Q If $u = \log_e \sin\left(\dfrac{\pi\,(2x^2+y^2+xz)^{\frac{1}{2}}}{2\,(x^2+xy+2yz+z^2)^{1/3}}\right)$

then show that

$$xu_x + yu_y + zu_z = \frac{\pi}{12} \text{ at } x=0,\ y=1,\ z=2.$$

Sol:

$$\Rightarrow \sin^{-1}(e^u) = \frac{\pi\,(2x^2+y^2+xz)^{1/2}}{2\,(x^2+xy+2yz+z^2)^{1/3}}$$

$$\Rightarrow v = \frac{\pi\,(2x^2+y^2+xz)^{1/2}}{2\,(x^2+xy+2yz+z^2)^{1/3}}, \quad v = \sin^{-1}(e^u)$$

here u is homogeneous of degree $1/3$.

$$\Rightarrow x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}=\frac{1}{3}v$$

$$\Rightarrow x\frac{\partial}{\partial x}\left(\sin^{-1}(e^u)\right)+y\frac{\partial}{\partial y}\left(\sin^{-1}(e^u)\right)+z\frac{\partial}{\partial z}\left(\sin^{-1}(e^u)\right)=\frac{1}{3}\sin^{-1}(e^u)$$

$$\Rightarrow x\times\frac{1}{\sqrt{1-e^{2u}}}\times e^u\times\frac{\partial u}{\partial x}+y\times\frac{1}{\sqrt{1-e^{2u}}}\times e^u\times\frac{\partial u}{\partial y}+z\times\frac{1}{\sqrt{1-e^{2u}}}\times e^u\times\frac{\partial u}{\partial z}$$

$$=\frac{1}{3}\sin^{-1}(e^u)$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{\sqrt{1-e^{2u}}}{e^u}\times\frac{1}{3}\sin^{-1}(e^u) \quad \text{--- } (1)$$

At $x=0,\ y=1,\ z=2$, we have

$$u = \log_e \sin\left(\frac{\pi}{2}\,\frac{(2x^2+y^2+xz)^{1/2}}{(x^2+xy+2yz+z^2)^{1/3}}\right)$$

$$= \log_e \sin\left(\frac{\pi}{2}\times\frac{1}{2}\right)$$

$$= \log_e \sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \boxed{e^u = \frac{1}{\sqrt{2}}}$$

Hence by (1), $xu_x + yu_y + zu_z = \dfrac{\sqrt{1-\frac{1}{2}}}{\frac{1}{\sqrt{2}}} \times \dfrac{1}{3}\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$

$$= \frac{\pi}{12} \ \#.$$

Differentiation of composite functions

Let $u = f(x, y)$ and $x = \phi(t)$, $y = \psi(t)$

Then the total derivative of u w.r.t 't' is defined as

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt}$$

$\rightarrow$ If $f(x, y) = c$, then

$$\frac{dc}{dx} = \frac{\partial f}{\partial x} \times \frac{dx}{dx} + \frac{\partial f}{\partial y} \times \frac{dy}{dx}$$

$$\Rightarrow \quad 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}}$$

Hence if $f(x, y) = c$

then $\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$

$u = \cancel{x^2 + y^2}$

$x = t^2 \sin t$

$f = x^3 y + y^{1/2} x^{t^3} + x y^{2} = 1$

→ Chain rule of partial differentiation

let $z = f(x, y)$, $x = \phi(u, v)$, $y = \psi(u, v)$

then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}$$

Q If $u = f(y-z, z-x, x-y)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol! Let $u = f(r, s, t)$ where $r = y-z$, $s = z-x$, $t = x-y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \times \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \times \frac{\partial t}{\partial x}$$

$$= -u_s + u_t$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \times \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \times \frac{\partial t}{\partial y}$$

$$= u_r - u_t$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \times \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \times \frac{\partial t}{\partial z} = -u_r + u_s$$

$\therefore$ $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

$= 0.$

Q If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$

then show that

$$\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = 0$$

Q If $u = f(x^2+2yz,\ y^2+2zx)$, show that

$$(y^2-zx)u_x + (x^2-yz)u_y + (z^2-xy)u_z = 0$$

Q If $f(x,y,z)=0$, then show that $\left(\frac{dy}{dz}\right)_x \left(\frac{dz}{dx}\right)_y \left(\frac{dx}{dy}\right)_z = -1$

Soln

Let x is constant, then

$$f(y,z) = x = \text{constant}$$

$$\frac{dy}{dz} = \frac{-f_z}{f_y}$$ ✓

If $$f(x,y) = z = \text{constant}$$

$$\frac{dx}{dy} = -\frac{f_y}{f_x}$$

then $$f(z,x) = y = \text{constant}$$

$$\frac{dz}{dx} = \frac{-f_x}{f_z}$$

So

$$\left(\frac{dy}{dz}\right)_x \left(\frac{dx}{dy}\right)_z \left(\frac{dz}{dx}\right)_y$$

$$= \frac{-\cancel{f_z}}{\cancel{f_y}} \times \frac{-\cancel{f_y}}{\cancel{f_x}} \times \frac{-\cancel{f_x}}{\cancel{f_z}}$$

$$= -1.$$ Ans.

Q

If $x = r\cos\theta$, $y = r\sin\theta$

then show that

a) $\dfrac{\partial^2 r}{\partial x^2} \cdot \dfrac{\partial^2 r}{\partial y^2} = \left(\dfrac{\partial^2 r}{\partial x \partial y}\right)^2$

b) $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right]$

Sol

As $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x^2 + y^2 = r^2$, $\tan\theta = \dfrac{y}{x}$

$\Rightarrow r^2 = x^2 + y^2$

$\Rightarrow 2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ and similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

$\because \quad \frac{\partial r}{\partial x} = \frac{x}{r}$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \frac{1 \times r - x \times \frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - x \times \frac{x}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r^3}$$

Similarly

$$\frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3}$$

Also $\frac{\partial r}{\partial y} = \frac{y}{r}$

$$\Rightarrow \frac{\partial^2 r}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{y}{r}\right)$$

$$= y \frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$= y \times \frac{-1}{r^2}\frac{\partial r}{\partial x}$$

$$= \frac{-xy}{r^3}$$

(a)

