# Karmarkar's Method

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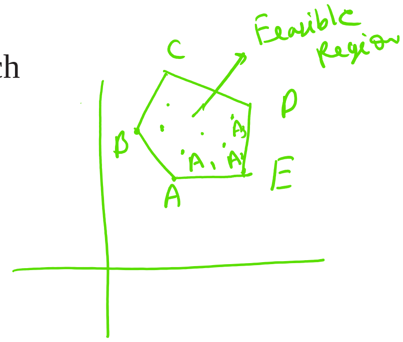


The method is known as an *interior method* since it finds improved search directions strictly in the interior of the feasible space.

while simplex method, which searches along the boundary of the feasible space by moving from one feasible vertex to an adjacent one until the optimum point is found.

It was reported [4.19] that Karmarkar's method solved problems involving 150,000 design variables and 12,000 constraints in 1 hour while the simplex method required 4 hours for solving a smaller problem involving only 36,000 design variables and 10,000 constraints.

**Statement of the Problem**

Karmarkar's method requires the LP problem in the following form:

$$\text{Minimize } f = c^T X$$

Subject to

$$[a]X = 0 \qquad (1)$$
$$x_1 + x_2 + \cdots + x_n = 1$$
$$X \geq 0$$

minimize $f = c_1x_1 + c_2x_2 + c_3x_3$
subject

$$2x_1 + 3x_2 + 4x_3 = 0$$
$$x_1 + 5x_2 + 4x_3 = 0$$
$$x_1 + x_2 + x_3 = 1$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[a]X = 0$$

Where, $X = \{x_1, x_2, \ldots, x_n\}^T$ → cost coefficients.

$c = \{c_1, c_2, \ldots, c_n\}^T$,

and [a] is an m × n matrix. In addition, an interior feasible starting solution to Eqs. (1) must be known. Usually,

$$X = \left\{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right\}^T = \text{interior feasible} \qquad (2)$$

$$f_{min} = 0$$

Although most LP problems may not be available in the form of Eq. (1) while satisfying the conditions of Eq. (2), it is possible to put any LP problem in a form that satisfies Eqs. (1) and (2) as indicated below.

**Algorithm**

Starting from an interior feasible point $\mathbf{X}^{(1)}$, Karmarkar's method finds a sequence of points $\mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \cdots$ using the following iterative procedure:

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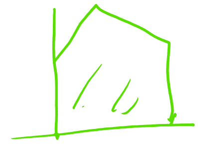
$$\mathbf{X}^{(1)} = \left\{ \frac{1}{n} \; \frac{1}{n} \cdots \frac{1}{n} \right\}^{\mathrm{T}}.$$



Set the iteration number as $k = 1$.

2. Test for optimality. Since $f = 0$ at the optimum point, we stop the procedure if the following convergence criterion is satisfied:

$$||\mathbf{c}^T \mathbf{X}^{(k)}|| \le \varepsilon$$

where ε is a small number. If above Eq. is not satisfied, go to step 3.

3. Compute the next point, $\mathbf{X}^{(k+1)}$. For this, we first find a point $\mathbf{Y}^{(k+1)}$ in the transformed unit simplex as

weight term / functions

$$\mathbf{Y}^{(k+1)} = \left\{ \frac{1}{n} \; \frac{1}{n} \cdots \frac{1}{n} \right\}^{\mathrm{T}} - \frac{\alpha([I] - [P]^{\mathrm{T}}([P][P]^{\mathrm{T}})^{-1}[P])[D(\mathbf{X}^{(k)})]\mathbf{c}}{||\mathbf{c}|| \sqrt{n(n-1)}} \qquad \text{— (1)}$$

where $||\mathbf{c}||$ is the length of the vector $\mathbf{c}$, $[I]$ the identity matrix of order $n$, $[D(\mathbf{X}^{(k)})]$ an $n \times n$ matrix with all off-diagonal entries equal to 0, and diagonal entries equal to the components of the vector $\mathbf{X}^{(k)}$ as

$$[D(\mathbf{X}^{(k)})]_{ii} = x_i^{(k)}, \qquad i = 1, 2, \ldots, n$$

$[P]$ is an $(m+1) \times n$ matrix whose first $m$ rows are given by $[a]\,[D(\mathbf{X}^{(k)})]$ and the last row is composed of 1's:

$$[P] = \begin{bmatrix} & [a][D(\mathbf{X}^{(k)})] & & \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

and the value of the parameter $\alpha$ is usually chosen as $\alpha = \frac{1}{4}$ to ensure convergence. Once $\mathbf{Y}^{(k+1)}$ is found, the components of the new point $\mathbf{X}^{(k+1)}$ are determined as

$$x_i^{(k+1)} = \frac{x_i^{(k)} y_i^{(k+1)}}{\sum_{r=1}^{n} x_r^{(k)} y_r^{(k+1)}}, \qquad i = 1, 2, \ldots, n$$

Set the new iteration number as $k = k + 1$ and go to step 2.

