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Numerical Technique

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \text{linear system}$$

Unknowns, $x_1, x_2, x_3, \cdots\cdots x_n$

Gauss Elimination Method :-

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Step-1 ⇒ Eliminate n from ① & ②

$$a_1x + b_1y + c_1z = d_1$$
$$b_2'y + c_2'z = d_2'$$
$$c_3''z = d_2''$$

Step-2 ⇒ Eliminate y from ③ -

$$a_1x + b_1y + c_1z = d_1$$
$$b_2'y + c_2'z = d_2'$$
$$c_3''z = d_2''$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & b_2' & c_2' \\ 0 & 0 & c_2'' \end{bmatrix}$$

Ques- Using Gauss elimination method solve the equations-

$$x+2y+3z-u=10$$
$$2x+3y-3z-u=1$$
$$2x-y+2z+3u=7$$
$$3x+2y-4z+3u=2$$

Sol^n ⇒

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & -3 & -1 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 7 \\ 2 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - 2R_1$

$R_3 \longrightarrow R_3 - 2R_1$

$R_4 \longrightarrow R_4 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & -5 & -4 & 5 \\ 0 & -4 & -13 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ -19 \\ -13 \\ -28 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 5R_2$

$R_4 \rightarrow R_4 - 4R_2$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & 0 & 41 & 0 \\ 0 & 0 & 23 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ -19 \\ 82 \\ 48 \end{bmatrix}$$

$R_3 \rightarrow \frac{1}{41} R_3$

$R_4 \rightarrow R_4 - 23R_3$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ -19 \\ 2 \\ 2 \end{bmatrix}$$

$u=1$, $z=2$

$$-y-9z+u=-19$$
$$-y-18+1=-19$$
$$-y=-19+17=-2 \quad , y=2$$
$$x+2y+3z-u=10$$
$$x+4+6-1=10$$
$$x=1$$

$x=1$, $y=2$, $z=2$, $u=1$ Ans

~~Gro~~ Gauss- Jordan Method :-

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \Rightarrow \text{transform} \begin{bmatrix} a'' & 0 & 0 \\ 0 & b'' & 0 \\ 0 & 0 & c'' \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ -19 \\ 2 \\ 2 \end{bmatrix}$$

$R_4 \rightarrow \frac{1}{2} R_4$

$R_2 \longrightarrow R_2 - R_4$

$R_1 \longrightarrow R_1 + R_4$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -9 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 11 \\ -20 \\ 2 \\ 1 \end{bmatrix}$$

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$R_1 \to R_1 - 3R_3$
$R_2 \to R_2 + 9R_3$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

$R_1 \to R_1 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

$x=1$, $y=2$, $z=2$, $u=1$ Ans

Ques- Gauss Jordan methods to solve equation-

$x+y+z=9$
$2x-3y+4z=13$
$3x+4y+5z=40$

Solv⇒

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix}$$

$R_1 = 5R_1 + R_2$, $R_3 = 5R_3 + R_2$

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$$\begin{bmatrix} 5 & 0 & 7 \\ 0 & -5 & 2 \\ 0 & 0 & 12 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 40 \\ -5 \\ 60 \end{bmatrix}$$

$R_3 \to \frac{1}{12} R_3$

$$\begin{bmatrix} 5 & 0 & 7 \\ 0 & -5 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 40 \\ -5 \\ 5 \end{bmatrix}$$

$R_1 \to R_1 - 7R_3$
$R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \\ 5 \end{bmatrix}$$

$x=1$, $y=3$, $z=5$ Ans

LU factorization :-

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{B}$$

Cofficient Matrix $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$

$|a_{11}| \neq 0 \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$

$$A = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}}_{U}$$

Let $L.V = B$

$AX = B$

$LUX = B$

$LUX = L.V$

$\boxed{UX = V}$

Ques- Apply LU factorization methods to solve the equations.

$2x + 3y + z = 9$

$x + 2y + 3z = 6$

$3x + y + 2z = 8$

Solu ⇒ Cofficient Matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

Unknown $= \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

Now. $|2| \neq 0$, $\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \neq 0 \quad |A| \neq 0$

Now $A = LU$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_1 & U_2 & U_3 \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12} + U_{22} & l_{21}U_{13} + U_{23} \\ l_{31}U_{11} & l_{31}U_{12} + l_{32}U_{22} & l_{31}U_{13} + l_{32}U_{23} + U_{33} \end{bmatrix}$$

$U_{11} = 2$, $U_{12} = 3$, $U_{13} = 1$

$U_{23} = \frac{5}{2}$, $U_{33} = 18$, $U_{22} = 1/2$

$l_{21} = 1/2$, $l_{31} = 3/2$, $l_{32} = -7$

Let $L.V = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$V_1 = 9$

$V_2 = 3/2$

$V_3 = 5$

$UX = V$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

$z = 5/18$, $y = 11/18$, $x = 31/9$ Ans

Eigen Value & Eigen Vectors :—

1) Power Method —

Ques- Determine the eigen largest value and corresponding eigen vectors.

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

⇒ ~~Ist~~ Let initial eigen vector $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

1st interaction ⇒

$$A\,X = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0.20 \end{bmatrix} = \cancel{A_1 X_1}\ \lambda_1 X_1$$

2nd interaction ⇒

$$AX_1 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.20 \end{bmatrix} = \begin{bmatrix} 5.8 \\ 1.4 \end{bmatrix} = 5.8 \begin{bmatrix} 1 \\ 0.2414 \end{bmatrix}$$
$$= \lambda_2 X_2$$

$$AX_3 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2486 \end{bmatrix} = \begin{bmatrix} 5.9944 \\ 1.4972 \end{bmatrix} = 5.9944 \begin{bmatrix} 1 \\ 0.2498 \end{bmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2498 \end{bmatrix} = \begin{bmatrix} 5.9992 \\ 1.4996 \end{bmatrix} = 5.9992 \begin{bmatrix} 1 \\ 0.24996 \end{bmatrix}$$
$$= \lambda_5 X_4$$

$$AX_5 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.24996 \end{bmatrix} = \begin{bmatrix} 5.9996 \\ 1.4999 \end{bmatrix} = 5.9996 \begin{bmatrix} 1 \\ 0.2499 \end{bmatrix}$$

