Unit-III

Beta function : Let $m, n > 0$ then Beta function is defined as

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

Properties (i) $B(m,n) = B(n,m)$

$$\because B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

Let $x = 1-y$

$$\Rightarrow dx = -dy$$

$$\therefore B(m,n) = \int_1^0 (1-y)^{m-1} \, y^{n-1} (-dy)$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1} \, dy$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1} \, dx = B(n,m)$$

(2) $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Proof:- $\because$ $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$1+y = \frac{1}{0} = \infty$$
$$1+y = \infty$$
$$y = \infty - 1 = \infty$$

let $x = \frac{1}{1+y}$ $\left(\Rightarrow 1+y = \frac{1}{x}\right)$

$$\Rightarrow dx = \frac{-1}{(1+y)^2}\,dy$$

$$B(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\left(\frac{-1}{(1+y)^2}\,dy\right)$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}}\,\frac{y^{n-1}}{(1+y)^{n-1}} \times \frac{1}{(1+y)^2}\,d$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \quad \#$$

(3) $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Pf: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

$\therefore B(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \times 2\sin\theta\cos\theta\,d\theta$

$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2n-2}\theta \sin\theta\cos\theta\,d\theta$

$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$ #

$\boxed{\Gamma(-\tfrac{1}{2})}$ ✗

Gamma function

if n is +ve then

$$\Gamma(n)=\int_0^\infty e^{-x}\,x^{n-1}\,dx \quad , \; \Gamma(0) = \text{is not defined.}$$

$\cancel{0\times\infty}$ $\cancel{0 \times \infty = 0}$

Properties:- (i) $\Gamma(n+1) = n\Gamma(n) = n!$

Pf:

$$\Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1}\,dx$$

$$\Rightarrow \Gamma(n+1) = \int_0^\infty e^{-x}\,x^{n}\,dx$$

$$= \left[-e^{-x}\,x^n\right]_0^\infty + n\int_0^\infty e^{-x}\,x^{n-1}\,dx$$

$$= (0-0) + n\,\Gamma(n) = n\,\Gamma(n)$$

$\left(\frac{\infty}{\infty}\right)$

$$\left\{\because \lim_{x\to\infty}\left[e^{-x}x^{n}\right] = \lim_{x\to\infty}\left[\frac{x^{n}}{e^{x}}\right]\right.$$

$$\left. = \lim_{n\to\infty}\left[\frac{n(n-1)\times\cdots\times 1}{e^{x}}\right] = 0\right\}$$

$$\therefore \boxed{\Gamma(n+1) = n\Gamma(n)}$$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$= n(n-1)(n-2)\times\cdots\times 1\,\Gamma(1)$$

$$= n! \qquad \left\{\because \Gamma(1) = \int_0^\infty e^{-x}1^{n-1}\,dx = 1\right\}$$

$$\therefore \Gamma(n+1) = \begin{cases} n\Gamma(n) \\ n!, & n \text{ is +ve integer.} \end{cases}$$

Note that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

$$\therefore \boxed{\Gamma(n+1)=n\Gamma(n)}$$

$$\Rightarrow \Gamma\left(\frac{1}{2}+1\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Rightarrow \boxed{\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\sqrt{\pi}}$$

$$\Gamma(5/2)=\Gamma(3/2+1)$$

$$=\frac{3}{2}\Gamma(3/2)$$

$$=\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}$$

$$=\frac{3}{4}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right)=\Gamma\left(\frac{5}{2}+1\right)$$

$$=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)$$

$$=\frac{5}{2}\times\frac{3}{4}\sqrt{\pi}$$

$$\boxed{\Gamma\left(\frac{7}{2}\right)=\frac{15}{8}\sqrt{\pi}}$$

$$\Gamma(6)=\Gamma(5+1)=5!=120$$

$$\Gamma(1)=\Gamma(0+1)=0!=1.$$

$$\Gamma(4)=\Gamma(3+1)=3!=6$$

Property 2

$$\frac{\Gamma(n)}{k^n} = \int_0^\infty e^{-kx} x^{n-1} dx$$

$$\therefore \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

put $x = ky$

$dx = k\,dy$

$$\therefore \Gamma(n) = \int_0^\infty e^{-ky} (ky)^{n-1} k\,dy$$

$$\Gamma(n) = k^n \int_0^\infty e^{-ky} y^{n-1} dy$$

$$\Rightarrow \boxed{\frac{\Gamma(n)}{k^n} = \int_0^\infty e^{-kx} x^{n-1} dx}$$

(3) $\Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$$

put $x = \log\frac{1}{y}$

$\Rightarrow e^{x} = \frac{1}{y} \Rightarrow y = e^{-x}$

$dy = -e^{-x} dx$

$\Rightarrow e^{-x} dx = -dy$

$$\therefore \Gamma(n) = \int_1^0 \left(\log\frac{1}{y}\right)^{n-1} (-dy)$$

$$= \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx \quad \#.$$

(4) $\boxed{n\Gamma(n) = \int_0^\infty e^{-x^{\frac{1}{n}}}\, dx}$

e^{-x}

$x \to y^{\frac{1}{n}}$

$\because \quad \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

let $x = y^{\frac{1}{n}}$

$\cancel{dx} \Rightarrow x^n = y$

$\Rightarrow n x^{n-1}\, dx = dy$

$\Rightarrow x^{n-1}\, dx = \frac{dy}{n}$

$\Gamma(n) = \int_0^\infty e^{-y^{\frac{1}{n}}} \frac{dy}{n}$

$\Rightarrow \boxed{n\Gamma(n) = \int_0^\infty e^{-x^{\frac{1}{n}}}\, dx}$

$\boxed{\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}}$

$\Rightarrow$ put $n = \frac{1}{2}$ $\Rightarrow$ $\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x^2}\, dx$

$\Rightarrow \boxed{\int_0^\infty e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}}$

Relationship between Beta & Gamma function

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Deductions

① $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

Pf

$$B(m,n) = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

or

$$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$\Rightarrow$

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

let $m+n=1$

$\Rightarrow m = 1-n$

Hence

$$\Gamma(1-n)\,\Gamma(n) = \boxed{\int_0^\infty \frac{x^{n-1}}{1+x}\,dx}$$

$$\Rightarrow \boxed{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$$

$$\left\{ \because \int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi} \right\}$$

$\because \Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

(a) If $n = \frac{1}{4}$

$$\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right) = \frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$\Rightarrow \boxed{\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right) = \pi\sqrt{2}}$$

(b) $n = \frac{1}{3}$

$$\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right) = \frac{\pi}{\sin\frac{\pi}{3}}$$

$$\therefore \boxed{\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}}$$

(2) $\because$

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\Rightarrow \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} = \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Let $2m-1 = p$, $2n-1 = q$

$$\Rightarrow m = \frac{p+1}{2}, \quad n = \frac{q+1}{2}$$

Hence

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\,\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}}$$

(a) If $p = 0 = q$

$$\frac{\pi}{\cancel{2}} = \frac{1}{\cancel{2}}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

$$\Rightarrow \pi = \left\{\Gamma\left(\tfrac{1}{2}\right)\right\}^2 \Rightarrow \boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

$\cancel{\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta}$

$\cancel{\int_0^{\frac{\pi}{2}} \sqrt{\tan}}$

(b) If $q = 0$ then

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right) \times \sqrt{\pi}}{2\,\Gamma\left(\frac{p+2}{2}\right)}$$

(c) If $p = n$, $q = -n$

$$\int_0^{\frac{\pi}{2}} \sin^n\theta \cos^{-n}\theta \, d\theta = \frac{1}{2}\,\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{\Gamma\left(\frac{2}{2}\right)}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \tan^n\theta \, d\theta = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(1-\left(\frac{1+n}{2}\right)\right)$$

$$= \frac{1}{2}\,\frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi}$$

$$= \frac{1}{2}\,\frac{\pi}{\sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right)}$$

$$= \frac{\pi}{2}\sec\left(\frac{n\pi}{2}\right) \quad \#$$

Legendre's duplication formula:-

$$\Gamma(n)\,\Gamma\left(n+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma(2n)$$

or

$$\boxed{\sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\,\Gamma(n)\,\Gamma\left(n+\tfrac{1}{2}\right)}$$

If $a=0$, $b=1$

$$\int_0^\infty \cos x \; x^{n-1}\,dx = \Gamma(n)\cos\frac{n\pi}{2}$$

and

$$\int_0^\infty \sin x \; x^{n-1}\,dx = \Gamma(n)\sin\frac{n\pi}{2}$$

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Formulae:-

(1) $$\Gamma\left(\tfrac{1}{n}\right)\Gamma\left(\tfrac{2}{n}\right)\times\cdots\times\Gamma\left(\tfrac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$$

(2) $$\int_0^\infty e^{-ax}\cos bx\; x^{n-1}\,dx = \frac{\Gamma(n)\cos n\theta}{(a^2+b^2)^{n/2}}$$

$$\int_0^\infty e^{-ax}\sin bx\; x^{n-1}\,dx = \frac{\Gamma(n)\sin n\theta}{(a^2+b^2)^{n/2}}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right).$$

Q Evaluate

(a) $\int_0^\infty \frac{x\,dx}{1+x^6}$ (c) $\int_0^1 \frac{dx}{\sqrt{1-x^n}}$

(b) $\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx$ (d) $\int_0^3 \frac{dx}{\sqrt{3x-x^2}}$

$$\boxed{B(m,n) = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx}$$

$$\rightarrow \quad = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$$

Sol:

ⓐ $I = \int_0^\infty \frac{x\,dx}{1+x^6}$

let $x^6 = t$

$x = t^{\frac{1}{6}}$

$dx = \frac{1}{6} t^{\frac{1}{6}-1}\,dt$

$$\therefore I = \int_0^\infty \frac{t^{\frac{1}{6}}\ \frac{1}{6}\ t^{\frac{1}{6}-1}\,dt}{1+t} = \frac{1}{6}\int_0^\infty \frac{t^{\frac{1}{3}-1}}{(1+t)^1}\,dt$$

$$= \frac{1}{6}\int_0^\infty \frac{t^{\frac{1}{3}-1}}{(1+t)^{\frac{1}{3}+\frac{2}{3}}}\,dt$$

$n = \frac{1}{3}$

$m+n = 1$

$m = 1 - \frac{1}{3}$

$= \frac{2}{3}$

$$= \frac{1}{6}\, B\left(\frac{1}{3}, \frac{2}{3}\right)$$

$$= \frac{1}{6}\, \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)}$$

$$= \frac{1}{6}\, \Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$$

$$= \frac{1}{6}\, \frac{\pi}{\sin\frac{\pi}{3}} = \frac{1}{6} \times \frac{2\pi}{\sqrt{3}}$$

$$= \frac{\pi}{3\sqrt{3}} \text{ Ans.}$$

(ii) $I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$

let $x^n = t$

$\Rightarrow x = t^{\frac{1}{n}}$

$dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

$$\therefore I = \int_0^1 \frac{\frac{1}{n} t^{\frac{1}{n}-1} dt}{(1-t)^{\frac{1}{2}}}$$

$$= \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1} (1-t)^{\frac{1}{2}-1} dt$$

$$= \frac{1}{n}\, B\left(\frac{1}{n}, \frac{1}{2}\right)$$

$$= \frac{1}{n}\, \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} \; \#$$

(a) $\int_0^\infty \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$, $c>1$.

$$\int_0^\infty e^{-x} x^{n-1} dx = \Gamma(n)$$

$$\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$$

Sol:-

$$I = \int_0^\infty \frac{x^c}{c^x} dx \quad \int_0^\infty c^{-x}\, x^c\, dx$$

$$\{x = e^{\log x}\}$$

$$= \int_0^\infty e^{\log c^{-x}}\, x^c\, dx$$

$$= \int_0^\infty e^{-(\log c)x}\, x^c\, dx$$

$$= \int_0^\infty e^{-(\log c)x}\, x^{c+1-1}\, dx$$

$$= \frac{\Gamma(c+1)}{(\log c)^{c+1}} \quad \left\{\because \int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}\right\}$$

