Unit - 3

Numerical Solution of Ordinary Differential E_q^n

$$y' = 1 - 2xy, \quad y(x_0) = y_0$$

$$y' + 2xy = 1$$

$$I.F. = e^{\int 2x\,dx} = e^{x^2}$$

$$y \cdot I.F. = \int_0^1 1 \cdot e^{x^2}\,dx + c$$

$$y e^{x^2} = \int e^{x^2}\,dx + c$$

① Taylor's series exponsion method

$$\left.\begin{aligned} y' &= f(x_0, y_0) \\ y(x_0) &= y_0 \end{aligned}\right\} \text{Initial Value Problem (IVP)}$$

$$y' = f(x, y)$$
$$y(x_0) = y_0$$

$$\boxed{y(x) = y_0 + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2!}\, y''(x_0) + \dots}$$

Q) $y' = x - y^2$ & $y(0) = 1$

$$x_0 = 0, \quad y_0 = 1$$

$$y(0) = 1$$
$$y'(0) = 0 - 1^2 = -1$$
$$y''(0) = 1 - 2yy' = 1 - 2 \times 1 \times -1 = 3$$
$$y'''(0) = 0 - 2(yy'' + y'^2) = -8$$

$$y = 1 + x(-1) + \frac{x^2}{2} \times 3 + \frac{x^3}{3!} \times (-8) + \dots$$

② Picards iterative method

$$y' = f(x,y)$$
$$y(x_0) = y$$

$$y' = f(x,y) \quad -①$$

Integrating ① from 'x_0' to 'x'

$$\int_{x_0}^{x} y' \, dx = \int_{x_0}^{x} f(x,y) \, dx$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x,y) \, dx$$

$$\boxed{y(x) = y(x_0) + \int_{x_0}^{x} f(x,y) \, dx}$$ Integral equation

$$y(x_0) = y_0$$

$$y^{(1)}(x) = y(x_0) + \int_{x_0}^{x} f(x, y_0) \, dx$$

$$y^{(2)}(x) = y(x_0) + \int_{x_0}^{x} f(x, y^{1}) \, dx$$

$$y^{(3)}(x) = y(x_0) + \int_{x_0}^{x} f(x, y^{2}) \, dx$$

$$y^{(n+1)}(x) = y(x_0) + \int_{x_0}^{x} f(x, y^{(n)}) \, dx$$

with $y(x_0) = y_0$

Q) Solve the eqⁿ

$$y' = x + y^2$$

Subject to $y(0) = 1$

using PIT.

A) $$y^{(1)} = y_0 + \int_0^x f(x, y_0)\,dx$$

$$= 1 + \int_0^x (1 + x)\,dx$$

$$= 1 + \left[x + \frac{x^2}{2}\right]_0^x$$

$$= 1 + x + \frac{x^2}{2}$$

$$y^{(2)} = y_0 + \int_0^x f(x, y')\,dx$$

$$= 1 + \int_0^x x + \left(1 + x + \frac{x^2}{2}\right)^2 dx$$

$$= 1 + \int_0^x \left(x + (1+x)^2 + \frac{x^4}{4} + \frac{2(1+x)x^2}{2} \right) dx$$

