

Unit - IIIBeta & Gamma functions $\beta(m, n)$ Beta function

Let  $m > 0, n > 0$  be positive numbers then

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Beta & Gamma funct'

Let  $n > 0$  then

$$\gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$\gamma(0) = \text{undefined}$

## Inferiorities of Beta fun<sup>n</sup>

①  $B(m, n) = B(n, m)$

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

ut  $x = 1-y \Rightarrow dx = -dy$

$$= - \int_1^0 (1-y)^{m-1} (1-1+y)^{n-1} (-dy)$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1} dy$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1} dx$$

$$= B(n, m)$$

$$(2) \quad B(m, n) = \int_0^\infty \frac{x^{m-1} dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Sol:

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}$$

Put  $x = \frac{1}{1+y} \quad (\Rightarrow 1+y = \frac{1}{x}, y = \frac{1}{x} - 1)$

$$\Rightarrow dx = \frac{-1}{(1+y)^2} dy$$

$$= \int_0^\infty \left(\frac{1}{1+y}\right)^{m-1} \left(1 - \frac{1}{1+y}\right)^{n-1} \times \frac{-1}{(1+y)^2} dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}} \times \frac{y^{n-1}}{(1+y)^{n-1}} \times \frac{1}{(1+y)^2} dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$(3) \quad B(m, n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

Pf:

$$\therefore B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

put  $x = \sin^2 \theta$

$$dx = 2 \sin \theta \cos \theta d\theta$$

$$\begin{aligned}
 B(m, n) &= \int_0^{\frac{\pi}{2}} (x)^{m-1} (1-x)^{n-1} \times 2 \sin \theta \cos \theta d\theta \\
 &= \int_0^{\frac{\pi}{2}} \sin^{2m-2} \theta \cos^{2n-2} \theta \times 2 \sin \theta \cos \theta d\theta \\
 &= 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta
 \end{aligned}$$

## Properties of Gamma func

(ii)  $\Gamma(n+1) = n!$  if  $n$  is the integer  
 $= n\Gamma(n)$  if otherwise

$$\left| \begin{array}{l} \lim_{x \rightarrow \infty} \frac{-e^{-x} x^n}{e^{-x}} \quad (\infty \times \infty) \\ = \lim_{x \rightarrow \infty} \frac{x^n}{e^x} \quad \left( \frac{\infty}{\infty} \right) \\ = \lim_{x \rightarrow \infty} \frac{n!}{e^x} = \frac{n!}{e^\infty} = 0 \end{array} \right.$$

PF:  $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

$$\begin{aligned} \Rightarrow \Gamma(n+1) &= \int_0^\infty e^{-x} x^n dx \quad - \textcircled{1} \\ &= \left\{ -e^{-x} x^n \right\}_0^\infty + n \int_0^\infty e^{-x} x^{n-1} dx \\ &= 0 - 0 + n \int_0^\infty e^{-x} x^{n-1} dx \\ &= n \int_0^\infty e^{-x} x^{n-1} dx \quad - \textcircled{11} \\ &= n \Gamma(n) \end{aligned}$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$$

$$\left[ -e^{-x} \right]_0^\infty = 0 + 1 = 1$$

$$= n \int_0^\infty e^{-x} x^{n-1} dx$$

$$= n(n-1) \int_0^\infty e^{-x} x^{n-2} dx$$

$$= n(n-1)(n-2) \int_0^\infty e^{-x} x^{n-3} dx$$

$$= n(n-1)(n-2) \dots (n-(n-1)) \int_0^\infty e^{-x} x^{n-n} dx$$

$$= n!$$

$$\therefore \Gamma(n+1) = n \Gamma(n)$$

$$= n!$$

$$\left| \begin{array}{l} \Gamma(1) = 0! = 1 \\ \Gamma(2) = 1! \\ \Gamma(6) = 5! \\ \Gamma(11) = 10! \text{ and so on.} \end{array} \right.$$

As we know that

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\begin{aligned}\Gamma\left(\frac{3}{2}\right) &= \Gamma\left(\frac{1}{2} + 1\right) \\ &= \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \sqrt{\pi}\end{aligned}$$

$$\begin{aligned}\Gamma\left(\frac{5}{2}\right) &= \Gamma\left(\frac{3}{2} + 1\right) = \frac{3}{2} \Gamma\left(\frac{3}{2}\right) \\ &= \frac{3}{4} \sqrt{\pi}\end{aligned}$$

$$\begin{aligned}\Gamma\left(\frac{7}{2}\right) &= \Gamma\left(\frac{5}{2} + 1\right) \\ &= \frac{5}{2} \Gamma\left(\frac{5}{2}\right) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) \\ &= \frac{15}{8} \sqrt{\pi}.\end{aligned}$$

$$(2) \quad \frac{\Gamma(n)}{k^n} = \int_0^\infty e^{-kx} x^{n-1} dx$$

∴  $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$

If:  $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$   
 put  $x = ky \Rightarrow dx = kdy$

$$\begin{aligned}\Gamma(n) &= \int_0^\infty e^{-ky} (ky)^{n-1} (kdy) \\ &= \int_0^\infty e^{-ky} k^{n-1} y^{n-1} kx dy \\ &= k^n \int_0^\infty e^{-ky} y^{n-1} dy \\ &= k^n \int_0^\infty e^{-kx} x^{n-1} dx\end{aligned}$$

$$(3) \int_0^\infty e^{-x^{\frac{1}{n}}} dx = n \Gamma(n)$$

$$= \frac{\int_0^\infty -y^{\frac{1}{n}} e^{-y} dy}{n}$$

$$\therefore \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\text{Put } x = y^{\frac{1}{n}}$$

$$dx = \frac{1}{n} y^{\frac{1}{n}-1} dy$$

$$= \frac{1}{n} \int_0^\infty e^{-y^{\frac{1}{n}}} dy$$

$$\Rightarrow \int_0^\infty -x^{\frac{1}{n}} e^{-x} dx = n \Gamma(n)$$

$$\gamma_{n=\frac{1}{2}}$$

$$\int_0^\infty e^{-x} dx = \frac{1}{2} \pi i \Gamma(\frac{1}{2})$$

$$= \frac{1}{2} \sqrt{\pi}$$

$$\int_0^\infty e^{-y^{\frac{1}{n}}} (y^{\frac{1}{n}})^{n-1} \frac{1}{n} y^{\frac{1}{n}-1} dy$$

$$= \int_0^\infty e^{-y^{\frac{1}{n}}} y^{1-\frac{1}{n}} \times \frac{1}{n} y^{\frac{1}{n}-1} dy$$

Relationship with Beta & Gamma func'

$$B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

Proof:  $\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx, \quad \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

Also  $\frac{\Gamma(m)}{z^n} = \int_0^\infty e^{-zx} z^{n-1} dx$

$$\Rightarrow \Gamma(m) = \int_0^\infty e^{-zx} z^m z^{n-1} dx$$

Multiplying both sides by  $z^{m-1} e^z$  and integrating w.r.t 'z' from 0 to  $\infty$ , we get

$$\Gamma(m) \Gamma(n) = \int_0^\infty e^{-z} z^{m-1} \left[ \int_0^\infty e^{-zx} z^m z^{n-1} dz \right] dz$$

$$= \int_0^\infty \int_0^\infty e^{-z-zx} z^{m+n-1} x^{n-1} dx dz$$

$$= \int_0^\infty \int_{-\infty}^\infty e^{-z(1+x)} z^{m+n-1} x^{n-1} dx dz$$

$$\text{Let } z(1+x) = t$$

$$\Rightarrow dz = \frac{dt}{1+x}$$

$$= \int_0^\infty \int_0^\infty e^{-t} \left( \frac{t}{1+x} \right)^{m+n-1} x^{n-1}$$

$$x dx \times \frac{dt}{1+x}$$



Put  $n = \frac{1}{2}$

$$r\left(\frac{1}{2}\right) r\left(\frac{1}{2}\right) = \frac{\pi}{\sin \frac{\pi}{2}}$$

$$\Rightarrow \left\{ r\left(\frac{1}{2}\right) \right\}^2 = \pi$$

$$\Rightarrow \boxed{r\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

If  $n = \gamma_3$

$$r(\gamma_3) r(2/\sqrt{3}) = \frac{\pi}{\sin \frac{\pi}{3}} = \frac{\pi}{\sqrt{3}/2}$$

$$\Rightarrow \boxed{r\left(\frac{1}{3}\right) r(2/\sqrt{3}) = \frac{2\pi}{\sqrt{3}}}$$

