5.2 Algebraic Structure [M.K.U. (B.E.) 2002, 2005, 2009; P.T.U. (B

A non-empty set G equipped with one or more binary operations is called on algebraic stru

Let symbols $*, +, \cdot, o, \oplus, \cup, \cap, \vee, \wedge$ etc. denote binary operations on a set G. Then $(G, *, o, \vee, \wedge)$ etc. are algebraic structure. $(N, +), (I, +), (I, -), (R, +), (R, +, .)$ are all algeb addition $(+)$ and multiplication $(\cdot)$ are both binary operation on the set R of real numb

$(R, +, .)$ is on algebraic structures, equipped with two operation.

5.3 Group

[R.G.P.V. (B.E.) Bhopal 2006; 2009; U.P.T.U. (B.Tech.) 2006, 2007; Rohtak (B.E.) 2007, 2009]

An algebraic structure $(G, *)$ where G is a non-empty set and '$*$' is a binary operation defined on G, is called ***a group*** if the binary operation $*$ satisfies the following postulates (group axioms):

(1) ***Closure Property***: If a and b belong to G to then $a * b$ also belongs to G, *i.e.*,

$$a \in G,\ b \in G \Rightarrow a * b \in G \ \forall\ a, b \in G.$$

(2) ***Associativity***: The binary operation $*$ is associative *i.e.*, if a, b, c. are the elements of G then

$$(a * b) * c = a * (b * c) \ \forall\ a, b, c \in G$$

(3) ***Existence of Identity***: There exists an element e in G such that $e * a = a = a * e \ \forall\ a \in G$. The element e is called the identity.

If $a * e = a \ \forall\ a \in G$ then e is the right identity and when $e * a = a$ then e is the left identity in G.

(4) ***Existence of Inverse***: For each element $a \in G$, there exists an element of G called the inverse of a and denoted by a^{-1} such that

$$a^{-1} * a = e = a * a^{-1},$$

where e is the identity element for $*$.

5.3.1 Abelian Group or Commutative Group

A Group $(G, *)$ is said to be abelian or commutative if in addition to four group axioms th
postulate is also satisfied.

5. ***Commutativity:*** The binary operation " $*$ " in commutative in G *i.e.*

$$a * b = b * a \ \forall\ a, b \in G$$

The algebraic structure $(G, *)$ satisfying postulates 2 to 5 is called on abelian group in the
great mathematician Niels Abel. Thus for an abelian group, we have

$$a + b = b + a \qquad \text{[additive}$$
$$a \cdot b = b \cdot a \qquad \text{[multiplicative}$$

A abelian group under addition is sometimes called module.

A group which is not abelian is called ***non-abelian***.

Note 1: If we use the additive notation '+' to denote the composition in G, then the inverse of an ele
is denoted by the symbol $-a$ *i.e.*,

$$(-a) + a = 0 = a + (-a)$$

Note 2: If we use the multiplicative notation ' . ' to denote the composition in G, then often we
identity by the symbol '1', Thus 1 is an element of G such that

$$1 \cdot a = a = a.1 \ \forall\ a \in G$$

In multiplicative notation, we often denote the inverse of a by $1/a$. Thus $1/a$ is an element of G s

$$\frac{1}{a} \cdot a = 1 = a \cdot \frac{1}{a}$$

In addition notation, we often denote the identity by the symbol '0'. Then 0 is an element of G s

$$0 + a = a = a + 0.$$

Note 3: In addition notation, the element $a + (-b) \in G$ is denoted by $a - b$. In multiplicative n
element $a \cdot b^{-1} \in G$ is denoted by a/b.

5.4 Some Definitions

5.4.1 Groupoid

An algebraic structure $(G, *)$ is called a groupoid or a binary algebra or quasi-group is the bina
satisfies only postulate G. Thus groupoid is a set G with a binary operation 'sign' defined on G
closed under the operation '$*$' *i.e.* if

$$a, b \in G \Rightarrow a * b \in G \; \forall \, a, b, \in G.$$

Illustration: *If N is the set of natural number, the set $(N, +)$ is a groupoid because the set N respect to addition (sum of two natural number is a natural number). Thus $2+7=9 \in N$.*

5.2.1 Semi-Group

[R.G.P.V. (B.E.) Bhopal, U.P.T.U. (B.Tech.) 2008; M.K.U. (M.C.A.) 2004, 2008, M.K.U.

