CHAPTER

63

BINOMIAL DISTRIBUTION

63.1 BINOMIAL DISTRIBUTION P (r) = $^nC_r\ p^r \cdot q^{n-r}$

To find the probability of the happening of an event once, twice, thrice, ... r times ...... exactly in n trials.

Let the probability of the happening of an event A in one trial be p and its probability of not happening be $1 - p = q$.

We assume that there are n trials and the happening of the event A is r times and its not happening is $n - r$ times.

$$\underbrace{A\ A.......A}_{r \text{ times}} \qquad \underbrace{\bar{A}.\bar{A}......\bar{A}}_{n-r \text{ times}} \qquad ...(1)$$

A indicates its happening, $\bar{A}$ its failure and $P(A) = p$ and $P(\bar{A}) = q$.

We see that (1) has the probability

$$\underbrace{pp\ ....\ p}_{r \text{ times}}\ \underbrace{q\,.\,q\ ....\ q}_{n-r \text{ times}} = p^r \cdot q^{n-r} = p^r q^{n-r} \qquad ...(2)$$

Clearly (1) is merely one order of arranging $r\,A$'s.

The probability of (1) $= p^r q^{n-r} \times$ Number of different arrangements of $r\,A$'s and $(n-r)\,\bar{A}$'s.

The number of different arrangements of $r\,A$'s and $(n-r)\,\bar{A}$'s $= {}^nC_r$

$\therefore$ Probability of the happening of an event r times ${}^nC_r\ p^r \cdot q^{n-r}$.

$$P(r) = {}^nC_r\ p^r \cdot q^{n-r} \quad (r = 0, 1, 2, ...,n).$$

$$= (r+1)\text{th term of } (q+p)^n$$

If $r = 0$, probability of happening of an event 0 times $= {}^nC_0\ q^n p^0 = q^n$

If $r = 1$, probability of happening of an event 1 time $= {}^nC_1\, q^{n-1}\, p$

If $r = 2$, probability of happening of an event 2 times $= {}^nC_2\ q^{n-2} p^2$

If $r = 3$, probability of happening of an event 3 times $= {}^nC_3\ q^{n-3} p^3$ and so on.

These terms are clearly the successive terms in the expansion of $(q+p)^n$.

Hence it is called Binomial Distribution.

Example 1. *Find the probability of getting 4 heads in 6 tosses of a fair coin.*

Solution. $p = \frac{1}{2},\ q = \frac{1}{2},\ n = 6,\ r = 4.$

We know that $P(r) = {}^nC_r\ q^{n-r}\ p^r$

$$P(4) = {}^6C_4\ q^{6-4}\ p^4 = \frac{6\times 5}{1\times 2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 = 15\times\left(\frac{1}{2}\right)^6 = \frac{15}{64} \qquad \textbf{Ans.}$$

Example 2. *If on an average one ship in every ten is wrecked, find the probability that out of 5 ships expected to arrive, 4 at least will arrive safely.*

Solution. Out of 10 ships, one ship is wrecked.

i.e., Nine ships out of ten ships are safe.

$$p\text{ (safety)} = \frac{9}{10}$$

P (At least 4 ships out of 5 are safe) = P (4 or 5) = P (4) + P (5)

$$= {}^5C_4\, p^4 q^{5-4} + {}^5C_5\, p^5 q^0 = 5\left(\frac{9}{10}\right)^4\left(\frac{1}{10}\right) + \left(\frac{9}{10}\right)^5 = \left(\frac{9}{10}\right)^4\left(\frac{5}{10}+\frac{9}{10}\right) = \frac{7}{5}\left(\frac{9}{10}\right)^4 \qquad \textbf{Ans.}$$

Example 3. *The overall percentage of failures in a certain examination is 20. If six candidates appear in the examination, what is the probability that at least five pass the examination?*

Solution. Probability of failures = 20% = $\frac{20}{100} = \frac{1}{5}$

Probability of pass $(P) = 1 - \frac{1}{5} = \frac{4}{5}$

Probability of at least five pass = P (5 or 6)

$$= P\,(5) + P(6) = {}^6C_5\, p^5 q + {}^6C_6 p^6 q^0 = 6\left(\frac{4}{5}\right)^5\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^6 = \left(\frac{4}{5}\right)^5\left[\frac{6}{5}+\frac{4}{5}\right] = 2\left(\frac{4}{5}\right)^5 = \frac{2048}{3125}$$

= 0.65536 **Ans.**

Example 4. *Ten percent of screws produced in a certain factory turn out to be defective. Find the probability that in a sample of 10 screws chosen at random, exactly two will be defective.*

Solution. $p = \frac{1}{10}$, $\quad q = \frac{9}{10}$, $n = 10$, $r = 2$

$$P\,(r) = {}^nC_r\;\; p^r\; q^{n-r}$$

$$P(2) = {}^{10}C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{10-2} = \frac{10\times 9}{1\times 2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^8 = \frac{1}{2}\cdot\left(\frac{9}{10}\right)^9 = 0.1937 \qquad \textbf{Ans.}$$

Example 5. *The probability that a man aged 60 will live to be 70 is 0.65. What is the probability that out of 10 men, now 60, at least 7 will live to be 70 ?*

Solution. The probability that a man aged 60 will live to be 70 = p = 0.65

$$q = 1 - p = 1 - 0.65 = 0.35$$

Number of men = n = 10

Probability that at least 7 men (7 or 8 or 9 or 10) will live to 70

$$= P\;(7) + P\,(8) + P\,(9) + P\,(10) = {}^{10}C_7\, q^3\, p^7 + {}^{10}C_8 q^2 p^8 + {}^{10}C_9\, q\, p^9 + p^{10}$$

$$= \frac{10\times 9\times 8}{1\times 2\times 3}(0.35)^3(0.65)^7 + \frac{10\times 9}{1\times 2}(0.35)^2(0.65)^8 + 10(0.35)(0.65)^9 + (0.65)^{10}$$

$$= (0.65)^7\,[\,120\;(0.35)^3 + 45\,(0.35)^2\,(0.65) + 10\,(0.35)\,(0.65)^2 + (0.65)^3]$$

$$= (0.65)^7 \times 125\,[120 \times (0.07)^3 + 45 \times (0.07)^2\,(0.13) + 10\,(0.07)\,(0.13)^2 + (0.13)^3]$$

$$= 0.04902 \times 125\;[0.04 + 0.028665 + 0.011830 + 0.002197]$$

$$= 6.1275 \times 0.082692 = 0.5067 \qquad \textbf{Ans.}$$

Example 6. *If 10 % of bolts produced by a machine are defective. Determine the probability that out of 10 bolts, chosen at random (i) 1 (ii) none (iii) at most 2 bolts will be defective.*

Solution. Probability of defective bolts = p = 10% = 0.1

Probability of not defective bolts = $q = 1 - p = 1 - 0.1 = 0.9$

Total number of bolts = n = 10

(i) Probability of 1 defective bolt $= {}^{10}C_1 (0.1)^1 (0.9)^9 = 0.3874$

(ii) Probability that none is defective = Probability of 0 defective bolt

$= P(0) = {}^{10}C_0 (0.1)^0 (0.9)^{10} = 0.3487$

(iii) Probability of 2 defective $= {}^{10}C_2 (0.1)^2 (0.9)^8 = 0.1937$

Probability of at most 2 defective $= P(0 \text{ or } 1 \text{ or } 2)$

$= P(0) + P(1) + P(2) = 0.3487 + 0.3874 + 0.1937$

$= 0.9298$ **Ans.**

Example 7. *An underground mine has 5 pumps installed for pumping out storm water, the probability of any one of the pumps failing during the storm is* $\frac{1}{8}$. *What is the probability that* (i) *at least 2 pumps will be working;* (ii) *all the pumps will be working during a particular storm ?*

Solution. (i) Probability of pump failing $= \frac{1}{8}$

Probability of pump working $= 1 - \frac{1}{8} = \frac{7}{8}, \quad P = \frac{7}{8}, \quad q = \frac{1}{8}, \quad n = 5$

(i) P (At least 2 pumps working) $= P$ (2 or 3 or 4 or 5 pumps working)

$= P(2) + P(3) + P(4) + P(5) = {}^5C_2\, p^2 q^3 + {}^5C_3\, p^3 q^2 + {}^5C_4\, p^4 q + {}^5C_5\, p^5 q^0$

$$= 10\left(\frac{7}{8}\right)^2\left(\frac{1}{8}\right)^3 + 10\left(\frac{7}{8}\right)^3\left(\frac{1}{8}\right)^2 + 5\left(\frac{7}{8}\right)^4\left(\frac{1}{8}\right) + \left(\frac{7}{8}\right)^5$$

$$= \frac{1}{8^5}[10\times 49 + 10\times 343 + 5\times 2401 + 16807]$$

$$= \frac{1}{8^5}[490 + 3430 + 12005 + 16807] = \frac{32732}{8^5} = \frac{8183}{8192}$$

(ii) P (All the 5 pumps working) $= P(5) = {}^5C_5\, p^5 q^0 = \left(\frac{7}{8}\right)^5 = \frac{16807}{32768}$ **Ans.** (i) $\frac{8183}{8192}$ (ii) $\frac{16807}{32768}$

Example 8. *Write two-three areas where binomial distribution is applied. The probability of entering student in chartered accountant will be graduate 0.5. Determine the probability that out of 10 students (i) none (ii) one or (iii) at least one will graduate.*

(*R.G.P.V., Bhopal, Dec., 2003*)

Solution. Given, the probability of an entering student in chartered accountant will graduate is $p = 0.5$

$\therefore$ The probability of an entering student in charactered accountant will not graduate is $q = 0.5$.

Therefore

(i) The probability of none will graduate out of 10 students

$P(0) = {}^{10}C_0\, p^0 q^{10} = {}^{10}C_0 (0.5)^0 (0.5)^{10} = 9.765625 \times 10^{-4}$ **Ans.**

(ii) The probability of exactly one student will graduate out of 10 students.

$P(1) = {}^{10}C_1 (0.5)^1 (0.5)^9 = 10 \times 0.5 \times (0.5)^9 = 9.765625 \times 10^{-3}$ **Ans.**

(iii) The probability of at least one will graduate out of 10 students

P (At least one) = 1 – (probability of none will graduats)

$= 1 - 9.765625 \times 10^{-4} = 0.99$ **Ans.**

Example 9. *The probability that a bomb dropped from a plane will strike the target is* $\frac{1}{5}$. *If six bombs are dropped, find the probability that:*

(i) Exactly two will strike the target.

(ii) At least two will strike the target. (*R.G.P.V., Bhopal, II Semester, Feb. 2006*)

Solution. Here, $p = \frac{1}{5}$, $q = 1 - \frac{1}{5} = \frac{4}{5}$, $n = 6$

We know that $P(r) = {}^nC_r\, p^r\, q^{n-r}$

$$P(2) = {}^6C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{6-2} = 15\left(\frac{256}{15625}\right) = \frac{768}{3125} = 0.24576$$

$$\begin{aligned} P(\text{at least } 2) &= P(2, 3, 4, 5, 6) = P(2) + P(3) + P(4) + P(5) + P(6) \\ &= P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) - P(0) - P(1) \\ &= 1 - [P(0) + P(1)] \\ &= 1 - \left[{}^6C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^6 + {}^6C_1 \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^5\right] = 1 - \left[\frac{4096}{15625} + 6\left(\frac{1024}{15625}\right)\right] \\ &= 1 - \frac{10240}{15625} = \frac{5385}{15625} = \frac{1077}{3125} = 0.34464 \end{aligned}$$

Hence (*i*) $P = 0.24576$ (*ii*) $P = 0.34464$ **Ans.**

EXERCISE 63.1

1. If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen at random
(*a*) 1 (*b*) 0 (*c*) At most 2
bolts will be defective. **Ans.** (*a*) 0.4096, (*b*) 0.4096, (*c*) 0.9728.

2. Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or a six ? **Ans.** 233

3. Find the probability of getting a total of 7 at least once in 4 tosses of a pair of fair dice.? (*A.M.I.E., Winter 2002*) **Ans.** $\frac{671}{1296}$

4. If the chance that any one of the 10 telephone lines is busy at any instant is 0.2, what is the chance that 5 of the lines are busy ? What is the probability that all the lines are busy ?
Ans. ${}^{10}C_5\,(0.2)^5\,(0.8)^5$, $(0.2)^{10}$

5. An insurance salesman sells policies to 5 men, all of identical age in good health. According to the actuarial tables the probability that a man of this particular age will be alive 30 years hence is $\frac{2}{3}$. Find the probability that in 30 years.
(*a*) All 5 men (*b*) At least 3 men (*c*) Only 2 men (*d*) At least 1 man will be alive.
Ans. (*a*) $\frac{32}{243}$ (*b*) $\frac{192}{243}$ (*c*) $\frac{40}{243}$ (*d*) $\frac{242}{243}$

6. Consider an urn in which 4 balls have been placed by the following scheme : A fair coin is tossed; if the coin falls head, a white ball is placed in the urn, and if the coin falls tail, a red ball is placed in urn. (*i*) What is the probability that the urn will contain exactly 3 white balls ? (*ii*) What is the probability that the urn will contain exactly 3 red balls, given that the first ball placed was red? **Ans.** (*i*) $\frac{1}{8}$, (*ii*) $\frac{3}{8}$

7. A box contains 10 screws, 3 of which are defective. Two screws are drawn at random without replacement. Find the probability that none of the two screws is defective. **Ans.** $\frac{7}{15}$

8. Out of 800 families with four children each, how many families would be expected to have :
(*i*) 2 boys and 2 girls; (*ii*) at least one boy; (*iii*) no girl; (*iv*) at most two girls ?
Assume equal probabilities for boys and girls. **Ans.** (*i*) 300, (*ii*) 750, (*iii*) 50, (*iv*) 550.

9. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is 5/6. What is the probability that he will knock down less than 2 hurdles ? **Ans.** $\frac{8}{3}\left(\frac{5}{6}\right)^9$

10. An electronic component consists of three parts. Each part has probability 0.99 of performing satisfactorily. The component fails if 2 or more parts do not perform satisfactorily. Assuming that the parts perform independently, determine the probability that the component does not perform satisfactorily.

Ans. 0.000298

11. The incidence of occupational disease in an industry is such that the workers have 20% chance of suffering from it. What is the probability that out of 6 workers 4 or more will catch the disease ?

(*A.M.I.E., Winter 2005*) **Ans.** $\dfrac{53}{2125}$

12. Among 10,000 random digits, find the probability p that the digit 3 appears at most 950 times.

(*A.M.I.E., Summer 2003*) Ans. $\sum\limits_{n=0}^{950} 1000\, C_r \left(\dfrac{1}{10}\right)^r \left(\dfrac{9}{10}\right)^{1000-r}$

13. A fair coin is tossed 400 times. Using normal approximation to the binomial, find the probability that a head will occur (a) more than 180 times and (b) less than 195 times.

