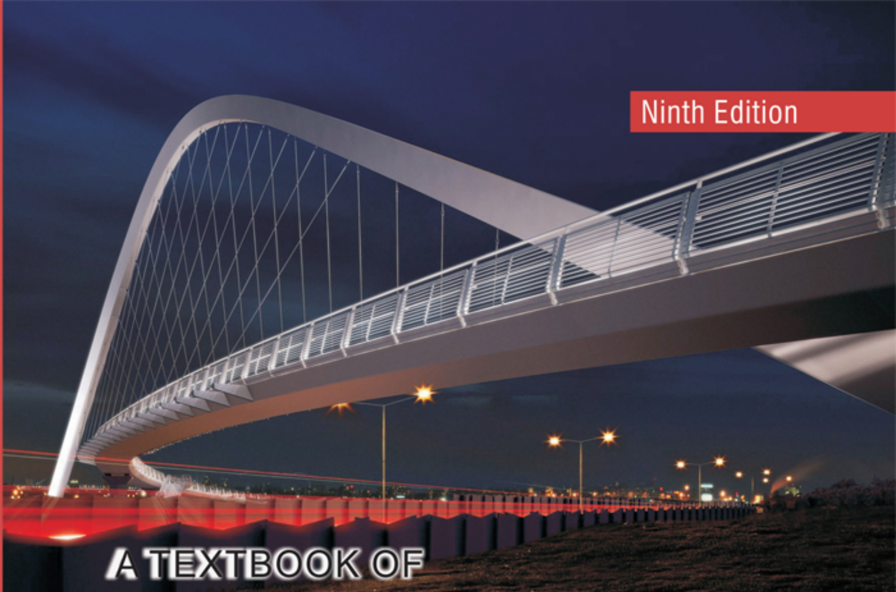

Ninth Edition

A TEXTBOOK OF ENGINEERING MATHEMATICS

For Punjab Technical University, Jalandhar

(Strictly According to the Latest Revised Syllabus)

SEMESTER-II

N.P. BALI

USHA PAUL

A TEXTBOOK OF

ENGINEERING MATHEMATICS

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ENGINEERING MATHEMATICS

[B.Tech./B.E., Semester–II]

(Strictly According to the Latest Revised Syllabus of Punjab Technical University, Jalandhar)

For all Branches

By

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Preface to the Tenth Edition

It is with the grace of God and over whelming response given by learned professors and discerning students that our book 'A Textbook of Engineering Mathematics' is gaining increasing popularity. We thank the almighty that our hardwork paid off and also place on record our sense of gratitude to the esteemed readers for their so much appreciation of our work.

Every year we revise the book as per university and our readers' requirements and bring the necessary alterations in the subject matter of the book. This edition of the book is strictly according to the revised syllabus of Engineering Mathematics-II. Authors observed that these days the trend of paper setting of various universities is to test the general understanding of the subject. So keeping in view this trend, this new edition is revised with lot of care, dedication and patience.

The following are the salient features of this edition :

(*i*) Many new solved examples and problems are added in each chapter. Also hints are given alongside typical unsolved problems.

(*ii*) Where needed, recapitulation of the topic is given in the beginning of the chapter. Working rules for lengthy formulae are also given.

(*iii*) At the end of each chapter **'Review of the Chapter'** is introduced so that students can revise the chapter at a glance. Also at the end of each chapter **'Short Answer Type Questions'** are given which form compulsory section (containing ten questions each of two marks) of the paper.

The present edition includes all the questions set in last ten years university papers mostly in the form of solved examples—these will certainly make students familiar with university pattern.

We have tried our best to make the book 'mistake-free' but inspite of our best efforts some errors might have crept in the book. Report of any such error and all suggestions for improving the future edition of the book are welcome and will be gratefully acknowledged.

It is hoped that book in its new form will attract more readers and will be found to be of much more utility.

We wish our readers the very best of luck for their brilliant success in life.

—AUTHORS

Preface to the First Edition

This book of Mathematics has been specially written to meet the requirements of B.E./B. Tech. first year students of various institutions, universities and Engineering Courses.

The salient features of the book are :

(*i*) The book presents the subject matter in a very systematic, simple and lucid style, so that students themselves will be able to understand the solutions of the problems.

(*ii*) Each chapter starts with necessary definitions and complete proofs of the standard theorems followed by solved examples. For convenience of students, working rules for the applications of theorems in questions are given. Also lists of important results are given at the end of chapters, where needed.

(*iii*) For convenience of students, lengthy chapters are divided into small units.

(*iv*) In the beginning of some chapters, some reference topics are discussed in detail inspite of the fact that these topics are not in the syllabus of certain universities. It is done because without the knowledge of these topics students cannot understand the main topic of the syllabus.

(*v*) *The most distinguished and outstanding feature* of this book is that each topic contains a large number of solved examples (Simple as well as typical). Many examples have been selected from various university papers so as to make students familiar with university pattern.

This book serves triple purpose *viz. textbook, help book and solved university* papers and authors are sure that the study of this book will instill confidence in students.

The authors of the book possess more than three decades of rich experience of teaching Mathematics to graduate as well as Postgraduate classes and have first hand experience of the problems and difficulties faced by students.

Suggestions for improvement of the book will be most gratefully received.

—AUTHORS

Syllabus

PUNJAB TECHNICAL UNIVERSITY, JALANDHAR
BTAM102, Engineering Mathematics-II

Objective/s and Expected outcome:

The learning objectives of core mathematics courses can be put into three categories: **Content Objectives:** Students should learn fundamental mathematical concepts and how to apply them. **Skill Objectives:** Students should learn critical thinking, modeling/problem solving and effective uses of technology. **Communication Objectives:** Students should learn how to read mathematics and use it to communicate knowledge. The students are expected to understand the fundamentals of the mathematics to apply while designing technology and creating innovations.

PART-A

1. **Ordinary Differential Equations of First Order**
   Exact Differential equations, Equations reducible to exact form by integrating factors; Equations of the first order and higher degree. Clairaut's equation. Leibniz's linear and Bernoulli's equation. (7)
2. **Linear Ordinary Differential Equations of Second and Higher Order**
   Solution of linear Ordinary Differential Equations of second and higher order; methods of finding complementary functions and particular integrals. Special methods for finding particular integrals: Method of variation of parameters, Operator method. Cauchy's homogeneous and Legendre's linear equation, Simultaneous linear equations with constant coefficients. (7)
3. **Applications of Ordinary Differential Equations**
   Applications to electric R-L-C circuits, Deflection of beams, Simple harmonic motion, Simple population model. (7)

PART-B

4. **Linear Algebra**
   Rank of a matrix, Elementary transformations, Linear independence and dependence of vectors, Gauss-Jordan method to find inverse of a matrix, reduction to normal form, Consistency and solution of linear algebraic equations, Linear transformations, Orthogonal transformations, Eigen values, Eigen vectors, Cayley-Hamilton Theorem, Reduction to diagonal form, orthogonal, unitary, Hermitian and similar matrices. (7)
5. **Infinite Series**
   Convergence and divergence of series, Tests of convergence (without proofs): Comparison test, Integral test, Ratio test, Raabe's test, Logarithmic test, Cauchy's root test and Gauss test. Convergence and absolute convergence of alternating series. (7)
6. **Complex Numbers and Elementary Functions of Complex Variable**
   De-Moivre's theorem and its applications. Real and Imaginary parts of exponential, logarithmic, circular, inverse circular, hyperbolic, inverse hyperbolic functions of complex variables. Summation of trigonometric series. (C+iS method). (7)

PART-A

1. Ordinary Differential Equations of First Order
2. Linear Ordinary Differential Equations of Second and Higher Order
3. Application of Ordinary Differential Equations

1

Ordinary Differential Equations of First Order

1.1. DEFINITIONS

(*i*) A differential equation is an equation involving differentials or differential coefficients. Thus

$$\frac{dy}{dx} = x^2 - 1 \qquad \ldots(1) \qquad\qquad \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0 \qquad \ldots(2)$$

$$(x + y^2 - 3y)\, dx = (x^2 + 3x + y)\, dy \qquad \ldots(3) \qquad\qquad y = x\frac{dy}{dx} + \frac{c}{\frac{dy}{dx}} \qquad \ldots(4)$$

$$\frac{d^3y}{dx^3} + 2\,\frac{d^2y}{dx^2} \cdot \frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = 0 \qquad \ldots(5) \qquad\qquad \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = k \cdot \frac{d^2y}{dx^2} \qquad \ldots(6)$$

$$x\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = nu \qquad \ldots(7) \qquad\qquad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y \qquad \ldots(8)$$

are all differential equations.

(*ii*) Differential equations which involve only one independent variable and the differential coefficients with respect to it are called **ordinary differential equations.**

Thus equations (1) to (6) are all ordinary differential equations.

(*iii*) Differential equations which involve two or more independent variables and partial derivatives with respect to them are called **partial differential equations.**

Thus equations (7) and (8) are partial differential equations.

(*iv*) The **order** of a differential equation is the order of the highest order derivative occurring in the differential equation. **(P.T.U., Jan. 2009)**

Thus equations (1), (3) and (4) are of first order ; equations (2) and (6) are of the second order while equation (5) is of the third order.

(*v*) The **degree** of a differential equation is the degree of the highest order derivative which occurs in the differential equation provided the equation has been made free of the radicals and fractions as far as the derivatives are concerned. **(P.T.U., Jan. 2009)**

Thus, equations (1), (2), (3) and (5) are of the first degree.

Equation (4) is $$y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + c$$

It is of the second degree.

Equation (6) is $$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$$

It is of the second degree.

(*vi*) **Solution of a Differential Equation.** A solution (or integral) of a differential equation is a relation, free from derivatives, between the variables which satisfies the given equation.

Thus if $y = f(x)$ be the solution, then by replacing y and its derivatives with respect to x, the given differential equation will reduce to an identity.

For example, $$y = c_1 \cos x + c_2 \sin x$$

is the solution of the differential equation $\dfrac{d^2 y}{dx^2} + y = 0$

Since $$\frac{dy}{dx} = -c_1 \sin x + c_2 \cos x$$

$$\frac{d^2 y}{dx^2} = -c_1 \cos x - c_2 \sin x = -y$$

$$\frac{d^2 y}{dx^2} + y = 0$$

The **general (or complete) solution** of a differential equation is that in which the number of independent arbitrary constants is equal to the order of the differential equation. **(P.T.U., Dec. 2005)**

Thus, $y = c_1 \cos x + c_2 \sin x$ (involving two arbitrary constants c_1, c_2) is the general solution of the differential equation $\dfrac{d^2 y}{dx^2} + y = 0$ of second order.

A **particular solution** of a differential equation is that which is obtained from its general solution by giving particular values to the arbitrary constants.

For example, $y = c_1 e^x + c_2 e^{-x}$ is the general solution of the differential equation $\dfrac{d^2 y}{dx^2} - y = 0$, whereas $y = e^x - e^{-x}$ or $y = e^x$ are its particular solutions.

The solution of a differential equation of nth order is its particular solution if it contains less than n arbitrary constants.

A singular solution of a differential equation is that solution which satisfies the equation but cannot be derived from its general solution.

1.2. GEOMETRICAL MEANING OF A DIFFERENTIAL EQUATION OF THE FIRST ORDER AND FIRST DEGREE

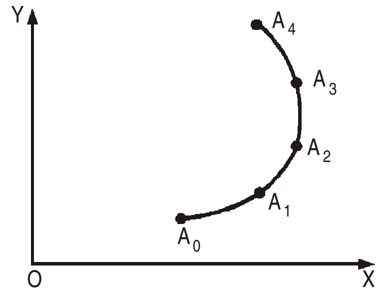
Let $$f\left(x,\ y,\ \frac{dy}{dx}\right) = 0 \qquad \ldots(1)$$

be a differential equation of the first order and first degree.

We know that the direction of a curve at a particular point is determined by drawing a tangent line at that point, *i.e.*, its slope is given by $\dfrac{dy}{dx}$ at that particular point.

Let $A_0\ (x_0, y_0)$ be any point in the plane. Let $m_0 = \dfrac{dy_0}{dx_0}$ be the slope of the curve at A_0 derived from (1).

Take a neighbouring point $A_1\ (x_1, y_1)$ such that the slope of $A_0 A_1$ is m_0. Let $m_1 = \dfrac{dy_1}{dx_1}$ be the slope of the curve at A_1 derived from (1). Take a neighbouring point $A_2\,(x_2\,,y_2)$ such that the slope of $A_1\,A_2$ is m_1. Continuing like this, we get a succession of points. If the points are taken sufficiently close to each other, they approximate a smooth curve C : $y = \phi(x)$ which is a solution of (1) corresponding to the initial point $A_0\,(x_0\,,\ y_0)$. Any point on C and the slope of the tangent at that point satisfy (1). If the moving point starts at any other point, not on C and moves as before, it will describe another curve. The equation of each such curve is a *particular solution* of the differential equation (1). The equation of the system of all such curves is the general solution of (1).



1.3. FORMATION OF A DIFFERENTIAL EQUATION

Differential equations are formed by elimination of arbitrary constants. To eliminate two arbitrary constants, we require two more equations besides the given relation, leading us to second order derivatives and hence a differential equation of the second order. Elimination of n arbitrary constants leads us to nth order derivatives and hence a differential equation of the nth order.

Let
$$f\,(x, y, c_1, c_2, ..., c_n) = 0 \qquad \ldots(1)$$
be an equation containing n arbitrary constants $c_1, c_2, ..., c_n$ (sometimes called parameters)

Differentiating (1) w.r.t. x successively n times, we get

$$\left.\begin{aligned} f_1\,(x, y, c_1, c_2, \cdots, c_n, \frac{dy}{dx}) &= 0 \\ f_2\,(x, y, c_1, c_2, \cdots, c_n, \frac{dy}{dx}, \frac{d^2y}{dx^2}) &= 0 \\ \text{and} \quad f_n\,(x, y, c_1, c_2, \cdots, c_n, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}) &= 0 \end{aligned}\right\} \qquad \ldots(2)$$

Eliminating $c_1, c_2, \cdots, c_n$ from (1) and (2), we get

$$\phi\,(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}) = 0$$

which is required nth order differential equation.

Hence nth order differential equation has exactly n arbitrary constants in its general solution.

ILLUSTRATIVE EXAMPLES

Example 1. *Eliminate the constants from the following equations:*

(*i*) $y = e^x\,(A \cos x + B \sin x)$ …(1) **(P.T.U., June 2003)**

(*ii*) $y = cx + c^2$ **(P.T.U., Dec. 2003)**

(*iii*) $y = Ae^x + Be^{-x} + C$ **(P.T.U., May 2004)**

and obtain the differential equation.

Sol. (*i*) There are two arbitrary constants A and B in equation (1).

Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = e^x\,(\text{A} \cos x + \text{B} \sin x) + e^x\,(-\,\text{A} \sin x + \text{B} \cos x) = y + e^x\,(-\,\text{A} \sin x + \text{B} \cos x) \qquad \ldots(2)$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x\left(-\text{A}\sin x + \text{B}\cos x\right) + e^x\left(-\text{A}\cos x - \text{B}\sin x\right) = \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y$$

[Using (1) and (2)]

or $\quad \dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$, which is the required differential equation.

(*ii*) $\quad y = cx + c^2 \quad$ ...(1)

Equation has only one parameter 'c'

$$\frac{dy}{dx} = c \quad \text{...(2)}$$

Eliminate c from (1) and (2), we get

$$y = x \cdot \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

or $\quad \left(\dfrac{dy}{dx}\right)^2 + x\dfrac{dy}{dx} - y = 0$; required differential equation.

(*iii*) $y = \text{A}e^x + \text{B}e^{-x} + \text{C} \quad$ ...(1)

Equation has three arbitrary constants so differentiate (1) thrice

$$\frac{dy}{dx} = \text{A}e^x - \text{B}e^{-x} \quad \text{...(2)}$$

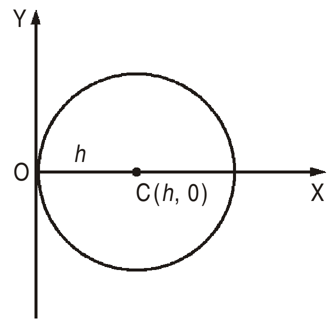
$$\frac{d^2y}{dx^2} = \text{A}e^x + \text{B}e^{-x}$$

$$\frac{d^3y}{dx^3} = \text{A}e^x - \text{B}e^{-x} = \frac{dy}{dx} \quad \text{[From (2)]}$$

$\therefore$ Required differential equation is

$$\frac{d^3y}{dx^3} = \frac{dy}{dx}.$$

Example 2. *Find the differential equation of all circles passing through the origin and having centres on the axis of x.*



Sol. The equation of such a circle is $(x - h)^2 + y^2 = h^2$

or $\quad x^2 + y^2 - 2hx = 0 \quad$ ...(1)

where h is the only arbitrary constant.

Differentiating (1) w.r.t. x, we have $2x + 2y\dfrac{dy}{dx} - 2h = 0$

or $\quad h = x + y\dfrac{dy}{dx}$

Substituting the value of h in (1), we have $x^2 + y^2 - 2x\left(x + y\dfrac{dy}{dx}\right) = 0$

or $\quad 2xy\dfrac{dy}{dx} + x^2 - y^2 = 0$

which is the required differential equation.

Example 3. *Form the differential equation of all circles of radius a.*

Sol. The equation of any circle of radius a is $(x-h)^2+(y-k)^2=a^2$ …(1)

where (h, k), the coordinates of the centre are arbitrary.

Differentiating (1) w.r.t. x, we have $2(x-h)+2(y-k)\dfrac{dy}{dx}=0$

or $$(x-h)+(y-k)\frac{dy}{dx}=0 \quad \text{…(2)}$$

Differentiating again, we have $1+(y-k)\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=0$ …(3)

From (3), $$y-k=-\frac{1+\left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}$$

and from (2), $$x-h=-(y-k)\frac{dy}{dx}=\frac{\dfrac{dy}{dx}\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}{\dfrac{d^2y}{dx^2}}$$

Substituting the values of $(x-h)$ and $(y-k)$ in (1), we get

$$\frac{\left(\dfrac{dy}{dx}\right)^2\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^2}{\left(\dfrac{d^2y}{dx^2}\right)^2}+\frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^2}{\left(\dfrac{d^2y}{dx^2}\right)^2}=a^2$$

or $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left[\left(\frac{dy}{dx}\right)^2+1\right]=a^2\left(\frac{d^2y}{dx^2}\right)^2 \quad \text{or} \quad \left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

Example 4. *Find the differential equations of all parabolas whose axes are parallel to y-axis.*

(P.T.U., May 2002)

Sol. Equations of the parabolas whose axes are parallel to y-axis is $(x-h)^2=4a(y-k)$ …(1)

where a, h, k are three parameters.

Differentiating (1) w.r.t. x three times, we get

$$2(x-h)=4a\frac{dy}{dx}$$

or $$x-h=2a\frac{dy}{dx}$$

Differentiate again $$1=2a\frac{d^2y}{dx^2}$$

Differentiate third time, we get

$$0 = 2a\frac{d^3y}{dx^3} \quad \text{or} \quad \frac{d^3y}{dx^3} = 0 \qquad (\because \quad a \neq 0)$$

Hence differential equation of given parabolas is

$$\frac{d^3y}{dx^3} = 0 \quad \text{or} \quad y_3 = 0.$$

TEST YOUR KNOWLEDGE

Eliminate the arbitrary constants and obtain the differential equations :

1. $y = cx + c^2$ **2.** $y = A + Bx + Cx^2$ **3.** $y = A\cos 2t + B\sin 2t$

4. $y = Ae^{3x} + Be^{2x}$ **5.** $y = Ae^{x} + Be^{-x} + C$ **6.** $y = ax^3 + bx^2$

7. $xy = Ae^{x} + Be^{-x} + x^2$ **8.** $x = A\cos(nt + \alpha)$ **9.** $y = ae^{2x} + be^{-3x} + ce^{x}$

10. $Ax^2 + By^2 = 1$ **11.** $y^2 - 2ay + x^2 = a^2$ **12.** $e^{2y} + 2ax\,e^{y} + a^2 = 0$

Find the differential equations of:

13. All straight lines in a plane. [**Hint:** Equation of the lines are $y = mx + c$]

14. All circles of radius r whose centres lie on the x-axis. [**Hint:** $(x - a)^2 + y^2 = r^2$ only a is parameter]

15. All parabolas with x-axis as the axis and $(a, 0)$ as focus. [**Hint:** $y^2 = 4ax$]

16. All conics whose axes coincide with the axes of co-ordinates. [**Hint:** $ax^2 + by^2 = 1$]

17. All circle in a plane.

18. All circles in the first quadrant which touch the co-ordinate axes [**Hint:** $(x - a)^2 + (y - a)^2 = a^2$]

19. All circles touching the axis of y at the origin and having centres on the x-axis.

[**Hint:** Same as solved example 2]

20. All parabolas with latus rectum '$4a$' and axis parallel to the x-axis.

[**Hint:** $(y - k)^2 = 4a\,(x - 4)$ two parameters h and k]

ANSWERS

1. $x\dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2 = y$ **2.** $\dfrac{d^3y}{dx^3} = 0$ **3.** $\dfrac{d^2y}{dt^2} + 4y = 0$

4. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$ **5.** $\dfrac{d^3y}{dx^3} - \dfrac{dy}{dx} = 0$ **6.** $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = 0$

7. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + x^2 - xy - 2 = 0$ **8.** $\dfrac{d^2x}{dt^2} + n^2x = 0$ **9.** $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} + 6y = 0$

10. $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} = 0$ **11.** $\left(x^2 - 2y^2\right)\left(\dfrac{dy}{dx}\right)^2 - 4xy\dfrac{dy}{dx} - x^2 = 0$

12. $\left(1 - x^2\right)\left(\dfrac{dy}{dx}\right)^2 + 1 = 0$ **13.** $\dfrac{d^2y}{dx^2} = 0$ **14.** $y^2\left[1 + \left(\dfrac{dy}{dx}\right)^2\right] = r^2$

15. $y\dfrac{dy}{dx} = 2a$ **16.** $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 = y\dfrac{dy}{dx}$

17. $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]\dfrac{d^3y}{dx^3} - 3\dfrac{dy}{dx}\left(\dfrac{d^2y}{dx^2}\right)^2 = 0$ **18.** $(x - y)^2\left[1 + \left(\dfrac{dy}{dx}\right)^2\right] = \left(x + y\dfrac{dy}{dx}\right)^2$

19. $x^2 - y^2 + 2xy\dfrac{dy}{dx} = 0$ **20.** $2a\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 = 0.$

1.4. SOLUTION OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

All differential equations of the first order and first degree cannot be solved. Only those among them which belong to (or can be reduced to) one of the following categories can be solved by the standard methods.

(*i*) Equations in which variables are separable.

(*ii*) Differential equation of the form $\dfrac{dy}{dx} = f(ax + by + c)$.

(*iii*) Homogeneous equations. (*iv*) Linear equations. (*v*) Exact equations.

1.4(a). VARIABLES SEPARABLE FORM

If a differential equation of the first order and first degree can be put in the form where dx and all terms containing x are at one place, also dy and all terms containing y are at one place, then the variables are said to be separable.

Thus the general form of such an equation is $f(x)\, dx + \phi(y)\, dy = 0$

Integrating, we get $\int f(x)\, dx + \int \phi(y)\, dy = c$ which is the general solution, c being an arbitrary constant.

Note. Any equation of the form $f_1(x)\, \phi_2(y)\, dx + f_2(x)\, \phi_1(y)\, dy = 0$ can be expressed in the above form by dividing throughout by $f_2(x)\, \phi_2(y)$.

Thus $$\frac{f_1(x)}{f_2(x)}\, dx + \frac{\phi_1(y)}{\phi_2(y)}\, dy = 0 \quad \text{or} \quad f(x)\, dx + \phi(y)\, dy = 0.$$

1.4(b). DIFFERENTIAL EQUATIONS OF THE FORM $\dfrac{dy}{dx} = f(ax + by + c)$

It is a differential equation of the form

$$\frac{dy}{dx} = f(ax + by + c) \qquad \ldots(1)$$

It can be reduced to a form in which the variables are separable by the substitution $ax + by + c = t$.

so that $$a + b\frac{dy}{dx} = \frac{dt}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$$

$\therefore$ Equation (1) becomes $$\frac{1}{b}\left(\frac{dt}{dx} - a\right) = f(t) \quad \text{or} \quad \frac{dt}{dx} = a + b\, f(t).$$

or $$\frac{dt}{a + b\, f(t)} = 2$$

After integrating both sides, t is to be replaced by its value.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$.

Sol. The given equation can be written as $y(1 - ay) - (x + a)\dfrac{dy}{dx} = 0$

or
$$\frac{dx}{x+a} = \frac{dy}{y(1-ay)}$$

Integrating both sides, we have $$\int \frac{dx}{x+a} = \int \left(\frac{1}{y} + \frac{a}{1-ay} \right) dy + c \qquad \text{[Partial fractions]}$$

$$\Rightarrow \qquad \log (x+a) = \left[\log y + a \, . \frac{\log(1-ay)}{-a} \right] + c$$

$$\Rightarrow \qquad \log (x+a) - \log y + \log (1-ay) = \log C, \text{ where } c = \log C$$

$$\Rightarrow \qquad \log \frac{(x+a)\,(1-ay)}{y} = \log C \;\Rightarrow\; (x+a)\,(1-ay) = Cy$$

which is the general solution of the given equation.

Note. Here c is replaced by log C to get a neat form of the solution.

Example 2. *Solve : $3\,e^x \tan y\, dx + (1 + e^x) \sec^2 y\, dy = 0$, given $y = \frac{\pi}{4}$ when $x = 0$.*

Sol. The given equation can be written as $\frac{3e^x}{1+e^x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$

Integrating, we have $3 \log (1 + e^x) + \log \tan y = \log c$

$\Rightarrow \quad \log (1 + e^x)^3 \tan y = \log c$

$\Rightarrow \quad (1 + e^x)^3 \tan y = c$ …(1)

which is the general solution of the given equation.

Since $y = \frac{\pi}{4}$ when $x = 0$, we have from (1)

$(1 + 1)^3 \times 1 = c \qquad \Rightarrow \quad c = 8$

$\therefore$ The required particular solution is $(1 + e^x)^3 \tan y = 8$.

Example 3. *Solve $x \cos x \cos y + \sin y \frac{dy}{dx} = 0$.* **(P.T.U., Dec. 2002)**

Sol.
$$x \cos x \cos y + \sin y \frac{dy}{dx} = 0$$

or
$$x \cos x \cos y = - \sin y \frac{dy}{dx}$$

or
$$x \cos x\, dx = - \tan y\, dy$$

Integrating both sides,

$$\int x \cos x\, dx = - \int \tan y\, dy + c$$

$$x \sin x - \int 1 \,.\, \sin x\, dx = \log \cos y + c$$

or
$$x \sin x + \cos x = \log \cos y + c.$$

Example 4. *Solve* $xy\dfrac{dy}{dx} = 1 + x + y + xy.$ **(P.T.U., Dec. 2003)**

Sol.
$$xy\frac{dy}{dx} = (1+x) + y(1+x)$$
$$= (1+x)(1+y)$$
$$\frac{y\,dy}{1+y} = \frac{1+x}{x}\,dx$$

Integrating both sides,
$$\int \frac{y}{1+y}\,dy = \int \frac{1+x}{x}\,dx + c$$
$$\int\left(1 - \frac{1}{1+y}\right)dy = \int\left(\frac{1}{x} + 1\right)dx + c$$

or $$y - \log(1+y) = \log x + x + c$$

or $$x - y + \log x(1+y) = -c = c'.$$

Example 5. *Solve* $(x + y + 1)^2 \dfrac{dy}{dx} = 1.$

Sol. Putting $x + y + 1 = t$, we get $1 + \dfrac{dy}{dx} = \dfrac{dt}{dx}$ or $\dfrac{dy}{dx} = \dfrac{dt}{dx} - 1$

$\therefore$ The given equation becomes $t^2\left(\dfrac{dt}{dx} - 1\right) = 1$ or $\dfrac{dt}{dx} = \dfrac{1+t^2}{t^2}$

$\Rightarrow$ $$\frac{t^2}{1+t^2}\,dt = dx$$

Integrating, we have $\displaystyle\int\left(1 - \frac{1}{1+t^2}\right)dt = \int dx + c = dx$ or $t - \tan^{-1} t = x + c$

or $$(x + y + 1) - \tan^{-1}(x + y + 1) = x + c$$

or $$y = \tan^{-1}(x + y + 1) + \mathrm{C},\ \text{where } \mathrm{C} = c - 1.$$

Example 6. *Solve* $\dfrac{dy}{dx} = \sin(x + y).$ **(P.T.U., May 2006)**

Sol. Put $x + y = t$

$\therefore$ $$1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

$\therefore$ Given equation changes to
$$\frac{dt}{dx} - 1 = \sin t$$
$$\frac{dt}{dx} = 1 + \sin t$$

or $$\frac{dt}{1 + \sin t} = dx$$

Integrate both sides,

$$\int \frac{dt}{1+\sin t} = \int dx + c$$

or
$$\int \frac{1-\sin t}{\cos^2 t}\, dt = x + c$$

or
$$\int \left(\sec^2 t - \tan t \sec t\right) dt = x + c$$

or
$$\tan t - \sec t = x + c$$

or
$$\sin t - 1 = (x + c) \cos t$$

Substitute back the value of t

$$\sin (x + y) - 1 = (x + c) \cos (x + y).$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{dy}{dx} = e^{2x+3y}$ **2.** $\dfrac{dy}{dx} = \dfrac{y}{x}$ **(P.T.U., Dec. 2005)**

3. (*a*) $(x + y)(dx - dy) = dx + dy$ (*b*) $\dfrac{dy}{dx} = \dfrac{x(2 \log x + 1)}{\sin y + y \cos y}$

4. $x\dfrac{dy}{dx} + \cot y = 0$ if $y = \dfrac{\pi}{4}$ when $x = \sqrt{2}$ **5.** $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$

6. $y\sqrt{1-x^2}\, dy + x\sqrt{1-y^2}\, dx = 0$ **7.** $\dfrac{y}{x} \cdot \dfrac{dy}{dx} = \sqrt{1 + x^2 + y^2 + x^2 y^2}$

8. $e^y \left(1 + x^2\right) \dfrac{dy}{dx} - 2x\left(1 + e^y\right) = 0$ **9.** $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$ **(P.T.U., Dec. 2002)**

[**Hint:** $\dfrac{\sec^2 x}{\tan x} dx = -\dfrac{\sec^2 y}{\tan y} dy$ Integrate, $\log \tan x = -\log \tan y + \log c \therefore \tan x \tan y = c$]

10. $\left(x^2 - yx^2\right)\dfrac{dy}{dx} + y^2 + xy^2 = 0$ **11.** $\left(1 + x^3\right) dy - x^2 y\, dx = 0$, if $y = 2$ when $x = 1$

12. $a(x\, dy + y\, dx) = xy\, dy$ **13.** $\dfrac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ **14.** $\dfrac{dy}{dx} = (4x + y + 1)^2$

15. $(x + y)^2 \dfrac{dy}{dx} = a^2$ **16.** $\sin (x + y)\, dy = dx$ **17.** $\dfrac{dy}{dx} = \cos (x + y)$

[**Hint:** Consult S.E. 6] **(P.T.U., June 2003)**

18. $\dfrac{dy}{dx} = \sin (x + y) + \cos (x + y)$ **19.** $\tan y \dfrac{dy}{dx} = \sin (x + y) + \sin (x - y)$

[**Hint:** $\tan y \dfrac{dy}{dx} = 2 \sin x \cos y$; Integrate $\int \tan y \sec y\, dy = \int 2 \sin x\, dx + c$, $\sec y = -2 \cos x + c$]

20. $\dfrac{dy}{dx} - x \tan (y - x) = 1$. [**Hint:** Put $y - x = t$]

ANSWERS

1. $3e^{2x} + 2e^{-3y} = c$
2. $y = cx$
3. (a) $x + y = c\,e^{x-y}$, (b) $y \sin y = x^2 \log x + c$
4. $x \sec y = 2$
5. $y\sqrt{1-x^2} + x\sqrt{1-y^2} = c$
6. $\sqrt{1-x^2} + \sqrt{1-y^2} = c$
7. $\sqrt{1+y^2} = \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + c$
8. $1 + e^y = c\left(1+x^2\right)$
9. $\tan x \tan y = c$
10. $\log\left(\frac{x}{y}\right) - \frac{1}{x} - \frac{1}{y} = c$
11. $y^3 = 4\,(x^3 + 1)$
12. $y = a \log (xy) + c$
13. $e^y = e^x + \frac{x^3}{3} + c$
14. $4x + y + 1 = 2 \tan (2x + c)$
15. $x + y = a \tan\left(\frac{y-c}{a}\right)$
16. $\tan (x + y) - \sec (x + y) = y + c$
17. $x + c = \tan\left(\frac{x+y}{2}\right)$
18. $\log\left[1 + \tan\left(\frac{x+y}{2}\right)\right] = x + c$
19. $2 \cos x + \sec y = c$
20. $\log \sin (y - x) = \frac{x^2}{2} + c$.

1.5. HOMOGENEOUS DIFFERENTIAL EQUATION AND ITS SOLUTION

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f_1(x, y)}{f_2(x, y)}$ …(1)

is called a homogeneous differential equation if $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of the same degree in x and y.

If $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of degree r in x and y, then

$$f_1(x, y) = x^r\,\phi_1\left(\frac{y}{x}\right) \text{ and } f_2(x, y) = x^r\,\phi_2\left(\frac{y}{x}\right)$$

$\therefore$ Equation (1) reduces to $\dfrac{dy}{dx} = \dfrac{\phi_1\left(\frac{y}{x}\right)}{\phi_2\left(\frac{y}{x}\right)} = F\left(\dfrac{y}{x}\right)$ …(2)

Putting $\dfrac{y}{x} = v$ *i.e.*, $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Equation (2) becomes $v + x\dfrac{dv}{dx} = F(v)$

Separating the variables, $\dfrac{dv}{F(v) - v} = \dfrac{dx}{x}$

Integrating, we get the solution in terms of v and x. Replacing v by $\dfrac{y}{x}$, we get the required solution.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* (*i*) $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$, **(P.T.U., June 2003)**

(*ii*) $(3x^2y - y^3)dx - (2x^2y - xy^2)dy = 0.$ **(P.T.U., May 2007)**

Sol. (*i*) The given equation can be written as $\dfrac{dy}{dx} = \dfrac{y + \sqrt{x^2 + y^2}}{x}$ ...(1)

The numerator and denominator on RHS of (1) are homogeneous functions of degree one.

Putting $y = vx$, so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Equation (1) becomes $v + x\dfrac{dv}{dx} = v + \sqrt{1+v^2}$ or $x\dfrac{dv}{dx} = \sqrt{1+v^2}$

Separating the variables, $\dfrac{dv}{\sqrt{1+v^2}} = \dfrac{dx}{x}$

Integrating both sides $\log\left(v + \sqrt{1+v^2}\right) = \log x + \log c \quad \because \int \dfrac{1}{\sqrt{1+v^2}}dv = \cosh^{-1} v = \log\left(v + \sqrt{1+v^2}\right)$

or $\log\left(v + \sqrt{1+v^2}\right) = \log(cx)$ or $v + \sqrt{1+v^2} = cx$

or $\dfrac{y}{x} + \sqrt{1 + \dfrac{y^2}{x^2}} = cx$ $\left[\because v = \dfrac{y}{x}\right]$

or $y + \sqrt{x^2+y^2} = cx^2$

which is the required solution.

(*ii*)
$$\frac{dy}{dx} = \frac{3xy^2 - y^3}{2x^2y - xy^2} = \frac{3\dfrac{y^2}{x^2} - \dfrac{y^3}{x^3}}{2\dfrac{y}{x} - \dfrac{y^2}{x^2}}$$

Put $\dfrac{y}{x} = v$ $\quad\therefore\quad y = vx;\ \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{3v^2 - v^3}{2v - v^2} = \frac{3v - v^2}{2 - v}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{3v - v^2}{2 - v} - v = \frac{3v - v^2 - 2v + v^2}{2 - v} = \frac{v}{2 - v}$$

$$\therefore \quad \frac{2 - v}{v}dv = \frac{1}{x}dx$$

Integrate both sides :

$$\int\left(\frac{2}{v} - 1\right)dv = \int\frac{1}{x}dx + c$$

or $$2 \log |v| - v = \log |x| + c$$

or $$2 \log |v| - \log |x| = v + c$$

or $$\log v^2 - \log x = v + c$$

or $$\log \frac{v^2}{x} = v + c \quad \text{or} \quad \log \frac{y^2}{x^2 . x} = \frac{y}{x} + c$$

or $$\frac{y^2}{x^3} = e^{\frac{y}{x} + c} = e^c . e^{\frac{y}{x}} = \text{A}e^{\frac{y}{x}}, \text{ where A} = e^c$$

$\therefore$ $$y^2 = \text{A}x^3 \, e^{\frac{y}{x}}.$$

Example 2. *Solve :* $\left(x \tan \frac{y}{x} - y \sec^2 \frac{y}{x} \right) dx + x \sec^2 \frac{y}{x} dy = 0.$ **(P.T.U., Dec. 2003)**

Sol. The given equation can be written as $\frac{dy}{dx} = \frac{y \sec^2 \frac{y}{x} - x \tan \frac{y}{x}}{x \sec^2 \frac{y}{x}} = \frac{y}{x} - \frac{\tan \frac{y}{x}}{\sec^2 \frac{y}{x}}$ ...(1)

Putting $\frac{y}{x} = v$ *i.e.,* $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Equation (1) becomes $v + x\frac{dv}{dx} = v - \frac{\tan v}{\sec^2 v}$ or $\frac{\sec^2 v}{\tan v} dv + \frac{dx}{x} = 0$

Integrating, we get $\log \tan v + \log x = \log c$

or $$\log (x \tan v) = \log c \text{ or } x \tan v = c$$

or $$x \tan \frac{y}{x} = c \qquad \left[\because \; v = \frac{y}{x} \right]$$

which is the required solution.

Example 3. *Find the general solution of the differential equation* $(2xy + x^2) y' = 3y^2 + 2xy.$ **(P.T.U., May 2006)**

Sol. Given equation is $(2xy + x^2) y' = 3y^2 + 2xy$

or $$\frac{dy}{dx} = \frac{3y^2 + 2xy}{2xy + x^2} = \frac{3\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)}{2 . \frac{y}{x} + 1}$$, which is homogeneous equation of order 2.

Put $\frac{y}{x} = v \quad \therefore \; y = vx \quad ; \quad \frac{dy}{dx} = v + x \frac{dv}{dx}$

$\therefore$ $$v + x \frac{dv}{dx} = \frac{3v^2 + 2v}{2v + 1} \quad \text{or} \quad x \frac{dv}{dx} = \frac{3v^2 + 2v}{2v + 1} - v$$

or $$x \frac{dv}{dx} = \frac{v^2 + v}{2v + 1} \quad \text{or} \quad \frac{2v + 1}{v(v + 1)} dv = \frac{1}{x} dx$$

Integrate both sides,

$$\int \frac{2v + 1}{v(v + 1)} dv = \int \frac{1}{x} dx + \log c$$

or $$\int\left(\frac{1}{v}+\frac{1}{v+1}\right)dv = \log x + \log c \qquad \text{(By partial fractions)}$$

or $$\log v + \log(v+1) - \log x = \log c \quad \text{or} \quad \log\frac{v(v+1)}{x} = \log c$$

or $$v(v+1) = cx \quad \text{or} \quad \frac{y}{x}\left(\frac{y}{x}+1\right) = cx$$

or $$y(y+x) = cx^3.$$

Example 4. *Solve:* $(1 + e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0.$ **(P.T.U., Dec. 2006)**

Sol. $(1 + e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

or $$\frac{dx}{dy} = -\frac{e^{x/y}\left(1-\frac{x}{y}\right)}{1+e^{x/y}}$$, which is homogeneous equation in $\frac{x}{y}$ form

∴ Put $$\frac{x}{y} = v \quad i.e., x = vy \quad \therefore \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$v + y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} \quad \text{or} \quad y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} - v$$

or $$y\frac{dv}{dy} = \frac{-(e^v+v)}{1+e^v} \quad \text{or} \quad \frac{1+e^v}{v+e^v}\,dv = -\frac{1}{y}\,dy$$

Integrate both sides,

$$\log(v+e^v) = -\log y + \log c$$

∴ $$\log(v+e^v)\,y = \log c$$

∴ $$(v+e^v)y = c$$

∴ $$y\left(\frac{x}{y}+e^{x/y}\right) = c \text{ or } x + ye^{x/y} = c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $(x+y)dx + (y-x)dy = 0$ **(P.T.U., May 2004, 2011)**
2. $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = 2xy$
3. $(x^2-y^2)dx = 2xydy$
4. $(y^2-2xy)dx = (x^2-2xy)dy$
5. $x(x-y)\dfrac{dy}{dx} = y(x+y)$
6. $\left(\sqrt{xy}-x\right)dy + y\,dx = 0$
7. $(y^2+2xy)dx + (2x^2+3xy)dy = 0$
8. $x^2y\,dx - (x^3+y^3)dy = 0$
9. $(x^2+2y^2)dx - xy\,dy = 0$, given that $y = 0$ when $x = 1$
10. $x\dfrac{dy}{dx} = y(\log y - \log x)$
11. $x\dfrac{dy}{dx} = y + x\cos^2\dfrac{y}{x}$
12. $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin\dfrac{y}{x}$

13. $\left(x\cos\frac{y}{x}+y\sin\frac{y}{x}\right)y-\left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)x\frac{dy}{dx}=0$

14. $(1+e^{x/y})\,dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$

15. $ye^{x/y}\,dx=(xe^{x/y}+y)\,dy$

16. $xy\log\left(\frac{x}{y}\right)dx+\left[y^2-x^2\log\left(\frac{x}{y}\right)\right]dy=0.$

ANSWERS

1. $\log(x^2+y^2)=2\tan^{-1}\frac{y}{x}+c$

2. $cx=e^{x/y}$

3. $x(x^2-3y^2)=c$

4. $xy(x-y)=c$

5. $\frac{x}{y}+\log\frac{x}{y}=c$

6. $2\sqrt{\frac{x}{y}}+\log y=c$

7. $xy^2(x+y)=c$

8. $y=ce^{\frac{x^2}{3y^3}}$

9. $x^2+y^2=cx^4$

10. $y=xe^{1+cx}$

11. $\tan\frac{y}{x}=\log(cx)$

12. $y=2x\tan^{-1}(cx)$

13. $xy\cos\frac{y}{x}=c$

14. $x+y\,e^{x/y}=c$

15. $e^{x/y}=\log y+c$

16. $\log y-\frac{x^2}{4y^2}\left(2\log\frac{y}{x}+1\right)=c.$

1.6. EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

A differential equation of the form $\frac{dy}{dx}=\frac{ax+by+c}{a'x+b'y+c'}$ ...(1)

can be reduced to the homogeneous form as follows :

Case I. When $\frac{a}{a'}\neq\frac{b}{b'}$ **(P.T.U., Dec. 2002)**

Putting $x=\mathrm{X}+h,\ y=\mathrm{Y}+k$ (h, k are constants)

so that $dx=d\mathrm{X},\ dy=d\mathrm{Y}$

Equation (1) becomes

$$\frac{d\mathrm{Y}}{d\mathrm{X}}=\frac{a(\mathrm{X}+h)+b(\mathrm{Y}+k)+c}{a'(\mathrm{X}+h)+b'(\mathrm{Y}+k)+c'}$$

$$=\frac{a\mathrm{X}+b\mathrm{Y}+(ah+bk+c)}{a'\mathrm{X}+b'\mathrm{Y}+(a'h+b'k+c')} \quad ...(2)$$

Choose h and k such that (2) becomes homogeneous.

This requires $ah+bk+c=0$ and $a'h+b'k+c'=0$

so that $\frac{h}{bc'-b'c}=\frac{k}{ca'-c'a}=\frac{1}{ab'-a'b}$ or $h=\frac{bc'-b'c}{ab'-a'b},\ k=\frac{ca'-c'a}{ab'-a'b}$

Since $\frac{a}{a'}\neq\frac{b}{b'}$ $\therefore$ $ab'-a'b\neq 0$ so that h, k are finite.

$\therefore$ Equation (2) becomes $\frac{d\mathrm{Y}}{d\mathrm{X}}=\frac{a\mathrm{X}+b\mathrm{Y}}{a'\mathrm{X}+b'\mathrm{Y}}$

which is homogeneous in X, Y and can be solved by putting $\mathrm{Y}=v\mathrm{X}$.

Case II. When $\frac{a}{a'} = \frac{b}{b'}, ab' - a'b = 0$ and the above method fails

Now, $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$ (say) so that $a' = ma,\ b' = mb$

Equation (1) becomes $\frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c'} = f(ax+by)$

which can be solved by putting $ax + by = t$.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve : (3y – 7x +7)dx + (7y – 3x + 3) dy = 0.*

Sol. The given equation can be written as $\frac{dy}{dx} = -\frac{3y-7x+7}{7y-3x+3}$ $\left[\text{Here } \frac{a}{a'} \neq \frac{b}{b'}\right]$ ...(1)

Putting $x = X + h,\ y = Y + k$ so that $dx = dX,\ dy = dY$ (h, k are constants)

Equation (1) becomes $\frac{dY}{dX} = -\frac{3(Y+k)-7(X+h)+7}{7(Y+k)-3(X+h)+3} = -\frac{3Y-7X+(-7h+3k+7)}{7Y-3X+(-3h+7k+3)}$ ...(2)

Now, choosing, h, k such that $-7h + 3k + 7 = 0$ and $-3h + 7k + 3 = 0$

Solving these equations $h = 1,\ k = 0$.

With these values of h, k equation (2) reduces to $\frac{dY}{dX} = -\frac{3Y-7X}{7Y-3X}$ ...(3)

Putting $Y = vX$ so that $\frac{dY}{dX} = v + X\frac{dv}{dX}$

Equation (3) becomes $v + X\frac{dv}{dX} = -\frac{3vX-7X}{7vX-3X}$ or $X\frac{dv}{dX} = \frac{7-3v}{7v-3} - v = \frac{7-7v^2}{7v-3}$

Separating the variables $\frac{7v-3}{1-v^2}\,dv = 7\frac{dX}{X}$ or $\left(\frac{2}{1-v} - \frac{5}{1+v}\right)dv = 7\frac{dX}{X}$

Integrating $-2\log(1-v) - 5\log(1+v) = 7\log X + c$

or $7\log X + 2\log(1-v) + 5\log(1+v) = -c$

or $\log[X^7(1-v)^2(1+v)^5] = -c$ or $X^7\left(1-\frac{Y}{X}\right)^2\left(1+\frac{Y}{X}\right)^5 = e^{-c}$

or $(X-Y)^2(X+Y)^5 = C$, where $C = e^{-c}$ ...(4)

Putting $X = x - h = x - 1,\ Y = y - k = y$

Equation (4) becomes $(x-y-1)^2(x+y-1)^5 = C$, which is the required solution.

Example 2. *Solve : (3y + 2x + 4) dx – (4x + 6y + 5) dy = 0.* **(P.T.U., May 2011)**

Sol. The given equation can be written as $\frac{dy}{dx} = \frac{(2x+3y)+4}{2(2x+3y)+5}$ ...(1)

Here, $\frac{a}{a'} = \frac{b}{b'}$

Putting $2x + 3y = t$ so that $2 + 3\frac{dy}{dx} = \frac{dt}{dx}$

or $$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dt}{dx} - 2\right)$$

Equation (1) becomes $$\frac{1}{3}\left(\frac{dt}{dx} - 2\right) = \frac{t+4}{2t+5}$$

or $$\frac{dt}{dx} = \frac{3t+12}{2t+5} + 2 = \frac{7t+22}{2t+5}$$

Separating the variables $$\frac{2t+5}{7t+22}\,dt = dx \quad \text{or} \quad \left(\frac{2}{7} - \frac{9}{7}\cdot\frac{1}{7t+22}\right)dt = dx$$

Integrating both sides $$\frac{2}{7}t - \frac{9}{49}\log(7t+22) = x + c$$

or $$14\,t - 9\log(7t+22) = 49x + 49c$$

Putting $t = 2x + 3y$, we have $14\,(2x+3y) - 9\log(14x+21y+22) = 49x + 49c$

or $$21x - 42y + 9\log(14x+21y+22) = -49c$$

or $$7\,(x-2y) + 3\log(14x+21y+22) = \mathrm{C} \qquad \left(\text{where } \mathrm{C} = -\frac{49}{3}c\right)$$

which is the required solution.

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{dy}{dx} = \dfrac{y+x-2}{y-x-4}$

2. $\dfrac{dy}{dx} = \dfrac{x+2y-3}{2x+y-3}$

3. $\dfrac{dy}{dx} + \dfrac{2x+3y+1}{3x+4y-1} = 0$

4. $(x+2y+3)\,dx - (2x-y+1)\,dy = 0$

5. $(2x-2y+5)\dfrac{dy}{dx} = x - y + 3$

6. $\dfrac{dy}{dx} = \dfrac{6x-4y+3}{3x-2y+1}$

7. $(2x+y+1)\,dx + (4x+2y-1)\,dy = 0$

8. $(x+y)(dx-dy) = dx + dy.$

ANSWERS

1. $x^2 + 2xy - y^2 - 4x + 8y = \mathrm{C}$

2. $(x-y)^3 = \mathrm{C}\,(x+y-2)$

3. $x^2 + 3xy + 2y^2 + x - y = \mathrm{C}$

4. $\log\,[(x+1)^2 + (y+1)^2] = 4\tan^{-1}\dfrac{y+1}{x+1} + \mathrm{C}$

5. $x - 2y + \log(x-y+2) = \mathrm{C}$

6. $2x - y = \log(3x-2y+3) + \mathrm{C}$

7. $x + 2y + \log(2x+y-1) = \mathrm{C}$

8. $x - y = \log(x+y) + \mathrm{C}.$

1.7. EXACT DIFFERENTIAL EQUATIONS (P.T.U., Jan. 2009)

Definition : A differential equation obtained from its primitive directly by differentiation, without any operation of multiplication, elimination or reduction etc. is said to be **an exact differential equation.**

Thus a differential equation of the form $\mathrm{M}(x, y)\,dx + \mathrm{N}(x, y)\,dy = 0$ is an exact differential equation if it can be obtained directly by differentiating the equation $u(x, y) = \mathrm{C}$ which is its primitive *i.e.*, if $du = \mathrm{M}\,dx + \mathrm{N}\,dy$.

1.8. THEOREM

The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. **(P.T.U., 2002, Jan. 2010, Dec. 2012, May 2014)**

Proof. The condition is necessary

The equation $Mdx + Ndy = 0$ will be exact, if $du = Mdx + Ndy$

But
$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$\therefore \quad Mdx + Ndy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Equating coefficients of dx and dy, we get

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$$

But
$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is the necessary condition of exactness.

The condition is sufficient.

Let
$$u = \int_{y\,\text{constant}} M\,dx$$

$$\therefore \quad \frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial M}{\partial y}$$

But
$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y} \quad \text{and} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

Integrating both sides w.r.t. x treating y as constant, we have $N = \frac{\partial u}{\partial y} + f(y)$

$$\therefore \quad Mdx + Ndy = \frac{\partial u}{\partial x}\,dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\}dy \qquad \left[\because\ M = \frac{\partial u}{\partial x},\ \ N = \frac{\partial u}{\partial y} + f(y)\right]$$

$$= \left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right) + f(y)\,dy = du + f(y)\,dy = d\left[u + \int f(y)\,dy\right]$$

which shows that $Mdx + Ndy$ is an exact differential and hence $Mdx + Ndy = 0$ is an exact differential equation.

Note. Since $\quad Mdx + Ndy = d\,[u + \int f(y)\,dy]$

$\therefore \quad Mdx + Ndy = 0 \;\Rightarrow\; d\,[u + \int f(y)\,dy] = 0$

Integrating $\quad u + \int f(y)\, dy = c$

But $\quad u = \int_{y\,\text{constant}} \mathrm{M}dx$ and $f(y)$ = terms of N not containing x

Hence the solution of Mdx + Ndy = 0 is $\displaystyle\int_{y\,\text{constant}} \mathbf{M}dx + \int (\textbf{terms of N not containing } x)\, dy = c.$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the initial value problem $e^x(\cos y\, dx - \sin y\, dy) = 0$; $y(0) = 0$.* **(P.T.U., May 2008)**

Sol. $\quad e^x \cos y\, dx - e^x \sin y\, dy = 0$

Compare it with $\mathrm{M}\, dx + \mathrm{N}\, dy = 0$

$$\mathrm{M} = e^x \cos y,\ \mathrm{N} = -\, e^x \sin y$$

$$\frac{\partial \mathrm{M}}{\partial y} = -\, e^x \sin y\ ;\ \frac{\partial \mathrm{N}}{\partial x} = -\, e^x \sin y$$

$$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$$

$\therefore$ Given equation is exact and its solution is $\displaystyle\int_{y\text{ constant}} \mathrm{M}\, dx + \int (\text{terms of N not containing } x)\, dy = c$

or $\quad \displaystyle\int_{y\text{ constant}} e^x \cos y\, dx + \int 0 \cdot dy = c$

or $\quad \cos y \cdot e^x = c \qquad \ldots(1)$

Given $\quad y(0) = 0$, *i.e.*, $y = 0$ when $x = 0$

Substituting in (1), we get $\quad 1 = c$

$\therefore$ Solution of the given equation is

$$e^x \cos y = 1$$

Example 2. *Solve : $(x^2 - ay)\, dx = (ax - y^2)\, dy$.* **(P.T.U., 2005)**

Sol. $\quad (x^2 - ay)\, dx - (ax - y^2)\, dy = 0 \qquad \ldots(1)$

Compare it with $\quad \mathrm{M}dx + \mathrm{N}dy = 0$

$$\mathrm{M} = x^2 - ay, \quad \mathrm{N} = -\, ax + y^2$$

$$\frac{\partial \mathrm{M}}{\partial y} = -\, a,\ \frac{\partial \mathrm{N}}{\partial x} = -\, a$$

$$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$$

$\therefore$ (1) is exact differential equation. Therefore, its solution is

$$\int_y \mathrm{M}\, dx + \int (\text{terms of N not containing } x)\, dy = c$$

or $\quad \displaystyle\int_y (x^2 - ay)\, dx + \int y^2\, dy = c$

or $\quad \dfrac{x^3}{3} - ayx + \dfrac{y^3}{3} = c$

or $\quad x^3 + y^3 - 3axy = c'$, where $c' = 3c$.

Example 3. *Solve : $(\sec x \tan x \tan y - e^x)\, dx + \sec x \sec^2 y\, dy = 0$.* **(P.T.U., Dec. 2005)**

Sol. $(\sec x \tan x \tan y - e^x)\, dx + \sec x \sec^2 y\, dy = 0$ ...(1)

Compare it with $\mathrm{M}\, dx + \mathrm{N}\, dy = 0$

$$\mathrm{M} = \sec x \tan x \tan y - e^x \text{ and } \mathrm{N} = \sec x \sec^2 y$$

$$\therefore \qquad \frac{\partial \mathrm{M}}{\partial y} = \sec x \tan x \sec^2 y$$

$$\frac{\partial \mathrm{N}}{\partial x} = \sec x \tan x \sec^2 y$$

$$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$$

$\therefore$ Equation (1) is exact.

$\therefore$ Its solution is

$$\int_{y \text{ constant}} \left(\sec x \tan x \tan y - e^x\right) dx + \int \left(\text{terms of N not containing } x\right) dy = c$$

$$\tan y \int \left(\sec x \tan x - e^x\right) dx + \int 0\, dy = c$$

$$\therefore \qquad \tan y\, (\sec x) - e^x = c.$$

Example 4. *For what value of k, the differential equation $\left(1 + e^{\frac{kx}{y}}\right) dx + e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) dy = 0$ is exact?*

(P.T.U., May 2010)

Sol. Given differential equation is

$$\left(1 + e^{\frac{kx}{y}}\right) dx + e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) dy = 0 \qquad \text{...(1)}$$

Compare it with $\qquad \mathrm{M}dx + \mathrm{N}dy = 0$

$$\mathrm{M} = 1 + e^{\frac{kx}{y}} \qquad \mathrm{N} = e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right)$$

(1) will be exact if $\qquad \dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$

i.e.,
$$\frac{\partial}{\partial y}\left(1 + e^{\frac{kx}{y}}\right) = \frac{\partial}{\partial x}\left\{e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\right\}$$

or
$$e^{\frac{kx}{y}}\left(\frac{-kx}{y^2}\right) = e^{\frac{x}{y}}\, \frac{1}{y}\left(1 - \frac{x}{y}\right) + e^{\frac{x}{y}}\left(-\frac{1}{y}\right) = e^{\frac{x}{y}}\left(\frac{1}{y} - \frac{x}{y^2} - \frac{1}{y}\right) = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)$$

or
$$k\, e^{\frac{kx}{y}} = e^{\frac{x}{y}}, \text{ which holds when } k = 1$$

$\therefore$ For $k = 1$, (1) is an exact differential equation.

Example 5. *Under what conditions on a, b, c, d, the differential equation (a sinh x cos y + b cosh x sin y) dx + (c sinh x cos y + d cosh x sin y) dy = 0 is exact?* **(P.T.U., May 2012)**

Sol. Given differential equation is

$(a \sinh x \cos y + b \cosh x \sin y)\, dx + (c \sinh x \cos y + d \cosh x \sin y)\, dy = 0$ …(1)

Compare it with $\mathrm{M}\, dx + \mathrm{N}\, dy = 0$

$$\mathrm{M} = a \sinh x \cos y + b \cosh x \sin y$$

$$\mathrm{N} = c \sinh x \cos y + d \cosh x \sin y$$

$$\frac{\partial \mathrm{M}}{\partial y} = -a \sinh x \sin y + b \cosh x \cos y$$

$$\frac{\partial \mathrm{N}}{\partial x} = c \cosh x \cos y + d \sinh x \sin y$$

Equation (1) will be exact

if $$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$$

i.e., $-a \sinh x \sin y + b \cosh x \cos y$

$$= c \cosh x \cos y + d \sinh x \sin y$$

which holds when

$-a = d$ and $b = c$ *i.e.*, $a = -d$ and $b = c$.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. (*a*) $(x^2 - 4xy - 2y^2)\, dx + (y^2 - 4xy - 2x^2)\, dy = 0$ (*b*) $(5x^4 + 3x^2y^2 - 2xy^3)\, dx + (2x^3y - 3x^2y^2 - 5y^4)\, dy = 0$

2. $(3x^2 + 6xy^2)\, dx + (6x^2 y + 4y^3)\, dy = 0$ **(P.T.U., May 2009)**

3. $(x^2 + y^2 - a^2)\, x dx + (x^2 - y^2 - b^2)\, y dy = 0$

4. $\left(1 + e^{\frac{x}{y}}\right) dx + \left(1 - \frac{x}{y}\right) e^{\frac{x}{y}}\, dy = 0$

5. $(y \cos x + 1)\, dx + \sin x dy = 0$

6. $(\sec x \tan x \tan y - e^x)\, dx + \sec x \sec^2 y dy = 0$

7. $ye^{xy}\, dx + (xe^{xy} + 2y)\, dy = 0$

8. $\left(y^2 e^{xy^2} + 4x^3\right) dx + \left(2xye^{xy^2} - 3y^2\right) dy = 0$

9. $(2xy \cos x^2 - 2xy + 1)\, dx + (\sin x^2 - x^2)\, dy = 0$

10. $(\sin x \cos y + e^{2x})\, dx + (\cos x \sin y + \tan y)\, dy = 0$

11. $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx + (x + \log x - x \sin y)\, dy = 0$

12. $\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$ **(P.T.U., May 2011)**

13. $x\, dy + y\, dx + \frac{x\, dy - y\, dx}{x^2 + y^2} = 0$

14. $[\cos x \tan y + \cos (x + y)]\, dx + [\sin x \sec^2 y + \cos (x + y)]\, dy = 0.$

ANSWERS

1. (*a*) $x^3 - 6x^2 y - 6xy^2 + y^3 = c$ (*b*) $x^5 + x^3y^2 - x^2y^3 - y^3 = c$

2. $x^3 + 3x^2 y^2 + y^4 = c$

3. $x^4 + 2x^2 y^2 - 2a^2 x^2 - y^4 - 2b^2 y^2 = c$

4. $x + ye^{x/y} = c$

5. $y \sin x + x = c$

6. $\sec x \tan y - e^x = c$

7. $e^{xy} + y^2 = c$

8. $e^{xy^2} + x^4 - y^3 = c$

9. $y \sin x^2 - x^2 y + x = c$

10. $-\cos x \cos y + \frac{1}{2} e^{2x} + \log \sec y = c$

11. $y(x + \log x) + x \cos y = c$

12. $y \sin x + (\sin y + y) x = c$

13. $xy + \tan^{-1} \frac{y}{x} = c$

$\left[\textbf{Hint:} \text{ The given equation is } d(xy) + d\left(\tan^{-1}\frac{y}{x}\right) = 0.\right]$

14. $\sin x \tan y + \sin(x + y) = c$

1.9. EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Integrating factor

Differential equations which are not exact can sometimes be made exact after multiplying by a suitable factor (a function of x or y or both) called the **integrating factor.**

For example, consider the equation $y\,dx - x\,dy = 0$ ...(1)

Here, $M = y$ and $N = -x$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore the equation is not exact.

(*i*) Multiplying the equation by $\frac{1}{y^2}$, it becomes $\frac{y\,dx - x\,dy}{y^2} = 0$ or $d\left(\frac{x}{y}\right) = 0$ which is exact.

(*ii*) Multiplying the equation by $\frac{1}{x^2}$, it becomes $\frac{y\,dx - x\,dy}{x^2} = 0$ or $d\left(\frac{y}{x}\right) = 0$ which is exact.

(*iii*) Multiplying the equation by $\frac{1}{xy}$, it becomes $\frac{dx}{x} - \frac{dy}{y} = 0$ or $d(\log x - \log y) = 0$ which is exact.

$\therefore$ $\frac{1}{y^2}, \frac{1}{x^2}$ and $\frac{1}{xy}$ are integrating factors of (1).

Note. *If a differential equation has one integrating factor, it has an infinite number of integrating factors.*

1.9(a). I.F. FOUND BY INSPECTION

In a number of problems, a little analysis helps to find the integrating factor. The following differentials are useful in selecting a suitable integrating factor.

(*i*) $y\,dx + x\,dy = d(xy)$

(*ii*) $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

(*iii*) $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

(*iv*) $\frac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

(*v*) $\frac{x\,dy - y\,dx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$

(*vi*) $\frac{y\,dx + x\,dy}{xy} = d\left[\log(xy)\right]$

(*vii*) $\frac{x\,dy + y\,dy}{x^2 + y^2} = d\left[\frac{1}{2}\log\left(x^2 + y^2\right)\right]$

(*viii*) $\frac{x\,dy - y\,dx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x + y}{x - y}\right)$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $y(2xy + e^x)dx - e^x dy = 0.$ **(P.T.U., May 2014)**

Sol. $$y(2xy + e^x)dx - e^x dy = 0$$

or $$2xy^2 dx + ye^x\, dx - e^x dy = 0 \qquad ...(1)$$

Since $$2x\, dx = \frac{1}{2} d(x^2)$$

$\therefore$ The term $2xy^2\, dx$ should not involve y^2

$\therefore$ It suggests that $\frac{1}{y^2}$ is the I.F.

$\therefore$ Multiplying (1) by $\frac{1}{y^2}$, we get

$$2x\, dx + \frac{y e^x dx - e^x\, dy}{y^2} = 0$$

or $$\frac{1}{2} d(x^2) + d\left(\frac{e^x}{y}\right) = 0$$

or $$d\left(\frac{1}{2}x^2 + \frac{e^x}{y}\right) = 0$$

Integrating; $\frac{1}{2}x^2 + \frac{e^x}{y} = c$, which is the required solution.

Example 2. *Find the integrating factor of the differential equation* $(y - 1)dx - xdy = 0$ *and hence solve it.* **(P.T.U., May 2006)**

Sol. Given equation is

$$(y - 1)dx - xdy = 0 \qquad \ldots(1)$$

Compare it with $\text{M}\, dx + \text{N}\, dy = 0$

$$\text{M} = y - 1, \text{N} = -x$$

$$\frac{\partial \text{M}}{\partial y} = 1, \frac{\partial \text{N}}{\partial x} = -1 ; \frac{\partial \text{M}}{\partial y} \neq \frac{\partial \text{N}}{\partial x}$$

$\therefore$ Equation (1) is not exact.

$\therefore$ Write (1) as $y\, dx - x\, dy = dx$

Multiply it by $\frac{1}{x^2}$, we get

$$\frac{ydx - xdy}{x^2} = \frac{1}{x^2} dx$$

or $$d\left(-\frac{y}{x}\right) = d\left(-\frac{1}{x}\right) \qquad \ldots(2)$$

$\therefore$ Factor $\frac{1}{x^2}$ makes the equation (1) exact differential equation

$\therefore$ $\frac{1}{x^2}$ is the I.F.

Integrate (2) ; $$\frac{-y}{x} = -\frac{1}{x} + c$$

or $y = 1 - cx$ is the required solution.

Example 3. *Solve* : $xdy - ydx = x\sqrt{x^2 - y^2}\, dx$. **(P.T.U., Jan. 2009)**

Sol. The given equation is $xdy - ydx = x^2\sqrt{1-\left(\frac{y}{x}\right)^2}\, dx$ or $\dfrac{\frac{x\,dy - y\,dx}{x^2}}{\sqrt{1-\left(\frac{y}{x}\right)^2}} = dx$

or $d\left(\sin^{-1}\frac{y}{x}\right) = dx$, which is exact.

Integrating, we get $\sin^{-1}\frac{y}{x} = x + c$ or $y = x\sin(x + c)$, which is the required solution.

1.9(b). I.F. FOR A HOMOGENEOUS EQUATION

If $\text{M}dx + \text{N}dy = 0$ is a homogeneous equation in x and y, then $\dfrac{1}{\text{M}x+\text{N}y}$ is an I.F. provided $\text{M}x + \text{N}y \neq 0$.

Proof. If $\dfrac{1}{\text{M}x+\text{N}y}$ is an integrating factor of $\text{M}dx + \text{N}dy = 0$ ...(1)

Then $\dfrac{\text{M}\,dx}{\text{M}x+\text{N}y} + \dfrac{\text{N}dy}{\text{M}x+\text{N}y} = 0$ is an exact equation.

$$\therefore \quad \frac{\partial}{\partial y}\left[\frac{\text{M}}{\text{M}x+\text{N}y}\right] = \frac{\partial}{\partial x}\left[\frac{\text{N}}{\text{M}x+\text{N}y}\right]$$

$$\frac{(\text{M}x+\text{N}y)\frac{\partial \text{M}}{\partial y} - \text{M}\left[x\frac{\partial \text{M}}{\partial y} + \text{N} + y\frac{\partial \text{N}}{\partial y}\right]}{(\text{M}x+\text{N}y)^2} - \frac{(\text{M}x+\text{N}y)\frac{\partial \text{N}}{\partial x} - \text{N}\left[\text{M} + x\frac{\partial \text{M}}{\partial x} + y\frac{\partial \text{N}}{\partial x}\right]}{(\text{M}x+\text{N}y)^2} = 0$$

or $$\text{M}x\frac{\partial \text{M}}{\partial y} + \text{N}y\frac{\partial \text{M}}{\partial y} - \text{M}x\frac{\partial \text{M}}{\partial y} - \text{MN} - \text{M}y\frac{\partial \text{N}}{\partial y} - \text{M}x\frac{\partial \text{N}}{\partial x} - \text{N}y\frac{\partial \text{N}}{\partial x} + \text{NM} + \text{N}x\frac{\partial \text{M}}{\partial x} + \text{N}y\frac{\partial \text{N}}{\partial x} = 0$$

or $$\text{N}y\frac{\partial \text{M}}{\partial y} - \text{M}y\frac{\partial \text{N}}{\partial y} - \text{M}x\frac{\partial \text{N}}{\partial x} + \text{N}x\frac{\partial \text{M}}{\partial x} = 0$$

or $$\text{N}\left\{x\frac{\partial \text{M}}{\partial x} + y\frac{\partial \text{M}}{\partial y}\right\} - \text{M}\left\{x\frac{\partial \text{N}}{\partial x} + y\frac{\partial \text{N}}{\partial y}\right\} = 0 \qquad \text{...(2)}$$

Now $\because$ M, N are homogeneous functions of x and y of order n therefore, we have

$x\dfrac{\partial \text{M}}{\partial x} + y\dfrac{\partial \text{M}}{\partial y} = n\text{M}$ and $x\dfrac{\partial \text{N}}{\partial x} + y\dfrac{\partial \text{N}}{\partial y} = n\text{N}$ (By Euler's theorem of homogeneous partial differential equations)

$\therefore$ From (2) $\text{N}\cdot n\text{M} - \text{M}\cdot n\text{N} = 0$ *i.e.*, $0 = 0$, which is true.

Hence $\dfrac{1}{\text{M}x+\text{N}y}$ is the I.F. of (1)

Case of failure. When $\text{M}x + \text{N}y = 0 \quad \therefore \; \text{N} = -\dfrac{\text{M}x}{y}$

From (1) $\text{M}dx - \dfrac{\text{M}\,x\,dy}{y} = 0$ or $\dfrac{dx}{x} = \dfrac{dy}{y}$; Integrate $\log x = \log y + \log c \quad \therefore \quad x = cy$

Note. If $\text{M}x + \text{N}y$ consists of only one term, use the above method of I.F. otherwise, proceed by putting $y = vx$.

Another Method. Consider $\mathrm{M}dx + \mathrm{N}dy = \frac{1}{2}\left[(\mathrm{M}x + \mathrm{N}y)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (\mathrm{M}x - \mathrm{N}y)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$

Divide by $\mathrm{M}x + \mathrm{N}y \ (\neq 0)$

$$\frac{\mathrm{M}dx + \mathrm{N}dy}{\mathrm{M}x + \mathrm{N}y} = \frac{1}{2}\left[d(\log|xy| + \frac{\mathrm{M}x - \mathrm{N}y}{\mathrm{M}x + \mathrm{N}y}\, d\log\left|\frac{x}{y}\right|\right]$$

Now, $\dfrac{\mathrm{M}x - \mathrm{N}y}{\mathrm{M}x + \mathrm{N}y} = \dfrac{\mathrm{M}\dfrac{x}{y} - \mathrm{N}}{\mathrm{M}\dfrac{x}{y} + \mathrm{N}}$ $\because$ M, N are homogeneous functions of x and y

$\therefore$ They can also be expressed in the form $\dfrac{x}{y}$.

$$\therefore \qquad \frac{\mathrm{M}x - \mathrm{N}y}{\mathrm{M}x + \mathrm{N}y} = \phi\left(\frac{x}{y}\right)$$

$$\therefore \qquad \frac{\mathrm{M}dx + \mathrm{N}dy}{\mathrm{M}x + \mathrm{N}y} = \frac{1}{2}\left[d\left\{\log|xy|\right\} + \phi\left(\frac{x}{y}\right) d\left\{\log\left|\frac{x}{y}\right|\right\}\right]$$

which is an exact derivative

Hence $\dfrac{1}{\mathrm{M}x + \mathrm{N}y}$ is the I.F.

Example 4. *Solve:* $(x^2 y - 2xy^2)\, dx - (x^3 - 3x^2 y)\, dy = 0.$ **(P.T.U., Dec. 2003, Dec. 2010)**

Sol. The given equation is homogeneous in x and y with $\mathrm{M} = x^2 y - 2xy^2$ and $\mathrm{N} = -x^3 + 3x^2 y$

Now, $\mathrm{M}x + \mathrm{N}y = x^3 y - 2x^2 y^2 - x^2 y + 3x^2 y^2 = x^2 y^2 \neq 0$

$\therefore$ I.F. $= \dfrac{1}{\mathrm{M}x + \mathrm{N}y} = \dfrac{1}{x^2 y^2}$

Multiplying throughout by $\dfrac{1}{x^2 y^2}$, the given equation becomes $\left(\dfrac{1}{y} - \dfrac{2}{x}\right)dx - \left(\dfrac{x}{y^2} - \dfrac{3}{y}\right)dy = 0$, which is exact. The solution is $\displaystyle\int_{y \text{ constant}} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\, dy = c$

or $\qquad \dfrac{x}{y} - 2\log x + 3\log y = c.$

1.9(c). I.F. FOR AN EQUATION OF THE FORM $yf_1(xy)dx + xf_2(xy)dy = 0$

(P.T.U., Dec. 2004)

Let the given differential equation be of the form

$$\mathrm{M}\,dx + \mathrm{N}\,dy = 0, \text{ where } \mathrm{M} = y f_1(xy),\ \mathrm{N} = x f_2(xy) \qquad \ldots(1)$$

$\dfrac{1}{\mathrm{M}x - \mathrm{N}y}$ will be the I.F of (1) if $\dfrac{\mathrm{M}}{\mathrm{M}x - \mathrm{N}y}dx + \dfrac{\mathrm{N}}{\mathrm{M}x - \mathrm{N}y}dy = 0$ is an exact equation

$$\therefore \qquad \frac{\partial}{\partial y}\left[\frac{\mathrm{M}}{\mathrm{M}x - \mathrm{N}y}\right] = \frac{\partial}{\partial x}\left[\frac{\mathrm{N}}{\mathrm{M}x - \mathrm{N}y}\right]$$

i.e., $$\frac{(\mathrm{M}x-\mathrm{N}y)\dfrac{\partial \mathrm{M}}{\partial y}-\mathrm{M}\left[x\dfrac{\partial \mathrm{M}}{\partial y}-\mathrm{N}-y\dfrac{\partial \mathrm{N}}{\partial y}\right]}{(\mathrm{M}x-\mathrm{N}y)^2}-\frac{(\mathrm{M}x-\mathrm{N}y)\dfrac{\partial \mathrm{N}}{\partial x}-\mathrm{N}\left[\mathrm{M}+x\dfrac{\partial \mathrm{M}}{\partial x}-y\dfrac{\partial \mathrm{N}}{\partial x}\right]}{(\mathrm{M}x-\mathrm{N}y)^2}=0$$

or $$\mathrm{M}x\frac{\partial \mathrm{M}}{\partial y}-\mathrm{N}y\frac{\partial \mathrm{M}}{\partial y}-\mathrm{M}x\frac{\partial \mathrm{M}}{\partial y}+\mathrm{MN}+\mathrm{M}y\frac{\partial \mathrm{N}}{\partial y}-\mathrm{M}x\frac{\partial \mathrm{N}}{\partial x}+\mathrm{N}y\frac{\partial \mathrm{N}}{\partial x}+\mathrm{NM}+\mathrm{N}x\frac{\partial \mathrm{M}}{\partial x}-\mathrm{N}y\frac{\partial \mathrm{N}}{\partial x}=0$$

or $$-\mathrm{N}y\frac{\partial \mathrm{M}}{\partial y}+\mathrm{M}y\frac{\partial \mathrm{N}}{\partial y}-\mathrm{M}x\frac{\partial \mathrm{N}}{\partial x}+\mathrm{N}x\frac{\partial \mathrm{M}}{\partial x}+2\mathrm{MN}=0$$

or $$\mathrm{N}\left\{x\frac{\partial \mathrm{M}}{\partial x}-y\frac{\partial \mathrm{M}}{\partial y}\right\}-\mathrm{M}\left[x\frac{\partial \mathrm{N}}{\partial x}-y\frac{\partial \mathrm{N}}{\partial y}\right]+2\mathrm{MN}=0 \qquad \ldots(2)$$

$\because$ $\mathrm{M}=y f_1(xy)$

$\therefore$ $$\frac{\partial \mathrm{M}}{\partial x}=y f_1'(xy).\,y$$

$$\frac{\partial \mathrm{M}}{\partial y}=f_1(xy)\,.1+y f_1'(xy).\,x$$

$\therefore$ $$x\frac{\partial \mathrm{M}}{\partial x}-y\frac{\partial \mathrm{M}}{\partial y}=xy^2 f_1'-y f_1-xy^2 f_1'$$

or $$x\frac{\partial \mathrm{M}}{\partial x}-y\frac{\partial \mathrm{M}}{\partial y}=-f_1\,y=-\mathrm{M}$$

Similarly, $\mathrm{N}=x f_2(xy)$

$$\frac{\partial \mathrm{N}}{\partial x}=f_2(xy)+x\cdot f_2'(xy)\;y$$

$$\frac{\partial \mathrm{N}}{\partial y}=x\cdot f_2'(xy)\;x$$

$$x\frac{\partial \mathrm{N}}{\partial x}-y\frac{\partial \mathrm{N}}{\partial y}=x f_2+x^2 y f_2'-x^2 y f_2'$$

$$=x f_2=f_2(xy)\,x=\mathrm{M}$$

$\therefore$ From (2), $\mathrm{N}(-\mathrm{M})-\mathrm{M}(\mathrm{N})+2\,\mathrm{MN}=0$

or $-2\,\mathrm{MN}+2\mathrm{MN}=0 \Rightarrow 0=0$ which is true. Hence $\dfrac{1}{\mathrm{M}x-\mathrm{N}y}$ is the I.F. of (1)

Case of failure. I.F. fails when $\mathrm{M}x-\mathrm{N}y=0$ *i.e.,* $\mathrm{N}=\dfrac{\mathrm{M}x}{y}$

$\therefore$ From (1), $$\mathrm{M}dx+\frac{\mathrm{M}x}{y}\,dy=0$$

or $$\frac{dx}{x}+\frac{dy}{y}=0$$

Integrating both sides $\log x+\log y=c_1$

or $\log(xy)=c_1$ or $xy=c$

If $\mathrm{M}dx+\mathrm{N}dy=0$ is of the form $f_1(xy)\,y\,dx+f_2(xy)\,x\,dy=0$, then $\dfrac{1}{\mathrm{M}x-\mathrm{N}y}$ is an I.F. provided

$\mathrm{M}x-\mathrm{N}y\neq 0$.

Example 5. *Solve :* $y(xy+2x^2y^2)\,dx+x(xy-x^2y^2)\,dy=0.$

Sol. The given equation is of the form $y f_1(xy)\,dx+x f_2(xy)\,dy=0$

Here $\mathrm{M}=xy^2+2x^2y^3$ and $\mathrm{N}=x^2y-x^3y^2$

Now, $\mathrm{M}x-\mathrm{N}y=x^2y^2+2x^3y^3-x^2y^2+x^3y^3=3x^3y^3\neq 0$

$\therefore$ $$\text{I.F.}=\frac{1}{\mathrm{M}x-\mathrm{N}y}=\frac{1}{3x^3y^3}$$

Multiplying throughout by $\dfrac{1}{3x^3y^3}$, the given equation becomes

$$\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy=0$$

which is exact. The solution is $\displaystyle\int_{y\text{ constant}}\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\int -\frac{1}{3y}\,dy=c$

or $$-\frac{1}{3xy}+\frac{2}{3}\log x-\frac{1}{3}\log y=c$$

or $$-\frac{1}{xy}+2\log x-\log y=C,\text{ where } C=3c.$$

1.9(d). I.F. FOR THE EQUATION Mdx + Ndy = 0

(*i*) If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$ is a function of x only, say $f(x)$, then $e^{\int f(x)\,dx}$ is an I.F.

(*ii*) If $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}$ is a function of y only, we say $g(y)$, then $e^{\int g(y)\,dy}$ is an I.F.

Proof. (*ii*) $Mdx+Ndy=0$ ...(1)

If $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=g(y)$ *i.e.*, a function of y alone then $e^{\int g(y)\,dy}$ is an I.F. of (1).

Now, $e^{\int g(y)\,dy}$ will be I.F. of (1)

If $M\,e^{\int g(y)dy}\,dx+N\,e^{\int g(y)\,dy}dy=0$ is exact

i.e., $$\frac{\partial}{\partial y}\left\{M\,e^{\int g(y)\,dy}\right\}=\frac{\partial}{\partial x}\left\{N\,e^{\int g(y)\,dy}\right\}$$

$$M\,e^{\int g(y)dy}\cdot g(y)+\frac{\partial M}{\partial y}\,e^{\int g(y)dy}=e^{\int g(y)dy}\,\frac{\partial N}{\partial x}$$

or $$Mg(y)=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$$

or $$\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=g(y)\text{, which is true}$$

Hence $e^{\int g(y)\,dy}$ is the I.F. of (1).
Students can easily prove (*i*) part themselves.

Example 6. *Solve:* $\left(xy^2-e^{1/x^3}\right)dx-x^2y\,dy=0$. **(P.T.U., Dec. 2003, May 2011)**

Sol. Here $M=xy^2-e^{1/x^3}$ and $N=-x^2y$

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{2xy-(-2xy)}{-x^2y}=-\frac{4}{x}\text{, which is a function of } x \text{ only.}$$

$\therefore$ $$\text{I.F.} = e^{\int -\frac{4}{x}\,dx} = e^{-4\log x} = \frac{1}{x^4}$$

Multiplying throughout by $\frac{1}{x^4}$, we have $\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right)dx - \frac{y}{x^2}\,dy = 0$

which is exact. The solution is $\int\limits_{y \text{ constant}} \left(\frac{y^2}{x^3} - \frac{1}{x^4}\,e^{1/x^3}\right)dx + \int 0.dy = c$

or $$-\frac{y^2}{2x^2} + \frac{1}{3}\int -\frac{3}{x^4}\,e^{1/x^3}\,dx = c \quad \text{or} \quad -\frac{y^2}{2x^2} + \frac{1}{3}\int e^t\,dt = c, \quad \text{where } t = \frac{1}{x^3} \quad \therefore\ dt = -\frac{3}{x^4}\,dx$$

or $$-\frac{y^2}{2x^2} + \frac{1}{3}\,e^t = c \quad \text{or} \quad -\frac{3y^2}{x^2} + 2e^{1/x^3} = \text{C}, \text{ where } \text{C} = 6c.$$

Example 7. *Find the general solution of the differential equation*

$(3x^2y^3e^y + y^3 + y^2)\,dx + (x^3y^3\,e^y - xy)\,dy = 0$ ...(1) **(P.T.U., May 2012, Dec. 2013)**

Sol. Given differential equation is $(3x^2y^3e^y + y^3 + y^2)\,dx + (x^3y^3e^y - xy)\,dy = 0$

compare it with $\text{M}dx + \text{N}dy = 0$

$$\text{M} = 3x^2y^3\,e^y + y^3 + y^2$$

$$\text{N} = x^3y^3e^y - xy$$

$$\frac{\partial \text{M}}{\partial y} = 3x^2\,(y^3e^y + 3y^2e^y) + 3y^2 + 2y$$

$$\frac{\partial \text{N}}{\partial x} = 3x^2\,y^3e^y - y$$

$$\frac{\partial \text{M}}{\partial y} \neq \frac{\partial \text{N}}{\partial x}$$

$\therefore$ Given differential equation is not exact

$$\frac{\dfrac{\partial \text{N}}{\partial x} - \dfrac{\partial \text{M}}{\partial y}}{\text{M}} = \frac{3x^2y^3e^y - y - 3x^2y^3e^y - 9x^2y^2e^y - 3y^2 - 2y}{3x^2y^3e^y + y^3 + y^2}$$

$$= \frac{-3(3x^2y^2\,e^y + y^2 + y)}{y(3x^2y^2e^y + y^2 + y)} = \frac{-3}{y}$$

which is a function of y only

$\therefore$ $$\text{I.F} = e^{\int -\frac{3}{y}\,dy} = e^{-3\log y} = e^{\log y^{-3}} = \frac{1}{y^3}$$

Multiply (1) by $\frac{1}{y^3}$

$$\left(3x^2e^y + 1 + \frac{1}{y}\right)dx + \left(x^3e^y - \frac{x}{y^2}\right)dy = 0 \text{ is an exact equation}$$

Its solution is $\int\limits_{y \text{ constant}} \left(3x^2e^y + 1 + \frac{1}{y}\right)dx + \int 0.dy = c$

$$x^3e^y + x + \frac{x}{y} = c$$

Example 8. *Find the integrating factor of the differential equation $(5x^3 + 12x^2 + 6y^2)dx + 6xy\,dy = 0$ which will make it exact. Hence solve the equation.* **(P.T.U., Dec. 2013)**

Sol. Differential equation is $(5x^3 + 12x^2 + 6y^2)dx + 6xydy = 0$ ...(1)

Compare it with $\quad \text{M}dx + \text{N}dy = 0$

$$\text{M} = 5x^3 + 12x^2 + 6y^2;\ \text{N} = 6xy$$

$$\frac{\partial \text{M}}{\partial y} = 12y\,;\ \frac{\partial \text{N}}{\partial x} = 6y$$

$$\frac{\partial \text{M}}{\partial y} \neq \frac{\partial \text{N}}{\partial x}$$

$\therefore$ Equation is not exact

$$\frac{\frac{\partial \text{M}}{\partial y} - \frac{\partial \text{N}}{\partial x}}{\text{N}} = \frac{12y - 6y}{6xy} = \frac{6y}{6xy} = \frac{1}{x} = f(x)$$, which is a function of x only.

$\therefore$ $\quad$ I.F. = of (1) is $e^{\int \frac{1}{x}\,dx} = e^{\log x} = x$

$\therefore$ Multiply (1) by x

$(5x^4 + 12x^3 + 6y^2x)dx + 6x^2y\,dy = 0$ is an exact equation

$\therefore$ Its solution is

$$\int_y \left(5x^4 + 12x^3 + 6y^2x\right) dx + \int 0\,dy = c$$

or
$$\frac{5x^5}{5} + 12\,\frac{x^4}{4} + 6y^2\,\frac{x^2}{2} = c$$

or
$$x^5 + 3x^4 + 3x^2y^2 = c$$

1.9(e). I.F. FOR THE EQUATION OF THE FORM $x^a y^b$ ($my\ dx + nx\ dy$) + $x^c y^d$($py\ dx + qx\ dy$) = 0 (where a, b, c, d, m, n, p, q are all Constants)

The above form of the equation has I.F. x^hy^k, where h, k can be obtained by after multiplication of the given equation by x^hy^k and the equation becomes exact. Apply the conditions of exactness, comparing the coefficients of the corresponding terms, we will get two linear equations in h, k which will give us the values of h, k.

Example 9. *Solve :* $(2x^2y^2 + y)\,dx + (3x - x^3\,y)\,dy = 0.$

Sol. The equation can be written as $2\,(x^2y^2dx - x^3ydy) + (y\,dx + 3xdy) = 0$

or
$$x^2y\,(2ydx - xdy) + x^0\,y^0\,(y\,dx + 3xdy) = 0$$

which is of the form $x^ay^b\,(mydx + nxdy) + x^cy^d\,(pydx + qxdy) = 0$. Therefore, it has an I.F. of the form $x^h\,y^k$.

Multiplying the given equation by $x^h\,y^k$, we have

$$(2x^{h+2}\,y^{k+2} + x^h\,y^{k+1})\,dx + (3x^{h+1}\,y^k - x^{h+3}\,y^{k+1})\,dy = 0$$

For this equation to be exact, we must have

$$\frac{\partial \text{M}}{\partial y} = \frac{\partial \text{N}}{\partial x} \quad i.e.,\ \ 2(k+2)\,x^{h+2}\,y^{k+1} + (k+1)\,x^h\,y^k = 3\,(h+1)\,x^h\,y^k - (h+3)\,x^{h+2}\,y^{k+1}$$

which holds when $2\,(k+2) = -\,(h+3)$ and $k + 1 = 3\,(h+1)$

i.e., when $h + 2k + 7 = 0$ and $3h - k + 2 = 0$

Solving these equation, we have $h = -\dfrac{11}{7},\ k = -\dfrac{19}{7}$

$\therefore$ $\quad$ I.F. $= x^{-\frac{11}{7}}\ y^{-\frac{19}{7}}$

Multiplying the given equation by $x^{-\frac{11}{7}}\ y^{-\frac{19}{7}}$, we have

$$\left(2x^{\frac{3}{7}} y^{-\frac{5}{7}} + x^{-\frac{11}{7}} y^{-\frac{12}{7}}\right) dx + \left(3x^{-\frac{4}{7}} y^{-\frac{19}{7}} - x^{\frac{10}{7}} y^{-\frac{12}{7}}\right) dy = 0$$

which is exact. The solution is $\displaystyle\int_{y \text{ constant}} \left(2x^{\frac{3}{7}} y^{-\frac{5}{7}} + x^{-\frac{11}{7}} y^{-\frac{12}{7}}\right) dx = c$

or $\quad \dfrac{7}{5} x^{\frac{10}{7}} y^{-\frac{5}{7}} - \dfrac{7}{4} x^{-\frac{4}{7}} y^{-\frac{12}{7}} = c$ or $4x^{\frac{10}{7}} y^{-\frac{5}{7}} - 5x^{-\frac{4}{7}} y^{-\frac{12}{7}} = \text{C}$

where $\text{C} = \dfrac{20}{7} c$.

Note. The values of h and k can also be determined from the relations

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \text{ and } \frac{c+h+1}{p} = \frac{d+k+1}{q}.$$

Example 10. *Solve:* $(2y^2 + 4x^2y)\, dx + (4xy + 3x^3)\, dy = 0.$ **(P.T.U., May 2011)**

Sol. Given equation is

$(2y^2 + 4x^2y)\, dx + (4xy + 3x^3)\, dy = 0$ ...(1)

The equation in not exact

$\therefore$ Rewrite the equation in the form:

$x^2 (4y\, dx + 3x\, dy) + y(2y\, dx + 4x\, dy) = 0$ ...(2)

which is of the form

$x^a y^b (my\, dx + nx\, dy) + x^c y^d (py\, dx + qx\, dy) = 0$

Let $x^h y^k$ be the I.F. of (1)

$\therefore \quad (2x^h y^{k+2} + 4x^{h+2} y^{k+1})dx + (4x^{h+1} y^{k+1} + 3x^{h+3} y^k)\, dy = 0$

is an exact equation

$$\therefore \quad \frac{\partial \text{M}}{\partial y} = \frac{\partial \text{N}}{\partial x}$$

i.e., $$\frac{\partial}{\partial y}\left(2x^h y^{k+2} + 4x^{h+2} y^{k+1}\right) = \frac{\partial}{\partial x}\left(4x^{h+1} y^{k+1} + 3x^{h+3} y^k\right)$$

or $\quad 2x^h (k+2)y^{k+1} + 4x^{h+2} (k+1)y^k = 4(h+1)\, x^h y^{k+1} + 3(h+3)x^{h+2} y^k$

which holds when $\quad 2(k+2) = 4(h+1)$

and $\quad 4(k+1) = 3(h+3)$

i.e., $\quad k + 2 = 2h + 2$ or $2h = k$

and $\quad 4k = 3h + 5$ or $8h - 3h = 5$

$\therefore \quad h = 1, \quad k = 2$

$\therefore \quad xy^2$ is the I.F. of (1)

Multiply (1) by xy^2, we get

$(2xy^4 + 4x^3y^3)dx + (4x^2 y^3 + 3x^4 y^2)dx = 0$

which is an exact solution

$\therefore$ Its solution is $\quad \displaystyle\int_y \left(2xy^4 + 4x^3 y^3\right) dx + \int 0\, dy = \text{C}$

or $\quad x^2y^4 + x^4y^3 = \text{C}$ is the solution of the given equation.

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $x\,dy - y\,dx = (x^2 + y^2)\,dx$ **(P.T.U., Jan. 2010)**

   [**Hint:** I.F. $= \dfrac{1}{x^2 + y^2}$]

2. $y\,dx - x\,dy + 3x^2y^2 e^{x^3}\,dx = 0$ **(P.T.U., Dec. 2013)**

   [**Hint:** I.F. $= \dfrac{1}{y^2}$]

3. $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$
4. $x\,dy - y\,dx = xy^2\,dx$
5. $\left(xye^{\frac{x}{y}} + y^2\right)dx - x^2 e^{\frac{x}{y}}\,dy = 0$
6. $x^2 y\,dx - (x^3 + y^3)\,dy = 0$
7. $(3xy^2 - y^3)\,dx - (2x^2 y - xy^2)\,dy = 0$
8. $(x^2 y^2 + xy + 1)\,y\,dx + (x^2 y^2 - xy + 1)\,x\,dy = 0$ **(P.T.U., May 2010, Dec. 2012)**
9. $y\,(2xy + 1)\,dx + x\,(1 + 2xy - x^3y^3)\,dy = 0$
10. $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$ **(P.T.U., Dec. 2012)**
11. $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0$
12. $\left(y + \dfrac{y^3}{3} + \dfrac{x^2}{2}\right)dx + \dfrac{1}{4}\left(x + xy^2\right)dy = 0$
13. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$
14. $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0$
15. $y\,dx - x\,dy + \log x\,dx = 0$
16. $(xy^2 + 2x^2 y^3)\,dx + (x^2 y - x^3 y^2)\,dy = 0$
17. $(2x^2 y - 3y^4)\,dx + (3x^3 + 2xy^3)\,dy = 0.$
18. $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0.$

ANSWERS

1. $y = x \tan (x + c)$
2. $\dfrac{x^2}{y} + e^{x^3} = c$
3. $-\dfrac{1}{xy} + \log \dfrac{x}{y} = c$
4. $\dfrac{x^2}{2} + \dfrac{x}{y} = c$
5. $e^{\frac{x}{y}} + \log x = c$
6. $\log y - \dfrac{x^3}{3y^3} = c$
7. $3 \log x - 2 \log y + \dfrac{y}{x} = c$
8. $xy + \log \dfrac{x}{y} - \dfrac{1}{xy} = c$
9. $\dfrac{1}{x^2 y^2} + \dfrac{1}{3x^3 y^3} + \log y = c$
10. $e^x (x^2 + y^2) = c$
11. $x - \dfrac{y^2}{x} - \dfrac{1}{x} = c$
12. $x^4 y + x^4 y^3 + x^6 = c$
13. $y + \dfrac{2}{y^2} x + y^2 = c$
14. $x^2 (ay^2 - xy) = c$
15. $cx + y \log x + 1 = 0$
16. $-\dfrac{1}{xy} + 2\log x - \log y = c$
17. $5x^{-\frac{36}{13}} y^{\frac{24}{13}} - 12x^{-\frac{10}{13}} y^{-\frac{15}{13}} = c$.
18. $\dfrac{x^2 y^4}{2} + xy^2 + \dfrac{y^6}{3} = c$

1.10. DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

So far, we have discussed differential equations of the first order and first degree. Now we shall study differential equations of the first order and degree higher than the first. For convenience, we denote $\dfrac{dy}{dx}$ by p.

A differential equation of the first order and nth degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots\ldots + P_n = 0 \qquad \ldots(1)$$

where $P_1, P_2, \ldots\ldots\ldots, P_n$ are functions of x and y.

Since it is a differential equation of the first order, its general solution will contain only one arbitrary constant.

In the various cases which follow, the problem is reduced to that of solving one or more equations of the first order and first degree.

1.11. EQUATIONS SOLVABLE FOR *p*

Resolving the left hand side of (1) into n linear factors, we have

$$[p - f_1(x, y)]\,[p - f_2(x, y)], \ldots\ldots\ldots, [p - f_n(x, y)] = 0$$

which is equivalent to $p - f_1(x, y) = 0,\ p - f_2(x, y) = 0,\ \ldots\ldots\ldots,\ p - f_n(x, y) = 0$

Each of these equations is of the first order and first degree and can be solved by the methods already discussed.

If the solutions of the above n component equations are

$$F_1(x, y, c) = 0,\ F_2(x, y, c) = 0,\ \ldots\ldots\ldots, F_n(x, y, c) = 0$$

then the general solution of (1) is given by $F_1(x, y, c).\ F_2(x, y, c) \ldots\ldots\ldots F_n(x, y, c) = 0.$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}.$ **(P.T.U., May 2014)**

Sol. $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$

$$p - \frac{1}{p} = \frac{x^2 - y^2}{xy}, \text{ where } p = \frac{dy}{dx}$$

$$xyp^2 - (x^2 - y^2)p - xy = 0$$

$$p = \frac{(x^2 - y^2) \pm \sqrt{(x^2 - y^2)^2 + 4x^2y^2}}{2xy}$$

$$= \frac{(x^2 - y^2) \pm (x^2 + y^2)}{2xy}$$

$\therefore$
$$p = \frac{2x^2}{2xy} \quad ; \quad p = -\frac{2y^2}{2xy}$$

or
$$p = \frac{x}{y} \quad ; \quad p = -\frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{y} \quad ; \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$ydy = xdx \quad ; \quad \frac{1}{y}dy = -\frac{1}{x}dx$$

Integrating both sides

$$\frac{y^2}{2} = \frac{x^2}{2} + c \quad ; \quad \log y = -\log x + c$$

or
$$y^2 - x^2 - 2c = 0 \quad \text{or} \quad \log xy = c \quad \text{or} \quad xy = e^c$$

$\therefore$ The general solution of given equation is $(y^2 - x^2 - 2c)\,(xy - e^c) = 0.$

Example 2. *Solve: $p(p + y) = x(x + y)$.* **(P.T.U., May 2007)**

Sol. $p^2 + py = x^2 + xy$ ...(1)

or $p^2 + py - (x^2 + xy) = 0$, which is quadratic in p

$\therefore \qquad p = \dfrac{-y \pm \sqrt{y^2 + 4\left(x^2 + xy\right)}}{2} = \dfrac{-y \pm \sqrt{\left(y + 2x\right)^2}}{2}$

$\therefore \qquad p = \dfrac{-y + y + 2x}{2}$

or $\qquad p = x$

or $\qquad \dfrac{dy}{dx} = x$

Integrating both sides,

$$y = \frac{x^2}{2} + c$$

or $\qquad y - \dfrac{x^2}{2} - c = 0 \qquad$ ...(2)

and $\qquad p = \dfrac{-y - y - 2x}{2}$

or $\qquad p = -y - x$

or $\qquad \dfrac{dy}{dx} = -y - x$

or $\qquad \dfrac{dy}{dx} + y = -x$

which is linear equation in y

Its $\qquad$ I.F. $= e^{\int 1 \cdot dx} = e^x$

$\therefore$ Its solution is

$$y\, e^x = \int e^x \left(-x\right) dx + c = -\int x e^x dx + c$$

or $\qquad y\, e^x = -(x-1)\, e^x + c \qquad$ [Integrating by parts]

or $\qquad y = -(x-1) + ce^{-x}.$

or $\qquad y + x - 1 - ce^{-x} = 0 \qquad$ ...(3)

Combining (2) and (3), general solution is

$$\left(y - \frac{x^2}{2} - c\right)\left(y + x - 1 - ce^{-x}\right) = 0.$$

Note. $\dfrac{dy}{dx} = -(x+y)$ can also be solved by putting $x + y = t$, but that is a lengthy solution.

Example 3. *Solve :* $p^2 + 2py \cot x = y^2$. **(P.T.U., Jan. 2009, Dec. 2012)**

Sol. The given equation can be written as $(p + y \cot x)^2 = y^2 (1 + \cot^2 x)$

or $\qquad p + y \cot x = \pm\, y \operatorname{cosec} x$

$\therefore$ The component equations are

$p = y(-\cot x + \operatorname{cosec} x) \qquad$ ...(1)

and $\qquad p = y(-\cot x - \operatorname{cosec} x) \qquad$ ...(2)

From (1), $\qquad \dfrac{dy}{dx} = y(-\cot x - \operatorname{cosec} x)$

or $$\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)\, dx$$

Integrating $$\log y = -\log \sin x + \log \tan \frac{x}{2} + \log c$$

$$= \log \frac{c \tan \frac{x}{2}}{\sin x}$$

or $$y = \frac{c \tan \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \frac{c}{2 \cos^2 \frac{x}{2}}$$

or $$y \cos^2 \frac{x}{2} = \text{C, where } \text{C} = \frac{c}{2}$$

From (2), $$\frac{dy}{dx} = y(-\cot x - \operatorname{cosec} x)$$

or $$\frac{dy}{y} = (-\cot x - \operatorname{cosec} x)\, dx$$

Integrating $\log y = -\log \sin x - \log \tan \frac{x}{2} + \log c = \log \frac{c}{\sin x \tan \frac{x}{2}}$

or $$y = \frac{c}{2 \sin^2 \frac{x}{2}} \quad \text{or} \quad y \sin^2 \frac{x}{2} = \text{C}$$

$\therefore$ The general solution of the given equation is $\left(y \cos^2 \frac{x}{2} - \text{C} \right) \left(y \sin^2 \frac{x}{2} - \text{C} \right) = 0$

TEST YOUR KNOWLEDGE

Solve the following equations :

1. $p^2 - 7p + 12 = 0$ **(P.T.U., Dec. 2006)**

[**Hint:** Solve $p = 3$, $p = 4$]

2. $xy \left(\frac{dy}{dx} \right)^2 - \left(x^2 + y^2 \right) \frac{dy}{dx} + xy = 0$

3. $yp^2 + (x - y)\, p - x = 0$

4. $x^2 \left(\frac{dy}{dx} \right)^2 + 3xy \frac{dy}{dx} + 2y^2 = 0$ **(P.T.U., Dec. 2011)**

5. $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

6. $p^2 - 2p \sinh x - 1 = 0$

7. $xy\, p^2 + p\, (3x^2 - 2y^2) - 6xy = 0$

[**Hint:** Quadratic in p and values of p are $\frac{2y}{x}, -\frac{3x}{y}$]

8. $4y^2 p^2 + 2pxy\, (3x + 1) + 3x^3 = 0.$

ANSWERS

1. $(y - 4x - c)\,(y - 3x - c) = 0$

2. $(y^2 - x^2 - c)\,(y - cx) = 0$

3. $(y - x - c)\,(x^2 + y^2 - c) = 0$

4. $(xy - c)\,(x^2 y - c) = 0$

5. $(xy - c)\,(x^2 - y^2 - c) = 0$

6. $(y - e^x - c)\,(y - e^{-x} - c) = 0$

7. $(y - cx^2)\,(y^2 + 3x^2 - c) = 0$

8. $\left(y^2 + x^3 - c \right) \left(y^2 + \frac{1}{2} x^2 - c \right) = 0.$

1.12. EQUATIONS SOLVABLE FOR y

If the equation is solvable for y, we can express y explicitly in terms of x and p. Thus, the equations of this type can be put as $y = f(x, p)$ ...(1)

Differentiating (1) w.r.t. x, we get $\dfrac{dy}{dx} = p = F\left(x, p, \dfrac{dp}{dx}\right)$ ...(2)

Equation (2) is a differential equation of first order in p and x.

Suppose the solution of (2) is $\phi(x, p, c) = 0$ ...(3)

Now, elimination of p from (1) and (3) gives the required solution.

If p cannot be easily eliminated, then we solve equations (1) and (3) for x and y to get

$$x = \phi_1(p, c), \ y = \phi_2(p, c)$$

These two relations together constitute the solution of the given equation with p as parameter.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $y + px = x^4 p^2$.

Sol. Given equation is $y = -px + x^4 p^2$ ...(1)

Differentiating both sides w.r.t. x,

$$\frac{dy}{dx} = p = -p - x\frac{dp}{dx} + 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

or $$2p + x\frac{dp}{dx} - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0 \quad \text{or} \quad \left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$$

Discarding the factor $(1 - 2px^3)$, we have $2p + x\dfrac{dp}{dx} = 0$ or $\dfrac{dp}{p} + 2\dfrac{dx}{x} = 0$

Integrating $\log p + 2\log x = \log c$ or $\log px^2 = \log c$ or $px^2 = c$

or $$p = \frac{c}{x^2}.$$

Putting this value of p in (1), we have $y = -\dfrac{c}{x} + c^2$, which is the required solution.

Example 2. *Solve :* $y = 2px - p^2$.

Sol. The given equation is $y = 2px - p^2$ ...(1)

Differentiating both sides w.r.t. x, $\dfrac{dy}{dx} = 2p + 2x\dfrac{dp}{dx} - 2p\dfrac{dp}{dx}$

or $$p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

or $$p + (2x - 2p)\frac{dp}{dx} = 0 \quad \text{or} \quad p\frac{dx}{dp} + 2x - 2p = 0$$

or $$\frac{dx}{dp} + \frac{2}{p}x = 2 \qquad \text{...(2)}$$

which is a linear equation.

$$\text{I.F.} = e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2$$

$\therefore$ The solution of (2) is $$x\,(\text{I.F.}) = \int 2(\text{I.F.})\,dp + c \quad \text{or} \quad xp^2 = \int 2p^2\,dp + c$$

or $$xp^2 = \frac{2}{3}p^3 + c \quad \text{or} \quad x = \frac{2}{3}p + cp^{-2} \qquad \text{...(3)}$$

p cannot be easily eliminated from (1) and (3)

$\therefore$ Putting the value of x in (1), we have $y = 2p\left(\dfrac{2}{3}p + cp^{-2}\right) - p^2$

or $$y = \frac{1}{3}p^2 + 2c\,p^{-1} \qquad \text{...(4)}$$

Equations (3) and (4) together constitute the general solution of (1).

TEST YOUR KNOWLEDGE

Solve the following equations :

1. $xp^2 - 2yp + ax = 0$ **(P.T.U., May 2011)**
2. $y - 2px = \tan^{-1}(xp^2)$ **(P.T.U., May 2010)**
3. $16x^2 + 2p^2y - p^3x = 0$
4. $y = x + 2\tan^{-1} p$
5. $y = 3x + \log p$
6. $x - yp = ap^2$.
7. $x^2\left(\dfrac{dy}{dx}\right)^4 + 2x\dfrac{dy}{dx} - y = 0$
8. $3x^4p^2 - px - y = 0$ **(P.T.U., May 2010)**

   [**Hint:** See S.E. 1]

ANSWERS

1. $2y = cx^2 + \dfrac{a}{c}$
2. $y = 2\sqrt{cx} + \tan^{-1} c$
3. $16 + 2c^2y - c^3x^2 = 0$
4. $x = \log\dfrac{p-1}{\sqrt{p^2+1}} - \tan^{-1}p + c,\ y = \log\dfrac{p-1}{\sqrt{p^2+1}} + \tan^{-1}p + c$
5. $y = 3x + \log\dfrac{3}{1-ce^{3x}}$
6. $x = \dfrac{p}{\sqrt{1-p^2}}\left(c + a\sin^{-1}p\right),\ y = \dfrac{1}{\sqrt{1-p^2}}\left(c + a\sin^{-1}p\right) - ap$
7. $y = c^2 + 2\sqrt{cx}$
8. $y = 3c^2 - \dfrac{c}{x}$.

1.13. EQUATIONS SOLVABLE FOR x

If the equation is solvable for x, we can express x explicitly in terms of y and p. Thus, the equations of this type can be put as $x = f(y, p)$ ...(1)

Differentiating (1) w.r.t. y, we get $\dfrac{dx}{dy} = \dfrac{1}{p} = \mathrm{F}\left(y, p, \dfrac{dp}{dy}\right)$ ...(2)

Equation (2) is a differential equation of first order in p and y.

Suppose the solution of (2) is $\phi(y, p, c) = 0$ ...(3)

Now, elimination of p from (1) and (3) gives the required solution.

If p cannot be easily eliminated, then we solve equations (1) and (3) for x and y to get

$$x = \phi_1(p, c),\ y = \phi_2(p, c)$$

These two relations together constitute the solution of the given equation with p as parameter.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $y = 2px + p^2y$ **(P.T.U., Dec. 2012)**

Sol. Given differential equation is

$$y = 2px + p^2y \qquad \text{...(1)}$$

Solving for x, we have

$$x = \frac{y}{2p} - \frac{py}{2}$$

Differentiate w.r.t., y
$$\frac{dx}{dy} = \frac{1}{2}\,\frac{p\cdot 1 - y\dfrac{dp}{dy}}{p^2} - \frac{1}{2}\left(p\cdot 1 + y\frac{dp}{dy}\right)$$

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy}$$

$$\frac{1}{2p} + \frac{p}{2} + \frac{y}{2p^2}\frac{dp}{dy} + \frac{y}{2}\frac{dp}{dy} = 0$$

$$\frac{1+p^2}{p} + \left(\frac{1}{p^2}+1\right) y \frac{dp}{dy} = 0$$

$$\frac{1+p^2}{p} + \frac{1+p^2}{p^2}\, y \frac{dp}{dy} = 0$$

or
$$\frac{1+p^2}{p}\left\{1 + \frac{y}{p}\frac{dp}{dy}\right\} = 0$$

Discarding the factor $\dfrac{1+p^2}{p}$, we have

$$1 + \frac{y}{p}\frac{dp}{dy} = 0$$

or
$$\frac{dp}{p} = -\frac{dy}{y}$$

Integrating both sides

$$\log p = -\log y + \log c$$

or
$$py = c \qquad \ldots(2)$$

Eliminate p from (1) and (2)

$$y = 2x \cdot \frac{c}{y} + \frac{c^2}{y^2}\, y$$

or
$$y = \frac{2cx}{y} + \frac{c^2}{y}$$

or
$$y^2 = 2cx + c^2 \text{; required solution.}$$

Example 2. *Solve :* $p = tan\left(x - \dfrac{p}{1+p^2}\right)$.

Sol. Solving for x, we have $x = \tan^{-1} p + \dfrac{p}{1+p^2}$ …(1)

Differentiating both sides w.r.t. y, $\dfrac{dx}{dy} = \dfrac{1}{p} = \dfrac{1}{1+p^2} \cdot \dfrac{dp}{dy} + \dfrac{(1+p^2) - 2p^2}{(1+p^2)^2} \cdot \dfrac{dp}{dy}$

or
$$\frac{1}{p} = \frac{2(1+p^2) - 2p^2}{(1+p^2)^2}\frac{dp}{dy} \quad \text{or} \quad dy = \frac{2p}{(1+p^2)^2}\, dp$$

Integrating $y = c - \dfrac{1}{1+p^2}$ …(2)

Equations (1) and (2) together constitute the general solution.

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $y = 3px + 6p^2 y^2$

2. $y = 2px + y^2 p^3$

3. $p^3 - 4xyp + 8y^2 = 0$

4. $y^2 \log y = xyp + p^2$.

ANSWERS

1. $y^3 = 3cx + 6c^2$ **2.** $y^2 = 2cx + c^3$

3. $64y = c\,(c - 4x)^2$ **4.** $\log y = cx + c^2$.

1.14. CLAIRAUT'S EQUATION **(P.T.U., May 2007, Jan. 2009)**

An equation of the form $y = px + f(p)$ ...(1)
is known as Clairaut's equation.

Differentiating (1) w.r.t. x, we get $p = p + x\dfrac{dp}{dx} + f'(p)\dfrac{dp}{dx}$ or $\left[x + f'(p)\right]\dfrac{dp}{dx} = 0$

Discarding the factor $\left[x + f'(p)\right]$, we have $\dfrac{dp}{dx} = 0$

Integrating $p = c$

Putting $p = c$ in (1), the required solution is $y = cx + f(c)$

Thus, the solution of Clairaut's equation is obtained by writing c for p.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following equations :*

(i) $p = \log\,(px - y)$ *or* $\dfrac{dy}{dx} = \log\left(x\dfrac{dy}{dx} - y\right)$ **(P.T.U., Dec. 2005, 2011, 2013)**

(ii) $\sin px \cos y = \cos px \sin y + p$. **(P.T.U., Dec. 2006)**

Sol. (*i*) $p = \log\,(px - y)$

or $e^p = px - y$ or $y = px - e^p$, which is Clairaut's equation where $f(p) = -e^p$

$\therefore$ Its solution is obtained by putting $p = c$

$\therefore$ solution is $y = cx - e^c$.

(*ii*) $\sin px \cos y = \cos px \sin y + p$

or $\sin px \cos y - \cos px \sin y = p$

or $\sin\,(px - y) = p$

or $px - y = \sin^{-1} p$

or $y = px - \sin^{-1} p$, which is Clairaut's form

$\therefore$ Its solution is (put $p = c$) $y = cx - \sin^{-1} c$.

Note. Many differential equations can be reduced to Clairaut's form by suitably changing the variables.

Example 2. *Solve:* $e^{4x}(p - 1) + e^{2y} p^2 = 0$.

Sol. [In problems involving e^{lx} and e^{my}, put $X = e^{kx}$ and $Y = e^{ky}$, where k is the H.C.F. of l and m].

Put $X = e^{2x}$ and $Y = e^{2y}$

so that $dX = 2e^{2x}\,dx$ and $dY = 2e^{2y}\,dy$

$$\therefore \quad p = \frac{dy}{dx} = \frac{e^{2x}}{e^{2y}}\frac{dY}{dX} = \frac{X}{Y}P, \quad \text{where} \quad P = \frac{dY}{dX}$$

The given equation becomes $X^2\left(\dfrac{X}{Y}P - 1\right) + Y \cdot \dfrac{X^2}{Y^2}P^2 = 0$

or $XP - Y + P^2 = 0$ or $Y = PX + P^2$, which is of Clairaut's form

$\therefore$ Its solution is $Y = cX + c^2$ and hence $e^{2y} = ce^{2x} + c^2$.

Example 3. *Solve :* $(px - y)(py + x) = 2p.$ **(P.T.U., Jan. 2009, May 2009)**

Sol. Put $X = x^2$ and $Y = y^2$

so that $dX = 2x\,dx$ and $dY = 2y\,dy$

$$\therefore \quad p = \frac{dy}{dx} = \frac{x}{y}\frac{dY}{dX} = \frac{\sqrt{X}}{\sqrt{Y}}P, \text{ where } P = \frac{dY}{dX}$$

The given equation becomes $\left(\frac{\sqrt{X}}{\sqrt{Y}}P\,.\sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P\,.\sqrt{Y} + \sqrt{X}\right) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$

or $$(PX - Y)(P + 1) = 2P \text{ or } PX - Y = \frac{2P}{P+1}$$

or $$Y = PX - \frac{2P}{P+1}, \text{ which is of Clairaut's form.}$$

$\therefore$ Its solution is $Y = cX - \dfrac{2c}{c+1}$ and hence $y^2 = cx^2 - \dfrac{2c}{c+1}\cdot$

Example 4. *Solve:* $(x^2 + y^2)(1 + p)^2 = 2(x + y)(1 + p)(x + yp) - (x + yp)^2.$ **(P.T.U., May 2011)**

Sol. Given equation can be written as:

$$x^2 + y^2 = \frac{2(x+y)(x+py)}{1+p} - \left(\frac{x+py}{1+p}\right)^2 \quad \ldots(1)$$

Put $X = x + y,\ Y = x^2 + y^2$

and Let $P = \dfrac{dY}{dX} = \dfrac{dY}{dx} \Big/ \dfrac{dX}{dx}$

$$\therefore \quad P = \frac{2x + 2y\dfrac{dy}{dx}}{1 + \dfrac{dy}{dx}} = \frac{2(x+py)}{1+p}$$

Substituting in (1)

$$Y = 2X\frac{P}{2} - \left(\frac{P}{2}\right)^2$$

or $Y = PX - \dfrac{P^2}{4}$, which is Clairaut's differential equation

$\therefore$ Its solution is $Y = CX - \dfrac{C^2}{4}$

or $$x^2 + y^2 = CX - \frac{C^2}{4}$$

TEST YOUR KNOWLEDGE

Solve the following equations :

1. $y = xp + \dfrac{a}{p}$

2. (*a*) $y = px + \sqrt{a^2p^2 + b^2}$

(*b*) $(y - px)(p - 1) = p$

3. $p = \log(px - y)$

4. $p = \sin(y - px)$ **(P.T.U., May 2007)**

[**Hint:** See Example 1 (*ii*)]

5. $p^2(x^2 - 1) - 2pxy + y^2 - 1 = 0$

6. $e^{3x}(p - 1) + p^3 e^{2y} = 0$

7. $x^2(y - px) = yp^2$

[**Hint:** Put $x^2 = X$, $y^2 = Y$]

8. $(y + px)^2 = x^2 p.$

[**Hint:** Put $xy = v$]

ANSWERS

1. $y = cx + \dfrac{a}{c}$	**2.** (*a*) $y = cx + \sqrt{a^2 c^2 + b^2}$, (*b*) $y = cx + \dfrac{c}{c-1}$
3. $y = cx - e^c$	**4.** $y = cx + \sin^{-1} c$
5. $(y - cx)^2 = 1 + c^2$	**6.** $e^y = ce^x + c^2$
7. $y^2 = cx^2 + c^2$	**8.** $xy = cx - c^2$.

1.15. DEFINITION OF LEIBNITZ'S LINEAR DIFFERENTIAL EQUATION

A differential equation is said to be ***Leibnitz's linear*** *or simply linear if the dependent variable and its derivatives occur only in the first degree and are not multiplied together.*

(P.T.U., June 2003, May 2005, 2007)

The general form of a linear differential equation of the first order is $\dfrac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$ ...(1)

where P and Q are functions of x only or may be constants.

1.16. SOLVE THE LINEAR DIFFERENTIAL EQUATION $\dfrac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$ (P.T.U., Dec. 2006)

To solve it, we multiply both sides by $e^{\int \mathrm{P}\,dx}$, we get

$$\frac{dy}{dx}\, e^{\int \mathrm{P}dx} + y\left(e^{\int \mathrm{P}dx}\ \mathrm{P}\right) = \mathrm{Q}e^{\int \mathrm{P}dx}$$

or
$$\frac{d}{dx}\left(y\, e^{\int \mathrm{P}dx}\right) = \mathrm{Q}e^{\int \mathrm{P}dx}$$

Integrating both sides, we have $y\, e^{\int \mathrm{P}dx} = \int \mathrm{Q}e^{\int \mathrm{P}dx} dx + c$

which is the required solution.

Note 1. In the general form of a linear differential equation, the coefficient of $\dfrac{dy}{dx}$ is unity.

The equation $\mathrm{R}\dfrac{dy}{dx} + \mathrm{S}y = \mathrm{T}$, where R, S and T are functions of x only or constants, must be divided by R to bring it to the general linear form.

Note 2. The factor $e^{\int \mathrm{P}\,dx}$ on multiplying by which the LHS of (1) becomes the differential coefficient of a single function is called the integrating factor (briefly written as I.F.) of (1).

Thus I.F. $= e^{\int \mathrm{P}\,dx}$ and the solution is $y\,(\text{I.F.}) = \int \mathrm{Q}(\text{I.F.})\, dx + c$.

Note 3. Sometimes a differential equation takes linear form if we regard x as dependent variable and y as independent variable. The equation can then be put as $\dfrac{dx}{dy} + \mathrm{P}x = \mathrm{Q}$, where P, Q are functions of y only or constants.

The integrating factor in this case is $e^{\int \mathrm{P}\,dy}$ and the solution is $x\ (\text{I.F.}) = \int \mathrm{Q}\,(\text{I.F.})\,dy + c\,\cdot$

Note 4. $e^{\log f(x)} = f(x)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $x\left(1-x^2\right)\dfrac{dy}{dx}+\left(2x^2-1\right)\,y=x^3$.

Sol. Dividing by $x\,(1-x^2)$ to make the coefficient of $\dfrac{dy}{dx}$ unity, the given equation becomes

$$\frac{dy}{dx}+\frac{2x^2-1}{x\left(1-x^2\right)}\,y=\frac{x^2}{1-x^2}$$

Comparing it with $\dfrac{dy}{dx}+\mathrm{P}y=\mathrm{Q}$, we have $\mathrm{P}=\dfrac{2x^2-1}{x\left(1-x^2\right)}$, $\mathrm{Q}=\dfrac{x^2}{1-x^2}$

Now, $\mathrm{P}=\dfrac{2x^2-1}{x(1-x)\,(1+x)}=-\dfrac{1}{x}+\dfrac{1}{2(1-x)}-\dfrac{1}{2(1+x)}$ by partial fractions

$$\int \mathrm{P}\,dx = -\log x-\frac{1}{2}\log(1-x)-\frac{1}{2}\log(1+x)=-\log\left[x\left(1-x\right)^{\frac{1}{2}}\left(1+x\right)^{\frac{1}{2}}\right]$$

$$=-\log\left[x\left(1-x^2\right)^{\frac{1}{2}}\right]=\log\frac{1}{x\sqrt{1-x^2}}$$

$$\text{I.F.}=e^{\int \mathrm{P}\,dx}=e^{\log\frac{1}{x\sqrt{1-x^2}}}=\frac{1}{x\sqrt{1-x^2}}$$

Thus the solution is

$$y\,(\text{I.F.})=\int \mathrm{Q}\,(\text{I.F.})\,dx+c \quad \text{or} \quad y\,.\frac{1}{x\sqrt{1-x^2}}=\int\frac{x^2}{1-x^2}\times\frac{1}{x\sqrt{1-x^2}}\,dx+c=\int\frac{x}{\left(1-x^2\right)^{\frac{3}{2}}}\,dx+c$$

$$=-\frac{1}{2}\int\left(1-x^2\right)^{-\frac{3}{2}}(-2x)\,dx+c=\left(1-x^2\right)^{-\frac{1}{2}}+c \qquad \because \int\left[f(x)\right]^n f'(x)\,dx=\frac{\left[f(x)\right]^{n+1}}{n+1}\;(n\neq-1)$$

or $$y=x+cx\sqrt{1-x^2}\,.$$

Example 2. *Solve :* $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}-\dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy}=1$.

Sol. The given equation can be written as $\dfrac{dy}{dx}+\dfrac{y}{\sqrt{x}}=\dfrac{e^{2\sqrt{x}}}{\sqrt{x}}$

Comparing it with $\dfrac{dy}{dx}+\mathrm{P}y=\mathrm{Q}$, we have $\mathrm{P}=\dfrac{1}{\sqrt{x}}$, $\mathrm{Q}=\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\text{I.F.}=e^{\int \mathrm{P}\,dx}=e^{\int\frac{1}{\sqrt{x}}dx}=e^{\frac{x^{1/2}}{1/2}}=e^{2\sqrt{x}}$$

$\therefore$ The solution is $\quad y\,(\text{I.F.})=\displaystyle\int \mathrm{Q}\,(\text{I.F.})\,dx+c \quad \text{or} \quad ye^{2\sqrt{x}}=\int\frac{e^{2\sqrt{x}}}{\sqrt{x}}\,.\,e^{2\sqrt{x}}dx+c$

or $$y e^{2\sqrt{x}} = \int x^{\frac{1}{2}}\, dx + c = 2\sqrt{x} + c$$

Example 3. *Solve :* $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$. **(P.T.U., Dec. 2011, 2013)**

Sol. The given equation can be written as $(1+y^2)\dfrac{dx}{dy} + x = \tan^{-1} y$

Dividing by $(1 + y^2)$, we get $\dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{\tan^{-1} y}{1+y^2}$

which is of the form $\dfrac{dx}{dy} + \mathrm{P}x = \mathrm{Q}$

Here $\mathrm{P} = \dfrac{1}{1+y^2}$, $\mathrm{Q} = \dfrac{\tan^{-1} y}{1+y^2}$

$$\text{I.F.} = e^{\int \mathrm{P}\,dy} = e^{\int \frac{1}{1+y^2}\,dy} = e^{\tan^{-1} y}$$

$\therefore$ The solution is $x\,(\text{I.F.}) = \int \mathrm{Q}\,(\text{I.F.})\,dy + c$

or $$x e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2}\, e^{\tan^{-1} y}\, dy + c = \int t\, e^t\, dt + c, \text{ where } t = \tan^{-1} y$$

$$= t\,e^t - \int 1 \,.\, e^t\, dt + c = t\,e^t - e^t + c = (\tan^{-1} y - 1)\, e^{\tan^{-1} y} + c$$

or $$x = \tan^{-1} y - 1 + c\, e^{-\tan^{-1} y}.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3 - 3$

2. $x \log x \dfrac{dy}{dx} + y = 2 \log x$

3. $(x+1)\dfrac{dy}{dx} - y = e^x (x+1)^2$

4. $(x^2+1)\dfrac{dy}{dx} + 2xy = x^2$

5. $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

6. $(1 + x^3)\dfrac{dy}{dx} + 6x^2 y = 1 + x^2$

7. $\dfrac{dy}{dx} + y \cot x = \cos x$

8. $(1 + x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1} x}$

9. $\dfrac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$, if $y = 0$ when $x = \dfrac{\pi}{2}$

10. $\dfrac{dy}{dx} + y \cot x = 5e^{\cos x}$, if $y = -4$ when $x = \dfrac{\pi}{2}$

11. $\dfrac{dy}{dx} - y \tan x = 3e^{-\sin x}$, if $y = 4$ when $x = 0$

12. $(1 - x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{1 - x^2}$

13. $x\dfrac{dy}{dx} + y = e^x - xy$

14. $(1 + y^2) + \left(x - e^{-\tan^{-1} y}\right)\dfrac{dy}{dx} = 0$

15. $e^{-y} \sec^2 y\, dy = dx + x\, dy$

16. $(x + 2y^3)\dfrac{dy}{dx} = y$

17. $y e^{y}\, dx = \left(y^{2}+2x e^{y}\right) dy$

18. $\dfrac{dy}{dx} = \dfrac{y}{2y\log y + y - x}$

19. $x^{2}\dfrac{dy}{dx} = 3x^{2}-2xy+1$

20. $\sqrt{1-y^{2}}\; dx = \left(\sin^{-1} y - x\right) dy$.

21. $x\left(\dfrac{dy}{dx}+y\right) = 1-y$ **(P.T.U., May 2012)**

[Hint: $\dfrac{dy}{dx} + \dfrac{x+1}{x}\, y = \dfrac{1}{x}$; $\dfrac{dy}{dx}$ + Py = Q form]

ANSWERS

1. $10\, xy = 2x^5 - 15x^2 + c$

2. $y \log x = (\log x)^2 + c$

3. $y = (x+1)\,(e^x + c)$

4. $y\,(x^2+1) = \dfrac{x^3}{3} + c$

5. $y = \tan x - 1 + c\, e^{-\tan x}$

6. $y\left(1+x^{3}\right)^{2} = y + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \dfrac{x^6}{6} + c$

7. $y\sin x = \dfrac{1}{2}\sin^{2} x + c$

8. $2\, y\, e^{\tan^{-1} x} = e^{2\tan^{-1} x} + c$

9. $y\sin x = 2x^{2} - \dfrac{\pi^{2}}{2}$

10. $y \sin x = 5e^{\cos x} - 9$

11. $y\cos x = 7 - 3e^{-\sin x}$

12. $y = \sqrt{1-x^{2}} + c\left(1-x^{2}\right)$

13. $xy = \dfrac{1}{2}\, e^{x} + c\, e^{-x}$

14. $x = e^{-\tan^{-1} y}\left(\tan^{-1} y + c\right)$

15. $x\, e^{y} = \tan y + c$

16. $x + y^3 + cy$

17. $x = y^2\,(c - e^{-y})$

18. $x = \dfrac{c}{y} + y\log y$

19. $y = x + x^{-1} + cx^{-2}$

20. $x = \sin^{-1} y - 1 + c\, e^{-\sin^{-1} y}$.

21. $(xy-1)\, e^{x} = c$

1.17. EQUATIONS REDUCIBLE TO THE LINEAR FORM (Bernoulli's Equation)

(P.T.U., May 2007, Jan 2009)

(*a*) An equation of the form $\dfrac{dy}{dx} + \mathbf{P}y = \mathbf{Q}\, y^{n}$ …(1)

where P and Q are functions of *x* only or constants is known as *Bernoulli's* equation. Though not linear, it can be made linear.

Dividing both sides of (1) by y^n, we have $y^{-n}\dfrac{dy}{dx} + \mathrm{P}\, y^{1-n} = \mathrm{Q}$ …(2)

Putting $y^{1-n} = z$ so that $(1-n)\, y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

or $$y^{-n}\frac{dy}{dx}=\frac{1}{1-n}\frac{dz}{dx}$$

Equation (2) becomes $\frac{1}{1-n}\frac{dz}{dx}+Pz=Q$ or $\frac{dz}{dx}+(1-n)Pz=(1-n)Q$

which is a linear differential equation with z as the dependent variable.

(*b*) General equation reducible to linear form is $f'(y)\frac{dy}{dx}+P\,f(y)=Q$ ...(1)

where P and Q are functions of *x* only or constants.

Putting $f(y)=z$ so that $f'(y)\frac{dy}{dx}=\frac{dz}{dx}$

Equation (1) becomes $\frac{dz}{dx}+Pz=Q$, which is linear.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following differential equations:*

(*i*) $x\frac{dy}{dx}+y=x^3y^6$ **(P.T.U., May 2007, 2011)**

(*ii*) $y'+y=y^2$ **(P.T.U., May 2008)**

(*iii*) $\left(xy^2-e^{1/x^3}\right)dx-x^2y\,dy=0.$ **(P.T.U., Dec. 2012)**

Sol. (*i*) $x\frac{dy}{dx}+y=x^3y^6$

Dividing by xy^6, we get

$$y^{-6}\frac{dy}{dx}+\frac{1}{x}y^{-5}=x^2 \qquad \text{...(1)}$$

Put $y^{-5}=z$ so that

$$-5y^{-6}\frac{dy}{dx}=\frac{dz}{dx}$$

Equation (1) becomes

$$-\frac{1}{5}\frac{dz}{dx}+\frac{1}{x}z=x^2$$

or $$\frac{dz}{dx}-\frac{5}{x}z=-5x^2$$

which is linear in z, where $P=-\frac{5}{x}$, $Q=-5x^2$

$$\text{I.F.}=e^{\int-\frac{5}{x}dx}=e^{-5\log x}=e^{\log x^{-5}=x^{-5}=\frac{1}{x^5}}$$

$\therefore$ Solution in z is

$$z(\text{I.F.})=\int Q\cdot\text{I.F.}\,dx+c$$

$$z\cdot\frac{1}{x^5}=\int-5x^2\cdot\frac{1}{x^5}dx+c=-5\int x^{-3}dx+c=-5\frac{x^{-2}}{-2}+c=\frac{5}{2x^2}+c$$

Substituting the value of z, we get

$$\frac{1}{y^5}\cdot\frac{1}{x^5} = \frac{5}{2x^2} + c$$

or $$\frac{1}{y^5} = \frac{5x^3}{2} + cx^5$$

(*ii*) $$y' + y = y^2$$

or $$\frac{dy}{dx} + y = y^2$$

Divide by y^2; $$y^{-2}\frac{dy}{dx} + \frac{1}{y} = 1 \qquad ...(1)$$

Put $$\frac{1}{y} = z \quad \therefore \quad -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

Substituting in (1); $$-\frac{dz}{dx} + z = 1 \quad \text{or} \quad \frac{dz}{dx} - z = -1$$

which is linear differential equation in z

where $P = -1, Q = -1$

$$\text{I.F.} = e^{\int -1\,dx} = e^{-x}$$

Solution in z is $$z\,(\text{I.F.}) = \int Q\cdot(\text{I.F.})\,dx + c$$

or $$z\,e^{-x} = \int (-1)\,e^{-x}\,dx + c = e^{-x} + c$$

or $$z = 1 + ce^x$$

Substituting the value of z; $\dfrac{1}{y} = 1 + ce^x$

or $$y = \frac{1}{1 + ce^x}$$

(*iii*) $$\left(xy^2 - e^{\frac{1}{x^3}}\right) dx - x^2 y\,dy = 0$$

or $$x^2 y\frac{dy}{dx} - xy^2 + e^{\frac{1}{x^3}} = 0$$

or $$x^2 y\frac{dy}{dx} - xy^2 = -e^{\frac{1}{x^3}}$$

or $$y\frac{dy}{dx} - \frac{1}{x}y^2 = -\frac{1}{x^2}e^{\frac{1}{x^3}}$$

Put $y^2 = z$; $$2y\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{1}{2}\frac{dz}{dx} - \frac{1}{x}z = -\frac{1}{x^2}e^{\frac{1}{x^3}}$$

or $$\frac{dz}{dx} - \frac{2}{x} z = -\frac{2}{x^2} e^{\frac{1}{x^3}}$$

which is linear differential equation in z, where $P = -\frac{2}{x}, Q = -\frac{2}{x^2} e^{\frac{1}{x^3}}$

$$\text{I.F.} = e^{\int \frac{-2}{x} dx} = e^{-2 \log x} = e^{\log x^{-2}} = \frac{1}{x^2}$$

Solution in z is; $$z\,(\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + c$$

or $$z\left(\frac{1}{x^2}\right) = \int \frac{1}{x^2}\left(-\frac{2}{x^2} e^{\frac{1}{x^3}}\right) dx + c = -2\int \frac{1}{x^4} e^{\frac{1}{x^3}}\, dx + c$$

Put $\frac{1}{x^3} = t$ $\quad\therefore\quad$ $\frac{-3}{x^4} dx = dt$

$\therefore$ $$z \cdot \frac{1}{x^2} = (-2)\int e^t \frac{dt}{-3} + c = \frac{2}{3} e^t + c = \frac{2}{3} e^{\frac{1}{x^3}} + c$$

or $$\frac{y^2}{x^2} = \frac{2}{3} e^{\frac{1}{x^3}} + c \quad \text{or} \quad 3y^2 = 2x^2 e^{\frac{1}{x^3}} + cx^2$$

Example 2. *Solve :* $xy\left(1 + xy^2\right)\frac{dy}{dx} = 1$. **(P.T.U., May 2009)**

Sol. The given equation can be written as $\frac{dx}{dy} - yx = y^3 x^2$

Dividing by x^2, we have $x^{-2}\frac{dx}{dy} - y\,x^{-1} = y^3$ ...(1)

Putting $x^{-1} = z$ so that $-x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$ or $x^{-2}\frac{dx}{dy} = -\frac{dz}{dy}$

Equation (1) becomes $-\frac{dz}{dy} - yz = y^3$

or $\frac{dz}{dy} + yz = -y^3$, which is linear in z.

$$\text{I.F.} = e^{\int y\,dy} = e^{\frac{1}{2}y^2}$$

$\therefore$ The solution is $z\,(\text{I.F.}) = \int -y^3\,(\text{I.F.})\, dy + c$

or $$z\,.e^{\frac{1}{2}y^2} = \int -y^3 e^{\frac{1}{2}y^2}\, dy + c$$

or $$z.e^{\frac{1}{2}y^2} = -\int y^2 e^{\frac{1}{2}y^2}\, .y\, dy + c = -\int 2te^t\, dt + c, \text{ where } t = \frac{1}{2}y^2$$

or $$z.e^{\frac{1}{2}y^2} = -2\left[t\,e^t - \int 1 - e^t\, dt\right] + c = -2\left(t\,e^t - e^t\right) + c = -2e^{\frac{1}{2}y^2}\left(\frac{1}{2}y^2 - 1\right) + c$$

or $$z = -2\left(\frac{1}{2}y^2 - 1\right) + c\,e^{-\frac{1}{2}y^2}$$

or $$\frac{1}{x} = 2 - y^2 + c\,e^{-\frac{1}{2}y^2}. \qquad \left(\because\ z = \frac{1}{x}\right)$$

Example 3. *Solve :* $\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$. **(P.T.U., May 2002)**

Sol. Dividing by $\cos^2 y$, we have $\sec^2 y \frac{dy}{dx} + x\,\frac{2\sin y\cos y}{\cos^2 y} = x^3$

or $$\sec^2 y\,\frac{dy}{dx} + 2x\tan y = x^3 \qquad \ldots(1)$$

Putting $\tan y = z$ so that $\sec^2 y\,\frac{dy}{dx} = \frac{dz}{dx}$

Equation (1) becomes $\frac{dz}{dx} + 2xz = x^3$, which is linear in z.

I.F. $= e^{\int 2x\,dx} = e^{x^2}$;

$\therefore$ The solution is $z\,.\,e^{x^2} = \int x^3\,.\,e^{x^2}\,dx + c = \int x^2\,e^{x^2}\,.\,x\,dx + c$

$$= \frac{1}{2}\int t\,e^t\,dt + c, \quad \text{where } t = x^2$$

$$= \frac{1}{2}(t-1)e^t + c = \frac{1}{2}\left(x^2 - 1\right)e^{x^2} + c$$

or $$z = \frac{1}{2}\left(x^2 - 1\right) + c\,e^{-x^2}$$

or $$\tan y = \frac{1}{2}\left(x^2 - 1\right) + c\,e^{x^2}. \qquad (\because\ z = \tan y)$$

Example 4. *Solve :* $e^y y' = e^x(e^x - e^y)$. **(P.T.U., May 2004)**

Sol. $$e^y y' = e^x(e^x - e^y) \qquad \ldots(1)$$

Put $$e^y = z$$

Differentiating w.r.t. x $$e^y\frac{dy}{dx} = \frac{dz}{dx}$$

i.e., $$e^y y' = \frac{dz}{dx} \qquad \ldots(2)$$

Substituting in (1)

$$\frac{dz}{dx} = e^{2x} - e^x\,.\,z$$

or $\frac{dz}{dx} + e^x\,.\,z = e^{2x}$, which is a linear differential equation in z.

$$\text{I.F.} = e^{\int e^x dx} = e^{e^x}$$

Solution is $z \,.\, e^{e^x} = \int e^{2x} \,.\, e^{e^x} dx \;\; c$

Put $e^x = t$ $\qquad \therefore \quad e^x\, dx = dt$

$\therefore \quad z \,.\, e^{e^x} = \int t\, e^t\, dt + c$ Integrate by parts

$= (t-1)\, e^t + c$

$\therefore \quad e^y \,.\, e^{e^x} = \left(e^x - 1\right) e^{e^x} + c$ or $e^{e^x}\left(1 - e^x + e^y\right) = c$.

Example 5. *Solve : (2x log x – xy) dy = – 2y dx.* **(P.T.U., Dec. 2004)**

Sol. $$2x \log x - xy = -2y\, \frac{dx}{dy}$$

or $$2y\, \frac{dx}{dy} - y \,.\, x + 2x \log x = 0$$

or $$\frac{dx}{dy} - \frac{1}{2}\, x + \frac{x}{y} \log x = 0$$

Divide by x; $$\frac{1}{x} \frac{dy}{dx} + \frac{1}{y} \log x = \frac{1}{2}$$

Put $\log x = z$ $\qquad \therefore \qquad \dfrac{1}{x} \dfrac{dx}{dy} = \dfrac{dz}{dy}$

or $\dfrac{dz}{dy} + \dfrac{1}{y}\, z = \dfrac{1}{2}$, which is linear differential equation in z.

$$\text{I.F.} = e^{\int \frac{1}{y}\, dy} = e^{\log y} = y$$

$\therefore$ Solution is $$z \,.\, y = \int y \,.\, \frac{1}{2}\, dy + c = \frac{y^2}{4} + c$$

or $$y \log x = \frac{y^2}{4} + c.$$

Example 6. *Solve :* $\dfrac{dy}{dx} - \tan xy = -y^2 \sec^2 x.$ **(P.T.U., Dec. 2004)**

Sol. $\dfrac{dy}{dx} - \tan x \,.\, y = -y^2 \sec^2 x$

Divide by y^2 ; $\dfrac{1}{y^2} \dfrac{dy}{dx} - \tan x\, \dfrac{1}{y} = -\sec^2 x$

Put $\dfrac{1}{y} = z$ $\qquad \therefore \quad -\dfrac{1}{y^2} \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ $$-\frac{dz}{dx} - \tan x \cdot z = -\sec^2 x$$

or $$\frac{dz}{dx} + \tan x \cdot z = \sec^2 x$$

which is linear differential equation in z

$$\text{I.F.} = e^{\int \tan x\, dx} = e^{-\log \cos x} = e^{\log \sec x} = \sec x$$

$\therefore$ Its solution is

$$z \sec x = \int \sec^2 x \,.\, \sec x\, dx + c = \int \sec^3 x\, dx + c \qquad \ldots(1)$$

Let $\quad \text{I} = \int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx$ Integrate by parts

$$= (\sec x)(\tan x) - \int \sec x \tan x \tan x\, dx$$

$$= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx$$

$$= \sec x \tan x - \text{I} + \int \sec x\, dx$$

$\therefore \quad 2\text{I} = \sec x \tan x + \log(\sec x + \tan x)$

$\therefore \quad \text{I} = \frac{1}{2}\left[\sec x \tan x + \log\left(\sec x + \tan x\right)\right]$

Substituting in equation (1),

$$z \,.\, \sec x = \frac{1}{2}\, [\sec x \tan x + \log(\sec x + \tan x)] + c$$

Substitute the value of z,

$$\frac{1}{y} \sec x = \frac{1}{2}\, [\sec x \tan x + \log(\sec x + \tan x)] + c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

2. $\frac{dy}{dx} - x^2 y = y^2 e^{-\frac{1}{3}x^3}$

3. (*a*) $(x^3 y^2 + xy)\, dx = dy$

(*b*) $\frac{dy}{dx} + y = xy^3$ **(P.T.U., May 2012)**

4. (*a*) $e^y\left(\frac{dy}{dx} + 1\right) = e^x$ **(P.T.U., May 2002)**

[**Hint:** Put $e^y = z$]

(*b*) $(x + 1)\frac{dy}{dx} + 1 = 2e^{-y}$

[**Hint:** Divide by $(x + 1)e^{-y}$ and Put $e^y = z$]

5. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)\, e^x \sec y$

6. $\frac{dy}{dx} + y \tan x = y^3 \cos x$

7. $\frac{dy}{dx} + \frac{x}{1 - x^2}\, y = x\sqrt{y}$

8. $\frac{dy}{dx} + \frac{y \log y}{x} = \frac{y(\log y)^2}{x^2}$

9. $y - \cos x \frac{dy}{dx} = y^2(1 - \sin x) \cos x$, given that $y = 2$ when $x = 0$.

10. $y\left(2xy + e^x\right) dx = e^x\, dy$

11. $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

ANSWERS

1. $\frac{x}{y} = 1 + c\sqrt{x}$

2. $y(c - x) = e^{\frac{1}{3}x^3}$

3. (*a*) $\frac{1}{y} = -x^2 + 2 + ce^{-\frac{1}{2}x^2}$

(*b*) $\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}$

4. (*a*) $e^{x+y} = \frac{1}{2}e^{2x} + c$

(*b*) $(x+1)\,e^y = 2x + c$

5. $\sin y = (1 + x)(e^x + c)$

6. $\cos^2 x = y^2\left(c - 2\sin x + \frac{2}{3}\sin^3 x\right)$

7. $\sqrt{y} = -\frac{1}{3}\left(1 - x^2\right) + c\left(1 - x^2\right)^{\frac{1}{4}}$

8. $\frac{1}{x\log y} = \frac{1}{2x^2} + c$

9. $2(\tan x + \sec x) = y(2\sin x + 1)$

10. $e^x = y\left(c - x^2\right)$

11. $\sec y = (c + \sin x)\cos x$.

REVIEW OF THE CHAPTER

1. **Ordinary Differential Equation:** Differential equations which involve only one independent variable and the differential co-efficients w.r.t. it are called ordinary differential equations.
2. **Order and Degree of a Differential Equations: The order** of a differential equation is the order of the highest order derivative occurring in the differential equation. **The degree** of a differential equation is the degree of the highest order derivative which occurs in the differential equation.
3. **The general solution, the particular and the singular solution of a differential equation**.

   The general solution of a differential equation is that in which the number of independent arbitrary constants is equal to the order of differential equation.

   The particular solution of a differential equation is that which is obtained from the general solution by giving particular values to the arbitrary constants.

   The singular solution of a differential equation is that which satisfies the equation but cannot be derived from its general solution.
4. **Solution of differential equations of first order and first degree.**

   (*a*) **Variable separable form:** Put *dx* and all the terms containing *x* on one side, also *dy* and all the terms containing *y* on other side and integrate.

   (*b*) **If** $\boldsymbol{\frac{dy}{dx} = -f(ax + by + c), f}$ then put $ax + by + c = t$ equation will be changed to variable separable form.
5. **Homogeneous Differential Equation:** A differential equation of the form $\frac{dy}{dx} = \frac{f_1(x, y)}{g_1(x, y)}$ is called a homogeneous differential equation if $f_1(x, y)$ and $g_1(x, y)$ are homogeneous functions of the same degree in *x* and *y.* To solve homogeneous differential equation put $\frac{y}{x}$ or $\frac{x}{y} = v$, equation will be changed to variable separable form.
6. For solution of the differential equation of the form $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$

   Case I. If $\frac{a}{a'} \neq \frac{b}{b'}$, put $x = X + h$, $y = Y + k$ such that $ah + bk + c = 0$, $a'h + b'k + c' = 0$, equation will change to homogeneous form, then put $\frac{Y}{X} = V$ and in the end change X, Y to *x*, *y*.

   Case II. If $\frac{a}{a'} = \frac{b}{b'}$ then put $ax + by = t$ differential equation is changed to variable separable form.

7. Exact Differential Equation: A differential equation obtained from its primitive directly by differentiation, without any operation of multiplication, elimination or reduction, etc., is called an exact differential equations.

8. Necessary and Sufficient Condition for the Exactness of a Differential Equation: The necessary and sufficient conditions for the exactness of Mdx + Ndy = 0 is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and the solution is

$$\int_{y \text{ constant}} M\,dx + \int \left(\text{terms of N not containing } x\right)\, dy = c.$$

9. Integrating Factor: If a differential equation is not exact but can be made exact after multiplying by a suitable function of (x or y or both) then that function is called integrating factor (I.F.). If a differential equation has one I.F., it has an infinite number of integrating factors.

10. The I.F. of Mdx + Ndy = 0 are

(a) If Mdx + Ndy = 0 is homogeneous differential equation then I.F. = $\dfrac{1}{Mx + Ny}$ provided Mx + Ny ≠ 0.

If Mx + Ny = 0, then equation can be reduced to variable separable from by putting N = $-\dfrac{Mx}{y}$

(b) If Mdx + Ndy = 0 is of the form $y f_1(xy)\, dx + x f_2(xy)\, dy = 0$, then I.F. = $\dfrac{1}{Mx - Ny}$ provided Mx – Ny ≠ 0.

If Mx – Ny = 0, then it reduces to variable separable form.

(c) If Mdx + Ndy = 0; $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x say $f(x)$ then I.F. = $e^{\int f(x)\, dx}$ and if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is a function of y say $g(y)$, then I.F. = $e^{\int g(y)\, dy}$

(d) I.F. of the differential equation of the form $x^a y^b (mydx + nxdy) + x^c y^d (pydx + qxdy) = 0$ is $x^h y^k$, where h, k are so chosen that when multiplied with the given equation, changes the equation to exact equation.

11. If the differential equation is of first order and higher degree then to solve the equation replace $\dfrac{dy}{dx}$ by p

(a) If equation is solvable for p then find different values of p *i.e.*, $\dfrac{dy}{dx}$ and integrate each separately (all solutions having one arbitrary constant only) multiply all the factors formed by different solutions. That is the solution of the given differential equation.

(b) If equation is solvable for y: Express y as a function of x and p *i.e.*, $y = f(x, p)$ then differentiate w.r.t. x, equation will reduce to differential equation of first order in x and p. Solve and eliminate p with the help of given equation.

(c) If equation is solvable for x: Express x as a function of y and p *i.e.*, $x = f(y, p)$; then differentiate w.r.t. y, then equation will reduce to first order equation is y and p. Solve and eliminate p.

12. Clairaut's Equation: An equation of the form $y = px + f(p)$ is known as Clairaut's equation.

Its solution is $y = cx + f(c)$ *i.e.,* replace p by an arbitrary constant c.

13. Leibnitz's Linear Equation: A differential equation is said to the linear if the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

The general form of linear differential equation is $\dfrac{dy}{dx}$ + Py = Q, where P, Q are functions of x or constants.

Solution of linear differential equation is y (I.F.) = $\int$ Q (I.F.) $dx + c$, where I.F. = $e^{\int P\, dx}$

Similar method for $\frac{dy}{dx} + Px = Q$, where P, Q are functions of y or constant.

14. **Bernoulli's Form:** Any equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P, Q are functions of x only is called Bernoulli's equation. To solve it, divide by y^n and put $y^{1-n} = z$ j; it will reduce to linear equation in x and z whose solution is

$$z \text{ I.F.} = \int Q \text{ I.F. } dx + c, \text{ where I.F.} = e^{\int P\,dx}$$

Replace z by y^{1-n}

15. Differential equation $f'(y) \frac{dy}{dx} + Pf(y) = Q$ can be reduced to linear differential equation by putting $f(y) = z$.

SHORT ANSWER TYPE QUESTIONS

1. Distinguish between order and degree of a differential equation. **(P.T.U., Jan. 2010)**

   [**Hint:** See art. 1.1 (*iv*, *v*)]

2. Define complete solution of a differential equation.

   Or

   When a solution of a differential equation is called its general solution. **(P.T.U., Dec. 2005)**

   [**Hint:** See art 1.1 (*vi*)]

3. How will you form a differential equation whose solution contains n parameters ? What will be the order of that differential equation ?

4. Verify that $y = cx + \frac{a}{c}$ and $y^2 = 4ax$ both are solutions of the same differential equation;

$$y = x\frac{dy}{dx} + a\frac{dx}{dy}.$$

5. Define a singular solution of a differential equation.

   [**Hint:** Consult art. 1.1 (*vi*)]

6. Show that $y = x\,e^{2x}$ is a solution of $\frac{dy}{dx} = y\left(2 + \frac{1}{x}\right)$.

7. Obtain the differential equations from the following equations:

   (*i*) $y = Cx + C - C^2$

   (*ii*) $y = A\cos mx + B\sin mx$, where m is fixed ; A, B are parameters.

   (*iii*) $y = Ae^x + Be^{-x} + C$ **(P.T.U., May 2004)**

   (*iv*) $y = e^x (A\cos x + B\sin x)$ **(P.T.U., June 2003)**

   (*v*) $y = cx + c^2$ **(P.T.U., Dec. 2003)**

   [**Hint:** See S.E. 1 (*i*, *ii*, *iii*) art. 1.3]

8. Find the differential equation of all circles passing through the origin and having centres on x-axis.

   [**Hint:** See S.E. 2 art. 1.3]

9. Find the differential equation of all parabolas whose axes are parallel to y-axis. **(P.T.U., May 2002)**

   [**Hint:** See S.E. 4 art. 1.3]

10. Solve the following differential equations:

(*i*) $e^y\left(1+x^2\right)\dfrac{dy}{dx}-2x\left(1+e^y\right)=0$ (*ii*) $\dfrac{dy}{dx}+\sqrt{\dfrac{1-y^2}{1-x^2}}=0$

(*iii*) $xy\dfrac{dy}{dx}=1+x+y+xy$ [**Hint:** See S.E. 4 art. 1.4] **(P.T.U., Dec. 2003)**

(*iv*) $(1+x^3)\,dy-x^2y\,dx=0$

(*v*) $x\cos x\cos y+\sin y\dfrac{dy}{dx}=0$ [**Hint:** See S.E. 3 art. 1.4] **(P.T.U., May 2003)**

(*vi*) $\dfrac{dy}{dx}-x\tan\left(y-x\right)=1$ [**Hint:** Put $y-x=t$]

(*vii*) $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy=0$ [**Hint:** Separate variables and integrate]

(*viii*) $\dfrac{dy}{dx}=\dfrac{y}{x}$ **(P.T.U., Dec. 2005)** (*ix*) $(y+x)\,dy=(y-x)\,dx$ **(P.T.U., May 2011)**

11. Explain briefly how to solve the differential equation:

(*i*) $\dfrac{dy}{dx}=\dfrac{ax+by+c}{a_1x+b_1y+c_1}$, where $\dfrac{a}{a_1}\neq\dfrac{b}{b_1}$ **(P.T.U., May 2003)**

(*ii*) $\dfrac{dy}{dx}=\dfrac{ax+by+c}{a_1x+b_1y+c_1}$, where $\dfrac{a}{a_1}=\dfrac{b}{b_1}$

12. (*i*) What is an exact differential equation? Check the exactness of the equation $(3x^2+2e^y)\,dx+(2xe^y+3y^2)\,dy=0$.
(P.T.U., Jan. 2009, May 2010)

(*ii*) State necessary and sufficient conditions for the differential equation M dx + N dy = 0 to be exact.
(P.T.U., Jan. 2009, May 2014)

(*iii*) Under what conditions on a, b, c and d, the differential equations

$(a\sinh x\cos y+b\cosh x\sin y)\,dx+(c\sinh x\cos y+d\cosh x\sin y)\,dy=0$ is exact? **(P.T.U., May 2012)**

[**Hint:** See S.E. 5 art. 1.8]

13. Solve the following differential equations:

(*i*) $(x^2-ay)\,dx=(ax-y^2)\,dy$ [**Hint:** See S.E. 2 art. 1.8] **(P.T.U., May 2005)**

(*ii*) $(y\cos x+1)\,dx+\sin x\,dy=0$

(*iii*) $\dfrac{dy}{dx}+\dfrac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$ **(P.T.U., May 2011)**

14. (*a*) Define integrating factor of a differential equation and find I.F. of $(y-1)\,dx-x\,dy=0$. **(P.T.U., May 2006)**

[**Hint:** See S.E. 2 art. 1.9(*a*)]

(*b*) Solve $y(2xy+e^x)dx-e^x dy=0$ [**Hint:** See S.E. 1 art. 1.9(*a*)] **(P.T.U., May 2014)**

15. Find I.F. of the following differential equations:

(*i*) $y\,dx-x\,dy+3x^2y^2\,e^{x^3}\,dx=0$ **(P.T.U., Dec. 2003)**

(*ii*) $(x^2+y^2+x)\,dx+xy\,dy=0$. [**Hint:** $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\dfrac{1}{x}$; I.F. $=e^{\int\frac{1}{x}dx}=x$]

(*iii*) $(x^2y-2xy^2)dx-(x^3-3x^2y)\,dy=0$ [**Hint:** See S.E. 4 art. 1.9(*b*)] **(P.T.U., Dec. 2003)**

(*iv*) $y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0$ [**Hint:** See S.E. 5 art. 1.9(*c*)]

(*v*) $(xy^2-e^{1/x^2})dx-x^2y\,dy=0$ [**Hint:** See S.E. 6 art. 1.9(*d*)] (*vi*) $(3x^2y^3e^y + y^3 + y^2)dx + (x^3y^3e^y - xy)dy = 0$ [**Hint:** See S.E. 7 art. 1.9(*d*)]

(*vii*) $(5x^3 + 12x^2 + 6y^2)dx + 6xy\,dy = 0$ [**Hint:** See S.E. 8 art. 1.9(*d*)] **(P.T.U., Dec. 2013)**

16. (*a*) Define Clairaut's equation and write its solution. **(P.T.U., May 2007, 2012)**

(*b*) Find the general solution of the equation $y = xy' + y'^2$. What is the name of this type of equation? **(P.T.U., Dec. 2013)**

17. Solve the following differential equations:

(*i*) $y = 2px + p^2y$ [**Hint:** See S.E. 1 art. 1.13] **(P.T.U., Dec. 2012)**

(*ii*) $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$ [**Hint:** See S.E. 1 art. 1.11] **(P.T.U., May 2014)**

(*iii*) $p = \log(px - y)$ [**Hint:** See S.E. 1(*i*) art. 1.14] **(P.T.U., Dec. 2005, 2011)**

(*iv*) $\sin px \cos y = \cos px \sin y + p$. [**Hint:** See S.E. 1(*ii*) art. 1.14] **(P.T.U., May 2007)**

(*v*) $p = \sin(y - px)$. [**Hint:** Same as (*iv*) part] **(P.T.U., May 2011)**

(*vi*) $(y - px)(p - 1) = p$ [**Hint:** Clairaut's form]

18. (*i*) For the differential equation of the type $yf(xy)\,dx + xg(xy)\,dy = 0$, the I.F. is $\dfrac{1}{xy\left[f(xy) - g(xy)\right]}$. Justify it. **(P.T.U., Dec. 2004)**

(*ii*) For the differential equation M dx + N dy = 0 ; where M, N are homogeneous functions of x and y, the I.F. is $\dfrac{1}{Mx + Ny}$ $(Mx + Ny \neq 0)$. Justify it.

