BMS-01 ENGINEERING MATHEMATICS-I Course category : Basic Sciences & Maths (BSM) Number of Credits : 4

UNIT-I Differential Calculus: Leibnitz theorem, Partial derivatives, Euler's theorem for homogenous function, Total derivative, Change of variable. Taylor's and Maclaurin's theorem. Expansion of function of two variables, Jacobian, Extrema of function of several variables.

UNIT-II Linear Algebra: Rank of Matrix, Inverse of a Matrix, Elementary transformation, Consistency of linear system of equations and their solution. Characteristic equation, Eigenvalues, Eigen-vectors, Cayley-Hamilton theorem.

UNIT-III Multiple Integrals: Double and triple integrals, change of order of integration, change of variables. Application of multiple integral to surface area and volume. Beta and Gamma functions, Dirichlet integral.

UNIT-IV Vector Calculus: Gradient, Divergence and Curl. Directional derivatives, line, surface and volume integrals. Applications of Green's, Stoke's and Gauss divergence theorems (without Proofs).

Books&References

1. B.S. Grewal: Higher Engineering Mathematics; Khanna Publishers.
2. B.V. Ramana: Higher Engineering Mathematics, Tata Mc. Graw Hill Education Pvt. Ltd., New Delhi.
3. H.K. Dass and Rama Verma: Engineering Mathematics; S. Chand Publications.
4. N.P. Bali and Manish Goel: Engineering Mathematics; Laxmi Publications.

$\frac{dy}{dx} = \lim_{h\to 0} \cancel{\frac{f(x+h)-f(x)}{h}}$

Unit - I

(successive differentiation)

Let $y = f(x)$ is differentiable then

$\frac{dy}{dx}$ is the first order derivative of y w.r.t 'x'.

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) =$ 2^{nd} order derivative of y w.r.t 'x'.

and $\frac{d^ny}{dx^n}$ is known as n^{th} order derivative of y w.r.t 'x'

We can also write

$$y' = \frac{dy}{dx} = y_1 \quad , \quad y'' = \frac{d^2y}{dx^2} = y_2$$

In general

$y^{(n)} = \frac{d^n y}{dx^n} = y_n$. If $D = \frac{d}{dx}$, then $Dy = \frac{dy}{dx}$, $D^n y = \frac{d^n y}{dx^n}$

is the nth order derivative of y w.r.t x.

nth order derivative

(1) $y = e^{ax}$, $D^n(y) = D^n(e^{ax}) = a^n e^{ax}$.

Proof:- $y = e^{ax}$

$y_1 = \frac{d}{dx}(e^{ax}) = a e^{ax}$

$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dx}(a e^{ax}) = a^2 e^{ax}$

In general

$y_n = a^n e^{ax}$.

(ii) $y = a^{mx}$, $D^n(y) = D^n(a^{mx}) = m^n a^{mx} (\log a)^n$

Proof:

$$y = a^{mx}$$

$$y_1 = a^{mx} (\log a) \times m$$

$$= m^1 (\log a)^1 a^{mx}$$

Again differentiating

$$y_2 = m(\log a) \frac{d}{dx}(a^{mx})$$

$$= m(\log a)\, a^{mx} \log a \times m$$

$$= m^2 (\log a)^2 a^{mx}$$

In general

$$y_n = m^n (\log a)^n a^{mx}.$$

(iii) If $y = (ax+b)^m$, $D^n y = \dfrac{a^n\, m!\, (ax+b)^{m-n}}{(m-n)!}$ if $m > n$

$= n!\, a^n$, if $m = n$

$= 0$, $m < n$.

$\frac{d^3}{dx^3}(x^2) = 0$

→ H.K. Das

→ N.P. Bali

B.S. Grewal

Proof :-

$y = (ax+b)^m$

Differentiating

$y_1 = m(ax+b)^{m-1} \times a$

$y_2 = m(m-1)(ax+b)^{m-2}\, a^2$

$y_3 = m(m-1)(m-2)(ax+b)^{m-3}\, a^3$

Hence

$y_n = m(m-1)(m-2) \times \cdots \times (m-n+1)(ax+b)^{m-n} a^n$

$$= \frac{m(m-1) \times \cdots \times (m-n+1) \times (m-n) \times \cdots \times 2 \times 1 \times (ax+b)^{m-n} a^n}{1 \times 2 \times 3 \times \cdots \times (m-n)}$$

$$= \frac{m!\,(ax+b)^{m-n}\, a^n}{(m-n)!}$$

If $m = n$

$y_n = n!\, a^n$

(iv) If $y = \frac{1}{ax+b}$, then find y_n.

$$y = \frac{1}{(ax+b)(cx+d)}$$

Sol:

$$y = \frac{1}{ax+b} = (ax+b)^{-1}$$

Differentiating w.r.t 'x'

$$y_1 = \frac{-1}{(ax+b)^2} \times a = \frac{-a}{(ax+b)^2} = -a(ax+b)^{-2}$$

Again differentiating we get

$$y_2 = (-1)^2\, 2\, \frac{a^2}{(ax+b)^3} = (-1)^2\, 2!\, \frac{a^2}{(ax+b)^3}$$

$$\Rightarrow y_3 = (-1)^3\, 3!\, \frac{a^3}{(ax+b)^4}$$

$$y_n = \frac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$$

(v) If $y = \sin(ax+b)$

then $y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$.

and

$y = \cos(ax+b)$

then $y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$

Proof :-

$y = \sin(ax+b)$

$\Rightarrow y_1 = \cos(ax+b)$

$= \sin\left(ax+b+\frac{\pi}{2}\right) \times a$

Again diff :-

$y_2 = \cos\left(ax+b+\frac{\pi}{2}\right) \times a^2$

$= a^2 \sin\left(ax+b+\frac{2\pi}{2}\right)$

In general

$y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

(vi) If

$y = e^{ax} \sin(bx+c)$

then $y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax} \sin(bx+c+n\phi)$

when $\phi = \tan^{-1}(b/a)$

Proof:-

$$y = e^{ax} \sin(bx+c)$$

$$y_1 = e^{ax} \times a \sin(bx+c) + e^{ax} \cos(bx+c) \times b$$

$$= e^{ax} \left(a \sin(bx+c) + b \cos(bx+c) \right)$$

let $a = r\cos\phi$, $b = r\sin\phi$

$$\Rightarrow r^2 = a^2 + b^2, \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

Hence

$$y_1 = r e^{ax} \left(\cos\phi \sin(bx+c) + \sin\phi \sin(bx+c) \right)$$

$$= r e^{ax} \sin(bx+c+\phi)$$

Again differentiating we get

$$y_2 = r\left[a e^{ax} \sin(bx+c+\phi) + e^{ax} b \cos(bx+c+\phi) \right]$$

$$y_2 = r^2 e^{ax} \sin(bx + c + 2\phi)$$

$$\begin{array}{r|l} & x \\ x^2-1 & x^3 \\ & \underline{x^3 - x} \\ & \quad - \;\; + \\ & \quad x \end{array}$$

$y = \cancel{\frac{1}{ax+b}}$

In general

$$y_n = r^n e^{ax} \sin(bx + c + n\phi)$$

where $r = (a^2+b^2)^{\frac{1}{2}}$, $\phi = \tan^{-1}\left(\frac{b}{a}\right)$.

Ques:- If $y = \dfrac{x^3}{x^2-1}$ then prove that $(y_n)_0 = \begin{cases} -(n!) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

Sol

$$y = \frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$

$$= x + \frac{x}{(x-1)(x+1)}$$

$$= x + \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x+1}\right]$$

so

$$y = x + \frac{1}{2}\left[\frac{1}{(x-1)} + \frac{1}{(x+1)}\right]$$

Differentiating n-times w.r.t 'x', we get

$$y_n = \frac{1}{2}\left[D^n (x-1)^{-1} + D^n (x+1)^{-1}\right]$$

$$= \frac{1}{2}\left[(-1)^n n! (x-1)^{-(n+1)} + (-1)^n n! (x+1)^{-(n+1)}\right]$$

$$\boxed{y_n = \frac{1}{2}(-1)^n n! \left[(x-1)^{-(n+1)} + (x+1)^{-(n+1)}\right]}$$

$\forall\, n \geq 1.$

Put $x = 0$

$$(y_n)_0 = \frac{1}{2}(-1)^n n! \left[(-1)^{-(n+1)} + 1^{-(n+1)}\right]$$

$$\Rightarrow (y_n)_0 = \frac{1}{2}(-1)^n n! \left[\frac{1}{(-1)^{n+1}} + 1\right]$$

If n is even

$\Rightarrow (y_n)_0 = 0$

If n is odd

$$(y_n)_0 = \frac{1}{2}(-1)^n n! [1+1]$$

$$= (-1)^n n!$$

$$= -(n!)$$ Ans

Qus:-

(i) If $y = \dfrac{x^2}{(x-1)(x-2)}$ $\rightarrow \dfrac{x^2}{x^2-3x+2}$ find (y_n).

(ii) If $y = \dfrac{x^2}{(x-2)^2(x+1)}$ find $y_n \rightarrow \dfrac{8}{9}\dfrac{(-1)^n n!}{(x-2)^{n+1}} + \dfrac{4}{3}\dfrac{(-1)^n (n+1)!}{(x-2)^{(n+2)}} + \dfrac{1}{9}\dfrac{(-1)^n n!}{(x+1)^{(n+1)}}$

(iii) If $y = \dfrac{x}{1+3x+2x^2}$ find y_n.

(iv) If $y = \dfrac{x^3}{x^2-1}$ show that $(y_4)_0 = 0$

$$y = \frac{3}{2x^2+3x+1} = \frac{3}{(2x+1)(x+1)} = 3\left[\frac{2}{2x+1} - \frac{1}{x+1}\right]$$

Q Find the n^{th} derivative of

(a) $\dfrac{x}{x^2+a^2}$

(b) $\dfrac{a}{x^2+a^2}$

(c) $\tan^{-1}\left(\dfrac{x}{a}\right)$

(d) $\tan^{-1}x$

(e) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

(f) $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

(vi) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

(e) $x=\tan\theta$

$$y=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\tan\theta/2\right)$$

$$=\theta/2$$

$$=\frac{1}{2}\tan^{-1}x$$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ $\qquad x + \cancel{a\theta} + x - \cancel{a\theta}$

Sol:-

$$y = \tan^{-1}\left(\frac{x}{a}\right)$$

Differentiating w.r.t 'x'

$$y_1 = \frac{1}{1+\frac{x^2}{a^2}} \times \frac{1}{a}$$

$$= \frac{a}{a^2+x^2}$$

$$= \frac{a}{x^2+a^2}$$

$$= \frac{a}{x^2 - i^2 a^2}$$

$$= \frac{a}{(x-ai)(x+ai)}$$

$\cancel{(x-ai)(x+ai)}$

$a^2 - i^2 x^2$

$(a-ix)(a+ix)$

$$= \frac{1}{2i}\left[\frac{1}{x-ai} - \frac{1}{x+ai}\right]$$

$$\therefore y_1 = \frac{1}{2i}\left[(x-ai)^{-1} - (x+ai)^{-1}\right]$$

$y = (x-ai)^{-1}$

$y_1 = -(x-ai)^{-2}$

$y_2 = (-1)(-2)(x-ai)^{-3}$

$= (-1)^2 2!\,(x-ai)^{-3}$

Differentiating $(n-1)$ times

$$y_n = \frac{1}{2i}\left[D^{n-1}(x-ai)^{-1} - D^{n-1}(x+ai)^{-1}\right]$$

$$= \frac{1}{2i}\left[(-1)^{n-1}(n-1)!\,(x-ai)^{-n} - (-1)^{n-1}(n-1)!\,(x+ai)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\left[(x-ai)^{-n} - (x+ai)^{-n}\right]$$

Put $x = r\cos\theta$, $a = r\sin\theta$

$\Rightarrow \theta = \tan^{-1}\left(\frac{a}{x}\right)$

$$\boxed{[\cos\theta + i\sin\theta]^n = \cos n\theta + i\sin n\theta}$$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[(r\cos\theta - ir\sin\theta)^{-n} - (r\cos\theta + ir\sin\theta)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i} r^{-n}\left[(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin\theta)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i} r^{-n}\left[\cancel{\cos n\theta} + i\sin n\theta - \cancel{\cos n\theta} + i\sin n\theta\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{\cancel{2i}} r^{-n} \times \cancel{2i}\sin n\theta$$

$$= (-1)^{n-1}(n-1)!\left(\frac{a}{\sin\theta}\right)^{-n}\sin n\theta = \frac{(-1)^{n-1}(n-1)!}{a^n}\sin^n\theta\sin n\theta$$

$\theta = \tan^{-1}\left(\frac{a}{x}\right)$

#

Demoivre's law

(f) $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$y = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$

$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)$

$= \frac{\pi}{4}+\theta$

$= \frac{\pi}{4}+\tan^{-1}x$

$$\boxed{y_1 = \frac{1}{1+x^2}}$$

$y_1 = \frac{1}{(x+i)(x-i)}$

$= \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$

$y_1 = \frac{1}{2i}\left((x-i)^{-1} - (x+i)^{-1}\right)$

Again differentiating '$n-1$' times we get

$y_n = \frac{1}{2i}\left[D^{n-1}(x-i)^{-1} - D^{n-1}(x+i)^{-1}\right]$

$= \frac{1}{2i}\left[(-1)^{n-1}(n-1)!\,(x-i)^{-n} - (-1)^{n-1}(n-1)!\,(x+i)^{-n}\right]$

$= \frac{1}{2i}(-1)^{n-1}(n-1)!\left[(x-i)^{-n} - (x+i)^{-n}\right]$

put $x = r\cos\theta \Rightarrow \theta = \tan^{-1}\left(\frac{1}{x}\right)$

$1 = r\sin\theta$

$r = \left(\frac{1}{\sin\theta}\right)$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left((r\cos\theta - r\sin\theta\, i)^{-n} - (r\cos\theta + i r\sin\theta)^{-n}\right)$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\, r^{-n}\left(\cancel{\cos n\theta} + i\sin n\theta - \cancel{\cos n\theta} + i\sin n\theta\right)$$

$$= (-1)^{n-1}(n-1)!\left(\frac{1}{\sin\theta}\right)^{-n}\sin n\theta$$

$$= (-1)^{n-1}(n-1)!\,\sin^{n}\theta\,\sin n\theta \quad \text{where } \theta = \tan^{-1}\left(\frac{1}{x}\right).$$

#

Q.

If a) $y = \cos x \cos 2x \cos 3x$ then find y_n.

b) $y = \sin^5 x \cos^3 x$ then find y_n.