Here $A_5 = A_6$ & $X_5 = X_6$

Largest eigen value $5.9996 \sim 6$ Ans

Corresponding eigen value $x = \begin{bmatrix} 1 \\ 0.2499 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}$ Ans

Eigen Value Problem —

2) Jacobi's Method :—

A be any real symmetric matrix

B_1 be any orthogonal matrix

Then $B_1^{-1} A B_1 = C$

B_2 such that

$B_2^{-1} B_1^{-1} A B_1 B_2 = D$

repeat it until get diagonal matrix.

$$A = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}_{2\times 2}$$

where $a_{ij} = a_{ji}$

Let $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be an orthogonal matrix

such that $B^{-1} = B^T$

$$B^{-1}AB = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{ii}\cos\theta + a_{ji}\sin\theta & a_{ij}\cos\theta + a_{jj}\sin\theta \\ -a_{ii}\sin\theta + a_{ji}\cos\theta & -a_{ij}\sin\theta + a_{jj}\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{ii}\cos^2\theta + a_{ji}\sin\theta\cos\theta + a_{ij}\sin\theta\cos\theta + a_{jj}\sin^2\theta & -a_{ii}\cos\theta\sin\theta - a_{ji}\sin^2\theta + a_{ij}\cos^2\theta + a_{jj}\sin\theta\cos\theta \\ -a_{ii}\sin\theta\cos\theta + a_{ji}\cos^2\theta - a_{ij}\sin^2\theta + a_{jj}\cos\theta\sin\theta & a_{ii}\sin^2\theta - a_{ji}\sin\theta\cos\theta - a_{ij}\sin\theta\cos\theta + a_{jj}\cos^2\theta \end{bmatrix}$$

$= \cancel{-a_{ii}}$

$= a_{ii}a_{jj} - a_{ji}a_{ij}$

Q- Using jacobi's Method, find out all eigen values and eigen vector of matrix.

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

$\Rightarrow$ Numerical largest non diagonal enteries $a_{13} = a_{31} = 2$

Orthogonal matrix in plane (1, 3)

$$B_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$ such that $B^{-1} = B^T$

$$\tan 2\theta = \frac{2a_{13}}{(a_{11} - a_{33})} = \frac{2.2}{1-1}$$

$$\tan 2\theta = \infty = \tan \pi/2$$

$$\theta = \frac{\pi}{4}$$

$$B_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$B_1^T A B_1 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Now numerical largest non diagonal entries

$a_{12} = a_{21} = 2$

Orthoganal matrix in (1, 2) plane,

$$B_2'' = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, ~~tha~~ $\tan 2\theta = \frac{2a_{12}}{(a_{11}-a_{22})} = \frac{2 \cdot 2}{3-3} = \infty = \tan\frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

$$B_2^{-1} = \begin{bmatrix} \frac{1}{\sqrt2} & \frac{-1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $B_2^{-1} C B_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (diagonal matrix)

eigen values are 5, 1, -1 Ans
and corresponding eigen vector is column vector of $B = B_1 B_2$ respectively -

$$B = B_1 B_2 = \begin{bmatrix} 1/\sqrt2 & 0 & -1/\sqrt2 \\ 0 & 1 & 0 \\ 1/\sqrt2 & 0 & 1/\sqrt2 \end{bmatrix} \begin{bmatrix} 1/\sqrt2 & -1/\sqrt2 & 0 \\ 1/\sqrt2 & 1/\sqrt2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 & -1/\sqrt2 \\ 1/\sqrt2 & \frac{1}{\sqrt2} & 0 \\ 1/2 & -1/2 & 1/\sqrt2 \end{bmatrix}$$ Ans

3) Given's Method -

Q- $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ to ~~trigo~~ tridigonal matrix using given's method.

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

non diagonal elements is $a_{13} = 1$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

where $\tan\theta = \frac{a_{13}}{a_{12}} = \frac{1}{1} = 1 = \tan\frac{\pi}{4}$

$\theta = \pi/4$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt2} & \frac{-1}{\sqrt2} \\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{bmatrix}$$

$P^{-1} = P' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ 0 & -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{bmatrix}$ P is an orthogonal matrix $\therefore P' = P^{-1}$

$$B = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ 0 & -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 1 \\ 2/\sqrt{2} & 5/\sqrt{2} & 5/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

$$= \begin{bmatrix} 3 & \frac{3}{\sqrt{2}} & 0 \\ \frac{2}{\sqrt{2}} & 10/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & \sqrt{2} & 0 \\ \sqrt{2} & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & \sqrt{2} & 0 \\ \sqrt{2} & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for eigen values-

$$|B - \lambda I| = 0 \qquad \begin{vmatrix} 3-\lambda & \sqrt{2} & 0 \\ \sqrt{2} & 5-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \begin{vmatrix} 3-\lambda & \sqrt{2} \\ \sqrt{2} & 5-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[15 - 8\lambda + \lambda^2 - 2] = 0$$

$$(1-\lambda)(\lambda^2 - 8\lambda - 13) = 0$$

$$\lambda = \frac{8 \pm \sqrt{64-52}}{2} = 4 \pm \sqrt{3}$$

$\lambda = 1, 4 \pm \sqrt{3}$ eigen value.

Error Analysis of a linear System—

ILL and well conditioned of linear System—

In a practical application, one usually encounters equation in which small changes in the situations, system is said to be ill conditioned.

On the other hand, if the corresponding changes in the solution are also small, the system is well conditioned.

Ex-(1) $2x + y = 2$

$2x + 1.01y = 2.01$

has solution $x = \frac{1}{2}$, $y = 1$

where as

$$2x + y = 2$$
$$2.01x + y = 2.01$$

has solution $x = 5$, $y = -8$

Thus system is in ill conditioned.