L.H.S

$$= \frac{\partial^2 r}{\partial x^2} \times \frac{\partial^2 r}{\partial y^2}$$

$$= \frac{r^2 - x^2}{r^3} \times \frac{r^2 - y^2}{r^3}$$

$$= \frac{y^2}{r^3} \times \frac{x^2}{r^3}$$

$$= \frac{x^2 y^2}{r^6}$$ ✓

$$R.H.S = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2 = \left(\frac{-xy}{r^3}\right)^2 = \frac{x^2 y^2}{r^6}$$ ✓

$$\therefore \frac{\partial^2 r}{\partial x^2} \times \frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2$$

(ii)

L.H.S $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}$

$$= \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3}$$

$$= \frac{2r^2 - (x^2 + y^2)}{r^3}$$

$$= \frac{r^2}{r^3} = \frac{1}{r}$$

$$R.H.S = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

$$= \frac{1}{r}\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right] = \frac{1}{r}\left[\frac{x^2+y^2}{r^2}\right] = \frac{1}{r}$$

$$\therefore \frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$$

Q If $u = f(r)$ then show that

$$u_{xx}+u_{yy} = f''(r)+\frac{1}{r}f'(r)$$

Sol: $u = f(r)$, $r^2 = x^2+y^2$ $\left(\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}\right)$

$$\Rightarrow \frac{\partial u}{\partial x} = f'(r)\times\frac{\partial r}{\partial x}$$

$$= f'(r)\times\frac{x}{r}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{f'(r)\,x}{r}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\left\{f''(r)\times\frac{\partial r}{\partial x}\times x + f'(r)\right\}r - f'(r)\times x\times\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{\left\{f''(r)\times\frac{x^2}{r} + f'(r)\right\}r - f'(r)\times x\times\frac{x}{r}}{r^2}$$

$$= \frac{f''(r)\times x^2}{r^2} + \frac{f'(r)}{r} - \frac{f'(r)x^2}{r^3}$$

$$\boxed{u_{xx} = \frac{f''(r)\times x^2}{r^2} + \frac{f'(r)\left\{r^2 - x^2\right\}}{r^3}}$$

$r^2 = x^2 + y^2 + z^2$

Similarly

$$u_{yy} = \frac{f''(r)\, y^2}{r^2} + \frac{f'(r)}{r^3}\left[r^2 - y^2\right].$$

On addition we get

$$u_{xx} + u_{yy} = \frac{f''(r)}{r^2}\left\{x^2 + y^2\right\} + \frac{f'(r)}{r^3}\left\{r^2 - x^2 + r^2 - y^2\right\}$$

$$= f''(r) + \frac{f'(r)}{r^3} \times r^2$$

$$= f''(r) + \frac{f'(r)}{r}$$

$\neq 1$

Q If $f(x,y)=0$, $\phi(y,z)=0$, then show that

$$\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z}\frac{dz}{dx}=\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}$$

Sol: As $$\frac{dz}{dx}=\left(\frac{dz}{dy}\right)\left(\frac{dy}{dx}\right)$$

Now $f(x,y)=0 \Rightarrow \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}$

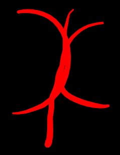
and $\phi(y,z)=0 \Rightarrow \frac{dz}{dy}=-\frac{\partial \phi/\partial y}{\partial \phi/\partial z}$

$\therefore$ $$\frac{dz}{dx}=\frac{-\partial \phi/\partial y}{\partial \phi/\partial z}\times\frac{-\partial f/\partial x}{\partial f/\partial y}$$

$$\Rightarrow \frac{\partial \phi}{\partial z}\times\frac{\partial f}{\partial y}\frac{dz}{dx}=\frac{\partial \phi}{\partial y}\times\frac{\partial f}{\partial x}$$

#.

Q. If the curves $f(x,y)=0$, $\phi(x,y)=0$ touch each other



then show that $\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x}=0$.

Sol: As $f(x,y)=0$, $\phi(x,y)=0$ touch each other

$\Rightarrow \frac{dy}{dx}=\frac{-\partial f/\partial x}{\partial f/\partial y}$ and $\frac{dy}{dx}=\frac{-\partial \phi/\partial x}{\partial \phi/\partial y}$ must be equal

$$\therefore \frac{-\partial f/\partial x}{\partial f/\partial y}=\frac{-\partial \phi/\partial x}{\partial \phi/\partial y}$$

$$\Rightarrow \frac{\partial f}{\partial x}\times\frac{\partial \phi}{\partial y}=\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x} \text{ Ans.}$$

Q

① If $z = f(x, y)$

$x = e^u \cos v$, $y = e^u \sin v$

show that $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right]$

② If $x = e^r \cos\theta$, $y = e^r \sin\theta$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\right\}$$

when $u = f(x, y)$

$z = f(x, y)$

$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial u}$

$u = f(x, y)$

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$

Sol:

(2) $x = e^r \cos\theta, \quad y = e^r \sin\theta$

$u = f(x, y)$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x} \times e^r \cos\theta + \frac{\partial u}{\partial y} \times e^r \sin\theta$$

$$= x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial r} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}}$$

Now $$\frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right) = \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)$$

$$= x \frac{\partial}{\partial x}\left(x \frac{\partial}{\partial x}\right) + x \frac{\partial}{\partial x}\left(y \frac{\partial}{\partial y}\right) + y \frac{\partial}{\partial y}\left(x \frac{\partial}{\partial x}\right) + y \frac{\partial}{\partial y}\left(y \frac{\partial}{\partial y}\right)$$

$$= x\left\{\frac{\partial}{\partial x} + x \times \frac{\partial^2}{\partial x^2}\right\} + 2xy \frac{\partial^2}{\partial x \partial y} + y\left\{\frac{\partial}{\partial y} + y \frac{\partial^2}{\partial y^2}\right\}$$

$$= x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$\therefore \frac{\partial^2 u}{\partial r^2} = x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \quad \text{— (1)}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\times\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\times\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}\times(-e^r\sin\theta) + \frac{\partial u}{\partial y}\times(e^r\cos\theta)$$

$$= -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial \theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$-y\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right)$$
$$= -y\left\{\frac{\partial}{\partial y} + x\frac{\partial^2}{\partial x\partial y}\right\}$$

$$\frac{\partial^2}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial \theta}\right)$$
$$= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)$$
$$= \cancel{y^2\frac{\partial^2}{\partial x^2} - 2xy\frac{\partial^2}{\partial x\partial y} + x^2\frac{\partial^2}{\partial y^2}}$$
$$= y^2\frac{\partial^2}{\partial x^2} - y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x} - 2xy\frac{\partial^2}{\partial x\partial y} + x^2\frac{\partial^2}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial \theta^2} = y^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} - x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} \quad \text{--- (2)}$$