Sol:-

$$y = \cos x \cos 2x \cos 3x$$

$$= \frac{1}{2}\left[\cos 3x + \cos x\right] \cos 3x$$

$$= \frac{1}{2}\left[\frac{1}{2}\left(\cos 6x + 1\right) + \frac{1}{2}\left(\cos 4x + \cos 2x\right)\right]$$

$$= \frac{1}{4}\left[\cos 6x + \cos 4x + \cos 2x + 1\right]$$

Differentiating n-times we get:-

$$y_n = \frac{1}{4}\left[6^n \cos\left(6x + \frac{n\pi}{2}\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \frac{n\pi}{2}\right)\right] \text{ Ans.}$$

(b) $y = \sin^5 x \cos^3 x = \sin^2 x \left(\sin x \cos x\right)^3$

$$= \sin^2 x \left(\frac{1}{2} \sin 2x\right)^3 = \frac{1}{8} \sin^2 x \sin^3 2x$$

Q. If $y = x\log\left(\frac{x-1}{x+1}\right)$. Prove that

$$y_n = (-1)^n (n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$$

Sol:

$$y = x\left[\log(x-1) - \log(x+1)\right]$$

$$= x\log(x-1) - x\log(x+1)$$

$$\Rightarrow y_1 = \log(x-1) + \frac{x}{x-1} - \log(x+1) - \frac{x}{x+1}$$

$$= \log(x-1) + \left(1+\frac{1}{x-1}\right) - \log(x+1) - \left(1-\frac{1}{x+1}\right)$$

$$\boxed{y_1 = \log(x-1) - \log(x+1) + \frac{1}{x-1} + \frac{1}{x+1}} \quad — (1)$$

$y = (x-1)^{-1}$

$y_1 = (-1)(x-1)^{-2}, \quad y_2 = (-1)^2 2!(x-1)^{-3}$

$y = \log(x-1)$

$y_1 = \frac{1}{x-1} = (x-1)^{-1}$

$y_2 = (-1)(x-1)^{-2}$

$y_3 = (-1)(-2)(x-1)^{-3}$

$= (-1)^2 2!(x-1)^{-3}$

$(-1)^{-2} = \frac{1}{(-1)^2} = 1$

Differentiating equation ① for '$n-1$' times

$$y_n = (-1)^{n-2}(n-2)!\,(x-1)^{-(n-1)} - (-1)^{n-2}(n-2)!\,(x+1)^{-(n-1)}$$
$$+ (-1)^{n-1}(n-1)!\,(x-1)^{-n} + (-1)^{n-1}(n-1)!\,(x+1)^{-n}$$

$$= (-1)^n (n-2)! \left[\frac{1}{(x-1)^{n-1}} - \frac{1}{(x+1)^{n-1}} - \frac{n-1}{(x-1)^n} - \frac{n-1}{(x+1)^n}\right]$$

$$= (-1)^n (n-2)! \left[\frac{x-1}{(x-1)^n} - \frac{(x+1)}{(x+1)^n} - \frac{(n-1)}{(x-1)^n} - \frac{(n-1)}{(x+1)^n}\right]$$

$$= (-1)^n (n-2)! \left[\frac{x-n}{(x-1)^n} - \frac{n+x}{(x+1)^n}\right] \#$$

Q. If $y = \sin px + \cos px$ show that

$$y_n = p^n \left[1 + (-1)^n \sin 2px \right]^{\frac{1}{2}}$$

Sol:

$$y = \sin px + \cos px$$

Differentiating n-times we get

$$y_n = p^n \sin\left(px + \frac{n\pi}{2}\right) + p^n \cos\left(px + \frac{n\pi}{2}\right)$$

$$= p^n \left[\sin\left(px + \frac{n\pi}{2}\right) + \cos\left(px + \frac{n\pi}{2}\right) \right]$$

$$= p^n \left\{ \left[\sin\left(px + \frac{n\pi}{2}\right) + \cos\left(px + \frac{n\pi}{2}\right) \right]^2 \right\}^{\frac{1}{2}}$$

$$= p^n \left[\sin^2\left(px + \frac{n\pi}{2}\right) + \cos^2\left(px + \frac{n\pi}{2}\right) + 2\sin\left(px + \frac{n\pi}{2}\right)\cos\left(px + \frac{n\pi}{2}\right) \right]^{\frac{1}{2}}$$

$$= p^n \left[1 + \sin 2\left(px + \frac{n\pi}{2}\right) \right]^{\frac{1}{2}}$$

$$= p^n \left[1 + \sin(2px + n\pi) \right]^{\frac{1}{2}}$$

$$= p^n \left[1 + (-1)^n \sin 2px \right]^{\frac{1}{2}}$$

$\cos^2 x = 1 + \cos 2\theta$

Q Find n^{th} derivative of

(i) $e^{ax} \cos^2 x \sin x$

(ii) $2^x \sin(3x+1)$

(iii) $y = \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

$\left(y_n = 2(-1)^{n-1}(n-1)!\, \sin^n\theta \sin n\theta\right)$

$a^x = e^{\log a^x}$

$= e^{(\log a)x}$

$\{X = e^{\log X}\}$

$e^{ax} \sin bx$

Sol: (ii) $y = 2^x \sin(3x+1)$

$= e^{\log 2^x} \sin(3x+1)$

$= e^{(\log 2)x} \sin(3x+1)$

$y = e^{ax} \sin bx$

$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax} \sin(bx + n\phi)$

on diff. n times we get

$y_n = ((\log 2)^2 + 9)^{\frac{n}{2}} 2^x \sin(3x+1+n\phi)$

$\phi = \tan^{-1}\left(\frac{3}{\log 2}\right)$

Leibnitz theorem :-

If u and v are two functions of x whose n^{th} derivative is known then

$$D^n(uv) = D^n(u) \times v + {}^nC_1 D^{n-1}(u) \times D(v) + {}^nC_2 D^{n-2}(u) \times D^2(v) + \cdots + u \cdot D^n(v).$$

where ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$

$${}^nC_1 = \frac{n!}{1! \times (n-1)!} = n$$

$${}^nC_2 = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2}$$

Ex:-

If $y = e^x x^2$

Differentiating n times by Leibnitz rule, we get

$$y_n = D^n(e^x x^2)$$

$$= D^n(e^x) \times x^2 + {}^nC_1 D^{n-1}(e^x) \times D(x^2) + {}^nC_2 D^{n-2}(e^x) \times D^2(x^2) + \cdots$$

$$= e^x x^2 + n e^x \times 2x + \frac{n(n-1)}{\cancel{2}} \times e^x \times \cancel{2}$$

$$= e^x \left(x^2 + 2nx + n(n-1)\right) \text{ Ans}$$

Proof :

We will prove the theorem by Mathematical induction on 'n'.

For $n = 1$

$$D(uv) = Du \times v + u \times Dv$$

$\Rightarrow$ Leibnitz theorem is true for $n = 1$.

$D = \frac{d}{dx}$

Let the theorem is true for $n = k$ i.e.,

$$D^k(uv) = D^k(u) \cdot v + {}^kC_1 D^{k-1}(u) \times D(v) + {}^kC_2 D^{k-2}(u) \cdot D^2(v) + \cdots + u \times D^k(v) \quad \text{—} (1)$$

Now we will prove the result for $n = k+1$.

Differentiating equation (1) w.r.t 'x'

$$D^{k+1}(uv) = [D^{k+1}(u) \times v + D^k(u) \cdot D(v)] + {}^kC_1 [D^k(u) \cdot D(v) + D^{k-1}(u) \times D^2(v)] + \cdots + [D(u) \times D^k(v) + u \times D^{k+1}(v)]$$

$$1 \to {}^{k}c_0$$
$$ {}^{k}c_0 + {}^{k}c_1 - $$

$$= D^{k+1}(u)\cdot v + [1 + {}^{k}c_1]\, D^{k}(u)\cdot D(v) + \cdots + u \times D^{k+1}(v)$$

$$\because \; {}^{n}c_r + {}^{n}c_{r-1} = {}^{n+1}c_r$$

$$\Rightarrow {}^{n}c_1 + {}^{n}c_0 = {}^{n+1}c_1$$

$$D^{k+1}(uv) = D^{k+1}(u)\cdot v + {}^{k+1}c_1\, D^{k}(u)\cdot D(v) + \cdots + u \times D^{k+1}(v). \quad -(2)$$

$\Rightarrow$ our result is true for $n = k+1$

Hence by induction our theorem is true for all positive integer value of n.

Q

If $y = (x^2-1)^m$ then show that

$$y_{2m} = (2m)!$$

$y = (x-1)^m + (x+1)^m$

Sol:-

$\because$ $y = (x^2-1)^m$

$$= x^{2m} + {}^mC_1 (x^2)^{m-1}(-1) + {}^mC_2 (x^2)^{m-2}(-1)^2 + \cdots + (-1)^m$$

$$= x^{2m} - {}^mC_1 x^{2m-2} + {}^mC_2 x^{2m-4} - \cdots + (-1)^m$$

on differentiating '2m' times

$\therefore y_{2m} = (2m)!$ #

$y = \frac{1}{x-1}$

$y = (x-1)^{-1}$

$y_1 = (-1)(x-1)^{-2}$

$y_2 = (-1)(-2)(x-1)^{-3}$

$= (-1)^2 2! (x-1)^{-3}$

$y_3 = (-1)(-2)(-3)(x-1)^{-4}$

$= (-1)^3 3! (x-1)^{-4}$

$y_n = (-1)^n n! (x-1)^{-(n+1)}$

$(x-a)^n = x^n + {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 + \cdots + a^n$

Q. If $y = \sin(m\sin^{-1}x)$ then show that

i) $(1-x^2)y_2 - xy_1 + m^2y = 0$

ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$

Sol:-

$$y = \sin(m\sin^{-1}x)$$

$$\Rightarrow y_1 = \cos(m\sin^{-1}x) \times \frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\,y_1 = m\cos(m\sin^{-1}x)$$

Again differentiating, we get

$$\frac{-x}{\sqrt{1-x^2}} \times y_1 + \sqrt{1-x^2} \times y_2 = -m\sin(m\sin^{-1}x) \times \frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow -xy_1 + (1-x^2)y_2 = -m^2y$$

$$\Rightarrow \boxed{(1-x^2)y_2 - xy_1 + m^2y = 0}$$

Differentiating this equation n times by Leibnitz rule :-

$$D^n\{(1-x^2)y_2\} - D^n(xy_1) + m^2 D^n(y) = 0$$

$$\Rightarrow \{D^n(y_2)\times(1-x^2) + {}^nC_1 D^{n-1}(y_2)\times D(1-x^2) + {}^nC_2 D^{n-2}(y_2)\times D^2(1-x^2)\}$$

$$- \{D^n(y_1)\times x + {}^nC_1 D^{n-1}(y_1)\times D(x)\} + m^2 y_n = 0$$

$$\boxed{D^n(uv) = D^n(u)\times v + {}^nC_1 \boxed{D^{n-1}(u)}\times D(v) + {}^nC_2 \boxed{D^{n-2}(u)}\times D^2(v) + \ldots}$$

$$\boxed{{}^nC_1 = n}$$

$$\Rightarrow y_{n+2}(1-x^2) + n\,y_{n+1}\times(-2x) + \frac{n(n-1)}{\cancel{2}}\times y_n\times(-\cancel{2})\}$$

$$- \{y_{n+1}\times x + n\,y_n\times 1\} + m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} + y_n\{-n^2 + \cancel{n} - \cancel{n} + m^2) = 0$$

Ans

(y_2)

Q If $y = a\cos(\log x) + b\sin(\log x)$ then show that

(i) $x^2 y_2 + x y_1 + y = 0$

(ii) $x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$

Sol:-

$$y = a\cos(\log x) + b\sin(\log x)$$

$$y_1 = a \times -\sin(\log x) \times \frac{1}{x} + b\cos(\log x) \times \frac{1}{x}$$

$$\Rightarrow x y_1 = -a\sin(\log x) + b\cos(\log x)$$

Again differentiating, we get

$$1 \cdot y_1 + x y_2 = -a\cos(\log x) \times \frac{1}{x} - b\sin(\log x) \times \frac{1}{x}$$

$$\Rightarrow x y_1 + x^2 y_2 = -[a\cos(\log x) + b\sin(\log x)] = -y$$

$$\boxed{x^2 y_2 + x y_1 + y = 0}$$

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

x^2

$$D^n(u \times v) = D^n(u) \times v + {}^nC_1\, D^{n-1}(u) \times D(v) + {}^nC_2 \times D^{n-2}(u) \times D^2(v) + {}^nC_3 \times D^{n-3}(u) \times D^3(v) + \cdots$$

Differentiating n-times by Leibnitz rule, we get

$$D^n\{x^2 y_2\} + D^n\{x y_1\} + D^n\{y\} = 0$$

$$\Rightarrow \left\{ y_{n+2} \times x^2 + n \times y_{n+1} \times 2x + \frac{n(n-1)}{\cancel{2}} \times y_n \times \cancel{2} \right\}$$

$$+ \left\{ y_{n+1} \times x + n \times y_n \times 1 \right\} + y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + y_{n+1}(2nx + x) + y_n\{n^2 - \cancel{n} + \cancel{n} + 1\} = 0$$

$$\Rightarrow \boxed{x^2 y_{n+2} + (2n+1)x\, y_{n+1} + (1+n^2)\, y_n = 0}$$

Q If $x = \cosh\left(\frac{1}{m}\log y\right)$

prove that $(x^2-1)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2-m^2)\,y_n = 0$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\cosh^2\theta - \sinh^2\theta = 1$$

$$\cosh 2\theta = \cosh^2\theta + \sinh^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$e^{\theta} = \cosh\theta + \sinh\theta$$

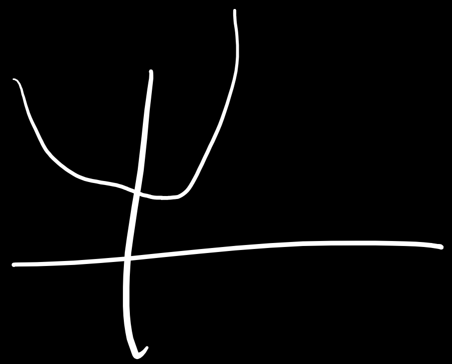
$$e^{-\theta} = \cosh\theta - \sinh\theta$$

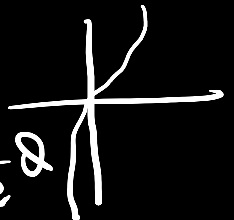
$$y = \cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$$

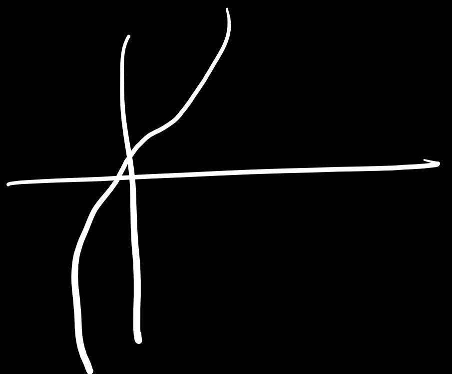
$$\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

$\cosh\theta$

$\cosh\theta$







$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$$

$$\boxed{\cosh^{-1}x = \log\left(x+\sqrt{x^2-1}\right)}$$

$$\cosh^{-1}x = y$$

$$x = \cosh y$$

$$\frac{d}{d\theta}(\cosh\theta) = \frac{e^{\theta}-e^{-\theta}}{2} = \sinh\theta$$

$$\frac{d}{d\theta}(\sinh\theta) = \frac{d}{d\theta}\left(\frac{e^{\theta}-e^{-\theta}}{2}\right) = \frac{e^{\theta}+e^{-\theta}}{2} = \cosh\theta$$

$$\cosh^2 y - \sinh^2 y = 1$$

$$\cosh^2 y = 1+\sinh^2 y$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1$$

$$\coth^2 x = 1 + \operatorname{cosech}^2 x$$

$$x = \cosh\left(\frac{1}{m}\log y\right)$$

$$\Rightarrow \cosh^{-1} x = \frac{1}{m}\log y$$

$$\Rightarrow m\cosh^{-1} x = \log y$$

$$\Rightarrow \boxed{y = e^{m\cosh^{-1} x}}$$

$$\Rightarrow y_1 = e^{m\cosh^{-1} x} \times m\frac{1}{\sqrt{x^2-1}}$$

$$\Rightarrow y_1 \times \sqrt{x^2-1} = m e^{m\cosh^{-1} x}$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

Again differentiating we get.

$$\frac{y_2 \times \sqrt{x^2-1} + y_1 \times \frac{x}{\sqrt{x^2-1}}}{1}$$

$$= m e^{m\cosh^{-1} x} \times m \times \frac{1}{\sqrt{x^2-1}}$$

$$\Rightarrow (x^2-1)y_2 + xy_1 = m^2 y$$

$$\Rightarrow \boxed{(x^2-1)y_2 + xy_1 - m^2 y = 0}$$

Differentiating for n-times we get

$$D^n\{(x^2-1)y_2\} + D^n\{xy_1\} - m^2 D^n\{y\} = 0$$

$$\{y_{n+2}\times(x^2-1) + n\times y_{n+1}\times(2x) + \frac{n(n-1)}{\cancel{2}}\times y_n\times \cancel{2}\}$$
$$+ \{y_{n+1}\times x + n\times y_n \times 1.\} - m^2 y_n = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + y_n\{n^2-n+n-m^2\} = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

Q (1) If $x = \tan(\log y)$, Show that

(1) $(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$

(2) $(1+x^2)y_{n+2} + [2(n+1)x-1]y_{n+1} + n(n+1)y_n = 0$

(2) $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ show that

$$(x^2-1)\, y_{n+2} + (2n+1)\, x y_{n+1} + (n^2-m^2)\, y_n = 0$$

(3) $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{m}\right)^m$ show that

$$x^2 y_{n+2} + (2n+1)\, x y_{n+1} + (n^2+m^2)\, y_n = 0 .$$

(4) If $y = \log\left(x + \sqrt{x^2+1}\right)$ show that

$$(1+x^2)\, y_{n+2} + (2n+1)\, x y_{n+1} + n^2 y_n = 0$$

① If $x = \tan(\log y)$ show that

(i) $(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$

(ii) $(1+x^2)y_{n+2} + [2(n+1)x-1]y_{n+1} + n(n+1)y_n = 0$.

$\tan^{-1}x = \log y$

$\Rightarrow \boxed{y = e^{\tan^{-1}x}} \Rightarrow y_1 = e^{\tan^{-1}x} \times \frac{1}{1+x^2}$

$(1+x^2)$

$\Rightarrow \boxed{(1+x^2)y_1 = y}$

$\cos 2\theta = 2\cos^2\theta - 1$

Q $\quad y = e^{ax}\cos^2 x \sin x$

$= e^{ax}\left(\frac{1+\cos 2x}{2}\right)\sin x$

$\sin x \cos 2x$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$= \frac{1}{2}e^{ax}(\sin x + \cos 2x \sin x)$

$= \frac{1}{2}e^{ax}\left(\sin x + \frac{1}{2}(\sin 3x - \sin x)\right\}$?

