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Representation of Sets
⇒ Roaster or Tabular
⇒ Rule or Set builder.

Set

Finite | Disjoint
Infinite | Null
Singleton
Subset
Universal

Operation on set

⇒ Union 'U'
⇒ Intersection '∩'
⇒ Complement 'A^c / A'' $= U - A$
⇒ Relative Complement $(A-B)$
↳ Relative complement of B wrt A
⇒ Symmetric Difference $A \Delta B$ or $A \oplus B = (A-B) \cup (B-A)$

e.g. $A = \{-3, 0, 1, 2\}$ $B = \{1, 2, 3, 4\}$
$A \oplus B = \{-3, 0, 3, 4\}$

Algebra of Sets

⇒ Idempotent law

$A \cap A = A$

$A \cup A = A$

⇒ Associative law

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

⇒ Cammutative law

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

⇒ Distributive law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

⇒ Identity

$$A \cup \phi = A$$
$$A \cup U = U$$
$$A \cap \phi = \phi$$
$$A \cap U = A$$

⇒ Involution law

$$(A')' = A$$

⇒ Complement law

$$A \cup A' = U$$
$$A \cap A' = \phi$$

⇒ D' Morgan's law

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

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$(A\cup B)' = A'\cap B'$

L.H.S: $(A\cup B)'$

Let $x \in (A\cup B)'$

$x \in U-(A\cup B)$

$\Rightarrow x \in U$ & $x \notin A\cup B$

$\Rightarrow x\in U$ & $(x\notin A$ & $x\notin B)$

$\Rightarrow (x\in U$ & $x\notin A)$ & $(x\in U$ & $x\notin B)$

$\Rightarrow x\in U-A$ & $x\in U-B$

$\Rightarrow x\in A'$ & $x\in B'$

$\Rightarrow x\in A'\cap B'$

$\Rightarrow (A\cup B)' \subseteq A'\cap B'$ — ① & $(A\cup B)' \supseteq A'\cap B'$ — ②

From ① & ②

$(A\cup B)' = A'\cap B'$

2) $(A\cap B)' = A'\cup B'$

L.H.S. $(A\cap B)'$

Let $x\in(A\cap B)'$

$\Rightarrow x\in U-(A\cap B)$

$\Rightarrow x\in U$ & $x\notin A\cap B$.

$\Rightarrow x\in U$ & $(x\notin A$ or $x\notin B)$

$\Rightarrow (x\in U$ & $x\notin A)$ or $(x\in U$ & $x\notin B)$

$\Rightarrow x\in U-A$ or $x\in U-B$

$\Rightarrow x\in A'$ or $x\in B'$

$\Rightarrow x\in A'\cup B'$

$(A\cap B)' \subseteq A'\cup B'$ — ① & $(A\cup B)' \supseteq A'\cup B'$ — ②

From ① & ②

$(A\cap B)' = A'\cup B'$

Cardinal No: OR Cardinality ($|A|$ / $n(A)$)

Properties of Cardinal no.

* $|A \cup B| = |A| + |B| - |A \cap B|$

* $|A - B| = |A| - |A \cap B|$

* $|B - A| = |B| - |B \cap A|$

* $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

Power set $|P(A)| = 2^{|A|}$

Theorem :- If $A \subseteq B$ then $P(A) \subseteq P(B)$

Countable set

* Finite & infinite set with one-one correspondance with natural numbers

e.g. $Z = \{0, \pm 1, \pm 2, \pm 3, \dots\}$
↳ countable

$\aleph_0$ (Aleph not)

$|N| = |Z| = |Q| = \aleph_0$

* $\aleph_0 + \aleph_0 = \aleph_0$
* $\aleph_0 \times \aleph_0 = \aleph_0$

$|A| = \aleph_0$ $\quad |P(A)| = 2^{\aleph_0} = C$ (Canters No)
(Uncountable)

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Theorem:- Union of countable families of countable sets is countable

Proof:- Let $\{A_1, A_2, A_3, \ldots A_n\}$ be countable family of countable sets

$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots \cancel{a_{1n}}\}$
$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots \cancel{a_{2n}}\}$
$A_3 = \{a_{31}, a_{32}, a_{33}, \ldots \cancel{a_{3n}}\}$
$\ldots$
$A_n = \{a_{n1}, a_{n2}, a_{n3} \ldots\}$

then $\bigcup_{i=1}^{n} A_i$ can be found in following way

$A_1 = a_{11}, a_{12}, a_{13}, \ldots$
$A_2 = a_{21}, a_{22}, a_{23}, \ldots$
$A_3 = a_{31}, a_{32}, a_{33}, \ldots$
$A_4 = a_{41}, a_{42}, a_{43}, \ldots$

List the elements a_{11}

a_{21} a_{12}

a_{31} a_{22} a_{13}

a_{41} a_{32} a_{23} a_{14}

$\ldots$

a_{n1} $a_{(n-1)2}$ $a_{(n-2)3}$ $a_{(n-3)4}$

It is clear that a_{ij} is the jth element of $(i+j-1)$th row. Thus all the ele. have been counted. Thus $\bigcup_{i=1}^{n} A_i$ is countable.

