Orthogonal matrix :-

If A is real orthogonal matrix then

$$A \cdot A' = I$$

$$\Rightarrow |A \cdot A'| = |I| = 1$$

$$\Rightarrow |A| \cdot |A'| = 1$$

$$\Rightarrow |A|^2 = 1 \qquad \{\because |A| = |A'|\}$$

$$\Rightarrow |A| = \pm 1$$

Unitary Matrix :- A matrix is said to be unitary if $A^\theta A = A A^\theta = I$,

where A^θ = transpose of complex conjugate of A.

→ If λ is eigen value of A, there exist a nonzero vector X such that

$$AX = \lambda X$$

$$\Rightarrow (AX)^\theta = (\lambda X)^\theta$$

$$\Rightarrow X^\theta A^\theta = \bar{\lambda} X^\theta$$

$$\Rightarrow X^\theta A^\theta AX = \bar{\lambda} X^\theta \lambda X$$

$$\Rightarrow X^\theta X = \bar{\lambda}\lambda X^\theta X$$

$$\Rightarrow (1 - \bar{\lambda}\lambda) X^\theta X = 0$$

$$\Rightarrow |\lambda| = 1 \qquad (\because \lambda\bar{\lambda} = |\lambda|^2)$$

$\Rightarrow$ eigen value of unitary matrices are of unit modulo.

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→ If λ is eigen value of a matrix A then $a\lambda$ will be the eigen value of $aA = B$

Let λ is eigen value of A, there exist a non-zero vector X such that

$$AX = \lambda X$$

$$\Rightarrow \quad aAX = (a\lambda)X$$

$$\Rightarrow \quad BX = (a\lambda)X$$

$\Rightarrow$ $a\lambda$ is an eigen value of $aA = B$.

→ Symmetric matrix :-
Eigen values of real symmetric matrices are real.
Consider λ is an eigen value of A which is complex, there exist a nonzero vector X such that

$$AX = \lambda X$$

$$\Rightarrow \quad \overline{AX} = \bar{\lambda}\bar{X} \qquad \text{(taking conjugate}$$

$$\Rightarrow \quad \bar{A}\bar{X} = \bar{\lambda}\bar{X}$$

$$\Rightarrow \quad A\bar{X} = \bar{\lambda}\bar{X} \qquad \{\because A \text{ is real symmetric matrix}\}$$

$$\Rightarrow \quad (A\bar{X})^T = \bar{\lambda}\bar{X}^T$$

$$\Rightarrow \quad \bar{X}^T A = \bar{\lambda}(\bar{X})^T \qquad \{\because A^T = A, \text{ as symmetric.}$$

Now
$$\bar{X}^T AX = (\bar{X})^T(AX) = (\bar{X}^T)(\lambda X) = \lambda(\bar{X}^T X) \quad \text{--- (1)}$$

again
$$\bar{X}^T AX = ((\bar{X})^T A)X = \bar{\lambda}\bar{X}^T X \quad \text{--- (2)}$$

from (1) & (2)

$$\lambda = \bar{\lambda} \quad \Rightarrow \lambda \text{ is a real number.}$$

Skew Symmetric matrix :-

Let S be a real skew symmetric matrix

i.e.

$$S^T = -S$$

Let λ be an ~~complex~~ eigen value of S, there exist a nonzero vector X such that

$$SX = \lambda X$$

$\Rightarrow \overline{(SX)} = \overline{(\lambda X)}$ (taking complex conjugate

$\Rightarrow \bar{S}\bar{X} = \bar{\lambda}\bar{X}$

$\Rightarrow S\bar{X} = \bar{\lambda}\bar{X}$ ($\because$ S is real skew symm-

$\Rightarrow (S\bar{X})^T = (\bar{\lambda}\bar{X})^T$

$\Rightarrow (\bar{X})^T S^T = \bar{\lambda}(\bar{X})^T$

$\Rightarrow -(\bar{X})^T S = \bar{\lambda}(\bar{X})^T$

$\cancel{\Rightarrow -(\bar{X})^T S X = \bar{\lambda}(\bar{X})^T X}$

Now

$$(\bar{X})^T S X = (\bar{X})^T (SX) = (\bar{X})^T(\lambda X) = \lambda \bar{X}^T X \quad -(1)$$

again

$$(\bar{X})^T S X = ((\bar{X})^T S) X = -\bar{\lambda}(\bar{X})^T X \quad -(2)$$

from (1) & (ii)

$$\lambda(\bar{X})^T X = -\bar{\lambda}(\bar{X})^T X$$

$\Rightarrow \lambda + \bar{\lambda} = 0$

Let λ is a real no. (i.e. $\bar{\lambda} = \lambda$)

$\Rightarrow \boxed{\lambda = 0}$

If λ is complex i.e. $\lambda = a+ib$

$\Rightarrow \quad \bar{\lambda} = a-ib$

$\because \quad \lambda + \bar{\lambda} = 0$

$\Rightarrow \quad a+ib+a-ib = 0$

$\Rightarrow \quad 2a = 0$

$\Rightarrow \quad a = 0$

i.e. $\quad \lambda = 0+ib = ib$

$\Rightarrow$ eigen values of real symmetric matrices are ~~eig~~ either purely imaginary or zero.

Involutary Matrix :- A matrix is said to be involutary if

$$\boxed{A^2 = I}$$

Let λ be eigen value of A, i.e. there exist X a non zero vector, such that

$$AX = \lambda X$$

$\Rightarrow \quad A(AX) = A(\lambda X)$

$\Rightarrow \quad A^2 X = \lambda (AX)$

$\Rightarrow \quad IX = \lambda (\lambda X)$

$\Rightarrow \quad X = \lambda^2 X$

$\Rightarrow \quad (\lambda^2 - 1) X = 0$

$\Rightarrow \quad \lambda = \pm 1$

$\Rightarrow$ eigen values of involutory matrices are either 1 or -1.