Also reduce $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)dy = 0$ to exact differential equation. **(P.T.U., Dec. 2009)**

[**Hint:** See S.E. 1 art. 1.9(*b*)]

(*iii*) For the differential equation Mdx + Ndy = 0 if $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$, then $e^{\int f(x)\,dx}$ is the I.F. justify it.

(*iv*) For the differential equation Mdx + Ndy = 0 if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$, then $e^{\int g(y)\,dy}$ is the I.F. justify it.

19. For what value of k, the differential equation $\left(1 + e^{\frac{kx}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)dy = 0$ is exact. **(P.T.U., May 2010)**

[**Hint:** See Solved Example 5 art. 1.8]

20. Define Leibnitz's linear differential equation of first order. Also give an example. **(P.T.U., May 2005, 2007)**
[**Hint:** See Art. 1.15]

21. Solve $\dfrac{dy}{dx} + Py = Q$, where P, Q are functions of x or constants. **(P.T.U., June 2003, Dec. 2006)**
[**Hint:** See Art. 1.16]

22. Define Bernoulli's linear differential equation and write its standard form. **(P.T.U., May 2007, Jan. 2009)**
[**Hint:** See Art. 1.17(*a*)]

23. How will you reduce $f'(y)\dfrac{dy}{dx} + P\,f(y) = Q$ to linear differential equation where, P, Q are function of x or constant?

[**Hint:** See Art. 1.17(*b*)]

24. Solve the following differential equations:

(*i*) $(x + 1)\dfrac{dy}{dx} - y = e^x(x + 1)^2$ (*ii*) $2\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2}$ **(P.T.U., May 2007)**

(*iii*) $x\dfrac{dy}{dx} + y = x^3y^6$ **(P.T.U., May 2011)**
[**Hint:** See S.E. 1(*i*) art. 1.17]

(*iv*) $(x+1)\dfrac{dy}{dx}+1=2e^{-y}$ [**Hint:** Divide by $(x+1)e^{-y}$; $e^y\dfrac{dy}{dx}+\dfrac{1}{x+1}e^y=\dfrac{2}{x+1}$; put $e^y=t$]

(*v*) $y'+y=y^2$ **(P.T.U., May 2008)**

[**Hint:** See S.E. 1(*ii*) art. 1.17]

(*vi*) $(1+y^2)\,dx=(\tan^{-1}y-x)dy$ **(P.T.U., Dec. 2011)**

[**Hint:** See S.E. 3 art. 1.16]

(*vii*) $\dfrac{dy}{dx}+y=xy^3$ **(P.T.U., May 2012)**

ANSWERS

3. By differentiating the equation n times and then eliminating n parameters from $n+1$ equations. (One is given equation and remaining n are the differential equations obtained by differentiating given equation n times)

7. (*i*) $y=(x+1)\dfrac{dy}{dx}-\left(\dfrac{dy}{dx}\right)^2$ (*ii*) $y_2+m^2y=0$ (*iii*) $y_3=y_1$

(*iv*) $y_2-2y_1+2y=0$ (*v*) $y=x\dfrac{dy}{dx}+\left(\dfrac{dy}{dx}\right)^2$

8. $x^2-y^2+2xy\dfrac{dy}{dx}=0$ **9.** $y_3=0$

10. (*i*) $(1+e^y)=A\,(1+x^2)$ (*ii*) $y\sqrt{1-x^2}+x\sqrt{1-y^2}=c$

(*iii*) $y=x+\log[x(1+y)]+c$ (*iv*) $y^3=4(x^3+1)$

(*v*) $x\sin x+\cos x=\log(\cos y)+c$ (*vi*) $\log\sin(y-x)=\dfrac{x^2}{2}+c$

(*vii*) $\tan x\tan y=c$ (*viii*) $y=cx$

(*ix*) $\log(x^2+y^2)=2\tan^{-1}\dfrac{y}{x}+c$

12. (*i*) Exact equation (*iii*) $a=-d,\ b=c$

13. (*i*) $x^3+y^3-3axy=c$ (*ii*) $y\sin x+x=c$ (*iii*) $y\sin x+(\sin y+y)\,x=c$

14. (*a*) $\dfrac{1}{x^2}$ (*b*) $\dfrac{1}{2}x^2+\dfrac{e^y}{y}=c$

15. (*i*) $\dfrac{1}{y^2}$ (*ii*) x (*iii*) $\dfrac{1}{x^2y^2}$

(*iv*) $\dfrac{1}{3x^3y^3}$ (*v*) $\dfrac{1}{x^4}$ (*vi*) $\dfrac{1}{y^3}$

(*vii*) x.

16. (*b*) $y=px+p^2$, Clairaut's equation

17. (*i*) $y^2=2cx+c^2$ (*ii*) $(y^2-x^2-2c)(xy-e^c)=0$ (*iii*) $y=cx-e^c$

(*iv*) $y=cx-\sin^{-1}c$. (*v*) $y=cx+\sin^{-1}c$. (*vi*) $y=cx+\dfrac{c}{c-1}$

18. $d\left(\dfrac{x}{y}\right)-d(2\log x)+d(3\log y)=0$ or $d\left(\dfrac{x}{y}-\log x^2+\log y^3\right)=0$

24. (*i*) $y=(x+1)(e^x+c)$ (*ii*) $\dfrac{x}{y}=1+c\sqrt{x}$ (*iii*) $\dfrac{1}{y^5}=\dfrac{5}{2}x^3+cx^5$

(*iv*) $(x+1)e^y=2x+c$ (*v*) $y=\dfrac{1}{1+ce^x}$ (*vi*) $x=\tan^{-1}y-1+ce^{-\tan^{-1}y}$

(*vii*) $\dfrac{1}{y^2}=x+\dfrac{1}{2}+ce^{2x}$

2

Linear Ordinary Differential Equations of Second and Higher Order

2.1. DEFINITIONS

A **linear differential equation** of nth order is that in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus, the general linear differential equation of the nth order is of the form $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_{n-1} \frac{dy}{dx} + P_n y = X$, where $P_1, P_2, \ldots\ldots, P_{n-1}, P_n$ and X are functions of x only.

A linear differential equation with constant coefficients is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = X \qquad \ldots(1)$$

where $a_1, a_2, \ldots\ldots, a_{n-1}, a_n$ are constants and X is either a constant or a function of x only.

First of all we discuss solution of linear differential equation with constant coefficients.

2.2. THE OPERATOR D

The part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ may be regarded as an operator such that when it operates on y, the result is the derivative of y.

Similarly, $\frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \cdots, \frac{d^n}{dx^n}$ may be regarded as operators.

For brevity, we write $\frac{d}{dx} \equiv D, \frac{d^2}{dx^2} \equiv D^2, \cdots, \frac{d^n}{dx^n} \equiv D^n$

Thus, the symbol D is a **differential operator or simply an operator.**

Written in symbolic form, equation (1) becomes $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_{n-1} D + a_n) y = X$

or $\qquad f(D) y = X$

where $\qquad f(D) = D^n + a_1 D^{n-1} + a_1 D^{n-2} + \ldots\ldots + a_{n-1} D + a_n$

i.e., $\quad f(D)$ is a polynomial in D.

The operator D can be treated as an algebraic quantity.

i.e.,

$$\begin{aligned} D(u+v) &= Du + Dv \\ D(\lambda u) &= \lambda Du \\ D^p D^q u &= D^{p+q} u \\ D^p D^q u &= D^q D^p u \end{aligned}$$

The polynomial $f(D)$ can be factorised by ordinary rules of algebra and the factors may be written in any order.

2.3. THEOREMS

Theorem 1. *If* $y = y_1$, $y = y_2$, ..., $y = y_n$ *are n linearly independent solutions of the differential equation*

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n)\, y = 0 \qquad \ldots(i)$$

then $u = c_1 y_1 + c_2 y_2 + \ldots\ldots\ldots + c_n y_n$ *is also its solution, where* $c_1, c_2, \ldots\ldots\ldots, c_n$ *are arbitrary constants.*

Proof. Since $y = y_1$, $y = y_2$, ..., $y = y_n$ are solution of equation (*i*).

$$\therefore \quad \left.\begin{array}{l} D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \cdots + a_n y_1 = 0 \\ D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_2 + \cdots + a_n y_2 = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + \cdots + a_n y_n = 0 \end{array}\right\} \qquad \ldots(ii)$$

Now, $D^n u + a_1 D^{n-1} u + a_2 D^{n-2} u + \ldots\ldots + a_n u$

$$\begin{aligned} &= D^n (c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n) + a_1 D^{n-1} (c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n) \\ &\quad + a_2 D^{n-2} (c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n) + \ldots \quad \ldots \quad \ldots \quad \ldots \quad + a_n (c_1 y_1 + c_2 y_2 + \ldots + c_n y_n) \\ &= c_1 (D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \ldots + a_n y_1) + c_2 (D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_n + \ldots + a_n y_2) \\ &\quad + \ldots \quad \ldots \quad \ldots \quad \ldots \quad + c_n (D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + \ldots + a_n y_n) \\ &= c_1 (0) + c_2 (0) + \ldots + c_n (0) \qquad [\because \text{ of } (ii)] \\ &= 0 \end{aligned}$$

which shows that $u = c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n$ is also the solution of equation (*i*).

Since this solution contains *n* arbitrary constants, it is the general or complete solution of equation (*i*).

Theorem 2. *If* $y = u$ *is the complete solution of the equation* $f(D)\, y = 0$ *and* $y = v$ *is a particular solution (containing no arbitrary constants) of the equation* $f(D)\, y = X$, *then the complete solution of the equation*

$$f(D)\, y = X \text{ is } y = u + v.$$

Proof. Since $y = u$ is the complete solution of the equation $f(D)\, y = 0$ ...(*i*)

$$\therefore \qquad f(D)\, u = 0 \qquad \ldots(ii)$$

Also, $y = v$ is a particular solution of the equation $f(D)\, y = X$ ...(*iii*)

$$\therefore \qquad f(D)\, v = X \qquad \ldots(iv)$$

Adding (*ii*) and (*iv*), we have $f(D)\,(u + v) = X$

Thus $y = u + v$ satisfies the equation (*iii*), hence it is the **complete solution (C.S.)** because it contains *n* arbitrary constants.

The part $y = u$ is called the **complementary function (C.F.)** and the part $y = v$ is called the **particular integral (P.I.)** of the equation (*iii*). **(P.T.U., Jan. 2010)**

$\therefore$ The complete solution of equation (*iii*), is $\mathbf{y = C.F. + P.I.}$

Thus in order to solve the equation (*iii*), *we first find the C.F. i.e., the C.S. of equation* (*i*) *and then the P.I. i.e., a particular solution of equation* (*iii*).

2.4. AUXILIARY EQUATION (A.E.)

Consider the differential equation $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots + a_n)\, y = 0$ ...(*i*)

Let $y = e^{mx}$ be a solution of (*i*), then $Dy = m\, e^{mx}$, $D^2 y = m^2 e^{mx}$, ..., $D^{n-2} y = m^{n-2} e^{mx}$

$D^{n-1} y = m^{n-1} e^{mx}$, $D^n y = m^n e^{mx}$

Substituting the values of y, Dy, D^2y, ... , D^ny in (i), we get

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_n) e^{mx} = 0$$

or $$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n = 0, \text{ since } e^{mx} \neq 0 \qquad \ldots(ii)$$

Thus $y = e^{mx}$ will be a solution of equation (i) if m satisfies equation (ii).

Equation (ii) is called the auxiliary equation for the differential equation (i).

Replacing m by D in (ii), we get $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n = 0 \qquad \ldots(iii)$

Equation (ii) gives the same values of m as equation (iii) gives of D. In practice, we take equation (iii) as the auxiliary equation which is obtained by equating to zero the symbolic co-efficient of y in equation (i).

Definition. The equation obtained by equating to zero the symbolic coefficient of y is called the **auxiliary equation,** briefly written as A.E.

2.5. RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n) y = 0 \qquad \ldots(i)$

where all the a_i's are constant.

Its auxiliary equation is $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n = 0 \qquad \ldots(ii)$

Let $D = m_1, m_2, m_3, \ldots\ldots, m_n$ be the roots of the A.E. The solution of equation (i) depends upon the nature of roots of the A.E. The following cases arise :

Case I. If all the roots of the A.E. are real and distinct, then equation (ii) is equivalent is

$$(D - m_1)(D - m_2) \ldots\ldots (D - m_n) = 0 \qquad \ldots(iii)$$

Equation (iii) will be satisfied by the solutions of the equations

$$(D - m_1) y = 0, \; (D - m_2) y = 0, \; \ldots\ldots, (D - m_n) y = 0$$

Now, consider the equation $(D - m_1) y = 0$, *i.e.*, $\dfrac{dy}{dx} - m_1 y = 0$

It is a linear equation and I.F. $= e^{\int -m_1 dx} = e^{-m_1 x}$

$\therefore$ Its solution is $y . e^{-m_1 x} = \int 0 . e^{-m_1 x} dx + c_1$ or $y = c_1 e^{m_1 x}$

Similarly, the solution of $(D - m_2) y = 0$ is $y = c_2 e^{m} 2^x$

....

The solution of $(D - m_n) y = 0$ is $y = c_n e^m n^x$

Hence the complete solution of equation (i) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^m n^x \qquad \ldots(iv)$$

Case II. If two roots of the A.E. are equal, let $m_1 = m_2$

The solution obtained in equation (iv) becomes

$$y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^m n^x = c e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^m n^x$$

It contains $(n - 1)$ arbitrary constants and is, therefore, not the complete solution of equation (i).

The part of the complete solution corresponding to the repeated root is the complete solution of

$$(D - m_1)(D - m_1) y = 0$$

Putting $(D - m_1) y = v$, it becomes $(D - m_1) v = 0$ *i.e.*, $\dfrac{dv}{dx} - m_1 v = 0$

As in case I, its solution is $v = c_1 e^{m_1 x}$

$\therefore \quad (D - m_1)\, y = c_1 e^{m_1 x} \quad$ or $\quad \dfrac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$

which is a linear equation and I.F. = $e^{-m_1 x}$

$\therefore$ Its solution is $y \,.\, e^{-m_1 x} = \int c_1 e^{m_1 x} \,.\, e^{-m_1 x}\, dx + c_2 = c_1 x + c_2$

or $\qquad y = (c_1 x + c_2)\, e^{m_1 x}$

Thus, the complete solution of equation (*i*) is

$$y = (c_1 x + c_2)\, e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

If, however, three roots of the A.E. are equal, say $m_1 = m_2 = m_3$, then proceeding as above, the solution becomes

$$y = \left(c_1 x^2 + c_2 x + c_3\right) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

Case III. If two roots of the A.E. are imaginary,

Let $\quad m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta \quad$ ($\because$ in a real equation imaginary roots occur in conjugate pair)

The solution obtained in equation (*iv*) becomes

$$\begin{aligned}
y &= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \\
&= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \\
&= e^{\alpha x}\left[c_1 (\cos\beta x + i \sin\beta x) + c_2 (\cos\beta x - i\sin\beta x)\right] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}
\end{aligned}$$

$$\left[\because \text{ By Euler's Theorem, } e^{i\theta} = \cos\theta + i\sin\theta\right]$$

$$\begin{aligned}
&= e^{\alpha x}\left[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x\right] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \\
&= e^{ax}\left(C_1 \cos\beta x + C_2 \sin\beta x\right) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}
\end{aligned}$$

[Taking $c_1 + c_2 = C_1$, $i\,(c_1 - c_2) = C_2$]

Case IV. If two pairs of imaginary roots be equal

Let $m_1 = m_2 = \alpha + i\beta$ and $m_3 = m_4 = \alpha - i\beta$

Then by case II, the complete solution is

$$y = e^{\alpha x}\left[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x\right] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve: $9y''' + 3y'' - 5y' + y = 0$.* **(P.T.U., May 2008)**

Sol. Symbolic form of given equation is

$$(9D^3 + 3D^2 - 5D + 1)\, y = 0$$

A.E. is $9D^3 + 3D^2 - 5D + 1 = 0$

or $\qquad (D + 1)(3D - 1)^2 = 0$

or $\qquad D = -1, \dfrac{1}{3}, \dfrac{1}{3}$

$\therefore \qquad$ C.S is $y = c_1 e^{-x} + (c_2 + c_3 x)\, e^{\frac{1}{3}x}$.

Example 2. *Solve :* $\dfrac{d^4 x}{dt^4} + 4x = 0$.

Sol. Given equation in symbolic form is $(D^4 + 4)\, x = 0$, where $D = \dfrac{d}{dt}$

Its A.E. is $\quad D^4+4=0 \quad$ or $\quad (D^4+4D^2+4)-4D^2=0$

or $\quad (D^2+2)^2-(2D)^2=0 \quad$ or $\quad (D^2+2D+2)(D^2-2D+2)=0$

whence
$$D = \frac{-2\pm\sqrt{-4}}{2} \text{ and } \frac{2\pm\sqrt{-4}}{2} \quad i.e., \ D = -1 \pm i \text{ and } 1 \pm i$$

Hence the C.S. is $\quad x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^t (c_3 \cos t + c_4 \sin t)$.

Example 3. *If* $\dfrac{d^2x}{dt^2} + \dfrac{g}{b}(x-a) = 0$; $(a > 0,\ b > 0,\ g > 0)$ *and* $x = \alpha$, $\dfrac{dx}{dt} = 0$ *when* $t = 0$, *show that* $x = a + (\alpha - a) \cos \sqrt{\dfrac{g}{b}}\, t$. **(P.T.U., May 2002)**

Sol. $$\frac{d^2x}{dt^2} + \frac{g}{b}(x-a) = 0$$

Put $x - a = y \quad \therefore \quad \dfrac{d^2x}{dt^2} = \dfrac{d^2y}{dt^2}$

$$\therefore \quad \frac{d^2y}{dt^2} + \frac{g}{b}y = 0 \quad \text{A.E. is } m^2 + \frac{g}{b} = 0 \quad \therefore \ m^2 = -\frac{g}{b} \ (\text{–ve})$$

$$\therefore \quad m = \pm i\sqrt{\frac{g}{b}}; \quad \therefore \ y = c_1 \cos\sqrt{\frac{g}{b}}\,t + c_2 \sin\sqrt{\frac{g}{b}}\,t$$

$$x - a = c_1 \cos\sqrt{\frac{g}{b}}\,t + c_2 \sin\sqrt{\frac{g}{b}}\,t$$

when $\quad x = \alpha,\ t = 0; \quad \alpha - a = c_1$

$$\therefore \quad x - a = (\alpha - a)\cos\sqrt{\frac{g}{b}}\,t + c_2 \sin\sqrt{\frac{g}{b}}\,t$$

$$\therefore \quad \frac{dx}{dt} = -(\alpha - a)\sqrt{\frac{g}{b}}\sin\sqrt{\frac{g}{b}}\,t + c_2\sqrt{\frac{g}{b}}\cos\sqrt{\frac{g}{b}}\,t$$

$$t = 0,\ \frac{dx}{dt} = 0 \quad \therefore \ 0 = c_2\sqrt{\frac{g}{b}} \quad \therefore \ c_2 = 0$$

$$\therefore \quad x - a = (\alpha - a)\cos\sqrt{\frac{g}{b}}\,t$$

Hence, $$x = a + (\alpha - a)\cos\sqrt{\frac{g}{b}}\,t.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = 0$

2. $\dfrac{d^2y}{dx^2} + (a+b)\dfrac{dy}{dx} + aby = 0$

3. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$

4. $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0$

5. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0$

6. $\dfrac{d^3y}{dx^3} + y = 0$ **(P.T.U., May 2012)**

7. $\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0$

8. $\frac{d^4y}{dx^4}-5\frac{d^2y}{dx^2}+4y=0$

9. $\frac{d^4y}{dx^4}+8\frac{d^2y}{dx^2}+16y=0$ **(P.T.U., Dec. 2010)**

10. $\left(D^2+1\right)^3\left(D^2+D+1\right)^2 y=0$

11. $\frac{d^2y}{dt^2}-3\frac{dx}{dt}+2x=0$, given that when $t=0,\ x=0$ and $\frac{dx}{dt}=0$

12. $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+29y=0$, given that when $x=0,\ y=0$ and $\frac{dy}{dx}=15$

13. If $\frac{d^4x}{dt^4}=m^4x$, show that $x=c_1\cos mt+c_2\sin mt+c_3\cosh mt+c_4\sinh mt$.

ANSWERS

1. $y=c_1e^{4x}+c_2e^{-x}$

2. $y=c_1e^{-ax}+c_2e^{-bx}$

3. $y=c_1e^{\left(2+\sqrt{3}\right)x}+c_2e^{\left(2-\sqrt{3}\right)x}$

4. $x=(c_1+c_2t)\,e^{-3t}$

5. $y=(c_1+c_2x+c_3x^2)\,e^x$

6. $y=c_1e^{-x}+e^{\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)$

7. $y=c_1e^{-x}+c_2e^{-2x}+c_3e^{-3x}$

8. $y=c_1e^x+c_2e^{-x}+c_3e^{2x}+c_4e^{-2x}$

9. $y=(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x$

10. $y=\left(c_1+c_2x+c_3x^2\right)\cos x+\left(c_4+c_5x+c_6x^2\right)\sin x+e^{-\frac{1}{2}x}\left[\left(c_7+c_8x\right)\cos\frac{\sqrt{3}}{2}x+\left(c_9+c_{10}x\right)\sin\frac{\sqrt{3}}{2}x\right]$

11. $x=0$

12. $y=3e^{-2x}\sin 5x$.

2.6. THE INVERSE OPERATOR $\frac{1}{f(D)}$

Definition: $\frac{1}{f(D)}X$ is that function of x, *free from arbitrary constants,* which when operated upon by $f(D)$ gives X.

Thus, $f(D)\left\{\frac{1}{f(D)}X\right\}=X$

$\therefore$ $f(D)$ and $\frac{1}{f(D)}$ are inverse operators.

Theorem 1. $\frac{1}{f(D)}X$ *is the particular integral of* $f(D)\,y=X$.

Proof. The given equation is $f(D)\,y=X$ ...(1)

Putting $y=\frac{1}{f(D)}X$ in (1), we have $f(D)\left\{\frac{1}{f(D)}X\right\}=X$ or $X=X$

which is true.

$\therefore$ $y=\frac{1}{f(D)}X$ is a solution of (1).

Since it contains no arbitrary constants, it is the particular integral of $f(D)\,y=X$.

Theorem 2. *Prove that :* $\frac{1}{D} X = \int X \, dx$. **(P.T.U., May 2002)**

Proof. Let $\frac{1}{D} X = y$

Operating both sides by D, we have $D\left(\frac{1}{D} X\right) = Dy$ or $X = \frac{dy}{dx}$

Integrating both sides w.r.t. x

$$y = \int X \, dx,$$

no arbitrary constant being added since $y = \frac{1}{D} X$ contains no arbitrary constant.

$$\therefore \qquad \frac{1}{D} X = \int X \, dx.$$

Theorem 3. *Prove that :* $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} \, dx$.

Proof. Let $\frac{1}{D-a} X = y$

Operating on both sides by $(D - a)$, $(D - a)\left(\frac{1}{D-a} X\right) = (D - a)\, y$

or $\qquad X = \frac{dy}{dx} - ay \quad i.e., \quad \frac{dy}{dx} - ay = X$

which is a linear equation and I.F. $= e^{\int -a\,dx} = e^{-ax}$

$\therefore$ Its solution is $ye^{-ax} = \int X e^{-ax} \, dx$, no constant being added

or $\qquad y = e^{ax} \int X e^{-ax} \, dx$

Hence, $\frac{1}{D-a} X = e^{ax} \int e^{-ax} X \, dx$.

2.7. RULES FOR FINDING THE PARTICULAR INTEGRAL **(P.T.U., Dec. 2004)**

Consider the differential equation, $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n)\, y = X$

It can be written as $f(D)\, y = X$, where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n$

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D)} X$$

Case I. When $\mathbf{X = e^{ax}}$

Since

$$D e^{ax} = a e^{ax}$$
$$D^2 e^{ax} = a^2 e^{ax}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D^{n-1} e^{ax} = a^{n-1} e^{ax}$$
$$D^n e^{ax} = a^n e^{ax}$$

$$\therefore \quad f(D)e^{ax} = (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n)\, e^{ax}$$
$$= (a^n + a_1 a^{n-1} + a_2 a^{n-2} + \cdots + a_{n-1} a + a_n)\, e^{ax}$$
$$= f(a)\, e^{ax}.$$

Operating on both sides by $\frac{1}{f(D)}$.

$$\frac{1}{f(\mathrm{D})}\left(f(\mathrm{D})\, e^{ax}\right) = \frac{1}{f(\mathrm{D})}\left(f(a)\, e^{ax}\right) \quad \text{or} \quad e^{ax} = f(a)\, \frac{1}{f(\mathrm{D})} e^{ax}$$

Dividing both sides by $f(a)$, $\dfrac{1}{f(a)}\, e^{ax} = \dfrac{1}{f(\mathrm{D})}\, e^{ax}$, provided $f(a) \neq 0$

Hence, $$\frac{1}{f(\mathrm{D})}\, e^{ax} = \frac{1}{f(a)}\, e^{ax}, \text{ provided } f(a) \neq 0.$$

Case of failure. If $f(a) = 0$, the above method fails.
Since $f(a) = 0$, $\mathrm{D} = a$ is a root of A.E. $f(\mathrm{D}) = 0$
$\therefore$ $\mathrm{D} - a$ is a factor of $f(\mathrm{D})$.
Let $f(\mathrm{D}) = (\mathrm{D} - a)\, \phi(\mathrm{D})$, where $\phi(a) \neq 0$ ...(*i*)

Then $$\frac{1}{f(\mathrm{D})}\, e^{ax} = \frac{1}{(\mathrm{D}-a)\,\phi(\mathrm{D})}\, e^{ax} = \frac{1}{\mathrm{D}-a} \cdot \frac{1}{\phi(\mathrm{D})}\, e^{ax} = \frac{1}{\mathrm{D}-a} \cdot \frac{1}{\phi(a)}\, e^{ax}$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{\mathrm{D}-a}\, e^{ax} = \frac{1}{\phi(a)}\, e^{ax} \int e^{ax} \cdot e^{-ax}\, dx \qquad \text{[By Theorem 3]}$$

$$= \frac{1}{\phi(a)} e^{ax} \int 1\, dx = x \cdot \frac{1}{\phi(a)}\, e^{ax} \qquad \ldots(ii)$$

Differentiating both sides of (*i*) w.r.t. D, we have $f'(\mathrm{D}) = (\mathrm{D} - a)\, \phi'(\mathrm{D}) + \phi(\mathrm{D})$
$\Rightarrow$ $f'(a) = \phi(a)$

$\therefore$ From (*ii*), we have $\dfrac{1}{f(\mathrm{D})}\, e^{ax} = x \cdot \dfrac{1}{f'(a)}\, e^{ax}$, provide $f'(a) \neq 0$

If $f'(a) = 0$, then $\dfrac{1}{f(\mathrm{D})}\, e^{ax} = x^2 \cdot \dfrac{1}{f''(a)} e^{ax}$, provided $f''(a) \neq 0$

and so on.

Example 1. *Find the P.I. of* $(D^3 - 3D^2 + 4)\, y = e^{2x}$.

Sol. $$\text{P.I.} = \frac{1}{\mathrm{D}^3 - 3\mathrm{D}^2 + 4}\, e^{2x}.$$

The denominator vanishes when D is replaced by 2. It is a case of failure.
We multiply the numerator by x and differentiate the denominator w.r.t. D.

$\therefore$ $$\text{P.I.} = x \cdot \frac{1}{3\mathrm{D}^2 - 6\mathrm{D}}\, e^{2x}$$

It is again a case of failure. We multiply the numerator by x and differentiate the denominator w.r.t. D.

$\therefore$ $$\text{P.I.} = x^2 \cdot \frac{1}{6\mathrm{D} - 6}\, e^{2x} = x^2 \cdot \frac{1}{6(2) - 6}\, e^{2x} = \frac{x^2}{6}\, e^{2x}.$$

Case II. When X = sin (*ax* + *b*) or cos (*ax* + *b*) **(P.T.U., Dec. 2005)**

$$\mathrm{D} \sin(ax + b) = a \cos(ax + b)$$
$$\mathrm{D}^2 \sin(ax + b) = (-a^2) \sin(ax + b)$$
$$\mathrm{D}^3 \sin(ax + b) = -a^3 \cos(ax + b)$$
$$\mathrm{D}^4 \sin(ax + b) = a^4 \sin(ax + b)$$

or $$(\mathrm{D}^2)^2 \sin(ax + b) = (-a^2)^2 \sin(ax + b)$$

....

....

In general, $(\mathrm{D}^2)^n \sin(ax + b) = (-a^2)^n \sin(ax + b)$

$\therefore \quad f(D^2) \sin(ax+b)$

$= [(D^2)^n + a_1 (D^2)^{n-1} + a_2 (D^2)^{n-2} + ... + a_{n-1} (D^2) + a_n] \sin(ax+b)$

$= [(-a^2)^n + a_1 (-a^2)^{n-1} + a_2 (-a^2)^{n-2} + ... + a_{n-1} (-a^2) + a_n] \sin(ax+b)$

$= f(-a^2) \sin(ax+b)$

Operating on both sides by $\dfrac{1}{f(D^2)}$,

$$\frac{1}{f(D^2)}\left[f(D^2)\sin(ax+b)\right] = \frac{1}{f(D^2)}\left[f(-a^2)\sin(ax+b)\right]$$

or $$\sin(ax+b) = f(-a^2)\frac{1}{f(D^2)}\sin(ax+b).$$

Dividing both sides by $f(-a^2)$,

$$\frac{1}{f(-a^2)}\sin(ax+b) = \frac{1}{f(D)^2}\sin(ax+b), \text{ provided } f(-a^2) \neq 0$$

Hence $$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b), \text{ provided } f(-a^2) \neq 0$$

Similarly, $$\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b), \text{ provided } f(-a^2) \neq 0$$

Case of failure. If $f(-a^2) = 0$, the above method fails.

Since $\cos(ax+b) + i\sin(ax+b) = e^{i(ax+b)}$ | Euler's Theorem

$$\therefore \quad \frac{1}{f(D^2)}\left[\cos(ax+b) + i\sin(ax+b)\right] = \frac{1}{f(D^2)} e^{i(ax+b)}$$

[If we replace D by ia, $f(D^2) = f(-a^2) = 0$, so that it is a case of failure]

$$= x \cdot \frac{1}{f'(D^2)} e^{i(ax+b)} = x \cdot \frac{1}{f'(D^2)}\left[\cos(ax+b) + i\sin(ax+b)\right]$$

Equating real parts $$\frac{1}{f(D^2)}\cos(ax+b) = x \cdot \frac{1}{f'(D^2)}\cos(ax+b), \text{ provided } f'(-a^2) \neq 0$$

Equating imaginary parts $$\frac{1}{f(D^2)}\sin(ax+b) = x \cdot \frac{1}{f'(D^2)}\sin(ax+b), \text{ provided } f'(-a^2) \neq 0$$

If $f'(-a^2) = 0$, then $$\frac{1}{f(D^2)}\sin(ax+b) = x^2 \cdot \frac{1}{f''(D^2)}\sin(ax+b), \text{ provided } f''(-a^2) \neq 0$$

$$\frac{1}{f(D^2)}\cos(ax+b) = x^2 \cdot \frac{1}{f''(D^2)}\cos(ax+b), \text{ provided } f''(-a^2) \neq 0$$

and so on.

Example 2. *Find the P.I. of* $y''' - y'' + 4y' - 4y = \sin 3x$. **(P.T.U., May 2008)**

Sol. Given equation in symbolic form is

$$(D^3 - D^2 + 4D - 4)\, y = \sin 3x$$

$$\text{P.I.} = \frac{1}{D^3 - D^2 + 4D - 4} \sin 3x$$

$$= \frac{1}{-9D + 9 + 4D - 4} \sin 3x \qquad [\because \text{ Put } D^2 = -9]$$

$$= \frac{1}{-5D + 5} \sin 3x = \frac{1}{5(1 - D)} \sin 3x$$

$$= \frac{1 + D}{5(1 - D^2)} \sin 3x = \frac{1 + D}{5(1 + 9)} \sin 3x \qquad [\because \text{ Putting } D^2 = -9]$$

$$= \frac{1}{50}\left[\sin 3x + D(\sin 3x)\right] = \frac{1}{50}\left[\sin 3x + 3\cos 3x\right]$$

Case III. When $X = x^m$, m being a positive integer

Here $\qquad \text{P.I.} = \dfrac{1}{f(D)} x^m$

Take out the lowest degree term from $f(D)$ to make the first term unity (so that Binomial Theorem for a negative index is applicable).

The remaining factor will be of the form $1 + \phi(D)$ or $1 - \phi(D)$

Take this factor in the numerator. It takes the form $[1 + \phi(D)]^{-1}$ or $[1 - \phi(D)]^{-1}$

Expand it in ascending powers of D as far as the term containing D^m, since $D^{m+1}(x^m) = 0$, $D^{m+2}(x^m) = 0$ and so on.

Operate on x^m term by term.

Example 3. *Find the P.I. of* $(D^2 + 5D + 4)\, y = x^2 + 7x + 9$.

Sol. $$\text{P.I.} = \frac{1}{D^2 + 5D + 4}(x^2 + 7x + 9) = \frac{1}{4\left(1 + \frac{5D}{4} + \frac{D^2}{4}\right)}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 + \left(\frac{5D}{4} + \frac{D^2}{4}\right)\right]^{-1}(x^2 + 7x + 9) = \frac{1}{4}\left[1 - \left(\frac{5D}{4} + \frac{D^2}{4}\right) + \left(\frac{5D}{4} + \frac{D^2}{4}\right)^2 - \cdots\right](x^2 + 7x + 9)$$

$$= \frac{1}{4}\left(1 - \frac{5D}{4} - \frac{D^2}{4} + \frac{25D^2}{16} \cdots\right)(x^2 + 7x + 9) = \frac{1}{4}\left(1 - \frac{5D}{4} + \frac{21D^2}{16} \cdots\right)(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[(x^2 + 7x + 9) - \frac{5}{4}D(x^2 + 7x + 9) + \frac{21}{16}D^2(x^2 + 7x + 9)\right]$$

$$= \frac{1}{4}\left[(x^2 + 7x + 9) - \frac{5}{4}(2x + 7) + \frac{21}{16}(2)\right] = \frac{1}{4}\left(x^2 + \frac{9}{2}x + \frac{23}{8}\right).$$

Case IV. When $X = e^{ax} V$, where V is a function of x

Let u be a function of x, then by successive differentiation, we have

$$D(e^{ax} u) = e^{ax} Du + a e^{ax} u = e^{ax}(D + a)\, u$$

$$D^2(e^{ax} u) = D[e^{ax}(D + a)\, u] = e^{ax}(D^2 + aD)\, u + ae^{ax}(D + a)\, u$$

$$= e^{ax}(D^2 + 2aD + a^2)\, u = e^{ax}(D + a)^2 u$$

Similarly, $\qquad D^3(e^{ax} u) = e^{ax}(D + a)^3 u$

In general, $\quad D^n (e^{ax} u) = e^{ax} (D + a)^n u$

$\therefore \quad f(D)(e^{ax} u) = e^{ax} f(D + a) u$

Operating on both sides by $\dfrac{1}{f(D)}$,

$$\frac{1}{f(D)}\left[f(D)\left(e^{ax}u\right)\right] = \frac{1}{f(D)}\left[e^{ax} f(D+a)u\right]$$

$$\Rightarrow \qquad e^{ax} u = \frac{1}{f(D)}\left[e^{ax} f(D+a)u\right]$$

Now, let $\quad f(D+a)\,u = V, \; i.e., \; u = \dfrac{1}{f(D+a)} V \qquad \ldots(i)$

$\therefore \quad$ From (i), we have $e^{ax} \dfrac{1}{f(D+a)} V = \dfrac{1}{f(D)}\left(e^{ax} V\right)$

or
$$\frac{1}{f(D)}\left(e^{ax} V\right) = e^{ax} \frac{1}{f(D+a)} V .$$

Thus, e^{ax} which is on the right of $\dfrac{1}{f(D)}$ may be taken out to the left provided D is replaced by D + a.

Example 4. *Find the P.I. of* $(D^2 - 4D + 3)\, y = e^x \cos 2x$.

Sol. P.I. $= \dfrac{1}{D^2 - 4D + 3} e^x \cos 2x = e^x \dfrac{1}{(D+1)^2 - 4(D+1) + 3} \cos 2x$

$= e^x \dfrac{1}{D^2 - 2D} \cos 2x = e^x \dfrac{1}{-2^2 - 2D} \cos 2x \qquad$ [Putting $D^2 = -2^2$]

$= -\dfrac{1}{2} e^x \dfrac{1}{2+D} \cos 2x = -\dfrac{1}{2} e^x \dfrac{2-D}{(2+D)(2-D)} \cos 2x$

$= -\dfrac{1}{2} e^x \dfrac{2-D}{4-D^2} \cos 2x = -\dfrac{1}{2} e^x \dfrac{2-D}{4-\left(-2^2\right)} \cos 2x$

$= -\dfrac{1}{16} e^x \left(2\cos 2x - D\cos 2x\right) = -\dfrac{1}{16} e^x \left(2\cos 2x + 2\sin 2x\right) = -\dfrac{1}{8} e^x \left(\cos 2x + \sin 2x\right).$

Case V. When X is any other function of x.

Resolve $f(D)$ into linear factors.

Let $f(D) = (D - m_1)(D - m_2) \ldots\ldots\ldots (D - m_n)\, X$

Then $\quad$ P.I. $= \dfrac{1}{f(D)} X = \dfrac{1}{(D - m_1)(D - m_2) \cdots (D - m_n)} X$

$= \left(\dfrac{A_1}{D - m_1} + \dfrac{A_2}{D - m_2} + \cdots + \dfrac{A_n}{D - m_2}\right) X \qquad$ (By Partial fractions)

$= A_1 \dfrac{1}{D - m_1} X + A_2 \dfrac{1}{D - m_2} X + \cdots + A_n \dfrac{1}{D - m_n} X$

$= A_1 e^{m_1 x} \displaystyle\int X e^{-m_1 x}\, dx + A_2 e^{m_2 x} \int X e^{-m_2 x}\, dx + \ldots + A_n e^{m_n x} \int X e^{-m_n x}\, dx$

See solved example 11 (art. 2.8) $\qquad \left[\because \dfrac{1}{D - m} X = e^{mx} \displaystyle\int X e^{-mx}\, dx\right].$

Example 5. *Find the particular solution of* $y'' + a^2y = \sec ax$. **(P.T.U., Dec. 2002, May 2012, Dec. 2013)**

Sol. Particular solution of $y'' + a^2y = \sec ax$

$$= \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{(D + ia)(D - ia)} \sec ax$$

$$= \left\{ \frac{\frac{1}{2ai}}{D - ia} - \frac{\frac{1}{2ai}}{D + ia} \right\} \sec ax \qquad \text{(By Partial fraction)}$$

$$= \frac{1}{2ai} \left\{ \frac{1}{D - ia} \sec ax - \frac{1}{D + ia} \sec ax \right\}$$

$$= \frac{1}{2ai} \left\{ e^{iax} \int e^{-iax} \sec ax \, dx - e^{-iax} \int e^{iax} \sec ax \, dx \right\}$$

$$= \frac{1}{2ai} \left\{ e^{iax} \int (\cos ax - i \sin ax) \sec ax \, dx - e^{-iax} \int (\cos ax + i \sin ax) \sec ax \, dx \right\}$$

$$= \frac{1}{2ai} \left\{ e^{iax} \int (1 - i \tan ax) \, dx - e^{-iax} \int (1 + i \tan ax) \, dx \right\}$$

$$= \frac{1}{2ai} \left\{ e^{iax} \left(x + i \frac{\log \cos ax}{a} \right) - e^{-iax} \left(x - i \frac{\log \cos ax}{a} \right) \right\}$$

$$= \frac{1}{2ai} \left\{ x (e^{iax} - e^{-iax}) + \frac{i}{a} \log \cos ax \left(e^{iax} + e^{-iax} \right) \right\}$$

$$= \frac{1}{2ai} \left\{ x \cdot 2i \sin ax + \frac{i}{a} \log \cos ax \cdot 2 \cos ax \right\}$$

$$= \frac{1}{a} \left\{ x \sin ax + \frac{\cos ax}{a} \log \cos ax \right\}$$

2.8. WORKING RULE TO SOLVE THE EQUATION

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} \cdots a_{n-1} \frac{dy}{dx} + a_n y = X$$

Step 1. Write the equation in symbolic form

$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots\ldots\ldots + a_{n-1} D + a_n) y = X$

Step 2. Solve the auxiliary equation

$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots\ldots\ldots + a_{n-1} D + a_n = 0$

Step 3. Write the complementary function with the help of following table.

Roots of the A.E.	**C.F.**
1. If roots are real and distinct say m_1, m_2, m_3	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots\ldots$
2. If two real roots are equal say $m_1 = m_2 = m$	C.F. $= (c_1 + c_2 x) e^{mx} + c_3 e^{m_3 x} + \ldots\ldots$
3. If three roots are equal $m_1 = m_2 = m_3 = m$	C.F. $= (c_1 + c_2 x + c_3 x^2) e^{mx} + c_4 e^{m_4 x} + \ldots\ldots$
4. If roots are a pair of imaginary (non-repeated) numbers (say) $\alpha \pm i\beta$.	C.F. $= e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$
5. If pair of imaginary roots is repeated twice, *i.e.*, $\alpha \pm i\beta$, $\alpha \pm i\beta$.	C.F. $= e^{\alpha x}\{(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x\}$

Step 4. Find the particular integral *i.e.*, P.I. $= \dfrac{1}{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n}$ X with the help of following rules.

Functions	**Particular Integrals**
1. When $X = e^{ax}$ then	P.I. $= \dfrac{e^{ax}}{f(D)}$ Put $D = a$ $= \dfrac{e^{ax}}{f(a)}$, provided $f(a) \neq 0$.

In case $f(a) = 0$ then multiply by x and differentiate the denominator w.r.t. D and continue this process untill denominator ceases to be zero on putting $D = a$.

2. When $X = \sin(ax + b)$ or $\cos(ax + b)$	P.I. $= \dfrac{1}{\phi(D^2)}$ $\sin(ax + b)$ or $\cos(ax + b)$ Put $D^2 = -a^2$ $= \dfrac{1}{\phi(-a^2)} \sin(ax + b)$ or $\cos(ax + b)$ provided $\phi(-a^2) \neq 0$

In case of failure apply to above mentioned rule of (1) case.

3. When $X = x^m$ then	P.I. $= [f(D)]^{-1} x^m$ expand $f(D)$ By binomial theorem up to D^m and then operate on x^m.
4. When $X = e^{ax} V$,	P.I. $= e^{ax} \dfrac{1}{f(D + a)} V$

5. If X is any other function of x, then P.I. $= \dfrac{1}{f(D)} X$. Resolve $\dfrac{1}{f(D)}$ into partial fractions and operate each partial fraction on X.

6. Remember $\dfrac{1}{D} X = \int X\, dx$ and $\dfrac{1}{D - a} X = e^{ax} \int X e^{-ax}\, dx$

Step 5. Then write the C.S. which is C.S. = C.F. + P.I.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $(D^2 + D + 1)y = (1 + \sin x)^2$. **(P.T.U., May 2007)**

Sol. $(D^2 + D + 1)\, y = (1 + \sin x)^2$

$$\text{A.E. is } D^2 + D + 1 = 0 \quad \therefore \quad D = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\text{C.F.} = e^{\frac{-x}{2}}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^2+D+1}(1+\sin x)^2 = \frac{1}{D^2+D+1}\left\{1 + 2\sin x + \sin^2 x\right\}$$

$$= \frac{1}{D^2+D+1}\left\{1 + 2\sin x + \frac{1-\cos 2x}{2}\right\} = \frac{1}{D^2+D+1}\left\{\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x\right\}$$

$$= \frac{3}{2}\cdot\frac{1}{D^2+D+1}\,.\,e^{0.x} + 2\frac{1}{D^2+D+1}\sin x - \frac{1}{2}\frac{1}{D^2+D+1}\cos 2x$$

$$\text{(Put } D = 0) \qquad \text{(Put } D^2 = -1) \qquad \text{(Put } D^2 = -4)$$

$$= \frac{3}{2}\cdot 1 + 2\cdot\frac{1}{D}\sin x - \frac{1}{2}\frac{1}{D-3}\cos 2x = \frac{3}{2} + 2(-\cos x) - \frac{1}{2}\frac{D+3}{D^2-9}\cos 2x$$

$$= \frac{3}{2} - 2\cos x - \frac{1}{2}\frac{D+3}{-13}\cos 2x = \frac{3}{2} - 2\cos x + \frac{1}{26}\left[-2\sin 2x + 3\cos 2x\right]$$

$$= \frac{3}{2} - 2\cos x - \frac{1}{13}\sin 2x + \frac{3}{26}\cos 2x$$

$$\text{C.S. is } y = e^{\frac{-x}{2}}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right] + \frac{3}{2} - 2\cos x - \frac{1}{13}\sin 2x + \frac{3}{26}\cos 2x.$$

Example 2. *Solve :* $(D-2)^2\, y = 8\,(e^{2x} + \sin 2x + x^2)$. **(P.T.U., May 2009)**

Sol. A.E. is $(D-2)^2 = 0 \quad \therefore \quad D = 2, 2$

C.F. $= (c_1 + c_2 x)\, e^{2x}$

$$\text{P.I.} = \frac{1}{(D-2)^2}\left[8\left(e^{2x} + \sin 2x + x^2\right)\right] = 8\left[\frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}\sin 2x + \frac{1}{(D-2)^2}x^2\right]$$

Now, $\dfrac{1}{(D-2)^2}e^{2x}$ Put D = 2; case of failure

$$= x\cdot\frac{1}{2(D-2)}e^{2x} \qquad \text{| Put D = 2. Case of failure}$$

$$= x^2\cdot\frac{1}{2}e^{2x} = \frac{x^2}{2}e^{2x}$$

$$\frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2-4D+4}\sin 2x = \frac{1}{-2^2-4D+4}\sin 2x \qquad [\because \text{ Putting } D^2 = -2^2]$$

$$= -\frac{1}{4D}\sin 2x = -\frac{1}{4}\int \sin 2x\,dx = -\frac{1}{4}\left(-\frac{\cos 2x}{2}\right) = \frac{1}{8}\cos 2x$$

$$\frac{1}{(D-2)^2}x^2 = \frac{1}{(2-D)^2}x^2 = \frac{1}{4\left(1-\frac{D}{2}\right)^2}x^2 = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}x^2$$

$$= \frac{1}{4}\left[1-2\left(-\frac{D}{2}\right)+\frac{(-2)(-3)}{2}\left(\frac{D}{2}\right)^2\cdots\right]x^2$$

$$= \frac{1}{4}\left[1+D+\frac{3}{4}D^2+\cdots\right]x^2 = \frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)$$

$\therefore$ $$\text{P.I.} = 8\left[\frac{x^2}{2}e^{2x}+\frac{1}{8}\cos 2x+\frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)\right] = 4x^2e^{2x}+\cos 2x+2x^2+4x+3$$

Hence the C.S. is $y = (c_1+c_2x)\,e^{2x}+4x^2e^{2x}+\cos 2x+2x^2+4x+3.$

Example 3. *Solve :* $(D+2)(D-1)^2y = e^{-2x}+2\sinh x.$

Sol. A.E. is $(D+2)(D-1)^2 = 0 \quad \therefore \quad D = -2, 1, 1$

$$\text{C.F.} = c_1e^{-2x}+(c_2+c_3x)\,e^x$$

$$\text{P.I.} = \frac{1}{(D+2)(D-1)^2}\left(e^{-2x}+2\sinh x\right)$$

$$= \frac{1}{(D+2)(D-1)^2}\left(e^{-2x}+e^x-e^{-x}\right) \qquad \left[\because \sinh x = \frac{e^x-e^{-x}}{2}\right]$$

Now, $$\frac{1}{(D+2)(D-1)^2}e^{-2x} = \frac{1}{D+2}\left[\frac{1}{(D-1)^2}e^{-2x}\right] = \frac{1}{D+2}\left[\frac{1}{(-2-1)^2}e^{-2x}\right]$$

$$= \frac{1}{9}\cdot\frac{1}{D+2}e^{-2x} \qquad \text{Put } D=-2 \text{ Case of failure}$$

$$= \frac{1}{9}x\cdot\frac{1}{1}e^{-2x} = \frac{x}{9}e^{-2x}$$

$$\frac{1}{(D+2)(D-1)^2}e^x = \frac{1}{(D-1)^2}\left[\frac{1}{D+2}e^x\right] = \frac{1}{(D-1)^2}\left[\frac{1}{1+2}e^x\right]$$

$$= \frac{1}{3}\cdot\frac{1}{(D-1)^2}e^x \qquad \text{Put } D=1 \text{ Case of failure}$$

$$= \frac{1}{3}\cdot x\frac{1}{2(D-1)}e^x \qquad \text{Put } D=1 \text{ Case of failure}$$

$$= \frac{1}{3}\cdot x^2\cdot\frac{1}{2}e^x = \frac{1}{6}x^2e^x$$

$$\frac{1}{(D+2)(D-1)^2}e^{-x} = \frac{1}{(-1+2)(-1-1)^2}e^{-x} = \frac{1}{4}e^{-x}$$

$\therefore$ $$\text{P.I.} = \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^x + \frac{1}{4}e^{-x}$$

Hence the C.S. is $$y = c_1 e^{-2x} + (c_2 + c_3 x)e^x + \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^x + \frac{1}{4}e^{-x}.$$

Example 4. *Solve :* $\dfrac{d^4 y}{dx^4} - y = \cos x \cosh x.$ **(P.T.U., May 2007, 2011)**

Sol. Given equation in symbolic form is $(D^4 - 1)y = \cos x \cosh x$

A.E. is $D^4 - 1 = 0$ or $(D^2 - 1)(D^2 + 1) = 0$ $\therefore$ $D = \pm 1, \pm i$

$$\text{C.F.} = c_1 e^x + c_2 e^{-x} + e^{0x}(c_3 \cos x + c_4 \sin x)$$
$$= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

$$\text{P.I.} = \frac{1}{D^4 - 1}\cos x \cosh x = \frac{1}{D^4 - 1}\cos x\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^4 - 1}e^x \cos x + \frac{1}{D^4 - 1}e^{-x}\cos x\right] = \frac{1}{2}\left[e^x \frac{1}{(D+1)^4 - 1}\cos x + e^{-x}\frac{1}{(D-1)^4 - 1}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^4 + 4D^3 + 6D^2 + 4D}\cos x + e^{-x}\frac{1}{D^4 - 4D^3 + 6D^2 - 4D}\cos x\right]$$

Put $D^2 = -1$

$$= \frac{1}{2}\left[e^x \frac{1}{(-1)^2 + 4D(-1) + 6(-1) + 4D}\cos x + e^{-x}\frac{1}{(-1)^2 - 4D(-1) + 6(-1) - 4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{-5}\cos x + e^{-x}\frac{1}{-5}\cos x\right] = -\frac{1}{5}\left(\frac{e^x + e^{-x}}{2}\right)\cos x = -\frac{1}{5}\cosh x \cos x$$

Hence the C.S. is $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \dfrac{1}{5}\cos x \cosh x$.

Example 5. *Solve:* $\dfrac{d^2 y}{dx^2} + 4y = x \sin 2x.$ **(P.T.U., Dec. 2002)**

Sol. S.F. of given equation is

$$(D^2 + 4)y = x \sin 2x$$
$$\text{A.E. is } D^2 + 4 = 0 \quad \therefore \quad D = \pm 2i$$

C.F. is $e^{0.x}(\cos 2x + i \sin 2x) = \cos 2x + i \sin 2x$

$$\text{P.I.} = \frac{1}{D^2 + 4}x \sin 2x = \text{Imaginary part of } \frac{1}{D^2 + 4}x e^{i2x}$$

$$= \text{Imaginary part of } e^{i2x}\frac{1}{(D + 2i)^2 + 4}x$$

$$= \text{Imaginary part of } e^{i2x}\frac{1}{D^2 + 4iD - 4 + 4}x$$

$$= \text{Imaginary part of } e^{i2x} \frac{1}{4i\mathrm{D}\left[1+\frac{\mathrm{D}^2}{4i\mathrm{D}}\right]} x$$

$$= \text{Imaginary part of } \frac{e^{i2x}}{4i\mathrm{D}}\left[1-\frac{i\mathrm{D}}{4}\right]^{-1} x$$

$$= \text{Imaginary part of } \frac{-i e^{i2x}}{4} \cdot \frac{1}{\mathrm{D}}\left[1+\frac{i\mathrm{D}}{4}\right] x$$

$$= \text{Imaginary part of } \frac{-i e^{i2x}}{4} \cdot \frac{1}{\mathrm{D}} \cdot \left[x+\frac{i}{4}\right]$$

$$= \text{Imaginary part of } \frac{-i\left(\cos 2x + i\sin 2x\right)}{4} \cdot \left(\frac{x^2}{2}+\frac{ix}{4}\right)$$

$$= -\frac{x^2}{8}\cos 2x + \frac{x}{16}\sin 2x$$

C.S. is $y = \text{C.F.} + \text{P.I.}$

$$= c_1 \cos 2x + c_2 \sin 2x - \frac{x^2}{8}\cos 2x + \frac{x}{16}\sin 2x.$$

Example 6. *Solve :* $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \sin x.$

(P.T.U., Dec. 2003, Jan. 2010, Dec. 2010, May 2011, Dec. 2012)

Sol. Given equation in symbolic form is $(\mathrm{D}^2 - 2\mathrm{D} + 1)\, y = x\, e^x \sin x$

A.E. is $\mathrm{D}^2 - 2\mathrm{D} + 1 = 0$ or $(\mathrm{D}-1)^2 = 0 \quad \therefore \quad \mathrm{D} = 1, 1$

C.F. $= (c_1 + c_2 x)\, e^x$

$$\text{P.I.} = \frac{1}{(\mathrm{D}-1)^2} e^x . x\sin x = e^x . \frac{1}{(\mathrm{D}+1-1)^2} x\sin x$$

$$= e^x \frac{1}{\mathrm{D}^2} x\sin x = e^x \frac{1}{\mathrm{D}} \int x\sin x\, dx$$

Integrating by parts $= e^x \dfrac{1}{\mathrm{D}}\left[x\,(\cos x) \int 1(\cos x)\, dx\right] = e^x \dfrac{1}{\mathrm{D}}\left(x\cos x \quad \sin x\right)$

$$= e^x \int (-x\cos x + \sin x)\, dx = e^x\left[-\left\{x\sin x - \int 1 .\sin x\, dx\right\} - \cos x\right]$$

$$= e^x\left[-x\sin x - \cos x - \cos x\right] = -e^x\left(x\sin x + 2\cos x\right)$$

Hence the C.S. is $\quad y = (c_1 + c_2 x)\, e^x - e^x\,(x\sin x + 2\cos x).$

Example 7. *Solve :* $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \sin x.$ **(P.T.U., May 2006, Dec. 2011)**

Sol. $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \sin x$

S.F. is $(\mathrm{D}^2 - 2\mathrm{D} + 1)\, y = e^x \sin x$

A.E. is $m^2 - 2m + 1 = 0$ *i.e.*, $m = 1, 1$.

C.F. is $(c_1 + c_2 x)\, e^x$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} e^x \sin x = \frac{1}{(D-1)^2} e^x \sin x$$

$$= e^x \frac{1}{(D+1-1)^2} \sin x = e^x \cdot \frac{1}{D^2} \sin x$$

Put $\qquad D^2 = -1$

$$= e^x \frac{\sin x}{-1} = -e^x \sin x$$

$\therefore \qquad$ C.S. is $y = \text{C.F.} + \text{P.I.}$

$$y = (c_1 + c_2 x) e^x - e^x \sin x = e^x [c_1 + c_2 x - \sin x].$$

Example 8. *Solve the differential equation* $(D^4 + D^2 + 1) y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2} x\right)$. **(P.T.U., May 2004)**

Sol. $(D^4 + D^2 + 1) y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2} x\right)$

A.E. is $m^4 + m^2 + 1 = 0$

$$m^2 = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$m^2 = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} \text{ and } \cos\frac{2\pi}{3} - i \sin\frac{2\pi}{3}$$

i.e., $\qquad m^2 = \text{cis}\,\frac{2\pi}{3}$ and $\text{cis}\left(-\frac{2\pi}{3}\right)$

when $\qquad m^2 = \text{cis}\,\frac{2\pi}{3}$

then $\qquad m = \left[\text{cis}\left(2k\pi + \frac{2\pi}{3}\right)\right]^{\frac{1}{2}}$

$$m = \text{cis}\,\frac{2k\pi + \frac{2\pi}{3}}{2}, \; k = 0, 1$$

$$m = \text{cis}\,\frac{2\pi}{6}, \; \text{cis}\,\frac{8\pi}{6} = \text{cis}\,\frac{\pi}{3}, \; \text{cis}\,\frac{4\pi}{3}$$

i.e., two values of m corresponding to $\text{cis}\,\frac{\pi}{3}, \text{cis}\,\frac{4\pi}{3}$ are $\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Other two values are obtained by changing i to $-i$.

$\therefore$ Four values of m are $\frac{1}{2} \pm i\frac{\sqrt{3}}{2},\ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$\text{C.F.} = e^{\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + e^{-\frac{1}{2}x}\left(c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^4 + D^2 + 1}\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-\frac{x}{2}} \frac{1}{\left(D - \frac{1}{2}\right)^4 + \left(D - \frac{1}{2}\right)^2 + 1} \cos\left(\frac{\sqrt{3}}{2}x\right) \qquad \text{[Using art. 2.7 Case IV]}$$

$$= e^{-\frac{x}{2}} \frac{1}{D^4 - 2D^3 + \frac{5}{2}D^2 - \frac{3}{2}D + \frac{21}{16}} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

Put $D^2 = -\frac{3}{4}$

$$= e^{-\frac{x}{2}} \frac{1}{\frac{9}{16} + \frac{3}{2}D - \frac{15}{8} - \frac{3}{2}D + \frac{21}{16}} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-\frac{x}{2}} \frac{1}{0} \cos\left(\frac{\sqrt{3}}{2}x\right) \qquad \textit{i.e.}, \text{ case of failure}$$

$$= e^{-\frac{x}{2}} \frac{x}{4D^3 - 6D^2 + 5D - \frac{3}{2}} \cos\left(\frac{\sqrt{3}}{2}\right)x$$

Put $D^2 = -\frac{3}{4}$

$$= e^{-\frac{x}{2}} \frac{x}{-3D + \frac{9}{2} + 5D - \frac{3}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= xe^{-\frac{x}{2}} \frac{1}{2D+3} \cos\left(\frac{\sqrt{3}}{2}x\right) = xe^{-\frac{x}{2}} \frac{2D-3}{4D^2-9} \cos\frac{\sqrt{3}}{2}x$$

$$= xe^{-\frac{x}{2}} \frac{(2D-3)}{-12} \cos\frac{\sqrt{3}}{2}x = \frac{-xe^{-\frac{x}{2}}}{12} \cdot \left\{-2.\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}x - 3\cos\frac{\sqrt{3}}{2}x\right\}$$

$$= \frac{x e^{-\frac{x}{2}}}{12}\left[\sqrt{3}\sin\frac{\sqrt{3}}{2}x + 3\cos\frac{\sqrt{3}}{2}x\right]$$

C.S. is $y = \text{C.F.} + \text{P.I.}$

$$y = e^{\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + e^{-\frac{x}{2}}\left(c_3\cos\frac{\sqrt{3}}{2}x + c_4\sin\frac{\sqrt{3}}{2}x\right)$$

$$+ \frac{x e^{-\frac{x}{2}}}{12}\cdot\left[\sqrt{3}\sin\frac{\sqrt{3}}{2}x + 3\cos\frac{\sqrt{3}}{2}x\right].$$

Example 9. *Solve : $(D^2 - 6D + 13)\, y = 8e^{3x}\sin 4x + 2^x$.* **(P.T.U., Dec. 2005)**

Sol. A.E. is $D^2 - 6D + 13 = 0$

$$D = \frac{6 \pm \sqrt{36 - 12}}{2} = 3 \pm 2i$$

$$\text{C.F.} = e^{3x}(c_1\cos 2x + c_2\sin 2x)$$

$$\text{P.I.} = \frac{1}{D^2 - 6D + 13}(8e^{3x}\sin 4x + 2^x)$$

$$= 8\,\frac{1}{D^2 - 6D + 13}\,e^{3x}\sin 4x + \frac{1}{D^2 - 6D + 13}\,2^x$$

$$= 8e^{3x}\,\frac{1}{(D+3)^2 - 6(D+3) + 13}\,\sin 4x + \frac{1}{D^2 - 6D + 13}\,e^{\log 2^x}$$

$$= 8e^{3x}\,\frac{1}{D^2 + 4}\,\sin 4x + \frac{1}{D^2 - 6D + 13}\,e^{x\log 2}$$

$$(\text{Put } D^2 = -16) \qquad (\text{Put } D = \log 2)$$

$$= 8e^{3x}\cdot\frac{1}{-16 + 4}\,\sin 4x + \frac{1}{(\log 2)^2 - 6\log 2 + 13}\,e^{x\log 2}$$

$$= \frac{8e^{3x}}{-12}\sin 4x + \frac{1}{(\log 2)^2 - 6\log 2 + 13}\,2^x$$

$$\text{C.S. is } y = e^{3x}(c_1\cos 2x + c_2\sin 2x) - \frac{2}{3}e^{3x}\sin 4x + \frac{1}{(\log 2)^2 - 6\log 2 + 13}\,2^x.$$

Example 10. *Solve : $(D^2 + 2D + 2)\, y = e^{-x}\sec x$.* **(P.T.U., Dec. 2002)**

Sol. A.E. is $D^2 + 2D + 2 = 0$

$$D = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\text{C.F.} = e^{-x}[c_1\cos x + c_2\sin x]$$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 2} \, e^{-x} \sec x = e^{-x} \frac{1}{(D-1)^2 + 2(D-1) + 2} \sec x$$

$$= e^{-x} \frac{1}{D^2 + 1} \sec x = e^{-x} \frac{1}{(D+i)(D-i)} \sec x$$

$$= e^{-x} \left[\frac{\frac{1}{2i}}{D - i} - \frac{\frac{1}{2i}}{D + i} \right] \sec x \qquad \text{[By Partial fractions]}$$

$$= \frac{e^{-x}}{2i} \left[\frac{1}{D-i} - \frac{1}{D+i} \right] \sec x$$

Now, $\frac{1}{D-i} \sec x = e^{ix} \int e^{-ix} \sec x \, dx = e^{ix} \int (\cos x - i \sin x) \frac{1}{\cos x} \, dx$

$$= e^{ix} \int (1 - i \tan x) \, dx = e^{ix} \left[x + i \log \cos x \right]$$

Similarly, $\frac{1}{D+i} \sec x = e^{-ix} [x - i \log \cos x]$

$\therefore$ $$\text{P.I.} = \frac{e^{-x}}{2i} \left[e^{ix} (x + i \log \cos x) - e^{-ix} (x - i \log \cos x) \right]$$

$$= \frac{e^{-x}}{2i} \left[x (e^{ix} - e^{-ix}) + i \log \cos x \, (e^{ix} + e^{-ix}) \right]$$

$$= \frac{e^{-x}}{2i} \left[x \cdot 2i \sin x + i \log \cos x \cdot 2 \cos x \right] = e^{-x} (x \sin x + \cos x \log \cos x)$$

C.S. is $y = e^{-x} [c_1 \cos x + c_2 \sin x + x \sin x + \cos x \log \cos x]$.

Example 11. *Solve :* $\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y = e^{e^x}$. **(P.T.U., Dec. 2003, 2012)**

Sol. $\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y = e^{e^x}$

S.F. is $(D^2 + 3D + 2) y = e^{e^x}$

A.E. is $m^2 + 3m + 2 = 0 \quad \therefore \quad m = -1, -2$

C.F. is $c_1 e^{-x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2 + 3D + 2} e^{e^x} = \frac{1}{(D+1)(D+2)} e^{e^x}$$

$$= \left[\frac{1}{D+1} - \frac{1}{D+2} \right] e^{e^x} \qquad \text{[By Partial fractions]}$$

$$= \frac{1}{D+1} e^{e^x} - \frac{1}{D+2} e^{e^x}$$

$$= e^{-x} \int e^x . e^{e^x} \, dx - e^{-2x} \int e^{2x} e^{e^x} \, dx \quad \because \frac{1}{D-a} X = e^{ax} \int e^{-ax} X \, dx$$

Put $e^x = t \quad \therefore \quad e^x \, dx = dt$

$$= e^{-x} \int e^t \, dt - e^{-2x} \int t \, e^t \, dt = e^{-x} e^t - e^{-2x} (t-1) e^t$$

$$= e^{-x} e^{e^x} - e^{-2x} (e^x - 1) \, e^{e^x}$$

$$= e^{e^x} [e^{-x} - e^{-x} + e^{-2x}] = e^{-2x} e^{e^x}.$$

Example 12. *Solve : $(D^2 - 4D + 4) y = 8x^2 e^{2x} \sin 2x$.* **(P.T.U., May 2002)**

Sol. $(D^2 - 4D + 4) y = 8x^2 e^{2x} \sin 2x$

A.E. is $m^2 - 4m + 4 = 0$ or $(m-2)^2 = 0 \quad \therefore \quad m = 2, 2$

$\therefore$ C.F. $= (c_1 + c_2 x) e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4} 8x^2 e^{2x} \sin 2x = \frac{1}{(D-2)^2} 8x^2 e^{2x} \sin 2x$$

$$= 8e^{2x} . \left(\frac{1}{(D+2-2)^2} \right) x^2 \sin 2x = 8e^{2x} . \frac{1}{D^2} x^2 \sin 2x$$

$$= 8e^{2x} \cdot \frac{1}{D} \int x^2 \sin 2x \, dx \text{ Integrate by parts}$$

$$= 8e^{2x} \cdot \frac{1}{D} \cdot \left\{ (x^2) \left(-\frac{\cos 2x}{2} \right) - (2x) \left(-\frac{\sin 2x}{4} \right) + 2 \left(\frac{\cos 2x}{8} \right) \right\}$$

$$= 8e^{2x} \cdot \frac{1}{D} \left\{ -\frac{x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} \right\}$$

$$= 4e^{2x} \int \left(-x^2 \cos 2x + x \sin 2x + \frac{\cos 2x}{2} \right) dx \text{ Integrate by parts}$$

$$= 4e^{2x} \left[-\left\{ x^2 \left(\frac{\sin 2x}{2} \right) - (2x) \left(-\frac{\cos 2x}{4} \right) + (2) \left(-\frac{\sin 2x}{8} \right) + x \left(-\frac{\cos 2x}{2} \right) - (1) \left(-\frac{\sin 2x}{4} \right) + \frac{\sin 2x}{4} \right\} \right]$$

$$= 2e^{2x} \left[-x^2 \sin 2x - x \cos 2x + \frac{\sin 2x}{2} - x \cos 2x + \frac{\sin 2x}{2} + \frac{\sin 2x}{2} \right]$$

$$= 2e^{2x} \left[-x^2 \sin 2x - 2x \cos 2x + \frac{3}{2} \sin 2x \right]$$

$$= -e^{2x} [(2x^2 - 3) \sin 2x + 4x \cos 2x]$$

C.S. is $y = (c_1 + c_2 x) e^{2x} - e^{2x} [(2x^2 - 3) \sin 2x + 4x \cos 2x]$.

Example 13. *Solve :* $(D^3 + 2D^2 + D)\,y = x^2 e^x + \sin^2 x$. **(P.T.U., June 2003)**

Sol. A.E. is $m^3 + 2m^2 + m = 0$

$$m(m^2 + 2m + 1) = 0 \quad \text{or} \quad m(m+1)^2 = 0 \quad \text{or} \quad m = 0, -1, -1$$

$$\text{C.F.} = c_1 e^{0x} + (c_2 + c_3 x)\, e^{-x} = c_1 + (c_2 + c_3 x)\, e^{-x}$$

$$\text{P.I.} = \frac{1}{D^3 + 2D^2 + D}\left(x^2 e^x + \sin^2 x\right) = \frac{1}{D^3 + 2D^2 + D}\left(x^2 e^x + \frac{1 - \cos 2x}{2}\right)$$

$$= \frac{1}{D^3 + 2D^2 + D}\, x^2 e^x + \frac{1}{D^3 + 2D^2 + D}\left(\frac{1}{2}\right) - \frac{1}{D^3 + 2D^2 + D}\left(\frac{1}{2}\cos 2x\right)$$

$$= e^x \frac{1}{(D+1)^3 + 2(D+1)^2 + (D+1)}\, x^2 + \frac{1}{2}\cdot\frac{1}{D^3 + 2D^2 + D}\, e^{0.x} - \frac{1}{2}\cdot\frac{1}{D^3 - 2D^2 + D}\cos 2x$$

(Put D = 0; Case of failure) (Put $D^2 = -4$)

$$= e^x \frac{1}{D^3 + 5D^2 + 8D + 4}\, x^2 + \frac{1}{2}\,\frac{x}{3D^2 + 4D + 1}\, e^{0.x} - \frac{1}{2}\cdot\frac{1}{-4D - 8 + D}\cos 2x$$

(Put D = 0)

$$= \frac{e^x}{4}\left[1 + \frac{8D + 5D^2 + D^3}{4}\right]^{-1} x^2 + \frac{x}{2} + \frac{1}{2}\cdot\frac{1}{3D + 8}\cos 2x$$

$$= \frac{e^x}{4}\left[1 - \frac{8D + 5D^2 + D^3}{4} + \left(\frac{8D + 5D^2 + D^3}{4}\right)^2\right] x^2 + \frac{x}{2} + \frac{3D - 8}{2\left(9D^2 - 64\right)}\cos 2x$$

$$= \frac{e^x}{4}\left[1 - \frac{8D}{4} - \frac{5D^2}{4} + 4D^2\right] x^2 + \frac{x}{2} + \frac{3D - 8}{2\left(-36 - 64\right)}\cos 2x$$

$$= \frac{e^x}{4}\left[x^2 - \frac{8}{4}(2x) + \frac{11}{4}(2)\right] + \frac{x}{2} - \frac{1}{200}\left[3(-2\sin 2x) - 8\cos 2x\right]$$

$$= \frac{e^x}{4}\left[x^2 - 4x + \frac{11}{2}\right] + \frac{x}{2} + \frac{3}{100}\sin 2x + \frac{\cos 2x}{25}$$

$$\text{C.S. } y = c_1 + (c_2 + c_3 x)\, e^{-x} + \frac{e^x}{4}\left[x^2 - 4x + \frac{11}{2}\right] + \frac{x}{2} + \frac{3}{100}\sin 2x + \frac{\cos 2x}{25}$$

which is the required solution.

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{d^3 y}{dx^3} + y = 3 + 5e^x$

2. $\dfrac{d^2 y}{dx^2} - 4y = \left(1 + e^x\right)^2$

3. $\dfrac{d^2 y}{dx^2} + 4\dfrac{dy}{dx} + 5y = -2\cosh x$

4. (*a*) $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + 5y = \sin 3x$

(*b*) $(D^2 + a^2)\, y = \sin ax$ **(P.T.U., May 2009)**

5. $\dfrac{d^3 y}{dx^3} + \dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} + y = \sin 2x$

6. $\dfrac{d^3 y}{dx^3} + y = \sin 3x - \cos^2 \dfrac{x}{2}$

7. $\left(D^2 - 4D + 3\right) y = \sin 3x \cos 2x$ **(P.T.U., Jan. 2009)**

8. $(D^2 - 3D + 2)\, y = 6e^{-3x} + \sin 2x$

9. $\dfrac{d^2y}{dx^2} + 4y = e^x + \sin 2x$

10. $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} = e^{2x} + \sin 2x$

11. $\dfrac{d^2y}{dx^2} - 4y = x^2$

12. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = 1 + x^2$

13. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 + 2x + 4$

14. $\dfrac{d^2y}{dx^2} + y = e^{2x} + \cosh 2x + x^3$

15. $\left(D^2 - 3D + 2\right) y = 2e^x \cos \dfrac{x}{2}$ **(P.T.U., Dec. 2003)**

$\left[\textbf{Hint: } \text{P.I.} = \dfrac{1}{D^2 - 3D + 2} 2e^x \cos\dfrac{x}{2} = 2e^x \dfrac{1}{(D+1)^2 - 3(D+1) + 2} = 2e^x \dfrac{1}{D^2 - D} \cos\dfrac{x}{2}\right]$

16. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

17. $\dfrac{d^4y}{dx^4} - y = e^x \cos x$

18. $(D^2 - 2D)\, y = e^x \sin x$

19. $(D^2 + 4D + 8)\, y = 12\, e^{-2x} \sin x \sin 3x$

20. $\dfrac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 2x$

21. $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x$

22. $(D - 1)^2 (D + 1)^2\, y = \sin^2 \dfrac{x}{2} + e^x + x$

23. $\dfrac{d^2y}{dx^2} - 4y = x \sinh x$ **(P.T.U., May 2012)**

24. $(D^2 - 1)\, y = x \sin x + (1 + x^2)\, e^x$

25. $\dfrac{d^2y}{dx^2} + 4y = 4 \tan 2x$

ANSWERS

1. $y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right) + 3 + \dfrac{5}{2}e^x$

2. $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{4} - \dfrac{2}{3}e^x + \dfrac{1}{4}xe^{2x}$

3. $y = e^{-2x}\left(c_1 \cos x + c_2 \sin x\right) - \dfrac{1}{10}e^x - \dfrac{1}{2}e^{-x}$

4. (*a*) $y = e^x\left(c_1 \cos 2x + c_2 \sin 2x\right) + \dfrac{1}{26}\left(3\cos 3x - 2\sin 3x\right)$

(*b*) $y = c_1 \cos ax + c_2 \sin ax - \dfrac{x \cos ax}{2a}$

5. $y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + \dfrac{1}{15}\left(2\cos 2x - \sin 2x\right)$

6. $y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right) + \dfrac{1}{730}\left(\sin 3x + 27\cos 3x\right) - \dfrac{1}{2} - \dfrac{1}{4}(\cos x - \sin x)$

7. $y = c_1 e^x + c_2 e^{3x} + \dfrac{1}{884}(10\cos 5x - 11\sin 5x) + \dfrac{1}{20}(\sin x + 2\cos x)$

8. $y = c_1 e^x + c_2 e^{2x} + \dfrac{3}{10}e^{-3x} + \dfrac{1}{20}(3\cos 2x - \sin 2x)$

9. $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{5}e^x - \dfrac{x}{4}\cos 2x$

10. $y = c_1 + e^x\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right) + \dfrac{1}{8}\left(e^{2x} + \sin 2x\right)$

11. $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right)$

12. $y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{18}\left(x^3 - \frac{x^2}{2} + \frac{25}{6}x\right)$

13. $y = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x$

14. $y = c_1 \cos x + c_2 \sin x + \frac{1}{5}e^{2x} + \frac{1}{5}\cosh 2x + x^3 - 6x$

15. $y = c_1 e^{x} + c_2 e^{2x} - \frac{8}{5}e^{x}\left(2\sin\frac{x}{2} + \cos\frac{x}{2}\right)$

16. $y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{4}e^{3x}(2x - 3) + \frac{1}{20}(3\cos 2x - \sin 2x)$

17. $y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}e^{x}\cos x$

18. $y = c_1 + c_2 e^{2x} - \frac{1}{2}e^{x}\sin x$

19. $y = e^{-2x}\left(c_1 \cos 2x + c_2 \sin 2x\right) + \frac{1}{2}e^{-2x}\left(3x\sin 2x + \cos 4x\right)$

20. $y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}\,x + \frac{e^{3x}}{11}\left(x^2 - \frac{12}{11}x + \frac{50}{121}\right) + \frac{e^{x}}{17}(4\sin 2x - \cos 2x)$

21. $y = c_1 \cos x + c_2 \sin x + \sin x \log \sin x - x\cos x$

22. $y = (c_1 + c_2 x)e^{x} + (c_3 + c_4 x)\,e^{-x} + \frac{1}{2} - \frac{1}{8}\cos x + \frac{x^2}{8}e^{x} + x$

23. $c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

24. $y = c_1 e^{x} + c_2 e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{1}{12}xe^{x}\left(2x^2 - 3x + 9\right)$

25. $y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$.

2.9. METHOD OF VARIATION OF PARAMETERS TO FIND P.I. (P.T.U., May 2004)

Consider the linear equation of **second order** with constant coefficients

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2\,y = \text{X} \qquad \ldots(1)$$

Let its C.F. be $y = c_1 y_1 + c_2 y_2$ so that y_1 and y_2 satisfy the equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2\,y = 0 \qquad \ldots(2)$$

Now, let us assume that the P.I. of (1) is $y = uy_1 + vy_2$, where u and v are unknown functions of x. ...(3)

Differentiating (3) w.r.t. x, we have $y' = uy_1' + v\,y_2' + u'\,y_1 + v'y_2 = uy_1' + vy_2'$ ...(4)

assuming that u, v satisfy the equation $u'\,y_1 + v'\,y_2 = 0$ ...(5)

Differentiating (4) w.r.t. x, we have $y'' = uy_1'' + u'\,y_1' + vy_2'' + v'\,y_2'$

Substituting the values of $y,\ y'$ and y'' in (1), we get

$$(uy_1'' + u'\,y_1' + vy_2'' + v'\,y_2') + a_1(uy_1' + vy_2') + a_2(uy_1 + vy_1) = \text{X}$$

or $$u(y_1'' + a_1 y_1' + a_2 y_1) + v(y_2'' + a_1 y_2' + a_2 y_2) + u'\,y_1' + v'\,y_2' = \text{X}$$

or $$u'\,y_1' + v'\,y_2' = \text{X} \quad \text{Since } y_1 \text{ and } y_2 \text{ satisfy (2).} \qquad \ldots(6)$$

Solving (5) and (6), we get $u' = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} \div \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\dfrac{y_2 X}{W}$

and $v' = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} \div \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \dfrac{y_1 X}{W}$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1, y_2.

Integrating $u = -\int \dfrac{y_2 X}{W}\, dx,\ v = \int \dfrac{y_1 X}{W}\, dx$

Substituting in (3), the P.I. is known.

Note 1. As the solution is obtained by varying the arbitrary constants c_1, c_2 of the C.F., the method is known as *variation of parameters*.

Note 2. Method of variation of parameters is to be used if instructed to do so.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the general solution of the equation y″ + 16y = 32 sec 2x ; using method of variation of parameters.* **(P.T.U., May 2008, 2010)**

Sol. Given equation in symbolic form is $(D^2 + 16)\, y = 32 \sec 2x$

A.E. is $D^2 + 16 = 0 \quad \therefore \quad D = \pm 4i$

C.F. is $y = c_1 \cos 4x + c_2 \sin 4x$

Here $y_1 = \cos 4x,\ y_2 = \sin 4x,\ X = 32 \sec 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 4x & \sin 4x \\ -4\sin 4x & 4\cos 4x \end{vmatrix} = 4$$

$$\text{P.I.} = uy_1 + vy_2 \text{ where } u = -\int \frac{y_2 X}{W}\, dx \text{ and } v = \int \frac{y_1 X}{W}$$

$$\therefore \quad \text{P.I.} = -\cos 4x \int \frac{\sin 4x \cdot 32 \sec 2x}{4}\, dx + \sin 4x \int \frac{\cos 4x \cdot 32 \sec 2x}{4}\, dx$$

$$= -8\cos 4x \int 2\sin 2x \cos 2x \cdot \frac{1}{\cos 2x}\, dx + 8 \sin 4x \int \frac{2\cos^2 2x - 1}{\cos 2x}\, dx$$

$$= -16 \cos 4x \int \sin 2x\, dx + 8 \sin 4x \int \left(2\cos 2x - \sec 2x\right) dx$$

$$= -16\cos 4x \left[-\frac{\cos 2x}{2}\right] + 8 \sin 4x \left[\frac{2\sin 2x}{2} - \frac{\log\left(\sec 2x + \tan 2x\right)}{2}\right]$$

$$= 8\cos 4x \cos 2x + 8 \sin 4x \sin 2x - 4 \sin 4x \log(\sec 2x + \tan 2x)$$

$$= 8\cos(4x - 2x) - 4\sin 4x \log(\sec 2x + \tan 2x)$$

$$= 8\cos 2x - 4 \sin 4x \log(\sec 2x + \tan 2x)$$

$\therefore$ C.S. is $y = \text{C.F.} + \text{P.I.}$

$$= c_1 \cos 4x + c_2 \sin 4x + 8\cos 2x - 4\sin 4x \log(\sec 2x + \tan 2x).$$

Example 2. *Solve:* $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$ *by variation of parameter method.*

(P.T.U., May 2010, 2012, Dec. 2012, 2013)

Sol. Equation in the symbolic form is

$$(D^2 - 6D + 9)\, y = \frac{e^{3x}}{x^2}$$

A.E. is $D^2 - 6D + 9 = 0$ *i.e.,* $(D-3)^2 = 0$ *i.e.,* $D = 3, 3$

C.F. $= (c_1 + c_2 x)\, e^{3x} = c_1 e^{3x} + c_2 x e^{3x} = c_1 y_1 + c_2 y_2$, where $y_1 = e^{3x}$, $y_2 = x e^{3x}$

and $$X = \frac{e^{3x}}{x^2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & (1+3x)\, e^{3x} \end{vmatrix} = e^{6x}$$

P.I. $= uy_1 + vy_2$, where $u = -\int \frac{y_2 X}{W}\, dx$ and $v = \int \frac{y_1 X}{W}\, dx$

$$\therefore \qquad \text{P.I.} = -e^{3x} \int \frac{x e^{3x} \cdot e^{3x}}{e^{6x} \cdot x^2}\, dx + x e^{3x} \int \frac{e^{3x} \cdot e^{3x}}{e^{6x} \cdot x^2}\, dx = -e^{3x} \int \frac{1}{x}\, dx + x e^{3x} \int \frac{1}{x^2}\, dx$$

$$= -e^{3x} \log x + x e^{3x} \left(-\frac{1}{x} \right) = -e^{3x} (1 + \log x)$$

C.S. is $y = (c_1 + c_2 x)_2\, e^{3x} - e^{3x} (1 + \log x)$

$= e^{3x} [c_1 + c_2 x - 1 - \log x] = e^{3x} [(c_1 - 1) + c_2 x - \log x]$

$= e^{3x} \left[c_1' + c_2 x - \log x\right]$, where $c_1' = c_1 - 1$ is the required solution.

Example 3. *Solve by method of variation of parameters the differential equation*

$$\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y = \sin(e^x). \qquad \textbf{(P.T.U., May 2012)}$$

Sol. Equation in symbolic form is

$$(D^2 + 3D + 2)y = \sin e^x$$

A.E. is $D^2 + 3D + 2 = 0$

$$(D+1)(D+2) = 0$$

$$D = -1, D = -2$$

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{-2x}$$

$$= c_1 y_1 + c_2 y_2$$

where $$y_1 = e^{-x},\ y_2 = e^{-2x},\ X = \sin e^x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$= -2e^{-3x} + e^{-3x} = -e^{-3x}$$

$$\text{P.I.} = uy_1 + vy_2, \text{ where } u = -\int \frac{y_2 X}{W}\, dx \text{ and } v = \int \frac{y_1 X}{W}\, dx$$

$$\therefore \qquad \text{P.I.} = -e^{-x} \int \frac{e^{-2x} \sin e^x}{-e^{-3x}}\, dx + e^{-2x} \int \frac{e^{-x} \sin e^x}{-e^{-3x}}\, dx$$

$$= e^{-x} \int e^x \sin e^x\, dx - e^{-2x} \int e^{2x} \sin e^x\, dx$$

Put $e^x = t \quad \therefore\ e^x\, dx = dt$

$$= e^{-x} \int \sin t\, dt - e^{-2x} \int t \sin t\, dt$$

$$= e^{-x}\{-\cos t\} - e^{-2x}\{t(-\cos t) - (1)(\sin t)\}$$

$$= -e^{-x}\cos e^x + e^{-2x}\{e^x \cos e^x + \sin e^x\}$$

$$= -e^{-x}\cos e^x + e^{-x}\cos e^x + e^{-2x}\sin e^x$$

$$= e^{-2x}\sin e^x$$

$$\text{C.S. is } y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x}\sin e^x$$

Example 4. *Solve the differential equation $(D^2 + 1)y = \text{cosec } x \cot x$.* **(P.T.U., May 2011)**

Sol. Given differential equation is

$$(D^2 + 1)y = \text{cosec } x \cot x$$

$$\text{A.E. is } D^2 + 1 = 0 \quad \therefore \quad D = \pm i$$

$$\text{C.F.} = c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2$$

where $y_1 = \cos x, y_2 = \sin x$

and $X = \text{cosec } x \cot x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$\text{P.I.} = uy_1 + vy_2$$

where
$$u = -\int \frac{y_2 X}{W}\, dx \text{ and } v = \int \frac{y_1 X}{W}\, dx$$

$\therefore$
$$u = -\int \frac{\sin x \,.\, \text{cosec } x \cot x}{1}\, dx = -\int \cot x\, dx = -\log|\sin x|$$

$$v = \int \frac{\cos x \,.\, \text{cosec } x \cot x}{1}\, dx = \int \cot^2 x\, dx = \int (\text{cosec}^2 x - 1)\, dx$$

$$= -\cot x - x = -(\cot x + x)$$

$\therefore$
$$\text{P.I.} = \{\log|\sin x|\}\cos x - (\cot x + x)\sin x$$

$$\text{C.S. is } y = C_1 \cos x + C_2 \sin x + \{\log|\sin x|\}\cos x - \cot x \sin x - x\sin x$$

$$= C_1 \cos x + C_2 \sin x + \{\log|\sin x|\}\cos x - \cos x - x \sin x$$

$$= (C_1 - 1)\cos x + C_2 \sin x + \{\log|\sin x|\}\cos x - x\sin x$$

$$= C_1' \cos x + C_2 \sin x + \{\log|\sin x|\}\cos x - x \sin x$$

where $C_1' = C_1 - 1$

TEST YOUR KNOWLEDGE

Solve by the method of variation of parameters :

1. $\dfrac{d^2y}{dx^2} + y = \text{cosec } x$

2. $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$ **(P.T.U., Dec. 2010)**

3. $\dfrac{d^2y}{dx^2} + 4y = \sec 2x$ **(P.T.U., May 2004)**

4. $\dfrac{d^2y}{dx^2} + 4y = 4\sec^2 2x$ **(P.T.U., Jan. 2009)**

[**Hint:** Consult S.E. 1]

5. $\dfrac{d^2y}{dx^2} + y = x\sin x$

6. $y'' - 2y' + 2y = e^x \tan x$.

7. $\dfrac{d^2y}{dx^2} + y = \sec x$ **(P.T.U., Dec. 2003, 2005)**

ANSWERS

1. $y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log \sin x$
2. $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log(\sec 2x + \tan 2x)$
3. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} \cos 2x \log \cos 2x + \frac{x}{2} \sin 2x$
4. $y = c_1 \cos 2x + c_2 \sin 2x - 1 + \sin 2x \log(\sec 2x + \tan 2x)$
5. $y = c_1 \cos x + c_2 \sin x + \frac{x}{2} \sin x - \frac{x^2}{4} \cos x$
6. $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$
7. $y = c_1 \cos x + c_2 \sin x + \cos x \log \cos x + x \sin x$

SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE CO-EFFICIENTS

2.10. OPERATOR METHOD

Consider a linear equation of second order with variable coefficients

$$\mathrm{P}\frac{d^2 y}{dx^2} + \mathrm{Q}\frac{dy}{dx} + \mathrm{R}y = \mathrm{S}, \text{ where P, Q, R, S are functions of } x \qquad \ldots(1)$$

writing D for $\frac{d}{dx}$, (1) becomes

$$\mathrm{PD}^2 y + \mathrm{QD}y + \mathrm{R}y = \mathrm{S}$$

or

$$(\mathrm{PD}^2 + \mathrm{QD} + \mathrm{R})y = \mathrm{S} \qquad \ldots(2)$$

Sometimes it will be possible to factorise the left hand side into two linear operators acting on y. In such a case the equation is integrated in two stages we illustrate the method by the following examples.

Important Remarks. *Note that the factors are non-commutative as these involve functions of x directly. Hence care should be taken while using the factorised operators in the correct order*

e.g.,

$$(\mathrm{D}-1)(x\mathrm{D}+1) \neq (x\mathrm{D}+1)(\mathrm{D}-1)$$

$$\begin{aligned}(\mathrm{D}-1)(x\mathrm{D}+1) &= \mathrm{D}(x\mathrm{D}+1) - x\mathrm{D} - 1\\ &= \mathrm{D}(x\mathrm{D}) + \mathrm{D} - x\mathrm{D} - 1\\ &= x\mathrm{D}^2 + 1\mathrm{D} + \mathrm{D} - x\mathrm{D} - 1\\ &= x\mathrm{D}^2 + (2-x)\mathrm{D} - 1\end{aligned}$$

But

$$\begin{aligned}(x\mathrm{D}+1)(\mathrm{D}-1) &= x\mathrm{D}(\mathrm{D}-1) + \mathrm{D} - 1\\ &= x\mathrm{D}^2 - x\mathrm{D} + \mathrm{D} - 1\\ &= x\mathrm{D}^2 + (1-x)\mathrm{D} - 1\\ &\neq (\mathrm{D}-1)(x\mathrm{D}+1)\end{aligned}$$

Example 1. *Solve:* $x\frac{d^2 y}{dx^2} + (x-2)\frac{dy}{dx} - 2y = x^3.$

Sol. In symbolic form equation is

$$[x\mathrm{D}^2 + (x-2)\mathrm{D} - 2]\,y = x^3 \qquad \ldots(1)$$

Now, $\quad xD^2 + (x-2)\,D - 2 = xD^2 + xD - 2D - 2$

$$= xD\,(D+1) - 2\,(D+1)$$

$$= (xD-2)\,(D+1)$$

$\therefore$ (1) becomes $\quad (xD-2)\,(D+1)\,y = x^3 \qquad \ldots(2)$

Let $\quad (D+1)\,y = v \qquad \ldots(3)$

$\therefore$ (2) becomes $\quad (xD-2)\,v = x^3$

or $$x\frac{dv}{dx} - 2v = x^3$$

or $$\frac{dv}{dx} - \frac{2}{x}v = x^2 \qquad \ldots(4)$$

which is a linear equation in x and v

Its $$\text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{2\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

$\therefore$ Solution of (4) is $$v \cdot \frac{1}{x^2} = \int \frac{1}{x^2}\,x^2\,dx + C_1$$

$$= x + C_1$$

$\therefore$ $$v = x^3 + C_1x^2$$

Substituting the value of v in (3)

$$(D+1)\,y = x^3 + C_1x^2$$

$$\frac{dy}{dx} + y = x^3 + C_1x^2;$$ again a linear differential equation

$$\text{I.F.} = e^{\int 1\,dx} = e^x$$

$\therefore$ Solution in $y\,e^x = \int e^x\,(x^3 + C_1\,x^2)\,dx + C_2$

Integrate by parts (Chain rule)

$\therefore$ $$y\,e^x = (x^3 + C_1x^2)\,(e^x) - (3x^2 + 2C_1x)\,(e^x) + (6x + 2C_1)\,e^x - (6)\,e^x + C_2$$

$\therefore$ $$y = x^3 + C_1x^2 - 3x^2 - 2C_1x + 6x + 2C_1 - 6 + C_2\,e^{-x}$$

$$= x^3 + (C_1-3)\,x^2 - 2\,(C_1-3)\,x + 2\,(C_1-3) + C_2\,e^{-x}$$

or $$y = x^3 + C'\,x^2 - 2C'\,x + 2C' + C_2\,e^{-x}, \text{ where } C_1 - 3 = C'$$

Example 2. *Factorise the operator on the LHS of* $[(x+2)\,D^2 - (2x+5)\,D + 2]\,y = (x+1)\,e^x$ *and hence solve.*

Sol. $(x+2)\,D^2 - (2x+5)\,D + 2 \quad = (x+2)\,D^2 - (2x+4+1)\,D + 2$

$$= (x+2)\,D^2 - 2\,(x+2)\,D - (D-2)$$

$$= (x+2)\,D\,(D-2) - (D-2)$$

$$= [(x+2)\,D - 1]\,[D-2]$$

$\therefore$ Given equation is

$$[(x+2)\,D - 1]\,[D-2]\,y = (x+1)\,e^x \qquad \ldots(1)$$

Let $\quad (D-2)\,y = v \qquad \ldots(2)$

$\therefore$ (1) becomes $[(x+2)\,D - 1]\,v = (x+1)\,e^x \qquad \ldots(3)$

or $$(x+2)\frac{dv}{dx} - v = (x+1)\,e^x$$

or $$\frac{dv}{dx} - \frac{1}{x+2} v = \frac{x+1}{x+2} e^x \qquad \ldots(4)$$

which is a linear differential equation

$$\text{I.F.} = e^{\int -\frac{1}{x+2}\, dx} = e^{-\log(x+2)} = e^{\log(x+2)^{-1}} = (x+2)^{-1} = \frac{1}{x+2}$$

$\therefore$ Solution of (4) is

$$v \cdot \frac{1}{x+2} = \int \frac{x+1}{x+2} e^x \cdot \frac{1}{x+2}\, dx + C_1 = \int \frac{x+1}{(x+2)^2} e^x\, dx + C_1$$

$$= \int \frac{x+2-1}{(x+2)^2} e^x\, dx + C_1 = \int \left[\frac{1}{x+2} - \frac{1}{(x+2)^2}\right] e^x\, dx + C_1$$

$$= \frac{1}{x+2} e^x + C_1 \qquad \left[\text{Using } \int [f(x) + f'(x)]\, e^x\, dx = f(x)\, e^x\right]$$

i.e., $$v = e^x + C_1 (x+2)$$

From (2), $$(D-2)\, y = e^x + C_1 (x+2)$$

or $$\frac{dy}{dx} - 2y = e^x + C_1 (x+2), \text{ which is again linear in } x \text{ and } y$$

$$\text{I.F.} = e^{-2x}$$

Solution is $$y\, e^{-2x} = \int e^{-2x} (e^x + C_1 x + 2C_1)\, dx + C_2$$

$$= \int \left(e^{-x} + C_1 x e^{-2x} + 2C_1 e^{-2x}\right) dx + C_2$$

$$= \frac{e^{-x}}{-1} + C_1 \left[x \frac{e^{-2x}}{-2} - (1)\left(\frac{e^{-2x}}{(-2)^2}\right)\right] + 2C_1 \frac{e^{-2x}}{-2} + C_2$$

$$= -e^{-x} - \frac{1}{2} C_1 x e^{-2x} - \frac{C_1}{4} e^{-2x} - C_1 e^{-2x} + C_2$$

$\therefore$ $$y = -e^x + \left(-\frac{C_1}{2} x - \frac{5}{4} C_1\right) + C_2 e^{2x}$$

$$= -e^x - \frac{C_1}{4}(2x+5) + C_2 e^{2x}$$

Example 3. *Solve:* $3x^2 y'' + (2 - 6x^2)\, y' - 4y = 0$.

Sol. Symbolic form of the given equation is

$$[3x^2 D^2 + (2 - 6x^2) D - 4]\, y = 0 \qquad \ldots(1)$$

Now, $$3x^2 D^2 + (2 - 6x^2) D - 4 = (3x^2 D^2 - 6x^2 D) + (2D - 4)$$

$$= 3x^2 D (D-2) + 2 (D-2)$$

$$= (3x^2 D + 2)(D-2)$$

$\therefore$ (1) is $$(3x^2 D + 2)(D-2)\, y = 0 \qquad \ldots(2)$$

Let $$(D-2)\, y = v \qquad \ldots(3)$$

$\therefore$ From (2) $$(3x^2 D + 2)\, v = 0$$

or $$3x^2 \frac{dv}{dx} + 2v = 0$$

Separate the variables

$$\frac{1}{v}\,dv = -\frac{2}{3x^2}\,dx$$

Integrate both sides,

$$\log v = -\frac{2}{3}\,\frac{x^{-1}}{-1} + C_1$$

or $$\log v = \frac{2}{3x} + C_1$$

or $$v = e^{\frac{2}{3x}}\,e^{C_1} = C_1'\,e^{\frac{2}{3x}}, \text{ where } C_1' = e^{C_1}$$

Substitute the value of v in (3)

$$(D-2)\,y = C_1\,e^{\frac{2}{3x}}$$

or $$\frac{dy}{dx} - 2y = C_1\,e^{\frac{2}{3x}}, \text{ which is a linear differential equation}$$

$\therefore$ $$\text{I.F.} = e^{\int -2\,dx} = e^{-2x}$$

Solution is $$y\,e^{-2x} = \int e^{-2x}.C_1\,e^{\frac{2}{3x}}\,dx + C_2$$

or $$y\,e^{-2x} = C_1\int e^{-2x}.\,e^{\frac{2}{3x}}\,dx + C_2$$

or $$y = C_1\,e^{2x}\int e^{-2x}.\,e^{\frac{2}{3x}}\,dx + C_2\,e^{2x}$$

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $y'' + (1-x)\,y' - y = e^x$
2. $xy'' + (x-1)\,y' - y = x^2$
3. $[(x+3)\,D^2 - (2x+7)\,D + 2]\,y = (x+3)^2\,e^x$
   [**Hint:** $[(x+3)\,D-1]\,[D-2]\,y = (x+3)^2\,e^x$]
4. $x^2y'' + y' - (1+x^2)\,y = e^{-x}$
5. $(x+1)\,y'' + (x-1)\,y' - 2y = 0$
6. $xy'' + (x-1)\,y' - y = 0$

ANSWERS

1. $y = C_1 e^x\int \frac{1}{x}\,e^{-x}\,dx + C_2\,e^x + e^x\log x$
2. $y = C_1\,(x-1) + C_2\,e^{-x} + x^2$
3. $y = C_1\,e^{2x} + C_2\,(2x+7) - e^x\,(x+4)$
4. $y = C_1 e^x\int e^{-2x+\frac{1}{x}}\,dx + C_2\,e^x - \frac{1}{2}\,e^{-x}$
5. $y = C_1\,(x^2+1) + C_2\,e^{-x}$
6. $y = C_1\,(x-1) + C_2\,e^{-x}$

2.11. CAUCHY'S HOMOGENEOUS LINEAR EQUATION (P.T.U., Dec. 2004)

An equation of the form $x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + ... + a_{n-1} x \frac{dy}{dx} + a_n y = \mathrm{X}$ ...(1)

where a_i' s are constant and X is a function of x, is called Cauchy 's homogeneous linear equation.

Such equation can be reduced to linear differential equations with constant coefficients by the substitution $x = e^z$ *i.e.*, $z = \log x$.

so that
$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \text{ or } x\frac{dy}{dx} = \frac{dy}{dz} = \mathrm{D}y \text{, where } \mathrm{D} = \frac{d}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x} \cdot \frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2 y}{dz^2} \qquad \left(\because \frac{dz}{dx} = \frac{1}{x}\right)$$

or
$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} = \mathrm{D}^2 y - \mathrm{D}y = \mathrm{D}(\mathrm{D}-1)y$$

Similarly, $x^3 \frac{d^3 y}{dx^3} = \mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)y$ and so on.

Substituting these values in equation (1), we get a linear differential equation with constant coefficients, which can be solved by the methods already discussed.

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the general solution of the equation* $2x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} - 6y = 0$. **(P.T.U., Dec. 2013)**

Sol. $2x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} - 6y = 0$ ...(1)

It is Cauchy's homogeneous linear differential equation

Put $x = e^z \quad \therefore \quad z = \log x$

$$x\frac{dy}{dx} = \mathrm{D}y; \quad x^2 \frac{d^2 y}{dx^2} = \mathrm{D}(\mathrm{D}-1)y, \text{ where } \mathrm{D} = \frac{d}{dz}$$

Equation (1) becomes

$$2\mathrm{D}(\mathrm{D}-1)y + \mathrm{D}y - 6y = 0$$

or $(2\mathrm{D}^2 - \mathrm{D} - 6)y = 0$

A.E. is $2\mathrm{D}^2 - \mathrm{D} - 6 = 0$

$(\mathrm{D}-2)(2\mathrm{D}+3) = 0$

i.e., $\mathrm{D} = 2, \mathrm{D} = -\frac{3}{2}$

Solution is $y = c_1 e^{2z} + c_2 e^{-\frac{3}{2}z} = c_1 x^2 + c_2 x^{-3/2}$.

Example 2. *Solve* $x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$. **(P.T.U., June 2003, May 2009, Dec. 2012)**

Sol. Given equation is Cauchy's homogeneous linear equation

Put $x = e^z$ *i.e.*, $z = \log x$

so that $$x\frac{dy}{dx} = \mathrm{D}y, \quad x^2\frac{d^2y}{dx^2} = \mathrm{D}(\mathrm{D}-1)y$$

$$x^3\frac{d^3y}{dx^3} = \mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)y, \text{ where } \mathrm{D} = \frac{d}{dz}$$

Substituting these values in the given equation, it reduces to

$$[\mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2) + 2\mathrm{D}(\mathrm{D}-1) + 2]y = 10(e^z + e^{-z})$$

or $$(\mathrm{D}^3 - \mathrm{D}^2 + 2)\,y = 10(e^z + e^{-z})$$

which is a linear equation with constant coefficients.

Its A.E. is $\mathrm{D}^3 - \mathrm{D}^2 + 2 = 0$ or $(\mathrm{D}+1)(\mathrm{D}^2 - 2\mathrm{D} + 2) = 0$

$$\therefore \quad \mathrm{D} = -1, \frac{2 \pm \sqrt{4-8}}{2} = -1, 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1e^{-z} + e^z(c_2\cos z + c_3\sin z) = \frac{c_1}{x} + x\,[c_2\cos(\log x) + c_3\sin(\log x)]$$

$$\text{P.I.} = 10\frac{1}{\mathrm{D}^3 - \mathrm{D}^2 + 2}(e^z + e^{-z}) = 10\left(\frac{1}{\mathrm{D}^3 - \mathrm{D}^2 + 2}e^z + \frac{1}{\mathrm{D}^3 - \mathrm{D}^2 + 2}e^{-z}\right)$$

$$= 10\left(\frac{1}{1^3 - 1^2 + 2}e^z + z.\frac{1}{3\mathrm{D}^2 - 2\mathrm{D}}e^{-z}\right) = 10\left(\frac{1}{2}e^z + z.\frac{1}{3(-1)^2 - 2(-1)}e^{-z}\right)$$

$$= 5e^z + 2ze^{-z} = 5x + \frac{2}{x}\log x$$

Hence the C.S. is $y = \dfrac{c_1}{x} + x\big[c_2\cos(\log x) + c_3\sin(\log x)\big] + 5x + \dfrac{2}{x}\log x.$

Example 3. *Solve the differential equation* $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$. **(P.T.U., Dec. 2005)**

Sol. $$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$$

Multiply by x^2 ;

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x, \text{ which is Cauchy's homogeneous linear equation.}$$

Put $$x = e^z \quad \therefore \quad z = \log x$$

$$x\frac{dy}{dx} = \mathrm{D}y\,;\; x^2\frac{d^2y}{dx^2} = \mathrm{D}(\mathrm{D}-1)\,y, \text{ we get}$$

$$\mathrm{D}(\mathrm{D}-1)\,y + \mathrm{D}y = 12z$$

$$\mathrm{D}^2 y = 12z$$

$$\text{A.E. is } \mathrm{D}^2 = 0 \quad \therefore \quad \mathrm{D} = 0, 0$$

$$\text{C.F.} = c_1 + c_2 z$$

$$\text{P.I.} = \frac{1}{\mathrm{D}^2}(12z) = 12\cdot\frac{1}{\mathrm{D}}\left(\frac{z^2}{2}\right) = 12\cdot\frac{z^3}{6} = 2z^3$$

$$\text{C.S. is } y = \text{C.F.} + \text{P.I.}$$

$$= c_1 + c_2 z + 2z^3$$

$$y = c_1 + c_2 \log x + 2 (\log x)^3.$$

Example 4. *Solve:* $x^2 \dfrac{d^2 y}{dx^2} - 3x \dfrac{dy}{dx} + y = \log x \dfrac{\sin(\log x) + 1}{x}$.

Sol. Given equation is Cauchy's homogeneous linear equation

$\therefore$ Put $x = e^z \quad \therefore \; z = \log x \,;\, x \dfrac{dy}{dx} = Dy, x^2 \dfrac{d^2 y}{dx^2} = D^2 y$, where $D = \dfrac{d}{dz}$

$$\therefore \quad [D(D-1) - 3D + 1]\, y = z \frac{\sin(z) + 1}{e^z}$$

or
$$[D^2 - 4D + 1]\, y = e^{-z} z (\sin z + 1)$$

$$\text{A.E. is } D^2 - 4D + 1 = 0 \quad i.e., D = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$$\therefore \quad \text{C.F.} = c_1 e^{(2+\sqrt{3})z} + c_2 e^{(2-\sqrt{3})z} = c_1 x^{2+\sqrt{3}} + c_2 x^{2-\sqrt{3}}$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 1} e^{-z} z(\sin z + 1) = e^{-z} \frac{1}{(D-1)^2 - 4(D-1) + 1} z(1 + \sin z)$$

$$= e^{-z} \left\{ \frac{1}{D^2 - 6D + 6} z + \frac{1}{D^2 - 6D + 6} z \sin z \right\}$$

$$= e^{-z} [I_1 + I_2] \qquad \ldots(1)$$

where
$$I_1 = \frac{1}{D^2 - 6D + 6} z = \frac{1}{6}\left[1 - D + \frac{D^2}{6}\right]^{-1} z = \frac{1}{6}\left[1 + D + \frac{D^2}{6}\right] z$$

$$= \frac{1}{6} [z + 1]$$

$$I_2 = \frac{1}{D^2 - 6D + 6} z \sin z$$

We know that $\dfrac{1}{f(D)}(xV) = x \dfrac{1}{f(D)} V + \left[\dfrac{d}{dD} f(D)\right] V$ [Note this P.I.]

$$= z \frac{1}{D^2 - 6D + 6} \sin z + \frac{d}{dD}\left(\frac{1}{D^2 - 6D + 6}\right) \sin z$$

Put $D^2 = -1$

$$= z \frac{1}{5 - 6D} \sin z - \frac{2D - 6}{(D^2 - 6D + 6)^2} \sin z$$

$$= z \frac{5 + 6D}{25 - 36D^2} \sin z - \frac{2D - 6}{(D^2 - 6D + 6)^2} \sin z$$

Put $\quad D^2 = -1$

$$= z\,\frac{(5+6D)}{61}\sin z - \frac{(2D-6)}{(5-6D)^2}\sin z$$

$$= z\,\frac{5\sin z + 6\cos z}{61} - \frac{1}{25+36D^2-60D}(2\cos z - 6\sin z)$$

Put $\quad D^2 = -1$

$$= \frac{z}{61}(5\sin z + 6\cos z) + \frac{1}{11+60D}(2\cos z - 6\sin z)$$

$$= \frac{z}{61}(5\sin z + 6\cos z) + \frac{11-60D}{121-3600D^2}(2\cos z - 6\sin z)$$

Put $\quad D^2 = -1$

$$= \frac{z}{61}(5\sin z + 6\cos z) + \frac{11-60D}{3721}(2\cos z - 6\sin z)$$

$$= \frac{z}{61}(5\sin z + 6\cos z) + \frac{1}{3721}[22\cos z - 66\sin z + 120\sin z + 360\cos z]$$

$$= \frac{z}{61}(5\sin z + 6\cos z) + \frac{1}{3721}(54\sin z + 382\cos z)$$

$\therefore$ From (1),

$$\text{P.I.} = e^{-z}\left[\frac{z+1}{6} + \frac{z}{61}(5\sin z + 6\cos z) + \frac{2}{3721}(27\sin z + 191\cos z)\right]$$

$$= \frac{1}{x}\left[\frac{1}{6}(\log x + 1) + \frac{\log x}{61}\{5\sin(\log x) + 6\cos(\log x)\} + \frac{2}{(61)^2}\{27\sin(\log x) + 191\cos(\log x)\}\right]$$

$$\text{C.S. is } y = c_1 x^{2+\sqrt{3}} + c_2 x^{2-\sqrt{3}} + \frac{1}{6x}(1+\log x) + \frac{1}{61x}\Big[\log x\{5\sin(\log x) + 6\cos(\log x)\}$$

$$+ \frac{2}{61}\{27\sin(\log x) + 191\cos(\log x)\}\Big]$$

Example 5. *Solve :* $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x \sin(\log x)$. **(P.T.U., Dec. 2003, Jan 2010)**

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $\quad x = e^z \quad$ *i.e.*, $\quad z = \log x \quad$ so that $\quad x\dfrac{dy}{dx} = Dy,\ x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

where $\quad D = \dfrac{d}{dz}$.

Substituting these values in the given equation, it reduces to $[D(D-1) + D + 1]\,y = z\sin z$

or $(D^2 + 1)\, y = z \sin z$

Its A.E. is $D^2 + 1 = 0$ so that $D = \pm i$

$$\text{C.F.} = c_1 \cos z + c_2 \sin z = c_1 \cos(\log x) + c_2 \sin(\log x)$$

$$\text{P.I.} = \frac{1}{D^2 + 1} z \sin z = \text{Imaginary part of } \frac{1}{D^2 + 1} z e^{iz}$$

$$= \text{I.P. of } e^{iz} \frac{1}{(D + i)^2 + 1} z = \text{I.P. of } e^{iz} \frac{1}{D^2 + 2iD} z$$

$$= \text{I.P. of } e^{iz} \frac{1}{2iD\left(1 + \dfrac{D}{2i}\right)} z = \text{I.P. of } e^{iz} \frac{1}{2iD\left(1 - \dfrac{iD}{2}\right)} z$$

$$= \text{I.P. of } \frac{1}{2i} e^{iz} \frac{1}{D}\left(1 - \frac{iD}{2}\right)^{-1} z = \text{I.P. of } \frac{1}{2i} e^{iz} \frac{1}{D}\left(1 + \frac{iD}{2} + \ldots\ldots\right) z$$

$$= \text{I.P. of } \frac{1}{2i} e^{iz} \frac{1}{D}\left(z + \frac{i}{2}\right) = \text{I.P. of } \frac{1}{2i} e^{iz} \int \left(z + \frac{i}{2}\right) dz$$

$$= \text{I.P. of } -\frac{i}{2} e^{iz} \left(\frac{z^2}{2} + \frac{i}{2} z\right) = \text{I.P. of } e^{iz} \left(-\frac{i}{4} z^2 + \frac{z}{4}\right)$$

$$= \text{I.P. of } (\cos z + i \sin z)\left(-\frac{i}{4} z^2 + \frac{z}{4}\right) = -\frac{z^2}{4} \cos z + \frac{z}{4} \sin z$$

$$= -\frac{1}{4} (\log x)^2 \cos(\log x) + \frac{1}{4} \log x \sin(\log x).$$

Hence the C.S. is $y = c_1 \cos(\log x) + c_2 \sin(\log x) - \dfrac{1}{4} (\log x)^2 \cos(\log x) + \dfrac{1}{4} \log x \sin(\log x)$.

Example 6. *Solve :* $x^2 \dfrac{d^2 y}{dx^2} + 3x \dfrac{dy}{dx} + y = \dfrac{1}{(1 - x)^2}$. **(P.T.U., Dec. 2002)**

Sol. Given equation is Cauchy's homogeneous linear equation

Put $x = e^z$ $\therefore$ $z = \log x,\ x \dfrac{dy}{dx} = \dfrac{dy}{dz} = Dy$

$$x^2 \frac{d^2 y}{dx^2} = D(D - 1)\, y, \text{ where } D = \frac{d}{dz}$$

Substituting the values in given equation

$$[D(D - 1) + 3D + 1]\, y = \frac{1}{(1 - e^z)^2}$$

$$(D^2 + 2D + 1)\, y = \frac{1}{(1 - e^z)^2}$$

A.E. is $D^2 + 2D + 1 = 0$, *i.e.*, $(D + 1)^2 = 0 \therefore D = -1, -1$

$$\text{C.F.} = (c_1 + c_2 z)\, e^{-z}$$

$$\text{P.I.} = \frac{1}{(D+1)^2} \cdot \frac{1}{\left(1-e^z\right)^2} = \frac{1}{D+1}\left\{\frac{1}{D+1} \cdot \frac{1}{\left(1-e^z\right)^2}\right\}$$

$$= \frac{1}{D+1}\left[e^{-z}\int e^z \cdot \frac{1}{\left(1-e^z\right)^2}\, dz\right] \qquad \Bigg| \text{By using } \frac{1}{D-a}X = e^{ax}\int e^{-ax} X\, dx$$

$$= \frac{1}{D+1}\left[e^{-z}\int \left(1-e^z\right)^{-2} e^z\, dz\right] = \frac{1}{D+1}\left[e^{-z}\int -\left(1-e^z\right)^{-2}\left(-e^z\right) dz\right]$$

$$= \frac{1}{D+1}\left[e^{-z}(-1)\frac{\left(1-e^z\right)^{-1}}{-1}\right] \qquad \Bigg| \text{By using} = \int \left[f(z)\right]^n f'(z)\, dz\, \frac{\left[f(z)\right]^{n+1}}{n+1};\ n \neq -1$$

$$= \frac{1}{D+1}\left(\frac{e^{-z}}{1-e^z}\right)$$

$$= e^{-z}\int e^z \cdot \frac{e^{-z}}{1-e^z}\, dz = e^{-z}\int \frac{dz}{1-e^z}$$

Put $e^z = t \therefore e^z\, dz = dt \therefore dz = \frac{1}{t}\, dt$

$$\text{P.I.} = e^{-z}\int \frac{1}{t(1-t)}\, dt$$

$$= e^{-z}\int \left(\frac{1}{t} + \frac{1}{1-t}\right) dt \qquad \text{[By Partial fractions]}$$

$$= e^{-z}\left[\log t - \log(1-t)\right] = e^{-z}\log\frac{t}{1-t} = e^{-z}\log\frac{e^z}{1-e^z}$$

C.S. is
$$y = (c_1 + c_2 z)\, e^{-z} + e^{-z}\log\frac{e^z}{1-e^z} = (c_1 + c_2 \log x)\frac{1}{x} + \frac{1}{x}\log\frac{x}{1-x}$$

$$= \frac{1}{x}\left[c_1 + c_2 \log x + \log\frac{x}{1-x}\right]$$

Example 7. *Solve :* $u = r\dfrac{d}{dr}\left(r\dfrac{du}{dr}\right) + ar^3$.

Sol.
$$u = r\left[r\frac{d^2u}{dr^2} + \frac{du}{dr}\right] + ar^3$$

or $$r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} - u = -ar^3,$$ which is Cauchy's homogeneous linear equation in u and r

Put $r = e^z$ and Let $D = \frac{d}{dz}$

$\therefore$ $$(D(D-1) + D - 1)u = -a e^{3z}$$

or $$(D^2 - 1)u = -a e^{3z}$$

A.E. is $$D^2 - 1 = 0 \qquad [\because D = -1, 1]$$

$$\text{C.F.} = c_1 e^z + c_2 e^{-z}$$

$$\text{P.I.} = \frac{1}{D^2-1}(-a e^{3z}) = \frac{-a e^{3z}}{8} \qquad | \text{ By putting } D = 3$$

$$\text{C.S. is } u = c_1 e^z + c_2 e^{-z} - \frac{a e^{3z}}{8} = c_1 r + \frac{c_2}{r} - \frac{a}{8} r^3.$$

Example 8. *Solve:* $x^2y'' - 4xy' + 8y = 4x^3 + 2\sin(\log x)$. **(P.T.U., May 2006)**

Sol. $$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 8y = 4x^3 + 2\sin(\log x)$$

which is Cauchy's homogeneous linear = x

Put $$x = e^z \text{ i.e., } z = \log x$$

$$x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore$ $$D(D-1)y - 4Dy + 8y = 4e^{3z} + 2\sin z$$

$$(D^2 - 5D + 8)y = 4e^{3z} + 2\sin z$$

$$\text{A.E. is } D^2 - 5D + 8 = 0 \quad \therefore\ D = \frac{5}{2} \pm i\frac{\sqrt{7}}{2}$$

$$\text{C.F.} = e^{\frac{5}{2}z}\left[c_1\cos\frac{\sqrt{7}}{2}z + c_2\sin\frac{\sqrt{7}}{2}z\right]$$

$$\text{P.I.} = \frac{1}{D^2 - 5D + 8}\left(4e^{3z} + 2\sin z\right)$$

$$= 4\underset{(\text{Put } D = 3)}{\frac{1}{D^2 - 5D + 8}}e^{3z} + 2\underset{(\text{Put } D^2 = -1)}{\frac{1}{D^2 - 5D + 8}}\sin z$$

$$= 4\cdot\frac{1}{2}e^{3z} + 2\cdot\frac{1}{-5D+7}\sin z = 2e^{3z} - 2\frac{5D+7}{25D^2 - 49}\sin z$$

$$= 2e^{3z} - 2\frac{5D+7}{-74}\sin z = 2e^{3z} + \frac{1}{37}[5\cos z + 7\sin z]$$

$$\text{C.S. } y = e^{\frac{5}{2}z}\left[c_1\cos\frac{\sqrt{7}}{2}z + c_2\sin\frac{\sqrt{7}}{2}z\right] + 2e^{3z} + \frac{5}{37}\cos z + \frac{7}{37}\sin z$$

$\therefore$ $$y = x^{\frac{5}{2}}\left[c_1\cos\left(\frac{\sqrt{7}}{2}\log x\right) + c_2\sin\left(\frac{\sqrt{7}}{2}\log x\right)\right] + 2x^3 + \frac{5}{37}\cos(\log x) + \frac{7}{37}\sin(\log x).$$

2.12. LEGENDRE'S LINEAR EQUATION (P.T.U., May 2007, Dec. 2005)

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1}}{dx^{n-1}} + \ldots.. + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = \mathrm{X} \qquad \ldots(1)$$

where a_i's are constants and X is a function x, is called Legendre's linear equation.

Such equations can be reduced to linear differential equations with constant coefficients, by the substitutions

$$a + bx = e^z \quad i.e., \quad z = \log(a+bx) \quad \text{so that } \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{b}{a+bx}\frac{dy}{dz}$$

or $$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz} = b\mathrm{D}y, \text{ where } \mathrm{D} = \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dz}\right) = -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\cdot\frac{d^2y}{dz^2}\cdot\frac{dy}{dx}$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d^2y}{dz^2}\cdot\frac{b}{a+bx} = \frac{b^2}{(a+bx)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

or $$(a+bx)^2\frac{d^2y}{dx^2} = b^2(\mathrm{D}^2y - \mathrm{D}y) = b^2\,\mathrm{D}(\mathrm{D}-1)\,y$$

Similarly, $(a+bx)^3\dfrac{d^3y}{dx^3} = b^3\,\mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)\,y.$

Substituting these values in equation (1), we get a linear differential equation with constant coefficient which can be solved by the methods already discussed.

Example 9. *Solve :* $(3x+2)^2\dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$.

Sol. Given equation is a Legendre's linear equation.

Put $\quad 3x+2 = e^z \quad i.e., \quad z = \log(3x+2)$ so that $(3x+2)\dfrac{dy}{dx} = 3\mathrm{D}y$.

$$(3x+2)^2\frac{d^2y}{dx^2} = 3^2\,\mathrm{D}(\mathrm{D}-1)y, \quad \text{where } \mathrm{D} = \frac{d}{dz}.$$

Substituting these values in the given equation, it reduces to

$$[3^2\,\mathrm{D}(\mathrm{D}-1) + 3.3\mathrm{D} - 36]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

or $$9(\mathrm{D}^2-4)y = \frac{1}{3}e^{2z} - \frac{1}{3} \text{ or } (\mathrm{D}^2-4)y = \frac{1}{27}(e^{2z}-1)$$

which is a linear equation with constant coefficients.

Its A.E. is $\mathrm{D}^2 - 4 = 0 \therefore \mathrm{D} = \pm 2$

$$\text{C.F.} = c_1e^{2z} + c_2e^{-z} = c_1(3x+2)^2 + c_2(3x+2)^{-2}$$

$$\text{P.I.} = \frac{1}{27}\cdot\frac{1}{D^2-4}\left(e^{2z}-1\right) = \frac{1}{27}\left[\frac{1}{D^2-4}e^{2z} - \frac{1}{D^2-4}e^{0z}\right]$$

$$= \frac{1}{27}\left[z.\frac{1}{2D}e^{2z} - \frac{1}{0-4}e^{0z}\right] = \frac{1}{27}\left[\frac{z}{2}\int e^{2z}dz + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[\frac{z}{4}e^{2z} + \frac{1}{4}\right] = \frac{1}{108}\left(ze^{2z}+1\right) = \frac{1}{108}[(3x+2)^2\log(3x+2)+1]$$

Hence the C.S. is $y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2\log(3x+2)+1]$.

Example 10. *Solve :* $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = \sin[2\log(1+x)]$.

(P.T.U., Dec. 2006, 2012, 2013, May 2012, 2014)

Sol. Given equation is Legendre's linear equation

$\therefore$ Put $1+x = e^z \quad \therefore \; z = \log(1+x)$

$$(1+x)\frac{dy}{dx} = Dy,\; (1+x)^2\frac{d^2y}{dx^2} = D(D-1)y, \text{ where } D = \frac{d}{dz}$$

$\therefore$ $D(D-1)y + Dy + y = \sin(2z)$

or $(D^2+1)y = \sin 2z$

which is linear differential equation with constant coefficients

A.E. is $D^2+1=0 \quad \therefore \; D = \pm i$

$$\text{C.F.} = c_1\cos z + c_2\sin z$$

$$\text{P.I.} = \frac{1}{D^2+1}\sin 2z$$

Put $D^2 = -4$

$\therefore$ $$\text{P.I.} = -\frac{1}{3}\sin 2z$$

$$\text{C.S. is } y = c_1\cos z + c_2\sin z - \frac{1}{3}\sin 2z$$

Put $z = \log(1+x)$

$\therefore$ $$y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] - \frac{1}{3}\sin[2\log(1+x)].$$

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = 0$ **(P.T.U., May 2006)**

2. $x^2\dfrac{d^2y}{dx^2} - 2y = x^2 + \dfrac{1}{x}$

3. (*i*) $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = (x+1)^2$

(*ii*) $x^2\dfrac{d^2y}{dx^2} + 9x\dfrac{dy}{dx} + 25y = 50$

4. $x^2\dfrac{d^3y}{dx^3} - 4x\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = 4$

[**Hint :** Multiply throughout by x]

5. (*i*) $x^4\dfrac{d^3y}{dx^3} + 2x^3\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = 1$

(*ii*) $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x^2$

6. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2\dfrac{d^2u}{dr^2} + r\dfrac{du}{dr} - u + kr^3 = 0$, where k is a constant. Solve the equation under the conditions $u = 0$ when $r = 0$, $u = 0$ when $r = a$.

7. $x^2 \dfrac{d^2y}{dx} - x\dfrac{dy}{dx} + y = \log x$ **(P.T.U., May 2009)**

8. $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x\log x$ **(P.T.U., May 2010)**

9. $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 12y = x^3 \log x$

10. $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^2 + 2\log x$

11. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = \sin(\log x)$

12. $x^3 \dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 8y = 65\cos(\log x)$

13. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = x^2 \sin(\log x)$

14. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \log x \dfrac{\sin(\log x) + 1}{x}$

15. $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2 \log x$

16. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\log(1+x)$

17. $(1+2x)^2 \dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2$

18. $(2x+3)^2 \dfrac{d^2y}{dx^2} - 2(2x+3)\dfrac{dy}{dx} - 12y = 6x \cdot$ **(P.T.U., Dec. 2013)**

ANSWERS

1. $y = c_1 \dfrac{1}{x} + c_2 \dfrac{1}{x^2}$

2. $y = c_1 x^2 + \dfrac{c_2}{x} + \dfrac{1}{3}\left(x^2 - \dfrac{1}{x}\right)\log x$

3. (*i*) $y = c_1x^4 + c_2x^{-5} - \dfrac{x^2}{14} - \dfrac{x}{9} - \dfrac{1}{20}$

(*ii*) $y = x^{-4}\,[c_1 \cos(3\log x) + c_2 \sin(3\log x) + 2$

4. $y = c_1 + c_2x^3 + c_3x^4 + \dfrac{2}{3}x$

5. (*i*) $y = (c_1 + c_2 \log x)x + c_3x^{-1} + \dfrac{1}{4x}\log x$

(*ii*) $y = c_1x^2 + c_2x^2 - x^2 \log x$

6. $u = \dfrac{kr}{8}(a^2 - r^2)$

7. $y = (c_1 + c_2 \log x)x + \log x + 2$

8. $y = x\,[c_1 \cos(\log x) + c_2 \sin(\log x) + x\log x$

9. $y = c_1x^3 + c_2x^{-4} + \dfrac{x^2}{98}\log x\,(7\log x - 2)$

10. $y = c_1x^{-1} + c_2x^4 - \dfrac{x^2}{6} - \dfrac{1}{2}\log x + \dfrac{3}{8}$

11. $y = x^2[c_1 \cos(\log x) + c_2 \sin(\log x)] + \dfrac{1}{8}[\sin(\log x) + \cos(\log x)]$

12. $y = c_1x^{-2} + x(c_2 \cos\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x) + 8\cos(\log x) - \sin(\log x)$

13. $y = x^2[c_1 \cos(\log x) + c_2 \sin(\log x)] - \dfrac{1}{2}x^2 \log x \cos(\log x)$

14. $y = c_1x^{2+\sqrt{3}} + c_2x^{2-\sqrt{3}} + \dfrac{1}{61x}$

$$\left[\log x\{5\sin(\log x) + 6\cos(\log x)\} + \frac{2}{61}\{21\sin(\log x) + 191\cos(\log x)\}\right] + \frac{1}{6x}(1 + \log x)$$

15. $y = c_1 x^3 + \dfrac{c_2}{x} - \dfrac{x^2}{3}\left(\log x + \dfrac{2}{3}\right)$

16. $y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] + 2\log(1+x) + \sin[\log(1+x)]$

17. $y = (1+2x)^2\,[c_1 + c_2 \log(1+2x) + \{\log(1+2x)\}^2]$

18. $y = c_1(2x+3)^{-1} + c^2(2x+3)^3 - \dfrac{3}{4}(2x+3) + 3\,.$

2.13. SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Now, we discuss differential equations in which there is one independent variable and two or more than two dependent variables. Such equations are called *simultaneous linear equations.* To solve such equations completely, we must have as many simultaneous equations as the number of dependent variables. Here, we shall consider simultaneous linear equations with constant coefficients only.

Let x, y be the two dependent variables and t the independent variable. Consider the simultaneous equations

$$f_1(\mathrm{D})\,x + f_2(\mathrm{D})\,y = \mathrm{T}_1 \quad \ldots(1) \qquad \phi_1(\mathrm{D})x + \phi_2(\mathrm{D})y = \mathrm{T}_2 \quad \ldots(2)$$

where $\mathrm{D} = \dfrac{d}{dt}$ and $\mathrm{T}_1, \mathrm{T}_2$ are functions of t.

To eliminate y, operating on both sides of (1) by ϕ_2 (D) and on both sides of (2) by f_2 (D) and subtracting, we get

$$[f_1(\mathrm{D})\,\phi_2(\mathrm{D}) - \phi_1(\mathrm{D})\,f_2(\mathrm{D})]\,x = \phi_2(\mathrm{D})\mathrm{T}_1 - f_2(\mathrm{D})\,\mathrm{T}_2 \text{ or } f(\mathrm{D})\,x = \mathrm{T}$$

which is a linear equation in x and t and can be solved by the methods already discussed, substituting the value of x in either (1) or (2), we get the value of y.

Note. We can also eliminate x to get a linear equation in y and t.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $\dfrac{dx}{dt} + 5x - 2y = t\,,\ \dfrac{dy}{dt} + 2x + y = 0$ *given that* $x = y = 0$ *when* $t = 0$.

(P.T.U., May 2010, Dec. 2012, 2013)

Sol. Writing D for $\dfrac{d}{dt}$, the given equations become

$$(\mathrm{D}+5)x - 2y = t \quad \ldots(1)$$

$$(\mathrm{D}+1)y + 2x = 0 \quad \ldots(2)$$

To eliminate y; operate on both sides of (1) by (D + 1) and (2) by 2 and add

$$(\mathrm{D}+5)\,(\mathrm{D}+1)x + 4x = (\mathrm{D}+1)t$$

$$(\mathrm{D}^2 + 6\mathrm{D} + 9)x = 1 + t$$

$$(\mathrm{D}+3)^2 x = 1 + t$$

$$\text{A.E. is } (\mathrm{D}+3)^2 = 0 \quad \therefore \mathrm{D} = -3, -3$$

$$\therefore \qquad \text{C.F.} = (\mathrm{C}_1 + \mathrm{C}_2 t)\,e^{-3t}$$

$$\text{P.I.} = \frac{1}{(\mathrm{D}+3)^2}(1+t)$$

$$= \frac{1}{(\mathrm{D}+3)^2}e^{0t} + \frac{1}{(\mathrm{D}+3)^2}t$$

$$= \frac{1}{9} + \frac{1}{9}\left(1 + \frac{\mathrm{D}}{3}\right)^{-2} t$$

$$= \frac{1}{9} + \frac{1}{9}\left(1 - \frac{2}{3}\mathrm{D}\right) t$$

$$= \frac{1}{9} + \frac{1}{9}\left(t - \frac{2}{3}\right)$$

$$= \frac{1}{9} + \frac{1}{9}t - \frac{2}{27} = \frac{1}{27} + \frac{t}{9}$$

$$\therefore \qquad x = \left(C_1 + C_2 t\right) e^{-3t} + \frac{1}{27} + \frac{t}{9} \qquad \ldots(3)$$

$$\frac{dx}{dt} = -3(C_1 + C_2 t)\, e^{-3t} + C_2 e^{-3t} + \frac{1}{9}$$

$$= e^{-3t}\left[-3C_1 + (C_2 - 3C_2 t)\right] + \frac{1}{9}$$

Substituting the values of x and $\frac{dx}{dt}$ in (1)

$$e^{-3t}(-3C_1 + C_2 - 3C_2 t) + \frac{1}{9} + 5\left\{(C_1 + C_2 t)\, e^{-3t} + \frac{1}{27} + \frac{t}{9}\right\} - t = 2y \qquad \ldots(4)$$

Put $x = 0, y = 0, t = 0$ in (3) and (4)

From (3) $0 = C_1 + \frac{1}{27}$; $C_1 = -\frac{1}{27}$

From (4) $\frac{1}{9} + C_2 + \frac{1}{9} + 5\left(-\frac{1}{27} + \frac{1}{27}\right) = 0$

$$\therefore \qquad C_2 = -\frac{2}{9}$$

$\therefore$ From (3)

$$x = \left(-\frac{1}{27} - \frac{2}{9}t\right) e^{-3t} + \frac{1}{27} + \frac{t}{9}$$

$$= -\frac{1}{27}\left\{(1 + 6t)\, e^{-3t} - 1 - 3t\right\}$$

From (4)

$$2y = e^{-3t}\left(\frac{1}{9} - \frac{2}{9} + \frac{2}{3}t\right) + \frac{1}{9} + 5\left[\left(-\frac{1}{27} - \frac{2t}{9}\right) e^{-3t} + \frac{1}{27} + \frac{t}{9}\right] - t$$

$$= \frac{e^{-3t}}{27}\left\{-3 + 18t - 5 - 30t\right\} + \frac{1}{9} + \frac{5}{27} + \frac{5t}{9} - t$$

$$= \frac{e^{-3t}}{27}(-8 - 12t) + \frac{1}{27}(8 + 15t - 27t)$$

$$y = \frac{e^{-3t}}{27}(-4 - 6t) + \frac{1}{27}(4 - 6t)$$

$$= \frac{-2e^{-3t}}{27}(2 + 3t) + \frac{2}{27}(2 - 3t)$$

Hence,

$$x = -\frac{1}{27}(1 + 6t)e^{-3t} + \frac{1}{27}(1 + 3t)$$

$$y = -\frac{2}{27}(2 + 3t)\, e^{-3t} + \frac{2}{27}(2 - 3t)$$

Example 2. *Solve :* $\frac{dx}{dt} + 2y = e^t$ *and* $\frac{dy}{dt} - 2x = e^{-t}$.

Sol.

$$\frac{dx}{dt} + 2y = e^t \qquad \ldots(1)$$

$$\frac{dy}{dt} - 2x = e^{-t} \qquad \ldots(2)$$

To eliminate y, differentiate (1) w.r.t. t

$$\frac{d^2x}{dt^2} + 2\frac{dy}{dt} = e^t$$

From (2),
$$\frac{dy}{dt} = 2x + e^{-t}$$

$$\therefore \qquad \frac{d^2x}{dt^2} + 4x + 2e^{-t} = e^t$$

$$\frac{d^2x}{dt^2} + 4x = e^t - 2e^{-t}$$

which is linear differential equation with constant coefficients.

Its S.F. is $(D^2 + 4)\,x = e^t - 2e^{-t}$ where $D = \frac{d}{dt}$

A.E. is $D^2 + 4 = 0$ $\quad \therefore \quad D = \pm 2i$

C.F. $= c_1 \cos 2t + c_2 \sin 2t$

P.I. $$= \frac{1}{D^2+4}\left(e^t - 2e^{-t}\right) = \frac{1}{D^2+4}e^t - 2\frac{1}{D^2+4}e^{-t}$$

(Put D = 1) (Put D = – 1)

$$= \frac{1}{5}e^t - \frac{2}{5}e^{-t}$$

$\therefore$ C.S. is $x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{5}e^t - \frac{2}{5}e^{-t}$

$$\frac{dx}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{1}{5}e^t + \frac{2}{5}e^{-t}$$

From (1),
$$2y = 2c_1 \sin 2t - 2c_2 \cos 2t - \frac{1}{5}e^t - \frac{2}{5}e^{-t} + e^t$$

$$\therefore \qquad y = c_1 \sin 2t - c_2 \cos 2t + \frac{2}{5}e^t - \frac{1}{5}e^{-t}$$

Hence,
$$x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{5}e^t - \frac{2}{5}e^{-t}$$

and
$$y = c_1 \sin 2t - c_2 \cos 2t + \frac{2}{5}e^t - \frac{1}{5}e^{-t}$$ is the required solution.

Example 3. *Solve the system of equations*

$$(2D - 4)\,y_1 + (3D + 5)y_2 = 3t + 2$$
$$(D - 2)\,y_1 + (D + 1)\,y_2 = t.$$ **(P.T.U., Jan 2008)**

Sol.
$$(2D - 4)y_1 + (3D + 5)y_2 = 3t + 2 \qquad \ldots(1)$$
$$(D - 2)y_1 + (D + 1)y_2 = t \qquad \ldots(2)$$

Multiply (2) by 2 and subtract from (1)

$$(D + 3)y_2 = t + 2$$

or
$$\frac{dy_2}{dt} + 3y_2 = t + 2,$$ which is linear differential equation in t.

Its $\qquad$ I.F. $= e^{\int 3dt} = e^{3t}$

Its solution is $\qquad y_2 e^{3t} = \int (t+2)\, e^{3t}\, dt + c_1$

Integrating by parts

$$= (t+2)\frac{e^{3t}}{3} - (1)\cdot\frac{e^{3t}}{9} + c_1$$

$$\therefore \qquad y_2 = \frac{t+2}{3} - \frac{1}{9} + c_1 e^{-3t} = \frac{3t+5}{9} + c_1 e^{-3t}$$

Substituting the value of y_2 in (2), we get

$$(\mathrm{D}-2)\, y_1 + (\mathrm{D}+1)\left[\frac{3t+5}{9} + c_1\, e^{-3t}\right] = t$$

or $$(\mathrm{D}-2)\, y_1 + \left[\frac{1}{3} - 3c_1 e^{-3t} + \frac{3t+5}{9} + c_1\, e^{-3t}\right] = t$$

or $$(\mathrm{D}-2)\, y_1 + \left[\frac{8+3t}{9} - 2c_1\, e^{-3t}\right] = t$$

or $$(\mathrm{D}-2) y_1 = t - \frac{8}{9} - \frac{1}{3}t + 2c_1 e^{-3t}$$

or $$\frac{dy_1}{dt} - 2y_1 = \frac{2}{3}t - \frac{8}{9} + 2c_1 e^{-3t}$$

which is linear differential equation in t

Its $\qquad$ I.F. $= e^{-2t}$

Its solution is $\qquad y_1 e^{-2t} = \int e^{-2t}\left(\frac{2}{3}t - \frac{8}{9} + 2c_1 e^{-3t}\right) dt + c_2$

$$= \frac{2}{3}\int t e^{-2t}\, dt - \frac{8}{9}\int e^{-2t}\, dt + 2c_1 \int e^{-5t}\, dt + c_2$$

$$= \frac{2}{3}\left[t\,\frac{e^{-2t}}{-2} - 1\left(\frac{e^{-2t}}{4}\right)\right] - \frac{8}{9}\,\frac{e^{-2t}}{-2} + 2c_1\,\frac{e^{-5t}}{-5} + c_2$$

$$= -\frac{t}{3}e^{-2t} - \frac{1}{6}e^{-2t} + \frac{4}{9}e^{-2t} - \frac{2}{5}c_1 e^{-5t} + c_2$$

$$= -\frac{t}{3}e^{-2t} + \frac{5}{18}e^{-2t} - \frac{2}{5}c_1 e^{-5t} + c_2$$

$$\therefore \qquad y_1 = -\frac{t}{3} + \frac{5}{18} - \frac{2}{5}c_1 e^{-3t} + c_2 e^{+2t}$$

Hence, $$y_1 = -\frac{1}{3}t + \frac{5}{18} - \frac{2}{5}c_1 e^{-3t} + c_2 e^{2t}$$

$$y_2 = \frac{3t+5}{9} + c_1 e^{-3t}.$$

Example 4. *Solve :* $\dfrac{d^2x}{dt^2} + 4x + 5y = t^2$ *and* $\dfrac{d^2y}{dt^2} + 5x + 4y = t + 1.$

Sol. Writing D for $\dfrac{d}{dt}$, the given equations becomes $(\mathrm{D}^2+4)\, x + 5y = t^2$ $\qquad$ …(1)

and $\qquad 5x + (D^2 + 4)y = t + 1 \qquad \ldots(2)$

To eliminate y, operating on both sides of (1) by $(D^2 + 4)$ and on both sides of (2) by 5 and subtracting, we get

$$[(D^2+4)^2 - 25]\,x = (D^2+4)\,t^2 - 5\,(t+1)$$

or $$(D^4 + 8D^2 - 9)x = 2 + 4t^2 - 5t - 5$$

or $$(D^4 + 8D^2 - 9)\,x = 4t^2 - 5t - 3$$

Its A.E. is $$D^4 + 8D^2 - 9 = 0$$

or $$(D^2+9)(D^2-1) = 0 \quad \therefore \quad D = \pm 1 \pm 3i$$

$$\text{C.F.} = c_1e^t + c_2e^{-t} + c_3\cos 3t + c_4 \sin 3t$$

$$\text{P.I.} = \frac{1}{D^4 + 8D^2 - 9}(4t^2 - 5t - 3) = \frac{1}{-9\left(1 - \frac{8D^2}{9} - \frac{D^4}{9}\right)}(4t^2 - 5t - 3)$$

$$= -\frac{1}{9}\left[1 - \left(\frac{8D^2}{9} + \frac{D^4}{9}\right)\right]^{-1}(4t^2 - 5t - 3)$$

$$= -\frac{1}{9}\left[1 + \left(\frac{8D^2}{9} + \frac{D^4}{9}\right) + \ldots\ldots\right](4t^2 - 5t - 3)$$

$$= -\frac{1}{9}\left[4t^2 - 5t - 3 + \frac{8}{9}(8)\right] = -\frac{1}{9}\left(4t^2 - 5t + \frac{37}{9}\right)$$

$$\therefore \qquad x = c_1e^t + c_2e^{-t} + c_3\cos 3t + c_4\sin 3t - \frac{4}{9}t^2 + \frac{5}{9}t - \frac{37}{81}$$

Now, $$\frac{dx}{dt} = c_1e^t - c_2e^{-t} - 3c_3\sin 3t + 3c_4\cos 3t - \frac{8}{9}t + \frac{5}{9}$$

$$\frac{d^2x}{dt^2} = c_1e^t + c_2e^{-t} - 9c_3\cos 3t - 9c_4\sin 3t - \frac{8}{9}$$

Substituting the values of x and $\frac{d^2x}{dt^2}$ in (1), we have from (1) $5y = t^2 - 4x - \frac{d^2x}{dt^2}$

$$\therefore \qquad 5y = t^2 - 4c_1e^t - 4c_2e^{-t} - 4c_3\cos 3t - 4c_4\sin 3t$$
$$+ \frac{169}{9}t^2 - \frac{20}{9}t + \frac{148}{81} - c_1e^t - c_2e^{-t} + 9c_3\cos 3t + 9c_4\sin 3t + \frac{8}{9}$$

$$\therefore \qquad y = \frac{1}{5}\left[-5c_1e^t - 5c_2e^{-t} + 5c_3\cos 3t + 5c_4\sin 3t + \frac{25}{9}t^2 - \frac{20}{9}t + \frac{220}{81}\right]$$

Hence, $$x = c_1e^t + c_2e^{-t} + c_3\cos 3t + c_4\sin 3t - \frac{1}{9}\left(4t^2 - 5t + \frac{37}{9}\right)$$

$$y = -c_1e^t - c_2e^{-t} + c_3\cos 3t + c_4\sin 3t - \frac{1}{2}\left(5t^2 - 4t + \frac{44}{9}\right).$$

Example 5. *Solve the simultaneous equations :* $t\frac{dx}{dt} + y = 0,\ t\frac{dy}{dt} + x = 0$ *given* $x(1) = 1,\ y(-1) = 0.$

Sol. The given equations are $\quad t\frac{dx}{dt} + y = 0 \qquad \ldots(1)$

$$t\frac{dy}{dt} + x = 0 \qquad \ldots(2)$$

Differentiating (1) w.r.t. t, we have

$$t\frac{d^2x}{dt^2}+\frac{dx}{dt}+\frac{dy}{dt}=0$$

Multiplying throughout by t

$$t^2\frac{d^2x}{dt^2}+t\frac{dx}{dt}+t\frac{dy}{dt}=0$$

or $$t^2\frac{d^2x}{dt^2}+t\frac{dx}{dt}-x=0 \quad \ldots(3) \text{ [Using (2)]}$$

which is Cauchy's homogeneous linear equation.

Putting $t=e^u$ *i.e.*, $u=\log t$, so that $t\frac{d}{dt}=\frac{d}{du}=\text{D}$, equation (3) becomes

$$[\text{D}(\text{D}-1)+\text{D}-1]\,x=0 \quad \text{or} \quad (\text{D}^2-1)\,x=0$$

Its A.E. is $\text{D}^2-1=0$ whence $\text{D}=\pm 1$

$$\therefore \quad x=c_1e^u+c_2e^{-u}=c_1t+\frac{c_2}{t} \quad \ldots(4)$$

From (1), $$y=-t\frac{dx}{dt}=-t\left(c_1-\frac{c_2}{t^2}\right)=-c_1t+\frac{c_2}{t} \quad \ldots(5)$$

Since $x(1)=1$, $\therefore$ from (4), we have $1=c_1+c_2$

Also, $y(-1)=0$ $\therefore$ from (5), we have $0=c_1-c_2$

Solving $$c_1=c_2=\frac{1}{2}$$

Hence, $$x=\frac{1}{2}\left(t+\frac{1}{t}\right),\ y=\frac{1}{2}\left(-t+\frac{1}{t}\right).$$

Example 6. *Solve the following simultaneous equations :* $\frac{dx}{dt}=2y, \frac{dy}{dt}=2z, \frac{dz}{dt}=2x$.

Sol. The given equations are

$$\frac{dx}{dt}=2y \quad \ldots(1) \qquad \frac{dy}{dt}=2z \quad \ldots(2) \qquad \frac{dz}{dt}=2x \quad \ldots(3)$$

Differentiating (1) w.r.t. t, $$\frac{d^2x}{dt^2}=2\frac{dy}{dt}=2(2z) \quad \text{[Using (2)]}$$

Differentiating again w.r.t. t, $$\frac{d^3x}{dt^3}=4\frac{dz}{dt}=4(2x) \quad \text{or} \quad (\text{D}^3-8)x=0$$

where $$\text{D}=\frac{d}{dt}$$

Its A.E. is $\text{D}^3-8=0$ or $(\text{D}-2)(\text{D}^2+2\text{D}+4)=0$

whence $$\text{D}=2, \frac{-2\pm 2i\sqrt{3}}{2} \quad \text{or} \quad \text{D}=2, -1\pm i\sqrt{3}$$

$$\therefore \quad x=c_1e^{2t}+c_2e^{-t}\cos\left(\sqrt{3}t-c_3\right)$$ (See note at the end of the questions)

From (1),
$$y = \frac{1}{2}\frac{dx}{dt}$$
$$= \frac{1}{2}\left[2c_1e^{2t} - c_2e^{-t}\cos\left(\sqrt{3}t - c_3\right) - c_2\sqrt{3}e^{-t}\sin\left(\sqrt{3}t - c_3\right)\right]$$
$$= c_1e^{2t} + c_2e^{-t}\left[-\frac{1}{2}\cos\left(\sqrt{3}t - c_3\right) - \frac{\sqrt{3}}{2}\sin\left(\sqrt{3}t - c_3\right)\right]$$
$$= c_1e^{2t} + c_2e^{-t}\left[\cos\frac{2\pi}{3}\cos\left(\sqrt{3}t - c_3\right) - \sin\frac{2\pi}{3}\sin\left(\sqrt{3}t - c_3\right)\right]$$
$$\Big|\ \because \cos\frac{2\pi}{3} = \frac{1}{2} \text{ and } \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$
$$= c_1e^{2t} + c_2e^{-t}\cos\left(\sqrt{3}t - c_3 + \frac{2\pi}{3}\right)$$

From (2),
$$z = \frac{1}{2}\frac{dy}{dt}$$
$$= \frac{1}{2}\left[2c_1e^{2t} - c_2e^{-t}\cos\left(\sqrt{3}t - c_3 + \frac{2\pi}{3}\right) - c_2\sqrt{3}\ e^{-t}\sin\left(\sqrt{3}t - c_3 + \frac{2\pi}{3}\right)\right]$$
$$= c_1e^{2t} + c_2e^{-t}\left[\cos\frac{2\pi}{3}\cos\left(\sqrt{3}t - c_3 + \frac{2\pi}{3}\right) - \sin\frac{2\pi}{3}\sin\left(\sqrt{3}t - c_3 + \frac{2\pi}{3}\right)\right]$$
$$= -c_3e^{2t} + c_2e^{-t}\cos\left(\sqrt{3}t - c_3 + \frac{4\pi}{3}\right).$$

Note. $c_1 \cos \beta x + c_2 \sin \beta x$ can be replaced by $c_1 \cos(\beta x - c_2)$ or $c_1 \cos(\beta x + c_2)$ or $c_1 \sin(\beta x - c_2)$ or $c_1 \sin(\beta x + c_2)$.

TEST YOUR KNOWLEDGE

Solve the following simultaneous equations :

1. $\frac{dx}{dt} = 5x + y, \frac{dy}{dt} = y - 4x.$ **(P.T.U., May 2014)**

2. $\frac{dx}{dt} + y = \sin t, \frac{dy}{dt} + x = \cot t$; given that $x = 2$ and $y = 0$ when $t = 0$.

3. $\frac{dx}{dt} + 4x + 3y = t, \frac{dy}{dt} + 2x + 5y = e^t.$ **(P.T.U., May 2010)**

4. $(D + 1)x + (2D + 1)y = e^t, \ (D - 1)x + (D + 1)y = 1.$

5. $\frac{dx}{dt} + 2x + 3y = 0, 3x + \frac{dy}{dt} + 2y = 2e^{2t}.$

6. $(D - 1)x + Dy = 2t + 1, \ (2D + 1)x + 2Dy = t.$

7. $\frac{d^2x}{dt^2} - 3x - 4y = 0, \frac{d^2y}{dt^2} + x + y = 0.$

8. $\frac{d^2x}{dt^2} - \frac{dy}{dt} = 2x + 2t, \frac{dx}{dt} + 4\frac{dx}{dt} = 3y.$

9. $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = \sin t, \ \frac{dx}{dt} + x - 3y = 0.$

10. A mechanical system with two degrees of freedom satisfies the equations $2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4, \ 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0.$

Obtained expressions for x and y is terms of t, given $x, y, \frac{dx}{dt}, \frac{dy}{dt}$ all vanish at $t = 0$.

11. $\frac{d^2x}{dt^2} + y = \sin t, \frac{d^2y}{dt^2} + x = \cos t.$

ANSWERS

1. $x = (c_1 + c_2 t)e^{3t}$, $y = e^{3t} (-2c_1 + c_2 - 2c_2 t)$
2. $x = e^t + e^{-t}$, $y = e^{-1} - e^t + \sin t$
3. $x = c_1 e^{-2t} + c_2 e^{-7t} + \frac{5}{14}t - \frac{31}{196} - \frac{1}{8}e^t$; $y = -\frac{2}{3}c_1 e^{-2t} + c_2 e^{-7t} - \frac{1}{7}t + \frac{9}{98} + \frac{5}{24}e^t$
4. $x = \frac{1}{2}\left[e^t - 1 - ac_1e^{at} + bc_2e^{bt}\right]$, $y = c_1e^{at} + c_2e^{bt} + \frac{1}{2}$, where $a = \frac{1}{2}(3 + \sqrt{17})$, $b = \frac{1}{2}(3 - \sqrt{17})$
5. $x = c_1e^t + c_2e^{-5t} - \frac{6}{7}e^{2t}$, $y = c_2e^{-5t} - c_1e^t + \frac{8}{7}e^{2t}$
6. $x = -t - \frac{2}{3}$, $y = \frac{1}{2}t^2 + \frac{4}{3}t + c$
7. $x = \left(c_1 + c_2t\right)e^{-t} + \left(c_3 + c_4t\right)e^t$, $y = -\frac{1}{2}\left[c_1 + c_2(1+t)\right]e^{-t} + \frac{1}{2}\left[c_4(1-t) - c_3\right]e^t$
8. $x = (c_1 + c_2t)\, e^t + c_3e^{-3/2t} - t$, $y = [c_2(3-t) - c_1]e^t - \frac{1}{6}c_3e^{-32t}$
9. $x = \frac{3}{2}c_1e^t - 3c_2e^{-2t} + c_3e^{-t} + \frac{3}{10}e^t(\cos t - 2\sin t)$; $y = c_1e^t + c_2e^{-2t} - \frac{1}{10}(\cos t + 3\sin t)$
10. $x = \frac{8}{9}\left(1 - \cos\frac{3t}{2}\right)$, $y = \frac{4}{3}t - \frac{8}{9}\sin\frac{3t}{3}$
11. $x = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t + \frac{t}{4}(\sin t - \cos t)$

    $y = -c_1e^t + c_2e^{-t} - c_3\cos t + c_4\sin t + \frac{1}{4}(2+t)(\sin t - \cos t)$.

REVIEW OF THE CHAPTER

1. **Linear Differential of *n*th Order:** A linear differential equation of *n*th order is that in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. It is of the form

   $$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} \ldots + P_n y = X,$$ where $P_1, P_2, \ldots, P_n$, X are functions of x only

   If $P_1, P_2, \ldots, P_n$ and all constants, then it is known as linear differential equation with constant coefficients.
2. If $y = y_1, y = y_2, \ldots, y = y_n$ are n linearly independent solutions of $(D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_n D^n)\, y = 0$, where D stands for $\frac{d}{dx}$, then $u = c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is also its solution (called general solution).
3. If $y = u$ is a general solution of $f(D)\, y = 0$ and $y = v$ is a particular solution of $f(D)\, y = X$, then $y = u + v$ is the complete solution of $f(D)\ y = X$.
4. **Auxiliary Equation:** In $f(D)\ y = X$, $f(D) = 0$ is called A.E. *i.e.*, A.E. is $D^n + a_1D^{n-1} + a_2D^{n-2} \ldots a_n = 0$.
5. **Rules to find Complementary Functions** of $(D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_{n-1}D + a_n)\ y = X$:
   (*i*) If the roots of the A.E. equation are real and distincts (say) m_1, m_2, m_3, then

   $$\text{C.F.} = c_1 e^{m_1x} + c_2 e^{m_2x} + c_3 e^{m_3x} \ldots$$

   (*ii*) If two roots are equal (say) $m_1 = m_2 = m$, then C.F. $= (c_1 + c_2x)\, e^{mx} + c_3\, e^{m_3x} + \ldots$

   (*iii*) If three roots are equal (say) $m_1 = m_2 = m_3 = m$, C.F. $= (c_1 + c_2x + c_3x^2)\, e^{mx} + c_4\, e^{m_4x} + \ldots$ and so on

   (*iv*) If roots are a pair of imaginary numbers (non repeated) $\alpha \pm i\beta$, C.F. $= e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

   (*v*) If pair of imaginary roots is repeated twice *i.e.*, $\alpha \pm i\beta, \alpha \pm i\beta$, C.F. $= e^{\alpha x}\{(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x\}$
6. Rule to find out Particular Integral *i.e.*, to find P.I. $= \frac{1}{f(D)}$ X, where $f(D) = D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots a_n$

(*i*) When $X = e^{ax}$, then P.I. $= \dfrac{e^{ax}}{f(a)}$ provided $f(a) \neq 0$

If $f(a) = 0$ then multiply numerator by x and differential $f(D)$ w.r.t. D and continue this process until $f(D)$ ceases to be zero at $D = a$.

(*ii*) When $X = \sin(ax + b)$ or $\cos(ax + b)$, then P.I. $= \dfrac{1}{\phi(-a^2)} \sin(ax + b)$ or $\cos(ax + b)$ provided $\phi(-a^2) \neq 0$

In case $\phi(-a^2) = 0$. Apply the same rule as discussed in (*i*).

(*iii*) When $X = x^m$ then P.I. $= [f(D)]^{-1} x^m$; expand $f(D)$ by Binomial theorem upto D^m and then operate on x^m

(*iv*) When $X = e^{ax} V$; P.I. $= e^{ax} \dfrac{1}{f(D + a)} V$.

(*v*) If X is any other function of x, then P.I. $= \dfrac{1}{f(D)} X$. Resolve $\dfrac{1}{f(D)}$ into partial fractions and operate each partial fraction on X.

(*vi*) Always remember $\dfrac{1}{D} X = \int X\, dx$ and $\dfrac{1}{D - a} X = e^{ax} \int e^{-ax} X\, dx$.

