Indirect Search Methods

Gradient of a function

The gradient of a function in an n-~~dimensional~~ component vector is given by

$$\nabla f_{n\times 1} = \begin{Bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \vdots \\ \partial f/\partial x_n \end{Bmatrix} \quad \ldots (6.56)$$

Steepest Descent (Cauchy) Method → the use of the negative of the gradient vector as a direction for minimization was first made by Cauchy in 1847. In this method, we start from an initial trial point X_1 and iteratively move along the steepest descent directions until the optimum point is found. This method is summarized as

1. Start with an arbitrary initial point X_1.
2. Find the search direction S_i as

$$S_i = -\nabla f_i = -\nabla f(X_i)$$

3. Determine the optimal step length λ_i^* in the direction S_i and set

$$X_{i+1} = X_i + \lambda_i^* S_i = X_i - \lambda_i^* \nabla f_i \quad (6.70)$$

$\approx \nabla f(X_{i+1})$

4. Test the new point, X_{i+1} for optimality. If X_{i+1} is optimum, stop the process, otherwise, go to Step 5.

⑤. Set new iteration number $i = i+1$ and go to Step 2.

Q. Minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ starting from the point $X_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Ans Iteration 1.

the gradient of f is given by

$$\nabla f = \begin{Bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{Bmatrix} = \begin{Bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{Bmatrix}$$

$$\Rightarrow \nabla f_1 = \begin{Bmatrix} 1 + 4 \cdot 0 + 2 \cdot 0 \\ -1 + 2 \cdot 0 + 2 \cdot 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

therefore

$$S_1 = -\nabla f_1 = -\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

To find X_2, we need to find the optimum step length λ_1^*. for this, we minimize

we minimize $f(X_1 + \lambda_1 S_1) = f\begin{pmatrix} -\lambda_1 \\ \lambda_1 \end{pmatrix}$

$$= -\lambda_1 - \lambda_1 + \cancel{2\lambda_1^2} - \cancel{2\lambda_1^2} + \lambda_1^2$$

$$= \lambda_1^2 - 2\lambda_1$$

now $\frac{\partial f}{\partial \lambda_1} = 2\lambda_1 - 2 = 0 \Rightarrow \lambda_1^* = 1$

we obtain

$$x_2 = x_1 + \lambda_1^* S_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + 1\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

as $\nabla f_2 = \nabla f(x_2) = \begin{Bmatrix} 1-4+2 \\ -1-2+2 \end{Bmatrix} = \begin{Bmatrix} -1 \\ -1 \end{Bmatrix} \neq \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, x_2 is not optimum.

Iteration 2.

$$S_2 = -\nabla f_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$f(x_2 + \lambda S_2) = f\left(\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} + \lambda_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right) = f(-1+\lambda_2, 1+\lambda_2)$$

$$= 5\lambda_2^2 - 2\lambda_2 - 1$$

$$\frac{df}{d\lambda_2} = 10\lambda_2 - 2 = 0 \Rightarrow \lambda_2^* = 1/5$$

$$\Rightarrow x_3 = x_2 + \lambda_2^* S_2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} + \frac{1}{5}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} -1+.2 \\ 1+.2 \end{Bmatrix} = \begin{Bmatrix} -0.80 \\ 1.20 \end{Bmatrix} \Rightarrow x_3 = \begin{Bmatrix} -0.80 \\ 1.20 \end{Bmatrix}$$

now, $\nabla f_3 = \nabla f(x_3) = \begin{Bmatrix} 1-3.20+2.4 \\ -1-1.6+2.40 \end{Bmatrix} = \begin{Bmatrix} 0.2 \\ -0.2 \end{Bmatrix}$

similarly we calculate x_4, and so on.

Newton's Method:-

consider the quadratic approximation of the function $f(X)$ at $X = X_i$ using the Taylor's series expansion

$$f(X) = f(X_i) + \nabla f_i^T (X - X_i) + \frac{1}{2}(X - X_i)^T [J_i](X - X_i)$$

(6.95)

where $[J_i] = [J]_{X_i}$ is the matrix of second partial derivative (Hessian Matrix) at a point X_i

By set the partial derivatives of Equation (6.95) equal to zero for minimum of $f(X)$, we obtain

$$\frac{\partial f(X)}{\partial x_j} = 0, \quad j = 1, 2, \ldots n \quad \text{--} (6.96)$$

Equation (6.96) and (6.95) give

$$\nabla f = \nabla f_i + [J_i](X - X_i) = 0 \quad \text{--} (6.97)$$

If $[J_i]$ is non-singular, equation (6.97) can be solved to obtain an improved approximation ($X = X_{i+1}$) as

$$X_{i+1} = X_i - [J_i]^{-1} \nabla f_i \quad \text{--} (6.98)$$

The sequence of points $X_1, X_2, \ldots, X_{i+1}$ can be shown to converge to the actual solution X^* from any initial point X_1 sufficiently close to the solution X^*, provided that $[J_1]$ is non singular

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Q. Minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ by taking the starting point as $\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = X_1$

A. To find X_2, we require $[J_1]^{-1}$

$$[J_1] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$[J_1]^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

$$g_1 = \nabla f_1 = \begin{Bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{Bmatrix}_{X_1} = \begin{Bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{Bmatrix}_{\{0,0\}}$$

$$= \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Now

$$X_2 = X_1 - [J_1]^{-1} \nabla f_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} \frac{1}{2} + \frac{1}{2} \\ -\frac{1}{2} - 1 \end{Bmatrix} = \begin{Bmatrix} -1 \\ +3/2 \end{Bmatrix}$$

To see X_2 is the optimum point, we evaluate

$$g_2 = \nabla f_2 = \begin{Bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{Bmatrix}_{\{-1, 3/2\}} = \begin{Bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 - 4 + 3 \\ -1 - 2 + 3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

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as $g = 0$, X_2 is the optimum point.