If  $n = \frac{1}{4}$

$$r\left(\frac{1}{4}\right) r(3/4) = \frac{\pi}{\sin \frac{\pi}{4}} = \pi \sqrt{2}$$

$$\therefore \boxed{r\left(\frac{1}{4}\right) r(3/4) = \pi \sqrt{2}}$$

If  $n = \gamma_5$

$$r(\gamma_5) r(5/\sqrt{6}) = \frac{\pi}{\sin \frac{\pi}{5}} = 2\pi$$

$$r(\gamma_5) r(5/\sqrt{6}) = 2\pi$$



Legendre's      Duplicatives formula

$$B(m,n) = B(n,m)$$

$$\Gamma(n) \Gamma(n + \frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n-1}} \times \Gamma(2n)$$

Pf: we know that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2n-1} \theta \, \cos^{2n-1} \theta \, d\theta = \frac{\Gamma(n) \Gamma(n)}{2 \Gamma(2n)} \quad \textcircled{1}$$

put  $m = \frac{1}{2}$  in  $\textcircled{1}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2n-1} \theta \, d\theta = \frac{\Gamma(\frac{1}{2}) \Gamma(n)}{2 \Gamma(n + \frac{1}{2})} \quad \textcircled{2}$$

if  $m = n$  in  $\textcircled{1}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2n-1} \theta \, \cos^{2n-1} \theta \, d\theta = \frac{\Gamma(n) \Gamma(n)}{2 \Gamma(2n)}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} (\sin \theta)^{2n-1} d\theta = \frac{1}{2} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)} \quad \left. \Rightarrow \cancel{\int_0^{\frac{\pi}{2}} \frac{1}{2^{2n-1}} \sin^{2n-1} \theta d\theta} \right\}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{2^{2n-1}} \sin^{2n-1} 2\theta d\theta = \frac{1}{2} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

$$w \quad 2\theta = t \\ \Rightarrow d\theta = \frac{dt}{2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{2^{2n-1}} \sin^{2n-1} t \frac{dt}{2} = \frac{1}{2} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{2^{2n-1}} \sin^{2n-1} t \frac{dt}{2} = \frac{1}{2} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

$$\left\{ \text{so } \int_0^{2a} f(x) dx = 2 \int_0^a f(2a-x) dy \right\}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2n-1} 2\theta d\theta \\ = \frac{2^{2n-1}}{2} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)} \quad \text{--- (3)}$$

By ② & ③

$$\frac{1}{2} \frac{\{\Gamma(n+1)\}^2}{\Gamma(2n+1)} = \frac{2^{2n-1}}{2^n} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

$$\Rightarrow \boxed{r(n) n(n+y_2) = \frac{\pi(2n) \times \sqrt{\pi}}{2^{2n-1}}}$$

Note:

$$\textcircled{1} \quad r(\frac{1}{n}) r(\frac{2}{n}) \times \dots \times r\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$$

$$\textcircled{2} \quad \int_{\delta}^{\infty} e^{-ax} x^{n-1} \cos bx dx = \frac{r(n) \cos \theta}{(a^2 + b^2)^{n/2}}$$

$$\int_{\delta}^{\infty} e^{-ax} x^{n-1} \sin bx dx = \frac{r(n) \sin \theta}{(a^2 + b^2)^{n/2}} \quad \text{where } \theta = \tan^{-1}(b/a)$$

Q Evaluate

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$$

$$\boxed{x^n = \sin^2 \theta} \quad \boxed{\int_0^1 t^{n-1} (1-t)^{m-1} dt = B(n, m)}$$

Sol. Let  $x^n = t$   
 $\Rightarrow x = t^{\frac{1}{n}}$   
 $\Rightarrow dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

$$I = \int_0^1 \frac{\frac{1}{n} t^{\frac{1}{n}-1} dt}{(1-t)^{\gamma_2}}$$

$$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{\frac{1}{2}-1} dt$$

$$= \frac{1}{n} B(\gamma_n, \gamma_2) = \frac{1}{n} \frac{\Gamma(\gamma_n) \Gamma(\gamma_2)}{\Gamma(\frac{n+1}{n} + \gamma_2)} = \frac{\sqrt{\pi} \Gamma(\gamma_n)}{n \Gamma(\gamma_n + \gamma_2)} \text{ No.}$$















$$\textcircled{1} \quad \text{Prove that } \int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+bx)^{m+n}} dx = \frac{\beta(m, n)}{a^n (a+b)^m}$$

$$\textcircled{2} \quad \int_0^{\frac{\pi}{2}} \frac{dx}{(a \cos^4 x + b \sin^4 x)^{\frac{1}{2}}} = \frac{\left\{ \Gamma\left(\frac{1}{4}\right) \right\}^2}{4(a b)^{\frac{1}{4}} \sqrt{\pi}}$$

sol: (2)

$$I = \int_0^{\frac{\pi}{2}} \frac{du}{(a \cos^4 u + b \sin^4 u)^{\frac{1}{2}}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 u \ du}{(a + b \tan^4 u)^{\frac{1}{2}}}$$

$$\tan u = t$$

$$\sec^2 u \ du = dt$$

$$\begin{aligned} I &= \int_0^{\infty} \frac{1+t^2}{a+b t^4} dt \\ &\quad \text{P} \quad b t^4 = a y \\ &\Rightarrow t = \left(\frac{a}{b}\right)^{\frac{1}{4}} y^{\frac{1}{4}} \\ &\Rightarrow dt = \left(\frac{a}{b}\right)^{\frac{1}{4}} \frac{1}{4} y^{\frac{1}{4}-1} dy \end{aligned}$$

$$= \int_1^\infty \frac{dt}{(a+bt^q)^{\frac{1}{2}}}$$

$$at + bt^q = ay$$

$$\Rightarrow t = \left(\frac{a}{b}\right)^{\frac{1}{q}} y^{\frac{1}{q}}$$

$$dt = \left(\frac{a}{b}\right)^{\frac{1}{q}} \frac{1}{q} y^{\frac{1}{q}-1} dy$$

$$\therefore I = \int_1^\infty \frac{\left(\frac{a}{b}\right)^{\frac{1}{q}} \frac{1}{q} y^{\frac{1}{q}-1} dy}{(a+ay)^{\frac{1}{2}}}$$

$$= \left(\frac{a}{b}\right)^{\frac{1}{q}} \frac{1}{a^{\frac{1}{2}}} \frac{1}{q} \int_1^\infty \frac{y^{\frac{1}{q}-1} dy}{(1+y)^{\frac{1}{q}+\frac{1}{q}}}$$

$$= \frac{1}{4(ab)^{\frac{1}{q}}} B\left(\frac{1}{q}, \frac{1}{q}\right)$$

$$= \frac{1}{4(ab)^{\frac{1}{q}}} \frac{\Gamma\left(\frac{1}{q}\right) \Gamma\left(\frac{1}{q}\right)}{\Gamma\left(\frac{2}{q} + \frac{1}{q}\right)}$$

$$= \frac{1}{4(ab)^{\frac{1}{q}}} \times \left\{ \Gamma\left(\frac{1}{q}\right) \right\}^2$$

Q6

$$\frac{x}{a+bz} = \frac{z}{(a+b)}$$

$$\Rightarrow \left\{ \frac{x(a+bz) - xb}{(a+bz)^2} \right\} dx = \frac{dz}{(a+b)}$$

$$\Rightarrow dx = \frac{1}{a} \frac{(a+bz)}{(a+b)} dz$$

At  $x=0, z=0$

At  $x=1, z=1$

Q Evaluate

$$(a) \int_0^\infty \frac{x dx}{1+x^6}$$

$$(b) \int_0^2 \frac{x^2}{\sqrt{2-x}} dx$$

$$(c) \int_0^1 \frac{dx}{\sqrt{1-x^n}}$$

$$(d) \int_0^3 \frac{dx}{\sqrt{3x-x^2}}$$

$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

$$\Rightarrow = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$r(n) = \int_0^\infty e^{-x} x^{n-1} dx$