7. Method of Variation of Parameter: To find solution of $\dfrac{d^2y}{dx^2} + a_1 \dfrac{dy}{dx} + a_2 y = X$ by method of variation of parameter.

Let
$$\text{C.F.} = c_1 y_1 + c_2 y_2$$
$$\text{P.I.} = u y_1 + v y_2$$

where
$$u = -\int \frac{y_2 X}{W} dx, \quad v = \int \frac{y_1 X}{W} dy$$

where
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \text{ (called Wronskian of } y_1, y_2)$$

$$\text{C.S.} = \text{C.F.} + \text{P.I.}$$

8. Operator Method: To find solution of $P \dfrac{d^2y}{dx^2} + Q \dfrac{dy}{dx} + Ry = S$, where P, Q, R, S are functions of x, write equation in symbolic form *i.e.*, $(PD^2 + QD + R)\, y = S$. Factorise $PD^2 + QD + R$ into two linear factors and integrate in two stages. Always remember that the factors are non-commutative.

9. Cauchy's Homogeneous Equation: An equation of the form $x^n \dfrac{d^n y}{dx^n} + a_1 x^{n-1} \dfrac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \dfrac{d^{n-2} y}{dx^{n-2}} + \cdots$

$$+ a_{n-1} x \frac{dy}{dx} + a_n y = X$$

where a's are constant and X is a function of x is called Cauchy's Homogeneous equation. To solve this equation put $x = e^z$ and replace $\dfrac{dy}{dx} = Dy$. Here D stand for $\dfrac{d}{dz}$, $\dfrac{d^2y}{dx^2} = D(D - 1)\, y$; $\dfrac{d^3y}{dx^3} = D(D - 1)(D - 2)\, y$ and so on. Equation will change to linear equation with constant coefficients.

10. Legendre's Linear Equation: An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 (a+bx)^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X$$

To solve it put $a + bx = e^z$ and replace $(a + bx) \dfrac{dy}{dx} = b\text{D}y$, where $\text{D} = \dfrac{d}{dz}$

$$(a+bx)^2 \frac{d^2 y}{dx^2} = b^2\text{D}(\text{D}-1)\, y$$

$$(a+bx)^3 \frac{d^3 y}{dx^3} = b^3\text{D}(\text{D}-1)(\text{D}-2)\, y$$

Again equation will change to linear differential equation with constant coefficients.

11. Simultaneous Linear Equations with Constant Coefficients: Consider the two simultaneous equations as $f_1(\text{D})x + f_2(\text{D})y + \text{T}_1$ and $\phi_1(\text{D})x + \phi_2(\text{D})y = \text{T}_2$

where $\text{D} = \dfrac{d}{dt}$ and T_1, T_2 and functions of t. First eliminate y from the two equations, the equations will become linear differential equation with constant coefficients in x and t and can be solved. Put the value of x in any one of the two equations and get the value of y.

SHORT ANSWER TYPE QUESTIONS

1. (*a*) What do you understand by complementary function? Explain. **(P.T.U., Jan. 2010)**

   (*b*) If $y = u$ is the complete solution of the equation $f(\text{D})\, y = 0$ and $y = v$ is a particular solution of the equation $f(\text{D})\, y = \text{X}$, then the complete solution of the equation $f(\text{D})\, y = \text{X}$ is $y = u + v$.

2. Define Auxiliary Equation of a linear differential equation.

3. What is the solution of the differential equation corresponding to roots of the A.E. if

   (*i*) roots are all real and distinct?

   (*ii*) roots are imaginary and distinct?

4. Solve the following differential equations :

   (*i*) $\dfrac{d^2y}{dx^2} + (a+b)\dfrac{dy}{dx} + aby = 0$

   [**Hint :** A.E. is $\text{D}^2 + (a + b)\,\text{D} + ab = 0$ $\therefore$ $\text{D} = -a$, $\text{D} = -b$ $\therefore$ $y = c_1 e^{-ax} + c_2 e^{-bx}$]

   (*ii*) $9y''' + 3y'' - 5y' + y = 0$ [**Hint :** S.E. 1 art. 2.5] **(P.T.U., May 2008)**

   (*iii*) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \sin x$. [**Hint :** S.E. 7 art. 2.8] **(P.T.U., June 2003, May 2006, Dec. 2011)**

   (*iv*) $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0$. [**Hint :** A.E. is $(\text{D}^2 + 1)^2 = 0$, $\text{D} = \pm i, \pm i$] **(P.T.U., Dec. 2010)**

   (*v*) $\dfrac{d^3y}{dx^3} + y = 0$ **(P.T.U., May 2012)**

   (*vi*) $y'' + 2y' + 2y = 0$ **(P.T.U., Dec. 2013)**

5. Find particular solutions of the following differential equations :

   (*i*) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{e^x}$ [**Hint :** S.E. 11 art. 2.8] **(P.T.U., Dec. 2003)**

   (*ii*) $(\text{D}^2 - 2\text{D} + 4)\, y = e^x \sin x$

   (*iii*) $(\text{D}^2 - 3\text{D} + 2)\, y = 2e^x \cos \dfrac{x}{2}$. **(P.T.U., Dec. 2003)** (*iv*) $(D^3 - 3D^2 + 4)\, y = e^{2x}$. [**Hint :** S.E. 1 art. 2.7]

(*v*) $y''' - y'' + 4y' - 4y = \sin 3x$ [**Hint :** S.E. 2 art. 2.7] **(P.T.U., May 2008)**

(*vi*) $(D^2 + a^2)y = \sin ax$ **(P.T.U., May 2009)**

(*vii*) $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = \sin 2x$ **(P.T.U., Dec. 2012)**

(*viii*) $(D - 2)^2 y = \sin 2x$ **(P.T.U., May 2014)**

6. Explain method of variation of parameters to find P.I. of a differential equation. **(P.T.U., May 2004)**

7. Explain briefly the method of operator for finding solution of a linear differential equation.

8. Solve $xy'' + (x - 1)\, y' - y = 0$ by operator method.

9. Define Cauchy's homogeneous linear differential equation and give one example. **(P.T.U., Dec. 2004)**

10. Solve the following differential equations :

(*i*) $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = 0$ **(P.T.U., May 2006)**

(*ii*) $2x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 6y = 0$ [**Hint :** S.E. 1 art. 2.11] **(P.T.U., Dec. 2013)**

(*iii*) $x^2 \dfrac{d^2y}{dx^2} - 2y = x^2 + \dfrac{1}{x}$.

11. Define Legendre's linear equation and give one example. **(P.T.U., Dec. 2005, May 2007)**

12. Solve the following simultaneous linear differential equations :

(*i*) $\dfrac{dx}{dt} + 2y = e^t$; $\dfrac{dy}{dt} - 2x = e^{-t}$ [**Hint :** S.E. 2 art. 2.13]

(*ii*) $\dfrac{dx}{dt} = -2x + y$; $\dfrac{dy}{dt} = -4x + 3y + 10\cos t$. **(P.T.U., Dec. 2002)**

ANSWERS

4. (*i*) $y = c_1 e^{-ax} + c_2 e^{-bx}$ (*ii*) $y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$

(*iii*) $y = (c_1 + c_2 x)\, e^x - e^x \sin x$ (*iv*) $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

(*v*) $y = c_1 e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right)$ (*vi*) $y = e^{-x}(c_1 \cos x + c_2 \sin x)$

5. (*i*) $e^{-2x} e^{e^x}$ (*ii*) $\dfrac{1}{2} e^x \sin x$

(*iii*) $-\dfrac{8}{5} e^x \left[\cos\dfrac{x}{2} + 2\sin\dfrac{x}{2}\right]$ (*iv*) $\dfrac{x^2}{6} e^{2x}$

(*v*) $\dfrac{1}{50}(\sin 3x + 3\cos 3x)$ (*vi*) $\dfrac{-x}{2a}\cos ax$

(*vii*) $-\dfrac{x}{8}\sin 2x$ (*viii*) $\dfrac{\cos 2x}{8}$

8. $y = c_1 (x - 1) + c_2 e^{-x}$

10. (*i*) $y = \dfrac{c_1}{x} + \dfrac{c_2}{x^2}$ (*ii*) $y = c_1 x^2 + c_2 x^{-3/2}$

(*iii*) $y = c_1 x^2 + \dfrac{c_2}{x} + \dfrac{1}{3}\left(x^2 - \dfrac{1}{x}\right)\log x$

12. (*i*) $x = c_1 \cos 2t + c_2 \sin 2t + \dfrac{1}{5}e^t - \dfrac{2}{5}e^{-t}$

$y = c_1 \sin 2t - c_2 \cos 2t + \dfrac{2}{5}e^t - \dfrac{1}{5}e^{-t}$

(*ii*) $x = c_1 e^{2t} + c_2 e^{-2t} - \sin t - 3\cos t$

$y = 4c_1 e^{2t} - 3c_2 e^{-t} - 7\cos t + \sin t$.

3

Application of Ordinary Differential Equations

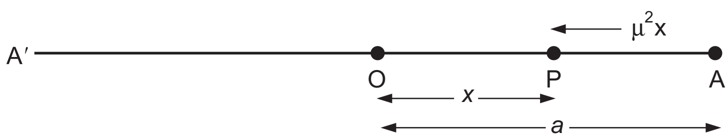
3.1. INTRODUCTION

In this chapter we shall study those physical problems which deal with the linear differential equations of first and higher order with constant coefficients. Such equations play a dominant role in unifying the theory of electrical and mechanical oscillatory system, simple harmonic motion, deflection of beam, simple pendulum and population models.

We shall begin by explaining the phenomenon of simple harmonic motion, then electric circuits, then simple pendulum then Deflection of Beam, and Population Models.

3.2. SIMPLE HARMONIC MOTION (S.H.M.)

A particle is said to execute **simple harmonic motion** if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point.



Let O be the fixed point in the line A′A. Let P be the position of the particle at any time t, where

$$OP = x.$$

Since the acceleration is always directed towards O, *i.e.*, the acceleration is in the direction opposite to that in which x increases, the equation of motion of the particle is $\dfrac{d^2x}{dt^2} = -\mu^2 x$

or $\quad (D^2 + \mu^2)\, x = 0, \quad$ where $\; D = \dfrac{d}{dt}$ ...(1)

It is a linear differential equation with constant coefficients.

Its A.E. is $\; D^2 + \mu^2 = 0\;$ so that $\; D = \pm\, i\mu$

$\therefore\;$ The solution of (1) is $\; x = c_1 \cos \mu t + c_2 \sin \mu t$ ...(2)

Velocity of particle at P $= \dfrac{dx}{dt} = -\, c_1\, \mu \sin \mu t + c_2\, \mu \cos \mu t$ ...(3)

If the particle starts from rest at A, where $\; OA = a$, then from (2), $\;$ (at $\; t = 0, \; x = a$) ; $\quad c_1 = a$

and from (3), $\; \left(\text{at } t = 0, \; \dfrac{dx}{dt} = 0\right); c_2 = 0$

$$\therefore \qquad x = a \cos \mu t \qquad ...(4)$$

and $$\frac{dx}{dt} = -a\mu \sin \mu\, t \qquad ...(5)$$

$$= -a\mu\sqrt{1-\cos^2\mu t} = -a\mu\sqrt{1-\frac{x^2}{a^2}} \qquad \left(\because \cos\mu t = \frac{x}{a}\right)$$

$$= -\mu\sqrt{a^2-x^2}\,. \qquad \ldots(6)$$

Equation (4) gives the displacement of the particle from the fixed point O at any time t.

Equation (6) gives the velocity of the particle at any time t, when its displacement from the fixed point O is x.

Equation (6) also shows that the velocity is directed towards O and decreases as x increases.

Now, equations (4) and (5) remain unaltered when t is replaced by $t + \frac{2\pi}{\mu}$, *i.e.*, when t is increased by $\frac{2\pi}{\mu}$ showing thereby that the particle occupies the same position and has the same velocity after a time $\frac{2\pi}{\mu}$.

The quantity $\frac{2\pi}{\mu}$, usually denoted by T, is called the **periodic time** *i.e.*, the time of complete oscillation.

Nature of Motion. At A, $x = a$ and $v = 0$. Since acceleration is directed towards O, the particle moves towards O. The acceleration gradually decreases and vanishes at O, when the particle has acquired maximum velocity. Thus the particle moves further towards A$'$ under retardation and comes at rest to A$'$, where OA$'$ = OA. It moves back towards O under acceleration and acquires maximum velocity at O. Thus the particle moves further towards A under retardation and comes to rest at A. It retraces its path and goes on oscillating between A at A$'$. The point O is called the **centre of motion** or the **mean position.** The maximum distance a which the particle covers on either side of the mean position is called the **amplitude** of the motion.

The number of complete oscillations per second is called the **frequency** of motion. If n is the frequency, then $\boldsymbol{n = \frac{1}{T} = \frac{\mu}{2\pi}}$.

ILLUSTRATIVE EXAMPLES

Example 1. *If the displacement of a particle in a straight line is given by x = a cos μt + b sin μt, then show that it describes S.H.M. with an amplitude* $\sqrt{a^2+b^2}$. **(P.T.U., Dec. 2013)**

Sol. $$x = a\cos\mu t + b\sin\mu t \qquad \ldots(1)$$

$$\frac{dx}{dt} = -a\mu\sin\mu t + b\mu\cos\mu t \qquad \ldots(2)$$

$$\frac{d^2x}{dt^2} = -a\mu^2\cos\mu t - b\mu^2\sin\mu t$$
$$= -\mu^2(a\cos\mu t + b\sin\mu t)$$
$$= -\mu^2 x$$

$\therefore$ Motion is simple harmonic

Let A be the amplitude of S.H.M.

then when $x = A$, $v = 0$ *i.e.*, $\frac{dx}{dt} = 0$

$\therefore$ From (2) $-a\mu\sin\mu t + b\mu\cos\mu t = 0$

$$\therefore \qquad \tan\mu t = \frac{b}{a}$$

$$\sin \mu t = \frac{b}{\sqrt{a^2+b^2}} \; ; \cos \mu t = \frac{a}{\sqrt{a^2+b^2}}$$

From (1) $$\text{A} = a \cdot \frac{a}{\sqrt{a^2+b^2}} + b \cdot \frac{b}{\sqrt{a^2+b^2}} = \frac{a^2+b^2}{\sqrt{a^2+b^2}} = \sqrt{a^2+b^2}$$

Example 2. *A particle is executing simple harmonic motion with amplitude 20 cm and time 4 seconds. Find the time required by the particle in passing between points which are at distances 15 cm and 5 cm from the centre of force and are on the same side of it.*

Sol. Here $a = 20\,\text{cm}$, $\text{T} = 4$ seconds

Since $\text{T} = \dfrac{2\pi}{\mu}$ $\therefore$ $\mu = \dfrac{\pi}{2}$

Let t_1 and t_2 seconds be the times when the particle is at distances 15 cm and 5 cm respectively from the centre of force.

Using $x = a \cos \mu t$, we have $15 = 20 \cos \dfrac{\pi}{2} t_1$

or $t_1 = \dfrac{2}{\pi} \cos^{-1} \dfrac{3}{4}$ and $5 = 20 \cos \dfrac{\pi}{2} t_2$ or $t_2 = \dfrac{2}{\pi} \cos^{-1} \dfrac{1}{4}$

Required time $= t_2 - t_1 = \dfrac{2}{\pi} \left(\cos^{-1} \dfrac{1}{4} - \cos^{-1} \dfrac{3}{4} \right) = 0.38$ sec.

Example 3. *A particle moving in a straight line with S.H.M. has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2 respectively. Show that the period of motion is $2\pi \sqrt{\dfrac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$.*

Sol. The velocity v of the particle when it is at a distance x from the mean position is given by

$v^2 = \mu^2 (a^2 - x^2)$, where a is the amplitude.

$\therefore$ $v_1^2 = \mu^2 (a^2 - x_1^2)$ ...(1) and $v_2^2 = \mu^2 (a^2 - x_2^2)$ ...(2)

Subtracting (1) from (2), we get $v_2^2 - v_1^2 = \mu^2 (x_1^2 - x_2^2)$ or $\mu^2 = \dfrac{v_2^2 - v_1^2}{x_1^2 - x_2^2}$

Periodic time $= \dfrac{2\pi}{\mu} = 2\pi \sqrt{\dfrac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$.

Example 4. *At the end of three successive seconds, the distances of a point moving with S.H.M. from its mean position are x_1, x_2, x_3 respectively. Show that the time of a complete oscillation is*

$$\frac{2\pi}{\cos^{-1}\left(\dfrac{x_1 + x_3}{2x_2} \right)}.$$ **(P.T.U., Dec. 2011)**

Sol. Let the moving point be at distances x_1, x_2, x_3 from the mean position at the end of $t, t+1, t+2$ seconds respectively.

Using $x = a \cos \mu t$, we have $x_1 = a \cos \mu t$ ...(1)

$x_2 = a \cos \mu (t+1)$ ...(2)

$x_3 = a \cos \mu (t+2)$ ...(3)

Adding (1) and (3), we get $x_1 + x_3 = a\,[\cos \mu\,(t+2) + \cos \mu t]$

$$= a\,.\,2\cos\frac{\mu(t+2)+\mu t}{2}\cos\frac{\mu(t+2)-\mu t}{2}$$

$$= 2a\cos\mu\,(t+1)\cos\mu = 2x_2\cos\mu \qquad \text{[Using (2)]}$$

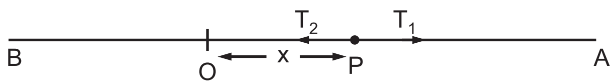
$$\Rightarrow \qquad \mu = \cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)$$

Hence the time of a complete oscillation $= \dfrac{2\pi}{\mu} = \dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1+x_3}{2x_2}\right)}$.

Example 5. *A particle of mass m executes simple harmonic motion in the line joining the points A and B on a smooth table and is connected with these points by elastic strings whose tensions in equilibrium are each T. If l and l′ be the extensions of the strings beyond their natural lengths find the time of an oscillation.*

(P.T.U., May 2013)

Sol. Let OA, OB be the two elastic strings with extensions l and l' and a particle of mass m is attached at O.



Let a be the natural length of the string OA and b be that of OB

Let P be the position of the particle at any time t

Let $\qquad \text{OP} = x$

$\text{OA} = a + l, \quad ; \quad \text{OB} = b + l'$

$\text{AP} = a + l - x \quad ; \quad \text{BP} = b + l' + x$

When particle is at P let tension in the string AP be T_1 and that in BP be T_2

Then by Hook's Law:

$$\text{Tension} = \frac{\lambda \cdot \text{extension}}{\text{natural length}}$$

Let λ_1 be the modulus of elasticity in the string AP and λ_2 be that of in string BP

Then $$T_1 = \frac{\lambda_1(\text{AP} - a)}{a}$$

$$T_2 = \frac{\lambda_2(\text{BP} - b)}{b}$$

$$\therefore \qquad T_1 = \frac{\lambda_1}{a}(a + l - x - a);\ T_2 = \frac{\lambda_2}{b}(b + l' + x - b)$$

or $$T_1 = \frac{\lambda_1}{a}(l - x);\ T_2 = \frac{\lambda_2}{b}(l' + x)$$

Since tensions are the only horizontal forces acting on the particle,

$\therefore$ Equation of motion is $m\dfrac{d^2x}{dt^2} = T_1 - T_2$

$$= \frac{\lambda_1}{a}(l - x) - \frac{\lambda_2}{b}(l' + x) \qquad \text{...(1)}$$

In equilibrium position, tension in each string is T

$$\therefore \qquad \mathrm{T} = \frac{\lambda_1 l}{a} \quad \text{and} \quad \mathrm{T} = \frac{\lambda_2 l'}{b}$$

$$\therefore \qquad \frac{\lambda_1}{a} = \frac{\mathrm{T}}{l} \quad ; \quad \frac{\lambda_2}{b} = \frac{\mathrm{T}}{l'}$$

Substitude in (1)

$$m\frac{d^2x}{dt^2} = \frac{\mathrm{T}}{l}(l-x) - \frac{\mathrm{T}}{l'}(l'+x)$$

$$= \mathrm{T} - \frac{\mathrm{T}}{l}x - \mathrm{T} - \frac{\mathrm{T}}{l'}x$$

$$= -\mathrm{T}\left(\frac{1}{l} + \frac{1}{l'}\right)x$$

or

$$\frac{d^2x}{dt^2} = -\frac{\mathrm{T}}{m}\left(\frac{1}{l} + \frac{1}{l'}\right)x$$

which shows S.H.M

$\therefore$ Time of oscillation

$$= \frac{2\pi}{\sqrt{\frac{\mathrm{T}}{m}\left(\frac{1}{l} + \frac{1}{l'}\right)}}$$

$$= \frac{2\pi\sqrt{m}}{\sqrt{\mathrm{T}\left(\frac{1}{l} + \frac{1}{l'}\right)}}.$$

TEST YOUR KNOWLEDGE

1. (*a*) A particle is executing S.H.M. with amplitude 5 metres and time 4 seconds. Find the time required by the particle in passing between points which are at distances 4 and 2 metres from the centre of force and are on the same side of it.

(*b*) A particle executing S.H.M. of amplitude 5 cm has a speed of 8 cm/sec when at a distance of 3 cm from the centre of the path. Find the period of motion of the particle. **(P.T.U., Dec. 2013)**

2. A particle of mass 4 gm vibrates through one centimetre on each side of the middle point of its making 330 complete vibrations per minute. Assuming its motion to be S.H.M. show that the maximum force upon the particle is 484 π^2 dynes.

3. A point executing S.H.M. passes through two points A and B, 2 metres apart, with the same velocity having occupied 4 seconds is passing from A to B. After another 4 seconds, it returns to B. Find the period and amplitude.

4. A particle of mass of 4 gm executing S.H.M. has velocities 8 cm/sec and 6 cm/sec respectively when it is at distances 3 cm and 4 cm from the centre of its path. Find its period and amplitude. Find also the force acting on the particle when it is a distance 1 cm from the centre.

5. At the end of three successive seconds, the distances of a point moving with S.H.M. from its mean position, measured in the same direction are 1, 5, 5. Show that the period of complete oscillation is $\frac{2\pi}{\theta}$, where $\cos\theta = \frac{3}{5}$.

6. A particle is performing S.H.M. of period T about a centre O and it passes through the position P (OP = *b*) with velocity *v* in the direction OP. Prove that the time which elapses before its return to P is $\frac{\mathrm{T}}{\pi}\tan^{-1}\left(\frac{v\mathrm{T}}{2\pi b}\right)$.

7. A particle moves with S.H.M. in a straight line. In the first second starting from rest, it travels a distance 'a' and in the next second it travels a distance 'b' in the same direction. Prove that the amplitude of motion is $\frac{2a^2}{3a-b}$.

8. An elastic string of natural length $2a$ and modulus λ is stretched between two points A and B distant $4a$ apart on a smooth horizontal table. A particle of mass m is attached to the middle of the string. Show that it can vibrate in line AB with period $\frac{2\pi}{\omega}$, where $\omega^2 = \frac{2\lambda}{am}$.

9. An elastic string of natural length $2l$ can just support a certain weight when it is stretched till its whole length is $3l$. One end of the string is now attached to a point on a smooth horizontal table and the same weight is attached to the other end. Prove that if the weight is pulled to any distance and then let go, the string will become slack again after a time $\frac{\pi}{2}\sqrt{\frac{l}{\gamma}}$.

10. In case of a stretched elastic horizontal string which has one end fixed and a particle of mass m attached to the other end. Find the equation of motion of the particle given that l is the natural length of the string and e is its elongation due to a weight mg. Also find the displacement of the particle when initially $s = s_o$, $v = 0$.

ANSWERS

1. (a) 0.33 seconds (b) π **3.** 16 sec, $\sqrt{2}$ m **4.** π sec, 5 cm, 16 dynes

10. $\frac{d^2s}{dt^2} + \frac{g}{e}s = \frac{gl}{e}$; $s = (s_o - l)\cos\left(\sqrt{\frac{g}{e}}\right)t + l$.

3.3. APPLICATION OF DIFFERENTIAL EQUATIONS TO ELECTRIC CIRCUITS

We shall consider circuits made up of

(i) Three passive elements – Resistance, Inductance, Capacitance

(ii) An active element – Voltage source which may be a battery or a generator

For the knowledge of the students we give a brief introduction of all those elements

(i) **Resistive circuit.** It is an electrical circuit in which a resistor and a source of electricity (e.m.f.) are connected in series. For example when we switch-off an electrical appliance a current 'i' will flow through the resistor and hence there will be a voltage drop across the resistor *i.e.*, the electrical potential at the two ends of the resistor will be different as some sort of cut will be dissipated through the air. The voltage drop across the resistor is proportional to instantaneous current 'i'

$\therefore$ $V_r = Ri$, where R, the constant of proportionality is called Resistance of Resistor.

(ii) **Inductive circuit.** It is the circuit consisting of an electric source and an inductor.

In inductive circuit, the voltage drop across the inductor is proportional to the instantaneous time rate of change of current 'i'

$\therefore$ $V_I = L\frac{di}{dt}$, where L, the constant of proportionality is called Inductance of the Inductor.

(iii) **Capacitative circuit.** It is the circuit consisting of a source of electrical energy and a capacitor C, which is a device that stores energy.

In capacitative circuit, the voltage drop across the capacitor is proportional to the instantaneous electric charge Q on the capacitor

$\therefore$ $V_c = \frac{1}{C}Q$, where C, the constant of proportionality is called capacitance.

3.4 BASIC RELATIONS BETWEEN ELEMENTS OF ELECTRIC CIRCUITS

From (3.2) we conclude four most important basic relations between the elements of electric circuits.

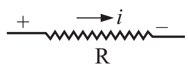
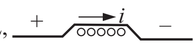
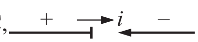
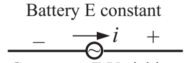
(*i*) $i = \dfrac{dQ}{dt}$ or $Q = \int i\,dt$

(*ii*) The potential voltage drop across the resistance R is Ri

(*iii*) The potential voltage drop across the induction L is $L\dfrac{di}{dt}$

(*iv*) The potential voltage drop across the capacitance C is $\dfrac{Q}{C}$.

3.5. SYMBOLS AND UNITS USED FOR THE ELEMENTS OF ELECTRIC CIRCUITS

S. No.	Element	Symbol	Unit
1.	Quantity of electricity	Q	coulomb
2.	Current	i	ampere (A)
3.	Resistance		ohm (Ω)
4.	Induction	L, 	henry (H)
5.	Capacitance	C, 	farad (F)
6.	Voltage or Electromotive force (e.m.f.)	E,  Battery E constant  Generator, E Variable	volt (V)
7.	Loop	It is any closed path formed by passing through two or more elements in series.	
8.	Nodes	Nodes are the terminals of any of these elements.	

3.6. KIRCHHOFF'S LAWS

Kirchhoff's Laws play an important role in the formation of differential equations for an electrical circuit which states as follows :

(*i*) The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit. This law is known as **Voltage Law**.

(*ii*) The algebraic sum of the currents flowing into (or from) any node is zero. This law is known as **Circuit Law.**

3.7. DIFFERENTIAL EQUATION OF AN ELECTRIC CIRCUIT IN SERIES CONTAINING RESISTANCE AND SELF INDUCTANCE (R,L SERIES CIRCUIT)

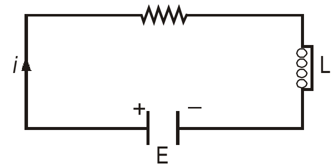
Consider a circuit containing resistance R and inductance L in series with a voltage source (battery) E. Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's voltage Law we have

$$Ri + L\frac{di}{dt} = E \qquad \text{[i.e., sum of voltage drops across R and L is E]}$$

or $$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \qquad \ldots(1)$$

which is linear differential equation of first order in i

i.e., of the type $\frac{dy}{dx} + Py = Q$



Its $$\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$$

$\therefore$ Solution is $$ie^{\frac{R}{L}t} = \int \frac{E}{L}e^{\frac{R}{L}t}\, dt + c$$

$$ie^{\frac{R}{L}t} = \frac{E}{L}\cdot\frac{e^{\frac{R}{T}t}}{\frac{R}{L}} + c$$

or $$ie^{\frac{R}{L}t} = \frac{E}{R}e^{\frac{R}{L}t} + c$$

or $$i = \frac{E}{R} + ce^{-\frac{R}{L}t} \qquad \ldots(2)$$

Initially if there is no current in the circuit *i.e.,* $i = 0$ when $t = 0$

Then $$0 = \frac{E}{R} + c \qquad \text{[From (2)]}$$

$\therefore$ $$c = -\frac{E}{R}$$

$\therefore$ (2) becomes $i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right]$, which shows that i increases with t and when $t \to \infty$; $i \to \frac{E}{R}$ *i.e., i* attains the maximum value $\frac{E}{R}$.

ILLUSTRATIVE EXAMPLES

Example 1. *A constant electromotive force E volts is applied to a circuit containing a constant resistance R ohms in series and a constant inductance L henries. If the initial current is zero, show that the current builds up to half its theoretical maximum in (L log 2)/R seconds.*

Sol. Let i be the current in the circuit at any time t.

By Kirchhoff's law, we have : $L\frac{di}{dt} + Ri = E$ or $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$ …(1)

which is Leibnit's linear differential equation, $\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$

$\therefore$ The solution of equation (1) is $i\,(\text{I.F.}) = \int \frac{E}{l}\,(\text{I.F.})\, dt + c$

or $$ie^{\frac{Rt}{L}} = \int \frac{E}{L}e^{\frac{Rt}{L}}dt + c = \frac{E}{L}\cdot\frac{L}{R}\, e^{\frac{Rt}{L}} + c$$

or $$i = \frac{E}{R} + ce^{-\frac{Rt}{L}} \qquad \ldots(2)$$

Initially, when $t = 0, i = 0$ so that $c = -\frac{E}{R}$

Thus (2) becomes, $i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$ ...(3)

This equation gives the current in the circuit at any time t.

Clearly, i increases with t and attains the maximum value $\frac{E}{R}$.

Let the current in the circuit be half its theoretical maximum after a time T seconds. Then

$$\frac{1}{2}\cdot\frac{E}{R} = \frac{E}{R}\left(1 - e^{-\frac{RT}{L}}\right) \quad \text{or} \quad e^{-\frac{RT}{L}} = \frac{1}{2} \quad \text{or} \quad -\frac{RT}{L} = \log\frac{1}{2} = -\log 2$$

$\therefore$ $T = (L \log 2)/R.$

Example 2. *The initial value problem governing the current 'i' flowing in series R.L. circuit when a voltage v (t) = t is applied is given by* $iR + L\frac{di}{dt} = t\ ;\ t \geq 0\ ;\ i(0) = 0$*, where R, L are constants. Find the current i (t) at time t.* **(P.T.U., May 2006)**

Sol. Given equation is $iR + L\frac{di}{dt} = t\ ;\ t \geq 0\ ;\ i(0) = 0$

or $$\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}t, \text{ which is a linear differential equation in } i$$

$\therefore$ $$\text{I.F.} = e^{\int\frac{R}{L}dt} = e^{\frac{R}{L}t}$$

Its solution is $$ie^{\frac{R}{L}t} = \int e^{\frac{R}{L}t}\frac{1}{L}t\,dt + c$$

$$= \frac{1}{L}\int t\,e^{\frac{R}{L}t}\,dt + c \qquad \text{(Integrate it by parts)}$$

$$= \frac{1}{L}\left[t\left(\frac{L}{R}e^{\frac{R}{L}t}\right) - (1)\left(\frac{L^2}{R^2}e^{\frac{R}{L}t}\right) + c\right] = \left(\frac{t}{R} - \frac{L}{R^2}\right)e^{\frac{R}{L}t} + c$$

$\therefore$ $$i = \frac{1}{R}t - \frac{L}{R^2} + c\,e^{-\frac{R}{L}t}$$

Given $$i = 0 \quad \text{when } t = 0 \quad \therefore\ c = \frac{L}{R^2}$$

$\therefore$ $$i = \frac{1}{R}t - \frac{L}{R^2} + \frac{L}{R^2}e^{-\frac{R}{L}t}$$

$$= \frac{1}{R}t - \frac{L}{R^2}\left[1 - e^{-\frac{R}{L}t}\right].$$

Example 3. *The equations of electromotive force in terms of current i for an electrical circuit having resistance R and a condenser of capacity C, in series, is* $E = Ri + \int\frac{i}{C}\,dt$*. Find the current i at any time t, when* $E = E_0 \sin \omega t$. **(P.T.U., Dec. 2006)**

Sol. The given equation can be written as $Ri + \int \frac{i}{C}\, dt = E_0 \sin \omega t$

Differentiating both sides w.r.t. t, we have $R\frac{di}{dt} + \frac{i}{C} = \omega E_0 \cos \omega t$

or $$\frac{di}{dt} + \frac{i}{RC} = \frac{\omega E_0}{R} \cos \omega t \qquad \ldots(1)$$

which is Leibnitz's linear equation.

$$\text{I.F.} = e^{\int \frac{1}{RC} dt} = e^{\frac{t}{RC}}$$

$\therefore$ The solution of equation (1) is

$$ie^{\frac{t}{RC}} = \int \frac{\omega E_0}{R} \cos \omega t \,.\, e^{\frac{t}{RC}}\, dt = \frac{\omega E_0}{R} \int e^{\frac{t}{RC}} \cos \omega t\, dt$$

$$= \frac{\omega E_0}{R} . \frac{e^{\frac{t}{RC}}}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}} \cos\left(\omega t - \tan^{-1} \frac{\omega}{\frac{1}{RC}}\right) + k$$

$$\left[\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos\left(bx - \tan^{-1} \frac{b}{a}\right)\right]$$

$$= \frac{\omega C E_0}{\sqrt{1 + R^2C^2\omega^2}} e^{\frac{t}{RC}} \cos(\omega t - \phi) + k \text{, where } \tan \phi = RC\omega$$

or $$i = \frac{\omega C E_0}{\sqrt{1 + R^2C^2\omega^2}} \cos(\omega t - \phi) + ke^{-\frac{t}{RC}}$$

which gives the current at any time t.

Example 4. *Solve the equation* $L\frac{di}{dt} + Ri = E_0 \sin \omega t$, *where L, R and* E_0 *are constants and discuss the case when t increases indefinitely.* **(P.T.U., May 2007)**

Sol. $$L\frac{di}{dt} + Ri = E_0 \sin \omega t \qquad \ldots(1)$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E_0}{L} \sin \omega t$$, which is linear differential equation of first order in 'i'

$$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}.$$

Its solution is

$$ie^{\frac{R}{L}t} = \int e^{\frac{R}{L}t} . \frac{E_0}{L} \sin \omega t\, dt + C$$

$$ie^{\frac{R}{L}t} = \frac{E_0}{L} \int e^{\frac{R}{L}t} \sin \omega t\, dt + C$$

$$= \frac{E_0}{L} \cdot \frac{e^{\frac{R}{L}t}}{\frac{R^2}{L^2} + \omega^2} \left[\frac{R}{L} \sin \omega t - \omega \cos \omega t \right] + C \quad \because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

$$= \frac{E_0 L e^{\frac{R}{L}t}}{R^2 + L^2 \omega^2} \left[\frac{R}{L} \sin \omega t - \omega \cos \omega t \right] + C$$

$$\therefore \qquad i = \frac{E_0 L}{R^2 + L^2 \omega^2} \left[\frac{R}{L} \sin \omega t - \omega \cos \omega t \right] + C e^{-\frac{R}{L}t}$$

Let $\frac{R}{L} = a \cos \phi, \quad \omega = a \sin \phi$

Squaring and adding

$$a^2 = \frac{R^2}{L^2} + \omega^2$$

Dividing, we get

$$\tan \phi = \frac{L\omega}{R}$$

$$\therefore \qquad i = \frac{E_0 L}{R^2 + L^2 \omega^2} . [a \cos \phi \sin \omega t - a \sin \phi \cos \omega t] + C e^{-\frac{R}{L}t}$$

$$= \frac{E_0 L}{R^2 + L^2 \omega^2} . a \sin (\omega t - \phi) + C e^{-\frac{R}{L}t}$$

$$= \frac{E_0 L}{R^2 + L^2 \omega^2} . \sqrt{\frac{R^2 + L^2 \omega^2}{L^2}} \sin \left\{ \omega t - \tan^{-1} \frac{L\omega}{R} \right\} + C e^{-\frac{R}{L}t}$$

$$i = \frac{E_0}{\sqrt{R^2 + L^2 \omega^2}} \sin \left\{ \omega t - \tan^{-1} \frac{L\omega}{R} \right\} + C e^{-\frac{R}{L}t}$$

Now, when t increases indefinitely *i.e.*, $t \to \infty$

Then $e^{-\frac{R}{L}t} \to 0 \therefore \quad i = \frac{E_0}{\sqrt{R^2 + L^2 \omega^2}} \sin \left(\omega t - \tan^{-1} \frac{L\omega}{R} \right).$

TEST YOUR KNOWLEDGE

1. When a resistance R ohms is connected in series with an inductance L henries, an e.m.f. of E volts, the current i amperes at time t is given $L\frac{di}{dt} + Ri = E$. If $E = 10 \sin t$ volts and $i = 0$, when $t = 0$, find i as a function of t.
2. A voltage Ee^{-at} is applied at $t = 0$ to a circuit containing inductance L and resistance R. Show that the current at any time t is $\frac{E}{R - aL} \left(e^{-at} - e^{-\frac{Rt}{L}} \right)$.

3. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i builds up at rate given by $L\frac{di}{dt} + Ri = E$. Find i as a function of t. How long will it be, before the current has reached one-half its maximum value if E = 6 volts, R = 100 ohms and L = 0.1 henry ?

4. Show that the differential equation for the current i in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force E sin ωt satisfies the equation. $L\frac{di}{dt} + Ri = E \sin \omega t$. Find the value of the current at any time t, if initially there is no current in the circuit. **(P.T.U., May 2006)**

[**Hint :** Consult example 4; in the result put $i = 0$ when $t = 0$, get $C = \frac{E \sin \phi}{\sqrt{R^2 + L^2 \omega^2}}$ when $\phi = \tan^{-1} \frac{L\omega}{R}$]

5. Solve the equation $L\frac{di}{dt} + Ri = 200 \cos 300\, t$, when R = 100, L = 0.05 and find i given that $i = 0$ when $t = 0$. What value does i approach after a long time?

6. A resistance R in series with inductance L is shunted by an equal resistance R with capacity C. An alternating e.m.f. sin pt produces currents i_1 and i_2 in the two branches. If initially there is no current, determine i_1 and i_2 from the equations :

$$L\frac{di_1}{dt} + Ri_1 = E \sin pt \quad \text{and} \quad \frac{i_2}{C} + R\frac{di_2}{dt} = p\, E \cos pt$$

Verify that if $R^2C = L$, the total current $i_1 + i_2$ will be $\frac{E}{R} \sin pt$.

ANSWERS

1. $i = \frac{10}{L^2 + R^2}(R \sin t - L \cos t + Le^{-\frac{Rt}{L}}$

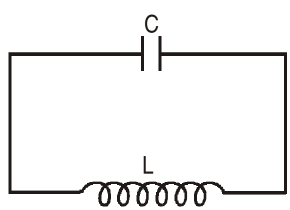
3. 0·0006931 sec

4. $i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}}\left[\sin(\omega t - \phi) + \sin \phi e^{-\frac{Rt}{L}}\right]$, where $\phi = \tan^{-1}\frac{\omega L}{R}$

5. $i = \frac{40}{409}\left[20 \cos 300t + 3 \sin 300t\right] - \frac{800}{409}e^{-200t}; \frac{40}{\sqrt{409}}$

6. $i_1 = \frac{E^2}{R^2 + L^2 p^2}\left[R \sin pt - pL \cos pt\right]; i_2 = \frac{pEC}{1 + p^2 R^2 c^2}\left[\cos pt + RC\, p \sin pt\right]$.

3.8(a). DIFFERENTIAL EQUATION OF AN ELECTRICAL OSCILLATORY CIRCUIT CONTAINING INDUCTANCE AND CAPACITANCE WITH NEGLIGIBLE RESISTANCE (L.C. CIRCUIT)



Consider an electrical circuit containing an inductance L and capacitance C. Let q be the electrical charge on the condenser plate and i be the current in the circuit at any time t. The voltage drop across L and C being $L\frac{di}{dt}$ and $\frac{q}{C}$ respectively and since there is no applied e.m.f. in the circuit, we have by Kirchhoff's Law,

$$L\frac{di}{dt}+\frac{q}{C}=0 \qquad \ldots(1)$$

Since $\quad i=\dfrac{dq}{dt}$, equation (1) becomes $L\dfrac{d^2q}{dt^2}+\dfrac{q}{C}=0$ or $\dfrac{d^2q}{dt^2}+\dfrac{q}{LC}=0$

Writing $\quad \omega^2=\dfrac{1}{LC}$, it becomes $\dfrac{d^2q}{dt^2}+\omega^2 q=0 \qquad \ldots(2)$

Equation (2) is of simple harmonic form. It represents free electrical oscillations of the current having period $\dfrac{2\pi}{\omega}=\dfrac{2\pi}{\sqrt{\dfrac{1}{LC}}}=2\pi\sqrt{LC}$.

$\therefore$ Solution of (2) is obtained as follows:

It is a linear differential equation with constant coefficients

$\therefore$ Its symbolic form is $(D^2+\omega^2)\,q=0$

A.E. is $D^2+\omega^2=0$ or $D=\pm i\omega$

$\therefore$ Solution of (2) is

$$q=c_1\cos\omega t+c_2\sin\omega t=c_1\cos\frac{1}{\sqrt{LC}}t+c_2\sin\frac{1}{\sqrt{LC}}t$$

This solution can also be put into the form

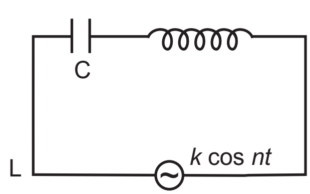
$$q=A\cos(\omega t+B)=A\cos\left(\frac{1}{\sqrt{LC}}t+B\right)$$

where c_1, c_2 or A, B can be determined from initial conditions of the problem.

3.8(b). DIFFERENTIAL EQUATION OF AN L.C. CIRCUIT WITH E.M.F. k COS nt

Consider an electrical circuit containing an inductance L and capacitance C. Let q be the electrical charge on the condenser plate and i be the current in the circuit at any time t. The voltage drops across L and C being $L\dfrac{di}{dt}$ and $\dfrac{q}{C}$ respectively and since e.m.f. $= k\cos nt$ $\therefore$ by Kirchhoff's Law

$$L\frac{di}{dt}+\frac{q}{C}=k\cos nt$$



Since $\quad i=\dfrac{dq}{dt} \quad \therefore \quad L\dfrac{d^2q}{dt^2}+\dfrac{q}{C}=k\cos nt$

or $$\frac{d^2q}{dt^2}+\frac{1}{LC}q=\frac{k}{L}\cos nt$$

Writing $\quad \omega^2=\dfrac{1}{LC}$ and $E=\dfrac{k}{L}$, we have

$\dfrac{d^2q}{dt^2}+\omega^2 q=E\cos nt$, which is a linear differential equation with constant coefficients $\quad \ldots(1)$

S.F. is $(D^2+\omega^2)\,q=E\cos nt$

A.E. $D^2 + \omega^2 = 0 \quad \therefore \quad D = \pm \omega i$

$$\text{C.F.} = c_1 \cos \omega t + c_2 \sin \omega t$$

$$\text{P.I.} = \frac{1}{D^2 + \omega^2} E \cos nt \qquad \ldots(2)$$

Put $D^2 = -n^2$

$$= \frac{1}{\omega^2 - n^2} E \cos nt \text{ when } \omega^2 \neq n^2$$

Case (*i*) When $\omega^2 \neq n^2$

$\therefore$ Solution of (1) is

$$q = c_1 \cos \omega t + c_2 \sin \omega t + \frac{1}{\omega^2 - n^2} E \cos nt$$

$c_1 \cos \omega t + c_2 \sin \omega t$ can be put into the form $A \cos (\omega t + B)$

$$\therefore \qquad q = A \cos (\omega t + B) + \frac{E}{\omega^2 - n^2} \cos nt$$

Substituting the values of ω and E

$$q = A \cos \left(\frac{t}{\sqrt{LC}} + B \right) + \frac{k}{L\left(\frac{1}{LC} - n^2 \right)} \cos nt$$

$$= A \cos \left(\frac{t}{\sqrt{LC}} + B \right) + \frac{k\,C}{1 - LCn^2} \cos nt$$

Case (*ii*) When $\omega^2 = n^2$

$\text{P.I.} = \dfrac{t}{2D} E \cos \omega t$ [*i.e.,* in (2) multiply numerator by t and differentiate the denominator w.r.t. D]

$$= \frac{t}{2} E \int \cos \omega t \, dt = \frac{t}{2} E \frac{\sin \omega t}{\omega} = \frac{Et}{2\omega} \sin \omega t$$

$\therefore$ C.S. of (1) is

$$q = c_1 \cos \omega t + c_2 \sin \omega t + \frac{Et}{2\omega} \sin \omega t$$

$$= c_1 \cos \omega t + \left(c_2 + \frac{Et}{2\omega} \right) \sin \omega t$$

which can again be reduced to the form

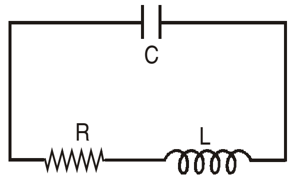
$$= r \sin (\omega t + \phi) \text{ by putting } c_1 = r \sin \phi \left(c_2 + \frac{Et}{2\omega} \right)$$

$$= r \cos \phi$$

where $r = \sqrt{c_1^2 + \left(c_2 + \dfrac{Et}{2\omega} \right)^2}$, $\tan \phi = \dfrac{c_1}{c_2 + \dfrac{Et}{2\omega}}$.

3.9(a). DIFFERENTIAL EQUATION OF ELECTRICAL SERIES CIRCUITS CONTAINING INDUCTION, CAPACITANCE AND RESISTANCE (L.C.R. CIRCUIT) WITH NEGLIGIBLE E.M.F. (P.T.U., Dec 2006, May 2007)

L.C.R. Circuit : Consider the discharge of a condenser C through an inductance L and the resistance R. Let q be the charge and i the current in the circuit at any time t. The voltage drop across L, C and R are respectively. $L\frac{di}{dt}, \frac{q}{C}, Ri$, which are same as $L\frac{d^2q}{dt^2}, \frac{q}{C}$ and $R\frac{dq}{dt}$ $\because\ i = \frac{dq}{dt}$



$\therefore$ By Kirchhoff's Law, we have

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \quad \text{or} \quad \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0$$

Writing $\frac{R}{L} = 2p$ and $\frac{1}{LC} = \omega^2$, it becomes $\frac{d^2q}{dt^2} + 2p\frac{dq}{dt} + \omega^2 q = 0$ ...(1)

which is a linear differential equation with constant coefficients

S.F. is $(D^2 + 2pD + \omega^2)\, q = 0$

A.E. is $D^2 + 2pD + \omega^2 = 0$

$$D = \frac{-2p \pm \sqrt{4p^2 - 4\omega^2}}{2} = -p \pm \sqrt{p^2 - \omega^2}$$

(1) If $p^2 > \omega^2$, then solution is

$$q = c_1 e^{\left(-p + \sqrt{p^2 - \omega^2}\right)t} + c_2 e^{\left(-p - \sqrt{p^2 - \omega^2}\right)t}$$

$$= e^{-pt}\left[c_1 e^{\sqrt{p^2 - \omega^2}\, t} + c_2 e^{-\sqrt{p^2 - \omega^2}\, t}\right]$$

$$= e^{-\frac{R}{2L}t}\left[c_1 e^{\left(\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t} + c_2 e^{-\left(\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\phi\right)t}\right]$$

$$q = e^{-\frac{R}{2L}t}\left[c_1 e^{\frac{1}{2L}\left(\sqrt{R^2 - \frac{4L}{C}}\right)t} + c_2 e^{-\frac{1}{2L}\left(\sqrt{R^2 - \frac{4L}{C}}\right)t}\right]$$

If $p^2 < \omega^2$, then $D = -p \pm i\sqrt{\omega^2 - p^2} = \alpha \pm i\beta$ type

$$\alpha = -p,\ \beta = \sqrt{\omega^2 - p^2}$$

$\therefore$ Solution is

$$q = e^{-pt}\left[c_1 \cos\left(\sqrt{\omega^2 - p^2}\right)t + c_2 \cos\left(\sqrt{\omega^2 - p^2}\right)t\right]$$

$$= e^{-\frac{R}{2L}t}\left[c_1 \cos\left(\frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}\right)t + c_2 \sin\left(\frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}\right)t\right]$$

It $p^2 = \omega^2$, then both roots of (1) are equal and each $= -p$

Then solution is

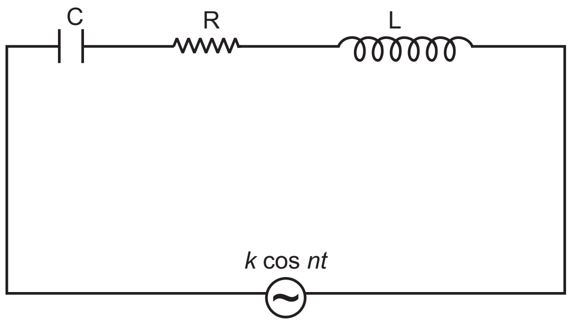
$$q = (c_1 + c_2 t)\, e^{-pt} = (c_1 + c_2 t)\; e^{-\frac{R}{2L}t}$$

3.9(b). DIFFERENTIAL EQUATION OF L.C.R. CIRCUIT WITH E.M.F. k COS nt

Consider an electrical circuit consisting of induction L, capacitance C and resistance R, which contains an alternating e.m.f. $k \cos nt$.

Let q be the charge and i the current in the circuit at any time t. The voltage drop across L, C and R are respectively $L\dfrac{di}{dt}, \dfrac{q}{C}, Ri$, which are same as $L\dfrac{d^2q}{dt^2}, \dfrac{q}{C}, R\dfrac{dq}{dt}$ as $i = \dfrac{dq}{dt}$

By Kirchhoff's Law we have



$$L\frac{di}{dt} + \frac{q}{C} + Ri = k \cos nt \quad \ldots(1)$$

or $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = k \cos nt$$

or $$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{k \cos nt}{L}$$

Writing $\dfrac{R}{L} = 2p, \dfrac{1}{LC} = \omega^2$ and $\dfrac{k}{L} = E$

$$\frac{d^2q}{dt^2} + 2p\frac{dq}{dt} + \omega^2 q = E \cos nt \qquad \ldots(2)$$

which is a linear differential equation with constant coefficients

S.F. is $(D^2 + 2pD + \omega^2)\, q = E \cos nt$

A.E. is $D^2 + 2pD + \omega^2 = 0$

$$\therefore \qquad D = \frac{-2p \pm \sqrt{4p^2 - 4\omega^2}}{2} = -p \pm \sqrt{p^2 - \omega^2}$$

$$\text{C.F.} = c_1 e^{\left(-p + \sqrt{p^2 - \omega^2}\right)t} + c_1 e^{\left(-p + \sqrt{p^2 - \omega^2}\right)t} \text{ if } p^2 > \omega^2$$

or $$\text{C.F.} = e^{-pt}\left[c_1 e^{\left(\sqrt{p^2 - \omega^2}\right)t} + c_2 e^{-\left(\sqrt{p^2 - \omega^2}\right)t}\right] \text{if } p^2 > \omega^2$$

or $$\text{C.F.} = e^{-pt}\left[c_1 \cos\sqrt{\omega^2 - p^2}\, t + c_2 \sin\sqrt{\omega^2 - p^2 t}\right] \text{if } p^2 < \omega^2$$

or $$\text{C.F.} = (c_1 + c_2 t)\, e^{-pt} \text{ if } p^2 = \omega^2 \qquad \ldots(3)$$

$$\text{P.I.} = \frac{1}{D^2 + 2pD + \omega^2} E \cos nt$$

Put $$D^2 = -n^2$$

$$= E\frac{1}{-n^2 + 2pD + \omega^2} \cos nt = E\frac{1}{2pD + (\omega^2 - n^2)} \cos nt$$

$$= \text{E}\,\frac{2p\text{D} - (\omega^2 - n^2)}{4p^2\text{D}^2 - (\omega^2 - n^2)^2}\cos nt$$

Put $\quad \text{D}^2 = -n^2$

$$= \text{E}\,\frac{2p\text{D}(\cos nt) - (\omega^2 - n^2)\cos nt}{-4p^2n^2 - (\omega^2 - n^2)^2} = \text{E}\,\frac{-2pn\sin nt - (\omega^2 - n^2)\cos nt}{-\left[4p^2n^2 + (\omega^2 - n^2)^2\right]}$$

$$= \text{E}\,\frac{2pn\sin nt + (\omega^2 - n^2)\cos nt}{4p^2n^2 + (\omega^2 - n^2)^2}$$

For simplification put $2pn = r\sin\phi$ and $\omega^2 - n^2 = r\cos\phi$

$$\therefore \qquad r^2 = 4p^2n^2 + (\omega^2 - n^2)^2 \text{ and } \tan\phi = \frac{2pn}{\omega^2 - n^2}$$

$$\therefore \qquad \text{P.I.} = \text{E}\,\frac{r\sin\phi\sin nt + r\cos\phi\cos nt}{r^2} = \frac{\text{E}}{r}\cos(nt - \phi)$$

$$= \frac{\text{E}}{\sqrt{4p^2n^2 + (\omega^2 - n^2)^2}}\cos\left(nt - \tan^{-1}\frac{2pn}{\omega^2 - n^2}\right) \qquad \ldots(4)$$

$\therefore$ Solution of (1) is

$$q = \text{C.F.} + \text{P.I.}$$

Substitute the value of C.F. and P.I. from (3) and (4) respectively.

ILLUSTRATIVE EXAMPLES

Example 1. *The voltage V and the current i at a distance x from the sending end of the transmission line satisfy the equations* $-\dfrac{dV}{dx} = Ai\,;\ \dfrac{-di}{dx} = BV$, *where A, B are constants. If* $V = V_0$ *at the sending end (x = 0) and V = 0 at the receiving end (x = l), then show that* $V = V_0\left\{\dfrac{\sinh n(l-x)}{\sinh nl}\right\}$, *where* $n^2 = AB$.

(P.T.U., May 2010)

Sol. Given equations are

$$-\frac{d\text{V}}{dx} = \text{A}i\,;\ -\frac{di}{dx} = \text{BV}$$

$$\Rightarrow \qquad i = -\frac{1}{\text{A}}\frac{d^2\text{V}}{dx^2} \qquad \therefore\ -\frac{d}{dx}\left(-\frac{1}{\text{A}}\frac{d\text{V}}{dx}\right) = \text{BV}$$

$$\text{or} \qquad \frac{1}{\text{A}}\frac{d^2\text{V}}{dx^2} = \text{BV or } \frac{d^2\text{V}}{dx^2} - \text{ABV} = 0$$

$$\text{or} \qquad \frac{d^2\text{V}}{dx^2} - n^2\text{V} = 0 \qquad (\text{given AB} = n^2)$$

which is a linear differential equation with constant coefficients

$\therefore$ Its Auxiliary equation is

$$D^2 - n^2 = 0 \quad \therefore \quad D = \pm n$$

Its solution is $\quad V = c_1 e^{nx} + c_2 e^{-nx}$ ...(1)

When $\quad x = 0, V = V_0$, then from (1) $V_0 = c_1 + c_2$

When $\quad x = l, V = 0$, then $0 = c_1 e^{nl} + c_2 e^{-nl}$

Solve the equations for c_1 and c_2, we get

$$c_1 = -\frac{V_0 e^{-nl}}{e^{nl} - e^{-nl}}, \qquad c_2 = \frac{V_0 e^{nl}}{e^{nl} - e^{-nl}}$$

$$= -\frac{V_0 e^{-nl}}{2 \sinh nl} \qquad\qquad = \frac{V_0 e^{nl}}{2 \sinh nl}$$

Substituting the values of c_1 and c_2 in (1)

$$V = \frac{V_0}{2 \sinh nl}\left[-e^{nx}.e^{-nl} + e^{-nx}.e^{nl}\right] = \frac{V_0}{2 \sinh nl}\left[-e^{-n(l-x)} + e^{n(l-x)}\right]$$

$$= \frac{V_0}{2 \sinh nl}\left[e^{n(l-x)} - e^{-n(l-x)}\right] = \frac{V_0}{2 \sinh nl} 2 \sinh n(l-x)$$

Hence $\quad V = \dfrac{V_0 \sinh n(l-x)}{\sinh nl}.$

Example 2. *Show that the frequency of free vibrations in a closed electrical circuit with inductance L and capacity C in series is* $\dfrac{30}{\pi\sqrt{LC}}$ *per minute.* **(P.T.U., Dec. 2001, May 2010)**

Sol. Let i be the current and q the charge in the condenser plate at any time t. The voltage drops across L and C are $L\dfrac{di}{dt} = L\dfrac{d^2q}{dt^2}$ and $\dfrac{q}{C}$ respectively

Since there is no e.m.f. in the circuit $\therefore$ by Kirchhoff's Law

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0 \quad \text{or} \quad \frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

Writing $\quad \dfrac{1}{LC} = \omega^2$ we have $\dfrac{d^2q}{dt^2} + \omega^2 q = 0$ or $\dfrac{d^2q}{dt^2} = -\omega^2 q$

It represents oscillatory current with period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\sqrt{\dfrac{1}{LC}}} = 2\pi\sqrt{LC}$

$\therefore$ Frequency $= \dfrac{1}{T}$ per second $= \dfrac{60}{2\pi\sqrt{LC}}$ per minute $= \dfrac{30}{\pi\sqrt{LC}}$ per minute.

Example 3. *An electric circuit consists of an inductance of 0.1 henry, a resistance of 20 ohms and a condenser of capacitance 25 micro-farads. Find the charge q and the current i at any time t, given that at* $t = 0,\ q = 0.05$ *coulomb,* $i = \dfrac{dq}{dt} = 0$ *when* $t = 0$.

Sol. The differential equation for the circuit can be written as $\mathrm{L}\dfrac{d^2q}{dt^2} + \mathrm{R}\dfrac{dq}{dt} + \dfrac{q}{\mathrm{C}} = 0$

or $$0.1\frac{d^2q}{dt^2} + 20\frac{dq}{dt} + \frac{q}{25 \times 10^{-6}} = 0 \qquad [\because\ 1 \text{ micro-farad} = 10^{-6} \text{ farads}]$$

or $$\frac{d^2q}{dt^2} + 200\frac{dq}{dt} + 400{,}000\, q = 0 \qquad \ldots(1) \quad \left|\ \because\ \frac{1}{25 \times 10^{-7}} = \frac{10000000}{25} = 400{,}000\right.$$

Its A.E. is $\mathrm{D}^2 + 200\mathrm{D} + 400{,}000 = 0$

$$\mathrm{D} = \frac{-200 \pm \sqrt{40000 - 1600000}}{2} = \frac{-200 \pm \sqrt{-1560000}}{2}$$

$$= \frac{-200 \pm 200\sqrt{39}\, i}{2} = -100 \pm 100\sqrt{39}\, i$$

$\therefore$ Its solution is $$q = e^{-100t}\left[c_1 \cos\left(100\sqrt{39}\, t\right) + c_2 \sin\left(100\sqrt{39}\, t\right)\right] \qquad \ldots(2)$$

Differentiating w.r.t. t, we have $$\frac{dq}{dt} = -100e^{-100t}\left[c_1 \cos\left(100\sqrt{39}\, t\right) + c_2 \sin\left(100\sqrt{39}\, t\right)\right]$$
$$+\ e^{-100t}\left[-100\sqrt{39}\, c_1 \sin\left(100\sqrt{39}\, t\right) + 100\sqrt{39}\, c_2 \cos\left(100\sqrt{39}\, t\right)\right] \qquad \ldots(3)$$

Since $q = 0.05$ when $t = 0$, $\therefore$ From (2), $c_1 = 0.05$

Also, $\dfrac{dq}{dt} = 0$ when $t = 0$, $\therefore$ From (3), $0 = -100\, c_1 + 100\sqrt{39}\, c_2$

or $$c_2 = \frac{c_1}{\sqrt{39}} = \frac{0.05}{\sqrt{39}} = 0.008$$

Hence $$q = e^{-100t}\left[0.05 \cos(624.5t) + 0.008 \sin(624.5t)\right]$$

and $$i = \frac{dq}{dt} = -100e^{-100t}\left[\left(c_1 - \sqrt{39}\, c_2\right)\cos(624.5t) + \left(\sqrt{39}\, c_1 + c_2\right)\sin(624.5t)\right]$$
$$= -0.32\, e^{-100t} \sin(624.5t). \qquad \left(\because\ \sqrt{39}\, c_2 = c_1\right)$$

Example 4. *Solve the differential equation* $L\dfrac{di}{dt} + \dfrac{1}{C}\displaystyle\int i\, dt = 0$, *which means that self inductance and capacitance in a circuit neutralize each other. Determine the constants in such a way that* i_0 *is the maximum current and* $i = 0$ *when* $t = 0$. **(P.T.U., Dec. 2011)**

Sol. $\mathrm{L}\dfrac{di}{dt} + \dfrac{1}{\mathrm{C}}\displaystyle\int i\, dt = 0$

Differentiate $L\dfrac{d^2i}{dt^2}+\dfrac{i}{C}=0$ or $\dfrac{d^2i}{dt^2}+\dfrac{i}{LC}=0$

which is linear differential equation with constant coefficients

$\therefore$ A.E. is $D^2+\dfrac{1}{LC}=0 \quad \Rightarrow \quad D=\pm i\dfrac{1}{\sqrt{LC}}$

Its solution is

$$i = c_1\cos\frac{1}{\sqrt{LC}}\,t + c_2\sin\frac{1}{\sqrt{LC}}\,t$$

When $\quad t=0,\ i=0 \ \Rightarrow c_1=0 \ \therefore \ i=c_2\sin\dfrac{1}{\sqrt{LC}}t \qquad \ldots(1)$

Now, Max value of $i=i_0$. Also $\sin\dfrac{1}{\sqrt{LC}}\,t$ has max value 1

$\therefore \quad i_0=c_2 \ \therefore$ From (1) solution is $i=i_0\sin\dfrac{1}{\sqrt{LC}}t$.

Example 5. *An uncharged condenser of capacity C is charged by applying an e.m.f. E* $\sin\dfrac{t}{\sqrt{LC}}$, *through leads of self-inductance L and negligible resistance. Prove that at time t, the charge on one of the plates is* $\dfrac{EC}{2}\left[\sin\dfrac{t}{\sqrt{LC}}-\dfrac{t}{\sqrt{LC}}\cos\dfrac{t}{\sqrt{LC}}\right]$.

Sol. Let q be the charge on the condenser at any time t. The differential equation for the circuit is

$$L\frac{d^2q}{dt^2}+\frac{q}{C}=E\sin\frac{1}{\sqrt{LC}} \qquad \ldots(1)$$

Its A.E. is $LD^2+\dfrac{1}{C}=0$ or $D^2=-\dfrac{1}{LC}$ so that $D=\pm\dfrac{i}{\sqrt{LC}}$

$$\text{C.F.} = c_1\cos\frac{t}{\sqrt{LC}}+c_2\sin\frac{t}{\sqrt{LC}}$$

and $$\text{P.I.} = \frac{1}{LD^2+\frac{1}{C}}\,E\sin\frac{t}{\sqrt{LC}}$$

Put $\quad D^2=-\dfrac{1}{LC}$; Case of failure

$$\therefore \quad \text{P.I.} = Et\,.\,\frac{1}{2LD}\sin\frac{t}{\sqrt{LC}}=\frac{Et}{2L}\left(-\sqrt{LC}\cos\frac{t}{\sqrt{LC}}\right)=-\frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}}$$

$\therefore$ The complete solution of (1) is $q=c_1\cos\dfrac{t}{\sqrt{LC}}+c_2\sin\dfrac{t}{\sqrt{LC}}-\dfrac{Et}{2}\sqrt{\dfrac{C}{L}}\cos\dfrac{t}{\sqrt{LC}} \qquad \ldots(2)$

Initially, when $\quad t=0,\ q=0 \ \therefore \ c_1=0$

Differentiating (2) w.r.t. t

$$\frac{dq}{dt}=-\frac{c_1}{\sqrt{LC}}\sin\frac{t}{\sqrt{LC}}+\frac{c_2}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}-\frac{E}{2}\sqrt{\frac{C}{L}}\left[\cos\frac{t}{\sqrt{LC}}-\frac{t}{\sqrt{LC}}\sin\frac{t}{\sqrt{LC}}\right]$$

Initially, $\frac{dq}{dt}=i=0$ when $t=0$

$\therefore \quad \frac{c_2}{\sqrt{LC}}-\frac{E}{2}\sqrt{\frac{C}{L}}=0$ or $c_2=\frac{EC}{2}$.

Substituting the values of c_1 and c_2 in equation (2), the charge q on the condenser plate, at any time t, is given by $q=\frac{EC}{2}\sin\frac{t}{\sqrt{LC}}-\frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}}$ or $q=\frac{EC}{2}\left[\sin\frac{t}{\sqrt{LC}}-\frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right]$.

Example 6. *In an L-C-R circuit, the charge q on a plate of the condenser is given by*

$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E\sin\omega t$, *where* $i=\frac{dq}{dt}$. *The circuit is turned to resonance so that*

$$\omega^2=\frac{1}{LC}.\ \text{If } R^2<\frac{4L}{C};\quad q=0=i.$$

when $t=0$, *show that* $q=\frac{E}{R\omega}\left[-\cos\omega t+e^{-\frac{Rt}{2L}}\left(\cos pt+\frac{R}{2Lp}\sin pt\right)\right]$

and $i=\frac{E}{R}\left[\sin\omega t-\frac{1}{p\sqrt{LC}}e^{-\frac{Rt}{2L}}\sin pt\right]$, *where* $p^2=\frac{1}{LC}-\frac{R^2}{4L^2}$. **(P.T.U., Jan. 2010, Dec. 2013)**

Sol. The given differential equation is $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E\sin\omega t$

or $\left(LD^2+RD+\frac{1}{C}\right)q=E\sin\omega t$, where $D=\frac{d}{dt}$ ...(1)

Its A.E. is $LD^2+RD+\frac{1}{C}=0$ so that $D=\frac{-R\pm\sqrt{R^2-\frac{4L}{C}}}{2L}=\frac{-R\pm i\sqrt{\frac{4L}{C}-R^2}}{2L}$, since $R^2<\frac{4L}{C}$

$$=-\frac{R}{2L}\pm i\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}=\frac{-R}{2L}\pm ip,\ \text{since}\ \frac{1}{LC}-\frac{R^2}{4L^2}=p^2$$

Its C.F. $=e^{-\frac{Rt}{2L}}(c_1\cos pt+c_2\sin pt)$

and P.I. $=\frac{1}{LD^2+RD+\frac{1}{C}}E\sin\omega t$

$$=E\cdot\frac{1}{-L\omega^2+RD+\frac{1}{C}}\sin\omega t=\frac{E}{R}\cdot\frac{1}{D}\sin\omega t,\ \text{since}\ \omega^2=\frac{1}{LC}$$

$$= -\frac{E}{R\omega} \cos \omega t$$

$\therefore$ The complete solution of equation (1) is

$$q = e^{-\frac{Rt}{2L}} \left(c_1 \cos pt + c_2 \sin pt\right) - \frac{E}{R\omega} \cos \omega t \qquad \ldots(2)$$

Initially, when $t = 0,\ q = 0 \ \therefore \ c_1 = \dfrac{E}{R\omega}$

Differentiating (2) w.r.t. t

$$\frac{dq}{dt} = e^{-\frac{Rt}{2L}} \left(-pc_1 \sin pt + pc_2 \cos pt\right) - \frac{R}{2L} e^{-\frac{Rt}{2L}} \left(c_1 \cos pt + c_2 \sin pt\right) + \frac{E}{R} \sin \omega t$$

Initially, when $t = 0,\ \dfrac{dq}{dt} = i = 0$

$$\therefore \quad pc_2 - \frac{R}{2L} c_1 = 0 \quad \text{or} \quad c_2 = \frac{R}{2pL} \cdot \frac{E}{R\omega} = \frac{E}{2pL\omega}$$

Substituting the values of c_1 and c_2 in equation (2), we get

$$q = e^{-\frac{Rt}{2L}} \left(\frac{E}{R\omega} \cos pt + \frac{E}{2pL\omega} \sin pt\right) - \frac{E}{R\omega} \cos \omega t$$

or
$$q = \frac{E}{R\omega} \left[-\cos \omega t + e^{-\frac{Rt}{2L}} \left(\cos pt + \frac{R}{2Lp} \sin pt\right)\right] \qquad \ldots(3)$$

Differentiating (3) w.r.t. t

$$\frac{dq}{dt} = \frac{E}{R\omega} \left[\omega \sin \omega t - \frac{R}{2L} e^{-\frac{Rt}{2L}} \left(\cos pt + \frac{R}{2Lp} \sin pt\right) + e^{-\frac{Rt}{2L}} \left(-p \sin pt + \frac{R}{2L} \cos pt\right)\right]$$

or
$$i = \frac{E}{R\omega} \left[\omega \sin \omega t - e^{-\frac{Rt}{2L}} \left(\frac{R^2}{4L^2 p} + p\right) \sin pt\right] = \frac{E}{R\omega} \left[\omega \sin \omega t - e^{-\frac{Rt}{2L}} \cdot \frac{R^2 + 4L^2 p^2}{4L^2 p} \sin pt\right]$$

$$= \frac{E}{R} \left[\sin \omega t - e^{-\frac{Rt}{2L}} \frac{1}{LC\, p\omega} \sin pt\right], \text{ since } \frac{R^2 + 4L^2 p^2}{4L^2} = \frac{1}{LC}$$

$$= \frac{E}{R} \left[\sin \omega t - e^{-\frac{Rt}{2L}} \cdot \frac{\sqrt{LC}}{LCp} \sin pt\right], \text{ since } \omega^2 = \frac{1}{LC}$$

or
$$i = \frac{E}{R} \left[\sin \omega t - \frac{1}{p\sqrt{LC}} e^{-\frac{Rt}{2L}} \sin pt\right].$$

Example 7. *A constant E (e.m.f.) at c = 0 is applied to a circuit consisting of an inductance L, resistance R and capacitance C in series. The initial values of the current and the charge being zero, find the current at any time t, if* $CR^2 < 4L$. *Show that the amplitudes of the successive vibrations are in geometrical progression.*

Sol. The differential equation for the circuit is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$

S.F. is $\left(LD^2 + RD + \frac{1}{C}\right) q = E$

A.E. is $LD^2 + RD + \frac{1}{C} = 0 \quad \therefore \quad D = \dfrac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$

Given $R^2 < \frac{4L}{C} \quad \therefore \; D = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

$$D = -\frac{R}{2L} \pm i\,p, \text{ where } p^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

$$\text{C.F.} = e^{-\frac{R}{2L}t}\left[c_1 \cos pt + c_2 \sin pt\right]$$

$$\text{P.I} = \frac{1}{LD^2 + RD + \frac{1}{C}} E = \frac{1}{LD^2 + RD + \frac{1}{C}} E\; e^{0.t} \quad \text{put } t = 0$$

$$= \frac{E}{\frac{1}{C}} e^{0\cdot t} = EC$$

$\therefore$ C.S. is $\quad q = e^{-\frac{Rt}{2L}}\left[c_1 \cos pt + c_2 \sin pt\right] + EC$

Now, when $\quad t = 0,\; q = 0 \;\therefore\; 0 = c_1 + EC$

$\therefore \quad c_1 = -EC$

$\therefore \quad q = e^{-\frac{R}{2L}t}\left[-EC \cos pt + c_2 \sin pt\right] + EC$

Now, $\quad i = \dfrac{dq}{dt} = e^{-\frac{R}{2L}t}\left[EC\, p \sin pt + c_2\, p \cos pt\right] - \dfrac{R}{2L} e^{-\frac{R}{2L}t}\left[-EC \cos pt + c_2 \sin pt\right]$

When $\quad t = 0,\; i = 0$

$\therefore \quad 0 = c_2\, p + \dfrac{R}{2L} EC \quad \therefore \quad c_2 = -\dfrac{REC}{2Lp}$

$$\therefore \qquad i = e^{-\frac{R}{2L}t}\left[ECp\sin pt - \frac{REC}{2L}\cos pt\right] - \frac{R}{2L}e^{-\frac{R}{2L}t}\left[-E\cos pt - \frac{REC}{2LP}\sin pt\right]$$

$$i = e^{-\frac{R}{2L}t}\left[ECp\sin pt - \frac{REC}{2L}\cos pt + \frac{REC}{2L}\cos pt + \frac{R^2EC}{4L^2p}\sin pt\right]$$

$$i = e^{-\frac{R}{2L}t}\left[ECp\sin pt + \frac{R^2EC}{4L^2p}\sin pt\right] = EC\,e^{-\frac{R}{2L}t}\left[p + \frac{R^2}{4L^2p}\right]\sin pt$$

$$= EC\frac{4L^2p^2 + R^2}{4L^2p}e^{-\frac{R}{2L}t}\sin pt$$

Substituting the values of p^2

$$= \frac{EC}{4L^2p}\cdot\left[4L^2\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right) + R^2\right]e^{-\frac{R}{2L}t}\sin pt$$

$$= \frac{EC}{4L^2p}\left[\frac{4L}{C}\right]e^{-\frac{R}{2L}t}\sin pt$$

$\therefore \quad i = \dfrac{E}{Lp}e^{-\frac{R}{2L}t}\sin pt$, where amplitude of successive vibration is given by $\dfrac{E}{Lp}e^{-\frac{R}{2L}t}$, which decreases as t increases.

$\therefore$ Amplitudes are $\dfrac{E}{LP}e^{-\frac{R}{2L}} + \dfrac{E}{LP}e^{-\frac{R}{2L}\cdot 2} + \dfrac{E}{LP}e^{-\frac{R}{2L}\cdot 3} + \cdots$

which forms a G.P. series.

Example 8. *An e.m.f. E sin pt is applied at t = 0 to a circuit containing a capacitance C and inductance L. The current i satisfies the equation* $L\dfrac{di}{dt} + \dfrac{1}{C}\displaystyle\int i\,dt = E\sin pt$. *If* $p^2 = \dfrac{1}{LC}$ *and initially the current i and the charge q are zero, show that the current at time t is* $\dfrac{Et}{2L}\sin pt$, *where* $i = \dfrac{dq}{dt}$.

(P.T.U., Dec. 2003, 2012, 2013, May 2012)

Sol. $L\dfrac{di}{dt} + \dfrac{1}{C}\displaystyle\int i\,dt = E\sin pt$

Substituting the value of $i = \dfrac{dq}{dt}$

$$\frac{Ld^2q}{dt^2} + \frac{1}{C}\int\frac{dq}{dt}dt = E\sin pt$$

or
$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = E\sin pt$$

S.F. is $\left(LD^2 + \dfrac{1}{C}\right)q = E\sin pt$

A.E. is $\quad LD^2 + \frac{1}{C} = 0 \quad \therefore \quad D = \pm\, i\frac{1}{\sqrt{LC}} = \pm\, i\,p$, where $p = \frac{1}{\sqrt{LC}}$

$$\text{C.F.} = c_1 \cos pt + c_2 \sin pt$$

$$\text{P.I.} = \frac{1}{LD^2 + \frac{1}{C}} E \sin pt \quad \text{Put } D^2 = -p^2$$

$$= \frac{1}{-Lp^2 + \frac{1}{C}} E \sin pt = \frac{EC \sin pt}{1 - LC\,p^2} \quad \text{But } p^2 = \frac{1}{LC}$$

$\therefore$ Case of failure

$$\therefore \quad \text{P.I.} = \frac{t}{2LD} E \sin pt \qquad \text{[Multiply by } t \text{ and differentiate the denominator]}$$

$$= \frac{Et}{2L} \cdot \frac{-\cos pt}{p} = -\frac{Et \cos pt}{2Lp}$$

$\therefore$ C.S. is $$q = c_1 \cos pt + c_2 \sin pt - \frac{Et \cos pt}{2Lp}$$

When $$t = 0,\ q = 0 \quad \therefore \quad 0 = c_1$$

$$\therefore \quad q = c_2 \sin pt - \frac{Et \cos pt}{2Lp}$$

$$i = \frac{dq}{dt} = c_2\, p \cos pt - \frac{E}{2Lp}\left[-t\, p \sin pt + \cos pt\right]$$

$$t = 0,\ i = 0$$

$$\therefore \quad 0 = c_2\, p - \frac{E}{2Lp} \quad \therefore \quad c_2 = \frac{E}{2Lp^2}$$

$$\therefore \quad i = \frac{E}{2Lp^2}\, p \cos pt - \frac{Et}{2p} \sin pt - \frac{E}{2Lp} \cos pt$$

$$\therefore \quad i = \frac{Et}{2p} \sin pt.$$

Example 9. *In an L-C-R circuit, the charge q on a plate of a condenser is given by* $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}$ *= E sin pt. The circuit is turned to resonance so that* $p^2 = \frac{1}{LC}$. *If initially the current i and the charge q be zero, show that, for small values of* $\frac{R}{L}$, *the current in the circuit at time t is given by* $\frac{Et}{2L}$ *sin pt.*

(P.T.U., May 2010)

Sol. Given differential equation is

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E\sin pt$$

S.F. is $\left(LD^2+RD+\frac{1}{C}\right)q=E\sin pt$

A.E. is $LD^2+RD+\frac{1}{C}=0$

$$\therefore \qquad D=\frac{-R\pm\sqrt{R^2-\frac{4L}{C}}}{2L}=-\frac{R}{2L}\pm\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}$$

$$D=-\frac{R}{2L}\pm\sqrt{-\frac{1}{LC}} \qquad \because \quad \frac{R}{L} \text{ is small} \quad \therefore \quad \frac{R^2}{L^2} \text{ is neglected}$$

$$=-\frac{R}{2L}\pm ip \qquad \because \quad p^2=\frac{1}{LC}$$

$$\therefore \qquad \text{C.F.}=e^{-\frac{R}{2L}t}\left[c_1\cos pt+c_2\sin pt\right]$$

$$\text{P.I.}=\frac{1}{LD^2+RD+\frac{1}{C}}\,E\sin pt \qquad \text{Put } D^2=-p^2$$

$$=\frac{1}{-Lp^2+RD+\frac{1}{C}}\,E\sin pt$$

$$=\frac{1}{-L\frac{1}{LC}+RD+\frac{1}{C}}\,E\sin pt=\frac{1}{RD}E\sin pt=\frac{E}{R}\cdot\frac{-\cos pt}{p}$$

C.S. is $$q=e^{-\frac{R}{2L}t}\left(c_1\cos pt+c_2\sin pt\right)-\frac{E}{pR}\cos pt$$

Now, $e^{-\frac{R}{2L}t}=1+\left(-\frac{R}{2L}t\right)+\frac{1}{2!}\left(-\frac{R}{2L}t\right)^2+\ldots\ldots..$ $\left[\text{Using } e^{-x}=1-x+\frac{1}{\lfloor 2}x^2-\frac{1}{\lfloor 3}x^3+\ldots\ldots\infty\right]$

$$=1-\frac{R}{2L}t \qquad \because \quad \frac{R}{L} \text{ is small}$$

$$\therefore \qquad q=\left(1-\frac{R}{2L}t\right)\left(c_1\cos pt+c_2\sin pt\right)-\frac{E}{pR}\cos pt$$

When $t=0,\ q=0\ \therefore\ 0=c_1-\frac{E}{pR}\ \therefore\ c_1=\frac{E}{pR}$

$$\therefore \qquad q=\left(1-\frac{R}{2L}t\right)\left(\frac{E}{pR}\cos pt+c_2\sin pt\right)-\frac{E}{pR}\cos pt$$

$$i = \frac{dq}{dt} = \left(1 - \frac{R}{2L}t\right)\left(-\frac{E}{pR} p \sin pt + c_2\, p \cos pt\right) - \frac{R}{2L}\left(\frac{E}{pR} \cos pt + c_2 \sin pt\right) + \frac{E}{R} \sin pt$$

$$t = 0, i = 0 \quad \therefore \quad 0 = c_2\, p - \frac{E}{2L\, p} \quad i.e., \quad c_2 = \frac{E}{2L\, p^2}$$

$$\therefore \quad i = \left(1 - \frac{R}{2L}t\right)\left[-\frac{E}{R}\sin pt + \frac{E}{2Lp}\cos pt\right] - \frac{R}{2L}\left[\frac{E}{pR}\cos pt + \frac{E}{2Lp^2}\sin pt\right] + \frac{E}{R}\sin pt$$

$$= -\frac{E}{R}\sin pt + \frac{E}{2Lp}\cos pt + \frac{Et}{2L}\sin pt - \frac{ERt}{4L^2 p}\cos pt - \frac{E}{2Lp}\cos pt - \frac{ER}{4L^2 p^2}\sin pt + \frac{E}{R}\sin pt$$

$$= \frac{Et}{2L}\sin pt - \frac{ERt}{4L^2 p}\cos pt - \frac{ER}{4L^2 p^2}\sin pt$$

$$= \frac{Et}{2L}\sin pt \quad \because \quad \frac{R}{L} \text{ is small}$$

Hence $i = \dfrac{Et}{2L} \sin pt$.

TEST YOUR KNOWLEDGE

1. The differential equation for a circuit in which self inductance and capacitance neutralize each other is $L\dfrac{d^2 i}{dt^2} + \dfrac{i}{C} = 0$. Find the current i as a function of t, given that I is the maximum current and $i = 0$ when $t = 0$. **(P.T.U., Dec. 2011)**
[**Hint:** Consult S.E. 4]

2. A condenser of capacity C discharged through an inductance L and resistance R in series and the charge q at time t satisfies the equation $\log\dfrac{40}{4} = \log 10 = 0$. Given that $L = 0.25$ henries, $R = 250$ ohms, $C = 2 \times 10^{-6}$ farads and that when $t = 0$, charge q is 0.002 coulombs and the current $\dfrac{dq}{dt} = 0$, obtain the value of q in terms of t.

3. If an e.m.f. E sin ωt is applied to a circuit containing a resistance R, an inductance L and a condenser of capacity C, the charge on the condenser at time t satisfies the equation $\log\dfrac{40}{4} = \log 10 = E \sin \omega t$. If $R = 2\sqrt{LC}$, solve the differential equation for q.

4. A circuit consists of an inductance L and condenser of capacity C in series. An alternating e.m.f. E sin nt is applied to the circuit at time $t = 0$, the initial current and the charge on the condenser being zero. Prove that the current at time t is $i = \dfrac{nE}{L\left(n^2 - \omega^2\right)}$ (cos ωt – cos nt, where $CL\omega^2 = 1$). Prove also that if $n = \omega$, the current at time t is $\dfrac{Et \sin \omega t}{2L}$.

5. An alternating e.m.f. E sin pt is applied to circuit at $t = 0$. Given the equation for the current i as

$L\dfrac{d^2 i}{dt^2} + R\dfrac{di}{dt} + \dfrac{i}{C} = p\,E \cos pt$, find the current i when

(*i*) $CR^2 > 4L$ (*ii*) $CR^2 < 4L$.

ANSWERS

1. $i = \mathrm{I}\sin\dfrac{1}{\sqrt{\mathrm{LC}}}$ **2.** $q = e^{-500t}(0.002\cos 1323t + 0.0008\sin 1323t)$

3. $q = (c_1 + c_2 t)\, e^{-\frac{\mathrm{R}t}{2\mathrm{L}}} + \mathrm{EC}\left[\dfrac{(1-\omega^2\mathrm{CL})\sin\omega t - 2\sqrt{\mathrm{LC}}\cos\omega t}{(1+\omega^2\mathrm{CL})^2}\right]$

5. (*i*) $i = \mathrm{A}e^{-\alpha t}\cosh(\beta t + \gamma)$

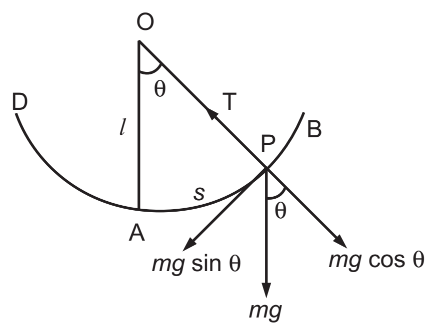
(*ii*) $i = \mathrm{A}\,e^{-\alpha t}\cos(\beta t + \gamma) + \dfrac{\mathrm{E}}{\mathrm{R}}\cos\phi\sin(pt + \phi)$, where

$\alpha = -\dfrac{\mathrm{R}}{2\mathrm{L}},\ \beta = \pm\left[\left(\dfrac{\mathrm{R}}{2\mathrm{L}}\right)^2 - \dfrac{1}{\mathrm{CL}}\right]$

$\phi = \tan^{-1}\left(\dfrac{1-\mathrm{CL}p^2}{\mathrm{CRP}}\right)$.

3.10. SIMPLE PENDULUM (P.T.U., May 2005)

If a heavy particle is attached to one end of a light inextensible string, the other end of which is fixed, and oscillates under gravity in a vertical plane, then the system is called a **simple pendulum.**



Let O be the fixed point, l the length of the string and m, the mass of the bob (heavy mass). Let P be the position of the bob at any time t. Let arc AP = s and $\angle$AOP = θ, where OA is the vertical line through O.

The forces acting on the bob are:

(*i*) its weight mg acting vertically downward

(*ii*) the tension T in the string.

The components of weight along and perpendicular to the path of motion are $mg\sin\theta$ and $mg\cos\theta$ respectively. The component $mg\cos\theta$ is balanced by the tension in the string.

$\therefore$ The equation of motion of the bob along the tangent is $m\dfrac{d^2s}{dt^2} = -mg\sin\theta$

or $$\frac{d^2(l\theta)}{dt^2} = -g\sin\theta \qquad (\because\ s = l\theta)$$

or $$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta = -\frac{g}{l}\left(\theta - \frac{\theta^3}{3!} + ...\right) = -\frac{g}{l}\theta \text{ to a first approx.}$$

or $$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \text{ where } \omega^2 = \frac{g}{l}$$

Its A.E. is $\mathrm{D}^2 + \omega^2 = 0$, whence $\mathrm{D} = \pm i\omega$.

$\therefore$ Its solution is $\theta = c_1\cos\omega t + c_2\sin\omega t$

or $$\theta = c_1\cos\sqrt{\frac{g}{l}}\,t + c_2\sin\sqrt{\frac{g}{l}}\,t$$

The motion of the bob is simple harmonic and the time of an oscillation is $\dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{l}{g}}$.

The motion of the bob from one extreme position B to the other extreme position B$'$ is called *a beat or a swing*.

$\therefore$ The time of one beat $= \dfrac{1}{2}$ (time of one oscillation) $= \pi\sqrt{\dfrac{l}{g}}$.

A simple pendulum which beats once every second is called a **second's pendulum.** Thus a second's pendulum beats 86400 times a day, the number of seconds in 24 hours.

Since the time for one beat in a second's pendulum is 1 second, taking $g = 981$ cm/sec^2, we have

$$1 = \pi\sqrt{\frac{l}{981}} \quad \text{or} \quad l = \frac{981}{\pi^2} = 99.4 \text{ cm}$$

Hence the length of a second's pendulum is 99.4 cm.

3.11. GAIN OR LOSS OF BEATS

Let a simple pendulum of length l make n beats in a time t, so that

$$t = n\pi\sqrt{\frac{l}{g}} \quad \text{or} \quad n = \frac{t}{\pi}\sqrt{\frac{g}{l}}$$

Taking logs, $\qquad \log n = \log\left(\dfrac{t}{\pi}\right) + \dfrac{1}{2} \cdot (\log g - \log l)$

Taking differentials on both sides, we get $\dfrac{dn}{n} = \dfrac{1}{2}\left(\dfrac{dg}{g} - \dfrac{dl}{l}\right)$ or $dn = \dfrac{n}{2}\left(\dfrac{dg}{g} - \dfrac{dl}{l}\right)$

which is the change in the number of beats.

If only g changes, l remaining constant, then $dn = \dfrac{n}{2} \cdot \dfrac{dg}{g}$ $\qquad (\because \; dl = 0)$

If only *l* changes, g remaining constant, then $dn = -\dfrac{n}{2} \cdot \dfrac{dl}{l}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find how many seconds a clock would lose per day, if the length of its pendulum were increased in the ratio 900 : 901*?

Sol. Let l be the original length and $l + dl$, the increased length of the pendulum, then

$$\frac{l}{l + dl} = \frac{900}{901} \quad \text{or} \quad \frac{l + dl}{l} = \frac{901}{900}$$

$$\therefore \qquad \frac{dl}{l} = \frac{901}{900} - 1 = \frac{1}{900}$$

Let n be the number of beats per day, then $n = 86400$.

If dn is the change in the number of beats, then $dn = -\dfrac{n}{2} \cdot \dfrac{dl}{l} = -\dfrac{86400}{2} \times \dfrac{1}{900} = -48$.

Since dn is negative, the clock will lose 48 seconds per day.

Example 2. *A pendulum oscillating seconds at one place is taken to another place where it loses 2 seconds per day. Compare the accelerations due to gravity at the two places.* **(P.T.U., Jan. 2009)**

Sol. Let g be the acceleration due to gravity of the pendulum which beats once every second at one place and $g + dg$ be the acceleration when pendulum loses 2 seconds per day at another place

$\therefore \quad n =$ 86400 seconds only and $dn = -2$

We know $$\frac{dn}{n} = \frac{1}{2}\frac{dg}{g}$$

$$\therefore \quad -\frac{2}{86400} \times 2 = \frac{dg}{g}$$

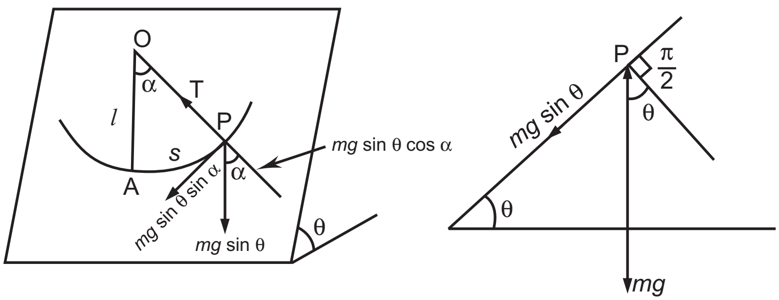
$$\therefore \quad \frac{g}{dg} = -\frac{21600}{1}$$

$$\frac{g}{g+dg} = \frac{-21600}{-21600+1} = \frac{21600}{21599}$$

Hence the required ratio of the accelerations due to gravity at the two places is 21600 : 21599.

Example 3. *A pendulum of length l hangs against a wall inclined at an angle* θ *to the horizontal. Show that the time of complete oscillation is* $2\pi\sqrt{\dfrac{l}{g\ sin\ \theta}}$.

Sol. Let the position of the bob of mass m, at any time t, be P and O be the point of suspension so that OP $= l$ and $\angle$AOP $= \alpha$, where OA is the line of greatest slope through O.



The component of weight of the bob along the plane is mg sin θ. The equation of motion of the bob along the tangent at P is $m\dfrac{d^2s}{dt^2} = -mg\sin\theta\sin\alpha.$

$$\frac{d^2(l\alpha)}{dt^2} = -g\sin\theta\left(\alpha - \frac{\alpha^3}{3!} + \cdots\right)$$

or $$l\frac{d^2a}{dt^2} = -g\sin\theta\,.\,\alpha \quad \text{to a first approx.}$$

or $$\frac{d^2\alpha}{dt^2} = -\frac{g\sin\theta}{l}\cdot\alpha \quad \text{or} \quad \frac{d^2\alpha}{dt^2} = -\omega^2\alpha\,, \quad \text{where} \quad \omega^2 = \frac{g\sin\theta}{l}$$

$\therefore$ The motion of the bob is simple harmonic and the time of one oscillation $= \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{l}{g\sin\theta}}$.

TEST YOUR KNOWLEDGE

1. A pendulum gains 10 seconds at one place and is taken to another place where it loses 10 seconds per day. Compare the accelerations due to gravity at the two places.

2. A clock with a second's pendulum is gaining 2 minutes a day. Prove that the length of the pendulum must be increased by 0.28 cm to make it go correctly.

3. Calculate the number of beats lost per day by a second's pendulum when the length of the string is increased by $\frac{1}{10,000}$ of itself.

4. If a pendulum clock loses 9 minutes per week, find in mm, what change is required in the length of the pendulum in order that the clock may keep correct time ?

5. A clock is taken from one place on the earth's surface to another. If the value of g is thus increased by one per cent, find what increase must be made in the length of the pendulum in order that the clock may keep correct time?

6. A pendulum whose length is l makes m oscillations in 24 hours. When its length is slightly altered, it makes $m + n$ oscillations in 24 hours. Show that the diminution in length is $\frac{2nl}{m}$ nearly.

7. A second's pendulum was too long on a day by a quantity k, it was then over corrected next day so as to become too short by k. Prove that the number of minutes gained in these two days is $1080 \frac{k^2}{l^2}$, where l is the true length of the pendulum.

8. A pendulum of length l has one end of the string fastened to a peg on a smooth plane inclined to horizon to an angle θ. With the string and the weight on the plane, its time of oscillation is t seconds. If the pendulum of length l' oscillates in one second when suspended vertically, prove that $\theta = \sin^{-1}\left(\frac{l}{l't^2}\right)$.

9. If l_1 be the length of an imperfectly adjusted second's pendulum which gains n seconds in one hour and l_2, the length of one which loses n seconds in one hour, at the same place, show that the true length of second's pendulum is $\frac{4l_1 l_2}{l_1 + l_2 + 2\sqrt{l_1 l_2}}$.

10. A simple pendulum has a period T. When the string is lengthened by a small fraction $\frac{1}{n}$ of its length, the period becomes T′. Show that approximately $\frac{1}{n} = \frac{2(\text{T}' - \text{T})}{\text{T}}$.

11. A simple pendulum of length l is oscillating through a small angle θ in a medium in which the resistance is proportional to the velocity. Find the differential equation of its motion. Discuss the motion and find the period of oscillation. [**Hint:** Consult art. 5.3 (*ii*) for oscillating motion]

12. The differential equation of a simple pendulum is $\frac{d^2x}{dt^2} + \omega_0{}^2 x = \text{F}_0 \sin nt$, where ω_0 and F_0 are constants. If initially $x = 0$, $\frac{dx}{dt} = 0$, determine the motion when $\omega_0 = n$.

ANSWERS

1. 4321 : 4319 **2.** 1.7 mm **5.** 1%

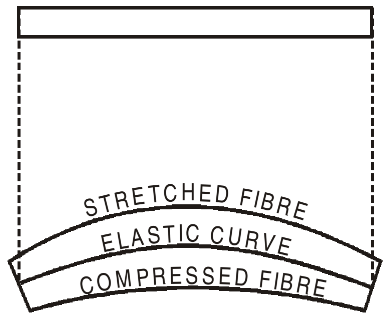
11. $\dfrac{d^2\theta}{dt^2} + 2p\dfrac{d\theta}{dt} + \omega\theta = 0$, where $2p = \dfrac{\lambda}{m}, \omega = \dfrac{g}{l}$.

Motion is oscillatory when $p < \omega$ and period = $\dfrac{2\pi}{\sqrt{\omega^2 - p^2}}$

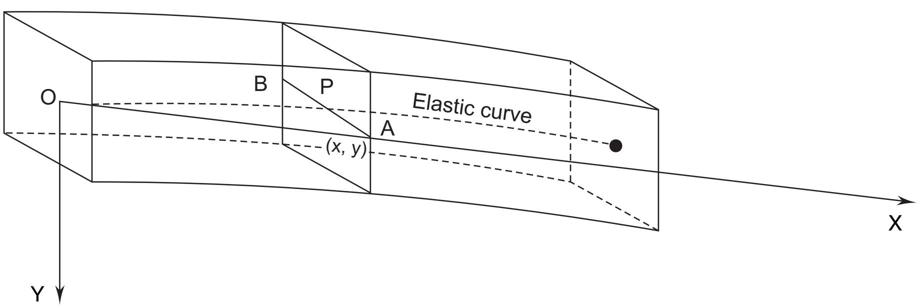
12. $x = \dfrac{F_0}{2n^2}(\sin nt - nt \cos nt)$.

3.12. DEFLECTION OF BEAMS

Consider a uniform beam as made up of fibres running lengthwise. When the beam is deflected, the fibres in some region are elongated and those in some region are contracted. In between these two, there is a region of the beam where the fibres are neither compressed nor stretched. This region is called the *neutral surface* of the beam and the curve of any particular fibre in this surface is called the *elastic curve* or *deflection curve* of the beam. The line in which any plane section of the beam cuts the neutral surface is called the *neutral axis* of that section.



Consider a cross-section of the beam cutting the elastic curve in P and the neutral surface in the line AB, the neutral axis of this section.



It is well-known from the mechanics of structure that the forces above and below the neutral surface acting in opposite directions have a tendency to restore the beam to its original position ; creating an interval bending moment M given by $\dfrac{EI}{R}$,

where E = modulus of elasticity of the material of the beam

I = moment of inertia of the cross-section of the beam about the neutral axis AB

R = radius of curvature of the elastic curve at P (x, y).

Now,
$$R = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$$

If the deflection (slope) of the beam is small, $\left(\dfrac{dy}{dx}\right)^2$ is very small and can be neglected, so that

$$R = \frac{1}{\dfrac{d^2y}{dx^2}}$$

Thus, $$M = EI\frac{d^2y}{dx^2} \qquad \ldots(1)$$

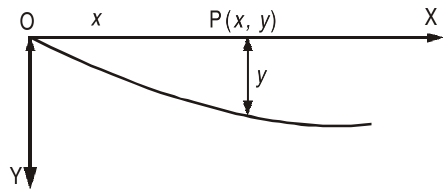
which is the differential equation of the elastic curve.

Note 1. The moment M with respect to AB of all external forces acting on either of the two segments into which the beam is separated by the cross-section is independent of the segment considered.

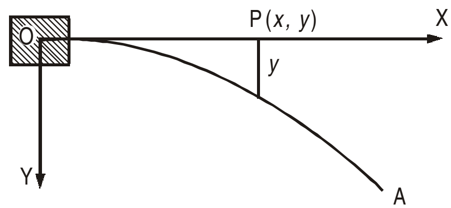
Note 2. The amount M is the algebraic sum of the moments of the external forces acting on the segment of the beam about the line AB. The upward forces give positive moments and the downward forces give negative moments.

3.13. BOUNDARY CONDITIONS

The general solution of the differential equation (1) will contain two arbitrary constants which, in any particular problem, are to be determined from the boundary (or end) conditions given below :



(*i*) *End freely supported*. At the freely supported end O, there is no deflection of the beam, so that $y = 0$. Also there is no bending moment at this end, so that $\dfrac{d^2y}{dx^2} = 0$.



(*ii*) *End fixed horizontally*. At the end fixed horizontally, the deflection and the slope of the beam are zero.

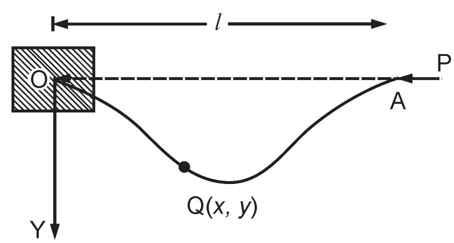
$\therefore \quad y = 0$ and $\dfrac{dy}{dx} = 0$.

(*iii*) *End perfectly free*. At the free end A in the above figure, there is no bending moment or shear force, so that

$$\frac{d^2y}{dx^2} = 0 \quad \text{and} \quad \frac{d^3y}{dx^3} = 0.$$

ILLUSTRATIVE EXAMPLES

Example 1. *The deflection of a strut of length l with one end* ($x = 0$) *built in and the other supported and subjected to end thrust P, satisfies the equation* $\dfrac{d^2y}{dx^2} + a^2y = \dfrac{a^2R}{P}(l - x)$. *Prove that the deflection curve is* $y = \dfrac{R}{P}\left(\dfrac{\sin ax}{a} - l\cos ax + l - x\right)$, *where* $al = \tan al$.



Sol. The given equation is $\dfrac{d^2y}{dx^2} + a^2y = \dfrac{a^2R}{P}\,(l - x)$

or $$(D^2 + a^2)\, y = \frac{a^2 R}{P}(l - x) \qquad \ldots(1)$$

Its A.E. is $D^2 + a^2 = 0$ so that $D = \pm\, ia$

$\therefore$ C.F. $= c_1 \cos ax + c_2 \sin ax$ and

$$\text{P.I.} = \frac{1}{D^2 + a^2}\, \frac{a^2 R}{P}\,(l - x)$$

$$= \frac{a^2 R}{P} \cdot \frac{1}{a^2\left(1 + \frac{D^2}{a^2}\right)}(l - x) = \frac{R}{P}\left(1 + \frac{D^2}{a^2}\right)^{-1}(l - x) = \frac{R}{P}\left(1 - \frac{D^2}{a^2} + \ldots\ldots\right)(l - x) = \frac{R}{P}\,(l - x)$$

$\therefore$ The complete solution of (1) is $y = c_1 \cos ax + c_2 \sin ax + \dfrac{R}{P}\,(l - x)$ $\qquad \ldots(2)$

Differentiating (2) w.r.t. x $\qquad \dfrac{dy}{dx} = -ac_1 \sin ax + ac_2 \cos ax - \dfrac{R}{P}$ $\qquad \ldots(3)$

Since the end O is build in, $\quad y = 0 \quad$ and $\quad \dfrac{dy}{dx} = 0$ at $\quad x = 0$

$\therefore$ From (2), $\qquad 0 = c_1 + \dfrac{Rl}{P} \qquad \Rightarrow \qquad c_1 = -\dfrac{Rl}{P}$

From (3), $\qquad 0 = ac_2 - \dfrac{R}{P} \qquad \Rightarrow \qquad c_2 = \dfrac{R}{aP}$

Substituting the values of c_1 and c_2 in (2), we have $y = \dfrac{R}{P}\left(\dfrac{\sin ax}{a} - l\cos ax + l - x\right)$ $\qquad \ldots(4)$

which is the equation of the deflection curve.

Also, at the end A, $\qquad y = 0$ when $x = l$.

$\therefore$ From (4), $\qquad 0 = \dfrac{R}{P}\left(\dfrac{\sin al}{a} - l\cos al\right)$

or $$\frac{\sin al}{a} = l\cos al \quad \therefore \quad al = \tan al.$$

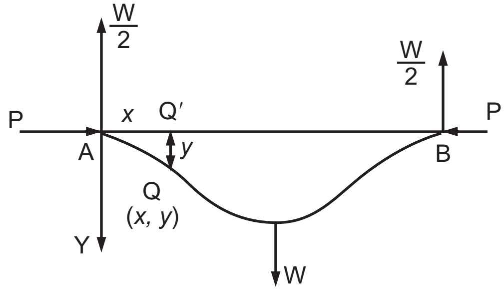
Example 2. *A light horizontal strut AB of length l is freely pinned at A and B and is under the action of equal and opposite compressive forces P at each of its ends and carries a load W at its centre.*

Prove that the deflection at the centre is $\dfrac{W}{2P}\left(\dfrac{1}{n}\tan\dfrac{nl}{2} - \dfrac{l}{2}\right)$, *where* $n^2 = \dfrac{P}{EI}$.

Sol. At each end there is a vertical reaction $\dfrac{W}{2}$. At any point Q (x, y) of the beam, the internal bending moment is EI $\dfrac{d^2y}{dx^2}$ and this must be equal to the moment of external forces taken to the left (or right) of the section QQ′.

$\therefore$ The differential equation for the elastic curve of the beam is EI $\dfrac{d^2y}{dx^2} = -\dfrac{W}{2}x - Py$

or $$\frac{d^2 y}{dx^2} + \frac{P}{EI} y = -\frac{W}{2EI} x$$



or $$\frac{d^2 y}{dx^2} + n^2 y = -\frac{n^2 W}{2P} x \quad \ldots(1) \quad \left(\because \frac{P}{EI} = n^2 \right)$$

Its A.E. is $D^2 + n^2 = 0$ so that $D = \pm in$.

$\therefore$ C.F. $= c_1 \cos nx + c_2 \sin nx$ and

$$\text{P.I.} = \frac{1}{D^2 + n^2} \left(-\frac{n^2 W}{2P} x \right) = -\frac{n^2 W}{2P} \cdot \frac{1}{n^2 \left(1 + \frac{D^2}{n^2}\right)} x = -\frac{W}{2P} \left(1 + \frac{D^2}{n^2}\right)^{-1} x$$

$$= -\frac{W}{2P} \left(1 - \frac{D^2}{n^2} + \ldots\ldots \right) x = -\frac{Wx}{2P}$$

$\therefore$ The complete solution of (1) is $y = c_1 \cos nx + c_2 \sin nx - \frac{W}{2P} x$ …(2)

Differentiating (2) w.r.t. x ; $\frac{dy}{dx} = -nc_1 \sin nx + nc_2 \cos nx - \frac{W}{2P}$ …(3)

At the end A, $x = 0, \; y = 0,$ from (2), $c_1 = 0$

$\therefore$ At $x = \frac{l}{2}$, $\frac{dy}{dx} = 0.$

$\therefore$ From (3), $0 = nc_2 \cos \frac{nl}{2} - \frac{W}{2P}$ $(\because \; c_1 = 0)$

or $$c_2 = \frac{W}{2Pn \cos \frac{nl}{2}}$$

Substituting the values of c_1 and c_2 in (2), we have

$$y = \frac{W}{2Pn \cos \frac{nl}{2}} \sin nx - \frac{W}{2P} x \quad \text{or} \quad y = \frac{W}{2P} \left(\frac{\sin nx}{n \cos \frac{nl}{2}} - x \right)$$

Deflection at the centre $= y\left(\text{at } x = \frac{l}{2}\right) = \frac{W}{2P} \left(\frac{\sin \frac{nl}{2}}{n \cos \frac{nl}{2}} - \frac{l}{2} \right) = \frac{W}{2P} \left(\frac{1}{n} \tan \frac{nl}{2} - \frac{l}{2} \right).$

TEST YOUR KNOWLEDGE

1. Find the equation of the elastic curve and its maximum deflection for the horizontal, simply supported, uniform beam of length $2l$ metres, having uniformly distributed load w kg/metre.
2. A horizontal strut of length l is clamped horizontally at one end and carries a vertical load W at the other end. If the horizontal thrust be P, prove that the deflect on at the free end is $\frac{W}{nP} (\tan nl - nl)$, where $n^2 = \frac{P}{EI}$.

3. A long uniform strut of length l is clamped at one end, the other end being free. If a thrust P be applied at the free end, at a distance 'a' from the neutral axis, prove that the deflection at the free end is $a\,(\sec nl - 1)$, where $n^2 = \dfrac{\text{P}}{\text{EI}}$.

4. A horizontal tie-rod is freely pinned at each end. It carries a uniform load w kg per unit length and has a horizontal pull P. Find the central deflection and the maximum bending moment, taking the origin at one of its ends.

5. A horizontal tie-rod of length $2l$ with concentrated load W at the centre and ends freely hinged, satisfies the differential equation $\text{EI}\dfrac{d^2y}{dx^2} = \text{P}y - \dfrac{\text{W}}{2}x$. With conditions $x = 0,\ y = 0$ and $x = l,\ \dfrac{dy}{dx} = 0$, prove that the deflection δ and the bending moment M at the centre $(x = l)$ are given by $\delta = \dfrac{\text{W}}{2\text{P}n}(nl - \tanh nl)$ and $\text{M} = -\dfrac{\text{W}}{2n}\tanh nl$, where $n^2\,\text{EI} = \text{P}$.

6. A light horizontal strut of length l is clamped at one end carries a vertical load W at the free end. If the horizontal thrust at the free is P, show that the strut satisfies the differential equation $\text{EI}\dfrac{d^2y}{dx^2} = (\delta - y)\text{P} + \text{W}(l - x)$, where y is the displacement of a point at a distance x from the fixed end and δ the deflection at the free end. Prove that the deflection at the free end is given by $\dfrac{\text{W}}{n\text{P}}(\tan nl - nl)$, where $n^2\text{EI} = \text{P}$.

ANSWERS

1. $y = \dfrac{w}{24\text{EI}}\left(4lx^3 - x^4 - 8l^3x\right);\ \dfrac{5wl^4}{24\text{EI}}$.

4. $\dfrac{w}{\text{P}a^2}\left(\text{sech}\dfrac{al}{2} - 1\right) + \dfrac{wl^2}{8\text{P}};\ \dfrac{w}{a^2}\left(\text{sech}\dfrac{al}{2} - 1\right)$.

3.14. CONDUCTION OF HEAT

The fundamental principles involved in the problems of heat conduction are:

(*a*) Heat flows from a higher temperature to the lower temperature.

(*b*) The quantity of heat in a body is proportional to its mass and temperature.

(*c*) The rate of heat flow across an area is proportional to the area and to the rate of change of temperature with respect to its distance normal to the area.

Thus if Q (cal/sec) be the quantity of heat that flows across a slab of area A (cm^2) and thickness δx in one second, with faces at temperatures T and T + δT, then by principle (3) ; $\text{Q} = -k\text{A}\dfrac{d\text{T}}{dx}$, where k is the coefficient of thermal conductivity and depends upon the material of the body.

ILLUSTRATIVE EXAMPLES

Example 1. *A pipe 20 cm in diameter contains steam at 150°C and is protected with a covering 5 cm thick for which k = 0·0025. If the temperature of the outer surface of the covering is 40°C find the temperature half-way through the covering under steady state conditions.*

Sol. Let Q cal/sec be the constant quantity of heat flowing out radially through a surface of the pipe having radius x cm and length 1 cm. Then area of this cylindrical surface A = $2\pi x$ sq cm.

Then $$Q = -KA\frac{dT}{dx} = -K \cdot 2\pi x\frac{dT}{dx}$$

or $$dT = \frac{-Q}{2\pi K}\frac{dx}{x}$$

Integrating both sides ; $T = -\frac{Q}{2\pi K}\log_e x + C$ given T = 150° when $x = 10$ cm.

$$\therefore \quad 150 = -\frac{Q}{2\pi K}\log_e 10 + C \qquad \ldots(1)$$

when $T = 40, x = 15$ we have

$$40 = -\frac{Q}{2\pi K}\log_e 15 + C \qquad \ldots(2)$$

Subtracting (2) from (1)

$$110 = -\frac{Q}{2\pi K}\cdot\{\log_e 10 - \log_e 15\}$$

$$= \frac{Q}{2\pi K}\log_e \frac{15}{10} = \frac{Q}{2\pi K}\log_e 1.5 \qquad \ldots(3)$$

Let $T = T_1$ when $x = 12.5$

$$\therefore \quad T_1 = -\frac{Q}{2\pi K}\log_e 12.5 + C \qquad \ldots(4)$$

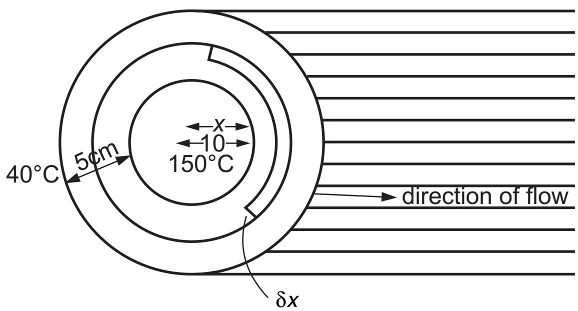
Subtracting (1) from (4)

$$T_1 - 150 = -\frac{Q}{2\pi K}\cdot\{\log_e 12.5 \quad \log_e 10\}$$

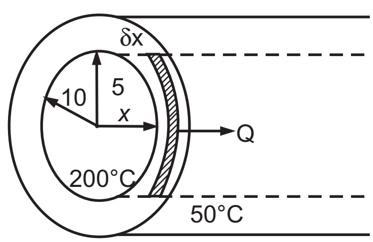
$$= \frac{110}{\log_e 1.5} \log_e \frac{12.5}{10} \text{ [From (3)]}$$

$$= 110\frac{\log_e 1.25}{\log_e 1.5}$$

$$\therefore \quad T_1 = 150 - 110\frac{\log_e 1.25}{\log_e 1.5} = 89.5°C.$$



Example 2. *A long hallow pipe has an inner diameter of 10 cm and outer diameter of 20 cm. The inner surface is kept at 200°C and the outer surface at 50°C. The thermal conductivity is 0.12. How mach heat is lost per minute from a portion of the pipe 20 metres long ? Find the temperature at a distance x = 7.5 cm from the centre of pipe.*



Sol. Here the isothermal surfaces are cylinders, the axis of each one of them is the axis of the pipe. **Consider one such cylinder of radius x cm length 1 cm.** The surface area of this cylinder is $A = 2\pi x$ sq cm. Let Q cal/sec be quantity of heat flowing across this surface, then

$$Q = -kA\frac{dT}{dx} = -k.2\pi x\frac{dT}{dx}$$

or $$dT = -\frac{Q}{2\pi k}\cdot\frac{dx}{x}$$

Integrating, we have $$T = -\frac{Q}{2\pi k}\log_e x + c \quad \ldots(1)$$

Since $T = 200$, when $x = 5$

$$\therefore \quad 200 = -\frac{Q}{2\pi k}\log_e 5 + c \quad \ldots(2)$$

Also, $T = 50$, when $x = 10$

$$\therefore \quad 50 = \frac{Q}{2\pi k}\log_e 10 + c \quad \ldots(3)$$

Subtracting (3) from (2), we have

$$150 = \frac{Q}{2\pi k}\left(\log_e 10 - \log_e 5\right)$$

or $$150 = \frac{Q}{2\pi k}\log_e 2 \quad \ldots(4)$$

$$\therefore \quad Q = \frac{2\pi k \times 150}{\log_e 2} = \frac{300\pi \times 0.12}{\log_e 2} = 163 \text{ cal/sec} \quad (\text{given } k = 0.12)$$

Hence the heat lost per minute through 20 metre length of the pipe

$$= 60 \times 2000Q = 120000 \times 163 = 1956000 \text{ cal.}$$

| 1 min = 60 seconds and 20 m = 2000 cm.

Now, let $T = t$, when $x = 7.5$

From (1), $$t = -\frac{Q}{2\pi k}\log_e 7.5 + c \quad \ldots(5)$$

Subtracting (2) from (5), $$t - 200 = -\frac{Q}{2\pi k}\left(\log_e 7.5 - \log_e 5\right)$$

or $$t - 200 = -\frac{Q}{2\pi k}\log_e 1.5 \quad \ldots(6)$$

Dividing (6) by (4), we have $$\frac{t-200}{150} = -\frac{\log_e 1.5}{\log_e 2}$$

or $$t = 200 - 150 \times 0.58 = 113$$

$\therefore$ When $x = 7.5$ cm, $T = 113°$ C.

3.15. NEWTON'S LAW OF COOLING

Newton's law of cooling states that the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.

If T_0 be the temperature of the surroundings and T that of the body at anytime t, then

$$\frac{dT}{dt} = -\boldsymbol{k}(\mathbf{T} - \mathbf{T_0}), \text{ where } k \text{ is the constant of proportionality.}$$

Example 3. *If the temperature of the air is 30°C and the substance cools from 100°C to 70°C in 15 minutes. Find when the temperature will be 40°C.*

Sol. Let the unit of time be minute and T the temperature of the substance of any instant t. Then by Newton's Law of cooling we have $\frac{dT}{dt} = -k(T-30)$

or
$$\frac{dT}{T-30} = -k\,dt$$

Integrate $\log(T-30) = -kt + C$ ...(1)

Initially when $t = 0$, $T = 100$

$\therefore$ $\log 70 = C$

Substituting the value of C in (1)

$$\log(T-30) = -kt + \log 70 \qquad \text{...(2)}$$

Also, when $t = 15$, $T = 70°$

$\therefore$ $\log 40 = -15k + \log 70$

$\therefore$ $15k = \log 70 - \log 40$

Subtracting the value of k in (2)

$$\log(T-30) = -\frac{t}{15}(\log 70 - \log 40) + \log 70$$

$\therefore$ When $T = 40$, we have

$$\log 10 = -\frac{t}{15}\log\frac{70}{40} + \log 70$$

or
$$\frac{t}{15}\log\frac{7}{4} = \log 70 - \log 10 = \log 7$$

$\therefore$
$$t = 15\frac{\log 7}{\log\frac{7}{4}} = 15\frac{\log_{10} 7}{\log_{10}\frac{7}{4}} = 15(3.48) = 52.20$$

Hence temperature will be 40°C after 52.2 minutes.

Example 4. *A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of air being 40°C. What will be the temperature of the body after 40 minutes from the original?*

Sol. Let T be the temperature of the body at any time t then by Newton's Law of cooling

$$\frac{dT}{dt} = -k(T-40)$$

or
$$\frac{dT}{T-40} = -k\,dt \text{ ; Integrate both sides}$$

$$\log(T-40) = -kt + C \qquad \text{...(1)}$$

Initially when $t = 0$, $T = 80$

$\therefore$ $\log 40 = C$

Substituting the value C in (1)

$$\log(T-40) = -kt + \log 40 \qquad \text{...(2)}$$

when $t = 20, T = 60°$

$\therefore$ From (2), $\log 20 = -20k + \log 40$

$\therefore$ $20k = \log 40 - \log 20 = \log 2$

$$k = \frac{1}{20}\log 2$$

$\therefore$ From (2), $\log(T-40) = -\frac{1}{20}\log 2.t + \log 40$

Now, when $t = 40$ min

$$\log(T-40) = -\frac{1}{20}\log 2\ .40 + \log 40$$

$$= -2\log 2 + \log 40 = -\log 4 + \log 40$$

$$= \log\frac{40}{4} = \log 10$$

$\therefore$ $T - 40 = 10$ $\therefore$ $T = 50°C$.

TEST YOUR KNOWLEDGE

1. Calculate the amount of heat passing through 1 sq cm of a refrigerator wall, if the thickness of the wall is 6 cm and the temperature inside the refrigerator is 0°C while outside it is 2°C. Assume $k = 0.0002$.

2. A pipe 10 cm in diameter contains steam at 100°C. It is covered with asbestos, 5 cm thick, for which $k = 0.0006$ and the outside surface is at 30°C. Find the amount of heat lost per hour from a metre long pipe.

3. A steam pipe 20 cm in diameter contains steam at 150°C and is covered by a layer of insulation 5 cm thick. The outside temperature is kept at 60°C. By how much should the thickness of the covering be increased in order that the rate of heat loss should be decreased by 25% ?

4. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference of a temperature between the object and the surroundings. Supposing water at a temperature 100°C cools to 80°C in 10 minutes, in a room maintained at a temperature of 30°C, find when the temperature of water will become 40°C. [**Hint :** Consult example 3]

5. Water at temperature 100°C cools in 10 minutes to 80°C in a room of temperature 25°C. Find (*i*) the temperature of water after 20 minutes, (*ii*) the time when the temperature is 40°C.

6. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes. Find the temperature of the body after 24 minutes.

ANSWERS

1. 0.000667 cal/sec	**2.** 140000 cal/hr	**3.** 2.16 cm
4. 57.9 minutes	**5.** (*i*) 65.5°C (*ii*) 51.9 minutes	**6.** 48°C.

3.16. RATE OF GROWTH OR DECAY

If the rate of change of a quantity y is proportional to the quantity present at any instant, then we have the following differential equation : $\frac{dy}{dt} = ky$

If k is positive, the problem is one of growth and if k is negative, the problem is one of decay.

Example 1. *Uranium disintegrates at a rate proportional to the amount present at any instant. If M_1 and M_2 gm of uranium are present at times T_1 and T_2 respectively, show that the half-life of uranium is*

$$\frac{(T_2 - T_1)\log 2}{\log\left(\frac{M_1}{M_2}\right)}.$$

Sol. Let M gm of uranium be present at any time t. Then the equation of disintegration of uranium is

$$\frac{dM}{dt} = -kM, \text{ where } k \text{ is a constant.}$$

or $$\frac{dM}{M} = -k\,dt$$

Integrating, $\log M = -kt + c$ ...(1)

Initially, when $t = 0, M = M_0$ (say)

$\therefore$ From (1), $c = \log M_0$

Substituting the value of c in (1), we have $\log M = \log M_0 - kt$

or $$kt = \log M_0 - \log M \qquad \text{...(2)}$$

Now, when $t = T_1, M = M_1$ and when $t = T_2, M = M_2$

$\therefore$ From (2), we have $kT_1 = \log M_0 - \log M_1$ ...(3)

and $$kT_2 = \log M_0 - \log M_2 \qquad \text{...(4)}$$

Subtracting (3) from (4), we get $k(T_2 - T_1) = \log M_1 - \log M_2$ or $k = \dfrac{\log \dfrac{M_1}{M_2}}{T_2 - T_1}$

Let T be the half-life of uranium *i.e.*, when $t = T, M = \frac{1}{2}M_0$.

$\therefore$ From (2), we get $kT = \log M_0 - \log \dfrac{M_0}{2} = \log 2$

$$\therefore \qquad T = \frac{\log 2}{k} = \frac{(T_2 - T_1)\log 2}{\log\left(\dfrac{M_1}{M_2}\right)}.$$

3.17. CHEMICAL REACTIONS AND SOLUTIONS

The problems of chemical reactions and solutions are especially important to chemical engineers. These problems can be well explained through the following example:

Example 2. *A tank contains 5000 litres of fresh water. Salt water which contains 100 gm of salt per litre flows into it at the rate of 10 litres per minute and the mixture kept uniform by stirring, runs out at the same rate. When will the tank contain 200000 gm of salt ?*

How long will it take for the quantity of salt in the tank to increase from 150000 gm to 250000 gm ?

Sol. Let Q gm be the quantity of salt present in the tank at time t, then $\dfrac{dQ}{dt}$ is the rate at which the salt content is changing and $\dfrac{dQ}{dt}$ = rate of salt entering the tank-rate of salt leaving the tank.

Now, the rate at which the salt increases due to the inflow = $100 \times 10 = 1000$ gm/min.

Let C gm be the concentration of salt at time t.

The rate at which the salt content decreases due to the outflow = $C \times 10 = 10\,C$ gm/min

Since the rate of inflow is the same as the rate of outflow, there is no change in the volume of water at any instant.

$$\Rightarrow \qquad C = \frac{Q}{5000}$$

The rate of decrease of salt content $= 10 \times \frac{Q}{5000} = \frac{Q}{500}$ gm/min $\cdot$

$$\therefore \qquad \frac{dQ}{dt} = 1000 - \frac{Q}{500} \text{ or } \frac{dQ}{dt} = \frac{500000 - Q}{500}$$

or
$$\frac{dQ}{500000 - Q} = \frac{dt}{500}$$

Integrating, $\quad -\log(500000 - Q) = \frac{t}{500} + c \qquad \text{...(1)}$

Initially, when $\quad t = 0, Q = 0$

$\therefore \quad c = -\log 500000$

From (1), we have $\quad \frac{t}{500} = \log 500000 - \log(500000 - Q)$

or
$$t = 500 \log \frac{500000}{500000 - Q} \qquad \text{...(2)}$$

Let $\quad t = \text{T}$, when $Q = 200000$

From (2),
$$T = 500 \log \frac{500000}{300000} = 500 \log_e \left(\frac{5}{3}\right)$$

$$= 500 \times 2.303 \log_{10}\left(\frac{5}{3}\right) = 500 \times 2.303 \times 0.2219$$

$= 255.5$ minutes $= 4$ hours 15.52 minutes.

Let $\quad t = T_1$, when $Q = 150000$ and $t = T_2$, when $Q = 250000$

From (2), we have
$$T_1 = 500 \log \frac{500000}{350000} = 500 \log \frac{10}{7}$$

$$T_2 = 500 \log \frac{500000}{250000} = 500 \log 2$$

$\therefore$ Required time
$$= T_2 - T_1 = 500\left(\log 2 - \log \frac{10}{7}\right) = 500 \log_e\left(\frac{7}{5}\right)$$

$$= 500 \times 2.303 \log_{10} 1.4 = 500 \times 2.303 \times 0.1461$$

$= 168.23$ minutes $= 2$ hours 48.23 minutes.

Example 3. *A tank contains 100 litres of fresh water. Two litres of brine, each containing 1 gm of dissolved salt, run into the tank per minute, and the mixture kept uniform by stirring runs out at the rate of 1 litre per minute. Find the amount of salt present when the tank contains 150 litres of brine.*

Sol. Let Q gm be the quantity of salt present in the brine at time t, then $\frac{dQ}{dt}$ is the rate at which the salt content is changing.

The rate at which the salt content increases due to the inflow = 2 × 1 = 2 gm/min.

Let C gm be the concentration of brine at time t.

The rate at which the salt content decreases due to the outflow = C × 1 = C gm/min

$$\therefore \qquad \frac{dQ}{dt} = 2 - C \qquad \ldots(1)$$

Now, the initial volume of liquid is 100 litres. In one minute, 2 litres of brine enter the tank and 1 litre of brine leaves the tank so that the volume of liquid in the tank increase at the rate of (2 – 1) = 1 litre/min.

The volume of liquid at time t is $(100 + t)$ litres containing Q gm of salt.

$$\therefore \qquad C = \frac{Q}{100 + t}$$

From (1), we have $\dfrac{dQ}{dt} = 2 - \dfrac{Q}{100+t}$ or $\dfrac{dQ}{dt} + \dfrac{Q}{100+t} = 2$ ...(2)

which is a linear equation in Q and t.

$$\text{I.F.} = e^{\int \frac{dt}{100+t}} = e^{\log(100+t)} = 100 + t$$

$\therefore$ The solution of (2) is $Q\,(100 + t) = \int 2\,(100 + t)\,dt + c$

$$\text{or} \qquad (100+t)\,Q = 2\left(100t + \frac{t^2}{2}\right) + c \qquad \ldots(3)$$

Initially, when $t = 0$, $\quad Q = 0$ so that $c = 0$

$$\therefore \quad \text{From (3)} \qquad Q = \frac{2\left(100t + \frac{t^2}{2}\right)}{100 + t} \qquad \ldots(4)$$

Now, if V is the volume of liquid at time t, then $V = 100 + t$

$\therefore$ When $\quad V = 150$ litres, $t = 150 - 100 = 50$ minutes

$$\text{and salt content} \qquad Q = \frac{2\left[100 \times 50 + \frac{(50)^2}{2}\right]}{100 + 50} = 83.3 \text{ gm}.$$

TEST YOUR KNOWLEDGE

1. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What was the value of N after $1\frac{1}{2}$ hours ?
2. In a culture of yeast, at each instant, the time rate change of active ferment is proportional to the amount present. If the active ferment doubles in two hours, how much can be expected at the end of 8 hours at the same rate of growth. Find also, how much time will elapse, before the active ferment grows to eight times its initial value.
3. Radium decomposes at a rate proportional to the amount present. If p per cent of the original amount disappears in l year, how much will remain at the end of $2l$ years ?

4. If 30% of a radioactive substance disappeared in 10 days, how long will it take for 90% of it to disappear.
5. Under certain conditions cane-sugar in water is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If of 75 gm at time $t = 0$, 8 gm are converted during the first 30 minutes, find the amount converted in $1\frac{1}{2}$ hours.
6. In a certain chemical reaction, the rate of conversion of a substance at time t is proportional to the quantity of substance still untransformed at that instant. The amount of substance remaining untransformed at the end of one hour and at the end of four hours are 60 gm and 21 gm respectively. How many grams of substance were present initially ?
7. A tank contains 1000 litres of fresh water. Salt water which contains 150 gm of salt per litre runs into it at the rate of 5 litres per minute and well stirred mixture runs out of it at the same rate. When will the tank contain 5000 gm of salf ?
8. A tank is initially filled with 100 litres of salt solution containing 1 gm of salt per litre. Fresh brine containing 2 gm of salt per litre runs into the tank at the rate of 5 litres per minute and the mixture, assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time, and determine how long it will take for this amount to reach 150 gm.
9. A tank contains 100 litres of an aqueous solution containing 10 kg of salt. Water is entering the tank at the rate of 3 litres per minute and the well stirred mixture runs out at 2 litres per minute. How much salt will the tank contain at the end of one hour ? After what time will the amount of salt in the tank be 625 gm?

ANSWERS

1. 604.9 **2.** 16 times the original, 6 hours

3. $\left(10 - \frac{p}{10}\right)^2$ **4.** 64.5 days **5.** 21.5 gm

6. 85 gm **7.** 6.77 minutes **8.** $100\left(2 - e^{-\frac{t}{20}}\right)$; 13.9 minutes

9. 3.6 kg ; 5 hours.

REVIEW OF THE CHAPTER

1. **Simple Harmonic Motion:** A particle is said to execute simple harmonic motion if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point. The differential equation of S.H.M. is $\frac{d^2x}{dt^2} = -\mu x$. Displacement of the particle from the fixed point at any time t is $x = a \cos \mu t$. Velocity of the particle at that point is $v = -\mu\sqrt{a^2 - x^2}$

   Period of motion = $\frac{2\pi}{\mu}$ and frequency of motion is $n = \frac{1}{T} = \frac{\mu}{2\pi}$

2. **Basic Relations between Elements of Electric Circuits:**

   (*i*) $i = \frac{dQ}{dt}$ or $Q = \int i\, dt$

   (*ii*) The potential voltage drop across the resistance, R = Ri

   (*iii*) The potential voltage drop across the inductance, L = $L\frac{di}{dt}$

   (*iv*) The potential voltage drop across the capacitance, C = $\frac{Q}{C}$

3. **Kirchhoff's Laws:**
   (*i*) The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit. This law is known as **Voltage Law**.
   (*ii*) The algebraic sum of the currents flowing into (or from) any node is zero. This law is known as **Circuit Law**.

4. **Differential Equation of an R, L Series Circuit** is $\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$ and its solution is $i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right]$.

5. **Differential Equation of an L, C Circuit** is $L\frac{di}{dt} + \frac{q}{C} = 0$ when there is no e.m.f. and its solution is

$$q = A\cos(\omega t + B), \text{ where } \omega = \frac{1}{\sqrt{LC}}$$

6. **Differential Equation of an L, C Circuit** with e.m.f. $k \cos nt$ is

$$L\frac{di}{dt} + \frac{q}{C} = k\cos nt \text{ and its solution is}$$

$$q = A\cos\left(\frac{t}{\sqrt{LC}} + B\right) + \frac{kC}{1 - LCn^2}\cos nt$$

when $\frac{1}{LC} \neq n^2$

and when $\frac{1}{LC} = n^2$, then solution is

$$q = r\sin(\omega t + \phi), \text{ where } \omega = \frac{1}{\sqrt{LC}};$$

$$r = \sqrt{C_1^2 + \left(C_2 + \frac{Et}{2\omega}\right)^2}, \ \tan\phi = \frac{C_1}{C_2 + \frac{Et}{2\omega}} \text{ and } E = \frac{k}{L}.$$

7. **Differential Equation of L.C.R. Circuit** is $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$ when there is no e.m.f. and its solution is

$$q = e^{-\frac{R}{2L}t}\left[C_1 e^{\frac{1}{2L}\left(\sqrt{R^2 - \frac{4L}{C}}\right)t} + C_2 e^{-\frac{1}{2L}\left(\sqrt{R^2 - \frac{4L}{C}}\right)t}\right] \text{ when } R^2 > \frac{4L}{C}$$

when $R^2 < \frac{4L}{C}$, then solution is

$$q = e^{-\frac{R}{2L}t}\left[C_1\cos\left(\frac{1}{2L}\sqrt{\frac{4L}{C} - R^2}\right)t + C_2\sin\left(\frac{1}{2L}\sqrt{\frac{4L}{C} - R^2}\right)t\right]$$

when $R^2 = \frac{4L}{C}$, then solution is

$$q = (C_1 + C_2 t)e^{-\frac{R}{2L}t}$$

8. **Differential Equation of L.C.R. Circuit with e.m.f. $k\cos nt$** is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = k\cos nt \text{ and its solution is}$$

$$q = e^{-pt}\left[C_1 e^{\sqrt{p^2 - \omega^2}\,t} + C_2 e^{-\sqrt{p^2 - \omega^2}\,t}\right]$$

$$+\frac{E}{\sqrt{4p^2n^2+\left(\omega^2-n^2\right)^2}}\cos\left(nt-\tan^{-1}\frac{2pn}{\omega^2-n^2}\right), \omega^2 \neq n^2 \text{ and } p^2 > \omega^2$$

or
$$q = e^{-pt}\left[C_1\cos\sqrt{\omega^2-p^2}\,t + C_2\sin\sqrt{\omega^2-p^2}\,t\right]$$
$$+\frac{E}{\sqrt{4p^2n^2+\left(\omega^2-n^2\right)^2}}\cos\left(nt-\tan^{-1}\frac{2pn}{\omega^2-n^2}\right) \text{ when } p^2 < \omega^2 \text{ and } \omega^2 \pm n^2$$

9. **Simple Pendulum:** If a heavy particle is attached to one end of a light inextensible string, the other end of which is fixed, and oscillates under gravity in a vertical plane, then the system is called a simple pendulum. The differential equation of the simple pendulum is $\frac{d^2Q}{dt^2} = -\frac{g}{l}Q$, where l is the length of the string

   Time of oscillation = $2\pi\sqrt{\frac{l}{g}}$

10. **Gain or Loss of Beats:** If a simple pendulum of length l makes n beats in a time t, then change in number of beats is given by
$$\frac{dn}{n} = \frac{n}{2}\left(\frac{dg}{g}-\frac{dl}{l}\right)$$
   If only g changes, l remains constant, then $dn = \frac{n}{2}\frac{dg}{g}$, there is gain in number of beats. **If only l changes,** g remains constant, then $dn = -\frac{n}{g}\frac{dl}{l}$, there is loss in number of beats.

11. **Deflection of Beam:** (Read art. 3.12)
12. **Conduction of Heat:** If Q (cal/sec) be the quantity of heat that flows across a slab of area A (cm^2) and thickness δx in one second with faces at temperatures T and $T + \delta T$, then $Q = -kA\frac{dT}{dx}$, where k is the coefficient of thermal conductivity and depends upon the material of the body.
13. **Newton's Law of Cooling:** It states that the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself *i.e.*, if T_0 is the temperature of the surroundings, and T that of the body at any time t, then $\frac{dT}{dt} = -k(T - T_0)$, where k is the constant of proportionality.
14. **Rate of Growth or Decay:** If the rate of change of a quantity y is proportional to the quantity present at any instant then $\frac{dy}{dt} = ky$

   If k is positive, then problem is one of growth.

   If k is negative, then problem is one of decay.

SHORT ANSWER TYPE QUESTIONS

1. (*a*) Define simple harmonic motion. Give one example.

   (*b*) If the displacement of a particle in a straight line is given by $x = a\cos\mu t + b\sin\mu t$, then show that it describes S.H.M. with an amplitude $\sqrt{a^2+b^2}$. **(P.T.U., Dec. 2013)**

   [**Hint:** S.E. 1 art 3.2]

2. A particle moving in a straight line with S.H.M. has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2 respectively. Show that the period of motion is $2\pi\sqrt{\dfrac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$.

[**Hint :** S.E. 2 art. 3.2]

3. Find differential equation of S.H.M. given by $x = \mathrm{A}\cos(nt + \alpha)$; n is constant. **(P.T.U., May 2007)**

[**Hint:** $\dfrac{dx}{dt} = -\mathrm{A}n\sin(nt+\alpha)$; $\dfrac{d^2x}{dt^2} = -\mathrm{A}n^2\cos(nt+\alpha) = -n^2x$ which is S.H.M.]

4. What is differential equation of R.L. series circuit ? Find its solution.

5. The initial value problem governing the current i, flowing in a series R-L circuit when a sinusoidal voltage $v(t) = \sin\omega t$ is applied is given by $i\mathrm{R} + \mathrm{L}\dfrac{di}{dt} = \sin\omega t,\ t \geq 0\ i(0) = 0$; find the current $\hat{i}$.

[**Hint :** Consult S.E. 4 art. 3.7 find the value of C by putting $i = 0$ when $t = 0$] **(P.T.U., May 2006, Dec. 2006)**

6. Show that the frequency of free vibration in a closed electrical circuit with inductance L and capacity C in series is $\dfrac{30}{\pi\sqrt{\mathrm{LC}}}$ per minute. [**Hint :** See S.E. 2 art. 3.9] **(P.T.U., May 2010)**

7. Solve the differential equation $\mathrm{L}\dfrac{di}{dt} + \dfrac{1}{c}\int i\,dt = 0$, which means that self inductance and capacitance in a circuit neutralize each other. Determine the constants in such a way that i_0 is the maximum current and $i = 0$ when $t = 0$

[**Hint:** S.E. 4 art 3.9] **(P.T.U., Dec. 2011)**

8. Define differential equation of L-R-C circuit. **(P.T.U., May 2007)**

9. A simple pendulum of length l is oscillating through a small angle θ in a medium in which the resistance is proportional to the velocity. Find the differential equation of motion. Discuss the motion and find period of oscillation.

10. Discuss the motion of a simple pendulum and find the period of oscillation.

11. A clock with a second's pendulum is gaining 2 minutes a day. Prove that the length of the pendulum must be increased by 0.28 cm to make it accurate.

12. How many seconds a clock would loose per day, if the length of the pendulum were increased in the ratio 900 : 901? [**Hint :** S.E. 1 art. 3.10]

13. A pendulum oscillating seconds at one place is taken to another place where it loses 2 seconds per day. Compare the accelerations due to gravity at the two places. **(P.T.U., Jan. 2009)**

[**Hint :** Consult S.E. 2 art. 3.11]

14. Explain Newton's law of cooling. Write its differential equation.

ANSWERS

1. (*a*) $\dfrac{d^2x}{dt^2} = -\dfrac{g}{e}x$

3. $\dfrac{d^2x}{dt^2} = -n^2x$

5. $i = \dfrac{\mathrm{L}}{\mathrm{L}^2\omega^2 + \mathrm{R}^2}\left\{we^{-\frac{\mathrm{R}}{\mathrm{L}}t} - \omega\cos\omega t + \dfrac{\mathrm{R}}{\mathrm{L}}\sin\omega t\right\}$

7. $i = i_0\sin\dfrac{1}{\sqrt{\mathrm{LC}}}t$

9. $\dfrac{d^2\theta}{dt^2} + 2p\dfrac{d\theta}{dt} + \omega\theta = 0$, where $2p = \dfrac{\lambda}{m}$, $\omega = \dfrac{g}{l}$ motion is oscillary when $p < \omega$ and period $= \dfrac{2\pi}{\sqrt{\omega^2 - p^2}}$

10. $2\pi\sqrt{\dfrac{l}{g}}$

12. 48 seconds

13. 21600 : 21599.

PART-B

4. Linear Algebra
5. Infinite Series
6. Complex Numbers and Elementary Functions of Complex Variable

4

Linear Algebra

4.1. WHAT IS A MATRIX?

A set of mn numbers (real or complex) arranged in a rectangular array having m rows (horizontal lines) and n columns (vertical lines), the numbers being enclosed by brackets [ ] or ( ) is called $m \times n$ matrix (read as "m and n" matrix). An $m \times n$ is usually written as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ a_{41} & a_{42} & a_{43} & \cdots & a_{4n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

. Here each element has two suffixes. The first suffix indicates the row and second suffix indicates the column in which element lies.

(*i*) Square Matrix : A matrix in which number of rows is equal to the number of columns is called a square matrix.

(*ii*) Multiplication of a Matrix by a Scalar : When each element of a matrix A (say) is multiplied by a scalar k (say), then k A is defined as multiplication of A by a scalar k.

(*iii*) Matrix Multiplication : Two matrices A and B are said to be conformable for the product AB if number of columns of A is equal to the number of rows of B.

Thus if $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$, where $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le p$. Then AB is defined as the matrix $C = [c_{ik}]_{m \times p}$.

where $c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + \ldots\ldots\ldots + a_{in} b_{nk}$.

$$= \sum_{j=1}^{n} a_{ij} b_{jk}$$

(*iv*) Properties of Matrix Multiplication :

(*a*) Matrix multiplication is not commutative in general *i.e.*, AB ≠ BA

(*b*) Matrix multiplication is associative *i.e.,* A (BC) = (AB)C.

(*c*) Matrix multiplication is distributive w.r.t. matrix addition *i.e.,* A (B + C) = AB + AC

(*d*) If A, I are square matrices of the same order, then AI = IA = A

(*e*) If A is a square matrix of order n, then $A \times A = A^2$; $A \times A \times A = A^3$; ....; $A \times A \times A \times \ldots\ldots\ldots\ldots \times A$ (m times) $= A^m$

(*v*) Transpose of a Matrix : Given a matrix A, then matrix obtained from A by changing its rows into columns and columns into rows is called the transpose of A and is denoted by A' or A^t.

(*vi*) Properties of Transpose of Matrix :

(*a*) (A')' = A,

(*b*) (A + B)' = A' + B',

(*c*) (A B)' = B' A' known as **Reversal Law of transposes.**

(*vii*) (*a*) **Symmetric Matrix :** A square matrix is said to symmetric if A' = A *i.e.,* if $A = [a_{ij}]$, then $a_{ij} = a_{ji}\ \forall\ i, j$

(*b*) **Skew Symmetric Matrix :** A square matrix is said to be skew symmetric if A' = – A *i.e.*, if $A = [a_{ij}]$, then $a_{ij} = -a_{ji}\ \forall\ i, j$ and when $i = j$, then $a_{ii} = 0$ for all values of *i*.

Thus in a skew symmetric matrix all diagonal elements are zero.

(*viii*) **Involutory Matrix :** A square matrix A is said to be involutory if $A^2 = I$

(*ix*) **Adjoint of a Square Matrix :** The adjoint of a square matrix is the transpose of the matrix obtained by replacing each element of A by its co-factors in |A|

A (Adj A) = (Adj A) A = $|A|\, I_n$; *n* being the order of matrix A

(*x*) **Singular and Non-Singular Matrices :** A square matrix is said to be singular if |A| = 0 and non-singular if $|A| \neq 0$.

(*xi*) (*a*) **Inverse of a Square Matrix :** Let A be a square matrix of order *n*. If there exists another matrix B of the same order such that AB = BA = I, then matrix **A is said to be invertible** and B is called **inverse of A.** Inverse of A is denoted by A^{-1}. Thus, $B = A^{-1}$ and $A\,A^{-1} = A^{-1}\,A = I$. From (*xv*) we see that A (Adj A) = |A| I

$$\therefore \qquad A^{-1} = \frac{\text{Adj A}}{|A|}.$$

(*b*) The inverse of a square matrix, if it exists, is unique.

(*c*) The necessary and sufficient condition for a square matrix A to posses inverse is that $|A| \neq 0$ *i.e.,* A is non-singular.

(*d*) If A is invertible, then so is A^{-1} and $(A^{-1})^{-1} = A$

(*xii*) **Reversal Law of Inverses :** If A and B are two non-singular matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

(*xiii*) **Reversal Law of Adjoints :** If A, B are two square matrices of the same order, then Adj (AB) = (Adj B) (Adj A).

4.2. ELEMENTARY TRANSFORMATIONS (OR OPERATIONS) (P.T.U., Dec. 2004)

Let $A = [a_{ij}]$ be any matrix of order $m \times n$ *i.e.*, $1 \le i \le m$, $1 \le j \le n$, then anyone of the following operations on the matrix is called an elementary transformation (or E-operation).

(*i*) **Interchange of two rows or columns.**

The interchange of *i*th and *j*th rows is denoted by R_{ij}.

The interchange of *i*th and *j*th columns is denoted by C_{ij}.

(*ii*) **Multiplication of (each element of) a row or column by a non-zero number *k*.**

The multiplication of *i*th row by *k* is denoted by kR_i.

The multiplication of *i*th column by *k* is denoted by kC_i.

(*iii*) **Addition of *k* times the elements of a row (column) to the corresponding elements of another row (or column), $k \neq 0$.**

The addition of *k* times the *j*th row to the *i*th row is denoted by $R_i + kR_j$.

The addition of *k* times the *j*th column to the *i*th column is denoted by $C_i + kC_j$.

If a matrix B is obtained from a matrix A by one or more E-operations, then B is said to be equivalent to A.

Two equivalent matrices A and B are written as A ~ B.

4.3. ELEMENTARY MATRICES

The matrix obtained from a unit matrix I by subjecting it to one of the E-operations is called an elementary matrix.

For example, let $\quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(*i*) Operating R_{23} or C_{23} on I, we get the same elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

It is denoted by E_{23}. Thus, the E-matrix obtained by either of the operations R_{ij} or C_{ij} on I is denoted by E_{ij}.

(*ii*) Operating $5R_2$ or $5C_2$ on I, we get the same elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

It is denoted by $5E_2$. Thus, the E-matrix obtained by either of the operations kR_i or kC_i is denoted by kE_i.

(*iii*) Operating $R_2 + 4R_3$ on I, we get the elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

It is denoted by E_{23} (4). Thus, the E-matrix obtained by the operation $R_i + kR_j$ is denoted by E_{ij} (k).

(*iv*) Operating $C_2 + 4C_3$ in I, we get the elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$, which is the transpose of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = E_{23}(4)$ and is, therefore, denoted by E'_{23} (4). Thus, the E-matrix obtained by the operation $C_i + kC_j$ is denoted by $E_{ij}'(k)$.

4.4. THE FOLLOWING THEOREMS ON THE EFFECT OF E-OPERATIONS ON MATRICES HOLD GOOD

(*a*) *Any E-row operation on the product of two matrices is equivalent to the same E-row operation on the prefactor.*

If the E-row operation is denoted by R, then R (AB) = R(A) · B

(*b*) *Any E-column operation on the product of two matrices is equivalent to the same E-column operation on the post-factor.*

If the E-column operation is denoted by C, then C(AB) = A · C(B)

(*c*) *Every E-row operation on a matrix is equivalent to pre-multiplication by the corresponding E-matrix.*

Thus the effect of E-row operation R_{ij} on A = $E_{ij} \cdot A$

The effect of E-row operation $k\mathrm{R}_i$ on $\mathrm{A} = k\mathrm{E}_i \cdot \mathrm{A}$

The effect of E-row operation $\mathrm{R}_i + k\mathrm{R}_j$ on $\mathrm{A} = \mathrm{E}_{ij}(k) \cdot \mathrm{A}$

(*d*) *Every E-column operation on a matrix is equivalent to post-multiplication by the corresponding E-matrix.*

Thus, the effect of E-column operation C_{ij} on $\mathrm{A} = \mathrm{A} \cdot \mathrm{E}_{ij}$

The effect of E-column operation $k\mathrm{C}_i$ on $\mathrm{A} = \mathrm{A} \,.\, (k\mathrm{E}_i)$

The effect of E-column operation $\mathrm{C}_i + k\mathrm{C}_j$ on $\mathrm{A} = \mathrm{A} \cdot \mathrm{E}'_{ij}(k)$.

4.5. INVERSE OF MATRIX BY E-OPERATIONS (Gauss-Jordan Method)

The elementary row transformations which reduce a square matrix A to the unit matrix, when applied to the unit matrix, gives the inverse matrix A^{-1}.

Let A be a non-singular square matrix. Then $\mathrm{A} = \mathrm{IA}$

Apply suitable E-row operations to A on the left hand side so that A is reduced to I.

Simultaneously, apply the same E-row operations to the prefactor I on right hand side. Let I reduce to B, so that
$$\mathrm{I} = \mathrm{BA}$$

Post-multiplying by A^{-1}, we get
$$\mathrm{IA}^{-1} = \mathrm{BAA}^{-1} \quad \Rightarrow \mathrm{A}^{-1} = \mathrm{B}(\mathrm{AA}^{-1}) = \mathrm{BI} = \mathrm{B}$$
$$\therefore \qquad \mathrm{B} = \mathrm{A}^{-1}.$$

Note. In practice, to find the inverse of A by E-row operations, we write A and I side by side and the same operations are performed on both. As soon as A is reduced to I, I will reduce to A^{-1}.

4.6. WORKING RULE TO REDUCE A SQUARE MATRIX TO A UNIT MATRIX I BY ELEMENTARY TRANSFORMATIONS (For Convenience We can Consider a Matrix A of Order 4 × 4)

(*i*) If in the first column, the principal element (*i.e.*, a_{11}) is not 'one' but 'one' is present some where else in the first column then first of all make 'one' as principal element (by applying row transformations R_{ij}).

(*ii*) Operate R_1 on $\mathrm{R}_2, \mathrm{R}_3, \mathrm{R}_4$ to make elements of C_1 all zero except first element.

(*iii*) Then operate R_2 on $\mathrm{R}_3, \mathrm{R}_4$ to make elements of C_2 all zero except first and second elements. Similarly, operate R_3 on R_4 to make elements of C_3 all zero except 1st, 2nd and 3rd.

(*iv*) Reduce each diagonal element to element 'one'.

Then reverse process starts :

(*v*) Operate R_4 on $\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3$ to make all elements of C_4 zero except last.

(*vi*) Then operate R_3 on R_1, and R_2 to make all elements of C_3 zero except last but one. Similarly operate R_2 on R_1 to make all elements of C_2 zero except last but second and the matrix is reduced to unit matrix.

Note. We can apply the above rule to any square matrix of any order.

ILLUSTRATIVE EXAMPLES

Example 1. *If* $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, *then show that*

$$A^n = A^{n-2} + A^2 - I \text{ for } n \geq 3. \text{ Hence find } A^{50}.$$ **(P.T.U., Jan. 2008)**

Sol.
$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We will use induction method to prove $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$

$\therefore$ for $n = 3$ we will prove $A^3 = A + A^2 - I$

Now,
$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Now,
$$A + A^2 - I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = A^3$$

$\therefore$ $A^3 = A + A^2 - I$ ...(1)

$\therefore$ Result is true for $n = 3$

Let us assume that the result is true for $n = k$

i.e., $A^k = A^{k-2} + A^2 - I$ ...(2)

To prove result is true for $n = k + 1$ *i.e.*, to prove

$$A^{k+1} = A^{k-1} + A^2 - I$$

Now, $A^{k+1} = A^k \cdot A = (A^{k-2} + A^2 - I)\, A$ [Using (2)]

$= A^{k-1} + A^3 - A$

$\therefore$ $A^{k+1} = A^{k-1} + A^2 - I$ [Using (1)]

Hence the result is true for $n = k + 1$, so the result is true for all values of $n \geq 3$

Now, $A^{50} = A^{48} + A^2 - I,$ [Using (2)]

or $A^{50} - A^{48} = A^2 - I$

or $A^{48}(A^2 - I) = (A^2 - I)$

or $A^{48}(A^2 - I) = I(A^2 - I)$

$\therefore$ $A^{48} = I$

$\therefore$
$$A^{50} = A^{48} \cdot A^2 = IA^2 = A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Example 2. *Reduce the following matrix to upper triangular form :* $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$.

Sol. (*Upper triangular matrix*) If in a square matrix, all the elements below the principal diagonal are zero, the matrix is called upper triangular.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix} \text{ by operations } \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \text{ by operation } R_3 + 5R_2$$

which is the upper triangular form of the given matrix.

Example 3. *Use Gauss-Jordan method to find inverse of the following matrices :*

(*i*) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ **(P.T.U., Dec. 2010)** (*ii*) $\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

Sol. (*i*) Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Consider $A = IA$

i.e.,
$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

To reduce LHS to a unit matrix

Operate $R_2 - R_1, R_3 + 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A$$

Operate $R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} A$$

Operate $R_2\left(\frac{1}{2}\right), R_3\left(-\frac{1}{4}\right)$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$$

Operate $R_1 - 3R_3, R_2 + 3R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} & \frac{3}{4} & \frac{3}{4} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$$

Operate $R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$$

$\therefore$ $I = BA$, where

$$B = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Hence,
$$A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

(*ii*) Let
$$A = \begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Operate R_{13}, then R_{23};
$$\begin{bmatrix} -1 & 4 & 1 & 2 \\ 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 + 2R_1, R_3 + 5R_1$;
$$\begin{bmatrix} -1 & 4 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 7 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_1\,(-1), R_2\left(\frac{1}{2}\right)$;
$$\begin{bmatrix} 1 & -4 & -1 & -2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 7 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_3 - 7R_2, R_4 - R_2$;
$$\begin{bmatrix} 1 & -4 & -1 & -2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -\frac{7}{2} & 1 & -2 & 0 \\ -\frac{1}{2} & 0 & -1 & 1 \end{bmatrix} A$$

Operate $R_4(2)$;
$$\begin{bmatrix} 1 & -4 & -1 & -2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -\frac{7}{2} & 1 & -2 & 0 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

Operate $R_1 + 2R_4,\ R_2 - \frac{1}{2}R_4,\ R_3 + \frac{1}{2}R_4$;
$$\begin{bmatrix} 1 & -4 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -5 & 4 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

Operate $R_1 + R_3$;
$$\begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 1 & -8 & 5 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

Operate $R_1 + 4R_2$;
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

$$\therefore \qquad I = BA \text{ and } B = A^{-1} = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix}$$

4.7. NORMAL FORM OF A MATRIX

Any non-zero matrix $A_{m \times n}$ can be reduced to anyone of the following forms by performing elementary (row, column or both) transformations :

(*i*) I_r (*ii*) $[I_r \ \ 0]$ (*iii*) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (*iv*) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where I_r is a unit matrix of order r

All those forms are known as Normal forms of the matrix

Note. The form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called **First Canonical Form of A.**

4.8. FOR ANY MATRIX A OF ORDER $m \times n$, FIND TWO SQUARE MATRICES P AND Q OF ORDERS m AND n RESPECTIVELY SUCH THAT PAQ IS IN THE NORMAL FORM $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Method : Write A = IAI

Reduce the matrix A on the LHS to normal form by performing elementary row and column transformations. Every row transformation on A must be accompanied by the same row transformation on the prefactor on RHS.

Every column transformation on A must be accompanied by the same column transformation on the post-factor on RHS.

Hence, A = IAI will transform to I = PAQ

Note. $A^{-1} = QP \quad \because P(AQ) = I \;\therefore\; AQ = P^{-1}$ or $(AQ)^{-1} = P$ *i.e.,* $Q^{-1}A^{-1} = P \quad \therefore \quad A^{-1} = QP$.

4.9. RANK OF A MATRIX

(P.T.U., May 2007, 2008, 2011, Dec. 2010)

Let A be any $m \times n$ matrix. It has square sub-matrices of different orders. The determinants of these square sub-matrices are called **minors of A.** If **all** minors of order $(r + 1)$ are zero but there **is at least one non-zero minor** of order r, then r is called **the rank of A.** Symbolically, rank of A = r is written as $\boldsymbol{\rho(A) = r}$.

From the definition of the rank of a matrix A, it follows that :

(*i*) If A is a null matrix, then $\rho(A) = 0$

[$\because$ every minor of A has zero value.]

(*ii*) If A is not a null matrix, then $\rho(A) \geq 1$.

(*iii*) If A is non-singular $n \times n$ matrix, then $\rho(A) = n$

[$\because$ $|A| \neq 0$ is largest minor of A.]

If I_n is the $n \times n$ unit matrix, then $|I_n| = 1 \neq 0 \Rightarrow \rho(I_n) = n$.

(*iv*) If A is an $m \times n$ matrix, then $\rho(A) \leq$ minimum of m and n.

(*v*) If all minors of order r are equal to zero, then $\rho(A) < r$.