Q. Minimize $f(x_1, x_2) = 100(x_1^2 - x_2)^2 + (1 - x_1)^2$ taking $X_1 = \begin{Bmatrix} -2 \\ -2 \end{Bmatrix}$

$$\nabla f = \begin{Bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{Bmatrix} = \begin{Bmatrix} 200(x_1^2 - x_2) \cdot 2x_1 + 2(1 - x_1) \cdot (-1) \\ 200(x_1^2 - x_2)(-1) \end{Bmatrix}_{(-2,-2)}$$

$$= \begin{Bmatrix} 400(4+2) \cdot (-2) + 2(1+2)(-1) \\ 200(4+2)(-1) \end{Bmatrix} = \begin{Bmatrix} -4806 \\ -1200 \end{Bmatrix}$$

Q2. $f = x_1^2 + x_2^2 - 2x_1 - 4x_2 + 5$, $X_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$ by Newton's Method.

Univariate Method

Univariate method can be summarized as follows

1) choose as arbitrary starting point X_1 and set $i = 1$.
2) Find the search direction S_i as

$$S_i^T = \begin{cases} (1, 0, 0 \ldots 0) & \text{for } i = 1, n+1, 2n+1 \ldots \\ (0, 1, \ldots 0) & \text{for } i = 2, n+2, 2n+2, \ldots \\ (0, 0, 1, \ldots 0) & \text{for } i = 3, n+3, 2n+3, \ldots \\ \vdots & \\ (0, 0, 0 \ldots 1) & \text{for } i = n, 2n, 3n, \ldots \end{cases}$$

3) Determine whether d_i should be +ve or negative. For the ~~curre~~ current direction S_i, this means find whether the function value decreases in the +ve or -ve direction. For this, we take a small Probe length (ε) and evaluate $f_i = f(X_i)$, $f^+ = f(X_i + \varepsilon S_i)$ and $f^{-1} = f(X_i - \varepsilon S_i)$. If $f^+ < f_i$, S_i will be the correct direction for decreasing the value of f and if $f^- < f_i$, $-S_i$ will be the correct one. If both f^+ and f^- are greater than f_i we take X_i as the minimum along the direction S_i.

4) Find the optimal step length d_i^* such that

$$f(X_i \pm d_i^* S_i) = \min_{d_i}(X_i \pm d_i S_i)$$

where + or − sign has to be used depending upon whether S_i or $-S_i$ is the direction for decreasing the function value.

⑤ Set $X_{i+1} = X_i \pm d_i^* S_i$ depending on the direction for decreasing the function value, and

$$f_{i+1} = f(X_{i+1}).$$

⑥ Set the new value of $i = i+1$ and go to Step 2. Continue this procedure until no significant change is achieved in the value of the objective function.

Q. Minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ with the starting point $(0,0)$

→ Let Probe length (ε) as 0.01.

iteration $i = 1$

Step 2. Choose the search direction S_1 as $S_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$

Step 3. To find whether the value of f decreases along S_1 or $-S_1$. We use the probe length ε. Since

$$f_1 = f(X_1) = f(0,0) = 0, \quad X_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$f^+ = f(X_1 + \varepsilon S_1) = f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \varepsilon\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = f(\varepsilon, 0)$$

$= 0.01 - 0 + 2(0.0001) + 0 + 0 = 0.0102 > f_1$

bcz $f^- = f(X_1 - \varepsilon S_1) = f(-\varepsilon, 0) = -0.01 - 0 + 2(0.0001)$

lali $+ 0 + 0 = -0.9998 < f_1$

$-S_1$ is the correct direction for minimizing f from X_1.

To find the optimum step length d_1^*, we minimize

$f(X_1 - d_1 S_1) = f\left\{\begin{pmatrix}0\\0\end{pmatrix} - d_1\begin{pmatrix}1\\0\end{pmatrix}\right\} = \cancel{f\{(0, -d_1)\}}$

$= f(-d_1, 0)$

$= (-d_1) - 0 + 2(-d_1)^2 + 0 + 0 = -d_1 + 2d_1^2$

as

$\frac{\partial f}{\partial d_1} = -1 + 4d_1 = 0 \Rightarrow d_1 = 1/4$ we have $d_1^* = 1/4$

Set $X_2 = X_1 - d_1^* S_1 = \begin{Bmatrix}0\\0\end{Bmatrix} - \frac{1}{4}\begin{Bmatrix}1\\0\end{Bmatrix} = \begin{Bmatrix}-1/4\\0\end{Bmatrix}$

$f_2 = f(X_2) = f(-1/4, 0) = -1/8$

iteration $i = 2$, choose the search direction S_2 as $S_2 = \begin{Bmatrix}0\\1\end{Bmatrix}$

$f_2 = f(X_2) = -\frac{1}{8} = -.125$

$f^+ = f(X_2 + \varepsilon S_2) = f\left(\begin{Bmatrix}-\frac{1}{4}\\0\end{Bmatrix} + 0.01\begin{Bmatrix}0\\1\end{Bmatrix}\right)$