Prove that

Q $\beta(m,m) = 2^{1-2m}\,\beta(m,\tfrac{1}{2})$

sol: R.H.S

$$= 2^{1-2m}\,\beta(m,\tfrac{1}{2})$$

$$= 2^{1-2m}\,\frac{\Gamma(m)\,\Gamma(\tfrac{1}{2})}{\Gamma(m+\tfrac{1}{2})}$$

$$= 2^{1-2m}\,\frac{\{\Gamma(m)\}^2\sqrt{\pi}}{\Gamma(m)\,\Gamma(m+\tfrac{1}{2})}$$

$$\left\{\because\ 2^{2n-1}\,\Gamma(n)\,\Gamma(n+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2n)\right.$$

$$= 2^{1-2m}\,\frac{\{\Gamma(m)\}^2\cancel{\sqrt{\pi}}}{\cancel{\sqrt{\pi}}\,\Gamma(2m)} \times 2^{2m-1}$$

$$= \frac{\Gamma(m)\times\Gamma(m)}{\Gamma(m+m)} = \beta(m,m)$$

$= L.H.S$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(m,n)$$

Evaluate

Q

(i) $I = \int_0^2 x\,(8-x^3)^{\frac{1}{3}}\,dx$

(ii) $I = \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx$

Sol:-

(i) $I = \int_0^2 x\,(8-x^3)^{\frac{1}{3}}\,dx$

let $x^3 = 8t$

$\Rightarrow x = (8t)^{\frac{1}{3}}$

$= 2t^{\frac{1}{3}}$

$dx = \frac{2}{3}\,t^{\frac{1}{3}-1}\,dt$

$\therefore$

$I = \int_0^1 2t^{\frac{1}{3}}\,(8-8t)^{\frac{1}{3}}\,\frac{2}{3}\,t^{\frac{1}{3}-1}\,dt$

$= \frac{4}{3}\times 2\int_0^1 t^{\frac{2}{3}-1}\,(1-t)^{\frac{4}{3}-1}\,dt$

$= \frac{8}{3}\,B\left(\frac{2}{3},\frac{4}{3}\right)$

$= \frac{8}{3}\,\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{6}{3}\right)}$

$= \frac{8}{3}\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)$

$= \frac{8}{3}\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(1+\frac{1}{3}\right)$

$= \frac{8}{9}\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)$

$$= \frac{8}{9}\,\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$$

$$= \frac{8}{9}\;\frac{\pi}{\sin\frac{\pi}{3}} = \frac{16\pi}{9\sqrt{3}}$$ Ans.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Show that $\displaystyle\int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\,dx = B(p,q)$

Sol:- $\displaystyle B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$

$$= \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\,dx$$

$$= \int_0^1 \frac{x^{p-1}}{(1+x)^{p+q}}\,dx + \int_1^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\,dx$$

$$= I_1 + I_2$$

Now $I_2 = \int_1^\infty \frac{x^{p-1}}{(1+x)^{p+q}} dx$

Let $x = \frac{1}{z}$ $\left(\Rightarrow z = \frac{1}{x} \right)$

$dx = \frac{-1}{z^2} dz$

$\therefore I_2 = \int_1^0 \frac{\left(\frac{1}{z}\right)^{p-1}}{\left(1+\frac{1}{z}\right)^{p+q}} \times \frac{-1}{z^2} dz$

$= \int_0^1 \frac{z^{p+q}}{z^{p-1}(z+1)^{p+q}} \times \frac{1}{z^2} dz$

$= \int_0^1 \frac{z^{q-1}}{(1+z)^{p+q}} dz = \int_0^1 \frac{x^{q-1}}{(1+x)^{p+q}} dx$

$\therefore$

$B(p,q) = I_1 + I_2$

$= \int_0^1 \frac{x^{p-1}}{(1+x)^{p+q}} dx + \int_0^1 \frac{x^{q-1}}{(1+x)^{p+q}} dx$

$= \int_0^1 \left(\frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}} \right) dx$

Ans

Q Evaluate

(a) $\int_0^\infty \cos x^2 \, dx$

(b) $\int_0^1 \log \Gamma(x) \, dx$

Sol: Sol:- (a) $I = \int_0^\infty \cos x^2 \, dx$

let $x^2 = t$

$\Rightarrow x = t^{\frac{1}{2}}$

$dx = \frac{1}{2} t^{\frac{1}{2}-1} dt$

$\therefore \quad I = \int_0^\infty \frac{1}{2} t^{\frac{1}{2}-1} \cos t \, dt$

$\therefore$

$I = \frac{1}{2} \int_0^\infty \cos t \; t^{\frac{1}{2}-1} dt$

$= \frac{1}{2} \Gamma(\tfrac{1}{2}) \cos \frac{\pi}{4}$

$\left\{ \because \int_0^\infty \cos x \; x^{n-1} dx = \Gamma(n) \cos \frac{n\pi}{2} \right\}$

$= \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{\sqrt{2}}$

$= \frac{1}{2}\sqrt{\frac{\pi}{2}}$ #

$\boxed{\cancel{\int_0^1 \log x\,dx} \quad \int_0^{\frac{\pi}{2}} \log \sin x\,dx \quad \int_0^1 \cancel{\log \sin x\,dx}}$

(b) $I = \int_0^1 \log \Gamma(x)\,dx$

or

$I = \int_0^1 \log \Gamma(1-x)\,dx \quad \left\{ \because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \right\}$

$$2I = \int_0^1 \{ \log \Gamma(x) + \log \Gamma(1-x) \}\,dx$$

$$= \int_0^1 \log(\Gamma(x)\,\Gamma(1-x))\,dx$$

$$= \int_0^1 \log\left(\frac{\pi}{\sin \pi x} \right) dx$$

$$= \int_0^1 \log \pi\,dx - \int_0^1 \log(\sin \pi x)\,dx$$

$$= \log \pi - \int_0^1 \log(\sin \pi x)\,dx \quad \text{— (1)}$$

If $\quad I_1 = \int_0^1 \log(\sin \pi x)\, dx$

$$\left\{ \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx \text{ if } f(2a-x) = f(x) \right\}$$

put $\pi x = t$

$\pi\, dx = dt$

$$\therefore I_1 = \int_0^{\pi} \log \sin t \,\frac{dt}{\pi}$$

$$= \frac{1}{\pi}\int_0^{\pi} \log \sin t\, dt$$

$$= \frac{1}{\pi}\int_0^{2\times\frac{\pi}{2}} \log \sin t\, dt$$

$$= \frac{2}{\pi}\underbrace{\int_0^{\frac{\pi}{2}} \log \sin t\, dt}$$

$$\left\{ \because \sin(\pi - t) = \sin t \right\}$$

$$= \frac{2}{\pi}\times\left(-\frac{\pi}{2}\log 2\right) = -\log 2$$

$\cancel{\int_0^\infty \frac{du\, y^{m-1}}{(1+u)^{m+n}}}$

Hence by ①

$$2I = \log\pi + \log 2$$

$$= \log(2\pi)$$

$$\boxed{I = \frac{1}{2}\log 2\pi}$$

$x - a = t$

$(b-(a+t))^n$

$(b-a-t)$

$t = (b-a)u$

Show that

(a) $\int_a^b (x-a)^m (b-x)^n\,dx = (b-a)^{m+n+1}\beta(m+1, n+1)$

(b) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{(a\cos^4\theta + b\sin^4\theta)^{1/2}} = \frac{\{\Gamma(\frac{1}{4})\}^2}{4(ab)^{\frac{1}{4}}\sqrt{\pi}}$

$\int \frac{\sec^2\theta}{(a+b\tan^4\theta)^{\frac{1}{2}}} \rightarrow b\tan^4\theta = au$

Q Evaluate

(a) $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

(b) $\frac{\Gamma(\frac{1}{3})\Gamma(5/6)}{\Gamma(\frac{2}{3})}$ $\left\{ 2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2}) = \sqrt{\pi}\,\Gamma(2n) \right.$

(c) $\int_0^1 \frac{dx}{\sqrt{-\log x}} \rightarrow (\log x = t)$

(d) $\Gamma(\frac{3}{2}-p)\Gamma(\frac{3}{2}+p)$

(e) $\int_0^1 x^m (\log x)^n\,dx \rightarrow (\log x = -t)$

(f) $\int_0^\infty x^n e^{-a^2x^2}\,dx$

(a)

$$I = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

$$= I_1 \times I_2$$

where $I_1 = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}}$

let $x^4 = t$

$\Rightarrow x = t^{\frac{1}{4}}$

$dx = \frac{1}{4} t^{\frac{1}{4}-1} dt$

$$\therefore I_1 = \int_0^1 \frac{t^{\frac{1}{2}} \frac{1}{4} t^{\frac{1}{4}-1} dt}{\sqrt{1-t}}$$

$$\therefore I_1 = \frac{1}{4} \int_0^1 t^{\frac{3}{4}-1} (1-t)^{\frac{1}{2}-1} dt$$

$$= \frac{1}{4} B\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \frac{\Gamma(3/4)\Gamma(\frac{1}{2})}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}$$

$$= \frac{1}{4} \frac{\Gamma(3/4)\sqrt{\pi}}{\Gamma(5/4)}$$

$$= \frac{1}{4} \frac{\Gamma(3/4) \times \sqrt{\pi}}{\Gamma\left(1+\frac{1}{4}\right)}$$

$$= \cancel{\frac{1}{4}} \frac{\Gamma(3/4)\sqrt{\pi}}{\cancel{\frac{1}{4}}\Gamma(1/4)} = \frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(\frac{1}{4})}$$

Now

$$I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

let $x^2 = \tan\theta$

$\Rightarrow x = \tan^{\frac{1}{2}}\theta$

$$dx = \frac{1}{2}\tan^{-\frac{1}{2}}\theta \sec^2\theta \, d\theta$$

$$\therefore I_2 = \int_0^{\frac{\pi}{4}} \frac{\frac{1}{2}\tan^{-\frac{1}{2}}\theta \sec^2\theta \, d\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\tan^{-\frac{1}{2}}\theta \cancel{\sec^2\theta}}{\cancel{\sec\theta}} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\cos^{\frac{1}{2}}\theta}{\sin^{\frac{1}{2}}\theta} \times \frac{1}{\cos\theta}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\frac{\sin 2\theta}{2}}}$$

let $2\theta = t$

$$d\theta = \frac{dt}{2}$$

$$\therefore I_2 = \frac{\sqrt{2}}{2\times 2}\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{\sin t}}$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}} t \cos^0 t \, dt$$

$$= \frac{1}{2\sqrt{2}} \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\times\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)}$$

$$= \frac{1}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}$$

$\therefore$

$$I = I_1 \times I_2$$

$$= \frac{\pi}{4\sqrt{2}}$$

(e) $I = \int_0^1 x^m (\log x)^n \, dx$

let $\log x = -t$

$\Rightarrow x = e^{-t}$

$dx = -e^{-t} dt$

$\therefore I = \int_\infty^0 (e^{-t})^m (-t)^n \, -e^{-t} dt$

$= \int_0^\infty e^{-mt} e^{-t} (-1)^n t^n dt$

$\therefore$

$I = (-1)^n \int_0^\infty e^{-(1+m)t} t^{n+1-1} dt$

$= (-1)^n \dfrac{\Gamma(n+1)}{(1+m)^{n+1}}$ Ans.