$$= 1 + \int_0^x \frac{x^4}{4} + x^3 + 2x^2 + 3x + 1$$

$$= 1 + \frac{x^5}{20} + \frac{x^4}{4} + \frac{2x^3}{3} + \frac{3x^2}{2} + x$$

Q) Solve $\frac{dy}{dx} = \frac{x^2}{1+y^2}$, $y(0) = 0$

& find $y(0.25)$, $y(0.5)$, $y(1)$ using PIT

A) $$y^{(1)} = y_0 + \int_0^x f(x, y_0)\,dx$$

$$= 0 + \int_0^x \frac{x^2}{1}\,dx$$

$$= \frac{x^3}{3}$$

$$y^{(2)} = y_0 + \int_0^x f(x, y') \, dx$$

$$= 0 + \int_0^x \left(\frac{x^2}{1 + \frac{x^6}{9}} \right)$$

$$= \int_0^x \frac{9x^2}{9 + x^6} \, dx$$

Let $x^3 = t$

$$y^{(2)} = \tan^{-1}\left(\frac{x^3}{3}\right)$$

$$= \frac{1}{3} x^3 - \frac{1}{81} x^9 + \ldots$$

At 0 to 1, $y^{(2)} \simeq y^{(1)}$

$$y(0.25) = \frac{(0.25)^3}{3}$$

$$y(0.5) = \frac{(0.5)^3}{3}$$

$$y(1) = \frac{(1)^3}{3}$$

*) Euler's method

$$\frac{dy}{dx} = f(x, y)$$

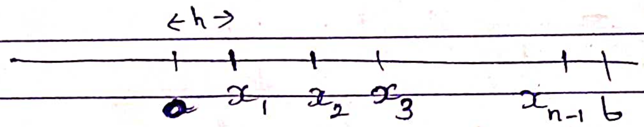
$$y(x_0) = y_0$$

$$y(x_1) = ?$$

$$y = y_0 + \int_0^x f(x, y)\, dx$$

$$f(x, y) = f(x_0, y_0)$$

$$x_0 \le x \le x_1$$



$$\left\{ \begin{array}{l} x_0 = a \\ x_n = b \end{array} \right\}$$

$$y = y_0 + \int_{x_0}^{x_1} f(x_0, y_0)\, dx$$

$$= y_0 + f(x_0, y_0) \int_{x_0}^{x_1} dx$$

$$y_1 = y_0 + h\, f(x_0, y_0) \qquad (\because x_1 - x_0 = h)$$

$$y_2 = y_1 + \int_{x_1}^{x_2} f(x_1, y_1)\, dx$$

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$\vdots$$

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

Q) Using euler's method find $y(0.04)$ for $y' = -y$, $y(0) = 1$

A) $h = \dfrac{b-a}{n}$

$0 \qquad 0.04$

Let $n = 4$,

$h = \dfrac{0.04 - 0}{4}$

$h = 0.01$

Euler's iterative scheme

$$Y_1 = Y_0 + h \, \cancel{f(a,b)} \, f(x_0, Y_0)$$

$$Y_1 = 1 + 0.01 \times (-1)$$

$$= 1 - 0.01$$

$$Y_1 = 0.99 \qquad (Y_1 = y(0.01))$$

$$Y_2 = Y_1 + h f(x_1, Y_1) \qquad (Y_2 = y(0.02))$$

$$Y_2 = 0.99 + 0.01 \times -0.99$$

$$= 0.99 - 0.0099$$

$$= 0.9801$$

$$Y_3 = Y_2 + h f(x_2, Y_2) \qquad (Y_3 = y(0.03))$$

$$= 0.9801 + 0.01 \times -0.9801$$

$$= 0.9801 - 0.009801$$

$$= 0.9703$$

$$Y_4 = 0.9703 - 0.009703 \qquad (Y_4 = y(0.04))$$

$$Y_4 = 0.9606$$

$$\boxed{y(0.04) = Y_4 = 0.9606}$$

*) Modified euler's method

$$\frac{dy}{dx} = f(x_0, y)$$

$$y(x_0) = y_0$$

Ist Iteration:-

i) Predictor

$$y_1 = y_0 + h f(x_0, y_0)$$

ii) Corrector

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$y_1 = -$$

Second iteration

i) Predictor

$$y_2 = y_1 + h f(x, y_1)$$

ii) Corrector

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2)\right]$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

Repeat untill value becomes similar

Q) Using modified euler method, find $y(0.2)$ and $y(0.4)$ given, $y' = y + e^x$, $y(0) = 0$

