2. Matrix multiplication is associative, if conformability is assured.

$$A(BC) = (AB)C$$

3. Matrix multiplication is distributive w.r.t. addition,

$$A\cdot(B+C) = AB + AC$$

4. Multiplication of square matrix A by unit matrix

$$AI = IA = A.$$

5. Adjoint of a matrix (square) :- Transpose of the cofactor matrix of a square matrix is, its cofactor matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Cofactor matrix $cof(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \quad A_{21} = (-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad A_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad A_{32} = (-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$Adj(A) = \text{transpose}\left(Cof(A)\right) = cof(A)'$

$$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

ex Find Adjoint of matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$A_{11} = (-1)^{1+1} \cdot (-12-12) = -24$

$A_{12} = (-1)^{1+2} \cdot (-4+6) = +10$

$A_{13} = (-1)^{1+3} \cdot (-4+6) = 2$

$A_{21} = (-1)^{2+1} \cdot (-4+12) = -8$

$A_{22} = (-1)^{2+2} \cdot (-4+6) = 2$

$A_{23} = (-1)^{2+3} \cdot (-4+2) = 2$

$A_{31} = (-1)^{3+1} \cdot (-3-9) = -12$

$A_{32} = (-1)^{3+2} \cdot (-3-3) = 6$

$A_{33} = (-1)^{3+3} \cdot (3-1) = 2$

$$Cof(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{bmatrix}$$

$$Adj(A) = cof A' = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

Inverse of a square matrix (A) :- If A be any matrix, then a matrix B if it exists, such that $AB = BA = I$, is called the inverse of A which is denoted by A^{-1} and $AA^{-1} = I$. Also $A^{-1} = \dfrac{adj(A)}{|A|}$, $|A| \neq 0$

eg: find inverse of matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$\because$ $adj(A) = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$

& $|A| = 1(-12-12) - 1(-4-6) + 3(-4+6)$

$= -24 + 10 + 6$

$= -8$

$A^{-1} = \dfrac{adj(A)}{|A|} = \dfrac{1}{-8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$ ✓

Rank of a matrix:- A ~~rank~~ non-zero number r is said to be rank of matrix A if

(i) there exist at least a minor of A of order r which is non zero

(ii) Every minor of higher order than r is zero

The rank of 'A' is denoted by $\rho(A) = r$

eg: find the rank of matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$

Sol^n $|A| = 2(-9+8) - 1(0+4) - 1(0-6)$

$= -2 - 4 + 6 = 0$

$\Rightarrow$ A can be transform into a matrix whose atleast one row or column is zero.

$\Rightarrow \rho(A) \neq 3$

On taking a minor of order 2

$$M_1 = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6 - 0 = 6 \neq 0$$

$\Rightarrow \rho(A) = 2.$

Ques: $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & -7 \end{bmatrix}$, find rank

Sol^n

$$|A| = -1\cdot(28+2) + 2(-14+1) + 3(-4+4)$$

$$= -30 - 30 = -60 \neq 0$$

$\Rightarrow \rho(A) = 3.$