Reduction to diagonal form:-

If a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

Let A be a square matrix of order 3. Let λ_1, λ_2 and λ_3 be its eigen values and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

be corresponding eigen vectors. Denoting the square

$$[X_1\ X_2\ X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$ by P, we have

$$AP = A[X_1, X_2, X_3] = [AX_1, AX_2, AX_3]$$

$$= [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$= PD$$

$$\Rightarrow \boxed{P^{-1}AP = D}$$

Observation :-

1. The matrix P which diagonalises A is called the modal matrix of A and the resulting diagonal matrix D is known as Spectral matrix of A.

2. The diagonal matrix D has eigenvalues of A as its diagonal element.

3. The matrix P, which diagonalise A, constitute the eigen vectors of A.

Working procedure :-

1. Find the eigen values of the square matrix A.
2. Find the corresponding eigen vectors and write the modal matrix P.
3. Find D using $D = P^{-1}AP$

Q Reduce the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the diagonal form.

Sol^n The characteristic equation of A is

$$\begin{bmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} = 0$$

or $\lambda^3 - \lambda^2 - 5\lambda - 5 = 0$

we have

$$\lambda_1 = 1, \quad \lambda_2 = \sqrt{5}, \quad \lambda_3 = -\sqrt{5}$$

For $\lambda = 1$

$$\begin{bmatrix} -1-1 & 2 & -2 \\ 1 & 2-1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-2x + 2y - 2z = 0$$
$$x + y + z = 0$$

$$\frac{x}{2+2} = \frac{y}{-2+2} = \frac{z}{-2-2}$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1} = k$$

$$x = k, \quad y = 0, \quad z = -k$$

for $k = 1$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $\lambda = \sqrt{5}$

$$\begin{bmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & -\sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(-1-\sqrt{5})x + 2y - 2z = 0$$
$$x + (2-\sqrt{5})y + z = 0$$
$$-x - y - \sqrt{5}z = 0$$

Solving 2^{nd} and 3^{rd} equation

$$\frac{x}{6-2\sqrt{5}} = \frac{y}{-1+\sqrt{5}} = \frac{z}{1-\sqrt{5}}$$

or $$\frac{x}{\sqrt{5}-1} = \frac{y}{1} = \frac{z}{-1} = k$$

$$x = (\sqrt{5}-1)k, \quad y = k, \quad z = -k$$

~~fear~~ for $k = 1$

$$X_2 = \begin{bmatrix} \sqrt{5}-1 \\ 1 \\ -1 \end{bmatrix}$$

Similarly for $\lambda = -\sqrt{5}$

$$X_3 = \begin{bmatrix} \sqrt{5}+1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow P = [X_1 \; X_2 X_3] = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Hence the diagonal matrix is

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$