Method of Solving Linear differential eqⁿ by changing the independent variables.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{--- (1)}$$

For changing the variable x to z we assume we assume

$$\left(\frac{dz}{dx}\right)^2 = |Q| \Rightarrow \left(\frac{dz}{dx}\right)^2 = cf(x) \quad \text{--- (2)}$$

where z is independent variable.

$$\frac{dz}{dx} = \sqrt{cf(x)}$$

$$z = \int \sqrt{cf(x)}\, dx \quad \text{--- (3)}$$

The relation (3) transform the d.e. (1) to

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad \text{--- (4)}$$

where

$$P_1 = \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right) \Big/ \left(\frac{dz}{dx}\right)^2$$

$$Q_1 = Q \Big/ \left(\frac{dz}{dx}\right)^2$$

$$R_1 = R \Big/ \left(\frac{dz}{dx}\right)^2$$

Working rule:-

calculate P_1, Q_1 and R_1 and the solve the eqⁿ (4). and then substitute $z = \sqrt{cf(x)}$.

Ques: Solve by changing the independent variable

$$(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + 4y = 0$$

Soln

changing it to standard form

$$\frac{d^2y}{dx^2} + \frac{2x}{(1+x^2)}\frac{dy}{dx} + \frac{4}{(1+x^2)^2}y = 0 \quad -(1)$$

Here $P = \frac{2x}{1+x^2}$, $Q = \frac{4}{(1+x^2)^2}$, $R = 0$

For changing the independent variable, assum

$$\left(\frac{dz}{dx}\right)^2 = Q(x) = \frac{4}{(1+x^2)^2}$$

$$\Rightarrow \frac{dz}{dx} = \frac{2}{1+x^2}$$

$$\Rightarrow z = 2\tan^{-1}x \quad -(2)$$

with relation (2) d.e. tranform to

$$\frac{d^2y}{dx_2} + P_1\frac{dy}{dx} + Q_1 y = R_1 \quad -(1)$$

where,

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{-4x}{(1+x^2)^2} + \frac{2x}{(1+x^2)}\cdot\frac{2}{(1+x^2)}}{\left(\frac{4}{1+x^2}\right)^2}$$

$$= 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4/(1+x^2)^2}{4/(1+x^2)^2} = 1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{0}{\left(\frac{dz}{dx}\right)^2} = 0$$

$\therefore$ d.e. (1) transform to

$$\frac{d^2y}{dz^2} + y = 0$$

$$(D^2+1)y = 0$$

auxiliary eqⁿ

$$m^2+1 = 0, \quad m = \pm i$$

$$\Rightarrow y = c_1 \cos z + c_2 \sin z$$

$$= c_1 \cos(2\tan^{-1}x) + c_2 \sin(2\tan^{-1}x).$$

Question Solve by changing the independent variable

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin x^2 \quad \text{--- (1)}$$

write ~~transform~~ it to standard form

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2\sin x^2$$

for changing the independent variable.
assume

$$\left(\frac{dz}{dx}\right)^2 = \cancel{Bx} = 4x^2 \quad \text{--- (2)}$$

$$\Rightarrow \frac{dz}{dx} = 2x,$$

$$\Rightarrow z = x^2$$

$$\frac{d^2z}{dx^2} = 2$$

eqⁿ (2) tranform eqⁿ (1) to

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$$

$$P_1 = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-4x^2}{4x^2} = -1$$

$$R_1 = \frac{8x^2 \sin x^2}{4x^2} = 2\sin z \qquad (\because x^2 = z)$$

$\therefore$ d.e. (1) transform to

$$\frac{d^2y}{dz^2} - y = 2\sin z$$

auxiliary eqn

$$m^2 - 1 = 0 \quad , \quad m = \pm 1$$

$$\text{C.F.} = c_1 e^{z} + c_2 e^{-z}$$

$$\text{P.I.} = \frac{1}{D^2-1} 2\sin z$$

$$= 2\frac{1}{D^2-1}\sin z \qquad D^2 \longrightarrow -1$$

$$= \frac{2}{-2}\sin z = -\sin z$$

$$\text{C.I} = \text{C.F.} + \text{P.I.}$$

$$= c_1 e^{z} + c_2 e^{-z} - \sin z$$

$$= c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2 \qquad (\because x^2 = z).$$

Variation of parameter:-

Let the given diff. eqⁿ in standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{--- (i)}$$

consider its homogeneous part

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad \text{--- (ii)}$$

Let $u(x)$ and $v(x)$ be the solⁿ of (ii)

C.F. is ~~$y = c_1 u_1(x) + c_2 u_2(x)$~~

$$y = c_1 u(x) + c_2 v(x) \quad \text{--- (i)}$$

where c_1 & c_2 are arbitrary constant.