$\left\{ \because \int_0^\infty e^{-kx} x^{n-1} dx = \dfrac{\Gamma(n)}{k^n} \right\}$

(c) $I = \int_0^1 \frac{dx}{\sqrt{-\log x}}$

Let $-\log x = t$

$\Rightarrow x = e^{-t}$

$\Rightarrow dx = -e^{-t}\,dt$

$\therefore I = \int_\infty^0 \frac{-e^{-t}\,dt}{\sqrt{t}}$

$= \int_0^\infty e^{-t}\, t^{\frac{1}{2}-1}\, dt$

$= \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\frac{3}{2} - p > 0$

$3 > 2p$

$-3 < 2p < 3$

$-1 < 2p < 1$

(d)

$I = \Gamma\left(\frac{3}{2} - p\right)\Gamma\left(\frac{3}{2} + p\right)$

$= \Gamma\left(1 + \frac{1}{2} - p\right)\Gamma\left(1 + \frac{1}{2} + p\right)$

$= \left(\frac{1}{2} - p\right)\Gamma\left(\frac{1}{2} - p\right)\left(\frac{1}{2} + p\right)\Gamma\left(\frac{1}{2} + p\right)$

$= \left(\frac{1}{4} - p^2\right)\Gamma\left(\frac{1}{2} - p\right)\Gamma\left(\frac{1}{2} + p\right)$

$= \left(\frac{1}{4} - p^2\right)\Gamma\left(\frac{1}{2} - p\right)\Gamma\left(1 - \left(\frac{1}{2} - p\right)\right)$

$= \left(\frac{1}{4} - p^2\right)\frac{\pi}{\sin\left(\frac{1}{2} - p\right)\pi}$

$\left\{\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right\}$

$$= \left(\frac{1}{4} - p^2\right) \frac{\pi}{\cos p\pi}$$

$$= \left(\frac{1}{4} - p^2\right) \pi \sec p\pi$$

Multiple integral

$\int_a^b f(x)\,dx$ | Change of order

Double integral

The integral $\iint_R f(x,y)\,dx\,dy$ is known as the double integral of $f(x,y)$ over the region R.

case i :- $\int_a^b \int_{x=\phi(y)}^{\psi(y)} f(x,y)\,dx\,dy$

To evaluate this integral we first integral w.r.t 'x' and then w.r.t 'y' i.e.,

$$I = \int_a^b \left[\int_{\phi(y)}^{\psi(x)} f(x,y)\,dx \right] dy$$

Case 2:- $I = \int_a^b \int_{y=\phi(x)}^{\psi(x)} f(x,y)\,dy\,dx$

Here we first integrate w.r.t 'y' and then w.r.t 'x'

$$I = \int_a^b \left[\int_{\phi(x)}^{\psi(x)} f(x,y)\,dy \right] dx$$

Case 3:- $I = \int_{x=a}^b \int_c^d f(x,y)\,dy\,dx$

In this case we can integrate first w.r.t 'x' then w.r.t 'y' or vice-versa.

Q. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

$\int \frac{dx \sec\theta\, d\theta}{\sec\theta}$

Sol:- $I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

$= \int_0^1 \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{(\sqrt{1+x^2})^2+y^2} \right] dx$

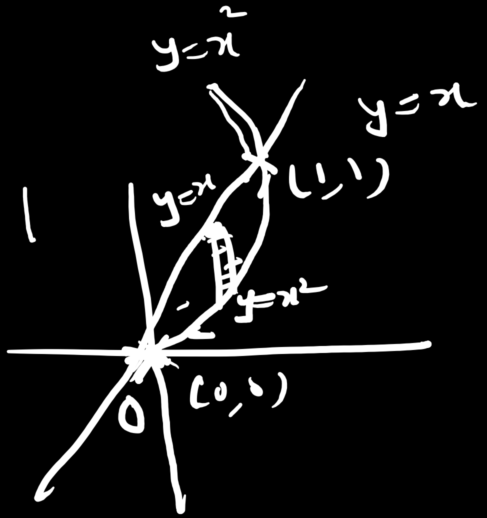
$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right) \right]_{y=0}^{\sqrt{1+x^2}} dx$

$= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}} = \frac{\pi}{4} \left[\log |x+\sqrt{1+x^2}| \right]_0^1$

$= \frac{\pi}{4} \log(1+\sqrt{2})$ Ans.

Q Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ and $y=x$.

Sol^n.



$$I=\iint xy(x+y)\,dx\,dy$$

$$=\int_{x=0}^{1}\left[\int_{y=x^2}^{y=x} xy(x+y)\,dy\right]dx$$

$$=\int_{x=0}^{1}\left[\int_{y=x^2}^{y=x}(x^2y+xy^2)\,dy\right]dx$$

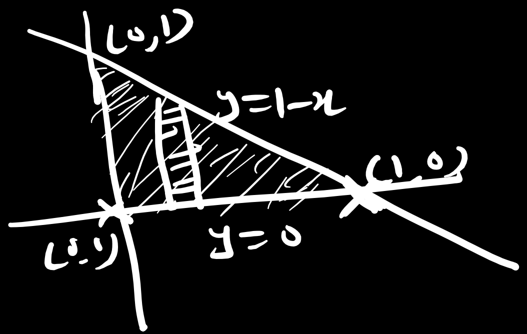
$$=\int_{x=0}^{1}\left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_{y=x^2}^{y=x}dx$$

$$=\int_0^1\left[\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right]dx$$

$$=\frac{1}{10}+\frac{1}{15}-\frac{1}{14}-\frac{1}{24}$$

$$=\frac{3}{56}$$

Q evaluate $\iint xy\,dx\,dy$ over the + ve quadrant of $x+y \le 1$



Sol:

$$I = \iint xy\,dx\,dy$$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{1-x} xy\,dy \right] dx$$

$$= \int_{x=0}^{1} \left[\frac{xy^2}{2} \right]_{y=0}^{1-x} dx$$

$$= \frac{1}{2} \int_{x=0}^{1} x(1-x)^2\,dx$$

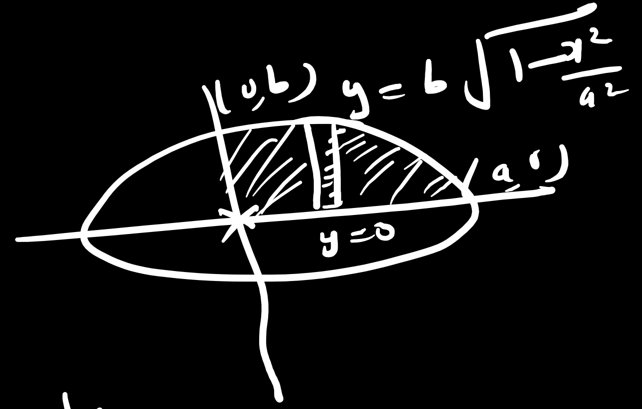
$$= \frac{1}{2} \int_{x=0}^{1} x(1+x^2-2x)\,dx = \frac{1}{2} \int_{x=0}^{1} (x+x^3-2x^2)\,dx$$

$$\frac{6+3-8}{12}$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) = \frac{1}{24}$$

Ans.

Q Evaluate $\iint_R \left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) dx\, dy$ over the +ve quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Sol:

$$I=\iint_R \left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) dx\, dy$$

$$=\int_0^a \left[\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) dy\right] dx$$

$$=\frac{\pi ab}{4}$$

Q Evaluate $\iint y \, dx\, dy$ over the part of the plane bounded by the lines $y = x$ and the parabola $y = 4x - x^2$

sol: Intersection points are given by

$$x = 4x - x^2$$

$$\Rightarrow 3x - x^2 = 0$$

$$\Rightarrow x(3 - x) = 0$$

$$\Rightarrow x = 0, 3$$

$$\therefore y = 0, 3$$

$\therefore$ $(0,0)$ & $(3,3)$ are two intersection pts.

$\because y = 4x - x^2$

$$x^2 - 4x = -y$$

$$x^2 - 4x + 4 = -(y - 4)$$

$$(x-2)^2 = -(y-4)$$

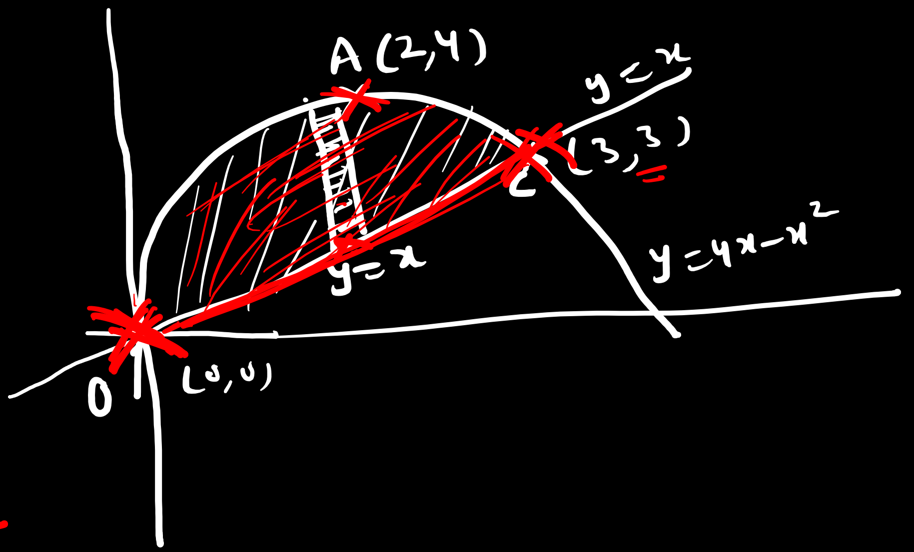
$$\Rightarrow \boxed{X^2 = -4AY}$$

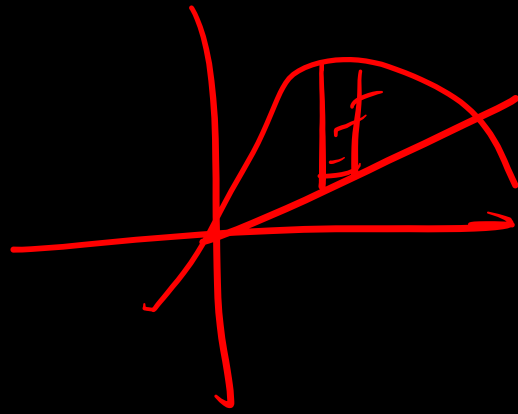
vertex $(X, Y) = (0, 0)$

$$\Rightarrow x = 2, \; y = 4$$

$\Rightarrow (2,4)$ is the vertex.

$$\to \iint dx\,dy$$





$$\therefore\ I = \iint y\,dx\,dy$$

$$= \int_{x=0}^{3}\left[\int_{y=x}^{4x-x^2} y\,dy\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{3}\left[(4x-x^2)^2 - x^2\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{3}\left[16x^2+x^4-8x^3-x^2\right]dx$$

$$= \frac{1}{2}\left[15\times\frac{9}{3}+\frac{1}{5}\times\cancel{32}-\frac{8}{4}\times\cancel{16}\right] = \frac{54}{5}\ \text{Ans}$$

Q Solve

$$I = \int_0^{\pi} \int_0^{a\sin\theta} r\,dr\,d\theta$$

Sol:-
$$I = \int_0^{\pi} \int_{r=0}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} \sin^2\theta\,d\theta$$

$$= \frac{a^2}{4} \int_0^{\pi} (1-\cos 2\theta)\,d\theta$$

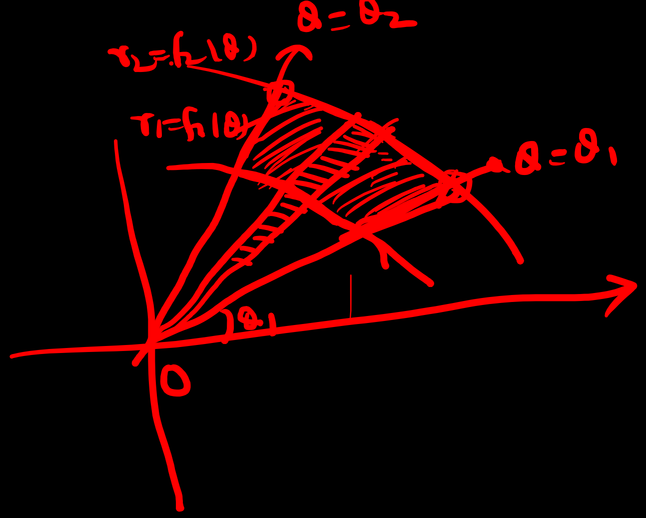
$$= \frac{a^2}{4} \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi}$$

$$= \frac{\pi a^2}{4}.$$

The double integral

$$\int_{\theta=\theta_1}^{\theta_2} \int_{r_1=f_1(\theta)}^{r_2=f_2(\theta)} f(r,\theta)\,dr\,d\theta$$

represents the double integral in polar co-ordinates.