Let G be a non-empty set and $*$ be a binary operation on G. $(G, *)$ is said to be semi group if t
$*$ is associative.

OR

$(G, *)$ is a semi group, if

(i) $x * y \in G$ for all x, y in G, and (ii) $(x * y) * z = x * (y * z)$ for all $x,$ y

5.2.2 Identity Element

Let $*$ be a binary operation on a non-empty set G. An element $e \in G$ is said to be an identity
operation $*$, if

$$a * e = e * a = a \ \forall\ a \in G$$

It may be observed that a semi group $(G, *)$ need not have an identity element with respect to

5.2.3 Monoid

[U.P.T

A semigroup $(M, *)$ with an identity element with respect to the operation $*$ is called mo

OR

An algebraic system $(M, *)$ is called a monoid, if

(i) M is closed with respect to $*$ *i.e.* if $x, y \in M$, then $x * y \in M$.

(ii) $*$ is an associative operation *i.e.* for any $x, y, z \in M$,

$$x * (y * z) = (x * y) * z.$$

(iii) Existence of identity element *i.e.* there exists a element $e \in M$ such that

$$e * x = x * e = x \quad \text{for any} \quad x \in M.$$

Illustration : *Let S be any non-empty set. Consider the power set $P(S)$ together with the ope
of two sets, then $(P(S), \cup)$ is a monoid with empty ϕ as identity element.*

5.5 Order of a Finite Group

The number of element in a finite group is called the ***Order of the Group***. An infinite group is said of infinite order. We shall denote the order a group G by the symbol $O\ (G)$.

Example 4: *Let $A = \{a, b\}$, which of the following tables define a semi group?. Which define a monoid on* **[U.P.T.U. (B.Tech.)**

*	a	b
a	b	a
b	a	b

(i)

*	a	b
a	b	b
b	a	a

(ii)

*	a	b
a	a	b
b	b	a

(iii)

Solution: (i) We have from (i)

(a) ***Closure:*** Since all entries in composition table are in A. Hence closure property satisfied.

(b) ***Associativity:*** Since there are only two element in the set A. Hence associativity is alway satisfied.

(c) ***Identity:*** $\forall$ elements of A $\exists$ an element $b \in B$ such that

$$b * a = a \text{ and } b * b = b$$

Hence (i) is semi group as well as monoid.

(ii) Again (ii) table satisfied closure and associativity and there is no identity. Hence it is semigroup but not monoid.

(iii) We have from (iii)

(a) ***Closure:*** Since all entries in composition table are in A. Hence closure property satisfied.

(b) ***Associativity:*** Since there are only two element in the set A. Hence associativity always satisfied.

(c) ***Identity:*** $\forall$ element of A there exist a, b, $\in A$ such that identity of A.

Therefore, it is semi group as well as monoid.

11.5. Subgroup

Let (G, *) be a group and H is a subset of G. (H, *) is said to be subgroup of G if (H
also group by itself.

Now every set is a subset of itself. Therefore, if G is a group, then G itself is a subgroup
Also if e is the identity element of G. Then the subset of G containing only identity element i
a subgroup of G. These two subgroups (G, *) and ({e}, *) of the group (G, *) are called **impr**
or **trivial** subgroups, others are called proper or nontrivial subgroups.

***Remark* 2:** Every group $(G, *)$ whose order is greater than one has at least two different s

(i) The subset consisting of the identity element alone with '$*$' *i.e.*, $\{(e), *\}$.

(ii) The entire set G with '$*$' *i.e.*, $(G, *)$.

These two subgroups are known as improper subgroups or trivial subgroups of G.

A subgroup other than these two subgroups is known as ***Proper Subgroup*** or non-trivial ***Sub***

***Remark* 3:** If $(G, *)$ is a group and $(H, *)$ is a subgroup of $(G, *)$, then we write as:

$$(H, *) \subseteq (G, *).$$

The symbol $\subseteq$ stands for " is the subgroup of ".