(2) $$A = \begin{bmatrix} -73 & 78 & 27 \\ 92 & 66 & 25 \\ -80 & 37 & 10 \end{bmatrix}.$$

$$|A| = 1$$

$$\begin{vmatrix} -73 & 78 & 27 \\ 92.01 & 66 & 25 \\ -80 & 37 & 10 \end{vmatrix} = 2.08$$

$$\Rightarrow \begin{vmatrix} -73 & 78.01 & 24 \\ 92 & 66 & 25 \\ -80 & 37 & 10 \end{vmatrix} = -28.20$$

Matrix is in ill-conditioned.

Conditioned Number—

Conditioned number is defined as—

$$\nu(A) = ||A|| \cdot ||A^{-1}||$$

Ex-① $A = \begin{bmatrix} 2 & 1 \\ 2 & 1.01 \end{bmatrix}$

taking Ecluidean norm—

$$\nu(A) = ||A||_2 \, ||A^{-1}||_2$$

Now, $||A||_2 = \sqrt{2^2+1^2+2^2+(1.01)^2} = 3.165$

$||A^{-1}||_2 = 158.273$

$\nu(A) = 3.165 \times 158.273 = 500.974$

$\Rightarrow$ A is in ill conditioned.

② $B = \begin{bmatrix} -0.6 & 0.6 \\ 0.4 & 0.2 \end{bmatrix}$

$||B||_2 = 0.959$

$||B^{-1}||_2 = 2.664$

$\nu(B) = ||B||_2 \cdot ||B^{-1}||_2$

$= 0.959 \times 2.664$

$= 2.555$

B is in well-conditioned.

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Gerschgorin Theorem :—

If λ be an eigen value of matrix A, then for some K $(1 \le K \le n)$ such that

$$|\lambda - a_{KK}| \le |a_{K_1}| + |a_{K_2}| + \cdots\cdots + |a_{Kn}| = \rho_K$$

i.e all the eigen value of A lie in the union of circles with center a_{KK} and radius ρ_K.

Proof :- Let λ be an eigen value of matrix A and corresponding eigen value X Then $AX = \lambda X$

Now
$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = \lambda x_1$$
$$a_{21}x_2 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = \lambda x_2$$
$$\cdots\cdots\cdots\cdots$$
$$a_{K_1}x_1 + a_{K_2}x_2 + \cdots + a_{KK}x_K + \cdots + a_{Kn}x_n = \lambda x_K$$
$$\cdots\cdots\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n = \lambda x_n$$

$x_1, x_2, \cdots, x_K, \cdots x_n$ are unknowns

Let $|x_K| \geq |x_m| \quad \forall \; m = 1, 2, \cdots n$

Then $\left|\frac{x_m}{x_K}\right| \le 1 \quad \forall \; m = 1, 2, \cdots n$

In K-th row of system of equation divided by x_K.

$$a_{K_1}\left(\frac{x_1}{x_K}\right) + a_{K_2}\left(\frac{x_2}{x_K}\right) + \cdots\cdots + a_{KK}\left(\frac{x_K}{x_K}\right) + \cdots a_{Kn}\left(\frac{x_n}{x_K}\right)$$

$$\Rightarrow |\lambda - a_{KK}| = \left|a_{K_1}\left(\frac{x_1}{x_K}\right) + \cdots + a_{KK}\left(\frac{x_K}{x_K}\right) + \cdots + a_{Kn}\left(\frac{x_n}{x_K}\right)\right|$$

$$|\lambda - a_{KK}| \le |a_{K_1}| + |a_{K_2}| + \cdots + |a_{Kn}| = \rho_K$$

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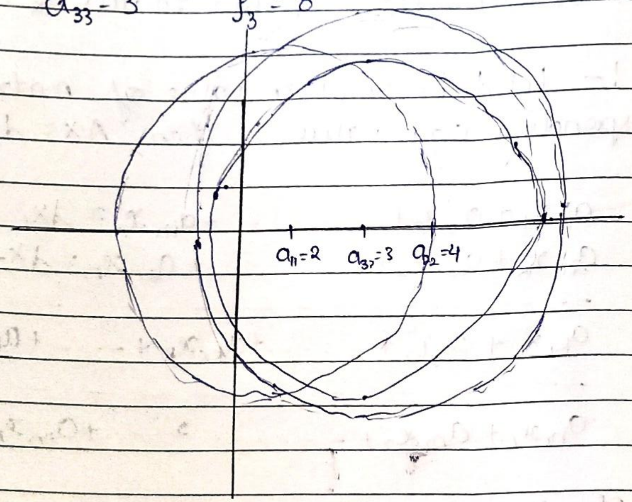
Q- $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix}$

$a_{11} = 2$, $\rho_1 = 9$

$a_{22} = 4$ $\rho_2 = 9$

$a_{33} = 3$ $\rho_3 = 8$



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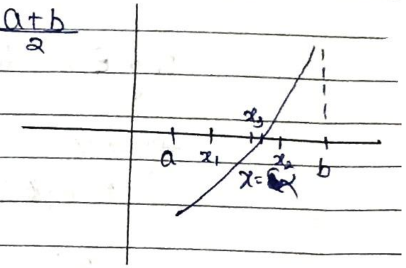
Bisection Method—

$f : [a,b] \rightarrow R$ is continous on $[a,b]$
and f is differentiable on (a,b)
such that $f(a) \cdot f(b) < 0$

$\Rightarrow \exists\ \alpha \in [a,b]$ such that
$f(\alpha) = 0$ (simple root)
if $f'(\alpha) \neq 0$

first approximation $x_1 = \frac{a+b}{2}$



$f(a) \cdot f(x_1) > 0$

or $f(x_1) \cdot f(b) < 0$

Now intravel
after first approximation
$[x, b]$

IInd approximation
$x_2 = \frac{x_1 + b}{2}$

$f(x_1) \cdot f(x_2) < 0$

$f(x_2) \cdot f(b) > 0$

$[x_1, x_2]$

ϵ_n be the error after nth iteration

then $\epsilon_n = x_n - \alpha$

~~let ϵ is permissible error~~

approximation	error
x_1	$b-a/2 = \varepsilon_1$
x_2	$b-a/2^2 = \varepsilon_2$
x_3	$b-a/2^3 = \varepsilon_3$
...	...
x_n	$b-a/2^n = \varepsilon_n$

Let be ε be permissible error

$$|\varepsilon| \geq |\varepsilon_n|$$
$$|\varepsilon| \geq \left|\frac{b-a}{2^n}\right|$$

$$\log \varepsilon \geq \log\left|\frac{b-a}{2^n}\right| = \log|b-a| - n\log 2$$

~~$\log \varepsilon \geq$~~

$$n\log 2 \geq \log|b-a| - \log \varepsilon$$

$$n \geq \frac{\log\left|\frac{b-a}{\varepsilon}\right|}{\log 2}$$

Ques → Find the root of the equation $x^3-ax-9=0$ using the bisection method up to three decimal places.