(1) + (ii) $\Rightarrow$

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} = (x^2+y^2) \frac{\partial^2 u}{\partial x^2} + (x^2+y^2) \frac{\partial^2 u}{\partial y^2}$$

$$\cancel{+ \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)}$$

$x = e^r \cos\theta, \quad y = e^r \sin\theta$

$\Rightarrow x^2 + y^2 = e^{2r}$

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} = e^{2r} \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r} \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} \right\}$$

Q. If $w = \sqrt{x^2+y^2+z^2}$ $\Rightarrow \{ w = f(x,y,z)$

$x = u\cos v$, $y = u\sin v$, $z = uv$

then $u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1+v^2}}$

Sol: $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \times \frac{\partial z}{\partial u}$

$= \frac{x}{w} \times 0 + \frac{y}{w} \times \sin v + \frac{z}{w} \times v$

$= \frac{y\sin v}{w} + \frac{zv}{w}$

$\Rightarrow u\frac{\partial w}{\partial u} = \frac{uy\sin v}{w} + \frac{zuv}{w}$

Taylor's & MacLaurin's series

let $f(x)$ is the function of x having derivation of different order then

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

This is known as Taylor's series.

If $h \to x-a$, $x \to a$

then $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) \cdots$

This is the Taylor's series in powers of "$x-a$" or we call it as Taylor series at $x=a$.

If $a=0$, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$$

or

$$y = y_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots$$

is known as Maclaurin's series in powers of 'x'.

Q Find the expansion of

(a) $f(x) = \sin x$

(b) $f(x) = \cos x$

(c) $f(x) = \tan^{-1}(x)$

(d) $f(x) = \log(1+x)$

(e) $f(x) = e^{a\sin^{-1}x}$ including its general term.

$(y_n)_0 = -((n-2)^2 + a^2)(y_{n-2})_0$

$y = e^{a\sin^{-1}x}$

$y_1 = \frac{a}{\sqrt{1-x^2}}\, y$

(c)

Let $y = \tan^{-1}x$ — (i) $(y)_0 = 0$

$y_1 = \frac{1}{1+x^2}$ — (ii)

$\Rightarrow (1+x^2)y_1 = 1 \rightarrow (y_1)_0 = 1$

$(1+x^2)y_2 + 2xy_1 = 0$ — (iii)

$\Rightarrow (y_2)_0 = 0$

$(1+x^2)y_3 + 2xy_2 + 2y_1 + 2xy_2 = 0$ — (iv)

$(y_3)_0 = -2$ and so on.

$y = (y_0) + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \cdots$

$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ Ans

Q. Expand $\sin x$ in powers of $x-\frac{\pi}{4}$

Sol:
$$f(x) = \sin x \Rightarrow f(\pi/4) = \frac{1}{\sqrt{2}}$$
$$f'(x) = \cos x \Rightarrow f'(\pi/4) = \frac{1}{\sqrt{2}}$$
$$f''(x) = -\sin x \Rightarrow f''(\pi/4) = -1/\sqrt{2}$$
$$f'''(x) = -\cos x \Rightarrow f'''(\pi/4) = -1/\sqrt{2}$$ and so on.

Now we have

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) - - -$$

$$\sin x = \frac{1}{\sqrt{2}} + \left(x - \frac{\pi}{4}\right) \times \frac{1}{\sqrt{2}} + \frac{1}{2}\left(x-\frac{\pi}{4}\right)^2 \times -\frac{1}{\sqrt{2}} + \frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3 \times \left(\frac{-1}{\sqrt{2}}\right) \cdots\cdots$$

Q expand $\tan(x+h)$ in powers of 'x'.

sol: As

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) - - - -$$

$\Rightarrow \tan(x+h) = f(x+h)$

$\Rightarrow f(x) = \tan x$

$\Rightarrow f'(x) = \sec^2 x$

$f''(x) = 2\sec^2 x \tan x$ and so on.

$$\therefore f(x+h) = \tan h + x(\sec^2 h) + \frac{x^2}{2!}(2\sec^2 h \tan h) - - - -$$

or

$$\tan(x+h) = \tan h + x(\sec^2 h) + x^2(\sec^2 h \tan h) - - - - -$$

Taylor's series for function of two variable

Let $z = f(x, y)$ be the function of two variable and suppose that z have continuous partial derivatives w.r.t 'x' and 'y' then

$$f(x+h, y+k) = f(x, y) + \left\{ h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right\} + \frac{1}{2!}\left\{ h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2} \right\}$$

$$+ \cdots\cdots$$

or

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$

$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \cdots$$

This series is know as Taylor series of $f(x,y)$ at (x,y).

If $x = a$, $h \to x-a$, $y \to b$, $k \to y-b$, then Taylor series at (a,b) is given by

$\boxed{f(x,y) = e^x \cos y}$

$$f(x,y) = f(a,b) + \left\{ (x-a)\left(\frac{\partial f}{\partial x}\right)_{(a,b)} + (y-b)\left(\frac{\partial f}{\partial y}\right)_{(a,b)} \right\}$$
$$+ \frac{1}{2!}\left\{ (x-a)^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)} + 2(x-a)(y-b)\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)} + (y-b)^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} \right\}$$
$$+ \cdots$$

If $(a,b) = (0,0)$ then the above series is known as Maclaurin's series

$$f(x,y) = f(0,0) + \left\{ x\left(\frac{\partial f}{\partial x}\right)_{(0,0)} + y\left(\frac{\partial f}{\partial y}\right)_{(0,0)} \right\} + \frac{1}{2!}\left\{ x^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} + 2xy\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} \right\} + \cdots$$

Q Expand $e^x \log(1+y)$ in Taylor series in the neighbourhood of $(0,0)$.