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Theorem :- Show $N \times N$ is countable

Proof :- $(1,1)$, $(1,2)$, $(1,3)$ . . . .
$(2,1)$, $(2,2)$, $(2,3)$ . . . .
$(3,1)$, $(3,2)$, $(3,3)$ - - . .
$(4,1)$, $(4,2)$, $(4,3)$ - -
– – – –

List the elements $(1,1)$
$(2,1)$, $(1,2)$
$(3,1)$, $(2,2)$, $(1,3)$
$(4,1)$, $(3,2)$, $(2,3)$, $(1,4)$

It is clear that a_{ij} is the jth ele of row. Thus all ele. have been counted.
Thus $\bigcup_{i=1}^{n} A_i$ is countable.

If A & B are countable the $A \times B$ is countable.

Relation

$R \subseteq A \times B = \{(a_1, b_1), (a_2, b_2), (a_3, b_3) \dots (a_n, b_n)\}$
$\phi \subset R$ (void)
$A \times B \subseteq R$ (universal)

Total no of relation $= 2^{mn}$
$|A| = m \quad |B| = n$
$|A \times B| = mn$

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Types of Relation

1. Inverse Relation

R be a relation from A to B.

R^{-1} be a relation from B to A

i.e. $R^{-1} = \{(b,a) : (a,b) \in R\}$

or, $x R y \Rightarrow y R^{-1} x$

e.g. $A = \{2,3,5\}$ $B = \{6,8,10\}$

$\forall (x,y) \in A$ or B, $(x,y) \in R \Leftrightarrow x$ divide y

R & $R^{-1} = ?$

$2R6$, $2R8$, $2R10$, $3R6$, $5R10$

$R = \{(2,6), (2,8), (2,10), (3,6), (5,10)\}$

$R^{-1} = \{(6,2), (8,2), (10,2), (6,3), (10,5)\}$

Domain (R) = Range $(R^{-1}) = \{2,3,5\}$

Domain (R^{-1}) = Range $(R) = \{6,8,10\}$

2. Identity Relation

$I_A = \{(x,x) : x \in A\}$

e.g. $A = \{1,2,3\}$

$I_A = \{(1,1), (2,2), (3,3)\}$

3. Reflexive Relation

$(a,a) \in R$ i.e. aRa $\forall a \in A$.

4. Irreflexive
$(a,a) \notin R$ i.e. $a \not R a$ $\forall a \in A$.

5. Symmetry
$(a,b) \in R \Rightarrow (b,a) \in R$
i.e. $aRb \Rightarrow bRa$ $\forall a,b \in A$

6. Asymmetric
$(a,b) \in R \Rightarrow (b,a) \notin R$
i.e. $aRb \Rightarrow b \not R a$ $\forall (a,b) \in A$.

7. Antisymmetric
$(a,b) \in R$ & $(b,a) \in R$
$\Rightarrow a = b$.
i.e. aRb & $bRa \Rightarrow a = b$ $\forall (a,b) \in A$.

8. Transitive
~~(a,b) ∈~~ aRb & $bRc \Rightarrow aRc$
$\forall a,b,c \in A$

e.g. Equivalence Relation

If R is relation in Z defined by
$R = \{(x,y) : x,y \in Z ; (x-y)$ divisible by $6\}$

Soln Reflexive : $xRx \Rightarrow x-x$ is divisible by 6
0 is " " "

Symmetric : $xRy \Rightarrow x-y$ is divisible by 6
$\Rightarrow -(y-x)$ is divisible by 6.
$\Rightarrow yRx$

Transitive: $x-y$ divisible by 6
$\Rightarrow$ $y-z$ divisible by 6.
$x-z$ divisible by 6

Note:

1. R is reflexive $\Rightarrow$ R^{-1} is reflexive
2. R is symmetric $\Leftrightarrow$ R^{-1} is symmetric
3. R is antisymmetric $\Leftrightarrow$ $R \cap R^{-1} \subseteq I$
4. R & S are reflexive $\Rightarrow$ $R \cup S$ & $R \cap S$ are also reflexive.
5. R & S are symmetric $\Rightarrow$ $R \cup S$ & $R \cap S$ are also symmetric
6. R & S are transitive $\Rightarrow$ $R \cap S$ is transitive.
7. R & S are equivalence $\Rightarrow$ $R \cap S$ is equivalence
8. R is equivalence $\Rightarrow$ R^{-1} is equivalence

Partial Order Rel.