Ans:- (2) Evaluate $\iint_A r^2\,dr\,d\theta$ where A is the area between $r = a\cos\theta$ and $r = 2a\cos\theta$.

Sol: $r = a\cos\theta$ is the circle with centre $(a/2, 0)$ and radius is $\frac{a}{2}$

and $r = 2a\cos\theta$ is the the circle with centre $(a, 0)$ and radius a

Intersection is given by

$$r = a\cos\theta, \quad r = 2a\cos\theta$$

$$\Rightarrow a\cos\theta = 2a\cos\theta$$

$$\Rightarrow a\cos\theta = 0$$

$$\Rightarrow \cos\theta = 0$$

$$\Rightarrow \theta = \pi/2, \; r = 0$$

$r = a\cos\theta$

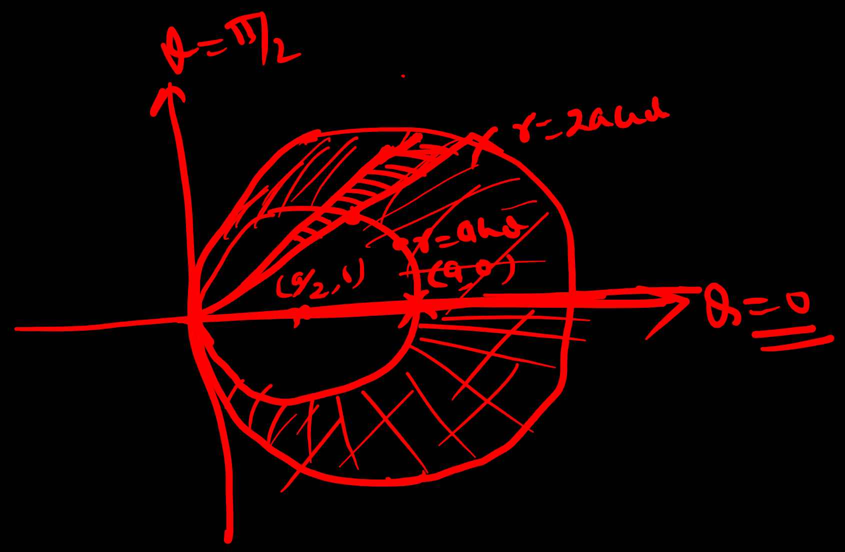
$\left\{ \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right.$

$r^2 = a\cos\theta\, r$

$r^2 = ax$

$x^2 - ax + y^2 = 0$

centre: $(a/2, 0)$



$$I = \iint_A r^2 \, dr \, d\theta$$

$$= 2 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=a\cos\theta}^{r=2a\cos\theta} r^2 \, dr \, d\theta$$

$$= \frac{2}{3} \int_0^{\frac{\pi}{2}} \left[r^3 \right]_{a\cos\theta}^{2a\cos\theta}$$

$$= \frac{2}{3} \int_0^{\frac{\pi}{2}} \left[8a^3 \cos^3\theta - a^3 \cos^3\theta \right] d\theta$$

$$= \frac{14a^3}{3} \int_0^{\frac{\pi}{2}} \cos^3\theta \, \sin^0\theta \, d\theta$$

$$= \frac{14a^3}{3} \; \frac{\Gamma\left(\frac{3+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{5}{2}\right)}$$

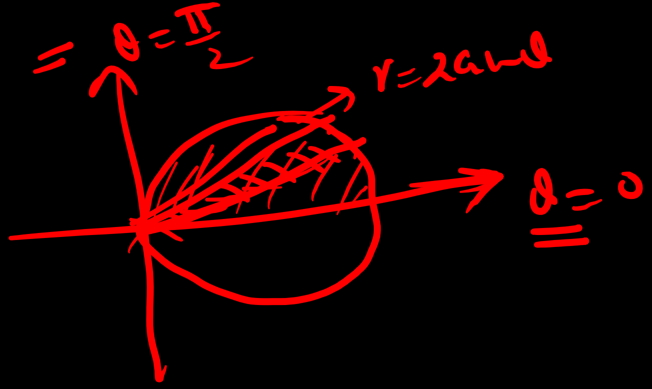
$$= \frac{14a^3}{3} \; \frac{\cancel{\sqrt{\pi}}}{\cancel{2} \times \frac{3}{\cancel{2}} \times \frac{1}{2} \cancel{\sqrt{\pi}}}$$

$$= \frac{28a^3}{9} \; \#$$

Q Show that $\iint_R r^2 \sin\theta \, dr \, d\theta = \frac{2a^3}{3}$

where R is the region bounded by the semi-circle $r = 2a\cos\theta$ and above the initial line.

Sol: $r = 2a\cos\theta$ is the circle with centre $(a, 0)$ and radius is a



$$\cancel{*}$$
$$r = 2a\cos\theta$$
$$r^2 = 2a\, r\cos\theta$$
$$r^2 = 2ax$$
$$x^2 + y^2 - 2ax = 0$$

$$I = \iint_R r^2 \sin\theta \, dr \, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2a\cos\theta} r^2 \sin\theta \, dr \, d\theta$$

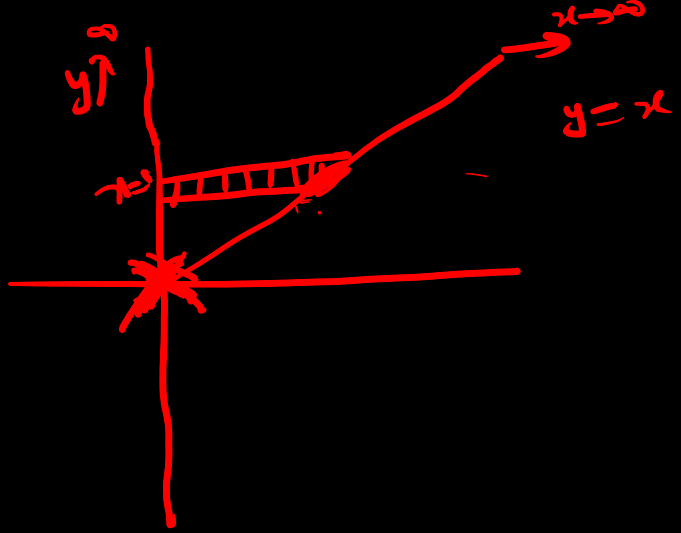
↳ solve it.

$\cancel{\int \frac{e^{-y}}{y} dy}$

$y=x$ to $y=\infty$

Solve:- $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

Sol: Here $y=x$ to $y=\infty$



Change of order is

$$I=\int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_{y=0}^{\infty} \left[\frac{e^{-y}}{y}\, x\right]_{x=0}^{y} dy$$

$$= \int_{y=0}^{\infty} e^{-y}\, dy = -\left[e^{-y}\right]_{y=0}^{\infty}$$

$$= 1$$

Q change the order of integrals in

$$I = \int_{0}^{a} \int_{\frac{y^2}{a}}^{y} f(x, y)\, dx\, dy$$

sol: given $x = \frac{y^2}{a}$ to $x = y$

$\Rightarrow y^2 = ax$ to $x = y$

Intersection pts are

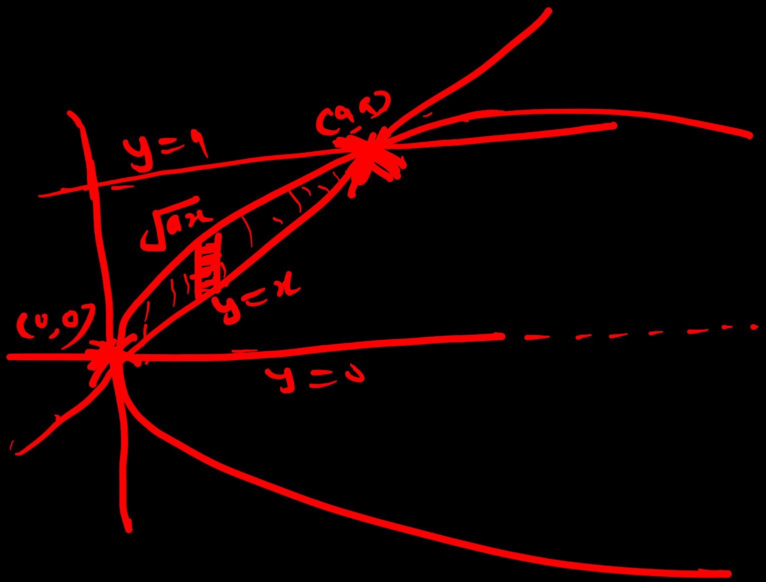
$y^2 = ay$

$\Rightarrow y(y-a) = 0$

$y = 0,\ y = a$

$\therefore x = y = 0, a$

$\therefore (0,0)$ & (a, a) are two intersection pts



change of order is

$$I = \int_{x=0}^{a} \int_{y=x}^{\sqrt{ax}} f(x, y)\, dy\, dx$$

#

Q Change the order of integration in $I = \int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x,y)\,dy\,dx$

Sol:- Given $y = \frac{x^2}{a}$ to $y = 2a - x$

Intersection pts are given by

$$\frac{x^2}{a} = 2a - x$$

$$\Rightarrow x^2 + ax - 2a^2 = 0$$

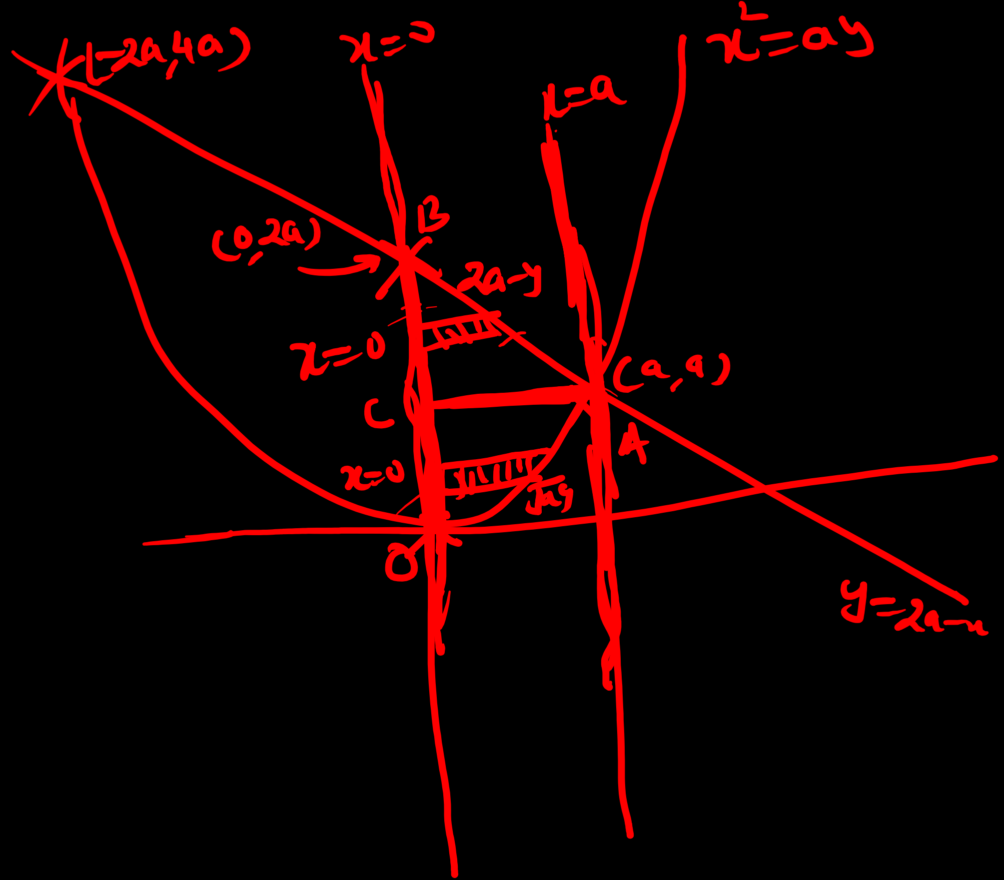
$$\Rightarrow x^2 + 2ax - ax - 2a^2 = 0$$

$$\Rightarrow (x+2a)(x-a) = 0$$

$$\Rightarrow x = a, -2a$$

$$y = 2a - x = a, 4a$$

$\therefore$ intersection pts are (a,a) & $(-2a, 4a)$.