Sol:

$$f(x,y) = e^x \log(1+y) \Rightarrow f(0,0) = 0$$

$$\Rightarrow f_x = e^x \log(1+y) \Rightarrow f_x(0,0) = 0$$

$$\Rightarrow f_{xx} = e^x \log(1+y) \Rightarrow (f_{xx})_{(0,0)} = 0$$

$$\Rightarrow f_{xy} = \frac{e^x}{1+y} \Rightarrow (f_{xy})_{(0,0)} = 1$$

$$f_y = \frac{e^x}{1+y} \Rightarrow (f_y)_{(0,0)} = 1$$

$$f_{yy} = \frac{-e^x}{(1+y)^2} \Rightarrow (f_{yy})_{(0,0)} = -1 \quad \text{and so on.}$$

$$\therefore f(x,y) = f(0,0) + \{x(f_x)_{(0,0)} + y(f_y)_{(0,0)}\} + \frac{1}{2!}\{x^2(f_{xx})_{(0,0)} + 2xy(f_{xy})_{(0,0)} + y^2(f_{yy})_{(0,0)}\} + \cdots$$

$$\boxed{e^x \log(1+y) = y + xy - \frac{y^2}{2} \cdots\cdots}$$

Expand y^x about $(1,1)$ and hence evaluate $(1.02)^{(1.03)}$

Sol: $f(x,y) = y^x \Rightarrow f(1,1) = 1$

$f_x = y^x \log y \Rightarrow f_x(1,1) = 0$

$f_{xx} = y^x (\log y)^2 \Rightarrow f_{xx}(1,1) = 0$

$f_{xy} = xy^{x-1} \log y + y^{x-1} \Rightarrow (f_{xy})_{(1,1)} = 1$

$f_y = xy^{x-1} \Rightarrow (f_y)_{(1,1)} = 1$

$f_{yy} = x(x-1)y^{x-2} \Rightarrow (f_{yy})_{(1,1)} = 0$ and so on.

$$f(x,y) = f(1,1) + \{(x-1)(f_x)_{(1,1)} + (y-1)(f_y)_{(1,1)}\} + \frac{1}{2!}\{(x-1)^2 (f_{xx})_{(1,1)} + 2(x-1)(y-1)(f_{xy})_{(1,1)} + (y-1)^2 (f_{yy})_{(1,1)}\} + \cdots$$

$y^x = 1 + (y-1) + (x-1)(y-1)$ ......

let $y = 1.02,\ x = 1.03$

$(1.02)^{(1.03)} \simeq 1 + .02 + 0.0006$

$\simeq 1.0206$

Jacobian

Defn: Let u and v are functions of two independent variables x and y then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is known as Jacobian of u and v w.r.t x and y and we write it as

$$\boxed{J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}}$$

If $u = f(x,y,z)$, $v = \phi(x,y,z)$, $w = \psi(x,y,z)$, then

$$J(u,v,w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties

① If u, v are functions of x, y and x, y are functions of r, s then

$u = f(x, y),\ x \to \phi(r, s)$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial(u, v)}{\partial(r, s)} = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(r, s)}$$

Pf R.H.S $= \dfrac{\partial(u, v)}{\partial(x, y)} \times \dfrac{\partial(x, y)}{\partial(r, s)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix}$$

$$= \frac{\partial(u, v)}{\partial(r, s)} = \text{L.H.S}$$

(2) If u, v is function of x, y and $J = \frac{\partial(u,v)}{\partial(x,y)}$

$J' = \frac{\partial(x,y)}{\partial(u,v)}$, then

$JJ' = 1$

sol:

$$JJ' = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

(3) (Jacobian of Implicit funcⁿ)

$f_1 = u^3 + v - x + y = 0$

$f_2 = u - v - x + y = 0$

Let u, v, w and x, y & z are connected by three relation

$$f_1(u, v, w, x, y, z) = 0$$

$$f_2(u, v, w, x, y, z) = 0$$

$$f_3(u, v, w, x, y, z) = 0$$

$u = x + y + z$

$v = x^2 + y - z$

$w = x + y + z$

$f_1 = u^2 + v + w - x^3 + y^3 + z = 0$

$f_2 = v^3 + w - y^3 + z$

$f_3 = u + v + w^3 - x + y + z^2$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}$$

(4) The variables u, v, w which are functions of x, y, z are functionally related or dependent iff

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

$u = x+y+z$

$v = x^2+y^2+z^2$

$w = xy+yz+zx$

$\boxed{u^2 = v + 2w}$

Q If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Sol:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2}\begin{vmatrix} -yz & zx & xy \\ yz & -zx & xy \\ yz & zx & -xy \end{vmatrix}$$

$$= \frac{x^2y^2z^2}{x^2y^2z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1(1-1) - 1(-1-1) + 1\,(1+1)$$

$$= 2+2 = 4.$$

Q. If

$$u = x+y+z$$

$$uv = y+z$$

$$z = uvw$$

then find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol:

$$z = uvw$$

$$y = uv - z$$

$$= uv - uvw$$

$$x = u - (y+z)$$

$$= u - uv$$

$\therefore$

$$x = u - uv$$

$$y = uv - uvw$$

$$z = uvw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)\{u^2v \cancel{- u^2vw} \cancel{+ u^2vw}\}$$

$$+ u\{uv^2 \cancel{- uv^2w} \cancel{+ uv^2w}\}$$

$$= u^2v - u^2v^2 + u^2v^2$$

$$= u^2v$$

$$\therefore \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$$

Also $$\frac{\partial(u,v,w)}{\partial(x,y,z)} \times \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

$$\Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{\frac{\partial(x,y,z)}{\partial(u,v,w)}} = \frac{1}{u^2 v} \quad \#$$

