series converges if $x<1$ and diverges if $x>1$.

Hence the radius of convergence R is 1 Ans

Question :- Find the value of x, for which

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$$

Converges or diverges.

Soln we have given

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}}\, x^{2(n-1)} = \frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \cdots$$

Here, $u_n = \dfrac{1}{(n+1)\sqrt{n}}\, x^{2(n-1)}$

$$u_{n+1} = \frac{1}{(n+2)\sqrt{n+1}}\, x^{2n}$$

Now $\dfrac{u_{n+1}}{u_n} = \dfrac{(n+1)\sqrt{n}}{(n+2)\sqrt{n+1}}\,(x^2)$

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$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)\sqrt{1+\frac{1}{n}}}\right) x^2$$

$$= x^2$$

By Ratio test

(i) If $x^2 < 1$, then $\sum u_n$ is convergent.

(ii) If $x^2 > 1$, then $\sum u_n$ is divergent.

(iii) If $x^2 = 1$, then D'alembert test fails.

Now for $x^2 = 1$

$$u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$$

let $v_n = \frac{1}{n^{3/2}}$.

$\therefore$ By Comparison test

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) = 1 \neq 0 \text{ (finite)}$$

$\Rightarrow \sum u_n$ and $\sum v_n$ are converges or diverges together.

$\because \sum v_n = \sum \frac{1}{n^{3/2}}$ $\left(p = \frac{3}{2} > 1\right)$ is converges by p-series test.

$\Rightarrow \sum u_n$ is convergent by Comparison test

Hence, the given series $\sum u_n$ converges for $x^2 \leq 1$ and divergent for $x^2 > 1$ Ans

Raabe's Test (Higher Ratio Test) -:

If $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, then

(i) The series is convergent if $l > 1$

(ii) The series is divergent if $l < 1$

(iii) The test fails if $l = 1$.

Question-: Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$

Solution-: Here,

$$u_n = \frac{(2n+1)}{(n+1)^2}$$

$$u_{n+1} = \frac{(2n+3)}{(n+2)^2}$$

Now

$$\frac{u_{n+1}}{u_n} = \frac{(2n+3)}{(n+2)^2}\left(\frac{(n+1)^2}{2n+1}\right)$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{(2+3/n)}{(1+2/n)^2}\,\frac{(1+1/n)^2}{(2+1/n)}\right)$$

$$= 1$$

$\Rightarrow$ D'Alembert's ratio test fails.

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Now

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+1)(n+2)^2}{(2n+3)(n+1)^2}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(2n+1)(n^2+4n+4)}{(2n+3)(n^2+2n+1)}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(2n^3+9n^2+12n+4)-(2n^3+7n^2+8n+3)}{2n^3+7n^2+8n+3}\right]$$

$$= \lim_{n\to\infty} \frac{n(2n^2+4n+1)}{2n^3+7n^2+8n+3} = \lim_{n\to\infty}\left(\frac{2n^3+4n^2+n}{2n^3+7n^2+8n+3}\right)$$

$$= \lim_{n\to\infty}\left[\frac{2+4/n+1/n^2}{2+7/n+8/n^2+3/n^3}\right] = 1$$

$\Rightarrow$ Raabe's test fails (Higher Ratio test).

Here $u_n = \frac{2n+1}{(n+1)^2} = \frac{n}{n^2}\left(\frac{2+1/n}{(1+1/n)^2}\right)$

$$u_n = \frac{1}{n}\frac{(2+1/n)}{(1+1/n)^2}$$

let $v_n = \frac{1}{n}$

$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{(2+1/n)}{(1+1/n)^2}\right) = 2 \neq 0 \text{ finite}$$

Then By Comparison test $\sum u_n$ & $\sum v_n$ Converges or diverges together.

$\because \sum u_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent $(p=1)$

(by p-series test)

$\Rightarrow \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ is divergent by Comparison test.