→ $f(x) = x^3-4x-9$

$f(2) = -9$

$f(3) = 6$

$f(2).f(3) < 0$

$\exists\ \alpha \in [2,3]$ such $f(\alpha) = 0$.

$x_1 = \frac{2+3}{2} = 2.5$

$f(2.5) = -3.375$

$f(2).f(2.5) > 0$

$f(2.5).f(3) < 0$

interval - $[2.5, 3]$

$x_2 = \frac{2.5+3}{2}$

$f(x_2) = f(2.75) = 0.79$

~~$f(2.5).f(2.75$~~

$f(2).f(2.5) > 0$

$f(2.5).f(2.75) < 0$

interval - $[2.5, 2.75]$

$x_3 = \frac{2.5+2.75}{2} = 2.625$

$f(x_3) = f(2.625) = -1.4121$

interval -

$[2.625, 2.75]$

$x_4 = \frac{2.625+2.75}{2}$

$= 2.6875$

$f(x_4) = f(2.6875) = -0.339$

interval

$[2.6875, 2.75]$

$x_5 = \frac{2.6875+2.75}{2} = 2.7187$

$f(x_5) = f(2.7187) = 0.22$

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interval —
[2.6875 , 2.7187]

$x_6 = \frac{2.6875 + 2.7187}{2}$

$= 2.7031$

$f(x_6) = f(2.7031) = -0.0615$

interval —
[2.7031 , 2.7187]

$x_7 = \frac{2.7031 + 2.7187}{2} = 2.7109$

$f(x_7) = f(2.7109) = 0.0787$

interval – [2.7031 , 2.7109]

$x_8 = \frac{2.7031 + 2.7109}{2} = 2.707$

$f(x_8) = f(2.707) = 0.0084$

interval – [2.707 , 2.7031]

$x_9 = \frac{2.707 + 2.7031}{2}$

$= 2.703$

$f(x_9) = f(2.703) = \cancel{-0.027}\ -0.063$

interval ⇒ [2.703 , 2.707]

$x_{10} = \frac{2.703 + 2.707}{2} = 2.705$

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$f(2.705) = -0.027$

interval [2.705 , 2.707]

$x_{11} = \frac{2.705 + 2.707}{2} = 2.706$

$f(2.706) = 3.633$

interval → [2.705 , 2.706]

$x_{12} = 2.7055$

$f(2.7055) = -0.0184$

interval · [2.7055 , 2.706]

$x_{13} = \frac{2.7055 + 2.706}{2}$

$= 2.70575$

$f(2.70575) = -0.013$

interval → [2.70575 , 2.706]

$x_{14} = \frac{2.70575 + 2.706}{2} = 2.7058$

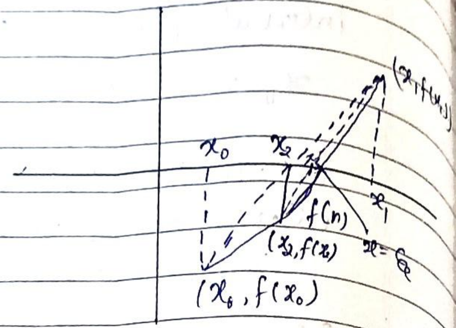
root $\alpha = 2.7058$ Ans

Regula falsi Method —

f(n) is continous and differentiable on $[x_0, x_1]$

$f(x_0).f(x_1) < 0$

$\Rightarrow \exists\ x = \alpha$ be the root of the equation



Equation of line -

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

when $x = x_2$

$\Rightarrow y = 0$

$$0 - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)$$

$$-\frac{(x_1 - x_0) f(x_0)}{f(x_1) - f(x_0)} = (x_2 - x_0)$$

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

II^{nd} approximation —

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

$x = x_3 \Rightarrow y = 0$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} \cdot f(x_2)$$

general formula —

$$x_n = x_{n-1} - \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} \cdot f(x_{n-1})$$

Q- Find the real root of the equation $x^3 - 2x - 5 = 0$ up to three decimal places.

Sol $\Rightarrow$ Here $f(x) = x^3 - 2x - 5$

$f(0) = -5 < 0$

$f(1) = -6$

$f(2) = -1$

$f(3) = 16$

Interval - $[2, 3]$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_1)$$

$$x_2 = 3 - \frac{3-2}{16+1} \times 16 = 3 - \frac{16}{17} = 2.05882$$

$f(x_2) = -0.39083 < 0$

~~Interval~~

$f(x_0).\ f(x_2) < 0$

Interval $\rightarrow [2.05882, 3]$

$$x_3 = 3 + \frac{2.05882 - 3}{16 + 0.39083}(0.39083)$$

$x_3 = \cancel{3.17573}\ 2.09655$

$f(x_3) = 0.0223 > 0$

interval — $[2.05882, 2.09655]$

$$x_4 = 2.09655 - \frac{2.05882 - 2.09655}{0.0223 + 0.39083} \times (0.39083)$$

Here $f(x) = \cancel{2}x^3 - 2x - 5$

$f(0) = -5$

$f(1) = -6$

$f(2) = -1$

$f(3) = 16$

Interval $[2, 3] = [x_0, x_1]$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_1)$$

$$= 3 - \frac{3-2}{16+1} \cdot 16 = 2.05882$$

$f(x_2) = \cancel{-0.93059}\ -0.39083$

interval - $[\overset{x_0}{2.05882}, \overset{x_1}{3}]$

$$x_3 = 3 - \frac{3 - 2.05882}{16 + 0.39083} \times 16$$

$x_3 = 2.08126$

$f(x_3) = -0.14746$

interval - $[\overset{x_0}{2.08126}, \overset{x_1}{3}]$

$$x_4 = 3 - \frac{3 - 2.08126}{16 + 0.14746} \cdot 16$$

$= \cancel{2.08963}\ 2.08969$

$f(x_4) = -0.05478$

interval - $\boxed{2.08969, 3}$

$$x_5 = 3 - \frac{3 - 2.08969}{16 + 0.05478} \cdot 16 = 3 - \frac{3 - 2.08969}{16 + 0.05411} \cdot 16$$

