Taylor's theorem for one variable :-

Assume that the function f has all derivative upto the order $(n+1)$ in some interval containing the point $x = x_0$

$$f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots + \frac{h^n}{n!} f^n(x_0) + R_n$$

$$R_n = \frac{h^{n+1}}{(n+1)!} f^{n+1}(\xi), \quad x_0 < \xi < x_0 + h$$

Taylor's theorem for two variables :-

Let a funcⁿ be defined in some domain D in R^2 and have continuous partial derivatives up to $(n+1)^{th}$ order in some nbd of a point $P(x_0, y_0)$ in D. Then

$$f(x_0+h, y_0+k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0)$$

$$+ R_n$$

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k),$$

$0<\theta<1$

Proof:- Let $x = x_0 + th$, $y = y_0 + tk$, where $t \in [0,1]$

Define $\phi(t) = f(x_0+th, y_0+tk)$

$$\phi'(t) = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0+th, y_0+tk)$$

$$\phi''(t) = h\left(\frac{\partial^2 f}{\partial x^2}h + \frac{\partial^2 f}{\partial y\partial x}k\right) + k\left(\frac{\partial^2 f}{\partial x\partial y}h + \frac{\partial^2 f}{\partial y^2}k\right)$$

$$= h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0+th, y_0+tk)$$

$$\phi'''(t) = h^2\left(\frac{\partial^3 f}{\partial x^3}h + \frac{\partial^3 f}{\partial y\partial x^2}k\right) + 2hk\left(\frac{\partial^3 f}{\partial x^2\partial y}h + \frac{\partial^3 f}{\partial x\partial y^2}k\right)$$

$$+ k^2\left(\frac{\partial^3 f}{\partial x\partial y^2}h + \frac{\partial^3 f}{\partial y^3}k\right)$$

$$= h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0+th, y_0+tk)$$

$\vdots$

$$\phi^n(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0+th, y_0+tk)$$

$$\phi^{n+1}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+th, y_0+tk)$$

Thus, using taylor's theorem for one variable

$$\phi(t) = \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(0) + \frac{1}{3!}\phi'''(0) + \dots + \frac{1}{n!}\phi^{n}(0)$$
$$+ \frac{1}{n+1!}\phi^{n}(\theta)$$

$\Rightarrow$

$$f(x_0+h, y_0+k) = f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(x_0,y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)$$
$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0,y_0) + \dots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)$$
$$+ \frac{1}{n+1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k).$$

★ Now putting $x_0+h = x$, $y_0+k = y$, so that $h = x-x_0$, $k = y-y_0$, we get

$$f(x,y) = f(x_0,y_0) + \left((x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\right)$$
$$+\frac{1}{2!}\left((x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}\right)$$
$$+ \dots$$ (This is called taylor series expansion in powers of $(x-x_0)$ & $(y-y_0)$.

If $x_0=0, y_0=0$ Then taylor expansion is called maclaurin expansion.

Ex Expand $e^x\log(1+y)$ in powers of x and y upto terms of third degree. $x_0=0$, $y_0=0$

$f(x,y) = e^x\log(1+y)$, $f(0,0) = 0$

$f_x = e^x\log(1+y)$, $f_x(0,0) = 0$

$f_y = \frac{e^x}{1+y}$, $f_y(0,0) = 1$

$f_{xx} = e^x\log(1+y)$, $f_{xx}(0,0) = 0$

$f_{xy} = \dfrac{e^x}{1+y}$ $\quad f_{xy}(0,0) = 1$

$f_{yy} = \dfrac{-e^x}{(1+y)^2}$ $\quad f_{yy}(0,0) = -1$

$f_{xxx} = e^x \log(1+y)$ $\quad f_{xxx}(0,0) = 0$

$f_{x^2y} = \dfrac{e^x}{1+y}$ $\quad f_{x^2y}(0,0) = 1$

$f_{xy^2} = \dfrac{-e^x}{(1+y)^2}$ $\quad f_{xy^2}(0,0) = -1$

$f_{yyy} = \dfrac{2e^x}{(1+y)^3}$ $\quad f_{yyy}(0,0) = 2$

Now the maclaurin expansion

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2y f_{x^2y}(0,0) + 3xy^2 f_{xy^2}(0,0) + y^3 f_{yyy}(0,0)]$$

$$+ \cdots$$

$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}(x^2y - xy^2) + \frac{1}{3}y^3 + \cdots$$

Ex Expand $x^2y + 3y - 2$ in power of $(x-1)$ and $(y+2)$ using Taylor's theorem.