Question-: Test the Convergence for the series

$$\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \frac{x^3}{5\cdot 6} + \frac{x^4}{7\cdot 8} + \cdots$$

Soln-: we have given

$$\sum_{n=1}^{\infty} \frac{x^n}{(2n-1)(2n)} = \frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \frac{x^3}{5\cdot 6} + \frac{x^4}{7\cdot 8} + \cdots$$

Here, $u_n = \frac{x^n}{(2n-1)(2n)}$ & $u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$

$$\therefore \lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} \left[\frac{x^{n+1}}{(2n+1)(2n+2)} \frac{(2n-1)(2n)}{x^n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{2(2-1/n)}{(2+1/n)(2+2/n)}\right] x$$

$$= x$$

By D'alembert's test, $\sum u_n$ is Convergent if $x<1$ and $\sum u_n$ is divergent if $x>1$

(iii) If $x=1$, D'Alembert's test fails.

Now for $x=1$, $u_n = \frac{1}{(2n-1)(2n)}$

Let us apply Raabe's test when $x=1$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)}{2n(2n-1)}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{4n^2+4n+2n+2-(4n^2-2n)}{2n(2n-1)}\right]$$

$$= \lim_{n\to\infty} \frac{n(8n+2)}{2n(2n-1)} = \lim_{n\to\infty} \frac{(8+2/n)}{2(2-1/n)}$$

$$= 2 > 1$$

Thus, by Raabe's test $\sum u_n$ is convergent when $x=1$.

Hence, the given series is convergent if $x \leq 1$ and divergent if $x > 1$. Ans

Question:- Test the convergence of the series

$$\frac{1\cdot 2}{3^2\cdot 4^2} + \frac{3\cdot 4}{5^2\cdot 6^2} + \frac{5\cdot 6}{7^2\cdot 8^2} + \cdots$$

Solution:- We have given

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2(n+3)^2} = \frac{1\cdot 2}{3^2\cdot 4^2} + \frac{3\cdot 4}{5^2\cdot 6^2} + \frac{5\cdot 6}{7^2\cdot 8^2} + \cdots$$

Here, $u_n = \frac{n(n+1)}{(n+2)^2(n+3)^2}$

$$u_{n+1} = \frac{(n+1)(n+2)}{(n+3)^2(n+4)^2}$$

By D'Alembert test

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left[\frac{\cancel{(n+1)}(n+2)}{\cancel{(n+3)^2}(n+4)^2}\cdot\frac{(n+2)^2\cancel{(n+3)^2}}{n\cancel{(n+1)}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(n+2)^3}{n(n+4)^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(1+2/n)^3}{(1+4/n)^2}\right] = 1$$

Thus, D'Alembert's ratio test fails.

By Raabe's test

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\frac{n(n+4)^2}{(n+2)^3} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(\cancel{n^3}+8n^2+16n) - (\cancel{n^3}+6n^2+12n+8)}{n^3+6n^2+12n+8}\right]$$

$$= \lim_{n\to\infty}\frac{n(2n^2+4n-8)}{n^3+6n^2+12n+8}$$

$$= \lim_{n\to\infty}\left(\frac{2+4/n-8/n^2}{1+6/n+12/n^2+8/n^3}\right) = 2 > 1$$

Hence, By Raabe's test $\sum u_n$ is Convergent.

Ans

Gauss's Test :

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms such that

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma}{n^2}, \quad \text{where } \alpha > 0$$

(i) If $\alpha > 1$, $\sum u_n$ is Convergent

If $\alpha < 1$, $\sum u_n$ is divergent whatever β may be.