(*A.M.I.E. Winter 2004*) **Ans.** (a) $1-\left(\dfrac{1}{2}\right)^{221}$ (b) $1-\left(\dfrac{1}{2}\right)^{115}$

14. In a bombing action there is 50% chance that any bomb will strike the target. Two direct hits are needed to destroy the target completely. How many bombs are required to be dropped to give a 99% chance or better of completely destroying the target. (*R.G.P.V., Bhopal, June 2008*) **Ans.** 11

63.2 MEAN OF BINOMIAL DISTRIBUTION

(*GBTU, Dec. 2012, AMIETE, Winter 2002, Summer 2000, A.M.I.E., Winter 2002*)

$$(q+p)^n = q^n + {}^nC_1\, q^{n-1}p^1 + {}^nC_2\, q^{n-2}p^2 + {}^nC_3\, q^{n-3}p^3 + \ldots\ldots + {}^nC_r\, q^{n-r}p^r + \ldots\ldots + p^n$$

Successes r	*Frequency f*	*Product r f*
0	q^n	0
1	$n\,q^{n-1}p$	$n\,q^{n-1}p$
2	$\dfrac{n(n-1)}{2} q^{n-2}p^2$	$n(n-1)\,q^{n-2}p^2$
3	$\dfrac{n(n-1)(n-2)}{6} q^{n-3}p^3$	$\dfrac{n(n-1)(n-2)}{2} q^{n-3}p^3$
....	......	....
n	p^n	np^n

$$\Sigma fr = nq^{n-1}p + n(n-1)q^{n-2}p^2 + \frac{n(n-1)(n-2)}{2}q^{n-3}p^3 + \ldots + n\,p^n$$

$$= np\left[q^{n-1} + \frac{(n-1)}{1!}q^{n-2}p + \frac{(n-1)(n-2)}{2}q^{n-3}p^2 + \ldots + p^{n-1}\right]$$

$$= n\,p\,(q+p)^{n-1} = np \qquad \text{(since } q+p=1)$$

$$\Sigma f = q^n + nq^{n-1}p + \frac{n(n-1)}{2}q^{n-2}p^2 + \ldots\ldots + p^n$$

$$= (q+p)^n = 1 \qquad \text{(since } q+p=1)$$

Hence, $\text{Mean} = \dfrac{\Sigma f r}{\Sigma f} = \dfrac{np}{1}$ **Ans.**

63.3 STANDARD DEVIATION OF BINOMIAL DISTRIBUTION

(*A. M. I. E.T. E., Winter 2002, A.M.I.E., Winter 2002*)

Successes r	*Frequency f*	*Product* $r^2 f$
0	q^n	0
1	$n\, q^{n-1} p$	$n\, q^{n-1} p$
2	$\frac{n\,(n-1)}{2} q^{n-2} p^2$	$2\, n\,(n-1)\, q^{n-2} p^2$
3	$\frac{n(n-1)\,(n-2)}{6} q^{n-3} p^3$	$\frac{3n\,(n-1)\,(n-2)}{2} q^{n-3} p^3$
.....	.....	......
n	p^n	$n^2 p^n$

We know that $$\sigma^2 = \frac{\Sigma f r^2}{\Sigma f} - \left(\frac{\Sigma f r}{\Sigma f}\right)^2 \qquad ...(1)$$

r is the deviation of items (successes) from 0.

$$\Sigma\, f = 1, \quad \Sigma\, fr = np$$

$$\Sigma\, fr^2 = 0 + nq^{n-1} p + 2n(n-1) q^{n-2} p^2 + \frac{3n(n-1)(n-2)}{2} q^{n-3} p^3 + ... + n^2 p^n$$

$$= np\,[q^{n-1} + \frac{2(n-1)}{1!} q^{n-2} p + \frac{3\,(n-1)\,(n-2)}{2!} q^{n-3} p^2 + ... + np^{n-1}]$$

$$= np\,[q^{n-1} + \frac{(n-1)\,q^{n-2} p}{1!} + \frac{(n-1)\,(n-2)}{2!} q^{n-3} p^2 + ... + p^{n-1}$$

$$+ \frac{(n-1) q^{n-2} p}{1!} + \frac{2\,(n-1)\,(n-2)}{2!} q^{n-3} p^2 + ... + (n-1)\, p^{n-1}\Bigg]$$

$$= n\,p\Bigg[q^{n-1} + (n-1)\, q^{n-2} p + \frac{(n-1)\,(n-2)}{2!} q^{n-3} p^2 + ... + p^{n-1}$$

$$+ (n-1)\, p\left\{q^{n-2} + (n-2)\, q^{n-3} p + \frac{(n-2)(n-3)}{2!} q^{n-4} p^2 + ... + p^{n-2}\right\}\Bigg]$$

$$= np\,[(q+p)^{n-1} + (n-1)\, p\,(q+p)^{n-2}] = np\,[1 + (n-1)\,p]$$

$$= np\,[np + (1-p)] = np[np+q] = n^2p^2 + npq$$

Putting these values in (1), we have

$$\text{Variance} = \sigma^2 = \frac{n^2 p^2 + n\,pq}{1} - \left(\frac{np}{1}\right)^2 = npq,$$

$$S.D. = \sigma = \sqrt{n\,p\,q}$$

Hence for the binomial distribution,

$$\text{Mean} = np, \quad \mu_2 = \sigma^2 = n\,p\,q$$

Example 10. *Find the first four moments of the binomial distribution.* (*AMIETE, Summer 2000*)

Solution. First moment about the origin

$$\mu_1' = \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r}\, r = \sum_{r=0}^{n} r.\frac{n(n-1)(n-2).......(n-r+1)}{r!} p^r q^{n-r}$$

$$= n\sum_{r=1}^{n} \frac{(n-1)(n-2).......(n-r+1)}{(n-r)!} p^r q^{n-r} = np\sum_{r=1}^{n} {}^{n-1}C_{r-1}\, p^{r-1} q^{n-r}$$

$$= np\,(q+p)^{n-1} = np$$

Thus, the mean of the Binomial distribution is np.

Second moment about the origin

$$\mu_2' = \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r} r^2 \qquad [r^2 = r(r-1) + r]$$

$$= \sum_{r=0}^{n} \{r(r-1)+r\}\, {}^nC_r\, p^r q^{n-r} = \sum_{r=0}^{n} r(r-1)\, {}^nC_r\, p^r q^{n-r} + \sum_{r=0}^{n} r\cdot {}^nC_r\, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} \frac{r(r-1)\,n(n-1)(n-2).......(n-r+1)}{r!}\, p^r q^{n-r}$$

$$\sum_{r=0}^{n} \frac{r\,n(n-1)(n-2).......(n-r+1)}{r!}\, p^r q^{n-r}$$

$$= n(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)(n-3).......(n-r+1)}{(r-2)!}\, p^{r-2} q^{n-r}$$

$$+ np\,\Sigma \frac{(n-1)(n-2).......(n-r+1)}{(r-1)!}\, p^{r-1} q^{n-r}$$

$$= n(n-1)\,p^2 (q+p)^{n-2} + np\,(q+p)^{n-1} = n(n-1)\,p^2 + np$$

Third moment about the origin

$$\mu_3' = \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r} \cdot r^3$$

[Let $r^3 = Ar(r-1)(r-2) + Br(r-1) + Cr$

By putting $r = 1, 2, 3$, we get A = 1, B = 3, C = 1]

$$\mu_3' = \sum_{r=0}^{n} \{r(r-1)(r-2) + 3r(r-1) + r\}\, {}^nC_r\, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r(r-1)(r-2)\, {}^nC_r\, p^r q^{n-r} + 3\sum_{r=0}^{n} r(r-1)\cdot {}^nC_r\, p^r q^{n-r} + \sum_{r=0}^{n} r\cdot {}^nC_r\, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} \frac{r(r-1)(r-2)\cdot n(n-1).......(n-r+1)}{r!}\, p^r q^{n-r}$$

$$+ 3\sum_{r=0}^{n} \frac{r(r-1)\cdot n(n-1).......(n-r+1)}{r!}\, p^r q^{n-r} + \sum_{r=0}^{n} r\,\frac{n(n-1).......(n-r+1)}{r!}\, p^r q^{n-r}$$

$$= \sum_{r=3}^{n} \frac{n(n-1)(n-2)(n-3).......(n-r+1)}{(r-3)!}\, p^r q^{n-r}$$

$$+ 3\sum_{r=2}^{n} \frac{n(n-1)(n-2)(n-3).......(n-r+1)}{(r-2)!}\, p^r q^{n-r}$$

$$+ \sum_{r=1}^{n} \frac{n(n-1)(n-2).......(n-r+1)}{(r-1)!}\, p^r q^{n-r}$$

$$= n(n-1)(n-2)p^3 \sum_{r=3}^{n} {}^{n-3}C_{r-3}\, p^{r-3} q^{n-3} + 3n(n-1)p^2 \sum_{r=2}^{n} {}^{n-2}C_{r-2}\, p^{r-2} q^{n-2}$$

$$+ np \sum_{r=1}^{n} {}^{(n-1)}C_{r-1}\, p^{r-1} q^{n-1}$$

$$= n(n-1)(n-2)\,p^3 (q+p)^{n-3} + 3\,n(n-1)\,p^2 (q+p)^{n-2} + np\,(q+p)^{n-1}$$

$$= n(n-1)(n-2)\,p^3 + 3\,n(n-1)\,p^2 + np$$

Fourth Moment

$$\mu_4' = \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r} \cdot r^4$$

[Let $r^4 = Ar\,(r-1)(r-2)(r-3) + Br\,(r-1)(r-2) + Cr\,(r-1) + Dr$

By putting r = 1, 2, 3, 4, we get A = 1, B = 6, C = 7, D = 1]

$$\mu_4' = \sum_{r=0}^{n} r(r-1)(r-2)(r-3)\cdot {}^nC_r\, p^r q^{n-r} + \sum_{r=0}^{n} 6r(r-1)(r-2)\cdot {}^nC_r\, p^r q^{n-r}$$

$$+ \sum_{r=0}^{n} 7r(r-1)\cdot {}^nC_r\, p^r q^{n-r} + \sum_{r=0}^{n} r\cdot {}^nC_r\, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} \frac{r(r-1)(r-2)(r-3)\cdot n(n-1).......(n-r+1)}{r!} p^r q^{n-r}$$

$$+ 6\sum_{r=0}^{n} \frac{r(r-1)(r-2)\cdot n(n-1).......(n-r+1)}{r!} p^r q^{n-r}$$

$$+7\sum_{r=0}^{n} \frac{r(r-1)\cdot n(n-1).......(n-r+1)}{r!} p^r q^{n-r} + \sum_{r=0}^{n} \frac{r\cdot n(n-1).......(n-r+1)}{r!} p^r q^{n-r}$$

$$= \sum_{r=4}^{n} \frac{n\,(n-1)(n-2)\,(n-3)(n-4)...(n-r+1)}{(r-4)!} p^r q^{n-r}$$

$$+6\sum_{r=3}^{n} \frac{n(n-1)(n-2)(n-3)...(n-r+1)}{(r-3)!} p^r q^{n-r}$$

$$+7\sum_{r=2}^{n} \frac{n(n-1)(n-2)...(n-r+1)}{(r-2)!} p^r q^{n-r} + \Sigma \frac{n(n-1)...(n-r+1)}{(r-1)!} p^r q^{n-r}$$

$$= n(n-1)(n-2)(n-3)\sum_{r=4}^{n} {}^{n-4}C_{r-4}\, p^r q^{n-r} + 6n(n-1)(n-2)\sum_{r=3}^{n} {}^{n-3}C_{r-3}\cdot p^r q^{n-r}$$

$$+7n(n-1)\sum_{r=2}^{n} {}^{n-2}C_{r-2}\cdot p^r q^{n-r} + n\sum_{r=1}^{n} {}^{n-1}C_{r-1}\cdot p^r q^{n-r}$$

$$= n\,(n-1)\,(n-2)\,(n-3)\,p^4 (q+p)^{n-4} + 6n\,(n-1)(n-2)\,p^3 (q+p)^{n-3}$$

$$+ 7n\,(n-1)\,p^2\,(q+p)^{n-2} + np\,(q+p)^{n-1}$$

$$= n\,(n-1)\,(n-2)\,(n-3)\,p^4 + 6n\,(n-1)(n-2)\,p^3 + 7n\,(n-1)\,p^2 + np$$

63.4 CENTRAL MOMENTS : (Moments about the mean)

Now, the first four central moments are obtained as follows:

Second Central Moment

$\mu_2 = \mu_2' - \mu_1' = [\,n\,(n-1)\,p^2 + np\,] - n^2 p^2 = np\,[\,(n-1)\,p + 1 - np\,] = np\,(1-p) = npq$

Variance of Binomial distribution is npq

Third Central Moment

$$\mu_3 = \mu_3' - 3\,\mu_2'\,\mu_1' + 2\,\mu_1^{3\prime}$$

$$= \{\,n\,(n-1)\,(n-2)\,p^3 + 3n\,(n-1)\,p^2 + np\} - 3\{\,n^2p^2 + npq\,)\,np\} + 2\,n^3 p^3$$

$$= np\,[\,-3\,n\,p^2 + 3np + 2\,p^2 - 3p + 1 - 3npq\,]$$

$$= np\,[\,3np\,(1-p) + 2p^2 - 3p + 1 - 3npq\,]$$

$$= np\,[\,3npq + 2p^2 - 3p + 1 - 3\,npq\,] = np\,[\,2p^2 - 3p + 1\,] = np[\,2p^2 - 2p + q\,]$$

$$= np\,[-2p\,(1-p) + q\,] = np\,(-2pq + q\,) = npq\,(1-2p\,) = npq\,(\,q-p\,)$$

Fourth Central Moment

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np - 4[n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np]np + 6[n(n-1)p^2 + np]n^2p^2 - 3n^4p^4$$

$$= np[(n-1)(n-2)(n-3)p^3 + 6(n-1)(n-2)p^2 + 7(n-1)p + 1 - 4\{n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np\} + 6\{n(n-1)p^2 + np\}np - 3n^3p^3]$$

$$= np[(n^3 - 6n^2 + 11n - 6)p^3 + (6n^2 - 18n + 12)p^2 + 7np - 7p + 1 + \{(-4n^3 + 12n^2 - 8n)p^3 - 4(3n^2 - 3n)p^2 - 4np\} + \{(6n^3 - 6n^2)p^3 + 6n^2p^2\} - 3n^3p^3]$$

$$= np[(n^3 - 6n^2 + 11n - 6 - 4n^3 + 12n^2 - 8n + 6n^3 - 6n^2 - 3n^3)p^3 + (6n^2 - 18n + 12 - 12n^2 + 12n + 6n^2)p^2 + (7n - 7 - 4n)p + 1]$$

$$= np[(3n-6)p^3 + (-6n+12)p^2 + (3n-7)p + 1]$$

$$= np[3np^3 - 6p^3 - 6np^2 + 12p^2 + 3np - 7p + 1]$$

$$= np[3np^3 - 3np^2 - 6p^3 + 6p^2 - 3np^2 + 3np + 6p^2 - 6p - p + 1]$$

$$= np[-3np^2(1-p) + 6p^2(1-p) + 3np(1-p) - 6p(1-p) + (1-p)]$$

$$= np[-3np^2q + 6p^2q + 3npq - 6pq + q] = npq[-3np^2 + 6p^2 + 3np - 6p + 1]$$

$$= npq[3np(1-p) - 6p(1-p) + 1] = npq[3npq - 6pq + 1]$$

$$= npq[1 + 3(n-2)pq]$$ **Ans.**

63.5 MOMENT GENERATING FUNCTIONS OF BINOMIAL DISTRIBUTION ABOUT ORIGIN

$$M_0(t) = E(e^{tx}) = \Sigma\, {}^nC_x p^x q^{n-x} \cdot e^{tx}$$

$$= \Sigma\, {}^nC_x (pe^t)^x q^{n-x} = (q + pe^t)^n$$

Differentiating w.r.t. 't' we get $Ma'(t) = n(q + pe^t)^{n-1}p \cdot e^t$

On putting $t = 0$, we get $\mu_1' = n(q+p)^{n-1}p$

$$\mu_1' = np$$

Since $\quad M_a(t) = e^{-at}M_0(t)$

Moment generating function of the Binomial distribution about its mean $(m) = np$ is given by

$$M_m(t) = e^{-npt}M_0(t)$$

$$M_m(t) = e^{-npt}(q + pe^t)^n = (qe^{-pt} + pe^{-pt+t})^n = (qe^{-pt} + pe^{(1-p)t})^n$$

$$= \left[q\left(1 - pt + \frac{p^2t^2}{2!} - \frac{p^3t^2}{2!} + \frac{p^4t^4}{4!} + \ldots\right) + p\left(1 + qt + \frac{q^2t^2}{2!} + \frac{q^3t^3}{3!} + \frac{p^4t^4}{4!} + \ldots\right)\right]^n$$

$$= \left[1 + pq\frac{t^2}{2!} + pq(q^2 - p^2)\frac{t^3}{3!} + pq(q^3 + p^3)\frac{t^4}{4!} + \ldots\right]^n$$

$$1 + \mu_1 t + \mu_2\frac{t^2}{2!} + \mu_3\frac{t^3}{3!} + \mu_4\frac{t^4}{4!} + \ldots\ldots$$

$$= 1 + npq\frac{t^2}{2!} + npq(q-p)\frac{t^3}{3!} + npq[1 + 3(n-2)pq]\frac{t^4}{4!} + \ldots\ldots$$

Equating the coefficients of like powers of t on both sides, we get

$$\mu_2 = npq, \quad \mu_3 = npq(q-p), \; \mu_4 = npq[1 + 3(n-2)pq]$$

Hence the moment coefficient of skewness is

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{[npq\,(q-p)]^2}{(npq)^3} = \frac{(q-p)^2}{npq} \quad ; \quad \gamma_1 = \sqrt{\beta_1} = \frac{q-p}{\sqrt{npq}}$$

Coefficient of Kurtosis is given by

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq\,[1+3pq\,(n-2)]}{(npq)^2} = 3 + \frac{1-6pq}{npq} \; ; \; \gamma_2 = \beta_2 - 3 = \frac{1-6pq}{npq}$$

Example 11. *If the probability of a defective bolt is 0.1, find (a) the mean (b) the standard deviation for the distribution bolts in a total of 400.*

Solution. $n = 400$, $p = 0.1$, Mean $= np = 400 \times 0.1 = 40$

$$\text{Standard deviation} = \sqrt{npq} = \sqrt{400 \times 0.1(1-0.1)}$$

$$= \sqrt{400 \times 0.1 \times 0.9} = 20 \times 0.3 = 6$$ **Ans.**

Example 12. *A die is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean and variance of the number of successes.* *(AMIETE, Dec. 2010)*

Solution. $$n = 3,\; p = \frac{1}{3},\; q = \frac{2}{3}$$

$$\text{Mean} = np = 3 \times \frac{1}{3} = 1$$

$$\text{Variance} = npq = 3 \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{3}$$ **Ans.**

Example 13. *If mean and variance of a binomial distribution are 4 and 2 respectively, find the probability of (i) exactly 2 successes (ii) less than 2 successes (iii) at least 2 successes.* *(R.G.P.V., Bhopal, II Semester, June 2005)*

Solution. Mean = 4 $\Rightarrow$ $np = 4$ ... (1)

Variance = 2 $\quad npq = 2$ ... (2)