Then the complete solⁿ of (i) is given by

$$\boxed{y = A(x)u(x) + B(x)v(x) + c} \quad \text{--- 3}$$

where

$$A(x) = -\int \frac{v(x)R}{uv' - u'v}\,dx$$

$$B(x) = \int \frac{u(x)R}{uv' - u'v}\,dx$$

Q Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ by variation of Parameter method.

Soln

$$\cancel{(D^2+1)}\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$$

The homogeneous part is

$$\frac{d^2y}{dx^2} + y = 0$$

$$(D^2+1)y = 0$$

auxiliary eqn is

$$m^2 + 1 = 0, \quad m = \pm i$$

$$\therefore \quad C.F. = \underbrace{c_1 \cos x}_{u(x)} + \underbrace{c_2 \sin x}_{v(x)}$$

Now the complete integral is

$$y = A(x)u(x) + B(x)v(x) + c \quad \text{--- (*)}$$

where

$$A(x) = -\int \frac{vR}{uv' - u'v}\,dx$$

$$= -\int \frac{\sin x \cdot \operatorname{cosec} x}{+\cos^2 x + \sin^2 x}\,dx$$

$$= -\int dx$$

$$= -x$$

$$B(x) = \int \frac{uR}{uv' - u'v}\,dx$$

$$= \int \frac{\cos x \cdot \operatorname{cosec} x}{+\cos^2 x + \sin^2 x}\,dx$$

$= + \int \cot x \, dx$

$= \log(\sin x)$

$\therefore$ complete sol^n is given by using ⊛

$y = -x\cos x + \log(\sin x) \cdot \sin x + c$

$$\boxed{y = -x\cos x + \sin x \cdot \log(\sin x) + c}$$

Cauchy Euler homogeneous linear equation:-

$$x^2 \frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = f(x) \quad \text{—}(1)$$

we substitute $x = e^z$

$\Rightarrow z = \log x$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$x^2 \frac{d^2y}{dz^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Thus eq^n (1) can be written as

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + a\frac{dy}{dz} + by = f(e^z)$$

$$\Rightarrow \boxed{\frac{d^2y}{dz^2} + (a-1)\frac{dy}{dz} + by = f(e^z)}$$

Linear d.e. with constant coefficient.

__Ques:-__ $$x^2 \frac{d^2y}{dz^2} - 2\cancel{x}\frac{dy}{dx} - 4y = x^4 \quad \text{--- (1)}$$

applying cauchy-Euler theorem

Substituting $x = e^z$, eq^n (1) is transformed to

$$\frac{d^2y}{dz^2} + (-2-1)\frac{dy}{dz} - 4y = (e^z)^4$$

$$\frac{d^2y}{dz^2} - 3\frac{dy}{dz} - 4y = e^{4z}$$

__For__ C.F.

$$(D^2 - 3D - 4)y = e^{4z}$$

auxiliary eq^n

$$m^2 - 3m - 4 = 0$$

$$m^2 - 4m + m - 4 = 0$$

$$m(m-4) + 1(m-4) = 0$$

$$m = 4, -1$$

$$Y = c_1 e^{4z} + c_2 e^{-z}$$

$$P.I. = \frac{1}{D^2 - 3D - 4} e^{4z}$$

$$\cancel{4^2 - 3\times4 - 4}$$

here $f(D) = D^2 - 3D - 4$

$f(4) = 4^2 - 3\times4 - 4 = 0$

$f'(D) = 2D - 3$

$f'(4) = 8 - 3 = 5$

$\therefore$ $P.I. = \frac{z}{5} e^{4z}$

There fore complete solⁿ is

$$Y = c_1 e^{4z} + c_2 e^{-z} + \frac{1}{5} z e^{4z}$$

$$= c_1 x^4 + c_2 x^{-1} + \frac{1}{5} x^4 \log x \quad (\because x = e^z)$$

Ques: find general solⁿ of

$x^2 y'' +$

$P.\,I. = \cos x \cdot \log \cos x + x \sin x$

Complete solution is $y = C.\,F. + P.\,I.$

$\Rightarrow \qquad y = A \cos x + B \sin x + \cos x \log \cos x + x \sin x$ Ans.

Example 15. *Obtain general solution of the differential equation* $x^2 y'' + xy' - y = x^3 e^x$

(Nagpur University, Summer 2004, U. P. II Semester, Summer 2002)

Solution. The given differential equation is $x^2 y'' + xy' - y = x^3 e^x$

$$\Rightarrow \qquad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^3 e^x \qquad \ldots(1)$$