$= f(-.25, 0.01) = -0.1399 < f_2$

$f^- = f(X_2 - \varepsilon S_2) = f(-0.25, -0.01) = -0.1099 > f_2$

$\Rightarrow S_2$ is the correct direction for decreasing the value of f from X_2.

we minimize $f(X_2+d_2S_2)$ to find d_2^*
here

$$f(X_2+d_2S_2) = f\left(\begin{pmatrix}-0.25\\0\end{pmatrix} + d_2\begin{pmatrix}0\\1\end{pmatrix}\right)$$
$$= f(-0.25, d_2) = d_2^2 - 1.5d_2 - 0.125$$
$$\frac{\partial f}{\partial d_2} = 2d_2 - 1.5 = 0 \Rightarrow d_2^* = \frac{1.5}{2} = 0.75$$

Set

$$X_3 = X_2 + d_2^* S_2 = \begin{Bmatrix}-0.25\\0\end{Bmatrix} + 0.75\begin{Bmatrix}0\\1\end{Bmatrix}$$
$$= \begin{Bmatrix}-0.25\\0.75\end{Bmatrix}$$
$$f(X_3) = -0.6875$$

Q. Minimize $= x_1 - x_2 + x_3 + 2x_1^2 + 2x_2^2 - x_3^2 + 2x_1x_3 + 4x_2x_3 - 6x_1x_2$

$X_1 = \begin{pmatrix}1\\2\\1\end{pmatrix}$, $S_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\varepsilon = 0.01$

$f_1 = f(X_1) = 7$

$f_1^+ = 6.9502 < f_1$

$f_1^- = 7.0502$

$\Rightarrow S_1$ is the correct direction

now

$$f(X_1 + d_1S_1) = f\begin{pmatrix}1+d_1\\2\\1\end{pmatrix} = 7 - 5d_1 + 2d_1^2$$
$$\frac{\partial f(X_1+d_1S_1)}{\partial d_1} = -5 + 4d_1 \Rightarrow d_1 = 5/4$$
$$\Rightarrow X_2 = \begin{pmatrix}9/4\\2\\1\end{pmatrix}$$
$$f(X_2) = f_2 = \frac{107}{25} = 4.28$$
$$f_2^+ = 4.2822$$
$$f_2^- = 4.2782 < f(X_2) = f_2$$

$-S_2$ is the correct direction

$$\Rightarrow X_3 = X_2 - d_2S_2 = \begin{pmatrix}9/5\\2-d_2\\1\end{pmatrix}$$

Geometric Programming.

Posynomial - the objective function $f(x)$ is given by the sum of several component costs $U_i(x)$ as

$$f(x) = U_1 + U_2 + \cdots + U_N$$

In many cases, the component cost U_i can be expressed as power functions of the type

$$U_i = c_i x_1^{a_{1i}} x_2^{a_{2i}} \cdots x_n^{a_{ni}}$$, where

the coefficients c_i are +ve constants, the co. exponent a_{ij} are real constants (+ve, zero, -ve) and the variables $x_1, x_2 \ldots x_n$ are taken to be +ve. function f because of +ve coefficients and variables and real exponents are called posynomials. For example

$$f(x_1, x_2, x_3) = 6 + 3x_1 - 8x_2 + 7x_3 + 2x_1x_3 - 3x_1x_3 + \frac{4}{3}x_2x_3 + \frac{8}{7}x_1^2 - 9x_2^2 + x_3^2$$ is a second-degree polynomial in variables of x_1, x_2, x_3

while

$$g(x_1, x_2, x_3) = x_1x_2x_3 + x_1^2x_2 + 4x_3 + \frac{2}{x_1x_2} + 5x_3^{-1/2}$$

is a posynomial

Unconstrained minimization Problems

Unconstrained minimizing problem

Find $X = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$ that minimizes the objective function

$$f(X) = \sum_{j=1}^{N} U_j(X) = \sum_{j=1}^{N} \left(c_j x_1^{a_{1j}} x_2^{a_{2j}} \ldots x_n^{a_{nj}} \right) \quad \ldots (8.3)$$

where $c_j \geq 0$, $x_j > 0$ and a_{ij} is real const.

Solution of an unconstraint Geometric Problem Using Differential calculus.

as minimizing objective function is

$$f(X) = \sum_{j=1}^{N} U_j(X) = \sum_{j=1}^{N} \left(c_j x_1^{a_{1j}} x_2^{a_{2j}} x_3^{a_{3j}} \ldots x_n^{a_{nj}} \right)$$

For maxima or minima

$$\frac{\partial f(x)}{\partial x_K} = \sum_{j=1}^{N} \frac{\partial U_j}{\partial x_K} = 0$$

$K = 1 \ldots n$, means n variables.

if we multiples

$$x_K \frac{\partial f}{\partial x_K} = \sum_{j=1}^{N} a_{Kj} U_j(X) = 0, \quad K = 1 \text{ to } n \quad \ldots (8.5)$$