Change of order is given by

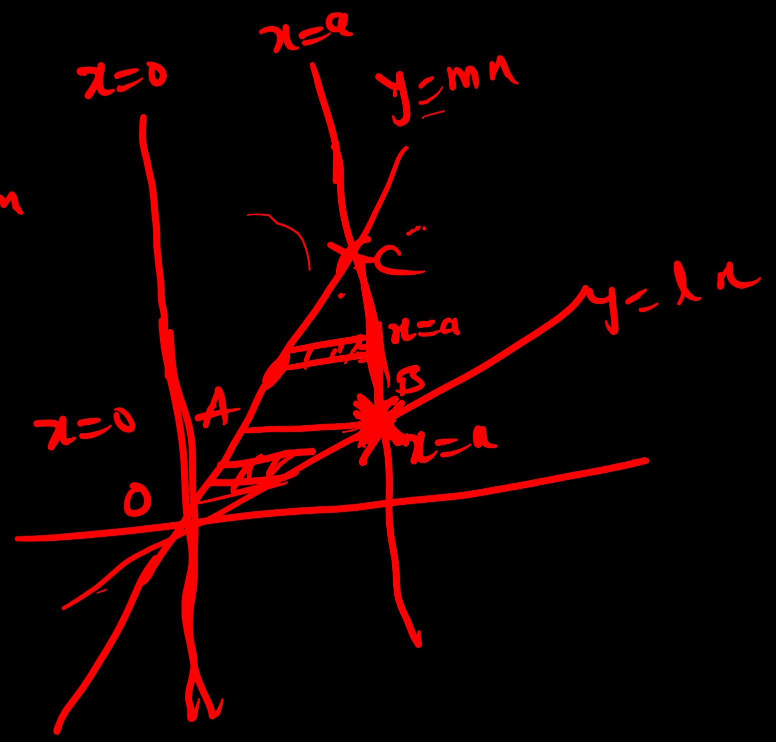
$$I = \iint_{OAC} f(x,y)\,dx\,dy + \iint_{ABC} f(x,y)\,dx\,dy$$

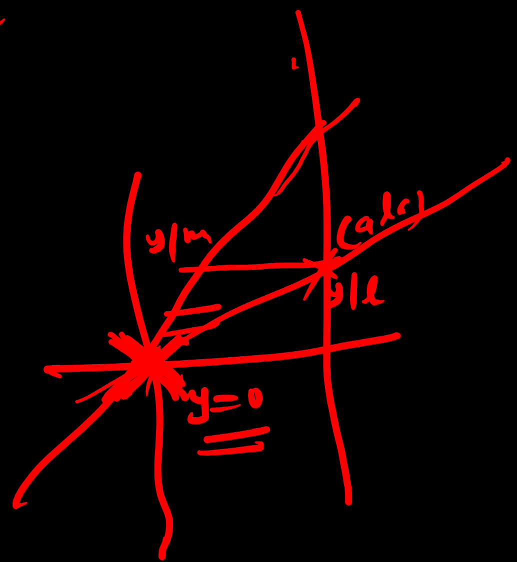
$$= \int_{y=0}^{y=a} \int_{x=0}^{\sqrt{ay}} f(x,y)\,dx\,dy + \int_{y=a}^{2a} \int_{x=0}^{x=2a-y} f(x,y)\,dx\,dy$$

#

Q) Change the order of integrals in

$$I = \int_{x=0}^{a} \int_{y=lx}^{mx} V(x,y)\,dy\,dx$$





Sol: Change of order is given by

$$I = \iint_{OAB} V(x,y)\,dx\,dy + \iint_{ABC} V(x,y)\,dx\,dy$$

$$= \int_{0}^{al} \int_{y/m}^{y/l} V(x,y)\,dx\,dy + \int_{al}^{am} \int_{y/m}^{x=a} V(x,y)\,dx\,dy$$

Change the order of integrals in

Q

① $\int_0^{2a}\int_0^{\frac{x^2}{4a}} f(x,y)\,dx\,dy$

② $\int_0^{2}\int_{\frac{x^2}{4}}^{3-x} xy\,dx\,dy$ and solve it

③ $\int_0^{\infty}\int_0^{x} x\,e^{\frac{-x^2}{y}}\,dy\,dx$ and solve it

④ $\int_0^{\infty}\int_0^{y} y\,e^{\frac{-y^2}{x}}\,dx\,dy$

⑤ $\int_0^{a}\int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2+y^2)\,dy\,dx$

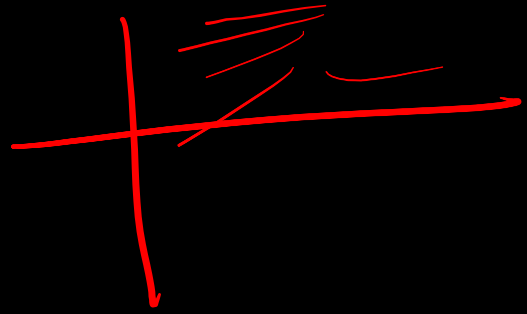
$y=\sqrt{\frac{x}{a}}$ $\quad y^2=\frac{x}{a}$

$\frac{y^2}{x}=t$

$y\,dy=\frac{dt\,x}{2}$

Q change the order of integration in

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{ax}} f(x,y)\,dy\,dx$$



Sol:- Given that $y=\sqrt{ax-x^2}$ to $y=\sqrt{ax}$

$\Rightarrow y^2+x^2-ax=0$, $y^2=ax \to$ parabola.

$\downarrow$

circle with centre $(a/2, 0)$ and radius $\frac{a}{2}$

Intersection pts are given by

$$\cancel{ax}+x^2-\cancel{ax}=0$$

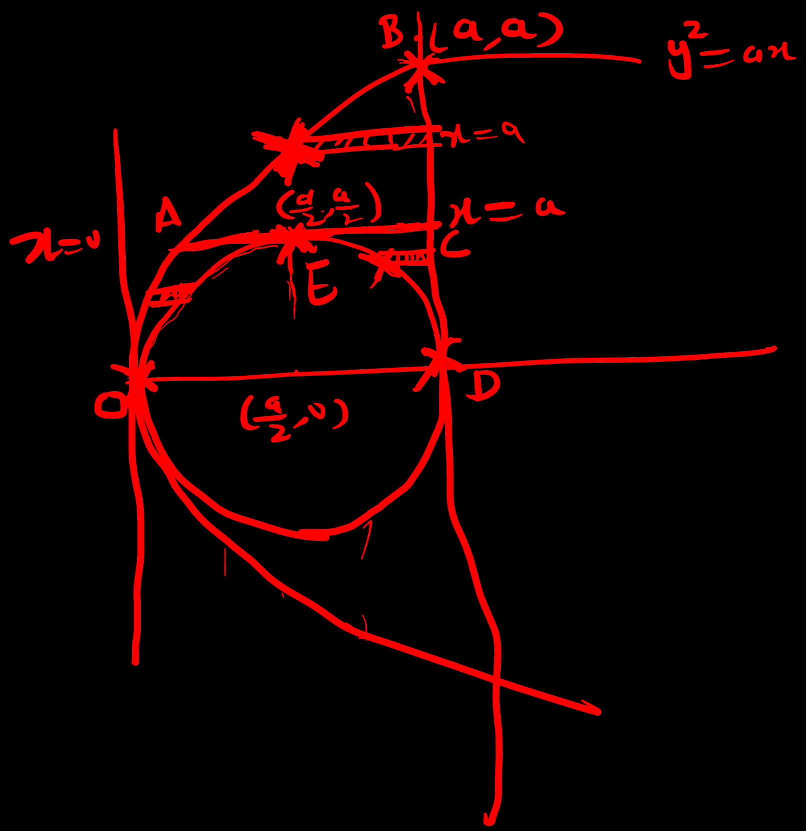
$$x=0 \Rightarrow y=0$$

$\therefore$ $(0,0)$ is the only one intersection pt.

$\therefore \; x^2 - ax + y^2 = 0$

$$x = \frac{a \pm \sqrt{a^2 - 4y^2}}{2}$$

$$= \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4y^2}$$



Change of order of integration is given by

$$I = \iint_{OAE} f(x,y)\,dx\,dy + \iint_{DEC} f(x,y)\,dx\,dy$$

$$+ \iint_{ABC} f(x,y)\,dx\,dy$$

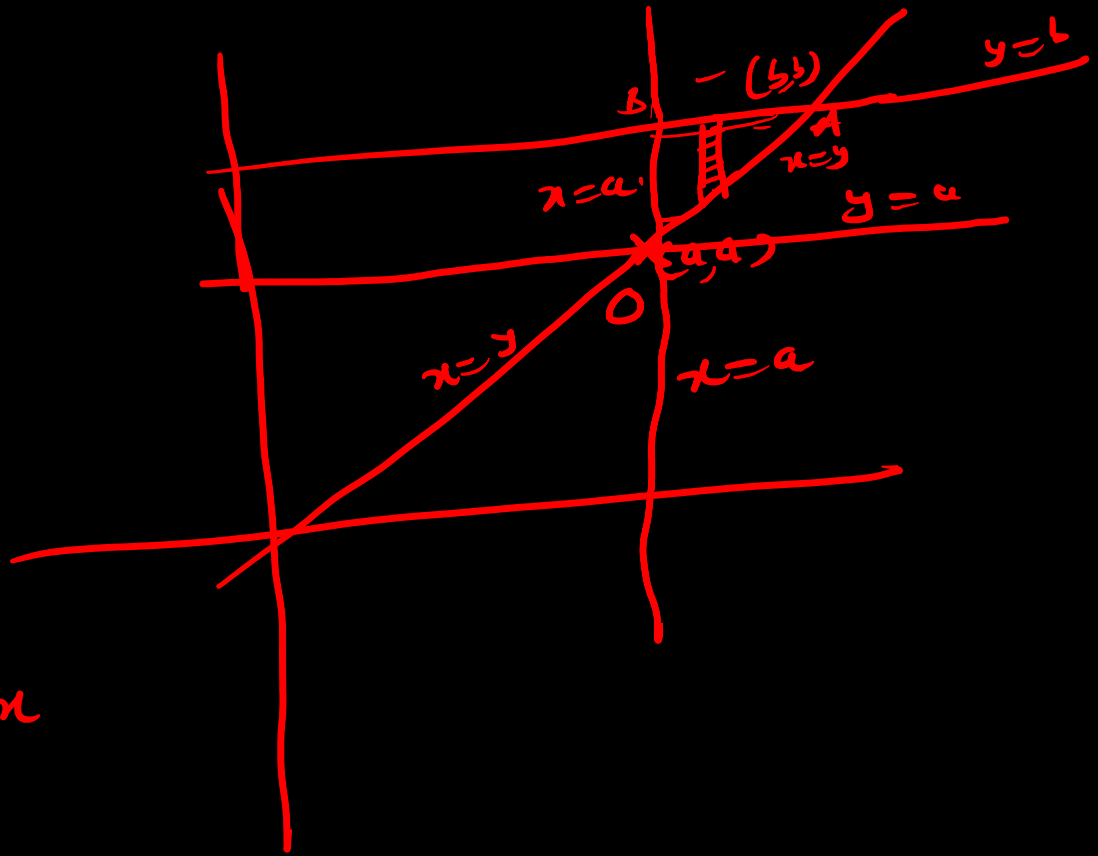
$$= \int_0^{\frac{a}{2}} \int_{\frac{y^2}{a}}^{\frac{a}{2} - \frac{1}{2}\sqrt{a^2 - 4y^2}} f(x,y)\,dx\,dy$$