***Remark* 4:** The identity of the subgroup H is the same as that of G. For if e' and e are the iden
G respectively, then

$$a \in H \Rightarrow e'\, a = a$$

and $$a \in G \Rightarrow e\, a = a$$

Now $$e'\, a = ea \Rightarrow e' = e.$$

***Remark* 5:** The inverse of $a \in H$ is the same as the inverse of an element of G. For if b an
inverse then

$$b\, a = e' \quad \text{and} \quad ca = e$$

$$ba = ca \Rightarrow b = c$$

[$\because$ by r

***Remark* 6:** The order of an element $a \in H$ is the same as the order of that as an element

***Remark* 7:** If H is a subgroup of G and K is a subgroup of H then K is a subgroup of G.

***Theorem* 16:** *If H and K are any elements of a group G, then*

$$(HK)^{-1} = K^{-1}\, H^{-1}$$

Proof: Let x be any arbitrary element of $(HK)^{-1}$, then $x = (hk)^{-1}\ \forall\ h \in H,\ k \in K$

$$= k^{-1}h^{-1} \in K^{-1}H^{-1}$$ [$\because$

$\therefore$ $$(HK)^{-1} \subseteq K^{-1}H^{-1}.$$

Again, Let y be any arbitrary element of $K^{-1}H^{-1}$, then $y = k^{-1}h^{-1},\ k \in K,\ h \in H$

$$= (hk)^{-1} \in (Hk)^{-1}$$

$\therefore$ $$K^{-1}H^{-1} \subseteq (HK)^{-1}$$

Hence $$(HK)^{-1} = K^{-1}H^{-1}.$$

Theorem 17: *If H is any subgroup, then $HH = H$.*

Proof: Let $h_1 h_2$ be any element of HH, where $h_1 \in H$ and $h_2 \in H$.

Since, H is a subgroup of G, therefore

$$h_1 h_2 \in H H \Rightarrow h_1 h_2 \in H.$$

$\therefore$ $$H H \subseteq H.$$

Now, let h be any element of H, then, we can write

: $h = he$, where e is the identity of G.

Now $$h \in H,\ e \in H \Rightarrow he \in HH$$

$\Rightarrow$ $$H \subseteq H H.$$

Hence $$H = HH.$$

Theorem 18: *If H is any subgroup of G, then $H^{-1} = H$. Also show that the converse is not true.*

Proof: Let h^{-1} be any arbitrary element of H^{-1}, then $h \in H$.

Now H is a subgroup of G, therefore.

$$h \in H \Rightarrow h^{-1} \in H$$

Thus $$h^{-1} \in H^{-1} \Rightarrow h^{-1} \in H$$

Therefore $$H^{-1} \subseteq H.$$

Again, $$h \in H \Rightarrow h^{-1} \in H$$ [$\because$ H itself

$$\Rightarrow (h^{-1})^{-1} \in H^{-1}$$ [by definit

$$\Rightarrow h \in H^{-1}$$

$\therefore$ $$H \subseteq H^{-1}$$

Hence $$H^{-1} = H.$$

If H is a complex of a group G and $H^{-1} = H$, then, it is not necessary that H is a subgroup of G. Fo $H = \{-1\}$ is a complex of the multiplicative group $G = \{-1, 1\}$. Also $H^{-1} = \{-1\}$, since -1 is the inv G. But $H = \{-1\}$ is not a subgroup of G. Since $((-1)-1) = 1 \notin H$, therefore H is not closed with multiplication.

5.10.1 Criterion For A Complex to be a Subgroup

Theorem 19: *A non-empty subset H of the group G is a subgroup of G, if and only if*

(i) $a \in H, b \in H \Rightarrow ab \in H$.

(ii) $a \in H \Rightarrow a^{-1} \in H$ where a^{-1} is the inverse of a in G.

Proof: *The condition are necessary*. Let H is a subgroup of G, then H must be closed with re
multiplication *i.e.* the composition in G. Therefore,

$$a \in H, b \in H \Rightarrow ab \in H.$$

Thus the condition (i) satisfies.

Let $a \in H$ and let a^{-1} be the inverse of a in G, then the inverse of a group, therefore each element of
possess inverse. Therefore.

$$a \in H \Rightarrow a^{-1} \in H$$

Thus the condition (ii) satisfies.

***The condition are sufficient*.** Let H be a non-empty subset of a group G such that the condition
(ii) satisfy. We are to prove that H is a subgroup of G.

1. ***Closure Property*:** Since $a \in H$, $b \in H \Rightarrow ab \in H$ [by condi
   Therefore, H is closed with respect to multiplication.