<u>H.W.</u>
<u>Q</u> If $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$, then

find $$\frac{\partial(u,v,w)}{\partial(x,y,z)}.$$

Q If $u^3+v^3+w^3 = x+y+z$

$u^2+v^2+w^2 = x^3+y^3+z^3$

$u+v+w = x^2+y^2+z^2$ then show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

sol: let

$f_1 = u^3+v^3+w^3-x-y-z$

$f_2 = u^2+v^2+w^2-x^3-y^3-z^3$

$f_3 = u+v+w-x^2-y^2-z^2$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix}$$

$$= -6 \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}$$

$$= -6 \begin{vmatrix} 1 & 0 & 0 \\ x^2 & (y-x)(y+x) & (z-x)(z+x) \\ x & y-x & z-x \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= -6\,(y-x)(z-x) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & x+y & x+z \\ x & 1 & 1 \end{vmatrix} = 6\,(y-z)(z-x)(x-y)$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)} = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 6 \begin{vmatrix} u^2 & v^2 & w^2 \\ u & v & w \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 6 \begin{vmatrix} u^2 & (v-u)(v+u) & (w-u)(w+u) \\ u & v-u & w-u \\ 1 & 0 & 0 \end{vmatrix} \qquad \begin{array}{l} C_3 \to C_3 - C_1 \\ C_2 \to C_2 - C_1 \end{array}$$

$$= 6(v-u)(w-u) \begin{vmatrix} u^2 & v+u & w+u \\ u & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = -6(u-v)(v-w)(w-u)$$

$\therefore$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -1 \times \frac{6(x-y)(y-z)(z-x)}{-6(u-v)(v-w)(w-u)}$$

$$= \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

#

Q If u, v, w are roots of the equation in λ and

$$\frac{x}{a+\lambda} + \frac{y}{b+\lambda} + \frac{z}{c+\lambda} = 1$$ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

$(a+\lambda)\{\lambda^2+(b+c)\lambda+bc\}$

Sol:

$$x(b+\lambda)(c+\lambda) + y(a+\lambda)(c+\lambda) + z(a+\lambda)(b+\lambda)$$

$$= (a+\lambda)(b+\lambda)(c+\lambda)$$

$$\Rightarrow xbc + x(b+c)\lambda + \lambda^2 x + yac + y(a+c)\lambda + y\lambda^2$$
$$+ abz + \lambda(a+b)z + \lambda^2 z = \lambda^3 + \lambda^2(a+b+c) + \lambda(ab+bc+ca) + abc$$

$$\Rightarrow \lambda^3 + \lambda^2(a+b+c-x-y-z) + \lambda\left(ab+bc+ca - x(b+c) - y(a+c) - z(a+b)\right)$$
$$+ abc - xbc - yac - abz = 0$$

$$u+v+w = x+y+z-a-b-c$$

$$uv+vw+wu = ab+bc+ca - x(b+c) - y(a+c) - z(a+b)$$

$$uvw = xbc + yac + abz - abc$$

Let $f_1 = u+v+w - x-y-z+a+b+c = 0$

$$f_2 = uv+vw+wu - ab-bc-ca+x(b+c)+y(a+c)+z(a+b)$$

$$f_3 = uvw - xbc - yac - abz + abc$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -1 & -1 \\ b+c & c+a & a+b \\ -bc & -ca & -ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ b+c & a-b & a-c \\ bc & c(a-b) & b(a-c) \end{vmatrix} \qquad \begin{matrix} C_3 \to C_3 - C_1 \\ C_2 \to C_2 - C_1 \end{matrix}$$

$$= (a-c)(a-b) \begin{vmatrix} 1 & 0 & 0 \\ b+c & 1 & 1 \\ bc & c & b \end{vmatrix}$$

$$= -(a-b)(b-c)(c-a)$$

$$\frac{\partial(r_1, r_2, r_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & uw & uv \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} \qquad \begin{matrix} C_3 \to C_3 - C_1 \\ C_2 \to C_2 - C_1 \end{matrix}$$

$$= (u-v)(u-w)(v-w)$$

$$= (u-v)(v-w)(w-u)$$

$$\therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)}$$

#

Q If $u = xy + yz + zx$

$v = x^2 + y^2 + z^2$

$w = x + y + z$ then show that u, v, w are functionally dependent and find a relationship between them.

Sol.

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} y+z & x+z & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ u, v, w are dependent.

$$u^2 = (x+y+z)^2$$

$$= \underline{x^2+y^2+z^2} + 2(\underline{xy+yz}+zx)$$

$$= \underline{v} + 2\underline{w}$$

$$\therefore \boxed{u^2 = v + 2w}$$

Q verify that $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$

are dependent and find a <u>relationship b/w</u> them.

Sol:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{(1+y^2)}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

$\Rightarrow$ u and v are dependent.

$$v = \tan^{-1}x + \tan^{-1}y$$

$$= \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$= \tan^{-1}u$$

$\therefore$ $\boxed{v = \tan^{-1}u}$

or

$\boxed{u = \tan v}$

#

Q If $u_1 = \frac{x_1}{x_n}$, $u_2 = \frac{x_2}{x_n}$, ---, $u_{n-1} = \frac{x_{n-1}}{x_n}$

and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, find $\frac{\partial(u_1, u_2, \ldots, u_{n-1})}{\partial(x_1, x_2, \ldots, x_{n-1})}$

Sol:

$$x_n^2 = 1 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2 \checkmark$$

$$\Rightarrow 2x_n \frac{\partial x_n}{\partial x_1} = -2x_1$$

$$\Rightarrow \frac{\partial x_n}{\partial x_1} = \frac{-x_1}{x_n}, \quad \frac{\partial x_n}{\partial x_2} = -\frac{x_2}{x_n} \text{ and so on.}$$

$$\frac{\partial(u_1, u_2, \ldots, u_{n-1})}{\partial(x_1, x_2, \ldots, x_{n-1})} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1}, & \frac{\partial u_1}{\partial x_2}, & \cdots, & \frac{\partial u_1}{\partial x_{n-1}} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_{n-1}}{\partial x_1} & \frac{\partial u_{n-2}}{\partial x_2} & \cdots, & \frac{\partial u_{n-1}}{\partial x_{n-1}} \end{vmatrix}$$

$$\frac{\partial u_1}{\partial x_1} = \frac{x_n - x_1 \times \frac{\partial x_n}{\partial x_1}}{x_n^2}$$

$$= \frac{x_n + \frac{x_1^2}{x_n}}{x_n^2} = \frac{x_n^2 + x_1^2}{x_n^3}$$

$$\frac{\partial u_1}{\partial x_2} = \frac{\partial}{\partial x_2}\left(\frac{x_1}{x_n}\right)$$