Dividing (2) by (1), we get

$$\frac{npq}{np} = \frac{2}{4} \quad \Rightarrow \quad q = \frac{1}{2}$$

$$p = 1 - q = 1 - \frac{1}{2} = \frac{1}{2}$$

Putting the value of p in (1), we get

$$n\left(\frac{1}{2}\right) = 4 \quad \Rightarrow \quad n = 8$$

(*i*) Probability of r successes = ${}^nC_r\, p^r\, q^{n-r}$

$$P(2) = {}^8C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{8-2} = {}^8C_2 \left(\frac{1}{2}\right)^8 = \frac{8 \times 7}{2}\,\frac{1}{256} = \frac{7}{64}$$

(*ii*) P (less than 2 successes) = $P(0) + P(1) = {}^8C_0\, p^0 q^8 + {}^8C_1\, p^1 q^7$

$$= \frac{1}{256} + 8\,\frac{1}{2}\left(\frac{1}{2}\right)^7 = \frac{9}{256}$$

(*iii*) P (at least 2 successes) = $P(2) + P(3) + \ldots + P(8)$

$$= P(0) + P(1) + P(2) + P(3) + \ldots + P(8) - P(0) - P(1)$$

$$= 1 - P(0) - P(1) = 1 - [P(0) + P(1)] = 1 - \frac{9}{256} = \frac{247}{256}$$ **Ans.**

Example 14. *Fit a Binomial distribution for the following data and compare the theoretical frequencies with actual ones :*

x	0	1	2	3	4	5
y	2	14	20	34	22	8

(*R.G.P.V., Bhopal, II Semester, June 2006*)

Solution.

x	$y = f$	fx	$P = {}^5C_r p^r q^{5-r}$	*Theoretical Frequency*
0	2	0	${}^5C_0\,(0.568)^0\,(0.432)^5 = 0.015$	$100 \times 0.015 = 1.5$
1	14	14	${}^5C_1\,(0.568)^1\,(0.432)^4 = 0.099$	$100 \times 0.099 = 9.9$
2	20	40	${}^5C_2\,(0.568)^2\,(0.432)^3 = 0.260$	$100 \times 0.260 = 26.0$
3	34	102	${}^5C_3\,(0.568)^3\,(0.432)^2 = 0.342$	$100 \times 0.342 = 34.2$
4	22	88	${}^5C_4\,(0.568)^4\,(0.432)^1 = 0.225$	$100 \times 0.225 = 22.5$
5	8	40	${}^5C_5\,(0.568)^5\,(0.432)^0 = 0.0591$	$100 \times 0.0591 = 5.91$
	100	284		

$$\Sigma f = 100, \qquad \Sigma f x = 284$$

$$\text{Mean} = \frac{\Sigma f x}{\Sigma f} = \frac{284}{100} = 2.84$$

$$\text{Mean} = np = 2.84$$

$$5p = 2.84 \quad \Rightarrow \quad p = \frac{2.84}{5} = 0.568$$

$$q = 1 - p = 1 - 0.568 = 0.432$$

Binomial Distribution $= 100\,(0.432 + 0.568)^5$ **Ans.**

63.6 RECURRENCE RELATION FOR THE BINOMIAL DISTRIBUTION

By Binomial distribution, $P(r) = {}^nC_r\; p^r\; q^{n-r}$ ...(1) (*A.M.I.E., Summer 2002*)

$$P(r+1) = {}^nC_{r+1}\; p^{r+1}\, q^{n-r-1} \qquad ...(2)$$

On dividing (2) by (1), we get

$$\frac{P(r+1)}{P(r)} = \frac{{}^nC_{r+1}}{{}^nC_r}\,\frac{p^{r+1}q^{n-r-1}}{p^r q^{n-r}}$$

$$= \frac{n(n-1)(n-2).....(n-r)}{(r+1)!}\,\frac{r!}{n(n-1)(n-2).....(n-r+1)}\,\frac{p}{q}$$

$$\frac{P(r+1)}{P(r)} = \frac{n-r}{r+1}\,\frac{p}{q} \quad \text{or} \quad P(r+1) = \frac{n-r}{r+1}\,\frac{p}{q}\,P(r)$$ **Ans.**

EXERCISE 63.2

1. Fit a binomial distribution to the following frequency data:

x	0	1	3	4
f	28	62	10	4

(*U. P III Sem. Dec. 2004*)

Ans. $P(r) = {}^{104}C_r (0.00999)^r (0.99111)^{104-r}$

2. Four coins were tossed 200 times. The number of tosses showing 0, 1, 2, 3 and 4 heads were found to be as under. Fit a binomial distribution to these observed results. Find the expected frequencies.

No. of heads:,	0	1	2	3	4
No. of tosses:	15	35	90	40	20

(*A.M.I.E. Winter 2004*)

3. Fill in the blanks :

(*a*) If three persons selected at random are stopped on a street, then the probability that all of them were born on Sunday is _______. (*A.M.I.E., Winter 2001*) **Ans.** $\dfrac{1}{343}$

(*b*) The mean, standard deviation and skewness of binomial distribution are ______, ____ and_____.

(*A.M.I.E., Summer 2001*) **Ans.** $n\,p, \sqrt{n\,p\,q}$

4. Tick √ the correct answer :

(*a*) The variance for a Binomial distribution is :

(*i*) $n\,p$ (*ii*) $\sqrt{n\,p}$ (iii) $n\,p\,q$ (*iv*) $\sqrt{n\,p\,q}$

(*R.G.P.V., Bhopal, II Semester, June 2007*) **Ans.** (*iii*)

(*b*) For the Binomial distribution $(p + q)^n$, he relation of mean and variance is :

(*i*) means = variance (*ii*) means < variance

(*iii*) mean > variance (*iv*) $(\text{mean})^2$ = variance

(*R.G.P.V., Bhopal, II Semester, June 2006*) **Ans.** (*iii*)

(*c*) In usual notation, for Binomial distribution, $n\ p\ q$, is

(*i*) $< n\,p$ (*ii*) $n\,p$ (*iii*) $> n\,p$ (*iv*) None of the above

(*A.M.I.E., Winter 2005*) **Ans.** (*i*)

CHAPTER

64

POISSON DISTRIBUTION

64.1 POISSON DISTRIBUTION

Poisson distribution is a particular limiting form of the Binomial distribution when p (or q) is very small and n is large enough.

Poisson distribution is

$$\boxed{P(r) = \frac{m^r e^{-m}}{r!}}$$

where m is the mean of the distribution.

Proof. In Binomial distribution.

$$P(r) = {}^nC_r\, q^{n-r} p^r = {}^nC_r (1-p)^{n-r} p^r$$

$$\left(\text{since mean} = m = np,\ p = \frac{m}{n}\right)$$

$$= {}^nC_r \left(1-\frac{m}{n}\right)^{n-r} \left(\frac{m}{n}\right)^r \qquad (m \text{ is constant})$$

$$= \frac{n(n-1)(n-2)\ldots(n-\overline{r-1})}{r!}\left(\frac{m}{n}\right)^r \left(1-\frac{m}{n}\right)^{n-r}$$

$$= \frac{\frac{n}{n}\left(\frac{n}{n}-\frac{1}{n}\right)\left(\frac{n}{n}-\frac{2}{n}\right)\ldots\left(\frac{n}{n}-\frac{r-1}{n}\right)m^r\left(1-\frac{m}{n}\right)^n}{r!\ \left(1-\frac{m}{n}\right)^r}$$

$$= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)m^r\left(1-\frac{m}{n}\right)^n}{r!\ \left(1-\frac{m}{n}\right)^r}$$

Taking limits, when n tends to infinity

$$\lim_{n\to\infty}\left(1-\frac{m}{n}\right)^n = \lim_{n\to\infty}\left[\left(1-\frac{m}{n}\right)^{-\frac{n}{m}}\right]^{-m} = e^{-m} \text{ and } \lim_{n\to\infty}\left(1-\frac{m}{n}\right)^r = 1$$

$$P(r) = \frac{m^r}{r!}e^{-m}$$

$$\boxed{P(r) = \frac{e^{-m}.m^r}{r!}}$$

64.2 MEAN OF POISSON DISTRIBUTION

$$P(r) = \frac{e^{-m}.m^r}{r!}$$ (*A.M.I.E.T.E., Summer 2004, 2002*)

Successes r	*Frequency f*	*f.r*
0	$\frac{e^{-m}m^0}{0!}$	0
1	$\frac{e^{-m}m^1}{1!}$	$e^{-m}. m$
2	$\frac{e^{-m}m^2}{2!}$	$e^{-m}. m^2$
3	$\frac{e^{-m}m^3}{3!}$	$\frac{e^{-m}.m^3}{2!}$
...	...	...
r	$\frac{e^{-m}m^r}{r!}$	$\frac{e^{-m}.m^r}{(r-1)!}$
...	...	...

$$\Sigma f\, r = 0 + e^{-m}.m + e^{-m}.m^2 + e^{-m}.\frac{m^3}{2!} + \ldots + e^{-m}\frac{m^r}{(r-1)!} + \ldots = e^{-m}.m\left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \ldots + \frac{m^{r-1}}{(r-1)!} + \ldots\right]$$

$$= m\,.\,e^{-m}.\,[e^m] = m$$

$$\text{Mean} = \frac{\Sigma fr}{\Sigma f} = \frac{m}{1},$$

$$\boxed{\textbf{Mean} = \boldsymbol{m}.}$$ **Ans.**

64.3 STANDARD DEVIATION OF POISSON DISTRIBUTION

$$P(r) = \frac{e^{-m}m^r}{r!}$$ (*A.M.I.E.T.E., Summer 2002*)

Successes r	*Frequency f*	*Product rf*	*Product r^2f*
0	$\frac{e^{-m}m^0}{0!}$	0	0
1	$\frac{e^{-m}m^1}{1!}$	$e^{-m}. m$	$e^{-m}.m$
2	$\frac{e^{-m}m^2}{2!}$	$e^{-m}. m^2$	$2e^{-m}. m^2$
3	$\frac{e^{-m}m^3}{3!}$	$\frac{e^{-m}.m^3}{2!}$	$3e^{-m}.\frac{m^3}{2!}$
....	....	....	....
r	$\frac{e^{-m}m^r}{r!}$	$\frac{e^{-m}.m^r}{(r-1)!}$	$\frac{re^{-m}.m^r}{(r-1)!}$
....	....	....	....

$\Sigma f = 1,$ $\Sigma fr = m$

$$\Sigma f\, r^2 = 0 + e^{-m}.m + 2e^{-m}.m^2 + 3.e^{-m}.\frac{m^3}{2} + \ldots + \frac{r\,e^{-m}.m^r}{(r-1)!} + ..$$

$$= m.e^{-m}\left[1+2m+\frac{3m^2}{2!}+\frac{4m^3}{3!}+\ldots\frac{r.m^{r-1}}{(r-1)!}+..\right]$$

$$= m.e^{-m}\left[1+m+\frac{m^2}{2!}+\frac{m^3}{3!}+\ldots\frac{m^{r-1}}{(r-1)!}+\ldots..+\ m+2\frac{m^2}{2!}+\frac{3m^3}{3!}+\ldots+\frac{(r-1)m^{r-1}}{(r-1)!}+\ldots\right]$$

$$= m.e^{-m}\left[\left\{1+m+\frac{m^2}{2!}+\frac{m^3}{3!}+\ldots+\frac{m^{r-1}}{(r-1)!}+\ldots\right\}\ +m\left\{1+\frac{m}{1!}+\frac{m^2}{2!}+\ldots+\frac{m^{r-2}}{(r-2)!}+\ldots\right\}\right]$$

$$= m.e^{-m}\,[e^m + m\,e^m] = m + m^2$$

$$\sigma^2 = \frac{\Sigma fr^2}{\Sigma f}-\left(\frac{\Sigma fr}{\Sigma f}\right)^2 = \frac{m+m^2}{1}-(m)^2 = m \quad \text{or} \quad \sigma = \sqrt{m}$$

$$\boxed{\textbf{S. D.} = \sqrt{m}}$$

Hence mean and variance of a Poisson distribution are each equal to m. Similarly we can obtain,

$$\mu_3 = m, \qquad \mu_4 = 3m^2 + m$$

$$\beta_1 = \frac{1}{m}, \qquad \beta_2 = 3+\frac{1}{m}$$

$$\gamma_1 = \frac{1}{\sqrt{m}}, \qquad \gamma_2 = \frac{1}{m}$$

64.4 MEAN DEVIATION

Show that in a Poisson distribution with unit mean, and the mean deviation about the mean is $\left(\frac{2}{e}\right)$ times the standard deviation.

Solution. $P(r) = \frac{m^r}{r!}e^{-m}$ But mean = 1 *i.e.* $m = 1$ and S.D. = $\sqrt{m}$ = 1

Hence, $$P(r) = \frac{e^{-m}}{r!} = \frac{e^{-1}}{r!} = \frac{1}{e}.\frac{1}{r!}$$

r	$P(r)$	$\lvert r-1\rvert$	$P(r)\,\lvert r-1\rvert$
0	$\frac{1}{e}$	1	$\frac{1}{e}$
1	$\frac{1}{e}$	0	0
2	$\frac{1}{e}\frac{1}{2!}$	1	$\frac{1}{e}\frac{1}{2!}$
3	$\frac{1}{e}\frac{1}{3!}$	2	$\frac{1}{e}\frac{2}{3!}$
4	$\frac{1}{e}\frac{1}{4!}$	3	$\frac{1}{e}\frac{3}{4!}$
....	....	....	....
r	$\frac{1}{e}\frac{1}{r!}$	$r-1$	$\frac{1}{e}\frac{r-1}{r!}$

$$\Sigma P(r)\,|r-1| = \frac{1}{e}+0+\frac{1}{e}\frac{1}{2!}+\frac{1}{e}\frac{2}{3!}+\frac{1}{e}\frac{3}{4!}+\ldots+\frac{1}{e}\frac{r-1}{r!}+\ldots = \frac{1}{e}\left[1+0+\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{r-1}{r!}+\ldots\right]$$

$$= \frac{1}{e}\left[1+\left(\frac{1}{1!}-\frac{1}{1!}\right)+\left(\frac{2}{2!}-\frac{1}{2!}\right)+\left(\frac{3}{3!}-\frac{1}{3!}\right)+\left(\frac{4}{4!}-\frac{1}{4!}\right)+\ldots+\left(\frac{r}{r!}-\frac{1}{r!}\right)+\ldots\right]$$

$$= \frac{1}{e}\left[1+\frac{1}{1!}+\frac{2}{2!}+\frac{3}{3!}+\frac{4}{4!}\times\ldots+\frac{r}{r!}+\ldots-\frac{1}{1!}-\frac{1}{2!}-\frac{1}{3!}-\frac{1}{4!}\ldots-\frac{1}{r!}-\ldots\right]$$

$$= \frac{1}{e}\left[1+\left\{1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{(r-1)!}+\ldots\right\}-\left\{1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots+\frac{1}{r!}\ldots\right\}+1\right]$$

$$= \frac{1}{e}[1+e-e+1]=\frac{2}{e} = \frac{2}{e}(1)=\frac{2}{e}\text{ S.D.}$$ **Proved.**

64.5 MOMENT GENERATING FUNCTION OF POISSON DISTRIBUTION

(*A.M.I.E., Summer 2000*)

Solution. $$P(r)=\frac{e^{-m}m^r}{r!}$$

Let $M_x(t)$ be the moment generating function, then

$$M_x(t)=\sum_{r=0}^{\infty}e^{tr}\frac{e^{-m}.m^r}{r!}=\sum_{r=0}^{\infty}e^{-m}.\frac{(me^t)^r}{r!} = e^{-m}\left[1+me^t+\frac{(me^t)^2}{2!}+\frac{(me^t)^3}{3!}+\ldots\right]=e^{-m}.e^{me^t} = e^{m(e^t-1)}$$

64.6 CUMULANTS

The cumulant generating function $K_x(t)$ is given by

$$K_x(t)=\log_e M_x(t)=\log_e e^{m(e^t-1)} = m(e^t-1)\log_e e$$

$$= m(e^t-1)=m\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots.+\frac{t^r}{r!}+\ldots-1\right] = m\left[t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots+\frac{t^r}{r!}+\ldots\right]$$

Now $K_r = r$ th cumulant = coefficient of $\frac{t^r}{r!}$ in $K(t) = m$

i.e., $k_r = m,$ where $r = 1, 2, 3, \ldots.$

Hence, all the cumulants of the Poisson distribution are equal. In particular, we have

Mean $= K_1 = m,\ \mu_2 = K_2 = m,\ \mu_3 = K_3 = m$

$$\mu_4 = K_4 + 3K_2^2 = m + 3m^2$$

$$\beta_1=\frac{\mu_3^2}{\mu_2^3}=\frac{m^2}{m^3}=\frac{1}{m},\quad \beta_2=\frac{\mu_4}{\mu_2^2}=\frac{m+3m^2}{m^2}=\frac{1}{m}+3$$

64.7 RECURRENCE FORMULA FOR POISSON DISTRIBUTION

Solution. By Poisson distribution

$$P(r)=\frac{e^{-m}.m^r}{r!} \qquad \ldots (1)$$

$$P(r+1)=\frac{e^{-m}m^{r+1}}{(r+1)!} \qquad \ldots (2)$$

By dividing (2) by (1), we get

$$\frac{P(r+1)}{P(r)}=\frac{e^{-m}m^{r+1}}{(r+1)!}\frac{r!}{e^{-m}.m^r}=\frac{m}{r+1}$$

$$P(r+1)=\frac{m}{r+1}P(r)$$ **Ans.**

Example 1. *If the variance of the Poisson distribution is 2, find the probabilities for $r = 1, 2, 3, 4$ from the recurrence relation of the Poisson distribution. Also find $P(r \geq 4)$.*

Solution. Variance $= m = 2$;

Mean = 2

$$P(r+1)=\frac{m}{r+1}P(r) \qquad \text{[Recurrence relation]}$$

Now $$P(r+1)=\frac{2}{r+1}P(r) \qquad (m=2)$$

If $r=0,\ P(1)=\dfrac{2}{0+1}P(0)=\dfrac{2}{0+1}(0.1353)=0.2706$ $\qquad P(0)=e^{-m}=e^{-2}=0.1353$

If $r=1,\ P(2)=\dfrac{2}{1+1}P(1) \quad =\dfrac{2}{2}(0.2706)=0.2706$

If $r=2,\ P(3)=\dfrac{2}{2+1}P(2) \quad =\dfrac{2}{3}(0.2706)=0.1804$

If $r=3,\ P(4)=\dfrac{2}{3+1}P(3) \quad =\dfrac{1}{2}(0.1804)=0.0902$

$$\begin{aligned}P(r\geq 4)&=P(4)+P(5)+P(6)+\ldots\\ &=1-[P(0)+P(1)+P(2)+P(3)]\\ &=1-[0.1353+0.2706+0.2706+0.1804]\\ &=1-0.8569=0.1431\end{aligned}$$ **Ans.**

Example 2. *Assume that the probability of an individual coal miner being killed in a mine accident during a year is* $\dfrac{1}{2400}$. *Use appropriate statistical distribution to calculate the probability that in a mine employing 200 miners, there will be at least one fatal accident in a year.* (*A.M.I.E.T.E., Summer 2001*)

Solution. $P=\dfrac{1}{2400},\quad n=200$

$$m=np=\frac{200}{2400}=\frac{1}{12}$$

P (At least one) $=P(1\text{ or }2\text{ or }3\text{ or }\ldots\ldots\text{ or }200)=P(1)+P(2)+P(3)+\ldots+P(200)$

$$=1-P(0)=1-\frac{e^{-m}.m^0}{0!}=1-e^{-\frac{1}{12}}=1-0.92=0.08$$ **Ans.**

Example 3. *Suppose 3% of bolts made by a machine are defective, the defects occurring at random during production. If bolts are packaged 50 per box, find*
(a) exact probability and
(b) Poisson approximation to it, that a given box will contain 5 defectives.