To find the minimizing vector $X^* = \begin{Bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{Bmatrix}$

we have

$$\sum_{j=1}^{N} a_{Kj} U_j(X^*) = 0, \quad K = 1, 2 \ldots n \quad (8.6)$$

now, divide eq (8.6) by f^* (minimum value of f)

$$\sum_{j=1}^{N} a_{Kj} \frac{U_j(X^*)}{f^*} = \sum_{j=1}^{N} a_{Kj} \Delta_j^* \quad - (8.7)$$

where $\Delta_j^* = \dfrac{U_j^*}{f^*}$

this relation (8.7) is orthogonality condition

$\boxed{\cancel{\Delta_j^* = \frac{U_j}{f}}}$

as

$$f(X) = \sum_{j=1}^{N} U_j(X)$$

$$\Rightarrow \sum_{j=1}^{N} \frac{U_j(X)}{f} = 1 \Rightarrow \sum_{j=1}^{N} \Delta_j^* = 1$$

$$\Rightarrow \Delta_1 + \Delta_2 + \ldots + \Delta_N = 1 \quad (8.8)$$

this condition (8.8) is called the normality condition. now,

$$f^* = \left(\frac{U_1^*}{\Delta_1^*}\right)^{\Delta_1^*} \left(\frac{U_2^*}{\Delta_2^*}\right)^{\Delta_2^*} \ldots \left(\frac{U_N^*}{\Delta_N^*}\right)^{\Delta_N^*} \quad \ldots (8.12)$$

or relation (8.12) can be written as

$$f^* = \left(\frac{c_1}{\Delta_1^*}\right)^{\Delta_1^*}\left(\frac{c_2}{\Delta_2^*}\right)^{\Delta_2^*}\left(\frac{c_3}{\Delta_3^*}\right)^{\Delta_3^*}\cdots\left(\frac{c_N}{\Delta_N^*}\right)^{\Delta_N^*}$$

n = no. of variables, N = no. of terms in the objective function

if $N = n+1$, there will be as many linear simultaneous equations as there are unknowns and we can find a unique solution.

if $N - n - 1 = 0$, the Problem is said to have a zero degree of difficulty. If $N > n+1 \Rightarrow$ we have more no. of variables than the equations, then sometimes this method is not applicable.

unknown Δ_j^* can be determined uniquely from the orthogonality and normality conditions

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Solve the Problem

$$f(X) = 80x_1x_2 + 40x_2x_3 + 20x_1x_3 + \frac{80}{x_1x_2x_3}$$ solve the Problem, it is a general Posynomial.

$c_1 = 80$, $c_2 = 40$, $c_3 = 20$, $c_4 = 80$

compare with

$$f(X) = \sum_{j=1}^{N} (c_j x_1^{a_{1j}} x_2^{a_{2j}} \cdots x_n^{a_{nj}})$$

$$= c_1 x_1^{a_{11}} x_2^{a_{21}} x_3^{a_{31}} + c_2 x_1^{a_{12}} x_2^{a_{22}} x_3^{a_{32}}$$

$$+ c_3 x_1^{a_{13}} x_2^{a_{23}} x_3^{a_{33}} + c_4 x_1^{a_{14}} x_2^{a_{24}} x_3^{a_{34}}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

~~the orthogonal condition~~

the orthogonality and normality conditions are given by

$$\Bigg[\; 1 \quad 1 \; 0$$

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$$\sum_{j=1}^{N} \Delta_j^{*} a_{kj} = 0 \qquad k = 1 \ldots n \quad (1.7)$$

$$\Delta_1 a_{k1} + \Delta_2 a_{k2} + \Delta_3 a_{k3} + \Delta_4 a_{k4} = 0$$

$$\Delta_1 a_{11} + \Delta_2 a_{12} + \Delta_3 a_{13} + \Delta_4 a_{14} = 0$$

$$\Delta_1 a_{21} + \Delta_2 a_{22} + \Delta_3 a_{23} + \Delta_4 a_{24} = 0$$

$$\Delta_1 a_{31} + \Delta_2 a_{32} + \Delta_3 a_{33} + \Delta_4 a_{34} = 0$$

(1.7) orthogonal conditions

(1.7) $\sum_{j=1}^{N} \Delta_j = 1$ is orthogonal condition.

$$\Rightarrow \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 1. \quad \text{— (a)}$$

$$\Delta_1 + 0 + \Delta_3 - \Delta_4 = 0$$

$$\Delta_1 + \Delta_2 + 0 - \Delta_4 = 0$$

$$0 + \Delta_2 + \Delta_3 - \Delta_4 = 0$$

Solving these eq. we get

$$\Delta_4 = \Delta_1 + \Delta_3 = \Delta_1 + \Delta_2 \Rightarrow \Delta_2 = \Delta_3$$

and $\Delta_4 = \Delta_1 + \Delta_3 = \Delta_2 + \Delta_3 \Rightarrow \Delta_1 = \Delta_2$

$$\Rightarrow \Delta_1 = \Delta_2 = \Delta_3 \Rightarrow \Delta_4 = 2\Delta_1$$

from eq (a), $\cancel{2\Delta_1 + \Delta_1 + \Delta_1}$ $\Delta_1 + \Delta_1 + \Delta_1 + 2\Delta_1 = 1$

$$\Rightarrow \Delta_1 = \frac{1}{5} \Rightarrow \Delta_1^{*} = \Delta_2^{*} = \Delta_3^{*} = \frac{1}{5}, \ \Delta_4^{*} = 2/5$$