(ii) If $\alpha = 1$ and $\begin{cases} \beta > 1, & \text{Convergent} \\ \beta \leq 1, & \text{divergent.} \end{cases}$

Question : Test the Convergence of the series

$$2 + \frac{3}{2}x + \frac{4}{3}x^2 + \frac{5}{4}x^3 + \cdots$$

Sol^n : We have given

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right) x^{n-1} = \frac{2}{1}x^0 + \frac{3}{2}x + \frac{4}{3}x^2 + \frac{5}{4}x^3 + \cdots$$

Here, $u_n = \left(\frac{n+1}{n}\right) x^{n-1}$

$$u_{n+1} = \left(\frac{n+2}{n+1}\right) x^n$$

By D'Alembert's test

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left[\frac{n+2}{n+1}x^n \frac{n}{(n+1)\, x^{n-1}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(1+2/n)}{(1+1/n)^2}\right] x$$

$$= x$$

(i) If $x<1$, then $\sum u_n$ is convergent.

(ii) If $x>1$, then $\sum u_n$ is divergent.

(iii) If $x=1$, D'Alembert's Ratio test fails.

Now we apply Raabe's Test. for $x=1$

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\left(\frac{n+1}{n}\right)\times\left(\frac{n+1}{n+2}\right) - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(\cancel{n^2}+\cancel{2n}+1)-(\cancel{n^2}+\cancel{2n})}{n(n+2)}\right]$$

$$= \lim_{n\to\infty}\left(\frac{n}{n^2+2n}\right) = \lim_{n\to\infty}\left[\frac{1}{n+2}\right]$$

$$= 0 < 1$$

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is divergent for $x = 1$, by Raabe's test

Hence $\sum_{n=1}^{\infty} u_n$ is convergent for $x<1$ and

divergent for $x \geq 1$

Ans

Now let us apply Gauss test

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma}{n^2}$$

Here, $\dfrac{(n+1)^2}{n(n+2)} = \alpha + \dfrac{\beta}{n} + \dfrac{\gamma}{n^2}$

$$n^2+2n \overline{\big)\ (n^2+2n+1)\ \big(} \ 1+\frac{1}{n^2}$$

$$\frac{-\ \cancel{n^2+2n}}{\quad}$$

$$+1$$

$$\frac{-1 + \frac{2}{n}}{-\frac{2}{n}}$$

$$\frac{n^2+2n+1}{n^2+2n} = 1 + \frac{1}{n(n+2)}$$

$$= 1 + \frac{1}{n^2}\left[1+\frac{2}{n}\right]^{-1}$$

$$= 1 + \frac{1}{n^2}\left[1 - \frac{2}{n} + \frac{4}{n^2} + - \right]$$

$$\alpha + \frac{\beta}{n} + \frac{\gamma}{n^2} = 1 + \frac{1}{n^2} - \frac{2}{n^3} + - - -$$

Here, $\alpha = 1$, $\beta = 0$, $\gamma = 1$

Then by Gauss test $\sum u_n$ is divergent.

Cauchy's integral test -:

A positive term series $f(1)+f(2)+f(3)+ \dots + f(n) + \dots$, where $f(n)$ decreases as n increases,

(i) $\sum_{n=1}^{\infty} f(n)$ is Convergent if

$$\lim_{n\to\infty} \int_1^n f(n)\,dn \text{ finite}$$

(ii) $\sum_{n=1}^{\infty} f(n)$ is divergent if

$$\lim_{n\to\infty} \int_1^n f(n)\,dn \text{ is not finite}$$

(either tends to ∞ or $-\infty$).

Question -: Test the convergence of $\sum_{n=1}^{\infty} \frac{2\tan^{-1}(n)}{1+n^2}$

By Cauchy test

$$\lim_{n\to\infty} \int_1^n f(n)\,dn = \lim_{n\to\infty} \int_1^n \frac{2\tan^{-1}(n)}{(1+n^2)}\,dn$$

$$= \lim_{n\to\infty} \left[\left(\tan^{-1}(n)\right)^2 \right]_1^n$$

$$= \lim_{n\to\infty} \left[\left(\tan^{-1}(n)\right)^2 - \left(\frac{\pi}{4}\right)^2 \right]$$

$$= \left(\tan^{-1}(\infty)\right)^2 - \left(\frac{\pi}{4}\right)^2 = \left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2$$

$$= \frac{3}{16}\pi^2 = \text{finite}.$$

Hence by cauchy integral test, $\sum u_n$ is convergent.