$x_5 = 2.09276$

$f(x_5) = \cancel{-0.0207}\ -0.01998$

interval — $[2.09276, 3]$

$$x_6 = 3 - \frac{3 - 2.0976}{16 + 0.01998} \cdot 16$$

$= 2.09873$

$f(x_6) = \cancel{0.04462}\ 0.04675$

interval - $[\overset{x_0}{2.09276}, \overset{x_1}{2.09873}]$

$$x_7 = 2.09873 - \frac{2.09873 - 2.09276}{0.04675 + 0.01998} \cdot 0.04675$$

$= 2.09455$

$f(x_7) = -1.65362$

interval - $[\overset{x_0}{2.09455}, \overset{x_1}{2.09854}]$

$$\cancel{x_8 = 2.09854 - \frac{2.09455 - 2.09854}{1.65363 + 0.04462} \cdot (1.65363)}$$

$$\cancel{x_8 = 2.09854 - \frac{2.09854 - 2.09455}{0.04462 + 1.65363} (0.04462)}$$

$\cancel{x_8 = 2.09843}$

$\cancel{x = 2.098}$ Ans

interval - $[2.09455, 2.09873]$

$$x_8 = 2.09873 - \frac{2.09873 - 2.09455}{0.04675 + 1.65362} (0.04675)$$

$x_8 = 2.0986$

$f(x_8) = 0.04529$

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Here $f(x) = x^3 - 2x - 5$

$f(0) = -5$

$f(1) = -6$

$f(2) = -1$

$f(3) = 16$

Interval $[2, 3] = [x_0, x_1]$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_1)$$

$$= 3 - \frac{3-2}{16+1} \cdot 16 = 2.05882$$

$f(x_2) = \cancel{-0.93059}\ -0.39083$

interval - $[\overset{x_0}{2.05882}, \overset{x_1}{3}]$

$$x_3 = 3 - \frac{3 - 2.05882}{16 + 0.39083} \times 16$$

$x_3 = 2.08120$

$f(x_3) = -0.14786$

interval - $[\overset{x_0}{2.08120}, \overset{x_1}{3}]$

$$x_4 = 3 - \frac{3 - 2.08120}{16 + 0.147846} \cdot 16$$

$= \cancel{2.08963}\ 2.08969$

$f(x_4) = -0.05478$

interval - $[2.08969, 3]$

$$x_5 = 3 - \frac{3 - 2.08969}{16 + \cancel{2.08478}} \cdot 16 = 3 - \frac{3 - 2.08969}{16 + 0.05411} \cdot 16$$

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$x_5 = 2.09276$

$f(x_5) = \cancel{-0.0202}\ -0.01998$

interval $- [2.09276, 3]$

$$x_6 = 3 - \frac{3 - 2.09276}{16 + 0.01998} \cdot 16$$

$= 2.09873$

$f(x_6) = \cancel{0.04462}\ 0.04675$

interval - $[\overset{x_0}{2.09276}, \overset{x_1}{2.09873}]$

$$x_7 = 2.09873 - \frac{2.09873 - 2.09276}{0.04675 + 0.01998} \cdot 0.04675$$

$= 2.09455$

$f(x_7) = -1.65362$

interval - $[\overset{x_0}{2.09455}, \overset{x_1}{2.09854}]$

$$\cancel{x_8 = 2.09854 - \frac{2.09455 - 2.09854}{1.65363 + 0.04462} \cdot (1.65363)}$$

$$\cancel{x_8 = 2.09854 - \frac{2.09854 - 2.09455}{0.04462 + 1.65363} (0.04462)}$$

$\cancel{x_8 = 2.09843}$

$x = 2.098$ Ans

interval - $[2.09455, 2.09873]$

$$x_8 = 2.09873 - \frac{2.09873 - 2.09455}{0.04675 + 1.65362} (0.04675$$

$x_8 = 2.0986$

$f(x_8) = 0.04529$

interval – $[2.09455, 2.0986]$

$$x_9 = 2.0986 - \frac{2.0986 - 2.09455}{0.04529 + 1.65362}(0.04529)$$

$$= 2.0984$$

$x = 2.098$ Ans

Secant Method — $\boxed{x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)}$

Q- $f(x) = x^3 - 2x - 5$

→ $f(x) = x^3 - 2x - 5$

$f(2) = -1$

$f(3) = 16$

Since $f(2) \cdot f(3) < 0$

Hence root lies b/w 2 & 3.

first approximation $n = 1$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$x_2 = 3 - \frac{3-2}{16-(-1)}(16) = 2.05882$$

$f(x_2) = -0.39083$

$n = 2$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} \cdot f(x_2)$$

$$= 2.05882 - \frac{3 - 2.05882}{-(0.39083) - 16} \cdot (0.39083)$$

$x_3 = 2.08126$

$f(x_3) = -0.14724$

$n = 3$

$$x_4 = x_3 - \frac{x_3 - x_2}{\cancel{-0.14724 + 0.39083}\ f(x_3) - f(x_2)} \times f(x_3)$$