Sol:-

$$(a) I = \int_0^\infty \frac{x dx}{1+x^6}$$

$$w \ x^6 = t$$

$$x = t^{\frac{1}{6}}$$

$$dx = \frac{1}{6} t^{\frac{1}{6}-1} dt$$

$$\therefore I = \int_0^\infty \frac{t^{\frac{1}{6}} \cdot \frac{1}{6} t^{\frac{1}{6}-1} dt}{1+t} = \frac{1}{6} \int_0^\infty \frac{t^{\frac{1}{3}-1}}{(1+t)^1} dt$$

$$= \frac{1}{6} \int_0^\infty \frac{t^{\frac{1}{2}-1}}{(1+t)^{\frac{1}{3}+\frac{2}{3}}} dt$$

$$n = \frac{1}{3}$$

$$m+n = 1$$

$$m = 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

$$= \frac{1}{\Gamma} \Beta(\frac{1}{3}, \frac{2}{3})$$

$$= \frac{1}{\Gamma} \frac{\Gamma(\frac{1}{3}) \Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3} + \frac{2}{3})}$$

$$= \frac{1}{\Gamma} \Gamma(\frac{1}{3}) \Gamma(1 - \frac{1}{3})$$

$$= \frac{1}{\Gamma} \frac{\pi}{\sin \frac{\pi}{3}} = \frac{1}{\Gamma} \times \frac{2\pi}{\sqrt{3}}$$

$$= \frac{\pi}{3\sqrt{3}} \text{ Ans.}$$

(ii)  $I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$

$$\text{w } x^n = t$$

$$\Rightarrow x = t^{\frac{1}{n}}$$

$$dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$$

$$\therefore I = \int_0^1 \frac{\frac{1}{n} t^{\frac{1}{n}-1} dt}{(1-t)^{\frac{1}{2}}}$$

$$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{\frac{1}{2}-1} dt$$

$$= \frac{1}{n} \Beta\left(\frac{1}{n}, \frac{1}{2}\right)$$

$$= \frac{1}{n} \frac{\Gamma(\frac{1}{n}) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{n} + \frac{1}{2})} \neq$$

$$\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = B(m, n)$$

evaluate

$$① I = \int_0^2 x (8-x^3)^{\frac{1}{3}} dx$$

$$② I = \int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx$$

$$\underline{\text{Solve:}}$$

$$① I = \int_0^2 x (8-x^3)^{\frac{1}{3}} dx$$

$$w \quad x^3 = 8t$$

$$\Rightarrow x = (8t)^{\frac{1}{3}}$$

$$= 2t^{\frac{1}{3}}$$

$$dx = \frac{2}{3} t^{\frac{1}{3}-1} dt$$

$$\begin{aligned} \therefore I &= \int_0^1 2t^{\frac{1}{3}} (8-8t)^{\frac{1}{3}} \frac{2}{3} t^{\frac{1}{3}-1} dt \\ &= \frac{4}{3} \times 2 \int_0^1 t^{\frac{2}{3}-1} (1-t)^{\frac{4}{3}-1} dt \\ &= \frac{8}{3} B\left(\frac{2}{3}, \frac{4}{3}\right) \\ &= \frac{8}{3} \frac{\Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{6}{3}\right)} \\ &= \frac{8}{3} \Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{4}{3}\right) \\ &= \frac{8}{3} \Gamma\left(\frac{2}{3}\right) \Gamma\left(1+\frac{1}{3}\right) \\ &= \frac{8}{9} \Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{1}{3}\right) \end{aligned}$$

Given by ①  

$$2\pi = \lg \pi + \gamma_2$$

$$= \lg(2\pi)$$

$$\boxed{I = \frac{1}{2} \lg 2\pi}$$
  

$$x-a=t$$

$$(b-(a+t))^n$$

$$(b-a-t)^n$$

$$t=(b-a)x$$

Given that

$$(a) \int_a^b (x-a)^m (b-x)^n dx$$

$$= (b-a)^{m+n+1} \Gamma(m+1, n+1)$$

$$(b) \int_0^{\frac{\pi}{2}} \frac{d\theta}{(a \cos^4 \theta + b \sin^4 \theta)^{\frac{1}{2}}} = \frac{\{\Gamma(\frac{1}{4})\}^2}{4(ab)^{\frac{1}{4}} \sqrt{\pi}}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{(a+b \tan^4 \theta)^{\frac{1}{2}}} \rightarrow b \tan^4 \theta = au$$

Evaluate

(a)  $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} =$

(b)  $\frac{\Gamma(\frac{1}{3}) \Gamma(\frac{5}{6})}{\Gamma(\frac{2}{3})} \left\{ 2^{2n-1} \Gamma(n) \Gamma(n+\frac{1}{2}) \right\} = \Gamma \Gamma(2n)$

(c)  $\int_0^1 \frac{dx}{\sqrt{-\lg x}} \rightarrow (\lg x = t)$

(d)  $\Gamma(\frac{3}{2}-p) \Gamma(\frac{3}{2}+p)$

(e)  $\int_0^1 x^m (y-x)^n dx \rightarrow (yx = -t)$

(f)  $\int_0^{\infty} x^n e^{-ax^2} dx$



$$\text{Now } I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

$$\text{Let } x^2 = \tan \theta$$

$$\Rightarrow x = \tan^{\frac{1}{2}} \theta$$

$$dx = \frac{1}{2} \tan^{\frac{1}{2}} \theta \sec^2 \theta d\theta$$

$$\therefore I_2 = \int_0^{\frac{\pi}{4}} \frac{\frac{1}{2} \tan^{\frac{1}{2}} \theta \sec^2 \theta d\theta}{\sqrt{1+\tan^2 \theta}}$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{\tan^{\frac{1}{2}} \theta \sec^2 \theta}{\sec \theta} d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{\cos^{\frac{1}{2}} \theta}{\sin^{\frac{1}{2}} \theta} \times \frac{1}{\cos \theta}$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\sin \theta \cos \theta}} = \frac{1}{4\sqrt{2}} \frac{\Gamma(\frac{1}{4}) \sqrt{\pi}}{\Gamma(\frac{3}{4})}$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\frac{\sin 2\theta}{2}}} = \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\sin 2\theta}}$$

$$\text{Let } 2\theta = t$$

$$d\theta = \frac{dt}{2}$$

$$\therefore I_2 = \frac{\sqrt{2}}{2\pi^2} \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{\sin t}}$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{\sin^{-\frac{1}{2}} t \cos t dt}{\sin^{\frac{1}{2}} t}$$

$$= \frac{1}{2\sqrt{2}} \frac{\Gamma\left(\frac{-1+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2 \times \Gamma\left(\frac{-1+0+2}{2}\right)}$$

$$\therefore I = I_1 + I_2$$

$$= \frac{\pi}{4\sqrt{2}}$$



$$(c) \quad I = \int_1^e \frac{dx}{\sqrt{-\log x}}$$

$$ut - \log u = t$$

$$\Rightarrow u = e^{-t}$$

$$\Rightarrow du = -e^{-t} dt$$

$$\begin{aligned} \therefore I &= \int_0^\infty \frac{-e^{-t} dt}{\sqrt{t}} \\ &= \int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt \\ &= \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \end{aligned}$$



$$\frac{3}{2} - p > 0$$

$$\begin{aligned} 3 &> 2p \\ 3 &< 2p & 3 &> \\ -1 &< 2p & & < 1 \end{aligned}$$

$$(d) \quad I = \Gamma\left(\frac{3}{2} + p\right) \Gamma\left(\frac{1}{2} + p\right)$$

$$= \Gamma\left(1 + \frac{1}{2} - p\right) \Gamma\left(1 + \frac{1}{2} + p\right)$$

$$= \left(\frac{1}{2} - p\right) \Gamma\left(\frac{1}{2} - p\right) \Gamma\left(\frac{1}{2} + p\right) \Gamma\left(\frac{1}{2} + p\right)$$

$$= \left(\frac{1}{4} - p^2\right) \Gamma\left(\frac{1}{2} + p\right) \Gamma\left(\frac{1}{2} + p\right)$$

$$= \left(\frac{1}{4} - p^2\right) \underbrace{\Gamma\left(\frac{1}{2} + p\right)}_{\Gamma\left(1 - \left(\frac{1}{2} + p\right)\right)}$$

$$= \left(\frac{1}{4} - p^2\right) \frac{\pi}{\sin\left(\frac{1}{2} + p\right)\pi}$$

$$\left\{ \therefore \Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin \pi} \right\}$$