$$= \frac{-x_1}{x_n^2} \times \frac{\partial x_n}{\partial x_2}$$

$$= \frac{x_1 x_2}{x_n^3} \text{ and so on.}$$

$$
= \begin{vmatrix}
\frac{x_1^2+x_n^2}{x_n^3} & \frac{x_1x_2}{x_n^3} & \frac{x_1x_3}{x_n^3} & \cdots & \frac{x_1x_{n-1}}{x_n^3} \\
\frac{x_2x_1}{x_n^3} & \frac{x_2^2+x_n^2}{x_n^3} & \frac{x_2x_3}{x_n^3} & \cdots & \frac{x_2x_{n-1}}{x_n^3} \\
\frac{x_3x_1}{x_n^3} & \frac{x_3x_2}{x_n^3} & \frac{x_3^2+x_n^2}{x_n^3} & \cdots & \frac{x_3x_{n-1}}{x_n^3} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{x_{n-1}\cdot x_1}{x_n^3} & \frac{x_{n-1}\cdot x_2}{x_n^3} & \cdots & \cdots & \frac{x_{n-1}^2+x_n^2}{x_n^3}
\end{vmatrix}
$$

$$
= \frac{1}{(x_n^3)^{n-1}} \begin{vmatrix}
x_1^2+x_n^2 & x_1x_2 & x_1x_3 & \cdots & x_1x_{n-1} \\
x_2x_1 & x_2^2+x_n^2 & x_2x_3 & \cdots & x_2x_{n-1} \\
x_3x_1 & x_3x_2 & x_3^2+x_n^2 & \cdots & x_3x_{n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
x_{n-1}x_1 & x_{n-1}x_2 & \cdots & \cdots & x_{n-1}^2+x_n^2
\end{vmatrix}
$$

$$= \frac{1}{x_n^{3n-3} \times x_1 \times x_2 \times \cdots \times x_{n-1}} \begin{vmatrix} x_1(x_1^2+x_n^2) & x_1^2 x_2 & x_1^2 x_3 & \cdots & x_1^2 x_{n-1} \\ x_2^2 x_1 & x_2(x_2^2+x_n^2) & x_2^2 x_3 & \cdots & x_2^2 x_{n-1} \\ x_3^2 x_1 & x_3^2 x_2 & x_3(x_3^2+x_n^2) & \cdots & x_3^2 x_{n-1} \\ - & - & - & - & - \\ x_{n-1}^2 x_1 & x_{n-1}^2 x_2 & - & \cdots & x_{n-1}(x_{n-1}^2+x_n^2) \end{vmatrix}$$

$$= \frac{1}{x_n^{3n-3}} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_2^2 & x_2^2+x_n^2 & x_2^2 & \cdots & x_2^2 \\ x_3^2 & x_3^2 & x_3^2+x_n^2 & \cdots & x_3^2 \\ - & - & - & - & - \\ - & - & - & - & - \\ x_{n-1}^2 & x_{n-1}^2 & - & \cdots & x_{n-1}^2+x_n^2 \end{vmatrix} \quad R_1 \to R_1+R_2+R_3+\cdots+R_{n-1}$$

$$= \frac{1}{x_1^{3n-3}} \begin{vmatrix} 1 & 0 & \cdots & 0 \\ x_2^2 & x_1^2 & \cdots & 0 \\ x_3^2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \\ x_{n-1}^2 & 0 & \cdots & x_1^2 \end{vmatrix}$$

$C_{n-1} \to C_{n-1} - C_1$

$C_{n-2} \to C_{n-2} - C_1$

$\cdots$

$C_3 \to C_3 - C_1$

$C_2 \to C_2 - C_1$

$$= \frac{1}{(x_1)^{3n-3}} \times (x_1^2)^{n-2}$$

$$= \frac{1}{x_1^{3n-3}} \times x_1^{2n-4} = \frac{1}{x_1^{n+1}}$$

Maxima & minima

let $z = f(x,y)$ be the function of two variable. we say that $f(x,y)$ has maxima at (a,b) if

$$f(a,b) > f(a+h, b+k)$$

and minima at (a,b) if

$$f(a,b) < f(a+h, b+k)$$

maximum or minimum value of $f(x,y)$ at (a,b) is said to be extremum.

Necessary condition for extremum

Let $z = f(x, y)$ be the function of two variables then the necessary cond^n for maxima or minima is

$$\frac{\partial z}{\partial x} = 0 = \frac{\partial z}{\partial y}$$

The solution of these equations is known as stationary points.

Sufficient conditions for extrema

Let $z = f(x, y)$ be the function of two variable then

if $r = \left(\frac{\partial^2 z}{\partial x^2}\right)_{(a,b)}$, $s = \left(\frac{\partial^2 z}{\partial x \partial y}\right)_{(a,b)}$, $t = \left(\frac{\partial^2 z}{\partial y^2}\right)_{(a,b)}$

(i) If $rt - s^2 > 0$, $r < 0$, then we say that (a,b) is a point of maxima.

(ii) If $rt - s^2 > 0$, $r > 0$, then at (a,b) we will have minima

(iii) If $rt - s^2 < 0$, then at (a,b) we have neither maxima nor minima and such points (a,b) are known as saddle points.

(iv) If $rt - s^2 = 0$, then further investigation is needed i.e we can have either maxima or minima

Q Find the extreme value of $xy(a-x-y)$.

Sol:- $z = xy(a-x-y)$

$= axy - x^2y - xy^2$

$\frac{\partial z}{\partial x} = ay - 2xy - y^2$, $\frac{\partial z}{\partial y} = ax - x^2 - 2xy$

$\frac{\partial^2 z}{\partial x^2} = -2y$, $\frac{\partial^2 z}{\partial y^2} = -2x$, $\frac{\partial^2 z}{\partial x \partial y} = a - 2x - 2y$

Now for extremum :-

$\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$

$y(a-2x-y) = 0$

$x(a-x-2y) = 0$

$\Rightarrow \quad y=0, \quad a-x-2y=0 \Rightarrow (a,0)$

$y=0, \quad x=0 \Rightarrow (0,0)$

$\left.\begin{matrix} (a,0) \\ (0,0) \end{matrix}\right\}$ two stationary points

$x=0, \quad a-2x-y=0 \Rightarrow (0,a)$

Also $\left.\begin{matrix} a-x-2y=0 \\ a-2x-y=0 \end{matrix}\right\} \Rightarrow \begin{matrix} x+2y=a \\ 2x+y=a \times 2 \end{matrix}$

$$\begin{array}{r} 2x+4y=2a \\ -\;(2x+y=a) \\ \hline \end{array}$$

$-3x=-a$

$\Rightarrow x=\frac{a}{3}, \quad 2y=a-\frac{a}{3}$

$\Rightarrow y=a/3$

Thus $(0,0), (0,a), (a,0), \left(\frac{a}{3},\frac{a}{3}\right)$ are 4 stationary points.