Solution. $$p=\frac{3}{100}=0.03$$

(*a*) $$q=1-p=1-0.03=0.97$$

Hence the probability for 5 defective bolts in a lot of 50

$$={}^{50}C_5\,(0.03)^5\,(0.97)^{45}=0.013074 \quad \text{(Binomial Distribution)}$$

(*b*) To get Poisson approximation $m=n\,p=50\times\dfrac{3}{100}=\dfrac{3}{2}=1.5$

Required Poisson approximation $=\dfrac{m^re^{-m}}{r!}=\dfrac{(1.5)^5e^{-1.5}}{5!}=0.01412$ **Ans.**

Example 4. *The number of arrivals of customers during any day follows Poisson distribution with a mean of 5. What is the probability that the total number of customers on two days selected at random is less than 2?*

Solution. $m=5$

$$P(r)=\frac{e^{-m}m^r}{r!},\quad P(r)=\frac{e^{-5}(5)^r}{r!}$$

If the number of customers on two days < 2 = 1 or 0

First day	*Second Day*	*Total*
0	0	0
0	1	1
1	0	1

Required probability $= P(0)\,P(0) + P(0)\,P(1) + P(1)\,P(0)$

$$= \frac{e^{-5}(5)^0}{0!}\cdot\frac{e^{-5}(5)^0}{0!} + \frac{e^{-5}(5)^0}{0!}\cdot\frac{e^{-5}(5)^1}{1!} + \frac{e^{-5}(5)^1}{1!}\,\frac{e^{-5}(5)^0}{0!}$$

$$= e^{-5}\cdot e^{-5} + e^{-5}\cdot e^{-5}\cdot 5 + e^{-5}\cdot 5\cdot e^{-5}$$

$$= e^{-10}[1+5+5] = 11e^{-10} = 11\times 4.54\times 10^{-5}$$

$$= 4.994\times 10^{-4}$$ **Ans.**

Example 5. *In a certain factory producing cycle tyres, there is a small chance of 1 in 500 tyres to be defective. The tyres are supplied in lots of 10. Using Poisson distribution, calculate the approximate number of lots containing no defective, one defective and two defective tyres, respectively, in a consignment of 10,000 lots.*

Solution. $p = \frac{1}{500}$, $n = 10$

$$m = np = 10\,.\,\frac{1}{500} = \frac{1}{50} = 0.02\,,\quad P(r) = \frac{e^{-m}.m^r}{r!}$$

S.No.	Proability of defective	Number of lots containing defective
1	$P(0) = \frac{e^{-0.02}(0.02)^0}{0!} = e^{-0.02} = 0.9802$	10,000 × 0.9802 = 9802 lots
2	$P(1) = \frac{e^{-0.02}(0.02)^1}{1!}$	10,000 × 0.019604 = 196 lots
	$= 0.9802 \times 0.02 = 0.019604$	
3.	$P(2) = \frac{e^{-0.02}(0.02)^2}{2!}$	10,000 × 0.000196 = 2 lots

$= 0.9802 \times 0.0002 = 0.00019604$ **Ans.**

Example 6. *A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the number of days in a year on which*

(i) neither car is on demand ($e^{-1.5} = 0.2231$)

(*ii*) *a car demand is refused.*

(*MDU, Dec. 2010, A.M.I.E., Summer 2004 Winter 2001, June 2009*)

Solution. $m = 1.5$

(*i*) If the car is not used, then demand $(r) = 0$

$$P(r) = \frac{e^{-m}.m^r}{r!},\quad P(0) = \frac{e^{-1.5}(1.5)^0}{0!} = e^{-1.5} = 0.2231$$

Number of days in a year when the demand is zero = 365 × 0.2231 = 81.4315 **Ans.** 81 days

(*ii*) Some demand is refused if the number of demands is more than two *i.e.* $r > 2$.

$$P(r>2) = P(3) + P(4) + \ldots = 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - \left[\frac{e^{-1.5}(1.5)^0}{0!} + \frac{e^{-1.5}(1.5)^1}{1!} + \frac{e^{-1.5}(1.5)^2}{2!}\right]$$

$= 1 - [e^{-1.5} + e^{-1.5} \times 1.5 + e^{-1.5} \times 1.125] \qquad = 1 - e^{-1.5}\,[1 + 1.5 + 1.125] = 1 - e^{-1.5} \times 3.625$

$= 1 - 0.2231 \times 3.625 = 1 - 0.8087375 \qquad = 0.1912625$ **Ans.**

Number of days in a year when some demand of car is refused

$= 365 \times 0.1912625 = 69.81 = 70$ days **Ans.**

Example 7. *If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the probability that out of 2000 individuals*

(a) exactly 3 (b) more than 2 individuals (c) None (d) More than one individual will suffer a bad reaction. (*A.M.I.E.T.E., Winter, 2002, 2000*)

Solution. $p = 0.001, \qquad n = 2000$

$$m = np = 2000 \times 0.001 = 2$$

$$\therefore \qquad P(r) = \frac{e^{-m} m^r}{r!} = e^{-2}\frac{2^r}{r!} = \frac{1}{e^2} \times \frac{2^r}{r!}$$

(*a*) P (more than 3) $P\,(3) = \dfrac{1}{e^2} \cdot \dfrac{2^3}{3!} = \dfrac{1}{(2.718)^2} \times \dfrac{8}{6} = (0.135) \times \dfrac{4}{3} = 0.18$

(*b*) P (more than 2) $= P\,(3) + P\,(4) + P\,(5) + \ldots + P\,(2000)$

$$= 1 - [P\,(0) + P\,(1) + P\,(2)] = 1 - \left[\frac{e^{-2}(2)^0}{0!} + \frac{e^{-2}(2)^1}{1!} + \frac{e^{-2}(2)^2}{2!}\right]$$

$$= 1 - e^{-2}[1 + 2 + 2] = 1 - \frac{5}{e^2} = 1 - 5 \times 0.135 = 1 - 0.675 = 0.325 \qquad \textbf{Ans.}$$

(*c*) $\qquad P$ (none) $= P\,(0) = \dfrac{e^{-2}(2)^0}{0!} = 0.135$

(*d*) P (more than 1) $= P\,(2) + P\,(3) + P\,(4) + \ldots + P\,(2000) = 1 - [P\,(0) + P\,(1)]$

$$= 1 - \left[\frac{e^{-2}(2)^0}{0!} + \frac{e^{-2}(2)^1}{1!}\right] = 1 - 3e^{-2} = 1 - 3 \times 0.135 = 1 - 0.405 = 0.595 \textbf{ Ans.}$$

Example 8. *In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. Use appropriate and suitable distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 50000 packets.* (*R.G.P.V., Bhopal, II Semester, June 2006*)

Solution. Here, $p = 0.002, \qquad n = 10$

$m = np \quad \Rightarrow \quad m = 10 \times 0.002 = 0.020$

$$P\,(r) = \frac{e^{-m}.\,(m)^r}{r!}$$

r	$p\,(r) = \dfrac{e^{-0.02}\,(0.02)^r}{r!}$	Number of packets = 50000 p
0	$p\,(0) = \dfrac{e^{-0.02}\,(0.02)^0}{0!} = 0.980$	$50000 \times (0.980) = 4900$
1	$p\,(1) = \dfrac{e^{-0.02}\,(0.02)^1}{1!} = 0.0196$	$50000 \times (0.0196) = 980$
2	$p\,(2) = \dfrac{e^{-0.02}\,(0.02)^2}{2!} = 0.000196$	$50000 \times (0.000196) = 9.8$

Hence, number of packets containing no defective razor blades = 49000.
Number of packets containing one defective razor blade = 980
Number of packets containing two defective razor blase = 9.8

Example 9. *If there are 3 misprints in a book of 1000 pages find the probability that a given page will contain*
(i) no misprint (ii) more than 2 misprints. *(U.P., III Semester, Dec. 2009)*

Solution. Total number of pages = 1000
No. of misprints =3

$$P = \frac{3}{1000} = 0.003, \qquad n = 1, \qquad m = nP = 1 \times 0.003 = 0.003$$

Poisson distribution

$$P(r) = \frac{e^{-m}.\, m^r}{r!}, \qquad P(0) = \frac{e^{-0.003}\,(0.003)^0}{0!} = e^{0.003} = 0.997$$

$$P(r > 2) = \text{P}(3) = \frac{e^{-0.003}(0.003)^3}{3!} = 0.0000000045$$

Hence (i) the probability that a page will contain no error = 0.997
(ii) the probability that a page will contain more than two misprints = 0.0000000045 **Ans.**

Example 10. *A manufacturer knows that the condensers he makes contain on an average 1% of defectives. He packs them in boxes of 100. What is the probability that a box picked out at random will contain 4 or more faulty condensers?*

Solution. $P = 1\% = 0.01, n = 100, \quad m = np = 100 \times 0.01 = 1$

$$P(r) = \frac{e^{-m}.(m)^r}{r!} = \frac{e^{-1}(1)^r}{r!} = \frac{e^{-1}}{r!}$$

$$P(\text{4 or more faulty condensers}) = P(4) + P(5) + \ldots + P(100) = 1 - [P(0) + P(1) + P(2) + P(3)]$$

$$= 1 - \left[\frac{e^{-1}}{0!} + \frac{e^{-1}}{1!} + \frac{e^{-1}}{2!} + \frac{e^{-1}}{3!}\right] = 1 - e^{-1}[1 + 1 + \frac{1}{2} + \frac{1}{6}] = 1 - \frac{8}{3e} = 1 - 0.981 = 0.019 \qquad \textbf{Ans.}$$

Example 11. *An insurance company found that only 0.01% of the population is involved in a certain type of accident each year. If its 1000 policy holders were randomly selected from the population, what is the probability that not more than two of its clients are involved in such an accident next year? (given that $e^{-0.1} = 0.9048$)*

Solution.

$$P = 0.01\% = \frac{1}{100} \times \frac{1}{100} = \frac{1}{10000}, \qquad n = 1000$$

$$m = np = (1000) \times \frac{1}{10000} = \frac{1}{10} = 0.1$$

$$P(r) = \frac{e^{-m} m^r}{r!}$$

$$P(\text{not more than 2}) = P(0, 1 \text{ and } 2) = P(0) + P(1) + P(2)$$

$$= \frac{e^{-0.1}(0.1)^0}{0!} + \frac{e^{-0.1}(0.1)^1}{1!} + \frac{e^{-0.1}(0.1)^2}{2!} = e^{-0.1}\left(1 + 0.1 + \frac{0.01}{2}\right)$$

$$= 0.9048 \times 1.105 = 0.9998 \qquad \textbf{Ans.}$$

Example 12. *Fit a Poisson distribution to the set of observations :*

x	0	1	2	3	4
f	122	60	15	2	1

(*R.G.P.V., Bhopal, II Semester, Dec. 2007, June 2007*)

Solution. The mean number $= \dfrac{\Sigma f \,.\, x}{\Sigma f}$.

x	f	fx
0	122	0
1	60	60
2	15	30
3	2	6
4	1	4
Total	200	100

$$\text{Mean} = \frac{\Sigma f x}{\Sigma f} = \frac{100}{200} = \frac{1}{2}$$

x	$P(x) = \dfrac{e^{-1/2}(1/2)^x}{x!}$	Theoretical frequency	Given frequency
0	$P(0) = \dfrac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^0}{0!} = 0.6065$	$0.6065 \times 200 = 121.3$	121
1	$P(1) = \dfrac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^1}{1!} = \dfrac{0.6065}{2} = 0.3033$	$0.3033 \times 200 = 60.7$	61
2	$P(2) = \dfrac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^2}{2!} = \dfrac{0.6065}{8} = 0.0758$	$0.0758 \times 200 = 15.2$	15
3	$P(3) = \dfrac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^3}{3!} = \dfrac{0.6065}{48} = 0.0126$	$0.0126 \times 200 = 2.5$	2
4	$P(4) = \dfrac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^4}{4!} = \dfrac{0.6065}{384} = 0.0016$	$0.0016 \times 200 = 0.32$	1

Ans.

EXERCISE 64.1

1. Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 2 per cent of such fuses are defective. **Ans.** 0.785

2. The number of accidents during a year in a factory has the Poisson distribution with mean 1.5. The accidents during different years are assumed independent. Find the probability that only 2 accidents take place during 2 years time. **Ans.** 0.224

3. A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantee that not more than 10 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality. [$e^{-5} = 0.006738$] **Ans.** 0.0136875

4. Suppose the number of telephone calls on an operator received from 9.00 to 9.05 follow a Poisson distribution with mean 3. Find the probability that

   (*i*) the operator will receive no calls in that time interval tomorrow,

   (*ii*) in the next three days the operator will receive a total of 1 call in that time interval.
   [$e^{-3} = 0.04978$] **Ans.** (*i*) e^{-3} (*ii*) $3 \times (e^{-3})^2 (e^{-3}.\ 3)$

5. On the basis of past record it has been found that there is 70% chance of power cut in a city on any particular day. What is the probability that from the first to the 10th date of the month, there are 5 or more days without power cut. (*A.M.I.E.T.E., Summer 2001*)

   Ans. $\left[\dfrac{3^5}{5!} + \dfrac{3^6}{6!} + \dfrac{3^7}{7!} + \dfrac{3^8}{8!} + \dfrac{3^9}{9!} + \dfrac{3^{10}}{10!}\right] e^{-3}$

6. The distribution of typing mistakes committed by a typist is given below. Assuming a Poisson model, find out the expected frequencies:

Mistakes per pages	0	1	2	3	4	5
No. of pages	142	156	69	27	5	1

Ans. 147, 147, 74, 25, 6, 1 pages.

7. Let x be the number of cars per minute passing a certain crossing of roads between 5.00 P.M. and 7.00 P.M. on a holiday. Assume x has a Poisson distribution with mean 4. Find the probability of observing atmost 3 cars during any given minute between 5.00 P.M. and 7 P.M. (given $e^{-4} = 0.0183$) **Ans.** 0.4331

8. Number of customers arriving at a service counter during a day has a Poisson distribution with mean 100. Find the probability that at least one customer will arrive on each day during a period of five days. Also find the probability that exactly 3 customers will arrive during two days.

Ans. $(1-e^{-100})^5,\ e^{-200} \times \dfrac{4(100)^3}{3}$

9. In a normal summer, a truck driver gets on an average one puncture in 1000 km. Applying Poisson distribution, find the probability that he will have

(*i*) no puncture, (*ii*) two punctures in a journey of 3000 kms. **Ans.** (*i*) e^{-3} (*ii*) 4.5 e^{-3}

10. Wireless sets are manufactured with 25 soldered joints each. On the average, 1 joint in 500 is defective. How many sets can be expected to be free from defective joints in a consignment of 10000 sets ?