So, optimal value of the objective function is

$$f^{*} = \prod_{j=1}^{N} \left(\frac{c_j}{\Delta_j^{*}}\right)^{\Delta_j^{*}} = \left(\frac{80}{1/5}\right)^{1/5} \left(\frac{40}{1/5}\right)^{1/5} \left(\frac{20}{1/5}\right)^{1/5} \left(\frac{80}{2/5}\right)^{2/5}$$

$$= (400)^{1/5} \times (200)^{1/5} \times (100)^{1/5} \times (200)^{2/5}$$

$$= (400 \times 200 \times 100 \times 200 \times 200)^{1/5}$$

$$= (32 \times 10^{10})^{1/5} = 200$$

now to find $x_1, x_2, x_3 \ldots$

we know

$$U_j^{*} = \Delta_j^{*} f^{*}$$

$$\Rightarrow U_1^{*} = 80 x_1^{*} x_2^{*} = \Delta_1^{*} f^{*} = \frac{1}{5} \times 200 = 40$$

$$U_2^{*} = 40 x_2^{*} x_3^{*} = \Delta_2^{*} f^{*} = \frac{1}{5} \times 200 = 40$$

$$U_3^{*} = 20 x_1^{*} x_3^{*} = \Delta_3^{*} f^{*} = \frac{1}{5} \times 200 = 40$$

$$U_4^{*} = \frac{80}{x_1^{*} x_2^{*} x_3^{*}} = \Delta_4^{*} f^{*} = \frac{2}{5} \times 200 = 80$$

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Solving these eqn is

$x_2^0 = \frac{1}{2x_1^0} \Rightarrow \frac{1}{x_2^0} = 2x_1$

$x_3^0 = \frac{1}{x_2^0} = 2x_1^0$

$x_3^0 = \frac{2}{x_1^0}$

$\Rightarrow \frac{2}{x_1^0} = \frac{1}{x_2^0} = 2x_1 \Rightarrow {x_1^0}^2 = 1 \Rightarrow x_1^0 = 1.$

$x_2^0 = \frac{1}{2}, \; x_3^0 = 2$

Solution of an unconstrained Geometric Problem using Arithmetic - Geometric inequality

Geometrical Primal Problem (Unconstraint)

Find $X = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$ so that $\text{Min } f(X) = \sum_{j=1}^{N} c_j x_1^{a_{1j}} x_2^{a_{2j}} \cdots x_n^{a_{nj}}$

$x_1 > 0, x_2 > 0 \ldots x_n \geq 0$

then Geometric dual of Primal Problem is

Find $\Delta = \begin{Bmatrix} \Delta_1 \\ \vdots \\ \Delta_N \end{Bmatrix}$ so that

$\text{Max } v(\Delta) = \prod_{j=1}^{N} \left(\frac{c_j}{\Delta_j} \right)^{\Delta_j}$ or

$\log \{ \text{Max } v(\Delta) \} = \log \left[\prod_{j=1}^{N} \left(\frac{c_j}{\Delta_j} \right)^{\Delta_j} \right]$

subject to the constraints

$\sum_{j=1}^{N} \Delta_j = 1$

$\sum_{j=1}^{N} a_{ij} \Delta_j = 0, \; i = 1, 2, \ldots n$

dual and Primal of Geometric Problem

Find $X = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$

so that

$g_0(X) = f(X) \to$ Minimum

subject to constraints

$g_1(X) \leq 1$

$g_2(X) \leq 1$

$\vdots$

$g_m(X) \leq 1$

with

$$g_0(X) = \sum_{j=1}^{N_0} c_{0j} \, x_1^{a_{01j}} x_2^{a_{02j}} \cdots x_n^{a_{0nj}}$$

$$g_1(X) = \sum_{j=1}^{N_1} c_{1j} \, x_1^{a_{11j}} x_2^{a_{12j}} \cdots x_n^{a_{1nj}}$$

$$g_2(X) = \sum_{j=1}^{N_2} c_{2j} \, x_1^{a_{21j}} x_2^{a_{22j}} \cdots x_n^{a_{2nj}}$$

$\vdots$

$$g_m(X) = \sum_{j=1}^{N_m} c_{mj} \, x_1^{a_{m1j}} x_2^{a_{m2j}} \cdots x_n^{a_{mnj}}$$

dual Problem / Geometric dual.

Find $d = \begin{Bmatrix} d_{01} \\ d_{02} \\ \vdots \\ d_{0N_0} \\ \hline d_{11} \\ d_{12} \\ \vdots \\ d_{1N_1} \\ \vdots \\ \hline d_{m1} \\ d_{m2} \\ \vdots \\ d_{mN_m} \end{Bmatrix}$

so that

$$v(d) = \prod_{k=0}^{m} \prod_{j=1}^{N_k} \left(\frac{c_{kj}}{d_{kj}} \sum_{l=1}^{N_k} d_{kl} \right)^{d_{kj}} \Rightarrow \text{maximum}$$

subject to the constraints

$d_{01} \geq 0, \; d_{02} \geq 0 \cdots d_{0N_0} \geq 0, \; d_{11} \geq 0, \; d_{12} \geq 0$

$d_{1N_1} \geq 0 \cdots d_{m1} \geq 0, \; d_{m2} \geq 0 \cdots d_{mN_m} \geq 0$

$$\sum_{j=1}^{N_0} d_{0j} = 1$$

$$\sum_{k=0}^{m} \sum_{j=1}^{N_k} a_{kij} \, d_{kj} = 0, \quad i = 1, 2, \cdots n$$

(c_{kj} are +ve and a_{kij} are real numbers)

here $k = 1$ to m (no. of constraints)

n = total no of variables

$g_0 = f$ = Primal function

m = no. of Primal constraints

$N = N_0 + N_1 + \cdots + N_m$ = total number of terms in the posynomials

$N - n - 1$ = degree of difficulty of Problem

v = dual function

$d_{01}, d_{02} \cdots d_{mN_m}$ = dual var

is the normality constraint

are the orthogonality constraints

$d_{kj} \geq 0, \; j = 1, 2, N_k$

$k = 0$ to m

$N = N_0 + N_1 + \cdots + N_m$

number of dual variables

$n+1$ number of dual constraints.