A) $h = \frac{0.4 - 0}{2} = 0.2$

① First iteration $y(0.2)$

(i) Predictor

$\cancel{y(0.2)} =$ $y_1 = y_0 + h f(x_0, y_0)$

$= 0 + 0.2 \times 1$

$y_1 = 0.2$

(ii) Corrector

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$= 0 + \frac{0.2}{2} \times [1 + 0.2 + 1.22]$$

$$= 0.1 \times 2.42$$

$$= 0.242$$

$$y_1^{(2)} = 0 + \frac{0.2}{2} \times [1 + 0.242 + 1.22]$$

$$= 0.1 \times 2.46$$

$$= 0.246$$

$$y_1^{(3)} = 0 + \frac{0.2}{2}[1 + 0.246 + 1.22]$$

$$= 0 + 0.1 \times 2.466$$

$$= 0.2466$$

$$y_1^{(4)} = 0.1 \times [1 + 0.2466 + 1.22]$$

$$= 0.1 \times 2.4666$$

$$= 0.2466$$

$$\boxed{y_1 = 0.246} = y(0.2)$$

(2) Second iteration $y(0.4)$

i) Predictor

$$y_2 = 0.246 + 0.2 \times [0.246 + e^{0.2}]$$

$$= 0.246 + 0.2 \times [0.246 + 1.221]$$

$$= 0.246 + 0.2 \times \cancel{1.738}\ 1.467$$

$$= 0.246 + \cancel{0.3476}\ 0.2934$$

$$= \cancel{0.5936}\ 0.5392$$

ii) Corrector

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 0.246 + 0.1 \times [0.246 + e^{0.2} + 0.5392 + e^{0.4}]$$

$$= 0.246 + 0.1 \times [0.246 + 1.221 + 0.5392 + 1.4918]$$

$$= 0.246 + 0.1 \times 3.498$$

$$= 0.246 + 0.3498$$

$$= 0.5958$$

$$y_2^{(2)} = 0.246 + 0.1 \times [0.246 + e^{0.2} + 0.5958 + e^{0.4}]$$

$$= 0.6015$$

$$y_2^{(3)} = 0.6020$$

$$y_2^{(4)} = 0.246 + 0.1 \times [0.246 + e^{0.2} + 0.6020 + e^{0.4}]$$

$$= 0.6021$$

$$\boxed{y(0.4) = 0.602}$$

22/11/22

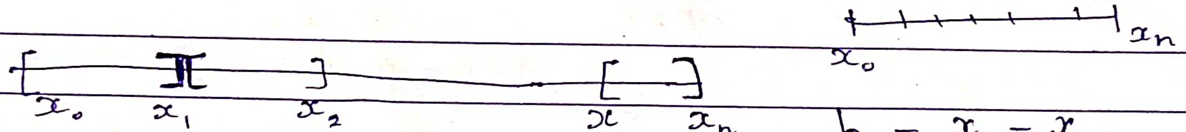
*) Runge-Kutta method of fourth order

$$\left.\begin{aligned} \frac{dy}{dx} &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}\right\} \quad (1)$$

From (1)

$$dy = f(x,y)\,dx$$

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$



$$h = \frac{x_n - x_0}{n}$$

$f(x,y) = f(x_0, y_0)$ in $[x_0, x_1]$

in then subinterval $[x_0, x_1]$

$$y = y_0 + f(x_0, y_0)\,h$$

Here, we define four parameter's which can be calculate successive by

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right)$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

Now, evaluate

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\boxed{y_1 = y_0 + k}$$

Q-1) Given $\frac{dy}{dx} = y - x$

where $y(0) = 2$

Find $y(0.1)$ & $y(0.2)$ using fourth order ~~Ra~~ Range-kutta method

A)