Ans $x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2$
$+ 2(x-1)(y+2) + (x-1)^2(y+2).$

Ques Obtain Taylor series expansion of $\tan^{-1}\left(\frac{y}{x}\right)$ about $(1,1)$ upto & including the second degree term. Hence compute $f(1.1, 0.9)$.

		$x=1, y=1$
$f(x,y)$	$\tan^{-1}\left(\frac{y}{x}\right)$	$\frac{\pi}{4}$
f_x	$\frac{-y}{x^2+y^2}$	$-\frac{1}{2}$
f_y	$\frac{x}{x^2+y^2}$	$\frac{1}{2}$
f_{xx}	$\frac{2xy}{(x^2+y^2)^2}$	$\frac{1}{2}$
f_{yy}	$\frac{-2xy}{(x^2+y^2)^2}$	$-\frac{1}{2}$
f_{xy}	$\frac{y^2-x^2}{(x^2+y^2)^2}$	0

By Taylor's theorem

$$f(x,y) = f(x_0,y_0) + \left[(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\right]$$

$$+ \frac{1}{2!}\left[(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\right]$$

$$+ \cdots$$

Here

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 \cdots$$

Putting $x = 1.1$ i.e. $x-1 = 0.1$, $y = 0.9$, $y-1 = -0.1$

$$f(1.1, 0.9) = \frac{\pi}{4} - \cancel{\frac{1}{2}(0.1)} - \cancel{\frac{1}{2}(-0.1)} + \cancel{\frac{1}{4}(0.1)^2} - \cancel{\frac{1}{4}(0.1)^2}$$

$$= \frac{\pi}{4} \text{ Ans}$$

Maxima and Minima of functions of two variables:

Working rule to find maxima & minima of functions of two variables:-

(i) Find f_x & f_y and equate each to zero. Solve these as simultaneous equations in x & y. Let (a,b), (c,d), ... be the pair of values

(ii) calculate the value of $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$ for each pair of values.

(iii)
(i) If $rt - s^2 > 0$ and $r < 0$ at (a,b), $f(a,b)$ is a maximum value.

(ii) If $rt - s^2 > 0$ and $r > 0$ at (a,b), $f(a,b)$ is minimum value

(iii) If $rt - s^2 < 0$ at (a,b) $f(a,b)$ is not an extreme value, i.e. (a,b) is a saddle point.

(iv) If $rt - s^2 = 0$ at (a,b), the case is doubtful needs further investigation.

Ques:- Examine the following funcⁿ for extreme values

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2.$$

$$f_x = 4x^3 - 4x + 4y = 0$$

$$f_y = 4y^3 + 4x - 4y = 0$$

$f_{xx} = 12x^2 - 4$, $f_{xy} = 4$, $f_{yy} = 12y^2 - 4$

$$x^3 - x + y = 0 \quad \text{--- (i)}$$
$$y^3 + x - y = 0 \quad \text{--- (ii)}$$

adding (i) & (ii)

$$x^3 + y^3 = 0$$
$$(x+y)(x^2+y^2-xy) = 0$$
$$x = -y \quad \text{--- (iii)}$$
$$y = -x \quad \text{--- (iv)}$$

Putting $y = -x$ in (i)

$$x^3 - 2x = 0$$
$$x(x^2-2) = 0$$

$x = 0, \sqrt{2}, -\sqrt{2}$ then from (iv) corresponding values of y

$$y = 0, -\sqrt{2}, 2$$

Pair of points $(0,0)$, $(\sqrt{2},-\sqrt{2})$, $(-\sqrt{2},\sqrt{2})$.

(i) ~~r(0,0)~~ at $(0,0)$

$$r = -4, \quad s = 4, \quad t = -4$$

$rt - s^2 = 16 - 16 = 0$ (i.e. the case is doubtful) further investigation is needed.

Now, $f(0,0) = 0$, and the point along x-axis ($y=0$)

$f(x,0) = x^2(x^2-2)$ is negative in some nbd of origine. Now along $y = x$

$f(x,x) = 2x^4$ which is +ve

i.e. in the nbd of $(0,0)$ function is positive as well as $\ominus$ve, hence at $(0,0)$ function has ~~point~~ saddle point.

(iii) at $(\sqrt{2}, -\sqrt{2})$

$rt - s^2 = 400 - 4^2 > 0$ r is +ve

hence func has minima at $(\sqrt{2}, -\sqrt{2})$

$f(\sqrt{2}, -\sqrt{2})$ is minimum value.

(iii) at $(-\sqrt{2}, \sqrt{2})$

$rt - s^2 > 0$, $r > 0$ hence $(-\sqrt{2}, \sqrt{2})$ is also point of minima & $f(-\sqrt{2}, \sqrt{2})$ is minimum value.