$\therefore \quad \dfrac{\partial^2 z}{\partial x^2} = -2y, \quad \dfrac{\partial^2 z}{\partial x \partial y} = a - 2x - 2y, \quad \dfrac{\partial^2 z}{\partial y^2} = -2x$

<u>At (0,0)</u>

$r = \left(\dfrac{\partial^2 z}{\partial x^2}\right)_{(0,0)} = 0, \quad s = \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)_{(0,0)} = a, \quad t = \left(\dfrac{\partial^2 z}{\partial y^2}\right)_{(0,0)} = 0$

$\therefore \quad rt - s^2 = -a^2 < 0 \Rightarrow$ At (0,0) we don't have any maxima or minima.

<u>At (a,0)</u>

$r = 0, \quad s = -a, \quad t = -2a$

$rt - s^2 = -a^2 < 0 \longrightarrow$ No maxima or minima

<u>At (0,a)</u>

$r = -2a, \quad s = a, \quad t = 0$

$rt - s^2 = -a^2 < 0 \longrightarrow$ No maxima or minima

At $\left(\frac{a}{3}, \frac{a}{3}\right)$

$r = -\frac{2a}{3}$, $s = a - \frac{2a}{3} - \frac{2a}{3} = a - \frac{4a}{3} = -\frac{a}{3}$, $t = -\frac{2a}{3}$

$rt - s^2 = \frac{4a^2}{9} - \frac{a^2}{9} = \frac{3a^2}{9} = \frac{a^2}{3} > 0$, $r = \frac{-2a}{3}$

If $\boxed{a > 0}$ then we have $rt - s^2 > 0$, $r = \frac{-2a}{3} \not< 0$

hence at $\left(\frac{a}{3}, \frac{a}{3}\right)$ we have maxima.

If $\boxed{a < 0}$, $\left(\frac{a}{3}, \frac{a}{3}\right)$ is minima and the minimum value

$$z = xy(a - x - y) = \frac{a}{3} \times \frac{a}{3}\left(a - \frac{a}{3} - \frac{a}{3}\right)$$

$$= \frac{a^3}{27}$$

Q Find maxima ~~or~~ minima of $Z = \sin x + \sin y + \sin(x+y)$

or

$Z = \cos x + \cos y + \cos(x+y)$

or

$Z = 8\sin x \sin y \sin(x+y)$

or

$Z = 8\cos x \cos y \cos(x+y)$

$\boxed{u = \sin x \sin y \sin z \text{ when } x, y, z \text{ are the angle of triangle}}$

$x+y+z=\pi$

$z = \pi - (x+y)$

Sol:

$$Z = \sin x + \sin y + \sin(x+y)$$

$$\frac{\partial Z}{\partial x} = \cos x + \cos(x+y), \quad \frac{\partial^2 Z}{\partial x^2} = -\sin x - \sin(x+y)$$

$$\frac{\partial Z}{\partial y} = \cos y + \cos(x+y), \quad \frac{\partial^2 Z}{\partial y^2} = -\sin y - \sin(x+y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\sin(x+y)$$

$\cos x = \cos y$

$x = 2n\pi \pm y$

$\cancel{\cos\alpha = \cos\beta}$

$\cancel{\alpha = 2n\pi \pm \beta}$

$\cancel{\cos x = \cos y}$

$\cancel{x = y}$

For maxima or minima

$$\frac{\partial z}{\partial x} = 0 = \frac{\partial z}{\partial y}$$

$$\cos x + \cos(x+y) = 0, \quad \cos y + \cos(x+y) = 0$$

$\downarrow$

$$\Rightarrow \cos x = -\cos(x+y)$$

$$\Rightarrow \cos x = \cos(\pi - (x+y))$$

$$\Rightarrow x = \pi - x - y \Rightarrow 2x + y = \pi$$

$\cancel{\cos x = \cos(\pi + x + y)}$

$\cancel{x = \pi + x + y}$

$\cancel{y = -\pi}$

$(-\sqrt{3}, -\sqrt{3})$

similarly

$$2y + x = \pi$$

on solving

$$(a,b) = \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$$

~~r =~~

$$r = \left(\frac{\partial^2 z}{\partial x^2}\right)_{\left(\frac{\pi}{3}, \frac{\pi}{3}\right)} = -\sqrt{3}$$

$$s = \left(\frac{\partial^2 z}{\partial x \partial y}\right)_{\left(\frac{\pi}{3}, \frac{\pi}{3}\right)} = \frac{\sqrt{3}}{2}$$

$$t = \left(\frac{\partial^2 z}{\partial y^2}\right)_{\left(\frac{\pi}{3}, \frac{\pi}{3}\right)} = -\sqrt{3}$$

$\because$ $rt - s^2 = \frac{9}{4} > 0$ and $r = -\sqrt{3} < 0$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ we have maxima.

and max^m value $= \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{2}$.

Q Find maximum or minimum value in $u = \cos A \cos B \cos C$ where A, B & C are the angles of the triangle.