Putting $x = e^z \qquad \Rightarrow \qquad D = \dfrac{d}{dz},\ x\dfrac{dy}{dx} = Dy, \quad x^2 \dfrac{d^2 y}{dx^2} = D(D-1)\,y,$ in (1), we get

$$D(D-1)\,y + D\,y - y = e^{3z} e^{e^z} \qquad \Rightarrow \qquad (D^2 - 1)y = e^{3z} e^{e^z}$$

$$\text{A. E. is } m^2 - 1 = 0 \quad \Rightarrow \quad m = \pm 1$$

$$\therefore \qquad \text{C. F.} = c_1 e^z + c_2 e^{-z}$$

$$= uy_1 + vy_2, \text{ where } y_1 = e^{-z},\ y = \left[y_1 = x_1\ y_2 = \frac{1}{x} \right]$$

$$\text{P.I.} = uy_1 + vy_2$$

$$\text{Also } u = -\int \frac{y_2 z}{y_1 y_2' - y_1' y_2}\, dz = -\int \frac{e^{-z} . e^{3z} . e^{e^z}}{e^z(-e^{-z}) - e^z(e^{-z})}\, dz = -\int \frac{e^{2z} e^{e^z}}{-1-1}\, dz \qquad \left[\begin{array}{l} x = e^z,\ dx = e^z dz \\ dz = \dfrac{dx}{e^z} = \dfrac{dx}{x} \end{array} \right.$$

$$= \frac{1}{2} \int e^{2z} e^{e^z}\, dz = \int x^2 e^x \frac{dx}{x} = \frac{1}{2} \int x e^x dx$$

$$= \frac{1}{2}\left[xe^x - (1)\, e^x \right] = \frac{1}{2}(xe^x - e^x)$$

$$\text{and } \quad v = \int \frac{y_1 z}{y_1 y_2' - y_1' y_2}\, dz = \int \frac{e^z . e^{3z} . e^{e^z}}{e^z(-e^{-z}) - e^z(e^{-z})}\, dz = \int \frac{e^{4z} e^{e^z}}{-1-1}\, dz = \int \frac{x^4 e^x}{-2} \frac{dx}{x} = -\frac{1}{2} \int x^3 e^x dx$$

$$= -\frac{1}{2}[x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x]$$

$$\text{P.I.} \quad = uy_1 + vy_2 = \frac{1}{2}(xe^x - e^x)\, x - \frac{1}{2}(x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x)\frac{1}{x}$$

$$= \frac{1}{2}\left[x^2 - x - x^2 + 3x - 6 + \frac{6}{x} \right] e^x = \frac{1}{2}\left(2x - 6 + \frac{6}{x} \right) e^x = \left(x - 3 + \frac{3}{x} \right) e^x$$

Complete solution = C.F. + P.I.

$$y = (c_1 e^z + c_2 e^{-z}) + \left(x - 3 + \frac{3}{x} \right) e^x$$

$$= c_1 x + \frac{c_2}{x} + \left(x - 3 + \frac{3}{x} \right) e^x \qquad \text{Ans.}$$

$$= \frac{e^x}{2}\left[x^4 - 3x^3 + 6x^2 - 6x - x^4 + 5x^3 - 20x^2 + 60x - 120 + \frac{120}{x}\right]$$

$$= \frac{e^x}{2}\left[2x^3 - 14x^2 + 54x - 120 + \frac{120}{x}\right]$$

$$y = C.\,F. + P.\,I.$$

$$= C_1\, x + \frac{C_2}{x} + (x^3 - 7x^2 + 27x - 60 + \frac{60}{x})\, e^x$$

Example 16. *Solve by method of variation of parameters:*

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

(Uttarakhand, II Semester, June 2007, A.M.I.E.T.E., Summer 2

(Nagpur University, Summer 2

Solution. $$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

A. E. is $(m^2 - 1) = 0$

$$m^2 = 1, \qquad m = \pm 1$$

$$C.\,F. = C_1\, e^x + C_2\, e^{-x}$$

$\therefore$ $$P.I. = uy_1 + vy_2$$

Here, $$y_1 = e^x, \qquad y_2 = e^{-x}$$

and $$y_1 \cdot y'_2 - y'_1 \cdot y_2 = -e^x \,.\, e^{-x} - e^x . e^{-x} = -2$$

$$u = \int \frac{-y_2 X}{y_1 . y'_2 - y'_1 . y_2}\, dx = -\int \frac{e^{-x}}{-2} \times \frac{2}{1+e^x}\, dx$$

$$= \int \frac{e^{-x}}{1+e^x}\, dx = \int \frac{dx}{e^x\,(1+e^x)} = \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right) dx$$

$$= \int e^{-x}\, dx - \int \frac{e^{-x}}{e^{-x}+1}\, dx = -e^{-x} + \log\,(e^{-x}+1)$$

$$v = \int \frac{y_1 X}{y_1 . y'_2 - y'_1 . y_2}\, dx = \int \frac{e^x}{-2}\, \frac{2}{1+e^x}\, dx = -\int \frac{e^x}{1+e^x}\, dx = -\log\,(1+e$$

$$\text{P.I.} = u.\, y_1 + v.\, y_2 = [-e^{-x} + \log\,(e^{-x}+1)\,]\, e^x - e^{-x} \log\,(1+e^x)$$

$$= -1 + e^x \log\,(e^{-x}+1) - e^{-x} \log\,(e^x+1)$$