## Multiple integral

$\int_a^b f(x) dx$  } Change of order

### Double integral

The integral  $\iint_R f(x,y) dxdy$  is known as  
the double integral of  $f(x,y)$  over the region  $R$ .

$$\text{Case i:- } \int_a^b \int_{x=\phi(y)}^{\psi(y)} f(x,y) dx dy$$

To evaluate this integral we first integrate w.r.t  $x$  and then  
w.r.t  $y$  i.e.,

$$I = \int_a^b \left[ \int_{\phi(y)}^{\psi(y)} f(x,y) dx \right] dy$$

$$\underline{\text{Case 2:-}} \quad I = \int_a^b \int_{y=\phi(x)}^{\psi(x)} f(x, y) dy dx$$

Here we first integrate w.r.t  $y$  and then w.r.t  $x$

$$I = \int_a^b \left[ \int_{\phi(x)}^{\psi(x)} f(x, y) dy \right] dx$$

$$\underline{\text{Case 3:-}} \quad I = \int_a^b \int_c^d g(x, y) dy dx$$

In this case we can integrate first w.r.t  $x$  then w.r.t  $y$   
or vice-versa.

Q2 Evaluate

$$\int_0^1 \int_{\sqrt{1+x^2}}^{\sqrt{1+x^2}} \frac{dy dx}{1+x^2+y^2}$$

$$\int \frac{dx \sqrt{1+x^2} dx}{x^2}$$

Sol:

$$\begin{aligned}
 I &= \int_0^1 \int_{\sqrt{1+x^2}}^{\sqrt{1+x^2}} \frac{dy dx}{1+x^2+y^2} \\
 &= \int_0^1 \left[ \int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{(\sqrt{1+x^2})^2 + y^2} \right] dx \\
 &= \int_0^1 \left[ \int_{\sqrt{1+x^2}}^{\infty} \tan^{-1} \left( \frac{y}{\sqrt{1+x^2}} \right) \right]_{y=0}^{\sqrt{1+x^2}} du \\
 &= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}} = \frac{\pi}{4} \left[ x \ln(x + \sqrt{1+x^2}) \right]_0^1 \\
 &= \frac{\pi}{4} \ln(1 + \sqrt{2}) \quad \underline{\text{Ans}}
 \end{aligned}$$

Q Evaluate  $\iint xy(x+y) dx dy$  over the area between  $y=x^2$  and  $y=x$ .

Sol:

$$I = \iint xy(x+y) dx dy$$

$$= \int_{x=0}^1 \left\{ \begin{array}{l} y=x \\ y=x^2 \end{array} \right. xy(x+y) dy \} dx$$

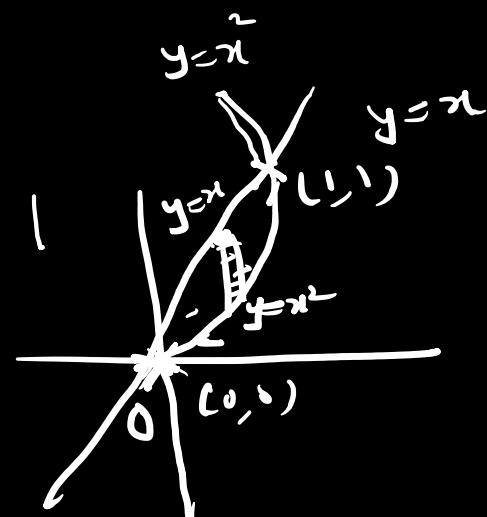
$$= \int_{x=0}^1 \left\{ \begin{array}{l} y=x \\ y=x^2 \end{array} \right. (x^2y + xy^2) dy \} dx$$

$$= \int_{x=0}^1 \left[ \frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 \left[ \frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right] dx$$

$$= \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{3}{56}$$



Q evaluate  $\iint xy \, dx \, dy$  over the +ve quadrant of  $x+y \leq 1$

Sol:

$$I = \iint xy \, dx \, dy$$

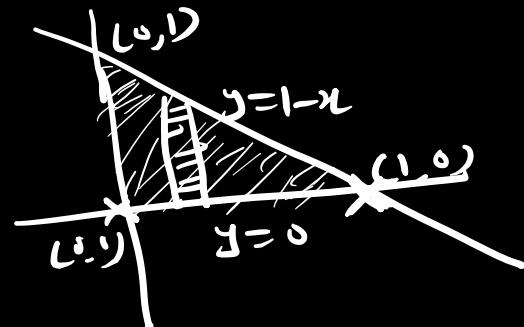
$$= \int_0^1 \left[ \int_{y=0}^{1-x} xy \, dy \right] dx$$

$$= \int_0^1 \left[ \frac{xy^2}{2} \right]_{y=0}^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 x(1-x)^2 dx$$

$$= \frac{1}{2} \int_0^1 x(1+x^2 - 2x) dx = \frac{1}{2} \int_0^1 (x + x^3 - 2x^2) dx$$

$$= \frac{1}{2} \left( \frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) = \frac{1}{24}$$

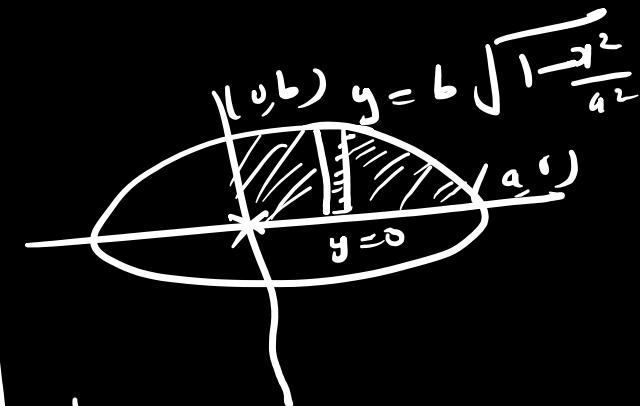


(+3-8)  
12

Q Evaluate  $\iint_R \left(1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}\right) dx dy$  over the quadrant of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol:

$$\begin{aligned}
 I &= \iint_R \left(1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}\right) dx dy \\
 &= \int_0^a \left[ \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left(1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}\right) dy \right] dx \\
 &= \frac{\pi ab}{4}
 \end{aligned}$$



Q Evaluate  $\iint y \, dy \, dx$  over the part of the plane bounded by the lines  $y=x$  and the parabola  $y=4x-x^2$

Sol: Intersection points are given by

$$x = 4x - x^2$$

$$\Rightarrow 3x - x^2 = 0$$

$$\Rightarrow x(3-x) = 0$$

$$\Rightarrow x = 0, 3$$

$$\therefore y = 0, 3$$

$\therefore (0,0)$  &  $(3,3)$  are two intersection pts.

