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UNIT-3 Combinators

Counting Techniques

Basic counting Techniques

(1) Permutation $^nP_r = n!/(n-r)!$

(2) Combination $^nC_r = n!/r!(n-r)!$

Theorem:

$$^nP_n = \frac{\lfloor n}{\lfloor n-n} = n!$$

Theorem:

$$\lfloor 0 = 1$$

Proof We know $^nP_r = \frac{n!}{(n-r)!}$

Put $n=r$

$$^nP_n = \frac{\lfloor n}{\lfloor 0} \Rightarrow \lfloor n = \frac{\lfloor n}{\lfloor 0}$$

$$\lfloor 0 = \frac{\lfloor n}{\lfloor n}$$

$$\lfloor 0 = 1.$$

$$^{12}P_4 = ?$$

$$\frac{(12)!}{(12-4)!} = \frac{12!}{8!} = 12 \times 11 \times 10 \times 9$$

$$= 11880$$

$$^{20}P_r = 6840 \qquad r = ?$$

$$\frac{20!}{(20-r)!} = 6840$$

$$r = 3$$

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$r = 4$

$$^nP_r = 20 \times {}^nP_2 \qquad n = ?$$

$$\frac{n!}{(n-r)!} = 20 \frac{n!}{(n-2)!}$$

$$\frac{(n-2)!}{(n-r)!} = 20$$

$$\frac{(n-2)!}{(n-4)!} = 20$$

$$\frac{(n-2)(n-3)\cancel{(n-4)!}}{\cancel{(n-4)!}} = 20$$

$$n^2 - 2n - 3n + 6 - 20 = 0$$

$$n^2 - 5n - 14 = 0$$

$$n^2 - 7n + 2n - 14 = 0$$

$$(n-7)(n+2) = 0$$

$$n = 7$$

Prove that $^nP_r = {}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$

$$\frac{(n-1)!}{(n-1-r)!} + r \times \frac{(n-1)!}{(n-r-1-1)!}$$

$$\frac{(n-1)!}{(n-r-2)!} \left(\frac{1}{(n-r-1)} + r \right)$$

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1.1

Permutation with Repetetion

→ Out of 'n' objects 'P' objects are of one kind 'q' objects are of second types and 'r' objects are of third kind and remaining objects are different. Then the number of permutation of n objects taken all at a time $\frac{n!}{(P)!\,q!\,r!}$

→ The number of permutation of 'n' different objects taken 'r' at a time when each objects may be repeated then, the ~~per~~ no of permutation is equal to n^r

~~ex Ways to ar~~

Ways to arrange the letters in

'ALLAHABAD'

$n = 9$; $p = 4$; $q = 2$

$$\text{ways} = \frac{9!}{4!\,2!} = 7560$$

Find the numb of different match that can be represented by sequences of 4 dashes and three dots?

$n = 7$; $p = 4$; $q = 3$

$$\frac{7!}{4!\,3!} = \frac{7 \times \cancel{6} \times 5}{3 \times \cancel{2}} = \underline{\underline{35}}$$

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Circular Permutation

The number of circular permutation of n diffrent object is $(n-1)!$

Theorem: Show no. of ways in which n diffrent beads can be arranged to form necklace is $\frac{(n-1)!}{2}$

Proof: Fixed the position of one wheel the remaining $(n-1)$ wheels can arranged in $(n-1)!$ ways

There is no restrictions for clockwise or anti clockwise so the required no. of ways are $\frac{(n-1)!}{2}$.

COMBINATION

Selection ~~of~~ which can be formed by taking sum of all is called combination

$${}^{n}C_r \left(\begin{matrix} n \\ r \end{matrix}\right)$$

$$\frac{(n)!}{(n-r)!\,(r)!}$$

Diffrence b/w P & C

In combination only a group is made and order does not matter.
In ~~combination~~ permutation not only a grp is formed but also arrangement is considered

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$$\frac{1}{r\,(n-r)!} = \frac{(n+1)}{(n-r+1)\,r}$$

$$\frac{1}{r} = \frac{n+1}{(n-r+1)}$$

$$\frac{1}{\cancel{r}} = \frac{n}{\cancel{r}}$$

$$1 = \frac{n}{}$$

Theorem: If $1 \le r \le n$ t

$$\frac{{}^{n}C_r}{{}^{n}C_{r-1}} = \frac{n}{}$$

$$LHS = \frac{\dfrac{n!}{(n-r)!\,r!}}{\dfrac{n!}{(n-r+1)!\,(r-1)!}} =$$

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Theorem: Let $0 \le r \le n$ Prove that ${}^{n}C_{r} = {}^{n}C_{n-r}$