$= \frac{1}{4}e^{ax}(\sin x + \sin 3x)$

$y = \frac{1}{4}\left(\underline{\underline{e^{ax}\sin x}} + \underline{\underline{e^{ax}\sin 3x}}\right)$

Q If $y = \frac{\log x}{x}$ then show that

$$y_n = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right]$$

Sol:- $y = \frac{\log x}{x}$

Differentiating n-times using Leibnitz theorem:

$$y_n = D^n\left(\log x \times \frac{1}{x}\right)$$

$$= D^n(x^{-1}) \times \log x + {}^nC_1\, D^{n-1}(x^{-1}) \times D(\log x)$$

$$+ {}^nC_2\, D^{n-2}(x^{-1}) \times D^2(\log x)$$

$$+ \cdots + x^{-1} \times D^n(\log x)$$

(i) $y = \log x$

$y_1 = \frac{1}{x} = x^{-1}$

$y_2 = (-1)\, x^{-2}$

$y_3 = (-1)(-2)\, x^{-3}$

$= (-1)^2\, 2!\, x^{-3}$

$y_4 = (-1)^3\, 3!\, x^{-4}$

$y_n = (-1)^{n-1}(n-1)!\, x^{-n}$

$y = \frac{1}{x} = x^{-1}$

$y_1 = (-1)\, x^{-2}$

$y_2 = (-1)(-2)\, x^{-3}$

$= (-1)^2\, 2!\, x^{-3}$

$y_n = (-1)^n n!\, x^{-(n+1)}$

$$= \frac{(-1)^n n!\, x^{-(n+1)}}{} \times \log x + n \times (-1)^{n-1} (n-1)!\, \times x^{-n} \times \frac{1}{x}$$

$$+ \frac{n(n-1)}{2} \times (-1)^{n-2} (n-2)!\, \times x^{-(n-1)} \times \left(\frac{-1}{x^2}\right)$$

$$+ x^{-1} \times (-1)^{n-1} (n-1)!\, x^{-n}$$

$$= \frac{(-1)^n n!}{x^{n+1}} \left[\log x - 1 - \frac{1}{2} \cdots\cdots - \frac{1}{n} \right]$$

Ans.

Q If $y = e^x \log x$ then

$$y_n = e^x \left[\log x + \frac{n}{x} - \frac{n(n-1)}{2} \times \frac{1}{x^2} \cdots \frac{(-1)^{n-1} (n-1)!}{x^n} \right]$$

Q If $\underline{I_n} = \frac{d^n}{dx^n}(x^n \log x)$ then prove that

$$\boxed{D^n(x^n) = n!}$$

$$D^3(x^3) = 3!$$

$$I_n = n I_{n-1} + (n-1)!$$

and hence show that

$$\frac{I_n}{n!} = \log x + 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Sol:-

$$I_n = \frac{d^n}{dx^n}(x^n \log x) = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{d}{dx}(x^n \log x)\right)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(n x^{n-1} \log x + x^n \times \frac{1}{x}\right)$$

$D^{n-1}(x^{n-1}) = (n-1)!$

$D^{n}(x^{n}) = n!$

$$= n\frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\log x\right) + \frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\right)$$

$$= n I_{n-1} + (n-1)!$$

$$\therefore \quad I_n = \underline{n I_{n-1}} + \underline{(n-1)!} \quad \text{——} \ (1)$$

Replace n by '$n-1$' in (1)

$$I_{n-1} = (n-1) I_{n-2} + (n-2)!$$

Hence (1) $\Rightarrow$

$$I_n = n\left[(n-1) I_{n-2} + (n-2)!\right] + (n-1)!$$

$$= n(n-1)\underline{\underline{I_{n-2}}} + n(n-2)! + (n-1)! \quad \text{——} \ (2)$$

I_{n-1}

I_{n-2}

I_{n-3}

$\vdots$

$I_{n-n} = I_0$

$$I_{n-3} = (n-3) I_{n-4} + (n-4)!$$

Replace n by $n-2$ in (1)

$$I_{n-2} = (n-2) I_{n-3} + (n-3)!$$

Hence by (2)

$$I_n = n(n-1)\left[(n-2) I_{n-3} + (n-3)!\right] + n(n-2)! + (n-1)!$$

$$= n(n-1)(n-2) I_{n-3} + n(n-1)(n-3)! + n(n-2)! + (n-1)!$$

Repeating this process we get

$$I_n = n(n-1)(n-2)\times \cdots (n-(n-1)) I_{n-n} + \cdots + n(n-1)(n-3)! + n(n-2)! + (n-1)!$$

$$= n!\, I_0 + \cdots + \frac{n!}{(n-2)} + \frac{n!}{(n-1)} + \frac{n!}{n}$$

$$= n! \left[\log n + 1 + \cdots + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n} \right]$$

Ans.

$$y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$$

$$\frac{y^{\frac{1}{m}}}{1} + \frac{1}{y^{\frac{1}{m}}} = 2x$$

$$\Rightarrow \quad y^{\frac{2}{m}} + 1 = 2x\, y^{\frac{1}{m}}$$

$$y^{\frac{2}{m}} - 2x\, y^{\frac{1}{m}} + 1 = 0$$

$$y^{\frac{1}{m}} = u$$

$$\boxed{u^2 - 2xu + 1 = 0}$$

$$u = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$u = x \pm \sqrt{x^2-1}$$

$$y^{\frac{1}{m}} = x \pm \sqrt{x^2-1}$$

$$\boxed{y = \left[x \pm \sqrt{x^2-1}\right]^m}$$

$$y_1 = m\left[x \pm \sqrt{x^2-1}\right]^{m-1} \times \left[1 \pm \frac{x}{\sqrt{x^2-1}}\right]$$

$$= m\left[x \pm \sqrt{x^2-1}\right]^{m-1} \times \frac{\left[\sqrt{x^2-1} \pm x\right]}{\sqrt{x^2-1}}$$

$$= \pm m\left[x \pm \sqrt{x^2-1}\right]^{m-1} \times \frac{\left[x \pm \sqrt{x^2-1}\right]}{\sqrt{x^2-1}}$$

$$y_1 = \frac{\pm m y}{\sqrt{x^2-1}}$$

$$\boxed{y_1^2 (x^2-1) = m^2 y^2}$$

Again diff

$$2y_1 y_2 (x^2-1) + y_1^2 \times 2x = m^2 \times 2y \times y_1$$

$$\Rightarrow \boxed{(x^2-1)y_2 + x y_1 - m^2 y = 0}$$

$$D^n(uv) = D^n(u)\times v + {}^nC_1\, D^{n-1}(u)\times D(v) + {}^nC_2\, D^{n-2}(u)\times D^2(v) + \cdots + u\times D^n(v)$$

Q

If $x+y=1$ prove that

$$\frac{d^n}{dx^n}(x^n y^n) = n!\left[y^n - ({}^nC_1)^2 y^{n-1}\cdot x + ({}^nC_2)^2 y^{n-2}\times x^2 - \cdots + (-1)^n x^n\right]$$

Sol:- L.H.S

$$D^n(x^n y^n) = \boxed{D^n(x^n)}\times y^n + {}^nC_1\,\boxed{D^{n-1}(x^n)}\times D(y^n) + {}^nC_2\, D^{n-2}(x^n)\times D^2(y^n) + \cdots + x^n\times D^n(y^n)$$

$D^{n-2}(x^n)$

$\frac{n(n-1)}{2}$

$D^2(1-x)^2 \longrightarrow 2\underbrace{(1-x)}_{y}\times(-1)$

$D(x^n) = n x^{n-1}$

$D^2(x^n) = n(n-1)x^{n-2}$

$D^3(x^n) = n(n-1)(n-2)x^{n-3}$

$- - - - -$

$- - - -$

$D^{n-1}(x^n) = n(n-1)(n-2) \times \cdots \times (n-(n-2))x^{n-(n-1)}$

$= n!\, x =$

$\boxed{D^n(x^n) = n!}$

$\boxed{D^n(x^m) = \frac{m!}{(m-n)!} x^{m-n}}$ #

$D^{n-2}(x^n)$

$= \frac{n!}{2!} x^2$

$D(y^n) = D(1-x)^n = n(1-x)^{n-1}(-1) = (-1) \times n \times y^{n-1}$

$D^2(y^n) = D^2(1-x)^n = n(n-1)(1-x)^{n-2}(-1)^2 = n(n-1)(-1)^2 y^{n-2}$

$- - - -$

$D^n(y^n) = (-1)^n n!$

$\boxed{{}^nc_1 = n}$

$$D^n(x^n y^n) = n!\, y^n + {}^nc_1 \times n! \times x \times (-1) \times n \times y^{n-1} + \cdots + x^n \times (-1)^n n!$$

$$= n!\left[y^n - ({}^nc_1)^2 y^{n-1} + \cdots + (-1)^n x^n \right] \#$$

Q $\quad D^n\left[\tan^{-1}\left(\dfrac{x\sin\alpha}{1-x\cos\alpha}\right)\right]$

Let $\quad y = \tan^{-1}\left(\dfrac{x\sin\alpha}{1-x\cos\alpha}\right)$

$$\Rightarrow \quad y_1 = \frac{1}{1+\left(\dfrac{x\sin\alpha}{1-x\cos\alpha}\right)^2}\,\frac{d}{dx}\left(\frac{x\sin\alpha}{1-x\cos\alpha}\right)$$

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$$= \frac{1}{\frac{(1-x\cos\alpha)^2 + x^2\sin^2\alpha}{(1-x\cos\alpha)^2}} \times \frac{\left[\sin\alpha \times (1-x\cos\alpha) - x\sin\alpha \times (-\cos\alpha)\right]}{(1-x\cos\alpha)^2}$$

$$\boxed{y_1 = \frac{\sin\alpha}{1+x^2-2x\cos\alpha}}$$

$$= \frac{\sin\alpha}{(x-\cos\alpha)^2+\sin^2\alpha}$$

$$= \frac{\sin\alpha}{(x-\cos\alpha)^2-i^2\sin^2\alpha}$$

$$y_1 = \frac{\sin\alpha}{(x-\cos\alpha-i\sin\alpha)(x-\cos\alpha+i\sin\alpha)}$$

$$= \frac{1}{2i}\left[\frac{1}{x-\cos\alpha-i\sin\alpha} - \frac{1}{x-\cos\alpha+i\sin\alpha}\right]$$

$$\boxed{y_1 = \frac{1}{2i}\left[(x-\cos\alpha-i\sin\alpha)^{-1} - (x-\cos\alpha+i\sin\alpha)^{-1}\right]}$$

#

As $\boxed{y = (x-a)^{-1}}$

$y_1 = (-1)(x-a)^{-2}$

$y_2 = (-1)(-2)\,(x-a)^{-3}$

$y_3 = (-1)(-2)(-3)(x-a)^{-4}$

$y_3 = (-1)^3\, 3!\, (x-a)^{-4}$

$\boxed{y_{n-1} = (-1)^{n-1}(n-1)!\,(x-a)^{-n}}$

Also

$(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$

Let

$$\left.\begin{aligned} x-\cos\alpha &= r\cos\theta \\ \sin\alpha &= r\sin\theta \end{aligned}\right\} \Rightarrow \tan\theta = \frac{\sin\alpha}{x-\cos\alpha}$$

$\Rightarrow x^2 - 2x\cos\alpha + 1 = r^2$ $\quad \theta = \tan^{-1}\left(\frac{\sin\alpha}{x-\cos\alpha}\right)$

Differentiating '$n-1$' times w.r.t 'x', we get

$$y_n = \frac{1}{2i}\left[D^{n-1}(x-\cos\alpha - i\sin\alpha)^{-1} - D^{n-1}(x-\cos\alpha + i\sin\alpha)^{-1}\right]$$

$$= \frac{1}{2i}\left[(-1)^{n-1}(n-1)!\,(x-\cos\alpha - i\sin\alpha)^{-n} - (-1)^{n-1}(n-1)!\,(x-\cos\alpha + i\sin\alpha)^{-n}\right] \quad -(1)$$

$$\sin(-n\theta) = -\sin n\theta$$

Hence

$$y_n = \frac{1}{2i}(-1)^{n-1}(n-1)!\left[(r\cos\theta - ir\sin\theta)^{-n} - (r\cos\theta + ir\sin\theta)^{-n}\right]$$

$$= \frac{1}{2i}(-1)^{n-1}(n-1)!\,r^{-n}\left[\cancel{\cos n\theta} + i\sin n\theta - \cancel{\cos n\theta} + i\sin n\theta\right]$$

$$= \frac{1}{2i}(-1)^{n-1}(n-1)!\,r^{-n} \times 2i\sin n\theta$$

$$\boxed{y_n = \frac{(-1)^{n-1}(n-1)!}{(x^2 - 2x\cos\alpha + 1)^{\frac{n}{2}}}\sin\left(n\tan^{-1}\left(\frac{\sin\alpha}{x-\cos\alpha}\right)\right)}$$

Ans.

$$\because\ r^2 = x^2 - 2x\cos\alpha + 1$$

Q Prove that

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2 \cdot 2^2} + \cdots \cdots = \frac{2n!}{(n!)^2}$$

(Hint :- $D^n(x^n \times x^n)$ $\longrightarrow$ calculate it using Leibnitz theorem)

$$\longrightarrow D^n(x^{2n}) = \boxed{\frac{(2n)!}{(n)!}} x^n =$$

$$D^n(x^m) = \frac{m!}{(m-n)!} x^{m-n}$$

$$D^n(x^m) = \frac{m!}{(m-n)!} x^{m-n}$$

$$D^n(x^n \times x^n) = n!\, x^n + n \times n! \times x \times n x^{n-1} + \cdots + x^n \times n!$$

$$= n!\, x^n \left[1 + \frac{n^2}{1^2} + \cdots \cdots \cdots\right]$$

"n+2"

Q If $y = e^{m\cos^{-1}x}$

then show that

$$(1-x^2)\boxed{y_{n+2}} - (2n+1)x\,y_{n+1} - (n^2+m^2)\,y_n = 0$$

and hence calculate $(y_n)_0$.

Sol:- $y = e^{m\cos^{-1}x}$ —— (A)

$\Rightarrow y_1 = \boxed{e^{m\cos^{-1}x}} \times \frac{-m}{\sqrt{1-x^2}}$ ①

$\Rightarrow y_1 = \frac{-my}{\sqrt{1-x^2}}$

On squaring, we get

$y_1^2(1-x^2) = m^2y^2$

Again differentiating, we get

$2y_1 y_2 \times (1-x^2) + y_1^2(-2x) = m^2 \times 2y \times y_1$

$D(y_1^2) = 2y_1 \times y_2$

$\Rightarrow \boxed{(1-x^2)y_2 - xy_1 - m^2 y = 0}$ —— (2)

(y_{n+2})

Differentiating this equation for n-times using Leibnitz theorem:-

$D^n\{(1-x^2)y_2\} - D^n\{xy_1\} - m^2 D^n\{y\} = 0$

using Leibnitz rule:-

$D^n\{uv\} = D^n(u) \times v + {}^nC_1 D^{n-1}(u) \times D(v)$
$+ {}^nC_2 D^{n-2}(u) \times D^2(v)$
$+ {}^nC_3 D^{n-3}(u) \times D^3(v) + \cdots + u \times D^n(v)$

$=$

$$\left\{y_{n+2}\times(1-x^2)+{}^nC_1\, y_{n+1}\times(-2x)+{}^nC_2\times y_n\times(-2)\right\}$$

$$-\left\{x\,y_{n+1}+{}^nC_1\times y_n\times 1\right\}-m^2 y_n=0$$

$$\left\{{}^nC_2=\frac{n(n-1)}{2}\right.$$

$$\Rightarrow (1-x^2)\,y_{n+2}-x(2n+1)\,y_{n+1}+y_n\{-n^2+n-n-m^2\}=0$$

$$\Rightarrow \boxed{(1-x^2)\,y_{n+2}-(2n+1)x\,y_{n+1}-(n^2+m^2)\,y_n=0} \quad -③$$

Put $x=0$

$(y_{n+2})_0$

$$\boxed{(y_{n+2})_0=(n^2+m^2)\,(y_n)_0} \quad -④$$

Replace n by "$n-2$"

$$\boxed{(y_n)_0=\left((n-2)^2+m^2\right)(y_{n-2})_0} \quad -⑤ \quad ,\ n\geq 2$$

By (A) $\quad (y)_0 = e^{\frac{m\pi}{2}}$ $\qquad \cos^{-1} 0 = \frac{\pi}{2}$

By ① $\quad (y_1)_0 = -m e^{\frac{m\pi}{2}}$ ✓

By ② $\quad (y_2)_0 = m^2 e^{\frac{m\pi}{2}}$ ✓

By (5) $\quad (y_3)_0 = -(1^2+3^2)\, m e^{\frac{m\pi}{2}}$ ✓

$(y_4)_0 = (2^2+m^2)\, m^2 e^{\frac{m\pi}{2}}$ ✓

$(y_5)_0 = -(3^2+m^2)(1^2+m^2)\, m e^{\frac{m\pi}{2}}$ ✓

$(y_6)_0 = (4^2+m^2)(2^2+m^2)\, m^2 e^{\frac{m\pi}{2}}$ ✓

Continuing in this way

$$(y_n)_0 = \begin{cases} -me^{\frac{m\pi}{2}}(1^2+m^2)(3^2+m^2)\cdots\times((n-2)^2+m^2), & n \text{ is odd} \\ m^2e^{\frac{m\pi}{2}}(2^2+m^2)(4^2+m^2)\times\cdots\times((n-2)^2+m^2), & n \text{ is even.} \end{cases}$$

#

Q

$$y = e^{m\cos^{-1}x}$$

If $y = \sin(a\sin^{-1}x)$ — (1) find $(y_n)_0$.