$$\because y = 4x - x^2$$

$$x^2 - 4x = -y$$

$$x^2 - 4x + 4 = -(y-4)$$

$$(x-2)^2 = -(y-4)$$

$$\Rightarrow x^2 = -4(y-4)$$

$$\text{vertex } (x,y) = (0,4)$$

$$\Rightarrow x = 2, y = 4$$

$\Rightarrow (2,4)$  is the vertex

$$\rightarrow \iint d\sigma \text{?}$$

$$I = \iint y \underline{d\sigma} dy$$

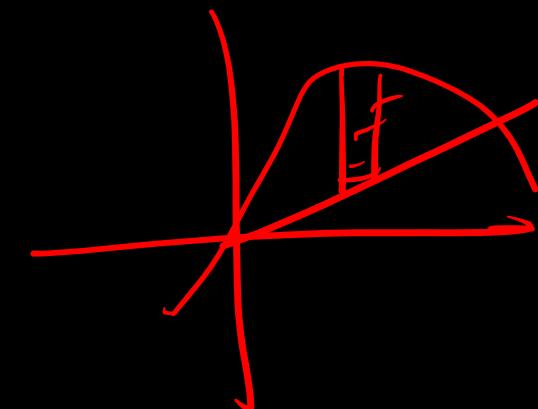
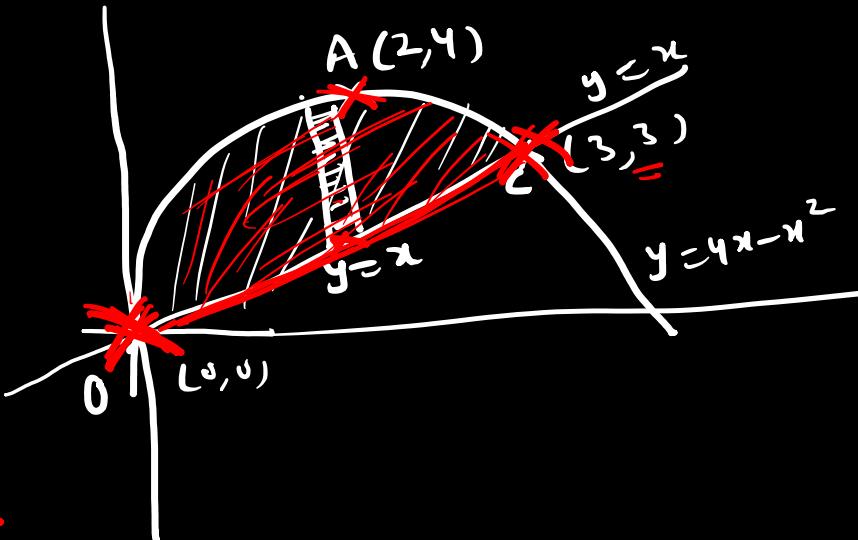
$$= \int_{x=0}^3 \left[ \iint_{y=x}^{4x-x^2} y \underline{dy} \right] dx$$

$$= \frac{1}{2} \int_{x=0}^3 \left[ (4x-x^2)^2 - x^2 \right] dx$$

$$= \frac{1}{2} \int_{x=0}^3 \left[ 16x^2 + x^4 - 8x^3 - x^5 \right] dx$$

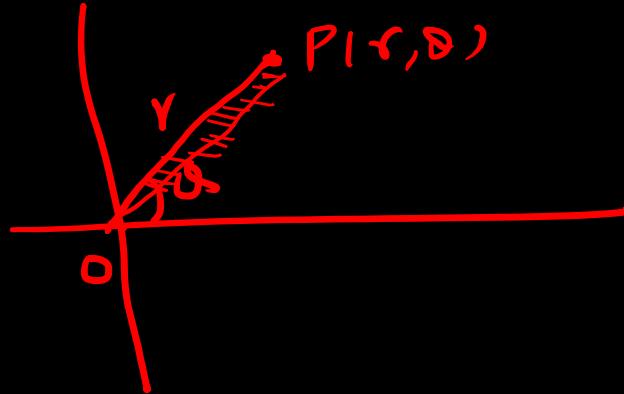
$$= \frac{1}{2} \left[ 15 \times \frac{9}{3} + \frac{1}{5} \times \cancel{22} - \frac{8}{4} \times \cancel{16} \right] = \frac{54}{5} \text{ m}$$

$$= \frac{1}{2} \left[ 15 \times \frac{9}{3} + \frac{1}{5} \times \cancel{22} - \frac{8}{4} \times \cancel{16} \right] = \frac{54}{5} \text{ m}$$



$$\int_0^{\pi} \left( \int_{f_1(\theta)}^{r = \sqrt{f_2(\theta)}} dr \right) d\theta =$$

$$\int_a^b \left( \int_{f(x)}^{y = \psi(x)} h(u, y) dy \right) dx$$



Q Solve

$$I = \int_0^{\pi} \int_0^{a \sin \theta} r \sin \theta \, dr \, d\theta$$

$\omega^2 \theta = 1 - 2 \sin^2 \theta$

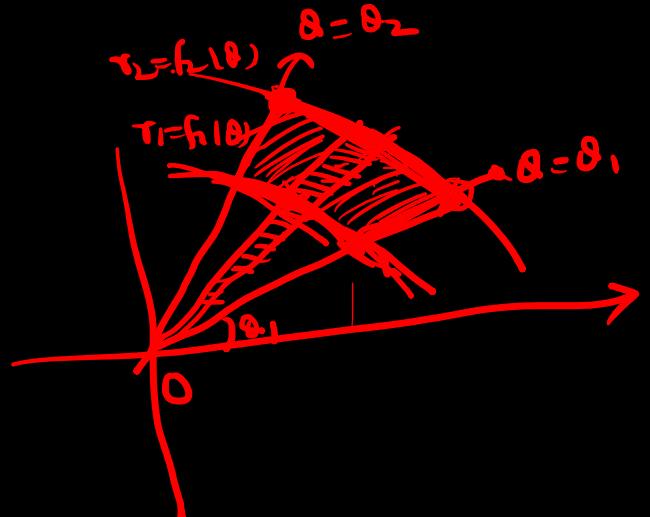
Sol:-

$$\begin{aligned} I &= \int_0^{\pi} \int_{r=0}^{a \sin \theta} r \, dr \, d\theta \\ &= \int_0^{\pi} \left[ \frac{r^2}{2} \right]_0^{a \sin \theta} d\theta \\ &= \frac{a^2}{2} \int_0^{\pi} \sin^2 \theta \, d\theta \\ &= \frac{a^2}{4} \int_0^{\pi} (1 - \cos 2\theta) \, d\theta \\ &= \frac{a^2}{4} \left( \theta - \frac{\sin 2\theta}{2} \right)_0^{\pi} \\ &= \frac{\pi a^2}{4}. \end{aligned}$$

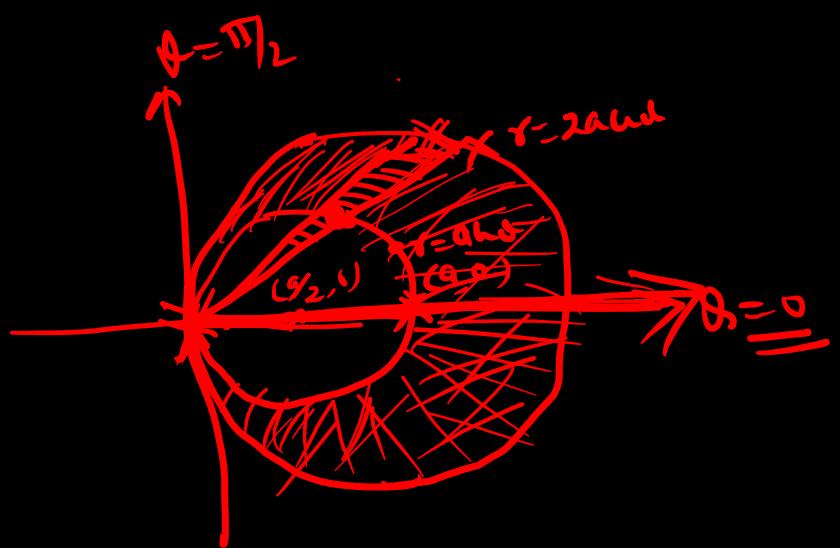
The double integral

$$\int_{\theta_1}^{\theta_2} \int_{r_1=f_1(\theta)}^{r_2=f_2(\theta)} f(r, \theta) \, dr \, d\theta$$

represents the double integral in polar coordinates.







$$\begin{aligned}
 I &= \iint_A r^2 dr d\theta \\
 &= 2 \int_{\theta=0}^{\pi/2} \int_{r=a\cos\theta}^{r=2a\sin\theta} r^2 dr d\theta \\
 &= \frac{2}{3} \int_0^{\pi/2} [r^3]_{a\cos\theta}^{2a\sin\theta} d\theta
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{3} \int_0^{\pi/2} [8a^3 \sin^3\theta - a^3 \cos^3\theta] d\theta \\
 &= \frac{14a^3}{3} \int_0^{\pi/2} \cos^3\theta \sin\theta d\theta \\
 &= \frac{14a^3}{3} \frac{r(\frac{3+1}{2}) r(\frac{0+1}{2})}{2r(\frac{5}{2})} \\
 &= \frac{14a^3}{3} \cancel{\frac{5!}{2! \times \frac{3}{2} \times \frac{1}{2}}} \cancel{\frac{5!}{2!}} \\
 &= \frac{28a^3}{9} + 1
 \end{aligned}$$

Q Show that  $\iint_R r^2 \sin \theta dr d\theta = \frac{2a^3}{3}$   
 where R is the region bounded by the semi-circle  $r=2a \cos \theta$  and  
 above the initial line.