$$+ \int_0^{\frac{a}{2}} \int_{\frac{a}{2} + \frac{1}{2}\sqrt{a^2 - 4y^2}}^{a} f(x,y)\,dx\,dy$$

$$+ \int_{\frac{a}{2}}^{a} \int_{\frac{y^2}{a}}^{a} f(x,y)\,dx\,dy$$

Q If $n \geq 0$, show that

$$\int_a^b \int_a^y (y-x)^n f(x)\,dx\,dy = \frac{1}{n+1}\int_a^b (b-x)^{n+1} f(x)\,dx$$



Sol: Let $I = \int_a^b \int_a^y (y-x)^n f(x)\,dx\,dy$

Given that $x=a$ to $x=y$

Also $y=a$ to $y=b$

change of order is

$$I = \int_a^b \int_x^b \boxed{(y-x)^n} f(x)\,dy\,dx$$

$$= \int_a^b \left[\int_{y=x}^{b} (y-x)^n \, dy \right] f(x)\, dx$$

$$= \int_a^b \left[\frac{(y-x)^{n+1}}{n+1} \right]_{y=x}^{b} f(x)\, dx$$

$$= \frac{1}{n+1} \int_a^b (b-x)^{n+1} f(x)\, dx$$

#

$\boxed{xy = a}$

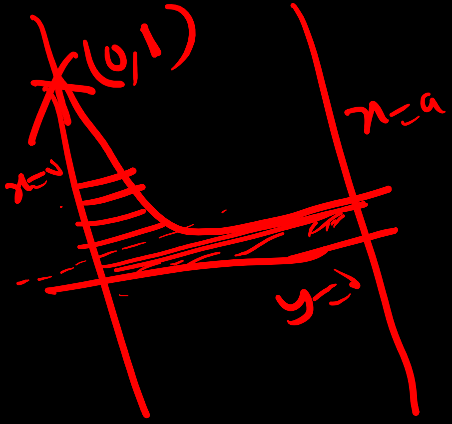
Q. Change the order of integrals in

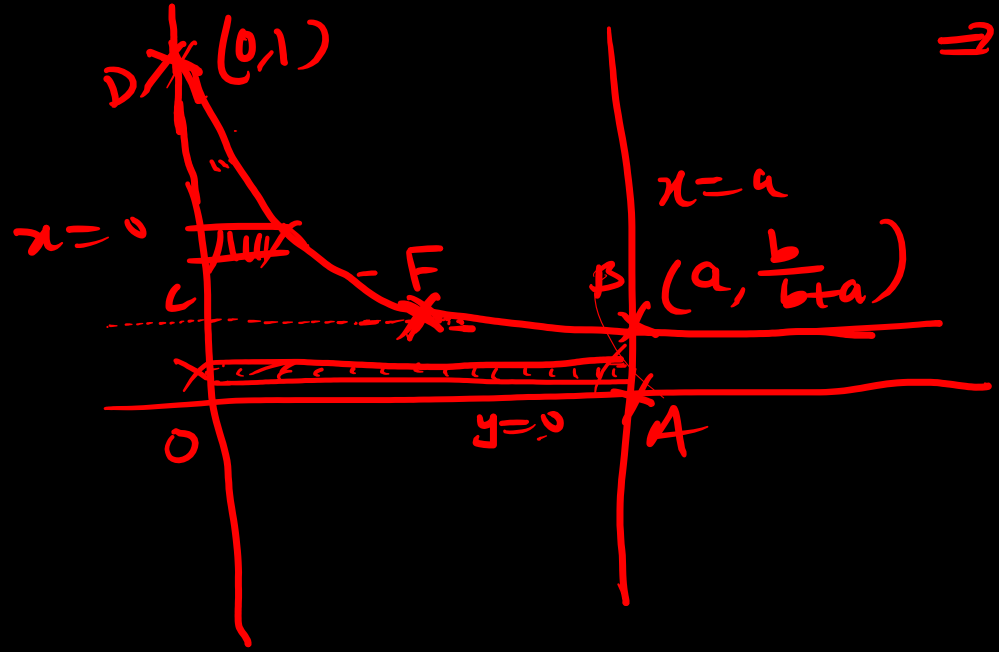
$$I = \int_0^a \int_0^{\frac{b}{b+x}} f(x,y)\,dy\,dx$$

Sol:- Given that $y = 0$ to $\boxed{y = \frac{b}{b+x}}$ $\rightarrow$ $\boxed{y = 1}$

$\Rightarrow b + x = \frac{b}{y} \Rightarrow x = \frac{b}{y} - b$

$\Rightarrow x = b\left(\frac{1-y}{y}\right)$





At $x = 0$

$\Rightarrow \boxed{y = 1}$

Change of order is given by

$$= \iint_{OABFC} f(x,y)\,dx\,dy + \iint_{DCF} f(x,y)\,dx\,dy$$

$$= \int_{\frac{b}{b+a}}^{1} \int_{x=0}^{\frac{b(1-y)}{y}} f(x,y)\,dx\,dy + \int_{0}^{\frac{b}{b+a}} \int_{x=0}^{a} f(x,y)\,dx\,dy$$

#

$$\cancel{z = \sqrt{1-x^2-y^2}}$$

$$\cancel{z^2 + x^2 + y^2 = 1}$$

Triple integral

Consider

$$I = \int_{x=a}^{b} \int_{y=\phi_1(x)}^{\phi_2(x)} \int_{z=f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

This integral is known as triple integral. If $f(x,y,z) = 1$ then the integral $\iiint dx\,dy\,dz$ is known as volume integral.

Q Solve :- (a) $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dx\,dy\,dz$

(b) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, \boxed{dz\,dy\,dx}$.

Change of variable

Consider the double integral

$$\iint_R f(x,y)\,dx\,dy$$

let $x = f_1(u,v)$, $y = f_2(u,v)$

$\boxed{dx\,dy = |J|\,du\,dv}$, where $J = \dfrac{\partial(x,y)}{\partial(u,v)}$

$$\therefore \iint_R f(x,y)\,dx\,dy = \iint_{R_1} F(u,v)\,|J|\,du\,dv$$

In particular,

$x = r\cos\theta$, $y = r\sin\theta$, $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

$$\therefore \boxed{\iint_R f(x,y)\,dx\,dy = \iint_{R_1} F(r,\theta)\,r\,dr\,d\theta}$$

In case of triple integral

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R_1} F(u,v,w)\,|J|\,du\,dv\,dw$$

when $x = f_1(u,v,w)$

$y = f_2(u,v,w)$

$z = f_3(u,v,w)$

Solve the following by changing into polar coordinates

(i) $\displaystyle\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

(ii) $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$

$$\int_0^\infty e^{-x^2}\,dx$$

$x^2 = t$

$x = t^{1/2}$

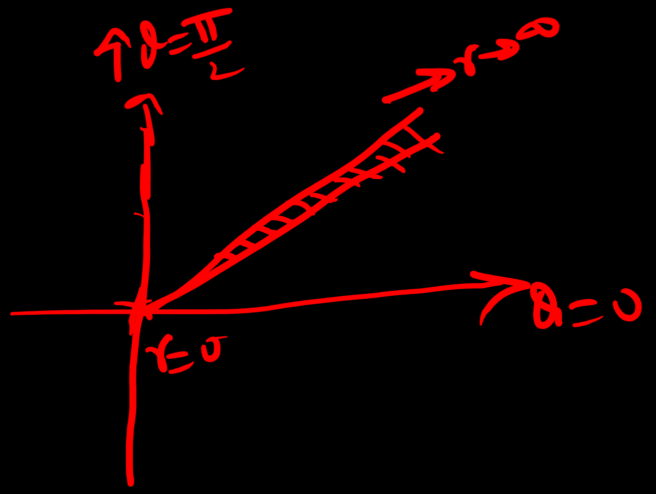
$dx = \frac{1}{2}t^{\frac{1}{2}-1}\,dt$

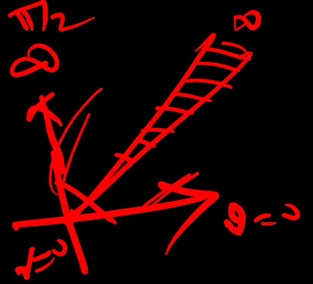
$\frac{1}{2}\int_0^\infty t^{\frac{1}{2}-1}e^{-t}\,dt$

$\frac{1}{2}\Gamma(1/2)$

$= \frac{\sqrt{\pi}}{2}$

Sol: (i)





Let $x = r\cos\theta$, $y = r\sin\theta$ $\Big\} \Rightarrow x^2+y^2 = r^2$

$dx\,dy = |J|\,dr\,d\theta$ where

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$\therefore\ dx\,dy = r\,dr\,d\theta$

$$\therefore \quad I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

$$= \frac{\pi}{2} \int_0^{\infty} e^{-r^2}\, r\, dr$$

let $r^2 = t$

$$\Rightarrow r\,dr = \frac{dt}{2}$$

$$\therefore \quad I = \frac{\pi}{2} \times \frac{1}{2} \int_0^{\infty} e^{-t}\, dt$$

$$= \frac{\pi}{4} \left[-e^{-t} \right]_0^{\infty} = \frac{\pi}{4}$$

(2)

$$I = \int_0^2 \int_0^{\sqrt{2x - x^2}} \frac{x\, dy\, dx}{\sqrt{x^2 + y^2}}$$

Given that

$y = 0$ to $y = \sqrt{2x - x^2}$

$\Rightarrow \underline{y = 0}$, $y^2 + x^2 - 2x = 0$

this is a circle with center $(1, 0)$ and radius 1.