$x_4 = 2.094824$

$f(x_4) = 0.003042$

$n = 4$

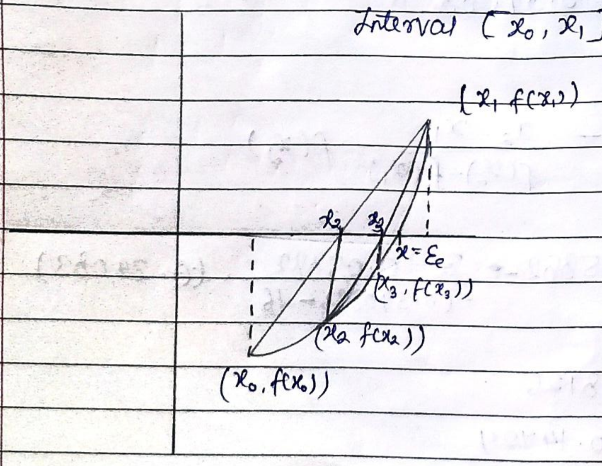
$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} \times f(x_4)$$

$$= 2.0945$$

$x = 2.094$ Ans

Rate of Convergence —

Interval $[x_0, x_1]$



1st approxn

$\varepsilon_2 = \xi - x_2$

$\varepsilon_3 = \xi - x_3$

$\varepsilon_n = \xi - x_n$

$x_n = \xi - \varepsilon_n$

Formula of Regula Falsi Method

$$x_{n+1} = x_n - \frac{x_n - x_0}{f(x_n) - f(x_0)} \cdot f(x_n)$$

$$\xi - \varepsilon_{n+1} = \xi - \varepsilon_n - \frac{(\xi - \varepsilon_n) - (\xi - \varepsilon_0)}{f(\xi - \varepsilon_n) - f(\xi - \varepsilon_0)} \cdot f(\xi - \varepsilon_n)$$

$$f(\xi + \varepsilon_n) = f(\xi) + \varepsilon_n f'(\xi) + \frac{\varepsilon_n^2}{2!} f''(\xi) + \dots$$

After solving we get

$$\varepsilon_{n+1} = \frac{1}{2} \varepsilon_n \varepsilon_0 \frac{f''(\xi)}{f'(\xi)} + O(\varepsilon_n^3)$$

Since $\varepsilon_{n+1} = C \varepsilon_n^p$

where C — Asymptotic Constant

& p — rate of convergence

Hence

$$C = \frac{1}{2} \varepsilon_0 \frac{f''(\xi)}{f'(\xi)}$$

Rate of convergence

$$\boxed{p = 1}$$

Rate of Convergent of Secant Method —

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

let ξ be exact root then $f(\xi) = 0$

Also ε_n be error after nth approximation

Then $x_n = \varepsilon_n + \xi$

$$\varepsilon_{n+1} + \xi = (\varepsilon_n + \xi) - \frac{(\varepsilon_n + \xi) - (\varepsilon_{n-1} + \xi)}{f(\varepsilon_n + \xi) - f(\varepsilon_{n-1} + \xi)} f(\varepsilon_n + \xi$$

$$\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n \varepsilon_{n-1} \frac{f''(\varepsilon)}{f'(\varepsilon)} \quad \text{——— (1)}$$

Since $\varepsilon_{n+1} = K\varepsilon_n^{\,p}$

$\Rightarrow \varepsilon_n = K(\varepsilon_{n-1})^p$

$(\varepsilon_n)^{1/p} = (K\varepsilon_{n-1}^{\,p})^{1/p}$

$(\varepsilon_n)^{1/p} = K^{1/p}\varepsilon_{n-1}$

$\varepsilon_{n-1} = K^{-1/p} \cdot \varepsilon_n^{1/p}$ using in eq (1)

$$\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n K^{-1/p}\varepsilon_n^{1/p} \cdot \frac{f''(\varepsilon)}{f'(\varepsilon)}$$

$$\varepsilon_{n+1} = \frac{1}{2}K^{-1/p}\varepsilon_n^{1+\frac{1}{p}} \frac{f''(\varepsilon)}{f'(\varepsilon)} \quad \text{——— (2)}$$

$$\varepsilon_{n+1} = \frac{1}{2}K\varepsilon_n^{\,p} \quad \text{——— (3)}$$

On comparing for rate of convergence

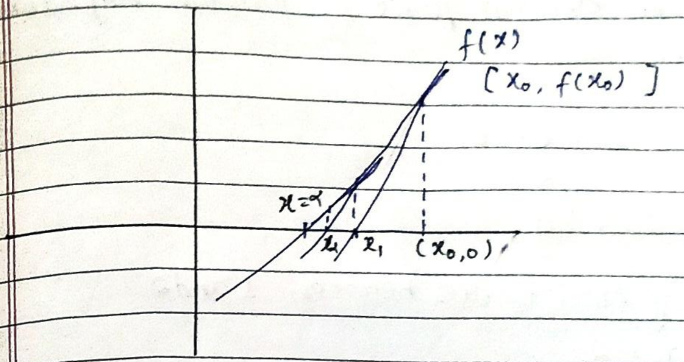
$p = 1 + \frac{1}{p}$

$p^2 - p - 1 = 0$

$p = \frac{-(-1) \pm \sqrt{1+4}}{2.1}$

$p = \frac{1 \pm \sqrt{5}}{2} = 1.618$

Newton Raphson Method :—



Suppose $x = \alpha$ be exact root of $f(x) = 0$ Then, $f(\alpha) = 0$.

Let x_0 be any point in the neighbourhood of $x = \alpha$

Now equation of tangent line at A $(x_0, f(x_0))$

$$y - f(x_0) = f'(x_0)(x - x_0) \quad \text{——— (1)}$$

equation (1) passing through $(x_1, 0)$

Then

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Repeating this process, the general formula for NRM —

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Q- Find the positive root of $x^4 - x = 10$ correct upto three decimal place by Newton Raphson Method.