Theorem:

$${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$$

$${}^{n}C_{r} = {}^{n+1}C_{n-r+1} - {}^{n}C_{n-r+1}$$

$${}^{n}C_{r} =$$

$$\frac{n!}{r!\,(n-r)!} = \frac{(n+1)!}{(n-r+1)!\,r!} - \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$\frac{1}{r\,(n-r)!} = \frac{(n+1)}{(n-r+1)\,r} - \frac{1}{(n-r+1)\,!}$$

$$\frac{1}{r} = \frac{n+1}{(n-r+1)\,r} - \frac{1}{(n-r+1)}$$

$$\frac{1}{\cancel{r}} = \frac{n+1-r}{\cancel{r}(n-r+1)}$$

$$1 = \frac{\cancel{n+1-r}}{\cancel{n+1-r}} = 1$$

Proved

theorem: If $1 \le r \le n$ then

$$\frac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = \frac{n-r+1}{r}$$

$$LHS = \frac{\dfrac{\cancel{n!}}{(n-r)!\,r!}}{\dfrac{\cancel{n!}}{(n-r+1)!\,(r-1)!}} = (r-1)\,\frac{n-r+1}{r}$$

LHS = RHS **Proved**

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Theorem: $\quad {}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

$$\frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r-1)!(r+1)!} = \frac{(n+1)!}{(n-r)!(r+1)!}$$

$$\frac{1}{(n-r)} + \frac{1}{(r+1)} = \frac{(n+1)}{(n-r)\,(r+1)}$$

$$LHS = \frac{\cancel{r}+1+n-\cancel{r}}{(n-r)(r+1)} = \frac{n+1}{(n-r)(r+1)}$$

$\underline{\underline{LHS = RHS}}$ Proved

Theorem $\quad {}^{n}C_{p} = {}^{n}C_{q} \Rightarrow p = q$ or $p+q = n$

$$\frac{n!}{(n-p)!\,p!} = \frac{n!}{(n-q)\,q!}$$

$$\frac{1}{(n-p)!\,p!} = \frac{1}{(n-q)!\,q!}$$

$$\underline{\underline{p = q}}$$

$${}^{n}C_{p} = {}^{n}C_{n-q}$$

for ~~E~~ being equal

$$n-q = p$$

$$\underline{\underline{n = p+q}} \text{ Proved}$$

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$^nP_r = 720$; $^nC_r = 120$ then $r = ?$

$^nP_r = r! \; ^nC_r$

$720 = r! \; 120$

$r! = \frac{\cancel{72}^{6}}{\cancel{12}} = 6$

$r! = 3!$

$\underline{\underline{r = 3}}$

In how many ways can a committee of five members been selected from 6 men and five women. consisting of 3 men and 2 women

$^6C_3 \times {^5C_2} = \frac{6!}{3!\,3!} \times \frac{5!}{3!\,2!}$

$\frac{2\cancel{6}\times\cancel{5}\times4}{\cancel{3}\times\cancel{2}} \times \frac{5\times4}{\cancel{2}}$

$= 10 \times 20$

$\underline{\underline{= 200}}$

Pigeonhole Principle / Drichlet Drewer Principal or Shoe box Principle

If n pegions are assgned to n pigeon holes and $m < n$ then atleast one pegion holes contains two or more pigeons

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Proof: $H_1, H_2, H_3 - - - - H_m$ and pigeon holes and $P_1, P_2, P_3 - - P_n$ are n pigeon to m pigeonholes as follows.

Let P_1 is assigned to H_1
P_2 H_2
P_m H_m

$m<n$ then there are $(n-m)$ pigeons are left.

At least one pigeon-holes will be arranged to a second pigeon.

Ex Show that among 13 people there are atleast ~~4~~ 2 people who were ~~ball~~ born in same month.

We consider 13 peoples as pigeons and 12 months ~~am~~ as pigeonholes (Jan, feb - - dec)

Here no ~~of~~ of pigeons > no of pigeon holes

By pigeon holes Principle.

