Second order Partial diff. eq^n

(1) $$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = f(x),\ a_0(x) \neq 0 \quad \text{---(1)}$$

is second order ordinary differential eq^n

→ If $a_0(x)$, $a_1(x)$ & $a_2(x)$ are real constant then (1) is called d.e. with constant coefficient otherwise, variable coefficient.

→ If $f(x) = 0$, then eq^n (i) homogeneous otherwise non homogeneous.

Complementary function of differential eq^n of Second order & with constant coefficient:—

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),\ a, b, c \in \mathbb{R}$$

To find C.F. we put

$$(aD^2 + bD + c)y = 0 \quad \text{where } D = \frac{d}{dx}$$

we find auxiliary func^n by putting $D = m$ ~~we have~~ as follows

$$am^2 + bm + c = 0$$

find the root of this auxiliary eq^n

① If roots are real say m_1 & m_2

(a) if $m_1 = m_2 = m$ then

$$C.F = (c_1 x + c_2) e^{mx}$$

(b) if $m_1 \neq m_2$ then

$$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

② If roots are complex then both will be conjugate to each other say

$$m_1 = \alpha + i\beta \quad , \quad m_2 = \alpha - i\beta$$

then $C.F = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Q $$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 0$$

auxiliary func^n

$$m^2 + m + 2 = 0$$

$$m_1, m_2 = \frac{-1 \pm \sqrt{1^2 - 4\times 2}}{2\times 1}$$

$$= -\frac{1}{2} \pm 7i$$

$$C.f. = e^{-\frac{1}{2}x}(c_1 \cos 7x + c_2 \sin 7x)$$

Q $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

auxiliary eq^n

$$m^2 - 3m + 2 = 0$$

$$m_1, m_2 = 1, 2$$

$$C.F = c_1 e^{x} + c_2 e^{2x}$$

Q $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

auxiliary eqn

$$m^2 - 2m + \quad = 0$$

$$m = 1, 1 \quad \text{(repeated)}$$

$$C.F. = (c_1 x + c_2) e^{x}$$

Particular integral :- If $f(x) \neq 0$ i.e. Non homogeneous eqn then we have to find particular integral.

Let given diff. eqn is

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

$$(aD^2 + bD + c)y = f(x)$$

then P.I. is defined as

$$\boxed{P.I. = \frac{1}{aD^2 + bD + C} f(x)}$$

* Evaluation of the P.I. depends on the function f(x)

(i) If $f(x) = e^{ax}$

$$P.I. = \frac{1}{aD^2 + bD + C} e^{ax} = \frac{1}{F(D)} e^{ax}$$

substitute $D = a$

(i) If $F(a) \neq 0$ then

$$P.I. = \frac{1}{F(a)} e^{ax}$$

(ii) If $F(a) = 0$ then differentiate $F(D)$ w.r.t. 'D' i.e. $F'(D)$. now if $F'(a) \neq 0$ then

$$P.I. = \frac{x}{F'(a)} e^{ax}$$

(ii) If $f(x) = \sin ax$

$$P.I. = \frac{1}{f(D^2)} \sin ax$$

$$= \begin{cases} \frac{1}{f(-a^2)} \sin ax, & \text{if } f(-a^2) \neq 0 \\ \frac{x}{f'(-a^2)} \sin ax, & \text{if } f(-a^2) = 0 \end{cases}$$

(iii) If $f(x) = \cos ax$

$$P.I. = \frac{1}{f(D^2)} \cos ax$$

$$= \begin{cases} \frac{1}{f(-a^2)} \cos ax, & f(a^2) \neq 0 \\ \frac{x}{f'(-a^2)} \cos ax, & f(a^2) = 0 \end{cases}$$

ex $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$

auxiliary eqn

$$m^2 + 6m + 9 = 0$$
$$(m+3)^2 = 0$$
$$m = -3, -3$$

$$C.F. = (C_1x + C_2)e^{-3x}$$

$$P.I. = \frac{1}{D^2+6D+9} 5e^{3x}$$
$$= 5\frac{1}{D^2+6D+9} e^{3x}$$
$$= 5\frac{1}{3^2+6\times3+9} e^{3x}$$
$$= \frac{5}{36} e^{3x}$$

complete soln.

$$y = C.F + P.I. = (C_1x+C_2)e^{-3x} + \frac{5}{36}e^{3x}$$

Quest $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 4y = e^x$

$$(D^3 - D^2 + 4D - 4)y = e^x$$

auxiliary funct is

$$m^3 - m^2 + 4m - 4 = 0$$
$$m = 1, 2i, -2i$$

$$C.F = C_1e^x + C_2\cos 2x + C_3\sin 2x$$

$$P.I. = \frac{1}{D^3 - D + 4D - 4} e^{1.x}$$

$\because f(a) = f(1) = 0$

$f'(D) = 3D^2 - 2D + 4$

$f'(1) = 3 - 2 + 4 = 5 \neq 0$

$\therefore$ P.S. $= \dfrac{x}{f'(1)} e^x$

$= \frac{1}{5} x e^x$

C.S.

$y = $ C.F. + P.S. $= c_1 e^x + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{5} x e^x$

ex $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x + \cos x$

$(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$

auxiliary eq^n is

$m^3 - 3m^2 + 4m - 2 = 0$

$m = 1, 1 \pm i$

$C.F = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$

$P.S. = \dfrac{1}{D^3 - 3D^2 + 4D - 2} (e^x + \cos x)$

$= \dfrac{1}{D^3 - 3D^2 + 4D - 2} e^x + \dfrac{1}{D^3 - 3D^2 + 4D - 2} \cos x$

Now, $\dfrac{1}{D^3 - 3D^2 + 4D - 2} e^x$, $F(D) = D^3 - 3D^2 + 4D - 2$

$f(1) = 0$

$f'(D) = 3D^2 - 6D + 4$

$f'(1) = 3 - 6 + 4 = 1$

$P.S_1 = \dfrac{x}{1} e^x$

$= x e^x$

again

$$P.I._2 = \frac{1}{D^3 - 3D^2 + 4D - 2} \cos x \qquad a = 1$$

$D^2 \rightarrow -a^2 = -1$

$$= \frac{1}{D(-1) - 3(-1) + 4D - 2} \cos x$$

$$= \frac{1}{3D+1} \cos x$$

$$= \frac{3D-1}{(3D)^2-1} \cos x = \frac{3D-1}{9D^2-1} = \frac{3D-1}{-10} \cos x$$

$$= -\frac{1}{10}(3D-1)\cos x$$

$$= -\frac{1}{10}(-3\sin x - 1) = \frac{1}{10}(3\sin x + \cos x)$$

complete integral is

$$y = C.F. + P.I._1 + P.I._2$$

$$= c_1 e^x + c_2 e^x \cos x + c_3 e^x \sin x + x e^x + \frac{1}{10}(3\sin x + \cos x)$$