0 0.1 0.2

$h = 0.1$

$\cancel{y = y_0}$ $k_1 = 0.1 \times 2 = 0.2$

$k_2 = 0.1 \times f(0.05, 2.1)$

$= 0.1 \times 2.05$

$= 0.205$

$k_3 = 0.1 \times f(0.05, 2.1025)$

$= 0.1 \times 2.0525$

$= 0.20525$

$k_4 = 0.1 \times f(0.1, 2.20525)$

$= 0.1 \times 2.15525$

$= 0.215525$

$k = \frac{1}{6}(0.2 + 0.41 + 0.41 + 0.2155)$

$= \frac{1}{6} \times (1.2355)$

$= 0.206$

Value of y at $x_1 = 0.1$

$y_1 = y_0 + k = 2 + 0.206$

$= 2.206$

$x_1 = 0.1$, $y_1 = 2.206$

$k_1 = h f(x_1, y_1)$

$= 0.1 \times 2.106 = 0.2106$

$$k_2 = 0.1 \times f(0.15, 2.311)$$
$$= 0.1 \times 2.161$$
$$= 0.21613$$
$$k_3 = 0.1 \times f(0.15, 2.314)$$
$$= 0.1 \times 2.164$$
$$= 0.2164$$
$$k_4 = 0.1 \times f(0.15, 2.4224)$$
$$= 0.1 \times 2.2824$$
$$= 0.2222$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$+ 0.4322$$
$$= \frac{1}{6}(0.206 + 0.432 + 0.222)$$
$$= 0.2163$$

$$y_2 = y_1 + k$$
$$= 2.206 + 0.2163$$
$$= 2.4223$$

$$\boxed{\begin{aligned} y(0.1) &= 2.206 \\ y(0.2) &= 2.4223 \end{aligned}}$$

Q-2) Given $\frac{dy}{dx} = x + y$

$y(0) = 1$

then $y(0.2) = ?$

A) $h = 0.2$

$k_1 = h \times f(x_0, y_0)$

$= 0.2 \times 1$

$= 0.2$

$k_2 = 0.2 \times f(0.1, 1.1)$

$= 0.2 \times 1.2$

$= 0.24$

$k_3 = 0.2 \times f(0.1, 1.12)$

$= 0.2 \times 1.22$

$= 0.244$

$k_4 = 0.2 \times f(0.2, 1.244)$

$= 0.2 \times 1.444$

$= 0.2888$

$k = \frac{1}{6}(0.2 + 0.48 + 0.488 + 0.2888)$

$= 0.2428$

$y_1 = y_0 + k$

$= 1 + 0.2428$

$\boxed{y_1 = 1.2428}$

24/11/22

*) Difference Equation

An eq^n which expresses a relation b/w independent variable & successive difference of dependent variable is called difference eq^n.

$$y(x)$$

$$y_x,\ \Delta y_x,\ \Delta^2 y_x \ldots,\ \nabla y_x,\ \nabla^2 y_x \ldots$$

E.g. $$\boxed{\Delta^2 y_x + \Delta y_x = x^2 + 2x + 1}$$

$$\Delta = E - 1$$

$$(E-1)^2 y_x + (E-1) y_x = x^2 + 2x + 1$$

$$(E^2 - 2E + 1 + E - 1) y_x = x^2 + 2x + 1$$

$$\boxed{(E^2 - E) y_x = x^2 + 2x + 1} \Rightarrow \text{Standard form}$$

$$E^2 y_x - E y_x = x^2 + 2x + 1$$

$$\boxed{y_{x+2} - y_{x+1} = x^2 + 2x + 1}$$

• Order of difference eq^n

The order of the d.e. is difference b/w the highest and the lowest subscript of the dependent variable by making the difference equation free from any difference operator.

E.g. $\Rightarrow$ $\Delta^2 y_x + \Delta y_x = x^2 + 2x + 1$

$\Rightarrow$ $y_{\boxed{x+2}} - y_{\boxed{x+1}} = x^2 + 2x + 1$

$\Rightarrow 1 \rightarrow$ order of d.e.

- Degree of difference eqⁿ

The highest degree of dependent variable after making the difference eqⁿ free from any operators.

e.g. $\Delta^2 Y_x + Y_x = x^2$

$$(E-1)^2 Y_x + Y_x = x^2$$

$$\Rightarrow E^2 Y_x - 2E Y_x + Y_x + Y_x = x^2$$

$$\Rightarrow Y_{x+2} - 2Y_{x+1} + 2Y_x = x^2$$

Order $= x+2-x = 2$

Degree $= 1$

- General form of a difference eqⁿ

$$a_0 Y_{x+n} + a_1 Y_{x+(n-1)} + a_2 Y_{x+(n-2)} + \dots + a_n Y_x = \phi(x) \quad \text{—(1)}$$

is a general form of d.e.