$\Rightarrow$ $y_1 = \cos(a\sin^{-1}x) \times \dfrac{a}{\sqrt{1-x^2}}$ — (2)

$\Rightarrow$ $\sqrt{1-x^2}\, y_1 = a\cos(a\sin^{-1}x)$

on diff.

$$\frac{-x}{\sqrt{1-x^2}} \times y_1 + \sqrt{1-x^2} \times y_2 = -a\sin(a\sin^{-1}x) \times \frac{a}{\sqrt{1-x^2}}$$

$\Rightarrow$ $-xy_1 + (1-x^2)y_2 = -a^2y$

$\Rightarrow$ $\boxed{(1-x^2)y_2 - xy_1 + a^2y = 0}$ — (3)

Again by Leibnitz rule,

$$\left\{(1-x^2)y_{n+2} + n(-2x)\times y_{n+1} + \frac{n(n-1)}{\cancel{2}}\times(-\cancel{2})\times y_n\right\}$$

$$-\{x y_{n+1} + n y_n\} + a^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x y_{n+1} + (-n^2+n-n+a^2)y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x y_{n+1} - (n^2-a^2)y_n = 0 \quad \text{--- (4)}$$

put $x=0$

$$(y_{n+2})_0 = (n^2-a^2)(y_n)_0$$

Replace n by "$n-2$"

$$(y_n)_0 = \left((n-2)^2-a^2\right)(y_{n-2})_0 \quad \text{--- (5)}$$

$n \geq 2$

Put $x=0$ in ①, ②, ③, ④ and ⑤, we get

$n \times n$

$nc_1 \times nc_1$

$nc_1 = n$

$$(y)_0 = 0$$

$$(y_1)_0 = a$$

$$(y_2)_0 = 0.$$

$$(y_3)_0 = (1^2 - a^2)\,a$$

$$(y_4)_0 = 0, \quad (y_6)_0 = (y_8)_0 = \cdots = 0$$

$$(y_5)_0 = (3^2 - a^2)(1^2 - a^2)\,a$$

$$(y_7)_0 = (5^2 - a^2)(3^2 - a^2)(1^2 - a^2)\,a$$

In general :-

$$(y_n)_0 = a(1^2-a^2)(3^2-a^2)(5^2-a^2) \cdots \times ((n-2)^2-a^2),$$
if n is odd

$= 0$ if n is even.

Q ① If $y = \tan^{-1}x$ show that $(y_n)_0 = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}}(n-1)! & \text{if } n \text{ is odd} \end{cases}$

② If $y = \log(x+\sqrt{1+x^2})$ find $(y_n)_0$

$$\boxed{\sinh^{-1}x = \log(x+\sqrt{x^2+1})}$$

(3) $y = (\sinh^{-1}x)^2$

find $(y_n)_0$.

Q $y = \tan^{-1}x \Rightarrow (y)_0 = 0$

$y_n = (-1)^{n-1}(n-1)!\, \sin^n\phi \sin n\phi$

$\phi = \cot^{-1}x$

$\Rightarrow y_1 = \dfrac{1}{1+x^2} \Rightarrow (y_1)_0 = 1$

$\Rightarrow (1+x^2)y_1 = 1$

Again diff, we get

$\boxed{(1+x^2)y_2 + 2xy_1 = 0} \Rightarrow (y_2)_0 = 0$

Now applying Leibnitz rule for n-times differentiation :-

$D^n((1+x^2)y_2) + 2D^n(xy_1) = 0$

$\Rightarrow \{(1+x^2)y_{n+2} + n \times 2x \times y_{n+1} + \frac{n(n-1)}{2} \times 2 \times y_n\}$

$+ 2\{x y_{n+1} + n \times 1 \times y_n\} = 0$

$$\Rightarrow \boxed{(1+x^2)y_{n+2} + (n+1)2x\,y_{n+1} + n(n+1)y_n = 0}$$

put $x=0$

$(y_n)_0 =$ (y_{n-2})

$$(y_{n+2})_0 = -n(n+1)(y_n)_0$$

Replace n by "$n-2$"

$$\boxed{(y_n)_0 = -(n-1)(n-2)(y_{n-2})_0}, \quad n \geq 2$$

$$(y_3)_0 = -2\times 1\,(y_1)_0 = -2\times 1\times 1 = (-1)\times 2!_0 = (-1)^{\frac{3-1}{2}}\,2!_0$$

$(y_4)_0 = 0$, $(y_6)_0 = 0$, $(y_8)_0 = 0$ --- and so on.

$$(y_5)_0 = -4\times 3\times (y_3)_0$$

$$= -4 \times 3 \times (-1) \times 2 \times 1$$

$$(y_5)_0 = (-1)^2 \, 4!$$

$$= (-1)^{\frac{5-1}{2}} \, 4!$$

In general

$$(y_n)_0 = (-1)^{\frac{n-1}{2}} (n-1)! \quad \text{when } n \text{ is odd.}$$

$$(y_n)_0 = 0 \quad , n \text{ is even}$$

(3) $y = (\sinh^{-1}x)^2$

$\Rightarrow (y)_0 = 0$

$\Rightarrow y_1 = 2(\sinh^{-1}x) \times \frac{1}{\sqrt{1+x^2}}$

on squaring

$y_1^2 (1+x^2) = 4 (\sinh^{-1}x)^2$

$\Rightarrow (1+x^2) y_1^2 = 4y \Rightarrow (y_1)_0 = 0$

Again differentiating, we get

$(1+x^2) \times 2y_1 \times y_2 + 2x \times y_1^2 = 4y_1$

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cosh x = \frac{e^x + e^{-x}}{2}$, $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\cosh^2 x - \sinh^2 x = 1$, $\cos^2\theta + \sin^2\theta = 1$

$\cosh x = \sqrt{1 + \sinh^2 x}$

$\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\sinh x) = \cosh x$

$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$ ✓

$$\Rightarrow \boxed{(1+x^2)y_2 + xy_1 = 2} \Rightarrow \boxed{(y_2)_0 = 2}$$

Differentiating n-times using Leibnitz rule :-

$$D^n\{(1+x^2)y_2\} + D^n\{xy_1\} = 0$$

$$\left\{(1+x^2)y_{n+2} + n \times (2x) \times y_{n+1} + \frac{n(n-1)}{2} \times 2 \times y_n\right\}$$

$$+ \{x \times y_{n+1} + n \times 1 \times y_n\} = 0$$

$$\Rightarrow \boxed{(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0}$$

but $x=0$:-

$$(y_{n+2})_0 = -n^2 (y_n)_0$$

replace n by "n-2"

$$\boxed{(y_n)_0 = -(n-2)^2 (y_{n-2})_0} \quad , n > 2 \quad 1$$

By (1)

$$(y_3)_0 = -1^2 (y_1)_0 = 0$$

$$(y_5)_0 = (y_7)_0 = \cdots = 0$$

$$(y_4)_0 = -(4-2)^2 (y_2)_0$$

$$= -2^2 \times 2 = (-1)^{\frac{4}{2}-1} 2^2 \times 2$$

$$(y_6)_0 = -(6-2)^2 (y_4)_0$$

$$= -4^2 \times -2^2 \times 2 = (-1)^{\frac{6}{2}-1} 4^2 \cdot 2^2 \cdot 2$$

$$(y_8)_0 = -6^2 \times -4^2 \times -2^2 \times 2 = (-1)^{\frac{8}{2}-1} 6^2 \cdot 4^2 \cdot 2^2 \cdot 2$$

In general, $(y_n)_0 = (-1)^{\frac{n}{2}-1} \times (n-2)^2 \times \cdots \times 6^2 \times 4^2 \times 2^2 \times 2$

$\rightarrow y = \cos^5 x$

$\rightarrow \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{\pm in\theta} = \cos n\theta \pm i\sin n\theta$

$$\boxed{{}^nC_r = {}^nC_{n-r}}$$

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$$

$$= x^n + {}^nC_1 x^{n-1} \times y^1 + {}^nC_2 x^{n-2} \times y^2 + \dots + y^n$$

$\cos n\theta = \dfrac{e^{ni\theta} + e^{-ni\theta}}{2}$ ✓

$$\cos^5 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^5$$

$$= \frac{1}{2^5}\Big(e^{5ix} + {}^5C_1 \times e^{4ix} \times e^{-ix} + {}^5C_2 \times e^{3ix} \times e^{-2ix} + {}^5C_3\, e^{2ix} \times e^{-3ix} + {}^5C_4\, e^{ix} \times e^{-4ix} + {}^5C_5\, e^{-5ix}\Big)$$

$$= \frac{1}{2^5}\left[\left(e^{5ix} + e^{-5ix}\right) + 5C_1\left(e^{3ix} + e^{-3ix}\right) + 5C_2\left(e^{ix} + e^{-ix}\right)\right]$$

$$= \frac{1}{2^4}\left[\left(\frac{e^{5ix} + e^{-5ix}}{2}\right) + 5 \times \left(\frac{e^{3ix} + e^{-3ix}}{2}\right) + 10\left(\frac{e^{ix} + e^{-ix}}{2}\right)\right]$$

$$\boxed{y = \frac{1}{2^4}\left[\cos 5x + 5\cos 3x + 10\cos x\right]}$$

$$D^n(\cos ax) = a^n \cos\left(ax + \frac{n\pi}{2}\right)$$

$$y_n = \frac{1}{2^4}\left[5^n \cos\left(5x + \frac{n\pi}{2}\right) + 5 \times 3^n \cos\left(3x + \frac{n\pi}{2}\right) + 10\cos\left(x + \frac{n\pi}{2}\right)\right] \text{ Ans}$$

$f(x) = \frac{\sin x}{\cos x}$

Q If $f(x) = \tan x$, prove that

$$f^n(0) - {}^nC_2 f^{n-2}(0) + {}^nC_4 f^{n-4}(0) \ldots\ldots = \sin\frac{n\pi}{2}$$

$\sin(4p+1)\frac{\pi}{2}$

$\sin(4p+3)\frac{\pi}{2}$

$\cos x f(x) = \sin x$

$f^n(x)\cos x + {}^nC_1 f^{n-1} \times (-\sin x) \ldots = \sin\left(x + \frac{n\pi}{2}\right)$

Q If $y = e^{a\sin^{-1}x}$ then deduce that

$$\lim_{x\to 0} \frac{y_{n+2}}{y_n} = n^2 + a^2$$

$\boxed{\tan^{-1}(\infty) = \frac{\pi}{2}}$

$\left(\sin\frac{n\pi}{2}\right)$

or

$$(y_{n+2})_0 = (n^2+a^2)(y_n)_0$$

$D^n(\tan^{-1}x)$

$= (-1)^{n-1}(n-1)!\, \sin^n\theta \sin n\theta, \quad \theta = \cot^{-1}\left(\frac{1}{x}\right)$

Q Show that $D^n(\tan^{-1}x) = 0$, $(n-1)!$ or $-(n-1)!$ according as $n = 2p$ or $4p+1$ or $(4p+3)$ respectively at $x=0$

#

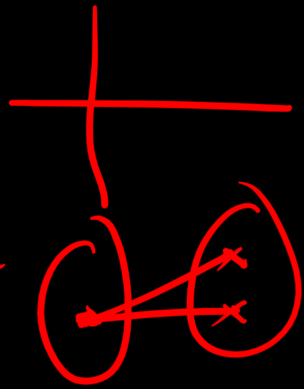
$z = f(x, y)$

~~$y = f(x)$~~

$z = f(x,y)$ $\boxed{x=1}$

Function of two variables

A function $z = f(x, y)$ is said to be function of two independent variables x & y defined for all pair of values (x, y) in the x-y plane.



$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$z = f(x,y)$

??

$\lim_{x \to a^+} f(x) = R\text{-}H\text{-}L$

$\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$

①

$$\frac{\partial z}{\partial x} \rightarrow \frac{\text{del } z}{\text{del } x}$$

Partial differentiation

let $z = f(x, y)$ i.e. z is function of two independent variable x & y.

$$z_x = \frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

is known as partial derivative of z w.r.t 'x' and we kept y as constant.

similarly, $z_y = \frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$

it is the partial derivative of z w.r.t 'y'.

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$z_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = z_{yx}$$

Q If $z = e^{ax+by} f(ax-by)$ then show that

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$$

$$z = e^{ax+by} f(ax-by)$$

$$\frac{\partial z}{\partial x} = e^{ax+by} \times a \, f(ax-by) + e^{ax+by} \times f'(ax-by) \times a \quad \text{— (i)}$$

$$\frac{\partial z}{\partial y} = e^{ax+by} \times b \, f(ax-by) + e^{ax+by} \times f'(ax-by) \times -b \quad \text{— (ii)}$$

$\text{(i)} \times b + \text{(ii)} \times a \Rightarrow$

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\, e^{ax+by} f(ax-by)$$

$$= 2abz \ \#$$

Q) If $x^x y^y z^z = c$ show that $\boxed{\frac{\partial^2 z}{\partial x \partial y}} = -(x \log ex)^{-1}$

at $x = y = z$.

Sol:- $x^x y^y z^z = c$

Taking log we get

$\log(x^x y^y z^z) = \log c$

$\log x^x + \log y^y + \log z^z = \log c$

$\Rightarrow x \log x + y \log y + z \log z = \log c$ —— (1)

Differentiating (1) partially w.r.t 'x'

$$\left(\log x + x \times \frac{1}{x}\right) + 0 + \left(\frac{\partial z}{\partial x}\log z + z \times \frac{1}{z}\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow (\log x + 1) + (1 + \log z)\frac{\partial z}{\partial x} = 0$$

$z = f(x, y)$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{(1+\log z)}}$$

Similarly

$$\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{(1+\log z)}$$

L.H.S

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-(1+\log y)}{(1+\log z)}\right)$$

$$= -(1+\log y)\frac{\partial}{\partial x}\left(\frac{1}{1+\log z}\right)$$

$$= -(1+\log y) \times \frac{-1}{(1+\log z)^2} \times \frac{1}{z} \times \frac{\partial z}{\partial x}$$

$$= \frac{-1}{z}\frac{(1+\log y)(1+\log x)}{(1+\log z)^3}$$

At $x = y = z$

$$\left(\frac{\partial^2 z}{\partial x \partial y}\right) = -\frac{1}{x}\frac{1}{(1+\log x)}$$

$$= -\frac{1}{x}\frac{1}{(\log e+\log x)}$$

$$= -\left(x\log ex\right)^{-1}$$ #

Q If $u = \log(x^3+y^3+z^3-3xyz)$

then show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

Sol:- $u = \log(x^3+y^3+z^3-3xyz)$ $\qquad -(x, y, z)$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

on adding we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3\,\cancel{\left(x^2+y^2+z^2-xy-yz-zx\right)}}{(x+y+z)\,\cancel{\left(x^2+y^2+z^2-xy-yz-zx\right)}}$$

$$=\frac{3}{x+y+z}$$

L.H.S

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\underline{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} \quad \frac{-3}{(x+y+z)^2} \quad \frac{-3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2} \quad \text{Ans}$$

Q If $\dfrac{x^2}{a^2+u} + \dfrac{y^2}{b^2+u} + \dfrac{z^2}{c^2+u} = 1$ $\qquad u = f(x, y, z)$

prove that

$$u_x^2 + u_y^2 + u_z^2 = 2(xu_x + yu_y + zu_z)$$

sol:-

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \quad \text{———} \ (1)$$

Differentiating partially w.r.t 'x'

$$2x \times \frac{1}{(a^2+u)} - x^2 \times \frac{1}{(a^2+u)^2} \times \frac{\partial u}{\partial x} - y^2 \frac{1}{(b^2+u)^2} \times \frac{\partial u}{\partial x}$$

$$- z^2 \times \frac{1}{(c^2+u)^2} \times \frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{2x}{(a^2+u)} = \underbrace{\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]}\times\frac{\partial u}{\partial x}$$

Let $\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2} = A$

then $\frac{\partial u}{\partial x} = \frac{\frac{2x}{(a^2+u)}}{A}$

$$\Rightarrow \boxed{u_x = \frac{2x}{A(a^2+u)}}$$

Similarly, $u_y = \frac{2y}{A(b^2+u)}$, $u_z = \frac{2z}{A(c^2+u)}$

$$\underline{\text{L.H.S}} = u_x^2 + u_y^2 + u_z^2$$

$$= \frac{4}{A^2}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

$$= \frac{4}{A^2} \times A = \frac{4}{A}.$$

$$\text{R.H.S} = 2(x u_x + y u_y + z u_z)$$

$$= 2\left(\frac{2x^2}{A(a^2+u)} + \frac{2y^2}{A(b^2+u)} + \frac{2z^2}{A(c^2+u)}\right)$$

$$= \frac{4}{A}\left(\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}\right) = \frac{4}{A}$$

$$\therefore \text{L.H.S} = \text{R.H.S}$$

Q If $u = e^{xyz}$ then show that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)\,u$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x \partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)\right)$$

Sol:-

$$u = e^{xyz}$$

$$\frac{\partial u}{\partial z} = e^{xyz} \times xy$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = e^{xyz} \times xz \times xy + e^{xyz} \times x$$

$$\Rightarrow \frac{\partial^2 u}{\partial y \partial z} = e^{xyz}\, x^2yz + x e^{xyz}$$

Differentiating w.r.t 'x'

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} \times yz \times x^2yz + e^{xyz} \times 2xyz + e^{xyz} + x e^{xyz} \times yz$$

$$= e^{xyz}\left(x^2y^2z^2 + 3xyz + 1\right)$$

$$= (1 + 3xyz + x^2y^2z^2)u$$ Ans.