To determine the rank of a matrix A, we adopt the following different methods :

4.10. WORKING RULE TO DETERMINE THE RANK OF A MATRIX

Method I : Start with the highest order minor (or minors) of A. Let their order be r. **If anyone of them** is non-zero, then $\rho(A) = r$.

If **all of them** are zero, start with minors of next lower order $(r - 1)$ and continue this process till you get a non-zero minor. The order of that minor is the rank of A.

This method usually involves a lot of computational work since we have to evaluate several determinants.

Method II : Reduce the matrix to the upper triangular form of the matrix by elementary row transformations, then number of non-zero rows of triangular matrix is equal to rank of the matrix.

Method III : Reduce the matrix to the normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by performing elementary transformations (row and column both), then r is the rank of the matrix. [$\because$ rth order minor $|I_r| = 1 \neq 0$ and each $(r + 1)$th order minor = 0]

4.11. PROPERTIES OF THE RANK OF A MATRIX

(*i*) Elementary transformations of a matrix do not alter the rank of the matrix.

(*ii*) $\rho(A') = \rho(A)$; $\rho(A^\theta) = \rho(A)$

(*iii*) $\rho(A)$ = number of non-zero rows in upper triangular form of the matrix A.

Example 4. *If A is a non-zero column matrix and B is a non-zero row matrix, then* $\rho(AB) = 1$.

Sol. Let A be a non-zero column matrix

Let $\quad A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}_{m \times 1}$ and B be a non-zero row matrix

Let $\quad B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}_{1 \times n}$, where at least one of a's and at least one of b's is non-zero

Now AB will be a matrix of order $m \times n$

$$\therefore \quad AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \cdots & a_2 b_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_m b_1 & a_m b_2 & \cdots & a_{mn} \end{bmatrix}$$

$$= b_1 b_2 \cdots b_n \begin{bmatrix} a_1 \; a_1 & \cdots & a_1 \\ a_2 \; a_2 & \cdots & a_2 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_m \; a_m & \cdots & a_m \end{bmatrix}$$

AB has at least one element non-zero and all minors of order ≥ 2 are zero because all lines are identical

$\therefore \quad \rho(AB) = 1.$

Example 5. *Find the rank of the following matrices:*

(*i*) $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$ **(P.T.U., Dec. 2004)** (*ii*) *Diag. matrix* $[-1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 4]$.

Sol. (*i*) Let $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$

A is of order 3×3 $\quad \therefore \quad \rho(A) \leq 3$

Reduce the matrix to triangular form

Operate $R_2 - 2R_1, R_3 - 3R_1$, $A \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -2 & -2 \\ 0 & -5 & 7 \end{bmatrix}$

Operate $\quad R_2\left(-\frac{1}{2}\right) \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & -5 & 7 \end{bmatrix}$

Operate $\quad R_3 + 5R_1 \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 12 \end{bmatrix}$

Now, minor of order $3 = \begin{vmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 12 \end{vmatrix} = 12 \neq 0$

$\therefore \quad \rho(A) = 3$

(*ii*) Let A = diag. matrix [–1 0 1 0 0 4]

$$= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \text{ which is a square matrix of order } 6 \times 6 \quad \therefore \quad \rho(A) \le 6$$

Also, it is a diagonal matrix so triangular matrix $\therefore$ $\rho(A)$ = Number of non-zero rows of triangular matrix = 3 hence $\rho(A) = 3$.

Example 6. *Reduce the following matrices to normal form and find their ranks.*

(*i*) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **(P.T.U., May 2012, Dec. 2012)** (*ii*) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ **(P.T.U., May 2007)**

Sol. (*i*) Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Operate R_{12} ; $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Operate $R_3 - 3R_1, R_4 - R_1$; $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

Operate $R_3 - R_2, R_4 - R_2$; $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operate $C_3 - C_1, C_4 - C_1$; $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operate $C_3 + 3C_2, C_4 + C_2$; $\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$= \begin{bmatrix} I_{2\times2} & O_{2\times2} \\ O_{2\times2} & O_{2\times2} \end{bmatrix}$$

which is the required normal form and $\rho(A) = 2$

$$(ii) \text{ Let } A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Operate R_{12} ; $\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Operate $R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1$; $\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

Operate $C_2 + C_1, C_3 + 2C_1, C_4 + 4C_1$; $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

[Now to change 5 to 1, instead of operating by $R_2\left(\frac{1}{5}\right)$, operate $R_2 - R_3$ and similarly to change 9 (in 2nd column) to 1 operate $R_4 - 2R_3$]

Operate $R_2 - R_3, R_4 - 2R_3$; $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 1 & -6 & -3 \end{bmatrix}$

Operate $R_3 - 4R_2, R_4 - R_2$; $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operate $C_3 + 6C_2, C_4 + 3C_2$; $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operate $C_3\left(\frac{1}{33}\right), C_4\left(\frac{1}{22}\right)$; $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operate $C_4 - C_3$; $\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{bmatrix} I_{3\times 3} & O_{3\times 1} \\ O_{1\times 3} & O_{1\times 1} \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

which is the required normal form and rank of A = 3.

Example 7. *Reduce the following matrix to normal form and hence find its rank* $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$.

(P.T.U., May 2004)

Sol. Let
$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$$

Operate $R_3 - R_1, R_4 + 2R_1$;
$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 12 & 2 \end{bmatrix}$$

Operate $C_3 - 2C_1, C_4 - C_1$;
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 12 & 2 \end{bmatrix}$$

Operate $R_3 + R_2, R_4 - 2R_2$;
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \end{bmatrix}$$

Operate $C_3 + 2C_2, C_4 - C_2$;
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \end{bmatrix}$$

Operate R_{34};
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operate $R_3\left(\frac{1}{16}\right)$;
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_3 & O_{3\times 1} \\ O_{3\times 1} & O_{1\times 1} \end{bmatrix}$$

$= \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$, which is the required normal form and $\rho(A) = 3$.

Example 8. *For a matrix* $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$*, find non-singular matrices P and Q such that PAQ is in the normal form. Also find* A^{-1} *(if it exists).* **(P.T.U., Dec. 2013)**

Sol. Consider A = IAI

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 - R_1$; (Subjecting prefactor the same operation)

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $C_2 - C_1$, $C_3 - 2C_1$; (Subjecting post-factor the same operation)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $R_3 + R_2$; (Subjecting same operation on prefactor)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $C_3 - C_2$; (Subjecting same operation on post-factor)

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & O_{2\times 1} \\ O_{1\times 2} & O_{1\times 1} \end{bmatrix} = PAQ, \text{ where } P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\rho(A) = 2$ $\therefore$ A is singular and A^{-1} does not exist.

TEST YOUR KNOWLEDGE

1. (*a*) Reduce to triangular form $\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$.

(*b*) If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$, find $|AB|$. **(P.T.U., May 2009)**

2. Use Gauss-Jordan method to find the inverse of the following:

(*i*) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ **(P.T.U., May 2012, Dec. 2013)**

(*v*) $\begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$ (*vi*) $\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & -3 \\ -1 & 2 & 1 & -1 \\ 2 & -3 & -1 & 4 \end{bmatrix}$.

3. Find rank of the following matrices:

(*i*) $\begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 3 & 1 & 4 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & 0 \\ 3 & -3 & 1 \end{bmatrix}$ **(P.T.U., Dec. 2013)**

(*iii*) $\begin{bmatrix} 2 & 4 & 3 & -2 \\ -3 & -2 & -1 & 4 \\ 6 & -1 & 7 & 2 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 2 & 5 & 3 & 3 \\ 2 & 3 & 3 & 4 \\ 3 & 6 & 3 & 2 \\ 4 & 12 & 0 & 8 \end{bmatrix}$

(*v*) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ **(P.T.U., June 2003)** (*vi*) $\begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$

(*vii*) $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ **(P.T.U., May 2009)** (*viii*) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$ **(P.T.U., Dec. 2011)**

4. Reduce the following matrices to normal form and hence find their ranks:

(*i*) $\begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ **(P.T.U., Dec. 2010)**

(*iii*) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

(*v*) $\begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}$ **(P.T.U., May 2012)** (*vi*) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ **(P.T.U., Dec. 2012)**

5. If A = $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$; find two non-singular matrices P and Q such that PAQ = I. Hence find A^{-1}.

[**Hint:** $\rho(A) = 3 \therefore A^{-1}$ exists = QP]

6. For a matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, find non-singular matrices P and Q such that PAQ is in the normal form. Also find A^{-1} (if it exists). **(P.T.U., Dec. 2003)**

ANSWERS

1. (*a*) $\begin{bmatrix} 3 & 4 & -5 \\ 0 & 13 & -11 \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 3 & 0 & 0 \\ -9 & 13 & 0 \\ -15 & 29 & 11 \end{bmatrix}$ (*b*) 16

Upper triangular Lower triangular

2. (*i*) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$

(*iii*) $\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

(*v*) $\frac{1}{25}\begin{bmatrix} -23 & 29 & -64 & -18 \\ 10 & -12 & 26 & 7 \\ 1 & -2 & 6 & 2 \\ 2 & -2 & 3 & 1 \end{bmatrix}$ (*vi*) $\frac{1}{18}\begin{bmatrix} 2 & 5 & -7 & 1 \\ 5 & -1 & 5 & -2 \\ -7 & 5 & 11 & 10 \\ -1 & -2 & 10 & 5 \end{bmatrix}$

3. (*i*) 2 (*ii*) 2 (*iii*) 3 (*iv*) 4 (*v*) 3 (*vi*) 3 (*vii*) 2 (*viii*) 2

4. (*i*) $\begin{bmatrix} I_2 & O_{2\times1} \\ O_{1\times2} & O_{1\times1} \end{bmatrix}$; rank = 2 (*ii*) $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$; $\rho = 2$

(*iii*) $[I_3 \times O_{3\times1}]$; $\rho = 3$ (*iv*) $\begin{bmatrix} I_2 & O_{2\times2} \\ O_{2\times2} & O_{2\times2} \end{bmatrix}$; $\rho = 2$

(*v*) $\begin{bmatrix} I_2 & O_{2\times1} \\ O_{1\times2} & O_{1\times1} \end{bmatrix}$; $\rho = 2$ (*vi*) $\begin{bmatrix} I_{3\times3} & O_{3\times1} \end{bmatrix}$; $\rho = 3$

5. $P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & -3 \end{bmatrix}$; $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$.

6. $P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -2 & 1 \end{bmatrix}$; $Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$; A^{-1} does not exist as $\rho(A) = 2$.

4.12. CONSISTENCY AND SOLUTION OF LINEAR ALGEBRAIC EQUATIONS

Proof. Consider the system of equations $\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\}$ (3 equations in 3 unknowns)

In matrix notation, these equations can be written as

$$\begin{bmatrix} a_1x+b_1y+c_1z \\ a_2x+b_2y+c_2z \\ a_3x+b_3y+c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ or } \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or $$AX = B$$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is called the coefficient matrix, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the column matrix of unknowns

$B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ is the column of constants.

If $d_1 = d_2 = d_3 = 0$, then B= O and the matrix equation AX = B reduce to AX = O.
Such a system of equation is called a system of **homogeneous linear equation.**
If at least one of d_1, d_2, d_3 is non-zero, then B $\neq$ O.
Such a system of equation is called a system of **non-homogeneous linear equation.**

Solving the matrix equation AX = B means finding X, *i.e.*, finding a column matrix $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ such that

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}. \text{ Then } x = \alpha, y = \beta, z = \gamma.$$

The matrix equation AX = B need not always have a solution. It may have no solution or a unique solution or an infinite number of solutions.

(*a*) Consistent Equations : A system of equations having **one or more solutions** is called a **consistent** system of equations.

(*b*) Inconsistent Equations : A system of equations having **no solutions** is called an **inconsistent** system of equations.

(*c*) State the conditions in terms of rank of the coefficient matrix and rank of the augmented matrix for a unique solution; no solution ; infinite number of solutions of a system of linear equations.

(P.T.U., May 2005, Dec. 2010)

For a system of non-homogeneous linear equation AX = B.

(*i*) if $\rho\,[A \vdots B] \neq \rho(A)$, the system is inconsistent.
(*ii*) if $\rho\,[A \vdots B] = \rho(A)$ = number of the unknowns, the system has a unique solution.
(*iii*) if $\rho\,[A \vdots B] = \rho(A)$ < number of unknowns, the system has an infinite number of solutions.

The matrix $[A \vdots B]$ in which the elements of A and B are written side by side is called the **augmented matrix.**

For a system of homogeneous linear equations AX = O.

(*i*) X = O is always a solution. This solution in which each unknown has the value zero is called the **Null Solution** or the **Trivial Solution.** Thus a homogeneous system is always consistent.
(*ii*) if $\rho(A)$ = number of unknown, the system has only the trivial solution.
(*iii*) if $\rho(A)$ < number of unknown, the system has an infinite number of non-trivial solutions.

4.13. IF A IS A NON-SINGULAR MATRIX, THEN THE MATRIX EQUATION AX = B HAS A UNIQUE SOLUTION

The given equation is $AX = B$ ...(1)

$\because$ A is a non-singular matrix, $\therefore$ A^{-1} exists.

Pre-multiplying both sides of (1) by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B \quad \text{or} \quad (A^{-1}A)X = A^{-1}B$$

or
$$IX = A^{-1}B \quad \text{or} \quad X = A^{-1}B$$

which is the required unique solution (since A^{-1} is unique).

Another Method to find the solution of AX = B:

Write the augmented matrix $[A \,\vdots\, B]$. By **E-row operations** on A and B, reduce A to a diagonal matrix thus getting

$$[A \,\vdots\, B] \sim \begin{bmatrix} p_1 & 0 & 0 & \vdots & q_1 \\ 0 & p_2 & 0 & \vdots & q_2 \\ 0 & 0 & p_3 & \vdots & q_3 \end{bmatrix}$$

Then $p_1 x = q_1,\ p_2 y = q_2,\ p_3 z = q_3$ gives the solution of AX = B.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the system of equations :*

$$5x + 3y + 7z = 4, \qquad 3x + 26y + 2z = 9, \qquad 7x + 2y + 11z = 5$$

with the help of matrix inversion. **(P.T.U., Dec. 2004, May 2007, Jan. 2010, May 2014)**

Sol. In matrix notation, the given system of equations can be written as

$$AX = B \qquad ...(1)$$

where
$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Let
$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{vmatrix} = 5\begin{vmatrix} 26 & 2 \\ 2 & 11 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 7 & 11 \end{vmatrix} + 7\begin{vmatrix} 3 & 26 \\ 7 & 2 \end{vmatrix}$$

$$= 5(286 - 4) - 3(33 - 14) + 7(6 - 182) = 1410 - 57 - 1232 = 121 \neq 0$$

$\Rightarrow$ A is non-singular $\therefore$ A^{-1} exists and the unique solution of (1) is

$$X = A^{-1}B \qquad ...(2)$$

Now co-factors of the elements of | A | are as follows:

$$A_1 = \begin{vmatrix} 26 & 2 \\ 2 & 11 \end{vmatrix} = 282,\ A_2 = -\begin{vmatrix} 3 & 7 \\ 2 & 11 \end{vmatrix} = -19,\ A_3 = \begin{vmatrix} 3 & 7 \\ 26 & 2 \end{vmatrix} = -176$$

$$B_1 = \begin{vmatrix} 3 & 2 \\ 7 & 11 \end{vmatrix} = -19,\ B_2 = \begin{vmatrix} 5 & 7 \\ 7 & 11 \end{vmatrix} = 6,\ B_3 = -\begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} = 11$$

$$C_1 = \begin{vmatrix} 3 & 26 \\ 7 & 2 \end{vmatrix} = -176, C_2 = -\begin{vmatrix} 5 & 3 \\ 7 & 2 \end{vmatrix} = 11, C_3 = \begin{vmatrix} 5 & 3 \\ 3 & 26 \end{vmatrix} = 121$$

$$\therefore \quad \text{adj A} = \text{transpose of} \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & 121 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj A} = \frac{1}{121}\begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & 121 \end{bmatrix}$$

$$\text{From (2), } X = A^{-1}B = \frac{1}{121}\begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & 121 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} = \frac{1}{121}\begin{bmatrix} 282(4)-19(9)-176(5) \\ -19(4)+6(9)+11(5) \\ -176(4)+11(9)+121(5) \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{121}\begin{bmatrix} 77 \\ 33 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{77}{121} \\ \frac{33}{121} \\ \frac{0}{121} \end{bmatrix} = \begin{bmatrix} \frac{7}{11} \\ \frac{3}{11} \\ 0 \end{bmatrix}$$

Hence, $x = \frac{7}{11},\ y = \frac{3}{11},\ z = 0.$

Example 2. *Use the rank method to test the consistency of the system of equations $4x - y = 12$, $-x - 5y - 2z = 0$, $-2y + 4z = -8$.* **(P.T.U., Dec. 2012)**

Sol. In matrix notation, the given equations can be written as

$$AX = B$$

where
$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & -5 & -2 \\ 0 & -2 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 12 \\ 0 \\ -8 \end{bmatrix}$$

$$\text{Augmented matrix } [A \vdots B] = \left[\begin{array}{ccc:c} 4 & -1 & 0 & 12 \\ -1 & -5 & -2 & 0 \\ 0 & -2 & 4 & -8 \end{array}\right]$$

$$\text{Operate } R_{12}; \quad \sim \left[\begin{array}{ccc:c} -1 & -5 & -2 & 0 \\ 4 & -1 & 0 & 12 \\ 0 & -2 & 4 & -8 \end{array}\right]$$

$$\text{Operate } R_2 + 4R_1; \quad \sim \left[\begin{array}{ccc:c} -1 & -5 & -2 & 0 \\ 0 & -21 & -8 & 12 \\ 0 & -2 & 4 & -8 \end{array}\right]$$

$$\text{Operate } R_1(-1), R_2\left(-\frac{1}{21}\right), R_3\left(-\frac{1}{2}\right);$$

$$\sim \left[\begin{array}{ccc:c} 1 & 5 & 2 & 0 \\ 0 & 1 & \frac{8}{21} & -\frac{4}{7} \\ 0 & 1 & -2 & 4 \end{array}\right]$$

Operate $R_3 - R_2$;
$$\sim \begin{bmatrix} 1 & 5 & 2 & \vdots & 0 \\ 0 & 1 & \frac{8}{21} & \vdots & -\frac{4}{7} \\ 0 & 0 & -\frac{50}{21} & \vdots & \frac{32}{7} \end{bmatrix}$$

Operate $R_3\left(-\frac{21}{50}\right)$;
$$\sim \begin{bmatrix} 1 & 5 & 2 & \vdots & 0 \\ 0 & 1 & \frac{8}{21} & \vdots & -\frac{4}{7} \\ 0 & 0 & 1 & \vdots & -\frac{48}{25} \end{bmatrix}$$

Operate $R_2 - \frac{8}{21} R_3, R_1 - 2R_3$;
$$\sim \begin{bmatrix} 1 & 5 & 0 & \vdots & \frac{96}{25} \\ 0 & 1 & 0 & \vdots & \frac{4}{25} \\ 0 & 0 & 1 & \vdots & -\frac{48}{25} \end{bmatrix}$$

Operate $R_1 - 5R_2$;
$$\sim \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{76}{25} \\ 0 & 1 & 0 & \vdots & \frac{4}{25} \\ 0 & 0 & 1 & \vdots & -\frac{48}{25} \end{bmatrix}$$

$\therefore$ $\rho(A) = 3 = \rho(A \vdots B) =$ number of unknowns

$\therefore$ The given system of equations is consistent and have a unique solution

Hence the solution is $x = \frac{76}{25}$, $y = \frac{4}{25}$, $z = -\frac{48}{25}$

Example 3. *For what values of* λ *and* μ *do the system of equations :* $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ *have (i) no solution, (ii) unique solution, (iii) more than one solution ?*

(P.T.U., Dec. 2002, May 2010)

Sol. In matrix notation, the given system of the equations can be written as

$$AX = B$$

where
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Augmented matrix $[A \vdots B]$

$$= \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 10 \\ 1 & 2 & \lambda & \vdots & \mu \end{bmatrix} \text{Operating } R_2 - R_1, R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 1 & \lambda-1 & \vdots & \mu-6 \end{bmatrix} \text{Operating } R_1 - R_2, R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & \vdots & 2 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 0 & \lambda-3 & \vdots & \mu-10 \end{bmatrix}$$

Case I. If $\lambda = 3, \mu \neq 10$

$\rho(A) = 2, \rho(A \vdots B) = 3$

$\because \quad \rho(A) \neq \rho(A \vdots B)$

$\therefore$ The system has no solution.

Case II. If $\lambda \neq 3$, μ may have any value

$\rho(A) = \rho(A \vdots B) = 3 =$ number of the unknowns

$\therefore$ System has unique solution.

Case III. If $\lambda = 3, \mu = 10$

$\rho(A) = \rho(A \vdots B) = 2 <$ number of the unknowns

$\therefore$ The system has an infinite number of solution.

Example 4. *(a) Solve the equations $x_1 + 3x_2 + 2x_3 = 0$, $2x_1 - x_2 + 3x_3 = 0$, $3x_1 - 5x_2 + 4x_3 = 0$, $x_1 + 17x_2 + 4x_3 = 0$.*

(b) Find the real value of λ for which the system of equations $x + 2y + 3z = \lambda x$, $3x + y + 2z = \lambda y$, $2x + 3y + z = \lambda z$ have non-trivial solution. **(P.T.U., May 2010, Dec. 2012)**

Sol. (*a*) In matrix notation, the given system of equations can be written as

$$AX = O$$

where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Operating $R_2 - 2R_1, R_3 - 3R_1, R_4 - R_1$ $A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$

Operating $R_3 - 2R_2, R_4 + 2R_2$ $\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Operating $R_1 + 2R_2$ $\sim \begin{bmatrix} 1 & -11 & 0 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2 <$ number of unknowns

$\Rightarrow$ The system has an infinite number of non-trivial solutions given by

$$x_1 - 11x_2 = 0, \quad -7x_2 - x_3 = 0$$

i.e., $x_1 = 11k, \quad x_2 = k, x_3 = -7k$, where k is any number.

Different values of k give different solutions.

(b) Given equations are

$$x + 2y + 3z = \lambda x$$
$$3x + y + 2z = \lambda y$$
$$2x + 3y + z = \lambda z$$

or
$$(1-\lambda)x + 2y + 3z = 0$$
$$3x + (1-\lambda)y + 2z = 0$$
$$2x + 3y + (1-\lambda)z = 0$$

These equations are homogeneous in x, y, z and will have a non-trivial solution if $\rho(A) < 3$ (the number of unknowns)

i.e., determinant of order 3 = 0

i.e.,
$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

or $$(1-\lambda)\left[(1-\lambda)^2 - 6\right] - 2\left[3(1-\lambda) - 4\right] + 3\left[9 - 2(1-\lambda)\right] = 0$$

or $$(1-\lambda)^3 - 6 + 6\lambda - 6 + 6\lambda + 8 + 27 - 6 + 6\lambda = 0$$

or $$1 - 3\lambda + 3\lambda^2 - \lambda^3 + 18\lambda + 17 = 0$$

or $$\lambda^3 - 3\lambda^2 - 15\lambda - 18 = 0$$

or $$(\lambda - 6)(\lambda^2 + 3\lambda + 3) = 0$$

$\therefore$ either $\lambda = 6$ or $\lambda^2 + 3\lambda + 3 = 0$

or $$\lambda = \frac{-3 \pm \sqrt{9-12}}{2} = \frac{-3 \pm i\sqrt{3}}{2}$$

$\therefore$ The only real value of λ is 6.

Example 5. *Discuss the consistency of the following system of equations. Find the solution if consistent :*

(*i*)
$$x + y + z = 4$$
$$2x + 5y - 2z = 3$$

(*ii*)
$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5.$$

(P.T.U., May 2005)

Sol. (*i*) $x + y + z = 4$

$2x + 5y - 2z = 3$, which can be written as

$$AX = B, \text{ where } A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Consider augmented matrix $[A \vdots B]$

$$= \begin{bmatrix} 1 & 1 & 1 & \vdots & 4 \\ 2 & 5 & -2 & \vdots & 3 \end{bmatrix}$$

Operating $R_2 - 2R_1$ $\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 4 \\ 0 & 3 & -4 & \vdots & -5 \end{bmatrix}$

$$\text{Operating } R_2\left(\frac{1}{3}\right) \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 4 \\ 0 & 1 & \frac{-4}{3} & \vdots & \frac{-5}{3} \end{bmatrix}$$

$$\text{Operating } R_1 - R_2 \sim \begin{bmatrix} 1 & 0 & \frac{7}{3} & \vdots & \frac{17}{3} \\ 0 & 1 & \frac{-4}{3} & \vdots & \frac{-5}{3} \end{bmatrix}$$

$$\rho(A) = 2 ; \rho(A : B) = 2$$

$$\rho(A) = \rho(A \vdots B) < \text{number of unknowns}$$

$\therefore$ Given system of equations are consistent and have infinite number of solutions given by

$$x + \frac{7}{3}z = \frac{17}{3}$$

$$y - \frac{4}{3}z = -\frac{5}{3}$$

Take $z = k$ we have $x = \dfrac{17-7k}{3}$, $y = \dfrac{4k-5}{3}$

Hence solutions is $x = \dfrac{17-7k}{3}$, $y = \dfrac{4k-5}{3}$, $z = k$, where k is any arbitrary constant.

(*ii*) Given equations can be put into the form AX = B

$$\text{where } A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}.$$

Consider augmented matrix

$$[A \vdots B] = \begin{bmatrix} 5 & 3 & 7 & \vdots & 4 \\ 3 & 26 & 2 & \vdots & 9 \\ 7 & 2 & 10 & \vdots & 5 \end{bmatrix}$$

Operating $R_2 - \frac{3}{5}R_1, R_3 - \frac{7}{5}R_1$

$$\sim \begin{bmatrix} 5 & 3 & 7 & \vdots & 4 \\ 0 & \frac{121}{5} & -\frac{11}{5} & \vdots & \frac{33}{5} \\ 0 & -\frac{11}{5} & \frac{1}{5} & \vdots & -\frac{3}{5} \end{bmatrix}$$

$$\text{Operate } R_3 + \frac{1}{11}R_2 ; \sim \begin{bmatrix} 5 & 3 & 7 & \vdots & 4 \\ 0 & \frac{121}{5} & -\frac{11}{5} & \vdots & \frac{33}{5} \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\text{Operate } R_2\left(\frac{5}{11}\right) \sim \begin{bmatrix} 5 & 3 & 7 & \vdots & 4 \\ 0 & 11 & -1 & \vdots & 3 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\rho(A) = 2, \rho(A:B) = 2$$
$$\rho(A) = \rho(A \vdots B) = 2 < \text{Number of unknowns}$$

$\therefore$ Given equations are consistent and have infinite number of solutions given by

$$5x + 3y + 7z = 4$$
$$11y - z = 3$$

Let $z = k\,;\; y = \dfrac{k+3}{11} \quad \therefore \quad x = \dfrac{4 - \dfrac{3(k+3)}{11} - 7k}{5} = \dfrac{7-16k}{11}$

Hence solution is $x = \dfrac{7-16k}{11}$, $y = \dfrac{k+3}{11}$, $z = k$.

Example 6. *For what value of k, the equations x + y + z = 1, 2x + y + 4z = k, 4x + y + 10z = k^2 have a solution and solve them completely in each case?* **(P.T.U., Dec. 2005)**

Sol.
$$x + y + z = 1$$
$$2x + y + 4z = k$$
$$4x + y + 10z = k^2$$

which can be put into matrix form AX = B

where
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

Consider the augmented matrix

$$[A \vdots B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 2 & 1 & 4 & \vdots & k \\ 4 & 1 & 10 & \vdots & k^2 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 4R_1$;
$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & 2 & \vdots & k-2 \\ 0 & -3 & 6 & \vdots & k^2-4 \end{bmatrix}$$

Operate $R_3 - 3R_2$;
$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & 2 & \vdots & k-2 \\ 0 & 0 & 0 & \vdots & k^2-3k+2 \end{bmatrix}$$

$$\rho(A) = 2 < \text{number of unknowns}$$

$\therefore$ System of equations cannot have a unique solution.

These will have an infinite number of solution only if $\rho(A : B) = \rho(A) = 2$, which is only possible if $k^2 - 3k + 2 = 0$ *i.e.*, $k = 1$ or $k = 2$.

when $k = 1$; the augmented matrix $= \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & 2 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$

$\therefore$ Equations are
$$x + y + z = 1$$
$$-y + 2z = -1$$

Let $z = \lambda,\ y = 1 + 2\lambda,\ x = -3\lambda$, where λ is an arbitrary constant

when $k = 2$; Augmented matrix $= \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & 2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$

Equations are $x + y + z = 1$ and $-y + 2z = 0$.

Take $z = \lambda',\ y = 2\lambda',\ x = 1 - 3\lambda'$

where λ' is any arbitrary constant.

Example 7. *Find the values of a and b for which the equations $x + ay + z = 3$; $x + 2y + 2z = b$, $x + 5y + 3z = 9$ are consistent. When will these equations have a unique solution?* **(P.T.U., Dec. 2003)**

Sol. Given equations are

$$x + ay + z = 3$$
$$x + 2y + 2z = b$$
$$x + 5y + 3z = 9.$$

The matrix equation is AX = B, where

$$A = \begin{bmatrix} 1 & a & 1 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{bmatrix},\ B = \begin{bmatrix} 3 \\ b \\ 9 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Augmented matrix is $\begin{bmatrix} 1 & a & 1 & \vdots & 3 \\ 1 & 2 & 2 & \vdots & b \\ 1 & 5 & 3 & \vdots & 9 \end{bmatrix}$

Operate $R_2 - R_1, R_3 - R_1 \sim \begin{bmatrix} 1 & a & 1 & \vdots & 3 \\ 0 & 2-a & 1 & \vdots & b-3 \\ 0 & 5-a & 2 & \vdots & 6 \end{bmatrix}$

Operate $R_3 - \dfrac{5-a}{2-a} R_2 \sim \begin{bmatrix} 1 & a & 1 & \vdots & 3 \\ 0 & 2-a & 1 & \vdots & b-3 \\ 0 & 0 & 2 - \frac{5-a}{2-a} & \vdots & 6 - \frac{5-a}{2-a}(b-3) \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & a & 1 & \vdots & 3 \\ 0 & 2-a & 1 & \vdots & b-3 \\ 0 & 0 & \frac{-1-a}{2-a} & \vdots & 6 - \frac{5-a}{2-a}(b-3) \end{bmatrix}$$

Case I. If $\dfrac{-1-a}{2-a} = 0$ and $6 - \dfrac{5-a}{2-a}(b-3) = 0$

Then $a = -1 \therefore b = 6$ *i.e.*, $a = -1, b = 6$

$\rho(A) = \rho(A : B) = 2 <$ number of unknowns

$\therefore$ Given equations have infinite number of solutions given by augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & \vdots & 3 \\ 0 & 3 & 1 & \vdots & 3 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Equations are $x - y + z = 3$

$$3y + z = 3$$

Let $z = k\,;\, y = \dfrac{3-k}{3} \quad \therefore\ x = 3 + \dfrac{3-k}{3} - k = 4 - \dfrac{4}{3}k$

$\therefore \quad x = 4 - \dfrac{4}{3}k\,,\ y = \dfrac{3-k}{3}\,,\ z = k$ is the solution.

Case II. If $a = -1$, but $b \neq 6$

Then ρ (A) = 2, but ρ (A : B) = 3

ρ (A) $\neq$ ρ (A : B) $\therefore$ Equations are inconsistent, *i.e.,* having no solution.

Case III. If $a \neq -1$, b can have any value, then ρ (A) = 3 = ρ (A : B)

$\therefore$ Given equations have a unique solution which is given by the equation.

$$x + ay + z = 3$$

$$(2-a)\,y + z = b - 3$$

$$\frac{-(1+a)}{2-a}\,z = 6 - \frac{5-a}{2-a}(b-3)$$

$$\therefore \quad z = \frac{2-a}{-(1+a)}\left\{6 - \frac{(5-a)(b-3)}{2-a}\right\} = \frac{6(2-a)-(5-a)(b-3)}{-(1+a)}$$

$$y = \frac{1}{2-a}\left[(b-3) - \frac{6(2-a)-(5-a)(b-3)}{-(1+a)}\right]$$

$$= \frac{1}{(2-a)(1+a)}\left[(b-3)(1+a) + 6(2-a) - (5-a)(b-3)\right]$$

$$\therefore \quad y = \frac{2(6-b)}{1+a}\ ;\ x = \frac{(5-3a)(6-b)}{1+a}.$$

TEST YOUR KNOWLEDGE

1. Write the following equations in matrix form AX = B and solve for X by finding A^{-1}.

(*i*) $2x - 2y + z = 1$, $x + 2y + 2z = 2$, $2x + y - 2z = 7$

(*ii*) $2x_1 - x_2 + x_3 = 4$, $x_1 + x_2 + x_3 = 1$, $x_1 - 3x_2 - 2x_3 = 2$

2. Using the loop current method on a circuit, the following equations were obtained :

$7i_1 - 4i_2 = 12,\ -4i_1 + 12i_2 - 6i_3 = 0,\ 6i_2 + 14i_3 = 0.$

By matrix method, solve for i_1, i_2 and i_3

3. Solve the following system of equations by matrix method :

(*i*) $x + y + z = 8$, $x - y + 2z = 6$, $3x + 5y - 7z = 14$

(*ii*) $x + y + z = 6$, $x - y + 2z = 5$, $3x + y + z = 8$

(*iii*) $x + 2y + 3z = 1$, $2x + 3y + 2z = 2$, $3x + 3y + 4z = 1$.

(*iv*) $3x + 3y + 2z = 1$, $x + 2y = 4$, $10y + 3z = -2$, $2x - 3y - z = 5$

4. Show that the equations $x + 2y - z = 3$, $3x - y + 2z = 1$, $2x - 2y + 3z = 2$, $x - y + z = -1$ are consistent and solve them.

5. Test for consistency the equations $2x - 3y + 7z = 5$, $3x + y - 3z = 13$, $2x + 19y - 47z = 32$.

(P.T.U., May 2012, Dec. 2012)

6. Solve the equations $x + 3y + 2z = 0$, $2x - y + 3z = 0$, $3x - 5y + 4z = 0$, $x + 17y + 4z = 0$.

7. (*a*) For what values of a and b do the equations $x + 2y + 3z = 6$, $x + 3y + 5z = 9$, $2x + 5y + az = b$ have (*i*) no solution, (*ii*) a unique solution, (*iii*) more than one solution ?

(*b*) For what value of k the system of equations $x + y + z = 2$, $x + 2y + z = -2$, $x + y + (k-5)z = k$ has no solution ?

(P.T.U., May 2012)

8. (*a*) Find the value of k so that the equations $x + y + 3z = 0, 4x + 3y + kz = 0, 2x + y + 2z = 0$ have a non-trivial solution.

(*b*) For what values of λ do the equations $ax + by = \lambda x$ and $cx + dy = \lambda y$ have a solution other than $x = 0, y = 0$. **(P.T.U., May 2003)**

9. Show that the equations $3x + 4y + 5z = a$, $4x + 5y + 6z = b$, $5x + 6y + 7z = c$ do not have a solution unless $a + c = 2b$. **(P.T.U., Dec. 2011)**

10. Investigate the value of λ and μ so that the equations $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ have (*i*) no solution, (*ii*) a unique solution, and (*iii*) an infinite number of solution.

11. Determine the value of λ for which the following set of equations may possess non-trivial solution.

$3x_1 + x_2 - \lambda x_3 = 0, 4x_1 - 2x_2 - 3x_3 = 0, 2\lambda x_1 + 4x_2 + \lambda x_3 = 0$. For each permissible value of λ, determine the general solution.

12. Investigate for consistency of the following equations and if possible find the solutions.

$4x - 2y + 6z = 8,\ x + y - 3z = -1,\ 15x - 3y + 9z = 21.$ **(P.T.U., Jan. 2009)**

13. Show that if $\lambda \neq -5$, the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ have a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solutions in each case.

14. Show that the equations $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$, $6y - 18z + 1 = 0$ are not consistent.

[**Hint:** To prove $\rho[A : B] \neq \rho(A)$] **(P.T.U., Dec. 2003)**

15. Solve the system of equations $2x_1 + x_2 + 2x_3 + x_4 = 6$; $6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$

$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$; $2x_1 + 2x_2 - x_3 + x_4 = 10$.

ANSWERS

1. (*i*) $x = 2, y = 1, z = -1$ (*ii*) $x_1 = 1, x_2 = -1, x_3 = 1$ **2.** $i_1 = \frac{396}{175}, i_2 = \frac{24}{25}, i_3 = \frac{72}{175}$

3. (*i*) $x = 5, y = \frac{5}{3}, z = \frac{4}{3}$ (*ii*) $x = 1, y = 2, z = 3$ (*iii*) $x = -\frac{3}{7}, y = \frac{8}{7}, z = -\frac{2}{7}$

(*iv*) $x = 2, y = 1, z = -4$

4. $x = -1, y = 4, z = 4$ **5.** Inconsistent **6.** $x = 11k, y = k, z = -7k$, where k is arbitrary

7. (*a*) (*i*) $a = 8, b \neq 15$ (*ii*) $a \neq 8$, b may have any value (*iii*) $a = 8, b = 15$

(*b*) $k = 6$

8. (*a*) $k = 8$, (*b*) $\lambda = a, b = 0, \lambda = d, c = 0$

10. (*i*) $\lambda = 5, \mu \neq 9$ (*ii*) $\lambda \neq 5$, μ arbitrary (*iii*) $\lambda = 5, \lambda = 9$

11. $\lambda = 1, -9$ for $\lambda = 1$ solution is $x = k, y = -k, z = 2k$. For $\lambda = -9$ solution is $x = 3k, y = 9k, z = -2k$

12. Consistent: $x = 1, y = 3k - 2, z = k$, where k is arbitrary

13. $\lambda \neq -5, x = \frac{4}{7}, y = -\frac{9}{7}, z = 0$; $\lambda = -5, x = \frac{4 - 5k}{7}, y = \frac{13k - 9}{7}, z = k$, where k is arbitrary

15. $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$.

4.14. VECTORS

Any ordered n-tuple of numbers is called an n-vector. By an ordered n-tuple, we mean a set consisting of n numbers in which the place of each number is fixed. If $x_1, x_2, \ldots\ldots\ldots, x_n$ be any n numbers then the ordered n-tuple $X = (x_1, x_2, \ldots\ldots\ldots\ldots, x_n)$ is called an n-vector. Thus the co-ordinates of a point in space can be represented by a 3-vector (x, y, z). Similarly $(1, 0, 2, -1)$ and $(2, 7, 5, -3)$ are 4-vectors. The n numbers $x_1, x_2, \ldots\ldots\ldots, x_n$ are called the components of the n-vector $X = (x_1, x_2, \ldots\ldots, x_n)$. A vector may be written either as a *row vector or* as a *column vector.* If A be a matrix of order $m \times n$, then each row of A will be an n-vector and each column of A will be an m-vector. A vector whose components are all zero is called a zero vector and is denoted by O. Thus $O = (0, 0, 0, \ldots., 0)$.

Let $\quad X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be two vectors.

Then X = Y if and only if their corresponding components are equal.

i.e., If $\quad x_i = y_i$ for $i = 1, 2, ..., n$

If k be a scalar, then $kX = (kx_1, kx_2, ..., kx_n)$.

4.15. LINEAR DEPENDENCE AND LINEAR INDEPENDENCE OF VECTORS

(P.T.U., May 2004, 2006, Jan. 2009)

A set of r, n-tuple vectors $X_1, X_2, ...,X_r$ is said to be *linearly dependent* if there exists r scalars (numbers) $k_1, k_2, ..., k_r$ not all zero, such that

$$k_1X_1 + k_2X_2 + ... + k_rX_r = O$$

A set of r, n-tuple vectors $X_1, X_2, ..., X_r$ is said to be *linearly independent* if every relation of the type

$$k_1X_1 + k_2X_2 + ... + k_rX_r = O \text{ implies } k_1 = k_2 = ... = k_r = 0$$

Note. If a set of vectors is linearly dependent, then at least one member of the set can be expressed as a linear combination of the remaining vectors.

Example 1. *Show that the vectors $x_1 = (1, 2, 4)$, $x_2 = (2, -1, 3)$, $x_3 = (0, 1, 2)$ and $x_4 = (-3, 7, 2)$ are linearly dependent and find the relation between them.*

Sol. Consider the matrix equation

$$k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 = 0$$

i.e., $\quad k_1 (1, 2, 4) + k_2 (2, -1, 3) + k_3 (0, 1, 2) + k_4 (-3, 7, 2) = 0$

i.e.,
$$\begin{aligned} k_1 + 2k_2 + 0\cdot k_3 - 3k_4 &= 0 \\ 2k_1 - k_2 + k_3 + 7k_4 &= 0 \\ 4k_1 + 3k_2 + 2k_3 + 2k_4 &= 0 \end{aligned}$$

which is a system of homogeneous linear equations and can be put in the form AX = 0.

i.e.,
$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 4R_1$;

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 - R_2$;

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad k_1 + 2k_2 - 3k_4 = 0 \qquad \ldots(i)$$

$$-5k_2 + k_3 + 13k_4 = 0 \qquad \ldots(ii)$$

$$k_3 + k_4 = 0 \qquad \ldots(iii)$$

$$\therefore \quad k_4 = -k_3$$

From (*ii*), $5k_2 = k_3 - 13k_3 = -12k_3 \therefore k_2 = -\frac{12}{5}k_3$

From (*i*), $k_1 = +3k_4 - 2k_2 = -3k_3 + \frac{24}{5}k_3 = \frac{9}{5}k_3$

Let $k_3 = t$

$\therefore \quad k_1 = \frac{9}{5}t,\ k_2 = -\frac{12}{5}t,\ k_3 = t,\ k_4 = -t$

$\therefore$ Given vectors are L.D. and the relation between them is $\frac{9}{5}t\,x_1 - \frac{12}{5}t\,x_2 + tx_3 - tx_4 = 0$

or $9x_1 - 12x_2 + 5x_3 - 5x_4 = 0.$

Example 2. *Show that the column vectors of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \end{bmatrix} \text{ are linearly dependent.}$$ **(P.T.U., Dec. 2002)**

Sol. Let $X_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Consider the matrix equation

$$k_1 X_1 + k_2 X_2 + k_3 X_3 = 0$$

$$k_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 0$$

i.e.,
$$k_1 + 2k_2 + 3k_3 = 0$$
$$-2k_1 + k_2 + 2k_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

Operate $R_2 + 2R_1$; $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$

or
$$k_1 + 2k_2 + 3k_3 = 0$$
$$5k_2 + 8k_3 = 0$$

$\therefore \quad k_2 = -\frac{8}{5}k_3 \; ; k_1 = \frac{16}{5}k_3 - 3k_3 = \frac{1}{5}k_3$

Let $k_3 = \lambda \neq 0$

$\therefore \quad k_1 = \frac{1}{5}\lambda, k_2 = -\frac{8}{5}\lambda, k_3 = \lambda$

$\therefore$ Given column vectors are L.D.

Example 3. *Determine whether the vectors $(3, 2, 4)^t, (1, 0, 2)^t, (1, -1, -1)^t$ are linearly dependent or not. (where 't' denotes transpose)* **(P.T.U., May 2006)**

Sol. Let $X_1 = (3, 2, 4)^t = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, X_2 = (1, 0, 2)^t = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, X_3 = (1, -1, -1)^t = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

Consider $k_1 X_1 + k_2 X_2 + k_3 X_3 = 0$

i.e.,
$$k_1 \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 0$$

or
$$\begin{aligned} 3k_1 + k_2 + k_3 &= 0 \\ 2k_1 + 0.k_2 - k_3 &= 0 \\ 4k_1 + 2k_2 - k_3 &= 0 \end{aligned}$$

or
$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & -1 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_1 - R_2$;
$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 4R_1$;
$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & -2 & -9 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 - R_2$;
$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \left.\begin{aligned} k_1 + k_2 + 2k_3 &= 0 \\ -2k_2 - 5k_3 &= 0 \\ -4k_3 &= 0 \end{aligned}\right] \Rightarrow \quad k_3 = 0, k_2 = 0, k_1 = 0$$

$\therefore$ Given vectors are not linearly dependent. These are linearly independent.

4.16. LINEAR TRANSFORMATIONS (P.T.U., May 2014)

Let a point P(x, y) in a plane transform to the point P' (x', y') under reflection in the co-ordinate axes, or reflection in the line $y = x \tan \theta$ or rotation of OP through an angle θ about the origin or rotation of axes, through an angle θ etc. Then the co-ordinates of P' can be expressed in terms of those of P by the linear relations of the form

$$\left.\begin{aligned} x' &= a_1 x + b_1 y \\ y' &= a_2 x + b_2 y \end{aligned}\right]$$

which in matrix notation is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ or $X' = AX$

such transformations are called linear transformation in two dimensions.

Similarly, relations of the form

$$\begin{aligned} x' &= a_1 x + b_1 y + c_1 z \\ y' &= a_2 x + b_2 y + c_2 z \\ z' &= a_3 x + b_3 y + c_3 z \end{aligned}$$

which in matrix notation is $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ or $X' = AX$ gives a linear transformation $(x, y, z) \to (x', y', z')$ in three dimensions.

In general, the relation $Y = AX$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

defines a linear transformation which carries any vector X into another vector Y over the matrix A which is called the linear operator of the transformation.

This transformation is called linear because $Y_1 = AX_1$ and $Y_2 = AX_2$ implies $aY_1 + bY_2 = A(aX_1 + bX_2)$ for all values of a and b.

Thus, if $X = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, then $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

so that $(2, -3) \to (5, -5)$ under the transformation defined by A.

If the transformation matrix A is non-singular, *i.e.*, if $|A| \neq 0$, then the linear transformation is called **non-singular or regular.**

If the transformation matrix A is singular, *i.e.*, if $|A| = 0$, then the linear transformation is also called **singular.**

For a non-singular transformation $Y = AX$, since A is non-singular, A^{-1} exists and we can write the inverse transformation, which carries the vector Y back into the vector X, as $X = A^{-1} Y$.

Note. If a transformation from $(x_1, x_2, \ldots\ldots\ldots\ldots, x_n)$ to $(y_1, y_2, \ldots\ldots\ldots\ldots, y_n)$ is given by $Y = AX$ and another transformation from $(y_1, y_2, \ldots\ldots\ldots\ldots, y_n)$ to $(z_1, z_2, \ldots\ldots\ldots\ldots, z_n)$ is given by $Z = BY$, then the transformation from $(x_1, x_2, \ldots\ldots\ldots\ldots, x_n)$ to $(z_1, z_2, \ldots\ldots\ldots\ldots, z_n)$ is given by $Z = BY = B(AX) = (BA)X$.

4.17. ORTHOGONAL TRANSFORMATION (P.T.U., Dec. 2012)

The linear transformation $Y = AX$, where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \; A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is said to be *orthogonal* if it transforms $y_1^2 + y_2^2 + \ldots\ldots\ldots + y_n^2$ into $x_1^2 + x_2^2 + \ldots\ldots\ldots + x_n^2$.

4.18(a). ORTHOGONAL MATRIX (P.T.U., Jan. 2009)

The matrix A of the above transformation is called ***an orthogonal matrix.***

Now, $$X'X = \begin{bmatrix} x_1 x_2 \ldots\ldots x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \ldots\ldots + x_n^2$$

and similarly $\quad Y'Y = y_1^2 + y_2^2 + \ldots.. + y_n^2$.

$\therefore$ If Y = AX is an orthogonal transformation, then

$$X'X = x_1^2 + x_2^2 + \ldots.. + x_n^2 = y_1^2 + y_2^2 + \ldots.. + y_n^2.$$
$$= Y'Y = (AX)'(AX) = (X'A')(AX)$$
$$= X'(A'A)X \qquad [\because (AB)' = B'A']$$

which holds only when A'A = I or when $A'A = A^{-1}A$

or when $\quad A' = A^{-1} \qquad [\because A^{-1}A = 1]$

Hence *a real square matrix A is said to be orthogonal if AA' =A' A = I*

Also, *for an orthogonal matrix* A, $A' = A^{-1}$.

4.18(b). PROPERTIES OF AN ORTHOGONAL MATRIX

(*i*) **The transpose of an orthogonal matrix is orthogonal.**

Proof. Let A be an orthogonal matrix

$$\therefore \quad AA' = I = A'A$$

Taking transpose of both sides of AA' = I

$(AA')' = I'$ or $(A')'A' = I$ *i.e.*, product of A' and its transpose *i.e.*, $(A')' = I$

$\therefore$ A' is an orthogonal matrix.

(*ii*) **The inverse of an orthogonal matrix is orthogonal**

Proof. Let A be an orthogonal matrix $\therefore$ AA' = I

Take inverse of both sides $(AA')^{-1} = I^{-1}$

or $\quad (A')^{-1}A^{-1} = I \quad$ or $(A^{-1})'(A^{-1}) = I$

i.e., Product of A^{-1} and its transpose *i.e.*, $(A^{-1})'$ is I

$\therefore$ A^{-1} is orthogonal.

(*iii*) **If A is an orthogonal matrix, then | A | = ± 1**

Proof. A is an orthogonal matrix

$\therefore$ AA' = I

Take determinant of both sides

$|AA'| = |I| \quad$ or $\quad |A||A'| = 1 \qquad [\because |I| = 1]$

i.e., $\quad |A|^2 = 1 \qquad [\because |A'| = |A|]$

i.e., $\quad |A| = \pm 1$

Note. An orthogonal matrix A is called proper or improper according as | A | = 1 or – 1.

(*iv*) **The product of two orthogonal matrices of the same order is orthogonal**

Proof. Let A, B be two orthogonal matrices of the same order so that

$$AA' = BB' = I$$

Now, $\quad (AB)(AB)' = (AB)(B'A') = A(BB')A'$
$$= A(I)A' = (AI)A' = AA' = I$$

$\therefore$ AB is also an orthogonal matrix.

Example 1. *Let T be the transformation from R^1 to R^3 defined by $T(x) = (x, x^2, x^3)$. Is T linear or not?* **(P.T.U., May 2010)**

Sol. Given $\quad T(x) = (x, x^2, x^3)$

$$\mathrm{T}(x_1) = (x_1, x_1^2, x_1^3)$$
$$\mathrm{T}(x_2) = (x_2, x_2^2, x_2^3)$$
$$\alpha\mathrm{T}(x_1) + \beta\mathrm{T}(x_2) = \alpha(x_1, x_1^2, x_1^3) + \beta(x_2, x_2^2, x_2^3)$$
$$= (\alpha x_1 + \beta x_2, \alpha x_1^2 + \beta x_2^2, \alpha x_1^3 + \beta x_2^3)$$

Now, $\mathrm{T}(\alpha x_1 + \beta x_2) = [(\alpha x_1 + \beta x_2), (\alpha x_1 + \beta x_2)^2, (\alpha x_1 + \beta x_2)^3]$

$$\neq (\alpha x_1 + \beta x_2, \alpha x_1^2 + \beta x_2^2, \alpha x_1^3 + \beta x_2^3)$$

$\therefore \quad \alpha\mathrm{T}(x_1) + \beta\mathrm{T}(x_2) \neq \mathrm{T}(\alpha x_1 + \beta x_2)$

$\therefore$ T is not linear

Example 2. *Show that the transformation $y_1 = x_1 + 2x_2 + 5x_3$; $y_2 = -x_2 + 2x_3$; $y_3 = 2x_1 + 4x_2 + 11x_3$ is regular. Write down the inverse transformation.* **(P.T.U., May 2011)**

Sol. The given transformation in the matrix form is Y = AX, where

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{bmatrix}; \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \mathrm{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$|\mathrm{A}| = \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{vmatrix}$$

$$= 1(-11-8) + 2(4+5)$$
$$= -19 + 18 = -1 \neq 0$$

$\therefore$ Matrix A is non-singular.

Hence given transformation is non-singular or regular.

The inverse transformation of Y = AX is $\mathrm{X} = \mathrm{A}^{-1}\mathrm{Y}$

To find A^{-1}

Consider A = IA

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{A}$$

Operate $\mathrm{R}_3 - 2\mathrm{R}_1$;

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \mathrm{A}$$

Operate $\mathrm{R}_1 - 5\mathrm{R}_3$, $\mathrm{R}_2 - 2\mathrm{R}_3$;

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & -5 \\ 4 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix} \mathrm{A}$$

Operate $\mathrm{R}_1 + 2\mathrm{R}_2$;

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 2 & -9 \\ 4 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix} \mathrm{A}$$

Operate $R_2(-1)$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix} A$

$$I = BA, \text{ where } B = \begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = B$$

$$\therefore \qquad X = \begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix} Y$$

$$\therefore \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = 19y_1 + 2y_2 - 9y_3$$
$$x_2 = -4y_1 - y_2 + 2y_3$$
$$x_3 = -2y_1 + y_3$$

Example 3. *(a) Prove that the following matrix is orthogonal*

$$\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$ **(P.T.U., May 2007)**

(b) Find the values of a, b, c if the matrix

$$A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \text{ is orthogonal.}$$ **(P.T.U., May 2009)**

Sol. (*a*) Denoting the given matrix by A, we have

$$A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Now, $$AA' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since AA' = I, A is an orthogonal matrix.

(*b*) Matrix A will be orthogonal if AA′ = I

i.e., $$\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $\quad 4b^2 + c^2 = 1 \quad$ and $\quad 2b^2 - c^2 = 0$

$a^2 + b^2 + c^2 = 1 \qquad a^2 - b^2 - c^2 = 0$

Solving $\quad c^2 = 2b^2 \quad \therefore \quad 4b^2 + 2b^2 = 1 \quad$ or $\quad b^2 = \dfrac{1}{6}$

or $\quad b = \pm\dfrac{1}{\sqrt{6}}$

$\therefore \quad c^2 = 2 \cdot \dfrac{1}{6} = \dfrac{1}{3} \quad \therefore \quad c = \pm\dfrac{1}{\sqrt{3}}$

$a^2 = b^2 + c^2 = \dfrac{1}{6} + \dfrac{1}{3} = \dfrac{1}{2} \quad \therefore \quad a = \pm\dfrac{1}{\sqrt{2}}$

Hence, $\quad a = \pm\dfrac{1}{\sqrt{2}}, b = \pm\dfrac{1}{\sqrt{6}}, c = \pm\dfrac{1}{\sqrt{3}}$.

TEST YOUR KNOWLEDGE

1. Are the following vectors linearly dependent ? If so, find a relation between them.

(*i*) $x_1 = (1, 2, 1), x_2 = (2, 1, 4), x_3 = (4, 5, 6), x_4 = (1, 8, -3)$ **(P.T.U., Jan. 2010)**

(*ii*) $x_1 = (2, -1, 4), x_2 = (0, 1, 2), x_3 = (6, -1, 16), x_4 = (4, 0, 12)$

(*iii*) $x_1 = (2, -1, 3, 2), x_2 = (1, 3, 4, 2), x_3 = (3, -5, 2, 2)$

(*iv*) $x_1 = (2, 3, 1, -1), x_2 = (2, 3, 1, -2), x_3 = (4, 6, 2, 1)$

(*v*) $x_1 = (2, 2, 1)^t, x_2 = (1, 3, 1)^t, x_3 = (1, 2, 2)^t$, where '$t$' stands for transpose. [**Hint:** See S.E. 3]

(*vi*) $x_1 = (1, 1, 1), x_2 = (1, -1, 1), x_3 = (3, -1, 3)$ **(P.T.U., Dec. 2012)**

2. For what value(s) of k, do the set of vectors $(k, 1, 1), (0, 1, 1), (k, 0, k)$ in R^3 are linearly independent? **(P.T.U., May 2010, 2012)**

3. (*a*) Show that the transformation $y_1 = x_1 - x_2 + x_3, y_2 = 3x_1 - x_2 + 2x_3, y_3 = 2x_1 - 2x_2 + 3x_3$ is non-singular. Find the inverse transformation.

(*b*) Show that the transformation $y_1 = 2x_1 + x_2 + x_3; y_2 = x_1 + x_2 + 2x_3; y_3 = x_1 - 2x_3$ is regular. Write down the inverse transformation.

4. Represent each of the transformation $x_1 = 3y_1 + 2y_2, \; y_1 = z_1 + 2z_2, x_2 = -y_1 + 4y_2$ and $y_2 = 3z_1$ by the use of matrices and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2.

5. A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by Y = AX, and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by Z = BY, where $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 1 & 3 & 5 \end{bmatrix}$. Obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3.

6. Which of the following matrices are orthogonal?

(*i*) $\dfrac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$. **(P.T.U., Jan. 2009)**

7. Prove that the following matrix is orthogonal:

$$\begin{bmatrix} -\dfrac{2}{3} & \dfrac{1}{3} & \dfrac{2}{3} \\ \dfrac{2}{3} & \dfrac{2}{3} & \dfrac{1}{3} \\ \dfrac{1}{3} & -\dfrac{2}{3} & \dfrac{2}{3} \end{bmatrix}$$

(P.T.U., May 2011)

ANSWERS

1. (*i*) Yes ; $x_3 = 2x_1 + x_2$ and $x_4 = 5x_1 - 2x_2$ (*ii*) Yes ; $x_3 = 3x_1 + 2x_2$ and $x_4 = 2x_1 + x_2$
(*iii*) Yes ; $2x_1 - x_2 - x_3 = 0$ (*iv*) Yes ; $5x_1 - 3x_2 - x_3 = 0$
(*v*) No; L.I. (*vi*) No; L.I.

2. For all non-zero values of k

3. (*a*) $x_1 = \frac{1}{2}(y_1 + y_2 - y_3),\ x_2 = \frac{1}{2}(-5y_1 + y_2 + y_3),\ x_3 = -4y_1 + 2y_3$
(*b*) $x_1 = 2y_1 - 2y_2 - y_3,\ x_2 = -4y_1 + 5y_2 + 3y_3,\ x_3 = y_1 - y_2 - y_3$

4. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ **5.** Z = (BA) X, where BA = $\begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix}$

6. (*i*) Orthogonal (*ii*) Not orthogonal.

4.19. COMPLEX MATRICES

If *all* the elements of a matrix are real numbers, then it is called *a real matrix* or a matrix over R. On the other hand, if *at least one* element of a matrix is a complex number $a + ib$, where a, b are real and $i = \sqrt{-1}$, then the matrix is called a *complex matrix.*

4.20(a). CONJUGATE OF A MATRIX

The matrix obtained by replacing the elements of a complex matrix A by the corresponding conjugate complex numbers is called the *conjugate of the matrix A* and is denoted by $\bar{A}$.

Thus, if $$A = \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix}, \text{ then } \bar{A} = \begin{bmatrix} 2-3i & 7i \\ 5 & 1+i \end{bmatrix}.$$

4.20(b). CONJUGATE TRANSPOSE OF A MATRIX

It is easy to see the *conjugate of the transpose of A i.e.,* $(\overline{A'})$ and the *transpose conjugate of A i.e.,* $(\bar{A})'$ are equal. Each of them is denoted by A^{θ}.

Thus $$(\overline{A'}) = (\bar{A})' = A^{\theta}.$$

4.21. HERMITIAN AND SKEW HERMITIAN MATRIX (P.T.U., May 2002, 2007, Dec. 2010)

A square matrix A is said to be **Hermitian if $A^{\theta} = A$.** *i.e.,* if $A = [a_{ij}]$, then $\overline{a_{ij}} = a_{ji}\ \forall\ i, j$ and when $i = j$, then $\overline{a_{ii}} = a_{ii} \Rightarrow a_{ii}$ is purely real *i.e.,* **all diagonal elements of a Hermitian matrix are purely real** while every other element is the conjugate complex of the element in the transposed position.

For example, $A = \begin{bmatrix} 5 & 2+i & -3i \\ 2-i & -3 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$ is a Hermitian matrix.

A square matrix A is said to be **Skew Hermitian** if $A^{\theta} = -A$ *i.e.,* if $A = [a_{ij}]$, then $\overline{a_{ij}} = -a_{ji}\ \forall\ i, j$ and when $i = j$, then $\overline{a_{ii}} = -a_{ii}$ *i.e.,* if $a_{ii} = a + ib$, then $\overline{a_{ii}} = a - ib$ and $\overline{a_{ii}} = -a_{ii}$

$\Rightarrow\ a - ib = -(a + ib) \Rightarrow a = 0$

$\therefore\ a_{ii}$ is either purely imaginary or zero.

In a Skew Hermitian matrix, **the diagonal elements are zero or purely imaginary number** of the form $i\beta$, where β is real. Every other element is the negative of the conjugate complex of the element in the transposed position.

For example, $B = \begin{bmatrix} 3i & 1+i & 7 \\ -1+i & 0 & -2-i \\ -7 & 2-i & -i \end{bmatrix}$ is a Skew Hermitian matrix.

Note. The following result hold :

(*i*) $\overline{(\overline{A})} = A$ (*ii*) $\overline{A+B} = \overline{A}+\overline{B}$ (*iii*) $\overline{\lambda A} = \overline{\lambda}\,\overline{A}$ (*iv*) $\overline{AB} = \overline{A}\,\overline{B}$

(*v*) $(A^{\theta})^{\theta} = A$ (*vi*) $(A+B)^{\theta} = A^{\theta} + B^{\theta}$ (*vii*) $(lA)^{q} = \overline{l}A^{q}$ (*viii*) $(AB)^{\theta} = B^{\theta}A^{\theta}$.

4.22(a). UNITARY MATRIX

A complex square matrix A is said to unitary if $A^{\theta}A = I$

or we can say $(\overline{A}')A = I$

Taking conjugate of both sides $A'\overline{A} = I$

Incase of real matrices : If A is a real matrix there $\overline{A} = A$, then A will be unitary if $A'\overline{A} = I \Rightarrow A'A = I$ which clearly shows that A is also an orthogonal matrix.

Hence every orthogonal matrix is unitary.

4.22(b). PROPERTIES OF A UNITARY MATRIX

(*i*) Determinant of a unitary matrix is of modulus unity

Proof. Let A be a unitary matrix

Then $AA^{\theta} = I$

Taking determinant of both sides $|AA^{\theta}| = |I|$

or $|A|\,|\overline{A}'| = 1$ or $|A|\,|\overline{A}| = 1$ $[\because\ |A'| = |A|]$

or $|A^2| = 1$ hence the result.

(*ii*) The product of two unitary matrices of the same order is unitary

Proof. Let A, B be two unitary matrices $\therefore\ AA^{\theta} = A^{\theta}A = I$ and $BB^{\theta} = B^{\theta}B = I$

Now, $(AB)(AB)^{\theta} = AB(B^{\theta}A^{\theta}) = A(BB^{\theta})A^{\theta} = (AI)A^{\theta} = AA^{\theta} = I$

Hence AB is unitary matrix.

(*iii*) The inverse of a unitary matrix is unitary **(P.T.U., May 2012)**

Proof. Let A be a unitary matrix $\therefore AA^{\theta} = A^{\theta}A = 1$

$AA^{\theta} = I$

Take inverse of both sides $(AA^{\theta})^{-1} = I$ or $(A^{\theta})^{-1}.\ A^{-1} = I$

or $(A^{-1})^{\theta}(A^{-1}) = I$ $\therefore A^{-1}$ is also unitary.

ILLUSTRATIVE EXAMPLES

Example 1. *If* $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$, *verify that $A^{\theta}A$ is a Hermitian matrix.*

Sol. $A' = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$

$$A^\theta = \overline{(A')} = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$\therefore \quad A^\theta A = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix} \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 6-8i & -19+17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5-5i & 30 \end{bmatrix} = B\text{(say)}$$

Now,
$$B' = \begin{bmatrix} 30 & 6+8i & -19-17i \\ 6-8i & 10 & -5-5i \\ -19-17i & -5-5i & 30 \end{bmatrix}$$

$$B^\theta = \overline{(B')} = \begin{bmatrix} 30 & 6-8i & -19+17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5-5i & 30 \end{bmatrix} = B$$

$\Rightarrow$ B $(= A^\theta A)$ is a Hermitian matrix.

Example 2. *If A and B are Hermitian, show that AB – BA is Skew Hermitian.*

Sol. A and B are Hermitian. $\Rightarrow A^\theta = A$ and $B^\theta = B$

Now, $(AB - BA)^\theta = (AB)^\theta - (BA)^\theta$

$= B^\theta A^\theta - A^\theta B^\theta = BA - AB = -(AB - BA)$

$\Rightarrow$ AB – BA is Skew Hermitian.

Example 3. *(a) If A is a Skew Hermitian matrix, then show that iA is Hermitian.*

(b) If A is Hermitian, then A^θ A is also Hermitian. **(P.T.U., May 2007)**

Sol. (*a*) A is a Skew Hermitian matrix $\Rightarrow A^\theta = -A$

Now, $(iA)^\theta = \bar{i}A^\theta = (-i)(-A) = iA$

$\Rightarrow$ *i*A is a Hermitian matrix.

(*b*) A is a Hermitian Matrix $\therefore A^\theta = A$

$A^\theta A$ will be Hermitian if $(A^\theta A)^\theta = A^\theta A$

Now, $(A^\theta A)^\theta = A^\theta (A^\theta)^\theta = A^\theta \cdot A$

Hence A^θ A is Hermitian

Example 4. *If* $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, *obtain the matrix* $(I - N)(I + N)^{-1}$, *and show that it is unitary.*

Sol.
$$I - N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$I + N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$|I + N| = \begin{vmatrix} 1 & 1+2i \\ -1+2i & 1 \end{vmatrix} = 1 - (4i^2 - 1) = 6 \therefore \text{ I + N is non-singular and } (I+N)^{-1} \text{ exists}$$

$$\text{adj}(I + N) = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$(I+N)^{-1} = \frac{1}{|I+N|}\text{adj}\,(I+N) = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$\therefore \quad (I-N)\,(I+N)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = A \text{ (say)}$$

$$A' = \frac{1}{6}\begin{bmatrix} -4 & 2-4i \\ -2-4i & -4 \end{bmatrix}$$

$$\overline{(A)}' = A^{\theta} = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$A^{\theta}A = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$A = (I-N)\,(I+N)^{-1}$ is unitary.

Example 5. *Prove that every Hermitian matrix can be written as A + iB, where A is real and symmetric and B is real and skew-symmetric.*

Sol. Let P be any Hermitian matrix.

Then $\quad P^{\theta} = P$

Consider, $\quad P = \dfrac{P+\overline{P}}{2} + i\,\dfrac{P-\overline{P}}{2i} = A + iB$, where

$$A = \frac{P+\overline{P}}{2},\ B = \frac{P-\overline{P}}{2i}$$

To prove A and B are real.

We know that $\quad z = x+iy \therefore \bar{z} = x-iy$, then $\dfrac{z+\bar{z}}{2} = 2x$ (real)

and $\quad \dfrac{z-\bar{z}}{2i} = \dfrac{2iy}{2i} = y$ (real)

Similarly,

$\dfrac{P+\overline{P}}{2}$ is a real matrix and $\dfrac{P-\overline{P}}{2i}$ is also real

$\therefore$ A, B are real.

To prove A is symmetric

$$A' = \left(\frac{P+\overline{P}}{2}\right)' = \frac{P'+P^{\theta}}{2} = \frac{P'+P}{2} = A \qquad (\because \quad P^{\theta} = P)$$

$\therefore$ A is symmetric.

Similarly, $B' = \left(\dfrac{P-\overline{P}}{2i}\right)' = \dfrac{P'-P^{\theta}}{2i} = \dfrac{P'-P}{2i} = -\dfrac{P-P'}{2i} = -B \therefore$ B is skew-symmetric.

TEST YOUR KNOWLEDGE

1. If $A = \begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{bmatrix}$, show that A is a Hermitian matrix and iA is a Skew-Hermitian matrix.

(P.T.U., May 2009, Jan. 2010)

2. If A is any square matrix, prove that $A + A^\theta$, AA^θ, $A^\theta A$ are all Hermitian and $A - A^\theta$ is Skew-Hermitian.

3. If A, B are Hermitian or skew-Hermitian, then so is A + B.

4. Show that the matrix $B^\theta AB$ is Hermitian or Skew-Hermitian according as A is Hermitian or skew-Hermitian.

5. Prove that $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix. **(P.T.U., Jan. 2009)**

6. If A is a Hermitian matrix, then show that iA is a skew-Hermitian matrix.

7. Show that every square matrix is uniquely expressible as the sum of a Hermitian and a Skew-Hermitian matrix.

[**Hint:** (*i*) Let $A = \frac{A + A^\theta}{2} + \frac{A - A^\theta}{2} = P + Q$, prove $P^\theta = P$ and $Q^\theta = -Q$ (*ii*) to prove uniqueness : Let A = R + S where $R^\theta = R$, $S^\theta = -S$ to prove R = P, S = Q]

4.23. CHARACTERISTIC EQUATION, CHARACTERISTIC ROOTS OR EIGEN VALUES, TRACE OF A MATRIX

If A is square matrix of order n, we can form the matrix $A - \lambda I$, where λ is a scalar and I is the unit matrix of order n. The determinant of this matrix equated to zero, *i.e.,*

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$ is called the ***characteristics equation*** **of A.**

On expanding the determinant, the characteristic equation can be written as a polynomial equation of degree n in λ of the form $(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \dots\dots\dots + k_n = 0$.

The roots of this equation are called the ***characteristic roots or latent roots or eigen-values*** **of A.**

(P.T.U., Jan. 2009, May 2014)

Note. The sum of the eigen-values of a matrix **A is equal to trace of A.**

[The trace of a square matrix is the sum of its diagonal elements].

4.24. EIGEN VECTORS **(P.T.U., Jan. 2009, May 2014)**

Consider the linear transformation Y = AX ...(1)

which transforms the column vector X into the column vector Y. In practice, we are often required to find those vectors X which transform into scalar multiples of themselves.

Let X be such a vector which transforms into λX (λ being a non-zero scalar) by the transformation (1).

Then $\quad Y = \lambda X$ ...(2)

From (1) and (2), $AX = \lambda X \Rightarrow AX - \lambda IX = O \Rightarrow (A - \lambda I) X = O$ ...(3)

This matrix equation gives n homogeneous linear equations

$$\left.\begin{aligned} (a_{11} - l)x_1 + a_{12}x_2 + \dots\dots + a_{1n} x_n &= 0 \\ a_{21}x_1 + (a_{22} - l)x_2 + \dots\dots + a_{2n} x_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ a_{n1} x_1 + a_{n2}x_2 + \dots\dots + (a_{nn} - l) x_n &= 0 \end{aligned}\right] \quad \text{...(4)}$$

These equations will have a non-trivial solution only if the coefficient matrix $|A - \lambda I|$ is singular *i.e.*, if $$|A - \lambda I| = 0 \quad ...(5)$$

This is the characteristic equation of the matrix A and has n roots which are the eigen-values of A. Corresponding to each root of (5), the homogeneous system (3) has a non-zero solution

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$, which is called an ***eigen vector or latent vector.***

Note. If X is a solution of (3), then so is kX, where k is an arbitrary constant. Thus, the eigen vector corresponding to an eigen-value is not unique.

4.25. PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS (P.T.U., May 2008)

If λ is an eigen value of A and X be its corresponding eigen vector then we have the following properties:

(*i*) $\alpha\lambda$ is an eigen value of αA and the corresponding eigen vector remains the same.

$$AX = \lambda X \Rightarrow \alpha (AX) = \alpha (\lambda X) \Rightarrow (\alpha A) X = (\alpha\lambda) X$$

$\therefore$ $\alpha\lambda$ is an eigen value of αA and eigen vector is X.

(*ii*) λ^m is an eigen value of A^m and corresponding eigen vector remains the same **(P.T.U., Dec. 2004)**

$$AX = \lambda X \Rightarrow A (AX) = A (\lambda X) \Rightarrow (AA) X = \lambda (AX)$$

$\therefore$ $A^2X = \lambda (\lambda X) = \lambda^2 X \Rightarrow \lambda^2$ is an eigen value of A^2

and eigen vector is X.

Pre-multiply successively m times by A, we get the result.

(*iii*) $\lambda - k$ is an eigen-value of $A - k$I and corresponding eigen vector is X.

$AX = \lambda X \Rightarrow AX - kIX = \lambda X - kIX$

or $(A - kI) X = (\lambda - k) X \Rightarrow \lambda - k$ is the eigen vector of $A - k$I and eigen vector is X.

(*iv*) $\dfrac{1}{\lambda}$ is an eigen value of A^{-1} (if it exists) and the corresponding eigen vector is X.

(P.T.U., May 2005)

$$AX = \lambda X \text{ ; Pre-multiply by } A^{-1}$$

$$A^{-1} (AX) = A^{-1} (\lambda X) \Rightarrow (A^{-1} A) X = \lambda (A^{-1} X)$$

or $$IX = \lambda (A^{-1} X) \quad \text{or} \quad A^{-1} X = \frac{1}{\lambda} X$$

$\therefore$ $\dfrac{1}{\lambda}$ is an eigen value of A^{-1} and eigen vector is X.

(*v*) $\dfrac{1}{\lambda - k}$ is an eigen value of $(A - kI)^{-1}$ and corresponding eigen vector is X

$$AX = \lambda X \Rightarrow (A - kI) X = (\lambda - k) X$$

Pre-multiply by $(A - kI)^{-1}$, we get

$$X = (A - kI)^{-1} (\lambda - k) X$$

Divide by $\lambda - k$, we get $\dfrac{1}{\lambda - k} X = (A - kI)^{-1} X$

$$\therefore \quad (A - kI)^{-1} X = \left(\frac{1}{\lambda - k}\right) X$$

$\therefore$ $\dfrac{1}{\lambda - k}$ is an eigen value of $(A - kI)^{-1}$ and the eigen vector is X.

(*vi*) $\dfrac{|A|}{\lambda}$ is an eigen value of adj A. **(P.T.U., Dec. 2003)**

$$AX = \lambda X \text{ ; Pre-multiply both sides by adj A}$$

$$(\text{adj } A)(AX) = (\text{adj } A) \lambda X$$

$\Rightarrow \qquad [(\text{adj A})\,A]\,X = \lambda\,[(\text{adj A})\,X]$

$\Rightarrow \qquad |A|\,X = \lambda\,[(\text{adj A})\,X] \qquad | \because A\,(\text{adj A}) = (\text{adj A})\,A = |A|\,I|$

$$\Rightarrow \qquad (\text{adj A})\,X = \left[\frac{|A|}{\lambda}\right] X$$

$\Rightarrow \dfrac{|A|}{\lambda}$ is an eigen value of adj A and eigen vector is X.

(*vii*) A and A^T have the same eigen values

$\because$ eigen values of A are given by $|A - \lambda I| = 0$

We know that $|A| = |A^T|$

$\therefore \qquad |A - \lambda I| = |(A - \lambda I)^T| = |A^T - (\lambda I)^T| = |A^T - \lambda I|$

$\therefore$ eigen values of A and A^T are same.

(*viii*) For a real matrix A, if $\alpha + i\beta$ is an eigen value, then its conjugate $\alpha - i\beta$ is also an eigen value of A.

Since eigen values of A are given by its characteristic equation $|A - \lambda I| = 0$ and if A is real, then characteristic equation is also a real polynomial equation and in a real polynomial equation, imaginary roots always occur in conjugate pairs. If $\alpha + i\beta$ is an eigen value, then $\alpha - i\beta$ is also an eigen value.

ILLUSTRATIVE EXAMPLES

Example 1. *(i) Find the eigen values and eigen vectors of the matrix* $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

(ii) Find the eigen values of the matrix $\begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$. **(P.T.U., Dec. 2006)**

Sol. (*i*) The characteristic equation of the given matrix is

$$|A - \lambda I| = 0 \qquad \text{or} \qquad \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

or $\quad (1-\lambda)(4-\lambda) - 10 = 0 \qquad$ or $\quad \lambda^2 - 5\lambda - 6 = 0$

or $\quad (\lambda - 6)(\lambda + 1) = 0 \qquad \therefore \quad \lambda = 6, -1.$

Thus, the eigen values of A are 6, – 1

Corresponding to $\lambda = 6$, the eigen vectors are given by $(A - 6I)\,X = O$

$$\text{or} \quad \begin{bmatrix} 1-6 & -2 \\ -5 & 4-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O \qquad \text{or} \quad \begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O$$

we get only one independent equation $-5x_1 - 2x_2 = 0$

$\therefore \quad \dfrac{x_1}{2} = \dfrac{x_2}{-5}$ gives the eigen vector (2, – 5)

Corresponding to $\lambda = -1$, the eigen vectors are given by $\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O$

We get only one independent equation $2x_1 - 2x_2 = 0$.

$\therefore \quad x_1 = x_2$ gives the eigen vector (1, 1).

(*ii*) The characteristic equation of the given matrix is $|A - \lambda I| = 0$

i.e., $\quad \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & -4-\lambda & 2 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$, expand w.r.t. 1st column, we get

$(1-\lambda)(-4-\lambda)(7-\lambda)=0$, *i.e.*, $\lambda=1, \lambda=-4, \lambda=7$.

Hence the eigen values are $-4, 1, 7$.

Example 2. *Find the eigen values and eigen vectors of the following matrices:*

(*i*) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **(P.T.U., May 2012)** (*ii*) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. **(P.T.U., Dec. 2012, 2013)**

Sol. (*i*) The characteristic equation of the given matrix is $|A-\lambda I|=0$

or $$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}=0$$

or $$(-2-\lambda)[-\lambda(1-\lambda)-12]-2[-2\lambda-6]-3[-4+1(1+\lambda)=0$$

or $$\lambda^3+\lambda^2-21\lambda-45=0$$

By trial, $\lambda=-3$ satisfies it.

$$\therefore \quad (\lambda+3)(\lambda^2-2\lambda-15)=0 \Rightarrow (\lambda+3)(\lambda+3)(\lambda-5)=0 \Rightarrow \lambda=-3,-3,5$$

Thus, the eigen values of A are $-3, -3, 5$.

Corresponding to $\lambda=-3$, eigen vectors are given by

$$(A+3I)X=O \quad \text{or} \quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=O$$

We get only one independent equation $x_1+2x_2-3x_3=0$

Choosing $x_2=0$, we have $x_1-3x_3=0$

$\therefore \quad \dfrac{x_1}{3}=\dfrac{x_2}{0}=\dfrac{x_3}{1}$ giving the eigen vector (3, 0, 1)

Choosing $x_3=0$, we have $x_1+2x_2=0$

$\therefore \quad \dfrac{x_1}{2}=\dfrac{x_2}{-1}=\dfrac{x_3}{0}$ giving the eigen vector (2, –1, 0)

Any other eigen vector corresponding to $\lambda=-3$ will be a linear combination of these two.

Corresponding to $\lambda=5$, the eigen vectors are given by $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=O$

$$\Rightarrow \quad \begin{aligned} -7x_1+2x_2-3x_3&=0 \\ 2x_1-4x_2-6x_3&=0 \\ -x_1-2x_2-5x_3&=0 \end{aligned}$$

From first two equations, we have $\dfrac{x_1}{-12-12}=\dfrac{x_2}{-6-42}=\dfrac{x_3}{28-4}$

or $\dfrac{x_1}{1}=\dfrac{x_2}{2}=\dfrac{x_3}{-1}$ giving the eigen vector (1, 2, – 1).

(*ii*) The characteristic equation of the given matrix is $|A-\lambda I|=0$

or $$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}=0$$

or $$(1-\lambda)\{(5-\lambda)(1-\lambda)-1\}-1\{1-\lambda-3\}+3\{1-15+3\lambda\}=0$$

or $$(1-\lambda)\{4-6\lambda+\lambda^2\}+\lambda+2-42+9\lambda=0$$

or $$4 - 10\lambda + 7\lambda^2 - \lambda^3 + 10\lambda - 40 = 0$$

or $$\lambda^3 - 7\lambda^2 + 36 = 0$$

or $$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0$$

or $$(\lambda + 2)(\lambda - 6)(\lambda - 3) = 0$$

$\therefore$ $$\lambda = -2, 3, 6$$

Thus the eigen values of A are – 2, 3, 6

Corresponding to $\lambda = -2$, eigen vectors are given by $(A + 2I)X = 0$

or $$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

We get two independent equations

$$3x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 7x_2 + x_3 = 0$$
$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$
$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$\therefore$ Eigen vector corresponding to $\lambda = -2$ is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 3$ is given by $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$-2x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$3x_1 + x_2 - 2x_3 = 0$$

From first two equations $$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

or $$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

It satisfies third equation

$\therefore$ Eigen vector corresponding to $\lambda = 3$ is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 6$ is given by $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$-5x_1 + x_2 + 3x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$3x_1 + x_2 - 5x_3 = 0$$

From first two equations

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

or $$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

The values of x_1, x_2, x_3 satisfy third equation

$\therefore$ Eigen vector corresponding to $\lambda = 6$ is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Hence the eigen vectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Example 3. *If λ is an eigen value of the matrix A, then prove that $g(\lambda)$ is an eigen value of $g(A)$, where g is polynomial.* **(P.T.U., May 2010)**

Sol. Given λ is an eigen value of matrix A

$\therefore$ There exists a non zero vector X such that

$$AX = \lambda X \qquad \ldots(1)$$

Now, $$A(AX) = A(\lambda X) \Rightarrow A^2X = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X \qquad \ldots(2)$$

$\therefore$ λ^2 is an eigen value of matrix A^2

Again $$A(A^2X) = A(\lambda^2 X)$$

$\Rightarrow$ $$A^3X = \lambda^2(AX) = \lambda^2(\lambda X) = \lambda^3 X \qquad \ldots(3)$$

$\therefore$ λ^3 is an eigen value of matrix A^3

Continue this process we can prove that

$$A^nX = \lambda^n X \text{ } i.e., \lambda^n \text{ is an eigen value of } A^n \qquad \ldots(4)$$

As g is a polynomial

Let $$g(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots a_n\lambda^n$$

$\Rightarrow$ $$g(A) = a_0I + a_1A + a_2A^2 + \ldots a_nA^n$$

$$g(A)X = [a_0I + a_1A + a_2A^2 + \ldots a_nA^n]X$$
$$= a_0(IX) + a_1(AX) + a_2(A^2X) + \ldots a_n(A^nX)$$
$$= a_0X + a_1(\lambda X) + a_2(\lambda^2 X) + \ldots a_n(\lambda^n X) \qquad \text{[By using (1), (2), (3), (4)]}$$

$\therefore$ $$g(A)X = (a_0 + a_1\lambda + a_2\lambda^2 + \ldots a_n\lambda^n)X$$

i.e., $$g(A)X = g(\lambda)X$$

$\Rightarrow$ $g(\lambda)$ is an eigen value of $g(A)$.

Example 4. *Show that eigen values of a Skew Hermitian matrix are either zero or purely imaginary.* **(P.T.U., Dec. 2012, 2013)**

Sol. Let A be a Skew Hermitian matrix

$\therefore$ $$A^\theta = -\hat{A} \qquad \ldots(1)$$

Let λ be an eigen value of A, then there exists a non-zero vector X such that

$$AX = \lambda X$$

$$(AX)^\theta = (\lambda X)^\theta \text{ or } X^\theta A^\theta = \bar{\lambda} X^\theta$$

or $$-X^\theta A = \bar{\lambda} X^\theta \qquad \text{[By using (1)]}$$

Post multiply both sides by X

$$-(X^\theta A)X = \left(\bar{\lambda} X^\theta\right) X$$

or $$-X^\theta (AX) = \bar{\lambda}\,(X^\theta X)$$

$$-X^\theta (\lambda X) = \bar{\lambda}\ (X^\theta X)$$

or $$-\lambda (X^\theta X) = \bar{\lambda}\ (X^\theta X) \Rightarrow \bar{\lambda} = -\lambda$$

$\Rightarrow$ $$\lambda + \bar{\lambda} = 0$$

Now if $$\lambda = a + ib$$

then $$\bar{\lambda} = a - ib$$

$$\lambda + \bar{\lambda} = 0 \quad \Rightarrow \quad a + ib + a - ib = 0 \text{ or } a = 0$$

i.e., $\lambda = ib$, *i.e.,* λ is purely imaginary.

Hence either eigen values are zero or purely imaginary.

4.26. CAYLEY HAMILTON THEOREM

(P.T.U., May 2004, 2006, 2007, Jan. 2009, May 2011)

Every square matrix satisfies its characteristic equation.

i.e., if the characteristic equation of the *n*th order square matrix A is

$$|A - \lambda I| = (-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \cdots + k_n = 0 \qquad \ldots(1)$$

then $$(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \cdots + k_n I = O \qquad \ldots(2)$$

Let $$P = \text{adj}\ (A - \lambda I)$$

Since the elements of $A - \lambda I$ are at most of first degree in λ, the elements of $P = \text{adj}\ (A - \lambda I)$ are polynomials in λ of degree $(n-1)$ or less. We can, therefore, split up P into a number of matrices each containing the same power of λ and write

$$P = P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \cdots + P_{n-2} \lambda^2 + P_{n-1} \lambda + P_n$$

Also, we know that if M is a square matrix, then M (adj M) = |M| × I

$\therefore$ $$(A - \lambda I) P = |A - \lambda I| \times I$$

By (1) and (2), we have

$$(A - \lambda I)(P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \cdots + P_{n-2} \lambda^2 + P_{n-1} \lambda + P_n) = [(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_i \lambda^{n-2} + \cdots + k_{n-2} \lambda^2 + k_{n-1} \lambda + k_n] I$$

Equating coefficients of like powers of λ on both sides, we have

$$-P_1 = (-1)^n I \qquad [\because IP_1 = P_1]$$
$$AP_1 - P_2 = k_1 I$$
$$AP_2 - P_3 = k_2 I$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$AP_{n-2} - P_{n-1} = k_{n-2} I$$
$$AP_{n-1} - P_n = k_{n-1} I$$
$$AP_n = k_n I$$

Pre-multiplying these equations by $A^n, A^{n-1}, A^{n-2}, \ldots\ldots\ldots\ldots, A^2, A, I$ respectively and adding, we get

$O = (-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \cdots + k_{n-2} A^2 + k_{n-1} A + k^n I$ terms on the LHS Cancel in pairs

or $$(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \cdots + k^{n-1} A + k_n I = O \qquad \ldots(3)$$

which proves the theorem.

Note 1. Multiplying (3) by A^{-1}, we have $(-1)^n A^{n-1} + k_1 A^{n-2} + \cdots + k_{n-1} I + k_n A^{-1} = O$

$\Rightarrow$ $$A^{-1} = -\frac{1}{k_n}[(-1)^n A^{n-1} + k_1 A^{n-2} + \cdots + k_{n-1} I]$$

Thus Cayley Hamilton theorem gives another method for computing the inverse of a matrix. Since this method express the inverse of a matrix of order *n* in terms of $(n-1)$ powers of A, it is most suitable for computing inverses of large matrices.

Note 2. If *m* be a positive integer such that $m > n$, then multiplying (3) by A^{m-n}, we get

$$(-1)^n A^m + k_1 A^{m-1} + k_2 A^{m-2} + \dots\dots\dots + k_{n-1} A^{m-n+1} + k_n A^{m-n} = O$$

showing that any positive integral power A^m $(m > n)$ of A is linearly expressible in terms of those of lower degree.

Example 5. *Verify Cayley Hamilton Theorem for the following matrices and find A^{-1} in each case*

(*i*) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ **(P.T.U., May 2014)** (*ii*) $\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$. **(P.T.U., Dec. 2006)**

Sol. (*i*) Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Characteristic equation of A is

$$|A - \lambda I| = 0$$

or $$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{bmatrix} = 0$$

or $$(1-\lambda)\{-(3-\lambda)(4+\lambda) - 12\} - 1\{-4-\lambda-6\} + 3\{-4+2(3-\lambda)\} = 0$$

$$(1-\lambda)(\lambda^2+\lambda-24) + \lambda + 10 + 6 - 6\lambda = 0$$

or $$\lambda^3 - 20\lambda + 8 = 0$$

To verify Cayley Hamilton Theorem, we have to show that $A^3 - 20A + 8I = 0$

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix}$$

$$A^3 - 20A + 8I = \begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix} - 20\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore \quad A^3 - 20A + 8I = 0$

Operate both sides by A^{-1}

$$A^2 - 20I = -8A^{-1}$$

$$\therefore \quad 8A^{-1} = \begin{bmatrix} 4 & 8 & 12 \\ -10 & -22 & -6 \\ -2 & -2 & -22 \end{bmatrix} + \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

(*ii*) Let $A = \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 4 & 3-\lambda & 2 \\ 2 & 4 & 3-\lambda \end{vmatrix} = 0 \text{ or } (3-\lambda)\{(3-\lambda)^2 - 8\} - 2\{12 - 4\lambda - 4\} + 4\{16 - 6 + 2\lambda\} = 0$$

or $$\lambda^3 - 9\lambda^2 + 3\lambda - 27 = 0$$

To verify Cayley Hamilton Theorem, we have to prove

$A^3 - 9A^2 + 3A - 27I = 0$

$$A^2 = \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 28 & 28 \\ 28 & 25 & 28 \\ 28 & 28 & 25 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 25 & 28 & 28 \\ 28 & 25 & 28 \\ 28 & 28 & 25 \end{bmatrix}\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 243 & 246 & 240 \\ 240 & 243 & 246 \\ 246 & 240 & 243 \end{bmatrix}$$

$$A^3 - 9A^2 + 3A - 27I = \begin{bmatrix} 243 & 246 & 240 \\ 240 & 243 & 246 \\ 246 & 240 & 243 \end{bmatrix} - 9\begin{bmatrix} 25 & 28 & 28 \\ 28 & 25 & 28 \\ 28 & 28 & 25 \end{bmatrix} + 3\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} - 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence theorem is verified.

$$A^3 - 9A^2 + 3A - 27I = 0$$

Operate both sides with A^{-1}

$$A^2 - 9A + 3I = 27A^{-1}$$

$$\therefore \quad 27A^{-1} = \begin{bmatrix} 25 & 28 & 28 \\ 28 & 25 & 28 \\ 28 & 28 & 25 \end{bmatrix} - 9\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 10 & -8 \\ -8 & 1 & 10 \\ 10 & -8 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{27}\begin{bmatrix} 1 & 10 & -8 \\ -8 & 1 & 10 \\ 10 & -8 & 1 \end{bmatrix}$$

Example 6. *If* $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$*, then use Cayley Hamilton Theorem to find the matrix represented by* A^5.

Sol. Characteristic equation of A is

$$|A - \lambda I| = 0$$

i.e.,
$$\begin{vmatrix} 2-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0 \text{ or } \lambda^2 - 7\lambda + 1 = 0$$

By Cayley Hamilton Theorem $A^2 - 7A + I = 0$

$$\therefore \quad A^2 = 7A - I \quad \ldots(1)$$
$$A^4 = 49A^2 - 14A + I$$
$$= 49(7A - I) - 14A + I \quad \text{[Using (1)]}$$
$$= 329A - 48I$$
$$A^5 = A^4 \cdot A = (329A - 48I)A$$
$$= 329A^2 - 48A = 329(7A - I) - 48A = 2255A - 329I$$
$$= 2255\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} - 329\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4181 & 6765 \\ 6765 & 10946 \end{bmatrix}.$$

Example 7. *Verify Cayley Hamilton Theorem for the matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ *and hence find* $B = A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$*; also find* A^{-1} *and* A^4. **(P.T.U., May 2011)**

Sol. The characteristic equation of A is

$$|A - \lambda I| = 0; \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$$

or
$$(1-\lambda)\{(4-\lambda)(6-\lambda) - 25\} - 2\{2(6-\lambda) - 15\} + 3\{10 - 3(4-\lambda)\} = 0$$

or
$$\lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0 \quad \ldots(1)$$

Cayley Hamilton Theorem is verified if A satisfies the characteristic equation *i.e.*, (1)

$$\therefore \quad A^3 - 11A^2 - 4A + I = 0 \quad \ldots(2)$$

Now,
$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

i.e.,
$$A^2 = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$$

Verification: $\begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - 11 \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore$ Cayley Hamilton Theorem is satisfied.

Now, $B = A^5 (A^3 - 11A^2 - 4A + I) + A (A^3 - 11A^2 - 4A + I) + A^2 + A + I$

$= A^5 \cdot 0 + A \cdot 0 + A^2 + A + I$ [Using (2)]

$= A^2 + A + I$

$$= \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 27 & 34 \\ 27 & 50 & 61 \\ 34 & 61 & 77 \end{bmatrix}$$

From (2), $A^{-1} = -A^2 + 11A + 4I$

$$\therefore \quad A^{-1} = -\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + 11 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

From (2), $A^4 = 11A^3 + 4A^2 - A$

$$= 11 \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} + 4 \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$\therefore \quad A^4 = \begin{bmatrix} 1782 & 3211 & 4004 \\ 3211 & 5786 & 7215 \\ 4004 & 7215 & 8997 \end{bmatrix}.$$

Example 8. *Using Cayley Hamilton Theorem find the inverse of* $\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$. **(P.T.U., Dec. 2012)**

Sol. Let $A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$

Characteristics equation of A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

or $(4 - \lambda) \{(1 - \lambda)^2 + 4\} - 3\{2(1 - \lambda) + 2\} + 1\{4 - 1 + \lambda\} = 0$

or $(4 - \lambda)(5 - 2\lambda + \lambda^2) - 3(4 - 2\lambda) + (3 + \lambda) = 0$

or $20 - 13\lambda + 6\lambda^2 - \lambda^3 - 12 + 6\lambda + 3 + \lambda = 0$

or $\lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0$

By Cayley Hamilton Theorem

$$A^3 - 6A^2 + 6A - 11\,I = 0$$

or
$$11\,I = A^3 - 6A^2 + 6A$$

Operate both sides by A^{-1}

$$11A^{-1} = A^2 - 6A + 6\,I$$

$$A^2 = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}$$

$\therefore$
$$11A^{-1} = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} - 6\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

$\therefore$
$$A^{-1} = \frac{1}{11}\begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

TEST YOUR KNOWLEDGE

1. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

2. Find the eigen values and eigen vectors of the matrices

(*i*) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **(P.T.U., June 2003, Jan. 2010)** (*ii*) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **(P.T.U., Dec. 2013)**

(*iii*) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (*v*) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(*vi*) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ **(P.T.U., May 2006)** (*vii*) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ **(P.T.U., May 2012)**

3. Prove that the characteristic roots of a diagonal matrix are the diagonal elements of the matrix.

4. Show that 0 is a characteristic root of a matrix if and only if the matrix is singular.

5. Show that if λ is a characteristic root of the matrix A, then $\lambda + k$ is a characteristic root of the matrix $A + kI$.

6. If $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ are the given values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots\ldots\ldots, \lambda_n^m$ (m being a positive integer).

7. Show that eigen values of a Hamilton matrix are real.

8. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that the equation is satisfied by A and hence obtain the inverse of the given matrix.

9. Find the characteristic equation of the matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$. Show that the equation is satisfied by A.

10. If A = $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ use Cayley Hamilton Theorem to find A^8. [**Hint:** $A^2 = 5I$] **(P.T.U., Dec. 2003, May 2010)**

11. Using Cayley Hamilton Theorem, find the inverse of

(*i*) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ **(P.T.U., Dec. 2013)** (*ii*) $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(P.T.U., Dec. 2005, Jan. 2009)

(*iv*) $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ **(P.T.U., May 2010)** (*v*) $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ **(P.T.U., Dec. 2005)**

(*vi*) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ **(P.T.U., May 2006)**

12. Find the characteristic equation of matrix A = $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

ANSWERS

1. 1, 6 ; (1, – 1), (1, 4)

2. (*i*) 0, 3, 15 ; (1, 2, 2), (2, 1 , –2) (2, –2, 1) (*ii*) 2, 2, 8 ; (1, 0, –2), (1, 2, 0), (2, –1, 1)

(*iii*) 1, 2, 3 ; (1, 0, 1), (1, 0, –1), (0, 1, 0) (*iv*) 1, 1, 3 ; (1, –2, 1) (1, 1, 0)

(*v*) 2, 3, 5 ; (1, –1, 0), (1, 0, 0), (2, 0, 1) (*vi*) 1, 2, 3 ; (1, – 1, 0), (– 2, 1, 2), (1, –1, – 2)

(*vii*) 1, 2, 3 ; (4, 3, 2), (3, 2, 1), (2, 1, 1)

8. $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0,\ \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ -6 & 1 & -10 \end{bmatrix}$

9. $\lambda^3 - \lambda^2 - 18\lambda - 40 = 0$ **10.** 625I

11. (*i*) $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$ (*ii*) $\begin{bmatrix} \frac{2}{65} & \frac{2}{13} & -\frac{9}{130} \\ -\frac{21}{65} & \frac{5}{13} & -\frac{3}{130} \\ \frac{2}{13} & -\frac{3}{13} & \frac{2}{13} \end{bmatrix}$ (*iii*) $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ (*iv*) $\frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$

(*v*) $\frac{1}{4}\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$ (*vi*) $\frac{1}{18}\begin{bmatrix} 1 & 7 & -5 \\ 7 & -5 & 1 \\ -5 & 1 & 7 \end{bmatrix}$

12. $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0;\ \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$.

4.27. DIAGONALIZABLE MATRICES

A matrix A is said to be diagonalizable if there exists an invertible matrix B. Such that $B^{-1}AB = D$, where D is a diagonal matrix and the diagonal elements of D are the eigen values of A.

Theorem. *A square matrix A of order n is diagonalizable if and only if it has n linearly independent eigen vectors.*

Proof. Let $X_1, X_2, ..., X_n$ be n linearly independent eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, ..., \lambda_n$ (not necessarily distinct) of matrix A

$\therefore$ $$AX_1 = \lambda_1 X_1, AX_2 = \lambda_2 X_2, ..., AX_n = \lambda_n X_n$$

Let $B = [X_1, X_2, ..., X_n]$ and $D = \text{Diag.} [\lambda_1, \lambda_2, ..., \lambda_n]$ formed by eigen values of A.

then
$$\begin{aligned} AB &= A [X_1, X_2, ..., X_n] = [AX_1, AX_2, ..., AX_n] \\ &= [\lambda_1 X_1, \lambda_2 X_2, ..., \lambda_n X_n] \\ &= [X_1 X_2, ..., X_n] \text{ Diag } [\lambda_1 \lambda_2, ..., \lambda_n] \\ AB &= BD \qquad ... (1) \end{aligned}$$

Since columns of B and L.I. $\therefore$ $p(B) = n$ $\therefore$ B is invertible

Pre-multiply both sides by B^{-1}

$\therefore$ $$B^{-1}AB = (B^{-1}B)D = D$$

$\therefore$ The matrix B, formed by eigen vectors of A, reduces the matrix A to its diagonal form.

Post multiply (1) by B^{-1}

$$A(BB^{-1}) = BDB^{-1} \quad \text{or} \quad A = BDB^{-1}.$$

Note 1. The matrix B which diagonalizes A is called the **Modal Matrix of A,** obtained by grouping the eigen values of A into a square matrix and matrix D is called **Spectral Matrix of A.**

Note 2. We have $A = BDB^{-1}$

$\therefore$
$$\begin{aligned} A^2 &= A \cdot A = (BDB^{-1})(BDB^{-1}) = BD(B^{-1}B)DB^{-1} \\ &= B(DID)B^{-1} = BD^2B^{-1} \end{aligned}$$

Repeating this process m times, we get

$$A^m = BD^m B^{-1} \ (m, a \text{ +ve integer}).$$

$\therefore$ If A is diagonalizable so is A^m.

Note 3. If D is a diagonal matrix of order n and

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots\cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots\cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots\cdots & \lambda_n \end{bmatrix}, \text{ then } D^m = \begin{bmatrix} \lambda_1^m & 0 & 0 & \cdots\cdots & 0 \\ 0 & \lambda_2^m & 0 & \cdots\cdots & 0 \\ 0 & 0 & \lambda_3^m & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots\cdots & \lambda_n^m \end{bmatrix}$$

$\therefore$ $$A^m = BD^m B^{-1}$$

Similarly if Q (D) is a polynomial in D, then

$$Q(D) = \begin{bmatrix} Q(\lambda_1) & 0 & 0 & \cdots\cdots & 0 \\ 0 & Q(\lambda_2) & 0 & \cdots\cdots & 0 \\ 0 & 0 & Q(\lambda_3) & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots\cdots & Q(\lambda_n) \end{bmatrix}$$

$\therefore$ $$Q(A) = B\,[Q(D)]B^{-1}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the matrix* $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ *is diagonalizable. Hence find P such that* $P^{-1}AP$ *is a diagonal matrix, then obtain the matrix* $B = A^2 + 5A + 3I$. **(P.T.U., May 2008, 2012)**

Sol. Characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

or $\lambda = 1, 2, 3.$

Since the matrix has three distinct eigen values

$\therefore$ It has three linearly independent eigen values and hence A is diagonalizable.

The eigen vector corresponding to $\lambda = 1$ is given by

$$(A - \lambda I)\,X = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $2x_1 + y_1 - z_1 = 0\,;\ -2x_1 + 2z_1 = 0$ and $y_1 + z_1 = 0$

which gives the solution.

$$x_1 = 1, \quad y_1 = -1, \quad z_1 = 1$$

The eigen vector corresponding to $\lambda = 2$ is

$$(A - 2I)\,X = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e.,

$$\begin{aligned} x_1 + y_1 - z_1 &= 0 \\ -2x_1 - y_1 + 2z_1 &= 0 \\ y_1 &= 0 \end{aligned}$$

which gives the solution

$$x_1 = 1, \quad y_1 = 0, \quad z_1 = 1.$$

Eigen vector corresponding to $\lambda = 3$ is given by

$$(A - 3I)\,X = \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e.,

$$\begin{aligned} y_1 - z_1 &= 0 \\ -2x_1 - 2y_1 + 2z_1 &= 0 \\ y_1 - z_1 &= 0 \end{aligned}$$

which gives the solution

$$x_1 = 0, \quad y_1 = 1, \quad z_1 = 0.$$

$\therefore$ This modal matrix $P = [X_1, X_2, X_3]$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Now, $$P^{-1} = \frac{\text{Adj. P}}{|P|} = \frac{1}{1}\begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$\therefore$ $$P^{-1}AP = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \text{diag}\,[1, 2, 3]$$

Hence A is diagonalizable and its diagonal form matrix contains the eigen values only as its diagonal elements.

Now to obtain $B = A^2 + 5A + 3I$ we use $Q(A) = P\,[Q(D)]P^{-1}$ [Art. 4.27 Note 3]

$$D = \text{diag}\,(1, 2, 3)$$

$$D^2 = \text{diag}\,(1, 4, 9)$$

$$A^2 + 5A + 3I = P\,(D^2 + 5D + 3I)\,P^{-1}$$

$$D^2 + 5D + 3I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 9 \\ 0 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

$\therefore$ $$B = A^2 + 5A + 3I = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 8 & -8 \\ -18 & 9 & 18 \\ -2 & 8 & 19 \end{bmatrix}$$

Example 2. *Find a matrix P which transforms the matrix* $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ *into a diagonal form.*

(P.T.U., Dec. 2003)

Sol. Let $$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0 \text{ or } \lambda^3 - 7\lambda^2 + 36 = 0$$

or $$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0$$

or $(\lambda + 2)(\lambda - 3)(\lambda - 6) = 0$ *i.e.*, $\lambda = -2, 3, 6$ are the eigen values.

When $\lambda = -2$; eigen vectors are given by

$$3x_1 + y_1 + 3z_1 = 0$$

$$x_1 + 7y_1 + z_1 = 0$$
$$3x_1 + y_1 + 3z_1 = 0$$

Solving first and second equations (3rd is same as first)

$$\frac{x_1}{-20} = \frac{y_1}{0} = \frac{z_1}{20} \quad \therefore \quad X_1 = k(-1, 0, 1)$$

When $\lambda = 3$; eigen vectors are given by

$$2x_1 + y_1 + 3z_1 = 0$$
$$x_1 + 2y_1 + z_1 = 0$$
$$3x_1 + y_1 - 2z_1 = 0$$

Solving first and second equations :

$$\frac{x_1}{-5} = \frac{y_1}{5} = \frac{z_1}{-5} \quad \therefore \quad X_2 = k(-1, 1, -1)$$

When $\lambda = 6$; eigen vectors are given by

$$-5x_1 + y_1 + 3z_1 = 0$$
$$x_1 - y_1 + z_1 = 0$$
$$3x_1 + y_1 - 5z_1 = 0$$

Solving first and second equations

$$\frac{x_1}{4} = \frac{y_1}{8} = \frac{z_1}{4} \quad \therefore \quad X_3 = k(1, 2, 1).$$

Modal matrix $\quad P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

$$P^{-1} = \frac{\text{Adj. } P}{|P|} = \frac{-1}{6}\begin{bmatrix} 3 & 2 & -1 \\ 0 & -2 & -2 \\ -3 & 2 & -1 \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix}$$

$\therefore$ Required diagonal form $D = P^{-1} AP$

$$= -\frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} 12 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -36 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$, which is formed by the eigen values of A.

Example 3. *Diagonalize the matrix* $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ *and obtain its Modal Matrix.*

Sol. Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$ *i.e.*, $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

or $(-2-\lambda)[-\lambda(1-\lambda)-12]-2[-2\lambda-6]-3[-4+1-\lambda] = 0$

or $-(2+\lambda)(\lambda^2-\lambda-12)+4(\lambda+3)+3(\lambda+3) = 0$

or $-(2+\lambda)(\lambda+3)(\lambda-4)+7(\lambda+3) = 0$

or $(\lambda+3)(-\lambda^2+2\lambda+8+7) = 0$

or $-(\lambda+3)(\lambda+3)(\lambda-5) = 0$

$\therefore$ $\lambda = -3, -3, 5.$

Characteristic vectors corresponding to $\lambda = -3$ are

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Operating $R_2 - 2R_1$, $R_3 + R_1$, we get

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ $x_1 + 2x_2 - 3x_3 = 0$

$\therefore$ $x_1 = -2x_2 + 3x_3$

$x_2 = 1\cdot x_2 + 0\cdot x_3$

$x_3 = 0\cdot x_2 + 1\cdot x_3$

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen vectors are $X_1 = (-2, 1, 0)$ and $X_2 = (3, 0, 1)$.

Characteristic vector corresponding to $\lambda = 5$ is

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

or $$\begin{bmatrix} -1 & -2 & -5 \\ -7 & 2 & -3 \\ 2 & -4 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$ by operating R_{32} and R_{21}

Operate $R_2 - 7R_1$, $R_3 + 2R_1$, we get

$$\begin{bmatrix} -1 & -2 & -5 \\ 0 & 16 & 32 \\ 0 & -8 & -16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Operating $R_3 + \frac{1}{2}R_2$, we get

$$\begin{bmatrix} -1 & -2 & -5 \\ 0 & 16 & 32 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ $-x_1 - 2x_2 - 5x_3 = 0$

$16x_2 + 32x_3 = 0$

or
$$x_1 + 2x_2 + 5x_3 = 0$$
$$x_2 + 2x_3 = 0$$
or
$$x_1 = -x_3$$
$$x_2 = -2x_3$$

$$\therefore \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = -x_3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\therefore$ Eigen vector $\qquad X_3 = (1, 2, -1)$

$$\therefore \qquad \text{Modal Matrix } P = \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}; |P| = 8$$

$$P^{-1} = \frac{\text{Adj. } P}{|P|} = \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}$$

Now Diagonal Matrix $D = P^{-1} AP$

$$= \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -24 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & 40 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$ *i.e.*, the diagonal matrix formed by eigen values of A.

Example 4. *Diagonalize* $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ *and hence find* A^8*. Find the Modal Matrix.* **(P.T.U., May 2011)**

Sol. The characteristic equation of A is $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

Expand the determinant w.r.t. R_3
$$(3-\lambda)\{(1-\lambda)(2-\lambda - 6\} = 0$$
or
$$(3-\lambda)\{(\lambda^2 - 3\lambda - 4\} = 0$$
or
$$(3-\lambda)(\lambda - 4)(\lambda + 1)\} = 0$$

$\therefore$ Eigen values are $\lambda = -1, 3, 4$

For $\lambda = -1$; the eigen vector is given by

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

or
$$2x_1 + 6x_2 + x_3 = 0$$
$$x_1 + 3x_2 = 0$$
$$4x_3 = 0$$

$\therefore \quad x_3 = 0, x_1 = -3x_2$

$$\therefore \quad X_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 3$; eigen vector is given by

$$\begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

or

$$-2x_1 + 6x_2 + x_3 = 0$$
$$x_1 - x_2 = 0$$

$\therefore \quad x_2 = x_1$ and $x_3 = -4x_1$

$$\therefore \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$$

For $\lambda = 4$; eigen vector is

$$\begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

$$\therefore \quad -3x_1 + 6x_2 + x_3 = 0$$
$$x_1 - 2x_2 = 0$$
$$-x_3 = 0$$
$$x_1 = 2x_2$$

$\therefore \quad x_3 = 0, x_1 = 2x_2$

$$\therefore \quad X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Thus the Modal Matrix P is

$$P = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{\text{Adj } P}{|P|}$$

$$|P| = \begin{vmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{vmatrix}$$

Expand w.r.t. R_3, we get $\quad |P| = -20$

$$\text{Adj } P = \begin{bmatrix} -4 & 0 & -4 \\ -8 & 0 & -12 \\ -1 & 5 & -4 \end{bmatrix}'$$

$$\therefore \quad P^{-1} = -\frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & 5 \\ -4 & -12 & -4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

Now the diagonal matrix D is given by

$$D = P^{-1} AP = \frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -15 \\ 16 & 48 & 16 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} -20 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 80 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The diagonal matrix formed by eigen values of A

To find A^8; $A = PDP^{-1}$

$$\therefore \qquad A^8 = PD^8 P^{-1}$$

$$= \frac{1}{20}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6561 & 0 \\ 0 & 0 & 65536 \end{bmatrix}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} -3 & 6561 & 131072 \\ 1 & 6561 & 65536 \\ 0 & -26244 & 0 \end{bmatrix}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 524300 & 1572840 & 491480 \\ 262140 & 786440 & 229340 \\ 0 & 0 & 131220 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} 26215 & 78642 & 24574 \\ 13107 & 39322 & 11467 \\ 0 & 0 & 6561 \end{bmatrix}.$$

4.28. SIMILAR MATRICES **(P.T.U., May 2007)**

Let A and B be square matrices of the same order. The matrix A is said to be similar to B if there exists an invertible matrix P such that $A = P^{-1}BP$ or $PA = BP$

Post multiply both sides by P^{-1}, we have

$$PAP^{-1} = B(PP^{-1}) = BI = B \qquad \therefore B = PAP^{-1}$$

$\therefore$ A is similar to B if and only if B is similar to A. The matrix P is called the **similarity matrix.**

4.29. Theorem, Similar Matrices have the same Characteristic Equation (and hence the same Eigen Values). Also if X is an Eigen Vector of A Corresponding to Eigen Value λ, then P^{-1} X is an Eigen Vector of B Corresponding to the Eigen Value λ, where P is Similarity Matrix

Proof. $\because$ B is similar to A and P is similarity matrix. $\therefore$ $AP = PB$ or $P^{-1}AP = B$

Let λ be the eigen value and X be the corresponding eigen vector of A

$$\therefore \qquad AX = \lambda X \qquad \ldots(1)$$

Now, $B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - P^{-1}(\lambda IP) = P^{-1}(A - \lambda I)P$

$$\therefore \quad |B - \lambda I| = |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P|$$
$$= |A - \lambda I||P^{-1}P| = |A - \lambda I||I|$$
$$= |A - \lambda I|$$

$\therefore$ Similar matrices have same characteristic polynomials.

Pre-multiply (1) both sides by an invertible matrix P^{-1}.

$$\therefore \quad P^{-1}(AX) = P^{-1}(\lambda X) = \lambda P^{-1}X$$

Let $X = PY \quad \therefore \quad P^{-1}(APY) = \lambda P^{-1}(PY)$

or $(P^{-1}AP)Y = \lambda(P^{-1}P)Y$

or $BY = \lambda Y$, where $B = P^{-1}AP \qquad \ldots(2)$

$\therefore$ B has the same eigen value λ as that of A which shows that eigen values of similar matrices are same.

$\therefore$ Similar matrices have the same characteristic equation and hence the same eigen values.

Now, from (2) Y is an eigen vector of B corresponding to λ, the eigen value of B.

$\therefore$ Eigen vector of $B = Y = P^{-1}X$

Hence, the result.

Note 1. Converse of the above theorem is not always true *i.e.*, two matrices which have the same characteristic equation need not always be similar.

Note 2. If A is similar to B, B is similar to C, then A is similar to C

Let there be two invertible matrices P and Q.

Such that $A = P^{-1}BP$ and $B = Q^{-1}CQ$

Thus, $A = P^{-1}(Q^{-1}CQ)P = (P^{-1}Q^{-1})C(QP) = (QP)^{-1}C(QP)$

$\therefore \quad A = R^{-1}CR$, where $R = QP$

Hence A is similar to C.

4.30. The Necessary and Sufficient Condition for an *n* Rowed Square Matrix A to be Similar to a Diagonal Matrix is that the Set of Characteristic Vectors of A Includes a Set of *n* Linearly Independent Vectors

Proof. Necessary Condition : A is similar to a diagonal matrix D(say) $\therefore$ there exists a non-singular matrix P such that $P^{-1}AP = D = \text{diag}(\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots, \lambda_n)$

$$\therefore \quad AP = PD$$

Let $P = [C_1, C_2, \ldots\ldots\ldots\ldots, C_n]$

$$\therefore \quad A[C_1, C_2, \ldots\ldots\ldots\ldots, C_n] = [C_1, C_2, \ldots\ldots\ldots\ldots, C_n]\,\text{Diag}\,[\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots, \lambda_n]$$

$$\therefore \quad AC_1 = \lambda_1 C_1\,;\ AC_2 = \lambda_2 C_2\,;\ AC_3 = \lambda_3 C_3, \ldots\ldots\ldots\ldots, AC_n = \lambda_n C_n$$

which shows that $C_1, C_2, \ldots\ldots\ldots\ldots, C_n$ are n characteristic vectors corresponding to eigen values $\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots, \lambda_n$ of A. As $C_1, C_2, \ldots\ldots\ldots\ldots, C_n$ are columns of a non-singular matrix $\therefore$ they form a L.I. set of vectors.

Sufficient Conditions : Let $C_1, C_2, \ldots\ldots\ldots\ldots, C_n$ be n L.I. set of n characteristic vectors and let $\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots, \lambda_n$ be the corresponding characteristic roots.

We have $AC_1 = \lambda_1 C_1, AC_2 = \lambda_2 C_2, \ldots\ldots\ldots\ldots, AC_n = \lambda_n C_n \qquad \ldots(1)$

If we take $P = [C_1, C_2, \ldots\ldots\ldots\ldots, C_n]$

Then system (1) is equivalent to $AP = PD \qquad \ldots(2)$

where $D = \text{Diag}\,[\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots, \lambda_n]$

Also matrix P is non-singular as its columns are L.I. $\therefore$ P^{-1} exists and we may write (2) as

$$P^{-1}AP = D$$

Hence A is similar to D.

Example 5. *Examine whether A is similar to B, where*

$(i)\quad A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ $\quad (ii)\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. **(P.T.U., May 2010)**

Sol. We know that A will be similar to B if there exists a non-singular matrix P such that $A = P^{-1}BP$ or $PA = BP$

Let $$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(i) $$PA = BP \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

or $$\begin{bmatrix} 5a-2b & 5a \\ 5c-2d & 5c \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ -3a+4c & -3b+4d \end{bmatrix}$$

$$\therefore \quad 5a-2b = a+2c \qquad 5a = b+2d$$
$$5c-2d = -3a+4c \qquad 5c = -3b+4d$$

or $$4a = 2b+2c \qquad i.e., 2a = b+c$$
$$3a = -c+2d \qquad i.e., 3a = -c+2d$$

or equations are
$$2a-b-c+0.d = 0$$
$$3a+0.b+c-2d = 0$$
$$5a-b+0.c-2d = 0$$
$$0.a+3b+5c-4d = 0$$

which is a set of homogeneous equation

$$\therefore \quad \begin{bmatrix} 2 & -1 & -1 & 0 \\ 3 & 0 & 1 & -2 \\ 5 & -1 & 0 & -2 \\ 0 & 3 & 5 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \text{ or } \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & -2 \\ -1 & 5 & 0 & -2 \\ 3 & 0 & 5 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \text{ by operating } C_{12}$$

Operate $R_3 - R_1, R_4 + 3R_1$; $$\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & 6 & 2 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

Operate $R_3 - R_2, R_4 - 2R_2$; $$\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$$\therefore \quad -a+2b-c = 0$$
$$3b+c-2d = 0$$

$\therefore$ If $a = 1, b = 1$, we get $c = 1$ and $d = 2$

$$\therefore \quad P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \text{ which is non-singular}$$

Hence A, B are similar

(ii) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

or $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

$$\therefore \quad a = a+c \Rightarrow c = 0$$

$b = b + d \Rightarrow d = 0$

$\therefore \quad P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, which is a singular matrix

$\therefore$ A, B are not similar matrices.

Example 6. *Examine which of the following matrices are similar to diagonal matrices*

(*i*) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Sol. (*i*) Characteristic equation of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is $|\lambda I - A| = 0$

i.e., $\begin{vmatrix} \lambda-8 & 6 & -2 \\ 6 & \lambda-7 & 4 \\ -2 & 4 & \lambda-3 \end{vmatrix} = 0$ *i.e.,* $\lambda^3 - 18\lambda^2 + 45\lambda = 0$; $\lambda = 0, 3, 15$

Characteristic vectors corresponding to $\lambda = 0$ is given by $(\lambda I - A)X = 0$ Put $\lambda = 0$

$$\begin{bmatrix} -8 & 6 & -2 \\ 6 & -7 & 4 \\ -2 & 4 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Operate R_{13}
$$\begin{bmatrix} -2 & 4 & -3 \\ 6 & -7 & 4 \\ -8 & 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Operate $R_2 + 3R_1, R_3 - 4R_1$
$$\begin{bmatrix} -2 & 4 & -3 \\ 0 & 5 & -5 \\ 0 & -10 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0;$$

Operate $R_3 + 2R_2$;
$$\begin{bmatrix} -2 & 4 & -3 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

or
$$-2x + 4y - 3z = 0$$
$$5y - 5z = 0$$

$\therefore$
$$y = z, \quad x = \frac{z}{2}$$

$\therefore$
$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{z}{2} \\ z \\ z \end{bmatrix} = \frac{z}{2}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \Rightarrow \quad X = \frac{z}{2}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$\therefore$ We may take single L.I. solution $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Similarly for $\lambda = 3$;
$$\begin{bmatrix} -5 & 6 & -2 \\ 6 & -4 & +4 \\ -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Operate R_{13}; $$\begin{bmatrix} -2 & 4 & 0 \\ 6 & -4 & 4 \\ -5 & 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Operate $R_1\left(-\frac{1}{2}\right), R_2\left(\frac{1}{2}\right)$; $$\begin{bmatrix} 1 & -2 & 0 \\ 3 & -2 & 2 \\ -5 & 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Operate $R_2 - 3R_1,\ R_3 + 5R_1$; $$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & -4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Operate $R_3 + R_2$; $$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

i.e., $$x - 2y = 0 \quad \text{or} \quad x = 2y$$
$$4y + 2z = 0 \quad\quad z = -2y$$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y \\ y \\ -2y \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$\therefore$ Eigen vector corresponding to $\lambda = 3$ is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

For $\lambda = 15$ $$\begin{bmatrix} 7 & 6 & -2 \\ 6 & 8 & 4 \\ -2 & 4 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \text{ or } \begin{bmatrix} -1 & 2 & 6 \\ 3 & 4 & 2 \\ 7 & 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$ by operating $R_3\left(\frac{1}{2}\right); R_2\left(\frac{1}{2}\right); R_{13}$

Operate $R_2 + 3R_1, R_3 + 7R_1$; $$\begin{bmatrix} -1 & 2 & 6 \\ 0 & 10 & 20 \\ 0 & 20 & 40 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Operate $R_3 - 2R_2,\ R_2\left(\frac{1}{10}\right)$; $$\begin{bmatrix} -1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\therefore \quad -x + 2y + 6z = 0,\ y + 2z = 0$$

$$\therefore \quad y = -2z,\ x = 2z \quad \therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2z \\ -2z \\ z \end{bmatrix} = z\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 15$ is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\therefore$ Set of L.I. characteristic vectors is

$$P = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Now, $$P^{-1} = -\frac{1}{27}\begin{bmatrix} -3 & -6 & -6 \\ -6 & -3 & 6 \\ -6 & 6 & -3 \end{bmatrix} \qquad \left(\because\ P^{-1} = \frac{\text{Adj } P}{|P|}\right)$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

= diag (0, 3, 15) *i.e.*, diagonal matrix formed by eigen values

Hence A is similar to diagonal matrix.

(*ii*) $$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Characteristic roots of A are $|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda-2 & -3 & -4 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & \lambda-1 \end{vmatrix} = 0 \quad \text{or} \quad (\lambda-2)^2(\lambda-1) = 0$$

$$\lambda = 1, 2, 2$$

Eigen vector corresponding to $\lambda = 1$ is $\begin{bmatrix} -1 & -3 & -4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

i.e., $$-x - 3y - 4z = 0$$
$$-y + z = 0$$

$\therefore$ $$y = z,\ x = -3y - 4z = -7z$$

$\therefore$ $$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7z \\ z \\ z \end{bmatrix} = z\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Single eigen vector corresponding to $\lambda = 1$ is $\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 2$, $$\begin{bmatrix} 0 & -3 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-3y - 4z = 0$$
$$z = 0$$

$\therefore$ $$y = 0$$

$\therefore$ $$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Corresponding to $\lambda = 2$, we get only one vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

As there are only two L.I. eigen vectors corresponding to three eigen values.

$\therefore$ There does not exist any non-singular matrix P.

Hence A is not similar to diagonal matrix.

Example 7. *Prove that if A is similar to a diagonal matrix, then A' is similar to A.*

Sol. Let A be similar to diagonal matrix D, then there exists a non-singular matrix P such that

$$P^{-1}AP = D$$

or $$A = PDP^{-1}$$

$$A' = (PDP^{-1})' = (P^{-1})' D'P' = (P')^{-1} DP' \qquad (\because \text{ D is a diagonal matrix } \therefore D' = D)$$

$\Rightarrow$ A' is similar to D

$\Rightarrow$ D is similar to A'

Now A is similar to D ; D is similar to A'

$\Rightarrow$ A is similar to A'

i.e., A' is similar to A.

Example 8. *Show that the rank of every matrix similar to A is the same as that of A.*

Sol. Let B be similar to A. Then there exists a non-singular matrix P such that

$$B = P^{-1}AP$$

Now, rank of B = rank of $(P^{-1}AP)$

= rank of A

$\because$ We know that rank of a matrix does not change on multiplication by a non-singular matrix.

Hence rank of B = rank of A.

4.31. MUTUAL RELATIONS BETWEEN CHARACTERISTIC VECTORS CORRESPONDING TO DIFFERENT CHARACTERISTIC ROOTS OF SOME SPECIAL MATRICES

Before discussing these relations, we first give some definitions.

(*a*) Inner Product of two Vectors : We consider the vector space V_n (C) of *n*-tuples over the field C of complex numbers.

Let X, Y be any two members of V_n (C) written as column vectors then the scalar $X^\theta Y$ is called inner product of vectors X and Y.

Thus if $$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \; Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Then $$X^\theta Y = \begin{bmatrix} \bar{x}_1, \bar{x}_2, \ldots\ldots\ldots, \bar{x}_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \bar{x}_1 y_1 + \bar{x}_2 y_2 + \ldots\ldots\ldots + \bar{x}_n y_n \end{bmatrix}$$, which is a single element matrix

Hence inner product of X and Y is $\bar{x}_1 y_1 + \bar{x}_2 y_2 + \ldots\ldots + \bar{x}_n y_n$.

Note 1. $X^{\theta}Y \neq XY^{\theta}$; infact one side is complex conjugate of other side.

Note 2. In case vectors are real, then we have $X^{\theta}Y = X'Y = XY' = XY^{\theta}$

$\therefore$ inner product concides.

Hence inner product of two real n-tuple vectors is

$$x_1y_1 + x_2y_2 + \ldots\ldots + x_n y_n$$

(*b*) **Length of a Vector :** The positive square root of the inner product $X^{\theta}X$ is called length of X . Thus the length of an n-vector with components $x_1, x_2, \ldots\ldots, x_n$ is positive square root of $\bar{x}_1x_1 + \bar{x}_2x_2 + \bar{x}_3x_3 \ldots\ldots \bar{x}_nx_n$, which is always positive except when X = 0 and when X = 0, then length is also zero.

In case of real vectors length of the vector $= x_1^{\,2} + x_2^{\,2}, \ldots\ldots, x_n^{\,2}$.

(*c*) **Normal Vector:** A vector whose length is 1, is called a normal vector.

(*d*) **Orthogonal Vectors:** A vector X is said to be orthogonal to a vector Y, if the inner product of X and Y is 0 *i.e.*, $X^{\theta}Y = 0 \Leftrightarrow XY^{\theta} = 0$

i.e., $\quad \bar{x}_1y_1 + \bar{x}_2y_2 \ldots\ldots \bar{x}_ny_n = 0$

or $\quad x_1\bar{y}_1 + x_2\bar{y}_2 \ldots\ldots x_n\bar{y}_n = 0$

In case of real vectors the condition of orthogonality becomes $x_1y_1 + x_2y_2 + \ldots\ldots + x_ny_n = 0$.

(*e*) **Condition for a Linear Transformation X = PY to Preserve length is that $P^{\theta}P = I$:**

[Lengths of the vectors preserved means length of vector X = length of vector Y]

We have $\quad X = PY$

$\Rightarrow \quad X^{\theta} = (PY)^{\theta} = Y^{\theta}P^{\theta}$

$\therefore \quad X^{\theta}X = (Y^{\theta}P^{\theta})(PY)$

$\quad = Y^{\theta}(P^{\theta}P)Y$

Given $\quad P^{\theta}P = I \qquad \therefore \quad X^{\theta}X = Y^{\theta}Y$

$\therefore$ Length of the vectors is presեved.

(*f*) **Every Unitary Transformation X = PY Preserves Inner Products:**

$\because$ X = PY is unitary transformation

$\therefore$ P is a unitary matrix $\therefore PP^{\theta} = I$

If $\quad X_1 = PY_1$

and $\quad X_2 = PY_2, \quad$ then $\quad X_2^{\theta} = (PY_2)^{\theta} = Y_2^{\theta}P^{\theta}$

$\therefore \quad X_2^{\theta}X_1 = (Y_2^{\theta}P^{\theta})(PY_1) = Y_2^{\theta}(P^{\theta}P)Y_1 = Y_2^{\theta}\,IY_1$

or $\quad X_2^{\theta}X_1 = Y_2^{\theta}Y_1$

Hence inner product is preserved.

4.32. COLUMN VECTORS OF A UNITARY MATRIX ARE NORMAL AND ORTHOGONAL IN PAIRS

Proof. Let $P = [X_1, X_2, \ldots\ldots, X_n]$ be a unitary matrix (where $X_1, X_2, \ldots\ldots, X_n$ represent columns of P)

$$P^{\theta}P = \begin{bmatrix} X_1^{\theta} \\ X_2^{\theta} \\ \vdots \\ X_n^{\theta} \end{bmatrix} [X_1, X_2, \ldots\ldots, X_n]$$

$$= \begin{bmatrix} X_1^{\theta}X_1 & X_1^{\theta}X_2 & \ldots\ldots & X_1^{\theta}X_n \\ X_2^{\theta}X_1 & X_2^{\theta}X_2 & \ldots\ldots & X_2^{\theta}X_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ X_n^{\theta}X_1 & X_n^{\theta}X_2 & \ldots\ldots & X_n^{\theta}X_n \end{bmatrix}$$

Now, $$P^{\theta}P = I \Rightarrow \begin{bmatrix} X_1^{\theta}X_1 & X_1^{\theta}X_2 & \cdots\cdots & X_1^{\theta}X_n \\ X_2^{\theta}X_1 & X_2^{\theta}X_2 & \cdots\cdots & X_2^{\theta}X_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ X_n^{\theta}X_1 & X_n^{\theta}X_2 & \cdots\cdots & X_n^{\theta}X_n \end{bmatrix} = \begin{bmatrix} I & 0 & \cdots\cdots & 0 \\ 0 & I & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \;\cdots & I \end{bmatrix}$$

$$\Rightarrow \quad X_1^{\theta}X_1 = X_2^{\theta}X_2 = X_3^{\theta}X_3, \ldots\ldots, X_n^{\theta}X_n = I$$

whereas all other sub matrices are zero matrices

i.e., $$X_i^{\theta}X_j = 0 \qquad i \neq j$$
$$= I \qquad i = j$$

which shows that column vectors $X_1, X_2, \ldots\ldots, X_n$ of P are normal $\left(\because X_i^{\theta}X_j = 1 \;;\; \forall\, i = j\right)$ and orthogonal $\left(\because X_i^{\theta}X_j = 0 \;;\; i \neq j\right)$

Cor. Similarity we can prove that the row vectors of a unitary matrix are also normal and orthogonal in pairs we will write $P = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$ and employ $PP^{\theta} = I$.

4.33(a). ORTHONORMAL SYSTEM OF VECTORS

A set of normal vectors which are orthogonal in pairs is called an orthonormal set.

4.33(b). EVERY ORTHONORMAL SET OF VECTORS IS LINEARLY INDEPENDENT

Proof. Let $X_1, X_2, \ldots\ldots, X_k$ be the set of orthonormal vectors of n-tuple

Consider the relation. $a_1X_1 + a_2X_2 + \ldots\ldots + a_kX_k = 0$

Pre-multiply by X_1^{θ}, we get

$$a_1\left(X_1^{\theta}X_1\right) + a_2\left(X_1^{\theta}X_2\right) + a_3\left(X_1^{\theta}X_3\right) + \ldots\ldots + a_k\left(X_1^{\theta}X_k\right) = 0$$

$$\Rightarrow \quad a_1\left(X_1^{\theta}X_1\right) = 0 \qquad [\because \text{ the set is that of orthogonal vectors}]$$

$$\Rightarrow \quad a_1 I = 0 \qquad [\because \text{ set of vectors is normal}]$$

$$\Rightarrow \quad a_1 = 0$$

Similarly, by pre-multiplying by $X_2^{\theta}, X_3^{\theta}, \ldots\ldots, X_k^{\theta}$ successively, we get $a_2 = 0, a_3 = 0, \ldots\ldots, a_k = 0$ hence the set is linearly independent.

4.34. ANY TWO CHARACTERISTIC VECTORS CORRESPONDING TO TWO DISTINCT CHARACTERISTIC ROOTS OF A HERMITIAN MATRIX ARE ORTHOGONAL

Proof. Let X_1, X_2 be any two characteristic vectors corresponding to two distinct characteristic roots λ_1 and λ_2 respectively of a Hermitian matrix

$$\therefore \quad AX_1 = \lambda_1 X_1 \qquad \lambda_1, \lambda_2 \text{ being real scalars} \qquad \ldots(1)$$

$$AX_2 = \lambda_2 X_2 \qquad \ldots(2)$$

Pre-multiply (1) by X_2^{θ} and (2) by X_1^{θ} we have

$$X_2^{\theta}AX_1 = \lambda_1 X_2^{\theta}X_1 \quad \ldots(3)$$

$$X_1^{\theta}AX_2 = \lambda_2 X_1^{\theta}X_2 \quad \ldots(4)$$

Take conjugate transpose of (3)

$$X_1^{\theta}A^{\theta}X_2 = \lambda_1 X_1^{\theta}X_2 \quad i.e., \quad X_1^{\theta}AX_2 = \lambda_1 X_1^{\theta}X_2 \quad [\because \text{ A is Hermitian } \therefore \ A^{\theta} = A]$$

or $$\lambda_1 X_1^{\theta}X_2 = X_1^{\theta}(AX_2) = X_1^{\theta}(\lambda_2 X_2) \quad \because \text{ of (2)}$$

$\therefore$ $$\lambda_1 X_1^{\theta}X_2 = \lambda_2 X_1^{\theta}X_2$$

or $$(\lambda_1 - \lambda_2)(X_1^{\theta}X_2) = 0 \text{ but } \lambda_1 - \lambda_2 \neq 0 \because \lambda_1, \lambda_2 \text{ are distinct}$$

$\therefore$ $$X_1^{\theta}X_2 = 0 \quad \Rightarrow X_1, X_2 \text{ are orthogonal vectors.}$$

Cor. Any two characteristic vectors corresponding to two distinct characteristic roots of a real symmetric matrix are orthogonal.

4.35. ANY TWO CHARACTERISTIC VECTORS CORRESPONDING TO TWO DISTINCT CHARACTERISTIC ROOTS OF A UNITARY MATRIX ARE ORTHOGONAL

(P.T.U., May 2004)

Proof. Let X_1, X_2 be two characteristic vectors corresponding to two distinct characteristic roots λ_1 and λ_2

$\therefore$ $$AX_1 = \lambda_1 X_1, \quad \text{where A is a unitary matrix } \therefore \ A^{\theta}A = I \quad \ldots(1)$$

$$AX_2 = \lambda_2 X_2 \quad \ldots(2)$$

Take conjugate transpose of (2)

$$(AX_2)^{\theta} = (\lambda_2 X_2)^{\theta} \text{ or } X_2^{\theta}A^{\theta} = \bar{\lambda}_2 X_2^{\theta} \quad \ldots(3)$$

From (1) and (3) by multiplication

$$(X_2^{\theta}A^{\theta})(AX_1) = (\bar{\lambda}_2 X_2^{\theta})(l_1 X_1)$$

or $$X_2^{\theta}(A^{\theta}A)X_1 = \bar{\lambda}_2 \lambda_1 (X_2^{\theta}X_1)$$

or $$X_2^{\theta}(IX_1) = \bar{\lambda}_2 \lambda_1 (X_2^{\theta}X_1) \quad (\because \text{ A is unitary matrix})$$

or $$X_2^{\theta}X_1 = \bar{\lambda}_2 \lambda_1 (X_2^{\theta}X_1)$$

or $$(1 - \bar{\lambda}_2 \lambda_1) X_2^{\theta}X_1 = 0 \quad \ldots(4)$$

Since in a unitary matrix modulus of each of the characteristic roots is unity $\therefore$ $\bar{\lambda}_2 \lambda_2 = 1$

$\therefore$ From (4), $(\bar{\lambda}_2 \lambda_2 - \bar{\lambda}_2 \lambda_1) X_2^{\theta} X_1 = 0$

or $$\bar{\lambda}_2 (\lambda_2 - \lambda_1) X_2^{\theta}X_1 = 0$$

Now, $$\bar{\lambda}_2 (\lambda_2 - \lambda_1) \neq 0 \quad \because \lambda_1, \lambda_2 \text{ are distinct} \quad \therefore \ \lambda_2 - \lambda_1 \neq 0 \text{ also } \lambda_2 \neq 0$$

$\therefore$ $$X_2^{\theta}X_1 = 0$$

Hence X_1, X_2 are orthogonal.

TEST YOUR KNOWLEDGE

1. Diagonalize the following matrices and obtain the modal matrix in each case

(*i*) $\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 9 & -1 & -9 \\ 3 & -1 & 3 \\ -7 & 1 & -7 \end{bmatrix}$

2. Show that the following matrices are similar to diagonal matrices. Also find the transforming matrices and the diagonal matrices

(*i*) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

3. Show that the following matrices are not similar to diagonal matrices

(*i*) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & -1 \\ 0 & 5 & 3 & -1 \end{bmatrix}$

4. If A and B are non-singular matrices of order n, show that the matrices AB and BA are similar.

5. Prove that every orthogonal set of vectors is linearly independent.

6. Prove that any two characteristic vectors corresponding to two distinct characteristic roots of a real symmetric matrix are orthogonal.

7. Show that characteristic vectors corresponding to different characteristic roots of a normal matrix are orthogonal.

8. If X is characteristic vector of a normal matrix A corresponding to characteristic root λ, then X is also a characteristic vector of A^{θ}, the corresponding characteristic root being λ.

ANSWERS

1. (*i*) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 1 \\ -1 & -1 & -3 \end{bmatrix}$

2. (*i*) $P = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ (*ii*) $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4.36. QUADRATIC FORM

Definition. A homogeneous polynomial of second degree in any number of variables is called a quadratic form. For example,

(*i*) $ax^2 + 2hxy + by^2$ (*ii*) $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx$ and

(*iii*) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2lxw + 2myw + 2nzw$

are quadratic forms in two, three and four variables.

In n-variables $x_1, x_2, \ldots\ldots\ldots, x_n$, the general quadratic form is $\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}x_ix_j$, where $b_{ij} \neq b_{ji}$

In the expansion, the coefficient of $x_ix_j = (b_{ij} + b_{ji})$.

Suppose $2a_{ij} = b_{ij} + b_{ji}$, where $a_{ij} = a_{ji}$ and $a_{ii} = b_{ii}$.

$$\therefore \qquad \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}x_ix_j = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j \text{, where } a_{ij} = \frac{1}{2}\left(b_{ij} + b_{ji}\right).$$

Hence every quadratic form can be written as $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j = X'AX$, so that the **matrix A is always symmetric**, where $A = [a_{ij}]$ and $X = [x_1\ x_2 \ldots x_n]$.

Now, writing the above examples of quadratic forms in matrix form, we get

(*i*) $ax^2 + 2hxy + by^2 = [x \;\; y]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(*ii*) $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx = [x \;\; y \;\; z]\begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and (*iii*) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2lxw + 2myw + 2nzw$

$$= [x \;\; y \;\; z \;\; w]\begin{bmatrix} a & h & f & l \\ h & b & g & m \\ f & g & c & n \\ l & m & n & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

Example. *Find a real symmetric matrix C such that* $Q = X'CX$ *equals* : $(x_1 + x_2)^2 - x_3^2$.

(P.T.U., Dec. 2002)

Sol.
$$Q = X'CX = (x_1 + x_2)^2 - x_3^2 = x_1^2 + x_2^2 - x_3^2 + 2x_1x_2$$
$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\therefore$ $C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, which is a real symmetric matrix.

4.37. LINEAR TRANSFORMATION OF A QUADRATIC FORM

Let $X'AX$ be a quadratic form in *n*-variables and let $X = PY$ …(1)

(where P is a non-singular matrix) be the non-singular transformation.

From (1), $X' = (PY)' = Y'P'$ and hence

$$X'AX = Y'P'APY = Y'(PAP)Y = Y'BY \quad \text{…(2)}$$

where $B = P'AP$. Therefore $Y'BY$ is also a quadratic form in *n*-variables. Hence it is a linear transformation of the quadratic form $X'AX$ under the linear transformation $X = PY$ and $B = P'AP$.

Note. (*i*) Here $B' = (P'AP)' = P'AP = B$ (*ii*) $\rho(B) = \rho(A)$.

$\therefore$ **A and B are congruent matrices.**

4.38. CANONICAL FORM

If a real quadratic form be expressed as a sum or difference of the squares of new variables by means of any real non-singular linear transformation, then the later quadratic expression is called a *canonical form* of the given quadratic form.

i.e., if the quadratic form $X'AX = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$ can be reduced to the quadratic form

$Y'BY = \sum_{i=1}^{n} \lambda_i y_i^2$ by a non-singular linear transformation X = PY, then

Y' BY is called the canonical form of the given one.

$\therefore$ If B = P'AP = diag $(\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n)$, then X'AX = Y'BY = $\sum_{j=1}^{n} \lambda_i y_i^2$.

Note. (*i*) Here some of λ_i (eigen values) may be positive or negative or zero.

(*ii*) A quadratic form is said to be real if the elements of the symmetric matrix are real.

(*iii*) If $\rho(A) = r$, then the quadratic form X′AX will contain only r terms.

4.39. INDEX AND SIGNATURE OF THE QUADRATIC FORM

The number p of positive terms in the canonical form is called the *index* of the quadratic form.

(The number of positive terms) – (The number of negative terms) *i.e.*, $p - (r - p) = 2p - r$ is called *signature* of the quadratic form, where $\rho(A) = r$.

4.40. DEFINITE, SEMI-DEFINITE AND INDEFINITE REAL QUADRATIC FORMS

Let X' AX be a real quadratic form in n-variables $x_1, x_2, \ldots\ldots\ldots, x_n$ with rank r and index p. Then we say that the quadratic form is

(*i*) *Positive definite* if $r = n, p = r$ (*ii*) *negative definite* if $r = n, p = 0$

(*iii*) *Positive semi-definite* if $r < n$, $p = r$ and (*iv*) *negative semi-definite* if $r < n, p = 0$

If the canonical form has both positive and negative terms, the quadratic form is said to be *indefinite*.

Note. If X'AX is positive definite, then $| A | > 0$.

4.41. LAW OF INERTIA OF QUADRATIC FORM

The index of a real quadratic form is invariant under real non-singular transformations.

4.42. LAGRANGE'S METHOD OF REDUCTION OF A QUADRATIC FORM TO DIAGONAL FORM

Let the quadratic form be in three variables x, y, z.

Step 1. Reduce the quadratic form to X′AX and find matrix A, where X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Step 2. In the quadratic form *i.e.*, in Q collect all the terms of x and express them as a perfect square in x, y, z by adding or subtracting the terms of y and z.

Step 3. In the next group collect all terms of y and express them as a perfect square by adding or subtracting the terms of z. Now only terms of z^2 will be left which will form 3rd group.

Step 4. Equating terms on the R.H.S. to $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$ and write down the values of y_1, y_2, y_3 in terms of x, y, z.

Step 5. Then express x, y, z in terms of y_1, y_2, y_3 and the linear transformation X = PY is known where P will be formed by coefficients of y_1, y_2, y_3 and Y = $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Step 6. Then Q will be transformed to the diagonal matrix $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, which will be same as P'AP.

ILLUSTRATIVE EXAMPLES

Example 1. *Reduce $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into canonical form.*

Sol. The given quadratic form can be written as X′ AX, where X′ = $[x, y, z]'$ and the symmetric matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}.$$

Let us reduce A into diagonal matrix. We know that $A = I_3 AI_3$

i.e.,
$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 - \frac{2}{3}R_1$, $R_3 - \frac{4}{3}R_1$ on A and prefactor on RHS, we get

$$\begin{bmatrix} 3 & 2 & 4 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 - \frac{2}{3}C_1$, $C_3 - \frac{4}{3}C_1$ on A and the post-factor on RHS, we get

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 + R_2$ on LHS and prefactor on RHS

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 + C_2$ on LHS and post-factor on RHS

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

or
$$\text{Diag}\left(3, -\frac{4}{3}, -1\right) = P'AP$$

$\therefore$ The canonical form of the given quadratic form is

$$Y'(P'AP)Y = \begin{bmatrix} y_1, & y_2, & y_3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_1^2 - \frac{4}{3}y_2^2 - y_3^2 .$$

Here $\rho(A) = 3$, index $= 1$, signature $= 1 - (2) = -1$.

Note 1. In this problem the non-singular transformation which reduces the given quadratic form into the canonical form is X = PY *i.e.*, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Note 2. The above example can also be questioned as 'Diagonalize' the quadratic form $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ by linear transformations and write the linear transformation.

Or

Reduce the quadratic form $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into the sum of squares.

Example 2. *Reduce the quadratic form* $x^2 - 4y^2 + 6z^2 + 2xy - 4xz + 2w^2 - 6zw$ *into sum of squares.*

Sol. The matrix form of the given quadratic is X'AX, where $X = (x, y, z, w)'$

and

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

Let us reduce A to the diagonal matrix. We know that $A = I_4 A I_4$.

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 - R_1$, $R_3 + 2R_1$, also on prefactor on RHS

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 - C_1$, $C_3 + 2C_1$, also on post-factor on RHS

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 + \frac{2}{5}R_2$, also on prefactor on RHS

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 + \frac{2}{5}C_2$, also on post-factor on RHS

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_4 + \frac{15}{14}R_3$, also on prefactor on RHS

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & -3 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ \frac{12}{7} & \frac{3}{7} & \frac{15}{14} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $C_4 + \frac{15}{14}C_3$, also on post-factor on RHS

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & 0 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ \frac{12}{7} & \frac{3}{7} & \frac{15}{14} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & \frac{12}{7} \\ 0 & 1 & \frac{2}{3} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{15}{14} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e., $$\text{diag.}\left(1, -5, \frac{14}{5}, -\frac{17}{14}\right) = P'AP$$

$\therefore$ The canonical form of the given quadratic form is

$$Y'(P'AP)Y = Y'\text{diag}\left(1, -5, \frac{14}{5}, -\frac{17}{14}\right)Y$$

$$= \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & 0 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$= y_1^2 - 5y_2^2 + \frac{14}{5}y_3^2 - \frac{17}{14}y_4^2,$$ which is the sum of the squares.

Example 3. *Show that the form* $5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x_1 + 6x_1x_2$ *is a positive semi-definite and find a non-zero set of values of* x_1, x_2, x_3 *which makes the form zero.* **(P.T.U., Dec. 2002)**

Sol. The matrix form of the given quadratic is $X'AX$, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

Let us reduce A to the diagonal form

$\therefore$ $$A = I\,A\,I$$

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[To avoid fractions first of all operate $R_2(5)$, $R_3(5)$, then $C_2(5)$, $C_3(5)$] Note that row transformations will effect prefactor and column transformation will affect post-factor on RHS

$$\begin{bmatrix} 5 & 15 & 35 \\ 15 & 650 & 50 \\ 35 & 50 & 250 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Operate $R_2 - 3R_1$, $R_3 - 7R_1$

$$\begin{bmatrix} 5 & 15 & 35 \\ 0 & 605 & -55 \\ 0 & -55 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 5 & 0 \\ -7 & 0 & 5 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Operate $C_2 - 3C_1$, $C_3 - 7C_1$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 605 & -55 \\ 0 & -55 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 5 & 0 \\ -7 & 0 & 5 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -7 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Operate $R_3 + \frac{1}{11} R_2$, $C_3 + \frac{1}{11} C_2$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 605 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 5 & 0 \\ -\frac{80}{11} & \frac{5}{11} & 5 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -\frac{80}{11} \\ 0 & 5 & \frac{5}{11} \\ 0 & 0 & 5 \end{bmatrix}$$

$\therefore$ Diagonal matrix $(5, 605, 0) = P'AP$

The quadratic form reduces to the diagonal form $5y_1^2 + 605y_2^2$

$$\rho(A) = 2;$$

Index p = Number of positive terms in the diagonal form = 2

n = The number of variables in quadratic form = 3

$$\rho(A) < 3 \text{ and } p = \rho(A)$$

$\therefore$ Given quadratic form is positive semi-definite

Now, $X = PY$ gives $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -\frac{80}{11} \\ 0 & 5 & \frac{5}{11} \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\therefore$

$$x_1 = y_1 - 3y_2 - \frac{80}{11} y_3$$

$$x_2 = 5y_2 + \frac{5}{11} y_3$$

$$x_3 = 5y_3$$

Assume, $y_1 = 0$, $y_2 = 0$, $y_3 = 11$, we get

$x_1 = -80$, $x_2 = 5$, $x_3 = 55$; Clearly this set of values of x_1, x_2, x_3 makes the given form zero.

Example 4. *Use Lagrange's method to diagonalize the quadratic form :* $2x^2 + 2y^2 + 3z^2 - 4yz + 2xy - 4xz$.

(P.T.U., May 2002)

Sol. Step I. The given quadratic form is $Q = 2x^2 + 2y^2 + 3z^2 - 4yz + 2xy - 4xz$, which can be expressed as

$$X'AX = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ where } A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

Step II. In Q collect all the terms of x and express them as a perfect square

$$= 2\,(x^2 + xy - 2xz) + 2y^2 + 3z^2 - 4yz$$

$$= 2\,[x^2 + x\,(y - 2z)] + 2y^2 + 3z^2 - 4yz$$

$$= 2\left\{\left(x + \frac{y-2z}{2}\right)^2\right\} + 2y^2 + 3z^2 - 4yz - \frac{(y-2z)^2}{2}$$

$$= 2\left\{x + \frac{1}{2}y - z\right\}^2 + 2y^2 + 3z^2 - 4yz - \frac{y^2 - 4yz + 4z^2}{2}$$

Step III. Collect the terms of y and express them as a perfect square

$$= 2\left\{x + \frac{1}{2}y - z\right\}^2 + \left(\frac{3y^2}{2} - 2yz\right) + z^2$$

$$= 2\left\{x + \frac{1}{2}y - z\right\}^2 + \frac{3}{2}\left\{y^2 - \frac{4}{3}yz + \frac{4}{9}z^2\right\} + z^2 - \frac{2}{3}z^2$$

$$= 2\left\{x + \frac{1}{2}y - z\right\}^2 + \frac{3}{2}\left\{y - \frac{2}{3}z\right\}^2 + \frac{1}{3}z^2$$

$$= 2y_1^2 + \frac{3}{2}\,y_2^2 + \frac{1}{3}\,y_3^2, \text{ where } y_1 = x + \frac{1}{2}y - z,\ y_2 = y - \frac{2}{3}z,\ y_3 = z$$

Step IV. Express x, y, z in terms of y_1, y_2, y_3, we get $z = y_3$; $y = y_2 + \frac{2}{3}y_3$, $x = y_1 - \frac{1}{2}y_2 + \frac{2}{3}y_3$

$$\therefore \quad \left.\begin{aligned} x &= y_1 - \frac{1}{2}y_2 + \frac{2}{3}y_3 \\ y &= y_2 + \frac{2}{3}y_3 \\ z &= y_3 \end{aligned}\right\}$$

Step V. Express these in the matrix form X = PY

$$\text{where} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix};\ Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The linear transformation is X = PY

Step VI. Reduces the quadratic form Q to the diagonal form.

$$B = P'AP = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{2}{3} & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -2 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$= \text{diag.}\left(2, \frac{3}{2}, \frac{1}{3}\right)$$

Note. Here $\rho(A) = 4$, index = 2, signature = 2 – 2 = 0.

4.43. REDUCTION TO CANONICAL FORM BY ORTHOGONAL TRANSFORMATION

Let $X'AX$ be a given quadratic form. The modal matrix B of A is that matrix whose columns are characteristic vectors of A. If B represents the orthogonal matrix of A (the normalised modal matrix of A whose column vectors are pairwise orthogonal), then X = BY will reduce $X'AX$ to Y' diag $(\lambda_1, \lambda_2, \ldots\ldots\ldots., \lambda_n)$ Y, where $\lambda_1, \lambda_2, \ldots\ldots\ldots., \lambda_n$ are characteristic values of A.

Note. This method works successfully if the characteristic vectors of A are linearly independent which are pairwise orthogonal.

Example 5. *Reduce $8x^2 + 7y^2 + 3z^2 + 12xy + 4xz - 8yz$ into canonical form by orthogonal reduction.*

Sol. The matrix of the quadratic form is $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic roots of A are given by $|A - \lambda I| = 0$

i.e., $$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda(\lambda - 3)(\lambda - 15) = 0$$

$\therefore \qquad \lambda = 0, 3, 15$

Characteristic vector for $\lambda = 0$ is given by $[A - (0)I]\, X = 0$

i.e.,
$$\begin{aligned} 8x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 + 7x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + 3x_3 &= 0 \end{aligned}$$

Solving first two, we get $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$ giving the eigen vector $X_1 = (1, 2, 2)$

When $\lambda = 3$, the corresponding characteristic vector is given by $[A - 3I]\, X = 0$

i.e.,
$$\begin{aligned} 5x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 + 4x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 &= 0 \end{aligned}$$

Solving any two equations, we get $X_2 = (2, 1, -2)$.

Similarly characteristic vector corresponding to $\lambda = 15$ is $X_3 = (2, -2, 1)$.

Now, X_1, X_2, X_3 are pairwise orthogonal *i.e.,* $X_1 \cdot X_2 = X_2 \cdot X_3 = X_3 \cdot X_1 = 0$.

$\therefore$ The normalised modal matrix is

$$B = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \frac{X_3}{\|X_3\|}\right] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$BB' = I$$

Now, B is orthogonal matrix and | B | = – I

i.e., $B^{-1} = B'$ and $B^{-1}AB = D = \text{diag}\{3, 0, 15\}$

i.e., $$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} A \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$X'AX = Y'(B^{-1}AB)Y = Y'DY$

$$= \begin{bmatrix} y_1, & y_2, & y_3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_1^2 + 0.y_2^2 + 15y_3^2$$, which is the required canonical form.

Note. Here the orthogonal transformation is X = BY, rank of the quadratic form = 2 ; index = 2, signature = 2. It is positive definite.

Example 6. *Reduce* $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ *into canonical form.*

Sol. The matrix of the quadratic form is $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic values are given by $|A - \lambda I| = 0$

i.e., $$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

or $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$, which on solving gives $\lambda = 2, 2, 8$.

The characteristic vector for $\lambda = 2$ is given by $[A - 2I]X = 0$, which reduces to single equation

$$2x_1 - x_2 + x_3 = 0.$$

Putting $x_2 = 0$, we get $\frac{x_1}{1} = \frac{x_3}{-2}$ or the vector is [1, 0, – 2]. Again by putting $x_1 = 0$, we get $\frac{x_2}{1} = \frac{x_3}{1}$ or the vector is [0, 1, 1].

The vector corresponding to $\lambda = 8$ is given by $[A - 8I]X = 0$

i.e., $$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{aligned} -2x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 - 5x_2 - x_3 &= 0 \\ 2x_1 - x_2 - 5x_3 &= 0 \end{aligned}$$

Solving any two of the equations, we get the vector as [2, –1, 1].

Now, $X_1 = [2, -1, 1]$; $X_2 = [0, 1, 1]$ and $X_3 = [1, 0, -2]$

Here X_1, X_2, X_3 are not pairwise orthogonal

$\because$ $X_1 \cdot X_2 = 0$; $X_2 \cdot X_3 \neq 0$ and $X_3 \cdot X_1 = 0$

To get X_3 orthogonal to X_2 assume a vector [*u*, *v*, *w*] orthogonal to X_2 also satisfying

$2x_1 - x_2 + x_3 = 0$; *i.e.,* $2u - v + w = 0$ and $0.u + 1.v + 1.w = 0$

Solving $[u, v, w] = [1, 1, -1] = X_3$ so that $X_1 \cdot X_2 = X_2 \cdot X_3 = X_3 \cdot X_1 = 0$

$\therefore$ The normalised modal matrix is $B = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$

Now B is orthogonal matrix and $|B| = 1$

i.e., $B' = B^{-1}$ and $B^{-1}AB = D$, where $D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$\therefore \quad X'AX = Y'(B^{-1}AB)Y = Y'DY$$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 8y_1^2 + 2y_2^2 + 2y_3^2$$

which is the required canonical form.

Note. In the above form rank of the quadratic form is 3, index = 3, signature = 3. It is positive definite.

TEST YOUR KNOWLEDGE

1. Write down the matrices of the following quadratic forms:

(*i*) $2x^2 + 3y^2 + 6xy$ (*ii*) $2x^2 + 5y^2 - 6z^2 - 2xy - yz + 8zx$

(*iii*) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 2x_1x_2 + 4x_1x_3 - 6x_1x_4 - 4x_2x_3 - 8x_2x_4 + 12x_3x_4$

(*iv*) $x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3 - 5x_2^2 + 4x_3^2$ **(P.T.U., May 2010)**

(*v*) $3x^2 + 7y^2 - 8z^2 - 4yz + 3xz$. **(P.T.U., Dec. 2011)**

2. Write down the quadratic form corresponding to the following matrices:

(*i*) $\begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$.