*) Linear and Non-Linear difference eqⁿ

If all the value $Y_x, Y_{x+1}, Y_{x+2}, \dots$ & Y_{x+n} are in degree '1' & none of these are multiplied then d.e (1) is called linear d.e.

*) Homogeneous & non-homogeneous difference eqn

The d.e. ① is such that be homogeneous if $\phi(x) = 0$, otherwise non-homogeneous.

*) Solution of linear homogeneous d.e.

Let $a_0 y_{x+n} + a_1 y_{x+(n-1)} + a_2 y_{x+(n-2)} + \dots + a_n y_x = 0$

be the given linear hom. d.e.
then we want it as

$$(f(E))\, y_x = 0$$

then we put

$$f(E) = 0 \quad \text{—} ②$$

Let $E_1, E_2, E_3, \dots, E_n$ be the roots of ②

① If all the roots are real & distinct

$$y_x = c_1 E_1^x + c_2 E_2^x + \dots + c_n E_n^x$$

② If some roots are repeated
Let $E_1 = E_2 = E$, $E_3, E_4, \dots E_n$

$$y_x = (c_1 + x c_2) E^x + c_3 E_3^x + c_4 E_4^x + \dots + c_n E_n^x$$

③ If some roots are complex
$E_1 = \alpha + i\beta$, $E_2 = \alpha - i\beta$, $E_3, E_4, \dots E_n$

$$y_x = (c_1 \cos(x\theta) + c_2 \sin(\theta x))\, r^x + c_3 E_3^x + \dots + c_n E_n^x$$

$$\theta = \tan^{-1}(\beta/\alpha), \quad r = \sqrt{\alpha^2 + \beta^2}$$

(4) If some roots are complex & repeated

$E_1 = E_2 = \alpha + i\beta$, $E_3 = E_4 = \alpha - i\beta$, E_5, ... E_n

$$y_x = \left((C_1 + C_2 x)\cos\theta x + (C_3 + C_4 x)\sin\theta x \right) r^x + C_5 E_5^x + \dots + C_n E_n^x$$

Q) (1) $y_{x+3} - 2y_{x+2} - y_{x+1} + 2y_x = 0$

(2) $y_{x+2} - 4y_{x+1} + 4y_x = 0$

A) (1) $E^3 y_x - 2E^2 y_x - E y_x + 2y_x = 0$

$$y_x (E^3 - 2E^2 - E + 2) = 0$$

$$E^3 - 2E^2 - E + 2 = 0$$

$1, -1, 2$

$$y_x = C_1 (1)^x + C_2 (-1)^x + C_3 2^x$$

(2) $E^2 y_x - 4E y_x + 4y_x = 0$

$$y_x (E^2 - 4E + 4) = 0$$

$$E^2 - 4E + 4 = 0$$

$(2, 2)$

$$y_x = (C_1 + x C_2) 2^x$$

~~Unit 4~~

*) Particular Integral

$$a_0 y_{x+n} + a_1 y_{x+n-1} + \dots + a_n y_x = \phi(x) \Rightarrow f(E)\, y_x = \phi(x)$$

C.I. = C.F. + P.I.