Q If $u = f(r)$, $r^2 = x^2 + y^2$ then show that

$$u_{xx} + u_{yy} = f''(r) + \frac{1}{r} f'(r)$$

Sol:-

$$\boxed{u = f(r)},$$

$$\Rightarrow u_x = f'(r) \times \frac{\partial r}{\partial x}$$

$$r = f(x, y)$$

$$r^2 = x^2 + y^2$$

$$\Rightarrow 2r \frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$= f'(r) \times \frac{x}{r}$$

$$\Rightarrow \boxed{u_x = \frac{x f'(r)}{r}}$$

$u \times v$

Again diff. we get

$$u_{xx} = \left\{ f'(r) + x \times f''(r) \times \frac{\partial r}{\partial x} \right\} \times \frac{1}{r} + \boxed{x f'(r)} \times \frac{-1}{r^2} \times \frac{\partial r}{\partial x}$$

$$= \frac{f'(r)}{r} + x f''(r) \times \frac{x}{r} \times \frac{1}{r} - \frac{x^2 f'(r)}{r^3}$$

$$= \frac{x^2}{r^2} f''(r) + \frac{f'(r)}{r^3}(r^2 - x^2)$$

$$\therefore \quad u_{xx} = \boxed{\frac{x^2}{r^2} f''(r)} + \boxed{\frac{f'(r)}{r^3}}(r^2 - x^2)$$

Similarly,

$$u_{yy} = \boxed{\frac{y^2}{r^2} f''(r)} + \boxed{\frac{f'(r)}{r^3}}(r^2 - y^2)$$

On addition, we get

$$u_{xx} + u_{yy} = \frac{f''(r)}{r^2}(x^2 + y^2) + \frac{f'(r)}{r^3}(2r^2 - x^2 - y^2)$$

$$= f''(r) + \frac{f'(r)}{r} \quad \text{Ans.}$$

Q (1) If $\theta = t^n e^{-\frac{r^2}{4t}}$

Find 'n' for which

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$$

$\left(n = -\frac{3}{2}\right)$

(2) If $u = x^2 \boxed{\tan^{-1}\frac{y}{x}} - y^2 \boxed{\tan^{-1}\left(\frac{x}{y}\right)}$, find the value of $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$.

(3) If $x = r\cos\theta$, $y = r\sin\theta$ show that $r_{xx} + r_{yy} = \frac{1}{r}\left[r_x^2 + r_y^2\right]$

$$\frac{\partial^2 r}{\partial x^2} \times \frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2, \quad \cancel{r_{xx} + r_{yy} = \frac{1}{r}\left[r_x^2 + r_y^2\right]}$$

$x = r\cos\theta, \quad y = r\sin\theta,$

$$\Rightarrow r^2 = x^2 + y^2$$

$$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \boxed{r_x = \frac{x}{r}}, \quad \boxed{r_y = \frac{y}{r}}$$

$$\checkmark\ r_{xx} = \frac{r - x \times \frac{\partial r}{\partial x}}{r^2} = \frac{r - x \times \frac{x}{r}}{r^2} = \frac{r^2 - x^2}{r^3}$$

$$\checkmark\ r_{yy} = \frac{r^2 - y^2}{r^3} \quad \Rightarrow r_{xx} + r_{yy} = \frac{2r^2 - (x^2 + y^2)}{r^3}$$

$$= \frac{1}{r}$$

$$r_{xx} + r_{yy} = \frac{1}{r}\left[r_x^2 + r_y^2\right]$$

R.H.S

$$\frac{1}{r}\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right]$$

$$= \frac{1}{r}$$

Homogeneous function

let $u(x,y)$ is of the form

$$u(x,y) = x^n f(y/x)$$

then $u(x,y)$ is said to be homogeneous function of degree 'n'.

$$u = \frac{x^1+y^1}{x^2+y^2}$$

$$= \frac{x^1\boxed{\left(1+\frac{y}{x}\right)}}{x^2\boxed{\left(1+\frac{y^2}{x^2}\right)}}$$

$$= x^{-1} f(y/x)$$

e.g. :-

$$u(x,y) = \frac{x+y}{x^2+y^2} = \frac{x}{x^2}\,\frac{(1+y/x)}{\left(1+\frac{y^2}{x^2}\right)} = x^{-1} f(y/x)$$

$\Rightarrow u(x,y)$ is homogeneous of degree "-1".

Euler theorem :-

If $u(x,y)$ is a homogeneous function of degree 'n'

then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$

Proof :- $\because$ u is homogeneous function of degree 'n'

$$u(x,y) = x^n f(y/x) \quad -(1)$$

$$\frac{\partial u}{\partial x} = nx^{n-1} f(y/x) + x^n f'(y/x) \times \frac{-y}{x^2}$$

$$\Rightarrow \frac{\partial u}{\partial x} = nx^{n-1} f(y/x) - y\, x^{n-2} f'(y/x)$$

$$\Rightarrow x\frac{\partial u}{\partial x} = nx^n f(y/x) - y\, x^{n-1} f'(y/x) \quad -(2)$$

Again diff (1) w.r.t 'y'

$$\frac{\partial u}{\partial y} = x^n f'(y/x) \times \frac{1}{x}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = y\, x^{n-1} f'\left(\frac{y}{x}\right) \quad -(3)$$

1-2

Hence ② + ③ ⇒

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n x^n f(y/x)$$

$$= nu \ \#$$

Q Verify Euler theorem for

$$u = \frac{x+y}{x^2+y^2}.$$

Sol:- Euler theorem is

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$\because$ $u = \dfrac{x^1+y^1}{x^2+y^2}$ is homo. of degree -1

Hence $\boxed{x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -u}$

Sol:

$$u = \frac{x+y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2) - (x+y)\times 2x}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2 - 2xy}{(x^2+y^2)^2}$$

$$\Rightarrow x\frac{\partial u}{\partial x} = \frac{x(y^2 - x^2 - 2xy)}{(x^2+y^2)^2}$$

Similarly,

$$y\frac{\partial u}{\partial y} = \frac{y(x^2 - y^2 - 2xy)}{(x^2+y^2)^2}$$

Hence

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{xy^2 - x^3 - 2x^2y + yx^2 - y^3 - 2xy^2}{(x^2+y^2)^2}$$

$$= -\frac{[x^3 + xy^2 + x^2y + y^3]}{(x^2+y^2)^2}$$

$$= -\frac{(x+y)(x^2+y^2)}{(x^2+y^2)^2}$$

$$= -\frac{(x+y)}{(x^2+y^2)}$$

$$= -u$$

Hence Euler theorem is verified.

Q If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that

(i) $x u_x + y u_y = \sin 2u$

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u$

$u = \frac{\cancel{x^4+y^4}}{\cancel{x^3+y^3}}$

$\frac{x^2+y^2}{x^2y^2}$

$u = x^n f(y/x)$

$u = \tan^{-1}(\theta)$

Sol: $\because u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$\Rightarrow \tan u = \frac{x^3+y^3}{x-y}$

$\Rightarrow z = \frac{x^3+y^3}{x-y}$ is homo. of degree 2 where $z = \tan u$

$=$

Applying Euler theorem for z, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

$$\Rightarrow x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$\Rightarrow x\sec^2 u\boxed{\frac{\partial u}{\partial x}} + y\boxed{\sec^2 u}\times\boxed{\frac{\partial u}{\partial y}} = 2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = \frac{2\times\sin u}{\cos u}\cos^2 u$$

$$= \sin 2u$$

$$\therefore \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u} \quad \text{—— (1)}$$

$\phi(u)$

Differentiating (1) w.r.t x

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y\,\partial x} = 2\cos 2u \times \frac{\partial u}{\partial x}$$

$$\Rightarrow\quad x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial y\,\partial x} = 2\cos 2u \times x\frac{\partial u}{\partial x} \quad -(2)$$

Similarly

$$y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} + xy\frac{\partial^2 u}{\partial x\,\partial y} = 2\cos 2u \times y\frac{\partial u}{\partial y} \quad -(3)$$

(2) + (3) $\Rightarrow$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) + 2xy\frac{\partial^2 u}{\partial x\,\partial y}$$

$$= 2\cos 2u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= [\phi'(u)-1]\,\phi(u)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(2\cos 2u-1)\sin 2u$$

$$= 2\sin 2u\cos 2u-\sin 2u$$

$$= \sin 4u-\sin 2u$$

$$= 2\cos 3u\,\sin u$$

#

Q

If $u = \boxed{x^1 \phi\left(\frac{y}{x}\right)} + \boxed{\psi\left(\frac{y}{x}\right)}$. Then

evaluate

(i) $x u_x + y u_y \rightarrow$

(ii) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$

$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = 0$

$= 0$

Sol:- $u = x^1 \phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$

$= v + w$

then $v = x^1 \phi\left(\frac{y}{x}\right) =$ homogeneous of degree 1

$w = \psi\left(\frac{y}{x}\right) =$ homogeneous of degree 0

$\because v$ is homogeneous $\Rightarrow x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = v$, —(1)

$\because$ w is homogeneous of degree '0'

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0 \quad \text{— (ii)}$$

On addition, we get

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = u$$

$$\Rightarrow x\frac{\partial}{\partial x}(v+w) + y\frac{\partial}{\partial y}(v+w) = u$$

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u = x\phi\left(\frac{y}{x}\right)} \quad \text{— (iii)}$$

Differentiating ~~(i)~~ w.r.t 'x'

$$\cancel{x\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + y\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial v}{\partial x}}$$

$$\Rightarrow \cancel{x^2\frac{\partial^2 v}{\partial x^2} + x\frac{\partial v}{\partial x} + xy\frac{\partial^2 v}{\partial x \partial y} = x\frac{\partial v}{\partial x}} \quad \text{— (iv)}$$

$\therefore$

By (3)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = v \quad \text{—— (3)}$$

Differentiating w.r.t 'x'

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial v}{\partial x}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x \partial y} = x\frac{\partial v}{\partial x} \quad \text{—— (4)}$$

Again diff (3) w.r.t 'y'

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}$$

$$\Rightarrow xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = y\frac{\partial v}{\partial y} \quad \text{—— (5)}$$

On adding (4) & (5)

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \underbrace{\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)}_{}$$

$$= \underbrace{\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)}_{}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + v = v$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

Note:- (i) If u is homogeneous of degree 'n'

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

and $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

$z=\tan u=\frac{x^2+y^2}{x+y}$

(ii) If $z=f(u)$ is homo. of degree 'n'

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\boxed{\frac{nf(u)}{f'(u)}}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=[\phi'(u)-1]\phi(u)$$

when $\phi(u)=\frac{nf(u)}{f'(u)}$

Q. If $u = \log_e\left(\dfrac{x^4+y^4}{x+y}\right)$

evaluate $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$

Sol:- $u = \log_e\left(\dfrac{x^4+y^4}{x+y}\right)$

$\Rightarrow e^u = \dfrac{x^4+y^4}{x+y}$

$\Rightarrow z = \dfrac{x^4+y^4}{x+y} \Rightarrow z$ is homo. of deg $= 3$

and $z = e^u = f(u)$

Hence using formula

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n\,f(u)}{f'(u)}$$

$$= \frac{3 \times e^u}{e^u} = 3 = \phi(u)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

$$= [\phi'(u) - 1]\,\phi(u)$$

$$= [0-1]\,\phi(u)$$

$$= -3$$

Q $V = \log_e \sin\left(\dfrac{\pi\,(2x^2+y^2+xz)^{\frac{1}{2}}}{2\,(x^2+xy+2yz+z^2)^{\frac{1}{3}}}\right)$ and show that

$\rightarrow$ $x\dfrac{\partial V}{\partial x} + y\dfrac{\partial V}{\partial y} + z\dfrac{\partial V}{\partial z} = \dfrac{\pi}{12}$

at $x=0,\ y=1,\ z=2.$

Sol:- we have

$$\sin^{-1}(e^V) = \frac{\pi\,(2x^2+y^2+xz)^{\frac{1}{2}}}{2\,(x^2+xy+2yz+z^2)^{\frac{1}{3}}}$$

$$1-\frac{2}{3} = \frac{1}{3}$$

Here $w = \dfrac{\pi\,(2x^2+y^2+xz)^{\frac{1}{2}}}{2\,(x^2+xy+2yz+z^2)^{\frac{1}{3}}}$ is homo. of degree $=1-\dfrac{2}{3}$

$= \dfrac{1}{3}$

where $w = \sin^{-1}(e^V)$

~~taking sin~~

So using formula

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=\frac{n f(V)}{f'(V)} \quad \text{where } f(V)=\sin^{-1}(e^V)$$

$\frac{\partial}{\partial x} f(n)$

$$=\frac{1}{3}\times\frac{\sin^{-1}(e^V)}{\frac{1}{\sqrt{1-e^{2V}}}\times e^V}$$

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=\frac{1}{3}w$$

$$=\frac{1}{3}\sin^{-1}(e^V)$$

At $x=0,\ y=1,\ z=2$

$w=\frac{\pi}{4}$. Also $w=\sin^{-1}(e^V)$

$$\Rightarrow \boxed{e^V=\sin w=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}}$$

$$\therefore\ x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z}$$

$$= \frac{1}{3} \times \frac{\pi}{4} \times \sqrt{1-\frac{1}{2}} \times \frac{1}{\sqrt{2}}$$

$$= \frac{\pi}{12} \quad \#$$

Differentiation of composite functions

$$\cancel{z = f(u, v)}$$

Let $u = f(x, y)$ where $\boxed{x = \phi(t)}$, $y = \psi(t)$

Then total derivative of u w.r.t 't'

$$\rightarrow \frac{du}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt}$$

<u>case</u> ① $z = f(x, y)$ and $x = \phi(u, v)$, $y = \psi(u, v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v} \quad \#$$

case 2 :-

$\boxed{z = f(x, y) = c}$ and $y = \varphi(x)$ then

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \times \frac{dx}{dx} + \frac{\partial z}{\partial y} \times \frac{dy}{dx}$$

$$0 = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \times \frac{dy}{dx}$$

$$\frac{-\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{dy}{dx} \quad \Rightarrow \quad \boxed{\frac{dy}{dx} = -\frac{z_x}{z_y}} = \boxed{-\frac{f_x}{f_y}}$$

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{z_x}{z_y}\right)$$

$x^7 y^3 + y^2 x^{1/2} + c x^7 y^9 + y^2 x^2 + y x^2 - x = 0 \Rightarrow$

$\frac{dy}{dx} = ?$

$\frac{dy}{dx} = (\,)$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(-\frac{z_x}{z_y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-z_x}{z_y}\right) \times \frac{dx}{dx} + \frac{\partial}{\partial y}\left(\frac{-z_x}{z_y}\right) \times \frac{dy}{dx}$$

$$= \left(\frac{-z_{xx} \times z_y + z_x \times z_{xy}}{z_y^2}\right) + \frac{\left(-z_{xy} z_y + z_x z_{yy}\right)}{z_y^2} \times \left(\frac{-z_x}{z_y}\right)$$

$$= -\left(\frac{z_{xx} z_y^2 - 2 z_x z_y z_{xy} + z_{yy} z_x^2}{z_y^3}\right)$$

#

Q. $u = f(y-z,\ z-x,\ x-y)$

prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

sol:- $u = f(y-z,\ z-x,\ x-y)$

$\boxed{u = f(r, s, t)}$, $r = y-z$, $s = z-x$, $t = x-y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\times\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\times\frac{\partial t}{\partial x}$$

$$= -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\times\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\times\frac{\partial t}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\times\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\times\frac{\partial t}{\partial z}$$

$$= -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \quad \text{--- (3)}$$

(1) + (2) + (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Q If $\phi(x, y, z) = 0$ show that $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$

Sol:-

$$\phi(x, y, z) = 0$$

If x is constant then $\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\phi_z}{\phi_y}$