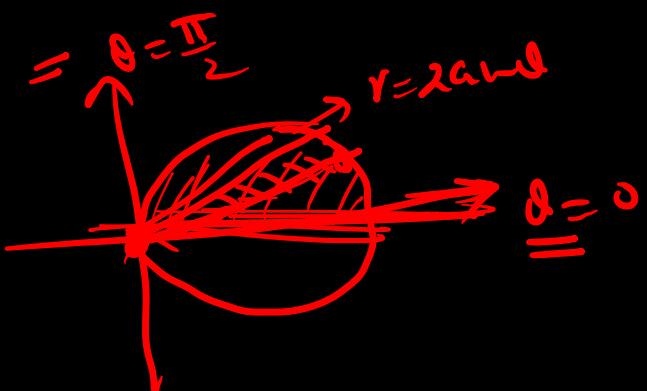
Sol:

$$I = \iint_R r^2 \sin \theta dr d\theta$$

$$= \int_0^{\pi} \int_0^{2a \cos \theta} r^2 \sin \theta dr d\theta$$

$$\text{where } \theta = 0 \text{ to } \pi, r = 0 \text{ to } 2a \cos \theta$$

Solve it.



*
$r = 2a \cos \theta$
$r = 2a \sin \theta$
$r^2 = 2ax$
$x^2 + y^2 - 2ax = 0$

Solu:-

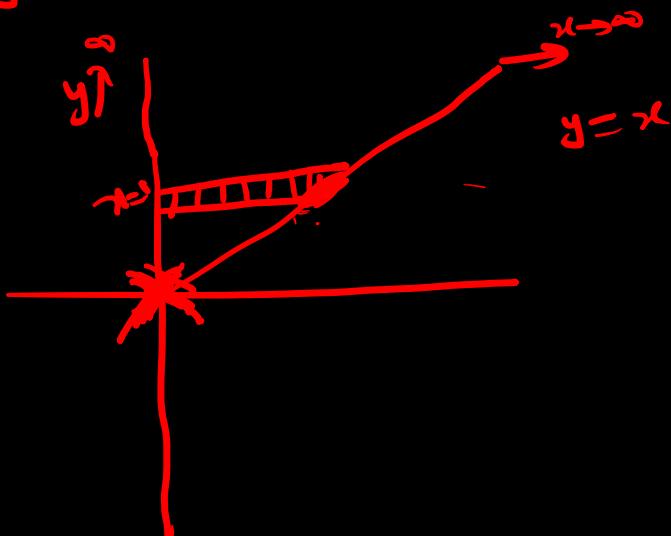
$$\int_0^{\infty} \int_{y=x}^{\infty} \frac{-e^y}{y} dy dx$$