2. ***Associativity*:** Let $a, b, c \in H$ arbitrarily, then $\forall\ a, b, c \in H \Rightarrow a, b, c \in G$

   $\Rightarrow$ $(ab)c = a(bc)$ [by associative

   Thus $(ab)c = a(bc)\ \forall\ a, b, c \in H$

3. ***Existence of Identity*:** The identity of the subgroup is the same as the identity of the gr

   Now $a \in H \Rightarrow a^{-1} \in H$ [by condi

   Again $a \in H, a^{-1} \in H \Rightarrow aa^{-1} \in H$ [by conc

   $\therefore$ The identity e is an element of H.

4. ***Existence of Inverse*.**

   Since, $a \in H \Rightarrow a^{-1} \in H$ [by con

   Therefore, each element of H possess inverse.

   Hence, H itself is a group for the composition in G.

   So, H is a subgroup of G.

Theorem 20: *A necessary and sufficient condition for a non-empty subset H of a group to be a s*
that

$$a \in H, b \in H \Rightarrow ab^{-1} \in H$$

Where b^{-1} is inverse of b in G.

Proof: The condition is necessary. Suppose H is a subgroup of G. Let $a \in H$, $b \in H$. Now ea
of H must possess inverse because H itself is a group.

$\therefore$ $$b \in H \Rightarrow b^{-1} \in H.$$

Further H must be closed with respect to multiplication *i.e.* the composition in G. Therefore

$$a \in H, b \in H \Rightarrow a \in H, b^{-1} \in H \Rightarrow ab^{-1} \in H.$$

Thus the condition (i), satisfies.

The condition is sufficient. Let H be the non-empty subset of group G such that

$$a \in H, b \in H \Rightarrow ab^{-1} \in H$$

We are to prove that H is subgroup of G.

1. ***Existence of Identity:*** We have $a \in H, a \in H \Rightarrow aa^{-1} \in H$ [by co

$\Rightarrow$ $$e \in H$$

Thus, the identity e is an element of H.

2. ***Existence of Inverse:*** Let a be any element of H, then by the condition (1), we hav

$$e \in H, a \in H \Rightarrow ea^{-1} \in H \Rightarrow a^{-1} \in H$$

Thus, each element of H possesses inverse.

3. ***Closure Property:***

Let $a, b \in H$ then $b \in H \Rightarrow b^{-1} \in H$

Thus $a, b \in H \Rightarrow a, b^{-1} \in H \Rightarrow a(b^{-1})^{-1} \in H$ [by the condition (1)]

$\Rightarrow$ $ab \in H$.

Therefore, H is closed with respect to the composition '.' in G

4. ***Associativity:*** The associative law satisfies in H, because

$\forall a, b, c \in H \Rightarrow a, b, c \in G$ [$\because H \subseteq G$]

$\Rightarrow$ $(ab)c = a(bc)$ [by associativity in G]

$\therefore$ In H, $(ab)c = a(bc) \; \forall \; a, b, c \in H$.

Hence, H itself is a group for the composition in G. Therefore H is a subgroup of G.

***Cor.* 1:** A necessary and sufficient condition for a non-empty subset H of a group to be a subg
$HH^{-1} \subseteq H$

Proof*: *The condition is necessary. It is given that H is a subgroup of G. Let ab^{-1} be eleme

Then $\quad a \in H, b \in H.$

Since H itself is a group, therefore

$$b \in H \Rightarrow b^{-1} \in H$$

Thus $\quad a \in H, b^{-1} \in H \Rightarrow ab^{-1} \in H \quad$ [by closur

$\therefore \quad ab^{-1} \in HH^{-1} \Rightarrow ab^{-1} \in H.$

Hence $\quad HH^{-1} \subseteq H.$

The condition is sufficient. Let H be a non-empty subgroup of G such that

$$HH^{-1} \subseteq H$$

Let a, $b \in H$. Then

$$ab \in H \Rightarrow ab^{-1} \in HH^{-1} \quad \text{[by definitio}$$

$\Rightarrow \quad ab^{-1} \in H \quad$ [by con

$\therefore \quad a \in H, b \in H \Rightarrow ab^{-1} \in H$. Hence, H is a subgroup of G.

Cor. 2: A necessary and sufficient condition for a non-empty subset H of a group G to be a subgrou

$$HH^{-1} = H$$

Proof: ***The condition is necessary.*** Suppose H is a subgroup of G. Therefore $(H, .)$ itself is a

Let $h \in HH^{-1}$ arbitrary, then

$h \in HH^{-1} \Rightarrow \exists\, a \in H,\, b^{-1} \in H^{-1}$ such that $h = ab^{-1}$.