⇒ $f(x) = x^4 - x - 10$

$f(1) = -10 < 0$

$f(2) = 4 > 0$

A root of $f(x) = 0$ lies between 1 and 2

let $x_0 = 2$

$f(2) = 4$

$f'(x) = 4x^3 - 1$

$f'(2) = 31$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{4}{31} = 1.87096$$

Again $f(x_1) = 0.382479$

$f'(x_1) = 25.19716$

Now $x_2 = 1.87096 - \dfrac{0.382479}{25.19716}$

$x_2 = 1.85578$

$f(x_2) = 0.0048$

$f'(x_2) = 24.5646$

$x_3 = 1.85578 - \dfrac{0.0048}{24.5646}$

$x_3 = 1.85558$

root $\boxed{\alpha = 1.856}$ Ans

Rate of Convergence of Newton Raphson Method :-

Since formula of Newton Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{———} \quad (1)$$

Since $x_n = \varepsilon_n + \alpha$ using in (1)

$$\varepsilon_{n+1} + \alpha = \varepsilon_n + \alpha - \frac{f(\varepsilon_n + \alpha)}{f'(\varepsilon_n + \alpha)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\varepsilon_n + \alpha)}{f'(\varepsilon_n + \alpha)}$$

$$f(\varepsilon_n + \alpha) = f(\alpha) + \frac{\varepsilon_n}{1!} f'(\alpha) + \frac{\varepsilon_n^2}{2!} f''(\alpha) + \text{------}$$

(with $f(\alpha) = 0$)

$$\varepsilon_{n+1} = \varepsilon_n - \frac{[f(\alpha) + \varepsilon_n f'(\alpha) + \frac{\varepsilon_n^2}{2!} f''(\alpha) + \text{-----}]}{[f'(\alpha) + \varepsilon_n f''(\alpha) + \frac{\varepsilon_n}{2!} f''(\alpha) + \text{---}]}$$

$$\varepsilon_{n+1} = \frac{\boxed{\cancel{\varepsilon_n f'(\alpha)} + \varepsilon_n^2 f''(\alpha) + \text{---}} - \boxed{\cancel{\varepsilon_n f'(\alpha)} + \frac{\varepsilon_n^2}{2!} f''(\alpha) + \text{--}}}{f'(\alpha) + \varepsilon_n f''(\alpha) + \text{---}}$$

$$= \frac{\frac{1}{2}\varepsilon_n^2 f''(\alpha)}{f'(\alpha)} \left[1 + \frac{\varepsilon_n f''(\alpha)}{f'(\alpha)} + \text{-----}\right]^{-1}$$

After Solving –

$$\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n^2 \frac{f''(\alpha)}{f'(\alpha)}$$

on comparing $\varepsilon_{n+1} = K\varepsilon_n^{p}$

Then rate of convergence $p = 2$

Rate of Convergence (NR method) :–

ε be any exact root of $f(n) = 0$

$\Rightarrow f(\varepsilon) = 0$ but $f'(\varepsilon) \neq 0$

If. $f(\varepsilon) = 0$ as well as $f'(\varepsilon) = 0$ but $f''(\varepsilon) \neq 0$

Since we know that $x_n = \varepsilon_n + \varepsilon$

formula for N.R method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\varepsilon_{n+1} + \varepsilon = \varepsilon_n + \varepsilon - \frac{f(\varepsilon_n+\varepsilon)}{f'(\varepsilon_n+\varepsilon)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\varepsilon_n+\varepsilon)}{f'(\varepsilon_n+\varepsilon)}$$

$$= \varepsilon_n - \frac{\left[f(\varepsilon) + \varepsilon_n f'(\varepsilon) + \frac{\varepsilon_n^2}{2!} f''(\varepsilon) + \cdots\right]}{\left[f'(\varepsilon) + \varepsilon_n f''(\varepsilon) + \frac{\varepsilon_n^2}{2!} f'''(\varepsilon) + \cdots\right]}$$

$$= \varepsilon_n - \frac{\varepsilon_n^2/2!\, f''(\varepsilon)}{\varepsilon_n f''(\varepsilon)}$$

$$= \frac{(1-\frac{1}{2})\,\varepsilon_n^2\, f''(\varepsilon)}{\varepsilon_n f''(\varepsilon)}$$

$$\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n$$

Since $\varepsilon_{n+1} = K\varepsilon_n^{p}$

Rate of convergence $= 1$

Q– root – $\sqrt{\alpha}$ $\quad \alpha \in \mathbb{R}^+$

$\Rightarrow$ $x = \sqrt{\alpha}$

$x^2 = \alpha$

$x^2 - \alpha = 0$

$f(x) = x^2 - \alpha$, $f'(x) = 2x$

formula of NR method –

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - \alpha}{2x_n}$$

$$\boxed{x_{n+1} = \frac{x_n^2 + \alpha}{2x_n}}$$ Ans

Q2. root – $\frac{1}{\alpha}$

$x = \frac{1}{\alpha}$

$\alpha x - 1 = 0$

$f(x) = \alpha x - 1 \qquad f'(x) = \alpha$

formula of NR method –

$$x_{n+1} = x_n - \frac{(\alpha x_n - 1)}{\alpha}$$

$$= \frac{\cancel{\alpha x_n} - \cancel{\alpha x_n} + 1}{\alpha} = \frac{1}{\alpha}$$ Ans

Modified Newton Raphson Method —

$f(n) = 0$

ε be exact root of $f(n) = 0$ of order m

$f(\varepsilon) = f'(\varepsilon) = f''(\varepsilon) = \cdots f^{m-1}(\varepsilon) = 0$

but $f^{m}(\varepsilon) \neq 0$

$f(x) = (x-\varepsilon)^m \cdot g(x)$ where $g(\varepsilon) \neq 0$

Suppose $g(x) = A$ in the neighbourhood of $x = \varepsilon$

$f(x) = A(x-\varepsilon)^m$

$f'(x) = Am(x-\varepsilon)^{m-1}$

Here $\frac{f(x)}{f'(x)} = \frac{A(n-\varepsilon)^m}{Am(x-\varepsilon)^{n-1}} = \frac{x-\varepsilon}{m}$