Solve the Problem

$$f(D, Q) = 100 D^1 Q^0 + 50 D^2 Q^0 + 20 D^0 Q^{-1} + 300 D^{-5} Q^2$$

Here. $c_1 = 100, c_2 = 50, c_3 = 20, c_4 = 300$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

the orthogonality and normality conditions are.

$$\begin{bmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{—(I)}$$

here $N = 4, \ n = 2, \ N > n+1 = 2+1$

$\Rightarrow$ these equations (I) do not yield the required Δ_j ($j = 1$ to 4). So, solving $\Delta_1, \Delta_2, \Delta_3$ in term of Δ_4, from (I)

$$\left.\begin{aligned} \Delta_1 + 2\Delta_2 - 5\Delta_4 &= 0 \\ -\Delta_3 + 2\Delta_4 &= 0 \\ \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 &= 1 \end{aligned}\right\} \text{—(II)}$$

from (II) $\Delta_3 = 2\Delta_4$

$$\Delta_1 = -2\Delta_2 + 5\Delta_4$$

$$-2\Delta_2 + 5\Delta_4 + \Delta_2 + 2\Delta_4 + \Delta_4 = 1$$

$$-\Delta_2 + 8\Delta_4 = 1$$

$$\Rightarrow \Delta_2 = 8\Delta_4 - 1$$

So

$$\Delta_1 = \cancel{-2\Delta_2 - 2(1 - 8\Delta_4) + 5\Delta_4}$$

$$= \cancel{-2 + 16\Delta_4 + 5\Delta_4 =}$$

$$\Delta_1 = -2(8\Delta_4 - 1) + 5\Delta_4 = -16\Delta_4 + 2 + 5\Delta_4$$

$$\Rightarrow \boxed{\begin{aligned} \Delta_1 &= 2 - 11\Delta_4 \\ \Delta_2 &= 8\Delta_4 - 1 \\ \Delta_3 &= 2\Delta_4 \end{aligned}}$$

the dual Problem can be written as

Maximize $v(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$

$$= \left(\frac{c_1}{\Delta_1}\right)^{\Delta_1} \left(\frac{c_2}{\Delta_2}\right)^{\Delta_2} \left(\frac{c_3}{\Delta_3}\right)^{\Delta_3} \left(\frac{c_4}{\Delta_4}\right)^{\Delta_4}$$

$$= \left(\frac{100}{2 - 11\Delta_4}\right)^{2 - 11\Delta_4} \left(\frac{50}{8\Delta_4 - 1}\right)^{8\Delta_4 - 1} \left(\frac{20}{2\Delta_4}\right)^{2\Delta_4} \left(\frac{300}{\Delta_4}\right)^{\Delta_4}$$

taking log on both sides, get

$\log v = (2-11D_4)[\log 100 - \ln(2-11D_4)]$ | for $\ln = \log$

$+(8D_4-1)[\ln 50 - \ln(8D_4-1)]$

$+2D_4[\log 20 - \log 2D_4] + D_4[\ln 300 - \ln D_4]$

necessary condition for the maximization

$$\frac{\partial}{\partial D_4}(\ln v) = -11[\ln 100 - \ln(2-11D_4)] + (2-11D_4)\frac{11}{2-11D_4} + 8[\ln 50 - \ln(8D_4-1)]$$

$$+(8D_4-1)\left(-\frac{8}{(8D_4-1)}\right) + 2[\ln 20 - \ln 2D_4)$$

$$+2D_4\left(\frac{-1}{2D_4}\cdot 2\right) + 1[\ln 300 - \ln D_4]$$

$$+ D_4\left[-\frac{1}{D_4}\right] = 0$$

$$\Rightarrow -11[2 - \ln(2-11D_4)] + 11 + 8\left[\ln\frac{50}{8D_4-1}\right] - 8$$

$$+2\log\left(\frac{20}{2D_4}\right) - 2 + \log\frac{300}{D_4} - 1 = 0$$

$\Rightarrow$ ~~ln~~

$$\Rightarrow -\ln\left[\frac{(100)^{11}}{(50)^8(20)^2(300)}\right] + \ln\left[\frac{(2-11D_4)^{11}}{(8D_4-1)^8(2D_4)^2 D_4}\right] = 0$$

$$\Rightarrow \frac{(2-11D_4)^{11}}{(8D_4-1)^8(2D_4)^2 D_4} = \frac{(100)^{11}}{(50)^8(20)^2(300)}$$

$$= \frac{(1\times10^2)^{11}}{5^8\times10^8\ 2^2\ 10^2\times3\times10^2} = \frac{1\times10^{22}}{5^8\times10^{12}\times12}$$

$$= \frac{10^{10}}{5^8\times12} = 2130$$

here get value of D_4^* by trial method.

$D_4^* \approx 0.147$, $D_1^* = 0.385$, $D_2^* = 0.175$

$D_3 = 0.294$

$$\Rightarrow v^* = f^* = \left(\frac{100}{0.385}\right)^{0.385}\left(\frac{50}{0.175}\right)^{0.175}\left(\frac{20}{0.294}\right)^{0.294}$$

$$\times\left(\frac{300}{0.147}\right)^{0.147} = 242$$

$U_1^* = D_1^* f^* = 0.385\times242 = 93.2$

$U_2^* = 42.4$

$U_3^* = 71.1$

$U_4^* = 35.6$

from

$y_1^* = 100\, D^* = 92.2$

$\Rightarrow D^* = 0.922$. $\delta^* = 0.281\, m^3/s$.