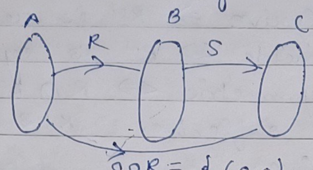
A relation R on set S is partial order if reflexive : aRa ; $\forall a \in S$
Antisymmetric : aRb & bRa $\Rightarrow$ $a=b$ $\forall a,b \in R$
Transitive : aRb & bRc $\Rightarrow$ aRc $\forall a,b,c \in S$

A set S together with partial order R i.e. (S,R) is partial order

Ex '$\geq$' relation is partial order on Z
Reflexive $a \geq a$
Antisymmetric : $a \geq b$ & $b \geq a$ $\Rightarrow$ $a=b$
Transitive : $a \geq b$ & $b \geq c$ $\Rightarrow$ $a \geq c$
(Z,R)

Composite Relation

If A, B & C are non empty sets & R be a relation from A to B & S be a relation from B to C then composite relation of R & S is relation from A to C



$S \circ R = \{(a,c) : a \in A,\ c \in C\}$

Ex. $A = \{1,2,3\}$; $B(p,q,r)$, $C = \{x,y,z\}$

$R = \{(1,p), (1,q) (2,q), (3,q)\}$

$S = \{\{(p,y); (q,x); (r,z)\}$

~~RoS = {(1,y); (1,x); (~~

$S \circ R = \{(1,y), (1,x); (2,x), (3,x)\}$

Function

Bijective ← [
- One-one (Injective) $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- Many-one → ~~co domain~~ में ele. की $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$
- Onto (Surjective)
]
- Into → co-domain का कोई ele. बच जाए.

Composition of Function

Let $f : A \to B$ & $g : B \to C$

Composition of f & g $g \circ f : A \to C$ defines as

$$g \circ f(x) = g[f(x)] \quad \forall\ x \in A$$

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Qu $A = \{1, 2, 3\}$; $B = \{a, b\}$; $C = \{r, s\}$

$f : A \to B$ as $f(1) = a$; $f(2) = a$, $f(3) = b$.

$g : B \to C$ as $g(a) = s$; $g(b) = r$

$gof \quad A \to C$

$$g(f(x)) = g(f(1)) = g(a) = s.$$
$$g(f(2)) = g(a) = s$$
$$g(f(3)) = g(b) = r$$

Property

⇒ Associative : $\underset{A \to B}{f} \circ (\underset{B \to C}{g} o \underset{C \to D}{h}) = (f o g) o h$

$f : A \to B \qquad g : B \to C \qquad h : C \to D$

$gof : A \to C$ & $hog : B \to D$

Hence : $h \circ (gof) : A \to D$

& $(hog) \circ f : A \to D.$

Dom. of $[h \circ (gof)]$ = dom. $[(hog) \circ f]$

let $x \in A$; $y \in B$, $z \in C$

i.e. $f(x) = y$ & $g(y) = z$

then

$$[(hog) \circ f](x) = h(z) \quad \text{— (1)}$$
$$[h \circ (gof)](x) = h(z) \quad \text{— (2)}$$

From (1) & (2)

$$[(hog) \circ f] = [h \circ (gof)]$$

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Theorem!

Let $f: A \to B$ & $g: B \to C$

(a) If f and g are injection then gof is injection

(b) If f & g are surjection then gof is surjection

PROOF :-

(a) Let $a_1, a_2 \in A$.

We have $(gof)a_1 = (gof)a_2$

$\Rightarrow g[f(a_1)] = g[f(a_2)]$

$\Rightarrow f(a_1) = f(a_2)$ [g is injection]

$a_1 = a_2$ [f is injection]

Thus, gof is injection

(b) Let $c \in C$, then we can find $a \in A$ s.t $(gof)(a) = c$

g is onto $C \Rightarrow b \in B$ s.t $g(b) = c$

then since f is onto B, there exist $a \in A$ s.t $F(a) = b$

Thus $(gof)(a) = g[f(a)] = g(b) = c$.