$\frac{f(x_n)}{f'(x_n)} = \frac{x_n - \varepsilon}{m}$

$\varepsilon = x_n - \frac{m f(x_n)}{f'(x_n)}$

let $\varepsilon = x_{n+1}$ Then

$$\boxed{x_{n+1} = x_n - \frac{m f(x_n)}{f'(x_n)}}$$

Rate of Convergence ~~of Newto~~ :—

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Fixed Point Iteration Method —

let $f(n)=0$ be an equation
and ξ be exact root of equation
$f(n)=0 \Rightarrow f(\xi)=0$
Now, $f(x) = x - \phi(x) = 0$
$\Rightarrow x = \phi(x)$.
$(n+1)$th approximation of Iteration Method
will be $\boxed{x_{n+1} = \phi(x_n)}$ (General form of Iterative method)
since ξ be exact root of $f(x)=0$

$$\phi(\xi) = \xi$$

Now $x_{n+1} - \xi = \phi(x_n) - \phi(\xi)$

$$\Rightarrow \xi_{n+1} = \left[\frac{\phi(x_n) - \phi(\xi)}{(x_n - \xi)}\right](x_n - \xi)$$

$$\Rightarrow \xi_{n+1} = \phi'(x_n) \cdot \xi_n$$

$$\Rightarrow \xi_{n+1} = \phi'(x_n) \cdot \phi'(x_{n-1})\, \xi_{n-1}$$

$$= \phi'(x_n)\,\phi'(x_{n-1}) - - - - - - - \phi(x_0)\,\xi_0$$

If ϕ' is bounded $\Rightarrow |\phi'(x)| \leq c \ \forall x$

$$\xi_{n+1} \leq c^{n+1}\xi_0$$

As $n \to \infty$ $c^{n+1}\xi_0 \to 0$ i.e. $\xi_{n+1} \to 0$

$$\Rightarrow x_{n+1} \to \xi$$

Result — $x_{n+1} = \phi(x_n)$ is convergent if $|\phi'(x_n)| \leq c < 1 \ \forall x$

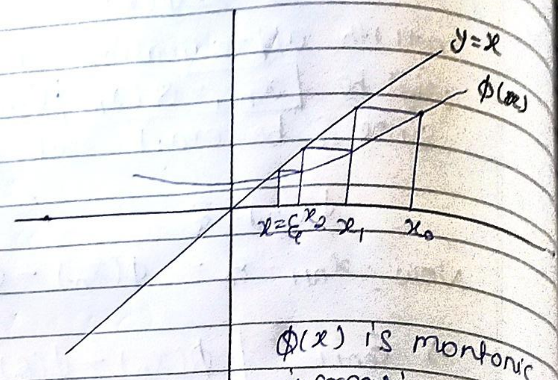
and divergent $|\phi'(x_n)| \geq 1$

Case I

When $0 < \phi'(x_n) < 1$

Since $\phi'(x_n) > 0 \Rightarrow \phi(x_n)$ is ~~hom~~ monotonic increasing.

$n = 0$

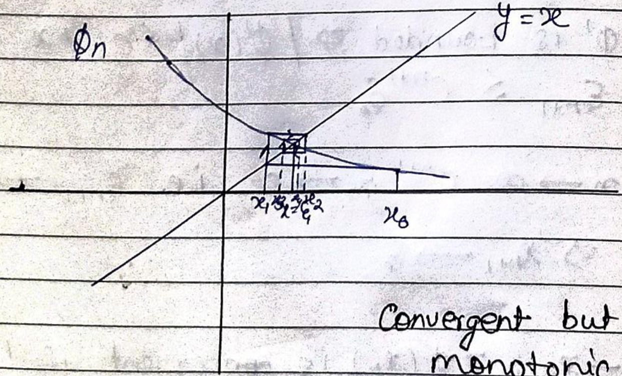


$\phi(x)$ is montonic increasing and convergent

Case II

$-1 < \phi'(x_n) < 0$

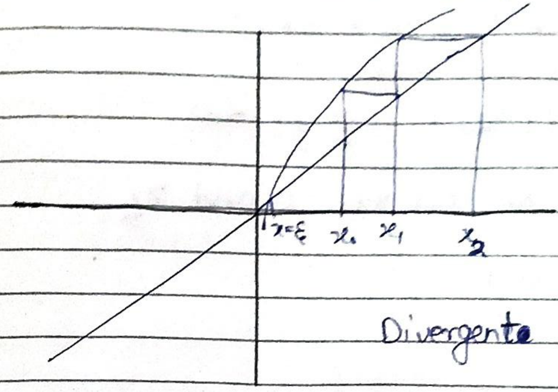
Since $\phi'(x_n) < 0 \Rightarrow \phi(x_n)$ is monotonic decreasing



Convergent but not monotonic

Case III $\phi'(x_n) \geq 1 \Rightarrow \phi'(x_n) \geq 0$

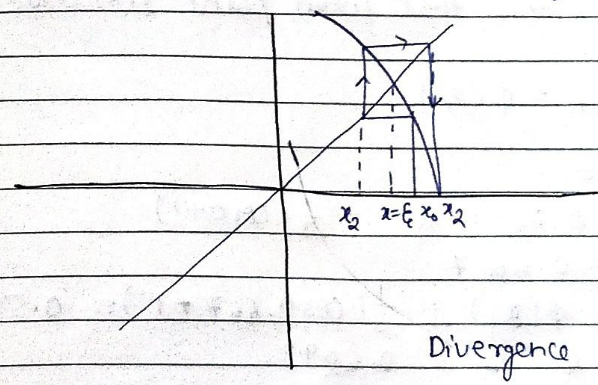
$\phi(x_n)$ is monotonic Increasing –



Divergente

Case IV $\phi'(x_n) \leq -1$

$\phi(x_n)$ is monotonic Decreasing –



Divergence

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Topic

Q-> Find the real root of the equation $\cos x = 3x - 1$ correct up to three decimal places using iteration method.

Sol-> Here $f(x) = \cos x - 3x + 1$

$f(0) = 2 > 0$

$f(\pi/2) = -\frac{3\pi}{2} + 1 < 0$

A root lies between 0 and $\pi/2$

$\cos x - 3x + 1 = 0$

$3x = 1 + \cos x$

$x = \frac{1}{3}(1 + \cos x$

Here $\phi_x = \frac{1}{3}(1 + \cos x)$

Since we know that fixed point Iterative formula.

$x_{n+1} = \phi(x_n)$

Let $x_0 = 0$

$x_1 = \phi(x_0) = \phi(0) = \frac{1}{3}(\cos 0 + 1)$

$x_1 = 0.6667$

$x_2 = \phi(x_1) = \frac{1}{3}(\cos 0.667 + 1) = 0.5953$

$x_3 = \phi(x_2) = 0.6093$

$x_4 = \phi(x_3) = 0.6067$

$x_5 = \phi(x_4) = 0.6072$

$x_6 = \phi(x_5) = 0.6071$

root is 0.607 Ans

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