~~$\int_0^{\infty} dy$~~

$$y = x + v \quad y = \infty$$

Sol:-

$$\text{Here } y = x \text{ to } y = \infty$$



Change of order is

$$I = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{-e^y}{y} dx dy$$

$$= \int_{y=0}^{\infty} \left[ \frac{e^{-y}}{y} \right]_{x=0}^y dy$$

$$= \int_{y=0}^{\infty} e^{-y} dy = -[e^{-y}]_{y=0}^{\infty} = 1$$

Q Change the order of integration in

$$I = \int_0^a \int_{\frac{y^2}{a}}^y f(x,y) dx dy$$

Sol: given  $x = \frac{y^2}{a}$  to  $x = y$

$$\Rightarrow y^2 = ax \text{ to } x = y$$

intersect pts are

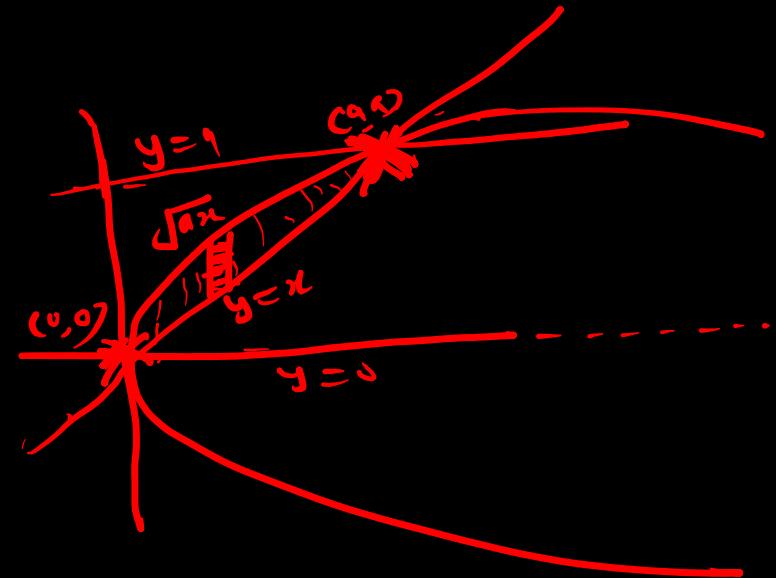
$$y^2 = ay$$

$$\Rightarrow y(y-a) = 0$$

$$y=0, y=a$$

$$\text{so } x=y=0, a$$

so  $(0,0)$  &  $(a,a)$  are two intersect pts



Change of order is

$$I = \int_{x=0}^a \int_{y=x}^{\sqrt{ax}} f(x,y) dy dx$$

#

Q Change the order of integration  $I = \int_0^a \int_{x-y}^{2a-x} f(x,y) dy dx$

Sol:- when  $y = \frac{x^2}{a}$  to  $y = 2a - x$

Intersecting pts are given by

$$\frac{x^2}{a} = 2a - x$$

$$\Rightarrow x^2 + ax - 2a^2 = 0$$

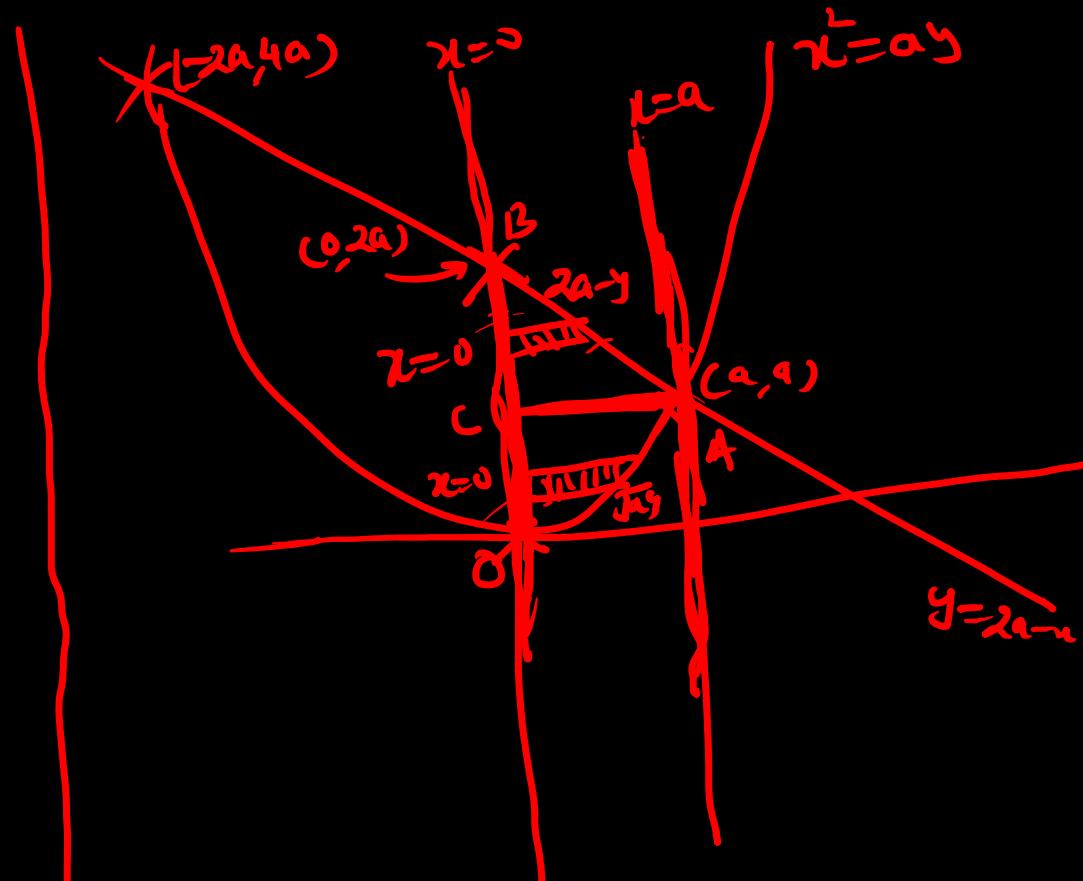
$$\Rightarrow x^2 + 2ax - ax - 2a^2 = 0$$

$$\Rightarrow (x+2a)(x-a) = 0$$

$$\Rightarrow x = a, -2a$$

$$y = 2a - x = a, 4a$$

$\therefore$  intersecting pts are  $(a, a)$  &  $(-2a, 4a)$ .



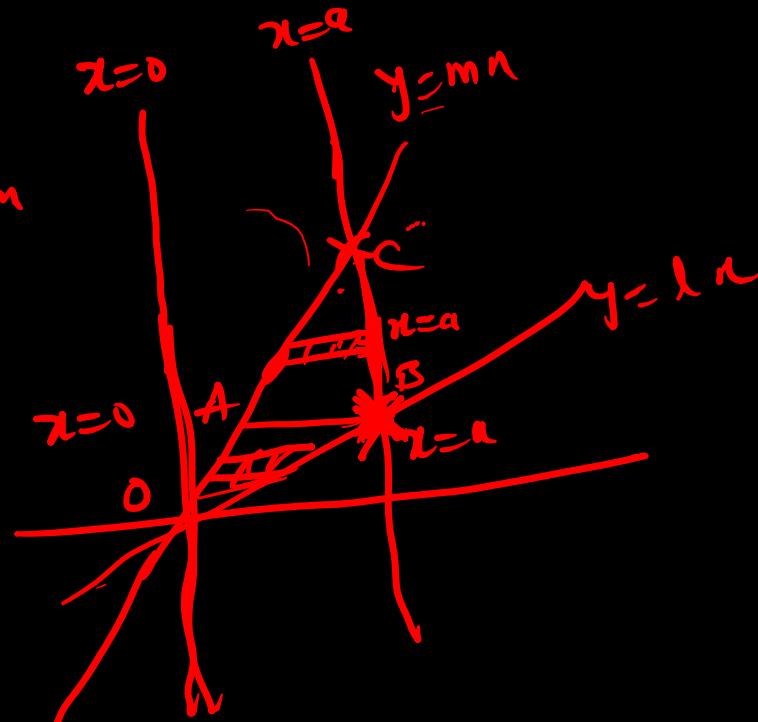


Q Change the order of integral in

$$I = \int_{x=0}^a \int_{y=lx}^{mx} v(x, y) dy dx$$

Sol: Change of order is given by

$$\begin{aligned} I &= \iint_{\partial A \cap S} v(x, y) dxdy + \iint_{A \setminus C} v(x, y) dxdy \\ &= \int_0^{al} \int_{y/m}^{y/l} v(x, y) dxdy + \int_{al}^{am} \int_{y/m}^{x/a} v(x, y) dxdy \end{aligned}$$





Q change the order of integrals in

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{ax}} f(x,y) dy dx =$$



Sol:- Given that

$\Rightarrow$

$$y = \sqrt{ax-x^2} \rightarrow y = \sqrt{ax}$$
$$y^2 + x^2 - ax = 0, y = ax \rightarrow \text{parabola.}$$

$\downarrow$

circle with centre  $(\frac{a}{2}, 0)$   
and radius  $\frac{a}{2}$

Integrals pts are given by

$$ax + x^2 - ax = 0$$

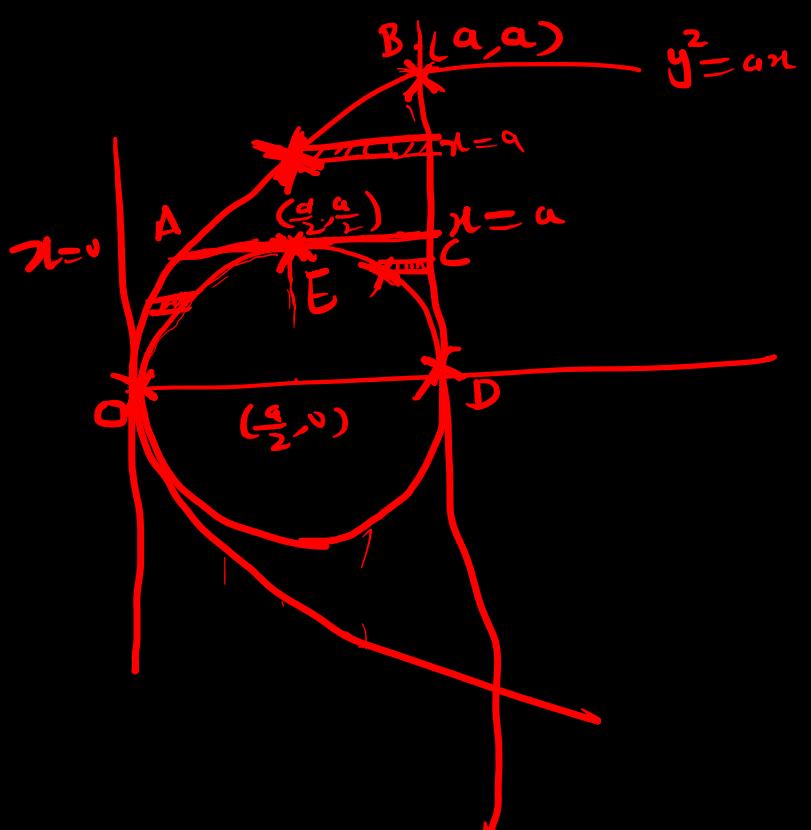
$$x=0 \Rightarrow y=0$$

∴  
 $(0,0)$  is the only one  
intersection pt.