3. Reduce the following quadratic forms to canonical forms or to sum of squares by linear transformation. Write also the rank, index and signature:

(*i*) $2x^2 + 2y^2 + 3z^2 + 2xy - 4yz - 4xz$ (*ii*) $12x_1^2 + 4x_2^2 + 5x_3^2 - 4x_2x_3 + 6x_1x_3 - 6x_1x_2$

(*iii*) $2x^2 + 6y^2 + 9z^2 + 2xy + 8yz + 6xz$ (*iv*) $x^2 + 4y^2 + z^2 + 4xy + 6yz + 2zx$.

4. Reduce the following quadratic forms to canonical forms or to sum of square by orthogonal transformation. Write also rank, index, signature:

(*i*) $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2zx$ (*ii*) $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_3 + 2x_1x_3 - 2x_2x_3$

(*iii*) $3x^2 - 2y^2 - z^2 - 4xy + 8xz + 12yz$ (*iv*) $x^2 + 3y^2 + 3z^2 - 2xy$.

5. Use Lagranges method to diagonalize the quadratic form $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$.

ANSWERS

1. (*i*) $\begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 5 & -\frac{1}{2} \\ 4 & -\frac{1}{2} & -6 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 1 & 1 & 2 & -3 \\ 1 & 2 & -2 & -4 \\ -2 & -2 & 3 & 6 \\ -3 & -4 & 6 & 4 \end{bmatrix}$

(*iv*) $\begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$ (*v*) $\begin{bmatrix} 3 & 0 & \frac{3}{2} \\ 0 & 7 & -2 \\ \frac{3}{2} & -2 & -8 \end{bmatrix}$.

2. (*i*) $2x^2 + 3y^2 + z^2 + 8xy + 2yz + 10xz$ (*ii*) $x_1^2 - 4x_2^2 + 6x_3^2 + 2x_4^2 + 2x_1x_2 - 4x_1x_3 - 6x_3x_4$.

3. (*i*) $3y_1^2 + \frac{3}{2}y_2^2 + \frac{1}{3}y_3^2$; Rank = 3, index = 3, Sig. = 3

(*ii*) $12y_1^2 + \frac{13}{4}y_2^2 + \frac{49}{13}y_3^2$; Rank = 3, index = 3, Sig. = 3

(*iii*) $2y_1^2 - 7y_2^2 - \frac{13}{14}y_3^2$; Rank = 3, index = 1, Sig. = – 1

(*iv*) $y_1^2 + y_2^2 + y_3^2$; Rank = 3 , index = 3, Sig. = 3.

4. (*i*) $2y_1^2 + 2y_2^2 + 6y_3^2$; Rank = 3, index = 3 , Sig. = 3

(*ii*) $4y_1^2 + y_2^2 + y_3^2$; Rank = 3, index = 3, Sig. = 3

(*iii*) $3y_2^2 + 6y_2^2 - 9y_3^2$; Rank = 3, index = 2, Sig. = 1

(*iv*) $y_1^2 + 2y_2^2 - 4y_3^2$; Rank = 3, index = 3, Sig. = 3.

5. diag. $\left(6, \frac{7}{3}, \frac{16}{7}\right)$.

REVIEW OF THE CHAPTER

1. **Matrix:** A set of mn numbers (real or complex) arranged in a rectangular array having m rows and n columns, enclosed by brackets [ ] or ( ) is called a $m \times n$ matrix.
2. **Elementary Transformations:** The following operations on a matrix are called elementary transformations:
   (*a*) Interchange of two rows or columns (R_{ij} or C_{ij})
   (*b*) Multiplication of each element of a row or column by a non-zero number k (kR_i or kC_i)
   (*c*) Addition of k times the elements of a row (column) to the corresponding elements of another row (column) ($k \neq 0$) ($R_i + kR_j$ or $C_i + kC_j$).
3. **Elementary Matrix:** The matrix obtained from a unit matrix I by subjecting it to one of the E-operations is called an elementary matrix.
4. **Gauss-Jordan Method:** It is the method to find inverse of a matrix by E-operations.
5. **Normal Form of a Matrix:** Any non-zero matrix $m \times n$ can be reduced to anyone of the following forms by performing E-operations (row, column or both)

   (*i*) I_r (*ii*) $\begin{bmatrix} I_r & 0 \end{bmatrix}$ (*iii*) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (*iv*) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

   where I_r is a unit matrix of order r. All these forms are known as normal forms of the matrix.
6. **Rank of a Matrix:** Let A be any $m \times n$ matrix. If all minors of order $(r + 1)$ are zero but there is at least one non-zero minor of order r, then r is called the rank of A, and is written as $\rho(A) = r$.
7. **Consistent, Inconsistent Equations:** A system of equations having one or more number of solutions is called a consistent system of equations. A system of equations having no solution is called inconsistent system of equations. For a system of *non-homogeneous linear equation* AX = B

(*i*) If $\rho[A : B] \neq \rho(A)$, the system is inconsistent.

(*ii*) If $\rho[A : B] = \rho(A)$ = number of unknowns, the system has a unique solution.

(*iii*) If $\rho[A : B] = \rho(A)$ < number of unknowns, the system has an infinite number of solutions.

The matrix [A : B] is called Augmented Matrix.

For a system of **Homogeneous Linear Equations** AX = 0.

(*i*) X = 0 is always a solution; called *trivial solution*

(*ii*) If $\rho(A)$ = number of unknowns, then the system has only the trivial solution

(*iii*) If $\rho(A)$ < number of unknowns, in system has an infinite number of non-trivial solutions.

Homogeneous equations are always consistent.

8. **Linear Dependence and Linear Independence of Vectors:** A set of n-tuple vectors $x_1, x_2, \ldots, x_r$ is said to be:

(*i*) Linearly dependent if $\exists$s r scalar $k_1, k_2, \ldots, k_r$ ***Not All Zero*** such that $k_1x_1 + k_2x_2 + \ldots + k_rx_r = 0$

(*ii*) Linearly independent if each one of $k_1, k_2, \ldots, k_r$ is zero *i.e.*, $k_1 = k_2 = \ldots\ldots = k_r = 0$.

If a set of vectors is linearly dependent then at least one member of the set can be expressed as a linear combination of the remaining vectors.

9. **Linear Transformation:** A transformation Y = AX is said to be linear transformation if $Y_1 = AX_1$ and $Y_2 = AX_2$ $\Rightarrow aY_1 + bY_2 = A(aX_1 + bX_2)\ \forall\ a, b$

If the transformation matrix A is non-singular, then the linear transformation is called non-singular or regular and if A is singular then linear transformation is also singular.

10. **Orthogonal Transformation:** The linear transformation Y = AX is orthogonal transformation if it transforms $y_1^2 + y_2^2 + \ldots + y_n^2$ to $x_1^2 + x_2^2 + \ldots + x_n^2$.

11. **Orthogonal Matrix:** A real square matrix A is said to be orthogonal if $AA' = A'A = I$

For an orthogonal matrix $A' = A^{-1}$

12. **Properties of an Orthogonal Matrix:**

(*i*) The transpose of an orthogonal matrix is orthogonal

(*ii*) The inverse of an orthogonal matrix is orthogonal

(*iii*) If A is orthogonal matrix then $|A| = \pm 1$

(*iv*) Product of two orthogonal matrices of the same order is an orthogonal matrix.

13. **Hermitian and Skew Hermitian Matrix:** A square matrix A is said to be Hermitian if $A^\theta = A$. All diagonal elements of a Hermitian matrix are purely real.

A square matrix A is said to be Skew Hermitian. If $A^\theta = -A$. All diagonal elements of a Skew Hermitian Matrix are zero or purely imaginary of the form $i\beta$.

14. **Unitary Matrix:** A complex matrix A is said to be unitary matrix if $A^\theta A = I$

15. **Properties of a Unitary Matrix:**

(*i*) Determinant of a unitary matrix is of modulus unity

(*ii*) The product of two unitary matrices of the same order is unitary

(*iii*) The inverse of a unitary matrix is unitary.

16. **Characteristic Equation, Characteristic Roots or Eigen Values, Trace of a Matrix:** If A is a square matrix of order, n is a scalar and I is a unit matrix of order n, then $|A - \lambda I| = 0$ is called characteristic equation of A. The roots of the characteristic equation are called characteristic roots or Eigen values of A.

The sum of Eigen values of A is equal to trace of A.

17. **Eigen Vectors:** Let A be a square matrix of order n, λ is a scalar. Consider the linear transformation Y = AX, where X be such a vector which transforms in λX. Then $Y = \lambda X$ and therefore, we have $AX = \lambda X$ or $(A - \lambda I)X = 0$ which gives n homogeneous linear equations and for non-trivial solutions of these linear homogeneous equation we must have $|A - \lambda I| = 0$, which gives n eigen values of A. Corresponding to each eigen value $(A - \lambda I)X = 0$ has a non-zero solution called eigen vector or latent vector.

18. **Cayley Hamilton Theorem:** Every square matrix satisfies its characteristic equation.

19. **Diagonalizable Matrices:** A matrix A is said to be diagonalizable if $\exists$s an invertible matrix B such that $B^{-1}AB = D$, where D is a diagonal matrix and the diagonal elements of D are the eigen values of A.

Note. A square matrix A is diagonalizable iff it has n linearly independent eigen vectors.

20. **Similar Matrices:** Let A and B be two square matrices of the same order. The matrix A is said to be similar to matrix B if $\exists$s an invertible matrix P such that PA = BP or $A = P^{-1}BP$

Similar matrices have the same characteristic equation.

21. Column vectors of a unitary matrix are normal and orthogonal.

22. Every orthonormal set of vectors is L.I.

23. Any two characteristic vectors corresponding to two distinct characteristic roots of a Hermitian/unitary matrix are orthogonal.

24. Quadratic Form: A homogeneous polynomial of second degree in any number of variables is called quadratic form. Every quadratic form can be expressed in the form X′ AX, where A is a symmetric matrix.

25. Linear Transformation of a Quadratic form: Let X′ AX be a quadratic form in n-variables and let X = PY be a non-singular transformation, then X′ AX = Y′BY, where B = P′ AP. Then Y′ BY is a linear transformation of the quadratic form X′ AX under the linear transformation X = PY. A and B are congruent matrices.

26. Canonical Form: If a real quadratic form X′ AX be expressed as a sum or difference of the squares of new variables by means of any real linear transformation then the later quadratic expression is called a canonical form of the given quadratic form non-singular. If $\rho(A) = r$, then quadratic form X′AX will contain only r terms.

27. Index and Signature of the Quadratic Form: The number p of the positive terms in the canonical form is called the index of the quadratic form.

The number of positive terms minus number of negative terms is called signature of the quadratic form *i.e.*, signature $= p - (r - p) = 2p - r$, where $\rho(A) = r$.

28. Definite, Semi-definite and Indefinite Real Quadratic Forms: Let X′ AX be a real quadratic form then it will be

(*i*) positive definite if $r = n, p = r$

(*ii*) negative definite if $r = n, p = 0$

(*iii*) positive semi-definite if $r < n, p = r$

(*iv*) negative semi-definite if $r < n, p = 0$

SHORT ANSWER TYPE QUESTIONS

1. (*a*) If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}; B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ prove that $|AB| = 16$ **(P.T.U., May 2009)**

(*b*) Prove by an example that AB can be zero matrix when neither A nor B is zero.

2. Explain elementary transformations on a matrix. **(P.T.U., Dec. 2004)**

3. What is Gauss Jordan Method of finding inverse of a non-singular matrix? Hence find the inverse of the matrix

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$ **(P.T.U., May 2012)**

4. Reduce the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. **(P.T.U., Dec. 2012)**

5. (*a*) Define rank of a matrix and give one example. **(P.T.U., Dec. 2005, 2006, May 2007, Jan. 2008, May 2011)**

(*b*) What is the rank of a non-singular matrix of order n? **(P.T.U., Dec. 2010)**

6. Find rank of the following matrices:

(*i*) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ **(P.T.U., May 2014)** (*ii*) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 1 & 6 \\ 2 & 4 & 2 & 4 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$ **(P.T.U., Dec. 2012)**

(*iv*) $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ **(P.T.U., May 2009)** (*v*) $\begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & 0 \\ 3 & -3 & 1 \end{bmatrix}$ **(P.T.U., Dec. 2013)**

(*vi*) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$ **(P.T.U., Dec. 2011)**

7. If A is a non-zero column and B is a non-zero row matrix, show that rank AB = 1.

8. State the conditions in terms of rank of the coefficient matrix and rank of the augmented matrix for a unique solution, no solution, infinite number of solutions of a system of linear equations. **(P.T.U., May 2005, Dec. 2010)**

9. (*a*) For what values of λ do the equations $ax + by = \lambda x$ and $cx + dy = \lambda y$ have a solution other than $x = 0$, $y = 0$?
[**Hint:** Consult S.E. 4(*b*) art. 4.13] **(P.T.U., May 2003)**

(*b*) Show that the equations $2x + 6y + 11 = 0$; $6x + 20y - 6z + 3 = 0$; $6y - 18z + 1 = 0$ are not-consistent.

[**Hint:** To prove $\rho(A) = 2$, $\rho(A \vdots B) = 3$ as $\rho(A \vdots B) \neq \rho(A)$ $\therefore$ equations are inconsistent] **(P.T.U., Dec. 2003)**

(*c*) For what value of K, the system of equations $x + y + z = 2$; $x + 2y + z = -2$; $x + y + (K - 5)z = K$ has no solution? **(P.T.U., May 2012)**

10. If A is a non-singular matrix, then the matrix equation AX = B has a unique solution.

11. (*a*) Define linear dependence and linear independence of vectors and give one example of each.
(P.T.U., May 2004, 2006, Jan. 2009)

(*b*) Test whether the subset S of R^3 is L.I. or L.D., given S = {(1, 0, 1), (1, 1, 0), (–1, 0, –1)}. **(P.T.U., May 2010)**

(*c*) Define linear dependence of vectors and determine whether the vectors (3, 2, 4), (1, 0, 2), (1, –1, –1) are linear dependent or not, where '*t*' denotes transpose. **(P.T.U., May 2006)**

[**Hint:** Consult S.E. 3 art. 4.15]

12. (*a*) Prove that $X_1 = (1, 1, 1)$, $X_2 = (1, -1, 1)$, $X_3 = (3, -1, 3)$ are linearly independent vectors.
(P.T.U., Dec. 2012)

(*b*) Are these vectors $x_1 = (1, 2, 1)$, $x_2 = (2, 1, 4)$, $x_3 = (4, 5, 6)$, $x_4 = (1, 8, -3)$ L.D? **(P.T.U., Jan. 2010)**

(*c*) For what value(s) of K do the set of vectors (K, 1, 1), (0, 1, 1), (K, 0, K) in R^3 are linearly independent?
(P.T.U., May 2010, 2012)

13. Show that column vectors of the matrix $A = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ are linearly dependent. **(P.T.U., May 2003)**

[**Hint:** S.E. 2 art. 4.15]

14. Define an orthogonal transformation. Derive the condition for the linear transformation on Y = AX to be orthogonal.
[**Hint:** See art 4.17] **(P.T.U., May 2012)**

15. (*a*) Define an orthogonal matrix and prove that

$$A = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \text{ is orthogonal.}$$ **(P.T.U., Jan. 2009, May 2011)**

(*b*) Prove that the matrix $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal. **(P.T.U., June 2003, May 2007, Jan. 2009)**

(*c*) Find the values of *a*, *b*, *c* if the matrix $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal. **(P.T.U., May. 2009)**

[**Hint:** S.E. 2 art. 4.17]

16. Prove that transpose of an orthogonal matrix is orthogonal.

17. Prove that inverse of an orthogonal matrix is orthogonal.

18. State the properties of an orthogonal matrix.

19. (*a*) Show that the transformation $y_1 = x_1 - x_2 + x_3$; $y_2 = 3x_1 - x_2 + 2x_3$ and $y_3 = 2x_1 - 2x_2 + 3x_3$ is non-singular (regular).

(*b*) Find the inverse transformation of $y_1 = x_1 + 2x_2 + 5x_3$; $y_2 = -x_2 + 2x_3$ and $y_3 = 2x_1 + 4x_2 + 11x_3$. **(P.T.U., May 2011)**
[**Hint:** S.E. 2 art. 4.18]

20. Prove that determinant of an orthogonal matrix is of modulus unity.

21. Define symmetric matrix and prove that inverse of a non-singular matrix is symmetric.

22. Define symmetric and skew symmetric matrix and express a square matrix A as the sum of a symmetric and a skew symmetric matrix.

23. Define a Hermitian matrix and prove that if A is Hermitian, then A^θ A is also Hermitian.
[**Hint:** S.E. 3(*b*) art. 4.22] **(P.T.U., May 2007, Dec. 2010)**

24. (*a*) Define a skew Hermitian matrix and prove that if A is Hermitian, then *i*A is skew Hermitian.

(*b*) Show that if $A = \begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{bmatrix}$, then *i*A is skew-Hermitian. **(P.T.U., Jan. 2010)**

25. (*a*) Define a unitary matrix and give one example of a unitary matrix.

(*b*) Show that $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is unitary. **(P.T.U., Jan. 2009)**

26. State properties of a unitary matrix.

27. Prove that inverse of a unitary matrix is unitary. **(P.T.U., May 2012)**

28. Prove that product of two unitary matrices of the same order is again a unitary matrix.

29. Prove that determinant of a unitary matrix is of modulus unity.

30. Define the following :

(*i*) Characteristic equation of a square matrix

(*ii*) Characteristic roots or latent roots or eigen values of a matrix **(P.T.U., Jan. 2009)**

(*iii*) Eigen vectors of a square matrix. **(P.T.U., Jan. 2009)**

31. (*a*) Find eigen values of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$. **(P.T.U., Dec. 2006)**

(*b*) Prove that eigen values of a diagonal matrix are given by its diagonal elements.

[**Hint:** Let $A = \begin{bmatrix} a_{11} & a_{22} & \ldots\ldots & a_{nn} \end{bmatrix}$; $|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & 0 & 0 & \ldots\ldots & 0 \\ 0 & a_{22}-\lambda & 0 & \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & \ldots\ldots & \vdots \\ \vdots & \vdots & \vdots & \ldots\ldots & \vdots \\ 0 & 0 & 0 & \vdots & a_{nn}-\lambda \end{vmatrix} = 0$

i.e., $(a_{11} - \lambda)\,(a_{22} - \lambda)\ \ldots\ldots\ (a_{nn} - \lambda) = 0$ *i.e.,* $\lambda = a_{11}, a_{22}, \ldots\ldots, a_{nn}$]

32. Show that if $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n$ are the latent roots of the matrix A, then A^3 has latent roots. $\lambda_1^3, \lambda_2^3, \ldots\ldots, \lambda_n^3$.
[**Hint:** S.E. 3 art. 4.25]

33. (*a*) Show that the eigen values of a Hermitian matrix are real.

(*b*) Show that eigen values of a Skew Hermitian matrix are either zero or purely imaginarly. **(P.T.U., Dec. 2012)**
[**Hint:** S.E. 4 art. 4.25]

34. Prove that matrix A and its transpose A′ have the same characteristic roots.
[**Hint:** Characteristic roots of A are $|A - \lambda I| = 0$ we have $|A - \lambda I| = |(A - \lambda I)'| = |A' - \lambda I'| = |A' - \lambda I|$
$\therefore$ A and A′ have same eigen roots]

35. (*a*) If λ is an eigen value of a non-singular matrix A prove the following :

(*i*) λ^{-1} is an eigen value of A^{-1} **(P.T.U., May 2005)**

(*ii*) $\frac{|A|}{\lambda}$ is an eigen value of Adj. A **(P.T.U., Dec. 2003)**

(*iii*) λ^2 is an eigen value of A^2. **(P.T.U., Dec. 2004)**

[**Hint:** See art. 4.25]

(*b*) Write four properties of eigen values **(P.T.U., May 2008)**

[**Hint:** See art. 4.25]

36. (*a*) State Cayley Hamilton Theorem. **(P.T.U., Dec. 2003, Jan. 2010, May 2011)**

(*b*) Use Cayley Hamilton Theorem to find A^8, where $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. **(P.T.U., Dec. 2003, May 2010)**

(*c*) If $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, then use Cayley Hamilton Theorem to find the matrix represented by A^5.

[**Hint:** S.E. 6 art. 4.26]

37. Verify Cayley Hamilton Theorem for the matrix **(P.T.U., Dec. 2013)**

$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Find inverse of the matrix also.

38. Test whether the following matrices are diagonalizable or not?

(*i*) $\begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ **(P.T.U., May 2012)** (*ii*) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ **(P.T.U., Dec. 2013)**

[**Hint:** S.E. 1 art 4.27]

39. (*a*) Define similar matrices. **(P.T.U., May 2007)**

(*b*) Examine whether the matrix A is similar to matrix B, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. **(P.T.U., May 2010)**

[**Hint:** See Solved Example 4(*ii*) art 4.30]

40. Prove that if A is similar to a diagonal matrix B, then A′ is similar to A.

[**Hint:** See Solved Example 6 Art. 4.30]

41. Show that rank of every matrix similar to A is same as that of A.

[**Hint:** S.E. 7 art. 4.30]

42. Show that two similar matrices have the same characteristic roots. **(P.T.U., May 2003)**

[**Hint:** Let A and B be two similar matrices $\therefore A = P^{-1} BP$; $| A - \lambda I | = | P^{-1} BP - \lambda I | = | P^{-1} BP - P^{-1} \lambda P |$ $= | P^{-1} (B - \lambda I) P | = | P^{-1} | | B - \lambda I | | P | = | B - \lambda I | | P^{-1} P | = | B - \lambda I | . 1$ $\therefore$ A, B have same characteristic roots]

43. Define index and signature of the quadratic form.

44. Find a real symmetric matrix such that Q = X′ CX equals ; $(x_1 + x_2)^2 - x_3^2$.

45. (*a*) Express the quadratic form $x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1 x_2 - 2x_2 x_3$ as the product of matrices.

(*b*) Obtain the symmetric matrix A for the quadratic forms

(*i*) $x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3 - 5x_2^2 + 4x_3^2$ **(P.T.U., May 2010)**

(*ii*) $3x^2 + 7y^2 - 8z^2 - 4yz + 3xz$. **(P.T.U., Dec. 2011)**

46. Write down quadratic form corresponding to the matrix $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 5 & -\frac{1}{2} \\ 4 & -\frac{1}{2} & -6 \end{bmatrix}$.

47. Define orthogonal set of vectors.

[**Hint:** See art. 4.33(*a*)]

48. Prove that every orthonormal set of vectors is linearly independent.

[**Hint:** See art. 4.33(*b*)]

49. Let T be a transformation from R^1 to R^3 defined by $T(x) = (x, x^2, x^3)$. Is T linear or not?

[**Hint:** See S.E. 1 art. 4.16] **(P.T.U., May 2010)**

ANSWERS

1. (*b*) $A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

5. (*b*) n

6. (*i*) 2 (*ii*) 2 (*iii*) 2 (*iv*) 2 (*v*) 3 (*vi*) 2

9. (*a*) $\lambda = a$, $b = 0$; $\lambda = d$, $c = 0$, (*c*) $K = 6$

11. (*b*) L.D. (*c*) Not dependent

12. (*b*) yes (*c*) for all non-zero values of K

15. (*c*) $a = \pm \dfrac{1}{\sqrt{2}}, b = \pm \dfrac{1}{\sqrt{6}}, c = \pm \dfrac{1}{\sqrt{3}}$

19. (*b*) $x_1 = 19y_1 + 2y_2 - 9y_3$; $x_2 = -4y_1 - y_2 + 2y_3$; $x_3 = -2y_1 + y_3$

31. (*a*) 1, – 4, 7

36. (*b*) 625I (*c*) $\begin{bmatrix} 4181 & 6765 \\ 6765 & 10946 \end{bmatrix}$

37. $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$

38. (*i*) diagonalizable (*ii*) not diagonalizable

39. (*b*) not similar

44. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

45. (*a*) $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(*b*) (*i*) $\begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 3 & 0 & \frac{3}{2} \\ 0 & 7 & -2 \\ \frac{3}{2} & -2 & -8 \end{bmatrix}$

46. $2x^2 + 5y^2 - 6z^2 - 2xy - yz + 8zx$

49. not linear.

5

Infinite Series

5.1. SEQUENCE

A sequence is a function whose domain is the set N of all natural numbers whereas the range may be any set S. In other words, a sequence in a set S is a rule which assigns to each natural number a unique element of S.

5.2. REAL SEQUENCE

A real sequence is a function whose domain is the set N of all natural numbers and range a subset of the set R of real numbers.

Symbolically $\quad f : \mathrm{N} \to \mathrm{R} \quad$ or $\quad (x : \mathrm{N} \to \mathrm{R}$ or $a : \mathrm{N} \to \mathrm{R})$

is a real sequence.

Note. If $x : \mathrm{N} \to \mathrm{R}$ be a sequence, the image of $n \in$ N instead of denoting it by $x(n)$, we shall generally denote it by x_n. Thus x_1, x_2, x_3 etc. are the real numbers associated to 1, 2, 3 etc. by this mapping. Also, the sequence $x : \mathrm{N} \to \mathrm{R}$ is denoted by $\{x_n\}$ or (x_n).

$x_1, x_2, ...$ are called the first, second terms of the sequence. The mth and nth terms x_m and x_n for $m \neq n$ are treated as distinct terms if $x_m = x_n$ *i.e.*, the terms occurring at different positions are treated as distinct terms even if they have the same value.

5.3. RANGE OF A SEQUENCE

The set of all **distinct** terms of a sequence is called its range.

Note. In a sequence $\{x_n\}$, since $n \in$ N and N is an infinite set, **the number of terms of a sequence is always infinite.** The range of a sequence may be a finite set. *e.g.,* if $x_n = (-1)^n$, then $\{x_n\} = \{-1, 1, -1, 1, ...\}$

The range of sequence $\{x_n\} = \{-1, 1\}$, which is a finite set.

5.4. CONSTANT SEQUENCE

A sequence $\{x_n\}$ defined by $x_n = c \in \mathrm{R} \quad \forall \; n \in$ N is called a constant sequence.

e.g., $\quad \{x_n\} = \{c, c, c, ...\}$ is a constant sequence with range $= \{c\}$.

5.5. BOUNDED AND UNBOUNDED SEQUENCES

Bounded above sequence. A sequence $\{a_n\}$ is said to be bounded above if $\exists$ a real number K such that $a_n \le \mathrm{K} \quad \forall \; n \in$ N.

Bounded below sequence. A sequence $\{a_n\}$ is said to be bounded below if $\exists$ a real number k such that $a_n \ge k \quad \forall \; n \in$ N.

Bounded sequence. A sequence $\{a_n\}$ is said to be bounded when it is bounded both above and below.

$\Rightarrow$ A sequence $\{a_n\}$ is bounded if $\exists$ two real numbers k and K ($k \le$ K) such that

$$k \le a_n \le \mathrm{K} \quad \forall \; n \in \mathrm{N}.$$

Choosing M = maximum {| k |, | K |}, we can also define a sequence $\{a_n\}$ to be bounded if

$$|a_n| \le M \quad \forall \quad n \in N.$$

Unbounded sequence. If $\exists$ no real number M such that $|a_n| \le M \ \forall\ n \in N$, then the sequence $\{a_n\}$ is said to be unbounded.

For examples (*i*). Consider the sequence $\{a_n\}$ defined by $a_n = \dfrac{1}{n}$.

Here $$\{a_n\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, ...\right\}$$

$\because$ $$0 < a_n \le 1 \quad \forall \quad n \in N$$

$\therefore$ $\{a_n\}$ is bounded.

(*ii*) Consider the sequence $\{a_n\}$ defined by $a_n = 2^{n-1}$

Here $$\{a_n\} = \{1, 2, 2^2, 2^3, ...\}.$$

Although $a_n \ge 1, \quad \forall \quad n \in N, \exists$ no real number K such that $a_n \le K$.

$\therefore$ The sequence is unbounded above.

5.6. CONVERGENT, DIVERGENT, OSCILLATING SEQUENCES

Convergent sequence. A sequence $\{a_n\}$ is said to be convergent if $\underset{n\to\infty}{\text{Lt}}\ a_n$ is finite.

For example, consider the sequence $\dfrac{1}{2}, \dfrac{1}{2^2}, \dfrac{1}{2^3}, ..., \dfrac{1}{2^n}, ...$

Here $a_n = \dfrac{1}{2^n}, \ \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \dfrac{1}{2^n} = 0$, which is finite.

$\Rightarrow$ The sequence $\{a_n\}$ is convergent.

Divergent sequence. A sequence $\{a_n\}$ is said to be divergent if $\underset{n\to\infty}{\text{Lt}}\ a_n$ is not finite, *i.e.*, if

$$\underset{n\to\infty}{\text{Lt}}\ a_n = +\infty \text{ or } -\infty.$$

For examples

(*i*) Consider the sequence $\{n^2\}$

Here $a_n = n^2, \ \underset{n\to\infty}{\text{Lt}}\ a_n = +\infty \quad \Rightarrow$ The sequence $\{n^2\}$ is divergent.

(*ii*) Consider the sequence $\{-2^n\}$.

Here $$a_n = -2^n, \ \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ (-2^n) = -\infty$$

$\Rightarrow$ The sequence $\{-2^n\}$ is divergent.

Oscillatory sequence. If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an oscillatory sequence. Oscillatory sequences are of two types :

(*i*) A bounded sequence which does not converge is said to **oscillate finitely.**

For example, consider the sequence $\{(-1)^n\}$.

Here $$a_n = (-1)^n$$

It is a bounded sequence, $\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n} = 1$

$$\underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n+1} = -1.$$

Thus, $\underset{n\to\infty}{\text{Lt}}\ a_n$ does not exist $\Rightarrow$ the sequence does not converge.

Hence this sequence oscillates finitely.

(*ii*) An unbounded sequence which does not diverge is said to **oscillate infinitely.**

For example, consider the sequence $\{(-1)^n n\}$.

Here $\qquad a_n = (-1)^n n.$

It is an unbounded sequence.

$$\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n} . 2n = \underset{n\to\infty}{\text{Lt}}\ 2n = +\infty$$

$$\underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n+1}(2n+1) = \underset{n\to\infty}{\text{Lt}}\ -(2n+1) = -\infty.$$

Thus the sequence does not diverge.

Hence this sequence oscillates infinitely.

Note. When we say $\underset{n\to\infty}{\text{Lt}}\ a_n = l$, it means $\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = l$

Similarly, $\qquad \underset{n\to\infty}{\text{Lt}}\ a_n = +\infty$ means $\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = +\infty.$

5.7. MONOTONIC SEQUENCES

(*i*) A sequence $\{a_n\}$ is said to be **monotonically increasing** if $a_{n+1} \geq a_n \quad \forall \quad n \in \text{N}.$

i.e., if $\qquad a_1 \leq a_2 \leq a_3 \leq \ldots\ldots \leq a_n \leq a_{n+1} \leq \ldots\ldots$

(*ii*) A sequence $\{a_n\}$ is said to be **monotonically decreasing** if $a_{n+1} \leq a_n \quad \forall \quad n \in \text{N}.$

i.e., if $\qquad a_1 \geq a_2 \geq a_3 \geq \ldots\ldots \geq a_n \geq a_{n+1} \geq \ldots\ldots$

(*iii*) A sequence $\{a_n\}$ is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.

(*iv*) A sequence $\{a_n\}$ is said to be **strictly monotonically increasing** if

$$a_{n+1} > a_n \quad \forall \quad n \in \text{N}.$$

(*v*) A sequence $\{a_n\}$ is said to be **strictly monotonically decreasing** if

$$a_{n+1} < a_n \quad \forall \quad n \in \text{N}.$$

(*vi*) A sequence $\{a_n\}$ is said to be **strictly monotonic** if it is either strictly monotonically increasing or strictly monotonically decreasing.

5.8. LIMIT OF A SEQUENCE

A sequence $\{a_n\}$ is said to approach the limit l (say) when $n \to \infty$, if for each $\varepsilon > 0$, $\exists$ a +ve integer m (depending upon ε) such that $|a_n - l| < \varepsilon \quad \forall \quad n \geq m$.

In symbols, we write $\quad \underset{n\to\infty}{\text{Lt}}\ a_n = l.$

Note. $|a_n - l| < \varepsilon \ \forall\ n \geq m \quad \Rightarrow \quad l - \varepsilon < a_n < l + \varepsilon \quad$ for $\quad n = m, m+1, m+2, \ldots\ldots$

5.9. EVERY CONVERGENT SEQUENCE IS BOUNDED

Let the sequence $\{a_n\}$ be convergent. Let it tend to the limit l.

Then given $\varepsilon > 0$, $\exists$ a +ve integer m, such that

$$|a_n - l| < \varepsilon \quad \forall \quad n \geq m$$

$$\Rightarrow \qquad l - \varepsilon < a_n < l + \varepsilon \quad \forall \quad n \geq m.$$

Let k and K be the least and the greatest of $a_1, a_2, a_3, ..., a_{m-1}, l - \varepsilon, l + \varepsilon$

Then $\qquad k \leq a_n \leq \text{K} \quad \forall \quad n \in \text{N},$

$\Rightarrow$ the sequence $\{a_n\}$ is bounded.

The converse is not always true *i.e.*, a sequence may be bounded, yet it may not be convergent. *e.g.*, Consider $a_n = (-1)^n$, then the sequence $\{a_n\}$ is bounded but not convergent since it does not have a unique limiting point.

5.10. CONVERGENCE OF MONOTONIC SEQUENCES

Theorem I. *The necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.*

A monotonic increasing sequence which is bounded above converges.

A monotonic decreasing sequence which is bounded below converges.

Theorem II. *If a monotonic increasing sequence is not bounded above, it diverges to* $+\infty$.

Theorem III. *If a monotonic decreasing sequence is not bounded below, it diverges to* $-\infty$.

Theorem IV. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences, then sequence* $\{a_n + b_n\}$ *is also convergent.*

Or

If $Lt\ a_n = A$ *and* $Lt\ b_n = B$, *then* $Lt\ (a_n + b_n) = A + B$.

Theorem V. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences such that* $Lt\ a_n = A$ *and* $Lt\ b_n = B$, *then*

(*i*) *sequence* $\{a_n b_n\}$ *is also convergent and converges to AB.*

(*ii*) *sequence* $\left\{\dfrac{a_n}{b_n}\right\}$ *is also convergent and converges to* $\dfrac{A}{B}$, $(B \neq 0)$.

Theorem VI. *The sequence* $\{|a_n|\}$ *converges to zero if and only if the sequence* $\{a_n\}$ *converges to zero.*

Theorem VII. *If a sequence* $\{a_n\}$ *converges to a and* $a_n \geq 0 \quad \forall\, n$, *then* $a \geq 0$.

Theorem VIII. *If* $a_n \to a$, $b_n \to b$ *and* $a_n \leq b_n \quad \forall\, n$, *then* $a \leq b$.

Theorem IX. *If* $a_n \to l$, $b_n \to l$, *and* $a_n \leq c_n \leq b_n$, $\forall\, n$, *then* $c_n \to l$. (*Squeeze Principle*)

ILLUSTRATIVE EXAMPLES

Example 1. *Give an example of a monotonic increasing sequence which is (i) convergent, (ii) divergent.* **(P.T.U., Dec. 2004)**

Sol. (*i*) Consider the sequence $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, ..., \dfrac{n}{n+1}, ...$

Since $\dfrac{1}{2} < \dfrac{2}{3} < \dfrac{3}{4} < ......$ the sequence is monotonic increasing.

$$a_n = \frac{n}{n+1}, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \frac{n}{n+1} = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{1+\frac{1}{n}} = 1$$

which is finite.

$\therefore$ The sequence is convergent.

(*ii*) Consider the sequence 1, 2, 3, ..., n, ...

Since $1 < 2 < 3 <$ ......, the sequence is monotonic increasing,

$$a_n = n, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ n = \infty$$

$\therefore$ The sequence diverges to $+\infty$.

Example 2. *Give an example of a monotonic decreasing sequence which is*

(*i*) *convergent,* (*ii*) *divergent.*

Sol. (*i*) Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...$

Since $1 > \frac{1}{2} > \frac{1}{3} >$ ......, the sequence is monotonic decreasing.

$$a_n = \frac{1}{n}, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{n} = 0.$$

$\therefore$ The sequence converges to 0.

(*ii*) Consider the sequence $-1, -2, -3, ..., -n, ...$

Since $-1 > -2 > -3 >$ ......, the sequence is monotonic decreasing.

$$a_n = -n, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ (-n) = -\infty$$

$\therefore$ The sequence diverges to $-\infty$.

Example 3. *Discuss the convergence of the sequence* $\{a_n\}$ *where*

(*i*) $a_n = \dfrac{n+1}{n}$ (*ii*) $a_n = \dfrac{n}{n^2+1}$ (*iii*) $a_n = 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + ... + \dfrac{1}{3^n}$.

Sol. (*i*)

$$a_n = \frac{n+1}{n}$$

$$a_{n+1} - a_n = \frac{n+2}{n+1} - \frac{n+1}{n} = \frac{-1}{n(n+1)} < 0\ \forall\ n$$

$\Rightarrow$ $a_{n+1} < a_n\ \forall\ n$

$\Rightarrow$ $\{a_n\}$ is a decreasing sequence.

Also,
$$a_n = \frac{n+1}{n} = 1 + \frac{1}{n} > 1\ \forall\ n$$

$\Rightarrow$ $\{a_n\}$ is bounded below by 1,

$\because$ $\{a_n\}$ is decreasing and bounded below, it is convergent.

$$\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\left(1 + \frac{1}{n}\right) = 1.$$

(*ii*)
$$a_n = \frac{n}{n^2+1}$$

$$a_{n+1} - a_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)}$$

$$= \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0 \quad \forall\, n \quad \Rightarrow \quad a_{n+1} < a_n \quad \forall \quad n$$

$\Rightarrow$ $\{a_n\}$ is a decreasing sequence.

Also, $\quad a_n = \dfrac{n}{n^2+1} > 0 \quad \forall\, n \quad \Rightarrow \quad \{a_n\}$ is bounded below by 0.

$\because$ $\{a_n\}$ is decreasing and bounded below, it is convergent.

$$\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \frac{n}{n^2+1} = \underset{n\to\infty}{\text{Lt}}\ \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0.$$

(*iii*) $\quad a_n = 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + \ldots\ldots + \dfrac{1}{3^n}$

= sum of $(n+1)$ terms of a G.P. whose first term is 1 and common ratio is $\frac{1}{3}$

$$= \frac{1\left(1 - \frac{1}{3^{n+1}}\right)}{1 - \frac{1}{3}} \qquad \left|\ S_n = \frac{a(1-r^n)}{1-r}\right.$$

$$= \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right)$$

Now, $\quad a_{n+1} = 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + \ldots\ldots + \dfrac{1}{3^n} + \dfrac{1}{3^{n+1}}$

$\therefore \quad a_{n+1} - a_n = \dfrac{1}{3^{n+1}} > 0 \quad \forall\, n \quad \Rightarrow \quad a_{n+1} > a_n \quad \forall\, n$

$\Rightarrow$ $\{a_n\}$ is an increasing sequence.

Also, $\quad a_n = \dfrac{3}{2}\left(1 - \dfrac{1}{3^{n+1}}\right) < \dfrac{3}{2} \quad \forall\, n \quad \Rightarrow \quad \{a_n\}$ is bounded above by $\dfrac{3}{2}$.

$\because$ $\{a_n\}$ is increasing and bounded above, it is convergent.

$$\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right) = \frac{3}{2}.$$

5.11. INFINITE SERIES

If $\{u_n\}$ is a sequence of real numbers, then the expression $u_1 + u_2 + u_3 + \ldots\ldots + u_n \ldots\ldots$ [*i.e.*, the sum of the terms of the sequence, which are infinite in number] is called an infinite series.

The infinite series $u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$ is usually denoted by $\sum\limits_{n=1}^{\infty} u_n$ or more briefly, by Σu_n.

5.12. SERIES OF POSITIVE TERMS

If all the terms of the series $\Sigma u_n = u_1 + u_2 + ... + u_n + ...$ are positive *i.e.*, if $u_n > 0$, $\forall\, n$, then the series Σu_n is called a series of positive terms.

5.13. ALTERNATING SERIES

A series in which the terms are alternate positive and negative is called an alternating series. Thus, the series $\Sigma (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + ... + (-1)^{n-1} u_n + ...$, where $u_n > 0$, $\forall\, n$, is an alternating series.

5.14. PARTIAL SUMS

If $\Sigma u_n = u_1 + u_2 + u_3 + ... + u_n + ...$ is an infinite series, where the terms may be +ve or –ve, then

$S_n = u_1 + u_2 + ... + u_n$ is called the nth partial sum of Σu_n. Thus, the nth partial sum of an infinite series is the sum of its first n terms.

$S_1, S_2, S_3, ...$ are the first, second, third, ... partial sums of the series.

Since $n \in N$, $\{S_n\}$ is a sequence called the **sequence of partial sums** of the infinite series Σu_n.

$\therefore$ To every infinite series Σu_n, there corresponds a sequence $\{S_n\}$ of its partial sums.

5.15. CONVERGENCE, DIVERGENCE AND OSCILLATION OF AN INFINITE SERIES (Behaviour of an Infinite Series) **(P.T.U., Dec. 2007)**

An infinite series Σu_n converges, diverges or oscillates (finitely or infinitely) according as the sequence $\{S_n\}$ of its partial sums converges, diverges or oscillates (finitely or infinitely).

(*i*) The series Σu_n converges (or is said to be convergent) if the sequence $\{S_n\}$ of its partial sums converges.

Thus, Σu_n is convergent if $\underset{n\to\infty}{\text{Lt}}\ S_n = \text{finite}.$

(*ii*) The series Σu_n diverges (or is said to be divergent) if the sequence $\{S_n\}$ of its partial sums diverges.

Thus, Σu_n is divergent if $\underset{n\to\infty}{\text{Lt}}\ S_n = +\infty \text{ or } -\infty$

(*iii*) The series Σu_n oscillates finitely if the sequence $\{S_n\}$ of its partial sums oscillates finitely.

Thus, Σu_n oscillates finitely if $\{S_n\}$ is bounded and neither converges nor diverges.

(*iv*) The series Σu_n oscillates infinitely if the sequence $\{S_n\}$ of its partial sums oscillates infinitely.

Thus, Σu_n oscillates infinitely if $\{S_n\}$ is unbounded and neither converges nor diverges.

Example 4. *Discuss the convergence or otherwise of the series*

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)} + ... \text{ to } \infty.$$

Sol. Here $$u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Putting $$n = 1, 2, 3, ..., n$$

$$u_1 = \frac{1}{1} - \frac{1}{2}$$

$$u_2 = \frac{1}{2} - \frac{1}{3}$$

$$u_3 = \frac{1}{3} - \frac{1}{4}$$

....

....

$$u_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding $$S_n = 1 - \frac{1}{n+1}$$

$$\underset{n\to\infty}{\text{Lt}} S_n = 1 - 0 = 1$$

$\Rightarrow$ $\{S_n\}$ converges to 1 $\Rightarrow$ Σu_n converges to 1.

Note. For another method, see solved example 8(*iii*) art 5.21.

Example 5. *Show that the series* $1^2 + 2^2 + 3^2 + ... + n^2 + ...$ *diverges to* $+\infty$.

Sol. $$S_n = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\underset{n\to\infty}{\text{Lt}} S_n = +\infty$$

$\Rightarrow$ $\{S_n\}$ diverges to $+\infty$

$\Rightarrow$ The given series diverges to $+\infty$.

5.16. NATURE OF GEOMETRIC SERIES $1 + x + x^2 + x^3 + ...$ to ∞

(*i*) *Converges if* $-1 < x < 1$ *i.e.,* $|x| < 1$ (*ii*) *Diverges if* $x \geq 1$

(*iii*) *Oscillates finitely if* $x = -1$

(*iv*) *Oscillates infinitely if* $x < -1$

Proof. (*i*) **When** $|x| < 1$

Since $|x| < 1$, $x^n \to 0$ as $n \to \infty$

$$S_n = 1 + x + x^2 + ... \text{ to } n \text{ terms} = \frac{1(1-x^n)}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}$$

$$\underset{n\to\infty}{\text{Lt}} S_n = \frac{1}{1-x} \quad \Rightarrow \quad \text{the sequence } \{S_n\} \text{ is convergent}$$

$\Rightarrow$ the given series is convergent.

(*ii*) **When** $x \geq 1$

Sub-case I. When $x = 1$

$$S_n = 1 + 1 + 1 + ... \text{ to } n \text{ terms} = n$$

$$\underset{n\to\infty}{\text{Lt}} S_n = \infty \quad \Rightarrow \quad \text{the sequence } \{S_n\} \text{ diverges to } \infty.$$

$\Rightarrow$ the given series diverges to ∞.

Sub-case II. When $x > 1$, $x^n \to \infty$ as $n \to \infty$

$$S_n = 1 + x + x^2 + ... \text{ to } n \text{ terms} = \frac{1(x^n - 1)}{x - 1}$$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = \infty \quad \Rightarrow \quad \text{the sequence } \{S_n\} \text{ diverges to } \infty$$

$\Rightarrow$ the given series diverges to ∞.

(*iii*) **When** $\boldsymbol{x = -1}$

$$S_n = 1 - 1 + 1 - 1 + \text{......} \text{ to } n \text{ terms}$$
$$= 1 \text{ or } 0 \text{ according as } n \text{ is odd or even.}$$

$\Rightarrow$ $\underset{n\to\infty}{\text{Lt}}\ S_n = 1 \text{ or } 0$ $\Rightarrow$ the sequence $\{S_n\}$ oscillates finitely.

$\Rightarrow$ the given series oscillates finitely.

(*iv*) **When** $\boldsymbol{x < -1}$

$$x < -1 \quad \Rightarrow \quad -x > 1$$

Let $r = -x$, then $r > 1$

$\therefore$ $r^n \to \infty$ as $n \to \infty$

$$S_n = 1 + x + x^2 + x^3 + \text{......} \text{ to } n \text{ terms} = \frac{1 - x^n}{1 - x} = \frac{1 - (-r)^n}{1 + r} \qquad [\because \quad x = -r]$$

$$= \frac{1 - r^n}{1 + r} \quad \text{or} \quad \frac{1 + r^n}{1 + r} \text{ according as } n \text{ is even or odd}$$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = \frac{1 - \infty}{1 + r} \quad \text{or} \quad \frac{1 + \infty}{1 + r} = -\infty \text{ or } +\infty$$

$\Rightarrow$ the sequence $\{S_n\}$ oscillates infinitely.

$\Rightarrow$ the given series oscillates infinitely.

5.17. NECESSARY CONDITION FOR CONVERGENCE OF A POSITIVE TERM SERIES

(P.T.U., May 2003, 2004, Dec. 2003, 2005, Jan. 2009, May 2011)

If a positive term series Σu_n is convergent, then $\underset{n\to\infty}{\text{Lt}}\ u_n = 0$

Proof. Let S_n denote the nth partial sum of the series Σu_n.
Then Σu_n is convergent $\Rightarrow$ $\{S_n\}$ is convergent

$\Rightarrow$ $\underset{n\to\infty}{\text{Lt}}\ S_n$ is finite and unique $= s$ (say). $\Rightarrow$ $\underset{n\to\infty}{\text{Lt}}\ S_{n-1} = s$

Now, $\qquad S_n - S_{n-1} = u_n$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ (S_n - S_{n-1}) = \underset{n\to\infty}{\text{Lt}}\ S_n - \underset{n\to\infty}{\text{Lt}}\ S_{n-1} = s - s = 0.$$

Hence $\boldsymbol{\Sigma u_n}$ is convergent $\Rightarrow$ $\boldsymbol{\underset{n\to\infty}{\text{Lt}}\ u_n = 0}$.

The converse of the above theorem is not always true, *i.e.*, the nth term may tend to zero as $n \to \infty$ even if the series is not convergent.

For example, the series $1 + \frac{1}{2} + \frac{1}{3} + \text{......} + \frac{1}{n} + \text{......}$ diverges, though

$$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{n} = 0.$$

Hence $\lim\limits_{n\to\infty} u_n = 0 \Rightarrow \Sigma u_n$ **may or may not be convergent.**

Note. $\lim\limits_{n\to\infty} u_n \neq 0 \Rightarrow \Sigma u_n$ **is not convergent.**

5.18. A POSITIVE TERM SERIES EITHER CONVERGES OR DIVERGES TO + ∞

Proof. Let Σu_n be a positive term series and S_n be its nth partial sum.

Then $S_{n+1} = u_1 + u_2 + \ldots\ldots + u_n + u_{n+1} = S_n + u_{n+1}$

$\Rightarrow \quad S_{n+1} - S_n = u_{n+1} > 0 \ \forall \ n \qquad [\because \ u_n > 0 \ \forall \ n]$

$\Rightarrow \quad S_{n+1} > S_n \ \forall \ n$

$\Rightarrow$ $\{S_n\}$ is a monotonic increasing sequence.

Two cases arise. The sequence $\{S_n\}$ may be bounded or unbounded above.

Case I. When $\{S_n\}$ is bounded above.

Since $\{S_n\}$ is monotonic increasing and bounded above, it is convergent $\Rightarrow \Sigma u_n$ is convergent.

Case II. When $\{S_n\}$ is not bounded above.

Since $\{S_n\}$ is monotonic increasing and not bounded above, it diverges to $+\infty \Rightarrow \Sigma u_n$ diverges to $+\infty$.

Hence a positive term series either converges or diverges to $+\infty$.

Cor. If $u_n > 0 \ \forall \ n$ **and** $\lim\limits_{n\to\infty} u_n \neq 0$**, then the series** Σu_n **diverges to** $+\infty$**.**

Proof. $u_n > 0 \ \forall \ n \Rightarrow \Sigma u_n$ is a series of +ve terms.

$\Rightarrow$ Σu_n either converges or diverges to $+\infty$

Since $\lim\limits_{n\to\infty} u_n \neq 0$ (given)

$\therefore$ Σu_n does not converge.

Hence Σu_n diverges to $+\infty$.

5.19 (a). THE NECESSARY AND SUFFICIENT CONDITION FOR THE CONVERGENCE OF A POSITIVE TERM SERIES Σu_n IS THAT THE SEQUENCE $\{S_n\}$ OF ITS PARTIAL SUMS IS BOUNDED ABOVE

Proof. Necessary Condition. Suppose the sequence $\{S_n\}$ is bounded above. Since the series Σu_n is of positive terms, the sequence $\{S_n\}$ is monotonically increasing. Since every monotonically increasing sequence which is bounded above, converges, therefore $\{S_n\}$ and hence Σu_n converges.

Sufficient Condition. Suppose Σu_n converges. Then the sequence $\{S_n\}$ of its partial sums also converges. Since every convergent sequence is bounded, $\{S_n\}$ is bounded. In particular, $\{S_n\}$ is bounded above.

5.19 (b). CAUCHY'S GENERAL PRINCIPLE OF CONVERGENCE OF SERIES

The necessary and sufficient condition for the infinite series $\sum\limits_{n=1}^{\infty} u_n$ to converge is that given $\varepsilon > 0$, however small, there exists a positive integer p such that $| S_{n+p} - S_n | < \varepsilon \ \forall \ n \geq m$; m and $p \in N$ *i.e.*, $| u_{n+1} + u_{n+2} \ldots u_{n+p} | < \varepsilon$ for $n \geq m, p, m \in N$.

Necessary Condition. Given $\sum_{n=1}^{\infty} u_n$ is convergent.

$\therefore$ $\underset{n \to \infty}{\text{Lt}} \; S_n$ = finite, where $\{S_n\}$ is the sequence of its partial sums

Let $\underset{n \to \infty}{\text{Lt}} \; S_n = l$, where l is a finite number.

$\therefore$ Given $\varepsilon > 0$, however small, $\exists s \; m \in N$ such that $| S_n - l | < \varepsilon/2 \; \forall \; n \geq m$ ...(1)

If $p \in N$ and $n \geq m$, then $n + p \geq m$

$\therefore$ From (1), $| S_{n+p} - l | < \dfrac{\varepsilon}{2} \quad \forall \; n \geq m$ ...(2)

Now,
$$\begin{aligned} | S_{n+p} - S_n | &= | (S_{n+p} - l) - (S_n - l) | \\ &\leq | S_{n+p} - l | + | S_n - l | \\ &< \varepsilon/2 + \varepsilon/2 \quad \text{for} \quad n \geq m, p \in N \end{aligned}$$

Hence $| S_{n+p} - S_n | < \varepsilon$ for $n \geq m, p \in N$.

Sufficient Condition. Given $| S_{n+p} - S_n | < \varepsilon \; \forall \; n \geq m, p \in N$

In particular $| S_{m+p} - S_m | < \varepsilon \quad \forall \; p \in N$

Now S_m, being the sum of first m terms of the sequence $\{S_n\}$ and S_{m+p} differs from S_m by a number $< \varepsilon \; \forall \; p \in N$.

$\therefore$ S_{m+p} cannot be infinite when $p \to \infty$ *i.e.*, $\underset{p \to \infty}{\text{Lt}} \; S_{m+p} \neq \infty$.

$\therefore$ $\underset{n \to \infty}{\text{Lt}} \; S_n \neq \infty$ (replace $m + p$ by n)

Also $\underset{n \to \infty}{\text{Lt}} \; S_n$ and $\underset{n \to \infty}{\text{Lt}} \; S_{n+p}$ have the same value S

Now, $| S_{n+p} - S_n | < \varepsilon \; \forall \; p \in N$

$$\Rightarrow \quad \underset{n \to \infty}{\text{Lt}} \; S_{n+p} = \underset{n \to \infty}{\text{Lt}} \; S_n = l \text{ (say)} \quad \forall \; p \in N$$

$\therefore$ $\sum_{n=1}^{\infty} u_n$ is convergent.

5.19 (c). IF m IS A GIVEN POSITIVE INTEGER, THEN THE TWO SERIES $u_1 + u_2 + \ldots\ldots + u_{m+1} + u_{m+2} + \ldots\ldots u_n$ AND $u_{m+1} + u_{m+2} + \ldots\ldots u_n$ CONVERGE OR DIVERGE TOGETHER

Proof. Let S_n and s_n denote the nth partial sums of the two series.

Then
$$\begin{aligned} S_n &= u_1 + u_2 + \ldots\ldots + u_n \\ s_n &= u_{m+1} + u_{m+2} + \ldots\ldots + u_n \\ &= (u_1 + u_2 + \ldots\ldots + u_n) - (u_1 + u_2 + \ldots\ldots + u_m) \\ &= S_n - S_m \quad \Rightarrow \quad s_n = S_n - S_m \end{aligned}$$ ...(1)

S_m being the sum of a finite number of terms of Σu_n is a fixed finite quantity.

(*i*) If $S_n \to$ a finite limit as $n \to \infty$, then from (1), so does s_n.

(*ii*) If $S_n \to +\infty$ as $n \to \infty$, so does s_n.

(*iii*) If $S_n \to -\infty$ as $n \to \infty$, so does s_n.

(*iv*) If S_n does not tend to any limit (finite or infinite), so does s_n.

$\Rightarrow$ The sequences $\{S_n\}$ and $\{s_n\}$ converge or diverge together.

$\Rightarrow$ The two given series converge or diverge together. Hence the result.

Note. The above theorem shows that the convergence, divergence or oscillation of a series is not affected by addition or omission of a finite number of its terms.

Example 6. *Prove that the series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *does not converge (by applying Cauchy's general principle of convergence).*

Sol. If possible suppose $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent.

$\therefore$ By Cauchy's general principle of convergence

$$| S_{n+p} - S_n | < \varepsilon \quad \forall\, n \geq m, p \in N$$

Take $\varepsilon = \frac{1}{2} \quad \therefore \quad | S_{n+p} - S_n | < \frac{1}{2} \quad \forall\, n \geq m, p \in N$

Put $n = m$; $\quad | S_{m+p} - S_m | < \frac{1}{2} \quad \forall \quad p \in N$

i.e., $\quad \left| \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{m+p} \right| < \frac{1}{2} \quad \forall \quad p \in N \qquad \left(\because \; S_n = \frac{1}{n}\right)$

or $\quad \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{m+p} < \frac{1}{2} \quad \forall \quad p \in N$

Put $p = m$, $\quad \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m} < \frac{1}{2} \qquad \ldots(1)$

But $\quad \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m} > \frac{1}{m+m} + \frac{1}{m+m} + \ldots + \frac{1}{2m} = \frac{m}{2m} = \frac{1}{2}$

$\therefore \quad \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m} > \frac{1}{2} \qquad \ldots(2)$

(2) contradicts (1)

$\therefore$ Our supposition is wrong

$\therefore$ Given series does not converge.

Example 7. *Prove that the series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is convergent (by applying Cauchy's general principle of convergence).*

Sol. Let $\quad S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}$

$$S_{n+p} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots \frac{1}{(n+p)^2}$$

$$|S_{n+p} - S_n| = \left| \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(n+p)^2} \right|$$

$$= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(n+p)^2}$$

$$< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots + \frac{1}{(n+p-1)(n+p)}$$

$$< \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \ldots + \left(\frac{1}{n+p-1} - \frac{1}{n+p}\right)$$

$$= \frac{1}{n} - \frac{1}{n+p}$$

$$< \frac{1}{n}$$

$\therefore \qquad |S_{n+p} - S_n| < \dfrac{1}{n}$

Let us choose m such that $m > \dfrac{1}{\varepsilon}$

$\therefore \qquad$ for $n \geq m > \dfrac{1}{\varepsilon}$, we have $\dfrac{1}{n} < \varepsilon$

$\therefore \qquad |S_{n+p} - S_n| < \varepsilon \quad$ for $\quad n \geq m, p \in N$

$\therefore \qquad$ by Cauchy's general principle of convergence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Note. These examples can be proved by applying p series test art. 5.21.

5.19 (d). IF Σu_n AND Σv_n CONVERGE TO u AND v RESPECTIVELY, THEN $\Sigma(u_n + v_n)$ CONVERGES TO $(u + v)$

Proof. Let $\quad U_n = u_1 + u_2 + \ldots\ldots + u_n$

$\qquad V_n = v_1 + v_2 + \ldots\ldots + v_n$

and $\qquad S_n = (u_1 + v_1) + (u_2 + v_2) + \ldots\ldots + (u_n + v_n)$

Then $\qquad S_n = (u_1 + u_2 + \ldots\ldots + u_n) + (v_1 + v_2 + \ldots\ldots + v_n) = U_n + V_n$.

Since Σu_n converges to u, $\displaystyle\operatorname*{Lt}_{n\to\infty} U_n = u$

Σv_n converges to v, $\displaystyle\operatorname*{Lt}_{n\to\infty} V_n = v$

$\therefore \qquad \displaystyle\operatorname*{Lt}_{n\to\infty} S_n = \operatorname*{Lt}_{n\to\infty} (U_n + V_n) = \operatorname*{Lt}_{n\to\infty} U_n + \operatorname*{Lt}_{n\to\infty} V_n = u + v.$

$\Rightarrow \quad \Sigma(u_n + v_n)$ converges to $(u + v)$.

5.20. COMPARISON TESTS

Test I. (a) If Σu_n and Σv_n are series of positive terms and Σv_n is convergent and there is a positive constant k such that $u_n \le kv_n$, $\forall\ n > m$, then Σu_n is also convergent.

Proof. Let $\quad U_n = u_1 + u_2 + \ldots\ldots + u_n \quad$ and $\quad V_n = v_1 + v_2 + \ldots\ldots + v_n$

Now $\qquad u_n \le kv_n\ \forall\ n > m$

$\Rightarrow \qquad u_{m+1} \le kv_{m+1}$

$\qquad\qquad u_{m+2} \le kv_{m+2}$

$\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Adding $\quad u_{m+1} + u_{m+2} + \ldots\ldots + u_n \le k(v_{m+1} + v_{m+2} + \ldots\ldots + v_n)$

$\Rightarrow \qquad U_n - U_m \le k\,(V_n - V_m) \quad \forall \quad n > m$

$\Rightarrow \qquad U_n \le kV_n + (U_m - kV_m) \quad \forall \quad n > m$

$\Rightarrow \qquad U_n \le kV_n + k_0 \quad \forall \quad n > m \qquad \ldots(1)$

where $k_0 = U_m - kV_m$ is a fixed number.

Since Σv_n is convergent, the sequence $\{V_n\}$ is convergent and hence bounded above.

$\therefore$ From (1), the sequence $\{U_n\}$ is bounded above.

$\because$ Σu_n is a series of +ve terms.

$\therefore$ $\{U_n\}$ is monotonic increasing.

$\because$ $\{U_n\}$ is monotonic increasing sequence and is bounded above.

$\therefore$ It is convergent.

$\Rightarrow$ Σu_n is convergent.

Test I. (b) If Σu_n and Σv_n are two series of positive terms and Σv_n is divergent and there is a positive constant k such that $u_n > kv_n$, $\forall\ n > m$, then Σu_n is also divergent.

Proof. Let $\quad U_n = u_1 + u_2 + \ldots\ldots + u_n$

and $\qquad V_n = v_1 + v_2 + \ldots\ldots + v_n$

Now, $\qquad u_n > kv_n \quad \forall \quad n > m$

$\Rightarrow \qquad u_{m+1} > kv_{m+1}$

$\qquad\qquad u_{m+2} > kv_{m+2}$

$\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots$

$\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots$

Adding $u_{m+1} + u_{m+2} + \ldots\ldots\ldots + u_n > k\,(v_{m+1} + v_{m+2} + \ldots\ldots + v_n)$

$\Rightarrow \qquad U_n - U_m > k\,(V_n - V_m) \qquad \forall \quad n > m$

$\Rightarrow \qquad U_n > kV_n + (U_m - kV_m) \qquad \forall \quad n > m$

$\Rightarrow \qquad U_n > kV_n + k_0 \qquad \forall \quad n > m \qquad \ldots(1)$

where $k_0 = U_m - kV_m$ is a fixed number.

Since Σv_n is divergent, the sequence $\{V_n\}$ is divergent.

$\Rightarrow$ For each positive real number k_1, however large, there exists a +ve integer m' such that

$$V_n > k_1 \qquad \forall \quad n > m'$$

Let $\quad m^* = \max.\ \{m, m'\}$, then $\quad V_n > k_1 \qquad \forall \quad n > m^*$

From (1), $\qquad U_n > kk_1 + k_0 = K \qquad \forall \quad n > m^*$

$\Rightarrow \qquad \{U_n\}$ is divergent

$\Rightarrow \qquad \Sigma u_n$ is divergent.

Test II. If Σu_n and Σv_n are two positive term series and there exist two positive constants H and K (independent of n) and a positive integer m such that $H < \dfrac{u_n}{v_n} < K \ \forall \ n > m$, then the two series Σu_n and Σv_n converge or diverge together.

Proof. Since Σv_n is a series of +ve terms, $v_n > 0, \forall \ n$

$$\therefore \qquad H < \frac{u_n}{v_n} < K \qquad \forall \ n > m$$

$$\Rightarrow \qquad Hv_n < u_n < Kv_n \qquad \forall \ n > m \qquad \ldots(1)$$

Case I. When Σv_n is convergent

From (1), $\quad u_n < Kv_n \quad \forall \ n > m$ and Σv_n is convergent.

$\Rightarrow \quad \Sigma u_n$ is convergent. [See test I(a)]

Case II. When Σv_n is divergent

From (1), $\quad u_n > Hv_n \quad \forall \ n > m$ and Σv_n is divergent.

$\Rightarrow \quad \Sigma u_n$ is divergent. [See test I(b)]

Case III. When Σu_n is convergent

From (1), $\quad Hv_n < u_n \quad \forall \ n > m$

$$\Rightarrow \qquad v_n < \frac{1}{H} u_n \qquad \forall \ n > m \qquad (\because \ H > 0)$$

Since Σu_n is convergent $\quad \therefore \quad \Sigma v_n$ is convergent. [See test I(a)]

Case IV. When Σu_n is divergent

From (1), $\quad Kv_n > u_n \quad \forall \ n > m$

$$\Rightarrow \qquad v_n > \frac{1}{K} u_n. \qquad \forall \ n > m \qquad (\because \ K > 0)$$

Since Σu_n is divergent $\quad \therefore \quad \Sigma v_n$ is divergent. [See test I(b)]

Particular Case of Test II (When $m = 0$)

If Σu_n and Σv_n are two positive term series and there exist two positive constants H and K (independent of n) such that $\quad H < \dfrac{u_n}{v_n} < K \qquad \forall \ n,$

then the two series Σu_n and Σv_n converge or diverge together.

Test III. (limit comparison test) Let Σu_n and Σv_n be two positive term series.

(*i*) **If $\displaystyle \mathop{\text{Lt}}_{n \to \infty} \frac{u_n}{v_n} = l$ (finite and non-zero), then Σu_n and Σv_n both converge or diverge together.**

(*ii*) **If $\displaystyle \mathop{\text{Lt}}_{n \to \infty} \frac{u_n}{v_n} = 0$ and Σv_n converges, then Σu_n also converges.**

(*iii*) **If $\displaystyle \mathop{\text{Lt}}_{n \to \infty} \frac{u_n}{v_n} = \infty$ and Σv_n diverges, then Σu_n also diverges.**

(*iv*) **If $\displaystyle \mathop{\text{Lt}}_{n \to \infty} \frac{u_n}{v_n} = \infty$ and Σu_n converges, then Σv_n also converges.** **(P.T.U., Dec. 2004)**

Proof. (*i*) Since $u_n > 0, v_n > 0 \quad \therefore \quad \dfrac{u_n}{v_n} > 0$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} \geq 0$$

$$\text{But} \qquad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = l \neq 0 \quad \Rightarrow \quad l > 0$$

$$\text{Now,} \qquad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = l$$

$\Rightarrow$ Given $\varepsilon > 0$, there exists a +ve integer m such that $\left|\dfrac{u_n}{v_n} - l\right| < \varepsilon \ \forall \ n > m$

$$\Rightarrow \qquad l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \qquad \forall \, n > m$$

$$\Rightarrow \qquad (l - \varepsilon)\, v_n < u_n < (l + \varepsilon)\, v_n \qquad \forall \, n > m \qquad (\because \quad v_n > 0)$$

Choose $\varepsilon > 0$ such that $l - \varepsilon > 0$.

Let $\quad l - \varepsilon = \text{H}, l + \varepsilon = \text{K}$, where H, K are > 0

$$\therefore \qquad \text{H}v_n < u_n < \text{K}v_n \qquad \forall \, n > m \qquad \ldots(1)$$

Case I. When Σu_n is convergent

From (1), $\qquad \text{H}v_n < u_n \qquad \forall \, n > m$

$$\Rightarrow \qquad v_n < \frac{1}{\text{H}} u_n \qquad \forall \, n > m \qquad (\because \quad \text{H} > 0)$$

Since Σu_n is convergent, Σv_n is also convergent.

Case II. When Σu_n is divergent.

From (1), $\qquad \text{K}v_n > u_n \qquad \forall \, n > m$

$$\Rightarrow \qquad v_n > \frac{1}{\text{K}} u_n \qquad \forall \, n > m \qquad (\because \quad \text{K} > 0)$$

Since Σu_n is divergent, Σv_n is also divergent.

Case III. When Σv_n is convergent.

From (1), $\qquad u_n < \text{K}v_n \qquad \forall \, n > m$

Since Σv_n is convergent, Σu_n is also convergent.

Case IV. When Σv_n is divergent.

From (1), $\qquad u_n > \text{H}v_n \qquad \forall \, n > m$

Since Σv_n is divergent, Σu_n is also divergent.

Hence Σu_n and Σv_n converge or diverge together.

(*ii*) Here $\quad \underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{v_n} = 0$

$\therefore$ Given $\varepsilon > 0$, there exists a +ve integer m such that $\left|\dfrac{u_n}{v_n} - 0\right| < \varepsilon \quad \forall \, n > m$

$$\Rightarrow \qquad -\varepsilon < \frac{u_n}{v_n} < \varepsilon \qquad \forall\ n > m$$

$$\Rightarrow \qquad u_n < \varepsilon\, v_n \qquad \forall\ n > m \qquad (\because \quad v_n > 0)$$

Since Σv_n is convergent, Σu_n is also convergent.

(*iii*) Here $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{v_n} = \infty$

$\therefore$ Given M > 0, however large, $\exists$s a +ve integer m such that $\dfrac{u_n}{v_n} > \text{M} \ \forall\ n > m$

$$\Rightarrow \qquad u_n > \text{M}v_n \qquad \forall\ n > m$$

Since Σv_n is divergent, Σu_n is also divergent.

(*iv*) Given $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{v_n} = \infty$

$\therefore$ Given M > 0, however large $\exists$s a positive integer m such that $\dfrac{u_n}{v_n} > \text{M}$ for $n > m$

$$\therefore \qquad u_n > \text{M}\ v_n$$

or $$\text{M}v_n < u_n \quad \text{or} \quad v_n < \frac{1}{\text{M}} u_n$$

As M is a large $\therefore$ $\dfrac{1}{\text{M}}$ is small.

Given $\displaystyle\sum_{n=1}^{\infty} u_n$ is convergent

$\therefore$ By comparison test $\Sigma\ v_n$ is also convergent.

Test IV. Let Σu_n and Σv_n be two positive term series.

(*i*) **If $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$ $\forall\ n > m$ and Σv_n is convergent, then Σu_n is also convergent.**

(*ii*) **If $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ $\forall\ n > m$ and Σv_n is divergent, then Σu_n is also divergent.**

Proof. (*i*) $$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \qquad \forall\ n > m$$

$$\Rightarrow \qquad \frac{u_{m+1}}{u_{m+2}} > \frac{v_{m+1}}{v_{m+2}}$$

$$\frac{u_{m+2}}{u_{m+3}} > \frac{v_{m+2}}{v_{m+3}}$$

$$\frac{u_{m+3}}{u_{m+4}} > \frac{v_{m+3}}{v_{m+4}}$$

....

....

$$\frac{u_{n-1}}{u_n} > \frac{v_{n-1}}{v_n}$$

Multiplying the corresponding sides of the above inequalities, we have

$$\frac{u_{m+1}}{u_n} > \frac{v_{m+1}}{v_n} \qquad \forall\ n > m$$

$$\Rightarrow \qquad u_n < \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n \qquad \forall\ n > m$$

$$\Rightarrow \qquad u_n < kv_n \qquad \forall\ n > m,$$

where $k = \dfrac{u_{m+1}}{v_{m+1}}$ is a fixed +ve quantity.

Since Σv_n is convergent, so is Σu_n.

(*ii*) Using $\quad \dfrac{u_{m+1}}{u_n} < \dfrac{v_{m+1}}{v_n} \qquad \forall\ n > m$

and proceeding as in part (*i*), we have $\dfrac{u_{m+1}}{u_n} < \dfrac{v_{m+1}}{v_n} \qquad \forall\ n > m$

$$\Rightarrow \qquad u_n > \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n \qquad \forall\ n > m$$

$$\Rightarrow \qquad u_n > kv_n \qquad \forall\ n > m,$$

where $k = \dfrac{u_{m+1}}{v_{m+1}}$ is a fixed +ve quantity.

Since Σv_n is divergent, so is Σu_n.

5.21. AN IMPORTANT TEST FOR COMPARISON KNOWN AS *p*-SERIES TEST FOR THE SERIES $\sum \frac{1}{n^p}$. [HYPER HARMONIC SERIES OR *p*-SERIES]

The series $\sum \dfrac{1}{n^p} = \dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \ldots\ldots + \dfrac{1}{n^p} + \ldots\ldots$ to ∞ **converges if** $p > 1$ **and diverges if** $p \le 1$.

Proof. Case I. *When p > 1*

$$\frac{1}{1^p} = 1$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}} \qquad \left[\because \frac{1}{3^p} < \frac{1}{2^p}\right]$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$$

$$= \frac{4}{4^p} = \frac{1}{4^{p-1}} = \frac{1}{(2^{p-1})^2} \qquad \left[\because \frac{1}{5^p} < \frac{1}{4^p}, \frac{1}{6^p} < \frac{1}{4^p} \text{ etc.}\right]$$

Similarly, the sum of next eight terms

$$= \frac{1}{8^p} + \frac{1}{9^p} + \ldots\ldots + \frac{1}{15^p} < \frac{1}{8^p} + \frac{1}{8^p} + \ldots\ldots + \frac{1}{8^p} = \frac{8}{8^p} = \frac{1}{8^{p-1}} = \frac{1}{(2^{p-1})^3}$$

and so on.

Now, $$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots$$

$$= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \frac{1}{9^p} + \ldots\ldots + \frac{1}{15^p}\right) + \ldots\ldots \quad \ldots(1)$$

$$< 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \frac{1}{(2^{p-1})^3} + \ldots\ldots \quad \ldots(2)$$

But (2) is a G.P. whose common ratio $= \frac{1}{2^{p-1}} < 1$ $\quad (\because \quad p > 1)$

$\therefore$ (2) is convergent $\Rightarrow$ (1) is convergent.

Hence the given series is convergent.

Case II. *When p = 1*

$$\sum \frac{1}{n^p} = \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$$

$$1 + \frac{1}{2} = 1 + \frac{1}{2}$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2} \quad \text{and so on.}$$

Now, $$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{n} + \ldots\ldots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots\ldots \quad \ldots(1)$$

$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots = 1 + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots \infty\right) \quad \ldots(2)$$

But $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots \infty$ is a G.P. whose common ratio = 1.

$\therefore$ (2) is divergent. $\Rightarrow$ (1) is divergent.

Hence the given series is divergent.

Case III. *When p < 1*

$$p < 1 \quad \Rightarrow \quad n^p < n \quad \Rightarrow \quad \frac{1}{n^p} > \frac{1}{n} \quad \forall \quad n$$

But the series $\sum \frac{1}{n}$ is divergent (Case II).

Hence $\sum \frac{1}{n^p}$ is also divergent.

Example 8. *Examine the convergence of the series:*

(*i*) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+ \text{......} \textit{ to } \infty$ (*ii*) $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+ \text{......} \textit{ to } \infty$

(*iii*) $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\infty$

Sol. (*i*) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+ \text{......} \textit{ to } \infty$

$$=\left(\frac{3}{5}+\frac{3}{5^3}+\text{...... to }\infty\right)+\left(\frac{4}{5^2}+\frac{4}{5^4}+\text{...... to }\infty\right)=\Sigma u_n+\Sigma v_n \text{ (say)}$$

Now, Σu_n is a G.P. with common ratio $=\frac{1}{5^2}$, which is numerically less than 1,

$\therefore$ Σu_n is convergent.

Σv_n is also a G.P. with common ratio $=\frac{1}{5^2}$, which is numerically less than 1.

$\therefore$ Σv_n is convergent.

$\therefore$ The given series *viz*. $\Sigma(u_n+v_n)$ is also convergent.

(*ii*) $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\text{...... to }\infty=1+\frac{1}{(2^2)^{2/3}}+\frac{1}{(3^2)^{2/3}}+\frac{1}{(4^2)^{2/3}}+\text{...... to }\infty$

$$=\frac{1}{1^{4/3}}+\frac{1}{2^{4/3}}+\frac{1}{3^{4/3}}+\frac{1}{4^{4/3}}+\text{...... to }\infty=\sum\frac{1}{n^{4/3}}=\sum\frac{1}{n^p} \text{ with } p=\frac{4}{3}>1$$

$\therefore$ By p-series test, the given series is convergent.

(*iii*) $\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ $\quad\therefore\quad u_n=\frac{1}{n(n+1)}=\frac{1}{n^2\left(1+\frac{1}{n}\right)}$

Let $v_n=\frac{1}{n^2}$

Compare Σu_n with Σv_n, we have $\frac{u_n}{v_n}=\frac{1}{1+\frac{1}{n}}$.

$$\operatorname*{Lt}_{n\to\infty}\frac{u_n}{v_n}=\operatorname*{Lt}_{n\to\infty}\frac{1}{1+\frac{1}{n}}=1, \text{ which is finite and} \neq 0.$$

$\therefore$ Σu_n and Σv_n behave together $\Sigma v_n=\Sigma\frac{1}{n^2}$ is of the form $\Sigma\frac{1}{n^p}$, where $p=2>1$

$\therefore$ by p-series test $\Sigma\frac{1}{n^2}$ converges.

By limit comparison test (art. 5.20 Test III)

Σu_n also converges *i.e.,* $\Sigma\frac{1}{n(n+1)}$ converges.

Hence $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\infty$ converges.

Example 9. *Test the convergence of the series* : $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+......$ **(P.T.U., May 2009)**

Sol. Here
$$u_n=\frac{T_n \text{ of } 1, 3, 5, ...}{n(n+1)(n+2)}=\frac{2n-1}{n(n+1)(n+2)}$$

(As 1, 3, 5, ... form an A.P. with $a = 1$, $d = 2$
$\therefore$ nth term $T_n = 1 + (n-1)^2 = 2n-1$)

$$=\frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=\frac{2-\frac{1}{n}}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let
$$v_n=\frac{1}{n^2}$$

Let us compare Σu_n with Σv_n,
$$\frac{u_n}{v_n}=\frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=\underset{n\to\infty}{\text{Lt}}\frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=\frac{2}{(1)(1)}=2, \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma\,\frac{1}{n^2}$ is of the form $\Sigma\,\frac{1}{n^p}$ with $p = 2 > 1$.

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

Example 10. *Test the convergence of the following series:*

(*i*) $\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\cdots\cdots\infty$ (*ii*) $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\cdots\cdots\infty$.

Sol. (*i*) Here
$$u_n=\frac{1}{(2n-1)^p}=\frac{1}{n^p\left(2-\frac{1}{n}\right)^p}$$

($\because$ 1, 3, 5, … are in AP and nth term $= 1 + (n-1)\,2 = 2n-1$)

Let
$$v_n=\frac{1}{n^p} \quad \therefore \quad \frac{u_n}{v_n}=\frac{1}{\left(2-\frac{1}{n}\right)^p}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=\frac{1}{2^p}, \text{ which is finite and } \neq 0.$$

$\therefore$ $\Sigma\, u_n$ and $\Sigma\, v_n$ behave alike

$$\Sigma\, v_n = \Sigma\, \frac{1}{n^p}\text{, which converges if } p > 1 \text{ and diverges if } p \le 1.$$

$\therefore$ Given series converges for $p > 1$ and diverges for $p \le 1$

(*ii*) Here $$u_n = \frac{n+1}{n^p} = \frac{1}{n^{p-1}}\left(1+\frac{1}{n}\right)$$

Let $$v_n = \frac{1}{n^{p-1}}\;;\; \frac{u_n}{v_n} = 1+\frac{1}{n}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{v_n} = 1\text{, which is finite and} \ne 0$$

$\therefore$ $\Sigma\, u_n$ and $\Sigma\, v_n$ behave alike

$$\Sigma\, v_n = \frac{1}{n^{p-1}}$$

converges if $p - 1 > 1$ *i.e.,* $p > 2$ and diverges if $p - 1 \le 1$ *i.e.,* $p \le 2$

$\therefore$ Given series converges for $p > 2$ and diverges for $p \le 2$.

Example 11. *Test the convergence of the following series :*

(*i*) $\dfrac{1}{\sqrt{1}+\sqrt{2}} + \dfrac{1}{\sqrt{2}+\sqrt{3}} + \dfrac{1}{\sqrt{3}+\sqrt{4}} + \ldots\ldots$ (*ii*) $\sqrt{\dfrac{1}{4}} + \sqrt{\dfrac{2}{6}} + \sqrt{\dfrac{3}{8}} + \ldots\ldots + \sqrt{\dfrac{n}{2(n+1)}} + \ldots\ldots$

(P.T.U., Dec. 2003)

Sol. (*i*) Here $$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}\left[1+\sqrt{1+\dfrac{1}{n}}\right]}$$

Let us compare Σu_n with Σv_n, where $v_n = \dfrac{1}{\sqrt{n}}$

$$\frac{u_n}{v_n} = \frac{1}{1+\sqrt{1+\dfrac{1}{n}}}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{1+\sqrt{1+\dfrac{1}{n}}} = \frac{1}{1+1} = \frac{1}{2}\text{, which is finite and} \ne 0.$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma\, \dfrac{1}{n^{1/2}}$ is of the form $\Sigma\, \dfrac{1}{n^p}$ with $p = \dfrac{1}{2} < 1$

$\therefore$ Σv_n is divergent $\Rightarrow$ Σu_n is divergent.

(*ii*) Here $$u_n = \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{1}{2\left(1+\dfrac{1}{n}\right)}}$$

$$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{\sqrt{2\left(1+\frac{1}{n}\right)}} = \frac{1}{\sqrt{2}} \neq 0$$

$\Rightarrow\quad \Sigma u_n$ does not converge.

Since the given series is a series of +ve terms, it either converges or diverges. Since it does not converge, it must diverge.

Hence the given series is divergent.

Example 12. *Test the convergence of* $\frac{1}{5}+\frac{\sqrt{2}}{7}+\frac{\sqrt{3}}{9}+\frac{\sqrt{4}}{11}+\ldots\infty$

Sol. Here
$$u_n = \frac{\sqrt{n}}{2n+3}$$

($\because$ 5, 7, 9, 11, … are in A.P. and nth term of A.P. $= 5 + (n-1)\,2 = 2n + 3$)

$$= \frac{\sqrt{n}}{n\left(2+\frac{3}{n}\right)} = \frac{1}{\sqrt{n}\left(2+\frac{3}{n}\right)}$$

Let
$$v_n = \frac{1}{\sqrt{n}} \qquad \therefore \qquad \frac{u_n}{v_n} = \frac{1}{2+\frac{3}{n}}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{v_n} = \frac{1}{2}\text{, which is finite and} \neq 0$$

$\therefore\qquad \Sigma\, u_n$ and $\Sigma\, v_n$ behave alike

$$\Sigma\, v_n = \frac{1}{n^{1/2}}\text{, which is } p \text{ series, where } p = \frac{1}{2} < 1$$

$\therefore\qquad \Sigma\, v_n$ diverges

Hence $\Sigma\, u_n = \frac{1}{5}+\frac{\sqrt{2}}{7}+\frac{\sqrt{3}}{9}+\ldots\infty$ also diverges.

Example 13. *Test the convergence of the following series :*

(*i*) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots\ldots$ (*ii*) $\Sigma\,\frac{n^2+1}{n^3+1}$. **(P.T.U., May 2006)**

Sol. (*i*) Leaving aside the first term ($\because$ *Addition or deletion of a finite number of terms does not alter the nature of the series*), we have

$$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{n\left(1+\frac{1}{n}\right)^{n+1}}$$

Take
$$v_n = \frac{1}{n}\ .$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\left(1+\frac{1}{n}\right)^{n}} \times \frac{1}{\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{e} \cdot \frac{1}{1} \qquad \left[\because \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e\right]$$

$$= \frac{1}{e}, \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 1$

$\therefore$ Σv_n is divergent. $\Rightarrow$ Σu_n is divergent.

(*ii*)
$$u_n = \frac{n^2+1}{n^3+1} = \frac{n^2\left(1+\frac{1}{n^2}\right)}{n^3\left(1+\frac{1}{n^3}\right)} = \frac{1}{n} \cdot \frac{1+\frac{1}{n^2}}{1+\frac{1}{n^3}}$$

Let
$$v_n = \frac{1}{n}$$

$\therefore$
$$\frac{u_n}{v_n} = \frac{1+\frac{1}{n^2}}{1+\frac{1}{n^3}}$$

When $n \to \infty$
$$\frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \frac{1+0}{1+0} = 1, \text{ which is finite and non-zero}$$

$\therefore$ Σu_n and Σv_n both converge or diverge together.

Since, $\Sigma v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = 1$.

$\therefore$ Σv_n is divergent

$\Rightarrow$ Σu_n is also divergent.

Hence $\sum \frac{n^2+1}{n^3+1}$ is divergent series.

Example 14. *Discuss the convergence or divergence of the following series:*

(*i*) $\sum \frac{1}{\sqrt{n}} \sin \frac{1}{n}$ (*ii*) $\Sigma \cot^{-1} n^2$.

Sol. (*i*) Here $u_n = \frac{1}{\sqrt{n}} \sin \frac{1}{n} = \frac{1}{\sqrt{n}} \cdot \frac{\sin \frac{1}{n}}{\frac{1}{n}} \times \frac{1}{n}$

$\therefore$ $$u_n = \frac{1}{n^{3/2}}\left(\frac{\sin 1/n}{\frac{1}{n}}\right)$$

Let $$v_n = \frac{1}{n^{3/2}} \quad \therefore \quad \frac{u_n}{v_n} = \frac{\sin\frac{1}{n}}{1/n}.$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \underset{\frac{1}{n}\to 0}{\text{Lt}} \frac{\sin\frac{1}{n}}{\sin\frac{1}{n}} = 1 \qquad \left|\ \underset{\theta\to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1\right.$$

which is finite and non-zero

$\therefore$ Σu_n and Σv_n behave alike

$$\Sigma v_n = \sum \frac{1}{n^{3/2}} \text{ is } p\text{-series, where } p = 3/2 > 1$$

$\therefore$ Σv_n converges

and hence $\Sigma u_n = \sum \frac{1}{\sqrt{n}} \sin\frac{1}{n}$ converges.

(*ii*) Here $$u_n = \cot^{-1} n^2 = \tan^{-1}\frac{1}{n^2} = \frac{1}{n^2}.\frac{\tan^{-1} 1/n^2}{1/n^2}.$$

Take $$v_n = \frac{1}{n^2} \quad \therefore \quad \frac{u_n}{v_n} = \frac{\tan^{-1}\frac{1}{n^2}}{1/n^2}.$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{\tan^{-1}\frac{1}{n^2}}{1/n^2} = \underset{h\to 0}{\text{Lt}} \frac{\tan^{-1} h}{h}\left(\text{where } h = \frac{1}{n^2}\right)$$

$= 1 \neq 0$, which is finite and non-zero

$\therefore$ Σu_n and Σv_n behave alike

$$\Sigma v_n = \sum \frac{1}{n^2} \text{ is } p\text{-series, where } p = 2$$

$\therefore$ Σv_n is convergent

So $\Sigma u_n = \Sigma \cot^{-1} n^2$ is also convergent.

Example 15. *Examine the convergence of the series:* $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \ldots\ldots$

Sol. Here $$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n}\left(\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}\right)}{n^3\left[\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}\right]} = \frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{n^{5/2}\left[\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}\right]}$$

Take $$v_n = \frac{1}{n^{5/2}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}} = \frac{\sqrt{1+0}-0}{(1+0)^3 - 0} = 1, \text{ which is finite and} \neq 0.$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma u_n = \Sigma \dfrac{1}{n^{5/2}}$ is of the form $\Sigma \dfrac{1}{n^p}$ with $p = \dfrac{5}{2} > 1$.

Σv_n is convergent. $\Rightarrow$ Σu_n is convergent.

Example 16. *Examine the convergence of the series:*

(*i*) $\sum \sqrt{n^4+1} - \sqrt{n^4-1}$ **(P.T.U., Dec. 2012, May 2012)**

(*ii*) $\sum \left(\sqrt[3]{n^3+1} - n\right)$. **(P.T.U., May 2007, Jan. 2010, May 2012)**

Sol. (*i*) Here $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

Rationalize $u_n = \dfrac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}}$

$$= \frac{2}{n^2\left[\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}\right]}$$

Let $$v_n = \frac{1}{n^2}$$

$\therefore$ $$\frac{u_n}{v_n} = \frac{2}{\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{2}{\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}}$$

$$= \frac{2}{2} = 1 \text{, which is finite and non-zero}$$

$\therefore$ u_n and v_n behave alike

Now, $\Sigma v_n = \sum \dfrac{1}{n^2}$ is a convergent series (by p series test $\because$ $p = 2 > 1$)

$\therefore$ Σu_n is also convergent

Hence the given series is a convergent series.

(*ii*) Here $$u_n = (n^3+1)^{1/3} - n = \left[n^3\left(1+\frac{1}{n^3}\right)\right]^{1/3} - n$$

$$= n\left(1+\frac{1}{n^3}\right)^{1/3} - n = n\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right]$$

$$= n\left[1 + \frac{1}{3}\cdot\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2\,!}\cdot\left(\frac{1}{n^3}\right)^2 + \ldots\ldots - 1\right]$$

$$= \frac{n}{n^3}\left[\frac{1}{3} - \frac{1}{9} \cdot \frac{1}{n^3} \ldots\ldots\right] = \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9} \cdot \frac{1}{n^3} \ldots\ldots\right]$$

Take $$v_n = \frac{1}{n^2}.$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{3} - \frac{1}{9} \cdot \frac{1}{n^3} \ldots\ldots\right) = \frac{1}{3}, \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\sum v_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2 > 1$

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

Note. Rationalization is effective only when square roots are involved whereas. Binomial Expansion is the general method.

Example 17. *Discuss the convergence or divergence of the following series:*

(*i*) $\sum \left(\frac{1}{n} - \log \frac{n+1}{n}\right)$ (*ii*) $\frac{1}{\log 2} + \frac{1}{\log 3} + \ldots + \frac{1}{\log n} + \ldots \infty.$

Sol. (*i*) $$u_n = \frac{1}{n} - \log \frac{n+1}{n} = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right)$$

$$= \frac{1}{n} - \left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \ldots \infty\right]$$

$$= \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} \ldots\ldots \infty$$

$$= \frac{1}{n^2} \cdot \left\{\frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} \ldots\ldots \infty\right\}$$

Let $$v_n = \frac{1}{n^2} \quad \therefore \quad \frac{u_n}{v_n} = \frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} \ldots \infty$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \frac{1}{2}, \text{ which is finite and } \neq 0$$

$\therefore$ Σu_n and Σv_n behave alike

$$\Sigma v_n = \sum \frac{1}{n^2} \text{ is } p\text{-series where } p = 2 > 1$$

$\therefore$ Σv_n converges and so given series Σu_n converges.

(*ii*) Given series is $$\sum_{n=2}^{\infty} u_n = \sum_{n=2}^{\infty} \frac{1}{\log n}$$

$\therefore$ $$u_n = \frac{1}{\log n}$$

We know that $$\log n < n \quad \therefore \quad \frac{1}{\log n} > \frac{1}{n}$$

$\therefore \quad u_n > \frac{1}{n}$. Take $v_n = \frac{1}{n}$

$\therefore \quad u_n > v_n$ and $\Sigma v_n = \sum \frac{1}{n}$ is of the type $\sum \frac{1}{n^p}$, where $p = 1$.

$\therefore \quad \Sigma v_n$ divergent

$\therefore$ By comparison test 5.20 I(b)

Σu_n is also divergent.

Hence $\sum_{n=2}^{\infty} \frac{1}{\log n}$ is divergent.

Example 18. *Test the convergence or divergence of the following series :*

(i) $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \infty$ (ii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$.

Sol. (i) $u_n = \frac{1}{n!}$

$$n! = 1.2.3.\ldots n \geq 1.2.2.2\ldots (n-1) \text{ times} = 2^{n-1}$$

$$\therefore \quad \frac{1}{n!} < \frac{1}{2^{n-1}} = v_n \text{ (say)}$$

$$\therefore \quad u_n = \frac{1}{n!} < v_n, \text{ where } v_n = \frac{1}{2^{n-1}}$$

Σv_n is a G.P. series with common ratio $\frac{1}{2} < 1$

$\therefore \quad \Sigma v_n$ is convergent

$\therefore$ By comparison test 5.20 I(a) Σu_n is also convergent

Hence $\Sigma u_n = \sum \frac{1}{n!}$ is convergent.

(ii) $\Sigma u_n = \sum \frac{1}{\sqrt{n!}} \quad \therefore \quad u_n = \frac{1}{\sqrt{n!}}$

As proved in (i) part $\frac{1}{n!} < \frac{1}{2^{n-1}} \quad \therefore \quad \frac{1}{\sqrt{n!}} < \frac{1}{2^{\frac{n-1}{2}}}$

$\therefore \quad u_n < v_n$, where $v_n = \frac{1}{2^{\frac{n-1}{2}}}$

Σv_n is an infinite G.P. with common ratio $\frac{1}{\sqrt{2}} < 1$

$\therefore \quad \Sigma v_n$ is convergent. Hence Σu_n *i.e.*, $\sum \frac{1}{\sqrt{n!}}$ is convergent

TEST YOUR KNOWLEDGE

Test the convergence or divergence of the following series :

1. $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+ \ldots\ldots \text{ to } \infty$

2. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+ \ldots\ldots \text{ to } \infty$

3. $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+ \ldots\ldots \text{ to } \infty$

4. $\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6}+ \ldots\ldots \text{ to } \infty$

5. $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+ \ldots\ldots \text{ to } \infty$ **(P.T.U., Jan. 2010)**

6. $\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\frac{1}{\sqrt{3.4}}+ \ldots\ldots \text{ to } \infty$

7. $\sum_{n=1}^{\infty} \frac{n+1}{n(2n-1)}$

8. $\sum_{n=1}^{\infty} \frac{1}{n^p (n+1)^p}$

9. $\frac{2^p}{1^q}+\frac{3^p}{2^q}+\frac{4^p}{3^q}+ \ldots\ldots \text{ to } \infty$

(p and q are positive numbers)

10. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

11. $\sum \frac{2n^3+5}{4n^5+1}$

12. $\sum \frac{\sqrt{n^2-1}}{n^3+1}$

13. $\sum \left(\sqrt{n^2+1}-n\right)$ **(P.T.U., Dec. 2006)**

14. $\sum \left(\sqrt{n^3+1}-\sqrt{n^3}\right)$

15. $\sum \left(\sqrt{n^4+1}-\sqrt{n-1}\right)$

16. $\frac{\sqrt{2}-\sqrt{1}}{1}+\frac{\sqrt{3}-\sqrt{2}}{2}+\frac{\sqrt{4}-\sqrt{3}}{3}+ \ldots\ldots$

17. $\sum \left\{(\sqrt[3]{n+1}-\sqrt[3]{n})\right\}$

18. $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n^p}.$

ANSWERS

1. Convergent **2.** Convergent **3.** Convergent **4.** Divergent

5. Divergent **6.** Divergent **7.** Divergent

8. Convergent for $p > \frac{1}{2}$, divergent for $p \le \frac{1}{2}$

9. Convergent for $q > p + 1$, divergent for $q \le p + 1$

10. Convergent **11.** Convergent **12.** Convergent **13.** Divergent

14. Convergent **15.** Convergent **16.** Convergent **17.** Divergent.

18. Convergent for $p > \frac{1}{2}$, divergent for $p \le \frac{1}{2}$.

5.22. D'ALEMBERT'S RATIO TEST **(P.T.U., Dec. 2004)**

Statement. *If Σu_n is a positive term series, and* $\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = l$, *then*

(*i*) *Σu_n is convergent if $l > 1$.* (*ii*) *Σu_n is divergent if $l < 1$.*

Note. If $l = 1$, the test fails, *i.e.*, no conclusion can be drawn about the convergence or divergence of the series. The series may converge, it may diverge.

Proof. Since $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{u_{n+1}} = l,$

$\therefore$ Given $\varepsilon > 0$, however small, there exists a positive integer m such that

$$\left| \frac{u_n}{u_{n+1}} - l \right| < \varepsilon \quad \forall \quad n \geq m$$

$$\Rightarrow \qquad l - \varepsilon < \frac{u_n}{u_{n+1}} < l + \varepsilon \quad \forall \quad n \geq m \qquad \ldots(1)$$

Case I. *When $l > 1$,* choose $\varepsilon > 0$ such that $l - \varepsilon = r > 1$

$$\therefore \qquad \text{for } n \geq m\,; \quad \frac{u_n}{u_{n+1}} > l - \varepsilon = r \quad i.e., \quad \frac{u_n}{u_{n+1}} > r \text{ for } n \geq m$$

Put $n = m, m+1, m+2, \ldots n-1$ (*i.e.,* $n - m$ terms)

$$\therefore \qquad \frac{u_m}{u_{m+1}} > r$$

$$\frac{u_{m+1}}{u_{m+2}} > r$$

$$\frac{u_{m+2}}{u_{m+3}} > r$$

....

....

$$\frac{u_{n-1}}{u_n} > r$$

Multiply these inequalities ; $\dfrac{u_m}{u_n} > r^{n-m}$

$$\therefore \qquad \frac{u_n}{u_m} < \frac{1}{r^{n-m}} \quad \text{or} \quad u_n < \frac{u_m}{r^{n-m}} = (r^m u_m)\frac{1}{r^n}$$

$$\therefore \qquad u_n < k.\frac{1}{r^n} \quad \forall\, n \geq m \text{ (where } k = r^m.\, u_m)$$

Let $v_n = \dfrac{1}{r^n}$, where $r > 1 \quad \therefore \quad \dfrac{1}{r} < 1$

$\Sigma\, v_n$ is a geometric series with common ratio < 1

$\therefore$ $\Sigma\, v_n$ is convergent and by comparison test 5.20 I(*i*)

$\displaystyle\sum_{n=1}^{\infty} u_n$ is also convergent.

Case II. *When $l < 1$*; choose $\varepsilon > 0$ such that $l + \varepsilon = \text{R} < 1$

$$\therefore \qquad \text{From (1)} \quad \frac{u_n}{u_{n+1}} < \text{R} \quad \forall\, n \geq m$$

Put $n = m, m+1, m+2, \ldots n-1$ and multiply (as in case I)

we get $$\frac{u_m}{u_n} < R^{n-m} \quad \text{or} \quad u_n > \frac{u_m}{R^{n-m}} = (R^m u_m) \cdot \frac{1}{R^n} = k' v_n$$

where $k' = R^m u_m$ and $v_n = \dfrac{1}{R^n}$

$\Sigma v_n = \sum \dfrac{1}{R^n}$, which is G.P. with common ratio $\dfrac{1}{R} > 1$

$\therefore$ Σv_n is divergent.

$\therefore$ Σu_n is also divergent.

Hence if Σu_n is a positive term series, and $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{u_{n+1}} = l$, then

(*i*) Σu_n is convergent if $l > 1$ (*ii*) Σu_n is divergent if $l < 1$.

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence of the following series:*

(*i*) $1 + \dfrac{2^p}{2!} + \dfrac{3^p}{3!} + \dfrac{4^p}{4!} + \ldots\ldots, (p > 0)$

(*ii*) $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{5} + \ldots\ldots + \dfrac{1}{2^{n-1}+1} + \ldots\ldots$

(*iii*) $\dfrac{1^2 \cdot 2^2}{1!} + \dfrac{2^2 \cdot 3^2}{2!} + \dfrac{3^2 \cdot 4^2}{3!} + \dfrac{4^2 \cdot 5^2}{4!} + \ldots\ldots$

Sol. (*i*) Here $$u_n = \frac{n^p}{n!} \qquad \left[\because \; 1 = \frac{1^p}{1!}\right]$$

$$\therefore \quad u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^p}{n!} \frac{(n+1)!}{(n+1)^p} = \frac{n^p \cdot (n+1)\, n!}{n!\,(n+1)^p} = \frac{n^p}{(n+1)^{p-1}}$$

$$= \frac{n^p}{n^{p-1}\left(1+\dfrac{1}{n}\right)^{p-1}} = \frac{n}{\left(1+\dfrac{1}{n}\right)^{p-1}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n}{\left(1+\dfrac{1}{n}\right)^{p-1}} = \infty > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(*ii*) Here $$u_n = \frac{1}{2^{n-1}+1} \quad \therefore \quad u_{n+1} = \frac{1}{2^n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^n + 1}{2^{n-1} + 1} = \frac{2^n\left(1 + \frac{1}{2^n}\right)}{2^{n-1}\left(1 + \frac{1}{2^{n-1}}\right)} = 2 \cdot \frac{1 + \frac{1}{2^n}}{1 + \frac{1}{2^{n-1}}}$$

$$\operatorname*{Lt}_{n\to\infty} \frac{u_n}{u_{n+1}} = \operatorname*{Lt}_{n\to\infty} 2 \cdot \frac{1 + \frac{1}{2^n}}{1 + \frac{1}{2^{n-1}}} = 2 > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(*iii*) Here $\qquad u_n = \dfrac{n^2 (n+1)^2}{n\,!} \qquad \therefore \quad u_{n+1} = \dfrac{(n+1)^2 (n+2)^2}{(n+1)\,!}$

$$\frac{u_n}{u_{n+1}} = \frac{n^2 (n+1)^2}{n\,!} \cdot \frac{(n+1)\,!}{(n+1)^2 (n+2)^2} = \frac{n^2\,(n+1)}{(n+2)^2}$$

$$= \frac{n^3\left(1 + \frac{1}{n}\right)}{n^2\left(1 + \frac{2}{n}\right)^2} = n \,.\, \frac{1 + \frac{1}{n}}{\left(1 + \frac{2}{n}\right)^2}$$

$$\operatorname*{Lt}_{n\to\infty} \frac{u_n}{u_{n+1}} = \operatorname*{Lt}_{n\to\infty} n \,.\, \frac{1 + \frac{1}{n}}{\left(1 + \frac{2}{n}\right)^2} = \infty > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

Example 2. *Discuss the convergence of the following series:*

(*i*) $\dfrac{2}{1} + \dfrac{2.5.8}{1.5.9} + \dfrac{2.5.8.11}{1.5.9.13} + \ldots \infty$ $\qquad$ (*ii*) $\left(\dfrac{1}{3}\right)^2 + \left(\dfrac{1.2}{3.5}\right)^2 + \left(\dfrac{1.2.3.}{3.5.7}\right)^2 + \ldots \infty$

Sol. (*i*) $\qquad u_n = \dfrac{2.5.8.11 \ldots (3n-1)}{1.5.9.13 \ldots (4n-3)}$

$$\left(\begin{array}{l} \text{As } 2, 5, 8, \ldots \text{ are in A.P.} \\ \therefore \ \text{ its } n\text{th term} = 2 + (n-1)\,3 = 3n - 1. \\ \text{Also } 1, 5, 9, \ldots \text{ are in A.P.} \\ \therefore \ \text{ its } n\text{th term} = 1 + (n-1)\,4 = 4n - 3 \end{array}\right)$$

$$u_{n+1} = \frac{2.5.8.11 \ldots (3n-1)\,(3n+2)}{1.5.9.13 \ldots (4n-3)\,(4n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{4n+1}{3n+2}$$

$$\operatorname*{Lt}_{n\to\infty} \frac{u_n}{u_{n+1}} = \operatorname*{Lt}_{n\to\infty} \frac{4 + \frac{1}{n}}{3 + \frac{2}{n}} = \frac{4}{3} > 1$$

$\therefore$ By D'Alembert's Ratio Test $\Sigma\, u_n$ is convergent.

(*ii*)
$$u_n = \left(\frac{1.2.3 \ldots n}{3.5.7 \ldots 2n+1}\right)^2$$

$$u_{n+1} = \left[\frac{1.2.3 \ldots n(n+1)}{3.5.7 \ldots (2n+1)\,(2n+3)}\right]^2$$

$$\frac{u_n}{u_{n+1}} = \left(\frac{2n+3}{n+1}\right)^2 = \left(\frac{2+\frac{3}{n}}{1+\frac{1}{n}}\right)^2$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{2}{1} = 2 > 1$$

$\therefore$ By D'Alembert's Ratio test $\Sigma\, u_n$ is convergent

Example 3. *Test the convergence of the following series:*

(*i*) $\sum \dfrac{n^3 + a}{2^n + a}$ (*ii*) $\sum \dfrac{n!\,2^n}{n^n}$

(*iii*) $\sum \dfrac{2^{n-1}}{3^n + 1}$ (*iv*) $\sum \dfrac{n^2 (n+1)^2}{n!}$.

Sol. (*i*) Here
$$u_n = \frac{n^3 + a}{2^n + a} \quad \therefore \quad u_{n+1} = \frac{(n+1)^3 + a}{2^{n+1} + a}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^3 + a}{(n+1)^3 + a} \cdot \frac{2^{n+1} + a}{2^n + a}$$

$$= \frac{n^3\left(1+\frac{a}{n^3}\right)}{n^3\left[\left(1+\frac{1}{n}\right)^3 + \frac{a}{n^3}\right]} \cdot \frac{2^{n+1}\left(1+\frac{a}{2^{n+1}}\right)}{2^n\left(1+\frac{a}{2^n}\right)} = \frac{1+\frac{a}{n^3}}{\left(1+\frac{1}{n}\right)^3 + \frac{a}{n^3}} \cdot \frac{2\left(1+\frac{a}{2^{n+1}}\right)}{1+\frac{a}{2^n}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1+0}{1+0} \cdot 2 \cdot \frac{1+0}{1+0} = 2 > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(*ii*) Here
$$u_n = \frac{n!\,2^n}{n^n} \quad \therefore \quad u_{n+1} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{n!\,2^n}{(n+1)!\,2^{n+1}} \cdot \frac{(n+1)^{n+1}}{n^n} = \frac{1}{2(n+1)} \cdot \frac{(n+1)^{n+1}}{n^n}$$

$$= \frac{1}{2} \cdot \frac{(n+1)^n}{n^n} = \frac{1}{2} \cdot \left(\frac{n+1}{n}\right)^n = \frac{1}{2}\left(1+\frac{1}{n}\right)^n$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{2}\left(1+\frac{1}{n}\right)^n = \frac{e}{2}$$

Now, $$2 < e < 3 \quad \Rightarrow \quad 1 < \frac{e}{2} < \frac{3}{2}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{e}{2} > 1 \quad \Rightarrow \quad \Sigma u_n \text{ is convergent.}$$

(*iii*) $$u_n = \frac{2^{n-1}}{3^n+1}\,;\; u_{n+1} = \frac{2^n}{3^{n+1}+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^{n-1}}{3^n+1} \times \frac{3^{n+1}+1}{2^n} = \frac{1}{2}\,\frac{3^{n+1}+1}{3^n+1} = \frac{1}{2}\,\frac{3^n\left(3+\frac{1}{3^n}\right)}{3^n\left(1+\frac{1}{3^n}\right)}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{2}\,\frac{3+\frac{1}{3^n}}{1+\frac{1}{3^n}} = \frac{3}{2} > 1$$

$\therefore$ By Ratio test $\Sigma\, u_n$ is convergent.

(*iv*) $$u_n = \frac{n^2\,(n+1)^2}{n\,!},\; u_{n+1} = \frac{(n+1)^2\,(n+2)^2}{(n+1)\,!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^2\,(n+1)^2}{n\,!}\;\frac{(n+1)\,!}{(n+1)^2\,(n+2)^2}$$

$$= \frac{n^2}{(n+2)^2}\,(n+1) = (n+1)\left(\frac{n}{n+2}\right)^2 = (n+1)\left(\frac{1}{1+\frac{2}{n}}\right)^2$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}}\; \infty\cdot 1 = \;\infty > 1$$

$\therefore$ By Ratio test $\Sigma\, u_n$ is convergent

Example 4. *Discuss the convergence of the series :*

(*i*) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}}\, x^n$ **(P.T.U., Dec. 2012)** (*ii*) $\displaystyle\sum \frac{x^n}{3^n \cdot n^2},\; x > 0$

(*iii*) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2n\,!}$ **(P.T.U., May 2012)** (*iv*) $\displaystyle\sum \frac{3^n-2}{3^n+1}\, x^{n-1},\; x > 0.$

Sol. (*i*) Here $$u_n = \sqrt{\frac{n}{n^2+1}}\; x^n$$

$\therefore \quad u_{n+1} = \sqrt{\dfrac{n+1}{(n+1)^2+1}} \, . \, x^{n+1}$

$$\frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n+1} \cdot \frac{n^2+2n+2}{n^2+1}} \cdot \frac{1}{x} = \sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n converges if $\dfrac{1}{x} > 1$ *i.e.*, $x < 1$

and diverges if $\dfrac{1}{x} < 1$ *i.e.*, $x > 1$

When $x = 1$, the Ratio Test fails.

$\therefore$ for $x = 1$,
$$u_n = \sqrt{\frac{n}{n^2+1}} = \sqrt{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

Take
$$v_n = \frac{1}{\sqrt{n}}, \ \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1, \text{ which is finite and } \neq 0.$$

$\therefore$ By Comparison Test, Σu_n and Σv_n converge or diverge together.

Since $\sum v_n = \sum \dfrac{1}{\sqrt{n}}$ is of the form $\sum \dfrac{1}{n^p}$ with $p = \dfrac{1}{2} < 1$

Σv_n diverges $\Rightarrow$ Σu_n diverges.

Hence the given series Σu_n converges if $x < 1$ and diverges if $x \geq 1$.

(*ii*)
$$u_n = \frac{x^n}{3^n . n^2} \ ; u_{n+1} = \frac{x^{n+1}}{3^{n+1} (n+1)^2} \qquad \therefore \quad \frac{u_n}{u_{n+1}} = \frac{x^n}{3^n . n^2} \cdot \frac{3^{n+1} (n+1)^2}{x^{n+1}}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = 3 \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{x} = 3\left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{3}{x}.$$

$\therefore$ By ratio test $\Sigma\, u_n$ converges if $\dfrac{3}{x} > 1$ and diverges *i.e.,* converges for $x < 3$ and diverges for $x > 3$

for $x = 3$,
$$u_n = \frac{3^n}{3^n \, . \, n^2} = \frac{1}{n^2}$$

$$\Sigma u_n = \sum \frac{1}{n^2}\text{, which is of the type } \sum \frac{1}{n^p}\text{, where } p = 2 > 1$$

$\therefore$ Σu_n converges for $x = 3$

Hence u_n is convergent for $x \le 3$ and diverges for $x > 3$.

(*iii*) $$u_n = \frac{x^n}{2n\,!},\ u_{n+1} = \frac{x^{n+1}}{2\,(n+1)\,!}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{2n\,!} \times \frac{(2n+2)\,!}{x^{n+1}} = \frac{(2n+2)(2n+1)}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{(2n+2)(2n+1)}{x} \to \infty > 1$$

$\therefore$ By Ratio test Σu_n is convergent.

(*iv*) $$u_n = \frac{3^n - 2}{3^n + 1}\, x^{n-1};\ u_{n+1} = \frac{3^{n+1} - 2}{3^{n+1} + 1}\, x^n$$

$$\frac{u_n}{u_{n+1}} = \frac{3^n - 2}{3^n + 1}\, x^{n-1} \cdot \frac{3^{n+1} + 1}{3^{n+1} - 2} \cdot \frac{1}{x^n} = \frac{(3^n - 2)(3^{n+1} + 1)}{(3^n + 1)(3^{n+1} - 2)} \cdot \frac{1}{x}$$

$$= \frac{3^n\left(1 - \frac{2}{3^n}\right) . 3^{n+1}\left(1 + \frac{1}{3^{n+1}}\right)}{3^n\left(1 + \frac{1}{3^n}\right) 3^{n+1}\left(1 - \frac{2}{3^{n+1}}\right)} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ By Ratio test Σu_n converges for $\frac{1}{x} > 1$ *i.e.*, for $x < 1$ and diverges for $x > 1$.

for $x = 1$ $$u_n = \frac{3^n - 2}{3^n + 1} = \frac{1 - \frac{2}{3^n}}{1 + \frac{2}{3^n}}$$

$$\underset{n\to\infty}{\text{Lt}} u_n = 1 \ne 0 \quad \therefore \quad \Sigma u_n \text{ is divergent.}$$

$\therefore$ Σu_n is convergent for $x < 1$ and divergent for $n \ge 1$.

Example 5. *Examine the convergence or divergence of the following series :*

(*i*) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots\ldots$ (*ii*) $\sum \frac{x^{n+1}}{(n+1)\sqrt{n}}$. **(P.T.U., Dec. 2007)**

(P.T.U., Dec. 2002, 2013)

Sol. (*i*) Here $$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \quad \therefore \quad u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \cdot \frac{1}{x^2} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}} \cdot \sqrt{1+\frac{1}{n}} \cdot \frac{1}{x^2}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x^2} > 1$, *i.e.,* $x^2 < 1$

and diverges if $\frac{1}{x^2} < 1$ *i.e.,* $x^2 > 1$.

When $x^2 = 1$, $$u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$$

Take $$v_n = \frac{1}{n^{3/2}} \ ; \ \frac{u_n}{v_n} = \frac{1}{1+\frac{1}{n}} \ ; \ \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = 1,$$ which is finite and $\neq 0$

Σv_n is convergent by p-series test $\because$ here $p = 3/2 > 1$

$\therefore$ By comparison test Σu_n is also convergent ($\because$ $p > 1$)

Hence Σu_n is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

(*ii*) $$u_n = \frac{x^{n+1}}{(n+1)\sqrt{n}} \ ; u_{n+1} = \frac{x^{n+2}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \cdot \frac{1}{x} = \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)} \sqrt{1+\frac{1}{n}} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

By D, Alembert's Ratio test Σu_n converges for $\frac{1}{x} > 1$ *i.e.,* $x < 1$ and diverges for $x > 1$

when $x = 1$, Ratio test fails

$\therefore$ For $x = 1$, $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$; Take $v_n = \frac{1}{n^{3/2}}$

$\therefore$ $\frac{u_n}{v_n} = \frac{1}{1+\frac{1}{n}}$; $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = 1$, which is finite and non-zero $\therefore$ Σu_n and Σv_n behave alike Σv_n is a cgt series (by p-test ; $p > 1$) $\therefore$ By comparison test Σu_n is also convergent. Hence Σu_n converges for $x \leq 1$ and diverges for $x > 1$

Example 6. *Discuss the convergence or divergence of the following series:*

(*i*) $x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \ldots + \frac{n^2-1}{n^2+1}x^n + \cdots \infty$

(*ii*) $\frac{x}{2\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^3}{4\sqrt{5}} + \cdots \infty.$

Sol. (*i*) $u_n = \frac{n^2-1}{n^2+1}\, x^n \,;\, u_{n+1} = \frac{(n+1)^2-1}{(n+1)^2+1}\, x^{n+1}$

$$\frac{u_n}{u_{n+1}} = \frac{n^2-1}{n^2+1}\,\frac{(n+1)^2+1}{(n+1)^2-1}\cdot\frac{1}{x}$$

$$= \frac{n^2\left(1-\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)}\,\frac{n^2\left\{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right\}}{n^2\left\{\left(1+\frac{1}{n}\right)^2-\frac{1}{n^2}\right\}}\cdot\frac{1}{x}$$

$$= \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}}\cdot\frac{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2-\frac{1}{n^2}}\cdot\frac{1}{x}$$

$$\underset{n\to\infty}{\mathrm{Lt}}\ \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ By Ratio test $\Sigma\, u_n$ converges if $\frac{1}{x} > 1$ *i.e.,* $x < 1$ and diverges if $x > 1$

When $x = 1,\ u_n = \frac{n^2-1}{n^2+1}$

$$\underset{n\to\infty}{\mathrm{Lt}}\ u_n = \underset{n\to\infty}{\mathrm{Lt}}\ \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}} = 1 \neq 0$$

$\therefore$ $\Sigma\, u_n$ diverges.

$\therefore$ $\Sigma\, u_n$ converges for $x < 1$ and diverges for $x \geq 1$.

(*ii*)

$$u_n = \frac{x^n}{(n+1)\sqrt{n+2}}\,;\, u_{n+1} = \frac{x^{n+1}}{(n+2)\sqrt{n+3}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{(n+1)\sqrt{n+2}} \times \frac{(n+2)\sqrt{n+3}}{x^{n+1}} = \frac{n+2}{n+1}\sqrt{\frac{n+3}{n+2}}\cdot\frac{1}{x}$$

$$\underset{n\to\infty}{\mathrm{Lt}}\ \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\mathrm{Lt}}\ \frac{1+\frac{2}{n}}{1+\frac{1}{n}}\sqrt{\frac{1+3/n}{1+2/n}}\cdot\frac{1}{n} = \frac{1}{x}$$

$\Sigma\, u_n$ converges for $\frac{1}{x} > 1$ *i.e.,* $x < 1$ and diverges for $x > 1$.

When $\quad x = 1,\, u_n = \dfrac{1}{(n+1)\sqrt{n+2}}$; $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\, \dfrac{1}{(n+1)\sqrt{n+2}} = 0$

As $\Sigma\, u_n$ is a +ve term series and $\underset{n\to\infty}{\text{Lt}}\, u_n = 0$

$\therefore \quad \Sigma\, u_n$ converges. Hence $\Sigma\, u_n$ converges for $x \le 1$, diverges for $x > 1$.

Example 7. *Examine the convergence or divergence of the following series :*

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \ldots\ldots + \frac{2^{n+1}-2}{2^{n+1}+1}\, x^n + \ldots\ldots\ (x > 0).$$ **(P.T.U., Jan. 2009)**

Sol. Here, leaving the first term, $\quad u_n = \dfrac{2^{n+1}-2}{2^{n+1}+1}\, x^n$

$$\therefore \qquad u_{n+1} = \frac{2^{n+2}-2}{2^{n+2}+1}\, x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^{n+1}-2}{2^{n+1}+1} \cdot \frac{2^{n+2}+1}{2^{n+2}-2} \cdot \frac{1}{x} = \frac{2^{n+1}\left(1-\dfrac{2}{2^{n+1}}\right)}{2^{n+1}\left(1+\dfrac{1}{2^{n+1}}\right)} \cdot \frac{2^{n+2}\left(1+\dfrac{1}{2^{n+2}}\right)}{2^{n+2}\left(1-\dfrac{2}{2^{n+2}}\right)} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}}\, \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\, \frac{1-\dfrac{1}{2^n}}{1+\dfrac{1}{2^{n+1}}} \cdot \frac{1+\dfrac{1}{2^{n+2}}}{1-\dfrac{1}{2^{n+1}}} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore \quad$ By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x} > 1 \quad$ *i.e.,* $\quad x < 1$

and diverges if $\frac{1}{x} < 1 \quad$ *i.e.,* $\quad x > 1$.

When $x = 1$,
$$u_n = \frac{2^{n+1}-2}{2^{n+1}+1} = \frac{2^{n+1}\left(1-\dfrac{2}{2^{n+1}}\right)}{2^{n+1}\left(1+\dfrac{1}{2^{n+1}}\right)} = \frac{1-\dfrac{1}{2^n}}{1+\dfrac{1}{2^{n+1}}}$$

$\underset{n\to\infty}{\text{Lt}}\, u_n = 1 \ne 0 \quad \Rightarrow \quad \Sigma u_n$ does not converge. Being a series of +ve terms, it must diverge.

Hence Σu_n is convergent if $x < 1$ and divergent if $x \ge 1$.

Example 8. *Test for convergence the positive term series :*

$$1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)} + \ldots\ldots$$

Sol. Leaving the first term $\quad u_n = \dfrac{(\alpha+1)(2\alpha+1)\ \ldots\ldots\ (n\alpha+1)}{(\beta+1)(2\beta+1)\ \ldots\ldots\ (n\beta+1)}$

$\therefore \qquad u_{n+1} = \dfrac{(\alpha+1)(2\alpha+1)\ \ldots\ldots\ (n\alpha+1)[(n+1)\ \alpha+1]}{(\beta+1)(2\beta+1)\ \ldots\ldots\ (n\beta+1)[(n+1)\ \beta+1]}$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)\ \beta+1}{(n+1)\ \alpha+1} = \frac{\left(1+\dfrac{1}{n}\right)\beta+\dfrac{1}{n}}{\left(1+\dfrac{1}{n}\right)\alpha+\dfrac{1}{n}}$$

$$\underset{n\to\infty}{\mathrm{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\mathrm{Lt}} \frac{\left(1+\dfrac{1}{n}\right)\beta+\dfrac{1}{n}}{\left(1+\dfrac{1}{n}\right)\alpha+\dfrac{1}{n}} = \frac{\beta}{\alpha}$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n converges if $\dfrac{\beta}{\alpha} > 1$ *i.e.,* $\beta > \alpha > 0$

and diverges if $\dfrac{\beta}{\alpha} < 1$ *i.e.,* $\beta < \alpha$ or $\alpha > \beta > 0$

When $\alpha = \beta$, the Ratio Test fails.

When $\alpha = \beta$, $u_n = 1$ $\underset{n\to\infty}{\mathrm{Lt}}\ u_n = 1 \neq 0$

$\Rightarrow$ Σu_n does not converge. Being a series of +ve terms, it must diverge.

Hence the given series is convergent if $\beta > \alpha > 0$ and divergent if $\alpha \geq \beta > 0$.

TEST YOUR KNOWLEDGE

Discuss the convergence of the following series :

1. $1 + \dfrac{2^2}{2!} + \dfrac{3^2}{3!} + \dfrac{4^2}{4!} + \ldots\ldots$ to ∞

2. $1 + \dfrac{2!}{2^2} + \dfrac{3!}{3^3} + \dfrac{4!}{4^4} + \ldots\ldots$ to ∞

3. $\dfrac{1}{1+2} + \dfrac{2}{1+2^2} + \dfrac{3}{1+2^3} + \ldots\ldots$ to ∞

4. (*i*) $\dfrac{2!}{3} + \dfrac{3!}{3^2} + \dfrac{4!}{3^3} + \ldots\ldots$ to ∞ (*ii*) $\sum \dfrac{n^2}{3^n}$

5. (*i*) $\sum \dfrac{1}{n!}$ (*ii*) $\sum \dfrac{n!}{n^n}$

6. $\sum \dfrac{x^n}{n}$, $x > 0$

7. $\sum \dfrac{n}{n^2+1} x^n$, $x > 0$

8. $\sum \sqrt{\dfrac{n+1}{n^3+1}} \cdot x^n$, $x > 0$

9. $x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots$ to ∞

10. $1 + \dfrac{x}{2} + \dfrac{x^2}{5} + \dfrac{x^3}{10} + \ldots\ldots \dfrac{x^n}{n^2+1} + \ldots\ldots$ to ∞

11. $\dfrac{x}{1.3} + \dfrac{x^2}{3.5} + \dfrac{x^3}{5.7} + \ldots\ldots$ to ∞

ANSWERS

1. Convergent **2.** Convergent **3.** Convergent
4. (*i*) Divergent (*ii*) Convergent **5.** (*i*) Convergent (*ii*) Convergent
6. Convergent for $x < 1$, divergent for $x \geq 1$ **7.** Convergent for $x < 1$, divergent for $x \geq 1$
8. Convergent for $x < 1$, divergent for $x \geq 1$ **9.** Convergent for $x < 1$, divergent for $x \geq 1$
10. Convergent for $x \leq 1$, divergent for $x > 1$ **11.** Convergent for $x \leq 1$, divergent for $x > 1$.

5.23. RAABE'S TEST

(P.T.U., May 2007)

Statement. *If Σu_n is a series of positive terms and $\underset{n\to\infty}{Lt}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, then the series is convergent if $l > 1$ and divergent if $l < 1$.*

Proof. Let us compare the given series Σu_n with an auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^p}$, which we know converges if $p > 1$ and diverges if $p \leq 1$.

Now,
$$\frac{v_n}{v_{n+1}} = \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}} = \left(\frac{n+1}{n}\right)^p = \left(1 + \frac{1}{n}\right)^p = 1 + \frac{p}{n} + \frac{p(p-1)}{2\,!} \cdot \frac{1}{n^2} + \ldots\ldots$$

Case I. *Let* $\Sigma v_n = \Sigma \frac{1}{n^p}$ *be convergent,* so that $p > 1$.

Then Σu_n will also converge if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$

or if
$$\frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2\,!} \cdot \frac{1}{n^2} + \ldots\ldots$$

or if
$$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2\,!} \cdot \frac{1}{n} + \ldots\ldots$$

or if
$$\underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) > p$$

or if
$$l > p$$

But p is itself greater than 1, $\therefore$ Σu_n is convergent if $l > 1$.

Case II. *Let* Σv_n *be divergent,* so that $p \leq 1$.

Then Σu_n will also diverge if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

or if
$$\frac{u_n}{u_{n+1}} < 1 + \frac{p}{n} + \frac{p(p-1)}{2\,!} \cdot \frac{1}{n^2} + \ldots\ldots$$

or if
$$n\left(\frac{u_n}{u_{n+1}} - 1\right) < p + \frac{p(p-1)}{2\,!} \cdot \frac{1}{n} + \ldots\ldots$$

or if $$\underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}}-1\right) < p$$

or if $$l < p.$$

But p itself ≤ 1. Thus the given series Σu_n diverges if $l < 1$. This proves the result.

Note 1. If $\underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}}-1\right) = 1$, then Raabe's test fails.

Note 2. Raabe's test is used when D'Alembert's Ratio test fails and when in the ratio test, $\frac{u_n}{u_{n+1}}$ does not involve the number *e*. When $\frac{u_n}{u_{n+1}}$ involves e, we apply logarithmic test after the ratio test and not Raabe's test.

5.24. LOGARITHMIC TEST (P.T.U., May 2007)

Statement. *A positive term series Σu_n converges or diverges according as*

$$\underset{n\to\infty}{Lt}\ n \log \frac{u_n}{u_{n+1}} > 1 \text{ or } < 1.$$

Proof. Let us compare the given series Σu_n with an auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^p}$, which we know converges if $p > 1$ and diverges if $p \le 1$.

Now, $$\frac{v_n}{v_{n+1}} = \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p.$$

Case I. *Let Σv_n be convergent,* so that $p > 1$.

Then Σu_n will also be convergent if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$

or if $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p$$

or if $$\log \frac{u_n}{u_{n+1}} > \log\left(1+\frac{1}{n}\right)^p = p \log\left(1+\frac{1}{n}\right)$$

or if $$\log \frac{u_n}{u_{n+1}} > p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\ldots\right] \qquad \left[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots\right]$$

or if $$n \log \frac{u_n}{u_{n+1}} > p\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \ldots\ldots\right]$$

or if $$\underset{n\to\infty}{\text{Lt}}\ n \log \frac{u_n}{u_{n+1}} > p$$

or if $$\underset{n\to\infty}{\text{Lt}}\ n \log \frac{u_n}{u_{n+1}} > 1 \qquad |\because\ p > 1$$

Case II. *Let Σv_n be divergent*, so that $p \le 1$.

Then Σu_n also diverges if $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$

or if $$\log \frac{u_n}{u_{n+1}} < \log \frac{v_n}{v_{n+1}}$$

or if $$\log \frac{u_n}{u_{n+1}} < \log \left(1 + \frac{1}{n}\right)^p = p \log \left(1 + \frac{1}{n}\right)$$

or if $$\log \frac{u_n}{u_{n+1}} < p \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\ldots\right)$$

or if $$n \log \frac{u_n}{u_{n+1}} < p \left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \ldots\ldots\right]$$

or if $$\underset{n \to \infty}{\text{Lt}}\ n \log \frac{u_n}{u_{n+1}} < p$$

or if $$\underset{n \to \infty}{\text{Lt}}\ n \log \frac{u_n}{u_{n+1}} < 1 \qquad | \because \quad p \le 1$$

Thus the series Σu_n converges or diverges according as $\underset{n \to \infty}{\text{Lt}}\ n \log \dfrac{u_n}{u_{n+1}} > 1$ or < 1.

Note 1. The test fails if $\underset{n \to \infty}{\text{Lt}}\ n \log \dfrac{u_n}{u_{n+1}} = 1$.

Note 2. The test is applied after the failure of Ratio test and generally when in Ratio test, $\dfrac{u_n}{u_{n+1}}$ involves '*e*'.

5.25. GAUSS TEST

Statement. *If for the series Σu_n of positive terms, $\dfrac{u_n}{u_{n+1}}$ can be expanded in the form*

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

then Σu_n converges if $\lambda > 1$ and diverges if $\lambda \le 1$.

Note. The test never fails as we know that the series diverges for $\lambda = 1$. Moreover the test is applied after the failure of ratio test and when it is possible to expand $\dfrac{u_n}{u_{n+1}}$ in powers of $\dfrac{1}{n}$ by Binomial Theorem or by any other method.

[Binomial theorem is $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!} x^2 + \ldots\ldots \infty$, where $|x| < 1$]

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence of the series* : $\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+......$

Sol. Here $$u_n=\frac{1.3.5......(2n-1)}{2.4.6.......2n}.$$

$\therefore$ $$u_{n+1}=\frac{1.3.5......(2n-1)(2n+1)}{2.4.6.......(2n)(2n+2)}$$

$\therefore$ $$\frac{u_n}{u_{n+1}}=\frac{1.3.5.......(2n-1)}{2.4.6......2n}\times\frac{2.4.6.......2n(2n+2)}{1.3.5......(2n-1)(2n+1)}$$

$$=\frac{2n+2}{2n+1}=\frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\to 1 \text{ as } n\to\infty.$$

$\therefore$ D'Alembert's Ratio test fails.

$$n\left[\frac{u_n}{u_{n+1}}-1\right]=n\left[\frac{2n+2}{2n+1}-1\right]=\frac{n}{2n+1}=\frac{1}{2+\frac{1}{n}}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\ n\left[\frac{u_n}{u_{n+1}}-1\right]=\frac{1}{2}<1.$$

$\therefore$ By Raabe's test, Σu_n diverges.

Example 2. *Discuss the convergence of the following series :*

(*i*) $\frac{1^2}{2^2}+\frac{1^2\cdot 2^2}{2^2\cdot 4^2}+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}+......$

(*ii*) $1+\frac{2^2}{3^2}+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}+......\infty.$ **(P.T.U., May 2004)**

Sol. (*i*) Here $$u_n=\frac{1^2\cdot 3^2\cdot 5^2......(2n-1)^2}{2^2\cdot 4^2\cdot 6^2......(2n)^2}$$

and $$u_{n+1}=\frac{1^2\cdot 3^2\cdot 5^2......(2n-1)^2(2n+1)^2}{2^2\cdot 4^2\cdot 6^2......(2n)^2(2n+2)^2}$$

$\therefore$ $$\frac{u_n}{u_{n+1}}=\frac{(2n+2)^2}{(2n+1)^2}=\frac{4n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)^2}=\frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}}=1.$$

Hence the ratio test fails

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{(2n+2)^2}{(2n+1)^2}-1\right]$$

$$=n\left[\frac{4n^2+8n+4-(4n^2+4n+1)}{(2n+1)^2}\right]=n\,\frac{(4n+3)}{(2n+1)^2}=\frac{4n^2+3n}{(2n+1)^2}$$

$$=\frac{1+\frac{3}{4n}}{\left(1+\frac{1}{2n}\right)^2}\to 1 \text{ as } n\to\infty$$

$\therefore$ Raabe's test also fails.

Now,
$$\frac{u_n}{u_{n+1}}=\frac{(2n+2)^2}{(2n+1)^2}=\frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}=\left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}$$

$$=\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{2}{2n}+\frac{3}{4n^2}-\ldots\ldots\right)=1+\frac{1}{n}+\frac{1}{n^2}\left(1-2+\frac{3}{4}\right)+\ldots\ldots$$

$$=1+\frac{1}{n}-\frac{1}{4n^2}+\ldots\ldots=1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

Comparing it with
$$\frac{u_n}{u_{n+1}}=1+\frac{\lambda}{n}+O\left(\frac{1}{n^2}\right)$$

We have $\lambda = 1$. Thus by Gauss test, the series Σu_n diverges.

Note that when D'Alembert ratio test fails, we can directly apply Gauss test.

(*ii*) Here
$$u_n=\frac{2^2\cdot 4^2\cdot 6^2\ldots\ldots(2n)^2}{3^2\cdot 5^2\cdot 7^2\ldots\ldots(2n+1)^2}$$

$$u_{n+1}=\frac{2^2\cdot 4^2\cdot 6^2\ldots\ldots(2n)^2\,(2n+2)^2}{3^2\cdot 5^2\cdot 7^2\ldots\ldots(2n+1)^2\,(2n+3)^2}$$

$$\frac{u_n}{u_{n+1}}=\frac{2^2\cdot 4^2\,.\,6^2\ldots\ldots(2n)^2}{3^2\cdot 5^2\cdot 7^2\ldots\ldots(2n+1)^2}\times\frac{3^2\cdot 5^2\cdot 7^2\ldots\ldots(2n+1)^2\,(2n+3)^2}{2^2\cdot 4^2\cdot 6^2\ldots\ldots(2n)^2\,(2n+2)^2}$$

$$=\frac{(2n+3)^2}{(2n+2)^2}=\frac{\left(1+\frac{3}{2n}\right)^2}{\left(1+\frac{2}{2n}\right)^2}=\frac{\left(1+\frac{3}{2n}\right)^2}{\left(1+\frac{1}{n}\right)^2}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\frac{\left(1+\frac{3}{2n}\right)^2}{\left(1+\frac{1}{n}\right)^2}=1$$

$\therefore$ D' Alemberts' Ratio Test cannot be applied

Apply Gauss Test

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{3}{2n}\right)^2\left(1+\frac{1}{n}\right)^{-2} = \left(1+\frac{3}{n}+\frac{9}{4n^2}\right)\left[1-\frac{2}{n}+0\left(\frac{1}{n^2}\right)\right]$$

$$= 1-\frac{2}{n}+\frac{3}{n}+0\left(\frac{1}{n^2}\right) = 1+\frac{1}{n}+0\left(\frac{1}{n^2}\right)$$

Compare it with $\dfrac{u_n}{u_{n+1}} = 1+\dfrac{\lambda}{n}+0\left(\dfrac{1}{n^2}\right)$

We have $\lambda = 1$

$\therefore$ By Gauss test given series is divergent.

Example 3. *Discuss the convergence of the series:*

$$x + \frac{2^2 x^2}{2\,!} + \frac{3^3 x^3}{3\,!} + \frac{4^4 x^4}{4\,!} + \cdots \infty .$$ **(P.T.U., May 2007, 2008, Dec. 2011)**

Sol. Here $\qquad u_n = \dfrac{n^n x^n}{n\,!}\,;\; u_{n+1} = \dfrac{(n+1)^{n+1}\, x^{n+1}}{(n+1)\,!}$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n\,!}\cdot\frac{(n+1)\,!}{(n+1)^{n+1}\, x^{n+1}} = \frac{n^n}{(n+1)^n}\cdot\frac{1}{x} = \frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x}$$

$$\operatorname*{Lt}_{n\to\infty}\frac{u_n}{u_{n+1}} = \operatorname*{Lt}_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x} = \frac{1}{e}\cdot\frac{1}{x} \qquad \because \operatorname*{Lt}_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$\therefore$ By Ratio test $\Sigma\, u_n$ converges if $\dfrac{1}{ex} > 1$ or $x < \dfrac{1}{e}$ and diverges for $x > \dfrac{1}{e}$

for $x = \dfrac{1}{e}$ $\qquad \dfrac{u_n}{u_{n+1}} = \dfrac{1}{\left(1+\frac{1}{n}\right)^n}\cdot e.$

As $\dfrac{u_n}{u_{n+1}}$ involves e

$\therefore$ Apply logarithmic test

$$n\log\frac{u_n}{u_{n+1}} = n.\log\frac{e}{\left(1+\frac{1}{n}\right)^n} = n\left\{\log e - \log\left(1+\frac{1}{n}\right)^n\right\} = n\left\{1 - n\log\left(1+\frac{1}{n}\right)\right\}$$

$$= n - n^2\left\{\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\ldots\infty\right\}$$

$$\left[\because \log(1+x) = x - \frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\infty\right]$$

$$= n - n + \frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} \ \ldots \infty$$

$$= \frac{1}{2} - \frac{1}{3n} + \frac{1}{4n} - \frac{1}{5n^3} + \ldots \infty$$

$$\underset{n \to \infty}{\text{Lt}} \ n \frac{u_n}{u_{n+1}} = \frac{1}{2} < 1$$

$\therefore \quad \Sigma\, u_n$ diverges for $x = \dfrac{1}{e}$.

Hence $\Sigma\, u_n$ converges for $x < \dfrac{1}{e}$ and diverges for $x \geq \dfrac{1}{e}$.

Example 4. *Discuss the convergence of the series :*

$$\frac{1}{2} x + x^2 + \frac{9}{8} x^3 + x^4 + \frac{25}{32} x^5 + \cdots\cdots\cdots \infty$$ **(P.T.U., May 2012)**

Sol. Given series is $\dfrac{1}{2} x + x^2 + \dfrac{9}{8} x^3 + x^4 + \dfrac{25}{32} x^5 + \cdots \infty$

or $$\frac{1^2}{2^1} x + \frac{2^2}{2^2} x^2 + \frac{3^2}{2^3} x^3 + \frac{4^2}{2^4} x^4 + \frac{5^2}{2^5} x^5 + \ldots\ldots. \infty$$

$$\therefore \qquad u_n = \frac{n^2}{2^n} x^n$$

$$u_{n+1} = \frac{(n+1)^2}{2^{n+1}} x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^2}{2^n} \cdot \frac{2^{n+1}}{(n+1)^2} \cdot \frac{1}{x} = 2 \cdot \left(\frac{n}{n+1}\right)^2 \cdot \frac{1}{x} = \frac{1}{\left(1 + \frac{1}{n}\right)^2} \cdot \frac{2}{x}$$

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{2}{x}$$

$\therefore$ By Ratio test Σu_n converges for $\dfrac{2}{x} > 1$ *i.e.*, for $x < 2$

and diverges for $x > 2$

For $x = 2$, Ratio test fails

$\therefore$ For $x = 2$, $\dfrac{u_n}{u_{n+1}} = \left(\dfrac{n}{n+1}\right)^2$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{n^2}{(n+1)^2} - 1 = \frac{n^2 - (n+1)^2}{(n+1)^2}$$

$$= \frac{(2n+1)(-1)}{(n+1)^2}$$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = -\frac{n(2n+1)}{(n+1)^2}$$

$$\underset{n\to\infty}{\text{Lt}}\ n\left[\frac{u_n}{u_{n+1}}-1\right] = -\underset{n\to\infty}{\text{Lt}}\ \frac{2+\frac{1}{n}}{\left(1+\frac{1}{n}\right)^2} = -2 < 1$$

$\therefore$ By Raabe's test series diverges

Hence Σu_n converges for $x < 2$ and diverges for $x \geq 2$.

Example 5. *Test for convergence the following series :*

$$1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\frac{5^4x^4}{5!}+\ldots\ldots+\infty$$ **(P.T.U., May 2005, 2006)**

Sol. Neglecting first term, we have

$$u_n = \frac{(n+1)^n}{(n+1)!}\ x^n$$

$$u_{n+1} = \frac{(n+2)^{n+1}}{(n+2)!}\ x^{n+1} \quad \therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)^n}{(n+1)!}x^n\ \frac{(n+2)!}{(n+2)^{n+1}\,.\,x^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^n}{x}\cdot\frac{(n+2)}{(n+2)^{n+1}} = \frac{(n+1)^n}{(n+2)^n}\cdot\frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}\cdot\frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\ \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}\cdot\frac{1}{x} = \underset{n\to\infty}{\text{Lt}}\ \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^{\frac{n}{2}\cdot 2}}\cdot\frac{1}{x}$$

$$= \frac{e}{e^2}\cdot\frac{1}{x} = \frac{1}{ex} \qquad \because \quad \underset{n\to\infty}{\text{Lt}}\left(1+\frac{a}{n}\right)^n = e^a$$

$\therefore$ By D'Alembert's Ratio test the given series is convergent if $\frac{1}{ex} > 1$ *i.e.*, $x < \frac{1}{e}$ and divergent if $x > \frac{1}{e}$

When $x = \frac{1}{e}$, Ratio test fails

then
$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}\ .\ e$$

Since $\frac{u_n}{u_{n+1}}$ involves e $\quad\therefore\quad$ Apply logarithm test

$$\log\frac{u_n}{u_{n+1}} = n\log\left(1+\frac{1}{n}\right) - n\log\left(1+\frac{2}{n}\right) + \log e$$

$$= n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + ...\right] - n\left[\frac{2}{n} - \frac{1}{2}\frac{4}{n^2} + \frac{1}{3}\cdot\frac{8}{n^3} ...... + 1\right] + 1$$

$$= \left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + ...\infty\right] + \left[-2 + \frac{2}{n} - \frac{8}{3n^2} + \frac{4}{n^3} - ......\infty\right] + 1$$

$$= \frac{3}{2n} - \frac{7}{3n^2} + \frac{15}{4n^3} - ......\infty$$

$$n \log \frac{u_n}{u_{n+1}} = \frac{3}{2} - \frac{7}{3n} + \frac{15}{4n^2} - ......\infty$$

$$\underset{n\to\infty}{\text{Lt}}\; n \log \frac{u_n}{u_{n+1}} = \frac{3}{2} > 1$$

$\therefore$ Series is convergent

$\therefore$ By logarithm test the given series is convergent when $x \le \frac{1}{e}$ and divergent for $x > \frac{1}{e}$.

Example 6. *Test for convergence the series:* $\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \ldots\infty.$

Sol.
$$u_n = \frac{(a+nx)^n}{n!}\,;\; u_{n+1} = \frac{\left[a+(n+1)^x\right]^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{(a+nx)^n}{n!}\cdot\frac{(n+1)!}{[a+(n+1)x]^{n+1}} = (n+1)\frac{(nx)^n\left(1+\frac{a}{nx}\right)^n}{[(n+1)x]^{n+1}\left[1+\frac{a}{(n+1)x}\right]^{n+1}}$$

$$= \frac{n^n}{(n+1)^n}\cdot\frac{1}{x}\,\frac{\left(1+\frac{a}{nx}\right)^{\frac{nx}{a}\cdot\frac{a}{x}}}{\left[1+\frac{a}{(n+1)\,x}\right]^{\frac{(n+1)x}{a}\cdot\frac{a}{x}}}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}\,\frac{\left[\left(1+\frac{a}{nx}\right)^{\frac{nx}{a}}\right]^{a/x}}{\left[\left(1+\frac{a}{(n+1)\,x}\right)^{\frac{(n+1)\,x}{a}}\right]^{a/x}}\cdot\frac{1}{x}$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}} = \frac{1}{e}\,\frac{e^{a/x}}{e^{a/x}}\cdot\frac{1}{x} = \frac{1}{ex} \qquad \because \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n = e.$$

$\therefore$ By Ratio test $\Sigma\, u_n$ converges for $\dfrac{1}{ex} > 1$, *i.e.,* $x < \dfrac{1}{e}$ and diverges for $x > \dfrac{1}{e}$, where $x = \dfrac{1}{e}$;

(as $\dfrac{u_n}{u_{n+1}}$ involves e $\therefore$ We apply logarithmic test)

$$\therefore \qquad n \log \frac{u_n}{u_{n+1}} = n \log \left\{ \frac{1}{\left(1+\frac{1}{n}\right)^n} \frac{\left(1+\frac{ae}{n}\right)^{\frac{n}{ae}ae}}{\left(1+\frac{ae}{n+1}\right)^{\frac{n+1}{ae}.ae}} . e \right\}$$

$$= n \log \left\{ \frac{1}{\left(1+\frac{1}{n}\right)^n} \frac{\left(1+\frac{ae}{n}\right)^{n}}{\left(1+\frac{ae}{n+1}\right)^{n+1}} . e \right\}$$

$$= n \left\{ \log \left(1+\frac{1}{n}\right)^{-n} + \log \left(1+\frac{ae}{n}\right)^{n} - \log \left(1+\frac{ae}{n+1}\right)^{(n+1)} + \log e \right\}$$

$$= n \left\{ -n \log \left(1+\frac{1}{n}\right) + n \log \left(1+\frac{ae}{n}\right) - (n+1) \log \left(1+\frac{ae}{n+1}\right) + 1 \right\}$$

$$= n \left\{ -n \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots \infty\right) + n \left(\frac{ae}{n} - \frac{a^2e^2}{2n^2} + \frac{a^3e^3}{3n^3} \ldots\right) \right.$$

$$\left. - (n+1) \left(\frac{ae}{n+1} - \frac{a^2e^2}{2(n+1)^2} + \frac{a^3e^3}{3(n+1)^3} \ldots \infty\right) \right\}$$

$$= \left(-n + \frac{1}{2} - \frac{1}{3n} + \ldots \infty\right) + \left(aen - \frac{a^2e^2}{2} + \frac{a^3e^3}{3n} \ldots\right)$$

$$- nae + \frac{na^2e^2}{2(n+1)} - \frac{n\,a^3e^3}{3\,(n+1)^2} + \ldots + n$$

$$\operatorname*{Lt}_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \frac{1}{2} - \frac{a^2e^2}{2} + \frac{a^2e^2}{2} = \frac{1}{2} < 1 \quad \therefore \quad \Sigma\, u_n \text{ is divergent at } x = \frac{1}{e}$$

$\therefore$ $\Sigma\, u_n$ converges for $x < \dfrac{1}{e}$ and diverges for $x \geq \dfrac{1}{e}$.

Example 7. *Discuss the convergence of the series :* $1 + \dfrac{x}{2} + \dfrac{2\,!}{3^2} x^2 + \dfrac{3\,!}{4^3} x^3 + \dfrac{4\,!}{5^4} x^4 + \ldots\ldots$

Sol. Neglecting the first term, we have

$$u_n = \frac{n\,!}{(n+1)^n}\; x^n \quad \text{and} \quad u_{n+1} = \frac{(n+1)\,!}{(n+2)^{n+1}} \cdot x^{n+1}$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n\,!}{(n+1)^n} \cdot \frac{(n+2)^{n+1}}{(n+1)\,!} \cdot \frac{1}{x}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{1}{n^n\left(1+\frac{1}{n}\right)^n} \cdot \frac{n^{n+1}\left(1+\frac{2}{n}\right)^{n+1}}{(n+1)} \cdot \frac{1}{x}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{2}{n}\right)^n\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)} \cdot \frac{1}{x} = \frac{e^2}{e} \cdot \frac{1}{x} = \frac{e}{x}$$

$$\left[\because \; \underset{n\to\infty}{\text{Lt}}\left(1+\frac{a}{n}\right)^n = \underset{n\to\infty}{\text{Lt}}\left[\left(1+\frac{a}{n}\right)^{n/a}\right]^a = e^a\right]$$

$\therefore$ By D'Alembert's ratio test, the series converges if $\frac{e}{x} > 1$ or if $x < e$ and diverges if $\frac{e}{x} < 1$ or if $x > e$.

If $x = e$, the ratio test fails, $\because \; \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1$.

Now, when $\qquad x = e$

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}} \cdot \frac{1}{e}\,.$$

Since the expression $\frac{u_n}{u_{n+1}}$ involves the number e, so we do not apply Raabe's test but apply logarithmic test.

$$\therefore \qquad \log \frac{u_n}{u_{n+1}} = (n+1)\log\left(1+\frac{2}{n}\right) - (n+1)\log\left(1+\frac{1}{n}\right) - \log e$$

$$= (n+1)\left[\log\left(1+\frac{2}{n}\right) - \log\left(1+\frac{1}{n}\right)\right] - 1$$

$$= (n+1)\left[\left(\frac{2}{n} - \frac{1}{2}\cdot\frac{4}{n^2} + \frac{1}{3}\cdot\frac{8}{n^3} - \ldots\ldots\right) - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\ldots\right)\right] - 1$$

$$= (n+1)\left[\frac{1}{n} - \frac{3}{2n^2} + \ldots\ldots\right] - 1$$

$$= 1 - \frac{3}{2n} + \frac{1}{n} - \frac{3}{2n^2} + \ldots\ldots - 1 = -\frac{1}{2n} - \frac{3}{2n^2} + \ldots\ldots$$

$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\; n \log \dfrac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\; n\left[-\dfrac{1}{2n} - \dfrac{3}{2n^2} + ...\right] = \underset{n\to\infty}{\text{Lt}}\left(-\dfrac{1}{2} - \dfrac{3}{2n} + ...\right) = -\tfrac{1}{2} < 1$

$\therefore$ By log test, the series *diverges*.

Hence the given series Σu_n converges if $x < e$ and diverges if $x \geq e$.

Example 8. *Discuss the convergence of the series:*

$$\frac{a}{b} + \frac{a(a+d)}{b(b+d)}x + \frac{a(a+d)(a+2d)}{b(b+d)(b+2d)}x^2 + \cdots \infty \qquad (a > 0,\ b > 0,\ x > 0)$$

Sol.
$$u_n = \frac{a\,(a+d)\,(a+2d)\ldots(a+\overline{n-1}\,d)}{b\,(b+d)\,(b+2d)\ldots(b+\overline{n-1}\,d)}\; x^n$$

$$u_{n+1} = \frac{a\,(a+b)\,(a+2d)\ldots(a+\overline{n-1}\,d)\,(a+nd)}{b\,(b+d)\,(b+2d)\ldots(b+\overline{n-1}\,d)\,(b+nd)}\; x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{b+nd}{a+nd}\cdot\frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\frac{nd\left(1+\dfrac{b}{nd}\right)}{nd\left(1+\dfrac{a}{nd}\right)}\cdot\frac{1}{x} = \underset{n\to\infty}{\text{Lt}}\frac{1+\dfrac{b}{nd}}{1+\dfrac{a}{nd}}\cdot\frac{1}{x} = \frac{1}{x}.$$

By Ratio test $\Sigma\, u_n$ converges if $\dfrac{1}{x} > 1$ *i.e.,* $x < 1$ and diverges if $x > 1$.

when $x = 1$, Ratio test fails

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{1+\dfrac{b}{nd}}{1+\dfrac{a}{nd}} = \left(1+\frac{b}{nd}\right)\left(1+\frac{a}{nd}\right)^{-1} = \left(1+\frac{b}{nd}\right)\left(1-\frac{a}{nd}+\ldots\right)$$

$$= 1 + \frac{b-a}{d}\cdot\frac{1}{n} + 0\left(\frac{1}{n^2}\right)$$

By Gauss test, $\Sigma\, u_n$ converges if $\dfrac{b-a}{d} > 1$ *i.e.,* $b > a + d$ and diverges if $b \leq a + d$.

$\therefore$ $\Sigma\, u_n$ converges if $x > 1$ and diverges when $x < 1$ when $x = 1$ then Σu_n converges if $b > a + d$ and diverges if $b \leq a + d$.

Example 9. *Discuss the convergence of the series :*

$$1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + ...$$

(P.T.U., Dec. 2003)

Sol. Neglecting the first term,

$$u_n = \frac{\alpha(\alpha+1)\,......\,(\alpha+n-1)\,\beta(\beta+1)\,......\,(\beta+n-1)}{1\cdot 2\cdot 3\,......\,n\,.\,\gamma(\gamma+1)\,......\,(\gamma+n-1)}\cdot x^n$$

$$u_{n+1} = \frac{\alpha(\alpha+1)\,......\,(\alpha+n-1)\,(\alpha+n)\,\beta(\beta+1)\,......\,(\beta+n-1)(\beta+n)}{1\cdot 2\cdot 3\,......\,n(n+1)\,\gamma(\gamma+1)(\gamma+2)\,......\,(\gamma+n-1)(\gamma+n)}\cdot x^{n+1}$$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} \cdot \frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)} \cdot \frac{1}{x}$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ By D'Alembert's Ratio test the series Σu_n converges if $\frac{1}{x} > 1$

i.e., if $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.,* if $x > 1$.

If $\qquad x = 1, \ \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1 \quad \therefore$ Ratio test fails.

Putting $\qquad x = 1$ in $\frac{u_n}{u_{n+1}}$, we have $\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1+\frac{\alpha}{n}\right)^{-1}\left(1+\frac{\beta}{n}\right)^{-1} \qquad \text{[Expand by Binomial Theorem]}$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1-\frac{\alpha}{n}+\frac{\alpha^2}{n^2}+\ldots\ldots\right)\left(1-\frac{\beta}{n}+\frac{\beta^2}{n^2}\ldots\ldots\right)$$

$$= \left(1+\frac{1}{n}+\frac{\gamma}{n}+\frac{\gamma}{n^2}\right)\left(1-\frac{\alpha}{n}-\frac{\beta}{n}+\frac{\alpha\beta}{n^2}+\frac{\alpha^2}{n^2}+\frac{\beta^2}{n^2}\ldots\ldots\right)$$

$$= 1 + \frac{1}{n}(1+\gamma-\alpha-\beta) + O\left(\frac{1}{n^2}\right).$$

$\therefore$ By Gauss test, the series Σu_n converges if $1 + \gamma - \alpha - \beta > 1$ *i.e.,* if $\gamma > \alpha + \beta$ and diverges if $1+\gamma-\alpha-\beta \le 1$ *i.e.,* if $\gamma \le \alpha+\beta$.

Thus the given series *converges if $x < 1$ and diverges if $x > 1$. If $x = 1$, then the series converges if* $\gamma > \alpha+\beta$ *and diverges if* $\gamma \le \alpha+\beta$.

Example 10. *Discuss the convergence of the series* $\sum \frac{n!}{x(x+1)(x+2)\ldots(x+n-1)}$.

Sol.
$$u_n = \frac{n!}{x(x+1)(x+2)\ldots(x+n-1)}$$

$$u_{n+1} = \frac{(n+1)!}{x(x+1)(x+2)\ldots(x+n-1)(x+n)}$$

$$\frac{u_n}{u_{n+1}} = \frac{x+n}{n+1} = \frac{1+\frac{x}{n}}{1+\frac{1}{n}}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\ \frac{1+x/n}{1+\frac{1}{n}} = 1$$

$\therefore$ Ratio test fails. We apply Gauss test

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \left(1+\frac{x}{n}\right)\left(1+\frac{1}{n}\right)^{-1} = \left(1+\frac{x}{n}\right)\left(1-\frac{1}{n}+\ldots\right)$$

$$= 1 + \frac{x-1}{n} + \text{O}\left(\frac{1}{n^2}\right) = 1 + (x-1)\,.\,\frac{1}{n} + \text{O}\left(\frac{1}{n^2}\right).$$

By Gauss test $\Sigma\, u_n$ converges if $x - 1 > 1$ and diverges for $x - 1 \le 1$ *i.e.,* $\Sigma\, u_n$ converges for $x > 2$ and diverges for $x \le 2$.

Example 11. *Verify the series* $\sum \dfrac{4\cdot 7\cdot 10\ldots\ldots(3n+1)}{1\cdot 2\cdot 3\ldots\ldots n}\, x^n$ *is convergent or divergent.*

(P.T.U., Dec. 2006, May 2006)

Sol. Here
$$u_n = \frac{4\cdot 7\cdot 10\ldots\ldots(3n+1)}{1\cdot 2\cdot 3\ldots\ldots n}\, x^n$$

$$u_{n+1} = \frac{4\cdot 7\cdot 10\ldots\ldots(3n+1)\,(3n+4)}{1\cdot 2\cdot 3\ldots\ldots n\,.\,(n+1)}\, x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{n+1}{3n+4}\cdot\frac{1}{x} = \frac{1+\frac{1}{n}}{3+\frac{4}{n}}\cdot\frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\ \frac{1+\frac{1}{n}}{3+\frac{4}{n}}\cdot\frac{1}{x} = \frac{1}{3x}$$

$\therefore$ By D'Alembert's ratio test, series converges if $\dfrac{1}{3x} > 1$

i.e. if $x < \dfrac{1}{3}$ and diverges for $x > \dfrac{1}{3}$.

When $x = \dfrac{1}{3}$ Ratio test fails, so apply Gauss test

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)}{3+\frac{4}{n}}\cdot 3 = \frac{1+\frac{1}{n}}{3\left(1+\frac{4}{3n}\right)}\cdot 3 = \left(1+\frac{1}{n}\right)\left(1+\frac{4}{3n}\right)^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{4}{3n}+\text{O}\left(\frac{1}{n^2}\right)\right) = 1-\frac{1}{3n}+\text{O}\left(\frac{1}{n^2}\right)$$

Compare it with $$\frac{u_n}{u_{n+1}} = 1 + \lambda \cdot \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

$$\lambda = -\frac{1}{3} < 1 \quad \therefore \quad \text{Series is divergent.}$$

Hence the given series is convergent for $x < \frac{1}{3}$ and divergent for $x \geq \frac{1}{3}$.