Question :- Examine the convergence of series

$$\sum_{n=2}^{\infty} \frac{1}{n\log n}$$

Here $f(x) = \frac{1}{x\log x}$

By cauchy's integral test

$$\lim_{n\to\infty} \int_2^n f(x)dx = \lim_{n\to\infty} \int_2^n \frac{1}{x\log x} dx$$

$$= \lim_{n\to\infty} \left[\log(\log x)\right]_2^n$$

$$= \lim_{n\to\infty} \left[\log(\log n) - \log(\log 2)\right]$$

$$= \text{not finite}$$

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n\log n}$ is divergent.

In General $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if $p > 1$

divergent if $p \leq 1$.

Question :- Test the Convergence of

$$\sum_{n=3}^{\infty} \frac{1}{n\log n \sqrt{(\log n)^2 - 1}}$$

Here

$$f(n) = \frac{1}{n\log n \sqrt{(\log n)^2 - 1}}$$

By cauchy's integral test

$$\lim_{n\to\infty} \int_3^n f(n)\,dn = \lim_{n\to\infty} \int_3^n \frac{1}{n\log n \sqrt{(\log n)^2 - 1}}\,dn$$

put $\log n = t$; $\frac{1}{n}dn = dt$

$$= \cancel{\lim_{n\to\infty} \int_{\log 3}^{\log n}}$$

$$= \int_{\log 3}^{\infty} \frac{dt}{t\sqrt{t^2-1}}$$

put $t = 1/z$

$dt = -1/z^2\, dz$

$$= \int_{1/\log(3)}^{0} \frac{-1/z^2\, dz}{1/z \sqrt{1/z^2 - 1}}$$

$$= \int_0^{1/\log 3} \frac{dz}{\sqrt{1-z^2}}$$

$$= \left[\sin^{-1}(z)\right]_0^{1/\log 3}$$

$$= \sin^{-1}\left(\frac{1}{\log 3}\right) - \sin^{-1}(0) = \sin^{-1}\left(\frac{1}{\log 3}\right)$$

= a finite quantity.

Hence by Cauchy's integral test, the given series is convergent **Ans**

Note that -: If $\{a_n\}$ and $\{b_n\}$ are two sequences such that $\lim_{n\to\infty}(a_n) = a$ & $\lim_{n\to\infty}(b_n) = b$, then

(i) $\lim_{n\to\infty}(a_n \pm b_n) = a \pm b$

(ii) $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty}(a_n) \lim_{n\to\infty}(b_n) = ab$.

(iii) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{a}{b}$, if $b \neq 0$, $b_n \neq 0 \;\forall n$.

Sandwitch theorem -: If $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are three sequences such that

(i) $a_n \leq b_n \leq c_n \quad \forall n$

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(c_n) = l$

$\Rightarrow \lim_{n\to\infty}(b_n) = l$.

Question-: Show that the sequence, $\{b_n\}$

$$b_n = \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} \cdots + \frac{1}{(n+n)^2}\right]$$

$$b_n < \frac{1}{(n+1)^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+1)^2}$$

$$< \frac{n}{(n+1)^2} =$$

and $$b_n > \frac{1}{(n+n)^2} + \frac{1}{(n+n)^2} + \cdots + \frac{1}{(n+n)^2}$$

$$b_n > \frac{n}{(n+n)^2} = \frac{1}{4n}$$

$$\Rightarrow a_n = \frac{1}{4n} < b_n < \frac{n}{(n+1)^2} = c_n$$

Here, $\lim_{n\to\infty}(a_n) = 0$ & $\lim_{n\to\infty}(c_n) = 0$

Then By sandwitch theorem

$$\boxed{\lim_{n\to\infty}(b_n) = 0}$$