Example 8.3 - Zero Degree of difficulty Problem

the optimization problem can be stated as

Find $X = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ so as to minimize

$f(X) = 20x_1x_3 + 40x_2x_3 + 80x_1x_2$ subject to

$\frac{80}{x_1x_2x_3} \leq 10$ or $\frac{8}{x_1x_2x_3} < 1$

Ans here, n = no. of variables = 3

N_0 = no. of terms in the objective function = 3

$N_1 = 1$

m = total no. of the constraint = 1

N_k = number of terms in kth constraint

mean

$N_1 = 1$ = no. of terms in 1st constraint.

$N = N_0 + N_1 + \cdots + N_m$ = Total number of terms in the posynomials, means here

$N = 3 + 1 = 4$

and $N - n - 1 = 4 - 3 - 1 = 0 \Rightarrow$ zero-degree of difficulty problems

so, dual Problem can be stated as

Find $d = \begin{Bmatrix} d_{01} \\ d_{02} \\ d_{03} \\ d_{11} \end{Bmatrix}$ to maximize

$$v(d) = \prod_{k=0}^{1} \prod_{j=1}^{N_k} \left(\frac{c_{kj}}{d_{kj}} \sum_{l=1}^{N_k} d_{kl} \right)^{d_{kj}}$$

$$= \prod_{j=1}^{N_0=3} \left(\frac{c_{0j}}{d_{0j}} \sum_{l=1}^{N_0} d_{0l} \right)^{d_{0j}} \prod_{j=1}^{N_1=1} \left(\frac{c_{1j}}{d_{1j}} \sum_{l=1}^{N_1=1} d_{1l} \right)^{d_{1j}}$$

$$= \prod_{j=1}^{3} \left[\frac{c_{0j}}{d_{0j}} (d_{01} + d_{02} + d_{03}) \right]^{d_{0j}} \prod_{j=1}^{1} \left(\frac{c_{1j}}{d_{1j}} (d_{11}) \right)^{d_{1j}}$$

$$= \frac{c_{00}}{d_{00}}$$

$$= \left(\frac{c_{01}}{d_{01}} (d_{01} + d_{02} + d_{03}) \right)^{d_{01}} \times$$

$$\left(\frac{c_{02}}{d_{02}} (d_{01} + d_{02} + d_{03}) \right)^{d_{02}} \times$$

$$\left(\frac{c_{03}}{d_{03}} (d_{01} + d_{02} + d_{03}) \right)^{d_{03}} \left(\frac{c_{11}}{d_{11}} d_{11} \right)^{d_{11}} \rightarrow E_1$$

subject to the constants are.

$$\sum_{j=1}^{N_0} d_{0j} = 1 \rightarrow \text{Normality } ~~condition~~ \text{. constraint}$$

$$\Rightarrow d_{01} + d_{02} + d_{03} = 1$$

and orthogonal constraints are

$$\sum_{K=0}^{m} \sum_{j=1}^{N_k} a_{kij} d_{kj} = 0, \quad i = 1, 2 .. n$$

$$\Rightarrow \cancel{\sum_{K=0}^{1} \sum_{j=1}^{10} a_{kij} d_{kj} = 0}$$

$$\cancel{\sum_{K=0}^{1} \sum_{j=1}^{N_0} a_{0ij} d_{0j} = 0}$$

$$\sum_{j=1}^{N_0} a_{0ij} d_{0j} + \sum_{j=1}^{N_1} a_{1ij} d_{ij} = 0$$

$$\sum_{j=1}^{3} a_{0ij} d_{0j} + \sum_{j=1}^{1} a_{1ij} d_{1j} = 0$$

$$a_{0i1} d_{01} + a_{0i2} d_{02} + a_{0i3} d_{03} + a_{1i1} d_{11} = 0$$

$$\Rightarrow$$

$$a_{011} d_{01} + a_{012} d_{02} + a_{013} d_{03} + a_{111} d_{11} = 0$$

$$a_{021} d_{01} + a_{022} d_{02} + a_{023} d_{03} + a_{121} d_{11} = 0$$

$a_{031} d_{01} + a_{032} d_{02} + a_{033} d_{03} + a_{131} d_{11} = 0$

$d_{0j} \geq 0,\ j = 1,2,3,\ d_{11} \geq 0$

in this Problem

$a_{011} = 1, a_{021} = 0, a_{031} = \phi,$

$a_{012} = 0, a_{022} = 1, a_{032} = 1, a_{013} = 1, a_{023} = 1,$

$a_{033} = 0, a_{111} = -1, a_{121} = -1, a_{131} = -1.$

So, Problem can be written

$$v(d) = \left[\frac{20}{d_{01}}(d_{01} + d_{02} + d_{03})\right]^{d_{01}}$$

subject to

$d_{01} + d_{02} + d_{03} = 1$

$d_{01} + d_{03} - d_{11} = 0$

$d_{02} + d_{03} - d_{11} = 0$

$d_{01} + d_{02} - d_{11} = 0$

$\Rightarrow d_{01}^* = d_{02}^* = d_{03}^* = \frac{1}{3},\ d_{11}^* = 2/3.$

thus the maximum value v or minimum value of x_0 is given by

$v^* = x_0^* = (60)^{1/3}(120)^{1/3}(240)^{1/3}(8)^{2/3} = 480$ Ans

Complementary Geometric Programming

Geometric Programming to include any rational function of posynomial terms and called the method of complementary geometric programming.