$$\text{or } x^2 - ax + y^2 = 0$$

$$x = \frac{a \pm \sqrt{a^2 - 4y^2}}{2}$$

$$= \frac{a}{2} \pm \frac{1}{2} \sqrt{a^2 - 4y^2}$$



Change of value of integrals is given by

$$I = \iint_{\text{DEC}} f(x, y) dx dy + \iint_{\text{ABC}} f(x, y) dx dy$$

OAE

$$+ \iint_{\text{ABC}} f(x, y) dx dy$$

$$= \int_0^{\frac{a}{2}} \int_{\frac{a}{2} - \frac{1}{2}\sqrt{a^2 - 4y^2}}^a f(x, y) dx dy$$

$$+ \int_0^{\frac{a}{2}} \int_{\frac{a}{2} + \frac{1}{2}\sqrt{a^2 - 4y^2}}^a f(x, y) dx dy$$

$$+ \int_{\frac{a}{2}}^a \int_{\frac{y^2}{a}}^a f(x, y) dx dy$$

Q If  $n \geq 0$ , show that

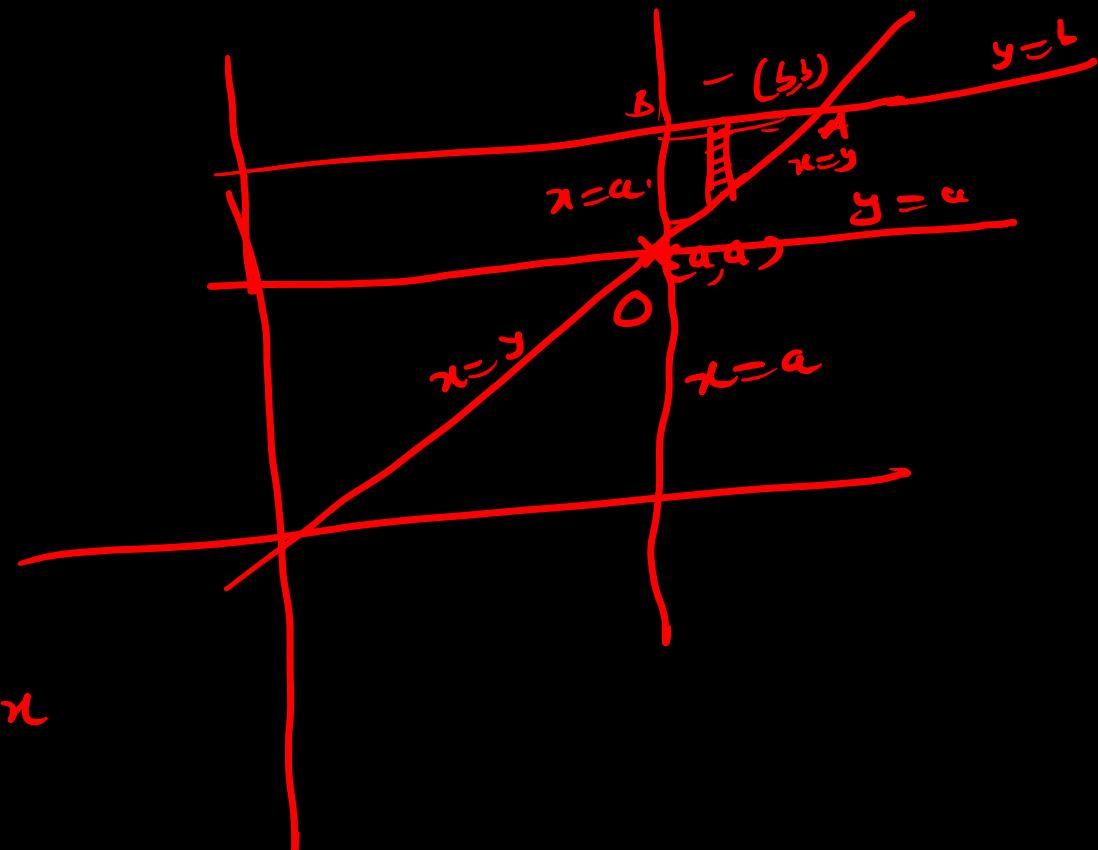
$$\int_a^b \int_a^y (y-x)^n f(x) dx dy = \frac{1}{n+1} \int_a^b (b-x)^{n+1} f(x) dx$$

Sol: Let  $I = \int_a^b \int_a^y (y-x)^n f(x) dx dy$   
Given that  $x=a$  to  $x=y$

Also  $y=a$  to  $y=b$

Change of order is

$$I = \int_a^b \int_x^b (y-x)^n f(x) dy dx$$





Q Change the order of integration in

$$xy = a$$

$$I = \int_0^a \int_0^{\frac{b}{b+x}} f(x,y) dy dx$$

Sol:-

Given that

$$y=0$$

+

$$y = \frac{b}{b+x}$$

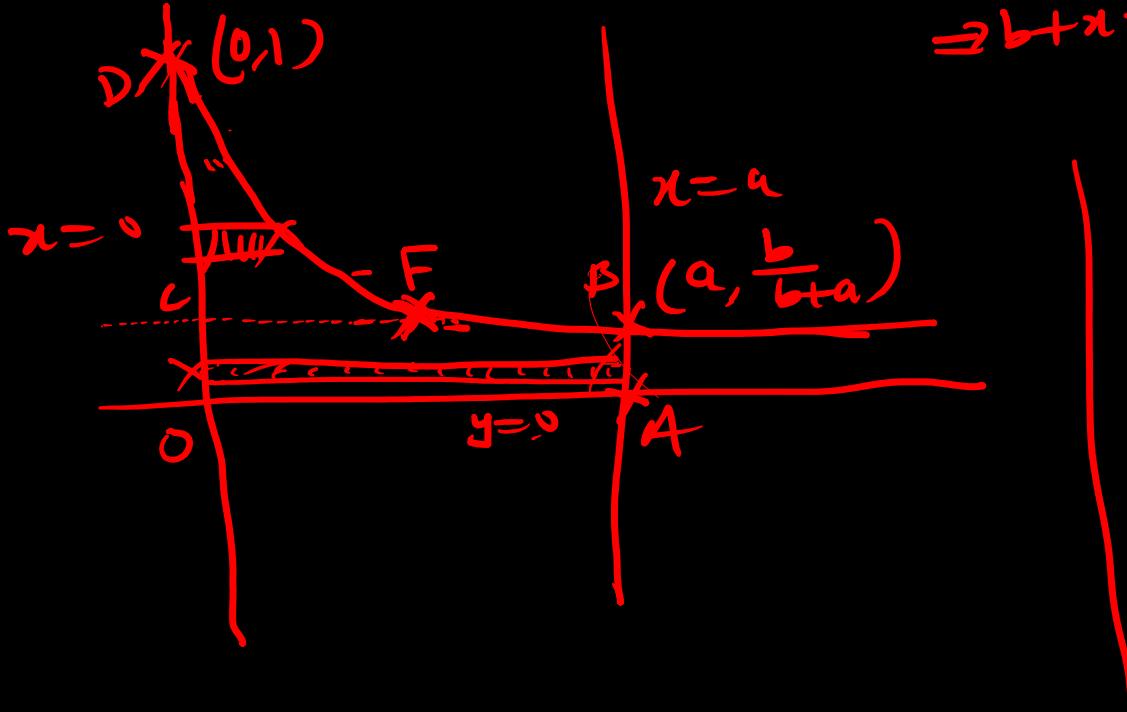
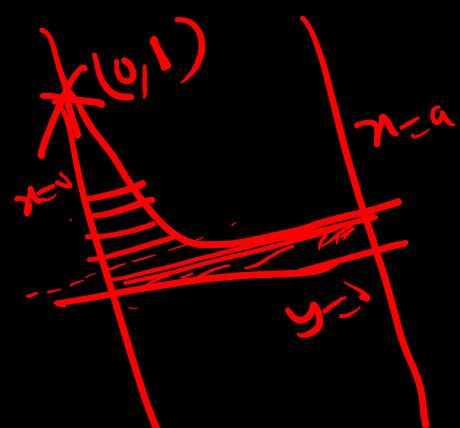
$$y=1$$

$$\Rightarrow b+x = \frac{b}{y} \Rightarrow x = \frac{b}{y} - b$$

$$\Rightarrow x = b \left( \frac{1-y}{y} \right)$$

At  $x=0$

$$\Rightarrow y=1$$





# Unit III (Multiple Integral)

- L1: <https://youtu.be/EUV1kpKS24c>
- L2: <https://youtu.be/zY9yf1N5Vbs>
- L3: <https://youtu.be/TabX9LzpBvM>
- L4: <https://youtu.be/HNNLmzOP8zc>
- L5: <https://youtu.be/KeXEzY1S7ic>
- L6: <https://youtu.be/pYE9HdvTj8s>
- L7: <https://youtu.be/McJFWZVvBvw>
- L8: <https://youtu.be/Kppi5PA0ppw>
- L9: <https://youtu.be/CDgDHLqaCwK>