**Example-** Find the solution of the following problem using Karmarkar's method:

$$\Rightarrow c^T = [2, 1, -1]$$

$$\text{Minimize } f = 2x_1 + x_2 - x_3$$

subject to

$$\rightarrow [a]X = 0 \quad \Big| \; ||c^T|| = \sqrt{4+1+1} = \sqrt{6}$$

$$x_2 - x_3 = 0$$

$$x_1 + x_2 + x_3 = 1$$

subject to

$$x_2 - x_3 = 0$$

$$x_1 + x_2 + x_3 = 1$$

$$x_i \geq 0, \quad i = 1, 2, 3$$

$\rightarrow$ $[a]X = 0$

$x_1 + x_2 + x_3 = 1$

$= \sqrt{6}$

Ans the given that $\varepsilon = 0.05$

Step 1 - choose initial feasible point as

$$x^{(1)} = \begin{Bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{Bmatrix}$$

and set $k = 1$

Step 2. $|C^T x^{(1)}| = \left| [2, 1, -1] \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \right| = \left| \frac{2}{3} + \frac{1}{3} - \frac{1}{3} \right|$

$= \frac{2}{3} = 0.667 \leq \varepsilon$

$\Rightarrow 0.667 \leq \varepsilon = 0.05$

$\Rightarrow$ this is not true, then goto next step.

now $D(x^1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

here $[a] = [0, 1, -1]$

so $[a] D(x^1) = [0, 1, -1] \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \left[0, \frac{1}{3}, -\frac{1}{3}\right]_{1 \times 3}$

now, $P = \begin{bmatrix} & a[x^{(1)}] & \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}$

$[P][P^T] = \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 1 \\ -\frac{1}{3} & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & 0 \\ 0 & 3 \end{bmatrix}$

$([P][P^T])^{-1} = \begin{bmatrix} 2/9 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 9/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

$$\begin{bmatrix} 0 & 3 \end{bmatrix} \quad - \quad \begin{bmatrix} 0 & 1/3 \end{bmatrix}$$

$$P^T([P][P^T])^{-1}[P] = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 1 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 9/2 & 0 \\ 0 & 1/3 \end{bmatrix}_{2\times 2} \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 1 \\ -\frac{1}{3} & 1 \end{bmatrix}_{3\times 2} \begin{bmatrix} 0 & \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}_{2\times 3}$$

$-\frac{1}{2}+\frac{1}{3}$

$\frac{-3+2}{6}$

$1/6$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 5/6 & -1/6 \\ \frac{1}{3} & -\frac{1}{6} & 5/6 \end{bmatrix}$$

$$I - P^T(P P^T)^{-1} P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 5/6 & -1/6 \\ \frac{1}{3} & -\frac{1}{6} & 5/6 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 1/6 & 1/6 \\ -1/3 & 1/6 & 1/6 \end{bmatrix}$$

now $D(x^1)\,c = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}_{3\times 3} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}_{3\times 1}$

$$= \begin{bmatrix} 2/3 \\ 1/3 \\ -1/3 \end{bmatrix}$$

now,

$$(I - P^T([P][P^T])^{-1} P)\, D(x^1) c = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \ldots & \ldots & \ldots \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \\ -1/3 \end{bmatrix}$$

$$\begin{bmatrix} \cdots & \cdots & \cdots \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & 1/6 \end{bmatrix}\begin{bmatrix} \cdots \\ \cdots \\ -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{9} - \frac{1}{9} + \frac{1}{9} = 4/9 \\ -\frac{2}{9} + \frac{1}{18} - \frac{1}{18} = -2/9 \\ -\frac{2}{9} + \frac{1}{18} - \frac{1}{18} = -2/9 \end{bmatrix}$$

If we write $\alpha = 1/4$, ① becomes

$$y^{(2)} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 4/9 \\ -2/9 \\ -2/9 \end{bmatrix} \frac{1}{\sqrt{3(3-1)}\sqrt{6}} = \begin{bmatrix} 34/108 \\ 37/108 \\ 37/108 \end{bmatrix} = \begin{bmatrix} y_1^{(2)} \\ y_2^{(2)} \\ y_3^{(2)} \end{bmatrix}$$

now $\sum_{r=1}^{3} x_r^{(1)} y_r^{(2)} + x_1^{(1)} y_1^{(2)} + x_2^{(1)} y_2^{(2)} + x_3^{(1)} y_3^{(1)}$

$$= \frac{1}{3}\left(\frac{34}{108}\right) + \frac{1}{3}\left(\frac{37}{108}\right) + \frac{1}{3} \times \frac{37}{108} = \frac{1}{3}$$

$$x_i^{(2)} = \begin{bmatrix} \dfrac{x_i^{(1)} y_i^{(2)}}{\sum_{r=1}^{3} x_r^{(1)} y_r^{(2)}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\left(\frac{34}{108}\right) / 1/3 \\ \frac{1}{3}\left(\frac{37}{108}\right) / 1/3 \\ \frac{1}{3}\left(\frac{37}{108}\right) / 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 34/108 \\ 37/108 \\ 37/108 \end{bmatrix} \quad \underline{\text{Ans}}$$

The End