Let the complementary geometric programming problem be stated as follows

Minimize $R_0(X)$ subject

$R_K(X) \leq 1 \qquad K = 1, 2, \ldots m$, where

$$R_K(X) = \frac{A_K(X) - B_K(X)}{C_K(X) - D_K(X)},\ K = 0, 1, 2, \ldots m$$

where $A_K(X)$, $B_K(X)$, $C_K(X)$ and $D_K(X)$ are posynomial and possibly some of them may be absent. To solve the problem stated, we introduce a new variable $x_0 > 0$, constrained to satisfy the relation $x_0 \geq R_0(X)$

i.e. $\frac{R_0(X)}{x_0} \leq 1$, so the Problem may be restated as

minimize x_0

subject to $\dfrac{A_K(X) - B_K(X)}{C_K(X) - D_K(X)} \leq 1, \; K = 0, 1, 2, \dots m$

where $A_0(X) = R_0(X)$, $C_0(X) = x_0$ & $B_0(X) = 0$

and $D_0(X) = 0$.

Thus any complementary geometric programming problem (CGP) can be stated in the standard form

minimize x_0 subject to

$$\frac{P_K(X)}{Q_K(X)} \leq 1, \quad K = 1, 2, \dots m \quad \dots (8.71)$$

$$X = \begin{Bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{Bmatrix} > 0, \text{ where} \quad \dots (8.72)$$

where $P_K(X)$ and $Q_K(X)$ are Polynomials of the form

$$P_K(X) = \sum_j C_{Kj} \prod_{i=0}^{n} (x_i)^{a_{kij}} = \sum_j P_{Kj}(X) \quad \dots (8.73)$$

$$Q_K(X) = \sum_j d_{Kj} \prod_{i=0}^{n} (x_i)^{b_{kij}} = \sum_j q_{Kj}(X) \dots (8.74)$$

or.

$$P_K(X) = \sum_j (C_{Kj})(x_1)^{a_{k1j}} x_2^{a_{k2j}} x_3^{a_{k3j}}$$

$$= \sum_j P_{Kj}(X)$$

$$= P_{K1}(X) + P_{K2}(X) + P_{K3}(X) + \dots$$

$$= C_{K1}(x_1)^{a_{k11}} (x_2)^{a_{k21}} (x_3)^{a_{k31}}$$

$$+ C_{K2}(x_1)^{a_{k12}} (x_2)^{a_{k22}} (x_3)^{a_{k32}}$$

$$Q_K(X) = q_{K1}(X) + q_{K2}(X) + q_{K3}(X)$$

$$Q_1(X) = q_{11}(X) + q_{12}(X) + q_{13}(X)$$

(8.6) Minimize x_1 subject to

$-4x_1^2 + 4x_2 \le 1$

$x_1 + x_2 \ge 1, \; x_1 > 0, \; x_2 > 0$

initial starting Point $X^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ (assume that)

Ans G.P.P. can be stated as

minimize x_1 subject to (1)

$4x_2 \le 1 + 4x_1^2$ (2)

$\dfrac{4x_2}{1+4x_1^2} \le 1$ (2)

$x_1 + x_2 \ge 1$

$\Rightarrow \dfrac{1}{x_1+x_2} \le \dfrac{1}{1}$

$\Rightarrow \dfrac{1/x_1}{1 + x_2/x_1} \le 1 \Rightarrow \dfrac{x_1^{-1}}{1 + x_1^{-1}x_2} \le 1$ (3)

$x_1 > 0, x_2 > 0$ (4) (5)

* now we start the Process that in eq^n (2) and (3), denominators convert into single term

2(a) Minimise $f(x_1, x_2) = 2x_1^2 + x_2^2$ $\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$ Univariate Method

Iteration -1, $S_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, $X_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

$f_1 = f(X_1) = f(1,2) = 2 \cdot 1^2 + 2^2 = 6, \; \varepsilon = 0.01$

$f^+ = f(X_1 + \varepsilon S_1) = f(1+\varepsilon, 2) = f(1.01, 2)$

$= 2(1.01)^2 + 4 = 2.0402 + 4 = 6.0402$

here $f^+ > f_1$

$f^- = f(X_1 - \varepsilon S_1) = f\{(1,2) - \varepsilon(1,0)\} = f(1-\varepsilon, 2)$

$= f(.99, 2) = 5.9602 < f_1$

$\Rightarrow -S_1$ will be the correct direction

$f^+ < f_i$, $f^- < f_i$

For optimum length, we minimize

$\cancel{f(X_1 +} f(X_1 - \lambda_1 S_1) = f(1-\lambda_1, 2)$

$= 2(1-\lambda_1)^2 + 4 = 2(1 + \lambda_1^2 - 2\lambda_1) + 4$

$\dfrac{\partial f}{\partial \lambda_1} = 2(2\lambda_1 - 2) = 0 \Rightarrow \lambda_1 = 1$

so, $X_2 = X_1 - \lambda_1 S_1 = (1,2) - 1(1,0) = (0,2)$

$\Rightarrow f(X_2) = 0 + 2^2 = 4$

2nd iteration, $S_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$

$f_2^+ = f(X_2 + \varepsilon S_2) = f\{(0,2) + 0.01(0,1)\} = f(0, 2.01)$

$= 0.0 + 4.0401 \Rightarrow f_2^+ > f_2$

$f_2^- = f(X_2 - 0.01 S_2) = f(0,2) - 0.01(0,1) = f(0, 1.99)$

$= 3.9601 \Rightarrow f_2^- < 4 = f_2$.