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Extended / generalize Pigeonhole Principle

Theorem: If n pigeons are assigns to n pigeon holes, then one of the pigeon holes must contain atleast $\left(\left(\frac{n-1}{m}\right)+1\right)$ or $\left(\frac{n}{m}+1\right)$

pigeons $= \left[\frac{p}{q}\right]$ (largest integers) $\leq \frac{p}{q} \in a$

Proof: We can prove this theorem by contradiction. Let, each pigeonhole contain no more

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than $\frac{n-m}{m}$ pigeons

then, there are atmost $m\left(\frac{n-1}{n}\right)$ pigeons

in all but $\frac{n-1}{m} \leq \frac{(n-1)m}{m} \Rightarrow \frac{n-1}{m} < n-1$

i.e. there are $(n-1)$ pegions in all which is contradiction.

Hence one of the pigeon holes must contain atleast $\left(\frac{n-1}{m}+1\right)$ or $\left(\frac{n}{m}+1\right)$ pigeons.

Show that if any 26 ~~pillows~~ peoples are selected then we may choose a subset of four so that all four were ~~ball~~ born on the same day of a week.

Sol^n We assign each person to the ~~to~~ day of the week on which he/she was born

The number of pigeons (are to be assigned to seven pigeon holes with $n=26$ and $m=7$ atleast $\frac{n-1}{m}+1$ or $\frac{n-1}{m}$

$$\frac{n-1}{m}+1 = \frac{26-1}{7}+1 = \frac{25+7}{7}$$

$$= \frac{32}{7}$$

$$= 4.57$$

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Ex: If 9 books are to be kept in 4 ~~selfs~~ shalfs their must be atleast ~~one~~ 1 shalf which contains atleast 3 books

9 books can be considered as pigeon 4 shalfs as pigeonholes.

$n = 9$ $m = 4$

By generalized ~~pieo~~ pigeon hole principle

$$\left(\frac{n-1}{m} + 1\right) = \frac{9-1}{4} + 1 = 3$$

Atleast 1 shalf will contain 3 books

RECURRANCE RELATION

A formulae which defines any term in terms of its previous term is called recurrance relation.

n^{th} term a_n of sequence 3, 8, 13, 18 - - is $\boxed{a_n = a_{n-1} + 5}$ for $n \geq 2$ and $a_1 = 3$

~~Diffrential eq~~^{n}

Sometimes recurrance relation is called differential eq^n as the beginning info a_1 is boundary cond^n.

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The recurrance relation can be written as the even

$f(x+3h) + 3f(x+2h) + 4f(x+h) + 8f(x) = 0$

$a_{n+3} + 3a_{n+2} + 4a_{n+1} + 8 = 0$

$y_{n+3} + 3y_{n+2} + 4y_{n+1} + 8 = 0$

$u_{x+3} + 3u_{x+2} + u_{x+2} + 8 = 0$

Order of reccurance relation

The diffrence b/w highest and lowest subscript.

Eg. $a_n + 3a_{n-1}$, order = 1

$7f(x) + f(x+1) + 5f(x+2) = K(x)$. order = ?
$= 2$

Degree of recurrance relation

Highest power of $f(x)$ or a_n or y_n

Eg. $y_{n+3}^{3} + 2y_{n+2}^{2} + 2y_{n+1} = 0$ degree = ?

degree = 3

$y_{n+4} + y_{n+1} + y_n = 0$

degree = 1

The ~~rel~~ reccurance relation is called
if it is of degree '1'.
quadratic if it is of degree '2'.

The reccurance relation is called homogeneous if it contains no terms and depends only on ~~vale~~ variables.