Ans. 9512

11. In a certain factory turning out razor blades, there is small chance $\dfrac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Using Poisson's distribution, calculate the approximate number of packets containing (i) no defective (ii) one defective and (iii) two defective blades respectively in a

12. If m and μ_r denote by the mean and central rth moment of a Poisson distribution, then prove that

$$\mu_{r+1} = r\,m\,\mu_{r-1} + m\frac{d\mu_r}{dm}. \qquad \left[\textbf{Hint. } \mu_r = \sum_{r=0}^{\infty}(x-m)^r \frac{e^{-m}m^x}{x!},\ \text{find } \frac{d\mu_r}{dm}\right]$$

13. A certain screw-making machine produces an average 2 defective screws out of 100, and pack them in boxes of 500. Find the probability that a box contains 15 defective screws.

(*A.M.I.E., Winter 2005*) **Ans.** 0.0347

14. The distribution of the number of road accidents per day in a city is Poisson with mean 4. Find the number of days out of 100 days when there will be:

(*i*) no accident (*ii*) at least 2 accidents (*iii*) at most 3 accidents (*iv*) between 2 and 5 accidents

Ans. (*i*) 2 days (*ii*) 91 days (*iii*) 43 days (*iv*) 39 days

15. Fill in the blanks.

(*a*) If x has a Poisson distribution such that $P(x = k) = P(x = k + 1)$ for some positive integer k then mean of x is .... **Ans.** $k + 1$

16. Choose the correct answer:

(*a*) In the Poisson distribution if $P(x = k) = P(x = k + 1)$, then the mean is :

(*i*) k (*ii*) 2k (*iii*) k + 1 (*iv*) $k - 1$ (*R.G.P.V. Bhopal, II Semester, June 2007*) **Ans.** (*iii*)

(*b*) The value of measure of skewness of Poisson distribution is :

(*i*) m (*ii*) $\sqrt{m}$ (*iii*) $\dfrac{1}{m}$ (*iv*) $\dfrac{1}{\sqrt{m}}$ (*R.G.P.V., Bhopal, II Semester, June 2006*) **Ans.** (*iii*)

(*g*) Poisson distribution with unit mean, mean-deviation about the mean is :

(*i*) $\dfrac{1}{e}$ (*ii*) $\dfrac{\sigma}{e}$ (*iii*) $\dfrac{2\sigma}{e}$ (*iv*) $\dfrac{2}{e}$ (*R.G.P.V., Bhopal, II Semester, Feb 2006*) **Ans.** (*iv*)

(h) In the Poisson distribution if $2P(x = 1) = P(x = 2)$, then the variance is :

(*i*) 0 (*ii*) –1 (*iii*) 4 (*iv*) 2 (*R.G.P.V., Bhopal, II Semester, June 2007*) **Ans.** (*iii*)

(*i*) In the Poisson distribution if $2p(x = 1) = p(x = 2)$

Then mean is

(*i*) 0 (*ii*) – 1 (*iii*) 4 (*iv*) 2 (*R.G.P.V., Bhopal, II Semester, June 2007*) **Ans.** (*iii*)

CHAPTER

65

NORMAL DISTRIBUTION

65.1 CONTINUOUS DISTRIBUTION

So far we have dealt with discrete distributions where the variate takes only the integral values. But the variates like temperature, heights and weights can take all values in a given interval. Such variables are called continuous variables.

Let $f(x)$ be a continuous function, then Mean $= \int_{-\infty}^{+\infty} x.f(x)\,dx$

$$\text{Variance} = \int_{-\infty}^{+\infty} (x-\bar{x})^2.f(x)\,dx \qquad (\bar{x} = \text{mean})$$

Note. $f(x)$ is called probability density function if

(1) $f(x) \geq 0$ for every value of x. (2) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

(3) $\int_a^b f(x)\,dx = P, \qquad (a < x < b)$

Example 1. *The probability density function f (x) of a continuous random variable x is defined by*

$$f(x) = \begin{cases} \dfrac{A}{x^3}, & 5 \leq x \leq 10 \\ 0, & otherwise \end{cases} \qquad \textit{Find the value of } A.$$

Solution. Here, $f(x) = \dfrac{A}{x^3}, \qquad 5 \leq x \leq 10$

Since $f(x)$ is probability density function, so

$$\int_5^{10} \frac{A}{x^3}\,dx = 1 \quad \Rightarrow \quad \left[-\frac{A}{2x^2}\right]_5^{10} = 1$$

$$\frac{A}{2}\left[-\frac{1}{100}+\frac{1}{25}\right] = 1$$

$$\frac{A}{2}\left(\frac{3}{100}\right) = 1 \quad \Rightarrow \quad A = \frac{200}{3}$$

Ans.

Example 2. *A function $f(x)$ is defined as follows*

$$f(x) = \begin{cases} 0, & x < 2 \\ \dfrac{1}{18}(2x+3), & 2 \leq x \leq 4 \\ 0, & x > 4 \end{cases}$$

Show that it is a probability density function.

Solution.

$$f(x) = \begin{cases} 0, & x < 2 \\ \dfrac{1}{18}(2x+3), & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

If $f(x)$ is a probability density function, then

(*i*) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Here $\int_2^4 \frac{1}{18}(2x+3)dx = \frac{1}{18}[x^2+3x]_2^4 = \frac{1}{18}(16+12-4-6) = 1$

(*ii*) $f(x) > 0$ for $2 \le x \le 4$

Hence, the given function is a probability density function. **Proved.**

Example 3. *The diameter of an electric cable is assumed to be continuous random variate with probability density function:*

$f(x) = 6x(1-x)$, $0 \le x \le 1$

(i) verify that above is a p.d.f. (ii) find the mean and variance.

Solution. (*i*) $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 6x(1-x)dx = \int_0^1 (6x-6x^2)dx$

$$= (3x^2-2x^3)_0^1 = 3-2 = 1$$

Secondly $f(x) > 0$ for $0 \le x \le 1$.

Hence the given function is a probability density function.

(*ii*) Mean $= \int_{-\infty}^{\infty} x.f(x)\,dx = \int_0^1 x.6x(1-x)dx$

$$= \int_0^1 (6x^2-6x^3)dx = \left(2x^3 - \frac{3}{2}x^4\right)_0^1 = 2 - \frac{3}{2} = \frac{1}{2}$$ **Ans.**

$$\text{Variance} = \int_{-\infty}^{\infty} (x-\overline{x})^2.f(x)\,dx = \int_0^1 \left(x-\frac{1}{2}\right)^2 .6x(1-x)\,dx$$

$$= \int_0^1 \left(x^2 - x + \frac{1}{4}\right)(6x-6x^2)dx = \int_0^1 \left(12x^3 - 6x^4 - \frac{15}{2}x^2 + \frac{3}{2}x\right)dx$$

$$= \left(3x^4 - \frac{6}{5}x^5 - \frac{5}{2}x^3 + \frac{3x^2}{4}\right)_0^1 = \left(3 - \frac{6}{5} - \frac{5}{2} + \frac{3}{4}\right) = \frac{1}{20}$$ **Ans.**

Example 4. *A continuous random variable has p.d.f.*

$$f(x) = ke^{-\frac{x}{5}},\ x \ge 0,$$
$$= 0,\ \textit{else where}$$

then the value of k is...... (*A.M.I.E., Winter 2002*)

Solution. $\int_{-\infty}^{\infty} p.d.f(x)\,dx = 1$

$\Rightarrow \int_{-\infty}^{0} p.d.f(x)\,dx + \int_0^{\infty} p.d.f(x)\,dx = 1$

$$0 + \int_0^{\infty} ke^{-\frac{x}{5}}\,dx = 1, \Rightarrow k\left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right]_0^{\infty} = 1 \Rightarrow -5k\left[\frac{1}{e^{\infty}} - 1\right] = 1 \Rightarrow 5k = 1 \Rightarrow k = \frac{1}{5}$$ **Ans.**

Example 5. *If the probability density function of a random variable x is*

$$f(x) = \begin{cases} kx^{\alpha-1}(1-x)^{\beta-1}, & \text{for } 0 < x < 1, \quad \alpha > 0, \quad \beta > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find k and mean of x.

Solution. If $f(x)$ is a probability density function,

Then $\int_{-\infty}^{\infty} x.f(x)\,dx = 1$

Here $\int_0^1 k\,x^{\alpha-1}(1-x)^{\beta-1}dx = 1$ [$f(x)$ is a Beta function.]

$$\Rightarrow \qquad k\frac{\overline{|\alpha}\ \overline{|\beta}}{\overline{|\alpha+\beta}} = 1 \quad \Rightarrow \quad k = \frac{\overline{|\alpha+\beta}}{\overline{|\alpha}\ \overline{|\beta}} \qquad \textbf{Ans.}$$

$$\text{Mean} = \int_{-\infty}^{\infty} x.f(x)\,dx = \int_0^1 x.k\,x^{\alpha-1}(1-x)^{\beta-1}dx$$

$$= k\int_0^1 x^{\alpha+1-1}(1-x)^{\beta-1}dx \qquad \ldots (1)$$

[$f(x)$ is Beta function]

Putting the value of k and the integral in (1), we get

$$\text{Mean} = \frac{\overline{|\alpha+\beta}}{\overline{|\alpha}\ \overline{|\beta}} \cdot \frac{\overline{|\alpha+1}\ \overline{|\beta}}{\overline{|\alpha+\beta+1}} = \frac{\overline{|\alpha+\beta}}{\overline{|\alpha}\ \overline{|\beta}}\ \frac{\alpha\overline{|\alpha}\ \overline{|\beta}}{(\alpha+\beta)\overline{|\alpha+\beta}} = \frac{\alpha}{\alpha+\beta} \qquad \textbf{Ans.}$$

EXERCISE 65.1

1. The probability density $p(x)$ of a continuous random variable is given by
$p(x) = y_0\,e^{-|x|}, -\infty < x < \infty$
Prove that $y_0 = 1/2$. Find mean and variance of the distribution. **Ans.** 0.2

2. If $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$ **Ans.** $\frac{1}{3}, \frac{2}{9}$

Find the mean and variance.

3. X is a random variable giving time (in minutes) during which a certain electrical equipment is used at maximum load in a specified time period. If the *pdf* is given by

$$f(x) = \begin{cases} \dfrac{x}{(1500)^2}, & 0 \le x \le 1500 \\ -\dfrac{(x-3000)}{(1500)^2}, & 1500 \le x \le 3000 \\ 0, & \text{elsewhere,} \end{cases}$$

represents the density of a variable x, find $E(x)$ and $Var(x)$
find the expected value of X. **Ans.** 1500, 375000

4. A function is defined as under :

$$f(x) = \frac{1}{k}, \ x_1 \le x \le x_2 \ = 0, \text{ elsewhere.}$$

Find the cumulative distribution of the variate x when k satisfies the requirements for $f(x)$ to be a density function.

Ans. $f(x) = 0, \ x < x_1; \quad (x - x_1)/(x_2 - x_1), \quad x_1 \le x \le x_2, 1; x \ge x_2.$

5. A continuous distribution of a variable x in the range $(-3, 3)$ is defined as

$$\begin{cases} f(x) = \dfrac{1}{16}(3+x)^2, & -3 \le x < -1 \\ \quad\;\; = \dfrac{1}{16}(6-2x^2), & -1 \le x < 1 \\ \quad\;\; = \dfrac{1}{16}(3-x)^2, & 1 \le x \le 3. \end{cases}$$

Verify that the area under the curve is unity. Show that the mean is zero.

65.2 MOMENT GENERATING FUNCTION OF THE CONTINUOUS PROBABILITY DISTRIBUTION ABOUT $x = a$ is given by

$$M_a(t) = \int_{-\infty}^{\infty} e^{t(x-a)} f(x)\,dx \qquad \text{where } f(x) \text{ is p.d.f.}$$

Example 6. *Find the moment generating function of the exponential distribution*

$$f(x) = \frac{1}{c} e^{-x/c} \qquad 0 \le x \le \infty,\ c > 0$$

Hence find its mean and S.D.

Solution. The moment generating function about origin is

$$M_0(t) = \int_0^{\infty} e^{tx} \frac{1}{c} e^{-x/c}\,dx = \frac{1}{c}\int_0^{\infty} e^{(t-1/c)x}\,dx = \frac{1}{c}\left[\frac{e^{(t-1/c)x}}{t-\frac{1}{c}}\right]_0^{\infty} = \frac{1}{c}\left[-\frac{1}{t-\frac{1}{c}}\right] = \frac{1}{1-ct} = (1-ct)^{-1}$$

$$= 1 + ct + c^2t^2 + c^3t^3 + c^4t^4 + \ldots.$$

$$\mu_1' = \frac{d}{dt}[M_0(t)]_{t=0} = [c + 2c^2t + 3c^3t^2 + 4c^4t^3 + \ldots]_{t=0} = c$$

$$\mu_2' = \frac{d^2}{dt^2}[M_0(t)]_{t=0} = [2c^2 + 6c^3t + 12c^4t^2 + \ldots]_{t=0} = 2c^2$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 2c^2 - c^2 = c^2$$

$$\text{S.D.} = c$$

Hence, $\quad$ Mean $= c$, S.D. $= c$ **Ans.**

65.3 NORMAL DISTRIBUTION

(*A. M. I. E., Summer 2002*)

Normal distribution is a continuous distribution. It is derived as the limiting form of the Binomial distribution for large values of n and p and q are not very small.

The normal distribution is given by the equation

$$f(x) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \ldots(1)$$

where μ = mean, σ = standard deviation, $\pi = 3.14159\ldots$, $\quad e = 2.71828\ldots$

$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

On substitution $z = \dfrac{x-\mu}{\sigma}$ in (1), we get $f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ $\qquad \ldots(2)$

Here mean = 0, standard deviation = 1.

(2) is known as standard form of normal distribution.

Theorem. To derive normal distribution as a limiting case of Binomial distribution where $p \ne q$ but $p \approx q$. (*U.P. III Semester, Dec. 2006; R.G.P.V., Dec. 2001*)

Statement. The limiting case of binomial distribution $(p+q)^n$, as $n \to \infty$ and neither p nor q are very small, generates the normal distribution.

Proof. The frequency for r and $(r+1)$ successes in binomial distribution are

$$f(r) = N.\,{}^nC_r\, p^r\, q^{n-r} \text{ and } f(r+1) = N.\,{}^nC_{r+1}\, p^{r+1}\, q^{n-(r+1)}$$

The frequency of r successes > frequency of $(r+1)$ successes if

$$f(r) > f(r+1) \Rightarrow \frac{f(r)}{f(r+1)} > 1$$

$$\Rightarrow \frac{N.\,{}^nC_r\, p^r\, q^{n-r}}{N.\,{}^nC_{r+1}\, p^{r+1}\, q^{n-r-1}} > 1 \quad \Rightarrow \frac{\frac{n!}{r!(n-r)!}\,.\,p^r\,.\,q^{n-r}}{\frac{n!}{(r+1)!\,(n-r-1)!}\,p^{r+1}\,q^{n-r-1}} > 1$$

$$\Rightarrow \frac{n!\,p^r\,.\,q^{n-r}\,(r+1)!\,(n-r-1)!}{r!(n-r)!\,.\,n!\,p^{r+1}\,.\,q^{n-r-1}} > 1 \quad \Rightarrow \frac{q\,.\,(r+1)}{(n-r)\,p} > 1 \quad \Rightarrow \quad q\,r + q > n\,p - p\,r$$

$$\Rightarrow \quad q > np - r\,(p+q)$$

$$\Rightarrow \quad r > n\,p - q \qquad \text{... (1)}$$

Again, similarly the frequency of r successes > the frequency of (r – 1) successes if

$$f(r) > f(r-1) \Rightarrow \frac{f(r)}{f(r-1)} > 1$$

$$\Rightarrow \frac{N.\,{}^nC_r\, p^r\, q^{n-r}}{N.\,{}^nC_{r-1}\, p^{r-1}\, q^{n-(r-1)}} > 1 \quad \Rightarrow \frac{\frac{n!}{r!(n-r)!}\,.\,p^r\,.\,q^{n-r}}{\frac{n!}{(r-1)!\,(n-r+1)!}\,p^{r-1}\,q^{n-r+1}} > 1$$

$$\Rightarrow \frac{n!\,p^r q^{n-r}\,(r-1)!\,(n-r+1)!}{r!\,(n-r)!\,n!\,p^{r-1}\,q^{n-r+1}} > 1 \quad \Rightarrow \quad \frac{p\,(n-r+1)}{r\,q} > 1$$

$$\Rightarrow \quad p\,n - pr + p > r\,q \quad \Rightarrow \quad p\,n + p > p\,r + qr$$

$$\Rightarrow \quad p\,n + p > r\,(p+q) \quad \Rightarrow \quad p\,n + p > r \qquad \text{... (2)}$$

$$[\because p + q = 1]$$

From (1) and (2), we have

$$p\,n + p > r > n\,p - q$$

$$p\,n + p + q > r > n\,p$$

$$n\,p + 1 > r > n\,p$$

Since a possible value of r is np, therefore, without loss of generality we can assume that np is an integer as $n \to \infty$. Hence the frequency of np successes can be assumed to be maximum frequency. Let y_0 be the frequency of np sccesses and y_x be the frequency of $(np + x)$ successes.

