Decomposition Principle

Monday, June 14, 2021 3:30 PM

Practically, some of linear programming problems may be very large in terms of the number of variables and or number of constraints.

If this L.P.P. has some special structure, then it solve by applying the Decomposition Principle developed by Dantzing and Wolfe.

In this principle, the original problem is decomposed into small subproblems and solved almost independently.

The special structure of L.P.P. can be generalized as

$$\text{Minimize } f(X) = C_1^T X_1 + C_2^T X_2 + \cdots + C_q^T X_q \tag{1.1}$$

Subject to

$$A_1X_1 + A_2X_2 + \cdots + A_qX_q = b_0 \rightarrow \text{m. Const.} \tag{1.2}$$

$$\left.\begin{array}{llll} B_1X_1 & & & = b_1 \\ & B_2X_2 & & = b_2 \\ & & \vdots & \\ & & B_qX_q & = b_q \end{array}\right\} \tag{1.3}$$

$A_1 x_1 \le b_1$

$\downarrow$

$A_1 x_1 + s_1 = b_1$

$$X_1 \ge 0, \quad X_2 \ge 0, \ldots, X_q \ge 0$$

Where,

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m1} \end{pmatrix}, \qquad X_2 = \begin{pmatrix} x_{m1+1} \\ x_{m1+2} \\ \vdots \\ x_{m1+m2} \end{pmatrix}, \ldots,$$

$$X_q = \begin{pmatrix} x_{m1+m2+\cdots+m_{q-1}+1} \\ x_{m1+m2+\cdots+m_{q-1}+2} \\ \vdots \\ x_{m1+m2+\cdots+m_{q-1}+m_q} \end{pmatrix}, \qquad X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_q \end{pmatrix}.$$

Here, size of matrix A_k is $(r_0 \times m_k)$ and that of B_k is $(r_k \times m_k)$. The problem has $\sum_{k=0}^{q} r_k$ constraints and $\sum_{k=1}^{q} m_k$ variables.

Now, the solution procedure using decomposition principle as the following steps-

Define q sudsidiary constraint sets using above general L.P.P. as

$$\left.\begin{aligned} B_1X_1 &= b_1 \\ B_2X_2 &= b_2 \\ &\vdots \\ B_kX_k &= b_k \\ &\vdots \\ B_qX_q &= b_q \end{aligned}\right\} \qquad (1.4)$$

1.5

Or, (1.4) write as

$$B_kX_k = b_k, \quad k = 1, 2, \dots, q\,, \quad X_k \geq 0. \qquad (1.5)$$

Represents r_k equality constraints. These constraints along with the $X_k \geq 0$ define the set of feasible solutions of equations (1.5). Assuming that this set of feasible solutions is bounded convex set.

Let t_k be the number of vertices of this feasible set. By definition of convex combination of a set of points, any point X_k satisfying (1.5) can be represented as

$$X_k = \beta_{k,1}X_1^{(k)} + \beta_{k,2}\,X_2^{(k)} + \cdots + \beta_{k,t_k}X_{t_k}^{(k)} \qquad (1.6)$$

$$\beta_{k,1} + \beta_{k,2} + \cdots + \beta_{k,t_k} = 1 \qquad (1.7)$$

$$0 \leq \beta_{k,j} \leq 1, \qquad k = 1, 2, \dots, q, \qquad j = 1,2, \dots, t_k \qquad (1.8)$$

Where, $X_1^{(k)}, X_2^{(k)}, \dots, X_{t_k}^{(k)}$ are the extreme points/ vertices of the feasible set defined by the equation (1.5). So, equations (1.5) can be written as-

$$B_k\left(\beta_{k,1}X_1^{(k)} + \beta_{k,2}\,X_2^{(k)} + \cdots + \beta_{k,t_k}X_{t_k}^{(k)}\right) = b_k, \quad k = 1, 2, \dots, q \quad (1.9)$$

$$\text{with} \qquad \beta_{k,1} + \beta_{k,2} + \cdots + \beta_{k,t_k} = 1$$

$$0 \leq \beta_{k,j} \leq 1, \qquad k = 1, 2, \dots, q, \qquad j = 1,2, \dots, t_k.$$

Now, with the help of equations (1.6) and (1.9), original problem is written as-

$$\text{Minimize } f(X) = C_1^T \left(\sum_{j=1}^{t_1} \beta_{1,j} X_j^{(1)}\right) + C_2^T \left(\sum_{j=1}^{t_2} \beta_{2,j} X_j^{(2)}\right) + \cdots + C_q^T \left(\sum_{j=1}^{t_q} \beta_{q,j} X_j^{(q)}\right) \tag{2.0}$$

Subject to

$$A_1\left(\sum_{j=1}^{t_1} \beta_{1,j} X_j^{(1)}\right) + A_2\left(\sum_{j=1}^{t_2} \beta_{2,j} X_j^{(2)}\right) + \cdots + A_q\left(\sum_{j=1}^{t_q} \beta_{q,j} X_j^{(q)}\right) = b_0 \tag{2.1}$$

$$\left.\begin{aligned} \sum_{j=1}^{t_1} \beta_{1,j} &= 1 \\ \sum_{j=1}^{t_2} \beta_{2,j} &= 1 \\ &\vdots \\ \sum_{j=1}^{t_q} \beta_{q,j} &= 1 \end{aligned}\right\} \tag{2.2}$$

$$\beta_{k,j} \geq 0, \qquad k = 1, 2, \dots, q, \qquad j = 1,2, \dots, t_k \tag{2.3}$$

Where, $X_1^{(k)}, X_2^{(k)}, \dots, X_{t_k}^{(k)}$ are the extreme points/ vertices of the feasible set defined by the equation (1.5) and known.

Also, C_k^T, A_k , $k = 1, 2, \dots, q$ are known in the problem. Only, $\beta_{k,j}$ are unknowns, means $\beta_{k,j}$ will be the new decision variables of the modified problem stated in equations (2.0) - (2.3).

So, after finding the $\beta_{k,j}$ by any known methods, the optimal solution of the original problem can be obtained.