Example 12. *Verify the series:* $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots \infty\ (x > 0)$.

Sol. $$u_n = \left(\frac{n+1}{n+2}\right)^n x^n \text{ (neglecting 1st term)}$$

$$u_{n+1} = \left(\frac{n+2}{n+3}\right)^{n+1} x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^n}{(n+2)^n} \cdot \frac{(n+3)^{n+1}}{(n+2)^{n+1}} \cdot \frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} \cdot \frac{(n+3)^n}{(n+2)^n} \cdot \frac{n+3}{n+2} \cdot \frac{1}{x}$$

$$= \frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2} \cdot \frac{\left[\left(1+\frac{3}{n}\right)^{\frac{n}{3}}\right]^3}{\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2} \cdot \frac{1+\frac{3}{n}}{1+\frac{2}{n}} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{e}{e^2} \cdot \frac{e^3}{e^2} \cdot \frac{1}{x} \qquad \because \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e$$

$$= \frac{1}{x}.$$

By Ratio test Σu_n converges when $\frac{1}{x} > 1$ *i.e.* if $x < 1$ and diverges if $x > 1$

When $x = 1$, Ratio test fails

$\therefore$ For $x = 1$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \frac{(n+1)^n}{(n+2)^n} = \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2}$$

$$= \frac{e}{e^2} = \frac{1}{e} \neq 0.$$

$\therefore$ for $x = 1$, as $\Sigma\, u_n$ is a +ve term series and $\underset{n \to \infty}{\text{Lt}}\; u_n \neq 0$

$\therefore$ $\Sigma\, u_n$ diverges.

$\therefore$ $\Sigma\, u_n$ converges for $x < 1$ and diverges for $x \geq 1$.

Note. This question can be done by applying Cauchy's root test see Example 3 art. 5.26.

Example 13. *Discuss for what values of x does the series* $\sum_{x=1}^{\infty} \frac{(n!)^2}{(2n)!} x^{2n}$ *converge/diverge ?*

(P.T.U., May 2002, 2003)

Sol. Here $u_n = \frac{(n!)^2}{2n!} x^{2n}$, $u_{n+1} = \frac{((n+1)!)^2}{(2n+2)!} x^{2n+2}$

$$\frac{u_n}{u_{n+1}} = \frac{(n!)^2}{(2n)!} . x^{2n} . \frac{(2n+2)!}{((n+1)!)^2} \frac{1}{x^{2n+2}} = \frac{1}{(n+1)^2} (2n+2)(2n+1) \frac{1}{x^2}$$

$$= \frac{2(2n+1)}{n+1} . \frac{1}{x^2} = \frac{4\left(1 + \frac{1}{2n}\right)}{1 + \frac{1}{n}} . \frac{1}{x^2}$$

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{4}{x^2} .$$

$\therefore$ By D' Alembert's ratio test the series $\Sigma\, u_n$ converges if $\frac{4}{x^2} > 1$ or $x^2 < 4$ or $|x| < 2$ or $-2 < x < 2$ and diverges for $x^2 > 4$ *i.e.*, for either $x > 2$ or $x < -2$.

When $x^2 = 4$ Ratio test fails $\therefore$ Apply Gauss test

$$\frac{u_n}{u_{n+1}} = 4\left(1 + \frac{1}{2n}\right)\left(1 + \frac{1}{n}\right)^{-1} . \frac{1}{4} = \left(1 + \frac{1}{2n}\right)\left(1 - \frac{1}{n} + 0\left(\frac{1}{n^2}\right)\right)$$

$$= 1 - \frac{1}{2n} + 0\left(\frac{1}{n^2}\right)$$

Compare it with $\frac{u_n}{u_{n+1}} = 1 + \lambda . \frac{1}{n} + 0. \left(\frac{1}{n^2}\right)$

$\therefore$ $\lambda = -\frac{1}{2} < 1$ $\therefore$ Given series is divergent for $x^2 = 4$ *i.e.*, $x = \pm 2$

Hence given series converges for $x^2 < 4$ *i.e.*, $-2 < x < 2$ and diverges for $x^2 \geq 4$ *i.e.*, for either $x \geq 2$ or $x \leq -2$.

Example 14. *Test the convergence of the series* $x^2 (\log 2)^q + x^3 (\log 3)^q + x^4 (\log 4)^q + \ldots \infty$.

Sol. $u_n = x^{n+1} [\log (n+1)]^q$

$u_{n+1} = x^{n+2} [\log (n+2)]^q$

$$\frac{u_n}{u_{n+1}} = \left[\frac{\log(n+1)}{\log(n+2)}\right]^q \cdot \frac{1}{x} = \left[\frac{\log(n+1)}{\log(1+n+1)}\right]^q \cdot \frac{1}{x}$$

$$= \left[\frac{\log(n+1)}{\log(n+1)\left[1+\frac{1}{n+1}\right]}\right]^q \cdot \frac{1}{x} = \left[\frac{\log(n+1)\left(1+\frac{1}{n+1}\right)}{\log(n+1)}\right]^{-q} \cdot \frac{1}{x}$$

$$= \left[\frac{\log(n+1)+\log\left[1+\frac{1}{n+1}\right]}{\log(n+1)}\right]^{-q} \cdot \frac{1}{x}$$

$$= \left[1+\frac{1}{\log(n+1)}\left(\frac{1}{n+1}-\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}\ldots\infty\right)\right]^{-q} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

By ratio test Σu_n converges if $\frac{1}{x} > 1$ or $x < 1$ diverges if $x > 1$

When $x = 1$; $u_n = [\log(n+1)]^q$

$$\underset{n\to\infty}{\text{Lt}} u_n = \infty \neq 0$$

Σu_n is a +ve term series and $\underset{n\to\infty}{\text{Lt}} u_n \neq 0$

$\therefore$ Σu_n is divergent for $x = 1$

$\therefore$ Σu_n is convergent for $x < 1$

and divergent for $x \geq 1$.

TEST YOUR KNOWLEDGE

Discuss the convergence of the following series :

1. $\frac{1}{1^2}+\frac{1+2}{1^2+2^2}+\frac{1+2+3}{1^2+2^2+3^2}+ \ldots\ldots$ to ∞

2. $\frac{1^2}{4^2}+\frac{1^2\cdot 5^2}{4^2\cdot 8^2}+\frac{1^2\cdot 5^2\cdot 9^2}{4^2\cdot 8^2\cdot 12^2}+ \ldots\ldots$ to ∞

3. $1+\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\ldots\ldots$ to ∞ **(P.T.U., Dec. 2010)**

4. $1+\frac{2}{1}\cdot\frac{1}{2}+\frac{2\cdot 4}{1\cdot 3}\cdot\frac{1}{3}+\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\cdot\frac{1}{4}+\ldots\ldots$ to ∞

5. $\frac{1^2}{2^2}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}x+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}x^2+\ldots\ldots$ to ∞

6. $1^p+\left(\frac{1}{2}\right)^p+\left(\frac{1\cdot 3}{2\cdot 4}\right)^p+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^p+\ldots\ldots$ to ∞

7. (*a*) $\frac{a}{b}+\frac{a(a+1)}{b(b+1)}+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\ldots\ldots$ to ∞

(*b*) $\frac{\alpha}{\beta}+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)}+\ldots\ldots$ to ∞ **(P.T.U., May 2011)**

8. $1+\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\ldots\ldots$ to ∞ $(x > 0)$

9. $\frac{x}{2}+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2.4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\cdots\infty$ $(x > 0)$

10. $\dfrac{x}{1.2}+\dfrac{x^2}{3.4}+\dfrac{x^3}{5.6}+\dfrac{x^4}{7.8}+ \ldots\ldots$ to $\infty \quad (x>0)$

11. $\dfrac{x}{1.2}+\dfrac{x^2}{2.3}+\dfrac{x^3}{3.4}+\dfrac{x^4}{4.5}+ \ldots\ldots$ to ∞

12. $\sum \dfrac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)}\cdot\dfrac{x^{2n}}{2n}$

13. $\sum \left[\dfrac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots 2n}\right]^p x^n$

14. $1+\dfrac{(1\,!)^2}{2\,!}x^2+\dfrac{(2\,!)^2}{4\,!}x^4+\dfrac{(3\,!)^2}{6\,!}$, then $x^6+\ldots..$ to ∞ $(x>0)$

[**Hint:** Neglect 1st term and see solved example 13]

ANSWERS

1. Divergent **2.** Convergent **3.** Convergent for $x \le 1$, divergent for $x > 1$

4. Divergent **5.** Convergent for $x < 1$, divergent for $x \ge 1$.

6. Convergent for $p > 2$, divergent for $p \le 2$

7. (*a*) Convergent for $b > a + 1$, divergent for $b \le a + 1$

(*b*) Convergent for $\beta > \alpha + 1$, divergent for $\beta \le \alpha + 1$

8. Convergent for $x < 1$, divergent for $x \ge 1$ **9.** Converges for $x^2 \le 1$, diverges for $x^2 > 1$

10. Convergent for $x \le 1$, divergent for $x > 1$ **11.** Convergent for $x \le 1$, divergent for $x > 1$

12. Convergent for $x^2 \le 1$, divergent for $x^2 > 1$ **13.** Convergent for $x \le 1$, $p > 2$, divergent for $x > 1$, $p \le 2$

14. Convergent for $x^2 < 4$, divergent for $x^2 \ge 4$.

5.26. CAUCHY'S ROOT TEST

Statement. *If Σu_n is a positive term series and $\underset{n\to\infty}{Lt}\,(u_n)^{1/n} = l$, then*

(*i*) *Σu_n is convergent if $l < 1$* (*ii*) *Σu_n is divergent if $l > 1$.*

Note. If $l = 1$, the test fails *i.e.,* no conclusion can be drawn about the convergence or divergence of the series. The series may converge, it may diverge.

Proof. Since $\underset{n\to\infty}{\text{Lt}}\,(u_n)^{1/n} = l$,

$\therefore$ Given $\varepsilon > 0$, however small, there exists a +ve integer m such that

$$|(u_n)^{1/n} - l| < \varepsilon \qquad \forall \quad n \ge m$$

$$\Rightarrow \quad l - \varepsilon < (u_n)^{1/n} < l + \varepsilon \qquad \forall \quad n \ge m$$

$$\Rightarrow \quad (l-\varepsilon)^n < u_n < (l+\varepsilon)^n \qquad \forall \quad n \ge m \qquad \ldots(1)$$

(*i*) **When $l < 1$**

Choose $\varepsilon > 0$ such that $\quad l < l + \varepsilon < 1$

Put $\quad l + \varepsilon = r,\quad$ then $\; 0 < r < 1$

From (1), $\quad u_n < r^n \qquad \forall \quad n \ge m$

Putting $n = m, m+1, m+2, \ldots\ldots$, we get $u_m < r^m$, $u_{m+1} < r^{m+1}$, $u_{m+2} < r^{m+2}$, ... and so on.

Adding $\quad u_m + u_{m+1} + u_{m+2} + \ldots\ldots + < r^m + r^{m+1} + r^{m+2} + \ldots\ldots$

$\Rightarrow$ each term of the given series Σu_n after leaving the first $(m - 1)$ terms, (*i.e.*, a finite number of terms) is less than the corresponding term of a geometric series which is convergent ($\because$ its common ratio $r < 1$). Hence the given series is also convergent.

(*ii*) **When $l > 1$**

Choose $\varepsilon > 0$ such that $\quad l - \varepsilon > 1$

Put $\quad l - \varepsilon = \text{R}$, then $\text{R} > 1$

From (1), $\quad u_n > \text{R}^n \quad \forall \quad n \geq m$

Putting $n = m, m+1, m+2$, ..., we get $u_m > \text{R}^m$, $u_{m+1} > \text{R}^{m+1}$, $u_{m+2} > \text{R}^{m+2}$, ... and so on.

Adding $\quad u_m + u_{m+1} + u_{m+2} + \ldots\ldots > \text{R}^m + \text{R}^{m+1} + \text{R}^{m+2} + \ldots\ldots$

$\Rightarrow$ each term of the series Σu_n after leaving the first $(m-1)$ terms, (*i.e.*, a finite number of terms) is greater than the corresponding term of a geometric series which is divergent. ($\because$ its common ratio $\text{R} > 1$). Hence the given series is also divergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Test the convergence of the following series:*

(*i*) $\displaystyle\sum \left(\frac{n}{n+1}\right)^{n^2}$ *or* $\displaystyle\sum \left(1 + \frac{1}{n}\right)^{-n^2}$ (*ii*) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ **(P.T.U., Dec. 2002)**

(P.T.U., May 2009, Dec. 2012)

(*iii*) $\displaystyle\sum \frac{(n - \log_n)^n}{2^n\, n^n}$.

Sol. (*i*) Here
$$u_n = \left(\frac{n}{n+1}\right)^{n^2}$$

$$\therefore \quad (u_n)^{1/n} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \left[\left(1 + \frac{1}{n}\right)^n\right]^{-1}$$

$$\underset{n \to \infty}{\text{Lt}} (u_n)^{1/n} = \underset{n \to \infty}{\text{Lt}} \left[\left(1 + \frac{1}{n}\right)^n\right]^{-1} = e^{-1} = \frac{1}{e} < 1 \qquad (\because \quad e = 2.7)$$

$\therefore$ By Cauchy's Root Test, the given series Σu_n is convergent.

(*ii*)
$$u_n = \frac{1}{(\log n)^n}$$

$$(u_n)^{\frac{1}{n}} = \frac{1}{\log n}$$

$$\underset{n \to \infty}{\text{Lt}} (u_n)^{\frac{1}{n}} = \underset{n \to \infty}{\text{Lt}} \frac{1}{\log n} = \frac{1}{\log \infty} = \frac{1}{\infty} = 0 < 1$$

$\therefore$ By Cauchy's root test the given series is convergent

Hence $\displaystyle\Sigma\, u_n = \sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ is convergent.

(*iii*)
$$u_n = \frac{(n-\log n)^n}{2^n\ n^n}$$

$$(u_n)^{\frac{1}{n}} = \frac{n-\log n}{2n} = \frac{1}{2} - \frac{1}{2}\frac{\log n}{n}$$

$$\underset{n\to\infty}{\text{Lt}}\ (u_n)^{\frac{1}{n}} = \frac{1}{2} - \frac{1}{2}\underset{n\to\infty}{\text{Lt}}\ \frac{\log n}{n} = \frac{1}{2} - 0$$

$$\because \quad \underset{n\to\infty}{\text{Lt}}\ \frac{\log n}{n} = \underset{n\to\infty}{\text{Lt}}\ \frac{\frac{1}{n}}{0} = 0 \text{ (by L' Hospital rule)}$$

$$\underset{n\to\infty}{\text{Lt}}\ (u_n)^{\frac{1}{n}} = \frac{1}{2} < 1$$

$\therefore$ By Cauchy's root test $\Sigma\, u_n$ is convergent.

Example 2. *Examine the convergence of the series :*

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \ldots\ldots$$

or
$$\sum\left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-n}$$
(P.T.U., Dec. 2013)

Sol. Here
$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$$

$$(u_n)^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1} = \left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= \left[\left(1+\frac{1}{n}\right)^{n} . \left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$\underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\left[\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= (e.1-1)^{-1} = \frac{1}{e-1} < 1 \qquad (\because \quad e = 2.7)$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent.

Example 3. *Discuss the convergence of the following series:*

(*i*) $\dfrac{1}{2} + \dfrac{2}{3}x + \left(\dfrac{3}{4}\right)^2 x^2 + \left(\dfrac{4}{5}\right)^3 x^3 + \ldots\infty$ (*ii*) $\sum\left(1+\dfrac{1}{\sqrt{n}}\right)^{-n^{3/2}}$. **(P.T.U., May 2008)**

Sol. (*i*) Neglecting first term

$$u_n = \left(\frac{n+1}{n+2}\right)^n x^n\ ;\ (u_n)^{\frac{1}{n}} = \frac{n+1}{n+2}\ x.$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{\frac{1}{n}} = \underset{n\to\infty}{\text{Lt}} \frac{1+\frac{1}{n}}{1+\frac{2}{n}}\, x = x.$$

By Cauchy's root test $\Sigma\, u_n$ converges for $x < 1$ and diverges for $x > 1$

When $x = 1$, Cauchy's root test fail

Then $$u_n = \left(\frac{n+1}{n+2}\right)^n = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}$$

$$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{n/2}\right]^2} = \frac{e}{e^2} \qquad \left|\ \because \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e\right.$$

$$= \frac{1}{e} \neq 0$$

$\because$ $\Sigma\, u_n$ is +ve term series and $\underset{n\to\infty}{\text{Lt}}\ u_n \neq 0$

$\therefore$ for $x = 1$; $\Sigma\, u_n$ diverges

$\therefore$ $\Sigma\, u_n$ converges for $x < 1$ and diverges for $x \geq 1$.

(*ii*) $$u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}} ;\ (u_n)^{\frac{1}{n}} = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{1/2}} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{1/2}}}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}} (u_n)^{\frac{1}{n}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$$

$\therefore$ $\Sigma\, u_n$ converges.

TEST YOUR KNOWLEDGE

Discuss the convergence of the following series :

1. $\sum \frac{1}{n^n}$

2. $\sum_{n=2}^{\infty} \frac{n}{(\log n)^n}$

3. $\sum \left(\frac{n+1}{3n}\right)^n$

4. $\sum \left(\frac{nx}{n+1}\right)^n$

5. $\sum 5^{-n-(-1)^n}$

6. $\sum \frac{(1+nx)^n}{n^n}$

7. $\sum \frac{(n+1)^n\, x^n}{n^{n+1}}$.

ANSWERS

1. Convergent **2.** Convergent **3.** Convergent

4. Convergent for $x < 1$, divergent for $x \geq 1$ **5.** Convergent

6. Convergent for $x < 1$, divergent for $x \geq 1$ **7.** Convergent for $x < 1$, divergent for $x \geq 1$.

5.27. CAUCHY'S INTEGRAL TEST (P.T.U., May 2002, 2014, Dec. 2005, Jan. 2009)

Statement. *If for $x \geq 1$, $f(x)$ is a non-negative, monotonic decreasing function of x such that $f(n) = u_n$, for all positive integral values of n, then the series Σu_n and the integral $\int_1^\infty f(x)\, dx$ converge or diverge together.*

Proof. Let r be a +ve integer. Choose x such that $r + 1 \geq x \geq r \geq 1$

Since $f(x)$ is a monotonic decreasing function of x.

$$\therefore \quad f(r+1) \leq f(x) \leq f(r) \quad \Rightarrow \quad u_{r+1} \leq f(x) \leq u_r \qquad [\because \; f(n) = u_n, n \in \text{N}]$$

$$\Rightarrow \quad \int_r^{r+1} u_{r+1}\, dx \leq \int_r^{r+1} f(x)\, dx \leq \int_r^{r+1} u_r\, dx$$

$$\Rightarrow \quad u_{r+1} \int_r^{r+1} dx \leq \int_r^{r+1} f(x)\, dx \leq u_r \int_r^{r+1} dx$$

$$\Rightarrow \quad u_{r+1} \Big[x \Big]_r^{r+1} \leq \int_r^{r+1} f(x)\, dx \leq u_r \Big[x \Big]_r^{r+1}$$

$$\Rightarrow \quad u_{r+1} \leq \int_r^{r+1} f(x)\, dx \leq u_r \qquad ...(1)$$

Putting $r = 1, 2, 3, ..., n$ in succession in (1), we have

$$u_2 \leq \int_1^2 f(x)\, dx \leq u_1$$

$$u_3 \leq \int_2^3 f(x)\, dx \leq u_2$$

$$..................................$$

$$..................................$$

$$u_{n+1} \leq \int_n^{n+1} f(x)\, dx \leq u_n$$

Adding the above inequalities, we have

$$u_2 + u_3 + ... + u_{n+1} \leq \int_1^2 f(x)\, dx + \int_2^3 f(x)\, dx + ... + \int_n^{n+1} f(x)\, dx \leq u_1 + u_2 + ... + u_n$$

$$\Rightarrow \quad S_{n+1} - u_1 \leq \int_1^{n+1} f(x)\, dx \leq S_n \quad \text{where} \quad S_n = \sum_1^n u_n = u_1 + u_2 + ... + u_n.$$

Proceeding to the limit as $n \to \infty$

$$\underset{n\to\infty}{\text{Lt}}\; S_{n+1} - u_1 \leq \underset{n\to\infty}{\text{Lt}} \int_1^{n+1} f(x)\, dx \leq \underset{n\to\infty}{\text{Lt}}\; S_n$$

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\; S_{n+1} - u_1 \leq \int_1^{\infty} f(x)\, dx \leq \underset{n\to\infty}{\text{Lt}}\; S_n \qquad ...(2)$$

(*i*) **If** $\int_1^{\infty} f(x)\,dx$ **converges,** then $\int_1^{\infty} f(x)\,dx$ = a fixed finite number = I (say).

Then from (2), we have $\quad \underset{n\to\infty}{\text{Lt}}\ S_{n+1} - u_1 \le I$

$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\ S_{n+1} \le I + u_1$ = a fixed finite number

$\Rightarrow$ $\{S_n\}$ is a convergent sequence

$\Rightarrow$ **the series Σu_n is convergent.**

(*ii*) **If** $\int_1^{\infty} f(x)\,dx$ **diverges,** then $\int_1^{\infty} f(x)\,dx = +\infty$

From (2), $\quad \underset{n\to\infty}{\text{Lt}}\ S_n \ge \underset{n\to\infty}{\text{Lt}} \int_1^{\infty} f(x)\,dx = +\infty$

$\Rightarrow$ $\{S_n\}$ is a divergent sequence

$\Rightarrow$ the series Σu_n is divergent.

Hence Σu_n and $\int_1^{\infty} f(x)\,dx$ converge or diverge together.

Note. If $x \ge k$, then Σu_n and $\int_k^{\infty} f(x)\,dx$ converge or diverge together.

ILLUSTRATIVE EXAMPLES

Example 1. *Test for convergence the series :* $\sum \dfrac{1}{n^2+1}$.

Sol. Here $\quad u_n = \dfrac{1}{n^2+1} = f(n)$

$\therefore \quad f(x) = \dfrac{1}{x^2+1}$

For $x \ge 1$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ Cauchy's Integral Test is applicable.

Now, $$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{dx}{x^2+1} = \left[\tan^{-1} x\right]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} = \text{finite}$$

$\Rightarrow$ $\int_1^{\infty} f(x)\,dx$ converges and hence by Integral Test, Σu_n also converges.

Example 2. *Using integral test discuss the convergence of* $\sum_{n=2}^{\infty} \dfrac{1}{n\sqrt{n^2-1}}$.

Sol. $\quad u_n = \dfrac{1}{n\sqrt{n^2-1}} = f(n)$

$\therefore \quad f(x) = \dfrac{1}{x\sqrt{x^2-1}}$

for $x \ge 2$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ Cauchy's Integral test is applicable.

Now, $$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx \qquad ...(1)$$

Put $\sqrt{x^2-1} = t \quad \therefore \quad x^2 = t^2 + 1$; Differentiate $\dfrac{x}{\sqrt{x^2-1}}dx = dt$

$$\therefore \qquad \frac{dx}{x\sqrt{x^2-1}} = \frac{dt}{x^2} = \frac{dt}{t^2+1} \text{ ; when } x = 2,\ t = \sqrt{3} \text{ ; when } x = \infty,\ t = \infty$$

$\therefore$ From (1), $$\int_2^\infty f(x)\,dx = \int_{\sqrt{3}}^\infty \frac{1}{t}\cdot\frac{dt}{t^2+1} = \int_{\sqrt{3}}^\infty \frac{1}{t\,(t^2+1)}\,dt$$

By partial fraction, let

$$\frac{1}{t\,(t^2+1)} = \frac{A}{t} + \frac{Bt+C}{t^2+1}$$

$$\therefore \qquad 1 = A\,(t^2+1) + t\,(Bt+C)$$

Put $t = 0$ on both sides, we get $1 = A$

Comparing coefficients of t^2 and t on both sides; $0 = A + B \quad \therefore \quad B = -1,\ 0 = C$

$$\therefore \qquad \frac{1}{t\,(t^2+1)} = \frac{1}{t} - \frac{t}{t^2+1}$$

$$\therefore \qquad \int_2^\infty f(x)\,dx = \int_{\sqrt{3}}^\infty \left(\frac{1}{t} - \frac{t}{t^2+1}\right)dt = \log t - \frac{1}{2}\log(t^2+1)\Bigg|_{\sqrt{3}}^\infty$$

$$= \log \frac{t}{\sqrt{t^2+1}}\Bigg|_{\sqrt{3}}^\infty = \log \frac{1}{\sqrt{1+\dfrac{1}{t^2}}}\Bigg|_{\sqrt{3}}^\infty$$

$$= \log 1 - \log \frac{\sqrt{3}}{2} = -\log \frac{\sqrt{3}}{2} = \text{finite}$$

$\therefore \quad \displaystyle\int_2^\infty f(x)\,dx$ converges and hence $\Sigma\, u_n$ converges.

Example 3. *Show that the series* $\displaystyle\sum_1^\infty \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $0 < p \le 1$. **(P.T.U., Dec. 2011)**

Sol. Here $$u_n = \frac{1}{n^p} = f(n)$$

$$\therefore \qquad f(x) = \frac{1}{x^p}$$

For $x \ge 1$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ Cauchy's Integral Test is applicable.

Case I. *When* $p \ne 1$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^\infty$$

Sub-Case 1. *When $p > 1$, $p - 1$* is +ve, so that $\int_1^\infty f(x)\,dx = -\frac{1}{p-1}\left[\frac{1}{x^{p-1}}\right]_1^\infty$

$$= -\frac{1}{p-1}[0-1] = \frac{1}{p-1} = \text{finite}$$

$\Rightarrow \quad \int_1^\infty f(x)\,dx$ converges $\Rightarrow$ Σu_n is convergent.

Sub-Case 2. *When $0 < p < 1$*, $1 - p$ is +ve, so that

$$\int_1^\infty f(x)\,dx = \frac{1}{1-p}\left[x^{1-p}\right]_1^\infty = \frac{1}{1-p}(\infty - 1) = \infty$$

$\Rightarrow \quad \int_1^\infty f(x)\,dx$ diverges $\Rightarrow$ Σu_n is divergent.

Case II. *When $p = 1$, $f(x) = \frac{1}{x}$*

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx = \left[\log x\right]_1^\infty = \infty - \log 1 = \infty - 0 = \infty$$

$\Rightarrow \quad \int_1^\infty f(x)\,dx$ diverges $\Rightarrow$ Σu_n is divergent.

Hence Σu_n converges if $p > 1$ and diverges if $p \le 1$.

Example 4. *Discuss the convergence of $\Sigma\, ne^{-n^2}$.*

Sol. Here $\qquad u_n = ne^{-n^2} = f(n)$

$\therefore \qquad f(x) = x\,e^{-x^2}$

For $x \ge 1$, $f(x)$ is +ve and monotonic decreasing

$\therefore \qquad$ Cauchy's Integral Test is applicable.

Now, $\qquad \int_1^\infty f(x)\,dx = \int_1^\infty xe^{-x^2}\,dx.$

Put $x^2 = t \quad \therefore \qquad 2xdx = dt$

$$= \int_1^\infty e^{-t}\frac{dt}{2} = \left.\frac{e^{-t}}{-2}\right|_1^\infty = 0 + \frac{1}{2e} = \frac{1}{2e} = \text{finite}$$

$\Rightarrow \qquad \int_1^\infty f(x)\,dx$ converges and hence by Integral Test $\Sigma\, u_n$ converges.

Example 5. *Discuss the convergence of the series : $\sum_{n=2}^\infty \frac{1}{n(\log n)^p}$, $(p > 0)$.*

Hence show that $\int_2^\infty \frac{dx}{x(\log x)^p}$ $(p > 0)$ converges if and only if $p > 1$. **(P.T.U., Dec. 2004)**

Sol. Here $$u_n = \frac{1}{n(\log n)^p} = f(x) \quad \therefore \quad f(x) = \frac{1}{x(\log x)^p}$$

For $x \geq 2, p > 0, f(x)$ is +ve and monotonic decreasing.

$\therefore$ By Cauchy's Integral Test $\sum_{n=2}^{\infty} u_n$ and $\int_2^{\infty} f(x)\,dx$ converge or diverge together.

Case I. *When* $p \neq 1$

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} (\log x)^{-p} \cdot \frac{1}{x}\,dx = \left[\frac{(\log x)^{-p+1}}{-p+1}\right]_2^{\infty}$$

$$\left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}, n \neq -1\right]$$

Sub-Case 1. *When* $p > 1$, $p - 1$ is +ve, so that $\int_2^{\infty} f(x)\,dx = -\frac{1}{p-1}\left[\frac{1}{(\log x)^{p-1}}\right]_2^{\infty}$

$$= -\frac{1}{p-1}\left[0 - \frac{1}{(\log 2)^{p-1}}\right] = \frac{1}{(p-1)(\log 2)^{p-1}} = \text{finite}$$

$$\Rightarrow \int_2^{\infty} f(x)\,dx \text{ converges } \Rightarrow \sum_{n=2}^{\infty} u_n \text{ converges.}$$

Sub-Case 2. *When* $p < 1$, $1 - p$ is +ve, so that

$$\int_2^{\infty} f(x)\,dx = \frac{1}{1-p}\left[(\log x)^{1-p}\right]_2^{\infty} = \frac{1}{1-p}\,[\infty - (\log 2)^{1-p}] = \infty$$

$$\Rightarrow \int_2^{\infty} f(x)\,dx \text{ diverges } \Rightarrow \sum_{n=2}^{\infty} u_n \text{ diverges.}$$

Case II. *When* $p = 1$, $$f(x) = \frac{1}{x \log x}$$

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{dx}{x \log x} = \int_2^{\infty} \frac{\frac{1}{x}}{\log x}\,dx = \left[\log \log x\right]_2^{\infty}$$

$$= \infty - \log \log 2 = \infty \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log f(x)\right]$$

$$\Rightarrow \int_2^{\infty} f(x)\,dx \text{ diverges } \Rightarrow \sum_{n=2}^{\infty} u_n \text{ diverges.}$$

Hence Σu_n converges if $p > 1$ and diverges if $0 < p \leq 1$.
By Cauchy's integration test we know that

$$\int_2^{\infty} \frac{dx}{x(\log x)^p} \text{ and } \sum_{n=2}^{\infty} u_n = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \text{ converge or diverge together}$$

Since $\sum_{n=2}^{\infty} u_n$ converges for $p > 1$ as discussed in case I $\therefore$ $\int_2^{\infty} \frac{dx}{x(\log x)^p}$ also converges for $p > 1$.

Example 6. *Using the integral test, discuss the convergence of* $\sum \frac{1}{(n \log n)(\log \log n)^p}$ *; p > 0.*

Sol.
$$u_n = \frac{1}{(n \log n)(\log \log n)^p} = f(n)$$

$$\therefore \quad f(x) = \frac{1}{(x \log x)(\log \log x)^p}$$

Clearly, for $x \geq 2, f(x)$ is +ve and monotonic decreasing $\quad \therefore \quad$ Integral test is applicable

$\therefore$ By Cauchy's Integral Test $\sum_{n=2}^{\infty} u_n$ and $\int_2^{\infty} f(x)\, dx$ behave alike

$$\therefore \quad \int_2^{\infty} f(x)\, dx = \int_2^{\infty} \frac{1}{(x \log x)(\log \log x)^p}\, dx \qquad \ldots(1)$$

$$= \int_2^{\infty} (\log \log x)^{-p} \left(\frac{1}{x \log x}\right) dx \quad \Bigg| \quad \because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= \left. \frac{(\log \log x)^{-p+1}}{-p+1} \right|_2^{\infty} \text{ when } p \neq 1.$$

Case 1. *When p < 1* ; $1 - p > 0$, $\int_2^{\infty} f(x)\, dx = \infty - \frac{(\log \log 2)^{1-p}}{1-p} = \infty =$ not finite

$\therefore \qquad \Sigma u_n$ diverges for $p < 1$.

Case 2. *When p > 1* ; $\quad \therefore \quad p - 1$ is +ve

$$\int_2^{\infty} f(x)\, dx = \left. \frac{(\log \log x)^{-(p-1)}}{-(p-1)} \right|_2^{\infty} = -\frac{1}{p-1} \left\{ \left. \frac{1}{(\log \log x)^{p-1}} \right|_2^{\infty} \right\}$$

$$= -\frac{1}{p-1} \left\{ \frac{1}{\infty} - \frac{1}{(\log \log 2)^{p-1}} \right\} = \frac{1}{(p-1)(\log \log 2)^{p-1}} = \text{finite}$$

$\therefore \qquad \Sigma u_n$ converges for $p > 1$.

Case 3. *When p = 1*, from (1), $\int_2^{\infty} f(x)\, dx = \int_2^{\infty} \frac{\frac{1}{x \log x}}{\log \log x}\, dx$, which is of the type

$$\int \frac{f'(x)}{f(x)}\, dx = \log f(x) \quad \therefore \quad \int_2^{\infty} f(x)\, dx = \log \log \log x \Big|_2^{\infty} = \infty \text{ (not finite)}$$

$\therefore \quad \Sigma u_n$ diverges.

Hence $\qquad \Sigma u_n$ converges for $p > 1$ and diverges for $p \leq 1$.

Example 7. *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1+n^2}$. **(P.T.U., May 2002)**

Sol. Here $$u_n = \frac{8 \tan^{-1} n}{1+n^2} = f(x)$$

$\therefore$ $$f(x) = \frac{8 \tan^{-1} x}{1+x^2}$$

Clearly for $x \geq 1$ $f(x)$ is +ve and monotonic decreasing

$\therefore$ Cauchy's Integral Test is applicable

Now, $$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{8 \tan^{-1} x}{1+x^2}\,dx = \int_1^{\infty} 8 \tan^{-1} x \left(\frac{1}{1+x^2}\right) dx$$

$$= \left. \frac{8 (\tan^{-1} x)^2}{2} \right|_1^{\infty} \qquad \left| \because \int f(x)\, f'(x)\,dx = \frac{[f(x)]^2}{2} \right.$$

$$= 4 \,.\, \{(\tan^{-1} \infty)^2 - (\tan^{-1} 1)^2\} = 4 \,.\, \left\{\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right\}$$

$$= 4\pi^2 \left[\frac{1}{4} - \frac{1}{16}\right] = 4\pi^2 \,.\, \frac{12}{16} = 3\pi^2, \text{ which is finite}$$

$\therefore$ By Cauchy's Integral Test

$$\int_1^{\infty} f(x)\,dx \quad \text{and} \quad \text{hence } \Sigma\, u_n \quad \textit{i.e.,} \quad \sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1+n^2} \text{ converges.}$$

TEST YOUR KNOWLEDGE

Using the integral test, discuss the convergence of the following series :

1. $\sum \frac{1}{2n+3}$ **2.** $\sum \frac{1}{n(n+1)}$ **3.** $\sum \frac{1}{\sqrt{n}}$

4. $\sum \frac{1}{(n+1)^2}$ **5.** $\sum \frac{2n^3}{n^4+3}$ **6.** $\sum \frac{n}{(n^2+1)^2}$.

ANSWERS

1. Divergent **2.** Convergent **3.** Divergent **4.** Convergent

5. Divergent **6.** Convergent.

5.28. LEIBNITZ'S TEST ON ALTERNATING SERIES **(P.T.U., Dec. 2007, May 2014)**

Statement. *The alternating series* $\Sigma(-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots\ldots$ $(u_n > 0\ \forall\ n)$ *converges if*

(*i*) $u_n > u_{n+1}\ \ \forall\ n$ (*ii*) $\underset{n\to\infty}{Lt}\ u_n = 0.$

Proof. Let S_n denote the nth partial sum of the series $\Sigma (-1)^{n-1} . u_n$.

$$S_{2n} = u_1 - u_2 + u_3 - u_4 + u_5 - ...... - u_{2n-2} + u_{2n-1} - u_{2n}$$
$$= u_1 - (u_2 - u_3) - (u_4 - u_5) ...... - (u_{2n-2} - u_{2n-1}) - u_{2n}$$
$$= u_1 - [(u_2 - u_3) + (u_4 - u_5) + ... + (u_{2n-2} - u_{2n-1}) + u_{2n}]$$
$$< u_1 \qquad [\because \quad u_n > u_{n+1} \text{ and } u_n > 0 \text{ for all } n]$$

$\Rightarrow$ The sequence $\{S_{2n}\}$ is bounded above.

Also $$S_{2n+2} = S_{2n} + u_{2n+1} - u_{2n+2}$$

$\Rightarrow$ $$S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} > 0 \quad \text{for all} \quad n \quad \Rightarrow \quad S_{2n+2} > S_{2n}$$

$\Rightarrow$ The sequence $\{S_{2n}\}$ is monotonically increasing.

Since every monotonically increasing sequence which is bounded above converges, therefore, the sequence $\{S_{2n}\}$ converges. Let it converge to S, then $\underset{n\to\infty}{\text{Lt}} \; S_{2n} = S$

Now $$\underset{n\to\infty}{\text{Lt}} \; S_{2n+1} = \underset{n\to\infty}{\text{Lt}} \; (S_{2n} + u_{2n+1}) = \underset{n\to\infty}{\text{Lt}} \; S_{2n} + \underset{n\to\infty}{\text{Lt}} \; u_{2n+1}$$
$$= S + 0 \qquad [\because \quad \underset{n\to\infty}{\text{Lt}} \; u_n = 0]$$
$$= S$$

$\Rightarrow$ The sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converge to the same real number S.

$\therefore$ Given $\varepsilon > 0$, there exist positive integers m_1 and m_2 such that

$$| S_{2n} - S | < \varepsilon \quad \forall \quad 2n > m_1 \quad \text{and} \quad | S_{2n+1} - S | < \varepsilon \quad \forall \quad 2n + 1 > m_2$$

Let $m = \max. \{m_1, m_2\}$, then

$$| S_{2n} - S | < \varepsilon \quad \forall \quad n > m \quad \text{and} \quad | S_{2n+1} - S | < \varepsilon \quad \forall \quad n > m$$

$\Rightarrow$ $$| S_n - S | < \varepsilon \quad \forall \quad n > m$$

$\therefore$ The sequence $\{S_n\}$ converges to S.

Hence the given series is convergent.

Note. The alternating series will not be convergent if any one of the two conditions is not satisfied.

ILLUSTRATIVE EXAMPLES

Example 1. *Examine the convergence of the following series :*

(*i*) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + ... \infty$. **(P.T.U., May 2004)**

(*ii*) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... \infty$ *or* $\sum \frac{(-1)^n}{n}$ **(P.T.U., May 2010, 2014)**

Sol. (*i*) It is an alternating series

$$u_n = \frac{1}{\sqrt{n}}, \; u_{n+1} = \frac{1}{\sqrt{n+1}}$$

$$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} \; \forall \; n \quad \therefore \quad u_n > u_{n+1} \; \forall \; n$$

$\therefore$ First condition of Leibnitz's test is true

$$\underset{n\to\infty}{\text{Lt}} \; u_n = \underset{n\to\infty}{\text{Lt}} \; \frac{1}{\sqrt{n}} = 0$$

Second condition of Leibnitz's test is also true.

$\therefore$ Both the conditions of Leibnitz's test are satisfied.

$\therefore$ The given series is convergent.

(*ii*) It is an alternating series

$$u_n = \frac{1}{n} \;;\; u_{n+1} = \frac{1}{n+1}$$

$$\frac{1}{n} > \frac{1}{n+1} \quad \forall\, n$$

$\therefore$ $$u_n > u_{n+1} \quad \forall\, n$$

$\therefore$ First condition of Leibnitz's test is true and satisfied

Now, $$\underset{n\to\infty}{\text{Lt}}\; u_n = \underset{n\to\infty}{\text{Lt}}\; \frac{1}{n} = 0$$

$\therefore$ Second condition of Leibnitz's test is also satisfied

$\therefore$ Given series is convergent.

Example 2. *Examine the convergence of the series :*

(*a*) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots\ldots$ (*b*) $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \ldots\ldots$

Sol. (*a*) It is an alternating series

(*i*) $$u_n = \frac{n+1}{n}, \quad u_{n+1} = \frac{n+2}{n+1}$$

$$u_n - u_{n+1} = \frac{n+1}{n} - \frac{n+2}{n+1} = \frac{(n+1)^2 - n(n+2)}{n(n+1)} = \frac{1}{n(n+1)} > 0 \quad \forall\, n$$

$\Rightarrow$ $$u_n > u_{n+1} \quad \forall \quad n$$

(*ii*) $$\underset{n\to\infty}{\text{Lt}}\; u_n = \underset{n\to\infty}{\text{Lt}}\; \frac{n+1}{n} = \underset{n\to\infty}{\text{Lt}}\left(1 + \frac{1}{n}\right) = 1 \neq 0.$$

Since the second condition of Leibnitz's Test is not satisfied, the series is not convergent.

(*b*) It is an alternating series.

(*i*) $$u_n = \frac{1}{(n+1)^3}\,[1+2+3+\ldots\ldots+n] = \frac{1}{(n+1)^3}\cdot\frac{n(n+1)}{2} = \frac{1}{2}\cdot\frac{n}{(n+1)^2}$$

$$u_{n+1} = \frac{1}{2}\cdot\frac{n+1}{(n+2)^2}$$

$$u_n - u_{n+1} = \frac{1}{2}\left[\frac{n}{(n+1)^2} - \frac{n+1}{(n+2)^2}\right] = \frac{1}{2}\,\frac{n(n+2)^2 - (n+1)^3}{(n+1)^2(n+2)^2}$$

$$= \frac{1}{2}\cdot\frac{n^2+n-1}{(n+1)^2(n+2)^2} > 0 \quad \forall\, n$$

$\Rightarrow$ $$u_n > u_{n+1} \quad \forall\, n$$

(*ii*) $$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{n}{2(n+1)^2} = \underset{n\to\infty}{\text{Lt}}\ \frac{\frac{1}{n}}{2\left(1+\frac{1}{n}\right)^2} = 0.$$

Since both the conditions of Leibnitz's Test are satisfied, the given series is convergent.

Example 3. *Test the convergence of the following series :* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n}{2n-1}$.

Sol. The given series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n}{2n-1} = \frac{1}{1} - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \ldots\ldots$

It is an alternating series

(*i*) $$u_n = \frac{n}{2n-1}, \quad u_{n+1} = \frac{n+1}{2n+1}$$

$$u_n - u_{n+1} = \frac{1}{4n^2-1} > 0 \quad \forall\, n \quad \Rightarrow \quad u_n > u_{n+1} \quad \forall\, n$$

(*ii*) $$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{n}{2n-1} = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{2-\frac{1}{n}} = \frac{1}{2} \neq 0.$$

Here the second condition of Leibnitz's Test is not satisfied. Hence the given series is not convergent.

Example 4. *Examine the convergence of the series*

$$1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \cdots \infty.$$

Sol. The given series can be rearranged in the form

$$\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots\infty\right) + \left(\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2}\ldots\infty\right)$$

$$= \Sigma\,(-1)^{n-1}\,u_n + \Sigma\,(-1)^{n-1}\,v_n$$

Consider $\Sigma(-1)^{n-1}\,u_n$, which is an alternating series

where $u_n = \frac{1}{(2n-1)^2}$

($\because$ 1, 3, 5, … are in A.P. and its *n*th term = $2n-1$)

$$\frac{1}{(2n-1)^2} > \frac{1}{(2n+1)^2} \qquad \therefore \quad u_n > u_{n+1}$$

and $$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{(2n-1)^2} = 0$$

$\therefore$ Both conditions of Leibnitz's test are satisfied

$\therefore$ $\Sigma\,(-1)^{n-1}\,u_n$ is convergent.

Now, for $\Sigma\,(-1)^{n-1}\,v_n$; $v_n = \frac{1}{(2n)^2}$

As $\dfrac{1}{(2n)^2} > \dfrac{1}{(2n+2)^2}$ $\quad\therefore\quad v_n > v_{n+1}$ and $\underset{n\to\infty}{\text{Lt}}\ v_n = \underset{n\to\infty}{\text{Lt}}\ \dfrac{1}{(2n)^2} = 0$

$\therefore$ Both conditions of Leibnitz's test are satisfied by $\Sigma\,(-1)^{n-1}\,v_n$. Therefore, both $\Sigma\,(-1)^{n-1}\,u_n$ and $\Sigma\,(-1)^{n-1}\,v_n$ are convergent $\quad\therefore\quad$ Their sum is also convergent.

Hence given series is convergent.

Example 5. *The series* $\dfrac{1}{3} - \dfrac{1}{2} + \dfrac{1}{9} - \dfrac{1}{4} + \dfrac{1}{27} - \dfrac{1}{8} + \ldots\ldots \infty$ *, does not meet one of the conditions of Leibnitz's test which one ? Find the sum of this series.* **(P.T.U., Dec. 2004)**

Sol. The given series is

$$\left(\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{9}-\frac{1}{4}\right)+\left(\frac{1}{27}-\frac{1}{8}\right)+\ldots.+\left(\frac{1}{3^n}-\frac{1}{2^n}\right)+\ldots.\infty$$

Here

$$u_n = \frac{1}{3^n} - \frac{1}{2^n}$$

$$u_{n+1} = \frac{1}{3^{n+1}} - \frac{1}{2^{n+1}}$$

$$u_n - u_{n+1} = \left(\frac{1}{3^n} - \frac{1}{3^{n+1}}\right) - \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right) = \frac{2}{3^{n+1}} - \frac{1}{2^{n+1}}$$

$$= \frac{2^{n+2} - 3^{n+1}}{6^{n+1}} < 0 \quad \forall\, n$$

$\therefore$ $\qquad u_n < u_{n+1} \quad \forall\, n \quad$ *i.e.*, $u_n \not> u_{n+1}\ \forall\, n$

$\therefore$ First condition of Leibnitz's test is not satisfied

whereas second condition *i.e.*, $\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \dfrac{1}{3^n} - \dfrac{1}{2^n} = 0$ is satisfied.

Now sum of the series

$$= \left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots..\infty\right) - \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots..\infty\right)$$

both are infinite G.Ps and sum of an infinite G.P. $= \dfrac{a}{1-r}$

$\therefore$ Sum of the given series $= \dfrac{\frac{1}{3}}{1-\frac{1}{3}} - \dfrac{\frac{1}{2}}{1-\frac{1}{2}} = \dfrac{1}{2} - 1 = -\dfrac{1}{2}$.

TEST YOUR KNOWLEDGE

Examine the convergence of the following series :

1. $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \ldots\ldots\ \infty$

2. $1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots\ldots$ to ∞

3. $1 - \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} - \dfrac{1}{4\sqrt{4}} + \ldots\ldots$ to ∞

4. $\dfrac{1}{a} - \dfrac{1}{a+b} + \dfrac{1}{a+2b} - \dfrac{1}{a+3b} + \ldots\ldots$ to ∞ $\quad (a > 0, b > 0)$

5. $\dfrac{1}{\log 2} - \dfrac{1}{\log 3} + \dfrac{1}{\log 4} - \dfrac{1}{\log 5} + \ldots\ldots$ to ∞

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,!}$

7. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n-2}$

8. $\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{5^n}$

9. $\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{5}+1} + \ldots\ldots$ to ∞

10. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$

11. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$ [**Hint:** $\cos n\pi = (-1)^n$] **(P.T.U., May 2012)**

ANSWERS

1. Convergent
2. Convergent
3. Convergent
4. Convergent
5. Convergent
6. Convergent
7. Convergent
8. Convergent
9. Convergent
10. Convergent
11. Convergent.

5.29(a). ABSOLUTE CONVERGENCE OF A SERIES **(P.T.U., Dec. 2003, 2011)**

Def. *If a convergent series whose terms are not all positive, remains convergent when all its terms are made positive, then it is called an* **absolutely convergent series**, *i.e.,*

The series Σu_n is said to be absolutely convergent if $\Sigma\, | u_n |$ is a convergent series.

5.29(b). CONDITIONAL CONVERGENCE OF A SERIES **(P.T.U., Dec. 2011)**

A series is said to be ***conditionally convergent*** *if it is convergent but does not converge absolutely.*

Example 1. *Test whether the following series are absolutely convergent or conditionally convergent.*

(*a*) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$ **(P.T.U., Dec. 2006)** (*b*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$. **(P.T.U., Dec. 2012)**

Sol. (*a*) The series is $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$

(*i*) In this alternating series, each term is less than the preceeding term numerically.

(*ii*) Moreover $u_n = \frac{1}{n^2}$, which $\to 0$ as $n \to \infty$.

Hence the series satisfies both the conditions of the test on alternating series and so the given series *converges*.

Again when all the term of the series are made positive, the series becomes

$$\sum | u_n | = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \Sigma \frac{1}{n^2}, \text{ which is a } p\text{-series, where } p = 2 > 1$$

$\therefore$ $\Sigma\, | u_n |$ is a convergent series.

Thus the given series converges absolutely.

(*b*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \Sigma\, (-1)^{n-1}\, u_n$ (say), where $u_n = \frac{1}{2n-1}$

Putting $n = 1, 2, 3, \ldots\ldots$, the series becomes $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

The series is clearly an alternating series.
The terms go on decreasing numerically and

$$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{2n-1} = 0$$

$\therefore$ By Leibnitz's Test, the series *converges*.
But when all terms are made positive, the series becomes,

$$\sum |u_n| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots\ldots$$

Here $u_n = \dfrac{1}{2n-1}$. Take $v_n = \dfrac{1}{n}$.

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\ \frac{n}{2n-1} = \underset{n\to\infty}{\text{Lt}}\left[\frac{1}{2-\frac{1}{n}}\right] = \frac{1}{2} = \text{finite} \neq 0$$

Hence by comparison test series Σu_n and Σv_n behave alike.

But $\Sigma v_n = \Sigma \dfrac{1}{n}$ is a divergent series ($\because$ here $p = 1$), $\therefore$ Σu_n also diverges.

Hence the given series converges, and the series of absolute terms diverges, therefore the given series converges conditionally.

5.30. EVERY ABSOLUTELY CONVERGENT SERIES IS CONVERGENT OR IF $\sum_{n=1}^{\infty} |u_n|$ IS CONVERGENT THEN $\sum_{n=1}^{\infty} u_n$ IS CONVERGENT

Proof. Let $\sum_{n=1}^{\infty} u_n$ be an absolutely convergent series.

$\therefore$ $\sum_{n=1}^{\infty} |u_n|$ is convergent.

By Cauchy's general principle of convergence, given $\varepsilon > 0$, $\exists$s a positive integer m such that

$$||u_{m+1}| + |u_{m+2}| \ldots\ldots |u_n|| < \varepsilon \quad \forall \quad n > m$$

or $$|u_{m+1}| + |u_{m+1}| + \ldots\ldots + |u_n| < \varepsilon \quad \forall \quad n > m \qquad \ldots(1)$$

Now, by triangle inequality, we have

$$|u_{m+1} + u_{m+2} + \ldots\ldots + u_n| \leq |u_{m+1}| + |u_{m+2}| + \ldots\ldots + |u_n| < \varepsilon \quad \forall \quad n > m \qquad \text{[Using (1)]}$$

$\therefore$ By Cauchy's general principle of convergence, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Hence $\sum |u_n|$ is convergent $\Rightarrow$ Σu_n is convergent.

Note 1. Absolute convergence $\Rightarrow$ Convergence, but convergence need not imply absolute convergence *i.e.,* the converse of above theorem need not be true.

For example, consider the series $\sum \dfrac{(-1)^{n-1}}{n} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \ldots\ldots$

It is convergent. [See Example 1 with Leibnitz's Test]

But the series $\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$ is divergent.

Note 2. The divergence of $\Sigma \mid u_n \mid$ does not imply the divergence of Σu_n.

For example, $\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$ is divergent whereas $\sum \frac{(-1)^{n-1}}{n}$ is convergent.

5.31. POWER SERIES

Def. A series of the form $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \infty$, where a's are independent of x, is called a power series in x. Such a series may converge for some or all values of x.

Interval of Convergence. If the power series is $\sum_{n=0}^{\infty} a_n x^n$, then take $u_n = a_n x^n$

$\therefore$ Series become $\sum_{n=0}^{\infty} u_n$, where $u_n = a_n x^n$.

As in power series a's can be +ve as well as –ve $\therefore$ for convergence of $\Sigma\, u_n$ we test the convergence of $\Sigma \mid u_n \mid$ $\because$ every absolutely convergent series is a convergent series.

$$\therefore \qquad \left| \frac{u_n}{u_{n+1}} \right| = \left| \frac{a_n\, x^n}{a_{n+1}\, x^{n+1}} \right| = \left| \frac{a_n}{a_{n+1}} \cdot \frac{1}{x} \right| = \left| \frac{a_n}{a_{n+1}} \right| \cdot \frac{1}{|x|}$$

Let $$\underset{n \to 0}{\text{Lt}} \left| \frac{a_n}{a_{n+1}} \right| = l$$

$$\therefore \qquad \text{Lt} \left| \frac{u_n}{u_{n+1}} \right| = \frac{l}{|x|}$$

$\therefore$ $\Sigma\, |u_n|$ converges if $\frac{l}{|x|} > 1$ (By Ratio test)

$\therefore$ $\Sigma\, u_n$ converges for $|x| < l$ *i.e.,* for $-l < x < l$

$\therefore$ The power series converges in the interval $(-l, l)$ and diverges outside this interval.

Interval $(-l, l)$ is called the Interval of Convergence of the power series.

Example 2. *Prove that the series* $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots\ldots$ *converges absolutely.*

Sol. The given series is $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin nx}{n^3}$

Since $| u_n | = \frac{|\sin nx|}{n^3} \le \frac{1}{n^3}$ $\forall\, n$ and $\sum \frac{1}{n^3}$ converges by p-series test $\because$ here $p = 3 > 1$.

$\therefore$ By comparison test, the series $\sum \mid u_n \mid$ converges.

$\Rightarrow$ The given series converges absolutely.

Example 3. *For what value of x does the series* $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$ *converges absolutely.*

(P. T.U., May 2003)

Sol. The series is $\displaystyle\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} (-1)^n (4x+1)^n$

$$|u_n| = |(-1)^n (4x+1)^n| = |(4x+1)^n|$$

and $$|u_{n+1}| = |(-1)^{n+1} (4x+1)^{n+1}| = |(4x+1)^{n+1}|$$

$$\frac{|u_n|}{|u_{n+1}|} = \frac{|(4x+1)^n|}{|(4x+1)^{n+1}|} = \frac{1}{|4x+1|}$$

$$\underset{n\to\infty}{\text{Lt}} \left|\frac{u_n}{u_{n+1}}\right| = \frac{1}{|4x+1|}$$

$\therefore$ By ratio test $\Sigma |u_n|$ converges if $\dfrac{1}{|4x+1|} > 1$

i.e., $$|4x+1| < 1 \quad \text{or} \quad |4x-(-1)| < 1$$

or $$-1-1 < 4x < -1+1$$

or $$-2 < 4x < 0 \qquad | \because \ |x-a| < l \Rightarrow a-l < x < a+l$$

or $$-\frac{1}{2} < x < 0$$

Hence the given series converges absolutely for $-\dfrac{1}{2} < x < 0$ *i.e.*, when $x \in \left(-\dfrac{1}{2}, 0\right)$.

Example 4. *Test the absolute convergence of the series* $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log x)^2}$. **(P.T.U., May 2012)**

Sol. Here $$u_n = \frac{(-1)^n}{n(\log x)^2}$$

$$|u_n| = \frac{1}{n(\log x)^2} = f(x) \text{ say}$$

Apply Cauchy's Integral test

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x(\log x)^2}\,dx = \int_2^{\infty} (\log x)^{-2}\left(\frac{1}{x}\right)dx$$

$$= \left.\frac{(\log x)^{-1}}{-1}\right|_2^{\infty} = -\left.\frac{1}{\log x}\right|_2^{\infty}$$

$$= 0 + \frac{1}{\log 2} = \frac{1}{\log 2} \text{ which is finite}$$

$\therefore$ $\displaystyle\int_2^{\infty} f(x)\,dx$ is convergent

$\therefore$ By Cauchy's Integral test

$\sum_{n=2}^{\infty} |u_n|$ and $\int_2^{\infty} f(x)\, dx$ converge or diverge together and

$\because$ $\int_2^{\infty} f(x)\, dx$ converges

$\therefore$ $\sum_{n=2}^{\infty} |u_n|$ converges

$\therefore$ Σu_n converges absolutely

Example 5. *Prove that the series* $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty$ *is convergent for* $-1 < x \le 1$. *Also write the interval of convergence.* **(P.T.U., Dec. 2013)**

Sol. The given series is

$$\Sigma u_n = \Sigma (-1)^{n-1} \frac{x^n}{n}$$

$$|u_n| = \left| \frac{x^n}{n} \right| = \frac{|x|^n}{n} ; |u_{n+1}| = \frac{|x|^{n+1}}{n+1}$$

$$\frac{|u_n|}{|u_{n+1}|} = \frac{|x|^n}{n} \cdot \frac{n+1}{|x|^{n+1}} = \frac{n+1}{n} \cdot \frac{1}{|x|}$$

$$\underset{n \to \infty}{\text{Lt}} \frac{|u_n|}{|u_{n+1}|} = \underset{n \to \infty}{\text{Lt}} \left(1 + \frac{1}{n}\right) \cdot \frac{1}{|x|} = \frac{1}{|x|}$$

By Ratio test $\Sigma\, |u_n|$ converges when

$$\frac{1}{|x|} > 1 \quad \textit{i.e.,} \quad |x| < 1 \quad \textit{i.e.,} \quad -1 < x < 1$$

and diverges when $\frac{1}{|x|} < 1$ *i.e.,* $|x| > 1$ *i.e.,* for $x > 1$ or $x < -1$

Ratio test fails when $|x| = 1$ *i.e.,* when $x = \pm 1$.

$\therefore$ When $x = 1$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \infty$, which is an alternating series and is convergent. (see S.E. 1 art. 5.28)

When $x = -1$, the series becomes

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \ldots \infty$$

$$= -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \infty\right)$$

$$= -\sum \frac{1}{n}, \text{ which is of the type } \sum \frac{1}{n^p}, \text{ where } p = 1$$

$\therefore$ by p-series test; it is divergent.

$\therefore$ Given series converges for $-1 < x \le 1$ and diverges for $x > 1$ or $x \le -1$

$\therefore$ The Interval of convergence $(-1, 1]$.

Example 6. *For what value of x the power series*

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \ldots.. + \left(-\frac{1}{2}\right)^n (x-2)^n + \ldots.. \infty$$

converges ? What is its sum ? **(P.T.U., May 2002)**

Sol. The given series is an alternating series

Here $$u_n = \left(-\frac{1}{2}\right)^n (x-2)^n$$

$$|u_n| = \frac{1}{2^n} |(x-2)^n| \ ; \ |u_{n+1}| = \frac{1}{2^{n+1}} |(x-2)^{n+1}|$$

$$\left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{1}{2^n}(x-2)^n \cdot \frac{2^{n+1}}{(x-2)^{n+1}}\right| = \frac{2}{|x-2|}$$

$$\underset{n\to\infty}{\text{Lt}} \left|\frac{u_n}{u_{n+1}}\right| = \frac{2}{|x-2|}$$

$\therefore$ By Ratio's test the series $\Sigma\, |u_n|$ is convergent

if $$\frac{2}{|x-2|} > 1 \quad \text{or} \quad |x-2| < 2 \quad \text{or} \quad 2-2 < x < 2+2 \quad \text{or} \quad 0 < x < 4$$

$\therefore$ $\Sigma |u_n|$ is convergent for $0 < x < 4$.

As every absolutely convergent series is convergent

$\therefore$ $$\sum u_n = \sum \left(-\frac{1}{2}\right)^n (x-2)^n \text{ is convergent for } 0 < x < 4$$

Now sum of the series $= 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \ldots\ldots \infty$ is an infinite G.P. with first term 1 and common ratio $-\frac{1}{2}(x-2)$ and $|\text{C . R}| = \left|-\frac{1}{2}(x-2)\right| < \frac{1}{2}\,|x-2| < 1$

$\therefore$ $$\text{Sum of the series} = \frac{1}{1-\left[-\frac{1}{2}(x-2)\right]} = \frac{1}{1+\frac{x-2}{2}} \qquad \left[S = \frac{a}{1-r}\right]$$

$$= \frac{2}{2+x-2} = \frac{2}{x} \quad \text{for } 0 < x < 4.$$

TEST YOUR KNOWLEDGE

1. Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos^2 nx}{n\sqrt{n}}$ converges absolutely.

[**Hint:** See S.E.1]

2. For what values of x are the following series convergent?

(*i*) $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + ...$ (*ii*) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

(P.T.U., Jan. 2010)

$$\left[\textbf{Hint:} (ii)\ u_n = \frac{x^n}{n!},\ u_{n+1} = \frac{x^{n+1}}{(n+1)!};\ \underset{u_n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} \to \infty \text{ irrespective of the values of } x \right.$$

$$\left. \therefore\quad \text{Lt} \frac{u_n}{u_{n+1}} > 1\ \forall\ x\ \therefore\ \Sigma u_n \text{ converges for all values of } x \right]$$

(*iii*) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$ (*iv*) $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + ...$

3. Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{1+n^2}$ **(P.T.U., May 2003)**

[**Hint:** Consult S.E. 4]

ANSWERS

2. (*i*) $-1 < x \le 1$ (*ii*) all x (*iii*) $-1 < x < 1$ (*iv*) $-1 < x \le 1$.

3. Convergent.

5.32. UNIFORM CONVERGENCE OF SERIES OF FUNCTIONS

(P.T.U., May 2004, 2005)

Let $u_n(x)$ be a real valued function defined on an interval I and for each $n \in$ N. Then $u_1(x) + u_2(x) + u_3(x) + ... = \sum_{n=1}^{\infty} u_n(x)$ is called an infinite series of functions each of which is defined on the interval I.

Let $S_n(x) = u_1(x) + u_2(x) + ... + u_n(x)$ be the nth partial sum of $\sum u_n(x)$.

Let $\alpha \in$ I and $\lim_{n\to\infty} S_n(\alpha) = S(\alpha)$, then the series $\sum u_n(x)$ is said to converge to $S(\alpha)$ at $x = \alpha$.

Thus, given $\varepsilon > 0$, there exists a positive integer m such that

$$|S_n(\alpha) - S(\alpha)| < \varepsilon \quad \forall \quad n > m.$$

The positive integer m depends on $\alpha \in$ I and the given value of $\varepsilon > 0$, *i.e.* $m = m(\alpha, \varepsilon)$. It is not always possible to find an m which works for each $x \in$ I. If we can find an m which depends only on ε and not on $x \in$ I, we say $\sum u_n(x)$ is uniformly convergent.

Definition. A series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformally to a function S(x) if for a given $\varepsilon > 0$, there exists a positive integer m depending only on ε and independent of x such that for every

$$x \in \text{I},\ |S_n(x) - S(x)| < \varepsilon\ \forall \quad n > m.$$

Note. The method of testing the uniform convergence of a series $\sum u_n(x)$, by definition, involves finding $S_n(x)$ which is not always easy. The following test avoids $S_n(x)$.

5.33. WEIERSTRASS'S M-TEST **(P.T.U., May 2003)**

Statement. *A series $\sum_{n=1}^{\infty} u_n(x)$ of functions converges uniformly and absolutely on an interval I if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of positive constants such that $|u_n(x)| \le M_n \quad \forall \quad n \in N$ and $\forall \quad x \in I$.*

Proof. Since $\sum_{n=1}^{\infty} M_n$ is convergent, by Cauchy's general principle of convergence, for each $\varepsilon > 0$, there exists a positive integer m such that

$$|M_{m+1} + M_{m+2} + ... + M_n| < \varepsilon \quad \forall \quad n > m$$

or
$$M_{m+1} + M_{m+2} + ... + M_n < \varepsilon \quad \forall \quad n > m \qquad ...(1)$$

Now, for all $x \in I, |u_n(x)| \le M_n$ ...(2)

$$\therefore \quad |u_{m+1}(x) + u_{m+2}(x) + ... + u_n(x)|$$
$$\le |u_{m+1}(x)| + |u_{m+2}(x)| + ... + |u_n(x)|$$
$$\le M_{m+1} + M_{m+2} + ... + M_n \qquad \text{[by (2)]}$$
$$< \varepsilon \quad \forall \quad n > m \qquad \text{[by (1)]}$$

$\Rightarrow$ By Cauchy's criterion, the series $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on I.

Also,
$$|u_{m+1}(x)| + |u_{m+2}(x)| + ... + |u_n(x)| < \varepsilon \quad \forall \quad n > m$$
$$\Rightarrow \quad \| u_{m+1}(x)| + |u_{m+2}(x)| + ... + |u_n(x)\| < \varepsilon \quad \forall \quad n > m$$

$\Rightarrow$ The series $\sum_{n=1}^{\infty} |u_n(x)|$ is uniformly convergent on I.

Hence the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly and absolutely on I.

Example 1. *Show that the following series are uniformly convergent :*

(*i*) $\sum_{n=1}^{\infty} \frac{\sin(x^2 + nx)}{n(n+2)}$ *for all real x.* **(P.T.U., May 2009)**

(*ii*) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$ *for all real x and p > 1.* **(P.T.U., Dec. 2005)**

(*iii*) $\sum_{n=1}^{\infty} \frac{1}{n^p + n^q x^2}$ *for all real x and p > 1.*

Sol. (*i*) Here
$$u_n(x) = \frac{\sin(x^2 + nx)}{n(n+2)}$$

$$\therefore \quad |u_n(x)| = \left| \frac{\sin(x^2 + nx)}{n(n+2)} \right| = \frac{|\sin(x^2 + nx)|}{n(n+2)} \le \frac{1}{n(n+2)} < \frac{1}{n^2} \;(= M_n) \quad \forall \quad x \in R$$

Since $\sum_{n=1}^{\infty} \text{M}_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, therefore, by M-test, the given series is uniformly convergent for all real x.

(*ii*) Here
$$u_n(x) = \frac{\cos nx}{n^p}$$

$$\therefore \qquad |u_n(x)| = \left|\frac{\cos nx}{n^p}\right| = \frac{|\cos nx|}{n^p} \le \frac{1}{n^p} \ (= \text{M}_n) \quad \forall \quad x \in \text{R}.$$

Since $\sum_{n=1}^{\infty} \text{M}_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$, therefore, by M-test, the given series is uniformly convergent for all real x and $p > 1$.

(*iii*) Here
$$u_n(x) = \frac{1}{n^p + n^q x^2}$$

Since $x^2 \ge 0$ for all real x

$$\therefore \qquad n^q x^2 \ge 0 \ \Rightarrow \ n^p + n^q x^2 \ge n^p \ \Rightarrow \ \frac{1}{n^p + n^q x^2} \le \frac{1}{n^p}$$

$$\therefore \qquad |u_n(x)| = \frac{1}{n^p + n^q x^2} \le \frac{1}{n^p} \ (= \text{M}_n) \quad \forall \quad x \in \text{R}.$$

Since $\sum_{n=1}^{\infty} \text{M}_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$.

Therefore, by M-test, the given series is uniformly convergent for all real x and $p > 1$.

TEST YOUR KNOWLEDGE

Test for uniform convergence the series :

1. $\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \ldots\ldots$ **(P.T.U., May 2003)**

[**Hint:** See S.E.1. (*ii*)]

2. $\sin x - \frac{\sin 2x}{2\sqrt{2}} + \frac{\sin 3x}{3\sqrt{3}} - \frac{\sin 4x}{4\sqrt{4}} + \ldots\ldots$

3. $\sum_{n=1}^{\infty} \frac{\cos (x^2 + n^2 x)}{m(n^2 + 2)}$

4. $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^3 x^2}$

5. Show that if $0 < r < 1$, then each of the following series is uniformly convergent on R:

(*i*) $\sum_{n=1}^{\infty} r^n \cos nx$ (*ii*) $\sum_{n=1}^{\infty} r^n \sin nx$

(*iii*) $\sum_{n=1}^{\infty} r^n \cos n^2 x$ (*iv*) $\sum_{n=1}^{\infty} r^n \sin a^n x$.

ANSWERS

1. Uniformly convergent for all real x **2.** Uniformly convergent for all real x

3. Uniformly convergent for all real x **4.** Uniformly convergent for all real x.

5.34. HOW TO TEST A SERIES FOR CONVERGENCE?

We have four types of infinite series,

(1) Positive term series
(2) Geometric series
(3) Alternating series
(4) Power series

(*i*) For positive terms series first find u_n and if possible evaluate $\underset{n\to\infty}{\text{Lt}}\ u_n$. If limit of u_n is $\neq 0$, the series is divergent. If limit of $u_n = 0$, then compare $\sum u_n$ with $\sum v_n$, where v_n is always of the type $\frac{1}{n^p}$; compare $\sum u_n$ with $\sum \frac{1}{n^p}$ and

apply comparison test ; if comparison test fails
apply Ratio Test ; if ratio test fails
apply Raabe's Test ; if Raabe's test fails
apply logarithmic Test ; If logarithmic test fails
apply Gauss Test

Special cases : (*a*) If in Ratio test $\frac{u_n}{u_{n+1}}$ involves e, we directly apply logarithmic test.

(*b*) If in Ratio test it is possible to expand $\frac{u_n}{u_{n+1}}$ in powers of $\frac{1}{n}$, then directly apply Gauss Test.

(*ii*) For alternating series apply Leibnitz's rule.

(*iii*) The geometric series $1 + x + x^2 \ldots\ldots \infty$
converges if $-1 < x < 1$ *i.e.*, $|x| < 1$
diverges if $x \geq 1$
oscillates finitely if $x = -1$
oscillates infinitely if $x < -1$

(*iv*) For power series apply the Ratio test. If Ratio test fails, then apply the same tests as applied in (*i*) case.

REVIEW OF THE CHAPTER

1. A sequence $\{a_n\}$ is said to be bounded if $\exists$s two real numbers, k and K such that

$$k \leq a_n \leq \text{K} \quad \forall\ n \in \text{N}.$$

2. A sequence $\{a_n\}$ is said to be:

(*i*) convergent if $\underset{n\to\infty}{\text{Lt}}\ a_n$ is finite

(*ii*) divergent if $\underset{n\to\infty}{\text{Lt}}\ a_n$ is not finite *i.e.*, $\underset{n\to\infty}{\text{Lt}}\ a_n = +\infty$ or $-\infty$

(*iii*) oscillary if $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$.

3. A sequence $\{a_n\}$ is said to:

(*i*) monotonic increasing if $a_{n+1} \geq a_n \quad \forall\ n \in \text{N}$

(*ii*) monotonic decreasing if $a_{n+1} \leq a_n \quad \forall\ n \in \text{N}$

(*iii*) a sequence is said to be monotonic if it is either monotonic increasing or monotonic decreasing.

4. Every convergent sequence is bounded.

5. Necessary and sufficient condition for the convergence of monotonic sequence is that it is bounded.

6. (*i*) A monotonic increasing sequence which is bounded above converges and if it is not bounded above it diverges to $+\infty$.

(*ii*) A monotonic decreasing sequence which is bounded below converges and if it not bounded below diverges to $-\infty$.

7. Infinite series: If $\{u_n\}$ is a sequence of real numbers, then the expression $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \ldots + u_n + \ldots \infty$ is called an infinite series.

8. Behaviour of infinite series: An infinite series $\Sigma\, u_n$ converges, diverges or oscillates (finitely or infinitely) according as the sequence $\{S_n\}$ of its partial sums converges, diverges or oscillates.

Thus (*i*) If series $\sum_{n=1}^{\infty} u_n$ is convergent, then $\underset{n\to\infty}{\text{Lt}}\ u_n = 0$ but converse is not true.

(*ii*) If $\underset{n\to\infty}{\text{Lt}}\ u_n \neq 0$, then $\sum_{n=1}^{\infty} u_n$ is not convergent. But if $\sum_{n=1}^{\infty} u_n$ is a +ve term series, then if $\underset{n\to\infty}{\text{Lt}}\ u_n \neq 0$, then

$\sum_{n=1}^{\infty} u_n$ diverges to $+\infty$ $\because$ a +ve term series either converges or diverges to $+\infty$.

9. Cauchy's general principle of convergence: The necessary and sufficient condition for the convergence of infinite series is that given $\varepsilon > 0$, however small $\exists$s a +ve integer m such that $|S_{n+p} - S_n| < \varepsilon$ for $n \geq m$ and $p \in N$. *i.e.*, $|u_{n+1} + u_{n+2} \ldots u_{n+p}| < \varepsilon$ for $n \geq m$ and $p \in N$.

10. Comparison tests: If $\Sigma\, u_n$ and $\Sigma\, v_n$ are two +ve term series, then:

Test 1. (*a*) If $u_n \leq K\, v_n$ $(K > 0)$ $\forall\ n > m$ and $\Sigma\, v_n$ is convergent, then $\Sigma\, u_n$ is also convergent

(*b*) $u_n \geq k\, v_n$ $(k > 0)$ $\forall\ n > m$ and $\Sigma\, v_n$ is divergent, then $\Sigma\, u_n$ is also divergent.

Test 2. $h < \dfrac{u_n}{v_n} < k$ $(h, k > 0)$ $\forall\ n > m$ both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge and diverge together.

Test 3. (*i*) If $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{v_n} = l$ (finite and non-zero), then $\Sigma\, u_n$ and $\Sigma\, v_n$ both converge and diverge together.

(*ii*) If $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{v_n} = 0$ and $\Sigma\, v_n$ converges, then $\Sigma\, u_n$ also converges.

(*iii*) If $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{v_n} = \infty$ and $\Sigma\, v_n$ diverges, then $\Sigma\, u_n$ also diverges.

(*iv*) If $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{v_n} = \infty$ and $\Sigma\, u_n$ converges, then $\Sigma\, v_n$ also converges.

Test 4. (*i*) If $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$ for $n > m$ and $\Sigma\, v_n$ is convergent, then $\Sigma\, u_n$ is also convergent.

(*ii*) If $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ for $n > m$ and $\Sigma\, v_n$ is divergent, then $\Sigma\, u_n$ is also divergent.

11. P-Series or Hyper-Harmonic Series: The series of the type $\sum \frac{1}{n^p}$ is known as p-series of hyper harmonic series and it converges if $p > 1$ and diverges if $p \le 1$.

12. D' Alembert's Ratio Test: If $\Sigma\, u_n$ is a +ve term series and $\underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = l$. Then $\Sigma\, u_n$ is convergent if $l > 1$ and divergent if $l < 1$ when $l = 1$; Ratio test fails.

13. Raabe's Test: If $\Sigma\, u_n$ is a +ve terms series and $\underset{n \to \infty}{\text{Lt}}\; n\left[\frac{u_n}{u_{n+1}} - 1\right] = l$, then $\Sigma\, u_n$ converges if $l > 1$ and diverges if $l < 1$ Raabe's test fails when $l = 1$.

14. Logarithmic Test: If $\Sigma\, u_n$ is a +ve term series and $\underset{n \to \infty}{\text{Lt}}\; n \log \frac{u_n}{u_{n+1}} = l$, then $\Sigma\, u_n$ converges if $l > 1$ and diverges if $l < 1$

Logarithmic test fails when $l = 1$.

15. Gauss Test: If $\Sigma\, u_n$ is a +ve term series $\frac{u_n}{u_{n+1}}$ can be expressed as $\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + 0\left(\frac{1}{n^2}\right)$, then $\Sigma\, u_n$ converges if $\lambda > 1$ and diverges if $\lambda \le 1$.

16. Cauchy's root Test: If $\Sigma\, u_n$ is a +ve term series and $\underset{n \to \infty}{\text{Lt}} (u_n)^{\frac{1}{n}} = l$, then $\Sigma\, u_n$ is a convergent series if $l < 1$ and divergent if $l > 1$. Cauchy's root test fails where $l = 1$.

17. Cauchy's Integral Test: If $f(x)$ is a non-negative, monotonic decreasing function of x such that $f(n) = u_n$ for all positive integral values of n, then $\Sigma\, u_n$ and $\int_1^\infty f(x)\, dx$ converge or diverge together.

18. Leibnitz's Test on Alternating Series: The alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges if

(*i*) $u_n > u_{n+1}$ (*ii*) $\underset{n \to \infty}{\text{Lt}}\; u_n = 0$.

19. Absolute Convergence of a series:

$\Sigma\, u_n$ is absolutely convergent if $\Sigma\, |\, u_n\, |$ is convergent

20. Conditional Convergence of a series:

$\Sigma\, u_n$ is conditionally convergent if it is convergent but does not converge absolutely

21. Every absolutely convergent series is convergent.

22. If in power series $\sum_{n=1}^{\infty} a_n x^n$; $\underset{n \to \infty}{\text{Lt}} \frac{a_n}{a_{n+1}} = l$, then interval of convergence of the power series is $(-l, l)$.

23. Uniform Convergence of Series of functions: A series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to a function $S(x)$ if for $\varepsilon > 0$; $\exists$s a +ve integer m depending on ε and independent of x such that for every $x \in I$, $|\, S_n(x) - S(x)\, | < \varepsilon \quad \forall\; n > m$.

24. Weierstrass's M-test: A series $\Sigma\, u_n(x)$ of functions converges uniformly and absolutely on an interval I if $\exists$s a convergent series $\sum_{n=1}^{\infty} M_n$ of +ve constants such that, $|\, u_n(x)\, | \le M_n \;\; \forall\; n \in N$ and $\forall\; x \in I$

SHORT ANSWER TYPE QUESTIONS

1. Prove that every convergent sequence is bounded.
[**Hint:** See art. 5.9]

2. Define monotonic increasing and decreasing sequence.
[**Hint:** See art. 5.7]

3. Give an example of a monotonic increasing sequence which is (*i*) convergent, (*ii*) divergent.
[**Hint:** S.E. 1 art. 5.10]

4. Give an example of a monotonic decreasing sequence which is (*i*) convergent, (*ii*) divergent.
[**Hint:** S.E. 2 art. 5.10]

5. Define convergence, divergence, oscillation of an infinite series. **(P.T.U., Dec. 2007)**
[**Hint:** See art. 5.15]

6. Prove that a positive term series either converges or diverges to $+\infty$. [**Hint:** See art. 5.18]

7. State Cauchy's general principle of convergence. [**Hint:** See art. 5.19(*b*)]

8. Prove that sequence $\{a_n\}$, where $a_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots \frac{1}{2^n} + \ldots \infty$ is convergent.

[**Hint:** S.E. 3(*iii*) art. 5.10]

9. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\underset{n\to\infty}{\text{Lt}}\ a_n = 0$. Give an example to show that converse is not true.

[**Hint:** See art. 5.17] **(P.T.U., Dec. 2003, May 2004, Jan. 2009, May 2011)**

10. Show that $\sum_{n=1}^{\infty} \sqrt{\frac{n}{2(n+1)}}$ is divergent. [**Hint:** S.E. 11(*ii*) art. 5.21]

11. Test the convergence of the following series:

(*i*) $\sum \frac{1}{n\,(n+1)}$ [**Hint:** S.E. 8 (*iii*) art. 5.21]

(*ii*) $\sum \frac{1}{n \log n}$ [**Hint:** S.E. 12 (*ii*) art. 5.21]

(*iii*) $\sum \frac{n^2+1}{n^3+1}$ [**Hint:** S.E. 13 (*ii*) art. 5.21] **(P.T.U., May 2006)**

(*iv*) $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ **(P.T.U., Dec. 2010)**

$$\left[\textbf{Hint:} \text{ Let } u_n = \sin\frac{1}{n};\ v_n = \frac{1}{n};\ \frac{u_n}{v_n} = \frac{\sin\frac{1}{n}}{\frac{1}{n}};\ \underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\ \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \neq 0 \text{ and } \sum v_n = \sum \frac{1}{n} \text{ diverges} \right.$$

$$\left. \therefore \sum u_n \text{ diverges} \right]$$

(*v*) $\sum \frac{n+1}{n^p}$ [**Hint:** S.E. 10 (*ii*) art. 5.21] (*vi*) $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\ldots\infty.$

$\left[\textbf{Hint:}\left(\frac{1}{2}+\frac{1}{2^3}+\ldots\infty\right)+\left(\frac{1}{3^2}+\frac{1}{3^4}+\ldots\infty\right)\right.$ both and G.Ps with C.R. < 1 $\therefore$ convergent]

(*vii*) $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\ldots\infty.$ $\left[\textbf{Hint:}\, u_n=\frac{1}{n(n+3)}=\frac{1}{n^2\left(1+\frac{3}{n}\right)}.\text{ Take } \Sigma\, v_n=\sum\frac{1}{n^2}\text{ which is convergent}\right]$

(*viii*) $\sum\frac{2n^3+5}{4n^5+1}.$ $\left[\textbf{Hint:}\text{ Take } v_n=\frac{1}{n^2}\right]$

(*ix*) $\sum_{n=1}^{\infty}\frac{1}{n^p\,(n+1)^p}.$ $\left[\textbf{Hint:}\, u_n=\frac{1}{n^{2p}\left(1+\frac{1}{n}\right)^p}.\text{ Take } v_n=\frac{1}{n^{2p}}\right]$

(*x*) $\sum_{n=1}^{\infty}\frac{n+1}{n\,(2n-1)}.$ [**Hint:** $u_n=\frac{n\left(1+\frac{1}{n}\right)}{n^2\left(2-\frac{1}{n}\right)}=\frac{1}{n}\,\frac{1+\frac{1}{n}}{2-\frac{1}{n}}$; Take $v=\frac{1}{n}$]

12. If $\Sigma\, u_n$ and $\Sigma\, v_n$ are two +ve term series then

(*i*) If $\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=l$ (finite and non-zero), then $\Sigma\, u_n$ and $\Sigma\, v_n$ both converge and diverge together.

[**Hint:** See art. 5.20 Test III(*i*)]

(*ii*) If $\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=\infty$ and $\Sigma\, v_n$ diverges, then $\Sigma\, u_n$ also diverges. [**Hint:** See art. 5.20 Test III(*iii*)]

(*iii*) $\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=0$ and $\Sigma\, v_n$ converges, then $\Sigma\, u_n$ also converges. [**Hint:** See art. 5.20 Test III(*ii*)]

13. Suppose $a_n>0$, $b_n>0$ $\forall\, n\in$ N. If $\underset{n\to\infty}{\text{Lt}}\frac{a_n}{b_n}=\infty$ and $\Sigma\, a_n$ converges. Can anything be said about $\Sigma\, b_n$? Give reason for your answer. [**Hint:** See art. 5.20 Test III (*iv*)]. **(P.T.U., Dec. 2004)**

14. Test for convergence of the series:

(*i*) $\sum\frac{n^2+1}{n^3+1}$ **(P.T.U., May 2006)**

[**Hint:** S.E. 13 (*ii*) art. 5.21]

(*ii*) $\sum\frac{1}{\sqrt{n}+\sqrt{n+1}}$ **(P.T.U., Dec. 2003)**

[**Hint:** S.E. 11 (*i*) art. 5.21]

(*iii*) $\sum\frac{1}{n\log n}$ **(P.T.U., Dec. 2002)**

[**Hint:** S.E. 12(*ii*) art. 5.21]

(*iv*) $\sum\sqrt[3]{n^3+1}-\sqrt[3]{n^3}$ **(P.T.U., May 2007)**

[**Hint:** S.E. 16 (*ii*) art. 5.21]

15. Test the convergence/divergence of the series $\sum_{n=1}^{\infty}\sqrt{n^4+1}-\sqrt{n^4-1}$. **(P.T.U., Dec. 2012)**

[**Hint:** S.E. 16(*i*) art 5.21]

16. Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{x^n}{2n!}$ **(P.T.U., May 2012)**

[**Hint:** S.E. 4(*iii*) art 5.22]

17. Show that $\sum \dfrac{n}{1+2^{-n}}$ is divergent.

$$\left[\textbf{Hint:} \operatorname*{Lt}_{n\to\infty} \frac{n}{1+2^{-n}} = \frac{\infty}{1+0} = \infty \neq 0 \text{ and } \Sigma\, u_n \text{ is +ve term series } \therefore \text{ divergent}\right]$$

18. Show that $\sum \left(\dfrac{n}{n+1}\right)^n$ is divergent.

$$\left[\begin{array}{l}\textbf{Hint:} \operatorname*{Lt}_{n\to\infty} \left(\dfrac{n}{n+1}\right)^n = \operatorname*{Lt}_{n\to\infty} \dfrac{1}{\left(1+\dfrac{1}{n}\right)^n} = \dfrac{1}{e} \neq 0. \text{ Also } \Sigma\, u_n \text{ is +ve} \\ \qquad\qquad \text{term series } \therefore\ \Sigma\, u_n \text{ is divergent}\end{array}\right].$$

19. Is the series $1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \ldots \infty$. summable ? If so find its sum. [**Hint:** See S.E. 3 (*iii*) art. 5.10]

20. State Cauchy's root test and prove the following:

(*i*) $\sum \dfrac{1}{n^n}$ is convergent

(*ii*) $\sum 5^{-n-(-1)^n}$ is convergent

$$\left[\textbf{Hint:} \operatorname*{Lt}_{n\to\infty} (u_n)^{\frac{1}{n}} = 5^{-1} \times \frac{1}{5^{\frac{(-1)^n}{n}}} = \frac{1}{5} \times 1 = \frac{1}{5} < 1 \ \therefore \text{ convergent} \left(\because \operatorname*{Lt}_{n\to\infty} \frac{(-1)^n}{n} = 0\right)\right]$$

(*iii*) $\sum \left(\dfrac{n+1}{3n}\right)^n$ is convergent

(*iv*) $\sum \left(1+\dfrac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ is convergent. [**Hint:** S.E. 3 (*ii*) art. 5.26] **(P.T.U., May 2008)**

(*v*) $\sum\limits_{n=2}^{\infty} \dfrac{1}{(\log n)^n}$. [**Hint:** S.E. 1 (*iii*) art. 5.26] **(P.T.U., Dec. 2002)**

(*vi*) $\sum \left(\dfrac{n}{n+1}\right)^{n^2}$. [**Hint:** S.E. 1 (*i*) art. 5.26] **(P.T.U., May 2009, Dec. 2012)**

21. State Cauchy's Integral test and prove the following:

(*i*) $\sum \dfrac{1}{n^2+1}$ is convergent. [**Hint:** See S.E. 1 art. 5.27] **(P.T.U., Dec. 2002)**

(*ii*) $\sum \dfrac{8 \tan^{-1} n}{1+n^2}$ is convergent. [**Hint:** See S.E. 7 art. 5.27] **(P.T.U., May 2002, Dec. 2005)**

22. Apply Cauchy's Integral test to test the convergence of the series $\sum_{n=1}^{\infty} 1/n^p$. [**Hint:** S.E. 3 art. 5.27]

23. Test the convergence of the series $\sum \dfrac{x^{n+1}}{(n+1)\sqrt{n}}$. [**Hint:** S.E. 10 art. 5.22]

24. The series $\dfrac{1}{3}-\dfrac{1}{2}+\dfrac{1}{9}-\dfrac{1}{4}+\dfrac{1}{27}-\dfrac{1}{8}+\ldots\dfrac{1}{3^n}-\dfrac{1}{2^n}+\ldots\infty$ does not meet one of the conditions of Leibnitz's test, which one ? Find the sum of the series. [**Hint:** S.E. 5 art. 5.28] **(P.T.U., Dec. 2004)**

25. Examine the convergence of the following series:

(*i*) $1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\ldots\infty$. [**Hint:** S.E. 1 (*ii*) art. 5.28] **(P.T.U., May 2010)**

(*ii*) $1-\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}-\dfrac{1}{\sqrt{4}}+\ldots\infty$. [**Hint:** S.E. 1 (*i*) art. 5.28]

(*iii*) $2-\dfrac{3}{2}+\dfrac{4}{3}-\dfrac{5}{4}+\ldots\infty$. [**Hint:** S.E. 2 (*i*) art. 5.28]

(*iv*) $1+\dfrac{1}{2^2}-\dfrac{1}{3^2}-\dfrac{1}{4^2}+\dfrac{1}{5^2}+\dfrac{1}{6^2}-\dfrac{1}{7^2}-\dfrac{1}{8^2}+\ldots\infty$. [**Hint:** S.E. 4 art. 5.28]

(*v*) $\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}\,n}{2n-1}$. [**Hint:** S.E. 3 art. 5.28]

(*vi*) $\sum_{n=1}^{\infty}\dfrac{1}{\log(n+1)}$

$$\left[\begin{array}{l}\textbf{Hint: } u_n=\dfrac{1}{\log(n+1)},\ u_{n+1}=\dfrac{1}{\log(n+2)},\ \log(n+1)<\log(n+2);\ \dfrac{1}{\log(n+1)}>\dfrac{1}{\log(n+2)}\\ \qquad u_n>u_{n+1}\ \forall\ n\ \underset{n\to\infty}{\text{Lt}}\ u_n=\underset{n\to\infty}{\text{Lt}}\ \dfrac{1}{\log(n+1)}=0.\ \therefore\ \text{Convergent.}\end{array}\right]$$

(*vii*) $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n^2+1}$ (*viii*) $\sum_{n=1}^{\infty}\dfrac{n\,(-1)^{n-1}}{5^n}$.

26. Explain (*i*) absolutely convergent infinite series. **(P.T.U., Dec. 2011)**

(*ii*) conditional convergence of a series by giving some examples.

27. (*a*) Prove that every absolutely convergent series is convergent. [**Hint:** See art. 5.30]

(*b*) Give an example of the series which is convergent but not absolutely convergent. Justify your statement. [**Hint:** Example art. 5.31] **(P.T.U., May 2014)**

28. For what value of x does the series $\sum_{n=0}^{\infty}(-1)^n\,(4x+1)^n$ converges absolutely? **(P.T.U., May 2003)**

[**Hint:** S.E. 3 art. 5.31]

29. Discuss the convergence of the series $\sum_{n=1}^{\infty}\dfrac{(-1)^n\tan^{-1}n}{1+n^2}$ [**Hint:** S.E. 4 art. 5.30] **(P.T.U., May 2003)**

30. Test whether the following series are absolutely convergent or conditionally convergent.

(*i*) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \infty$. [**Hint:** S.E. 1(*a*) art. 5.29] **(P.T.U., Dec. 2006)**

(*ii*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$. [**Hint:** S.E. 1(*b*) art. 5.29] **(P.T.U., Dec. 2012)**

(*iii*) $\sum (-1)^{n-1} \frac{n+1}{n^2}$. **(P.T.U., Dec. 2010)**

31. Test the absolute convergence of $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$ [**Hint:** S.E. 4 art. 5.31] **(P.T.U., May 2012)**

32. Prove that $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$ converges $\forall x$

$$\left[\textbf{Hint:}\ u_n = \frac{x^n}{n!}, u_{n+1} = \frac{x^{n+1}}{(n+1)!}; \frac{u_n}{u_{n+1}} = \frac{(n+1)}{x}; \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \infty \ \forall\ x \quad \therefore \quad \Sigma\, u_n \text{ converges } \forall n\right].$$

33. Find the interval of convergence of $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty$.

[**Hint:** S.E. 5 art. 5.31] **(P.T.U., Dec. 2013)**

34. For what values of x, the power series $1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 \ldots + \left(\frac{-1}{2}\right)^n (x-2)^n + \ldots \infty$ converges ?

[**Hint:** S.E. 6 art. 5.31]

35. What do you understand by uniform convergence of a series ? Explain with the help of an example.
[**Hint:** See art. 5.32] **(P.T.U., May 2004, 2005)**

36. State Weierstrass's M-Test for uniform convergence of $\Sigma\, u_n(x)$ in an interval and apply it to show that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ ($p > 1$) converges uniformly for all values of x. [**Hint:** S.E. 1 (*ii*) art. 5.33] **(P.T.U., May 2003)**

37. State the following:

(*i*) p-series test or Hyper Harmonic test

(*ii*) D'Alembert's Ratio test **(P.T.U., Dec. 2004)**

(*iii*) Raabe's test **(P.T.U., May 2007)**

(*iv*) Logarithmic test **(P.T.U., May 2007)**

(*v*) Gauss test

(*vi*) Cauchy's root test **(P.T.U., Dec. 2002)**

(*vii*) Cauchy's Integral test **(P.T.U., May 2003, Dec. 2005, Jan. 2009, May 2014)**

(*viii*) Leibnitz's test on alternating series **(P.T.U., Dec. 2007, May 2014)**

(*ix*) Power series

(*x*) Absolute convergence of a series **(P.T.U., Dec. 2003)**

(*xi*) Uniform convergence of a series of functions **(P.T.U., May 2004, 2005)**

(*xii*) Weierstrass's M-test.

ANSWERS

11. (*i*) Convergent (*ii*) Divergent (*iii*) Divergent
(*iv*) Divergent (*v*) Convergent for $p > 2$, divergent for $p \leq 2$
(*vi*) Convergent (*vii*) Convergent (*viii*) Convergent
(*ix*) Convergent for $p > \frac{1}{2}$; Divergent for $p \leq \frac{1}{2}$ (*x*) Divergent

14. (*i*) Divergent (*ii*) Divergent (*iii*) Divergent (*iv*) Convergent

15. Convergent

16. Convergent

19. Convergent

23. Convergent for $x \leq 1$; Divergent for $x > 1$

25. (*i*) Convergent (*ii*) Convergent (*iii*) not Convergent (*iv*) Convergent
(*v*) not Convergent (*vi*) Convergent (*vii*) Convergent (*viii*) Convergent

27. (*b*) $\sum \frac{(-1)^{n-1}}{n}$ **28.** $x \in \left(-\frac{1}{2}, 0\right)$ **29.** Convergent

30. (*i*) Converges absolutely (*ii*) Converges conditionally (*iii*) Converges conditionally

31. Converges absolutely

33. $(-1, 1)$ **34.** $0 < x < 4$.

6

Complex Numbers and Elementary Functions of Complex Variable

6.1. RECAPITULATION OF COMPLEX NUMBERS

Students have already studied complex numbers in lower classes and are familiar with the basic concepts of the subject but still we would like to revise the main principles and methods of complex numbers for the benefit of students.

6.2. COMPLEX NUMBERS

(*i*) Definition of a Complex Number

A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$, is called a complex number.

The set of complex numbers is denoted by C.

If $z = x + iy$ is a complex number, then

x is called the *real part* of z and we write $\text{Re}(z) = x$

y is called the *imaginary part* of z and we write $\text{Im}(z) = y$

If $x = 0$ and $y \neq 0$, then $z = 0 + iy = iy$ is called a *purely imaginary number*.

If $y = 0$, then $z = x + i.0 = x$ is a real number.

If $x = 0$ and $y = 0$, then $z = 0 + i\,.\,0 = 0$ is called the *zero complex number.*

(*ii*) Conjugate of a Complex Number

If $z = x + iy$, then $\bar{z} = x - iy$ is called conjugate of z

(*iii*) Properties of Complex Numbers

(*a*) The sum, difference, product and quotient of two complex numbers is a complex number.

(*b*) If a complex number is equal to zero, then its real and imaginary parts are separately equal to zero.

Thus, $\quad \boldsymbol{x + iy = 0 \quad \Rightarrow \quad x = 0}$ **and** $\boldsymbol{y = 0}$

(*c*) If two complex numbers are equal, then their real and imaginary parts are separately equal.

Thus, $\quad \boldsymbol{x + iy = a + ib \quad \Rightarrow \quad x = a}$ **and** $\boldsymbol{y = b}$

(*d*) If two complex numbers are equal, then their conjugates are also equal.

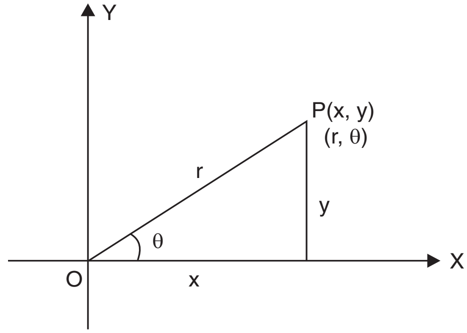
Thus, $\quad \boldsymbol{a + ib = c + id \quad \Rightarrow \quad a - ib = c - id}$

In general $\quad \boldsymbol{f(x + iy) = g(x + iy) \Rightarrow \quad f(x - iy) = g(x - iy)}$ $\qquad$ (*on changing i to – i*)

(*iv*) Polar-form of a Complex Number

Let P represents a non-zero complex number $z = x + iy$ in xy-plane. Then the ordered pair (x, y) represents z. Let (r, θ) be polar coordinates of P. Then $\text{OP} = r$, $\angle \text{XOP} = \theta$

$\therefore \qquad x = r\cos\theta, \quad y = r\sin\theta$



and $\quad z = r\cos\theta + i\, r\sin\theta$

$\quad\quad = r(\cos\theta + i\sin\theta)$

is polar representation of z.

Squaring and adding the values of x and y, we get

$$r^2 = x^2 + y^2$$

then OP $= r = \sqrt{x^2 + y^2}$ is called **Modulus or Norm** or **Absolute Value** of z and is represented by $|z|$.

Dividing y by x, we get $\tan\theta = \dfrac{y}{x}$ or $\theta = \tan^{-1}\dfrac{y}{x}$

'θ' is called **Amplitude** or **Argument** of z. The value of θ lying in the interval $-\pi < \theta \leq \pi$ is called **Principal Value** of amplitude.

(*v*) If z_1 and z_2 are Two Complex Numbers, then

(*a*) $|z_1 z_2| = |z_1||z_2|$

(*b*) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(*c*) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(*d*) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \; (z_2 \neq 0)$

(*e*) $|z_1 + z_2| \leq |z_1| + |z_2|$

(*f*) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

(*g*) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

(*h*) $|z|^2 = |\bar{z}|^2 = z\bar{z}$

(*i*) $\arg z + \arg \bar{z} = 0$

6.3. DE-MOIVRE'S THEOREM

(P.T.U., May 2002, 2004, Dec., 2005, May 2007, 2008, 2010, 2014)

Statement. (*i*) *If n is any integer, positive or negative, then*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

and *(ii) If n is a fraction, positive or negative, then one of the values of*

$(\cos\theta + i\sin\theta)^n$ *is* $\cos n\theta + i\sin n\theta$.

Proof. Case I. *When n is a positive integer.*

We shall prove the theorem by induction method.

When $n = 1$, the theorem becomes

$$(\cos\theta + i\sin\theta)^1 = \cos 1\,\theta + i\sin 1\,\theta$$

$\Rightarrow \quad \cos\theta + i\sin\theta = \cos\theta + i\sin\theta$ which is true.

Let us suppose the theorem is true for $n = m$

i.e., let $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$ ...(1)

Now, $(\cos\theta + i\sin\theta)^{m+1} = (\cos\theta + i\sin\theta)^m(\cos\theta + i\sin\theta)$

$= (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)$ [Using (1)]

$= (\cos m\theta\cos\theta - \sin m\theta\sin\theta) + i(\sin m\theta\cos\theta + \cos m\theta\sin\theta)$

$= \cos(m\theta + \theta) + i\sin(m\theta + \theta) = \cos(m+1)\theta + i\sin(m+1)\theta$

$\Rightarrow$ The theorem is true for $n = m + 1$.

Hence by the Principle of Mathematical Induction, the theorem is true for all positive integers n.

Case II. *When n is a negative integer.*

Let $n = -m$, where m is a positive integer.

$\therefore$ $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta} \qquad \text{[by Case I]}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} + \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta - i^2\sin^2 m\theta} = \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \qquad [\because\ i^2 = -1]$$

$= \cos m\theta - i\sin m\theta = \cos(-m)\theta + i\sin(-m)\theta$

[$\because$ $\cos(-\theta) = \cos\theta$; $\sin(-\theta) = -\sin\theta$]

$= \cos n\theta + i\sin n\theta.$ [$\because$ $-m = n$]

Case III. *When n is a fraction, positive or negative.*

Let $n = \dfrac{p}{q}$, where q is a positive integer and p is any integer, positive or negative. It follows from case I, that

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos\left(q\cdot\frac{\theta}{q}\right) + i\sin\left(q\cdot\frac{\theta}{q}\right) = \cos\theta + i\sin\theta$$

Taking qth root of both sides, $\cos\dfrac{\theta}{q} + i\sin\dfrac{\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{1/q}$

Raising to pth power, $\left(\cos\dfrac{\theta}{q} + i\sin\dfrac{\theta}{q}\right)^p$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$

or $\cos\dfrac{p}{q}\theta + i\sin\dfrac{p}{q}\theta$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$.

Since $\dfrac{p}{q} = n$, $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$

Hence De-Moivre's Theorem is completely established.

Cor. 1. $(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

Cor. 2. $(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

Cor. 3. $(\cos\theta - i\sin\theta)^{-n} = [\cos(-\theta) + i\sin(-\theta)]^{-n} = \cos n\theta + i\sin n\theta$

Cor. 4. $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta.$

Caution. For the application of De-Moivre's Theorem

1. Real part must be with cos and imaginary part with sin *i.e.,* De-Moivre's Theorem cannot be directly applied to $(\sin\theta + i\cos\theta)^n$.

Procedure to find the $(\sin\theta + i\cos\theta)^n$

$$(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$$

2. The angle with sin and cos must be the same *i.e.,* De-Moivre's Theorem cannot be applied to $(\cos\alpha + i\sin\beta)^n$.

Note. $(\text{cis}\,\theta_1)(\text{cis}\,\theta_2)\ .....\ (\text{cis}\,\theta_n) = \text{cis}\,(\theta_1 + \theta_2 + ... + \theta_n)$.

ILLUSTRATIVE EXAMPLES

Example 1. (*a*) *Is* $(\sin\theta + i\cos\theta)^n = \sin n\theta + i\cos n\theta$ *? If not justify it.* **(P.T.U., Dec. 2006)**

(*b*) *Simplify* : $\dfrac{(\cos 3\theta + i\sin 3\theta)^5\,(\cos\theta - i\sin\theta)^3}{(\cos 5\theta + i\sin 5\theta)^7\,(\cos 2\theta - i\sin 2\theta)^5}$.

Sol. (*a*) $(\sin\theta + i\cos\theta)^n \neq \sin n\theta + i\cos n\theta$

$$\because \qquad (\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n$$

$$= \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right) \neq \sin n\theta + i\cos n\theta.$$

(*b*) $$\frac{(\cos 3\theta + i\sin 3\theta)^5\,(\cos\theta - i\sin\theta)^3}{(\cos 5\theta + i\sin 5\theta)^7\,(\cos 2\theta - i\sin 2\theta)^5} = \frac{[(\cos\theta + i\sin\theta)^3]^5\,[(\cos\theta + i\sin\theta)^{-1}]^3}{[(\cos\theta + i\sin\theta)^5]^7\,[(\cos\theta + i\sin\theta)^{-2}]^5}$$

$$= \frac{(\cos\theta + i\sin\theta)^{15}\,(\cos\theta + i\sin\theta)^{-3}}{(\cos\theta + i\sin\theta)^{35}\,(\cos\theta + i\sin\theta)^{-10}} = (\cos\theta + i\sin\theta)^{15-3-35+10}$$

$$= (\cos\theta + i\sin\theta)^{-13} = \cos 13\theta - i\sin 13\theta.$$

Example 2. *Simplify* : $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4$.

Sol. $$\left[\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right]^4 = \frac{(\cos\theta + i\sin\theta)^4}{\left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{\cos 4\left(\frac{\pi}{2}-\theta\right) + i\sin 4\left(\frac{\pi}{2}-\theta\right)} = \frac{(\cos\theta + i\sin\theta)^4}{\cos(2\pi - 4\theta) + i\sin(2\pi - 4\theta)}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{\cos 4\theta - i\sin 4\theta} = \frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta + i\sin\theta)^{-4}} = (\cos\theta + i\sin\theta)^8 = \cos 8\theta + i\sin 8\theta.$$

Example 3. (*i*) *If* $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, *prove that* $\dfrac{x-y}{x+y} = i\tan\dfrac{\alpha-\beta}{2}$.

(P.T.U., Dec. 2012)

(*ii*) $\dfrac{(x+y)(xy-1)}{(x-y)(xy+1)} = \dfrac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}$.

Sol. (*i*) $$\text{LHS} = \frac{x-y}{x+y} = \frac{(\cos\alpha+i\sin\alpha)-(\cos\beta+i\sin\beta)}{(\cos\alpha+i\sin\alpha)+(\cos\beta+i\sin\beta)}$$

$$= \frac{(\cos\alpha-\cos\beta)+i(\sin\alpha-\sin\beta)}{(\cos\alpha+\cos\beta)+i(\sin\alpha+\sin\beta)}$$

$$= \frac{-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}+i\cdot 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+i\cdot 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}$$

$$= \frac{i^2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}+i\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+i\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}} \qquad [\because\ i^2=-1]$$

$$= \frac{i\sin\frac{\alpha-\beta}{2}\left[i\sin\frac{\alpha+\beta}{2}+\cos\frac{\alpha+\beta}{2}\right]}{\cos\frac{\alpha-\beta}{2}\left[\cos\frac{\alpha+\beta}{2}+i\sin\frac{\alpha+\beta}{2}\right]} = i\tan\frac{\alpha-\beta}{2} = \text{RHS}$$

(*ii*) $$\text{LHS} = \frac{[(\cos\alpha+i\sin\alpha)+(\cos\beta+i\sin\beta)]\,[(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)-1]}{[(\cos\alpha+i\sin\alpha)-(\cos\beta+i\sin\beta)]\,[(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)+1]}$$

$$= \frac{[(\cos\alpha+\cos\beta)]+i(\sin\alpha+\sin\beta)]\,[\cos(\alpha+\beta)+i\sin(\alpha+\beta)-1]}{[(\cos\alpha-\cos\beta)+i(\sin\alpha-\sin\beta)]\,[\cos(\alpha+\beta)+i\sin(\alpha+\beta)+1]}$$

$$= \frac{\left[2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2i\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right]\left[-[(1-\cos(\alpha+\beta)]+i\sin(\alpha+\beta)]\right]}{\left[-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}+2i\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right]\left[[1+\cos(\alpha+\beta)]+i\sin(\alpha+\beta)\right]}$$

$$= \frac{2\cos\frac{\alpha-\beta}{2}\left[\cos\frac{\alpha+\beta}{2}+i\sin\frac{\alpha+\beta}{2}\right]\left[-2\sin^2\frac{\alpha+\beta}{2}+2i\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}\right]}{2i\sin\frac{\alpha-\beta}{2}\left[\cos\frac{\alpha+\beta}{2}+i\sin\frac{\alpha+\beta}{2}\right]\left[2\cos^2\frac{\alpha+\beta}{2}+2i\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}\right]}$$

$$= \frac{\cos\frac{\alpha-\beta}{2}}{i\sin\frac{\alpha-\beta}{2}}\ \frac{2i\sin\frac{\alpha+\beta}{2}\left[\cos\frac{\alpha+\beta}{2}+i\sin\frac{\alpha+\beta}{2}\right]}{2\cos\frac{\alpha+\beta}{2}\left[\cos\frac{\alpha+\beta}{2}+i\sin\frac{\alpha+\beta}{2}\right]}$$

$$= \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}} = \frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta} = \text{RHS}$$

Example 4. *If 'a' denotes $\cos 2\alpha + i \sin 2\alpha$ with similar expressions for b, c, d, prove that*

(*i*) $\sqrt{abcd} + \dfrac{1}{\sqrt{abcd}} = 2\cos(\alpha+\beta+\gamma+\delta)$ (*ii*) $\sqrt{\dfrac{ab}{cd}} + \sqrt{\dfrac{cd}{ab}} = 2\cos(\alpha+\beta-\gamma-\delta)$.

Sol. (*i*)

$$a = \cos 2\alpha + i\sin 2\alpha,\ b = \cos 2\beta + i \sin 2\beta$$
$$c = \cos 2\gamma + i\sin 2\gamma,\ d = \cos 2\delta + i\sin 2\delta$$
$$abcd = (\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta)(\cos 2\gamma + i\sin 2\gamma)(\cos 2\delta + i\sin 2\delta)$$
$$= \cos(2\alpha+2\beta+2\gamma+2\delta) + i\sin(2\alpha+2\beta+2\gamma+2\delta)$$
$$\sqrt{abcd} = (abcd)^{1/2} = [\cos(2\alpha+2\beta+2\gamma+2\delta) + i\sin(2\alpha+2\beta+2\gamma+2\delta)]^{1/2}$$
$$= \cos(\alpha+\beta+\gamma+\delta) + i\sin(\alpha+\beta+\gamma+\delta)$$
$$\frac{1}{\sqrt{abcd}} = (\sqrt{abcd})^{-1} = [\cos(\alpha+\beta+\gamma+\delta) + i\sin(\alpha+\beta+\gamma+\delta)]^{-1}$$
$$= \cos(\alpha+\beta+\gamma+\delta) - i\sin(\alpha+\beta+\gamma+\delta)$$
$$\therefore\ \sqrt{abcd} + \frac{1}{\sqrt{abcd}} = 2\cos(\alpha+\beta+\gamma+\delta).$$

(*ii*)
$$\frac{ab}{cd} = \frac{(\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta)}{(\cos 2\gamma + i\sin 2\gamma)(\cos 2\delta + i\sin 2\delta)} = \frac{\cos(2\alpha+2\beta) + i\sin(2\alpha+2\beta)}{\cos(2\gamma+2\delta) + i\sin(2\gamma+2\delta)}$$
$$= \cos(2\alpha+2\beta-2\gamma-2\delta) + i\sin(2\alpha+2\beta-2\gamma-2\delta)$$
$$\therefore\ \sqrt{\frac{ab}{cd}} = \left[\frac{ab}{cd}\right]^{1/2} = [\cos(2\alpha+2\beta-2\gamma-2\delta) + i\sin(2\alpha+2\beta-2\gamma-2\delta)]^{1/2}$$
$$= \cos(\alpha+\beta-\gamma-\delta) + i\sin(\alpha+\beta-\gamma-\delta)$$
$$\sqrt{\frac{cd}{ab}} = \left(\sqrt{\frac{ab}{cd}}\right)^{-1} = [\cos(\alpha+\beta-\gamma-\delta) + i\sin(\alpha+\beta-\gamma-\delta)]^{-1}$$
$$= \cos(\alpha+\beta-\gamma-\delta) - i\sin(\alpha+\beta-\gamma-\delta)$$
$$\therefore\ \sqrt{\frac{ab}{cd}} + \sqrt{\frac{cd}{ab}} = 2\cos(\alpha+\beta-\gamma-\delta).$$

Example 5. *If $\sin\alpha + \sin\beta + \sin\gamma = \cos\alpha + \cos\beta + \cos\gamma = 0$, prove that*

(*i*) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$ **(P.T.U., Dec. 2002)**

(*ii*) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

(*iii*) $\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta) = 0$

(*iv*) $\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0$

(*v*) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$ (*vi*) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(*vii*) $\Sigma \cos 4\alpha = 2\,\Sigma \cos 2(\beta+\gamma)$ (*viii*) $\Sigma \sin 4\alpha = 2\,\Sigma \sin 2(\beta+\gamma)$

(*ix*) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 3/2.$ **(P.T.U., May 2003)**

Sol. Let $a = \cos\alpha + i\sin\alpha$; $b = \cos\beta + i\sin\beta$; $c = \cos\gamma + i\sin\gamma$.

$$a+b+c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$$
$$= (0) + i(0) = 0$$
$$\therefore\ a+b+c = 0$$

$\Rightarrow \quad a + b = -c$

Cubing both sides $\quad (a+b)^3 = -c^3$

or $\quad a^3 + b^3 + 3ab\,(a+b) = -c^3$

or $\quad a^3 + b^3 - 3abc = -c^3$

or $\quad a^3 + b^3 + c^3 = 3abc$

$$\Rightarrow \quad (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$$
$$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$

$$\Rightarrow \quad (\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma)$$
$$= 3[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$$

$$\Rightarrow \quad (\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i\,(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$$
$$= 3\cos(\alpha+\beta+\gamma) + i \cdot 3\sin(\alpha+\beta+\gamma)$$

Equating the real and imaginary parts on both sides,

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$$
$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$$

$\therefore$ Parts (*i*) and (*ii*) are proved.

Now, $\quad \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = a^{-1} + b^{-1} + c^{-1}$

$$= (\cos\alpha + i\sin\alpha)^{-1} + (\cos\beta + i\sin\beta)^{-1} + (\cos\gamma + i\sin\gamma)^{-1}$$
$$= (\cos\alpha - i\sin\alpha) + (\cos\beta - i\sin\beta) + (\cos\gamma - i\sin\gamma)$$
$$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$$
$$= 0 - i(0) \qquad \text{[From given conditions]}$$

$\therefore \quad \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = 0 \quad \Rightarrow \quad bc + ca + ab = 0$ or $\Sigma bc = 0$

$\Rightarrow \quad \Sigma(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma) = 0 \quad \Rightarrow \quad \Sigma[\cos(\beta+\gamma) + i\sin(\beta+\gamma)] = 0$

Equating the real and imaginary parts on both sides

$$\Sigma\cos(\beta+\gamma) = 0$$
$$\Sigma\sin(\beta+\gamma) = 0$$

$\therefore$ Parts (*iii*) and (*iv*) are proved.

Since $\quad a + b + c = 0$

Squaring $\quad a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$

But $\quad ab + bc + ca = 0 \qquad$ [Proved above]

$\therefore \quad a^2 + b^2 + c^2 = 0 \quad$ *i.e.*, $\quad \Sigma a^2 = 0$

$\Rightarrow \quad \Sigma(\cos\alpha + i\sin\alpha)^2 = 0$

$\Rightarrow \quad \Sigma(\cos 2\alpha + i\sin 2\alpha) = 0$

Equating the real and imaginary parts on both sides

$$\left.\begin{array}{l}\Sigma\cos 2\alpha = 0\\ \Sigma\sin 2\alpha = 0\end{array}\right\}$$

$\therefore$ Parts (*v*) and (*vi*) are proved

$\because \quad a + b + c = 0$

$\therefore \qquad a + b = -c$

Squaring $\qquad a^2 + b^2 + 2ab = c^2$ or $a^2 + b^2 - c^2 = -2ab$

Squaring again $\quad a^4 + b^4 + c^4 + 2a^2b^2 - 2b^2c^2 - 2c^2a^2 = 4a^2b^2$

$\Rightarrow \qquad \Sigma a^4 = 2\Sigma b^2c^2$

$\Rightarrow \qquad \Sigma(\cos\alpha + i\sin\alpha)^4 = 2\Sigma(\cos\beta + i\sin\beta)^2(\cos\gamma + i\sin\gamma)^2$

$\Rightarrow \qquad \Sigma(\cos 4\alpha + i\sin 4\alpha) = 2\Sigma(\cos 2\beta + i\sin 2\beta)(\cos 2\gamma + i\sin 2\gamma)$

$\Rightarrow \qquad \Sigma(\cos 4\alpha + i\sin 4\alpha) = 2\Sigma[\cos(2\beta + 2\gamma) + i\sin(2\beta + 2\gamma)]$

Equating the real and imaginary parts on both sides,

$$\left.\begin{aligned} \Sigma\cos 4\alpha &= 2\,\Sigma\cos 2(\beta+\gamma) \\ \Sigma\sin 4\alpha &= 2\Sigma\sin 2(\beta+\gamma) \end{aligned}\right\}$$

Parts (*vii*) and (*viii*) are proved

Now, $\qquad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \dfrac{1-\cos 2\alpha}{2} + \dfrac{1-\cos 2\beta}{2} + \dfrac{1-\cos 2\gamma}{2}$

$$= \frac{3}{2} - \frac{1}{2}(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) = \frac{3}{2} - \frac{1}{2}\cdot 0 \text{ from part `}v\text{'}$$

$$= \frac{3}{2}$$

Similarly, $\quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{1+\cos 2\alpha}{2} + \dfrac{1+\cos 2\beta}{2} + \dfrac{1+\cos 2\gamma}{2}$

$$= \frac{3}{2} + \frac{1}{2}(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) = \frac{3}{2} + \frac{1}{2}\cdot 0 \text{ from part `}v\text{'}$$

$$= \frac{3}{2}.$$

Hence (*ix*) part is proved.

Example 6. *If* $\cos\alpha + 2\cos\beta + 3\cos\gamma = 0$ *;* $\sin\alpha + 2\sin\beta + 3\sin\gamma = 0$ *prove that*

(*i*) $\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha + \beta + \gamma)$

(*ii*) $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha + \beta + \gamma)$

(*iii*) $\cos(2\alpha - \beta - \gamma) + 8\cos(2\beta - \gamma - \alpha) + 27\cos(2\gamma - \alpha - \beta) = 18$

(*iv*) $\sin(2\alpha - \beta - \gamma) + 8\sin(2\beta - \gamma - \alpha) + 27\sin(2\gamma - \alpha - \beta) = 0.$

Sol. Let $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$

By given conditions $a + 2b + 3c = (\cos\alpha + 2\cos\beta + 3\cos\gamma) + i(\sin\alpha + 2\sin\beta + 3\sin\gamma) = 0$

$\therefore \qquad a + 2b = -3c \qquad \ldots(1)$

Cubing both sides

$$(a + 2b)^3 = -27c^3$$

or $\qquad a^3 + 8b^3 + 6ab(a + 2b) = -27c^3$

$\qquad a^3 + 8b^3 + 6ab(-3c) = -27c^3 \qquad$ [Using (1)]

or $\qquad a^3 + 8b^3 + 27c^3 = 18abc \qquad \ldots(2)$

$(\cos\alpha + i\sin\alpha)^3 + 8(\cos\beta + i\sin\beta)^3 + 27(\cos\gamma + i\sin\gamma)^3$

$$= 18(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$

$$[\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma] + i[\sin 3\alpha + 8\sin 3\beta + 27\cos 3\gamma]$$

$$= 18[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$$

Comparing real and imaginary parts

$\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$; part (*i*) is proved

$\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$; part (*ii*) is proved

Now, from (2) $\quad a^3 + 8b^3 + 27c^3 = 18\,abc$

Divide by abc; $\quad \dfrac{a^2}{bc} + 8\dfrac{b^2}{ac} + 27\dfrac{c^2}{ab} = 18$

$$\therefore \quad \frac{(\cos\alpha + i\sin\alpha)^2}{(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)} + 8\frac{(\cos\beta + i\sin\beta)^2}{(\cos\alpha + i\sin\alpha)(\cos\gamma + i\sin\gamma)}$$

$$+ 27\frac{(\cos\gamma + i\sin\gamma)^2}{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)} = 18$$

$$\therefore \quad \frac{\text{cis } 2\alpha}{\text{cis }(\beta+\gamma)} + 8\frac{\text{cis } 2\beta}{\text{cis }(\gamma+\alpha)} + 27\frac{\text{cis } 2\gamma}{\text{cis }(\alpha+\beta)} = 18$$

or $\quad \text{cis }(2\alpha-\beta-\gamma) + 8\,\text{cis }(2\beta-\gamma-\alpha) + 27\,\text{cis }(2\gamma-\alpha-\beta) = 18$

Comparing real and Imaginary parts on both sides

$\cos(2\alpha-\beta-\gamma) + 8\cos(2\beta-\gamma-\alpha) + 27\cos(2\gamma-\alpha-\beta) = 18$

$\sin(2\alpha-\beta-\gamma) + 8\sin(2\beta-\gamma-\alpha) + 27\sin(2\gamma-\alpha-\beta) = 0$ (*iii*) and (*iv*) are proved.

Example 7. *Find the general value of* θ *which satisfies the equation*

$$(\cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta) \ldots\ldots [\cos(2r-1)\theta + i\sin(2r-1)\theta] = 1.$$

Sol. $\quad (\cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta) \ldots\ldots [\cos(2r-1)\theta + i\sin(2r-1)\theta] = 1$

$\Rightarrow \quad \cos[\theta + 3\theta + \ldots\ldots + (2r-1)\theta] + i\sin[\theta + 3\theta + \ldots\ldots + (2r-1)\theta] = 1$

$\Rightarrow \quad \cos[1 + 3 + \ldots\ldots + (2r-1)]\theta + i\sin[1 + 3 + \ldots\ldots + (2r-1)]\theta = 1$

$\Rightarrow \quad \cos\dfrac{r}{2}(1 + 2r - 1)\theta + i\sin\dfrac{r}{2}(1 + 2r - 1)\theta = 1$

[$\because$ 1, 3, 5, ..., $2r-1$ form an A.P. with r terms.

Their sum $= \dfrac{\text{Number of terms}}{2}$ (First term + Last term)]

$\Rightarrow \quad \cos(r^2\theta) + i\sin(r^2\theta) = 1.$

Equating the real and imaginary parts on both sides,

$$\left.\begin{array}{l}\cos(r^2\theta) = 1\\ \sin(r^2\theta) = 1\end{array}\right\} \quad \Rightarrow \quad r^2\theta = 2n\pi$$

Hence $\theta = \dfrac{2n\pi}{r^2}$, where n is any integer.

Example 8. (*a*) *If* $x_r = \cos \dfrac{\pi}{2^r} + i \sin \dfrac{\pi}{2^r}$, *prove that* $x_1 x_2 x_3 \ldots\ldots \infty = -1$.

(*b*) *If* $x_r = \cos \dfrac{\pi}{3^r} + i \sin \dfrac{\pi}{3^r}$, *show that* $x_1 x_2 x_3 \ldots x_n = \cos \left[\dfrac{\pi}{2}\left(1 - \dfrac{1}{3^n}\right)\right] + i \sin \left[\dfrac{\pi}{2}\left(1 - \dfrac{1}{3^n}\right)\right]$.

Hence show that $x_1 x_2 x_3 \ldots \infty = i$. **(P.T.U., May 2003)**

Sol. (*a*) $x_r = \cos \dfrac{\pi}{2^r} + i \sin \dfrac{\pi}{2^r}$

Putting $r = 1, 2, 3, \ldots$, we have

$$x_1 = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}, \; x_2 = \frac{\pi}{2^2} + i \sin \frac{\pi}{2^2}$$

$$x_3 = \cos \frac{\pi}{2^3} + i \sin \frac{\pi}{2^3} \text{ and so on.}$$

$\therefore$

$$x_1 x_2 x_3 \ldots \text{ to } \infty$$

$$= \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)\left(\cos \frac{\pi}{2^2} + i \sin \frac{\pi}{2^2}\right)\left(\cos \frac{\pi}{2^3} + i \sin \frac{\pi}{2^3}\right) \ldots \text{ to } \infty$$

$$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\ldots \text{ to } \infty\right) + i \sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\ldots \text{ to } \infty\right)$$

$$= \cos \frac{\frac{\pi}{2}}{1 - \frac{1}{2}} + i \sin \frac{\frac{\pi}{2}}{1 - \frac{1}{2}}$$

$\because \; \dfrac{\pi}{2} + \dfrac{\pi}{2^2} + \dfrac{\pi}{2^3} \ldots\ldots \infty$ is infinite G.P.

and sum of infinite G.P. $= \dfrac{a}{1 - r}$

$$= \cos \pi + i \sin \pi = -1.$$

(*b*)

$$x_r = \cos \frac{\pi}{3^r} + i \sin \frac{\pi}{3^r} = \cos \frac{\pi}{3^r}$$

Put $r = 1, 2, 3, \ldots, n$

$$x_1 x_2 x_3 \ldots x_n = \text{cis} \frac{\pi}{3} \text{ cis} \frac{\pi}{3^2} \text{ cis} \frac{\pi}{3^3} \ldots \text{cis} \frac{\pi}{3^n}$$

$$= \text{cis}\left[\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} \ldots \frac{\pi}{3^n}\right] = \text{cis} \frac{\pi}{3}\left[1 + \frac{1}{3} + \frac{1}{3^2} \ldots \frac{1}{3^n - 1}\right]$$

which is a G.P. with C.R. $= \dfrac{1}{3}$

$$= \text{cis} \frac{\pi}{3} \frac{1\left[1 - \frac{1}{3^n}\right]}{1 - \frac{1}{3}} = \text{cis} \frac{\pi}{2}\left[1 - \frac{1}{3^n}\right]$$

$\because$ for a G.P. $S_n = \dfrac{a(1 - r^n)}{1 - r}$

$$= \text{cis} \frac{\pi}{2}\left[1 - \frac{1}{3^n}\right] + i \sin \frac{\pi}{2}\left[1 - \frac{1}{3^n}\right]$$

When $n \to \infty$, $\dfrac{1}{3^n} \to 0$ *i.e.*,

$$x_1 x_2 x_3 \ldots \infty = \underset{n \to \infty}{\text{Lt}} \text{ cis } \frac{\pi}{2}\left[1 - \frac{1}{3^n}\right] = \text{cis } \frac{\pi}{2}$$

$$= \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i.\, 1 = i$$

Example 9. *If* $x + \dfrac{1}{x} = 2 \cos \theta$, $y + \dfrac{1}{y} = 2 \cos \phi$, *prove that one of the values of*

(*i*) $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m}$ *is* $2 \cos (m\theta - n\phi)$ (*ii*) $x^m y^n + \dfrac{1}{x^m y^n}$ *is* $2 \cos (m\theta + n\phi)$.

Sol. $x + \dfrac{1}{x} = 2\cos \theta$

$\Rightarrow$ $x^2 + 1 = 2x \cos \theta \;\Rightarrow\; x^2 - 2x \cos \theta + \cos^2\theta + \sin^2 \theta = 0$

$\Rightarrow$ $(x - \cos \theta)^2 = -\sin^2 \theta \;\Rightarrow\; x - \cos \theta = \pm\, i \sin \theta \;\Rightarrow\; x = \cos \theta \pm\, i \sin \theta$

$\therefore$ One of the values of x is $\cos \theta + i \sin \theta$.

Similarly, one of the values of y is $\cos \phi + i \sin \phi$.

(*i*) One of the values of $\dfrac{x^m}{y^n} = \dfrac{(\cos \theta + i \sin \theta)^m}{(\cos \phi + i \sin \phi)^n} = \dfrac{\cos m\theta + i \sin m\theta}{\cos n\phi + i \sin n\phi} = \cos (m\theta - n\phi) + i \sin (m\theta - n\phi)$

One of the values of $\dfrac{y^n}{x^m}$, *i.e.*, $\left[\dfrac{x^m}{y^n}\right]^{-1}$

$$= [\cos (m\theta - n\phi) + i \sin (m\theta - n\phi)]^{-1} = \cos (m\theta - n\phi) - i \sin (m\theta - n\phi).$$

Hence one of the values of $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m}$ is $2 \cos (m\theta - n\phi)$.

(*ii*) One of the values of $x^m y^n$

$$= (\cos \theta + i \sin \theta)^m (\cos \phi + i \sin \phi)^n = (\cos m\theta + i \sin m\theta)(\cos n\phi + i \sin n\phi)$$
$$= \cos (m\theta + n\phi) + i \sin (m\theta + n\phi)$$

One of the values of $\dfrac{1}{x^m y^n}$, *i.e.* $(x^m y^n)^{-1}$

$$= [\cos (m\theta + n\phi) + i \sin (m\theta + n\phi)]^{-1} = \cos (m\theta + n\phi) - i \sin (m\theta + n\phi).$$

Hence one of the values of $x^m y^n + \dfrac{1}{x^m y^n}$ is $2 \cos (m\theta + n\phi)$.

Example 10. (*a*) *If* $2 \cos \theta = x + \dfrac{1}{x}$, *prove that* $\dfrac{x^{2n} + 1}{x^{2n-1} + x} = \dfrac{\cos n\theta}{\cos (n-1)\theta}$.

(*b*) *If* $x^2 - 2x \cos \theta + 1 = 0$ *show that* $x^{2n} - 2x^n \cos n\theta + 1 = 0$.

(*c*) *Find an equation whose roots are the nth powers of the roots of the equation* $x^2 - 2x \cos \theta + 1 = 0$.

Sol. (*a*) $$2\cos\theta = x + \frac{1}{x} \quad \text{or} \quad x^2 - 2x\cos\theta + 1 = 0$$

Solve for x; $$x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

or $$x = \frac{2\cos\theta \pm 2i\sin\theta}{2} = \cos\theta \pm i\sin\theta \qquad \ldots(1)$$

$\therefore$ Two values of x are $\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$

Choose any one of the two values

Let $$x = \cos\theta + i\sin\theta$$

Then
$$\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{(\cos\theta + i\sin\theta)^{2n}+1}{(\cos\theta+i\sin\theta)^{2n-1}+(\cos\theta+i\sin\theta)}$$
$$= \frac{\cos 2n\theta + i\sin 2n\theta + 1}{\cos(2n-1)\theta + i\sin(2n-1)\theta + \cos\theta + i\sin\theta}$$
$$= \frac{(1+\cos 2n\theta) + i\sin 2n\theta}{[\cos(2n-1)\theta + \cos\theta] + i[\sin(2n-1)\theta + \sin\theta]}$$
$$= \frac{2\cos^2 n\theta + 2i\sin n\theta\cos n\theta}{2\cos n\theta\cos(n-1)\theta + i\,2\sin n\theta\cos(n-1)\theta}$$
$$= \frac{2\cos n\theta[\cos n\theta + i\sin n\theta]}{2\cos(n-1)\theta[\cos n\theta + i\sin n\theta]} = \frac{\cos n\theta}{\cos(n-1)\theta}$$

Part (*a*) is proved.

(*b*) Solving $x^2 - 2x\cos\theta + 1 = 0$ we get $x = \cos\theta \pm i\sin\theta$ from (1)

Let $$x = \cos\theta + i\sin\theta$$
$$\begin{aligned}
x^{2n} - 2x^n\cos n\theta + 1 &= (\cos\theta + i\sin\theta)^{2n} - 2(\cos\theta + i\sin\theta)^n\cos n\theta + 1\\
&= \cos 2n\theta + i\sin 2n\theta - 2\cos n\theta[\cos n\theta + i\sin n\theta] + 1\\
&= \cos 2n\theta + i\sin 2n\theta - 2\cos^2 n\theta - 2i\sin n\theta\cos n\theta + 1\\
&= [\cos 2n\theta - 2\cos^2 n\theta + 1] + i[\sin 2n\theta - 2\sin n\theta\cos n\theta]\\
&= [(1+\cos 2n\theta) - 2\cos^2 n\theta] + i[\sin 2n\theta - \sin 2n\theta]\\
&= [2\cos^2 n\theta - 2\cos^2 n\theta] + i[\sin 2n\theta - \sin 2n\theta]\\
&= 0 + i\,.\,0 = 0 = \text{RHS}
\end{aligned}$$

(*c*) Roots of $x^2 - 2x\cos\theta + 1 = 0$ are

$\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$ from (1)

Let $\alpha = \cos\theta + i\sin\theta,\ \beta = \cos\theta - i\sin\theta$ (Proved in part *a*)

We want to form an equation whose roots are α^n and β^n.

$\therefore$ Quadratic equation is $x^2 - (\alpha^n + \beta^n)x + (\alpha^n\beta^n) = 0$ $\quad | \because \quad x^2 - Sx + P = 0$
$$\begin{aligned}
\alpha^n + \beta^n &= (\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\\
&= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2\cos n\theta\\
\alpha^n\beta^n &= (\cos\theta + i\sin\theta)^n(\cos\theta - i\sin\theta)^n = [\cos^2\theta + \sin^2\theta]^n = 1
\end{aligned}$$

$\therefore$ Required equation is $\quad x^2 - 2\cos n\theta\, x + 1 = 0$

Example 11. *Prove that:* $(1+\sin\theta+i\cos\theta)^n+(1+\sin\theta-i\cos\theta)^n = 2^{n+1}\cos^n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)$.

(P.T.U., May 2008)

Sol. $(1+\sin\theta+i\cos\theta)^n+(1+\sin\theta-i\cos\theta)^n$

$$= \left[1+\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n + \left[1+\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n$$

$$= \left[2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+2i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^n$$

$$+\left[2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-2i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^n$$

$$= 2^n\cos^n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}^n + \left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}^n\right]$$

$$= 2^n\cos^n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)+i\sin\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)+\cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)-i\sin\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)\right]$$

$$= 2^n\cos^n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)2\cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right) = 2^{n+1}\cos^n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)$$

Example 12. *Prove that* $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$.

Sol. $(\sin\theta+i\cos\theta)(\sin\theta-i\cos\theta) = \sin^2\theta - i^2\cos^2\theta$

$= \sin^2\theta+\cos^2\theta$ $\quad [\because \ i^2=-1]$

$= 1$ $\quad \ldots(1)$

$$\therefore \quad \left[\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right]^n = \left[\frac{(\sin\theta+i\cos\theta)(\sin\theta-i\cos\theta)+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right]^n \quad \text{[Using (1)]}$$

[Note the step]

$$= \left[\frac{(\sin\theta+i\cos\theta)\,(\sin\theta-i\cos\theta+1)}{1+\sin\theta-i\cos\theta}\right]^n = (\sin\theta+i\cos\theta)^n$$

$$= \left[\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n$$

$$= \cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right).$$

Example 13. (*i*) *Prove that* $(a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}} = 2(a^2+b^2)^{\frac{m}{2n}} \cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right)$

(*ii*) $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^{n+1}\cos\frac{n\pi}{6}$ **(P.T.U., May 2004)**

(*iii*) $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$. **(P.T.U., May 2003)**

Sol. (*i*) Let $a = r\cos\theta$ and $b = r\sin\theta$

Squaring and adding, $r^2 = a^2+b^2 \quad \therefore \quad r = \sqrt{a^2+b^2}$

Dividing $\tan\theta = \frac{b}{a} \quad \therefore \quad \theta = \tan^{-1}\frac{b}{a}$

$$(a+ib)^{\frac{m}{n}} + (a-ib)^{\frac{m}{n}} = [r(\cos\theta + i\sin\theta)]^{\frac{m}{n}} + [r(\cos\theta - i\sin\theta)]^{\frac{m}{n}}$$

$$= r^{\frac{m}{n}}(\cos\theta + i\sin\theta)^{\frac{m}{n}} + r^{\frac{m}{n}}(\cos - i\sin\theta)^{\frac{m}{n}}$$

$$= r^{\frac{m}{n}}\left(\cos\frac{m}{n}\theta + i\sin\frac{m}{n}\theta\right) + r^{\frac{m}{n}}\left(\cos\frac{m}{n}\theta - i\sin\frac{m}{n}\theta\right)$$

$$= r^{\frac{m}{n}}\left(2\cos\frac{m}{n}\theta\right) = (\sqrt{a^2+b^2})^{\frac{m}{n}} \cdot 2\cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right)$$

$$= 2(a^2+b^2)^{\frac{m}{2n}}\cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right).$$

(*ii*) Let $\sqrt{3} = r\cos\theta$ and $1 = r\sin\theta$

Squaring and adding $4 = r^2 \quad \therefore \quad r = 2$

$$\cos\theta = \frac{\sqrt{3}}{2}, \quad \sin\theta = \frac{1}{2}$$

$$\therefore \qquad \theta = \frac{\pi}{6}$$

Now, $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = (r\cos\theta + ir\sin\theta)^n + (r\cos\theta - ir\sin\theta)^n$

$$= r^n(\cos\theta + i\sin\theta)^n + r^n(\cos\theta - i\sin\theta)^n$$

Apply De-Moivre's theorem

$$= r^n\begin{Bmatrix}\cos n\theta + i\sin n\theta \\ +\cos n\theta - i\sin n\theta\end{Bmatrix}$$

$$= r^n \cdot 2\cos n\theta$$

$$= 2^n \cdot 2\cos n\cdot\frac{\pi}{6} = 2^{n+1}\cos n\,\frac{\pi}{6}.$$

(*iii*) Put $1 = r\cos\theta$ and $1 = r\sin\theta$.

Squaring and adding $2 = r^2 \quad \therefore \quad r = \sqrt{2}$

Dividing $\tan\theta = 1 \quad \therefore \quad \theta = \dfrac{\pi}{4}$

$$\therefore \quad (1+i)^n + (1-i)^n = (r\cos\theta + i\, r\sin\theta)^n + (r\cos\theta - i\, r\sin\theta)^n$$

$$= r^n[\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta]$$

$$= \left(\sqrt{2}\right)^n . \, 2\cos n\theta = 2 \, . \, 2^{n/2}\cos\frac{n\pi}{4} = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}.$$

Example 14. *If* $(a_1 + ib_1)(a_2 + ib_2) \ldots\ldots (a_n + ib_n) = A + iB$, *prove that,*

(*i*) $(a_1^2 + b_1^2)(a_2^2 + b_2^2) \ldots\ldots (a_n^2 + b_n^2) = A^2 + B^2$.

(*ii*) $\tan^{-1}\dfrac{b_1}{a_1} + \tan^{-1}\dfrac{b_2}{a_2} + \ldots\ldots + \tan^{-1}\dfrac{b_n}{a_n} = \tan^{-1}\dfrac{B}{A}$.

Sol. Let $a_1 + ib_1 = r_1(\cos\theta_1 + i\sin\theta_1)$.

Equating real and imaginary parts on both sides $r_1\cos\theta_1 = a_1$; $r_1\sin\theta_1 = b_1$

Squaring and adding, $r_1^2 = a_1^2 + b_1^2$

Dividing, $\tan\theta_1 = \dfrac{b_1}{a_1}$ or $\theta_1 = \tan^{-1}\dfrac{b_1}{a_1}$

Similarly, $r_2^2 = a_2^2 + b_2^2$, $\theta_2 = \tan^{-1}\dfrac{b_2}{a_2}$

$r_3^2 = a_3^2 + b_3^2$, $\theta_3 = \tan^{-1}\dfrac{b_3}{a_3}$

....

....

$r_n^2 = a_n^2 + b_n^2$, $\theta_n = \tan^{-1}\dfrac{b_n}{a_n}$

Now it is given that $(a_1 + ib_1)(a_2 + ib_2) \ldots\ldots (a_n + ib_n) = A + iB$

$\Rightarrow \quad r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2) \ldots\ldots r_n(\cos\theta_n + i\sin\theta_n) = A + iB$

$\Rightarrow \quad r_1 r_2 \ldots\ldots r_n[(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots\ldots (\cos\theta_n + i\sin\theta_n)] = A + iB$

$\Rightarrow \quad r_1 r_2 \ldots\ldots r_n[\cos(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots\ldots + \theta_n)] = A + iB.$

Equating real and imaginary parts on both sides,

$$r_1 r_2 \ldots\ldots r_n \cos(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) = A \qquad \ldots(1)$$

$$r_1 r_2 \ldots\ldots r_n \sin(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) = B \qquad \ldots(2)$$

Squaring and adding (1) and (2),

$$r_1^2 r_2^2 \ldots\ldots r_n^2[\cos^2(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) + \sin^2(\theta_1 + \theta_2 + \ldots\ldots + \theta_n)] = A^2 + B^2$$

$$\Rightarrow \quad r_1^2 r_2^2 \ldots\ldots r_n^2 = A^2 + B^2$$

$$\Rightarrow \quad (a_1^2 + b_1^2)(a_2^2 + b_2^2) \ldots\ldots (a_n^2 + b_n^2) = A^2 + B^2 \qquad \ldots(\text{I})$$

Dividing (2) by (1), $\tan(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) = \dfrac{B}{A}$

$$\Rightarrow \qquad \theta_1 + \theta_2 + \ldots\ldots + \theta_n = \tan^{-1}\frac{B}{A}$$

$$\Rightarrow \qquad \tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \ldots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A} \qquad \ldots\text{(II)}$$

Example 15. *If* $(1+x)^n = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \ldots$ *show that*

(*i*) $p_0 - p_2 + p_4 \ldots\ldots = 2^{\frac{n}{2}} \cos\dfrac{n\pi}{4}$ (*ii*) $p_1 - p_3 + p_5 \ldots\ldots = 2^{\frac{n}{2}} \sin\dfrac{n\pi}{4}$.

Sol. $(1+x)^n = p_0 + p_1 x + p_2 x^2 + p_3 x^3 \ldots\ldots$

Put $x = i$ on both sides, $(1+i)^n = p_0 + p_1 i + p_2 i^2 + p_3 i^3 + p_4 i^4 + p_5 i^5 + \ldots\ldots = p_0 + ip_1 - p_2 - ip_3 + p_4 + ip_5 + \ldots\ldots$

$[\because \quad i^2 = -1,\ i^3 = i.i^2 = -i,\ i^4 = (i^2)^2 = 1,\ i^5 = i\, i^4 = i]$

$$\therefore \qquad (1+i)^n = (p_0 - p_2 + p_4 \ldots\ldots) + i\,(p_1 - p_3 + p_5 \ldots\ldots) \qquad \ldots(1)$$

Let $\qquad 1 + i = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts $r\cos\theta = 1$, $r\sin\theta = 1$ ...(2)

Squaring and adding, $\quad r^2 = 1 + 1 = 2 \qquad \therefore \qquad r = \sqrt{2}$

From (2), $\qquad \cos\theta = \dfrac{1}{r} = \dfrac{1}{\sqrt{2}}$, $\quad \sin\theta = \dfrac{1}{r} = \dfrac{1}{\sqrt{2}}$

Both these equations are satisfied when $\theta = \dfrac{\pi}{4}$

$$\therefore \qquad 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\Rightarrow \qquad (1+i)^n = (\sqrt{2})^n\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

$$\therefore \quad \text{From (1), } 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = (p_0 - p_2 + p_4 \ldots) + i\,(p_1 - p_3 + p_5 \ldots\ldots)$$

Equating real and imaginary parts on both sides,

$$p_0 - p_2 + p_4 \ldots\ldots = 2^{\frac{n}{2}}\cos\frac{n\pi}{4} \qquad \ldots\text{(I)}$$

$$p_1 - p_3 + p_5 \ldots\ldots = 2^{\frac{n}{2}}\sin\frac{n\pi}{4} \qquad \ldots\text{(II)}$$

Example 16. *If* $x = \cos\theta + i\sin\theta$ *and* $\sqrt{1-c^2} = nc - 1$. *Show that* $1 + c\cos\theta = \dfrac{c}{2n}(1+nx)\left(1+\dfrac{n}{x}\right)$.

Sol. Given $\qquad \sqrt{1-c^2} = nc - 1$ ...(1)

and $\qquad x = \cos\theta + i\sin\theta$

$$\therefore \qquad \frac{1}{x} = \cos\theta - i\sin\theta$$

$$\text{RHS} = \frac{c}{2n}(1+nx)\left(1+\frac{n}{x}\right) = \frac{c}{2n}\left[1 + n\left(x + \frac{1}{x}\right) + n^2\right]$$

$$= \frac{c}{2n}\,[1 + n \cdot 2\cos\theta + n^2] = \frac{c}{2}\left[\left(n + \frac{1}{n}\right) + 2\cos\theta\right] \qquad \ldots(2)$$

From (1), $\qquad n = \dfrac{1 + \sqrt{1 - c^2}}{c}$

Eliminate n from (2),

$$\text{RHS} = \frac{c}{2} \cdot \left[\frac{1 + \sqrt{1 - c^2}}{c} + \frac{c}{1 + \sqrt{1 - c^2}} + 2\cos\theta\right] = \frac{c}{2}\left[\frac{\left(1 + \sqrt{1 - c^2}\right)^2 + c^2}{c\left(1 + \sqrt{1 - c^2}\right)} + 2\cos\theta\right]$$

$$= \frac{c}{2}\left[\frac{1 + 1 - c^2 + 2\sqrt{1 - c^2} + c^2}{c\left(1 + \sqrt{1 - c^2}\right)} + 2\cos\theta\right] = \frac{c}{2}\left[\frac{2\left\{1 + \sqrt{1 - c^2}\right\}}{c\left(1 + \sqrt{1 - c^2}\right)} + 2\cos\theta\right]$$

$$= \frac{c}{2}\left[\frac{2}{c} + 2\cos\theta\right] = 1 + c\cos\theta = \text{LHS}$$

Example 17. *If* $\sin\psi = i\tan\theta$, *prove that* $\cos\theta + i\sin\theta = \tan\left(\dfrac{\pi}{4} + \dfrac{\psi}{2}\right)$.

Sol. $\qquad i\tan\theta = \sin\psi \;\Rightarrow\; \dfrac{i\sin\theta}{\cos\theta} = \dfrac{\sin\psi}{1} \;\Rightarrow\; \dfrac{\cos\theta}{i\sin\theta} = \dfrac{1}{\sin\psi}$

By componendo and dividendo, $\quad \dfrac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta} = \dfrac{1 + \sin\psi}{1 - \sin\psi}$

or $$(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)^{-1} = \frac{\cos^2\dfrac{\psi}{2} + \sin^2\dfrac{\psi}{2} + 2\cos\dfrac{\psi}{2}\sin\dfrac{\psi}{2}}{\cos^2\dfrac{\psi}{2} + \sin^2\dfrac{\psi}{2} - 2\cos\dfrac{\psi}{2}\sin\dfrac{\psi}{2}}$$

or $$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) = \left[\frac{\cos\dfrac{\psi}{2} + \sin\dfrac{\psi}{2}}{\cos\dfrac{\psi}{2} - \sin\dfrac{\psi}{2}}\right]^2$$

or $$\cos\theta + i\sin\theta = \frac{\cos\dfrac{\psi}{2} + \sin\dfrac{\psi}{2}}{\cos\dfrac{\psi}{2} - \sin\dfrac{\psi}{2}}$$

Dividing the numerator and denominator on RHS by $\cos\dfrac{\psi}{2}$, we get

$$\cos\theta + i\sin\theta = \frac{1 + \tan\dfrac{\psi}{2}}{1 - \tan\dfrac{\psi}{2}} = \tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right).$$

Example 18. *If α, β are the roots of $x^2 - 2x + 2 = 0$, then prove that:*

(*i*) $$\frac{(t+\alpha)^n - (t+\beta)^n}{\alpha - \beta} = \frac{\sin n\phi}{\sin^n \phi}$$

(*ii*) $$\frac{(t+\alpha)^n + (t+\beta)^n}{\alpha + \beta} = \frac{\cos n\phi}{\sin^n \phi}, \text{ where } t + 1 = \cot \phi.$$

Sol. $$x^2 - 2x + 2 = 0$$

$$x = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm i2}{2} = 1 \pm i$$

$\therefore$ $$\alpha = 1 + i, \beta = 1 - i$$

(*i*) $$\frac{(t+\alpha)^n - (t+\beta)^n}{\alpha - \beta} = \frac{(t+1+i)^n - (t+1-i)^n}{1+i-1+i}$$

$$= \frac{(\cot \phi + i)^n - (\cot \phi - i)^n}{2i} \qquad \text{given } t + 1 = \cot \phi$$

$$= \frac{(\cos \phi + i \sin \phi)^n - (\cos \phi - i \sin \phi)^n}{2i (\sin \phi)^n}$$

$$= \frac{(\cos n\phi + i \sin n\phi) - (\cos n\phi - i \sin n\phi)}{2i \sin^n \phi}$$

$$= \frac{2i \sin n\phi}{2i \sin^n \phi} = \frac{\sin n\phi}{\sin^n \phi}$$

(*ii*) Do it yourself.

Example 19. *If $a = cis\, \alpha$, $b = cis\, \beta$, $c = cis\, \gamma$, prove that*

$$\frac{(b+c)(c+a)(a+b)}{abc} = 8 \cos \frac{\beta - \gamma}{2} \cos \frac{\gamma - \alpha}{2} \cos \frac{\alpha - \beta}{2}.$$

Sol. $$\frac{(b+c)(c+a)(a+b)}{abc} = \left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right)\left(\frac{a+b}{c}\right) \qquad ...(1)$$

Now, $$\frac{b+c}{a} = \left(\frac{b}{a} + \frac{c}{a}\right) = \left[\frac{\text{cis}\,\beta}{\text{cis}\,\alpha} + \frac{\text{cis}\,\gamma}{\text{cis}\,\alpha}\right]$$

$$= \text{cis}\,(\beta - \alpha) + \text{cis}\,(\gamma - \alpha)$$

$$= \cos (\beta - \alpha) + i \sin (\beta - \alpha) + \cos (\gamma - \alpha) + i \sin (\gamma - \alpha)$$

$$= 2 \cos \frac{\beta + \gamma - 2\alpha}{2} \cos \frac{\beta - \gamma}{2} + 2i \sin \frac{\beta + \gamma - 2\alpha}{2} \cos \frac{\beta - \gamma}{2}$$

$$= 2 \cos \frac{\beta - \gamma}{2} \left\{\cos \frac{\beta + \gamma - 2\alpha}{2} + i \sin \frac{\beta + \gamma - 2\alpha}{2}\right\}$$

$$= 2 \cos \frac{\beta - \gamma}{2} \text{ cis } \frac{\beta + \gamma - 2\alpha}{2}$$

Similarly, $$\frac{c+a}{b} = 2\cos\frac{\gamma-\alpha}{2}\ \text{cis}\ \frac{\gamma+\alpha-2\beta}{2}$$

and $$\frac{a+b}{c} = 2\cos\frac{\alpha-\beta}{2}\ \text{cis}\ \frac{\alpha+\beta-2\gamma}{2}$$

$\therefore$ From (1),

$$\frac{(b+c)(c+a)(a+b)}{abc} = 2\cos\frac{\beta-\gamma}{2}\ \text{cis}\ \frac{\beta+\gamma-2\alpha}{2}\ 2\cos\frac{\gamma-\alpha}{2}\ \text{cis}\ \frac{\gamma+\alpha-2\beta}{2}$$
$$2\cos\frac{\alpha-\beta}{2}\ \text{cis}\ \frac{\alpha+\beta-2\gamma}{2}$$
$$= 8\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2}\ \text{cis}\ \frac{\beta+\gamma-2\alpha+\gamma+\alpha-2\beta+\alpha+\beta-2\gamma}{2}$$
$$= 8\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2}\ \text{cis}\ 0$$
$$= 8\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2}. \qquad \mid \because \ \text{cis}\ 0 = 1$$

TEST YOUR KNOWLEDGE

1. Prove that

(*i*) $\dfrac{(\cos 3\theta + i\sin 3\theta)^5 (\cos 2\theta - i\sin 2\theta)^3}{(\cos 4\theta + i\sin 4\theta)^{-9} (\cos 5\theta + i\sin 5\theta)^9} = 1$ (*ii*) $\dfrac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^5} = \sin(4\alpha+5\beta) - i\cos(4\alpha+5\beta)$.

2. If $a = \text{cis}\ \alpha$, $b = \text{cis}\ \beta$ and $c = \text{cis}\ \gamma$, prove that

(*i*) $\dfrac{ab}{c} + \dfrac{c}{ab} = 2\cos(\alpha+\beta-\gamma)$ (*ii*) $a^p b^q c^r + \dfrac{1}{a^p b^q c^r} = 2\cos(p\alpha + q\beta + r\gamma)$.

3. If $p = \text{cis}\ 2\theta$ and $q = \text{cis}\ 2\phi$, prove that $\sqrt{\dfrac{p}{q}} - \sqrt{\dfrac{q}{p}} = 2i\sin(\theta-\phi)$.

4. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, show that

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}.$$

5. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and $x + y + z = 0$, then prove that $x^{-1} + y^{-1} + z^{-1} = 0$.

6. Prove that the general value of θ which satisfies the equation

$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots\ldots (\cos n\theta + i\sin n\theta) = 1$ is $\dfrac{4m\pi}{n(n+1)}$, where m is any integer.

7. If $2\cos\theta = a + \dfrac{1}{a}$, $2\cos\phi = b + \dfrac{1}{b}$, prove that one of the values of

(*i*) $ab + \dfrac{1}{ab}$ is $2\cos(\theta+\phi)$ (*ii*) $a^p b^q + \dfrac{1}{a^p b^q}$ is $2\cos(p\theta + q\phi)$

8. Prove that

(*i*) $[(\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi)]^n + [(\cos\theta + \cos\phi) - i(\sin\theta + \sin\phi)]^n$

$$= 2^{n+1}\cos^n\left(\frac{\theta-\phi}{2}\right)\cos\frac{n(\theta+\phi)}{2}$$

(*ii*) $[(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)]^n + [(\cos\theta - \cos\phi) - i(\sin\theta - \sin\phi)]^n$

$$= 2^{n+1} \sin^n \frac{\theta - \phi}{2} \cos \frac{n(\pi + \theta + \phi)}{2}$$

(*iii*) $(1 + \cos\theta + i \sin\theta)^n + (1 + \cos\theta - i \sin\theta)^n = 2^{n+1} \cos^n \frac{\theta}{2} \cos \frac{n\theta}{2}$

(*iv*) $\left(\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right)^n = \cos n\theta + i \sin n\theta.$

9. If $\left(1 + i\frac{x}{a}\right)\left(1 + i\frac{x}{b}\right)\left(1 + i\frac{x}{c}\right)$ ...... $= A + iB$, prove that

(*i*) $\left(1 + \frac{x^2}{a^2}\right)\left(1 + \frac{x^2}{b^2}\right)\left(1 + \frac{x^2}{c^2}\right)$ ...... $= A^2 + B^2$

(*ii*) $\tan^{-1}\frac{x}{a} + \tan^{-1}\frac{x}{b} + \tan^{-1}\frac{x}{c} + ... = \tan^{-1}\frac{B}{A}$.

10. If α, β be the roots of $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1} \cos \frac{n\pi}{3}$.

6.4. ROOTS OF A COMPLEX NUMBER

As already discussed in De-Moivre's Theorem that when n is a rational number (*i.e.*, fraction positive or negative) then $\cos n\theta + i \sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$. Now we shall find all the values of $(\cos\theta + i\sin\theta)^n$, where $n = \frac{p}{q}$; p, q are both integers; $(p, q) = 1$ and $q \neq 0$

6.4(a). SHOW THAT THERE ARE q AND ONLY q DISTINCT VALUES OF $(\cos\theta + i \sin\theta)^{\frac{1}{q}}$, q BEING A POSITIVE INTEGER

By De-Moivre's Theorem, we know that $\cos\frac{\theta}{q} + i \sin\frac{\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$.

Let us find all the values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$

$$(\cos\theta + i\sin\theta)^{\frac{1}{q}} = [\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]^{\frac{1}{q}} = \cos\frac{2n\pi + \theta}{q} + i\sin\frac{2n\pi + \theta}{q}$$

Putting $n = 0, 1, 2, ..., q - 1$ in succession, we obtain the following q values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$

$$\left.\begin{array}{ll}
\cos\dfrac{\theta}{q} + i\sin\dfrac{\theta}{q} & \text{when } n = 0 \\
\cos\dfrac{2\pi+\theta}{q} + i\sin\dfrac{2\pi+\theta}{q} & \text{when } n = 1 \\
\cos\dfrac{4\pi+\theta}{q} + i\sin\dfrac{4\pi+\theta}{q} & \text{when } n = 2 \\
\dots\dots\dots\dots\dots\dots\dots\dots & \\
\dots\dots\dots\dots\dots\dots\dots\dots & \\
\cos\dfrac{2(q-1)\pi+\theta}{q} + i\sin\dfrac{2(q-1)\pi+\theta}{q} & \text{when } n = q-1
\end{array}\right] \quad ...(I)$$

Since no two of the angles in the q values in (I) are equal or differ by a multiple of 2π, therefore, their sines and cosines cannot be equal simultaneously.