Sol: $\because$ $A + B + C = \pi$

$\Rightarrow C = \pi - (A+B)$

$\therefore$ $u = \cos A \cos B \cos C$

$= \cos A \cos B \cos(\pi - (A+B))$

$\therefore \quad u = -\cos A \cos B \cos(A+B)$

$$\frac{\partial u}{\partial A} = -\cos B\left[-\sin A \cos(A+B) + \cos A \times -\sin(A+B)\right]$$

$$= \cos B \sin(2A+B)$$

$$\frac{\partial u}{\partial B} = \cos A \sin(2B+A)$$

✓ $$\frac{\partial^2 u}{\partial A^2} = 2\cos B \cos(2A+B)$$

✓ $$\frac{\partial^2 u}{\partial B^2} = 2\cos A \cos(2B+A)$$

✓ $$\frac{\partial^2 u}{\partial A \partial B} = \left[-\sin B \sin(2A+B) + \cos B \times \cos(2A+B)\right]$$

$$= \cos(2A+2B)$$

For maxima & minima

$$\frac{\partial u}{\partial A} = 0 = \frac{\partial u}{\partial B}$$

$$\cos B \sin(2A+B) = 0 \quad \text{--- (1)}$$

$$\cos A \sin(2B+A) = 0 \quad \text{--- (11)}$$

$$\cos B = 0$$

$$\boxed{B = \frac{\pi}{2}}$$

$$\sin(2B+A) = 0$$

$$\sin(\pi + A) = 0$$

$$-\sin A = 0, \ \cancel{\boxed{A = 0}}$$

$$\cancel{2A+B = 0}$$

$$\cancel{2B+A = 0}$$

$$\Rightarrow \sin(2A+B) = 0 = \sin\pi$$

$$\text{and } \sin(2B+A) = 0 = \sin\pi$$

$$\Rightarrow 2A+B = \pi$$

$$2B+A = \pi \times 2$$

$$-3B = -\pi \Rightarrow \boxed{B = \frac{\pi}{3}}$$

$$2A+B = \pi$$

$$2A = \pi - \frac{\pi}{3}$$

$$\boxed{A = \frac{\pi}{3}}$$

$$\boxed{C = \frac{\pi}{3}}$$

$\therefore\ r = \left(\frac{\partial^2 u}{\partial A^2}\right)_{\left(\frac{\pi}{3}, \frac{\pi}{3}\right)} = 2\cos\frac{\pi}{3} \times \cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)$

$= 2 \times \frac{1}{2} \times -1 = -1$

$s = \cos\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \cos\frac{4\pi}{3} = \frac{-1}{2}$

$t = -1$

$\therefore\ rt - s^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0$, $r = -1 < 0$

$\Rightarrow \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a point of maxima.

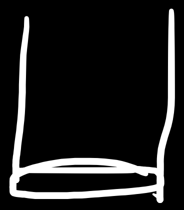
$\therefore$ max^m value $z = \cos\frac{\pi}{3} \times \cos\frac{\pi}{3} \times \cos\frac{\pi}{3} = \frac{1}{8}$ #

Q. Find the dimensions of the rectangular box, open at the top of maximum capacity whose surface is given.

Sol:- Let x, y & z are the dimension of the rectangular box

$$S = xy + 2yz + 2zx = \text{given} \Rightarrow 2z(y+x) = -xy + S$$

$$\boxed{z = \frac{-xy+S}{2(x+y)}}$$



Now we have to maximize

$$u = xyz$$

$$\boxed{u = xy\left(\frac{S-xy}{2(x+y)}\right)}$$

$$\therefore \quad u = \frac{5xy}{2(x+y)} - \frac{x^2y^2}{2(x+y)} = \frac{5xy - x^2y^2}{2(x+y)}$$

$$\cancel{\frac{\partial u}{\partial x} = \frac{5y}{2}\left[\frac{1}{x+y} + x \times \frac{1}{(x+y)^2}\right]} - \cancel{\frac{y^2}{2}\left[2x \times \frac{1}{x+y} + x^2 \times \frac{-1}{(x+y)^2}\right]}$$

$$= \frac{5y}{}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{5y - 2xy^2}{2(x+y)} - \frac{(5xy - x^2y^2)}{2(x+y)^2}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{5x - 2yx^2}{2(x+y)} - \frac{5xy - x^2y^2}{2(x+y)^2}$$

$$= 0$$

$$= 0$$

Now $\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}$

$$\Rightarrow \quad \frac{5y - 2xy^2}{2(x+y)} - \frac{5xy - x^2y^2}{2(x+y)^2} = 0 \quad \text{— (I)}$$

$$\frac{5x - 2yx^2}{2(x+y)} - \frac{5xy - x^2y^2}{2(x+y)^2} = 0 \quad \text{— (II)}$$

(I) — (II) $\Rightarrow$

$$5y - 2xy^2 = 5x - 2yx^2$$

$$\Rightarrow 5(y-x) = 2xy^2 - 2yx^2$$

$$\Rightarrow 5(y-x) = 2xy(y-x)$$

$$\Rightarrow (5-2xy)(x-y)=0$$

$$\therefore 5 \neq 2xy$$

$$\boxed{x=y}$$

$$\boxed{y=\sqrt{\frac{5}{3}}}$$

$$\sqrt{\frac{108}{3}} = \sqrt{36} = 6$$

Hence by ①

$$\frac{5\cancel{x}-2x^3}{2\times 2x} - \frac{(5x^2-x^4)}{2\times 4x^2} = 0$$

$$\Rightarrow 2x(5x-2x^3) - 5x^2+x^4=0$$

$$\Rightarrow 5x^2-3x^4=0$$

$$\Rightarrow 5=3x^2 \Rightarrow \boxed{x=\sqrt{5/3}}$$

$$z = \frac{s - xy}{2(x+y)} = \frac{s - \frac{s}{3}}{2 \times 2 \times \sqrt{s/3}} = \frac{2s}{3 \times 4 \times \sqrt{\frac{s}{3}}}$$

$$= \frac{1}{2}\sqrt{\frac{s}{3}}$$

$$\therefore \; x = \sqrt{\frac{s}{3}}, \quad y = \sqrt{\frac{s}{3}}, \quad z = \frac{1}{2}\sqrt{\frac{s}{3}}.$$

Now show that $\underline{rt - s^2 > 0,\ r < 0}$ at (x, y, z).

#

- Video Lecture (Unit I)

(1) https://youtu.be/OwJ8CvXejYs
(2) https://youtu.be/qtAibJ_N4bw
(3) https://youtu.be/qtAibJ_N4bw
(4) https://youtu.be/VlTors_ON0Y
(5) https://youtu.be/JHNO49RmSgw
(6) https://youtu.be/M-JE2-jXSaw
(7) https://youtu.be/0Fv2iMLL77A
(8) https://youtu.be/4ZyN99gNCOo
(9) https://youtu.be/Il-X_8BeFog
(10) https://youtu.be/FHWaoaPX4rk
(11) https://youtu.be/B2xwuEQRJxs
(12) https://youtu.be/jBukOH3HxhU