Then

$$y_0 = f(n\,p) = N\,.\,{}^nC_{np}\, p^{np}\, q^{n-np} \qquad \text{[From (1), for } r = n\,p]$$

$$= N\,\frac{n!}{(n\,p)!\,(n-np)!}\,p^{np}\,q^{n-np} = N\,\frac{n!}{(n\,p)!\,(n\,q)!}\,p^{np}\,q^{nq} \quad \text{... (3)} \quad [\because q = 1 - p]$$

and $$y_x = N\,.\,\frac{n!}{(n\,p+x)!\,(n\,q-x)!}\,p^{np+x}\,q^{nq-x} \qquad \text{... (4)}$$

Diving (4) by (3), we get $\dfrac{y_x}{y_0} = \dfrac{(n\,p)!\,(n\,q)!}{(n\,p+x)!\,(n\,q-x)!}\,p^x\,q^{-x}$ ... (5)

In n be large, then according to James Stirling, we have

$$n! = e^{-n}\,h^{n+\frac{1}{2}}\,\sqrt{(2\pi)},$$

From (5) $$\frac{y_x}{y_0} = \frac{e^{-np}(np)^{np+1/2}\sqrt{2\pi}\, e^{-nq}(nq)^{nq+1/2}\sqrt{2\pi}\; p^x q^{-x}}{e^{-(np+x)}(np+x)^{np+x+1/2}\sqrt{2\pi}\, e^{-(nq-x)}(nq-x)^{nq-x+1/2}\sqrt{2\pi}}$$

$$= \frac{(np)^{np+1/2}(nq)^{nq+1/2}(nq/nq)^x}{(np)^{np+x+1/2}\left\{1+\frac{x}{np}\right\}^{np+x+1/2}(np)^{nq-x+1/2}\left\{1-\frac{x}{nq}\right\}^{nq-x+1/2}}$$

$$= \frac{1}{\left\{1+\frac{x}{np}\right\}^{np+x+1/2}\left\{1-\frac{x}{nq}\right\}^{np+x+1/2}}$$

$$\therefore \quad \log\frac{y_x}{y_0} = -\left(np+x+\frac{2}{2}\right)\log\left(1+\frac{x}{np}\right)-\left(nq-x+\frac{1}{2}\right)\log\left(1-\frac{x}{nq}\right)$$

$$= -\left(np+x+\frac{1}{2}\right)\left(\frac{x}{nq}-\frac{x^2}{2n^2p^2}+\frac{x^3}{3p^3q^3}-\ldots\right)+\left(nq-x+\frac{1}{2}\right)\left(\frac{x}{nq}+\frac{x^2}{2n^2q^2}+\frac{x^3}{3n^3q^3}+\ldots\right)$$

$$= x\left(1-\frac{1}{2np}+1+\frac{1}{2np}\right)+x^2\left(\frac{1}{2np}-\frac{1}{np}+\frac{1}{4n^2p^2}+\frac{1}{2nq}-\frac{1}{nq}+\frac{1}{4n^2q^2}\right)$$

$$+\; x^3\left(\frac{1}{3n^2q^2}+\frac{1}{6n^3q^3}-\frac{1}{2n^2q^2}-\frac{1}{2n^2p^2}-\frac{1}{3n^2q^2}+\frac{1}{6n^3p^3}\right)-\ldots$$

$$= \frac{p-q}{2npq}x+\frac{p^2+q^2}{4n^2p^2q^2}x^2-\frac{x^2}{2npq}+\ldots \text{ + terms of higher orders.}$$

Neglecting terms containing $1/n^2$, we have

$$\therefore \quad \log\frac{y_x}{y_0} = -\frac{q-p}{2npq}x-\frac{x^2}{2npq}$$

Since p < 1, q < 1 and so q – p is very samll as compared with n. Therefore I[st] term may be neglected. $(q-p=0)$

$$\therefore \quad \log\frac{y_x}{y_0} = -\frac{x^2}{2npq} = -\frac{x^2}{2\sigma^2} \qquad [\because \sigma^2 = npq, \text{ the variance of Binomial distribution}]$$

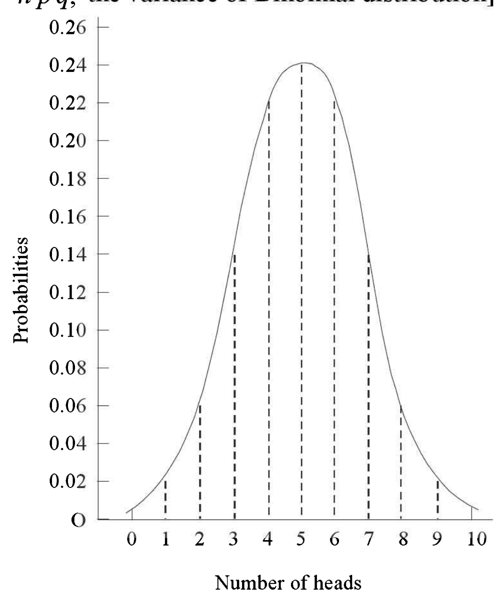
$$\Rightarrow \quad y_x = y_0\, e^{-x^2/2\sigma^2}$$

65.4 NORMAL CURVE

Let us show binomial distribution graphically.

The probabilities of heads in 1 tosses are
$^{10}C_0\, q^{10} p^0,\ ^{10}C_1\, q^9 p^1,\ ^{10}C_2\, q^8 p^2,$
$^{10}C_3\, q^7 p^3,\ ^{10}C_4\, q^6 p^4,\ ^{10}C_5\, q^5 p^5,$
$^{10}C_6\, q^4 p^6,\ ^{10}C_7\, q^3 p^7,\ ^{10}C_8\, q^2 p^8,$
$^{10}C_9\, q^1 p^9,\ ^{10}C_{10}\, q^0 p^{10}.$

$p = \frac{1}{2},\ q = \frac{1}{2}$. It is shown in the figure given.



If the variates (heads here) are treated as if they were continuous, the required probability curve will be a *normal curve* as shown in the above figure by dotted lines.

Properties of the normal curve. $y = y_0\, e^{-\frac{x^2}{2\sigma^2}}$

1. The curve is symmetrical about the y-axis. The mean, median and mode coincide at the origin.
2. The curve is drawn, if mean (origin of x) and standard deviation are given. The value of y_0 can be calculated from the fact that the area of the curve must be equal to the total number of observations.
3. y decreases rapidly as x increases numerically. The curve extends to infinity on either side of the origin.
4. (*a*) $P(\mu - \sigma < x < \mu + \sigma) = 68\%$
   (*b*) $P(\mu - 2\sigma < x < \mu + 2\sigma) = 95.5\%$
   (*c*) $P(\mu - 3\sigma < x < \mu + 3\sigma) = 99.7\%$

Hence (*a*) About $\frac{2}{3}$ of the values will lie between $(\mu - \sigma)$ and $\mu + \sigma$

(*b*) About 95% of the values will lie between $(\mu - 2\sigma)$ and $(\mu + 2\sigma)$.

(*c*) About 99.7 % of the values will lie between $(\mu - 3\sigma)$ and $(\mu + 3\sigma)$.

65.5 MEAN FOR NORMAL DISTRIBUTION

$$\text{Mean} = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{x^2}{2\sigma^2}} .x\, dx = \frac{1}{\sigma\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} (t\sigma)(\sigma\, dt) \qquad \left[\text{putting } \frac{x}{\sigma} = t\right]$$

$$= \frac{\sigma}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} t\, e^{-\frac{t^2}{2}} dt = \frac{\sigma}{\sqrt{2}\,\pi} \left[e^{-\frac{t^2}{2}}\right]_{-\infty}^{+\infty} = \frac{\sigma}{\sqrt{(2\pi)}} [0] = 0$$

65.6 STANDARD DEVIATION FOR NORMAL DISTRIBUTION

$$\mu_2' = \int x^2 . f(x)\, dx \quad \text{or} \quad \mu_2' = \int_{-\infty}^{+\infty} x^2 . \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{x^2}{2\sigma^2}} . dx$$

Put $\qquad \dfrac{x^2}{2\sigma^2} = t \quad \Rightarrow \quad x = \sqrt{2}\, \sigma t^{1/2}. \qquad \Rightarrow \qquad dx = \dfrac{\sqrt{2}\,\sigma}{2 t^{1/2}}\, dt$

$$\mu_2' = \int_{-\infty}^{+\infty} (2\sigma^2 t) \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{2\sigma^2 t}{2\sigma^2}} . \frac{\sqrt{2}\,\sigma}{2t^{\frac{1}{2}}} dt = \int_{-\infty}^{+\infty} (2\sigma^2 t) \frac{1}{\sigma\sqrt{(2\pi)}} e^{-t} \left(\frac{\sqrt{2}\,\sigma}{2t^{1/2}}\right) dt$$

$$= \frac{2\sigma^2}{\sigma\sqrt{(2\pi)}} \frac{\sqrt{2}\sigma}{2} \int_{-\infty}^{+\infty} t^{\frac{3}{2}-1} e^{-t} dt, \; = \frac{\sigma^2}{\sqrt{\pi}} . 2\int_0^{+\infty} t^{\frac{3}{2}-1} e^{-t} dt \qquad \left[\int_0^{+\infty} x^{n-1} e^{-x} dx = \Gamma(n)\right]$$

$$= \frac{\sigma^2}{\sqrt{\pi}} . 2\, \Gamma\!\left(\frac{3}{2}\right) = 2\frac{\sigma^2}{\sqrt{\pi}} . \frac{1}{2} \Gamma\!\left(\frac{1}{2}\right) = \frac{\sigma^2}{\sqrt{\pi}} \sqrt{\pi} = \sigma^2$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \sigma^2 - (0)^2 = \sigma^2$$

$$S.D. = \sigma$$

Ans.

65.7 MEDIAN OF THE NORMAL DISTRIBUTION

If a is the median, then it divides the total area into two equal halves so that

$$\int_{-\infty}^{a} f(x)\, dx = \frac{1}{2} = \int_{a}^{\infty} f(x)\, dx$$

where $\qquad f(x) = \dfrac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Suppose Median a > mean, μ then

$$\int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{a} f(x)\,dx = \frac{1}{2} \qquad \left[\text{But } \int_{-\infty}^{\mu} f(x)\,dx = \frac{1}{2}\right]$$

$$\frac{1}{2} + \int_{\mu}^{a} f(x)\,dx = \frac{1}{2} \qquad (\mu = \text{mean})$$

$$\int_{\mu}^{a} f(x)\,dx = 0$$

Thus $a = \mu$

Similarly, when a < mean, we have $a = \mu$.

Thus, median = mean = μ.

65.8 MEAN DEVIATION ABOUT THE MEAN μ (*U.P. III Semester, Dec. 2009*)

$$\text{Mean deviation} = E\,|x-\mu|$$

$$= \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \Rightarrow \; dz = \frac{1}{\sqrt{2}} t^{-\frac{1}{2}} dt \quad \text{where } z = \frac{x-\mu}{\sigma} \quad \Rightarrow \quad dz = \frac{dx}{\sigma}$$

$$= \sigma \frac{1}{\sqrt{(2\pi)}} \left[\int_{-\infty}^{0} -z e^{-\frac{z^2}{2}} dz + \int_{0}^{\infty} z e^{-\frac{z^2}{2}} dz\right]$$

$$= \frac{2\sigma}{\sqrt{(2\pi)}} \int_0^{\infty} z e^{-\frac{z^2}{2}} dz = \sigma\sqrt{\frac{2}{\pi}} \left[-e^{-\frac{z^2}{2}}\right]_0^{\infty} \quad \text{(as the function is even)}$$

$$= \sigma\sqrt{\frac{2}{\pi}}\,[-0+1] = \sigma\sqrt{\frac{2}{\pi}} = \frac{4}{5}\sigma \text{ approximately.}$$

65.9 MODE OF THE NORMAL DISTRIBUTION

We know that mode is the value of the variate x for which $f(x)$ is maximum. Thus, by differential calculus $f(x)$ is maximum if $f'(x) = 0$ and $f''(x) < 0$

where $$f(x) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Clearly $f(x)$ will be maximum when the exponent will be maximum which will be the case

$$\frac{(x-\mu)}{2\sigma^2} = 0 \qquad \Rightarrow \qquad (x-\mu)^2 = 0 \qquad \Rightarrow \qquad x = \mu$$

Thus mode is μ, and modal ordinate = $\dfrac{1}{\sigma\sqrt{(2\pi)}}$

65.10 MOMENT OF NORMAL DISTRIBUTION

$$\mu_{2n+1} = \int_{-\infty}^{\infty} (x-\mu)^{2n+1} f(x)\,dx = \frac{1}{\sigma\sqrt{(2\pi)}} \int_{-\infty}^{\infty} (x-\mu)^{2n+1} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} (\sigma z)^{2n+1} e^{-\frac{z^2}{2}} dz \qquad \left[z = \frac{x-\mu}{\sigma}\right]$$

$$= \frac{\sigma^{2n+1}}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} z^{2n+1} e^{-\frac{z^2}{2}} dz = 0 \qquad \text{(since } z^{2n+1} e^{-\frac{z^2}{2}} \text{ is an odd function)}$$

$$\mu_{2n} = \int_{-\infty}^{\infty} (x-\mu)^{2n} f(x)\, dx = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} (\sigma z)^{2n} e^{-\frac{z^2}{2}} dz = \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n} e^{-\frac{z^2}{2}} dz$$

$$= \frac{2\sigma^{2n}}{\sqrt{(2\pi)}} \int_0^{\infty} z^{2n} e^{-\frac{z^2}{2}} dz = \frac{2\sigma^{2n}}{\sqrt{(2\pi)}} \int_0^{\infty} (2t)^n e^{-t} \frac{1}{\sqrt{2}} t^{-\frac{1}{2}} dt$$

$$\left[z^{2n} \cdot e^{-\frac{z^2}{2}} \text{ is an even function}\right]$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^{\infty} t^{\left(n+\frac{1}{2}-1\right)} e^{-t\,dt} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{\left(n-\frac{1}{2}\right)} dt \qquad \left[\frac{z^2}{2} = t \Rightarrow dz = \frac{1}{\sqrt{2}} t^{-\frac{1}{2}} dt\right]$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \left|\overline{n+\frac{1}{2}}\right.$$

Changing n to $(n-1)$, we get

$$\mu_{2n-2} = \frac{2^{n-1}\sigma^{2n-2}}{\sqrt{\pi}} \left|\overline{n-\frac{1}{2}}\right.$$

On dividing, we get

$$\frac{\mu_{2n}}{\mu_{2n-2}} = 2\sigma^2 \frac{\left|\overline{n+\frac{1}{2}}\right.}{\left|\overline{n-\frac{1}{2}}\right.} = \frac{2\sigma^2\left(n-\frac{1}{2}\right)\left|\overline{n-\frac{1}{2}}\right.}{\left|\overline{n-\frac{1}{2}}\right.} = 2\sigma^2\left(n-\frac{1}{2}\right)$$

$$\mu_{2n} = \sigma^2 (2n-1)\mu_{2n-2}$$

which gives the recurrence relation for the moments of normal distribution

$$\mu_{2n} = [(2n-1)\sigma^2]\,[(2n-3)\sigma^2]\mu_{2n-4}$$

$$= [(2n-1)\sigma^2]\,[(2n-3)\sigma^2][(2n-5)\sigma^2]\mu_{2n-6}$$

$$= [(2n-1)\sigma^2]\,[(2n-3)\sigma^2][(2n-5)\sigma^2] \,.....\, (3\sigma^2)\;(1\cdot\sigma^2)\mu_0$$

$$= (2n-1)\,(2n-3)\,(2n-5) - ......1\cdot\sigma^{2n} \qquad (\mu_0 = 1)$$

$$= 1.\,3.\,5.\,7......\,(2n-5)\,(2n-3)\,(2n-1)\,\sigma^{2n}$$

65.11 MOMENT GENERATING FUNCTION OF NORMAL DISTRIBUTION

Normal distribution function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}; \qquad -\infty < x < \infty$$

Moment generating function about the origin $= M_0(t)$

$$= \int_{-\infty}^{\infty} e^{ix}\, \sigma \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad ...(1)$$

On putting $\frac{x-\mu}{\sigma} = z$ so that $dx = \sigma\, dz$ in (1), we get

$$M_0(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)} e^{-\frac{z^2}{2}} \sigma\, dz = \frac{\sigma e^{\mu t}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z} . e^{-\frac{z^2}{2}} dz = \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[z^2 - 2t\sigma z + t^2\sigma^2] + \frac{1}{2}t^2\sigma^2} dz = \frac{e^{\mu t + \frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} dz \qquad ...(2)$$

On putting $\frac{1}{2}\ (z - t\sigma)^2 = y^2$ in (2) so that $(z - t\,\sigma)\,dz \quad = 2y\,dy$

i.e. $\Rightarrow \qquad \sqrt{2}\ y\,dz \ = 2\,y\,dy$

$$\Rightarrow \qquad dz \ = \ \sqrt{2}\ d\,y \ \ M_0(t) = \frac{e^{\mu t + \frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2}\sqrt{2}\,dy \ = \frac{e^{\mu t + \frac{1}{2}t^2\sigma^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= e^{\mu t + \frac{1}{2}t^2\sigma^2}\ \frac{1}{\sqrt{\pi}}\,(\sqrt{\pi}) = e^{\mu t + \frac{1}{2}t^2\sigma^2} \qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi}\right]$$

65.12 AREA UNDER THE NORMAL CURVE

By taking $z = \dfrac{x - \bar{x}}{\sigma}$, the standard normal curve is formed.