If y is constant then $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\phi_x}{\phi_z}$

If z is constant then $\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\phi_y}{\phi_x}$

Hence $\left(\frac{\partial z}{\partial y}\right)_x \times \left(\frac{\partial z}{\partial x}\right)_y \times \left(\frac{\partial x}{\partial y}\right)_z = -1$

Q If $f(x,y)=0$, $\phi(y,z)=0$

Show that $\frac{\partial f}{\partial y} \times \frac{\partial \phi}{\partial z} \times \frac{dz}{dx} = \frac{\partial f}{\partial x} \times \frac{\partial \phi}{\partial y}$

Sol:

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx} \quad \text{--- (1)}$$

$\because f(x,y)=0 \Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}$

$\phi(y,z)=0 \Rightarrow \frac{dz}{dy} = \frac{-\phi_y}{\phi_z}$

Hence by (1)

$$\frac{dz}{dx} = -\frac{f_x}{f_y} \times \frac{-\phi_y}{\phi_z} \Rightarrow \boxed{f_y \phi_z \frac{dz}{dx} = f_x \phi_y}$$

Q ① $z = xy\, f\left(\frac{y}{x}\right)$ and z is constant show that

$$\frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x\left[y + x\frac{dy}{dx}\right]}{y\left[y - x\frac{dy}{dx}\right]}$$

② $z = x^2 y$, $\boxed{x^2 + xy + y^2 = 1}$. show that

$$\frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$$

$$\frac{dz}{dx} = 2xy - \frac{x^2(2x+y)}{x+2y}$$

$$\frac{dz}{dx} = 2xy + x^2\frac{dy}{dx}$$

$\frac{dy}{dx}$ $\cancel{x}$

$d = \boxed{y = f(x)}$

$z = f(x, y)$

$\frac{\partial z}{\partial x}$

(3) $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$

show that

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$$

(4) $u = f(x^2+2yz,\ y^2+2zx)$ show that

$$(y^2-zx)\frac{\partial u}{\partial x} + (x^2-yz)\frac{\partial u}{\partial y} + (z^2-xy)\frac{\partial u}{\partial z} = 0$$

Q. If $x+y = 2e^{\theta}\cos\phi$

$x-y = 2ie^{\theta}\sin\phi$

then show that

$$\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial \phi^2} = 4xy\,\frac{\partial^2 V}{\partial x \partial y}$$

$V = V(\theta, \phi)$ — $\theta, \phi \to x, y$

$V \to V(\theta, \phi)$, $\theta, \phi = F(x, y)$

$V \to V(x, y)$, $x, y \to F(\theta, \phi)$

Sol :-

$x+y = 2e^{\theta}\cos\phi$

$x-y = 2ie^{\theta}\sin\phi$

$\Rightarrow 2x = 2e^{\theta}(\cos\phi + i\sin\phi)$

$\Rightarrow x = e^{\theta} e^{i\phi}$

$\Rightarrow \boxed{x = e^{\theta + i\phi}}$

Also

$2y = 2e^{\theta}\cos\phi - 2ie^{\theta}\sin\phi$

$\Rightarrow y = e^{\theta}(\cos\phi - i\sin\phi)$

$= e^{\theta} e^{-i\phi}$

$\Rightarrow \boxed{y = e^{\theta - i\phi}}$

so let $\boxed{v = v(x, y)}$ when $x = \psi(\theta, \phi) = e^{\theta + i\phi}$

$y = F(\theta, \phi) = e^{\theta - i\phi}$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}\left(e^{\theta + i\phi}\right) + \frac{\partial v}{\partial y}\left(e^{\theta - i\phi}\right)$$

$$= x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

$$\frac{\partial^2 v}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial \theta}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial \theta}\right)\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial \theta}\right)\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial}{\partial x}\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) \times x + \frac{\partial}{\partial y}\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) \times y$$

$$= \left[x\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + y\frac{\partial^2 v}{\partial x \partial y}\right]x + \left(x\frac{\partial^2 v}{\partial y \partial x} + \frac{\partial v}{\partial y} + y\frac{\partial^2 v}{\partial y^2}\right)y$$

$$\frac{\partial^2 V}{\partial \theta^2} = x^2\frac{\partial^2 V}{\partial x^2} + x\frac{\partial V}{\partial x} + 2xy\frac{\partial^2 V}{\partial x \partial y} + y^2\frac{\partial^2 V}{\partial y^2} + y\frac{\partial V}{\partial y} \quad \text{——} \; (1)$$

Also $$\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial x}\left(\frac{\partial x}{\partial \phi}\right) + \frac{\partial V}{\partial y}\left(\frac{\partial y}{\partial \phi}\right) = i\left[x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y}\right]$$

$$\frac{\partial^2 V}{\partial \phi^2} = \frac{\partial}{\partial \phi}\left(\frac{\partial V}{\partial \phi}\right) = \frac{\partial}{\partial x}\left[\frac{\partial V}{\partial \phi}\right]\left(\frac{\partial x}{\partial \phi}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial \phi}\right)\left(\frac{\partial y}{\partial \phi}\right)$$

$$= i\frac{\partial}{\partial x}\left[x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y}\right] \times ix + i\frac{\partial}{\partial y}\left(x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y}\right)(-iy)$$

$$= -x\frac{\partial}{\partial x}\left[x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y}\right] + y\frac{\partial}{\partial y}\left(x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y}\right)$$

$$= -x^2\frac{\partial^2 V}{\partial x^2} - x\frac{\partial V}{\partial x} + xy\frac{\partial^2 V}{\partial x \partial y} + xy\frac{\partial^2 V}{\partial x \partial y} - y\frac{\partial V}{\partial y} - y^2\frac{\partial^2 V}{\partial y^2} \quad \text{——} \; (2)$$

on addition, result

$$\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial \phi^2} = 4xy\frac{\partial^2 V}{\partial x \partial y}$$

Expansion of variables

Taylor theorem :-

Let $f(x+h)$ be a function x and differentiable any number of times then, Taylor's series in powers of h is given by

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n}{n!} f^n(x) + \cdots \quad \text{—— (1)}$$

Note :- (i) $f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^n(a) + \cdots$

(ii) If $h \to x-a$ in (1)

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + \cdots \quad \text{—— (2)}$$

$\sin(0.1)$

Maclaurin's series

If we put $a=0$ in equation (2), we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$$

This is the expansion of $f(x)$ in ascending powers of x.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$$

Ex:-

$f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{iv}(x) = \sin x \Rightarrow f^{iv}(0) = 0$

$f^{v}(x) = \tan x \Rightarrow f^{v}(0) = 1$

and so on ...

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}$$

Q Expand :-

(a) $\sin^{-1}x$, $\tan^{-1}x$, $\log\left(\frac{1+x}{1-x}\right)$, $\log\sec x$, $\tan^{-1}(1+x)$, $e^{x}\sec x$

in powers of x.

(b) Show that

$$e^{a\sin^{-1}x} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a(a^2+1)}{3!}x^3 + \frac{a^2(a^2+2^2)}{4!}x^4$$

$$+ \frac{a(a^2+1)(a^2+3)}{5!}x^5 + \cdots$$

and find its general term.

Q. If $y = \cos(m\sin^{-1}x)$ then prove that $(y_{n+2})_0 = (n^2 - m^2)(y_n)_0$ and hence expand $\cos(m\sin^{-1}x)$.

$$D^n(uv) = D^n(u)\times v + {}^nC_1 D^{n-1}(u)\times D(v) + {}^nC_2 D^{n-2}(u)\times D^2(v) + \cdots$$

Sol:-

$$y = \cos(m\sin^{-1}x) \quad \text{--- (I)} \Rightarrow (y)_0 = 1$$

$$\Rightarrow y_1 = -\sin(m\sin^{-1}x)\times\frac{m}{\sqrt{1-x^2}} \Rightarrow (y_1)_0 = 0$$

$$\Rightarrow y_1\sqrt{1-x^2} = -\sin(m\sin^{-1}x)\times m \quad \text{--- (II)}$$

Again diff

$$y_2\times\sqrt{1-x^2} - \frac{x}{\sqrt{1-x^2}}y_1 = -\cos(m\sin^{-1}x)\times\frac{m^2}{\sqrt{1-x^2}}$$

$$\Rightarrow \boxed{(1-x^2)y_2 - xy_1 + m^2y = 0} \quad \text{--- (III)} \Rightarrow (y_2)_0 = -m^2$$

Differentiating n-times by Leibnitz rule, we get

$$\{(1-x^2)y_{n+2} + n\times(-2x)\times y_{n+1} + \frac{n(n-1)}{2}\times(-2)\times y_n\}$$

$$-\{x y_{n+1} + n y_n\} + m^2 y_n = 0$$

$$\Rightarrow \boxed{(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - (n^2-m^2)y_n = 0} \quad \text{— (IV)}$$

put $x=0$

$$\boxed{(y_{n+2})_0 = (n^2-m^2)(y_n)_0} \quad \text{— (V)}$$

$(y_3)_0 = 0 = (y_5)_0 \cdots = 0$

$(y_4)_0 = (2^2-m^2)(-m^2)$

and so on.

$(y_6)_0 = -m^2(2^2-m^2)(4^2-m^2)$

As

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

or

$$y(x) = y_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots$$

$$\cos(m\sin^{-1}x) = 1 + \frac{x^2}{2!}(-m^2) + \frac{x^4}{4!}\left(-m^2(2^2-m^2)\right) \cdots\cdots$$

$$\boxed{\cos(m\sin^{-1}x) = 1 - \frac{m^2x^2}{2} - \frac{m^2(2^2-m^2)}{4!}x^4 \cdots}$$

Q. Expand $\sin x$ in powers of $x-\frac{\pi}{4}$.

Sol:- Let $f(x) = \sin x \Rightarrow f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x \Rightarrow f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x \Rightarrow f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

and so on.

$$\therefore f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Let $a = \frac{\pi}{4}$

$\therefore \quad \sin x = \frac{1}{\sqrt{2}} + \left(x - \frac{\pi}{4}\right) \times \left(\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} \times \left(-\frac{1}{\sqrt{2}}\right) - - -$

<u>Ans</u>

<u>Q</u> Expand $\log(\sin(x+h))$

<u>Sol :-</u>

$$f(x+h) = \log(\sin(x+h))$$

$$\Rightarrow f(x) = \log \sin x$$

$$\Rightarrow f'(x) = \frac{\cos x}{\sin x} = \cot x$$

$$f''(x) = -\operatorname{cosec}^2 x \text{ and so on.}$$

$$\therefore f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + - - - - -$$

$\Rightarrow$

$$\log(\sin(x+h)) = \log\sin x + h(\cot x) + \frac{h^2}{2!}(-\csc^2 x) - - - -$$

Ans.

Q show that

$$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \left(\frac{x}{1+x}\right)^2 \frac{1}{2!} f''(x) - - - - -$$

Sol:-

$$f\left(\frac{x^2}{1+x}\right) = f\left(x + \left(\frac{-x}{1+x}\right)\right)$$

$$= f(x+h) \quad \text{where } h = \frac{-x}{1+x}$$

$$= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) - - -$$

$$= f(x) + \left(\frac{-x}{1+x}\right) f'(x) + \frac{1}{2} \frac{x^2}{(1+x)^2} f''(x) - - - -$$

Ans.

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2$$

Taylor's series for function of two variables

Let $f(x,y)$ be a function of two independent variables x and y and suppose that $f(x,y)$ has the derivatives w.r.t 'x' & 'y', then

$$f(x+h,y+k) = f(x,y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)$$
$$+ \frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\partial y^2}\right.$$
$$\left.+ k^3\frac{\partial^3 f}{\partial y^3}\right) + \cdots$$

$f(\cancel{a}+x-\cancel{a})$

$\cancel{(x, y)}$

OR

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \cdots$$

let $h \to x-a$, $k \to y-b$, $x \to a$, $y \to b$. Then Taylor series about (a,b)

$$\therefore \boxed{f(x,y) = f(a,b) + (x-a)\left(\frac{\partial f}{\partial x}\right)_{(a,b)} + (y-b)\left(\frac{\partial f}{\partial y}\right)_{(a,b)} + \frac{1}{2!}\left\{(x-a)^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)} + 2(x-a)(y-b)\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)} + (y-b)^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}\right\} + \cdots}$$

If $(a,b) = (0,0)$, then Maclaurin's series is given by

$$f(x,y) = f(0,0) + x\left(\frac{\partial f}{\partial x}\right)_{(0,0)} + y\left(\frac{\partial f}{\partial y}\right)_{(0,0)} + \frac{1}{2!}\left[x^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} + 2xy\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)}\right] + \cdots$$

Q If $f(x) = e^x \cos y$ then expand it at $(0,0)$.

$f(0,0) = 1$

$f_x = e^x \cos y \Rightarrow (f_x)_{(0,0)} = 1$

$f_y = -e^x \sin y \Rightarrow (f_y)_{(0,0)} = 0$

$$f_{xx} = e^x \cos y \Rightarrow (f_{xx})_{(0,0)} = 1$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left[e^x \cos y\right] = -e^x \sin y$$

$$\Rightarrow (f_{xy})_{(0,0)} = 0$$

$$f_{yy} = -e^x \cos y \Rightarrow (f_{yy})_{(0,0)} = -1$$

and so on.

$$\therefore \quad f(x,y) = f(0,0) + x (f_x)_{(0,0)} + y (f_y)_{(0,0)}$$

$$+ \frac{1}{2!}\left[x^2 (f_{xx})_{(0,0)} + 2xy (f_{xy})_{(0,0)} + y^2 (f_{yy})_{(0,0)}\right]$$

$$+ \cdots$$

$$\boxed{e^x \cos y = 1 + x + \frac{1}{2!}\left[x^2 + 0 - y^2\right] \cdots}$$

$$= \cdots$$

Ans.

Q ① expand $\tan^{-1}\left(\frac{y}{x}\right)$ about $(1,1)$ upto 2^{nd} degree term.

Hence calculate $\tan^{-1}\left(\frac{0.9}{1.1}\right)$

② expand $e^{x}\cos y$ at $\left(1,\frac{\pi}{4}\right)$.

③ expand $e^{x}\tan^{-1}y$ in powers of $(x-1)$ and $(y-1)$.

Q Expand x^y in powers of $(x-1)$ and $(y-2)$ and hence evaluate $(1.1)^{1.02}$.

Sol:- Here $a=1$ and $b=2$

let $f(x,y) = x^y \Rightarrow f(a,b) = 1$

$f_x = y x^{y-1}$

$f_y = x^y \log_e x$

$f_{xx} = y(y-1)x^{y-2}$

$f_{yy} = x^y (\log_e x)^2$

As $f_x = y x^{y-1}$

$f_{xy} = x^{y-1} + y x^{y-1} \log_e x$

$(f_x)_{(1,2)} = 2$

$(f_y)_{(1,2)} = 0$

$(f_{xx})_{(1,2)} = 2$

$(f_{yy})_{(1,2)} = 0$

$(f_{xy})_{(1,2)} = 1$

and so on.

$$f(x,y) = f(a,b) + (x-a)(f_x)_{(a,b)} + (y-b)(f_y)_{(a,b)} + \frac{1}{2!}\left[(x-a)^2 (f_{xx})_{(a,b)} + 2(x-a)(y-b)(f_{xy})_{(a,b)} + (y-b)^2 (f_{yy})_{(a,b)}\right] + \cdots$$

$$\Rightarrow \quad x^y = 1 + (x-1)\times 2 + \frac{1}{2}\left[(x-1)^2 \times 2 + 2(x-1)(y-2)\right] + \cdots$$

$$\Rightarrow \quad \boxed{x^y = 1 + 2(x-1) + (x-1)^2 + (x-1)(y-2) + \cdots}$$

$$\Rightarrow \quad (1.1)^{1.02} \approx 1 + 2(1.1-1) + (1.1-1)^2 + (1.1-1)(1.02-2)$$

$$\approx 1 + 0.2 + 0.01 + (0.1)(-0.98)$$

$$\approx 1.21 - 0.098$$

$$\approx 1.112 \quad \#$$

$$\begin{array}{r} 1.210 \\ 0.098 \\ \hline 1.112 \end{array}$$

$$\{\boxed{1.10}21\}$$

$u = x - y + z,\ v = x^2 + y^2 + z,\ w = x^2 + y^2 + z^2$

Jacobian

Defn:- If u, v & w are functions of independent variables x, y, z then Jacobian of u, v and w, with respect to x, y and z is defined as

$x = u + v + w$
$y = u - v + w$
$z = u^2 + v^2 - w$

$$J(u, v, w) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties of Jacobian

(i) If u, v are functions of r, s and r, s are function of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

$u = f(r,s)$, $r = \psi(x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\times\frac{\partial s}{\partial x}$$