$\therefore$ All the q values obtained in (I) are *distinct.*

If we put $n = q$, we get

$$\cos \frac{2q\pi + \theta}{q} + i \sin \frac{2q\pi + \theta}{q} = \cos\left(2\pi + \frac{\theta}{q}\right) + i \sin\left(2\pi + \frac{\theta}{q}\right) = \cos \frac{\theta}{q} + i \sin \frac{\theta}{q}$$

This value is the same as the one obtained by putting $n = 0$.

Similarly, if we put $n = q + 1, q + 2, ...$, we will get the same values as obtained in (I) by putting $n = 1, 2, ...$

Hence $(\cos\theta + i \sin\theta)^{\frac{1}{q}}$ has q and only q distinct values obtained by putting $n = 0, 1, 2, ..., q - 1$ in $\cos \frac{2n\pi + \theta}{q} + i \sin \frac{2n\pi + \theta}{q}$.

Working Rule for finding the *q*th roots of $x + iy$

Let $\qquad x + iy = r(\cos\theta + i \sin\theta)$

$$\therefore \qquad (x + iy)^{\frac{1}{q}} = r^{\frac{1}{q}} (\cos\theta + i \sin\theta)^{\frac{1}{q}} = r^{\frac{1}{q}} [\cos(2n\pi + \theta) + i \sin(2n\pi + \theta)]^{\frac{1}{q}}$$

$$= r^{\frac{1}{q}} \left[\cos \frac{2n\pi + \theta}{q} + i \sin \frac{2n\pi + \theta}{q}\right]$$

Putting $n = 0, 1, 2, ..., q - 1$, the q values of $(x + iy)^{\frac{1}{q}}$ are obtained.

6.4(b). SHOW THAT $(\cos\theta + i \sin\theta)^{p/q}$ HAS q AND ONLY q DISTINCT VALUES, p AND q BEING INTEGERS PRIME TO EACH OTHER

$$(\cos\theta + i \sin\theta)^{\frac{p}{q}} = [(\cos\theta + i \sin\theta)^p]^{\frac{1}{q}}$$

$$= (\cos p\theta + i \sin p\theta)^{\frac{1}{q}} \qquad [\because \ p \text{ is an integer}]$$

$$= [\cos(2n\pi + p\theta) + i \sin(2n\pi + p\theta)]^{\frac{1}{q}} = \cos \frac{2n\pi + p\theta}{q} + i \sin \frac{2n\pi + p\theta}{q}$$

To find all the values of the given expression, putting $n = 0, 1, 2, ..., (q - 1)$ in succession, we obtain the following q values of $(\cos\theta + i \sin\theta)^{\frac{p}{q}}$

$$\left.\begin{array}{ll}
\cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q} & \text{when } n = 0 \\
\cos \frac{2\pi + p\theta}{q} + i \sin \frac{2\pi + p\theta}{q} & \text{when } n = 1 \\
\cos \frac{4\pi + p\theta}{q} + i \sin \frac{4\pi + p\theta}{q} & \text{when } n = 2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
\cos \frac{2(q-1)\pi + p\theta}{q} + i \sin \frac{2(q-1)\pi + p\theta}{q} & \text{when } n = q - 1
\end{array}\right] \qquad ...(\text{I})$$

Since no two of the angles in the q values in (I) are equal or differ by a multiple of 2π, therefore, their sines and cosines cannot be equal simultaneously.

$\therefore$ All the q values obtained in (I) are *distinct.*

If we put $n = q$, we get

$$\cos \frac{2q\pi + p\theta}{q} + i \sin \frac{2q\pi + p\theta}{q} = \cos\left(2\pi + \frac{p\theta}{q}\right) + i \sin\left(2\pi + \frac{p\theta}{q}\right) = \cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q}$$

This value is the same as the one obtained by putting $n = 0$.

Similarly, if we put $n = q + 1, q + 2, \ldots\ldots.$, we will get the same values as obtained in (I) by putting $n = 1, 2, \ldots..$

Hence $(\cos \theta + i \sin \theta)^{p/q}$ has q and only q distinct values obtained by putting $n = 0, 1, 2, \ldots\ldots, q - 1$ in

$$\cos \frac{2n\pi + p\theta}{q} + i \sin \frac{2n\pi + p\theta}{q}.$$

Note 1. To find the distinct values of $(\cos \theta + i \sin \theta)^{p/q}$, p and q must be co-prime i.e., p and q should have no common factor > 1.

e.g., $(\cos \theta + i \sin \theta)^{9/12}$ does not have 12 distinct values but only 4, since $(\cos \theta + i \sin \theta)^{9/12} = (\cos \theta + i \sin \theta)^{3/4}$, here $q = 4$.

Note 2. The q distinct values of $(\cos \theta + i \sin \theta)^{p/q}$ are obtained by putting $n = 0, 1, 2, \ldots.., q - 1$ in

$$\cos \frac{2n\pi + p\theta}{q} + i \sin \frac{2n\pi + p\theta}{q} = \cos\left(\frac{p\theta}{q} + \frac{2n\pi}{q}\right) + i \sin\left(\frac{p\theta}{q} + \frac{2n\pi}{q}\right)$$

$$= \left(\cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q}\right)\left(\cos \frac{2n\pi}{q} + i \sin \frac{2n\pi}{q}\right)$$

$$= \text{cis}\,\frac{p\theta}{q}\ \text{cis}\,\frac{2n\pi}{q} = \text{cis}\,\frac{p\theta}{q} \cdot \left(\text{cis}\,\frac{2\pi}{q}\right)^n$$

$$= ar^n, \text{ where } a = \text{cis}\,\frac{p\theta}{q},\ r = \text{cis}\,\frac{2\pi}{q}.$$

Thus, the q distinct values of $(\cos \theta + i \sin \theta)^{p/q}$ are $a, ar, ar^2, \ldots\ldots., ar^{q-1}$

Their product $= a \,.\, ar \,.\, ar^2 \ldots\ldots.\ ar^{q-1} = a^q \,.\, r^{1 + 2 + \ldots\ldots + (q-1)}$

$$= a^q\ r^{\frac{q-1}{2}(1 + q - 1)} = a^q \,.\, r^{\frac{q(q-1)}{2}} = \left(\text{cis}\,\frac{p\theta}{q}\right)^q \left(\text{cis}\,\frac{2\pi}{q}\right)^{\frac{q(q-1)}{2}}$$

$$= \text{cis}\,p\theta \,.\, \left[\left(\text{cis}\,\frac{2\pi}{q}\right)^{q/2}\right]^{q-1} = \text{cis}\,p\theta \,.\, (\text{cis}\,\pi)^{q-1}$$

$$= (\cos p\theta + i \sin p\theta)(\cos \pi + i \sin \pi)^{q-1} = (-1)^{q-1}(\cos p\theta + i \sin p\theta).$$

6.4(c). SHOW THAT THE q VALUES OF $(\cos\theta + i\sin\theta)^{p/q}$ FORM A GEOMETRICAL PROGRESSION WHOSE SUM IS ZERO, p AND q BEING INTEGERS PRIME TO EACH OTHER OF $(\cos\theta + i\sin\theta)^{p/q}$

Proceeding as in 6.4(b), the sum of q values of $(\cos\theta + i\sin\theta)^{p/q}$ is

$$= a + ar + ar^2 + \ldots\ldots + ar^{q-1} = \frac{a(1-r^q)}{1-r} \qquad \left[r = \text{cis}\,\frac{2\pi}{q} \neq 1\right]$$

$$= \frac{a\left[1-\left(\text{cis}\,\frac{2\pi}{q}\right)^q\right]}{1-r} = \frac{a(1-\text{cis}\,2\pi)}{1-r} = \frac{a[1-(\cos 2\pi + i\sin 2\pi)]}{1-r} = \frac{a[1-(1+0)]}{1-r} = 0.$$

ILLUSTRATIVE EXAMPLES

Example 1. (*a*) *Find nth roots of unity and prove that these form a geometrical progession. Also show that the sum of these n roots is zero and their product is* $(-1)^{n-1}$. **(P.T.U., Dec. 2013)**

(*b*) *Solve* $x^7 = 1$ *and prove that the sum of the nth powers of the roots is 7 or zero, according as n is or is not a multiple of 7.*

Sol. (*a*) We have to evaluate $(1)^{1/n}$

$$(1)^{1/n} = (\cos 0 + i\sin 0)^{1/n} = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$$

$$= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \text{ where } k = 0, 1, 2, \ldots\ldots, (n-1)$$

$\therefore$ nth roots of unity are

$$1, \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}, \cos\frac{6\pi}{n} + i\sin\frac{6\pi}{n}, \ldots\ldots,$$

$$\cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}$$

Let $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = \lambda$

$\therefore$ nth roots of unity are

$1, \lambda, \lambda^2, \lambda^3 \ldots\ldots, \lambda^{n-1}$, which forms a G.P. with first term = 1 and C.R. = λ

Now sum of these roots $= 1 + \lambda + \lambda^2 + \ldots\ldots + \lambda^{n-1}$

$$= \frac{1(1-\lambda^n)}{1-\lambda}, \quad \text{where} \quad \lambda \neq 1 \text{ sum of G.P.} = \frac{a(1-r^n)}{1-r}$$

Now, $$\lambda^n = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = \cos 2\pi + i\sin 2\pi = 1$$

$\therefore$ $$\text{Sum} = \frac{1-1}{1-\lambda} = 0$$

Their product $$= 1\cdot\lambda\cdot\lambda^2 \ldots\ldots \lambda^{n-1} = \lambda^{1+2+3\,\ldots\ldots\,(n-1)} = \lambda^{\frac{(n-1)n}{2}} = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{\frac{(n-1)n}{2}}$$

$$= \cos\left(\frac{2\pi}{n}\cdot\frac{n(n-1)}{2}\right) + i\sin\left(\frac{2\pi}{n}\cdot\frac{n(n-1)}{2}\right) = \cos(n-1)\pi + i\sin(n-1)\pi$$

$$= (-1)^{n-1} + 0 = (-1)^{n-1}.$$

(*b*) We have to evaluate $(1)^{1/7}$

$$(1)^{1/7} = (\cos 0 + i \sin 0)^{1/7} = [\cos (2n\pi + 0) + i \sin (2n\pi + 0)]^{1/7}$$

$$= \cos \frac{2n\pi}{7} + i \sin \frac{2n\pi}{7}, \text{ where } n = 0, 1, 2, 3, 4, 5, 6$$

$\therefore$ The seventh roots of unity are

$$\cos 0 + i \sin 0; \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}; \cos \frac{4\pi}{7} + i \sin \frac{4\pi}{7} \cos \frac{6\pi}{7} + i \sin \frac{6\pi}{7};$$

$$\cos \frac{8\pi}{7} + i \sin \frac{8\pi}{7}; \cos \frac{10\pi}{7} + i \sin \frac{10\pi}{7}; \cos \frac{12\pi}{7} + i \sin \frac{12\pi}{7}.$$

The *n*th powers of the roots are

$$1; \cos \frac{2n\pi}{7} + i \sin \frac{2n\pi}{7}; \cos \frac{4n\pi}{7} + i \sin \frac{4n\pi}{7}; \cos \frac{6n\pi}{7} + i \sin \frac{6n\pi}{7}; \cos \frac{8n\pi}{7} + i \sin \frac{8n\pi}{7};$$

$$\cos \frac{10n\pi}{7} + i \sin \frac{10n\pi}{7}; \cos \frac{12n\pi}{7} + i \sin \frac{12n\pi}{7}$$

or $$1, x, x^2, x^3, x^4, x^5, x^6, \text{ where } x = \cos \frac{12n\pi}{7} + i \sin \frac{12n\pi}{7}$$

If n is not a multiple of 7, $x \neq 1$

$$\therefore \quad \text{Reqd. sum} = 1 + x + x^2 + ... + x^6 = \frac{1(1-x^7)}{1-x} = \frac{1}{1-x}\left[1 - \left(\cos \frac{2n\pi}{7} + i \sin \frac{2n\pi}{7}\right)^7\right]$$

$$= \frac{1}{1-x}[1 - (\cos 2n\pi + i \sin 2n\pi)] = \frac{1}{1-x}[1 - 1] = 0$$

If n is a multiple of 7, let $n = 7m$, where m is an integer.

$$x = \text{cis}\, \frac{2n\pi}{7} = \text{cis}\, 2m\pi = 1.$$

$$\therefore \quad \text{Required sum} = 1 + x + x^2 + ... + x^6 = 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7.$$

Example 2. *Find the values of* $(-1)^{1/6}$.

Sol. $$-1 = \cos \pi + i \sin \pi$$

$$\therefore \quad (-1)^{1/6} = (\cos \pi + i \sin \pi)^{1/6} = [\cos (2n\pi + \pi) + i \sin (2n\pi + \pi)]^{1/6}$$

$$= \cos \frac{(2n+1)\pi}{6} + i \sin \frac{(2n+1)\pi}{6}, \text{ where } n = 0, 1, 2, 3, 4, 5$$

Putting $n = 0, 1, 2, 3, 4, 5$, the required values are

$$\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}; \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}; \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6};$$

$$\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}; \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}; \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}$$

or $$\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}; 0 + i \cdot 1; \cos\left(\pi - \frac{\pi}{6}\right) + i \sin\left(\pi - \frac{\pi}{6}\right);$$

$$\cos\left(\pi + \frac{\pi}{6}\right) + i \sin\left(\pi + \frac{\pi}{6}\right); 0 + i(-1); \cos\left(2\pi - \frac{\pi}{6}\right) + i \sin\left(2\pi - \frac{\pi}{6}\right)$$

or $$\frac{\sqrt{3}+i}{2}; i; -\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}; -\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}; -i; \cos \frac{\pi}{6} - i \sin \frac{\pi}{6}$$

or $$\frac{\sqrt{3}+i}{2}; i; \frac{-\sqrt{3}+i}{2}; \frac{-\sqrt{3}-i}{2}; -i; \frac{\sqrt{3}-i}{2} \quad \text{or} \quad \pm i; \frac{\sqrt{3} \pm i}{2}; \frac{-\sqrt{3} \pm i}{2}.$$

Example 3. *Find all the values of* $\left(\frac{1}{2}+\frac{\sqrt{3}\,i}{2}\right)^{3/4}$ *and show that their continued product is 1.*

(P.T.U., Dec. 2011, 2012)

Sol. Let $\quad \frac{1}{2}+i\,.\,\frac{\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts $\quad r\cos\theta = \frac{1}{2}\ ,\ r\sin\theta = \frac{\sqrt{3}}{2} \qquad \ldots(1)$

Squaring and adding, $\quad r^2 = \frac{1}{4}+\frac{3}{4} = 1 \quad \therefore \quad r = 1$

From (1), $\quad \cos\theta = \frac{1}{2r} = \frac{1}{2}\ ;\ \sin\theta = \frac{\sqrt{3}}{2r} = \frac{\sqrt{3}}{2}$

Both these equations are satisfied when $\theta = \frac{\pi}{3}$

$$\therefore \quad \frac{1}{2} + i\cdot\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^{3/4} = \left[\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^3\right]^{1/4} = (\cos\pi + i\sin\pi)^{1/4}$$

$$= [\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)]^{1/4} = \cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}$$

Putting $n = 0, 1, 2, 3$, the required values are

$$\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\ ;\ \cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\ ;\ \cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\ ;\ \cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}$$

i.e., $\quad \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}} \quad$ *i.e.,* $\quad \frac{1\pm i}{\sqrt{2}}, \frac{-1\pm i}{\sqrt{2}}$.

The required product $= \cos\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right) + i\sin\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right)$

$= \cos 4\pi + i\sin 4\pi = 1.$

Example 4. *Prove that* $\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$ *has the value –1, if n = 3k ± 1 and the value 2, if n = 3k, where k is an integer.*

Sol. Let $\quad -\frac{1}{2} = r\cos\theta, \quad \frac{\sqrt{3}}{2} = r\sin\theta$

Squaring and adding $\quad \frac{1}{4}+\frac{3}{4} = r^2 \quad \therefore \quad r = 1$

$$\therefore \quad \cos\theta = -\frac{1}{2}, \sin\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = \frac{2\pi}{3}$$

$$\therefore \quad \left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = (r\cos\theta + i\,r\sin\theta)^n + (r\cos\theta - i\,r\sin\theta)^n$$

$$= r^n[\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta] = 2r^n\cos n\theta$$

$$= 2\cos\frac{2n\pi}{3}$$

If $n = 3k \pm 1$

Then
$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = 2\cos\frac{2\pi}{3}(3k\pm 1) = 2\cos\left(2k\pi \pm \frac{2\pi}{3}\right)$$

$$= 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

If $n = 3k$

Then
$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = 2\cos\frac{2\pi}{3}(3k)$$

$$= 2\cos 2k\pi = 2.1 = 2$$

Hence
$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = -1 \text{ if } n = 3k \pm 1 \text{ i.e., } n \text{ is not a multiple of } 3$$

$$= 2 \text{ if } n = 3k \text{ i.e., } n \text{ is a multiple of } 3.$$

Example 5. *Prove that* $(a+ib)^{\frac{1}{n}} + (a-ib)^{\frac{1}{n}}$ *has n real values and find those of* $(1+i\sqrt{3})^{1/3} + (1-i\sqrt{3})^{1/3}$.

Sol. Let $a = r\cos\theta,\ b = r\sin\theta$

Squaring and adding, $a^2 + b^2 = r^2 \quad \therefore \quad r = \sqrt{a^2+b^2}$

Dividing, $\tan\theta = \dfrac{b}{a} \quad \therefore \quad \theta = \tan^{-1}\dfrac{b}{a}$

$$\therefore \quad (a+ib)^{\frac{1}{n}} + (a-ib)^{\frac{1}{n}} = [r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} + [r(\cos\theta - i\sin\theta)]^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}[\cos(2r\pi+\theta) + i\sin(2r\pi+\theta)]^{\frac{1}{n}} + r^{\frac{1}{n}}[\cos(2r\pi+\theta) - i\sin(2r\pi+\theta)]^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}\left[\cos\frac{2r\pi+\theta}{n} + i\sin\frac{2r\pi+\theta}{n} + \cos\frac{2r\pi+\theta}{n} - i\sin\frac{2r\pi+\theta}{n}\right]$$

$$= r^{\frac{1}{n}}\,.\,2\cos\frac{2r\pi+\theta}{n} = 2\,[(a^2+b^2)^{1/2}]^{\frac{1}{n}}\cos\left[\frac{2r\pi+\theta}{n}\right]$$

$$= 2\,(a^2+b^2)^{\frac{1}{2n}}\cos\left[\frac{1}{n}\left(2r\pi + \tan^{-1}\frac{b}{a}\right)\right]$$

which is real and will give n real values corresponding to $r = 0, 1, 2, ..., (n-1)$.

Putting $a = 1$, $b = \sqrt{3}$ and $n = 3$, the three required values of $(1+i\sqrt{3})^{1/3} + (1-i\sqrt{3})^{1/3}$ are

$$2(1+3)^{1/6}\cos\left[\frac{1}{3}\left(2r\pi+\tan^{-1}\frac{\sqrt{3}}{1}\right)\right], \text{ where } r = 0, 1, 2$$

i.e., $$2\cdot(2^2)^{1/6}\cos\left[\frac{1}{3}\left(2r\pi+\frac{\pi}{3}\right)\right], \text{ where } r = 0, 1, 2$$

i.e., $$2.2^{1/3}\cos\frac{6r\pi+\pi}{9}, \text{ where } r = 0, 1, 2$$

i.e., $$2^{4/3}\cdot\cos\frac{\pi}{9}\ ;\ 2^{4/3}\cdot\cos\frac{7\pi}{9}\ ;\ 2^{4/3}\cdot\cos\frac{13\pi}{9}$$

i.e., $$2^{4/3}\cdot\cos\frac{r\pi}{9}, \text{ where } r = 1, 7, 13$$

Example 6. *If* $a = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$, $b = a + a^2 + a^4$, $c = a^3 + a^5 + a^6$, *show that b and c are the roots of the equation* $x^2 + x + 2 = 0$.

Sol. $$a^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7 = \cos 2\pi + i\sin 2\pi = 1$$

Now, $$b + c = a + a^2 + a^3 + a^4 + a^5 + a^6 = (1 + a + a^2 + a^3 + a^4 + a^5 + a^6) - 1$$

$$= \frac{1(1-a^7)}{1-a} - 1 \qquad \left[S_n = \frac{a(1-r^n)}{1-r}\right]$$

$$= \frac{1}{1-a}[1-1]-1 = -1 \qquad [\because \ a^7 = 1]$$

and $$bc = (a + a^2 + a^4)(a^3 + a^5 + a^6) = a^4 + a^6 + a^7 + a^5 + a^7 + a^8 + a^7 + a^9 + a^{10}$$

$$= a^4 + a^6 + 1 + a^5 + 1 + a + 1 + a^2 + a^3$$

$$[\because \ a^7 = 1 \quad \therefore \ a^8 = a^7 \,.\, a = a \text{ etc.}]$$

$$= (1 + a + a^2 + a^3 + a^4 + a^5 + a^6) + 2$$

$$= \frac{1(1-a^7)}{1-a} + 2 = \frac{1}{1-a}(1-1) + 2 = 2$$

$\therefore$ The equation whose roots are b and c is

$x^2 - (b + c)x + bc = 0$ or $x^2 - (-1)x + 2 = 0$ or $x^2 + x + 2 = 0$.

Example 7. *Solve the following equations:*

(*a*) $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$

(*b*) $x^4 - x^3 + x^2 - x + 1 = 0$ **(P.T.U., May 2003, 2005)**

(*c*) $x^7 + x^4 + x^3 + 1 = 0$ **(P.T.U., Dec., 2010)**

(*d*) $x^9 - x^5 + x^4 - 1 = 0.$

Sol. (*a*) Given equation is $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$ ...(1)

Multiplying both sides by $(x - 1)$, we have $x^7 - 1 = 0$ ...(2)

$\Rightarrow$ $x^7 = 1$

$$\therefore \quad x = (1)^{1/7} = (\cos 0 + i\sin 0)^{1/7} = (\cos 2n\pi + i\sin 2n\pi)^{1/7} = \cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}$$

Putting $n = 0, 1, 2, 3, 4, 5, 6$ the seven roots of (2) are

$$\cos 0 + i\sin 0\ ;\ \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\ ;\ \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}\ ;$$

$$\cos\frac{6\pi}{7}+i\sin\frac{6\pi}{7}\ ;\ \cos\frac{8\pi}{7}+i\sin\frac{8\pi}{7}\ ;$$

$$\cos\frac{10\pi}{7}+i\sin\frac{10\pi}{7}\ ;\ \cos\frac{12\pi}{7}+i\sin\frac{12\pi}{7}\ .$$

But $\cos\frac{8\pi}{7}+i\sin\frac{8\pi}{7}=\cos\left(2\pi-\frac{6\pi}{7}\right)+i\sin\left(2\pi-\frac{6\pi}{7}\right)=\cos\frac{6\pi}{7}-i\sin\frac{6\pi}{7}$

$$\cos\frac{10\pi}{7}+i\sin\frac{10\pi}{7}=\cos\left(2\pi-\frac{4\pi}{7}\right)+i\sin\left(2\pi-\frac{4\pi}{7}\right)=\cos\frac{4\pi}{7}-i\sin\frac{4\pi}{7}$$

$$\cos\frac{12\pi}{7}+i\sin\frac{12\pi}{7}=\cos\left(2\pi-\frac{2\pi}{7}\right)+i\sin\left(2\pi-\frac{2\pi}{7}\right)=\cos\frac{2\pi}{7}-i\sin\frac{2\pi}{7}$$

$\therefore$ Roots of (2) are $1,\ \cos\frac{2\pi}{7}\pm i\sin\frac{2\pi}{7}\ ;\ \cos\frac{4\pi}{7}\pm i\sin\frac{4\pi}{7}\ ;\ \cos\frac{6\pi}{7}\pm i\sin\frac{6\pi}{7}$

or $$1,\ \cos\frac{r\pi}{7}\pm i\sin\frac{r\pi}{7},\ \text{where } r=2,4,6.$$

The root 1 corresponds to $x-1=0$

$\therefore$ The remaining six roots are those of the given equation.

Hence the roots of (1) are given by $\cos\frac{r\pi}{7}\pm i\sin\frac{r\pi}{7}$, where $r=2, 4, 6$.

(*b*) Given equation is $x^4-x^3+x^2-x+1=0$

Multiply both sides by $x+1$ we have $(x+1)(x^4-x^3+x^2-x+1)=0$

i.e., $x^5+1=0$ *i.e.*, $x^5=-1=\cos\pi+i\sin\pi$

$\therefore$ $$x^5=\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)$$

$$x=(\cos(2n+1)\pi+i\sin(2n+1)\pi)^{1/5}$$

$$x=\cos\frac{(2n+1)\pi}{5}+i\sin\frac{(2n+1)\pi}{5},\ \text{where } n=0,1,2,3,4.$$

Putting $n=0, 1, 2, 3, 4$ five roots of $x^5+1=0$ are

$$\cos\frac{\pi}{5}+i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5},\ \cos\pi+i\sin\pi,$$

$$\cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5},\ \cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}$$

or $$\cos\frac{\pi}{5}+i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5},\ -1,$$

$$\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5},\ \cos\frac{\pi}{5}-i\sin\frac{\pi}{5}$$

or the roots are $-1,\ \cos\frac{\pi}{5}\pm i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5}\pm i\sin\frac{3\pi}{5}$.

root $x=-1$ corresponds to $x+1=0$

$\therefore$ The remaining four roots *i.e.*, $\cos\frac{\pi}{5}\pm i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5}\pm i\sin\frac{3\pi}{5}$ are roots of the given equation.

(*c*) Given equation is $x^7+x^4+x^3+1=0$ *i.e.*, $x^4(x^3+1)+(x^3+1)=0$

i.e., $$(x^4+1)(x^3+1)=0 \qquad \ldots(1)$$

either $$x^4+1=0 \quad \text{or} \quad x^3+1=0$$

Now, $x^4 = -1 \quad \Rightarrow \quad x = (-1)^{1/4}$

$\therefore \quad x = (\cos \pi + i \sin \pi)^{1/4}$

$= [\cos (2n\pi + \pi) + i \sin (2n\pi + \pi)]^{1/4}$

$$= \cos \frac{(2n+1)\pi}{4} + i \sin \frac{(2n+1)\pi}{4}$$

Putting $n = 0, 1, 2, 3$, the four roots of $x^4 + 1 = 0$ are

$$\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} ; \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$$

$$\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} ; \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}$$

i.e., $\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} ; \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$;

$$\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} ; \cos \frac{\pi}{4} - i \sin \frac{\pi}{4}$$

i.e., $\frac{1 \pm i}{\sqrt{2}} ; \frac{-1 \pm i}{\sqrt{2}}$

$x^3 = -1 \quad \Rightarrow \quad x = (-1)^{1/3}$

$\therefore \; x = (\cos \pi + i \sin \pi)^{1/3}$

$= [\cos (2n\pi + \pi) + i \sin (2n\pi + \pi)]^{1/3}$

$$= \cos \frac{(2n+1)\pi}{3} + i \sin \frac{(2n+1)\pi}{3}$$

Putting $n = 0, 1, 2$; the three roots of $x^3 + 1 = 0$ are

$$\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} ; \cos \pi + i \sin \pi ;$$

$$\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}$$

i.e., $\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} ; -1 ; \cos \frac{\pi}{3} - i \sin \frac{\pi}{3}$

i.e., $\frac{1 \pm i\sqrt{3}}{2} ; -1$

Hence the roots of (1) are $-1, \frac{1 \pm i}{\sqrt{2}} ; \frac{-1 \pm i}{\sqrt{2}} ; \frac{1 \pm i\sqrt{3}}{2}$.

(*d*) Given equation is $x^9 - x^5 + x^4 - 1 = 0$

$x^5 (x^4 - 1) + (x^4 - 1) = 0 \quad$ or $\quad (x^5 + 1)(x^4 - 1) = 0$

$\therefore \quad x^5 + 1 = 0 \quad$ or $\quad x^4 - 1 = 0$

$x^5 = -1$

$x^5 = \cos \pi + i \sin \pi$

$x^5 = \cos (2k\pi + \pi) + i \sin (2k\pi + \pi)$

$$\therefore \quad x = \cos \frac{2k+1}{5}\pi + i \sin \frac{2k+1}{5}\pi$$

where $\quad k = 0, 1, 2, 3, 4$

$$\therefore \quad x = \cos \frac{\pi}{5} + i \sin \frac{\pi}{5}, \cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5},$$

$$\cos \frac{5\pi}{5} + i \sin \frac{5\pi}{5}, \cos \frac{7\pi}{5} + i \sin \frac{7\pi}{5},$$

$$\cos \frac{9\pi}{5} + i \sin \frac{9\pi}{5}$$

$$x = \cos \frac{\pi}{5} + i \sin \frac{\pi}{5}, \cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5},$$

$$\cos \pi + i \sin \pi, \cos \frac{3\pi}{5} - i \sin \frac{3\pi}{5}$$

$$\cos \frac{\pi}{5} - i \sin \frac{\pi}{5}$$

$x^4 = 1 = \cos 0 + i \sin 0 = \cos 2k\pi + i \sin 2k\pi$

$$\therefore \quad x = \cos \frac{2k\pi}{4} + i \sin \frac{2k\pi}{4}$$

$k = 0, 1, 2, 3$

$$= \cos \frac{k\pi}{2} + i \sin \frac{k\pi}{2}$$

where $\quad k = 0, 1, 2, 3$

$\therefore \quad x = \cos 0 + i \sin 0,$

$$\cos \frac{\pi}{2} + i \sin \frac{\pi}{2},$$

$\cos \pi + i \sin \pi,$

$$\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$$

$\therefore \quad x = 1, i, -1, -i$

$= \pm 1, \pm i$

$$\therefore \; x = \cos \frac{\pi}{5} \pm i \sin \frac{\pi}{5}, \cos \frac{3\pi}{5} \pm i \sin \frac{3\pi}{5}, -1, \pm 1, \pm i$$

Example 8. *Solve $x^{12} - 1 = 0$ and find which of its roots satisfy the equation $x^4 + x^2 + 1 = 0$.*

Sol. $x^{12} - 1 = 0 \quad \therefore \quad x^{12} = 1$

$$x^{12} = \cos 0 + i \sin 0 = \cos 2n\pi + i \sin 2n\pi$$

$$\therefore \quad x = \cos \frac{2n\pi}{12} + i \sin \frac{2n\pi}{12}\ ; n = 0, 1, 2, 3, ...,11$$

$$\therefore \quad x = \cos \frac{n\pi}{6} + i \sin \frac{n\pi}{6}\ ; n = 0, 1, 2, 3, ..., 11$$

$$\therefore \quad x = \text{cis } 0, \text{cis } \frac{\pi}{6}, \text{cis } \frac{2\pi}{6}, \text{cis } \frac{3\pi}{6}, \text{cis } \frac{4\pi}{6}, \text{cis } \frac{5\pi}{6},$$

$$\text{cis } \frac{6\pi}{6}, \text{cis } \frac{7\pi}{6}, \text{cis } \frac{8\pi}{6}, \text{cis } \frac{9\pi}{6}, \text{cis } \frac{10\pi}{6}, \text{cis } \frac{11\pi}{6}$$

$$= 1, \text{cis } \frac{\pi}{6}, \text{cis } \frac{\pi}{3}, i, \text{cis } \frac{2\pi}{3}, \text{cis } \frac{5\pi}{6}, -1, \text{cis } \left(-\frac{5\pi}{6}\right),$$

$$\text{cis } \left(-\frac{2\pi}{3}\right), -i, \text{cis } \left(-\frac{\pi}{3}\right), \text{cis } \left(-\frac{\pi}{6}\right)$$

$$= \pm 1, \pm i, \text{cis } \left(\pm \frac{\pi}{6}\right), \text{cis } \left(\pm \frac{\pi}{3}\right), \text{cis } \left(\pm \frac{2\pi}{3}\right), \text{cis } \left(\pm \frac{5\pi}{6}\right)$$

$$= \pm 1, \pm i, \frac{\sqrt{3} \pm i}{2}, \frac{1 \pm i\sqrt{3}}{2}, \frac{-1 \pm i\sqrt{3}}{2}, \frac{-\sqrt{3} \pm i}{2}$$

$$= \pm 1, \pm i, \pm \frac{\sqrt{3} \pm i}{2}, \pm \frac{1 \pm i\sqrt{3}}{2}$$

Now, $\quad x^4 + x^2 + 1 = 0$

$$x^2 = \frac{-1 \pm i\sqrt{3}}{2} = \cos \frac{2\pi}{3} \pm i \sin \frac{2\pi}{3}$$

$$= \cos \left(2k\pi + \frac{2\pi}{3}\right) \pm i \sin \left(2k\pi + \frac{2\pi}{3}\right)$$

$$\therefore \quad x = \cos \frac{1}{2}\left(2k\pi + \frac{2\pi}{3}\right) \pm i \sin \frac{1}{2}\left(2k\pi + \frac{2\pi}{3}\right), \text{ where } k = 0, 1$$

$$\therefore \quad x = \cos \frac{\pi}{3} \pm i \sin \frac{\pi}{3}, \cos \frac{4\pi}{3} \pm i \sin \frac{4\pi}{3}$$

$$= \frac{1 \pm i\sqrt{3}}{2}, \frac{-1 \mp i\sqrt{3}}{2} = \frac{1 \pm i\sqrt{3}}{2}, -\frac{1 \pm i\sqrt{3}}{2} = \pm \frac{1 \pm i\sqrt{3}}{2}$$

$\therefore$ Last four roots of $x^{12} - 1 = 0$ are the roots of $x^4 + x^2 + 1 = 0$.

Example 9. *Prove by the use of De-Moivre's Theorem that the roots of the equation $(x - 1)^n = x^n$ (n being a +ve integer) are $\frac{1}{2}\left[1 + i \cot \frac{r\pi}{n}\right]$, where r has the values 1, 2, ..., (n – 1).*

Sol. $(x - 1)^n = x^n \quad \therefore \quad \left(\frac{x-1}{x}\right)^n = 1$

or $$\left(\frac{x-1}{x}\right)^n = \cos 0 + i \sin 0 = \cos 2r\pi + i \sin 2r\pi$$

Taking the nth root of both sides

$$\frac{x-1}{x} = (\cos 2r\pi + i \sin 2r\pi)^{\frac{1}{n}} = \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}$$

$$1 - \frac{1}{x} = \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n} \qquad r = 0, 1, 2, ..., (n-1)$$

When $r = 0$, we have $\frac{x-1}{x} = \text{cis } 0 = 1$ or $x - 1 = x$ or $-1 = 0$, which is impossible

Actually the given equation is of degree $(n-1)$ and not n since x^n cancels on both sides.

$\therefore \qquad r = 1, 2, 3, ..., (n-1)$

Putting $\quad \frac{2r\pi}{n} = \theta$, it becomes

$$\frac{1}{x} = 1 - \cos\theta - i \sin\theta$$

$$x = \frac{1}{1-\cos\theta - i\sin\theta} = \frac{1}{2\sin^2\frac{\theta}{2} - i \,.\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{1}{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)}$$

$$= \frac{\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(\sin^2\frac{\theta}{2} - i^2\cos^2\frac{\theta}{2}\right)} = \frac{\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \frac{1}{2}\left[1 + i\cot\frac{\theta}{2}\right]$$

Hence $\qquad x = \frac{1}{2}\left[1 + i\cot\frac{r\pi}{n}\right]$, where $r = 1, 2, ..., (n-1)$.

Example 10. *Use De-Moivre's Theorem to solve the equation $(z-1)^5 + z^5 = 0$.* **(P.T.U., Dec. 2012)**

Sol. $\qquad (z-1)^5 = -z^5$

$$\left(\frac{z-1}{z}\right)^5 = -1 = \cos\pi + i\sin\pi$$

$$= \cos(2k+1)\pi + i\sin(2k+1)\pi$$

$$\frac{z-1}{z} = \frac{\cos\frac{2k+1}{5}\pi + i\sin\frac{2k+1}{5}\pi}{1} \quad \text{where, } k = 0, 1, 2, 3, 4$$

By componendo dividendo $\qquad \frac{z-1-z}{z} = \frac{\cos\frac{2k+1}{5}\pi + i\sin\frac{2k+1}{5}\pi - 1}{1}$

$\therefore$

$$z = \frac{1}{1 - \cos\frac{2k+1}{5}\pi - i\sin\frac{2k+1}{5}\pi}$$

$$= \frac{1}{2\sin^2 \frac{2k+1}{10}\pi - 2\sin\frac{2k+1}{10}\pi\cos\frac{2k+1}{10}\pi}$$

$$= \frac{1}{2\sin^2 \frac{2k+1}{10}\pi\left[1 - i\cot\frac{2k+1}{10}\pi\right]}$$

$$= \frac{1 + i\cot\frac{2k+1}{10}\pi}{2\sin^2\frac{2k+1}{10}\pi\operatorname{cosec}^2\frac{2k+1}{10}\pi}$$

$$\therefore \qquad z = \frac{1}{2}\left(1 + i\cot\frac{2k+1}{10}\pi\right), \text{ where } k = 0, 1, 2, 3, 4$$

Put $z = 0, 1, 2, 3, 4$, we get

$$z = \frac{1}{2}\left(1 + i\cot\frac{\pi}{10}\right), \frac{1}{2}\left(1 + i\cot\frac{3\pi}{10}\right),$$

$$\frac{1}{2}\left(1 + i\cot\frac{5\pi}{10}\right), \frac{1}{2}\left(1 + i\cot\frac{7\pi}{10}\right), \frac{1}{2}\left(1 + i\cot\frac{9\pi}{10}\right)$$

$$\cot\frac{9\pi}{10} = \cot\left(\pi - \frac{\pi}{10}\right) = -\cot\frac{\pi}{10}$$

$$\cot\frac{7\pi}{10} = \cot\left(\pi - \frac{3\pi}{10}\right) = -\cot\frac{3\pi}{10}$$

$$\cot\frac{5\pi}{10} = \cot\frac{\pi}{2} = 0$$

$$\therefore \qquad z = \frac{1}{2}\left(1 \pm i\cot\frac{\pi}{10}\right), \frac{1}{2}\left(1 \pm i\cot\frac{3\pi}{10}\right), 0$$

Hence $z = 0, \frac{1}{2}\left(1 \pm i\cot\frac{\pi}{10}\right), \frac{1}{2}\left(1 \pm i\cot\frac{3\pi}{10}\right)$.

Example 11. *If $(3 + x)^3 - (3 - x)^3 = 0$, then prove that $x = 3i\tan\frac{r\pi}{3}$, $r = 0, 1, 2$.* **(P.T.U., May 2010)**

Sol. Given $(3 + x)^3 - (3 - x)^3 = 0$

or $$\left(\frac{3+x}{3-x}\right)^3 = 1 = \text{cis } 2r\pi$$

or $$\frac{3+x}{3-x} = (\text{cis } 2r\pi)^{\frac{1}{3}} = \text{cis }\frac{2r\pi}{3};\ r = 0, 1, 2$$

or $$\frac{3+x}{3-x} = \frac{\text{cis }\frac{2r\pi}{3}}{1}$$

Apply componendo dividendo

$$\frac{3+x+3-x}{3+x-3+x} = \frac{\text{cis }\frac{2r\pi}{3}+1}{\text{cis }\frac{2r\pi}{3}-1}$$

$$\frac{6}{2x} = \frac{1+\text{cis}\,\frac{2r\pi}{3}}{-\left(1-\text{cis}\,\frac{2r\pi}{3}\right)}$$

$$-\frac{3}{x} = \frac{1+\cos\frac{2r\pi}{3}+i\sin\frac{2r\pi}{3}}{1-\cos\frac{2r\pi}{3}-i\sin\frac{2r\pi}{3}} = \frac{2\cos^2\frac{r\pi}{3}+2i\sin\frac{r\pi}{3}\cos\frac{r\pi}{3}}{2\sin^2\frac{r\pi}{3}-2i\sin\frac{r\pi}{3}\cos\frac{r\pi}{3}}$$

$$= \frac{2\cos\frac{r\pi}{3}\left[\cos\frac{r\pi}{3}+i\sin\frac{r\pi}{3}\right]}{2\sin\frac{r\pi}{3}\left[\sin\frac{r\pi}{3}-i\cos\frac{r\pi}{3}\right]} = \frac{\cot\frac{r\pi}{3}}{-i}\cdot\frac{\text{cis}\,\frac{r\pi}{3}}{\left[i\sin\frac{r\pi}{3}+\cos\frac{r\pi}{3}\right]}$$

or $$-\frac{3}{x} = \frac{\cot\frac{r\pi}{3}}{-i}\,\frac{\text{cis}\,\frac{r\pi}{3}}{\text{cis}\,\frac{r\pi}{3}} = i\cot\frac{r\pi}{3}$$

or $$x = -\frac{3}{i}\tan\frac{r\pi}{3}$$

or $$x = 3i\tan\frac{r\pi}{3}, \text{ where } r = 0, 1, 2\,.$$

Example 12. *Show that the roots of* $(x+1)^6+(x-1)^6=0$ *are* $\pm\, i\cot\frac{2k+1}{12}\pi$, $(k=0, 1, 2)$ *and deduce that*

(*i*) $(x+1)^6+(x-1)^6 = 2\prod_{k=0}^{2}\left(x^2+\cot^2\frac{2k+1}{12}\pi\right)$

(*ii*) $\cot\frac{\pi}{12}\cot\frac{3\pi}{12}\cot\frac{5\pi}{12} = 1$ (*iii*) $\sin\frac{\pi}{12}\sin\frac{3\pi}{12}\sin\frac{5\pi}{12} = \frac{1}{4\sqrt{2}}$.

Sol. The given equation is $(x+1)^6 = -(x-1)^6$

$$\Rightarrow \quad \left(\frac{x+1}{x-1}\right)^6 = -1 = \text{cis}\,\pi$$

$$\Rightarrow \quad \frac{x+1}{x-1} = (\text{cis}\,\pi)^{1/6} = [\text{cis}\,(2n\pi+\pi)]^{1/6} = \text{cis}\,\frac{2n+1}{6}\pi, \text{ where } n = 0, 1, 2, \ldots\ldots, 5$$

$$= \cos\theta + i\sin\theta, \text{ where } \theta = \frac{2n+1}{6}\pi$$

By componendo and dividendo $\dfrac{x+1+x-1}{x+1-x+1} = \dfrac{\cos\theta+i\sin\theta+1}{\cos\theta+i\sin\theta-1}$

$$x = \frac{(1+\cos\theta)+i\sin\theta}{-(1-\cos\theta)+i\sin\theta} = \frac{2\cos^2\theta/2+2\,i\sin\theta/2\cos\theta/2}{-2\sin^2\theta/2+2\,i\sin\theta/2\cos\theta/2}$$

$$= \cot\frac{\theta}{2}\,\frac{\cos\theta/2+i\sin\theta/2}{-\sin\theta/2+i\cos\theta/2} = \frac{1}{i}\cot\frac{\theta}{2}\,\frac{\cos\theta/2+i\sin\theta/2}{\cos\theta/2+i\sin\theta/2}$$

$$= \frac{1}{i} \cot \frac{\theta}{2} = \frac{i}{i^2} \cot \frac{2n+1}{12} \pi = -i \cot \frac{2n+1}{12} \pi \ , \text{ where } n = 0, 1, 2, 3, 4, 5.$$

when $n = 0$, $x = -i \cot \dfrac{\pi}{12}$ when $n = 1$, $x = -i \cot \dfrac{3\pi}{12}$

when $n = 2$, $x = -i \cot \dfrac{5\pi}{12}$ when $n = 3$, $x = -i \cot \dfrac{7\pi}{12} = -i \cot \left(\pi - \dfrac{5\pi}{12}\right) = i \cot \dfrac{5\pi}{12}$

when $n = 4$, $x = -i \cot \dfrac{9\pi}{12} = -i \cot \left(\pi - \dfrac{3\pi}{12}\right) = i \cot \dfrac{3\pi}{12}$

when $n = 5$, $x = -i \cot \dfrac{11\pi}{12} = -i \cot \left(\pi - \dfrac{\pi}{12}\right) = i \cot \dfrac{\pi}{12}$

$\therefore$ The roots of the given equation are

$$\pm\ i \cot \frac{\pi}{12}, \pm\ i \cot \frac{3\pi}{12}, \pm\ i \cot \frac{5\pi}{12} \quad \text{or} \quad \pm\ i \cot \frac{2k+1}{12} \pi, \text{ where } k = 0, 1, 2.$$

(*i*) Since the roots of $(x+1)^6 + (x-1)^6 = 0$ are $\pm\ i \cot \dfrac{\pi}{12}, \pm\ i \cot \dfrac{3\pi}{12}, \pm\ i \cot \dfrac{5\pi}{12}$

$$\therefore \quad (x+1)^6 + (x-1)^6 = \lambda \left(x - i \cot \frac{\pi}{12}\right)\left(x + i \cot \frac{\pi}{12}\right)$$

$$\times \left(x - i \cot \frac{3\pi}{12}\right)\left(x + i \cot \frac{3\pi}{12}\right)\left(x - i \cot \frac{5\pi}{12}\right)\left(x + i \cot \frac{5\pi}{12}\right)$$

$$= \lambda \left(x^2 - i^2 \cot^2 \frac{\pi}{12}\right)\left(x^2 - i^2 \cot^2 \frac{3\pi}{12}\right)\left(x^2 - i^2 \cot^2 \frac{5\pi}{12}\right)$$

$$= \lambda \left(x^2 + \cot^2 \frac{\pi}{12}\right)\left(x^2 + \cot^2 \frac{3\pi}{12}\right)\left(x^2 + \cot^2 \frac{5\pi}{12}\right)$$

Equating co-efficients of x^6, $1 + 1 = \lambda \quad \Rightarrow \quad \lambda = 2$

$$\therefore \quad (x+1)^6 + (x-1)^6 = 2\left(x^2 + \cot^2 \frac{\pi}{12}\right)\left(x^2 + \cot^2 \frac{3\pi}{12}\right)\left(x^2 + \cot^2 \frac{5\pi}{12}\right) \qquad \ldots(1)$$

$$= 2 \prod_{k=0}^{2} \left(x^2 + \cot^2 \frac{2k+1}{12} \pi\right).$$

Note. Just as Σ represents sum, Π represents product.

(*ii*) Putting $x = 0$ in (1), $1 + 1 = 2 \cot^2 \dfrac{\pi}{12} \cot^2 \dfrac{3\pi}{12} \cot^2 \dfrac{5\pi}{12}$

$$\Rightarrow \qquad \cot \frac{\pi}{12} \cot \frac{3\pi}{12} \cot \frac{5\pi}{12} = 1$$

$$\left(\text{taking +ve sign with square root since all the angles involved are less than } \frac{\pi}{2}\right)$$

(*iii*) Putting $x = 1$ in (1), $2^6 = 2\left(1 + \cot^2 \dfrac{\pi}{12}\right)\left(1 + \cot^2 \dfrac{3\pi}{12}\right)\left(1 + \cot^2 \dfrac{5\pi}{12}\right)$

$$\Rightarrow \quad \operatorname{cosec}^2 \frac{\pi}{12} \operatorname{cosec}^2 \frac{3\pi}{12} \operatorname{cosec}^2 \frac{5\pi}{12} = 2^5 = 32$$

$$\Rightarrow \quad \sin^2 \frac{\pi}{12} \sin^2 \frac{3\pi}{12} \sin^2 \frac{5\pi}{12} = \frac{1}{32} \quad \Rightarrow \quad \sin \frac{\pi}{12} \sin \frac{3\pi}{12} \sin \frac{5\pi}{12} = \frac{1}{\sqrt{32}} = \frac{1}{4\sqrt{2}}.$$

TEST YOUR KNOWLEDGE

1. Find all the values of
(*i*) $(1)^{1/4}$ (*ii*) $(-1)^{1/5}$ (*iii*) $(i)^{1/3}$
(*iv*) $(-i)^{1/6}$ (*v*) $(32)^{1/5}$ (*vi*) $(-8i)^{1/3}$ **(P.T.U., May 2012)**
(*vii*) $(-1)^{1/4}$ **(P.T.U., Dec. 2003)**

2. Find the 5th roots of unity and prove that the sum of their *n*th powers always vanishes unless *n* be a multiple of 5, *n* being an integer, and then the sum is 5.

3. Find all the values of
(*i*) $(-1+i)^{2/5}$ (*ii*) $(1-i\sqrt{3})^{1/3}$
(*iii*) $(1+i)^{1/4}$ **(P.T.U., Dec. 2010)** (*iv*) $(-1+i\sqrt{3})^{3/2}$.

4. Find all the values of $(1+i\sqrt{3})^{3/4}$ and show that their continued product is 8.

5. If ω is a complex cube root of unity, prove that $1+\omega+\omega^2=0$. **(P.T.U., May 2011)**

6. Express $\rho = \dfrac{(\sqrt{3}-1)+i(\sqrt{3}+1)}{2\sqrt{2}}$ in the form $r(\cos\theta + i\sin\theta)$ and derive all the values of $\rho^{1/4}$.

7. Find all the values of $(1+i)^{1/3}$ and obtain their product.

8. Use De-Moivre's theorem to solve the following equations:
(*i*) $x^5-1=0$ (*ii*) $x^7+1=0$
(*iii*) $x^4+x^3+x^2+x+1=0$ (*iv*) $x^4+x^2+1=0$.

9. Solve the following equations:
(*i*) $(1+x)^n=(1-x)^n$ (*ii*) $(5+x)^5-(5-x)^5=0$
(*iii*) $z^4-(1-z)^4=0$
[**Hint:** Consult S.E. 9] **(P.T.U., May 2012)**
(*iv*) $(1+x)^3=i(1-x)^3$.

10. Show that the roots of the equation $(1+x)^{2n}+(1-x)^{2n}=0$ are given by $\pm i\tan\dfrac{(2r-1)\pi}{4n}$, where $r=1, 2, 3, \ldots\ldots, n$.

ANSWERS

1. (*i*) $\pm 1, \pm i$ (*ii*) $-1, \cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}$

(*iii*) $-i, \dfrac{\pm\sqrt{3}+i}{2}$ (*iv*) $\pm\left(\cos\frac{r\pi}{12}+i\sin\frac{r\pi}{12}\right), r = 3, 7, 11$

(*v*) $2, 2\left(\cos\frac{2\pi}{5} \pm i\sin\frac{2\pi}{5}\right), 2\left(\cos\frac{4\pi}{5} \pm i\sin\frac{4\pi}{5}\right)$ (*vi*) $2i, \pm\sqrt{3}-i$

(*vii*) $\dfrac{1\pm i}{\sqrt{2}}, \dfrac{-1\pm i}{\sqrt{2}}$.

2. $1, \operatorname{cis}\frac{2\pi}{5}, \operatorname{cis}\frac{4\pi}{5}, \operatorname{cis}\frac{6\pi}{5}, \operatorname{cis}\frac{8\pi}{5}$ or $1, \cos\frac{2\pi}{5} \pm i\sin\frac{2\pi}{5}, \cos\frac{4\pi}{5} \pm i\sin\frac{4\pi}{5}$

3. (*i*) $2^{1/5}\text{ cis }\dfrac{(4n+3)\pi}{10}$, $n = 0, 1, 2, 3, 4$ (*ii*) $2^{1/3}\left(\cos\dfrac{r\pi}{9} - i\sin\dfrac{r\pi}{9}\right)$, $r = 1, 7, 13$

(*iii*) $\pm 2^{1/8}\text{ cis }\dfrac{r\pi}{16}$, where $r = 1, 9$ (*iv*) $\pm 2\sqrt{2}$

4. $2^{\frac{3}{4}}\cdot\dfrac{1\pm i}{\sqrt{2}},\ 2^{\frac{3}{4}}\cdot\dfrac{-1\pm i}{\sqrt{2}}$ **6.** $r = \text{cis }\dfrac{5\pi}{12}$; $\pm\text{ cis }\dfrac{r\pi}{48}$, where $r = 5, 29$

7. $2^{1/6}\text{ cis }\dfrac{r\pi}{12}$; $r = 1, 9, 17$; product $= 1 + i$

8. (*i*) $1, \cos\dfrac{2\pi}{5} \pm i\sin\dfrac{2\pi}{5}, \cos\dfrac{4\pi}{5} \pm i\sin\dfrac{4\pi}{5}$ (*ii*) $-1, \cos\dfrac{r\pi}{7} \pm i\sin\dfrac{r\pi}{7}$, where $r = 1, 3, 5$

(*iii*) $\cos\dfrac{r\pi}{5} \pm i\sin\dfrac{r\pi}{5}$, where $r = 2, 4$ (*iv*) $\dfrac{1\pm i\sqrt{3}}{2}, \dfrac{-1\pm i\sqrt{3}}{2}$

9. (*i*) $i\tan\dfrac{r\pi}{n}$, where $r = 0, 1, 2, ..., (n-1)$ (*ii*) $5i\tan\dfrac{r\pi}{5}$, where $r = 0, 1, 2, 3, 4$

(*iii*) $\dfrac{1}{2}, \dfrac{1}{2}\left(1 \pm i\dfrac{1}{\sqrt{2}}\right)$ (*iv*) $i\tan\dfrac{(4r+1)\pi}{12}$, where $r = 0, 1, 2.$

6.5(a). EXPRESS COS^n θ IN TERMS OF COSINES OF MULTIPLES OF θ (*n* BEING A POSITIVE INTEGER)

Let $x = \cos\theta + i\sin\theta$, $\therefore$ $x^n = \cos n\theta + i\sin n\theta$

then $\dfrac{1}{x} = \cos\theta - i\sin\theta$, $\therefore$ $\dfrac{1}{x^n} = \cos n\theta - i\sin n\theta$

Adding $x + \dfrac{1}{x} = 2\cos\theta$; $x^n + \dfrac{1}{x^n} = 2\cos n\theta$

$$\therefore \quad (2\cos\theta)^n = \left(x + \frac{1}{x}\right)^n = {}^nC_0x^n + {}^nC_1x^{n-1}\cdot\frac{1}{x} + {}^nC_2x^{n-2}\cdot\frac{1}{x^2} + ...$$

$$+ {}^nC_{n-2}x^2\cdot\frac{1}{x^{n-2}} + {}^nC_{n-1}x^1\cdot\frac{1}{x^{n-1}} + {}^nC_n\cdot\frac{1}{x^n}$$

Since in a binomial expansion co-efficients of terms equidistant from the beginning and end are equal *i.e.*, ${}^nC_n = {}^nC_0$; ${}^nC_{n-1} = {}^nC_1$ etc. we combine the first term with the last, the second with the last but one and so on.

Case I. *If n is even*

Number of terms $= n + 1$ *i.e.,* odd

There is only one middle term which is left by itself as the last term after grouping in pairs.

$$\text{The middle term} = T_{\frac{n}{2}+1} = {}^nC_{n/2}\,x^{n-\frac{n}{2}}\cdot\frac{1}{x^{n/2}} = {}^nC_{n/2}\,x^{n/2}\cdot\frac{1}{x^{n/2}} = {}^nC_{n/2}$$

$$\therefore \quad (2\cos\theta)^n = {}^nC_0\left(x^n + \frac{1}{x^n}\right) + {}^nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + ... + {}^nC_{n/2}$$

But $\quad x^n + \dfrac{1}{x^n} = 2\cos n\theta,\ x^{n-2} + \dfrac{1}{x^{n-2}} = 2\cos(n-2)\,\theta$ etc.

$\therefore \quad 2^n \cos^n\theta = {}^nC_0 \cdot 2\cos n\theta + {}^nC_1 \cdot 2\cos(n-2)\theta + {}^nC_2 \cdot 2\cos(n-4)\theta + \ldots + {}^nC_{n/2}$

Hence if ***n* is even**

$$\cos^n\theta = \frac{1}{2^{n-1}}\left[{}^nC_0\cos n\theta + {}^nC_1\cos(n-2)\,\theta + {}^nC_2\cos(n-4)\,\theta + \ldots\ \frac{1}{2}\cdot {}^nC_{n/2}\right]$$

If n is even

Last term of $\quad \cos^n\theta = \dfrac{1}{2^n}\,{}^nC_{\frac{n}{2}}$

$$= \frac{1}{2^n}\cdot {}^nC_{\frac{n}{2}} = \frac{1}{2^n}\cdot\frac{n!}{\left(\frac{n}{2}\right)!\left(n-\frac{n}{2}\right)!} \qquad \left[\because\ {}^nC_r = \frac{n!}{r!\,(n-r)!}\right]$$

$$= \frac{1}{2^n}\cdot\frac{n!}{\left[\left(\frac{n}{2}\right)!\right]^2}$$

Case II. *If n is odd*

Number of terms $= n+1$ *i.e.*, even

There are two middle terms $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$

$$T_{\frac{n+1}{2}} = T_{\frac{n-1}{2}+1} = {}^nC_{\frac{n-1}{2}}\cdot x^{n-\frac{n-1}{2}}\cdot\frac{1}{x^{\frac{n-1}{2}}} = {}^nC_{\frac{n-1}{2}}\,x^{\frac{n+1}{2}}\cdot\frac{1}{x^{\frac{n-1}{2}}} = {}^nC_{\frac{n-1}{2}}\,x.$$

$$T_{\frac{n+3}{2}} = T_{\frac{n+1}{2}+1} = {}^nC_{\frac{n+1}{2}}\cdot x^{n-\frac{n+1}{2}}\cdot\frac{1}{x^{\frac{n+1}{2}}} = {}^nC_{n-\frac{n+1}{2}}\,x^{\frac{n-1}{2}}\cdot\frac{1}{x^{\frac{n+1}{2}}} = {}^nC_{\frac{n-1}{2}}\cdot\frac{1}{x}$$

Using ${}^nC_r = {}^nC_{n-r}$ we have ${}^nC_{\frac{n+1}{2}} = {}^nC_{n-\frac{n+1}{2}}$

$\therefore$ The two middle terms pair up together

$$\therefore \quad (2\cos\theta)^n = {}^nC_0\left(x^n + \frac{1}{x^n}\right) + {}^nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \ldots\ldots + {}^nC_{\frac{n-1}{2}}\left(x + \frac{1}{x}\right)$$

or $\quad 2^n\cos^n\theta = {}^nC_0\,.\,2\cos n\theta + {}^nC_1\,.\,2\cos(n-2)\theta + {}^nC_2.\,2\cos(n-4)\theta + \ldots\ldots + {}^nC_{\frac{n-1}{2}}\,.\,2\cos\theta$

Hence if ***n* is odd**

$$\cos^n\theta = \frac{1}{2^{n-1}}\,[{}^nC_0\cos n\theta + {}^nC_1\cos(n-2)\theta + {}^nC_2\cos(n-4)\theta + \ldots\ldots + {}^nC_{\frac{n-1}{2}}\cos\theta]$$

If n is odd

Last term of $\cos^n \theta = \dfrac{1}{2^{n-1}}\, {}^nC_{\frac{n-1}{2}} \cos\theta$

$$= \frac{1}{2^{n-1}} \cdot {}^nC_{\frac{n-1}{2}} \cos\theta = \frac{1}{2^{n-1}} \cdot \frac{n\,!}{\left(\frac{n-1}{2}\right)!\left(n-\frac{n-1}{2}\right)!} \cdot \cos\theta$$

$$= \frac{1}{2^{n-1}} \cdot \frac{n\,!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!} \cdot \cos\theta$$

Note 1. The expansion of $\cos^n \theta$ is in terms of cosines of multiples of θ.

Note 2. Pascal's Rule to write the binomial co-efficients.

Note 3. We see that $\cos^n \theta$ will contain factors of the type $\left(x+\dfrac{1}{x}\right)^n$. To find coefficients of various powers of x we will use Pascals rule of binomial coefficients which is as follow:

The series of co-efficients in successive powers of $x + \dfrac{1}{x}$ beginning with the power unity are as follows :

Index (Power)	*Binomial Co-efficients*								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

and so on.

Each figure is obtained by adding the figure just above it to the figure preceeding the latter (*i.e.,* upper + left hand)

e.g.,
$$\begin{array}{c} 5+10 \\ \downarrow \\ 15 \end{array} \qquad\qquad \begin{array}{c} 35+21 \\ \downarrow \\ 56 \end{array}$$

It may be observed that the expansion of $\left(x+\dfrac{1}{x}\right)^n$ starts with x^n, the powers decreasing by 2 every time.

Thus $\left(x+\dfrac{1}{x}\right)^5 = x^5 + 5x^3 + 10x + \dfrac{10}{x} + \dfrac{5}{x^3} + \dfrac{1}{x^5}$; binomial co-efficients being written by Pascal's Rule.

ILLUSTRATIVE EXAMPLES

Example 1. *Express* $\cos^8 \theta$ *in terms of cosines of multiples of* θ. **(P.T.U., May 2006, 2014)**

Sol. Let $x = \cos\theta + i \sin\theta$, then $\dfrac{1}{x} = \cos\theta - i\sin\theta$

So that $x + \dfrac{1}{x} = 2\cos\theta$ and $x^m + \dfrac{1}{x^m} = 2\cos m\theta$, where m is a +ve integer ...(1)

From (1) we have

$$(2\cos\theta)^8 = \left(x+\frac{1}{x}\right)^8$$

By Pascal's Rule

1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

$$\therefore \quad (2\cos\theta)^8 = x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + 56\,\frac{1}{x^2} + 28\,\frac{1}{x^4} + 8\,\frac{1}{x^6} + \frac{1}{x^8}$$

$$= \left(x^8 + \frac{1}{x^8}\right) + 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) + 56\left(x^2 + \frac{1}{x^2}\right) + 70$$

$$2^8 \cos^8\theta = 2\cos 8\theta + 8\,.\,2\cos 6\theta + 28\,.\,2\cos 4\theta + 56\,.\,2\cos 2\theta + 70$$

$$\therefore \quad \cos^8\theta = \frac{1}{2^7}\,[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$$

$$= \frac{1}{128}\,[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$$

6.5(b). EXPRESS SIN^n θ IN A SERIES OF COSINES OR SINES OF MULTIPLES OF θ ACCORDING AS *n* IS AN EVEN OR ODD INTEGER

Let $x = \cos\theta + i\sin\theta$; then $\dfrac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore \quad x - \frac{1}{x} = 2i\sin\theta$$

Also, $\quad x^n + \dfrac{1}{x^n} = 2\cos n\theta \quad$ and $\quad x^n - \dfrac{1}{x^n} = 2i\sin n\theta$

Case I. *When n is even.*

$$(2i\sin\theta)^n = \left[x - \frac{1}{x}\right]^n = {}^nC_0\,x^n - {}^nC_1\,x^{n-1}\,.\,\frac{1}{x} + {}^nC_2\,x^{n-2}\,.\,\frac{1}{x^2} - \ldots\ldots$$

$$+\,{}^nC_{n-2}\,x^2\,.\,\frac{1}{x^{n-2}} - {}^nC_{n-1}\,x\,.\,\frac{1}{x^{n-1}} + {}^nC_n\,\frac{1}{x^n}$$

Number of terms $= n + 1$, *i.e.,* odd

There is only one middle term which is left by itself as the last term after grouping in pairs.

$$\text{Middle term} \;=\; T_{\frac{n}{2}+1} = {}^nC_{\frac{n}{2}}\,x^{n-\frac{n}{2}}\left(-\frac{1}{x}\right)^{n/2}$$

$$= (-1)^{\frac{n}{2}}\,.\,{}^nC_{\frac{n}{2}}\,x^{\frac{n}{2}}\,\frac{1}{x^{n/2}} = (-1)^{\frac{n}{2}}\,.\,{}^nC_{\frac{n}{2}}$$

$$\therefore \quad (2i\sin\theta)^n = {}^nC_0\left(x^n + \frac{1}{x^n}\right) - {}^nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) - \ldots + (-1)^{\frac{n}{2}}\,.\,{}^nC_{\frac{n}{2}}$$

$$\Rightarrow \quad 2^n\,.\,(-1)^{\frac{n}{2}}\,\sin^n\theta = {}^nC_0\,.\,2\cos n\theta - {}^nC_1\,.\,2\cos(n-2)\theta + {}^nC_2\,.\,2\cos(n-4)\theta - \ldots\ldots + (-1)^{\frac{n}{2}}\,.\,{}^nC_{\frac{n}{2}}$$

$$\left[\because \quad i^n = (i^2)^{\frac{n}{2}} = (-1)^{\frac{n}{2}}\right]$$

$$\Rightarrow \quad \sin^n \theta = \frac{1}{2^n (-1)^{\frac{n}{2}}} \cdot 2\left[{}^nC_0 \cos n\theta - {}^nC_1 \cos (n-2)\theta + {}^nC_2 \cos (n-4)\theta - \ldots\ldots + (-1)^{\frac{n}{2}} \frac{1}{2} \cdot {}^nC_{\frac{n}{2}}\right]$$

$$= \frac{(-1)^{\frac{n}{2}}}{2^{n-1}}\left[{}^nC_0 \cos n\theta - {}^nC_1 \cos (n-2)\theta + {}^nC_2 \cos (n-4)\theta - \ldots\ldots + (-1)^{\frac{n}{2}} \frac{1}{2} \cdot {}^nC_{\frac{n}{2}}\right]$$

$$\left[\because \; n \text{ is even} \;\; \therefore \;\; (-1)^n = 1, \frac{1}{(-1)^{\frac{n}{2}}} = \frac{(-1)^n}{(-1)^{\frac{n}{2}}} = (-1)^{\frac{n}{2}}\right]$$

Case II. *When n is odd.*

Number of terms $= n + 1$ *i.e.,* even

There are two middle terms $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$

$$T_{\frac{n+1}{2}} = T_{\frac{n-1}{2}+1} = {}^nC_{\frac{n-1}{2}} \cdot x^{n-\frac{n-1}{2}} \left(-\frac{1}{x}\right)^{\frac{n-1}{2}}$$

$$= {}^nC_{\frac{n-1}{2}} \cdot x^{\frac{n+1}{2}} \cdot (-1)^{\frac{n-1}{2}} \frac{1}{x^{\frac{n-1}{2}}} = (-1)^{\frac{n-1}{2}} \; {}^nC_{\frac{n-1}{2}} \cdot x$$

$$T_{\frac{n+3}{2}} = T_{\frac{n+1}{2}+1} = {}^nC_{\frac{n+1}{2}} \cdot x^{n-\frac{n+1}{2}} \left(-\frac{1}{x}\right)^{\frac{n+1}{2}}$$

$$= {}^nC_{n-\frac{n-1}{2}} \cdot x^{\frac{n-1}{2}} \cdot (-1)^{\frac{n+1}{2}} \cdot \frac{1}{x^{\frac{n+1}{2}}} = (-1)^{\frac{n+1}{2}} \; {}^nC_{\frac{n-1}{2}} \cdot \frac{1}{x}$$

$$= (-1)(-1)^{\frac{n-1}{2}} \; {}^nC_{\frac{n-1}{2}} \cdot \frac{1}{x}$$

$\therefore$ The two middle terms pair up together.

$$\therefore \quad (2i \sin \theta)^n = {}^nC_0 \, x^n - {}^nC_1 \, x^{n-1} \cdot \frac{1}{x} + {}^nC_2 \cdot x^{n-2} \cdot \frac{1}{x^2} - \ldots\ldots - {}^nC_{n-2} \, x^2 \cdot \frac{1}{x^{n-2}} + {}^nC_{n-1} \, x \cdot \frac{1}{x^{n-1}} - {}^nC_n \cdot \frac{1}{x^n}$$

$$= {}^nC_0\left(x^n - \frac{1}{x^n}\right) - {}^nC_1\left(x^{n-2} - \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} - \frac{1}{x^{n-4}}\right) - \ldots\ldots + (-1)^{\frac{n-1}{2}} \cdot {}^nC_{\frac{n-1}{2}}\left(x - \frac{1}{x}\right)$$

Since $x^m - \dfrac{1}{x^m} = 2i \sin m\theta$

$$\therefore \quad 2^n \cdot i \cdot (-1)^{\frac{n-1}{2}} \sin^n \theta = {}^nC_0 \cdot 2i \sin n\theta - {}^nC_1 \cdot 2i \sin (n-2)\theta$$

$$+ {}^nC_2 \cdot 2i \sin (n-4)\theta - \ldots\ldots + (-1)^{\frac{n-1}{2}} \cdot {}^nC_{\frac{n-1}{2}} \cdot 2i \sin \theta$$

$$\left[\because \;\; i^n = i \cdot i^{n-1} = i(i^2)^{\frac{n-1}{2}} = i(-1)^{\frac{n-1}{2}}\right]$$

$$\Rightarrow \sin^n \theta = \frac{1}{2^n \cdot (-1)^{\frac{n-1}{2}}} \cdot 2\left[{}^nC_0 \sin n\theta - {}^nC_1 \sin (n-2)\theta + {}^nC_2 \sin (n-4)\theta - \ldots\ldots + (-1)^{\frac{n-1}{2}} \; {}^nC_{\frac{n-1}{2}} \sin \theta\right]$$

$$= \frac{1}{2^{n-1} \cdot (-1)^{\frac{n-1}{2}}} \left[{}^nC_0 \sin n\theta - {}^nC_1 \sin (n-2)\theta + {}^nC_2 \sin (n-4)\theta - \ldots\ldots (-1)^{\frac{n-1}{2}} \; {}^nC_{\frac{n-1}{2}} \sin \theta \right]$$

$$= \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} \left[{}^nC_0 \sin n\theta - {}^nC_1 \sin (n-2)\theta + {}^nC_2 \sin (n-4)\theta - \ldots\ldots + (-1)^{\frac{n-1}{2}} \; {}^nC_{\frac{n-1}{2}} \sin \theta \right]$$

$$\left[\because \quad n \text{ is odd}, n-1 \text{ is even} \quad \therefore \quad (-1)^{n-1} = 1, \; \frac{1}{(-1)^{\frac{n-1}{2}}} = \frac{(-1)^{n-1}}{(-1)^{\frac{n-1}{2}}} = (-1)^{\frac{n-1}{2}} \right]$$

Example 2. *Express $\sin^8 \theta$ in a series of cosines of multiples of θ.*

Sol. Let $\quad x = \cos \theta + i \sin \theta$; then $\dfrac{1}{x} = \cos \theta - i \sin \theta$

so that $x - \dfrac{1}{x} = 2i \sin \theta, x + \dfrac{1}{x} = 2 \cos \theta$ and $x^m + \dfrac{1}{x^m} = 2 \cos m\theta$

$$(2i \sin \theta)^8 = \left(x - \frac{1}{x} \right)^8 = x^8 - 8x^6 + 28x^4 - 56x^2 + 70 - \frac{56}{x^2} + \frac{28}{x^4} - \frac{8}{x^6} + \frac{1}{x^8} \qquad | \text{ By Pascal's Rule}$$

$$= \left(x^8 + \frac{1}{x^8} \right) - 8 \left(x^6 + \frac{1}{x^6} \right) + 28 \left(x^4 + \frac{1}{x^4} \right) - 56 \left(x^2 + \frac{1}{x^2} \right)$$

$$\Rightarrow \quad 2^8 \cdot i^8 \sin^8 \theta = 2 \cos 8\theta - 8 \cdot 2 \cos 6\theta + 28 \cdot 2 \cos 4\theta - 56 \cdot 2 \cos 2\theta + 70$$

$$\therefore \quad \sin^8 \theta = \frac{1}{2^7 \cdot i^8} [\cos 8\theta - 8 \cos 6\theta + 28 \cos 4\theta - 56 \cos 2\theta + 35]$$

$$= \frac{1}{128} [\cos 8\theta - 8 \cos 6\theta + 28 \cos 4\theta - 56 \cos 2\theta + 35].$$

Expansion of $\sin^m \theta \cos^n \theta$: We see that $\sin^m \theta \cos^n \theta$ will contain factors of the type $\left(x - \dfrac{1}{x} \right)^m \left(x + \dfrac{1}{x} \right)^n$.

To find coefficients of various powers of x we consider the following example :

For example. To obtain the coefficients of various powers of x in the product $\left(x - \dfrac{1}{x} \right)^4 \left(x + \dfrac{1}{x} \right)^2$, we have the following rule:

First write the coefficients of $\left(x - \dfrac{1}{x} \right)^4$ in a row $\because$ out of the two indices 4 is greater coefficient of $\left(x - \dfrac{1}{x} \right)^4$ are $1, -4, 6, -4, 1$ | By Pascal's Rule

Then to find coefficients of $\left(x - \dfrac{1}{x} \right)^4 \left(x + \dfrac{1}{x} \right)$ add in the upper number its proceeding number in the same line. First coefficient is always one. Repeat the process, the same number of times as is the index of $\left(x + \dfrac{1}{x} \right)$.

		1	–4	6	–4	1		
	I							
		1	–4+1	6–4	–4+6	1–4	0+1	
i.e.,		1	–3	2	2	–3	1	
	II							
		1	–3+1	2–3	2+2	–3+2	1–3	0+1
i.e.,		1	–2	–1	4	–1	–2	1

Similarly to multiply $\left(x+\frac{1}{x}\right)^4$ by $\left(x-\frac{1}{x}\right)^3$, we have the following rule. Write coefficients of $\left(x+\frac{1}{x}\right)^4$ and then to find coefficients of $\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)$, subtract in the upper number its proceeding number in the same line. First coefficient is always one. Repeat the process the same number of times as is the index of $\left(x-\frac{1}{x}\right)$.

		1	4	6	4	1			
	I								
		1	4–1	6–4	4–6	1–4	0–1		
i.e.,		1	3	2	–2	–3	–1		
	II								
		1	3–1	2–3	–2–2	–3+2	–1+3	0+1	
i.e.,		1	2	–1	–4	–1	2	1	
	III								
		1	2–1	–1–2	–4+1	–1+4	2+1	1–2	0–1
i.e.,		1	1	–3	–3	3	3	–1	–1

Example 3. *Show that* $2^5 \sin^4\theta \cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$. **(P.T.U., May 2011)**

Sol. Let $x = \cos\theta + i\sin\theta$; then $\frac{1}{x} = \cos\theta - i\sin\theta$

So that $x+\frac{1}{x} = 2\cos\theta,\quad x-\frac{1}{x} = 2i\sin\theta,\quad x^m+\frac{1}{x^m} = 2\cos m\theta.$

We have $(2i\sin\theta)^4(2\cos\theta)^2 = \left(x-\frac{1}{x}\right)^4\left(x+\frac{1}{x}\right)^2$ ...(1)

The co-efficients of the various powers of x in $\left(x-\frac{1}{x}\right)^4$ are (by Pascal's Table) 1, –4, 6, – 4, 1.

Multiplying $\left(x-\frac{1}{x}\right)^4$ by $\left(x+\frac{1}{x}\right)$ twice in succession as shown in the following scheme:

	1	–4	6	–4	1		
I	1	–3	2	2	–3	1	
II	1	–2	–1	4	–1	–2	1

$$\therefore \quad \left(x-\frac{1}{x}\right)^4\left(x+\frac{1}{x}\right)^2 = x^6 - 2x^4 - x^2 + 4 - \frac{1}{x^2} - \frac{2}{x^4} + \frac{1}{x^6}$$

$$= \left(x^6+\frac{1}{x^6}\right) - 2\left(x^4+\frac{1}{x^4}\right) - \left(x^2+\frac{1}{x^2}\right) + 4$$

$$= 2\cos 6\theta - 2\cdot 2\cos 4\theta - 2\cos 2\theta + 4$$

$\therefore$ From (1), $2^6 .\, i^4 \sin^4\theta\cos^2\theta = 2\,[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2]$

$\therefore$ $2^5 \sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2.$

Example 4. *Expand $\cos^5\theta\sin^7\theta$ in a series of sines of multiples of θ.*

Sol. Let $x = \cos\theta + i\sin\theta$; $\dfrac{1}{x} = \cos\theta - i\sin\theta$

So that $x + \dfrac{1}{x} = 2\cos\theta$, $x - \dfrac{1}{x} = 2i\sin\theta$, $x^m - \dfrac{1}{x^m} = 2i\sin m\theta$.

We have $$(2i\sin\theta)^7\cdot(2\cos\theta)^5 = \left(x-\frac{1}{x}\right)^7\left(x+\frac{1}{x}\right)^5 \qquad ...(1)$$

The co-efficients of the various powers of x in $\left(x-\dfrac{1}{x}\right)^7$ are (by Pascal's Table)

1 −7 21 −35 35 −21 7 −1

Multiplying $\left(x-\dfrac{1}{x}\right)^7$ by $\left(x+\dfrac{1}{x}\right)$ five times in succession, as shown in the following scheme.

	1	−7	21	−35	35	−21	7	−1					
I	1	−6	14	−14	0	14	−14	6	−1				
II	1	−5	8	0	−14	14	0	−8	5	−1			
III	1	−4	3	8	−14	0	14	−8	−3	4	−1		
IV	1	−3	−1	11	−6	−14	14	6	−11	1	3	−1	
V	1	−2	−4	10	5	−20	0	20	−5	−10	4	2	−1

$$\therefore \quad \left(x-\frac{1}{x}\right)^7\left(x+\frac{1}{x}\right)^5$$

$$= x^{12} - 2x^{10} - 4x^8 + 10x^6 + 5x^4 - 20x^2 + 0 + \frac{20}{x^2} - \frac{5}{x^4} - \frac{10}{x^6} + \frac{4}{x^8} + \frac{2}{x^{10}} - \frac{1}{x^{12}}$$

$$= \left[x^{12}-\frac{1}{x^{12}}\right] - 2\left[x^{10}-\frac{1}{x^{10}}\right] - 4\left[x^8-\frac{1}{x^8}\right] + 10\left[x^6-\frac{1}{x^6}\right] + 5\left[x^4-\frac{1}{x^4}\right] - 20\left[x^2-\frac{1}{x^2}\right].$$

$$= 2i\sin 12\theta - 2\,.\,2i\sin 10\theta - 4\,.\,2i\sin 8\theta + 10\,.\,2i\sin 6\theta + 5\,.\,2i\sin 4\theta - 20\,.\,2i\sin 2\theta.$$

$\therefore$ From (1),

$$2^{12}\,.\,i^7\,.\,\sin^7\theta\cos^5\theta = 2i\,[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta]$$

$$\sin^7\theta\cos^5\theta = \frac{1}{2^{11}\,.\,i^6}\,[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta]$$

$$= -\frac{1}{2^{11}}\,[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta].$$

Example 5. *In* $\sin^4\theta\cos^3\theta = A_1\cos\theta + A_3\cos 3\theta + A_5\cos 5\theta + A_7\cos 7\theta$, *prove that* $A_1 + 9A_3 + 25A_5 + 49A_7 = 0$.

Sol. Let $$x = \cos\theta + i\sin\theta\,;\ \frac{1}{x} = \cos\theta - i\sin\theta$$

$$\therefore\quad x + \frac{1}{x} = 2\cos\theta\,;\ x - \frac{1}{x} = 2i\sin\theta$$

$$(2\,i\sin\theta)^4\,(2\cos\theta)^3 = \left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^3$$

By Pascal's Table coefficients of various powers of x in $\left(x - \frac{1}{x}\right)^4$ are $1, -4, 6, -4, 1$

Multiplying $\left(x - \frac{1}{x}\right)^4$ by $\left(x + \frac{1}{x}\right)$ three times in succession as shown below:

	1	– 4	6	– 4	1			
I	1	– 3	2	2	– 3	1		
II	1	– 2	– 1	4	– 1	– 2	1	
III	1	– 1	– 3	3	3	– 3	– 1	1

$$\therefore\quad \left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^3 = x^7 - x^5 - 3x^3 + 3x + \frac{3}{x} - \frac{3}{x^3} - \frac{1}{x^5} + \frac{1}{x^7}$$

$$(2i\sin\theta)^4\,(2\cos\theta)^3 = \left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) - 3\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$$

$$2^7\sin^4\theta\cos^3\theta = 2\cos 7\theta - 2\cos 5\theta - 3\,.\,2\cos 3\theta + 3\,.\,2\cos\theta$$

$$\therefore\quad \sin^4\theta\cos^3\theta = \frac{1}{2^6}\,[\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta]$$

$$= \frac{3}{64}\cos\theta - \frac{3}{64}\cos 3\theta - \frac{1}{64}\cos 5\theta + \frac{1}{64}\cos 7\theta$$

$$= A_1\cos\theta + A_3\cos 3\theta + A_5\cos 5\theta + A_7\cos 7\theta \qquad \text{(given)}$$

$$\therefore\quad A_1 = \frac{3}{64},\ A_3 = -\frac{3}{64},\ A_5 = -\frac{1}{64},\ A_7 = \frac{1}{64}$$

$$A_1 + 9A_3 + 25A_5 + 49A_7 = \frac{3}{64} - \frac{27}{64} - \frac{25}{64} + \frac{49}{64} = \frac{52-52}{64} = 0.$$

TEST YOUR KNOWLEDGE

Prove that :

1. $\cos^7\theta = \frac{1}{64}\,(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$. **(P.T.U., Dec. 2011)**

2. $\cos^6\theta = \frac{1}{32}\,[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10]$.

3. $\sin^6\theta = \frac{1}{32}\,(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta$.

4. $2^6 \sin^7\theta = 35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta.$

5. $64(\cos^8\theta + \sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35.$

6. $\sin^7\theta\cos^3\theta = -\dfrac{1}{2^9}(\sin 10\theta - 4\sin 8\theta + 3\sin 6\theta + 8\sin 4\theta - 14\sin 2\theta).$

7. $\sin^6\theta\cos^2\theta = \dfrac{1}{2^7}(5 - 4\cos 2\theta - 4\cos 4\theta + 4\cos 6\theta - \cos 8\theta).$

8. $\cos^5\theta\sin^3\theta = -\dfrac{1}{2^7}(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta).$

9. $\cos^6\theta\sin^4\theta = \dfrac{1}{2^9}(\cos 10\theta + 2\cos 8\theta - 3\cos 6\theta - 8\cos 4\theta + 2\cos 2\theta + 6).$

6.6. EXPANSION OF COS $n\theta$ AND SIN $n\theta$ (n BEING A +ve INTEGER)

We know, from De-Moivre's Theorem, that $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

Expanding the right hand side by Binomial Theorem, we have

$$\cos n\theta + i\sin n\theta = (\cos\theta)^n + {}^nC_1(\cos\theta)^{n-1}(i\sin\theta) + {}^nC_2(\cos\theta)^{n-2}(i\sin\theta)^2 + {}^nC_3(\cos\theta)^{n-3}(i\sin\theta)^3 + {}^nC_4(\cos\theta)^{n-4}(i\sin\theta)^4 + \ldots + {}^nC_{n-1}(\cos\theta)(i\sin\theta)^{n-1} + {}^nC_n(i\sin\theta)^n$$

Now, $i^2 = -1,\ i^3 = i^2 . i = -i,\ i^4 = (i^2)^2 = 1$ and so on.

$${}^nC_{n-1} = {}^nC_1 = n,\ {}^nC_n = {}^nC_0 = 1$$

$\therefore$
$$\cos n\theta + i\sin n\theta = \cos^n\theta + i\,{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta - i\,{}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \ldots\ldots + i^{n-1}.\,n\cos\theta\sin^{n-1}\theta + i^n\sin^n\theta$$

Two cases arise, according as n is odd or even.

Case I. *If n is odd,* ($n - 1$ is even)

$$\cos n\theta + i\sin n\theta = \cos^n\theta + i\,{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta - i\,{}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \ldots\ldots + n.(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta + i.(-1)^{\frac{n-1}{2}}\sin^n\theta$$

$$\left[\because\ i^{n-1} = (i^2)^{\frac{n-1}{2}} = (-1)^{\frac{n-1}{2}};\ i^n = i.i^{n-1} = i(-1)^{\frac{n-1}{2}}\right]$$

Equating real and imaginary parts, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \ldots\ldots + n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta \quad \ldots(1)$$

and
$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots\ldots + (-1)^{\frac{n-1}{2}}\sin^n\theta \quad \ldots(2)$$

Case II. *If n is even,* [($n - 1$) is odd, ($n - 2$) is even]

$$\cos n\theta + i\sin n\theta = \cos^n\theta + i.{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta - i.{}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \ldots\ldots + i(-1)^{\frac{n-2}{n}}n\cos\theta\sin^{n-1}\theta + (-1)^{\frac{n}{2}}\sin^n\theta$$

$$\left[\because\ i^{n-1} = i.i^{n-2} = i(i^2)^{\frac{n-2}{2}} = i(-1)^{\frac{n-2}{2}};\ i^n = (i^2)^{\frac{n}{2}} = (-1)^{\frac{n}{2}}\right]$$

Equating real and imaginary parts, we get

$$\cos n\theta = \cos^n \theta - {}^nC_2 \cos^{n-2} \theta \sin^2 \theta + {}^nC_4 \cos^{n-4} \theta \sin^4 \theta - \ldots\ldots + (-1)^{\frac{n}{2}} \sin^n \theta \quad \ldots(3)$$

and $$\sin n\theta = {}^nC_1 \cos^{n-1} \theta \sin \theta - {}^nC_3 \cos^{n-3} \theta \sin^3 \theta + \ldots\ldots + (-1)^{\frac{n-2}{2}} n \cos \theta \sin^{n-1} \theta \quad \ldots(4)$$

ILLUSTRATIVE EXAMPLES

Example 1. (*a*) *Expand cos 7θ in descending powers of cos θ.* **(P.T.U., Dec. 2013)**

(*b*) *Expand sin 7θ in ascending powers of sin θ.*

Sol. We have $(\cos 7\theta + i \sin 7\theta) = (\cos \theta + i \sin \theta)^7$.

Expanding the RHS by Binomial Theorem, we have

$\cos 7\theta + i \sin 7\theta = (\cos \theta)^7 + {}^7C_1 (\cos \theta)^6 (i \sin \theta) + {}^7C_2 (\cos \theta)^5 (i \sin \theta)^2 + {}^7C_3 (\cos \theta)^4 (i \sin \theta)^3 + {}^7C_4 (\cos \theta)^3 (i \sin \theta)^4 + {}^7C_5 (\cos \theta)^2 (i \sin \theta)^5 + {}^7C_6 (\cos \theta) (i \sin \theta)^6 + {}^7C_7 (i \sin \theta)^7$

Now, $i^2 = -1, i^3 = i \,.\, i^2 = -i, i^4 = 1, i^5 = i, i^6 = -1$

$$i^7 = i^3 \,.\, i^4 = -i, {}^7C_7 = 1, {}^7C_6 = {}^7C_1 = 7, {}^7C_5 = {}^7C_2 = \frac{7.6}{1.2} = 21,$$

$${}^7C_4 = {}^7C_3 = \frac{7.6.5.}{1.2.3} = 35$$

$$\therefore \quad \cos 7\theta + i \sin 7\theta = \cos^7 \theta + 7i \cos^6 \theta \sin \theta - 21 \cos^5 \theta \sin^2 \theta - 35i \cos^4 \theta \sin^3 \theta + 35 \cos^3 \theta \sin^4 \theta + 21i \cos^2 \theta \sin^5 \theta - 7 \cos \theta \sin^6 \theta - i \sin^7 \theta$$

Equating real and imaginary parts,

$$\begin{aligned}(a)\ \cos 7\theta &= \cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta \\ &= \cos^7 \theta - 21 \cos^5 \theta (1 - \cos^2 \theta) + 35 \cos^3 \theta (1 - \cos^2 \theta)^2 - 7 \cos \theta (1 - \cos^2 \theta)^3 \\ &= \cos^7 \theta - 21 \cos^5 \theta + 21 \cos^7 \theta + 35 \cos^3 \theta (1 - 2 \cos^2 \theta + \cos^4 \theta) - 7 \cos \theta (1 - 3 \cos^2 \theta + 3 \cos^4 \theta - \cos^6 \theta) \\ &= 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta\end{aligned}$$

$$\begin{aligned}(b)\quad \sin 7\theta &= 7 \cos^6 \theta \sin \theta - 35 \cos^4 \theta \sin^3 \theta + 21 \cos^2 \theta \sin^5 \theta - \sin^7 \theta \\ &= 7(1 - \sin^2 \theta)^3 \sin \theta - 35(1 - \sin^2 \theta)^2 \sin^3 \theta + 21(1 - \sin^2 \theta) \sin^5 \theta - \sin^7 \theta \\ &= 7(1 - 3 \sin^2 \theta + 3 \sin^4 \theta - \sin^6 \theta) \sin \theta - 35(1 - 2 \sin^2 \theta + \sin^4 \theta) \sin^3 \theta + 21 \sin^5 \theta - 21 \sin^7 \theta - \sin^7 \theta \\ &= 7 \sin \theta - 56 \sin^3 \theta + 112 \sin^5 \theta - 64 \sin^7 \theta.\end{aligned}$$

Example 2. *Prove that* $2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$, *where* $x = 2 \cos \theta$.

Sol. $$2(1 + \cos 8\theta) = 2 \,.\, 2 \cos^2 4\theta = (2 \cos 4\theta)^2 \quad \ldots(1)$$

Now, $$\begin{aligned}\cos 4\theta &= \cos^4 \theta - {}^4C_2 \cos^2 \theta \sin^2 \theta + {}^4C_4 \sin^4 \theta \\ &= \cos^4 \theta - 6 \cos^2 \theta (1 - \cos^2 \theta) + (1 - \cos^2 \theta)^2 = 8 \cos^4 \theta - 8 \cos^2 \theta + 1\end{aligned}$$

∴ From (1), we have

$$\begin{aligned}2(1 + \cos 8\theta) &= (16 \cos^4 \theta - 16 \cos^2 \theta + 2)^2 \\ &= [(2 \cos \theta)^4 - 4(2 \cos \theta)^2 + 2]^2 = (x^4 - 4x^2 + 2)^2, \text{ where } x = 2 \cos \theta.\end{aligned}$$

Example 3. *Prove that* $\dfrac{1 + \cos 7\theta}{1 + \cos \theta} = (x^3 - x^2 - 2x + 1)^2$, *where* $x = 2 \cos \theta$.

Sol. $$\frac{1+\cos 7\theta}{1+\cos\theta} = \frac{2\cos^2\frac{7\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{2\cos^2\frac{7\theta}{2}}{2\cos^2\frac{\theta}{2}} \times \frac{2\sin^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$= \left(\frac{2\cos\frac{7\theta}{2}\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}\right)^2 = \left(\frac{\sin 4\theta - \sin 3\theta}{\sin\theta}\right)^2 \qquad ...(1)$$

Now, $\sin 4\theta = {}^4C_1 \cos^3\theta \sin\theta - {}^4C_3 \cos\theta \sin^3\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore$ From (1), we have $$\frac{1+\cos 7\theta}{1+\cos\theta} = (4\cos^3\theta - 4\cos\theta\sin^2\theta - 3 + 4\sin^2\theta)^2$$

$$= [4\cos^3\theta - 4\cos\theta(1-\cos^2\theta) - 3 + 4(1-\cos^2\theta)]^2$$
$$= (8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1)^2$$
$$= (x^3 - x^2 - 2x + 1)^2, \text{ where } x = 2\cos\theta.$$

6.7(a). EXPANSION OF TAN $n\theta$

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta\,...}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta\,...}$$

Dividing the numerator and denominator by $\cos^n\theta$

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - ...}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - ...}$$

6.7(b). EXPANSION OF TAN $(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n)$

We know that

$\cos(\theta_1+\theta_2+\theta_3+...+\theta_n) + i\sin(\theta_1+\theta_2+\theta_3+.....+\theta_n)$

$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)......(\cos\theta_n + i\sin\theta_n) \qquad ...(1)$

Now, $\cos\theta_1 + i\sin\theta_1 = \cos\theta_1\left(1 + i\frac{\sin\theta_1}{\cos\theta_1}\right) = \cos\theta_1(1 + i\tan\theta_1)$

$\cos\theta_2 + i\sin\theta_2 = \cos\theta_2(1 + i\tan\theta_2)$

$\cos\theta_3 + i\sin\theta_3 = \cos\theta_3(1 + i\tan\theta_3)$

....

....

$\cos\theta_n + i\sin\theta_n = \cos\theta_n(1 + i\tan\theta_n)$

$\therefore$ (1) may be written as

$\cos(\theta_1+\theta_2+\theta_3+......+\theta_n) + i\sin(\theta_1+\theta_2+\theta_3+......+\theta_n)$

$= \cos\theta_1\cos\theta_2\cos\theta_3......\cos\theta_n(1+i\tan\theta_1)(1+i\tan\theta_2)......(1+i\tan\theta_n)$

$= \cos\theta_1\cos\theta_2......\cos\theta_n[1 + is_1 + i^2s_2 + i^3s_3 + i^4s_4 + i^5s_5 + ...]$

where s_r denotes the sum of the products of the tangents of the angles $\theta_1, \theta_2 ......, \theta_n$ taken r at a time. *i.e.*, $s_1 = \Sigma\tan\theta_1$, $s_2 = \Sigma\tan\theta_1\tan\theta_2$ and so on.

$$= \cos\theta_1 \cos\theta_2 \ldots\ldots \cos\theta_n[1 + is_1 - s_2 - is_3 + s_4 + is_5 - \ldots\ldots]$$
$$= \cos\theta_1 \cos\theta_2 \ldots\ldots \cos\theta_n[(1 - s_2 + s_4 - \ldots\ldots) + i(s_1 - s_3 + s_5 - \ldots\ldots)]$$

Equating the real and imaginary parts

$$\cos(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) = \cos\theta_1 \cos\theta_2 \ldots\ldots \cos\theta_n(1 - s_2 + s_4 \ldots\ldots) \quad \ldots(2)$$
$$\sin(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) = \cos\theta_1 \cos\theta_2 \ldots\ldots \cos\theta_n(s_1 - s_3 + s_5 \ldots\ldots) \quad \ldots(3)$$

Dividing (3) by (2), $\tan(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) = \dfrac{s_1 - s_3 + s_5 - \ldots}{1 - s_2 + s_4 - \ldots}$

Example 4. *If α, β, γ be the roots of equation $x^3 + px^2 + qx + p = 0$ prove that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radians, except in one particular case and point it out.* **(P.T.U., May 2002)**

Sol. Let $\alpha = \tan\theta_1, \beta = \tan\theta_2, \gamma = \tan\theta_3$, then $\theta_1 = \tan^{-1}\alpha, \theta_2 = \tan^{-1}\beta, \theta_3 = \tan^{-1}\gamma$

Given equation is $x^3 + px^2 + qx + p = 0$

Its roots are α, β, γ *i.e.*, $\tan\theta_1, \tan\theta_2, \tan\theta_3$.

$$\therefore \quad s_1 = \Sigma\alpha = \Sigma\tan\theta_1 = -p$$
$$s_2 = \Sigma\alpha\beta = \Sigma\tan\theta_1\tan\theta_2 = q$$
$$s_3 = \alpha\beta\gamma = \tan\theta_1\tan\theta_2\tan\theta_3 = -p$$

Now, $\quad \tan(\theta_1 + \theta_2 + \theta_3) = \dfrac{s_1 - s_3}{1 - s_2} = \dfrac{-p+p}{1-q} = \dfrac{0}{1-q} = 0$

Unless $q = 1$ in which case the fraction takes the indeterminate form $\dfrac{0}{0}$.

Leaving out the exceptional case, we have $\tan(\theta_1 + \theta_2 + \theta_3) = 0$

$\therefore \quad \theta_1 + \theta_2 + \theta_3 = n\pi$ radians.

Hence $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radians except when $q = 1$.

Example 5. *If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$; prove that $xy + yz + zx = 1$.* **(P.T.U., May 2003)**

Sol. Let $\quad \tan^{-1}x = \theta_1, \tan^{-1}y = \theta_2, \tan^{-1}z = \theta_3$

given $\quad \theta_1 + \theta_2 + \theta_3 = \dfrac{\pi}{2}$

$$\therefore \quad \tan(\theta_1 + \theta_2 + \theta_3) = \tan\frac{\pi}{2} = \infty$$

$$\therefore \quad \frac{s_1 - s_3}{1 - s_2} = \infty \quad \therefore \quad 1 - s_2 = 0 \quad \text{or} \quad s_2 = 1$$

or $\quad \tan\theta_1\tan\theta_2 + \tan\theta_2\tan\theta_3 + \tan\theta_3\tan\theta_1 = 1$

or $\quad xy + yz + zx = 1.$

Example 6. *If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 - x^3 \sin 2\theta + x^2 \cos 2\theta - x\cos\theta - \sin\theta = 0$, prove that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma + \tan^{-1}\delta = n\pi + \dfrac{\pi}{2} - \theta$.*

Sol. Let $\tan^{-1}\alpha = \theta_1, \tan^{-1}\beta = \theta_2, \tan^{-1}\gamma = \theta_3, \tan^{-1}\delta = \theta_4$

$\therefore$ $\tan\theta_1 = \alpha, \tan\theta_2 = \beta, \tan\theta_3 = \gamma, \tan\theta_4 = \delta$

Given $= n$ is $x^4 - x^3\sin 2\theta + x^2\cos 2\theta - x\cos\theta - \sin\theta = 0$

Its roots are $\alpha, \beta, \gamma, \delta$ *i.e.,* $\tan\theta_1, \tan\theta_2, \tan\theta_3, \tan\theta_4$

$$\therefore \quad \mathbf{s_1} = \Sigma\alpha = \Sigma\tan\theta_1 = \sin 2\theta$$

$$\mathbf{s_2} = \Sigma\alpha\beta = \Sigma\tan\theta_1\tan\theta_2 = \cos 2\theta$$

$$\mathbf{s_3} = \Sigma\alpha\beta\gamma = \Sigma\tan\theta_1\tan\theta_2\tan\theta_3 = \cos\theta$$

$$\mathbf{s_4} = \alpha\beta\gamma\delta = \tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4 = -\sin\theta$$

Now, $$\tan(\theta_1+\theta_2+\theta_3+\theta_4) = \frac{\mathbf{s_1}-\mathbf{s_3}}{1-\mathbf{s_2}+\mathbf{s_4}} = \frac{\sin 2\theta - \cos\theta}{1-\cos 2\theta - \sin\theta}$$

$$= \frac{2\sin\theta\cos\theta - \cos\theta}{1-(1-2\sin^2\theta)-\sin\theta} = \frac{\cos\theta\,(2\sin\theta - 1)}{\sin\theta\,(2\sin\theta-1)} = \cot\theta$$

$$\therefore \quad \tan(\theta_1+\theta_2+\theta_3+\theta_4) = \tan\left(\frac{\pi}{2}-\theta\right)$$

$$\therefore \quad \theta_1+\theta_2+\theta_3+\theta_4 = n\pi + \frac{\pi}{2} - \theta.$$

Example 7. *If* $\theta_1, \theta_2, \theta_3$ *be the three values of* θ *which satisfy the equation tan* $2\theta = \lambda$ *tan* $(\theta+\alpha)$ *and be such that no two of these differ by a multiple of* π, *prove that* $\theta_1 + \theta_2 + \theta_3 + \alpha$ *is a multiple of* π. **(P.T.U., Dec. 2002)**

Sol. The given equation is $\tan 2\theta = \lambda\tan(\theta+\alpha)$

or $$\frac{2\tan\theta}{1-\tan^2\theta} = \lambda.\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

$$\Rightarrow \quad 2\tan\theta(1-\tan\theta\tan\alpha) = \lambda(1-\tan^2\theta)(\tan\theta+\tan\alpha)$$

$$\Rightarrow \quad 2\tan\theta - 2\tan^2\theta\tan\alpha = \lambda(\tan\theta+\tan\alpha - \tan^3\theta - \tan^2\theta\tan\alpha)$$

or $$\lambda\tan^3\theta - (2-\lambda)\tan\alpha\tan^2\theta + (2-\lambda)\tan\theta - \lambda\tan\alpha = 0.$$

This equation is a cubic in $\tan\theta$ and as such, its roots are $\tan\theta_1, \tan\theta_2, \tan\theta_3$

$$\mathbf{s_1} = \Sigma\tan\theta_1 = \frac{(2-\lambda)\tan\alpha}{\lambda}, \ \mathbf{s_2} = \Sigma\tan\theta_1\tan\theta_2 = \frac{2-\lambda}{\lambda},$$

$$\mathbf{s_3} = \tan\theta_1\tan\theta_2\tan\theta_3 = \frac{\lambda\tan\alpha}{\lambda} = \tan\alpha$$

Now, $$\tan(\theta_1+\theta_2+\theta_3) = \frac{\mathbf{s_1}-\mathbf{s_3}}{\mathbf{1}-\mathbf{s_2}} = \frac{\dfrac{(2-\lambda)\tan\alpha}{\lambda} - \tan\alpha}{1-\dfrac{2-\lambda}{\lambda}}$$

$$= \frac{(2-\lambda)\tan\alpha - \lambda\tan\alpha}{\lambda - 2 + \lambda} = \frac{2(1-\lambda)\tan\alpha}{-2(1-\lambda)}$$

$= -\tan\alpha$, except when $\lambda = 1$ in, which case the fraction takes the indeterminate form $\dfrac{0}{0}$.

$\Rightarrow \qquad \tan(\theta_1 + \theta_2 + \theta_3) = \tan(-\alpha)$

$\therefore \qquad \theta_1 + \theta_2 + \theta_3 = n\pi - \alpha \quad$ or $\quad \theta_1 + \theta_2 + \theta_3 + \alpha = n\pi$, a multiple of π.

In case $\lambda = 1$, the given equation becomes $\tan 2\theta = \tan(\theta + \alpha)$ which gives

$2\theta = n\pi + (\theta + \alpha)$ or $\theta = n\pi + \alpha$ so that the values of θ differ by multiples of π.

Example 8. *Prove that the equation ah sec* θ *– bk cosec* $\theta = a^2 - b^2$ *has four roots and that the sum of the four values of* θ *which satisfy it is equal to an odd multiple of* π *radians.*

Sol. Let $\qquad \tan\dfrac{\theta}{2} = t$

Now, $\qquad \cos\theta = \dfrac{1 - \tan^2\dfrac{\theta}{2}}{1 + \tan^2\dfrac{\theta}{2}} = \dfrac{1 - t^2}{1 + t^2} \qquad \therefore \quad \sec\theta = \dfrac{1 + t^2}{1 - t^2}$

$$\sin\theta = \frac{2\tan\dfrac{\theta}{2}}{1 + \tan^2\dfrac{\theta}{2}} = \frac{2t}{1 + t^2} \qquad \therefore \quad \operatorname{cosec}\theta = \frac{1 + t^2}{2t}$$

Making these substitutions in the given equation, we have

$$ah \cdot \frac{1 + t^2}{1 - t^2} - bk \cdot \frac{1 + t^2}{2t} = a^2 - b^2$$

Multiplying both sides by $2t(1 - t^2)$

$$2aht(1 + t^2) - bk(1 - t^2)(1 + t^2) = 2(a^2 - b^2)(1 - t^2)t$$

or $$2aht + 2aht^3 - bk(1 - t^4) = 2(a^2 - b^2)(t - t^3)$$

or $$bkt^4 + (2ah + 2a^2 - 2b^2)t^3 + (2ah - 2a^2 + 2b^2)t - bk = 0 \qquad \ldots(1)$$

It is a biquadratic in t and hence has four roots.

Let $\theta_1, \theta_2, \theta_3, \theta_4$ be the four values of θ satisfying the given equation then the roots of (1) are $\tan\dfrac{\theta_1}{2}, \tan\dfrac{\theta_2}{2}, \tan\dfrac{\theta_3}{2}, \tan\dfrac{\theta_4}{2}$.

Let us denote them by t_1, t_2, t_3, t_4.

$$s_1 = \Sigma t_1 = \Sigma \tan\frac{\theta_1}{2} = -\frac{2ah + 2a^2 - 2b^2}{bk}, \quad s_2 = \Sigma t_1 t_2 = \Sigma \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = 0$$

$$s_3 = \Sigma t_1 t_2 t_3 = \Sigma \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2} = -\frac{2ah - 2a^2 + 2b^2}{bk}$$

$$s_4 = t_1 t_2 t_3 t_4 = \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\tan\frac{\theta_4}{2} = \frac{-bk}{bk} = -1$$

Now, $$\tan\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2}\right)=\frac{\mathbf{s_1}-\mathbf{s_3}}{1-\mathbf{s_2}+\mathbf{s_4}}$$

$$=\frac{-\dfrac{2ah+2a^2-2b^2}{bk}+\dfrac{2ah-2a^2+2b^2}{bk}}{1-0-1}=\frac{\dfrac{4b^2-4a^2}{bk}}{0}=\infty=\tan\frac{\pi}{2}$$

$$\therefore \quad \frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2}=n\pi+\frac{\pi}{2}$$

or $$\theta_1+\theta_2+\theta_3+\theta_4=2n\pi+\pi=(2n+1)\pi=\text{an odd multiple of }\pi.$$

Example 9. *Prove that the equation sin 3*θ *= a sin* θ *+ b* cos θ *+ c has six roots and that the sum of the six values of* θ *which satisfy it is equal to an odd multiple of* π *radians.*

Sol. $$\sin 3\theta = a\sin\theta+b\cos\theta+c$$

$$3\sin\theta-4\sin^3\theta = a\sin\theta+b\cos\theta+c$$

$$\therefore \quad 4\sin^3\theta+(a-3)\sin\theta+b\cos\theta+c=0$$

or $$4\left(\frac{2\tan\theta/2}{1+\tan^2\theta/2}\right)^3+(a-3)\,\frac{2\tan\theta/2}{1+\tan^2\theta/2}+b\,\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}+c=0$$

Let $\tan\theta/2=t$

$$\therefore \quad \frac{32t^3}{(1+t^2)^3}+(a-3)\,\frac{2t}{1+t^2}+b\,\frac{1-t^2}{1+t^2}+c=0$$

or $$32t^3+(2a-6)\,t\,(1+t^2)^2+b(1-t^2)(1+t^2)^2+c\,(1+t^2)^3=0$$

or $$32t^3+(2a-6)(t+2t^3+t^5)+b(1-t^2)(1+2t^2+t^4)+c\,(1+3t^2+3t^4+t^6)=0$$

or $$32t^3+(2a-6)\,t+(4a-12)\,t^3+(2a-6)\,t^5+b+bt^2-bt^4-bt^6+c+3ct^2+3ct^4+ct^6=0$$

or $$(c-b)\,t^6+(2a-6)\,t^5+(3c-b)\,t^4+(32+4a-12)\,t^3+(b+3c)\,t^2+(2a-6)\,t+(b+c)=0.$$

It is sixth degree in t $\therefore$ it has six roots.

Roots of this equation are $\tan\frac{\theta_1}{2}$, $\tan\frac{\theta_2}{2}$, $\tan\frac{\theta_3}{2}$, $\tan\frac{\theta_4}{2}$, $\tan\frac{\theta_5}{2}$, $\tan\frac{\theta_6}{2}$ *i.e.,* $t_1, t_2, t_3, t_4, t_5, t_6$

$$\therefore \quad \mathbf{s_1}=\Sigma\,t_1=\Sigma\tan\frac{\theta_1}{2}=-\frac{2a-6}{c-b}$$

$$\mathbf{s_2}=\Sigma\,t_1t_2=\frac{3c-b}{c-b}\;;\ \mathbf{s_3}=\Sigma\,t_1t_2t_3=-\frac{20+49}{c-b}$$

$$\mathbf{s_4}=\Sigma\,t_1t_2t_3t_4=\frac{b+3c}{c-b},\ \mathbf{s_5}=\Sigma t_1t_2t_3t_4t_5=-\frac{2a-6}{c-b},\ \mathbf{s_6}=\frac{b+c}{c-b}$$

$$\therefore \quad \tan\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6}{2}\right)=\frac{\mathbf{s_1-s_3+s_5}}{\mathbf{1-s_2+s_4-s_6}}$$

$$= \frac{-\dfrac{(2a-b)}{c-b} + \dfrac{20+4a}{c-b} - \dfrac{2a-6}{c-b}}{1 - \dfrac{3c-b}{c-b} + \dfrac{b+3c}{c-b} - \dfrac{c+b}{c-b}}$$

$$= \frac{-2a+6+20+4a-2a+6}{c-b-3c+b+b+3c-c-b} = \frac{32}{0} = \infty = \tan\frac{\pi}{2}$$

$\therefore$ $$\frac{\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6}{2} = n\pi + \frac{\pi}{2}$$

or $$\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6 = 2n\pi+\pi = (2n+1)\,\pi = \text{odd multiple of } \pi.$$

TEST YOUR KNOWLEDGE

1. Prove that $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$ **(P.T.U., May 2012)**

2. (*a*) Express $\dfrac{\sin 6\theta}{\sin\theta}$ as a polynomial in $\cos\theta$.

(*b*) Prove that $\dfrac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$.

3. Prove that:

(*i*) $\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1.$

(*ii*) $\dfrac{\sin 8\theta}{\sin\theta} = 128\cos^7\theta - 192\cos^5\theta + 80\cos^3\theta - 8\cos\theta.$

(*iii*) $\tan 5\theta = \dfrac{5t-10t^3+t^5}{1-10t^2+5t^4}$, where $t = \tan\theta$.

(*iv*) $\tan 7\theta = \dfrac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$.

4. Prove that :

(*i*) $2^{n/2}\cos\dfrac{n\pi}{4} = 1 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \ldots \infty$

(*ii*) $2^{n/2}\sin\dfrac{n\pi}{4} = {}^nC_1 - {}^nC_3 + {}^nC_5 - \ldots \infty$

$\left[\textbf{Hint :} \text{ Put } \theta = \dfrac{\pi}{4} \text{ in } 1.5(c)\right].$

5. Express $\tan 5\theta$ in terms of powers of $\tan\theta$ and deduce that $5\tan^4\dfrac{\pi}{10} - 10\tan^2\dfrac{\pi}{10} + 1 = 0$.

6. Prove that $1 + \cos 9A = (1+\cos A)(16\cos^4 A - 8\cos^3 A - 12\cos^2 A + 4\cos A + 1)^2$.

$\left[\textbf{Hint :} \text{ Find the value of } \dfrac{1+\cos 9A}{1+\cos A} \text{ see solved example 3}\right]$

7. Prove that the equation $a^2\cos^2\theta + b^2\sin^2\theta + 2ga\cos\theta + 2fb\sin\theta + c = 0$ has four roots and that the sum of the values of θ which satisfy it is an even multiple of π radians.

ANSWERS

2. (*a*) $32\cos^5\theta - 24\cos^3\theta + 6\cos\theta$

5. $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$.

6.8. FORMATION OF EQUATIONS

We explain this method by examples given below :

ILLUSTRATIVE EXAMPLES

Example 1. *Form an equation whose roots are* $\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{6\pi}{9}, \cos\frac{8\pi}{9}$.

(*a*) *Also form an equation whose roots are* $\sec\frac{2\pi}{9}, \sec\frac{4\pi}{9}, \sec\frac{6\pi}{9}, \sec\frac{8\pi}{9}$.

(*b*) *Also form an equation whose roots are* $\sec^2\frac{2\pi}{9}, \sec^2\frac{4\pi}{9}, \sec^2\frac{6\pi}{9}, \sec^2\frac{8\pi}{9}$ *and prove that*

$$\tan^2\frac{2\pi}{9} + \tan^2\frac{4\pi}{9} + \tan^2\frac{6\pi}{9} + \tan^2\frac{8\pi}{9} = 36.$$

Sol. Let $\theta = \frac{2n\pi}{9}$, where n is an integer (zero, positive or negative)

Now give values to θ as 0, 1, 2, 3, 4, 5, 6, 7, 8, we see that

for $n = 0$, $\cos\theta = \cos 0 = 1$

for $n = 1$, $\cos\theta = \cos\frac{2\pi}{9}$

for $n = 2$, $\cos\theta = \cos\frac{4\pi}{9}$

for $n = 3$, $\cos\theta = \cos\frac{6\pi}{9}$

for $n = 4$, $\cos\theta = \cos\frac{8\pi}{9}$

for $n = 5$, $\cos\theta = \cos\frac{10\pi}{9} = \cos\left(2\pi - \frac{8\pi}{9}\right) = \cos\frac{8\pi}{9}$

for $n = 6$, $\cos\theta = \cos\frac{12\pi}{9} = \cos\left(2\pi - \frac{6\pi}{9}\right) = \cos\frac{6\pi}{9}$

for $n = 7$, $\cos\theta = \cos\frac{14\pi}{9} = \cos\left(2\pi - \frac{4\pi}{9}\right) = \cos\frac{4\pi}{9}$

for $n = 8$, $\cos\theta = \cos\frac{16\pi}{9} = \cos\left(2\pi - \frac{2\pi}{9}\right) = \cos\frac{2\pi}{9}$

We see that for $n = 5, 6, 7, 8$, we do not get any new value of $\cos\theta$.

$\therefore$ The only distinct values of $\cos\theta$ are $1, \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{6\pi}{9}, \cos\frac{8\pi}{9}$

Consider $9\theta = 2n\pi$ (Resolve 9θ into 5θ and 4θ)

($\because$ There are 5 distinct values of $\cos\theta$)

or $$5\theta + 4\theta = 2n\pi$$

or $$5\theta = 2n\pi - 4\theta$$

or $$\cos 5\theta = \cos (2n\pi - 4\theta)$$

or $$\cos (4\theta + \theta) = \cos 4\theta$$

$$\cos 4\theta \cos \theta - \sin 4\theta \sin \theta = \cos 4\theta.$$

or $$\cos 4\theta (\cos \theta - 1) - 2 \sin 2\theta \cos 2\theta \sin \theta = 0$$

or $$(\cos \theta - 1)(2 \cos^2 2\theta - 1) - 4 \sin^2 \theta \cos \theta (2 \cos^2 \theta - 1) = 0$$

or $$(\cos \theta - 1)[2 (2 \cos^2 \theta - 1)^2 - 1] - 4 \cos \theta (1 - \cos^2 \theta)(2 \cos^2 \theta - 1) = 0$$

or $$(\cos \theta - 1)[2 (4 \cos^4 \theta - 4 \cos^2 \theta + 1) - 1] - 4 \cos \theta (2 \cos^2 \theta - 1 - 2 \cos^4 \theta + \cos^2 \theta) = 0$$

or $$(\cos \theta - 1)[8 \cos^4 \theta - 8 \cos^2 \theta + 1] - 4 \cos \theta [- 2 \cos^4 \theta + 3 \cos^2 \theta - 1] = 0$$

or $$8 \cos^5 \theta - 8 \cos^3 \theta + \cos \theta - 8 \cos^4 \theta + 8 \cos^2 \theta - 1 + 8 \cos^5 \theta - 12 \cos^3 \theta + 4 \cos \theta = 0$$

or $$16 \cos^5 \theta - 8 \cos^4 \theta - 20 \cos^3 \theta + 8 \cos^2 \theta + 5 \cos \theta - 1 = 0$$

Put $\cos \theta = x$

$$16x^5 - 8x^4 - 20x^3 + 8x^2 + 5x - 1 = 0$$

$x = 1$ satisfies this equation

$$\therefore \quad (x - 1)(16x^4 + 8x^3 - 12x^2 - 4x + 1) = 0$$

$\therefore \quad x = 1$ corresponds to $\cos \theta = 1$

i.e., the value of $\cos \dfrac{2n\pi}{9}$ for $n = 0$

$\therefore$ If we delete $x - 1$, then the remaining equation $16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0$ will have the roots

$$\cos \frac{2\pi}{9}, \cos \frac{4\pi}{9}, \cos \frac{6\pi}{9}, \cos \frac{8\pi}{9} \qquad \ldots(1)$$

(*a*) Take $x = \dfrac{1}{y}$ in (1)

The equation changes to

$$\frac{16}{y^4} + \frac{8}{y^3} - \frac{12}{y^2} - \frac{4}{y} + 1 = 0 \text{ and roots change to } \frac{1}{\cos \frac{2\pi}{9}}, \frac{1}{\cos \frac{4\pi}{9}}, \frac{1}{\cos \frac{6\pi}{9}}, \frac{1}{\cos \frac{8\pi}{9}}$$

$\therefore$ The equation whose roots are $\sec \dfrac{2\pi}{9}, \sec \dfrac{4\pi}{9}, \sec \dfrac{6\pi}{9}, \sec \dfrac{8\pi}{9}$ is

$$y^4 - 4y^3 - 12y^2 + 8y + 16 = 0 \qquad \ldots(2)$$

Change y to x.

The required equations is

$$x^4 - 4x^3 - 12x^2 + 8x + 16 = 0$$

(*b*) In (2) put $y^2 = t$

$$t^2 - 4yt - 12\, t + 8y + 16 = 0$$

or $$(t^2 - 12t + 16)^2 = (4t - 8)^2\, y^2$$

or $$(t^2 - 12t + 16)^2 = (4t - 8)^2 \,.\, t$$

or $$t^4+144t^2+256-24t^3-384t+32t^2 = t(16t^2-64t+64)$$
$$t^4-40t^3+240t^2-448t+256 = 0$$

Its roots are $\sec^2\frac{2\pi}{9}, \sec^2\frac{4\pi}{9}, \sec^2\frac{6\pi}{9}, \sec^2\frac{8\pi}{9}$

Sum of the roots $= \sec^2\frac{2\pi}{9}+\sec^2\frac{4\pi}{9}+\sec^2\frac{6\pi}{9}+\sec^2\frac{8\pi}{9} = 40$

$\therefore \quad 1+\tan^2\frac{2\pi}{9}+1+\tan^2\frac{4\pi}{9}+1+\tan^2\frac{4\pi}{9}+1+\tan^2\frac{8\pi}{9} = 40$

$$\therefore \quad \tan^2\frac{2\pi}{9}+\tan^2\frac{4\pi}{9}+\tan^2\frac{6\pi}{9}+\tan^2\frac{8\pi}{9} = 36.$$

Example 2. *Form an equation whose roots are* $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$ *and hence evaluate* $\sec\frac{\pi}{7}+\sec\frac{3\pi}{7}+\sec\frac{5\pi}{7}$. *Also obtain the equation whose roots are* $\tan^2\frac{\pi}{7}$, $\tan^2\frac{3\pi}{7}$, $\tan^2\frac{5\pi}{7}$ *and hence evaluate* $\cot^2\frac{\pi}{7}\cot^2\frac{3\pi}{7}\cot^2\frac{5\pi}{7}$.

Sol. Let $\theta = \frac{(2n+1)\pi}{7}$, where $n = 0, 1, 2, 3, 4, 5, 6$

for $n = 0$, $\cos\theta = \cos\frac{\pi}{7}$

for $n = 1$, $\cos\theta = \cos\frac{3\pi}{7}$

for $n = 2$, $\cos\theta = \cos\frac{5\pi}{7}$

for $n = 3$, $\cos\theta = \cos\frac{7\pi}{7} = -1$

for $n = 4$, $\cos\theta = \cos\frac{9\pi}{7} = \cos\left(2\pi-\frac{5\pi}{7}\right) = \cos\frac{5\pi}{7}$

for $n = 5$, $\cos\theta = \cos\frac{11\pi}{7} = \cos\left(2\pi-\frac{3\pi}{7}\right) = \cos\frac{3\pi}{7}$

for $n = 6$, $\cos\theta = \cos\frac{13\pi}{7} = \cos\left(2\pi-\frac{\pi}{7}\right) = \cos\frac{\pi}{7}$

We see that for $n = 4, 5, 6$, we are not getting different values of $\cos\theta$.

$\therefore$ Distinct values of $\cos\theta$ are obtained for

$$n = 0, 1, 2, 3 \quad i.e., \quad \cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}, \cos\frac{7\pi}{7} = -1$$

$$\therefore \quad 7\theta = (2n+1)\pi$$

($\because$ There are only four distinct values of $\cos\theta$ $\therefore$ Resolve 7θ into 4θ and 3θ)

$\therefore \quad 4\theta + 3\theta = (2n+1)\pi$

$$4\theta = (2n+1)\pi - 3\theta$$

$$\cos 4\theta = \cos\{(2n+1)\pi - 3\theta\} = -\cos 3\theta$$

or $\quad 2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta)$

or $\quad 2[2\cos^2\theta - 1]^2 - 1 = -4\cos^3\theta + 3\cos\theta$

or $\quad 2[4\cos^4\theta - 4\cos^2\theta + 1] - 1 = -4\cos^3\theta + 3\cos\theta$

or $\quad 8\cos^4\theta - 8\cos^2\theta + 2 - 1 = -4\cos^3\theta + 3\cos\theta$

or $\quad 8\cos^4\theta + 4\cos^3\theta - 8\cos^2\theta - 3\cos\theta + 1 = 0$

Put $\cos\theta = x$

$$8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0 \qquad \ldots(1)$$

$$x = -1 \text{ satisfies (1)}$$

$\therefore \quad (x+1)(8x^3 - 4x^2 - 4x + 1) = 0$

$x = -1$ corresponds to $\cos\theta = \cos\dfrac{(2n+1)\pi}{7}$ for $n = 3$ *i.e.*, $\cos\dfrac{7\pi}{7}$.

$\therefore \quad 8x^3 - 4x^2 - 4x + 1 = 0$ has roots

$$\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$$

Change x to $\dfrac{1}{y}$

$$\frac{8}{y^3} - \frac{4}{y^2} - \frac{4}{y} + 1 = 0$$

or $\quad 8 - 4y - 4y^2 + y^3 = 0$

or $\quad y^3 - 4y^2 - 4y + 8 = 0 \qquad \ldots(2)$

Its roots are $\dfrac{1}{\cos\frac{\pi}{7}}, \dfrac{1}{\cos\frac{3\pi}{7}}, \dfrac{1}{\cos\frac{5\pi}{7}}$ *i.e.*, $\sec\dfrac{\pi}{7}, \sec\dfrac{3\pi}{7}, \sec\dfrac{5\pi}{7}$

$$\sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7} = \text{sum of the roots} = 4$$

Put $y^2 = t$ in (2)

$\therefore \quad ty - 4t - 4y + 8 = 0$

or $\quad (t-4)^2 y^2 = (4t-8)^2$

$$(t^2 - 8t + 16)t = 16t^2 - 64t + 64$$

or $\quad t^3 - 24t^2 + 80t - 64 = 0 \qquad \ldots(3)$

It has roots $\sec^2\dfrac{\pi}{7}, \sec^2\dfrac{3\pi}{7}, \sec^2\dfrac{5\pi}{7}$ here $t = \sec^2\dfrac{\pi}{7} = 1 + \tan^2\dfrac{\pi}{7}$ or $\tan^2\dfrac{\pi}{7} = t - 1$.

Put $z = t - 1$ or $t = z + 1$ in (3)

$$(z+1)^3 - 24(z+1)^2 + 80(z+1) - 64 = 0$$

$$z^3 + 3z^2 + 3z + 1 - 24z^2 - 48z - 24 + 80z + 80 - 64 = 0$$

or $\quad z^3 - 21z^2 + 35z - 7 = 0 \qquad \ldots(4)$

It has roots $\tan^2 \frac{\pi}{7}$, $\tan^2 \frac{3\pi}{7}$, $\tan^2 \frac{5\pi}{7}$

Now, $$\tan^2 \frac{\pi}{7} \tan^2 \frac{3\pi}{7} \tan^2 \frac{5\pi}{7} = \text{Product of the roots} = 7$$

$\therefore$ $$\cot^2 \frac{\pi}{7} \cot^2 \frac{3\pi}{7} \cot^2 \frac{5\pi}{7} = \frac{1}{7}. \quad \text{Proved.}$$

TEST YOUR KNOWLEDGE

1. (*a*) Prove that $\cos \frac{2\pi}{7}$, $\cos \frac{4\pi}{7}$, $\cos \frac{6\pi}{7}$ are the roots of $8x^3 + 4x^2 - 4x - 1 = 0$. Hence form an equation whose roots are $\sec \frac{2\pi}{7}$, $\sec \frac{4\pi}{7}$, $\sec \frac{6\pi}{7}$.

(*b*) Also form an equation whose roots are $\sec^2 \frac{2\pi}{7}$, $\sec^2 \frac{4\pi}{7}$, $\sec^2 \frac{6\pi}{7}$ and prove that

$$\tan^2 \frac{2\pi}{7} + \tan^2 \frac{4\pi}{7} + \tan^2 \frac{6\pi}{7} = 21.$$

2. Form an equation whose roots are $\cos \frac{\pi}{11}$, $\cos \frac{3\pi}{11}$, $\cos \frac{5\pi}{11}$, $\cos \frac{7\pi}{11}$, $\cos \frac{9\pi}{11}$.

Hence evaluate :

(*i*) $\cos \frac{\pi}{11} + \cos \frac{3\pi}{11} + \cos \frac{5\pi}{11} + \cos \frac{7\pi}{11} + \cos \frac{9\pi}{11}$.

(*ii*) $\sec \frac{\pi}{11} + \sec \frac{3\pi}{11} + \sec \frac{5\pi}{11} + \sec \frac{7\pi}{11} + \sec \frac{9\pi}{11}$.

ANSWERS

1. (*a*) $x^3 + 4x^2 - 4x - 8 = 0$. (*b*) $x^3 - 24x^2 + 80x - 64 = 0$.

2. $32x^5 - 16x^4 - 32x^3 + 12x^2 + 6x - 1 = 0$ (*i*) $\frac{1}{2}$ (*ii*) 6.

6.9. EXPONENTIAL FUNCTION OF A COMPLEX VARIABLE

Def. The exponential function of the complex variable $z = x + iy$, where x and y are real, is defined as

$$\text{Exp.}(z) = e^{x+iy} = e^x (\cos y + i \sin y) = e^x \text{ cis } y$$

Note. When $x = 0$, $e^{iy} = \cos y + i \sin y = \text{cis } y$

Changing i to $-i$, $e^{-iy} = \cos y - i \sin y = \text{cis } (-y)$.

Prove that e^z is a periodic function, where z is a complex variable. (P.T.U., May 2008)

Proof. Let $z = x + iy$

Then, by definition $e^z = e^{x+iy} = e^x (\cos y + i \sin y) = e^x [\cos (2n\pi + y) + i \sin (2n\pi + y)]$

$$= e^{x+i(2n\pi+y)} = e^{(x+iy)+2n\pi i} = e^{z+2n\pi i}$$

i.e., e^z remains unchanged when z is increased by any multiple of $2\pi i$.

$\Rightarrow$ e^z is a periodic function with period $2\pi i$.

Example 1. *Split up into real and imaginary parts:*

(*i*) $e^{3xy + 4iy^2}$ **(P.T.U., May 2014)** (*ii*) $e^{(5+3i)^2}$.

Sol. (*i*) $e^{3xy+4iy^2} = e^{3xy} . e^{i4y^2} = e^{3xy}(\cos 4y^2 + i \sin 4y^2)$

$$\mathrm{R}(e^{3xy+4iy^2}) = e^{3xy} \cos 4y^2$$

$$\mathrm{Im}\,(e^{3xy+4iy^2}) = e^{3xy} \sin 4y^2$$

(*ii*) $(5+3i)^2 = 25+9i^2+30i = 16+30i$ $[\because \quad i^2 = -1]$

$\therefore \quad e^{(5+3i)^2} = e^{16+30i} = e^{16}(\cos 30 + i \sin 30)$

$\therefore \quad \mathrm{Re}\,[e^{(5+3i)^2}] = e^{16} \cos 30, \mathrm{Im}\,[e^{(5+3i)^2}] = e^{16} \sin 30.$

Example 2. *Prove that* $[\sin(\alpha-\theta) + e^{-i\alpha} \sin \theta]^n = \sin^{n-1} \alpha\,[\sin(\alpha - n\theta) + e^{-i\alpha} \sin n\theta]$.

Sol. LHS $= [(\sin \alpha \cos \theta - \cos \alpha \sin \theta)\ 7+ (\cos \alpha - i \sin \alpha) \sin \theta]^n$

$= (\sin \alpha \cos \theta - i \sin \alpha \sin \theta)^n = [\sin \alpha\,(\cos \theta - i \sin \theta)]^n$

$= [\sin \alpha . e^{-i\theta}]^n = \sin^n \alpha . e^{-in\theta}$

RHS $= \sin^{n-1} \alpha\,[(\sin \alpha \cos n\theta - \cos \alpha \sin n\theta) + (\cos \alpha - i \sin \alpha) \sin n\theta]$

$= \sin^{n-1} \alpha\,[\sin \alpha \cos n\theta - i \sin \alpha \sin n\theta] = \sin^n \alpha\,[\cos n\theta - i \sin n\theta]$

$= \sin^n \alpha . e^{-in\theta}$

$\therefore \quad$ LHS = RHS

6.10. CIRCULAR FUNCTIONS OF A COMPLEX VARIABLE

1. Definitions. For all real values of x, we know that

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x$$

Adding and subtracting, we get $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$; $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

These are called Euler's Exponential values of $\sin x$ and $\cos x$, where $x \in \mathrm{R}$.

If $z = x + iy$ the circular functions of z are defined as follows :

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$\cot z = \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}, \qquad \sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}, \quad \operatorname{cosec} z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}}$$

2. Euler's Theorem.

For all values of θ, real or complex, $e^{i\theta} = \cos \theta + i \sin \theta$.

For all values of θ, real or complex $\cos \theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin \theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$$\therefore \quad \cos \theta + i \sin \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{2e^{i\theta}}{2} = e^{i\theta}.$$

Hence, $\quad e^{i\theta} = \cos \theta + i \sin \theta \quad$ for all values of θ.

3. Periodicity of Circular Functions.

(*a*) **To prove that sin *z* and cos *z* are periodic functions with period 2π.**

We know that $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

If n is any integer, then

$$\sin(z + 2n\pi) = \frac{e^{i(z+2n\pi)} - e^{-i(z+2n\pi)}}{2i}$$

$$= \frac{e^{iz} \,.\, e^{2n\pi i} - e^{-iz} \,.\, e^{-2n\pi i}}{2i} = \frac{e^{iz} - e^{-iz}}{2i} \qquad [\because\ e^{2n\pi i} = 1 = e^{-2n\pi i}]$$

$$= \sin z$$

$\Rightarrow$ sin z remains unchanged when z is increased by any multiple of 2π.

$\therefore$ **sin *z* is a periodic function with period 2π.**

Similarly, cos z can be shown to be a periodic function with period 2π.

(*b*) **To prove that tan *z* is a periodic function with period π.**

We know that $\tan z = \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$

If n is any integer, $\tan(z + n\pi) = \dfrac{e^{i(z+n\pi)} - e^{-i(z+n\pi)}}{i[e^{i(z+n\pi)} + e^{-i(z+n\pi)}]} = \dfrac{e^{iz} \,.\, e^{in\pi} - e^{-iz} \,.\, e^{-in\pi}}{i[e^{iz} \,.\, e^{in\pi} + e^{-iz} \,.\, e^{-n\pi}]}$

Multiplying the numerator and denominator by $e^{in\text{p}}$

$$= \frac{e^{iz} \,.\, e^{2n\pi i} - e^{-iz}}{i[e^{iz} \,.\, e^{2n\pi i} + e^{-iz}]} = \frac{e^{iz} - e^{-iz}}{i\,(e^{iz} + e^{-iz})} = \tan z \qquad |\because\ e^{2n\pi i} = 1 = e^{-2n\pi i}$$

$\Rightarrow$ tan z remains unchanged when z is increased by any multiple of π.

$\therefore$ **tan *z* is a periodic function with period** π.

6.11. TRIGONOMETRICAL FORMULAE FOR COMPLEX QUANTITIES

If *z* is a complex quantity, prove that

(*i*) $\sin^2 z + \cos^2 z = 1$ (*ii*) $\sin 2z = 2 \sin z \cos z$

(*iii*) $\cos 2z = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$

(*iv*) $\tan 2z = \dfrac{2 \tan z}{1 - \tan^2 z}$ (*v*) $\sin(-z) = -\sin z$

(*vi*) $\sin 3z = 3 \sin z - 4 \sin^3 z$ (*vii*) $\tan 3z = \dfrac{3 \tan z - \tan^3 z}{1 - 3\tan^2 z}$

Proof. (*i*) LHS $= \sin^2 z + \cos^2 z = \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2$

$$= -\tfrac{1}{4}(e^{2iz} + e^{-2iz} - 2) + \tfrac{1}{4}(e^{2iz} + e^{-2iz} + 2) = \tfrac{1}{2} + \tfrac{1}{2} = 1 = \text{RHS}$$

(*ii*) $\text{RHS} = 2 \sin z \cos z = 2.\ \dfrac{e^{iz} - e^{-iz}}{2i} \cdot \dfrac{e^{iz} + e^{-iz}}{2} = \dfrac{e^{2iz} - e^{-2iz}}{2i} = \sin 2z = \text{LHS}$

(*iii*) $\cos^2 z - \sin^2 z = \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2 - \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2$

$$= \frac{1}{4}\ [(e^{2iz} + e^{-2iz} + 2) + (e^{2iz} + e^{-2iz} - 2)] = \frac{e^{2iz} + e^{-2iz}}{2} = \cos 2z$$

$$2\cos^2 z - 1 = 2\left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 - 1 = \frac{1}{2}\ (e^{2iz} + e^{-2iz} + 2) - 1 = \frac{e^{2iz} + e^{-2iz}}{2} = \cos 2z$$

$$1 - 2\sin^2 z = 1 - 2\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = 1 + \frac{1}{2}\ (e^{2iz} + e^{-2iz} - 2) = \frac{e^{2iz} + e^{-2iz}}{2} = \cos 2z$$

Hence the result.

(*iv*) $\text{RHS} = \dfrac{2\tan z}{1 - \tan^2 z} = \dfrac{2 \cdot \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}}{1 - \left[\dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right]^2} = \dfrac{2(e^{iz} - e^{-iz})(e^{iz} + e^{-iz})}{i[(e^{iz} + e^{-iz})^2 + (e^{iz} - e^{-iz})^2]}$

$$= \frac{2(e^{2iz} - e^{-2iz})}{i.2(e^{2iz} + e^{-2iz})} = \frac{e^{2iz} - e^{-2iz}}{i(e^{2iz} + e^{-2iz})} = \tan 2z = \text{LHS}$$

(*v*) $\sin(-z) = \dfrac{e^{i(-z)} - e^{-i(-z)}}{2i} = \dfrac{e^{-iz} - e^{iz}}{2i} = -\dfrac{e^{iz} - e^{-iz}}{2i} = -\sin z.$

(*vi*) $\sin 3z = \dfrac{e^{3iz} - e^{-3iz}}{2i} = \dfrac{x^3 - y^3}{2i}$, where $x = e^{iz}, y = e^{-iz}$

$$= \frac{(x - y)^3 + 3xy(x - y)}{2i} = \frac{1}{2i}\ [(e^{iz} - e^{-iz})^3 + 3\ .\ e^{iz}\ .\ e^{-iz}\ (e^{iz} - e^{-iz})]$$

$$= \frac{1}{2i}\ [(2i\sin z)^3 + 3(2i\sin z)] = \frac{1}{2i}\ [-8i\sin^3 z + 6i\sin z] = 3\sin z - 4\sin^3 z.$$

(*vii*) $\text{RHS} = \dfrac{3\tan z - \tan^3 z}{1 - 3\tan^2 z}$

$$= \frac{3 \cdot \left[\dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right] - \left[\dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right]^3}{1 - 3 \cdot \left[\dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right]^2} = \frac{3\left[\dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right] + \dfrac{1}{i} \cdot \left[\dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right]^3}{1 + 3 \cdot \left[\dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right]^2}$$

$$= \frac{3\dfrac{x}{iy} + \dfrac{1}{i}\cdot\left(\dfrac{x}{y}\right)^3}{1 + 3\left(\dfrac{x}{y}\right)^2}, \text{ where } x = e^{iz} - e^{-iz},\ y = e^{iz} + e^{-iz}$$

$$= \frac{3xy^2 + x^3}{iy^3}\cdot\frac{y^2}{y^2 + 3x^2} = \frac{x(3y^2 + x^2)}{iy(y^2 + 3x^2)} = \frac{x(3e^{2iz} + 3e^{-2iz} + 6 + e^{2iz} + e^{-2iz} - 2)}{iy(e^{2iz} + e^{-2iz} + 2 + 3e^{2iz} + 3e^{-2iz} - 6)}$$

$$= \frac{x(4e^{2iz} + 4e^{-2iz} + 4)}{iy(4e^{2iz} + 4e^{-2iz} - 4)} = \frac{(e^{iz} - e^{-iz})(e^{2iz} + e^{-2iz} + 1)}{i(e^{iz} + e^{-iz})(e^{2iz} + e^{-2iz} - 1)}$$

$$= \frac{e^{3iz} - e^{-3iz}}{i(e^{3iz} + e^{-3iz})} \qquad \left[\begin{array}{l} \because\ (a - b)(a^2 + b^2 + ab) = a^3 - b^3 \\ \quad\ (a + b)(a^2 + b^2 - ab) = a^3 + b^3 \end{array}\right]$$

$$= \tan 3z.$$

Example 3. *If* α, β *are the imaginary cube roots of unity prove that*

$$\alpha e^{\alpha x} + \beta e^{\beta x} = -e^{-x/2}\left(\cos\frac{\sqrt{3}}{2}x + \sqrt{2}\sin\frac{\sqrt{3}}{2}x\right).$$

Sol. We know that imaginary cube roots of unity are ω and ω^2, where

$$\omega = \frac{-1 + i\sqrt{3}}{2}, \quad \omega^2 = \frac{-1 - i\sqrt{3}}{2}.$$

Here $$\alpha = \frac{-1 + i\sqrt{3}}{2}, \quad \beta = \frac{-1 - i\sqrt{3}}{2}$$

$$\alpha e^{\alpha x} + \beta e^{\beta x} = \alpha e^{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x} + \beta e^{\left(-\frac{1}{2} - i\sqrt{\frac{3}{2}}\right)x}$$

$$= \alpha e^{-x/2}.\ e^{i\sqrt{\frac{3}{2}}x} + \beta e^{-x/2},\ e^{-i\frac{\sqrt{3}}{2}x} = e^{-x/2}\left\{\alpha\, e^{i\frac{\sqrt{3}}{2}x} + \beta e^{-i\frac{\sqrt{3}}{2}x}\right\}$$

$$= e^{-x/2}\left\{\alpha\left(\cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x\right) + \beta\left(\cos\frac{\sqrt{3}}{2}x - i\sin\frac{\sqrt{3}}{2}x\right)\right\}$$

$$= e^{-x/2}\left\{(\alpha + \beta)\cos\frac{\sqrt{3}}{2}x + i(\alpha - \beta)\sin\frac{\sqrt{3}}{2}x\right\}$$

$$\alpha + \beta = -1, \quad \alpha - \beta = i\sqrt{3}$$

$\therefore$ $$\alpha e^{\alpha x} + \beta e^{\beta x} = e^{-x/2}\left\{-\cos\frac{\sqrt{3}}{2}x + i(i\sqrt{3})\sin\frac{\sqrt{3}}{2}x\right\}$$

$$= -e^{-x/2}\left\{\cos\frac{\sqrt{3}}{2}x + \sqrt{3}\sin\frac{\sqrt{3}}{2}x\right\}.$$

TEST YOUR KNOWLEDGE

1. If $z = x + iy$, find the real and imaginary parts of exp (z^2).

2. Prove that:

(*i*) $\sin(\alpha + n\theta) - e^{i\alpha} \sin n\theta = e^{-in\theta} \sin \alpha$ (*ii*) $[\sin(\alpha + \theta) - e^{i\alpha} \sin \theta]^n = \sin^n \alpha \,.\, e^{-in\theta}$

3. If z is a complex number, prove that:

(*i*) $\cos(-z) = \cos z$ (*ii*) $\tan(-z) = -\tan z$

(*iii*) $\cos 3z = 4\cos^3 z - 3\cos z$ (*iv*) $\tan z = \dfrac{\sin 2z}{1 + \cos 2z}$.

4. If z_1, z_2 are complex numbers, show that:

(*i*) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$ (*ii*) $\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$

(*iii*) $\tan(z_1 + z_2) = \dfrac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$ (*iv*) $\sin z_1 + \sin z_2 = 2 \sin \dfrac{z_1 + z_2}{2} \cos \dfrac{z_1 - z_2}{2}$

(*v*) $\cos z_1 - \cos z_2 = 2 \sin \dfrac{z_1 + z_2}{2} \sin \dfrac{z_2 - z_1}{2}$

5. Show that:

(*i*) $\cos(\alpha + i\beta) = \dfrac{1}{2}(e^{-\beta} + e^{\beta}) \cos \alpha + \dfrac{i}{2}(e^{-\beta} + e^{\beta}) \sin \alpha$

(*ii*) $\sin(\alpha + i\beta) = \dfrac{1}{2}(e^{-\beta} + e^{\beta}) \sin \alpha + \dfrac{i}{2}(e^{-\beta} + e^{\beta}) \cos \alpha$.

ANSWER

1. $e^{x^2 - y^2} \cos 2xy,\ e^{x^2 - y^2} \sin 2xy$.

6.12. LOGARITHMS OF COMPLEX NUMBERS

Definition. If $\omega = e^z$, where z and ω are complex numbers, then z is called a logarithm of ω to the base e. Thus $\log_e \omega = z$.

1. Prove that $\log_e \omega$ is a many-valued function.

We know that $e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1$

Let $e^z = \omega$, then $e^{z + 2n\pi i} = e^z \,.\, e^{2n\pi i} = e^z . 1 = \omega$

$\therefore$ by definition $\log_e \omega = z + 2n\pi i$, where n is zero, or any +ve or –ve integer.

Thus if z be a logarithm of ω, so is $z + 2n\pi i$.

Hence the logarithm of a complex number has infinite values and is thus a many-valued function.

Note. The value z + 2nπi is called the general value of $\log_e \omega$ and is denoted by $\text{Log}_e\, \omega$.

Thus $\textbf{Log}_e\, \omega = z + 2n\pi i = 2n\pi i + \log_e \omega$

If $\omega = x + iy$, then **Log (x + iy) = 2nπi + log (x + iy).**

If we put n = 0, in the general value, we get the principal value of z, i.e., $\log_e \omega$.

2. Prove that log (– N) = πi + log N, where N is positive.

Proof. $-N = N(-1) = N(\cos \pi + i \sin \pi) = N \,.\, e^{i\pi}$

$\therefore$ $\log(-N) = \log(N \,.\, e^{i\pi}) = \log N + \log e^{i\pi} = \log N + \pi i$.

3. Separate log $(\alpha + i\beta)$ into real and imaginary parts.

Proof. Let $\alpha + i\beta = r(\cos \theta + i \sin \theta)$ so that $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1} \dfrac{\beta}{\alpha}$

$$\therefore \quad \text{Log}(\alpha + i\beta) = 2n\pi i + \log(\alpha + i\beta) = 2n\pi i + \log[r(\cos\theta + i\sin\theta)]$$
$$= 2n\pi i + \log(r\,e^{i\theta}) = 2n\pi i + \log r + \log e^{i\theta} = 2n\pi i + \log r + i\theta$$
$$= 2n\pi i + \log\sqrt{\alpha^2+\beta^2} + i\tan^{-1}\frac{\beta}{\alpha} = 2n\pi i + \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$$
$$= \frac{1}{2}\log(\alpha^2+\beta^2) + i\left[2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right]$$

$$\therefore \quad R_e[\text{Log}(\alpha+i\beta)] = \tfrac{1}{2}\log(\alpha^2+\beta^2)$$
$$I_m[\text{Log}(\alpha+i\beta)] = 2n\pi + \tan^{-1}\frac{\beta}{\alpha}$$

Note. Putting $n = 0$, the principal value of $\mathbf{\log}\,(\alpha + \mathbf{i}\beta) = \frac{1}{2}\,\mathbf{\log}\,(\alpha^2+\beta^2) + \mathbf{i}\,\mathbf{\tan^{-1}}\frac{\beta}{\alpha}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that* $\log(1 + re^{i\theta}) = \frac{1}{2}\log(1 + 2r\cos\theta + r^2) + i\tan^{-1}\frac{r\sin\theta}{1 + r\cos\theta}$.

Deduce that $\log(1 + \cos\theta + i\sin\theta) = \log\left(2\cos\frac{\theta}{2}\right) + i\,\frac{\theta}{2}$

Sol. $\log(1 + re^{i\theta}) = \log[1 + r(\cos\theta + i\sin\theta)] = \log[(1 + r\cos\theta) + i(r\sin\theta)]$

$$= \frac{1}{2}\log[(1+r\cos\theta)^2 + (r\sin\theta)^2] + i\tan^{-1}\frac{r\sin\theta}{1+r\cos\theta}$$
$$= \frac{1}{2}\log[1 + 2r\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta] + i\tan^{-1}\frac{r\sin\theta}{1+r\cos\theta}$$
$$= \frac{1}{2}\log[1 + 2r\cos\theta + r^2] + i\tan^{-1}\frac{r\sin\theta}{1+r\cos\theta} \quad \ldots(1)$$

Putting $r = 1$ in (1),

$$\log(1 + e^{i\theta}) = \frac{1}{2}\log(1 + 2\cos\theta + 1) + i\tan^{-1}\frac{\sin\theta}{1+\cos\theta}$$

i.e.,
$$\log(1 + \cos\theta + i\sin\theta) = \frac{1}{2}\log[2(1+\cos\theta)] + i\tan^{-1}\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$
$$= \frac{1}{2}\log\left[2.\,2\cos^2\frac{\theta}{2}\right] + i\tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\log\left[\left(2\cos\frac{\theta}{2}\right)^2\right] + i\,\frac{\theta}{2}$$
$$= \frac{1}{2}.\,2\log\left(2\cos\frac{\theta}{2}\right) + i\,.\,\frac{\theta}{2} = \log\left(2\cos\frac{\theta}{2}\right) + i\,.\,\frac{\theta}{2}$$

Example 2. (*a*) *Find the general value of* $\log\left(-1 + i\sqrt{3}\right)$. **(P.T.U., May 2012)**

(*b*) *Prove that* $\log(-4) = 2\log 2 + (2n+1)\pi i$. **(P.T.U., May 2007)**

Sol. (*a*) $-1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$$r\cos\theta = -1 \text{ and } r\sin\theta = \sqrt{3}$$

Squaring and adding

$$r^2 = 1 + 3 = 4$$

$$\therefore \quad r = 2$$

$$\cos\theta = -\frac{1}{2} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\sin\theta = \frac{\sqrt{3}}{2} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{2\pi}{3}$$

$$\therefore \quad \theta = \frac{2\pi}{3}$$

$$-1 + i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$= 2e^{i\frac{2\pi}{3}}$$

General value of $-1 + i\sqrt{3}$

$$= 2n\pi i + \log\left(-1 + i\sqrt{3}\right)$$

$$= 2n\pi i + \log\left(2e^{\frac{i2\pi}{3}}\right)$$

$$= 2n\pi i + \log 2 + \log e^{\frac{i2\pi}{3}}$$

$$= 2n\pi i + \log 2 + i\frac{2\pi}{3}$$

$$= \log 2 + 2\pi i\left(n + \frac{1}{3}\right)$$

$$= \log 2 + 2\frac{3n+1}{3}\pi i$$

(*b*) $$-4 = 4(-1) = 4(\cos\pi + i\sin\pi) = 4e^{i\pi}$$

$$\text{Log}(-4) = \text{Log}(4e^{i\pi}) = 2n\pi i + \log(4e^{i\pi})$$

$$= 2n\pi i + \log 4 + \log e^{i\pi}$$

$$= 2n\pi i + \log 4 + i\pi$$

$$= \log 4 + (2n+1)\pi i$$

$$= \log 2^2 + (2n+1)\pi i$$

$$= 2\log 2 + (2n+1)\pi i.$$

Example 3. *Separate into real and imaginary parts Log (4 + 3i).*

Sol. Let $\quad 4 + 3i = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts $r\cos\theta = 4$; $r\sin\theta = 3$

Squaring and adding, $r^2 = 16 + 9 = 25 \quad \therefore \quad r = 5$

Dividing, $\quad \tan\theta = \frac{3}{4} \quad \therefore \quad \theta = \tan^{-1}\frac{3}{4}$

$$\therefore \quad \log(4+3i) = \text{Log}[r(\cos\theta + i\sin\theta)] = \text{Log}(re^{i\theta}) = 2n\pi i + \log(re^{i\theta})$$

$$= 2n\pi i + \log r + \log e^{i\theta} = 2n\pi i + \log 5 + i\theta = \log 5 + 2n\pi i + i\tan^{-1}\frac{3}{4}$$

$$\therefore \quad R_e[\log(4+3i)] = \log 5$$

$$I_m[\log(4+3i)] = \left(2n\pi + \tan^{-1}\frac{3}{4}\right).$$

Example 4. (*a*) *Prove that* $\tan\left(i \log \dfrac{a-ib}{a+ib}\right) = \dfrac{2ab}{a^2 - b^2}$

(*b*) *Prove that* $\sin\left\{i \log \dfrac{1+ie^{-i\theta}}{1-ie^{i\theta}}\right\}$ *is wholly real.*

Sol. (*a*) Let $a + ib = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts $r\cos\theta = a,\ r\sin\theta = b$

Dividing, $\tan\theta = \dfrac{b}{a}$ ...(1)

Also, $a - ib = r(\cos\theta - i\sin\theta)$

$$\text{L.H.S.} = \tan\left[i \log \frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)}\right] = \tan\left[i\log\frac{e^{-i\theta}}{e^{i\theta}}\right]$$

$$= \tan[i \log e^{-2i\theta}] = \tan[i(-2i\theta)\log e] \qquad |\log e = 1$$

$$= \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\,\dfrac{b}{a}}{1-\dfrac{b^2}{a^2}} \qquad \text{[Using (1)]}$$

$$= \frac{2ab}{a^2-b^2}.$$

(*b*) $$\sin\left\{i\log\frac{1+i(\cos\theta - i\sin\theta}{1-i(\cos\theta+i\sin\theta)}\right\}$$

$$= \sin\left\{i\log\frac{(1+\sin\theta)+i\cos\theta}{(1+\sin\theta)-i\cos\theta}\right\}$$

$$= \sin\left\{i\log\frac{1+\cos\left(\dfrac{\pi}{2}-\theta\right)+i\sin\left(\dfrac{\pi}{2}-\theta\right)}{1+\cos\left(\dfrac{\pi}{2}-\theta\right)-i\sin\left(\dfrac{\pi}{2}-\theta\right)}\right\}$$

$$= \sin\left\{i\log\frac{2\cos^2\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)+2i\sin\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)\cos\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)}{2\cos^2\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)-2\left(\sin\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)\cos\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)}\right\}$$

$$= \sin\left\{i\log\frac{\cos\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)+i\sin\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)}{\cos\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)-i\sin\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)}\right\} = \sin\left\{i\log\frac{e^{i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}}{e^{-i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}}\right\}$$

$$= \sin\left\{i\log e^{2i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right\} = \sin\left\{i^2\left(\frac{\pi}{2}-\theta\right)\right\} = -\sin\left(\frac{\pi}{2}-\theta\right) = -\cos\theta$$

which is wholly real.

Example 5. *Express log (log i) in the form A + iB.*

Sol. $$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$$

$\therefore$ $$\text{Log } i = 2n\pi i + \log e^{i\pi/2} = 2n\pi i + i\frac{\pi}{2} = i(4n+1)\frac{\pi}{2}$$

$\therefore$ $$\text{Log (Log } i) = \text{Log}\left[i(4\pi+1)\frac{\pi}{2}\right] = 2m\pi i + \log\left[i(4\pi+1)\frac{\pi}{2}\right]$$

$$= 2m\pi i + \log i + \log(4n+1)\frac{\pi}{2}$$

$$= 2m\pi i + \log e^{i\pi/2} + \log(4n+1)\frac{\pi}{2} \qquad \left(\because\ i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}\right)$$

$$= 2m\pi i + i\frac{\pi}{2} + \log(4n+1)\frac{\pi}{2} = \log(4n+1)\frac{\pi}{2} + i(4m+1)\frac{\pi}{2}.$$

Example 6. (*a*) *Show that* $\log\dfrac{x+iy}{x-iy} = 2i\tan^{-1}\dfrac{y}{x}$.

(*b*) *Prove that* $\tan^{-1}x = \dfrac{1}{2i}\log\dfrac{1+ix}{1-ix}$. **(P.T.U., May 2007)**

Sol. (*a*) Let $x = r\cos\theta,\quad y = r\sin\theta$

$\therefore$ $x^2 + y^2 = r^2$ and $\tan\theta = \dfrac{y}{x}$

$$\log\frac{x+iy}{x-iy} = \log\frac{r(\cos\theta + i\sin\theta)}{r(\cos\theta - i\sin\theta)} = \log\frac{e^{i\theta}}{e^{-i\theta}} = \log e^{2i\theta} = 2i\theta$$

$$= 2i\tan^{-1}\frac{y}{x}.$$

(*b*) Let $1 = r\sin\theta,\quad x = r\sin\theta \quad \therefore\quad r^2 = 1 + x^2, \tan\theta = x.$

$$\text{R.H.S.} = \frac{1}{2i}\log\frac{1+ix}{1-ix}$$

$$= \frac{1}{2i}\log\frac{r\cos\theta + i\,r\sin\theta}{r\cos\theta - i\,r\sin\theta} = \frac{1}{2i}\log\frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}$$

$$= \frac{1}{2i}\log\frac{e^{i\theta}}{e^{-i\theta}} = \frac{1}{2i}\log e^{2i\theta} = \frac{1}{2i}2i\theta = \theta = \tan^{-1}x.$$

6.13. GENERAL EXPONENTIAL FUNCTION

The exponential function a^z is defined by the equation $a^z = e^{z\log a}$, where a and z are any numbers, real or complex.

Since $\quad \text{Log } a = 2n\pi i + \log a$

$\therefore$ The general exponential function $a^z = e^{z \log a}$

$\therefore$ $$a^z = e^{z(2n\pi i + \log a)}$$

Hence a^z is a many valued function and its principal value is obtained by putting $n = 0$.

Example 7. (*a*) *Prove that i^i is wholly real and find its principal value. Also show that the values of i^i form a G.P.* **(P.T.U., Dec. 2007, May 2010, Dec. 2013)**

(*b*) *Prove that* $\log i^i = -\left(2n + \frac{1}{2}\right)\pi$. **(P.T.U., Dec. 2002)**

Sol. (*a*) $$i^i = e^{i \operatorname{Log} i}$$ [By definition]

$$= e^{i[2n\pi i + \log i]} = e^{i[2n\pi i + \log(\cos \pi/2 + i \sin \pi/2)]}$$

$$= e^{i[2n\pi i + \log e^{i\pi/2}]} = e^{i[2n\pi i + i\pi/2]} = e^{i^2(4n+1)\pi/2} = e^{-(4n+1)\pi/2}$$

which is wholly real.

The principal value of $i^i = e^{-\pi/2}$ (putting $n = 0$)

Putting $n = 0, 1, 2, ...$ the values of i^i are $e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, ...$

which form a G.P. whose common ratio is $e^{-2\pi}$.

(*b*) Log $$i^i = i \log i$$

$$= i[2n\pi\ i + \log i] = i\left[2n\pi i + \log\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)\right]$$

$$= i\left[2n\pi i + \log e^{i\frac{\pi}{2}}\right] = i\left[2n\pi i + i\frac{\pi}{2}\right) = i^2 \pi \left(2n + \frac{1}{2}\right)$$

$$= -\left(2n + \frac{1}{2}\right)\pi .$$

Example 8. *If* $i^{\alpha + i\beta} = \alpha + i\beta$, *prove that* $\alpha^2 + \beta^2 = e^{-(4n+1)\ \pi\beta}$.