Homogeneous

$a_n = a_{n-2}^2$ Homogeneous

order = 2 & degree = 2

$a_n = a_{n-1} + n^2$

Non-homogeneous, order = 1, ~~ord~~ degree = 1

Linear ~~rr~~ reccurrance relation with const. coefficient

Reccurance relation with degree '1'

General form:-

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_K a_{n-K} = f(n)$$

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Linear Homogeneous Recurrance relation with const. coeff

If and only if $f(r) = 0$

Sol^n of Homogeneous Recurrance Relation

consider $C_0 a_r + C_1 a_{r-1} + \dots + C_K a_{r-K} = 0$ — (1)

Sol^n of (1) is of the form $A\alpha_1^K$

where α_1 is root of characteristic eq^n

Substituting $a_r = A\alpha^r$, $a_{r-1} = A\alpha^{r-1}$

$C_0 A\alpha^{r-1} + C_1 A\alpha^{r-1} + C_2 A\alpha^{r-1} + \dots + C_K A\alpha^{r-K} = 0$

$A\alpha^{r-K}(C_0\alpha^K + C_1\alpha^{K-1} + C_2\alpha^{K-2} + \dots + C_K)$

$C_0\alpha^K + C_1\alpha^{K-1} + C_2\alpha^{K-2} + \dots + C_K = 0$

which is the characteristic eq^n of reccurance relation

$\therefore$ If α_1 is a root then the characteristic eq^n of K^{th} degree will give K^{th} characteristic

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CHARACTERISTIC EQ^n

Distinct Roots

If $r_1, r_2, r_3 --- r_k$ are distinct roots then general sol^n = a_n

$$a_n = b_1 r_1^n + b_2 r_2^n + b_3 r_3^n + - - + b_k r_k^n$$

where $b_1, b_2, b_3 - - b_k$ are constants

Solve $a_n = a_{n-1} + 2a_{n-2}$, $n \geq 2$ with $a_0 = 0$ and $a_1 = 1$

given $a_n - a_{n-1} - 2a_{n-2} = 0$ — ①

A second ~~pair~~ order home recurrance sol^n with constant coeff.

Let $a_n = r^n$ is a sol^n of ① then

$(r^n - r^{n-1} - 2r^{n-2} = 0)\, r^{2-n}$

$r^2 - r - 2 = 0 \Rightarrow (r-2)(r-1) = 0$

$r = 2, -1$

General sol^n is $a_n = b_1 2^n + b_2 (-1)^n$

for $n = 0$, $a_0 = b_1 + b_2 \Rightarrow b_1 + b_2 = 0$

for $n = 1$, $a_1 = b_1(2) + b_2(-1) \Rightarrow 2K_1 - K_2 = 1$

Solving we get $K_1 = 1/3$ $K_2 = 1/3$

so the sol^n is $\boxed{a_n = \frac{1}{3} 2^n - \frac{1}{3}(-1)^n}$

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Multiple root

If r is a root obsessive eq^n of mth order of a recurrance relation with multiplicity m then general sol^n is

$$a_n = (b_1 + n b_2 + n^2 b_3 + \cdots + n^{m-1} b_m) r^n$$

where $b_1, b_2, b_3, \cdots b_m$ are const.

Ques Solve $a_n = 4(a_{n-1} - a_{n-2})$ with initial cond^n $a_0 = a_1 = 1$

given $a_n - 4a_{n-1} + 4a_{n-2} = 0$
Let $a_n = r^n$ is a ~~root~~ sol^n then

$$(r^n - 4r^{n-1} + 4r^{n-2} = 0)\, r^{2-n}$$

$$r^2 - 4r + 4 = 0 \Rightarrow r = 2, 2$$

general sol^n is $a_n = (b_1 + n b_2) 2^n$
for $n = 0$, $a_0 = (b_1 + 0)\, 2^0 \Rightarrow b_1 = 1$
$n = 1$ $a_1 = (b_1 + b_2)\, 2 \Rightarrow b_1 + 2b_2 = 1$

$$\therefore b_2 = -1/2$$

Now sol^n. $a_n = \left(1 - \frac{n}{2}\right) 2^n$

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Non- Homogeneous Recurrance Relation

A second order non- ~~linear~~ homogeneous linear reference relat^n with const- Coeff is $a_{n-2} + b\, a_{n-1} + c\, a_n = f(n)$

the sol^n of non- homonegenous reccurance relation is

$\Longrightarrow$ sum of particular sol^n and sol^n of the associated linear homogeneous reccurance relation

there is no general method of finding particular sol^n of reccurance relation of function f(n)

this method for finding particular sol^n is called trial sequence method.