Proof: R.H.S $= \dfrac{\partial(u,v)}{\partial(r,s)} \times \dfrac{\partial(r,s)}{\partial(x,y)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial(u,v)}{\partial(x,y)}$$

$$= \frac{\partial(u,v)}{\partial(x,y)} = \underline{\underline{L.H.S}}$$

(2) let $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$

<u>Proof:-</u>

$$JJ' = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \times \frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \times \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore \boxed{JJ' = 1}$$

$u - x^3 - y^3 - v^4 = f_1$

$u^2 - v^{\frac{1}{4}} + x = f_2$

$u^3 + v^3 + w^3 + xyz = f_3$

(3) Let u, v, w are relation (implicit functions) of x, y & z such that

$$f_1(u, v, w, x, y, z) = 0 \text{ —}$$

$$f_2(u, v, w, x, y, z) = 0 \text{ —}$$

$$f_3(u, v, w, x, y, z) = 0 \text{ —}$$

Then $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \times \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}$$

Q If $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$ then evaluate

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}$$

Sol:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$C_1 \to C_1 - C_3$

$C_2 \to C_2 - C_3$

z

$$= \begin{vmatrix} y(z-x) & x(z-y) & xy \\ z-x & z-y & x+y \\ 0 & 0 & 1 \end{vmatrix}$$

$$= y(z-x)x(z-y) - x(z-y)(z-x)$$

$$= (y-z)(z-x)\{-y+x\} = (x-y)(y-z)(z-x)$$

#

$f_1 =$

Q. If $x+y+z=u$

$y+z=uv$

$z=uvw$ then evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

Sol:- method 1

$z=uvw$

$y=uv-uvw$

$x=u-(y+z)=u-uv$

$\therefore x=u(1-v),\quad y=uv(1-w),\quad z=uvw \rightarrow$

method 2:

$f_1 = x+y+z-u$

$f_2 = y+z-uv$

$f_3 = z-uvw$

$$\therefore \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -wu & -uv \end{vmatrix} = -u^2v$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\therefore \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = -1 \times \frac{-u^2v}{1} = u^2v \quad \#$$

$u = x+y+z$
$v = x^2+y^2+z^2$
$w = xy+yz+zx$

$J(u,v,w) = ? = 0$

$\boxed{u^2 = v + 2w}$

Theorem:- Let u, v, w be the functions of x, y & z. Then the necessary and sufficient condition for the existence of a relation between u, v, w and x, y, z is that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

Q If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$ then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

Sol:-

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2}\begin{vmatrix} -yz & zx & xy \\ yz & -zx & xy \\ yz & zx & -xy \end{vmatrix}$$

$$= \frac{\cancel{x^2y^2z^2}}{\cancel{x^2y^2z^2}}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2 & 0 \\ 0 & -2 & 2 \\ 1 & 1 & -1 \end{vmatrix} \qquad \begin{matrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - R_3 \end{matrix}$$

$$= 4$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 4 = J$$

$$\because JJ' = 1$$

$$\Rightarrow \boxed{J' = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{J} = \frac{1}{4}}$$

Q If $u^3+v^3+w^3 = x+y+z$

$u^2+v^2+w^2 = x^3+y^3+z^3$

and $u+v+w = x^2+y^2+z^2$. find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Sol:- $f_1 = u^3+v^3+w^3-x-y-z$

$f_2 = u^2+v^2+w^2-x^3-y^3-z^3$

$f_3 = u+v+w-x^2-y^2-z^2$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$$

$$= \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 \\ -3x^2 & 3(x^2-y^2) & 3(x^2-z^2) \\ 2x & 2(x-y) & 2(x-z) \end{vmatrix} \quad \begin{array}{l} C_3 \to C_3 - C_1 \\ C_2 \to C_2 - C_1 \end{array}$$

$$= -1\left[6(x^2-y^2)(x-z) - 6(x^2-z^2)(x-y)\right]$$

$$= -6(x-y)(x-z)(x+y-x-z)$$

$$= 6(x-y)(y-z)(z-x) \ \#$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3u^2 & 3(v^2-u^2) & 3(w^2-u^2) \\ 2u & 2(v-u) & 2(w-u) \\ 1 & 0 & 0 \end{vmatrix} \quad \begin{matrix} C_3 \to C_3 - C_1 \\ C_2 \to C_2 - C_1 \end{matrix}$$

$$= 6\left[(v^2-u^2)(w-u) - (w^2-u^2)(v-u)\right]$$

$$= 6(v-u)(w-u)\,[v+u-w-u)]$$

$$= -6(u-v)(v-w)(w-u)$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = -1 \times \frac{6(x-y)(y-z)(z-x)}{-6(u-v)(v-w)(w-u)}$$

$$= \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

\#

Q Show that

$\boxed{u = x+y+z}$

$v = x^2+y^2+z^2-2xy-2yz-2zx$

$w = x^3+y^3+z^3-3xyz$

are functionally related. Find the relationship b/w u, v & w.

Sol:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 2x-2y-2z & 2y-2x-2z & 2z-2y-2x \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$= 0$ (prove it)

$w = \underbrace{x^3+y^3+z^3} - 3xyz$

Now

$uv = (x+y+z)(x^2+y^2+z^2-2xy-2yz-2zx)$

$= x^3+xy^2+xz^2+yx^2+y^3+yz^2+zx^2+zy^2+z^3$

$+ (x+y+z)(-2xy-2yz-2zx)$

$=$

$$w = x^3 + y^3 + z^3 - 3xyz$$

$$= (x+y+z)(x^2+y^2+z^2 - xy - yz - zx)$$

$$= u\,(\underbrace{x^2+y^2+z^2 - 2xy - 2yz - 2zx} + xy + yz + zx)$$

$$= u\,(v + xy + yz + zx) \quad \text{---} \; ①$$

This equation ① $\Rightarrow$

$$w = u\left(v + \frac{u^2 - v}{4}\right)$$

$$= \frac{u}{4}(u^2 + 3v) \;\#$$

$\therefore \quad u = x + y + z$

$$u^2 = (x+y+z)^2$$

$$= x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

$$= v + 2xy + 2yz + 2zx + 2xy + 2yz + 2zx$$

$\{\therefore\; v = \underbrace{x^2+y^2+z^2} - 2xy - 2yz - 2zx\}$

$$\frac{u^2 - v}{4} = xy + yz + zx$$

$(ab + k(a+b) + k^2)(c+k)$

Q If u, v, w are roots of $\frac{x}{a+k} + \frac{y}{b+k} + \frac{z}{c+k} = 1$ in k then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

Sol:- $x(b+k)(c+k) + y(a+k)(c+k) + z(a+k)(b+k)$

$= (a+k)(b+k)(c+k)$

$\Rightarrow k^2(x+y+z) + k[x(b+c) + y(a+c) + z(a+b)]$

$+ xbc + yac + zab = k^3 + k^2(a+b+c) + k(ab+bc+ca) + abc$

$\Rightarrow k^3 + k^2[a+b+c - (x+y+z)] + k[ab + bc + ca - x(b+c) - y(a+c) - z(a+b)]$

$+ [abc - xbc - yac - zab] = 0$

$\therefore \quad u+v+w = x+y+z-a-b-c$

$uv+vw+wu = ab+bc+ca - x(b+c) - y(a+c) - z(a+b)$

$uvw = xbc + yac + zab - abc$

$f_1 = u+v+w - (x+y+z-a-b-c)$

$f_2 =$

$f_3 =$ ——

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}} \longrightarrow$$

Q (i) If $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$

show that $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$ and find relationship.

(Hint :- prove that $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$

$$v = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}u$$

$$\Rightarrow \boxed{u = \tan v}$$)

(ii) If $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$

find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

Let then

Maxima and minima of two variables

Let $f(x,y)$ be any function of two variable.

If $f(a,b) > f(a+h, b+k)$, then we say that at (a,b) we have maxima and if $f(a,b) < f(a+h, b+h)$, we say that at (a,b) we have minima.

maximum or minimum value of a function f is said to be extremum.

Necessary cond^n for extremum

Let $z = f(x,y)$ be a function of two variable, then the necessary cond^n for ~~maxi~~ ~~nor~~ extremum is

$$\frac{\partial z}{\partial x} = 0 = \frac{\partial z}{\partial y}$$

The ~~value of (a,b)~~. solution of these equalities are said to be stationary points.

Sufficient cond^n for extrema

Let $r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}$, $s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)}$, $t = \left(\frac{\partial^2 z}{\partial y^2}\right)_{(a,b)}$

$\boxed{\frac{\partial^2 y}{\partial x^2} < 0}$

Now

(i) If $rt - s^2 > 0$, $r < 0$, then we say that f is maximum at (a,b).

(ii) If $rt - s^2 > 0$, $r > 0$, then we say that f is minimum at (a,b).

(iii) If $rt - s^2 < 0$, then ~~we~~ there is no maxima or minima and such points are said to be saddle point.

(iv) If $rt - s^2 = 0$, it is doubtful case and further investigation is required for such points

Q Find the extreme value of $xy(a-x-y)$.

Sol:-

$$z = xy(a-x-y)$$

$$= axy - x^2y - xy^2$$

$$\frac{\partial z}{\partial x} = ay - 2xy - y^2, \quad \frac{\partial z}{\partial y} = ax - x^2 - 2xy$$

$$\frac{\partial^2 z}{\partial x^2} = -2y, \quad \frac{\partial^2 z}{\partial y^2} = -2x, \quad \frac{\partial^2 z}{\partial x \partial y} = a - 2x - 2y$$

Now for extremum :-

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0$$

$$y(a-2x-y) = 0$$

$$x(a-x-2y) = 0$$

$\Rightarrow \quad y=0, \quad a-x-2y=0 \Rightarrow (a,0)$
$y=0, \quad x=0 \Rightarrow (0,0)$ } two stationary points

$x=0, \quad a-2x-y=0 \Rightarrow (0,a)$

Also $\left.\begin{array}{l} a-x-2y=0 \\ a-2x-y=0 \end{array}\right\} \Rightarrow$

$$\begin{array}{l} x+2y=a \\ 2x+y=a \quad \times 2 \\ - \\ \hline -3x=-a \end{array}$$

$\Rightarrow x=\frac{a}{3}, \quad 2y=a-\frac{a}{3}$

$\Rightarrow y=a/3$

Thus $(0,0), (0,a), (a,0), \left(\frac{a}{3},\frac{a}{3}\right)$ are 4 stationary points.

$\therefore \quad \frac{\partial^2 z}{\partial x^2} = -2y, \quad \frac{\partial^2 z}{\partial x \partial y} = a - 2x - 2y, \quad \frac{\partial^2 z}{\partial y^2} = -2x$

<u>At $(0,0)$</u>

$$r = \left(\frac{\partial^2 z}{\partial x^2}\right)_{(0,0)} = 0, \quad s = \left(\frac{\partial^2 z}{\partial x \partial y}\right)_{(0,0)} = a, \quad t = \left(\frac{\partial^2 z}{\partial y^2}\right)_{(0,0)} = 0$$

$\therefore \quad rt - s^2 = -a^2 < 0 \Rightarrow$ At $(0,0)$ we don't have any maxima or minima.

<u>At $(a,0)$</u>

$$r = 0, \quad s = -a, \quad t = -2a$$

$rt - s^2 = -a^2 < 0 \longrightarrow$ No maxima or minima

<u>At $(0,a)$</u>

$$r = -2a, \quad s = a, \quad t = 0$$

$rt - s^2 = -a^2 < 0 \longrightarrow$ No maxima or minima

At $\left(\frac{a}{3}, \frac{a}{3}\right)$

$r = -\frac{2a}{3}$, $s = a - \frac{2a}{3} - \frac{2a}{3} = \frac{a-4a}{3} = -\frac{a}{3}$, $t = -\frac{2a}{3}$

$rt - s^2 = \frac{4a^2}{9} - \frac{a^2}{9} = \frac{3a^2}{9} = \frac{a^2}{3} > 0$, $r = \frac{-2a}{3}$

If $\boxed{a > 0}$ then we have $rt - s^2 > 0$, $r = \frac{-2a}{3} < 0$

hence at $\left(\frac{a}{3}, \frac{a}{3}\right)$ we have maxima.

If $\boxed{a < 0}$, $\left(\frac{a}{3}, \frac{a}{3}\right)$ is minima and the minimum value

$z = xy(a - x - y) = \frac{a}{3} \times \frac{a}{3}\left(a - \frac{a}{3} - \frac{a}{3}\right)$

$= \frac{a^3}{27}$

Q. Find maxima or minima of $Z = \sin x + \sin y + \sin(x+y)$

or

$Z = \cos x + \cos y + \cos(x+y)$

or

$Z = \sin x \sin y \sin(x+y)$

or

$Z = \cos x \cos y \cos(x+y)$

$\boxed{\begin{array}{l} u = \sin x \sin y \sin z \\ \text{when } x, y, z \text{ are the angle of triangle} \end{array}}$

$x + y + z = \pi$

$z = \pi - (x+y)$

Sol:

$$Z = \sin x + \sin y + \sin(x+y)$$

$$\frac{\partial Z}{\partial x} = \cos x + \cos(x+y), \quad \frac{\partial^2 Z}{\partial x^2} = -\sin x - \sin(x+y)$$

$$\frac{\partial Z}{\partial y} = \cos y + \cos(x+y), \quad \frac{\partial^2 Z}{\partial y^2} = -\sin y - \sin(x+y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\sin(x+y)$$

$\cos x = \cos y$

$x = 2n\pi \pm y$

$\cancel{\cos x = \cos y}$

$\cancel{\boxed{x = y}}$

For maxima or minima

$$\frac{\partial z}{\partial x} = 0 = \frac{\partial z}{\partial y}$$

$$\cos x + \cos(x+y) = 0 \quad , \quad \cos y + \cos(x+y) = 0$$

$\downarrow$

$$\Rightarrow \cos x = -\cos(x+y)$$

$$\Rightarrow \cos x = \cos(\pi - (x+y))$$

$$\Rightarrow x = \pi - x - y \Rightarrow 2x + y = \pi$$

similarly

$$2y + x = \pi$$

on solving

$$(a, b) = \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$$

~~$r =$~~

$$r = \left(\frac{\partial^2 z}{\partial x^2}\right)_{\left(\frac{\pi}{3}, \frac{\pi}{3}\right)} = -\sqrt{3}$$

$$s = \left(\frac{\partial^2 z}{\partial x \partial y}\right)_{\left(\frac{\pi}{3}, \frac{\pi}{3}\right)} = \frac{-\sqrt{3}}{2}$$

$$t = \left(\frac{\partial^2 z}{\partial y^2}\right)_{\left(\frac{\pi}{3}, \frac{\pi}{3}\right)} = -\sqrt{3}$$

$\therefore\ rt-s^2=\frac{9}{4}>0$ and $r=-\sqrt{3}<0$

At $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ we have maxima.

and max^m value $=\sin\frac{\pi}{3}+\sin\frac{\pi}{3}+\sin\left(\frac{2\pi}{3}\right)=\frac{3\sqrt{3}}{2}$.

Q Find maximum or minimum value in $u=\cos A\cos B\cos C$ where A, B & C are the angles of the triangle.

Sol:

$\because\ A+B+C=\pi$

$\Rightarrow C=\pi-(A+B)$

$\therefore\ u=\cos A\cos B\cos C$

$=\cos A\cos B\cos\left(\pi-(A+B)\right)$

$\therefore \quad u = -\cos A \cos B \cos(A+B)$

$$\frac{\partial u}{\partial A} = -\cos B\left[-\sin A \cos(A+B) + \cos A \times -\sin(A+B)\right]$$

$$= \cos B \sin(2A+B)$$

$$\frac{\partial u}{\partial B} = \cos A \sin(2B+A)$$

✓ $$\frac{\partial^2 u}{\partial A^2} = 2\cos B \cos(2A+B)$$

✓ $$\frac{\partial^2 u}{\partial B^2} = 2\cos A \cos(2B+A)$$

✓ $$\frac{\partial^2 u}{\partial A \partial B} = \left[-\sin B \sin(2A+B) + \cos B \times \cos(2A+B)\right]$$

$$= \cos(2A+2B)$$

For maxima & minima

$$\frac{\partial u}{\partial A} = 0 = \frac{\partial u}{\partial B}$$

$$\cos B \sin(2A+B) = 0 \quad \text{— (i)}$$

$$\cos A \sin(2B+A) = 0 \quad \text{— (ii)}$$

$$\Rightarrow \sin(2A+B) = 0 = \sin\pi$$

and $\sin(2B+A) = 0 = \sin\pi$

$$\Rightarrow 2A+B = \pi$$

$$2B+A = \pi \times 2$$

$$-3B = -\pi \Rightarrow \boxed{B = \frac{\pi}{3}}$$

$\cos B = 0$

$\boxed{B = \frac{\pi}{2}}$

$\cancel{\sin(2B+A) = 0}$

$\cancel{\sin(\pi + A) = 0}$

$-\sin A = 0$, $\cancel{\boxed{A = 0}}$

$\cancel{2A+B = 0}$

$\cancel{2B+A = 0}$

$2A+B = \pi$

$2A = \pi - \frac{\pi}{3}$

$\boxed{A = \frac{\pi}{3}}$

$\boxed{C = \frac{\pi}{3}}$

$\therefore \quad r = \left(\frac{\partial^2 u}{\partial A^2}\right)_{\left(\frac{\pi}{3}, \frac{\pi}{3}\right)} = 2 \cos\frac{\pi}{3} \times \cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)$

$$= 2 \times \frac{1}{2} \times -1 = -1$$

$$s = \cos\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \cos\frac{4\pi}{3} = \frac{-1}{2}$$

$$t = -1$$

$\therefore \quad rt - s^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0, \; r = -1 < 0$

$\Rightarrow \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a point of maxima.