Case 1: if $\phi(x) = a^x$

$$P.I. = \frac{1}{f(E)} a^x$$

$$E \to a$$

$$P.I. = \frac{1}{f(a)} a^x, \quad f(x) \neq 0$$

if $f(a) = 0$

i) if $f(E) = E - a$

$$P.I. = \frac{1}{E-a} a^x$$

$$= \frac{x}{1!} a^{x-1}$$

ii) if $f(E) = (E-a)^2$

$$P.I. = \frac{1}{(E-a)^2} a^x$$

$$= \frac{x(x-1)}{2!} a^{x-2}$$

Q) i) $y_{x+2} - 4y_{x+1} + 3y_x = 5^x$

A) $(E^2 - 4E + 3)y_x = 5^x$

$$E^2 - 4E + 3 = 0$$

$$(E-1)(E-3) = 0$$

$$E = 1, 3$$

$$C.F. = c_1 (1)^x + c_2 (3)^x$$

$$P.I. = \frac{1}{E^2 - 4E + 3} 5^x$$

$$= \frac{1}{5^2 - 4 \times 5 + 3} \times 5^x$$

$$= \frac{5^x}{8}$$

$$C.I. = c_1 (1)^x + c_2 (3)^x + \frac{5^x}{8}$$

ii) $(E^2 - 4E + 4)y_x = 2^x$

$$(E-2)^2 = 0$$

$$E = 2$$

$$C.F. = (c_1 + c_2 x) 2^x$$

$$P.I. = \frac{1}{(E-2)^2} 2^{x-2}$$

$$= \frac{x(x-1)}{2!} \cdot 2^{x-2}$$

Case II: if $\phi(x) = \sin kx$

$$= \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$$

$$= \frac{1}{2i}\left((e^{ik})^{x} - (e^{-ik})^{x}\right)$$

$$= \frac{1}{2i}\left(a^{x} - b^{x}\right)$$

$e^{ikx} = \cos kx + i\sin kx$

$e^{-ikx} = \cos kx - i\sin kx$

$\frac{e^{ikx} - e^{-ikx}}{2i} = \sin kx$

$$P.I. = \frac{1}{2i} \times \frac{1}{f(E)} \times (a^{x} - b^{x}), \text{ where } a = e^{ik} \text{ \& } b = e^{-ik}$$

Case III: if $\phi(x) = \cos x$

$$= \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$$

$$= \frac{1}{2}\left((e^{ik})^{x} + (e^{-ik})^{x}\right)$$

$$= \frac{1}{2}\left(a^{x} + b^{x}\right)$$

Q) $(E^2 - 2\cos\alpha E + 1)\, y_x = \cos x$

A) $E^2 - 2\cos\alpha E + 1 = 0$

$$E = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2}$$

$$= \frac{2\cos\alpha \pm 2i\sin\alpha}{2}$$

$$E = \cos\alpha \pm i\sin\alpha$$

$$C.F. = (c_1 \cos\theta x + c_2 \sin\theta x)\, r^{x}$$

$\theta = \tan^{-1}(\beta/\alpha)$, $r = \sqrt{\alpha^2 + \beta^2}$

$= \tan^{-1}(\tan\alpha)$, $r = \sqrt{\sin^2\alpha + \cos^2\alpha}$

$= \alpha$, $r = 1$

$$\text{C.F.} = (C_1 \cos x + C_2 \sin x)(1)^x$$

$$\text{P.I.} = \frac{1}{E^2 - 2\cos\alpha E + 1} \cos \alpha x$$

$$= \frac{1}{2} \times \frac{1}{E^2 - (e^{i\alpha} + e^{-i\alpha})E + 1} (e^{i\alpha x} + e^{-i\alpha x}) \quad \left[\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}\right]$$

$$= \frac{1}{2} \left\{ \frac{1}{(E - e^{i\alpha})(E - e^{-\alpha x})} (e^{i\alpha})^x + \frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})} (e^{-i\alpha})^x \right\}$$

$$= \frac{1}{2 \times 2i \sin x} \left\{ \frac{1}{(E - e^{i\alpha})} (e^{i\alpha})^x - \frac{1}{(E - e^{-i\alpha})} (e^{-i\alpha})^x \right\}$$

$$= \frac{1}{4i\sin x} \left\{ \frac{x}{1!} (e^{i\alpha})^{x-1} - \frac{x}{1!} (e^{-i\alpha})^{x-1} \right\}$$

$$= \frac{x}{2\sin x} \sin(\alpha(x-1))$$

Case IV: $\phi(x) = x^p$

$$x^{[n]}$$

$$x^{[1]} = x$$

$$x^{[2]} = x(x-1)$$

$$x^{[3]} = x(x-1)(x-2)$$

E.g. $x^2 = x(x-1) + x$

$$= x^{[2]} + x^{[1]}$$

$$\Delta x^{[n]} = n x^{[n-1]}$$

$$\phi(x) = x^p$$
$$= F\left(x^{[1]}, x^{[2]}, \ldots\right)$$

$$P.I. = \frac{1}{f(E)} x^p$$
$$= \frac{1}{f(1+\Delta)} x^p$$
$$= (1+\Delta)^{-q} F\left[x^{[1]}, x^{[2]}, \ldots\right]$$