$f(n)$	Trial form
b^n (If b is not a root of characteristics eq^n	Ab^n
$\Rightarrow$ Polynomial $P(n)$ · of degree m	$A_0 + + A_1 n + A_2 n^2 + \dots + A_m n^m$.
$c^n P(n)$ · (If c · is not a root of characteristics eq^n)	$c^m (A_0 + A_1 n + A_2 n^2 + \dots + A_m n^m)$

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b^n (If b is not a root of characteristic eq^n of multiplicaty	$A_0 n^r b^n$
$c^n P(n)$. (If c is a root of characteristic eq^n of multiplicity	$n^r(A_0 + A_1 n + A_2 n^2 + \dots)$

If $func^n$ of n is const that is polynomial is zero then trial sol^n is taken as A

If $func^n$ of n is linear combination then trial sol^n is taken as sum of corresponding trial $func^n$ with diffrent unknown constants that are to be determined

Case I:

$f(n) = \text{const}$.

(1) $a_n = P$

(2) $a_n = nP$ (If ① fails)

(3) $a_n = n^2P$ (If ② fails)

Case II:

$f(n)$ is polynomial

i.e. $f(n) = C_0 + C_1 n + C_2 n^2 + \dots + C_m n^m$

(1) Put $a_n = d_0 + d_1 n + d_2 n^2 + \dots + d_m n^m$

(2) Find $d_0, d_1, d_2, \dots d_m$ by companies coefficient

Case III

When $f(x) = Pa^n$

$a_n = d a^n$

Ex Solve $a_{n+2} = 5a_{n+1} + 6a_n = 2$ with initial condition $a_0 = 1$, $a_1 = -1$

Have reccurance relation is

$$a_{n+2} - 5a_{n+1} + 6a_n = 0$$

Let $a_n = r^n$ $\quad (r^{n+2} - 5r^{n+1} + 6r^n = 0) r^{-n}$

$$\Rightarrow r^2 - 5r + 6 = 0$$

$$r = 3, 2$$

Sol^n is $\quad a_n^{(P)} = C_1 3^n + C_2 2^n$

For particular sol^n $\quad a_n^{(P)} = P$ gives the eq^n

$$P - 5P + 6 = 2$$

$$\Rightarrow P = 1 \Rightarrow a_n^{(P)}$$

Hence general sol^n is

$$a_n = C_1 3^n + C_2 2^n + 1$$

$n = 0$ $\quad a_0 = C_1 3^0 + C_2 2^0 + 1 = C_1 + C_2 + 1 = 1$

$$C_1 + C_2 = 0$$

$n = 1$ $\quad a_1 = C_1 3 + C_2 2 + 1 = 3C_1 + 2C_2 + 1 = 1$

$$3C_1 + 2C_2 = -2$$

$C_1 = -2 \quad C_2 = 2$

$$\boxed{a_n = -2 \cdot 3^n + 2 \cdot 2^n + 1}$$

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Generating Func^n

The generating func^n for the sequence $a_0, a_1, a_2 - - - a_K + - - -$ of real no is infinite series $G(x) = a_0 + a_1 x + a_2 x^2 + - - - + a_K x^K +$

$$= \sum_{K=0}^{\infty} a_K x^K$$

Ex

Sequence	Generating Function
$a_K = 2$	$\sum_{K=0}^{\infty} 2x^K$
$a_K = 3^K$	$\sum_{K=0}^{\infty} 3^K x^K$
$a_K = K+1$	$\sum_{K=0}^{\infty} (K+1)(x^K)$

Some special Cases in Generating func^n

General Term of a_K	$G(x)$
(1) 1	$1/1-x$

Sequence $= 1, 1, 1, - - -$

$G(x) = 1 + x + x^2 + - - - + (1-x)^{-1} = \frac{1}{1-x}$

Sequence $= 1, 2, 3 - - -$ $G(x) = 1x + 2x + 3x - - -$

(2) $K+1$ $\quad 1/(1-x)^2$

$G(x) = 1 + 2x + 3x^2 + - - + (K+1)x^K + - - + = (1-x)^{-2} = \frac{1}{(1-x)^2}$

(3) K $\quad \frac{x}{(1-x)^2}$

(4) $K(K+1)$ $\quad \frac{2x}{(1-x)^3}$

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(5) $(K+1)(K+2)$ $\qquad$ $\dfrac{2}{(1-x)^3}$

(6) a^K $\qquad$ $\dfrac{1}{1-ax}$

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