let $x = r\cos\theta$

$y = r\sin\theta$

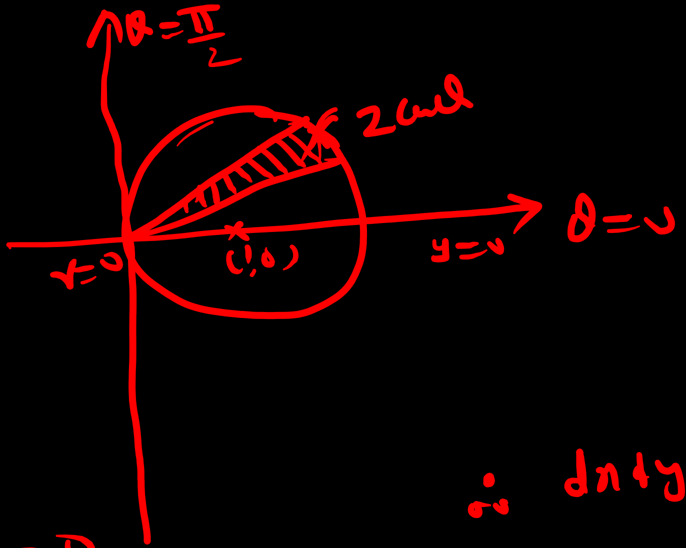
then

$x^2+y^2-2x=0$

$r^2 - 2r\cos\theta = 0$

$\Rightarrow r^2 - 2r\cos\theta = 0$

$\Rightarrow r=0,\ r=2\cos\theta$



$\therefore\ dx\,dy = |J|\,dr\,d\theta$

$= r\,dr\,d\theta$

$$\therefore\ I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2\cos\theta} \frac{r\cos\theta \times r\,dr\,d\theta}{\sqrt{r^2}}$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2\cos\theta} r\cos\theta\,dr\,d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} (2\cos\theta)^2 \cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^3\theta\,\sin^0\theta\,d\theta$$

$$= \cancel{2} \times \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\cancel{2}\,\Gamma\left(\frac{3+0+2}{2}\right)}$$

$$= \frac{\Gamma(1/2)}{\Gamma(5/2)}$$

$$= \frac{\cancel{\sqrt{\pi}}}{\frac{3}{2}\times\frac{1}{2}\times\cancel{\sqrt{\pi}}}$$

$$= \frac{4}{3}.$$

Q Evaluate $\iint_R (x+y)^2\,dx\,dy$ where R is the parallelogram in the xy-plane with vertices $\overset{A}{(1,0)}$, $\overset{B}{(3,1)}$, $\overset{C}{(2,2)}$, $\overset{D}{(0,1)}$ using the transformation $u = x+y$ and $v = x-2y$

Sol:- $dx\,dy = |J|\,du\,dv$ where $J = \dfrac{\partial(x,y)}{\partial(u,v)}$

$= \dfrac{1}{|J'|}\,du\,dv$ $\left\{ \because JJ' = 1,\ J' = \dfrac{\partial(u,v)}{\partial(x,y)} \right\}$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -2-1 = -3$$

$\therefore$ $dx\,dy = \dfrac{1}{|J'|}\,du\,dv = \dfrac{1}{3}\,du\,dv$

$\therefore$ $u = \boxed{x+y}$ $\quad v = x - 2y$

$1-(-2)$

3

At $A(1,0)$

$u = 1, \; v = 1, \; (u,v) = (1,1)$

At $(3,1)$

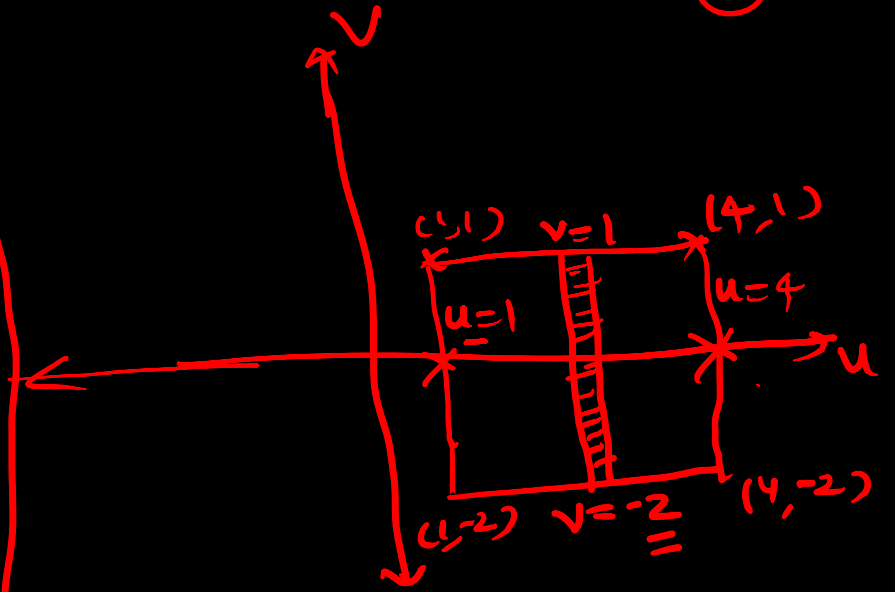
$u = 4, \; v = 1, \; (u,v) = (4,1)$

At $(2,2)$

$u = 4, \; v = -2, \; (u,v) = (4,-2)$

At $(0,1)$

$u = 1, \; v = -2, \; (u,v) = (1,-2)$



$$\therefore I = \iint_R (x+y)^2 \, dx\, dy$$

$$= \int_{u=1}^{4} \int_{v=-2}^{1} u^2 \frac{1}{3} \, du\, dv$$

$$= \int_{u=1}^{4} u^2 du = \frac{1}{3}\left[u^3\right]_{u=1}^{4}$$

$$= \frac{1}{3}\left[4^3 - 1^3\right] = \frac{63}{3} = 21 \; \#.$$

$u = 0$

Q Evaluate $\iint_R (x+y)^2 \, dx\,dy$ when R is the parallelogram bounded by $x+y=0$, $x+y=2$, $3x-2y=0$ and $3x-2y=3$

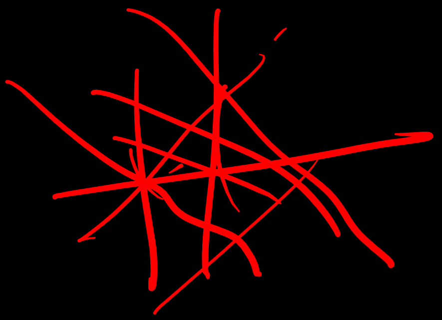
Sol:- If $u = x+y$, $v = 3x-2y$ then $u = 0$ to 2.

$v = 0$ to 3.

$$dx\,dy = |J| \, du\,dv, \quad J = \frac{\partial(x,y)}{\partial(u,v)}$$

$$= \frac{1}{|J'|} \, du\,dv, \quad J' = \frac{1}{J} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -5$$

$$= \frac{1}{5} \, du\,dv$$



So $\iint_R (x+y)^2 \, dx\,dy$

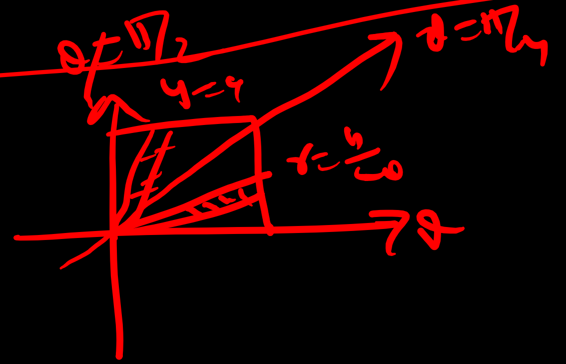
$$= \int_{u=0}^{2}\int_{v=0}^{3} u^2 \times \frac{1}{5} \, du\,dv$$

$$= \frac{1}{5} \times \frac{1}{3} \int_{v=0}^{3} [2]^3 \, dv$$

$$= \frac{8}{15} \times 3 = \frac{8}{5}.$$

$r\cos\theta = 4$

$r = \frac{4}{\cos\theta}$



$r = a$

$r\cos\theta = a$

$$\boxed{r = \frac{a}{\cos\theta}}$$

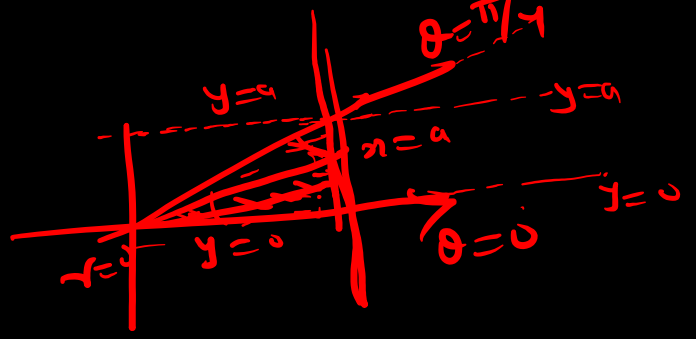
Q

Solve by change of variables

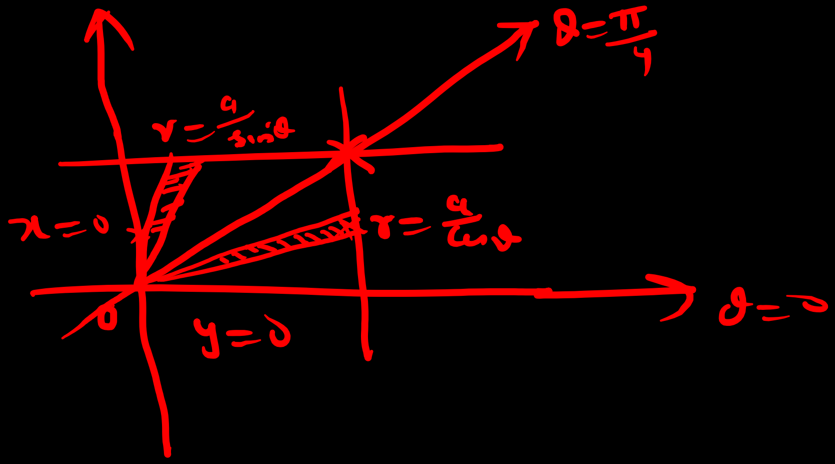
(1) $I = \int_0^a \int_y^a \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$

(2) $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$

(3) $\int_0^a \int_0^a \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$



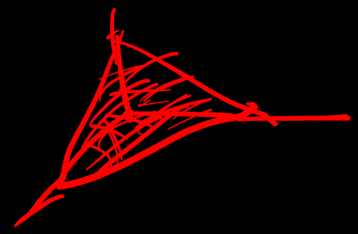
$$I = \int_0^a \int_0^a \frac{x\, dy\, dx}{\sqrt{x^2+y^2}}$$



$$= \boxed{\int_{\theta=0}^{\frac{\pi}{4}} \int_{r=0}^{\frac{a}{\cos\theta}} \frac{r\cos\theta \times r\, dr\, d\theta}{r}}$$

$$+ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{r=0}^{\frac{a}{\sin\theta}} \frac{r\cos\theta \times r\, dr\, d\theta}{r}$$

Dirichlet's Integral



$$\boxed{\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz} = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$$

when V is the region $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x+y+z \leq 1$

$\rho = \cancel{kxyz}$

Note:- $$\iint_D x^{l-1} y^{m-1}\, dx\,dy = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}\, a^{l+m}$$ when D is the domain $x \geq 0$, $y \geq 0$ and $x+y \leq a$.

$$\text{volume integral} = \iiint_V dx\,dy\,dz$$

$$\text{mass} = \iiint \rho \times dx\,dy\,dz\,.$$

$F(x+y+z) = e^{x+y+z}$

$F(u) = e^u$

Liouville's Integral

Let $x \geq 0, y \geq 0, z \geq 0$ such that $h_1 \leq x+y+z \leq h_2$, then

$$\iiint F(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)} \int_{h_1}^{h_2} F(u)\, u^{l+m+n-1}\, du$$

Q Find the volume and mass contained in the solid region in the first octant of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, if the density at any pt. is $\rho = kxyz$.