Q) $y_{x+2} - 4y_x = x^2$

A) $(E^2-4)\, y_x = x^2$

$$E = -2, 2$$

$$C.F. = c_1(-2)^x + c_2(2)^x$$

$$P.I. = \frac{1}{(E^2-4)} x^2$$
$$= \frac{1}{(1+\Delta)^2-4} x^2$$
$$= \frac{1}{1+2\Delta+\Delta^2-4} x^2$$
$$= \frac{1}{-3+2\Delta+\Delta^2} x^2$$
$$= \frac{1}{-3}\left(1 - \frac{2\Delta+\Delta^2}{3}\right)^{-1} x^2$$
$$= \frac{1}{-3}\left(1 + \frac{2\Delta+\Delta^2}{3} + \left(\frac{2\Delta+\Delta^2}{3}\right)^2 + \cdots\right)\left(x^{[2]} + x^{[1]}\right)$$
$$= \frac{1}{-3}\left(1 + \frac{1}{3}(2\Delta+\Delta^2) + \frac{1}{9}(4\Delta^2)\right)\left(x^{[2]} + x^{[1]}\right)$$

$$= \frac{1}{-3}\left(1 + \frac{2}{3}\Delta + \frac{7}{9}\Delta^2\right)\left(x^{[2]} + x^{[1]}\right)$$

$$= \frac{-1}{3}\left(\left(x^{[2]} + x^{[1]}\right) + \frac{2}{3}\left(2x^{[1]} + 1\right) + \frac{7}{9} \times 2\right)$$

$$= \frac{-1}{3}\left(x^{[2]} + \frac{7}{3}x^{[1]} + \frac{20}{9}\right)$$

$$= \frac{-1}{3}\left(x(x-1) + \frac{7}{3}x + \frac{20}{9}\right)$$

2) $Y_{x+2} - 4Y_x = x^2 + x - 1$

Case - V :- $\phi(x) = a^x f(x)$

Let $F(E)y_x = a^x f(x)$

$$P.I. = \frac{1}{F(E)} \cdot a^x f(x)$$

$$= a^x \frac{1}{F(aE)} f(x)$$

$\left\{ f(x) = \sin x / \cos x / P(x) \right\}$

Q) $y_{x+2} - 2y_{x+1} + y_x = x^2 2^x$

A) $C.F. = (c_1 x + c_2)(1)^x$

$$P.I. = \frac{1}{E^2 - 2E + 1} 2^x . x^2$$

$$= \frac{1}{(E-1)^2} . 2^x . x^2$$

$$= 2^x \frac{1}{(2E-1)^2} x^2$$

$$= 2^x \frac{1}{(1+2\Delta)^2} x^2$$

$$= 2^x \frac{1}{(1+4\Delta+4\Delta^2)} x^2$$

$$= 2^x (1 + (4\Delta + 4\Delta^2))^{-1} x^2$$

$$= 2^x (1 - (4\Delta + 4\Delta^2) + (4\Delta + 4\Delta^2)^2 + \ldots) x^2$$

$$= 2^x . (1 - (4\Delta + 4\Delta^2) + (4\Delta + 4\Delta^2)^2)(x^{[2]} + x^{[1]})$$

$$= 2^x . (x^{[2]} + x^{[1]} - (8x^{[1]} + 4 + 8) + 32)$$

$$= 2^x . (x^{[2]} - 7x^{[1]} - 4)$$

$$= 2^x . (x(x-1) - 4x + 20)$$

$$= 2^x (x^2 - 5x + 20)$$