Inverse

Let $f: A \to B$ ~~B~~ then $g: B \to A$ if

$gof = I_A$ & $fog = I_B$

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Q Show that $f(x) = x^3$ & $g(x) = \sqrt[3]{x}$ $\forall x \in R$ are inverse to each other

$$fog(x) = f[g(x)] = f[\sqrt[3]{x}] = x = Ix$$
$$gof(x) = g[f(x)] = g(x^3) = x = Ix$$

Theorem:

$f: A \to B$ is one-one & onto then
$f^{-1}: B \to A$ is one-one & onto

Proof:

Given :- $f: A \to B$ is one-one & onto

Let $a_1, a_2 \in A$ & $b_1, b_2 \in B$

so $b_1 = f(a_1)$; $b_2 = f(a_2)$

& $a_1 = f^{-1}(b_1)$; $a_2 = f^{-1}(b_2)$

f is one-one $f(a_1) = f(a_2) \Leftrightarrow a_1 = a_2$

$b_1 = b_2 \Leftrightarrow f^{-1}(b_1) = f^{-1}(b_2)$

i.e. $f^{-1}(b_1) = f^{-1}(b_2) \Rightarrow b_1 = b_2$

$\therefore f^{-1}$ is one-one

f is onto :-

all element at B is associated with unique ele. at A.

i.e.

For any $a \in A$ is pre image of $b \in B$

where $b = f(a) \Rightarrow a = f^{-1}(b)$

i.e. for $b_1 \in B$ f^{-1}

So f^{-1} is onto

Principle of Mathematical Induction

1) $S(n)$ is true for $n = 1, \cancel{2, 3}$ i.e. $S(1)$ is true
2) $S(k)$ is true $\Rightarrow S(k+1)$ is true

Qu $1 + 2 + 3 \cdots + n = \frac{n(n+1)}{2} = \Sigma N$

For $n = 1$

$S(1) = \frac{1(1+1)}{2} = 1$

For $n = k$

$S(k) = \frac{k(k+1)}{2}$

For $n = k+1$, we have to prove

$S(k+1) = \frac{(k+1)(k+2)}{2}$

$\Rightarrow$ $1 + 2 + 3 + \ldots k + (k+1) = \frac{k(k+1)}{2} + k+1$

$= (k+1)\left[\frac{k}{2} + 1\right]$

$= \frac{(k+1)(k+2)}{2}$

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Q $1^2+2^2+3^2+ \dots n^2 = \frac{1}{6}n(n+1)(2n+1)$

For $n=1$

$S(1) = \frac{1}{6}(2)(3)$

$= 1$

For $n=k$

$S(k) = \frac{k}{6}(k+1)(2k+1)$

For $n=k+1$

$1^2+2^2+3^2+ \dots k^2+(k+1)^2 = \frac{k}{6}(k+1)(2k+1) + (k+1)^2$

$\cancel{= (k+1)\left[\frac{k(2k+1)}{6} + k+1\right]}$

$\cancel{= (k+1)\left[\frac{2k(2k+1) + 6k+6}{6}\right]}$

$\cancel{= \frac{1}{6}(k+1)\left[2k^2+k+6\right]}$

$=$

$= (k+1)\left[\frac{k(2k+1)}{6} + k+1\right]$

$= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$

$= \frac{(k+1)}{6}\left[2k^2+7k+6\right]$

$= \frac{(k+1)}{6}\left[2k^2+4k+3k+6\right]$

$= \frac{(k+1)}{6}\left[2k(k+2)+3(k+2)\right] = \frac{(k+1)}{6}\left[(k+2)(2k+3)\right]$

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Q. Prove by mathematical induction $6^{n+2} + 7^{2n+1}$ is divisible by 43 $\forall\, n \in Z$

<u>Proof :</u>

For $n = 1$

$$6^{1+2} + 7^{2+1} = 6^3 + 7^3$$
$$= 559$$

which is divisible by 43.

For $n = k$

$6^{k+2} + 7^{2k+1}$ is divisible by 43

$\Rightarrow 6^{k+2} + 7^{2k+1} = 43m \quad , \forall\, m \in Z$

For $n = k+1$

$$6^{k+3} + 7^{2k+3}$$
$$= 6^{k+2+1} + 7^{2k+1+2}$$
$$= 6^{k+2} \cdot 6 + 7^{2k+1}\, 7^2$$
$$= 6 \cdot 6^{k+2} + 7^{2k+1}(43+6)$$
$$= \cancel{6 [6^k}$$
$$= 6 [6^{k+2} + 7^{2k+1}] + 43 \cdot 7^{2k+1}$$
$$= 6 \cdot 43m + 43 \cdot 7^{2k+1}$$
$$= 43 [6m + 7^{2k+1}]$$