Sol. $$\alpha + i\beta = i^{\alpha + i\beta} = e^{\log i^{\alpha + i\beta}} = e^{(\alpha + i\beta) \log i}$$

$$= e^{(\alpha + i\beta)[2n\pi i + \log i]} = e^{(\alpha + i\beta)[2n\pi i + \log(\cos \pi/2 + i \sin \pi/2)]}$$

$$= e^{(\alpha + i\beta)[2n\pi i + \log e^{i\pi/2}]} = e^{(\alpha + i\beta)[2n\pi i + i\pi/2]} = e^{-\beta(4n+1)\pi/2 + i\alpha(4n+1)\pi/2}$$

$$= e^{-\beta(4n+1)\pi/2} \cdot e^{i\alpha(4n+1)\pi/2} = e^{-\beta(4n+1)\pi/2}\left[\cos(4n+1)\frac{\alpha\pi}{2} + i \sin(4n+1)\frac{\alpha\pi}{2}\right]$$

$[\because \quad e^{i\theta} = \cos\theta + i \sin\theta]$

Equating real and imaginary parts

$$\alpha = e^{-(4n+1)\beta\pi/2} \cdot \cos(4n+1)\frac{\alpha\pi}{2} \ ; \quad \beta = e^{-(4n+1)\beta\pi/2} \cdot \sin(4n+1)\frac{\alpha\pi}{2}$$

Squaring and adding,

$$\alpha^2 + \beta^2 = e^{-(4n+1)\beta\pi}\left[\cos^2(4n+1)\frac{\alpha\pi}{2} + \sin^2(4n+1)\frac{\alpha\pi}{2}\right] = e^{-(4n+1)\beta\pi}.$$

Example 9. *Considering only the principal value, prove that the real part of*

$$(1+i\sqrt{3})^{1+i\sqrt{3}} \text{ is } 2e^{-\pi\sqrt{3}} \cos\left(\frac{\pi}{3}+\sqrt{3}\log 2\right).$$

Sol. $(1+i\sqrt{3})^{1+i\sqrt{3}} = e^{(1+i\sqrt{3})\log(1+i\sqrt{3})} = e^{(1+i\sqrt{3})[\frac{1}{2}\log(1+3)+i\tan^{-1}\sqrt{3}]}$

$$= e^{(1+i\sqrt{3})[\frac{1}{2}\log 4+i\pi/3]} = e^{(1+i\sqrt{3})(\frac{1}{2}.2\log 2+i\pi/3)}$$

$$= e^{(\log 2-\pi/\sqrt{3})+i(\pi/3+\sqrt{3}\log 2)} = e^{\log 2-\pi/\sqrt{3}} . e^{i(\pi/3+\sqrt{3}\log 2)}$$

$$= e^{\log 2}\, e^{-\pi/\sqrt{3}}\left[\cos\left(\frac{\pi}{3}+\sqrt{3}\log 2\right)+i\sin\left(\frac{\pi}{3}+\sqrt{3}\log 2\right)\right]$$

$$= 2e^{-\pi/\sqrt{3}}\left[\cos\left(\frac{\pi}{3}+\sqrt{3}\log 2\right)+i\sin\left(\frac{\pi}{3}+\sqrt{3}\log 2\right)\right] \qquad [\because \; e^{\log f(x)}=f(x)]$$

$\Rightarrow$ Real part of $(1+i\sqrt{3})^{1+i\sqrt{3}}$ is $2e^{-\pi/\sqrt{3}} \cos\left(\frac{\pi}{3}+\sqrt{3}\log 2\right)$.

Example 10. *If* $i^{i^{i\,\ldots\ldots\, ad.inf.}} = A + iB$ *and only principal values are considered, prove that*

(*a*) $tan\ \dfrac{\pi A}{2} = \dfrac{B}{A}$ (*b*) $A^2 + B^2 = e^{-B\pi}$.

Sol. $i^{i^{i\,\ldots\ldots\, ad.inf.}} = A + iB \qquad \Rightarrow \qquad i^{A+iB} = A + iB$

Now, $A + iB = i^{A+iB} = e^{(A+iB)\log i}$ (Taking principal values only)

$$= e^{(A+iB)\log(\cos\pi/2+i\sin\pi/2)} = e^{(A+iB)\log(e^{i\pi/2})}$$

$$= e^{(A+iB)(i\pi/2)} = e^{-(B\pi/2)+i.(A\pi/2)}$$

$$= e^{-B\pi/2} . e^{iA\pi/2} = e^{-B\pi/2}\left(\cos\frac{A\pi}{2}+i\sin\frac{A\pi}{2}\right)$$

Equating real and imaginary parts

$$A = e^{-B(\pi/2)}\cos\frac{A\pi}{2} \qquad \ldots(1)$$

$$B = e^{-B(\pi/2)}\sin\frac{A\pi}{2} \qquad \ldots(2)$$

Dividing (2) by (1), $\tan\dfrac{A\pi}{2} = \dfrac{B}{A}$ ...(I)

Squaring and adding (1) and (2), $A^2 + B^2 = e^{-B\pi}\left(\cos^2\dfrac{A\pi}{2}+\sin^2\dfrac{A\pi}{2}\right) = e^{-B\pi}$ ...(II)

Example 11. *If* $(a + ib)^p = m^{x+iy}$, *then prove that* $\dfrac{y}{x} = \dfrac{2\tan^{-1}\left(\dfrac{b}{a}\right)}{\log(a^2+b^2)}$ *when only principal values are considered.* **(P.T.U., Dec. 2006)**

Sol. $$(a+ib)^p = m^{x+iy}$$

Taking log of both sides, $$\log (a+ib)^p = \log m^{x+iy}$$

or $$p \log (a+ib) = (x+iy) \log m$$

or $$p\left[\tfrac{1}{2}\log (a^2+b^2)+i\tan^{-1}\frac{b}{a}\right] = x\log m + iy\log m$$

(Considering only the principal values)

Equating real and imaginary parts $$x\log m = \tfrac{1}{2}\,p\log (a^2+b^2) \qquad \ldots(i)$$

$$y\log m = p\tan^{-1}\frac{b}{a} \qquad \ldots(ii)$$

Dividing (*ii*) by (*i*), $$\frac{y}{x} = \frac{p\tan^{-1}\dfrac{b}{a}}{\tfrac{1}{2}\,p\log (a^2+b^2)} = \frac{2\tan^{-1}\dfrac{b}{a}}{\log (a^2+b^2)}.$$

Example 12. *If tan log (x + iy) = a + ib and $a^2 + b^2 \neq 1$, then prove that tan log $(x^2 + y^2) = \dfrac{2a}{1-a^2-b^2}$.*

Sol. $$\tan\log (x+iy) = a+ib \qquad \ldots(i)$$

$\Rightarrow$ $$\tan\log (x-iy) = a-ib \qquad \ldots(ii)$$

Now, $$\tan\log (x^2+y^2) = \tan\log (x+iy)(x-iy)$$

$$= \tan[\log (x+iy)+\log (x-iy)] = \frac{\tan\log (x+iy)+\tan\log (x-iy)}{1-\tan\log (x+iy)\,.\,\tan\log (x-iy)}$$

$$= \frac{a+ib+a-ib}{1-(a+ib)(a-ib)} = \frac{2a}{1-a^2-b^2}, \text{ where } a^2+b^2 \neq 1.$$

Example 13. *Show that* $\left(\sqrt{i}\right)^{\sqrt{i}} = e^{-\frac{\pi}{4\sqrt{2}}}\left(\cos\dfrac{\pi}{4\sqrt{2}} + i\sin\dfrac{\pi}{4\sqrt{2}}\right)$

Sol. $$\left(\sqrt{i}\right)^{\sqrt{i}} = e^{\log\left(\sqrt{i}\right)^{\sqrt{i}}} = e^{\sqrt{i}\log\sqrt{i}} = e^{\sqrt{i}\,\frac{1}{2}\log i}$$

We know that $$i = \text{cis}\,\frac{\pi}{2} = e^{i\frac{\pi}{2}}$$

$\therefore$ $$\left(\sqrt{i}\right)^{\sqrt{i}} = e^{\sqrt{i}\,\frac{1}{2}\log e^{\frac{i\pi}{2}}} = e^{\left(\text{cis}\,\frac{\pi}{2}\right)^{\frac{1}{2}}.\,\frac{1}{2}\left(i\,\frac{\pi}{2}\right)}$$

$$= e^{i\frac{\pi}{4}\,\text{cis}\,\frac{\pi}{4}} = e^{\frac{i\pi}{4}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}$$

$$= e^{i\frac{\pi}{4}\,\frac{1+i}{\sqrt{2}}} = e^{\frac{\pi}{4\sqrt{2}}(i-1)} = e^{\frac{\pi}{4\sqrt{2}}i}\,.\,e^{-\frac{\pi}{4\sqrt{2}}}$$

$$= e^{-\frac{\pi}{4\sqrt{2}}}\,\text{cis}\,\frac{\pi}{4\sqrt{2}} = e^{-\frac{\pi}{4\sqrt{2}}}\left[\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right].$$

Example 14. *If* $\dfrac{(1+i)^{x+iy}}{(1-i)^{x-iy}} = \alpha + i\,\beta$, *prove that one of the values of* $\tan^{-1} \dfrac{\beta}{\alpha} = \dfrac{1}{2}\,\pi x + y \log 2$.

Sol. First take $(1 + i)^{x+iy}$

$$(1+i)^{x+iy} = e^{\log (1+i)^{x+iy}} = e^{(x+iy)\log(1+i)}$$

One of the values of

$$\log(1+i) = \log|1+i| + i\tan^{-1}\frac{1}{1} \qquad \left[\log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}\right]$$

$$= \log\sqrt{2} + i\,\frac{\pi}{4}$$

$$\therefore \quad (1+i)^{x+iy} = e^{(x+iy)\left(\log\sqrt{2}+i\frac{\pi}{4}\right)} = e^{\left(x\log\sqrt{2} - y\frac{\pi}{4}\right) + i\left[y\log\sqrt{2} + x\frac{\pi}{4}\right]}$$

$$(1+i)^{x+iy} = e^{\left(x\log\sqrt{2} - \frac{y\pi}{4}\right) + i\left[y\log\sqrt{2} + \frac{x\pi}{4}\right]}$$

Changing i to $-i$

$$(1-i)^{x-iy} = e^{\left(x\log\sqrt{2} - \frac{y\pi}{4}\right) - i\left[y\log\sqrt{2} + \frac{x\pi}{4}\right]}$$

$$\frac{(1+i)^{x+iy}}{(1-i)^{x-iy}} = \frac{e^{\left(x\log\sqrt{2} - \frac{y\pi}{4}\right) + i\left[y\log\sqrt{2} + \frac{x\pi}{4}\right]}}{e^{\left(x\log\sqrt{2} - \frac{y\pi}{4}\right) - i\left[y\log\sqrt{2} + \frac{x\pi}{4}\right]}} = e^{2i\left[y\log\sqrt{2} + \frac{x\pi}{4}\right]}$$

$$= e^{i\left[2y\log 2^{1/2} + \frac{x\pi}{2}\right]} = e^{i\left[y\log 2 + \frac{x\pi}{2}\right]}$$

$$\therefore \quad \alpha + i\,\beta = \cos\left(y\log 2 + \frac{x\pi}{2}\right) + i\sin\left(y\log 2 + \frac{x\pi}{2}\right)$$

$$\therefore \quad \alpha = \cos\left(y\log 2 + \frac{x\pi}{2}\right) \quad \text{and} \quad \beta = \sin\left(y\log 2 + \frac{x\pi}{2}\right)$$

$$\therefore \quad \frac{\beta}{\alpha} = \tan\left[y\log 2 + \frac{x\pi}{2}\right]$$

$$\tan^{-1}\frac{\beta}{\alpha} = \frac{1}{2}\,\pi x + y\log 2 \text{ is one of the values.}$$

Example 15. *Find modulus and argument of* $(1 + i)^{1-i}$. **(P.T.U., May 2003)**

Sol.
$$(1+i)^{1-i} = e^{\log(1+i)^{1-i}} = e^{(1-i)\log(1+i)}$$

$$= e^{(1-i)\left[\log\sqrt{1+1} + i\tan^{-1}\frac{1}{1}\right]} = e^{(1-i)\left[\frac{1}{2}\log 2 + i\frac{\pi}{4}\right]}$$

$$= e^{\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)} = e^{\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right)} \cdot e^{i\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)}$$

$$= e^{\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right)}\left[\cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)\right]$$

Real part of $\quad (1+i)^{1-i} = e^{\left(\frac{1}{2}\log 2+\frac{\pi}{4}\right)} \cos\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)$

Img. part of $\quad (1+i)^{1-i} = e^{\left(\frac{1}{2}\log 2+\frac{\pi}{4}\right)} \sin\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)$

Modulus of $\quad (1+i)^{1-i} = \sqrt{(\text{Re})^2+(\text{Im})^2}$

$$= \sqrt{e^{2\left(\frac{1}{2}\log 2+\frac{\pi}{4}\right)}}\left[\cos^2\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)+\sin^2\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)\right]$$

$$= \sqrt{e^{2\left(\frac{1}{2}\log 2+\frac{\pi}{4}\right)}}\,.1 = e^{\frac{1}{2}\log 2+\frac{\pi}{4}} = e^{\log\sqrt{2}}\,.\,e^{\frac{\pi}{4}} = \sqrt{2}\,e^{\frac{\pi}{4}}$$

Argument of $\quad (1+i)^{1-i} = \tan^{-1}\dfrac{\text{Img. part}}{\text{Real part}} = \tan^{-1}\dfrac{\sin\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)}{\cos\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)} = \dfrac{\pi}{4}-\dfrac{1}{2}\log 2.$

Example 16. *Prove that* $\text{Log}_i\, i = \dfrac{4m+1}{4n+1}$*, where m, n are integers.*

Sol. $\quad \text{Log}_i\, i = \dfrac{\text{Log}_e\, i}{\text{Log}_e\, i}$

We know that $\quad i = \text{cis}\,\dfrac{\pi}{2} = e^{\frac{i\pi}{2}}$

$\therefore \quad \text{Log}_e\, i = 2m\pi\, i + \text{Log}\, i = 2m\,\pi\, i + \text{Log}\; e^{\frac{i\pi}{2}} = 2m\,\pi i + i\,\dfrac{\pi}{2}$

$$= (4m+1)\,\frac{i\,\pi}{2}, \text{ where } m \text{ is any integer}$$

Similarly, $\text{Log}_e\, i$ in the denominator

$$= 2n\pi\, i + \text{Log}\, i = (4n+1)\,\frac{i\,\pi}{2}, \text{ where } n \text{ is any integer}$$

$\therefore \quad \text{Log}_i\, i = \dfrac{(4m+1)\,\frac{i\,\pi}{2}}{(4n+1)\,\frac{i\,\pi}{2}} = \dfrac{4m+1}{4n+1}.$

TEST YOUR KNOWLEDGE

1. Find the general value of

(*i*) $\log(-i)$ (*ii*) $\log(1+i)$.

(*iii*) $\log(-3)$ **(P.T.U., Dec. 2002)**

2. Prove that

(*i*) $i\log\left(\dfrac{x-i}{x+i}\right) = \pi - 2\tan^{-1}x$ (*ii*) $\cos\left[i\log\left(\dfrac{a+ib}{a-ib}\right)\right] = \dfrac{a^2-b^2}{a^2+b^2}$

(*iii*) $i^i = e^{-(4n+1)\frac{\pi}{2}}$ (*iv*) $\text{Log}\, i^i = -\left(2n+\dfrac{1}{2}\right)\pi.$

3. Show that

(*i*) $\log(1 + i\tan\alpha) = \log\sec\alpha + i\alpha$ (*ii*) $\text{Log}_e\left(\dfrac{3-i}{3+i}\right) = 2i\left(n\pi - \tan^{-1}\dfrac{1}{3}\right)$

4. Prove that $\sin(\log i^i) = -1$.

5. If $\log\log(x + iy) = p + iq$, show that $y = x\tan[\tan q \log\sqrt{x^2 + y^2}\,]$.

6. Prove that the principal value of $\dfrac{(a+ib)^{p+iq}}{(a-ib)^{p-iq}}$ is $\cos 2(p\alpha + q\log r) + i\sin 2(p\alpha + q\log r)$, where $r = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\dfrac{b}{a}$.

7. Prove that $\dfrac{(1+i)^{1-i}}{(1-i)^{1+i}} = \sin(\log 2) + i\cos(\log 2)$.

8. Prove that the real part of the principal value of $i^{\log(1+i)}$ is $e^{-\frac{\pi^2}{8}}\cos\left(\dfrac{\pi}{4}\log 2\right)$.

ANSWERS

1. (*i*) $(4n-1)\dfrac{\pi i}{2}$, (*ii*) $\dfrac{1}{2}\log 2 + i(8n+1)\dfrac{\pi}{4}$.

(*iii*) $\log 3 + i(2n+1)\pi$

6.14. (a) HYPERBOLIC FUNCTIONS

(*i*) For all values of x, real or complex. The quantity $\dfrac{e^x - e^{-x}}{2}$ is called *hyperbolic sine of x* and is written as *sinh x* and

(*ii*) The quantity $\dfrac{e^x + e^{-x}}{2}$ is called *hyperbolic cosine of x* and is written as *cosh x*.

Thus $\sinh x = \dfrac{e^x - e^{-x}}{2}$; $\cosh x = \dfrac{e^x + e^{-x}}{2}$

The other hyperbolic functions are defined in terms of hyperbolic sine and cosine as follows :

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \;; \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \;; \quad \text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Note. $\sinh 0 = \dfrac{e^0 - e^{-0}}{2} = \dfrac{1-1}{2} = 0$; $\cosh 0 = \dfrac{e^0 + e^{-0}}{2} = \dfrac{1+1}{2} = 1$

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x \;;$$

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x}.$$

6.14(b). RELATIONS BETWEEN HYPERBOLIC AND CIRCULAR FUNCTIONS

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \ ; \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Putting $\theta = ix$ in these equations, we get

$$\cos(ix) = \frac{e^{i(ix)}+e^{-i(ix)}}{2} = \frac{e^{-x}+e^{x}}{2} = \cosh x$$

$$\sin(ix) = \frac{e^{i(ix)}-e^{-i(ix)}}{2i} = \frac{e^{-x}-e^{x}}{2i} = \frac{-(e^{x}-e^{-x})}{2i} = \frac{i^2(e^{x}-e^{-x})}{2i} = i\,.\,\frac{e^{x}-e^{-x}}{2} = i\sinh x$$

$$\tan(ix) = \frac{\sin(ix)}{\cos(ix)} = \frac{i\sinh x}{\cosh x} = i\tanh x$$

$$\cot(ix) = \frac{\cos(ix)}{\sin(ix)} = \frac{\cosh x}{i\sinh x} = \frac{i\cosh x}{i^2\sinh x} = -i\coth x$$

$$\sec(ix) = \frac{1}{\cos(ix)} = \frac{1}{\cosh x} = \text{sech}\, x$$

$$\text{cosec}(ix) = \frac{1}{\sin(ix)} = \frac{1}{i\sinh x} = \frac{i}{i^2\sinh x} = -i\,\text{cosech}\, x.$$

By definition, $\sinh\theta = \dfrac{e^{\theta}-e^{-\theta}}{2}$; $\cosh\theta = \dfrac{e^{\theta}+e^{-\theta}}{2}$; $\tanh\theta = \dfrac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$

Putting $\theta = ix$, we get

$$\sinh(ix) = \frac{e^{ix}-e^{-ix}}{2} = i\,.\,\frac{e^{ix}-e^{-ix}}{2i} = i\sin x \ ; \ \cosh(ix) = \frac{e^{ix}-e^{-ix}}{2} = \cos x$$

$$\tanh(ix) = \frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}} = i\,.\,\frac{\dfrac{e^{ix}-e^{-ix}}{2i}}{\dfrac{e^{ix}+e^{-ix}}{2}} = i\,.\,\frac{\sin x}{\cos x} = i\tan x.$$

6.14(c). PROVE THAT HYPERBOLIC FUNCTIONS ARE PERIODIC AND FIND THEIR PERIODS

(*i*) We know that $\sinh x = \dfrac{e^{x}-e^{-x}}{2}$

$\therefore \quad \sinh(x+2n\pi i) = \dfrac{e^{x+2n\pi i}-e^{-(x+2n\pi i)}}{2}$, where n is any integer

$$= \tfrac{1}{2}\,[e^{x}\,.\,e^{2n\pi i} - e^{-x}\,.\,e^{-2n\pi i}] = \frac{1}{2}\,[e^{x}\,.\,1 - e^{-x}\,.\,1] = \frac{e^{x}-e^{-x}}{2} = \sinh x$$

Thus $\sinh x$ remains unchanged when x is increased by any multiple of $2\pi i$.
Hence $\sinh x$ is a periodic function and its period is $2\pi i$.

(*ii*) $\cosh x = \dfrac{e^{x}+e^{-x}}{2}$

$\cosh(x+2n\pi i) = \dfrac{e^{x+2n\pi i}+e^{-(x+2n\pi i)}}{2}$, where n is any integer

$$= \tfrac{1}{2}\,[e^{x}\,.\,e^{2n\pi i} + e^{-x}\,.\,e^{-2n\pi i}] = \tfrac{1}{2}\,[e^{x}\,.\,1 + e^{-x}\,.\,1] = \frac{e^{x}+e^{-x}}{2} = \cosh x$$

Thus cosh x remains unchanged when x is increased by any multiple of $2\pi i$.
Hence **cosh x is a periodic function and its period is $2\pi i$.**

(*iii*) $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\tanh (x + n\pi i) = \frac{e^{x+n\pi i} - e^{-(x+n\pi i)}}{e^{x+n\pi i} + e^{-(x+n\pi i)}}, \text{ where } n \text{ is any integer}$$

$$= \frac{e^x \cdot e^{n\pi i} - e^{-x} \cdot e^{-n\pi i}}{e^x \cdot e^{n\pi i} + e^{-x} \cdot e^{-n\pi i}}$$

Multiplying the numerator and denominator by $e^{n\pi i}$

$$= \frac{e^x \cdot e^{2n\pi i} - e^{-x}}{e^x \cdot e^{2n\pi i} + e^{-x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x \qquad [\because \; e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1]$$

Thus tanh x remains unchanged when x is increased by any multiple of πi.

Hence **tanh x is a periodic function and its period is πi.**

Note. cosech x, sech x and coth x being reciprocals of sinh x, cosh x and tanh x respectively, are also periodic functions with periods $2\pi i$, $2\pi i$ and πi respectively.

6.15. FORMULAE OF HYPERBOLIC FUNCTIONS

1. Prove that (*a*) $\cosh^2 x - \sinh^2 x = 1$, (*b*) $\text{sech}^2 x + \tanh^2 x = 1$, (*c*) $\coth^2 x - \text{cosech}^2 x = 1$

Proof. (*a*) For all values of θ, $\cos^2 \theta + \sin^2 \theta = 1$

Putting $\theta = ix$, we get $\cos^2 (ix) + \sin^2 (ix) = 1$

or $(\cosh x)^2 + (i \sinh x)^2 = 1 \qquad [\because \; \cos ix = \cosh x \;;\; \sin (ix) = i \sinh x]$

or $\mathbf{\cosh^2 x - \sinh^2 x = 1} \qquad [\because \; i^2 = -1]$

(*b*) We know that $\cosh^2 x - \sinh^2 x = 1$

Dividing both sides by $\cosh^2 x$, we have

$$1 - \tanh^2 x = \text{sech}^2 x \qquad \Rightarrow \qquad \mathbf{sech^2 x + tanh^2 x = 1}$$

(*c*) We know that $\cosh^2 x - \sinh^2 x = 1$

Dividing both sides by $\sinh^2 x$, we have

$$\coth^2 x - 1 = \text{cosech}^2 x \qquad \Rightarrow \qquad \mathbf{coth^2 x - cosech^2 x = 1}$$

2. Prove that (*a*) $\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

(*b*) $\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(*c*) $\tanh (x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

Proof. (*a*) $\sinh (x \pm y) = \dfrac{1}{i} \sin i (x \pm y) \qquad \left[\because \; \sinh x = \dfrac{1}{i} \sin ix\right]$

$$= \frac{1}{i} (\sin ix \cos iy \pm \cos ix \sin iy)$$

$$= \frac{1}{i} (i \sinh x \cosh y \pm \cosh x \cdot i \sinh y)$$

$[\because \; \sin i\theta = i \sinh \theta \;;\; \cos i\theta = \cosh \theta]$

Hence $\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

(*b*) $\cosh(x \pm y) = \cos i(x \pm y)$ $[\because \cosh x = \cos ix]$

$= \cos ix \cos iy \mp \sin ix \sin iy = \cosh x \cosh y \mp i \sinh x \,.\, i \sinh y$

$= \cosh x \cosh y \mp (-\sinh x . \sinh y)$ $[\because i^2 = -1]$

Hence $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(*c*) $\tanh(x \pm y) = \dfrac{\sinh(x \pm y)}{\cosh(x \pm y)} = \dfrac{\sinh x \cosh y \pm \cosh x \sinh y}{\cosh x \cosh y \pm \sinh x \sinh y}$

Dividing the numerator and denominator by $\cosh x \cosh y$

$\therefore$ $$\boldsymbol{\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}}$$

3. Prove that (*a*) $\sinh 2x = 2 \sinh x \cosh x = \dfrac{2 \tanh x}{1 - \tanh^2 x}$

(*b*) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \dfrac{1 + \tanh^2 x}{1 - \tanh^2 x}$

(*c*) $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

Proof. (*a*) We know that $\sin 2\theta = 2 \sin\theta \cos\theta$

Putting $\theta = ix$, we get $\sin(2ix) = 2\sin(ix)\cos(ix)$ or $i \sinh 2x = 2 \,.\, i \sinh x . \cosh x$

or $\boldsymbol{\sinh 2x = 2 \sinh x \cosh x}$

Also, $\sin 2\theta = \dfrac{2 \tan\theta}{1 + \tan^2\theta}$

Putting $\theta = ix$, we get $\sin(2ix) = \dfrac{2 \tan ix}{1 + \tan^2 ix} = \dfrac{2 \,.\, i \tanh x}{1 + (i \tanh x)^2}$

or $i \sinh 2x = \dfrac{2i \tanh x}{1 - \tanh^2 x}$ or $\boldsymbol{\sinh 2x = \dfrac{2 \tanh x}{1 - \tanh^2 x}}$

(*b*) We know that $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Putting $\theta = ix$, we get $\cos(2ix) = \cos^2(ix) - \sin^2(ix)$ or $\cosh 2x = (\cosh x)^2 - (i \sinh x)^2$

or $\boldsymbol{\cosh 2x = \cosh^2 x + \sinh^2 x}$

We know that $\cos 2\theta = 2\cos^2\theta - 1$

Putting $\theta = ix$, we get $\cos(2ix) = 2\cos^2(ix) - 1$ or $\boldsymbol{\cosh 2x = 2\cosh^2 x - 1}$

Cor. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

We know that $\cos 2\theta = 1 - 2\sin^2\theta$

Putting $\theta = ix$, we get $\cos(2ix) = 1 - 2\sin^2(ix)$

or $\boldsymbol{\cosh 2x} = 1 - 2(i \sinh x)^2 = \boldsymbol{1 + 2\sinh^2 x}$

Cor. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

We know that $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$

Putting $\theta = ix$, we get $\cos(2ix) = \dfrac{1 - \tan^2(ix)}{1 + \tan^2(ix)} = \dfrac{1 - (i \tanh x)^2}{1 + (i \tanh x)^2}$ or $\boldsymbol{\cosh 2x = \dfrac{1 + \tanh^2 x}{1 - \tanh^2 x}}$

(*c*) We know that $$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Putting $\theta = ix$, we get $$\tan(2ix) = \frac{2\tan(ix)}{1-\tan^2(ix)}$$

$$i\tanh 2x = \frac{2i\tanh x}{1-(i\tanh x)^2} = \frac{2i\tanh x}{1+\tanh^2 x}$$

$\therefore$ $$\mathbf{\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}}.$$

4. Prove that (*a*) $\sinh 3x = 3\sinh x + 4\sinh^3 x$.

(*b*) $\cosh 3x = 4\cosh^3 x - 3\cosh x$ (*c*) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$.

Proof. (*a*) We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Putting $\theta = ix$, we get $\sin(3ix) = 3\sin(ix) - 4\sin^3(ix)$

or $i\sinh 3x = 3i\sinh x - 4(i\sinh x)^3$

or $i\sinh 3x = 3i\sinh x + 4i\sinh^3 x$ $[\because\ i^3 = -i]$

or $\mathbf{\sinh 3x = 3\sinh x + 4\sinh^3 x}$

(*b*) We know that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Putting $\theta = ix$, we get $\cos(3ix) = 4\cos^3(ix) - 3\cos(ix)$

or $\mathbf{\cosh 3x = 4\cosh^3 x - 3\cosh x}$

(*c*) We know that $$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

Putting $\theta = ix$, we get $$\tan(3ix) = \frac{3\tan(ix) - \tan^3(ix)}{1-3\tan^2(ix)}$$

or $$i\tanh 3x = \frac{3\cdot i\tanh x - (i\tanh x)^3}{1-3(i\tanh x)^2}$$

or $$i\tanh 3x = \frac{3\cdot i\tanh x + i\tanh^3 x}{1+3\tanh^2 x}$$ or $$\mathbf{\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}}.$$

5. Prove that:

(*i*) $2\sinh A\cosh B = \sinh(A+B) + \sinh(A-B)$

(*ii*) $2\cosh A\sinh B = \sinh(A+B) - \sinh(A-B)$

(*iii*) $2\cosh A\cosh B = \cosh(A+B) + \cosh(A-B)$

(*iv*) $2\sinh A\sinh B = \cosh(A+B) - \cosh(A-B)$

Proof. We shall prove only the last result.

The first three are left as an exercise for the student.

We know that $2\sin x\sin y = \cos(x-y) - \cos(x+y)$

Putting $x = iA$; $y = iB$, we get $2\sin(iA)\,.\,\sin(iB) = \cos i(A-B) - \cos i(A+B)$

or $2\,.\,i\sinh A\,.\,i\sinh B = \cosh(A-B) - \cosh(A+B)$

or $-2\sinh A\sinh B = \cosh(A-B) - \cosh(A+B)$ $[\because\ i^2 = -1]$

or $2\sinh A\sinh B = \cosh(A+B) - \cosh(A-B)$

6. Prove that:

(*i*) $\sinh C + \sinh D = 2 \sinh \dfrac{C+D}{2} \cosh \dfrac{C-D}{2}$

(*ii*) $\sinh C - \sinh D = 2 \cosh \dfrac{C+D}{2} \sinh \dfrac{C-D}{2}$

(*iii*) $\cosh C + \cosh D = 2 \cosh \dfrac{C+D}{2} \cosh \dfrac{C-D}{2}$

(*iv*) $\cosh C - \cosh D = 2 \sinh \dfrac{C+D}{2} \sinh \dfrac{C-D}{2}$

Proof. We shall prove only the last result. The first three are left as an exercise for the student.

We know that $\cos x - \cos y = 2 \sin \dfrac{x+y}{2} \sin \dfrac{y-x}{2}$

Putting $x = i$A and $y = i$B, we get

$$\cos(iA) - \cos(iB) = 2 \sin\left(i\,\frac{A+B}{2}\right) \sin\left(i\,\frac{B-A}{2}\right)$$

$$\Rightarrow \qquad \cosh A - \cosh B = 2i \sinh \frac{A+B}{2} \,.\, i \sinh \frac{B-A}{2}$$

$$= -2 \sinh \frac{A+B}{2} \sinh \frac{B-A}{2} = 2 \sinh \frac{A+B}{2} \sinh \frac{B-A}{2}$$

$[\because \quad \sinh(-x) = -\sinh x]$

7. Prove that:

$$\tanh(x+y+z) = \frac{\tanh x + \tanh y + \tanh z + \tanh x \tanh y \tanh z}{1 + \tanh x \tanh y + \tanh y \tanh z + \tanh z \tanh x}$$

Proof. We know that, $\tan(\alpha+\beta+\gamma) = \dfrac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}$

Putting $\alpha = ix$; $\beta = iy$; $\gamma = iz$, we get

$$\tan i(x+y+z) = \frac{\tan(ix) + \tan(iy) + \tan(iz) - \tan(ix)\tan(iy)\tan(iz)}{1 - \tan(ix)\tan(iy) - \tan(iy)\tan(iz) - \tan(iz)\tan(ix)}$$

$$i \tanh(x+y+z) = \frac{i\tanh x + i\tanh y + i\tanh z - i\tanh x \,.\, i\tanh y \,.\, i\tanh z}{1 - i\tanh x \,.\, i\tanh y - i\tanh y \,.\, i\tanh z - i\tanh z \,.\, i\tanh x}$$

or
$$\tanh(x+y+z) = \frac{\tanh x + \tanh y + \tanh z + \tanh x \tanh y \tanh z}{1 + \tanh x \tanh y + \tanh y \tanh z + \tanh z \tanh x}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Separate into real and imaginary parts*

(*a*) *sin (x + iy)* (*b*) *cos (x + iy)* **(P.T.U., Dec. 2004)**

(*c*) *tan (x + iy)* (*d*) *cot (x + iy)*

(*e*) *sec (x + iy)* **(P.T.U., May 2004)** (*f*) *cosec (x + iy).*

Sol. (*a*)
$$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + \cos x \cdot i \sinh y = \sin x \cosh y + i \cdot \cos x \sinh y$$

(*b*)
$$\cos(x+iy) = \cos x \cos iy - \sin x \sin iy$$
$$= \cos x \cosh y - \sin x \cdot i \sinh y = \cos x \cosh y - i \cdot \sin x \sinh y$$

(*c*)
$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$
$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} \qquad \left[\begin{array}{l} \because \ 2\sin A \cos B = \sin(A+B) + \sin(A-B) \\ \ \ \ 2\cos A \cos B = \cos(A+B) + \cos(A-B) \end{array}\right]$$
$$= \frac{\sin 2x + i \cdot \sinh 2y}{\cos 2x + \cosh 2y} = \frac{\sin 2x}{\cos 2x + \cosh 2y} + i \cdot \frac{\sinh 2y}{\cos 2x + \cosh 2y}.$$

(*d*)
$$\cot(x+iy) = \frac{\cos(x+iy)}{\sin(x+iy)} = \frac{2\cos(x+iy)\sin(x-iy)}{2\sin(x+iy)\sin(x-iy)}$$
$$= \frac{\sin 2x - \sin 2iy}{\cos 2iy - \cos 2x} \qquad \left[\begin{array}{l} \because \ 2\cos A \sin B = \sin(A+B) - \sin(A-B) \\ \ \ \ 2\sin A \sin B = \cos(A-B) - \cos(A+B) \end{array}\right]$$
$$= \frac{\sin 2x - i\,.\sinh 2y}{\cosh 2y - \cos 2x} = \frac{\sin 2x}{\cosh 2y - \cos 2x} - i\,.\,\frac{\sinh 2y}{\cosh 2y - \cos 2x}$$

(*e*)
$$\sec(x+iy) = \frac{1}{\cos(x+iy)} = \frac{2\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$
$$= \frac{2(\cos x \cos iy + \sin x \sin iy)}{\cos 2x + \cos 2iy} = \frac{2(\cos x \cosh y + \sin x \cdot i \sinh y)}{\cos 2x + \cosh 2y}$$
$$= \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y} + i \cdot \frac{2\sin x\,.\sinh y}{\cos 2x + \cosh 2y}$$

(*f*)
$$\operatorname{cosec}(x+iy) = \frac{1}{\sin(x+iy)} = \frac{2\sin(x-iy)}{2\sin(x+iy)\sin(x-iy)}$$
$$= \frac{2(\sin x \cos iy - \cos x \sin iy)}{\cos 2iy - \cos 2x} = \frac{2(\sin x \cosh y - \cos x\, i \sinh y)}{\cosh 2y - \cos 2x}$$
$$= \frac{2\sin x \cosh y}{\cosh 2y - \cos 2x} - i\,.\,\frac{2\cos x \sinh y}{\cosh 2y - \cos 2x}.$$

Example 2. *Separate the following into real and imaginary parts* :

(*a*) *sinh (x + iy)* (*b*) *cosh (x + iy)*
(*c*) *tanh (x + iy)* (*d*) *coth (x + iy)*
(*e*) *sech (x + iy)* (*f*) *cosech (x + iy).*

Sol. (*a*)
$$\sinh(x+iy) = \frac{1}{i}\sin i(x+iy) \qquad [\because \ i\sinh\theta = \sin i\theta]$$
$$= \frac{i}{i^2}\sin(ix-y) = -i(\sin ix \cos y - \cos ix \sin y)$$
$$= -i(i\sinh x \cos y - \cosh x \sin y) = \sinh x \cos y + i\cosh x \sin y$$

(*b*) $\cosh(x+iy) = \cos i(x+iy)$ $[\because \cosh\theta = \cos i\theta]$

$$= \cos(ix - y) = \cos ix \cos y + \sin ix \sin y = \cosh x \cos y + i \sinh x \sin y$$

(*c*) $\tanh(x+iy) = \dfrac{1}{i} \tan i(x+iy)$ $[\because i\tanh\theta = \tan i\theta]$

$$= \frac{i}{i^2}\tan(ix-y) = -i\,\frac{\sin(ix-y)}{\cos(ix-y)} = -i\cdot\frac{2\sin(ix-y)\cos(ix+y)}{2\cos(ix-y)\cos(ix+y)}$$

$$= -i\,.\,\frac{\sin 2ix - \sin 2y}{\cos 2ix + \cos 2y} = -i\cdot\frac{i\sinh 2x - \sin 2y}{\cosh 2x + \cos 2y}$$

$$= \frac{\sinh 2x}{\cosh 2x + \cos 2y} + i\cdot\frac{\sin 2y}{\cosh 2x + \cos 2y}.$$

(*d*) $\coth(x+iy) = \dfrac{\cosh(x+iy)}{\sinh(x+iy)} = \dfrac{\cos i(x+iy)}{\frac{1}{i}\sin i(x+iy)} = i\cdot\dfrac{\cos(ix-y)}{\sin(ix-y)}$

$$= i\cdot\frac{2\sin(ix+y)\cos(ix-y)}{2\sin(ix+y)\sin(ix-y)} = i\cdot\frac{\sin 2ix + \sin 2y}{\cos 2y - \cos 2ix} = i\cdot\frac{i\sinh 2x + \sin 2y}{\cos 2y - \cosh 2x}$$

$$= \frac{-\sinh 2x}{\cos 2y - \cosh 2x} + i\cdot\frac{\sin 2y}{\cos 2y - \cosh 2x}$$

$$= \frac{\sinh 2y}{\cosh 2x - \cos 2y} - i\cdot\frac{\sin 2y}{\cosh 2x - \cos 2y}.$$

(*e*) $\operatorname{sech}(x+iy) = \dfrac{1}{\cosh(x+iy)} = \dfrac{1}{\cos i(x+iy)}$

$$= \frac{1}{\cos(ix-y)} = \frac{2\cos(ix+y)}{2\cos(ix+y)\cos(ix-y)} = \frac{2(\cos ix\cos y - \sin ix \sin y)}{\cos 2ix + \cos 2y}$$

$$= \frac{2(\cosh x\cos y - i\sinh x\sin y)}{\cosh 2x + \cos 2y} = \frac{2\cosh x\cos y}{\cosh 2x + \cos 2y} - i\cdot\frac{2\sinh x \sin y}{\cosh 2x + \cos 2y}.$$

(*f*) $\operatorname{cosech}(x+iy) = \dfrac{1}{\sinh(x+iy)} = \dfrac{1}{\frac{1}{i}\sin i(x+iy)} = \dfrac{i}{\sin(ix-y)}$

$$= i\cdot\frac{2\sin(ix+y)}{2\sin(ix+y)\sin(ix-y)}$$

$$= i\cdot\frac{2(\sin ix\cos y + \cos ix \sin y)}{\cos 2y - \cos 2ix} = i\cdot\frac{2(i\sinh x\cos y + \cosh x\sin y)}{\cos 2y - \cosh 2x}$$

$$= -\frac{2\sinh x\cos y}{\cos 2y - \cosh 2x} + i\cdot\frac{2\cosh x \sin y}{\cos 2y - \cosh 2x}$$

$$= \frac{2\sinh x\cos y}{\cosh 2x - \cos 2y} - i\cdot\frac{2\cosh x\sin y}{\cosh 2x - \cos 2y}.$$

Example 3. *If* $u = \log \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, *then prove that:*

(*i*) $\tanh \frac{u}{2} = \tan \frac{\theta}{2}$ **(P.T.U., May 2006)** (*ii*) $\cosh u = \sec \theta$

(*iii*) $\tanh u = \sin \theta$ (*iv*) $\sinh u = \tan \theta$. **(P.T.U., Dec. 2005)**

(*v*) $\theta = -i \log\left(\tan \frac{\pi}{4} + i \frac{u}{2}\right)$. **(P.T.U., May 2003)**

Sol. $u = \log \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

(*i*) $e^u = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) \Rightarrow e^{u/2} \cdot e^{u/2} = \dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} \Rightarrow \dfrac{e^{u/2}}{e^{-u/2}} = \dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$

By componendo and dividendo

$$\frac{e^{u/2}-e^{-u/2}}{e^{u/2}+e^{-u/2}} = \frac{\left(1+\tan\frac{\theta}{2}\right)-\left(1-\tan\frac{\theta}{2}\right)}{\left(1+\tan\frac{\theta}{2}\right)+\left(1-\tan\frac{\theta}{2}\right)} \Rightarrow \tanh \frac{u}{2} = \tan \frac{\theta}{2}$$

(*ii*)
$$\cosh u = \frac{1+\tanh^2 \frac{u}{2}}{1-\tanh^2 \frac{u}{2}} = \frac{1+\tan^2 \frac{\theta}{2}}{1-\tan^2 \frac{\theta}{2}} \qquad \text{[Using part } (i)]$$
$$= \frac{1}{\cos\theta} = \sec\theta.$$

(*iii*) We know that tanh
$$u = \frac{2\tanh\frac{u}{2}}{1+\tanh^2\frac{u}{2}} = \frac{2\tan\theta/2}{1+\tan^2\theta/2} \qquad \text{[Using part } (i)]$$
$$= \sin\theta.$$

(*iv*) We know that sinh
$$u = \frac{2\tanh\frac{u}{2}}{1-\tanh^2\frac{u}{2}} = \frac{2\tan\theta/2}{1-\tan^2\theta/2} \qquad \text{[Using part } (i)]$$
$$= \tan\theta.$$

(*v*) From (*i*) part $\tanh \frac{u}{2} = \tan \frac{\theta}{2}$ (prove it)

$$\frac{1}{i}\tan\frac{iu}{2} = \frac{\dfrac{e^{i\theta/2}-e^{-i\theta/2}}{2i}}{\dfrac{e^{i\theta/2}+e^{-i\theta/2}}{2}} = \frac{1}{i}\,\frac{e^{i\theta/2}-e^{-i\theta/2}}{e^{i\theta/2}+e^{-i\theta/2}}$$

or
$$\frac{\tan \frac{iu}{2}}{1} = \frac{e^{i\theta/2} - e^{-i\theta/2}}{e^{i\theta/2} + e^{-i\theta/2}}$$

(By componendo-dividendo)

$$\frac{1 + \tan \frac{iu}{2}}{1 - \tan \frac{iu}{2}} = \frac{e^{i\theta/2} + e^{-i\theta/2} + e^{i\theta/2} - e^{-i\theta/2}}{e^{i\theta/2} + e^{-i\theta/2} - e^{i\theta/2} + e^{-i\theta/2}}$$

$$\tan\left(\frac{\pi}{4} + \frac{iu}{2}\right) = \frac{2e^{i\theta/2}}{2e^{-i\theta/2}} = e^{i\theta}$$

$\therefore$
$$i\theta = \log \tan\left(\frac{\pi}{4} + i\frac{u}{2}\right)$$

$\therefore$
$$\theta = \frac{1}{i} \log \tan\left(\frac{\pi}{4} + i\frac{u}{2}\right)$$

$\therefore$
$$\theta = -i \log \tan\left(\frac{\pi}{4} + i\frac{u}{2}\right).$$

Example 4. *If sin (A + iB) = x + iy, prove that*

(*i*) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$ **(P.T.U., Dec. 2002)**

(*ii*) $x^2 \operatorname{cosec}^2 A - y^2 \sec^2 A = 1$.

Sol. (*i*) $x + iy = \sin(A + iB) = \sin A \cos iB + \cos A \sin iB = \sin A \cosh B + i \cos A \sinh B$

Equating real and imaginary parts on both sides

$$x = \sin A \cosh B \; ; \; y = \cos A \sinh B \qquad \ldots(1)$$

From (1), $\dfrac{x}{\cosh B} = \sin A \; ; \quad \dfrac{y}{\sinh B} = \cos A$

Squaring and adding, $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A = 1$

(*ii*) Also from (1), $\dfrac{x}{\sin A} = \cosh B \; ; \; \dfrac{y}{\cos A} = \sinh B$

Squaring and subtracting, $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B = 1$

or $x^2 \operatorname{cosec}^2 A - y^2 \sec^2 A = 1$.

Example 5. *If x + iy = cosh (u + iv), show that*

(*i*) $\dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\sinh^2 u} = 1$ (*ii*) $x^2 \sec^2 v - y^2 \operatorname{cosec}^2 v = 1$. **(P.T.U., Jan. 2010)**

Sol. $x + iy = \cosh(u + iv)$

$= \cos i(u + iv)$ $[\because \cosh\theta = \cos i\theta]$

$= \cos(iu - v) = \cos iu \cos v + \sin iu \sin v = \cosh u \cos v + i \sinh u \sin v$

Equating real and imaginary parts, $x = \cosh u \cos v$; $y = \sinh u \sin v$ ...(1)

(*i*) From (1), $\dfrac{x}{\cosh u} = \cos v$, $\dfrac{y}{\sinh u} = \sin v$

Squaring and adding, $\dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\sinh^2 u} = \cos^2 v + \sin^2 v = 1$

(*ii*) From (1), $\dfrac{x}{\cos v} = \cosh u$; $\dfrac{y}{\sin v} = \sinh u$

Squaring and subtracting, $x^2 \sec^2 v - y^2 \operatorname{cosec}^2 v = \cosh^2 u - \sinh^2 u = 1$.

Example 6. *If $x + iy = \tan (A + iB)$; prove that*

(*i*) $x^2 + y^2 + 2x \cot 2A = 1$

(*ii*) $x^2 + y^2 - 2y \coth 2B + 1 = 0$

Sol. (*i*) $x + iy = \tan (A + iB)$

Changing i into $-i$, we get $x - iy = \tan (A - iB)$

Now $\tan 2A = \tan [(A + iB) + (A - iB)]$

$$= \frac{\tan (A + iB) + \tan (A - iB)}{1 - \tan (A + iB) \tan (A - iB)} = \frac{(x + iy) + (x - iy)}{1 - (x + iy)(x - iy)} = \frac{2x}{1 - (x^2 + y^2)}$$

or $\dfrac{1}{\cot 2A} = \dfrac{2x}{1 - (x^2 + y^2)}$ or $1 - (x^2 + y^2) = 2x \cot 2A$

or $x^2 + y^2 + 2x \cot 2A = 1$

(*ii*) $\tan (2iB) = \tan [(A + iB) - (A - iB)]$

$$= \frac{\tan (A + iB) - \tan (A - iB)}{1 + \tan (A + iB) \tan (A - iB)} = \frac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)} = \frac{2iy}{1 + x^2 + y^2}$$

or $i \tanh 2B = \dfrac{2iy}{1 + x^2 + y^2}$ or $\dfrac{1}{\coth 2B} = \dfrac{2y}{1 + x^2 + y^2}$

or $1 + x^2 + y^2 = 2y \coth 2B$

Hence $x^2 + y^2 - 2y \coth 2B + 1 = 0$.

Example 7. *If $\tan (\theta + i\phi) = \cos \alpha + i \sin \alpha = e^{i\alpha}$, prove that $\theta = \dfrac{n\pi}{2} + \dfrac{\pi}{4}$ and $\phi = \dfrac{1}{2} \log \tan \left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)$.*

(P.T.U., Dec. 2007)

Sol. $\tan (\theta + i\phi) = \cos \alpha + i \sin \alpha$ ...(1)

Changing i into $-i$, we get

$\tan (\theta - i\phi) = \cos \alpha - i \sin \alpha$ ...(2)

Now, $\tan 2\theta = \tan [(\theta + i\phi) + (\theta - i\phi)] = \dfrac{\tan (\theta + i\phi) + \tan (\theta - i\phi)}{1 - \tan (\theta + i\phi) \tan (\theta - i\phi)}$

$$= \frac{(\cos \alpha + i \sin \alpha) + (\cos \alpha - i \sin \alpha)}{1 - (\cos \alpha + i \sin \alpha)(\cos \alpha - i \sin \alpha)}$$

$$= \frac{2 \cos \alpha}{1 - (\cos^2 \alpha - i^2 \sin^2 \alpha)} = \frac{2 \cos \alpha}{1 - (\cos^2 \alpha + \sin^2 \alpha)}$$

$$= \frac{2\cos\alpha}{1-1} = \frac{2\cos\alpha}{0} = \infty = \tan\frac{\pi}{2}$$

$\therefore \qquad 2\theta = n\pi + \dfrac{\pi}{2}$ $\qquad [\because \tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha]$

or $\qquad \theta = \dfrac{n\pi}{2} + \dfrac{\pi}{4}$

Also $\qquad \tan 2i\phi = \tan[(\theta + i\phi) - (\theta - i\phi)]$

$$= \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)} = \frac{(\cos\alpha + i\sin\alpha) - (\cos\alpha - i\sin\alpha)}{1 + (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$= \frac{2i\sin\alpha}{1 + (\cos^2\alpha + \sin^2\alpha)} = \frac{2i\sin\alpha}{1+1} = i\sin\alpha$$

$$i\tanh 2\phi = i\sin\alpha \quad \text{or} \quad \tanh 2\phi = \sin\alpha$$

or $\qquad \dfrac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \dfrac{\sin\alpha}{1} \qquad$ or $\qquad \dfrac{e^{2\phi} + e^{-2\phi}}{e^{2\phi} - e^{-2\phi}} = \dfrac{1}{\sin\alpha}$

By componendo and dividendo

$$\frac{2e^{2\phi}}{2e^{-2\phi}} = \frac{1+\sin\alpha}{1-\sin\alpha} \quad \text{or} \quad e^{4\phi} = \frac{1+\sin\alpha}{1-\sin\alpha}$$

or $$e^{4\phi} = \frac{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} - 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}} = \left[\frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}\right]^2$$

or $$e^{2\phi} = \frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}} = \frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

Taking logarithms of both sides $\quad \log e^{2\phi} = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)$ or $2\phi = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)$

$\therefore \qquad \phi = \dfrac{1}{2}\log\tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right).$

Example 8. *Separate into real and imaginary parts log sin (x + iy).*

Sol. Log sin $(x + iy) = \log(\sin x\cos iy + \cos x\sin iy)$

$= \log(\sin x\cosh y + i\cos x\sinh y)$

$= \log(\alpha + i\beta)$, where $\alpha = \sin x\cosh y$, $\beta = \cos x\sinh y$

$$= \frac{1}{2}\log(\alpha^2 + \beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$$

$$= \frac{1}{2}\log(\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y) + i\tan^{-1}\left(\frac{\cos x\sinh y}{\sin x\cosh y}\right)$$

$$= \frac{1}{2}\log\left[\frac{1-\cos 2x}{2}\cdot\frac{\cosh 2y + 1}{2} + \frac{1+\cos 2x}{2}\cdot\frac{\cosh 2y - 1}{2}\right] + i\tan^{-1}(\cot x\tanh y)$$

$$= \frac{1}{2} \log \left[\frac{1}{4} (2 \cosh 2y - 2 \cos 2x) \right] + i \tan^{-1} (\cot x \tanh y)$$

$$= \frac{1}{2} \log \left[\frac{1}{2} (\cosh 2y - \cos 2x) \right] + i \tan^{-1} (\cot x \tanh y).$$

Example 9. (*a*) *Find all values of z such that* $\sinh z = e^{\frac{i\pi}{3}}$

(*b*) *Find all the roots of* $\sinh z = i$. **(P.T.U., May 2003)**

(*c*) *Find all values of z such that* $\sqrt{2} \sin z = \cosh \beta + i \sinh \beta$; β *real.*

Sol. (*a*) Let $z = x + iy$

$$\sinh (x + iy) = e^{\frac{i\pi}{3}}$$

$$\sinh x \cos y + i \cosh x \sin y = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + i \frac{\sqrt{3}}{2}$$

Equating real and imaginary parts,

$$\sinh x \cos y = \frac{1}{2} \quad \Rightarrow \quad \sinh x = \frac{1}{2 \cos y} \qquad \ldots(1)$$

$$\cosh x \sin y = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \cosh x = \frac{\sqrt{3}}{2 \sin y} \qquad \ldots(2)$$

Squaring and subtracting (1) from (2),

$$\cosh^2 x - \sinh^2 x = \frac{3}{4 \sin^2 y} - \frac{1}{4 \cos^2 y}$$

$$1 = \frac{3}{4} \cdot \frac{1}{\sin^2 y} - \frac{1}{4 \cos^2 y}$$

or $$4 \sin^2 y \cos^2 y = 3 \cos^2 y - \sin^2 y$$

$$4 \sin^2 y - 4 \sin^4 y = 3 - 4 \sin^2 y$$

$$4 \sin^4 y - 8 \sin^2 y + 3 = 0$$

$$\sin^2 y = \frac{8 \pm \sqrt{64 - 48}}{8} = \frac{8 \pm 4}{8}$$

$$\sin^2 y = \frac{12}{8} = \frac{3}{2} \; ; \quad \sin^2 y = \frac{4}{8} = \frac{1}{2}$$

$$\sin^2 y = \frac{3}{2} \text{ is impossible} \quad \because \quad \text{for real } y \, ; \sin^2 y \le 1$$

$\therefore$ $$\sin^2 y = \frac{1}{2} \quad \therefore \quad \sin y = \pm \frac{1}{\sqrt{2}}$$

$$\sin y \ne -\frac{1}{\sqrt{2}}$$

$\because$ If $\sin y$ is –ve, then from (2) $\cosh x$ is also –ve which is impossible

$\therefore$ $$\sin y = \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4}$$

$\therefore$ general value of $y = n\pi + (-1)^n \frac{\pi}{4}$

Case I. *If n is even*

Then $y = n\pi + \frac{\pi}{4}$, $\cos y$ is +ve $\because$ when n is even $\cos(n\pi + \theta) = \cos\theta$ and $\cos y = \frac{1}{\sqrt{2}}$

From (1) and (2)

$$\sinh x = \frac{1}{\sqrt{2}} \quad \therefore \quad x = \sinh^{-1}\frac{1}{\sqrt{2}} = \log\left(\frac{1}{\sqrt{2}} + \sqrt{\frac{1}{2}+1}\right) = \log\frac{\sqrt{3}+1}{\sqrt{2}}$$

$$\cosh x = \frac{\sqrt{3}}{2 \cdot \frac{1}{\sqrt{2}}} = \frac{\sqrt{3}}{\sqrt{2}} \quad \therefore \quad x = \cosh^{-1}\sqrt{\frac{3}{2}} = \log\left(\sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}-1}\right) = \log\frac{\sqrt{3}+1}{\sqrt{2}}$$

$$\therefore \qquad z = x + iy = \log\frac{\sqrt{3}+1}{\sqrt{2}} + i\left(n\pi + \frac{\pi}{4}\right)$$

Case II. *If n is odd*, $y = n\pi - \frac{\pi}{4}$, $\cos y$ is –ve $\because$ if n is odd. $\therefore$ $\cos(n\pi - \theta) = -\cos\theta$ and

$$\cos y = \cos\left(n\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

From (1) and (2) $\sinh x = -\frac{1}{\sqrt{2}}$, $\cosh x = \sqrt{\frac{3}{2}}$

$$\therefore \qquad x = \sinh^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \log\left(-\frac{1}{\sqrt{2}} + \sqrt{\frac{1}{2}+1}\right) = \log\frac{\sqrt{3}-1}{\sqrt{2}}$$

$$\therefore \qquad z = x + iy = \log\frac{\sqrt{3}-1}{\sqrt{2}} + i\left(n\pi - \frac{\pi}{4}\right).$$

(*b*) $\sinh z = i$

$$\frac{1}{i}\sin iz = i \quad \text{or} \quad \sin iz = i^2 = -1$$

or $\sin i(x+iy) = -1$ or $\sin(ix - y) = -1$

$$\sin ix \cos y - \cos ix \sin y = -1$$

or $i \sinh x \cos y - \cosh x \sin y = -1$

Comparing real and imaginary parts

$$\sinh x \cos y = 0 \qquad \text{...(1)}$$

$$\cosh x \sin y = 1 \qquad \text{...(2)}$$

From (1) either $\sinh x = 0$ or $\cos y = 0$

i.e., $x = 0$

Substitute in (2), we get

$$\sin y = 1 = \sin\frac{\pi}{2}$$

i.e., $y = n\pi + (-1)^n \frac{\pi}{2}$

Case I. *If n is even*

$$y = n\pi + \frac{\pi}{2}$$

$$\therefore \qquad z = 0 + i\left(n\pi + \frac{\pi}{2}\right) = i\left(\frac{2n+1}{2}\pi\right)$$

$$\therefore \qquad y = 2n\pi \pm \frac{\pi}{2}$$

But $y \neq 2n\pi - \frac{\pi}{2}$

$\because$ If $y = 2n\pi - \frac{\pi}{2}$ then from (2) $\cosh x = -1$ which is impossible

Case II. *If n is odd*

$$y = n\pi - \frac{\pi}{2}$$

$$\therefore \quad z = i\left(n\pi - \frac{\pi}{2}\right)$$

$$= i\left(\frac{2n-1}{2}\pi\right)$$

$$\therefore \quad y = 2n\pi + \frac{\pi}{2}$$

$\therefore$ From (2) $\cosh x = 1$

$$\therefore \quad x = 0$$

$$\therefore \quad z = i\left(2n\pi + \frac{\pi}{2}\right) = i\left(m\pi + \frac{\pi}{2}\right), \text{ where } m \text{ is even}$$

which is same as in case I.

Hence, $$z = i\,\frac{2n+1}{2}\,\pi \text{ if } n \text{ is even}$$

$$= i\,\frac{2n-1}{2}\,\pi \text{ if } n \text{ is odd.}$$

(*c*) $$\sqrt{2}\,\sin z = \cosh\beta + i\sinh\beta$$

$$\sin z = \frac{1}{\sqrt{2}}\cosh\beta + \frac{1}{\sqrt{2}}\sinh\beta \qquad \{\because \cos i\beta = \cosh\beta,\ \sin i\beta = i\sinh\beta\}$$

$$= \sin\frac{\pi}{4}\cos i\beta + \cos\frac{\pi}{4}\sin i\beta$$

$$= \sin\left(\frac{\pi}{4} + i\beta\right)$$

$$\therefore \quad z = n\pi + (-1)^n\left(\frac{\pi}{4} + i\beta\right)$$

$$= \left[n\pi + (-1)^n\frac{\pi}{4}\right] + i\beta(-1)^n;\ n \in \text{I}.$$

Example 10. *If* $\tan(x + iy) = \sin(u + iv)$ *prove that* $\dfrac{\sin 2x}{\sinh 2y} = \dfrac{\tan u}{\tanh v}$. **(P.T.U., Dec. 2003)**

Sol. $\tan(x + iy) = \sin(u + iv) = \sin u\cosh v + i\cos u\sinh v$

Change i to $-i$

$$\tan(x - iy) = \sin u\cosh v - i\cos u\sinh v$$

Adding $$\tan(x + iy) + \tan(x - iy) = 2\sin u\cosh v$$

Subtracting $$\tan(x + iy) - \tan(x - iy) = 2i\cos u\sinh v$$

Dividing the two $$\frac{\tan(x+iy) + \tan(x-iy)}{\tan(x+iy) - \tan(x-iy)} = \frac{\tan u}{i\tanh v}$$

$$\frac{\dfrac{\sin(x+iy)}{\cos(x+iy)} + \dfrac{\sin(x-iy)}{\cos(x-iy)}}{\dfrac{\sin(x+iy)}{\cos(x+iy)} - \dfrac{\sin(x-iy)}{\cos(x-iy)}} = \frac{1}{i}\,\frac{\tan u}{\tanh v}$$

or $$\frac{\sin(x+iy+x-iy)}{\sin(x+iy-x+iy)} = \frac{1}{i}\frac{\tan u}{\tan v} \quad \text{or} \quad \frac{\sin(2x)}{\sin(2iy)} = \frac{1}{i}\frac{\tan u}{\tanh v}$$

or $$\frac{\sin 2x}{i\sinh 2y} = \frac{1}{i}\frac{\tan u}{\tanh v} \quad \text{or} \quad \frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}$$

6.16. INVERSE TRIGONOMETRICAL FUNCTIONS

As for a real variable x, we define inverse sine function as $y = \sin^{-1} x$ when $x = \sin y$

Similarly we define inverse sine function for a complex variable z as

$$\omega = \sin^{-1} z \text{ when } z = \sin\omega$$

Now, $$z = \sin\omega = \frac{e^{i\omega} - e^{-i\omega}}{2i} \qquad \text{(by def. of } \sin\omega)$$

or $$2iz = e^{i\omega} - e^{-i\omega}$$

Solve for $e^{i\omega}$ $$2iz = e^{i\omega} - \frac{1}{e^{i\omega}}$$

or $$(2iz)\, e^{i\omega} = e^{i2\omega} - 1$$

or $$e^{2i\omega} - (2iz)\, e^{i\omega} - 1 = 0$$

$\therefore$ $$e^{i\omega} = \frac{2iz \pm \sqrt{4i^2z^2 + 4}}{2} = iz \pm \sqrt{1 - z^2}$$

$\therefore$ $$i\omega = \log(iz \pm \sqrt{1 - z^2})$$

$$i\omega = \log(iz + \sqrt{1 - z^2}) \qquad \left[\begin{array}{l}\text{only +ve sign is taken} \because \pm\sqrt{1-z^2} \text{ is} \\ \text{covered by double value function } \sqrt{1-z^2}\end{array}\right]$$

$\therefore$ $$\omega = \frac{1}{i}\log(iz + \sqrt{1 - z^2})$$

$\therefore$ $$\omega = \sin^{-1} z = -i\log(iz + \sqrt{1 - z^2})$$

$\sin^{-1} z$ is defined for all values of z except

$$iz + \sqrt{1 - z^2} = 0 \qquad i.e., \qquad iz = -\sqrt{1 - z^2}$$

or $$(iz)^2 = 1 - z^2 \quad \text{or} \quad -z^2 = 1 - z^2 \quad \text{or} \quad 0 = 1, \quad \text{which is impossible}$$

$\therefore$ $$\mathbf{\sin^{-1} z = -i\log(iz + \sqrt{1 - z^2})}$$

Similarly other complex inverse functions are defined by the following :

$$\mathbf{\cos^{-1} z = -i\log(z + \sqrt{1 - z^2})}$$

$$\mathbf{\tan^{-1}} z = -\frac{i}{2}\log\frac{1+iz}{1-iz} = \mathbf{\frac{i}{2}\log\frac{i+z}{i-z}\ ;\ z \neq \pm i}$$

$$\mathbf{cosec^{-1}} z = \sin^{-1}\frac{1}{z} = \mathbf{-i\log\left(\frac{i+\sqrt{z^2-1}}{z}\right);\quad z \neq 0}$$

$$\mathbf{sec^{-1}}\, z = \cos^{-1}\frac{1}{z} = -\boldsymbol{i}\,\mathbf{log}\left(\frac{\mathbf{1}+\sqrt{\mathbf{1}-z^{2}}}{z}\right);\ z \neq\ \mathbf{0}$$

$$\mathbf{cot^{-1}}\, z = \tan^{-1}\frac{1}{z} = -\frac{\boldsymbol{i}}{\mathbf{2}}\,\mathbf{log}\,\frac{z+\boldsymbol{i}}{z-\boldsymbol{i}},\ z \neq \pm\,\boldsymbol{i}$$

We will give proofs of $\tan^{-1} z$ and $\operatorname{cosec}^{-1} z$

Remaining three *i.e.*, $\cos^{-1} z$, $\sec^{-1} z$, $\cot^{-1} z$ students can easily prove themselves

Let $\tan^{-1} z = \omega \quad \therefore \quad z = \tan\omega = \dfrac{\sin\omega}{\cos\omega}$

$$z = \frac{e^{i\omega} - e^{-i\omega}}{i(e^{i\omega} + e^{-i\omega})} \quad \text{or} \quad \frac{iz}{1} = \frac{e^{i\omega} - e^{-i\omega}}{e^{i\omega} + e^{-i\omega}}.$$

Apply componendo-dividendo

$$\frac{1+iz}{1-iz} = \frac{2e^{i\omega}}{2e^{-i\omega}} = e^{2i\omega}$$

Taking log of both sides

$$\therefore \quad 2i\omega = \log\frac{1+iz}{1-iz}$$

$$\therefore \quad \omega = \frac{1}{2i}\log\frac{1+iz}{1-iz} = \frac{-i}{2}\log\frac{1+iz}{1-iz} \qquad \mid \text{When } iz \neq 1 \text{ or } z \neq -i$$

or
$$\omega = \frac{-i}{2}\log\frac{i(-i+z)}{i(-i-z)} = -\frac{i}{2}\log\frac{i-z}{i+z} = \frac{i}{2}\log\frac{i+z}{i-z}, \qquad \text{where } z \neq i$$

Hence, $\tan^{-1} z = -\dfrac{i}{2}\log\dfrac{1+iz}{1-iz}$ or $\dfrac{i}{2}\log\dfrac{i+z}{i-z}$, $z \neq \pm i$

To prove $\operatorname{cosec}^{-1} z = -i\log\dfrac{i+\sqrt{z^2-1}}{z}$, $z \neq 0$

Let $\operatorname{cosec}^{-1} z = \omega \quad \therefore \quad z = \operatorname{cosec}\omega = \dfrac{1}{\sin\omega}$

$$\therefore \quad z = \frac{1}{\dfrac{e^{i\omega} - e^{-i\omega}}{2i}} \quad \text{or} \quad z = \frac{2i}{e^{i\omega} - e^{-i\omega}}$$

or $\quad ze^{i\omega} - ze^{-i\omega} - 2i = 0 \qquad$ Multiply by $e^{i\omega}$

or $\quad ze^{i2\omega} - z - 2i\,e^{i\omega} = 0 \qquad$ or $\qquad ze^{2(i\omega)} - 2i\,e^{(i\omega)} - z = 0$

Solve for $e^{i\omega}$; $\quad e^{i\omega} = \dfrac{2i \pm \sqrt{4i^2 + 4z^2}}{2z}$, $\quad z \neq 0 \quad$ or $\quad e^{i\omega} = \dfrac{i \pm \sqrt{z^2-1}}{z}$

Taking +ve sign $\quad e^{i\omega} = \dfrac{i+\sqrt{z^2-1}}{z}$

Taking log of both sides,

$$i\omega = \log\frac{i+\sqrt{z^2-1}}{z}, \quad z \neq 0$$

$\therefore$ $$\omega = \frac{1}{i}\log\frac{i+\sqrt{z^2-1}}{z}$$

or $$\omega = -i\log\frac{i+\sqrt{z^2-1}}{z}, \quad z \neq 0. \quad \textbf{Proved.}$$

6.17. INVERSE HYPERBOLIC FUNCTION

For a complex variable z:

(*a*) To prove $\sinh^{-1} z = \log\left(z+\sqrt{z^2+1}\right)$

Let $\sinh^{-1} z = \omega \quad \therefore \quad z = \sinh\omega$

$\therefore$ $$z = \frac{e^{\omega}-e^{-\omega}}{2} \qquad \text{or} \qquad 2z = e^{\omega} - \frac{1}{e^{\omega}}$$

or $e^{2\omega} - 2ze^{\omega} - 1 = 0$

Solve for e^{ω} $$e^{\omega} = \frac{2z \pm \sqrt{4z^2+4}}{2} = z + \sqrt{z^2+1}$$ (Taking +ve sign only)

$\therefore$ $$\omega = \log\left(z+\sqrt{z^2+1}\right) \quad \text{or} \quad \boldsymbol{\sinh^{-1} z = \log\left(z+\sqrt{z^2+1}\right)}$$

(*b*) To prove $\cosh^{-1} z = \log\left(z+\sqrt{z^2-1}\right)$

Let $\cosh^{-1} z = \omega \quad \therefore \quad z = \cosh\omega$

$\therefore$ $$z = \frac{e^{\omega}+e^{-\omega}}{2} \qquad \text{or} \qquad 2z = e^{\omega} + \frac{1}{e^{\omega}}$$

or $e^{2\omega} - 2z\,e^{\omega} + 1 = 0$

Solve for e^{ω}, $$e^{\omega} = \frac{2z \pm \sqrt{4z^2-4}}{2} = z + \sqrt{z^2-1}$$ (Taking +ve sign only)

$\therefore$ $$\omega = \log\left(z+\sqrt{z^2-1}\right)$$

or $$\boldsymbol{\cosh^{-1} z = \log\left(z+\sqrt{z^2-1}\right)}.$$

(*c*) To prove $\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}\ ; z \neq \pm 1$

Let $\tanh^{-1} z = \omega \quad \therefore \quad z = \tanh\omega = \dfrac{e^{\omega}-e^{-\omega}}{e^{\omega}+e^{-\omega}}$

$\therefore$ $$\frac{z}{1} = \frac{e^{\omega}-e^{-\omega}}{e^{\omega}+e^{-\omega}}$$

Apply componendo-dividendo.

$$\frac{1+z}{1-z} = \frac{2e^{\omega}}{2e^{-\omega}} = e^{2\omega}$$

$\therefore \qquad e^{2\omega} = \dfrac{1+z}{1-z} \quad \text{or} \quad 2\omega = \log \dfrac{1+z}{1-z}, \quad \text{where} \quad z \neq 1$

$\therefore \qquad \omega = \dfrac{1}{2} \log \dfrac{1+z}{1-z}, \quad z \neq 1$

We can also be put in the form $-\dfrac{1}{2} \log \dfrac{1-z}{1+z}$, where $z \neq -1$

$\therefore \qquad \omega = \dfrac{1}{2} \log \dfrac{1+z}{1-z}, \quad \text{where } z \neq \pm 1$

$\therefore \qquad \tanh^{-1} z = \dfrac{1}{2} \log \dfrac{1+z}{1-z}, \quad \text{where } z \neq \pm 1$

Readers can easily prove the remaining inverse hyperbolic functions

i.e., $\quad \mathbf{cosech^{-1}}\, z = \sinh^{-1} \dfrac{1}{z} = \mathbf{log} \dfrac{\mathbf{1}+\sqrt{\mathbf{1}+z^2}}{z}\ ; z \neq \mathbf{0}$

$$\mathbf{sech^{-1}}\, z = \cosh^{-1} \frac{1}{z} = \mathbf{log} \frac{\mathbf{1}+\sqrt{\mathbf{1}-z^2}}{z}\ ; z \neq \mathbf{0}$$

$$\mathbf{coth^{-1}}\, z = \tanh^{-1} \frac{1}{z} = \frac{\mathbf{1}}{\mathbf{2}}\ \mathbf{log} \frac{z+\mathbf{1}}{z-\mathbf{1}}\ ; z \neq \pm \mathbf{1}.$$

Example 11. *Separate into real and imaginary parts*

(*i*) *$\tan^{-1}(x + iy)$.* **(P.T.U., May 2006)**

(*ii*) *$\cos^{-1}(e^{i\theta})$; θ is an acute angle* **(P.T.U., May 2002, 2003, Dec. 2010)**

Sol. (*i*) Let $\quad \tan^{-1}(x+iy) = u + iv \qquad \ldots(1)$

then $\quad \tan^{-1}(x-iy) = u - iv \qquad \ldots(2)$

Adding (1) and (2), we have

$$2u = \tan^{-1}(x+iy) + \tan^{-1}(x-iy) = \tan^{-1} \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \tan^{-1} \frac{2x}{1-x^2-y^2}$$

$\therefore$ Real part $\qquad u = \dfrac{1}{2} \tan^{-1} \dfrac{2x}{1-x^2-y^2}$

Subtracting (2) from (1), we have

$$2iv = \tan^{-1}(x+iy) - \tan^{-1}(x-iy)$$

$$= \tan^{-1} \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = \tan^{-1} \frac{2iy}{1+x^2+y^2}$$

$\Rightarrow \qquad \tan 2iv = \dfrac{2iy}{1+x^2+y^2} \quad \Rightarrow \quad i \tanh 2v = \dfrac{2iy}{1+x^2+y^2}$

$\Rightarrow \quad$ Imaginary part $\quad v = \dfrac{1}{2} \tanh^{-1} \dfrac{2y}{1+x^2+y^2}$.

Hence, $\qquad \tan^{-1}(x+iy) = \dfrac{1}{2} \tan^{-1} \dfrac{2x}{1-x^2-y^2} + \dfrac{i}{2} \tan^{-1} \dfrac{2y}{1+x^2+y^2}$

(*ii*) Let $\cos^{-1}(e^{i\theta}) = x + iy$

$$e^{i\theta} = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$$

$\therefore$ $\cos\theta + i\sin\theta = \cos x \cosh y - i \sin x \sinh y$

Comparing real and imaginary parts, we get

$$\cos x \cosh y = \cos\theta \qquad \ldots(1)$$

$$\sin x \sinh y = -\sin\theta \qquad \ldots(2)$$

Squaring and adding $\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = 1$

$$(1 - \sin^2 x)\cosh^2 y + \sin^2 x \sinh^2 y = 1$$

or $\cosh^2 y - \sin^2 x(\cosh^2 y - \sinh^2 y) = 1$

or $1 + \sinh^2 y - \sin^2 x = 1 \quad \therefore \quad \sin^2 x = \sinh^2 y \qquad \ldots(3)$

Squaring (2), we get $\sin^2 x \sinh^2 y = \sin^2\theta$

$$\sin^2 x \,.\, \sin^2 x = \sin^2\theta \qquad \text{[From (3)]}$$

$$(\sin^2 x)^2 = \sin^2\theta \quad \therefore \quad \sin^2 x = \sin\theta \qquad [+\text{ve sign only} \because \theta \text{ is acute}]$$

$\therefore$ $\sin x = \sqrt{\sin\theta} \quad \therefore \quad x = \sin^{-1}(\sqrt{\sin\theta})$

From (2) $\sin x \sinh y = -\sin\theta$

$$\sqrt{\sin\theta}\,\sinh y = -\sin\theta \quad \therefore \quad \sinh y = -\sqrt{\sin\theta} \quad \therefore \quad y = \sinh^{-1}(-\sqrt{\sin\theta})$$

$\therefore$ $y = \log[-\sqrt{\sin\theta} + \sqrt{1+\sin\theta}\,] = \log[\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\,]$

$\therefore$ real part $= \sin^{-1}(\sqrt{\sin\theta})$

Imaginary part $= \log[\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\,]$.

Hence $\cos^{-1}(e^{i\theta}) = \sin^{-1}(\sqrt{\sin\theta}) + i\log(\sqrt{1+\sin\theta} - \sqrt{\sin\theta})$.

Example 12. *If* $\sin^{-1}(u + iv) = \alpha + i\beta$, *prove that* $\sin^2\alpha$ *and* $\cosh^2\beta$ *are the roots of the equation* $x^2 - (1 + u^2 + v^2)x + u^2 = 0$. **(P.T.U., May 2004)**

Sol. $\sin^{-1}(u + iv) = \alpha + i\beta$

$$u + iv = \sin(\alpha + i\beta) = \sin\alpha\cos(i\beta) + \cos\alpha\sin(i\beta) = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta$$

Comparing real and imaginary parts,

$$\sin\alpha\cosh\beta = u \qquad \ldots(1)$$

$$\cos\alpha\sinh\beta = v \qquad \ldots(2)$$

$$\begin{aligned} 1 + u^2 + v^2 &= 1 + \sin^2\alpha\cosh^2\beta + \cos^2\alpha\sinh^2\beta \\ &= 1 + \sin^2\alpha\cosh^2\beta + (1 - \sin^2\alpha)(\cosh^2\beta - 1) \\ &= 1 + \sin^2\alpha\cosh^2\beta + \cosh^2\beta - 1 - \sin^2\alpha\cosh^2\beta + \sin^2\alpha \\ &= \cosh^2\beta + \sin^2\alpha \qquad \ldots(3) \end{aligned}$$

Equation whose roots are $\sin^2\alpha$, $\cosh^2\beta$ is

$x^2 - x(\sin^2\alpha + \cosh^2\beta) + \sin^2\alpha\cosh^2\beta = 0$ [Using $x^2 - 5x + P = 0$]

or $x^2 - x(1 + u^2 + v^2) + u^2 = 0$ $|\because$ of (1) and (3)

Hence $\sin^2\alpha$, $\cosh^2\beta$ are the roots of

$$x^2 - x(1 + u^2 + v^2) + u^2 = 0.$$

Example 13. *Find all the values of* $\sin^{-1} 2$ *treating 2 as a complex number.* **(P.T.U., Dec. 2004)**

Sol. We have

$$\sin^{-1} z = -i \log (iz + \sqrt{1 - z^2})$$

For $z = 2$; $\qquad \sin^{-1} 2 = -i \log (2i + \sqrt{1 - 4})$

$\therefore \qquad \sin^{-1} 2 = -i \log (2i + \sqrt{3}i) = -i \log (2 + \sqrt{3})i = -i \{\log (2 + \sqrt{3}) + \log i\}$

We know that

$$\log i = \log |i| + i [2n\pi + \arg i], \quad \text{where } n \text{ is an integer}$$

$$= \log 1 + i \left[2n\pi + \frac{\pi}{2} \right] = i \left[2n + \frac{1}{2} \right] \pi$$

$$\therefore \qquad \sin^{-1} 2 = -i \left\{ \log (2 + \sqrt{3}) + \frac{4n+1}{2} \pi i \right\}$$

$$= -i \log (2 + \sqrt{3}) - \frac{4n+1}{2} \pi i^2$$

$$= -i \cosh^{-1} 2 + \frac{4n+1}{2} \pi \qquad \Big| \because \cosh^{-1} 2 = \log [2 + \sqrt{2^2 - 1}] = \log (2 + \sqrt{3})$$

by def.

Hence $\quad \sin^{-1} 2 = \dfrac{4n+1}{2} \pi - i \cosh^{-1} 2.$

TEST YOUR KNOWLEDGE

1. Prove that
   (*i*) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$; n being a positive integer.
   (*ii*) $\left(\dfrac{1 + \tanh x}{1 - \tanh x} \right)^3 = \cosh 6x + \sinh 6x.$
2. If $y = \log \tan x$, show that $\sinh ny = \dfrac{1}{2} (\tan^n x - \cot^n x)$.
3. If $\tan y = \tan \alpha \tanh \beta$ and $\tan z = \cot \alpha \tanh \beta$, prove that $\tan (y + z) = \sinh 2\beta \operatorname{cosec} 2\alpha$.
4. If $\tan \theta = \tanh x \cot y$ and $\tan \phi = \tanh x \tan y$, prove that $\dfrac{\sin 2\theta}{\sin 2\phi} = \dfrac{\cosh 2x + \cos 2y}{\cosh 2x - \cos 2y}$.
5. If $c \cosh (\theta + i\phi) = x + iy$, prove that
   (*i*) $x^2 \operatorname{sech}^2 \theta + y^2 \operatorname{cosech}^2 \theta = c^2$ (*ii*) $x^2 \sec^2 \phi - y^2 \operatorname{cosec}^2 \phi = c^2$.
6. If $\tan (x + iy) = A + iB$, show that $\dfrac{A}{B} = \dfrac{\sin 2x}{\sinh 2y}$.
7. If $\sin (\theta + i\phi) = \rho(\cos \alpha + i \sin \alpha)$, prove that
   (*i*) $\rho^2 = \dfrac{1}{2} (\cosh 2\phi - \cos 2\theta)$ (*ii*) $\tan \alpha = \tanh \phi \cot \theta$.
8. If $\sin (\theta + i\phi) = \cos \alpha + i \sin \alpha$, prove that $\cos^2 \theta = \pm \sin \alpha$.
9. If $\cos (\theta + i\phi) = \cos \alpha + i \sin \alpha$, prove that
   (*i*) $\sin^2 \theta = \pm \sin \alpha$ (*ii*) $\cos 2\theta + \cosh 2\phi = 2$.
10. If $\sin (\theta + i\phi) = \tan \alpha + i \sec \alpha$, show that $\cos 2\theta \cosh 2\phi = 3$.
11. If $\cos (\theta + i\phi) = R(\cos \alpha + i \sin \alpha)$, prove that $e^{2\phi} = \dfrac{\sin (\theta - \alpha)}{\sin (\theta + \alpha)}$.

12. If $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$, show that $e^{2\phi} = \pm \cot\dfrac{\alpha}{2}$ and $2\theta = \left(n + \dfrac{1}{2}\right)\pi + \alpha$.

13. Prove that: $\tan\dfrac{u + iv}{2} = \dfrac{\sin u + i\sinh v}{\cos u + \cosh v}$.

14. If $\tan(x + iy) = \cosh(\alpha + i\beta)$, prove that $\tanh\alpha\tan\beta = \operatorname{cosec} 2x \sinh 2y$.

15. If $C\tan(x + iy) = A + iB$, prove that $\tan 2x = \dfrac{2CA}{C^2 - A^2 - B^2}$.

16. Prove that $(1 + \cosh x + \sinh x)^n = 2^n \cosh^n \dfrac{x}{2}\left(\cos\dfrac{nx}{2} + \sinh\dfrac{nx}{2}\right)$.

17. If $\cosh x = \sec\theta$, prove that

(*i*) $\tanh^2\dfrac{x}{2} = \tan^2\dfrac{\theta}{2}$ (*ii*) $x = \log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$.

18. If $\tan\dfrac{x}{2} = \tanh\dfrac{u}{2}$, prove that

(*i*) $\cos x \cosh u = 1$ (*ii*) $\tan x = \sinh u$ (*iii*) $u = \log\tan\left(\dfrac{\pi}{4} + \dfrac{x}{4}\right)$.

[**Hint:** see S.E. 3 (*i*), (*ii*), (*iv*) parts].

19. If $x = 2\cos\alpha\cosh\beta$, $y = 2\sin\alpha\sinh\beta$, prove that $\sec(\alpha + i\beta) + \sec(\alpha - i\beta) = \dfrac{4x}{x^2 + y^2}$.

20. If $\sin[\log(A + iB)] = x + iy$, show that $\dfrac{x^2}{\sin^2 u} - \dfrac{y^2}{\cos^2 u} = 1$, where $A^2 + B^2 = e^{2u}$.

21. Separate into real and imaginary parts :

(*i*) $e^{\cosh(x + iy)}$ (*ii*) $\sin^2(x + iy)$ (*iii*) $\log\cos(x + iy)$.

22. If $\tan(x + iy) = \theta + i\phi$, prove that $\theta^2 + \phi^2 = \dfrac{\cosh^2 y - \cos^2 x}{\cosh^2 y - \sin^2 x}$.

23. If $x + iy = \cos(u + iv)$, show that

(*i*) $(1 + x)^2 + y^2 = (\cosh v + \cos u)^2$ (*ii*) $(1 - x)^2 + y^2 = (\cosh v - \cos u)^2$.

24. If $\cos^{-1}(u + iv) = \alpha + i\beta$, prove that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation

$$x^2 - (1 + u^2 + v^2)x + u^2 = 0.$$

25. Prove that (*i*) $\tan^{-1}(e^{i\theta}) = \dfrac{n\pi}{2} + \dfrac{\pi}{4} - \dfrac{i}{2}\log\tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$.

(*ii*) $\sin^{-1}(\cos\theta + i\sin\theta) = \cos^{-1}(\sqrt{\sin\theta}) + i\log(\sqrt{\sin\theta} + \sqrt{1 + \sin\theta})$; $0 < \theta < \dfrac{\pi}{2}$

(P.T.U., Dec. 2006, 2012, May 2014)

[**Hint:** $\sin^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}(e^{i\theta})$; consult S.E. 11(*ii*)]

26. Find $\tanh x$ if $5\sinh x - \cosh x = 5$.

[**Hint:** Divide both sides by $\cosh x$, square, replace $\text{sech}^2 x$ by $(1 - \tanh^2 x)$ and solve for $\tanh x$]

27. If $\cos^{-1}(x + iy) = \alpha + i\beta$, show that

(*i*) $x^2\sec^2\alpha - y^2\operatorname{cosec}^2\alpha = 1$ (*ii*) $x^2\,\text{sech}^2\beta + y^2\,\text{cosech}^2\beta = 1$.

ANSWERS

21. (*i*) $e^{\cosh x \cos y}$ [cos (sinh x sin y) + i sin (sinh x sin y)]

(*ii*) $\frac{1}{2}$ $[(1 - \cos 2x \cosh 2y) + i \sin 2x \sinh 2y]$

(*iii*) $\frac{1}{2} \log \left[\frac{1}{2}(\cos 2x + \cosh 2y)\right] - i \tan^{-1} (\tan x \tanh y)\Bigg]$.

26. $\frac{4}{5}, -\frac{3}{5}$.

6.18. C + iS METHOD OF SUMMATION

This method can be applied in finding out the sums of the series of the form

$$a_0 \cos \alpha + a_1 \cos (\alpha + \beta) + a_2 \cos (\alpha + 2\beta) + ...$$

and
$$a_0 \sin \alpha + a_1 \sin (\alpha + \beta) + a_2 \sin (\alpha + 2\beta) + ...$$

only when the sum of the series $a_0 + a_1x + a_2x^2 + ...$ is known. The above series may be finite or infinite.

Method. Let $\quad C = a_0 \cos \alpha + a_1 \cos (\alpha + \beta) + a_2 \cos (\alpha + 2\beta) + ...$

and $\quad S = a_0 \sin \alpha + a_1 \sin (\alpha + \beta) + a_2 \sin (\alpha + 2\beta) + ...$

If we want to find the sum of the sine series, the series of cosines is called the *companion or auxiliary series*. In case, the sum of the cosine series is required, the series of sines is called the companion or auxiliary series.

Multiplying the series of sines by i and adding to the sum of cosines, we get the series of complex numbers as

$$\begin{aligned} C + iS &= a_0 (\cos \alpha + i \sin \alpha) + a_1 [\cos (\alpha + \beta) + i \sin (\alpha + \beta)] + a_2 [\cos (\alpha + 2\beta) + i \sin (\alpha + 2\beta)] + ... \\ &= a_0 e^{i\alpha} + a_1 e^{i(\alpha + \beta)} + a_2 e^{i(\alpha + 2\beta)} + ... \qquad [\because \ \cos \theta + i \sin \theta = e^{i\theta}] \\ &= e^{i\alpha} [a_0 + a_1 e^{i\beta} + a_2 e^{i2\beta} + ...] \\ &= e^{i\alpha} [a_0 + a_1x + a_2x^2 + ...], \text{ where } x = e^{i\beta} \\ &= e^{i\alpha} . f(x) \end{aligned}$$

The series represented by $f(x)$ can be summed up if it is in any one of the following forms :

(*i*) series in G.P. or its modification.

(*ii*) Binomial series or one which can be reduced to it.

(*iii*) exponential series, *i.e.*, depending on the expansion of e^x or e^{-x}

(*iv*) series which take the form of the expansions of either sin x, cos x, cosh x or sinh x.

(*v*) logarithmic series depending on the expansion of log $(1 + x)$ or log $(1 - x)$.

(*vi*) Gregory's series.

The sum so obtained can be expressed in the form X + iY, where X and Y are real. Equating the real and imaginary parts, we get C and S.

The following results will be frequently used:

1. Sum to n terms of an A.P.

$$a + (a + d) + (a + 2d) + ... + [a + (n - 1)d] = \frac{n}{2} [2a + (n - 1)d].$$

2. Sum to n terms of a G.P. $\quad a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(1 - r^n)}{1 - r}$

Sum to infinity of a G.P. (when $r < 1$ numerically) $= \dfrac{a}{1 - r}$.

3. $e^{i\theta} = \cos \theta + i \sin \theta$ $\qquad$ **4.** $e^{-i\theta} = \cos \theta - i \sin \theta$

5. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots \infty$

6. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots \infty$

7. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots \infty$

8. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots \infty$

9. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots \infty$

10. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots \infty$

11. $\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \infty$

12. $\log (1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots \infty\right)$

13. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots \infty$

14. $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\ldots \infty = \frac{1}{2} \log \frac{1+x}{1-x}$

15. $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots\ldots + x^n$ when n is a +ve integer.

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots\ldots \infty$ when n is a negative integer or a fraction and $|x| < 1$

$(1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \ldots\ldots$

$(1 - x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \ldots\ldots$

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots$

Note. The students should bear in mind that in forming auxiliary series, sines or cosines of multiple angles (*i.e.,* of the form $\sin n\theta$, $\cos n\theta$) should be replaced by cosines or sines respectively whereas sines or cosines with powers, if any, will remain the same.

ILLUSTRATIVE EXAMPLES

1. Series depending on expansion of e^x, e^{-x}

Example 1. *Sum the following series* $1 + \frac{\cos \alpha}{\cos \alpha} + \frac{\cos 2\alpha}{2!\cos^2 \alpha} + \frac{\cos 3\alpha}{3!\cos^3 \alpha} + \ldots\ldots \infty$.

Sol. Let
$$C = 1 + \frac{\cos \alpha}{\cos \alpha} + \frac{\cos 2\alpha}{2!\cos^2 \alpha} + \frac{\cos 3\alpha}{3!\cos^3 \alpha} + \ldots\ldots \infty$$

$\therefore$
$$S = 0 + \frac{\sin \alpha}{\cos \alpha} + \frac{\sin 2\alpha}{2!\cos^2 \alpha} + \frac{\sin 3\alpha}{3!\cos^3 \alpha} + \ldots\ldots \infty \qquad \text{(See Note above)}$$

$\therefore$
$$C + iS = 1 + \frac{1}{\cos \alpha} (\cos \alpha + i \sin \alpha) + \frac{1}{2!\cos^2 \alpha} \times (\cos 2\alpha + i \sin 2\alpha) + \frac{1}{3!\cos^3 \alpha} (\cos 3\alpha + i \sin 3\alpha) + \ldots\ldots \infty$$

$$= 1 + \frac{1}{\cos\alpha} \cdot e^{i\alpha} + \frac{1}{2!\cos^2\alpha} e^{2i\alpha} + \frac{1}{3!\cos^3\alpha} e^{3i\alpha} + \ldots\ldots \infty \quad [\because \quad \cos\theta + i\sin\theta = e^{i\theta}]$$

$$= 1 + \sec\alpha\, e^{i\alpha} + \frac{1}{2!} \sec^2\alpha \,.\, e^{2i\alpha} + \frac{1}{3!} \sec^3\alpha \,.\, e^{3i\alpha} + \ldots\ldots \infty$$

$$= 1 + (\sec\alpha \,.\, e^{i\alpha}) + \frac{1}{2!} (\sec\alpha \,.\, e^{i\alpha})^2 + \frac{1}{3!} (\sec\alpha \,.\, e^{i\alpha})^3 + \ldots\ldots \infty$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots. \infty, \text{ where } z = \sec\alpha \,.\, e^{i\alpha}$$

$$= e^z = e^{\sec\alpha \,.\, e^{i\alpha}} = e^{\sec\alpha(\cos\alpha + i\sin\alpha)}$$

$$= e^{1 + i\tan\alpha} = e \,.\, e^{i \,.\, \tan\alpha} = e\,[\cos(\tan\alpha) + i\sin(\tan\alpha)] \qquad [\because \quad e^{i\theta} = \cos\theta + i\sin\theta]$$

Equating real parts, we get $C = e\cos(\tan\alpha)$.

Example 2. *Sum to infinity*

(*i*) $\sin\alpha + x\sin(\alpha+\beta) + \dfrac{x^2}{2!}\sin(\alpha+2\beta) + \ldots..$

(*ii*) $\cos\alpha + x\cos(\alpha+\beta) + \dfrac{x^2}{2!}\cos(\alpha+2\beta) + \ldots\ldots$

(*iii*) $1 + x\cos\theta + \dfrac{x^2\cos 2\theta}{2!} + \dfrac{x^3\cos 3\theta}{3!} + \ldots\ldots$ **(P.T.U., Dec. 2003)**

(*iv*) $\cos\theta + \sin\theta \quad \cos 2\theta + \dfrac{\sin^2\theta}{1.2}\cos 3\theta + \ldots\ldots$ **(P.T.U., Dec. 2002)**

Sol. Let $\quad S = \sin\alpha + x\sin(\alpha+\beta) + \dfrac{x^2}{2!}\sin(\alpha+2\beta) + \ldots..$

and $\quad C = \cos\alpha + x\cos(\alpha+\beta) + \dfrac{x^2}{2!}\cos(\alpha+2\beta) + \ldots\ldots$

$$\therefore \quad C + iS = (\cos\alpha + i\sin\alpha) + x[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] + \frac{x^2}{2!}[\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)] + \ldots..$$

$$= e^{i\alpha} + xe^{i(\alpha+\beta)} + \frac{x^2}{2!} e^{i(\alpha+2\beta)} + \ldots\ldots. = e^{i\alpha}\left[1 + x \,.\, e^{i\beta} + \frac{x^2}{2!} e^{2i\beta} + \ldots\ldots\right]$$

$$= e^{i\alpha}\left[1 + z + \frac{z^2}{2!} + \ldots\ldots\ldots\right], \text{ where } z = xe^{i\beta}$$

$$= e^{i\alpha} \,.\, e^z = e^{i\alpha} \,.\, e^{xe^{i\beta}} = e^{i\alpha} \,.\, e^{x(\cos\beta + i\sin\beta)}$$

$$= e^{i\alpha + x\cos\beta + ix\sin\beta} = e^{x\cos\beta} \,.\, e^{i(\alpha + x\sin\beta)}$$

$$= e^{x\cos\beta}[\cos(\alpha + x\sin\beta) + i\sin(\alpha + x\sin\beta)]$$

Equating imaginary parts $\quad S = e^{x\cos\beta} \,.\, \sin(\alpha + x\sin\beta)$ …(1)

Equating real parts $\quad C = e^{e\cos\beta} \,.\, \cos(\alpha + x\sin\beta)$.

(*i*) and (*ii*) parts are proved.

(*iii*) Let $$C = 1 + x\cos\theta + \frac{x^2\cos 2\theta}{2\,!} + \frac{x^3\cos 3\theta}{3\,!} + \ldots\ldots \infty$$

$$S = x\sin\theta + \frac{x^2\sin 2\theta}{2\,!} + \frac{x^3\sin 3\theta}{3\,!} + \ldots\ldots \infty$$

$$C + iS = 1 + x(\cos\theta + i\sin\theta) + \frac{x^2}{2\,!}(\cos 2\theta + i\sin 2\theta) + \frac{x^3}{3\,!}(\cos 3\theta + i\sin 3\theta) + \ldots\ldots$$

$$= 1 + x\,e^{i\theta} + \frac{x^2}{2\,!}\,e^{i2\theta} + \frac{x^3}{3\,!}\,e^{i3\theta} + \ldots\ldots \infty$$

$$= 1 + t + \frac{t^2}{2\,!} + \frac{t^3}{3\,!} + \ldots\ldots \infty, \text{ where } xe^{i\theta} = t$$

$$= e^t = e^{xe^{i\theta}} = e^{x(\cos\theta + i\sin\theta)} = e^{x\cos\theta}\,.\,e^{ix\sin\theta}$$

$$= e^{x\cos\theta}\,[\cos(x\sin\theta) + i\sin(x\sin\theta)]$$

Equating real part, we get $C = e^{x\cos\theta}\cos(x\sin\theta)$.

(*iv*) Let $$C = \cos\theta + \sin\theta\cos 2\theta + \frac{\sin^2\theta\cos 3\theta}{1\cdot 2} + \ldots\ldots \infty$$

$$S = \sin\theta + \sin\theta\sin 2\theta + \frac{\sin^2\theta\sin 3\theta}{1\cdot 2} + \ldots\ldots \infty$$ (See Note art. 6.18)

$$C + iS = (\cos\theta + i\sin\theta) + \sin\theta(\cos 2\theta + i\sin\theta) + \frac{\sin^2\theta}{2\,!}(\cos 3\theta + i\sin 3\theta) + \ldots\ldots \infty$$

$$= e^{i\theta} + \sin\theta\,.\,e^{i\,2\theta} + \frac{\sin^2\theta}{2\,!}\,e^{i\,3\theta} + \ldots\ldots \infty$$

$$= e^{i\theta}\left[1 + \sin\theta\,e^{i\theta} + \frac{\sin^2\theta\,e^{i2\theta}}{2\,!} + \ldots\ldots \infty\right]$$

$$= e^{i\theta}\left(1 + t + \frac{t^2}{2\,!} + \ldots\ldots \infty\right), \text{ where } t = \sin\theta e^{i\theta}$$

$$= e^{i\theta}\,.\,e^t = e^{i\theta + \sin\theta\,e^{i\theta}}$$

$$= e^{i\theta + \sin\theta(\cos\theta + i\sin\theta)} = e^{\sin\theta\cos\theta + i(\theta + \sin^2\theta)} = e^{\sin\theta\cos\theta}\,.\,e^{i(\theta + \sin^2\theta)}$$

$$= e^{\sin\theta\cos\theta}\,[\cos(\theta + \sin^2\theta) + i\sin(\theta + \sin^2\theta)]$$

Comparing real parts on both sides

$$C = e^{\sin\theta\cos\theta}\cos(\theta + \sin^2\theta).$$

2. Series depending on expansion of sin *x*, cos *x* and sinh *x*, cosh *x*

Example 3. *Sum the series* $\sin\alpha - \dfrac{\sin(\alpha + 2\beta)}{2\,!} + \dfrac{\sin(\alpha + 4\beta)}{4\,!} - \ldots\ldots \infty$. **(P.T.U., May 2004)**

Sol. Let $$S = \sin\alpha - \frac{\sin(\alpha + 2\beta)}{2\,!} + \frac{\sin(\alpha + 4\beta)}{4\,!} - \ldots\ldots$$

$$C = \cos\alpha - \frac{\cos(\alpha+2\beta)}{2!} + \frac{\cos(\alpha+4\beta)}{4!} - \ldots..$$

$$\therefore \quad C + iS = (\cos\alpha + i\sin\alpha) - \frac{1}{2!}[\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)]$$
$$+ \frac{1}{4!}[\cos(\alpha+4\beta) + i\sin(\alpha+4\beta)] - \ldots\ldots.$$
$$= e^{i\alpha} - \frac{1}{2!}\cdot e^{i(\alpha+2\beta)} + \frac{1}{4!}\cdot e^{i(\alpha+4\beta)} \ldots..$$
$$= e^{i\alpha}\left[1 - \frac{e^{2i\beta}}{2!} + \frac{e^{4i\beta}}{4!} - \ldots\ldots..\right] = e^{i\alpha}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots..\right], \text{ where } x = e^{i\beta}$$
$$= e^{i\alpha}.\cos x = (\cos\alpha + i\sin\alpha).\cos(e^{i\beta})$$
$$= (\cos\alpha + i\sin\alpha)\cos(\cos\beta + i\sin\beta)$$
$$= (\cos\alpha + i\sin\alpha)[\cos(\cos\beta)\cos(i\sin\beta) - \sin(\cos\beta)\sin(i\sin\beta)]$$
$$[\because \quad \cos(A+B) = \cos A\cos B - \sin A\sin B]$$
$$= (\cos\alpha + i\sin\alpha)[\cos(\cos\beta)\cosh(\sin\beta) - \sin(\cos\beta).\,i\sinh(\sin\beta)]$$
$$= [\cos\alpha\cos(\cos\beta)\cosh(\sin\beta) + \sin\alpha\sin(\cos\beta)\sinh(\sin\beta)]$$
$$+ i[\sin\alpha\cos(\cos\beta)\cosh(\sin\beta) - \cos\alpha\sin(\cos\beta)\sinh(\sin\beta)]$$

Equating imaginary parts

$$S = \sin\alpha\cos(\cos\beta)\cosh(\sin\beta) - \cos\alpha\sin(\cos\beta)\sinh(\sin\beta).$$

Example 4. *Find the sum to infinity of the following series* $1 + \frac{x^2\cos 2\theta}{2!} + \frac{x^4\cos 4\theta}{4!} + \ldots..$

Sol. Let $\quad C = 1 + \frac{x^2\cos 2\theta}{2!} + \frac{x^4\cos 4\theta}{4!} + \ldots..;$

$$S = \frac{x^2\sin 2\theta}{2!} + \frac{x^4\sin 4\theta}{4!} + \ldots\ldots$$

$$C + iS = 1 + \frac{x^2}{2!}(\cos 2\theta + i\sin 2\theta) + \frac{x^4}{4!}(\cos 4\theta + i\sin 4\theta) + \ldots\ldots$$
$$= 1 + \frac{x^2}{2!}e^{2i\theta} + \frac{x^4}{4!}.e^{4i\theta} + \ldots.. = 1 + \frac{y^2}{2!} + \frac{y^4}{4!} + \ldots.., \text{ where } y = xe^{i\theta}$$
$$= \cosh y = \cosh(xe^{i\theta}) = \cosh[x(\cos\theta + i\sin\theta)]$$
$$= \cos i[x(\cos\theta + i\sin\theta)] = \cos[ix\cos\theta - x\sin\theta]$$
$$= \cos(ix\cos\theta)\cos(x\sin\theta) + \sin(ix\cos\theta)\sin(x\sin\theta)$$
$$= \cosh(x\cos\theta)\cos(x\sin\theta) + i\sinh(x\cos\theta)\sin(x\sin\theta)$$

Equating real parts $\quad C = \cosh(x\cos\theta)\cos(x\sin\theta).$

3. Series depending upon Binomial Series

Example 5. *Sum the series* $1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \ldots..\infty.$

(P.T.U., Dec. 2011, May 2012)

Sol. Let $$C = 1 - \frac{1}{2}\cos\theta + \frac{1\cdot 3}{2\cdot 4}\cos 2\theta - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 3\theta + ...$$

$$S = -\frac{1}{2}\sin\theta + \frac{1\cdot 3}{2\cdot 4}\sin 2\theta - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin 3\theta + ...$$

$\therefore$ $$C + iS = 1 - \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1\cdot 3}{2\cdot 4}(\cos 2\theta + i\sin 2\theta) - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(\cos 3\theta + i\sin 3\theta) + ...$$

$$= 1 - \frac{1}{2}e^{i\theta} + \frac{1\cdot 3}{2\cdot 4}\,.\,e^{2i\theta} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}e^{3i\theta} + ...$$

$$= 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + ..., \text{ where } x = e^{i\theta}$$

$$= (1+x)^{-1/2} = (1+e^{i\theta})^{-1/2} = (1+\cos\theta + i\sin\theta)^{-1/2}$$

$$= \left(2\cos^2\frac{\theta}{2} + i\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-1/2} = \left(2\cos\frac{\theta}{2}\right)^{-1/2} \times \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1/2}$$

$$= \left(2\cos\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}\right)$$ [De-Moivre's Theorem.]

Equating real parts, $C = \dfrac{\cos\frac{\theta}{4}}{\sqrt{2\cos\frac{\theta}{2}}}$.

Example 6. *Sum the following series* $n\sin\alpha + \dfrac{n(n+1)}{1\,.\,2}\sin 2\alpha + \dfrac{n(n+1)(n+2)}{1\,.\,2\,.\,3}\sin 3\alpha + ...\ \infty$.

(P.T.U., Dec. 2004, 2013)

Sol. Let $$S = n\sin\alpha + \frac{n(n+1)}{1\cdot 2}\sin 2\alpha + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\sin 3\alpha + ...\ \infty$$

Let $$C = 1 + n\cos\alpha + \frac{n(n+1)}{1\cdot 2}\cos 2\alpha + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\cos 3\alpha + ...\ \infty$$

$$C + iS = 1 + n\,e^{i\alpha} + \frac{n(n+1)}{1\,.\,2}e^{i2\alpha} + \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}e^{i3\alpha} + ...\ \infty$$

$$= (1-e^{i\alpha})^{-n} = \frac{1}{(1-e^{i\alpha})^n} \times \frac{(1-e^{-i\alpha})^n}{(1-e^{-i\alpha})^n}$$

$$= \frac{[1-(\cos\alpha - i\sin\alpha)]^n}{(1-e^{i\alpha}-e^{-i\alpha}+1)^n} = \frac{[(1-\cos\alpha)+i\sin\alpha]^n}{(2-2\cos\alpha)^n}$$

$$= \frac{[2\sin^2\alpha/2 + 2i\sin\alpha/2\cos\alpha/2]^n}{2^n(2\sin^2\alpha/2)^n}$$

$$= \frac{2^n\,.\,\sin^n\alpha/2\,[\sin\alpha/2 + i\cos\alpha/2]^n}{(2^n\sin^n\alpha/2)\,(2\sin\alpha/2)^n}$$

$$= \frac{\left[\cos\left(\pi/2 - \alpha/2\right) + i \sin\left(\pi/2 - \alpha/2\right)\right]^n}{2^n \sin^n \alpha/2}$$

$$= \frac{1}{2^n \sin^n \alpha/2}\left[\cos n\left(\frac{\pi - \alpha}{2}\right) + i \sin\left(\frac{n(\pi - \alpha)}{2}\right)\right]$$

Comparing imaginary parts on both sides

$$S = \frac{\sin \frac{n(\pi - \alpha)}{2}}{2^n \sin^n \alpha/2}.$$

4. Series depending on G.P.

Example 7. *Sum the series $1 + x \cos \alpha + x^2 \cos 2\alpha + x^3 \cos 3\alpha + ...$ to n terms where x is less than unity. Also find the sum to infinity.*

Sol. Let $C = 1 + x \cos \alpha + x^2 \cos 2\alpha + ... + x^{n-1} \cos (n-1)\alpha$

$S = 0 + x \sin \alpha + x^2 \sin 2\alpha + ... + x^{n-1} \sin (n-1)\alpha$

$\therefore$ $C + iS = 1 + x(\cos \alpha + i \sin \alpha) + x^2 (\cos 2\alpha + i \sin 2\alpha) + ...$

$...... + x^{n-1} [\cos (n-1)\alpha + i \sin (n-1)\alpha]$

$= 1 + xe^{i\alpha} + x^2 e^{2i\alpha} + ... + x^{n-1} . e^{i(n-1)\alpha}$

$= 1 + y + y^2 + ... + y^{n-1}$, where $y = xe^{i\alpha}$

$$= \frac{1 \cdot (1 - y^n)}{1 - y} = \frac{1 - x^n e^{ni\alpha}}{1 - xe^{i\alpha}} = \frac{1 - x^n \cdot e^{ni\alpha}}{1 - xe^{i\alpha}} \cdot \frac{1 - xe^{-i\alpha}}{1 - xe^{-i\alpha}}$$

$$= \frac{1 - xe^{-i\alpha} - x^n e^{ni\alpha} + x^{n+1} \cdot e^{i(n-1)\alpha}}{1 - x(e^{i\alpha} + e^{-i\alpha}) + x^2 \cdot e^{i\alpha} \cdot e^{-i\alpha}}$$

$$= \frac{1 - x(\cos \alpha - i \sin \alpha) - x^n(\cos n\alpha + i \sin n\alpha) + x^{n+1}[\cos (n-1)\alpha + i \sin (n-1)\alpha]}{1 - x \cdot 2 \cos \alpha + x^2}$$

Equating real parts $C = \dfrac{1 - x \cos \alpha - x^n \cos n\alpha + x^{n+1} \cos (n-1)\alpha}{1 - 2x \cos \alpha + x^2}$

$\because$ x is numerically less than 1

$\therefore$ $x^n, x^{n+1} \to 0$ as $n \to \infty$,

$\therefore$ For sum to infinity $C + iS = \dfrac{1 - x(\cos \alpha - i \sin \alpha)}{1 - 2x \cos \alpha + x^2}$

Equating real parts, $C = \dfrac{1 - x \cos \alpha}{1 - 2x \cos \alpha + x^2}$.

Example 8. *Solve the series : $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \sin (\alpha + 3\beta) + ... \sin (\alpha + \overline{n-1}\, \beta)$*

(P.T.U., May 2008, Jan. 2010)

Sol. Let $S = \sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + ... \sin (\alpha + \overline{n-1}\, \beta)$

$C = \cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + ... \cos (\alpha + \overline{n-1}\, \beta)$

$C + iS = (\cos \alpha + i \sin \alpha) + [\cos (\alpha + \beta) + i \sin (\alpha + \beta)] + [\cos (\alpha + 2\beta) + i \sin (\alpha + 2\beta)]$

$+ ... [\cos (\alpha + \overline{n-1}\, \beta) + i \sin (\alpha + \overline{n-1}\, \beta)]$

$$= e^{i\alpha} + e^{i(\alpha+\beta)} + e^{i(\alpha+2\beta)} + \dots\ e^{i(\alpha+\overline{n-1}\beta)}$$

$$= e^{i\alpha}\left[1 + e^{i\beta} + e^{i2\beta} + \dots e^{i(n-1)\beta}\right]$$

which is a G.P series with first term 1, common ratio $e^{i\beta}$ and number of terms $= n$

$$\therefore \qquad C + iS = e^{i\alpha}\,\frac{1\,(1-e^{in\beta})}{1-e^{i\beta}} \qquad \left[\text{Using } S_n = \frac{a\,(1-r^n)}{1-r}\right]$$

Multiply and divide by $1 - e^{-i\beta}$

$$C + iS = \frac{e^{i\alpha}\,(1-e^{in\beta})\,(1-e^{-i\beta})}{(1-e^{i\beta})\,(1-e^{-i\beta})} = \frac{e^{i\alpha}\left[1-e^{-i\beta}-e^{in\beta}+e^{i(n-1)\beta}\right]}{1-e^{i\beta}-e^{-i\beta}+1}$$

$$= \frac{e^{i\alpha} - e^{i(\alpha-\beta)} - e^{i(\alpha+n\beta)} + e^{i(\alpha+\overline{n-1}\beta)}}{2-2\cos\beta}$$

$$= \frac{\cos\alpha + i\sin\alpha - \cos(\alpha-\beta) - i\sin(\alpha-\beta) - \cos(\alpha+n\beta) - i\sin(\alpha+n\beta) + \cos(\alpha+\overline{n-1}\beta) + i\sin(\alpha+\overline{n-1}\beta)}{4\sin^2\beta/2}$$

Comparing imaginary parts

$$S = \frac{\sin\alpha - \sin(\alpha-\beta) - \sin(\alpha+n\beta) + \sin(\alpha+\overline{n-1}\beta)}{4\sin^2\beta/2}$$

$$= \frac{2\cos\left(\dfrac{2\alpha-\beta}{2}\right)\sin\dfrac{\beta}{2} - 2\cos\dfrac{2\alpha+(2n-1)\beta}{2}\sin\beta/2}{4\sin^2\beta/2}$$

$$= \frac{2\sin\dfrac{\beta}{2}\left[\cos\left(\alpha-\dfrac{\beta}{2}\right) - \cos\left(\alpha+\dfrac{2n-1}{2}\beta\right)\right]}{4\sin^2\beta/2}$$

$$= \frac{2\sin\left(\alpha+\dfrac{n-1}{2}\beta\right)\sin\dfrac{n}{2}\beta}{2\sin\beta/2} = \frac{\sin\left(\alpha+\dfrac{n-1}{2}\beta\right)\sin\dfrac{n\beta}{2}}{\sin\dfrac{\beta}{2}}$$

5. Series depending upon the expansion of log $(1 + x)$ or log $(1 - x)$ or $\tan^{-1} x$.

Example 9. *Sum the series*

(*a*) $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \dots\dots \infty$ (*b*) $\sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \dots\dots \infty$

Sol. Let $\qquad C = \cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \dots\dots \infty$

$$S = \sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \dots\dots \infty$$

$$C + iS = (\cos\theta + i\sin\theta) - \tfrac{1}{2}(\cos 2\theta + i\sin 2\theta) + \tfrac{1}{3}(\cos 3\theta + i\sin 3\theta) - \dots\dots \infty$$

$$= e^{i\theta} - \tfrac{1}{2}e^{2i\theta} + \tfrac{1}{3}e^{3i\theta} - \dots\dots \infty = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\dots \infty, \text{ where } x = e^{i\theta}$$

$$= \log(1+x) = \log(1+e^{i\theta}) = \log(1+\cos\theta + i\sin\theta)$$

$$= \frac{1}{2}\log[(1+\cos\theta)^2 + \sin^2\theta] + i\sin^{-1}\frac{\sin\theta}{1+\cos\theta}$$

$$\left[\because \log(x+iy) = \tfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}. \text{ Here } x = 1+\cos\theta\ ;\ y = \sin\theta\right]$$

$$= \frac{1}{2}\log[1+2\cos\theta+\cos^2\theta+\sin^2\theta] + i\tan^{-1}\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= \frac{1}{2}\log 2(1+\cos\theta) + i\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\log\left(2\cdot 2\cos^2\frac{\theta}{2}\right) + i\,\frac{\theta}{2} = \log\left(2\cos\frac{\theta}{2}\right) + i\,\frac{\theta}{2}$$

Equating real and imaginary parts

$$C = \log\left(2\cos\frac{\theta}{2}\right) \qquad ...(1) \qquad\qquad S = \frac{\theta}{2} \qquad ...(2)$$

Example 10. *Sum the series* $\sin\alpha\cos\beta - \frac{1}{2}\sin^2\alpha\cos 2\beta + \frac{1}{3}\sin^3\alpha\cos 3\beta - \ldots\ \infty$ **(P.T.U., May 2003)**

Sol. Let $\quad C = \sin\alpha\cos\beta - \frac{1}{2}\sin^2\alpha\cos 2\beta + \frac{1}{3}\sin^3\alpha\cos 3\beta - \ldots$

$$S = \sin\alpha\sin\beta - \tfrac{1}{2}\sin^2\alpha\sin 2\beta + \tfrac{1}{3}\sin^3\alpha\sin 3\beta - \ldots$$

$$C + iS = \sin\alpha\,.\,e^{i\beta} - \tfrac{1}{2}\sin^2\alpha\,.\,e^{2i\beta} + \tfrac{1}{3}\sin^3\alpha\,.\,e^{3i\beta} - \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \text{ where } x = \sin\alpha\,.\,e^{i\beta}$$

$$= \log(1+x) = \log(1+\sin\alpha\,.\,e^{i\beta}) = \log[1+\sin\alpha(\cos\beta + i\sin\beta)]$$

$$= \log(1+\sin\alpha\cos\beta + i\sin\alpha\sin\beta)$$

$$= \frac{1}{2}\log[(1+\sin\alpha\cos\beta)^2 + \sin^2\alpha\sin^2\beta] + i\tan^{-1}\left(\frac{\sin\alpha\sin\beta}{1+\sin\alpha\cos\beta}\right)$$

$$= \frac{1}{2}\log[1+2\sin\alpha\cos\beta + \sin^2\alpha(\cos^2\beta+\sin^2\beta)] + i\tan^{-1}\left(\frac{\sin\alpha\sin\beta}{1+\sin\alpha\cos\beta}\right)$$

$$= \frac{1}{2}\log(1+2\sin\alpha\cos\beta+\sin^2\alpha) + i\tan^{-1}\left(\frac{\sin\alpha\sin\beta}{1+\sin\alpha\cos\beta}\right)$$

Equating real parts $C = \dfrac{1}{2}\log(1+2\sin\alpha\cos\beta+\sin^2\alpha)$.

Example 11. *If* $C = \cos^2\theta - \dfrac{1}{3}\cos^3\theta\cos 3\theta + \dfrac{1}{5}\cos^5\theta\cos 5\theta - \ldots\ \infty$ *show that* $\tan 2C = 2\cot^2\theta$.

Sol. $\qquad C = \cos\theta\,.\,\cos\theta - \dfrac{1}{3}\cos^3\theta\cos 3\theta + \dfrac{1}{5}\cos^5\theta\cos 5\theta - \ldots$

Let $\quad \text{S} = \cos\theta \,.\, \sin\theta - \dfrac{1}{3}\cos^3\theta \,.\, \sin 3\theta + \dfrac{1}{5}\cos^5\theta \,.\, \sin 5\theta - .....$

$$\text{C} + i\text{S} = \cos\theta \,.\, e^{i\theta} - \frac{1}{3}\cos^3\theta \,.\, e^{3i\theta} + \frac{1}{5}\cos^5\theta \,.\, e^{5i\theta} - .... = x - \frac{x^3}{3} + \frac{x^5}{5} - ....$$

where $\quad x = \cos\theta \,.\, e^{i\theta}$

$$= \tan^{-1} x = \tan^{-1}(\cos\theta \,.\, e^{i\theta})$$

$$\therefore \qquad \text{C} + i\text{S} = \tan^{-1}[\cos\theta(\cos\theta + i\sin\theta)] \qquad ...(1)$$

Changing i into $-i$,

$$\text{C} - i\text{S} = \tan^{-1}[\cos\theta(\cos\theta - i\sin\theta)] \qquad ...(2)$$

Adding (1) and (2)

$$2\text{C} = \tan^{-1}[\cos\theta(\cos\theta + i\sin\theta)] + \tan^{-1}[\cos\theta(\cos\theta - i\sin\theta)]$$

$$= \tan^{-1}\frac{\cos\theta(\cos\theta + i\sin\theta) + \cos\theta(\cos\theta - i\sin\theta)}{1 - \cos\theta\left(\cos\theta + i\sin\theta\right).\cos\theta(\cos\theta - i\sin\theta)}$$

$$= \tan^{-1}\left(\frac{2\cos^2\theta}{1 - \cos^2\theta(\cos^2\theta + \sin^2\theta)}\right) = \tan^{-1}\frac{2\cos^2\theta}{1 - \cos^2\theta}$$

$$= \tan^{-1}\left(\frac{2\cos^2\theta}{\sin^2\theta}\right) = \tan^{-1}(2\cot^2\theta)$$

Hence $\qquad \tan 2\text{C} = 2\cot^2\theta.$

Example 12. *Sum the series :* $e^{\alpha}\cos\beta - \dfrac{e^{3\alpha}}{3}\cos 3\beta + \dfrac{e^{5\alpha}}{5}\cos 5\beta .... \infty$ **(P.T.U., Dec. 2005)**

Sol. Let $\quad \text{C} = e^{\alpha}\cos\beta - \dfrac{e^{3\alpha}}{3}\cos 3\beta + \dfrac{e^{5\alpha}}{5}\cos 5\beta .... \infty$

$$\text{S} = e^{\alpha}\sin\beta - \frac{e^{3\alpha}}{3}\sin 3\beta + \frac{e^{5\alpha}}{5}\sin 5\beta .... \infty$$

$$\text{C} + i\text{S} = e^{\alpha}(\cos\beta + i\sin\beta) - \frac{e^{3\alpha}}{3}(\cos 3\beta + i\sin 3\beta) + \frac{e^{5\alpha}}{5}(\cos 5\beta + i\sin 5\beta) .... \infty$$

$$= e^{\alpha}\,.\,e^{i\beta} - \frac{e^{3\alpha}}{3}e^{i3\beta} + \frac{e^{5\alpha}}{5}e^{i5\beta} .... \infty$$

$$= e^{(\alpha + i\beta)} - \frac{e^{3(\alpha + i\beta)}}{3} + \frac{e^{5(\alpha + i\beta)}}{5} .... \infty$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} .... \infty \quad \text{where } x = e^{\alpha + i\beta}$$

$$= \tan^{-1} x = \tan^{-1}(e^{\alpha + i\beta}) = \tan^{-1}[e^{\alpha}(\cos\beta + i\sin\beta)]$$

$$= \tan^{-1}(e^{\alpha}\cos\beta + i\,e^{\alpha}\sin\beta)$$

We know that $\tan^{-1}(x + iy) = \dfrac{1}{2}\tan^{-1}\dfrac{2x}{1 - x^2 - y^2} + i\dfrac{1}{2}\tanh^{-1}\dfrac{2y}{1 + x^2 + y^2}$

[Proved in example 11 (*i*) art. 6.17]

$$= \frac{1}{2} \tan^{-1} \frac{2e^{\alpha} \cos\beta}{1 - e^{2\alpha} \cos^2\beta - e^{2\alpha} \sin^2\beta} + i \frac{1}{2} \tanh^{-1} \frac{2e^{\alpha} \sin\beta}{1 + e^{2\alpha} (\cos^2\beta + \sin^2\beta)}$$

$$= \frac{1}{2} \tan^{-1} \frac{2e^{\alpha} \cos\beta}{1 - e^{2\alpha}} + i \cdot \frac{1}{2} \tanh^{-1} \frac{2e^{\alpha} \sin\beta}{1 + e^{2\alpha}}$$

Comparing real part $C = \frac{1}{2} \tan^{-1} \frac{2e^{\alpha} \cos\beta}{1 - e^{2\alpha}}$

or $$C = \frac{1}{2} \tan^{-1} \frac{2\cos\beta}{e^{-\alpha} - e^{\alpha}}$$

$$= -\frac{1}{2} \tan^{-1} \frac{\cos\beta}{\frac{e^{\alpha} - e^{-\alpha}}{2}} = -\frac{1}{2} \tan^{-1} \frac{\cos\beta}{\sinh\alpha}$$

$$= -\frac{1}{2} \tan^{-1} (\operatorname{cosech}\alpha \cos\beta).$$

6. Method of Hyperbolic Series

In Hyperbolic series, C + iS method is not applied. To sum up a series of hyperbolic sines or cosines.

(*i*) Replace $\sinh x$ by $\frac{e^x - e^{-x}}{2}$ and $\cosh x$ by $\frac{e^x + e^{-x}}{2}$.

(*ii*) Separate the series in e^x and e^{-x}.

(*iii*) Sum up each of these series by using results of standard series.

(*iv*) Put the result in terms of hyperbolic sines or cosines.

Example 13. *Sum the series*

(*a*) $\sinh\alpha - \frac{1}{2} \sinh 2\alpha + \frac{1}{3} \sinh 3\alpha - \ldots\ldots \infty$

(*b*) $1 + x\cosh\alpha + x^2 \cosh 2\alpha + x^3 \cosh 3\alpha + \ldots\ldots$ *to n terms.*

Sol. (*a*) $\sinh\alpha - \frac{1}{2} \sinh 2\alpha + \frac{1}{2} \sinh 3\alpha \ldots\ldots \infty$

$$= \frac{e^{\alpha} - e^{-\alpha}}{2} - \frac{1}{2} \cdot \frac{e^{2\alpha} - e^{-2\alpha}}{2} + \frac{1}{3} \cdot \frac{e^{3\alpha} - e^{-3\alpha}}{2} \ldots\ldots \infty$$

$$= \frac{1}{2} \left(e^{\alpha} - \frac{1}{2} e^{2\alpha} + \frac{1}{3} e^{3\alpha} \ldots\ldots \infty\right) - \frac{1}{2} \left(e^{-\alpha} - \frac{1}{2} e^{-2\alpha} + \frac{1}{3} e^{-3\alpha} \ldots\ldots \infty\right)$$

$$= \frac{1}{2} \log (1 + e^{\alpha}) - \frac{1}{2} \log (1 + e^{-\alpha})$$

$$= \frac{1}{2} \log \frac{1 + e^{\alpha}}{1 + e^{-\alpha}} = \frac{1}{2} \log \frac{e^{\frac{\alpha}{2}} \left(e^{-\frac{\alpha}{2}} + e^{\frac{\alpha}{2}}\right)}{e^{-\frac{\alpha}{2}} \left(e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}}\right)} = \frac{1}{2} \log e^{\alpha} = \frac{\alpha}{2}.$$

(*b*) $1 + x\cosh\alpha + x^2 \cosh 2\alpha + x^3 \cosh 3\alpha + \ldots\ldots$ to n terms.

$$= 1 + x \left(\frac{e^{\alpha} + e^{-\alpha}}{2}\right) + x^2 \left(\frac{e^{2\alpha} + e^{-2\alpha}}{2}\right) + x^3 \left(\frac{e^{3\alpha} + e^{-3\alpha}}{2}\right) + \ldots\ldots \text{ to } n \text{ terms.}$$

$$= \tfrac{1}{2} \cdot [2 + x(e^{\alpha} + e^{-\alpha}) + x^2(e^{2\alpha} + e^{-2\alpha}) + x^3(e^{3\alpha} + e^{-3\alpha}) + \ldots\ldots \text{ to } n \text{ terms}]$$

$$= \tfrac{1}{2} [(1 + xe^{\alpha} + x^2e^{2\alpha} + \ldots\ldots \text{ to } n \text{ terms}) + (1 + ex^{-\alpha} + x^2e^{-2\alpha} + \ldots\ldots \text{ to } n \text{ terms})]$$

$$= \frac{1}{2}\left[\frac{1(1 - x^n e^{n\alpha})}{1 - xe^{\alpha}} + \frac{1(1 - x^n e^{-n\alpha})}{1 - xe^{-\alpha}}\right] \qquad \text{(each series being a G.P.)}$$

$$= \frac{1}{2} \cdot \frac{(1 - x^n e^{n\alpha})(1 - xe^{-\alpha}) + (1 - x^n e^{-n\alpha})(1 - xe^{\alpha})}{(1 - xe^{\alpha})(1 - xe^{-\alpha})}$$

$$= \frac{1}{2} \cdot \frac{2 - x(e^{\alpha} + e^{-\alpha}) - x^n(e^{n\alpha} + e^{-n\alpha}) + x^{n+1}[e^{(n-1)\alpha} + e^{-(n-1)\alpha}]}{1 - x(e^{\alpha} + e^{-\alpha}) + x^2}$$

$$= \frac{1}{2} \cdot \frac{2 - x \cdot 2\cosh\alpha - x^n \cdot 2\cosh n\alpha + x^{n+1} \cdot 2\cosh(n-1)\alpha}{1 - x \cdot 2\cosh\alpha + x^2}$$

$$= \frac{1 - x\cosh\alpha - x^n\cosh n\alpha + x^{n+1}\cosh(n-1)\alpha}{1 - 2x\cosh\alpha + x^2}.$$

TEST YOUR KNOWLEDGE

Sum of the following series :

1. $\sin\alpha + \dfrac{\sin 2\alpha}{2!} + \dfrac{\sin 3\alpha}{3!} + \ldots\ldots \infty.$

2. $\cos\alpha + \dfrac{\cos\alpha}{1!}\cos 2\alpha + \dfrac{\cos^2\alpha}{2!}\cos 3\alpha + \dfrac{\cos^3\alpha}{3!}\cos 4\alpha + \ldots\ldots \infty.$

3. $\cos\theta + \dfrac{\sin\theta}{1!}\cos 2\theta + \dfrac{\sin^2\theta}{2!}\cos 3\theta + \ldots\ldots \infty.$

4. $\cos\alpha - \dfrac{\cos(\alpha + 2\beta)}{3!} + \dfrac{\cos(\alpha + 4\beta)}{5!} - \ldots\ldots \infty.$

5. $1 + \dfrac{\cos 2\theta}{2!} + \dfrac{\cos 4\theta}{4!} + \ldots\ldots \infty.$

6. $\sin\theta - \dfrac{\sin 3\theta}{3!} + \dfrac{\sin 5\theta}{5!} - \ldots\ldots \infty.$

7. $\dfrac{1}{2}\sin\alpha + \dfrac{1\cdot 3}{2\cdot 4}\sin 2\alpha + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin 3\alpha + \ldots\ldots \infty.$

8. $x\sin\alpha + x^2\sin 2\alpha + x^3\sin 3\alpha + \ldots\ldots \infty, |x| < 1.$

9. $\cos^2\alpha + \cos^2\alpha\cos 2\alpha + \cos^3\alpha\cos 3\alpha + \ldots\ldots \infty.$

10. $\sin\alpha\cos\alpha + \sin^2\alpha\cos 2\alpha + \sin^3\alpha\cos 3\alpha + \ldots\ldots \infty.$

11. $\sin\alpha\sin\beta + \sin 2\alpha\sin^2\beta + \sin 3\alpha\sin^3\beta + \ldots\ldots \infty.$

12. $\sin\alpha + \dfrac{1}{2}\sin 2\alpha + \dfrac{1}{2^2}\sin 3\alpha + \dfrac{1}{2^3}\sin 4\alpha + \ldots\ldots \infty.$

13. $c\cos\theta + \dfrac{c^2}{2}\cos 2\theta + \dfrac{c^3}{3}\cos 3\theta + \ldots\ldots \infty.$

14. $c \sin \alpha - \frac{c^2}{2} \sin 2\alpha + \frac{c^3}{3} \sin 3\alpha - \ldots\ldots \infty.$

15. $\cos^2 \alpha - \frac{1}{2} \cos^2 \alpha \cos 2\alpha + \frac{1}{3} \cos^3 \alpha \cos 3\alpha - \ldots\ldots \infty.$

16. $\sin^2 \theta - \frac{1}{2} \sin 2\theta \sin^2 \theta + \frac{1}{3} \sin 3\theta \sin^3 \theta - \ldots\ldots \infty.$

17. $c \cos \alpha + \frac{c^3}{3} \cos 3\alpha + \frac{c^5}{5} \cos 5\alpha + \ldots\ldots \infty.$

18. $c \sin \alpha + \frac{c^3}{3} \sin 3\alpha + \frac{c^5}{5} \sin 5\alpha + \ldots\ldots \infty.$

19. $e^a \cos \beta - \frac{e^{3\alpha}}{3} \cos 3\beta + \frac{e^{5\alpha}}{5} \cos 5\beta - \ldots\ldots \infty.$

20. $\cosh \alpha - \frac{1}{2} \cosh 2\alpha + \frac{1}{3} \cosh 3\alpha + \ldots\ldots \infty.$

21. $x \sinh \alpha + x^2 \sinh 2\alpha + x^3 \sinh 3\alpha + \ldots\ldots \infty.$

ANSWERS

1. $e^{\cos \alpha} \sin (\sin \alpha)$

2. $e^{\cos^2 \alpha} \cos (\alpha + \sin \alpha \cos \alpha)$

3. $e^{\sin \theta \cos \theta} \cos (\theta + \sin^2 \theta)$

4. $\cos (\alpha - \beta) \sin (\cos \beta) \cosh (\sin \beta) - \sin (\alpha - \beta) \cos (\cos \beta) \sinh (\sin \beta)$

5. $\cosh (\cos \theta) \cos (\sin \theta)$

6. $\cos (\cos \theta) \sinh (\sin \theta)$

7. $\dfrac{\sin \dfrac{\pi - \alpha}{4}}{\sqrt{2 \sin \dfrac{\alpha}{2}}}$

8. $\dfrac{x \sin \alpha}{1 - 2x \cos \alpha + x^2}$

9. 0

10. $\dfrac{\sin \alpha (\cos \alpha - \sin \alpha)}{1 - 2 \sin \alpha \cos \alpha + \sin^2 \alpha}$

11. $\dfrac{\sin \alpha \sin \beta}{1 - 2 \cos \alpha \sin \beta + \sin^2 \beta}$

12. $\dfrac{4 \sin \alpha}{5 - 4 \cos \alpha}$

13. $-\dfrac{1}{2} \log (1 - 2c \cos \theta + c^2)$

14. $\tan^{-1} \left(\dfrac{c \sin \alpha}{1 + c \cos \alpha} \right)$

15. $\dfrac{1}{2} \log (1 + 3 \cos^2 \alpha)$

16. $\tan^{-1} \left(\dfrac{\sin^2 \theta}{1 + \sin \theta \cos \theta} \right)$

17. $\dfrac{1}{4} \log \left(\dfrac{1 + 2c \cos \alpha + c^2}{1 - 2c \cos \alpha + c^2} \right)$

18. $\dfrac{1}{2} \tan^{-1} \left(\dfrac{2c \sin \alpha}{1 - c^2} \right)$

19. $-\dfrac{1}{2} \tan^{-1} (\text{cosech } \alpha \cos \beta)$

20. $\log \left(2 \cosh \dfrac{\alpha}{2} \right)$

21. $\dfrac{x \sinh \alpha}{1 - 2x \cosh \alpha + x^2}$.

REVIEW OF THE CHAPTER

1. **De-Moivre's theorem:** (*i*) If n is any integer, positive or negative, then $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ and (*ii*) if n is a fraction +ve or –ve, then one of the values of $(\cos \theta + i \sin \theta)^n$ is $\cos n\theta + i \sin n\theta$

Note: (*i*) $\cos \theta + i \sin \theta$ is represented by cis θ

(*ii*) $(\sin \theta + i \cos \theta)^n \neq \sin n\theta + i \cos n\theta$

(*iii*) $(\text{cis } \theta_1)(\text{cis } \theta_2) \ldots (\text{cis } \theta_n) = \text{cis } (\theta_1 + \theta_2 + \ldots + \theta_n)$

2. $(\cos\theta + i\sin\theta)^{p/q}$, where $(p, q) = 1$ has q and only q distinct values and the q values form a G.P. whose sum is zero (p, q being integers).

3. To express $\cos^n\theta$ in terms of cosines of multiples of θ, take $x = \cos\theta + i\sin\theta$;

$\frac{1}{x} = \cos\theta - i\sin\theta$, $x + \frac{1}{x} = 2\cos\theta$ $\quad\therefore\quad$ $(2\cos\theta)^n = \left(x + \frac{1}{x}\right)^n$; Expand $\left(x + \frac{1}{x}\right)^n$ by Binomial Theorem and collect the terms equidistant from beginning and end and use $x^n + \frac{1}{x^n} = 2\cos n\theta$.

4. To express $\sin^n\theta$ in terms of cosines or sines of the multiples of θ

Take $x = \cos\theta + i\sin\theta$, $\frac{1}{x} = \cos\theta - i\cos\theta$

$x - \frac{1}{x} = 2i\sin\theta$ and $(2i\sin\theta)^n = \left(x - \frac{1}{x}\right)^n$. Expand $\left(x - \frac{1}{x}\right)^n$ by Binomial theorem and collect the terms equidistant from beginning and end. Also use $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

5. To express $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$ use De-Moivre's theorem $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$; Expand by Binomial theorem and compare real and imaginary parts, we get $\cos n\theta$ and $\sin n\theta$.

6. (*i*) $\tan n\theta = \dfrac{{}^nc_1\tan\theta - {}^nc_3\tan^3\theta + {}^nc_5\tan^5\theta\ldots}{1 - {}^nc_2\tan^2\theta + {}^nc_4\tan^4\theta\ldots}$

(*ii*) $\tan(\theta_1 + \theta_2 + \ldots + \theta_n) = \dfrac{s_1 - s_3 + s_5\ldots}{1 - s_2 + s_4\ldots}$, where s_r denotes the sum of the products of the tangents of the angles $\theta_1, \theta_2, \ldots, \theta_n$ taken r at a time.

7. **Exponential function of a complex number:** Exp $z = e^z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y) = e^x \operatorname{cis} y$. **Period** of e^z is $2\pi i$.

8. **Circular functions of a complex number:** If $z = x + iy$, then circular functions of z are:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \tan z = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} = \frac{\sin z}{\cos z}$$

$$\cot z = \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}, \sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}$$

$$\operatorname{cosec} z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}}.$$

9. **Euler's theorem:** $\forall\ \theta$, real or complex $e^{i\theta} = \cos\theta + i\sin\theta$

Period of $\sin z$ and $\cos z$ is 2π

Period of $\tan z$ is π.

10. **Logarithms of complex numbers:** (*i*) If $\omega = e^z$, where z and w are complex numbers, then z is called logarithm of ω *i.e.*, $z = \log_e\omega$. It is many valued function. The general value of $\log_e\omega$ is $z + 2n\pi i$ and is denoted by $\log_e\omega$. Thus $\log_e\omega = 2n\pi i + \log\omega$.

(*ii*) $\log_e(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2 + \beta^2) + i\,[2n\pi + \tan^{-1}\beta/\alpha]$.

11. **General exponential function:** The general exponential function a^z is defined by $a^z = e^{z \log a}$, where a and z are any numbers real or complex.

$$\therefore \qquad a^z = e^{z(2n\pi i + \log a)}.$$

12. **Hyperbolic functions:** For all values of x, real or complex ;

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \qquad \text{cosech}\, x = \frac{2}{e^x - e^{-x}}$$

Period of $\sinh x$ and $\cosh x$ is $2\pi i$

Period of $\tanh x$ is πi

13. **Relation between hyperbolic and circular functions:**

$$\sin (ix) = i \sinh x \qquad \sinh (ix) = i \sin x$$

$$\cos (ix) = \cosh x \qquad \cosh (ix) = \cos x$$

$$\tan (ix) = i \tanh x \qquad \tanh (ix) = i \tan x$$

$$\cot (ix) = -i \coth x \qquad \coth (ix) = -i \cot x$$

$$\sec (ix) = \text{sech}\, x \qquad \text{sech}\, (ix) = \sec x$$

$$\text{cosec}\, (ix) = -i\, \text{cosech}\, x \qquad \text{cosech}\, (ix) = -i\, \text{cosec}\, x.$$

14. **Inverse trigonometrical functions:**

If z is a complex number, then

$$\sin^{-1} z = -i \log \left(i z + \sqrt{1 - z^2}\right)$$

$$\cos^{-1} z = -i \log \left(z + \sqrt{1 - z^2}\right)$$

$$\tan^{-1} z = -\frac{i}{2} \log \frac{1 + i z}{1 - iz} = \frac{i}{2} \log \frac{i + z}{i - z}\ ; z \neq \pm i$$

$$\cot^{-1} z = \tan^{-1} \frac{i}{z} = -\frac{i}{2} \log \frac{z + i}{z - i}\ z \neq \pm i$$

$$\sec^{-1} z = \cos^{-1} \frac{1}{z} = -i \log \left(\frac{1 + \sqrt{1 - z^2}}{z}\right); z \neq 0$$

$$\text{cosec}^{-1} z = \sin^{-1} \frac{1}{z} = -i \log \left(\frac{i + \sqrt{z^2 - 1}}{z}\right), z \neq 0.$$

15. **Inverse hyperbolic functions:** If z is a complex number, then

$$\sinh^{-1} z = \log \left(z + \sqrt{z^2 + 1}\right)$$

$$\cosh^{-1} z = \log \left(z + \sqrt{z^2 - 1}\right)$$

$$\tanh^{-1} z = \frac{1}{2} \log \frac{1 + 2}{1 - 2}\ ; z \neq \pm 1$$

$$\text{cosech}^{-1} z = \sinh^{-1} \frac{1}{z} = \log \frac{1 + \sqrt{1 + z^2}}{z}\ ; z \neq 0$$

$$\text{sech}^{-1} z = \cosh^{-1} \frac{1}{z} = \log \frac{1 + \sqrt{1 - z^2}}{z}\ ; z \neq 0$$

$$\coth^{-1} z = \tanh^{-1} \frac{1}{z} = \frac{1}{2} \log \frac{z + 1}{z - 1}\ ; z \neq \pm 1.$$

16. Summation of series:

To find sum of the series of the form

$$a_0 \cos \alpha + a_1 \cos (\alpha + \beta) + a_2 \cos (\alpha + 2\beta) + \ldots$$

and $$a_0 \sin \alpha + a_1 \sin (\alpha + \beta) + a_2 \sin (\alpha + 2\beta) + \ldots$$

Take $$C = a_0 \cos \alpha + a_1 \cos (\alpha + \beta) + a_2 \cos (\alpha + 2\beta) + \ldots$$

$$S = a_0 \sin \alpha + a_1 \sin (\alpha + \beta) + a_2 \sin (\alpha + 2\beta) + \ldots$$

Write $$C + i\,S = a_0 e^{i\alpha} + a_1 e^{i(\alpha+\beta)} + a_2 e^{i(\alpha+2\beta)} + \ldots$$

$$= e^{i\alpha} [a_0 + a_1 e^{i\beta} + a_2 e^{i2\beta} + \ldots]$$

$$= e^{i\alpha} [a_0 + a_1 x + a_2 x^2 + \ldots], \text{ where } x = e^{i\beta}$$

$$= e^{i\alpha} f(x)$$

The series $f(x)$ can be summed up by following methods (*i*) G.P. series, (*ii*) Binomial series, (*iii*) Exponential series, (*iv*) expansions of sin x, cos x, tan x, sinh x, cosh, (*v*) logarithmic series, and (*vi*) Gregory's series.

SHORT ANSWER TYPE QUESTIONS

1. (*a*) State De-Moivre's theorem and prove it for the most fundamental case.
[**Hint:** See Art. 6.3] **(P.T.U., May 2004, Dec. 2005, May 2014)**

(*b*) If $x = \text{cis}\,\theta$, $y = \text{cis}\,\phi$, show that $\dfrac{x-y}{x+y} = i \tan \dfrac{\theta - \phi}{2}$ **(P.T.U., Dec 2012)**

[**Hint:** S.E . 3(*i*) art 6.3]

2. Prove that $(1 + i)^n + (1 - i)^n = 2^{\frac{n}{2}+1} \cos \dfrac{n\pi}{4}$. **(P.T.U., May 2003)**

[**Hint:** (*i*) S.E. 13 (*iii*) art. 6.3]

3. Solve the following equations using De-Moivre's theorem

(*i*) $x^4 - x^3 + x^2 - x + 1 = 0$ **(P.T.U., Dec. 2002, May 2003, 2005)**

[**Hint:** Solved Example 8 (*b*) art. 6.4]

(*ii*) $x^7 + x^4 + x^3 + 1 = 0$

[**Hint:** S.E. 8 (*c*) art. 6.4]

(*iii*) $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$

[**Hint:** Solved Example 2 (*a*) art. 6.4]

4. If $\sin \alpha + \sin \beta + \sin \gamma = 0 = \cos \alpha + \cos \beta + \cos \gamma$, then prove that

(*i*) $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 3/2$ **(P.T.U., May 2003)**

[**Hint:** Solved Example 5 (*ix*) art. 6.3]

(*ii*) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos (\alpha + \beta + \gamma)$ **(P.T.U., Dec. 2002)**

[**Hint:** Solved Example 5 (*i*) art. 6.3]

5. Prove that $\left(\sqrt{3} + i\right)^n + \left(\sqrt{3} - i\right)^n = 2^{n+1} \cos \dfrac{n\pi}{6}$ **(P.T.U., May 2004)**

[**Hint:** Solved Example 13 (*ii*) art. 6.3]

6. Use De-Moivre's Theorem to find roots of $z^5 + 1 = 0$.

[**Hint:** $z^5 = -1 = \text{cis}\,\pi = \text{cis}\,(2n\pi + \pi)$, $z = \text{cis}\,\dfrac{2n+1}{5}\pi$, where $n = 0, 1, 2, 3, 4$]

7. If $2 \cos \theta = x + \dfrac{1}{x}$, prove that $\dfrac{x^{2n} + 1}{x^{2n-1} + x} = \dfrac{\cos n\theta}{\cos (n-1)\theta}$.

[**Hint:** Solved Example 10 (*a*) art. 6.3]

8. If $x_r = \cos \dfrac{\pi}{3^r} + i \sin \dfrac{\pi}{3^r}$ show that $x_1 x_2 x_3 \ldots x_n = \cos\left[\dfrac{\pi}{2}\left(1 - \dfrac{1}{3^n}\right)\right] + i \sin\left[\dfrac{\pi}{2}\left(1 - \dfrac{1}{3^n}\right)\right]$

Hence show that $x_1 x_2 x_3 \ldots \infty = i$. **(P.T.U., May 2003)**

[**Hint:**Solved Example 8 (*b*) art. 6.3]

9. Prove that $(1 + \cos\theta + i \sin\theta)^n + (1 + \cos\theta - i \sin\theta)^n = 2^{n+1} \cos^n \dfrac{\theta}{2} \cos \dfrac{n\theta}{2}$.

[**Hint:** Consult Solved Example 10 art. 6.3]

10. If ω is the complex cube root of unity prove that $1 + \omega + \omega^2 = 0$. **(P.T.U., May 2011)**

11. Find all the values of (*i*) $(1+i)^{\frac{1}{4}}$ **(P.T.U., Dec. 2010)**

(*ii*) $(-8i)^{\frac{1}{3}}$. **(P.T.U., May 2012)**

[**Hint:** Consult Solved Examples 4, 5 art. 6.4]

(*iii*) $\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2} i\right)^{3/4}$ **(P.T.U., Dec. 2011, 2012)**

[**Hint:** S.E. 3 art. 6.4]

12. (*a*) Find *n*th roots of unity and prove that these form a G.P. Also show that the sum of these *n* roots is zero and their product is $(-1)^{n-1}$.

[**Hint:** Solved Example 1(*a*) art. 6.4]

(*b*) If $(3 + x)^3 - (3 - x)^3 = 0$; Prove that $x = 3i \tan \dfrac{r\pi}{3}$; $r = 0, 1, 2$ **(P.T.U., May 2010)**

[**Hint:** S.E. 10(*a*) art. 6.4]

13. Find all the values of $(-1)^{\frac{1}{4}}$. **(P.T.U., May 2003)**

[**Hint:** Consult Solved Example 2 art. 6.4]

14. Find all the values of $(-1 + i)^{2/5}$.

[**Hint:** $-1 + i = r \sin\theta + i\, r \sin\theta \quad \therefore \quad r = \sqrt{2}, \theta = \dfrac{3\pi}{4}$

$$(-1+i)^{2/5} = r^{2/5} (\cos\theta + i \sin\theta)^{2/5} = 2^{1/5}\left[\text{cis}\left(2n\pi + \frac{3\pi}{4}\right)\right]^{2/5}$$

$$= 2^{1/5} \text{ cis } \frac{2}{5}\left(2n\pi + \frac{3\pi}{4}\right), n = 0, 1, 2, 3, 4]$$

15. (*a*) Prove that $(\cos^7\theta = \dfrac{1}{64}\left[\cos 7\theta + 7\cos 5\theta + 21 \cos 3\theta + 35 \cos 5\theta\right]$ **(P.T.U., Dec. 2011)**

(*b*) Express $\cos^8\theta$ in terms of cosines of multiples of θ **(P.T.U., May 2006, 2014)**

[**Hint:** Solved Example 1 (*ii*) art. 6.5]

16. (*a*) Prove that $\cos 6\theta = 32 \cos^6\theta - 48 \cos^4\theta + 18 \cos^2\theta - 1$. **(P.T.U., May 2012)**

(*b*) Expand $\sin 7\theta$ is powers of $\sin\theta$. [**Hint:** S.E. 1 (*b*) art. 6.6] **(P.T.U., Dec. 2013)**

17. If $u = \log \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$; prove that (*i*) $\tanh \dfrac{u}{2} = \tan \dfrac{\theta}{2}$ **(P.T.U., May 2006)**

[**Hint:** Solved Example 3 (*i*) art. 6.15]

(*ii*) $\theta = -i \log \tan\left(\dfrac{\pi}{4} + i\dfrac{u}{2}\right)$ **(P.T.U., May 2003)**

[**Hint:** Solved Example 3(*v*) art. 6.15]

18. (*a*) Find the general value of $(-1 + i\sqrt{3})$ **(P.T.U., May 2012)**

[**Hint:** S.E. 2(*a*) art. 6.12]

(*b*) Prove that Log (– 4) = 2 log 2 + (2*n* + 1) π*i*. **(P.T.U., May 2007)**

[**Hint:** Solved Example 2(*b*) art. 6.12]

19. Solve the equation $e^{2z-1} = 1 + i$ **(P.T.U., Dec. 2012)**

$$\left[\textbf{Hint: } 2z - 1 = \log(1+i) = \log\sqrt{2} + i\,\frac{\pi}{4} \therefore\ z = \frac{1}{2}\log\sqrt{2} + \frac{1}{2} + i\,\frac{\pi}{8}\right]$$

20. Prove that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ **(P.T.U., May 2007)**

$$\left[\begin{array}{l}\textbf{Hint: } \text{LHS } (\cosh x + \sinh x)^n = \left(\dfrac{e^x + e^{-x}}{2} + \dfrac{e^x - e^{-x}}{2}\right)^n = e^{nx} \\ \qquad \text{RHS } \cosh nx + \sinh nx = \dfrac{e^{nx} + e^{-nx}}{2} + \dfrac{e^{nx} - e^{-nx}}{2} = e^{nx}\end{array}\right]$$

21. (*a*) What is i^i? **(P.T.U., May 2010, Dec. 2013)**
[**Hint:** S.E. 7(*a*) art. 6.13]

(*b*) Find the value of $\log_i i$ **(P.T.U., Dec. 2002)**

[**Hint:** Solved Example 16 art. 6.12]

22. Find all the roots of $\sinh z = i$ **(P.T.U., May 2003)**

[**Hint:** Solved Example 9(*b*) art. 6.15]

23. Find all values of $\sin^{-1} 2$ treating 2 as complex number **(P.T.U., Dec 2004)**

[**Hint:** Solved Example 13 art. 6.15]

24. Separate real and Imaginary parts of the following :

(*i*) $e^{3xy + 4iy^2}$ [**Hint:** Solved Example 1(*ii*) art. 6.9] **(P.T.U., May 2014)**

(*ii*) $\log[\log i]$ [**Hint:** Solved Example 5 art. 6.12]

(*iii*) $\sec(x + iy)$ **(P.T.U., May 2004)**

(*iv*) $\cos(x + iy)$ [**Hint:** Solved Example 1 art. 6.15] **(P.T.U., Dec. 2004)**

(*v*) $\sin(x + iy)$

(*vi*) $\sinh(x + iy)$
(*vii*) $\text{cosech}(x + iy)$ } [**Hint:** Solved Example 2 art. 6.15]

(*viii*) $\log\sin(x + iy)$
(*ix*) $\cos^{-1}(e^{i\theta})$ } [**Hint:** Solved Example 8 art. 6.15] **(P.T.U., May 2003)**

(*x*) $\tan^{-1}(x + iy)$ [**Hint:** Solved Example 11 art. 6.17] **(P.T.U., May 2006)**

(*xi*) $\log(4 + 3i)$ **(P.T.U., Dec. 2010)**

(*xii*) $\sin^{-1}(e^{i\theta})$, where θ is acute. **(P.T.U., Dec. 2006, 2012, May 2014)**

25. Prove that the following:

(*i*) $\sinh^{-1} z = \log\left(z + \sqrt{z^2 + 1}\right)$ [**Hint:** Solved Example art. 6.17] **(P.T.U., Dec. 2002)**

(*ii*) $\tanh^{-1} z = \dfrac{1}{2}\log\dfrac{1+z}{1-z}$; $z \neq \pm 1$ [**Hint:** Solved Example art. 6.17]

(*iii*) $\sin^{-1} z = -i\log\left(iz + \sqrt{1 - z^2}\right)$ (*iv*) $\tan^{-1} z = \dfrac{i}{2}\log\dfrac{i+z}{i-z}$; $z \neq \pm i$. [**Hint:** See art. 6.15]

26. Find modulus and argument of $(1 + i)^{1-i}$

[**Hint:** Solved Example 15 art. 6.13]

27. (*i*) Prove that e^z is periodic function, *z* is complex number. **(P.T.U., May 2008)**

[**Hint:** Consult art. 6.9]

(*ii*) Prove that sin *z*, cos *z*, tan *z* are periodic functions and hence find their respective periods.

[**Hint:** See art. 6.10]

ANSWERS

3. (*i*) $\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}$ (*ii*) $-1,\ \frac{1\pm i}{\sqrt{2}},\ \frac{-1\pm i}{\sqrt{2}},\ \frac{1\pm i\sqrt{3}}{2}$

(*iii*) $1,\ \cos\frac{r\pi}{7} \pm i\sin\frac{r\pi}{7}$; $r = 2, 4, 6$

6. $-1,\ \cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}$

11. (*i*) $\pm 2^{\frac{1}{8}} \text{cis}\,\frac{r\pi}{16};\ r = 1, 9$ (*ii*) $2i, \pm\sqrt{3} - i$ (*iii*) $\frac{1\pm i}{\sqrt{2}},\ \frac{-1\pm i}{\sqrt{2}}$

12. $\text{cis}\,\frac{2(r-1)\pi}{n}$; $r = 1, 2, 3, \ldots, n$ **13.** $\frac{1\pm i}{\sqrt{2}},\ \frac{-1\pm i}{\sqrt{2}}$

14. $2^{\frac{1}{5}}\ \text{cis}\,\frac{(4n+3)\pi}{10}$; $n = 0, 1, 2, 3, 4$ **15.** (*b*) $\frac{1}{128}$ [$\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35$]

16. (*b*) $7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$]

18. (*a*) $\log 2 + i\,\frac{2(3n+1)}{3}\pi$ **19.** $\frac{1}{2} + \frac{1}{4}\log 2 + i\,\frac{\pi}{8}$

21. (*a*) $e^{-\frac{4n+1}{2}\pi}$ (*b*) $\frac{4m+1}{4n+1}$; m, n are integers

22. $z = \frac{2n+1}{2}\pi i$ if n is even **23.** $\frac{4n+1}{2}\pi - i\cosh^{-1} 2$

$= \frac{2n-1}{2}\pi i$ if n is odd.

24. (*i*) $R = e^{3xy}\cos 4y^2$; Img $= e^{3xy}\sin 4y^2$

(*ii*) $R = \log(4n+1)\ \frac{\pi}{2}$; Img $= (4n+1)\ \frac{\pi}{2}$

(*iii*) $R = \frac{2\cos x\cosh y}{\cos 2x + \cosh 2y}$; Img $= \frac{2\sin x\sinh y}{\cos 2x + \cosh 2y}$

(*iv*) $R = \cos x\cosh y$; Img $= -\sin x\sinh y$

(*v*) $R = \sin x\cosh y$; Img $= \cos x\sinh y$

(*vi*) $R = \sinh x\cos y$; Img $= \cosh x\sin y$

(*vii*) $R = \frac{2\sinh x\cos y}{\cosh 2x - \cos 2y}$; Img $= \frac{-2\cosh x\sin y}{\cosh 2x - \cos 2y}$

(*viii*) $R = \frac{1}{2}\log\left(\frac{1}{2}\cosh 2y \frac{-1}{2}\cos 2x\right)$; Img $= \tan^{-1}(\cot x\tanh y)$

(*ix*) $R = \sin^{-1}\sqrt{\sin\theta}$; Img $= \log\left[\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right]$

(*x*) $R = \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}$; Img $= \frac{1}{2}\tanh^{-1}\frac{2y}{1+x^2+y}$

(*xi*) $R = \log 5$; $I = 2n\pi + \tan^{-1}\frac{3}{4}$

(*xii*) $R = \cos^{-1}\sqrt{\sin\theta}$; Img $= \log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$

26. $\sqrt{2}\,e^{\frac{\pi}{4}}; \frac{\pi}{4} - \frac{1}{2}\log 2$.

ABOUT THE BOOK

The book 'A Textbook of Engineering Mathematics' (Semester-II) has been written keeping in mind the interest of all types of engineering students (meritorious, average and below average). As engineering students have to do maximum work in minimum times, so in this book emphasis is on the presentation of the fundamental and theoretical concepts in an intelligible and easy to understand manner. Simple as well as typical examples have been used to explain each theoretical concept. All questions from last ten year papers of various universities have been included in the book at proper places. For the convenience of students reference topics, working rules of lengthy formulae, hints to difficult problems and lists of important results are given in the chapters.

The book has been written after consulting a number of available textbooks by Indian as well as foreign authors, so the author assumes that this book is complete in itself.

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N.P. Bali is a prolific author of over 100 books for degree and engineering students. He has been writing books for more than forty years.

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