Sol:- Given that $x \geq 0, y \geq 0, z \geq 0$

Let $\frac{x^2}{a^2} = u$, $\frac{y^2}{b^2} = v$, $\frac{z^2}{c^2} = w$

$\Rightarrow x = a u^{\frac{1}{2}}$, $y = b v^{\frac{1}{2}}$, $z = c w^{\frac{1}{2}}$

$dx = \frac{a}{2} u^{\frac{1}{2}-1}\, du$, $dy = \frac{b}{2} v^{\frac{1}{2}-1}\, dv$, $dz = \frac{c}{2} w^{\frac{1}{2}-1}\, dw$

$\therefore\ u \geq 0,\ v \geq 0,\ w \geq 0,\quad u+v+w=1$

$$\text{Volume} = \iiint_V dx\,dy\,dz$$

$$= \iiint_V \frac{a}{2} u^{\frac{1}{2}-1}\,du\ \frac{b}{2} v^{\frac{1}{2}-1}\,dv\ \frac{c}{2} w^{\frac{1}{2}-1}\,dw$$

$$= \frac{abc}{8} \iiint_V u^{\frac{1}{2}-1}\ v^{\frac{1}{2}-1}\ w^{\frac{1}{2}-1}\ du\,dv\,dw$$

$$= \frac{abc}{8}\ \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1)} = \frac{abc}{8} \times \frac{\pi \times \sqrt{\pi}}{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}} = \frac{\pi abc}{6}.$$

$a=b=c=r$

$x^2+y^2+z^2=r^2$

$\downarrow$

$V = 8 \times \frac{\pi}{6} \times r^3$

$= \frac{4}{3}\pi r^3$ #

$$\text{mass} = \iiint \rho \, dx\,dy\,dz$$

$$= k \iiint xyz \, dx\,dy\,dz$$

$$= k \iiint a u^{\frac{1}{2}} \, b v^{\frac{1}{2}} \, c w^{\frac{1}{2}} \, \frac{a}{2} u^{\frac{1}{2}-1} \, \frac{b}{2} v^{\frac{1}{2}-1} \, \frac{c}{2} w^{\frac{1}{2}-1} \, du\,dv\,dw$$

$$= \frac{k a^2 b^2 c^2}{8} \iiint u^{1-1} v^{1-1} w^{1-1} \, du\,dv\,dw$$

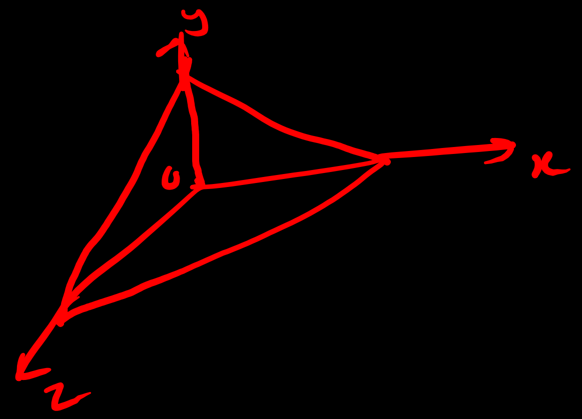
$$= \frac{k a^2 b^2 c^2}{8} \times \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{k a^2 b^2 c^2}{48}.$$

Q. Find the volume of the solid $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$

given that $x \geq 0$, $y \geq 0$, $z \geq 0$.

Q. Find the volume of the solid bounded by the coordinate planes and the surface $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=1$

Q. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B & C. Find the volume of tetrahedron OABC and mass, if the density $\rho = kxyz$.



Sol: let $\frac{x}{a}=u$, $\frac{y}{b}=v$, $\frac{z}{c}=w$

$\Rightarrow x=au,\ y=bv,\ z=cw$

$\Rightarrow dx=a\,du,\ dy=b\,dv,\ dz=c\,dw$

$$\therefore V=\iiint dx\,dy\,dz = abc\iiint du\,dv\,dw$$

$$= abc\iiint u^{1-1}v^{1-1}w^{1-1}\,du\,dv\,dw$$

$$= abc\,\frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{6}.$$

$$\text{mass} = \iiint \rho \, dx\,dy\,dz$$

$$= k \iiint xyz \, dx\,dy\,dz$$

$$= k \iiint au \; bv \; cw \; a\,du \; b\,dv \; c\,dw$$

$$= k a^2 b^2 c^2 \iiint uvw \, du\,dv\,dw$$

$$= k a^2 b^2 c^2 \iiint u^{2-1} \, v^{2-1} \, w^{2-1} \, du\,dv\,dw$$

$$= k a^2 b^2 c^2 \; \frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2+1)} = \frac{k a^2 b^2 c^2}{720} \;\#$$

$6 \times 5 \times 4 \times 3 \times 2 \times 1$

Solve:- $\iiint e^{x+y+z}\,dx\,dy\,dz$, taken over the +ve octant such that $x+y+z\le 1$

Sol:-

$$I = \iiint e^{x+y+z}\,dx\,dy\,dz$$

Given that $x\ge 0,\ y\ge 0,\ z\ge 0,\quad 0\le x+y+z\le 1$

$F(u)$, $u = x+y+z$

$$I = \iiint e^{x+y+z}\, x^{1-1}\, y^{1-1}\, z^{1-1}\,dx\,dy\,dz$$

$$= \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 e^u\, u^{1+1+1-1}\,du$$

$$= \frac{1}{2}\int_0^1 e^u u^2\,du = \frac{1}{2}\left[e^u u^2 - e^u \times 2u + e^u \times 2\right]_0^1$$

$$= \frac{1}{2}\left[e - \cancel{2e} + \cancel{2e} - 2\right] = \frac{e}{2} - 1 \ \#$$

Eg show that $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^2} = \frac{3}{4} - \log 2$, the integral being taken throughout the volume bounded by $x=0, y=0, z=0, x+y+z=1$

Sol:- $I = \iiint \frac{dx\,dy\,dz}{(x+y+z+1)^2}$, $x \geq 0, y \geq 0, z \geq 0$, $0 \leq x+y+z \leq 1$

$$= \iiint \frac{x^{1-1} y^{1-1} z^{1-1}}{(x+y+z+1)^2}\, dx\,dy\,dz$$

$$= \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(3)} \int_0^1 \frac{1}{(u+1)^2} u^{3-1}\, du$$

$$= \frac{1}{2} \int_0^1 \frac{u^2}{(u+1)^2}\, du$$

let $u+1=t$

$du = dt$

$$= \frac{1}{2} \int_1^2 \frac{(t-1)^2}{t^2}\, dt = \frac{1}{2} \int_1^2 \left(1 + \frac{1}{t^2} - \frac{2}{t}\right) dt$$

$$= \frac{1}{2}\left[t - \frac{1}{t} - 2\log t\right]_1^2$$

$$= \frac{1}{2}\left[2 - \frac{1}{2} - 2\log 2\right]$$

$$= \frac{1}{2}\left[\frac{3}{2} - 2\log 2\right]$$

$$= \frac{3}{4} - \log 2$$

#

13 Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$ where the integral being extended to all +ve values of x, y & z for which the expression is real.

Sol: Given that $x \geq 0, y \geq 0, z \geq 0$ such that $x^2+y^2+z^2 < a^2$

$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{a^2} < 1$

$\therefore I = \iiint \frac{dx\,dy\,dz}{\sqrt{a^2-(x^2+y^2+z^2)}}$

$= \frac{1}{a}\iiint \frac{dx\,dy\,dz}{\sqrt{1-\left(\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{a^2}\right)}}$

$= \frac{1}{a}\iiint \frac{\frac{a}{2} u^{\frac{1}{2}-1} \frac{a}{2} v^{\frac{1}{2}-1} \frac{a}{2} w^{\frac{1}{2}-1} du\,dv\,dw}{\sqrt{1-(u+v+w)}}$

Let $\frac{x^2}{a^2}=u, \frac{y^2}{a^2}=v, \frac{z^2}{a^2}=w$

$\Rightarrow x = a u^{\frac{1}{2}}, y = a v^{\frac{1}{2}}, z = a w^{\frac{1}{2}}$

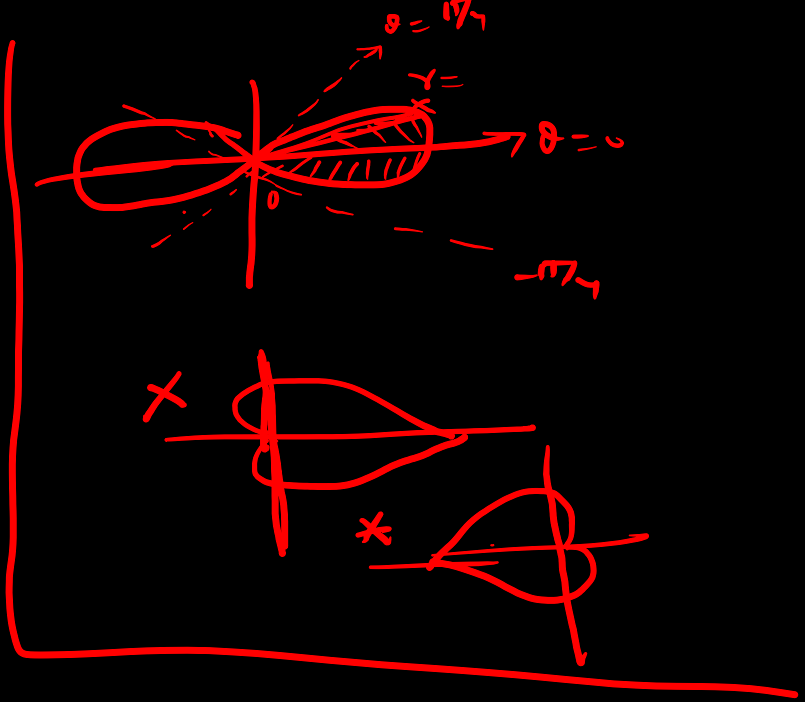
$dx = \frac{a}{2} u^{\frac{1}{2}-1} du, dy = \frac{a}{2} v^{\frac{1}{2}-1} dv$

$dz = \frac{a}{2} w^{\frac{1}{2}-1} dw$

and $u \geq 0, v \geq 0, w \geq 0$ such that

$0 \leq u+v+w < 1$

$$= \frac{a^2}{8}\iiint u^{\frac{1}{2}-1}\, v^{\frac{1}{2}-1}\, w^{\frac{1}{2}-1}\, \frac{1}{\sqrt{1-(u+v+w)}}\, du\,dv\,dw$$



$$= \frac{a^2}{8} \times \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})} \int_0^1 \frac{1}{\sqrt{1-t}}\, t^{\frac{3}{2}-1}\, dt$$

$$= \frac{\pi a^2}{4} \times \int_0^1 t^{\frac{3}{2}-1}\,(1-t)^{\frac{1}{2}-1}\, dt$$

$$= \frac{\pi a^2}{4}\, B\left(\frac{3}{2}, \frac{1}{2}\right)$$

$$= \frac{\pi a^2}{4}\, \frac{\Gamma(3/2)\,\Gamma(1/2)}{\Gamma(\frac{3}{2}+\frac{1}{2})} = \frac{\pi a^2}{4} \times \frac{1}{2} \times \sqrt{\pi} \times \sqrt{\pi} = \frac{\pi^2 a^2}{8}$$ #

Unit III (Multiple Integral)

- **L1: https://youtu.be/EUV1kpKS24c**
- **L2: https://youtu.be/zY9yf1N5Vbs**
- **L3: https://youtu.be/TabX9LzpBvM**
- **L4: https://youtu.be/HNNLmzOP8zc**
- **L5: https://youtu.be/KeXEzY1S7ic**
- **L6: https://youtu.be/pYE9HdvTj8s**
- **L7: https://youtu.be/McJFWZVvBvw**
- **L8: https://youtu.be/Kppi5PA0ppw**
- **L9: https://youtu.be/CDgDHLqaCWk**