The total area under this curve is 1. The area under the curve is divided into two equal parts by z = 0. Left hand side area and right hand side area to $z = 0$ is 0.5. The area between the ordinate $z = 0$ and any other ordinate can be noted from the table.

Example 7. *On a final examination in mathematics, the mean was 72, and the standard deviation was 15. Determine the standard scores of students receiving graders.*

(*a*) 60 (*b*) 93 (*c*) 72

Solution.

(*a*) $z = \dfrac{x - \bar{x}}{\sigma} = \dfrac{60 - 72}{15} = -0.8$ (*b*) $z = \dfrac{93 - 72}{15} = +1.4$ (*c*) $z = \dfrac{72 - 72}{15} = 0$ **Ans.**

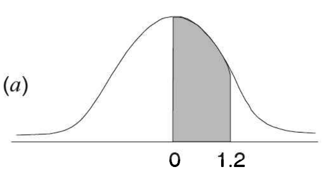
Example 8. *Find the area under the normal curve in each of the cases*

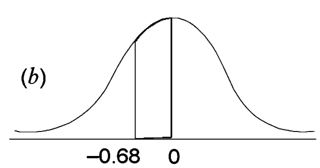
(*a*) *z = 0 and z = 1.2;* (*b*) *z = – 0.68 and z = 0;*
(*c*) *z = – 0.46 and z = 2.21;* (*d*) *z = 0.81 and z = 1.94;*
(*e*) *To the left of z = 0.6;* (*f*) *Right of z = –1.28.*

Solution.

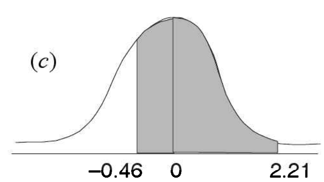
(*a*) Area between $z = 0$ and $z = 1.2$
$= .3849$

(*b*) Area between $z = 0$ and $z = -0.68$
$= 0.2518$

(*a*)


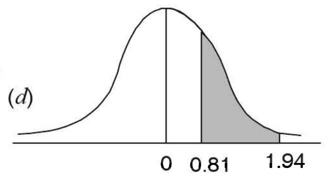
(*b*)


(*c*) Required area = (Area between $z = 0$ and $z = 2.21$)
+ (Area between $z = 0$ and $z = -0.46$)
= (Area between $z = 0$ and $z = 2.21$)
+ (Area between $z = 0$ and $z = 0.46$)
$= 0.4865 + 0.1772 = 0.6637.$

(*c*)


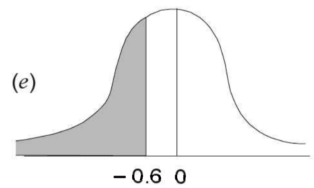
(*d*) Required area = (Area between $z = 0$ and $z = 1.94$) – (Area between $z = 0$ and $z = 0.81$)

$= 0.4738 - 0.2910 = 0.1828$



(*e*) Required area $= 0.5 -$ (Area between $z = 0$ and $z = 0.6$)

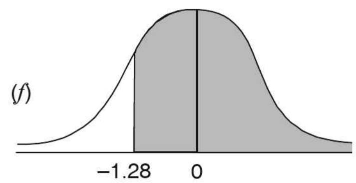
$= 0.5 - 0.2257 = 0.2743$



(*f*) Required area = (Area between $z = 0$ and $z = -1.28$) + 0.5

$= 0.3997 + 0.5$

$= 0.8997.$



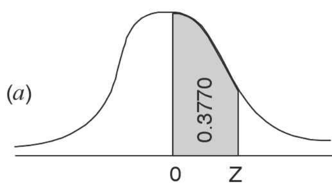
Example 9. *Find the value of z in each of the cases*

(*a*) Area between 0 and z is 0.3770

(*b*) Area to the left of z is 0.8621

Solution.

(*a*) $z = \pm 1.16$

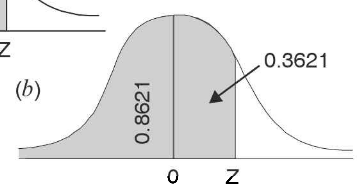


(*b*) Since the area is greater than 0.5.

Area between 0 and *z*.

$= 0.8621 - 0.5 = 0.3621$

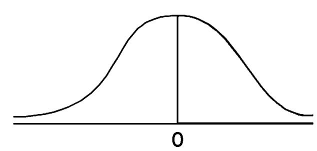
from which $z = 1.09$ **Ans.**



Example 10. *Students of a class were given an aptitude test Their marks were found to be normally distributed with mean 60 and standard deviation 5. What percentage of students scored more than 60 marks ?*

Solution. $x = 60,\ \bar{x} = 60,\ \sigma = 5$

$$z = \frac{x - \bar{x}}{\sigma} = \frac{60 - 60}{5} = 0$$



if $x > 60$ then $z > 0$

Area lying to the right of $z = 0$ is 0.5.

The percentage of students getting more than 60 marks = 50 % **Ans.**

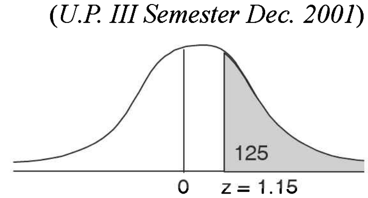
Example 11. *Assume mean height of soldiers to be 68.22 inches with a variance of 10.8 inches square. How many soldiers in a regiment of 1,000 would you expect to be over 6 feet tall, given that the area under the standard normal curve between x = 0 and x = 0.35 is 0.1368 and between x =0 and x = 1.15 is 0.3746.*

(*U.P. III Semester Dec. 2001*)

Solution. Mean $= \bar{x} = 68.22$ inches

variance $= \sigma^2 = 10.8$ inches squares

If $x = 72$ inches then $z = \dfrac{x - \mu}{\sigma} = \dfrac{72 - 68.22}{\sqrt{10.8}} = 1.15$

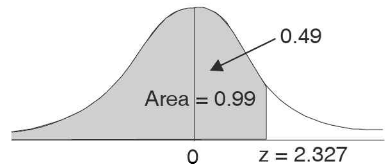


$$P(x > 72) = P(z > 1.15)$$

$$= 0.5 - P(0 \le z \le 1.15) = 0.5 - 0.3746 = 0.1254$$

Number of soldiers $= 1000 \times 0.1254 = 125.4 \approx 125$ (app.) **Ans.**

$$z = \frac{x-\mu}{\sigma} \quad \Rightarrow \quad z = \frac{x-64.5}{3.3} \quad \text{... (1)}$$



Area between 0 and $z = 0.99 - 0.5 = 0.49$

From the table, z for area 0.49 is 2.327.

Putting the value of z in (1), we get

$$\Rightarrow \quad \frac{x-64.5}{3.3} = 2.327 \quad \Rightarrow x - 64.5 = 3.3 \times 2.327$$

$$x - 64.5 = 7.68$$

$$\Rightarrow \quad x = 7.68 + 64.5 = 72.18 \text{ inches}$$

Hence 99% students are of height less than 72.18 inches. **Ans.**

Example 12. *A sample of 100 dry battery cells tested to find the length of life produced the following results:*

$\bar{x}$ = 12 hours, σ = 3 hours

Assuming the data to be normally distributed, what percentage of battery cells are expected to have life

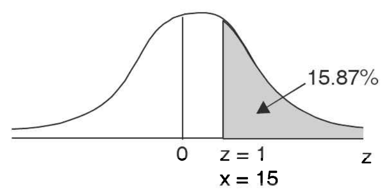
(i) more than 15 hours *(ii) less than 6 hours*

(iii) between 10 and 14 hours ? (*U.P. III Semester Dec. 2003*)

Solution. Here, Mean $= \bar{x} = 12$ hours

and Standard deviation $= \sigma = 3$ hours

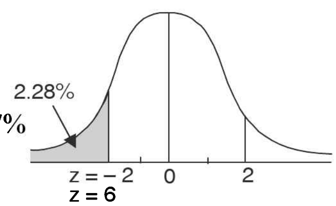
x denotes the length of life of dry battery cells.



$$z = \frac{x-\bar{x}}{\sigma}$$

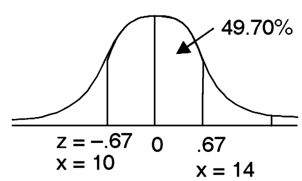
(*i*) When $x = 15$, then $z = \dfrac{15-12}{3} = 1$

$$\therefore \quad P(x > 15) = P(z > 1)$$
$$= P(0 < z < \infty) - P(0 < z < 1)$$
$$= 0.5 - 0.3413 = 0.1587 = 15.87\%$$



(*ii*) When x = 6, then $z = \dfrac{6-12}{3} = \dfrac{-6}{3} = -2$

$$P(x < 6) = \mathrm{P}(\mathrm{z} < -2)$$
$$= P(z > 2) = 0.5 - P(0 < z < 2)$$
$$= 0.5 - 0.4772 = 0.0228 = 2.28\%$$



(*iii*) When $x = 10$, then $z = \dfrac{10-12}{3} = \dfrac{-2}{3} = -0.67$

When $x = 14$, then $z = \dfrac{14-12}{3} = \dfrac{2}{3} = 0.67$

$$P(10 < x < 14) = P(-0.67 < z < 0.67)$$
$$= 2P(0 < z < 0.67) = 2 \times 0.2485 = 0.4970 = 49.70\%$$ **Ans.**

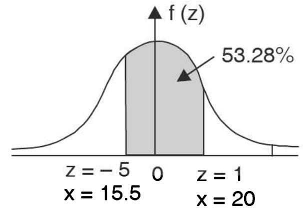
Example 13. *The mean yield per plot of a crop is 17 kg and standard deviation is 3 kg. If distribution of yield per plot is normal, find the percentage of plots giving yields:*

(*i*) *Between 15.5 kg and 20 kg ; and*

(*ii*) *More than 20 kg.* [*U.P. (MBA) 2005)*]

Solution. Mean $= \mu = 17$ kg

S.D. $= \sigma = 3$ kg



Standard Normal variable $\quad z = \dfrac{x-\mu}{\sigma}$

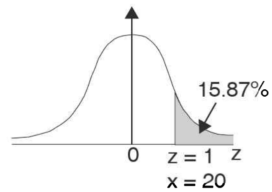
(*i*) When $x_1 = 15.5$, $\quad z_1 = \dfrac{x_1-\mu}{\sigma} = \dfrac{15.5-17}{3} = -0.5$

When $x_2 = 20$, $\quad z_2 = \dfrac{x_2-\mu}{\sigma} = \dfrac{20-17}{3} = 1$

$\therefore \quad P(15.5 < x < 20) = P(-0.5 < z < 1)$

$= P(0 < z < -0.5) + P(0 < z < 1) = 0.1915 + 0.3413 = 0.5328$

$\therefore$ Required percentage of plots = 53.28%



(*ii*) When x = 20, $z = \dfrac{20-17}{3} = 1$

$P(x > 20) = P(z > 1)$

$= 0.5 - P(0 < z < 1) = 0.5 - 0.3413$

$= 0.1587$ **Ans.**

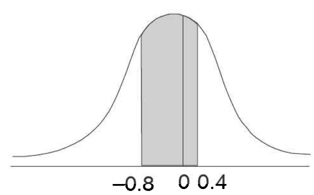
Example 14. *In a sample of 1000 cases, the mean of a certain test is 14 and standard deviation is 2.5. Assuming the distribution to be normal, find*

(*i*) *how many students score between 12 and 15 ?*

(*ii*) *how many score above 18 ?* (*iii*) *how many score below 8 ?*

(*iv*) *how many score 16 ?*

Solution. $n = 1000$, $\bar{x} = 14$, $\sigma = 2.5$



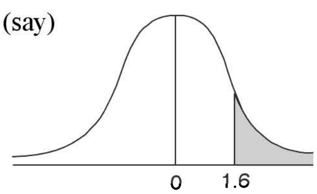
(*i*) $\quad z_1 = \dfrac{x-\bar{x}}{\sigma} = \dfrac{12-14}{2.5} = -0.8$

$z_2 = \dfrac{15-14}{2.5} = \dfrac{1}{2.5} = 0.4$

The area lying between – 0.8 to 0.4 = Area from 0 to –0.8 + area from 0 to 0.4

$= 0.2881 + 0.1554 = 0.4435$

The required number of students = $1000 \times 0.4435 = 443.5 = 444$ (say)



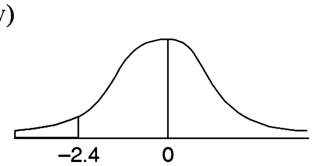
(*ii*) $\quad z_1 = \dfrac{18-14}{2.5} = \dfrac{4}{2.5} = 1.6$

Area right to 1.6 = 0.5 – Area between 0 and 1.6

$= 0.5 - 0.4452 = 0.0548$

The required number of students

$= 1000 \times 0.0548 = 54.8 = 55$ (say)



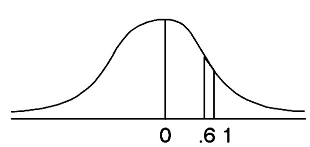
(*iii*) $\quad z = \dfrac{8-14}{2.5} = -\dfrac{6}{2.5} = -2.4$

Area left to –2.4 = 0.5 – area between 0 and –2.4

$= 0.5 - 0.4918 = 0.0082$

The required number of students = $1000 \times 0.0082 = 8.2 = 8$ (say)

(*iv*) Area between 15.5 and 16.5



$z_1 = \dfrac{15.5-14}{2.5} = 0.6$

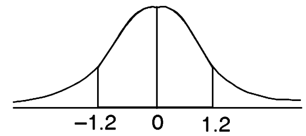
$z_2 = \dfrac{16.5-14}{2.5} = 1$

Area between 0.6 and 1 $= 0.3413 - 0.2257 = 0.1156$

The required number of students = $0.1156 \times 1000 = 115.6 = 116$ say **Ans.**

Example 15. *The mean inside diameter of a sample of 200 washers produced by a machine is 0.502 cm and the standard deviation is 0.005 cm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 0.496 to 0.508 cm, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed* (*A.M.I.E., Summer 2001*)

Solution.
$$z_1 = \frac{x-\bar{x}}{\sigma} = \frac{0.496-0.502}{0.005} = -1.2$$
$$z_2 = \frac{x-\bar{x}}{\sigma} = \frac{0.508-0.502}{0.005} = +1.2$$



Area for non-defective washers = Area between $z = -1.2$ and $z = +1.2$

$= 2$ Area between $z = 0$ and $z = 1.2$.

$= 2 \times (0.3849) = 0.7698 = 76.98\%$

Percentage of defective washers $= 100 - 76.98$

$= 23.02\%$ **Ans.**

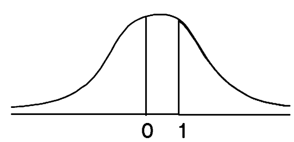
Example 16. *A manufacturer of envelopes knows that the weight of the envelopes is normally distributed with mean 1.9 gm and variance 0.01 gm. Find how many envelopes weighing (i) 2 gm or more, (ii) 2.1 gm or more, can be expected in a given packet of 1000 envelopes.*

[*Given : if t is the normal variable, then* $\phi\ (0 \le t \le 1) = 0.3413$ *and* $\phi\ (0 \le t \le 2) = 0.4772$]

Solution. $\mu = 1.9$ gm, Variance $= 0.01$ gm $\Rightarrow \sigma = 0.1$

(*i*) $x = 2$ gms or more

$$z = \frac{x-\mu}{\sigma} = \frac{2-1.9}{0.1} = \frac{0.1}{0.1} = 1$$



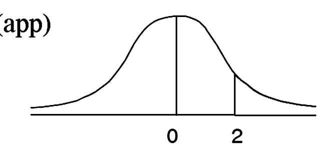
$P\ (z > 1) =$ Area right to $z = 1$

$= 0.5 - 0.3413 = 0.1587$

Number of envelopes heavier than 2 gm in a lot of 1000

$= 1000 \times 0.1587 = 158.7 = 159$ (app)

(*ii*)
$$z = \frac{2.1-1.9}{0.1} = \frac{0.2}{0.1} = 2$$



$P\ (z > 2) =$ Area right to $z = 2$

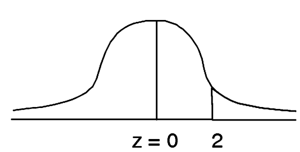
$= 0.5 - 0.4772 = 0.0228$

Number of envelopes heavier than 2.1 gm in a lot of 1000

$= 1000 \times 0.0228 = 22.8 = 23$ (app) **Ans.** (*i*) 159 (*ii*) 23

Example 17. *The life of army shoes is 'normally' distributed with mean 8 months and standard deviation 2 months. If 5000 pairs are issued how many pairs would be expected to need replacement after 12 months?*

[*Given that* $P\ (z \ge 2) = 0.0228$ *and* $z = \dfrac{(x-\mu)}{\sigma}$]



Solution. Mean $(\mu) = 8$

Standard deviation $(\sigma) = 2$

Number of pairs of shoes $= 5000$

Total months (x) = 12

When $z = \dfrac{x-\mu}{\sigma} = \dfrac{12-8}{2} = 2$

Area when ($z \geq 2$) = 0.0228

Number of pairs whose life is more than 12 months ($z > 2$)

$= 5000 \times 0.0228 = 114$

Replacement after 12 months = 5000 – 114 = 4886 pairs of shoes

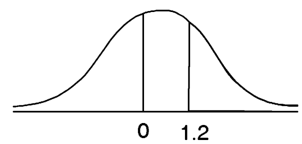
Example 18. *In a male population of 1000, the mean height is 68.16 inches and standard deviation is 3.2 inches. How many men may be more than 6 feet (72 inches) ?*
[ϕ (1.15) = *0.8749,* ϕ (1.2) = 0.8849, ϕ (1.25) = *0.8944*]
where the argument is the standard normal variable.

