Elementary transformation of a matrix:-

The following operations, three of which refer to rows and three of column known as elementary transformation

(i) The interchange of any two rows (columns)

(ii) The multiplication of any row (column) by a non-zero number.

(iii) The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Notation: The elementary transformations will be denoted by the following symbols:

(i) R_{ij} for the interchange of the i^{th} and j^{th} row

(ii) kR_i for multiplication of the ith row by k.

(iii) $R_i + pR_j$ for addition of i^{th} row, p times the j^{th} row.

* The corresponding column transformation will be denoted by writing C in place of R.

** Elementary transformation do not change either the order or rank of a matrix, while the value of the minors may get change by the transformation I & II, their zero or non-zero character remain unaffected.

Equivalent matrix:- The matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order and the same rank. The symbol ~ is used for equivalence.

Ques:- Determine the rank of the matrix

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \qquad C_3 \to C_3 - C_1 \ \& \ C_4 \to C_4 - C_1$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 - R_1, \ R_4 \to R_4 - R_1$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 - 3R_2, \ R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, we can observe that 4^{th} order minor is zero and every third order minor is zero One of the second order minor $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

$\Rightarrow \rho(A) = 2.$

Normal form (Canonical form):-

By performing elementary transformation any non-zero matrix A can be reduced to one of the following four forms, called normal form of

(i) I_r (ii) $[I_r \ \ 0]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The number r so obtained is called the rank of A and we write $\rho(A) = r$. The forms $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called first canonical form of A.

Q Find the rank of the following matrix A by reducing it into normal form:-

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad R_4 \to R_4 + \frac{1}{2} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 + C_1 \\ C_4 \to C_4 - 3C_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & +1 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2 \to R_2 + 3R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_4 \to C_4 + 2C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_4 \to C_4 + \frac{1}{7}C_2$$

$$\sim \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 0 \end{bmatrix} \qquad \begin{array}{l} C_2 \to -\frac{1}{7}C_2 \\ C_3 \to -\frac{1}{2}C_3 \end{array}$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\rho(A) = 3}$$

Q For what value of b the rank of the matrix

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix} \text{ is } 2.$$

$\Rightarrow$ third order minor is 0

$$|A| = 1(30-26) - 5(-2b) + 4(-3b) = 0$$

$$\Rightarrow 4 + 10b - 12b = 0$$

$$\Rightarrow 2b = 4 \quad \Rightarrow \boxed{b = 2}$$