$\Rightarrow$ $a, b \in H$ such that $h = ab^{-1}$ [$\because$ $x \in H \Leftrightarrow$ x

$\Rightarrow$ $a, b^{-1} \in H$ such that $h = ab^{-1}$ [$\because$ H is a su

$\Rightarrow$ $ab^{-1} \in H$ such that $h = ab^{-1}$ [$\because$ H is a su

$\Rightarrow$ $h \in H$

$\Rightarrow$ $HH^{-1} \subseteq H$

Now, H is a subgroup of G, therefore $e \in H$. If h is any arbitrary element of H, then

$$h = he = he^{-1} \in HH^{-1}$$ [$\because$ $h \in$

$\therefore$ $H \subseteq HH^{-1}$.

Hence $HH^{-1} = H$.

The condition is sufficient. Suppose H is a non-empty subset of group G such that

$$HH^{-1} = H$$

We have to prove that H is a subgroup of G, therefore,

$$a, b \in H \Rightarrow ab^{-1} \in H.$$

$$a, b \in H \Rightarrow a \in H,\, b^{-1} \in H^{-1}$$ [by defi

$$\Rightarrow ab^{-1} \in HH^{-1} \Rightarrow ab^{-1} \in H.$$

Hence. H is a subgroup of G.

5.10.2 Criterion In the Case of Finite Complexes

***Theorem* 21:** Let $(A, *)$ be a group and B a subset of A. If B is a finite set then $(B, *)$ is a subgro
$*$ is a closed operation on B. [R.G.P.V. (B.E.) Bh

***Proof*:** Suppose B is a non-empty finite subset of A and B is closed under $*$ operation *i.e.*,

$$a \in B,\ b \in B \in a * b \in B.$$

Then to prove that B is a subgroup of A.

1. ***Closure Property*:** Let $a, b \in B$ arbitrarily, then

$$a, b \in B \Rightarrow a * b \in B$$ [by the gi

Thus B is closed with respect to multiplication

2. ***Associativity*:** Let $a, b, c \in B$, then

$$\forall\ a, b, c \in B \Rightarrow a, b, c \in A$$

$\Rightarrow$ $$(a * b) * c = a * (b * c)$$ [by ass

$\therefore$ $$\text{In } B,\ (a * b) * c = a * (b * c)\ \forall\ a, b, c \in B$$

3. ***Existence of Identity*:** Let e be the identity of A. By the given condition, we have

$$a \in B,\ a \in B \Rightarrow a^2 = a * a \in B$$

$\Rightarrow$ $$a^3 = a^2 * a \in B$$

$\Rightarrow$ $$a^4 = a^3 * a \in B$$

Proceeding in this way, we get $a^m \in B$, where m is any positive integer. Thus the infin
elements $a, a^2\ a^3\ \ldots\ a^m \ldots$ belongs to B. If all these elements are distinct B will be an infini
finite positive integers r and $s\ (r > s)$ we must have,

$$a^r = a^s \Rightarrow a^r * a^{-s} = a^s * a^{-s}$$ [$\because a^s \in A \Rightarrow$ (c

$\Rightarrow$ $$a^{r-s} = a^0 = e, \quad \text{where } e \text{ is the identity of A}$$

$\Rightarrow$ $$a^{r-s} = e, \quad \text{where } r.\ s \text{ is a positive integer}$$

$\Rightarrow$ $$e = a^{r-s} \in B$$

$\Rightarrow$ $$e \in B.$$

Therefore, the identity e *i.e.*, a^0 is also on element of B.

4. ***Existence of Inverse***

Now $\qquad r-s \geq 1 \Rightarrow r-s-1 \geq 0$

$\Rightarrow \qquad a^{r-s-1} \in B$

$\Rightarrow \qquad a^{r-s} * a^{-1} \in B$

$\Rightarrow \qquad e * a^{-1} \in B$

$\Rightarrow \qquad a^{-1} \in B$

Thus $a \in B \Rightarrow a^{-1} \in B \ \forall \ a \in B$ *i.e.* each element of B possesses inverse.

Finally, the element of B are also the element of A. Therefore the composition in B must be associative. Hence B is a subgroup of A.