Solution. Male population = 1000

Mean height = 68.16 inches

Standard deviation = 3.2 inches

Men more than 72 inches = ?



ϕ (1.15) = 0.8749, ϕ (1.2) = 0.8849

ϕ (1.25) = 0.8944

$$z = \frac{x-\bar{x}}{\sigma} = \frac{72-68.16}{3.2} = 1.2$$

ϕ (1.2) = 0.8949

ϕ for more than 1.2 = 1 – 0.8849 = 0.1151

Men more than 72 inches = 1000 × 0.1151 = 115.1 = 115 (say) **Ans.**

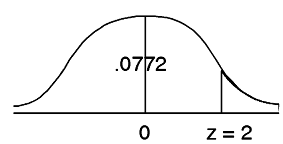
Example 19. *Pipes for tobacco are being packed in fancy plastic boxes. The length of the pipes is normally distributed with μ = 5" and a = 0.1". The internal length of the boxes is 5.2". What is the probability that the box would be small for the pipe?*

[*given that* ϕ (*1.8*) = *0.9641,* ϕ (2) = *0.9772,* ϕ (2.5) = 0.9938]

Solution. $\mu = 5''$, $\sigma = 0.1''$, $x = 5.2''$

ϕ (1.8) = 0.9641, ϕ (2) = 0.9722, ϕ (2.5) = 0.9938

$$z = \frac{x-\mu}{\sigma} = \frac{5.2-5}{0.1} = 2$$



ϕ (2) = 0.9772

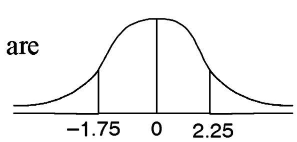
ϕ (z > 2) = 1– 0.9772 = 0.0228

The box will be small if the length of the pipe is more than 5.2" (z = 2).

Hence the probability is 0.0228 **Ans.**

Example 20. *Assuming that the diameters of 1,000 brass plugs taken consecutively from a machine form a normal distribution with mean* 0.*7515 cm and standard deviation* 0. *0020 cm, how many of the plugs are likely to be rejected if the approved diameter is* 0.*752 ± .004 cm ?*

Solution. Tolerance limits of the diameter of non-defective plugs are



0.752 – 0.004 = 0.748 cm and

0.752 + 0.004 = 0.756 cm

$$z = \frac{x - \mu}{\sigma}$$

If $x_1 = 0.748$, $\quad z_1 = \dfrac{0.748 - 0.7515}{0.002} = -1.75$

If $x_2 = 0.756$, $\quad z_2 = \dfrac{0.756 - 0.7515}{0.002} = 2.25$

Area under $\quad z_1 = -1.75$ to $z_2 = 2.25$

$= $ (Area from $z = 0$ to $z_1 = -1.75$) + (Area from $z = 0$ to $z_2 = 2.25$)

$= 0.4599 + 0.4878 = 0.9477$

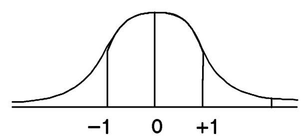
Number of plugs likely to be rejected

$= 1000\,(1 - 0.9477) = 1000 \times .0523 = 52.3$

Approximately 52 plugs are likely to be rejected. **Ans.**

Example 21. *A manufacturer knows from experience that the resistance of resistors he produces is normal with mean $\mu = 100$ ohms and standard deviation $\sigma = 2$ ohms. What percentage of resistors will have resistance between 98 ohms and 102 ohms ?*

Solution. $\mu = 100$ ohms, $\sigma = 2$ ohms



$x_1 = 98, \quad x_2 = 102$

$z = \dfrac{x - \mu}{\sigma}, \quad z_1 = \dfrac{98 - 100}{2} = -1$

$z_2 = \dfrac{102 - 100}{2} = +1$

Area between $\quad z_1 = -1$ and $z_2 = +1$

$= $ (Area between $z = 0$ and $z = -1$)

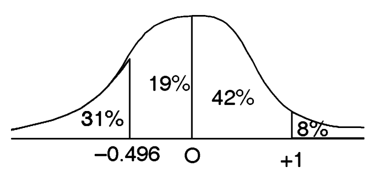
$+$ (Area between $z = 0$ and $z = +1$)

$= 2$ (Area between $z = 0$ and $z = +1$) $= 2 \times 0.3413 = 0.6826$

Percentage of resistors having resistance between 98 ohms and 102 ohms = 68.26 **Ans.**

Example 22. *In a normal distribution, 31% of the items are under 45 and* 8% *are over 64. Find the mean and standard deviation of the distribution.* (*AMIETE, Dec. 2010*)

Solution. Let μ. be the mean and σ the S.D.



If $x = 45$, $\quad z = \dfrac{45 - \mu}{\sigma}$

If $x = 64$, $\quad z = \dfrac{64 - \mu}{\sigma}$

Area between 0 and $\quad z = \dfrac{45 - \mu}{\sigma} = 0.50 - .31 = 0.19$

[From the table, for the area 0.19, $z = 0.496$]

$$\frac{45 - \mu}{\sigma} = -0.496 \qquad ...(1)$$

Area between $z = 0$ and $\quad z = \dfrac{64 - \mu}{\sigma} = 0.5 - 0.08 = 0.42.$

(From the table, for area 0.42, $\quad z = 1.405$)

$$\frac{64 - \mu}{\sigma} = 1.405 \qquad ...(2)$$

Solving (1) and (2) we get $\mu = 50$, $\sigma = 10$. **Ans.**

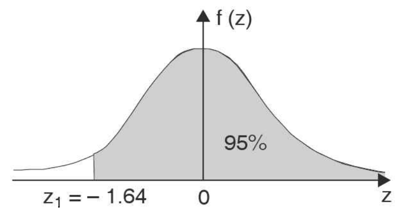
Example 23. *The income of a group of 10,000 persons was found to be normally distributed with mean Rs. 750 p.m. and standard deviation of Rs. 50. Show that, of this group, about 95% had income exceeding Rs. 668 and only 5% had income exceeding Rs. 832. Also find the lowest income among the richest 100.*

(*U.P. III Semester Dec. 2004*)

Solution. Mean $= \mu = 750$

Standard deviation $= \sigma = 50$

and $z = \dfrac{x-\mu}{\sigma}$



(i) If $x_1 = 668$, then $z_1 = \dfrac{668-750}{50} = -1.64$

$P(x_1 > 668) = P(z_1 < -1.64)$

$= 0.5 + P(-1.64 \le z \le 0) = 0.5 + P(0 \le z \le 1.64) = 0.5 + 0.4495 = 0.9495$

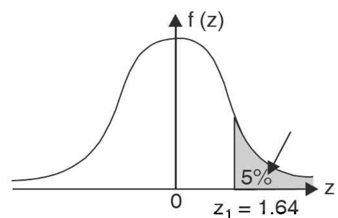
$\therefore$ Percentage of persons having income exceeding Rs. 668 = 94.95% $\approx$ 95% (approx.)

(*ii*) If $x = 832$, then $z = \dfrac{832-750}{50} = 1.64$

$P(x_2 > 832) = P(z_2 > 1.64)$

$= 0.5 - 0.4495$

$= 0.0505$



$\therefore$ Percentage of persons having income exceeding Rs. 832 = 5.05% = 5% (approx.)

(*iii*) Let x be the lowest income among the richest 100 persons.

100 persons = 1% of 10,000

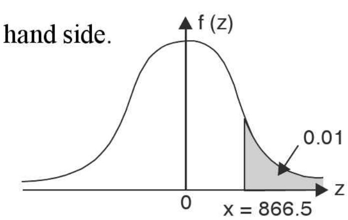
100 persons represents 1% area under the curve on the right hand side.

Thus the area between 0 and z

$= 0.5 - 0.01 = 0.49$

From the table z for area 0.49 is 2.33

$z = \dfrac{x-\mu}{\sigma}$



$\Rightarrow \quad 2.33 = \dfrac{x-750}{50} \Rightarrow x - 750 = 50 \times 2.33$

$\Rightarrow \quad x - 750 = 116.5 \Rightarrow x = 866.5$

Hence, the minimum income among the 100 richest persons is equal to Rs. 866.5. **Ans.**

Example 24. *Fit a normal curve to the following data :*

Length of line (*in cm*)	8.60	8.59	8.58	8.57	8.56	8.55	8.54	8.53	8.52
Frequency	2	3	4	9	10	8	4	1	1

Solution. Let assumed mean = 8.56 cm

x_i	f_i	$x_i - 8.56$	$f_i(x_i - 8.56)$	$f_i(x_i - 8.56)^2$
8.60	2	.04	.08	.0032
8.59	3	.03	.09	.0027
8.58	4	.02	.08	.0016
8.57	9	.01	.09	.0009

8.56	10	0	0	0
8.55	8	– .01	– .08	.0008
8.54	4	– .02	– .08	.0016
8.53	1	– .03	– .03	.0009
8.52	1	– .04	– .04	.0016
	$\Sigma f_i = 42$		$\Sigma f_i (x_i - 8.56) = 0.11$	$\Sigma f_i (x_i - 8.56)^2 = 0.0133$

$$\text{Mean} = a + \frac{\Sigma f_i\,(x_i - 8.56)}{\Sigma f_i} = 8.56 + \frac{0.11}{42} = 8.56 + 0.00262 = 8.56262 \quad \textbf{Ans.}$$

$$\text{Standard deviation} = \sqrt{\frac{\Sigma f_i\,(x_i - 8.56)^2}{\Sigma f_i} - \left(\frac{\Sigma f_i\,(x_i - 8.56)}{\Sigma f_i}\right)^2} = \sqrt{\frac{0.0133}{42} - \left(\frac{0.11}{42}\right)^2}$$

$$= \sqrt{0.000316666 - 0.000006859} = \sqrt{0.00030980} = 0.0176$$

Hence, the equation of the normal curve fitted to the given data is

$$P(x) = \frac{N}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma}}$$

where, $\mu = 8.56262$

$\sigma = 0.0176$ and $N = 42$ **Ans.**

EXERCISE 65.2

1. In a regiment of 1000, the mean height of the soldiers is 68.12 units and the standard deviation is 3.374 units. Assuming a normal distribution, how many soldiers could be expected to be more than 72 units? It is given that
   $P(z = 1.00) = 0.3413, P(z = 1.15) = 0.3749$ and
   $P(z = 1.25) = 0.3944$, where z is the standard normal variable. **Ans.** 125

2. The lifetime of radio tubes manufactured in a factory is known to have an average value of 10 years. Find the probability that the lifetime of a tube taken randomly (*i*) exceeds 15 years, (*ii*) is less than 5 years, assuming that the exponential probability law is followed. **Ans.** (*i*) 0.2231, (*ii*) 0.3935.

3. The breaking strength X of a cotton fabric is normally distributed with $E(x) = 16$ and $\sigma(x) = 1$. The fabric is said to be good if $X \geq 14$. What is the probability that a fabric chosen at random is good. Given that $\phi(2) = 0.9772$ **Ans.** 0.9772

4. A manufacturer knows from experience that the resistance of resistors he produces is normal with mean it $\mu = 140\ \Omega$ and standard deviation $\sigma = 5\Omega$. Find the percentage of the resistors that will have resistance between $138\ \Omega$ and $142\ \Omega$. (given $\phi(0.4) = 0.6554$, where z is the standard normal variate). **Ans.** 31.08%

5. A manufacturing company packs pencils in fancy plastic boxes. The length of the pencils is normally distributed with $\mu = 6''$ and $\sigma = 0.2''$. The internal length of the boxes is $6.4''$. What is the probability that the box would be too small for the pencils (Given that a value of the standardized normal distribution function is $\phi(2) = 0.9772$). **Ans.** 0.0228.

6. A manufacturer produces airmail envelopes, whose weight is normal with mean $\mu = 1.95$ gm and standard deviation $\sigma = 0.05$ gm. The envelopes are sold in lots of 1000. How many envelopes in a lot will be heavier than 2 gm? Use the fact that $\frac{1}{\sqrt{2\pi}}\int_2^1 \exp\left(\frac{-x^2}{2}\right) dx = 0.3413$ **Ans.** 159

7. The mean height of 500 students is 151 cm and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many student's height lie between 120 and 155 cm. **Ans.** 294

8. A large number of measurement is normally distributed with a mean of 65.5″ and S.D. of 6.2″. Find the percentage of measurements that fall between 54.8 and 68.8″. **Ans.** 66.01%

9. Find the mean and variance of the density function $f(x) = \lambda e^{-\lambda x}$ **Ans.** $\frac{1}{\lambda}, \frac{1}{\lambda^2}$

10. If x is normally distributed with mean 1 and variance 4,

(*i*) Find $Pr(-3 \le x \le 3)$; (*ii*) Obtain k if $Pr(x \le k) = 0.90$ **Ans.** (*i*) 0.8185, (*ii*) 3.56

11. A normal variable x has mean 1 and variance 4. Find the probability that $x \ge 3$. (Given: z is the standard normal variable and $\phi(0) = 0.5$, $\phi(0.5) = 0.6915$, $\phi(1) = 0.8413$, $\phi(1.5) = 0.9332$) **Ans.** 0.1587

12. The random variable x is normally distributed with $E(x) = 2$ and variance $V(x) = 4$. Find a number p (approximately), such that $P(x > p) = 2P(x \le p)$. [The values of the standard normal distribution are 0 (–0.43) = 0.3336, and 0 (– 0.44) = 0.3300]. **Ans.** 1.13834

If X ~ N (10, 4) find Pr [|X| ≥ 5]. **Ans.** $\frac{1}{e}, \frac{1}{5\sqrt{2\pi}} e^{-\frac{(x-75)^2}{2(0.5)}}, 0.062$

13. The continuous random variable x is normally distributed with $E(x) = \mu$ and $V(x) = \mu^2$. If $Y = cx + d$, then find $V(Y)$. **Ans.** $c^2 \mu^2$

14. The pdf of X is given by $f(X) = \lambda e^{-\lambda x}$ $x \ge 0$, $\lambda \ge 0$. Calculate $Pr[X > E(X)]$.

If $X \sim N(75, 25)$, find $Pr[X > 80/X > 77]$ (*A.M.I.E., Winter 2001*)

If $X \sim N(10, 4)$, find $Pr[|X| \ge 5$ **Ans.** $\frac{1}{e}, \frac{1}{5\sqrt{2\pi}} e^{-\frac{x-75)^2}{2(0.5)}}, 0.062$

15. A random variable x has a standard normal distribution ϕ. Prove : Pr (1| X | > k) = 2 [1– ϕ (k)]

16. The random variable x has the probability density function $f(x) = kx$ if $0 \le x \le 2$

Find k. Find x such that

(*i*) $Pr(X \le x) = 0.1$ (*ii*) $Pr(X \le x) = 0.95$ **Ans.** $k = \frac{1}{2}$ (*i*) $x = 0.632$ (*ii*) $x = 1.949$

17. For a normal curve, show that $\mu_{2n+1} = 0$ and $\mu_{2n} = (2n - 1)\sigma^2 \mu_{2n-2}$.

18. In a mathematics examination, the average grade was 82 and the standard deviation was 5. All the students with grades from 88 to 94 received a grade B. If the grades are normally distributed and 8 students received a B grade, find how many students took the examination. Given

$\frac{x}{\sigma}$	1.20	2.00	2.40	2.45
A	0.3849	0.4772	0.4918	0.4929

(*A.M.I.E.T.E., Winter 2001*) **Ans.** 75 students

19. Explain the characteristics and importance of a normal distribution. (*A.M.I.E., Summer 2004*)

20. The life time of a certain component has a mean life of 400 hours and standard deviation of 50 hours. Assuming normal distribution for the life time of 1000 components, determine approximately the number of components whose life time lies between 340 to 465 hours. You may use the following data? Where symbols have their usual meanings. (*A.M.I.E., Winter 2002*) **Ans.** 788

21. For standard normal variate mean μ. is (*A.M.I.E., Winter 2005*)

(*a*) 1 (*b*) 0 (*c*) 6 (*d*) none of the above **Ans.** (*b*)

22. Fill in the blanks :

(*f*) The mean, median and mode of a normal distribution are........... (*A.M.I.E., Summer 2000*) **Ans.** zero

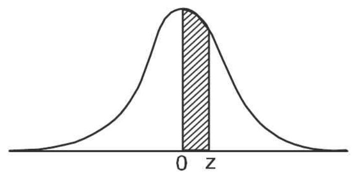
(*h*) The probability density function of Beta distribution with $\alpha = 1$, $\beta = 4$ is $f(x)$=......

(*AMIE., Summer 2000*) **Ans.** $4(1 - x)^3$

TABLE – 1

AREA UNDER STANDARD NORMAL CURVE FROM $z = 0$ T0 $z = \dfrac{x - \mu}{\sigma}$

An entry in the table is the proportion under the entire curve which is between $z = 0$ and a positive value of z. Area for negative values of z are obtained by symmetry.



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.8832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4415	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4841	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990