5.10.3 Criterion for the Product of two Subgroup to be a Subgroup

Theorem 22: *If H and K are subgroup of a group G then HK is a subgroup of G iff HK = KH.*

Proof: Suppose $HK = KH$. To prove that HK is a subgroup of G, it is sufficient to prove that

$$(HK)(HK)^{-1} = HK.$$

We have,

$$(HK)(HK)^{-1} = (HK)(K^{-1}H^{-1}) = H(KK^{-1})H^{-1}$$

$= (HK)H^{-1}$ [$\because$ K is a subgroup of $G \Rightarrow K$

$= (KH)H^{-1}$ [$\because$

$= K(HH^{-1})$ [by assoc

$= KH$ [$\because$ H is a subgroup of $G \Rightarrow H$

$\therefore$ $HK = KH \Rightarrow HK$ is a subgroup of G.

Converse: Let HK is a subgroup. Then

$$(HK)^{-1} = HK \Rightarrow K^{-1}H^{-1} = HK$$

$\Rightarrow$ $HK = HK$ [$\because$ K is a subgroup of $G \Rightarrow K^{-1} = K$ and similarly H

5.10.4 Intersection of Subgroup

***Theorem* 23:** If H_1 and H_2 are two subgroups of a group G, then $H_1 \cap H_2$ is also a subgroup of G

[U.P.T.U. (B.Tech.) 2004, 20

OR

Intersection of two subgroups of a group G, is *a* subgroup of G. **[Rohtak (B.E.) 20**

***Proof*:** Suppose H_1 and H_2 be any two subgroup of G. Then

$$H_1 \cap H_2 \neq \phi .$$

Since at least the identity element e is common to both H_1 and H_2 . In order to prove that $H_1 \cap$ subgroup it is sufficient to prove that

$$a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2 .$$

Now $$a \in H_1 \cap H_2 \Rightarrow a \in H_1 \text{ and } a \in H_2$$

$$b \in H_1 \cap H_2 \Rightarrow b \in H_1 \text{ and } b \in H_2.$$

But H_1, H_2 are subgroups, therefore

$$a \in H_1, b \in H_1 \Rightarrow ab^{-1} \in H_1$$

$$a \in H_2, b \in H_2 \Rightarrow ab^{-1} \in H_2$$

Finally $$ab^{-1} \in H_1, ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2.$$

Thus $$a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2.$$

Hence $H_1 \cap H_2$ is a subgroup of G.

Theorem 24: *Union of two subgroup is not necessarily a subgroup.* [U.P.T.U.(

Proof: Suppose G be the additive group of integer. Then

$$H_1 = \{0, \pm 2, \pm 4, \pm 6 \ldots) \text{ and } H_2 = \{0, \pm 3, \pm 6, \pm 9, \ldots]$$

are both subgroups of G.

We have

$$H_1 \cup H_2 = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9 \ldots\}$$

Obviously $H_1 \cup H_2$ is not closed with respect to addition since $3 \in H_1 \cup H_2$ is not closed wit addition since $3 \in H_1 \cup H_2$, $4 \in H_1 \cup H_2$ But $3+4=7 \in H_1 \cup H_2$

Therefore, $H_1 \cup H_2$ is not a subgroup of G.

However, $H_1 \cap H_2 = \{0, \pm 6, \pm 12, \pm 18, \ldots\}$ is a subgroup of G.

Example 36: *Let H be a subgroup of a group G, K is defined by* $K = \{x \in G : xH = Hx\}$.

Prove that K is a subgroup of G.

Solution: Let $x_1, x_2 \in K$ arbitrarily, then

$$x_1H = Hx_1,\ x_2H = Hx_2$$

Firstly we shall show that $x_2^{-1} \in K$.

Here $\quad x_2H = Hx_2 \Rightarrow x_2^{-1}(x_2H)x_2^{-1} = x_2^{-1}(Hx_2)x_2^{-1}$

$\Rightarrow \quad Hx_2^{-1} = x_2^{-1}H$

$\Rightarrow \quad x_2^{-1} \in K.$

Now, we shall show $\quad x_1x_2^{-1} \in K$

Here

$$\begin{aligned}(x_1x_2^{-1})H &= x_1(x_2^{-1}H) && \text{[by}\\ &= x_1(Hx_2^{-1}) \\ &= (x_1H)x_2^{-1} && \text{[by}\\ &= (Hx_1)x_2^{-1} \\ &= H(x_1x_2^{-1}) && \text{[by}\end{aligned}$$

$\therefore \quad x_1x_2^{-1} \in K.$

Thus $\quad x_1, x_2 \in K \Rightarrow x_1x_2^{-1} \in K.$

Hence, K is a subgroup of G.