$\therefore$ max^m value $z = \cos\frac{\pi}{3} \times \cos\frac{\pi}{3} \times \cos\frac{\pi}{3} = \frac{1}{8}$ #

Q Find the dimensions of the rectangular box, open at the top of maximum capacity whose surface is given.

Sol:- Let x, y & z are the dimension of the rectangular box

$$S = xy + 2yz + 2zx = \text{given} \Rightarrow 2z(y+x) = -xy + S$$

$$\boxed{z = \frac{-xy+S}{2(x+y)}}$$

Now we have to maximize

$$u = xyz$$

$$\boxed{u = xy\left(\frac{S - xy}{2(x+y)}\right)}$$

$$\therefore \quad u = \frac{5xy}{2(x+y)} - \frac{x^2y^2}{2(x+y)} = \frac{5xy - x^2y^2}{2(x+y)}$$

$$\cancel{\frac{\partial u}{\partial x} = \frac{5y}{2}\left[\frac{1}{x+y} + x \times \frac{1}{(x+y)^2}\right] - \frac{y^2}{2}\left[2x \times \frac{1}{x+y} + x^2 \times \frac{-1}{(x+y)^2}\right]}$$

$$= \frac{5y}{}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{5y - 2xy^2}{2(x+y)} - \frac{(5xy - x^2y^2)}{2(x+y)^2}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{5x - 2yx^2}{2(x+y)} - \frac{5xy - x^2y^2}{2(x+y)^2}$$

$= 0$

$= 0$

Now $\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}$

$$\Rightarrow \frac{5y - 2xy^2}{2(x+y)} - \frac{5xy - x^2y^2}{2(x+y)^2} = 0 \quad \text{— (I)}$$

$$\frac{5x - 2yx^2}{2(x+y)} - \frac{5xy - x^2y^2}{2(x+y)^2} = 0 \quad \text{— (II)}$$

(I) − (II) $\Rightarrow$

$$5y - 2xy^2 = 5x - 2yx^2$$

$$\Rightarrow 5(y-x) = 2xy^2 - 2yx^2$$

$$\Rightarrow 5(y-x) = 2xy(y-x)$$

$$\boxed{y = \sqrt{\frac{5}{3}}}$$

$$\Rightarrow (5 - 2xy)(x - y) = 0$$

$$\therefore 5 \neq 2xy$$

$$\boxed{x = y}$$

$$\sqrt{\frac{108}{3}} = \sqrt{36} = 6$$

Hence by ①

$$\frac{5\cancel{x} - 2x^3}{2 \times 2x} - \frac{(5x^2 - x^4)}{2 \cancel{\times} 4x^2} = 0$$

$$\Rightarrow 2x(5x - 2x^3) - 5x^2 + x^4 = 0$$

$$\Rightarrow 5x^2 - 3x^4 = 0$$

$$\Rightarrow 5 = 3x^2 \Rightarrow \boxed{x = \sqrt{5/3}}$$

$$z = \frac{s - xy}{2(x+y)} = \frac{s - \frac{s}{3}}{2 \times 2 \times \sqrt{s/3}} = \frac{2s}{3 \times 4 \times \sqrt{\frac{s}{3}}}$$

$$= \frac{1}{2}\sqrt{\frac{s}{3}}$$

$$\therefore \quad x = \sqrt{\frac{s}{3}}, \quad y = \sqrt{\frac{s}{3}}, \quad z = \frac{1}{2}\sqrt{\frac{s}{3}}.$$

Now show that $\underline{rt - s^2 > 0, \; r < 0}$ at (x, y, z).

\#

Lagrange's multiplier method

Let f is the given function which we have to extremize and let ϕ is the given condition.

Then consider

$$u = f + \lambda\phi$$ where λ is the scalar known as Lagrange's multiplier. For maxima & minima

$$\left\{\begin{aligned} \frac{\partial u}{\partial x} &= 0 \Rightarrow \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \\ \frac{\partial u}{\partial y} &= 0 \Rightarrow \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \\ \frac{\partial u}{\partial z} &= 0 \Rightarrow \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \end{aligned}\right.$$

Q Find the minimum value of $x+y+z$ subject to the condition $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$

$\left\{ \Rightarrow \frac{a}{x}+\frac{b}{y}+1=\frac{c}{z} \right)$

sol: Let $f = x+y+z$ and $\phi = \frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1$

and consider $\boxed{u = f+\lambda\phi}$

$$= (x+y+z)+\lambda\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-1\right)$$

For maxima & minima

$$\frac{\partial u}{\partial x}=0, \quad \frac{\partial u}{\partial y}=0, \quad \frac{\partial u}{\partial z}=0$$

$$1-\frac{\lambda a}{x^2}=0 \quad -(1), \qquad 1-\frac{\lambda b}{y^2}=0 \quad -(2), \qquad 1-\frac{\lambda c}{z^2}=0 \quad -(3)$$

$$(i)\times x + (ii)\times y + (iii)\times z \Rightarrow$$

$$(x+y+z) - \lambda\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right) = 0$$

$$\Rightarrow f - \lambda \times 1 = 0$$

$$\Rightarrow \boxed{\lambda = f}$$

By (i), (ii) & (iii)

$$\lambda = \frac{x^2}{a} = \frac{y^2}{b} = \frac{z^2}{c}$$

$$\Rightarrow x = \sqrt{a\lambda}, \quad y = \sqrt{b\lambda}, \quad z = \sqrt{c\lambda}$$

$$\because \lambda = f = x+y+z$$

$$\Rightarrow \lambda = \sqrt{a\lambda} + \sqrt{b\lambda} + \sqrt{c\lambda}$$

$$\Rightarrow \frac{\lambda}{\sqrt{\lambda}} = \sqrt{a}+\sqrt{b}+\sqrt{c}$$

$$\Rightarrow \boxed{\sqrt{\lambda} = \sqrt{a}+\sqrt{b}+\sqrt{c}}$$

$$\Rightarrow \boxed{\lambda = \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}$$

$\therefore\ x = \sqrt{a}\sqrt{\lambda}$

$\boxed{x = \sqrt{a}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)}$ ✓

$\boxed{y = \sqrt{b}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)}$ ✓

$\boxed{z = \sqrt{c}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)}$ ✓

$\boxed{\lambda<0}$ $\boxed{\frac{\partial^2 f}{\partial x^2}<0}$ $\boxed{x,y}$

x, y, z

$z = f(x,y)$

$\therefore\ f = x+y+z = \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$

$= \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2$

$\therefore\ f = x+y+z$

$\Rightarrow \frac{\partial f}{\partial x} = 1 + \frac{\partial z}{\partial x}$

$\therefore\ \frac{a}{x}+\frac{b}{y}+\frac{c}{z} = 1$

$\Rightarrow -\frac{a}{x^2} + 0 - \frac{c}{z^2}\frac{\partial z}{\partial x} = 0$

$\Rightarrow \frac{\partial z}{\partial x} = \frac{-az^2}{cx^2}$

$$\therefore \quad \frac{\partial f}{\partial x} = 1 + \frac{\partial z}{\partial x}$$

$$= 1 - \frac{az^2}{cx^2}$$

Again diff

$$\frac{\partial^2 f}{\partial x^2} = -\frac{a}{c}\left(2z \times \frac{\partial z}{\partial x} + z^2 \times \frac{-1}{x}\right)$$

$$= -\frac{a}{c}\left(2z \times \frac{-az^2}{cx^2} - \frac{z^2}{x}\right)$$

$$= \frac{a}{c}\left(\frac{2az^3}{cx^2} + \frac{z^2}{x}\right) > 0$$

$\Rightarrow$ f is minimum ~~at~~ at (x, y, z).

Q. (1) Find the maximum or minimum value of

(1) $x^3+y^3-3axy \rightarrow (a,a)$ (max if $a<0$)

(2) $xy+\frac{a^3}{x}+\frac{a^3}{y} \rightarrow$ (minimum value $3a^2$)

(3) Divide a into three parts such that their product is maximum $\left[\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)\right]$

$$x+y+z=a$$

$$u=xyz$$

$$\boxed{u=xy(a-x-y)}$$

Q Find the minimum value of $x^2+y^2+z^2$ subject to $ax+by+cz=p$

sol:- consider

$$u=(x^2+y^2+z^2)+\lambda(ax+by+cz-p)$$

when $x^2+y^2+z^2=f,\ \phi=ax+by+cz-p$

For maxima & minima

$$\frac{\partial u}{\partial x}=0,\quad \frac{\partial u}{\partial y}=0,\quad \frac{\partial u}{\partial z}=0$$

$$\Rightarrow 2x+\lambda a=0 \quad \text{—(i)},\qquad 2y+\lambda b=0 \quad \text{—(ii)},\qquad 2z+\lambda c=0 \quad \text{—(iii)}$$

(i) $\times x$ + (ii) $\times y$ + (iii) $\times z \Rightarrow$

$$2(x^2+y^2+z^2)+\lambda(ax+by+cz)=0$$

$$\Rightarrow 2f+\lambda p=0$$

$$\Rightarrow \boxed{\lambda=\frac{-2f}{p}} \quad \text{--- (iv)}$$

Also from (i), (ii) & (iii)

$$\lambda=\frac{-2x}{a}=\frac{-2y}{b}=\frac{-2z}{c}$$

$$\Rightarrow x=\frac{-\lambda a}{2},\quad y=\frac{-\lambda b}{2},\quad z=\frac{-\lambda c}{2}$$

By (iv), $\quad \lambda=\frac{-2}{p}(x^2+y^2+z^2)$

$$\therefore \lambda = \frac{-2}{p}\left(\frac{\lambda^2 a^2}{4} + \frac{\lambda^2 b^2}{4} + \frac{\lambda^2 c^2}{4}\right)$$

$$\Rightarrow \boxed{\frac{-2p}{a^2+b^2+c^2} = \lambda}$$

$$\therefore x = \frac{-\lambda a}{2} = \frac{ap}{a^2+b^2+c^2}$$

$$y = \frac{bp}{a^2+b^2+c^2}, \quad z = \frac{cp}{a^2+b^2+c^2}$$

$\therefore$ value of func^n $f = x^2+y^2+z^2 = \dfrac{p^2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2} = \dfrac{p^2}{a^2+b^2+c^2}$

#

$\because \quad f = x^2 + y^2 + z^2$

$$\frac{\partial f}{\partial x} = 2x + 2z\frac{\partial z}{\partial x}$$

$\because \quad ax + by + cz = p$

$$\Rightarrow a + 0 + c\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-a}{c}$$

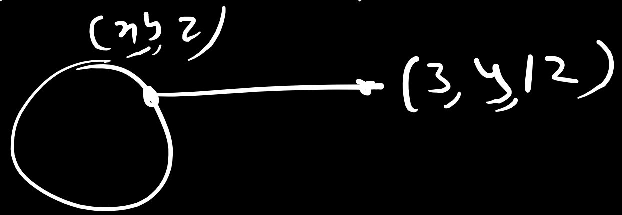
$$\therefore \frac{\partial f}{\partial x} = 2x - \frac{2a}{c}z$$

$$\frac{\partial^2 f}{\partial x^2} = 2 - \frac{2a}{c}\frac{\partial z}{\partial x} = 2 + \frac{2a^2}{c^2} > 0$$

$\Rightarrow f$ is minimum at (x, y, z).

Q

(1) Find the maximum & minimum distances of the point $(3,4,12)$ from the sphere $x^2+y^2+z^2=1$



$f = (x-3)^2+(y-4)^2+(z-12)^2 + \lambda(x^2+y^2+z^2-1)$

(2) ~~Find the ma the temp. T at the point (x,y,z)~~

Find the highest temp $T = 400xyz^2$ on the unit sphere $x^2+y^2+z^2=1$

$$u = 400xyz^2 + \lambda(x^2+y^2+z^2-1)$$

$$x = \pm\frac{1}{2},\quad y = \pm\frac{1}{2},\quad z = \pm\frac{1}{2}$$

$$T = 400xyz^2 = 50$$

(3) Find the volume of largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

~~$a=b=c$~~
~~$x^2+y^2+z^2=a^2$~~

~~$u = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$~~

~~$V = 8xyz$~~

$V^2 = 64x^2y^2z^2$

$u = x^2y^2z^2$

$= x^2y^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\times c^2$

Matrix Algebra

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \end{bmatrix}$$

$|A| = 0$

$\cancel{A^{-1} = \frac{\mathrm{Adj} A}{|A|}}$

symm → $A' = A$

skew-sym → $A' = -A$

Hermitian matrix → $(\bar{A})' = A$

Skew hermitian → $(\bar{A})' = -A$

orthogonal → $AA' = A'A = I$

Nilpotent matrix → $A^p = 0$, p is the least +ve integer

Idempotent matrix → $A^2 = A$

Involutory matrix → $A^2 = I$

$\boxed{AA' = I = A'A}$

?

Elementary operations

The following operations are known as Elementary operations

(i) Interchange of two rows or column

$$R_i \leftrightarrow R_j \quad \text{or} \quad C_i \leftrightarrow C_j$$

(ii) multiplication of ith row/column by a constant

$$R_i \to kR_i \quad \text{or} \quad C_i \to kC_i$$

(iii) Addition of k times of ith row (column) in jth row (column)

$$R_j \to kR_i + R_j \quad \text{or} \quad C_j \to kC_i + C_j$$

$$\begin{bmatrix}1 & 2\\ 0 & 2\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\ 2 & c\end{bmatrix} \sim R_2 \to 2R_1 + R_2$$

Inverse of the matrix using E-row operations

$A = AI$

Applying E-row operations we get

$I = AB$

$$\Rightarrow \boxed{A^{-1} = B}$$

$\cancel{AA^{-1}} = I$

Q Ex Find the inverse of the following matrix using E-row operations

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

Sol:-

$\therefore \quad A = AI$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$4 - 5$

$-6 + 5$

$R_3 \to 2R_3 - 5R_1$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 2 \end{bmatrix}$$

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} -4 & 0 & 2 \\ 0 & 1 & 0 \\ -5 & 0 & 2 \end{bmatrix}$$

$$\cancel{A \approx I \qquad \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} -4 & 0 & 2 \\ -5 & 1 & 2 \\ -5 & 0 & 2 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} -4 & 0 & 2 \\ -5 & 1 & 2 \\ -10 & 1 & 4 \end{bmatrix}$$

$R_1 \to R_1 - 2R_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 16 & -2 & -6 \\ -5 & 1 & 2 \\ -10 & 1 & 4 \end{bmatrix}$$

$R_1 \to \frac{1}{2}R_1$, $R_3 \to R_3 \times -1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

$\therefore \quad I = AB$

$$\Rightarrow A^{-1} = B = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} \text{ Ans.}$$

$$\left\{ \because AA^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$(0\ 0\ 0)$

$$A = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \quad \boxed{|A| = 1} \neq 0$$

$\rho(A) = 3$

$\boxed{\text{Echleon from}}$

Rank of the matrix

Rank of a matrix A is the order of the highest non-zero minor of the matrix.

i.e., A matrix 'A' is said to have rank 'r' if ∃ a minor of A whose order is r and determinant is non-zero and all other minors of order (r+1) are zero

Rank of the matrix is denoted by $\rho(A)$.

$A_{3\times 2}$

$\boxed{\rho(A) \leq 2}$

(i) I_n is the identity matrix of order $n\times n$

$\rho(I_n) = n$.

(ii) If A is $m\times n$ matrix then $\rho(A) \leq$ minimum of m and n

Ex: (1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$|A| = 1(0-1) - 2(0-0) + 3(1-0)$

$= -1 + 3 = 2 \neq 0 \Rightarrow \boxed{\rho(A) = 3}$

(2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$|A| = 0 \Rightarrow \boxed{\rho(A) < 3}$

$\because$ All the submatrix of order 2×2 have determinant 0, hence $\rho(A) < 2$

$\Rightarrow \boxed{\rho(A) = 1}$

$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_1$, $R_2 \to R_2 - R_1$

$\boxed{\rho(A) = 1}$ Ans

- Video Lecture (Unit I)

(1) https://youtu.be/OwJ8CvXejYs
(2) https://youtu.be/qtAibJ_N4bw
(3) https://youtu.be/qtAibJ_N4bw
(4) https://youtu.be/VlTors_ON0Y
(5) https://youtu.be/JHNO49RmSgw
(6) https://youtu.be/M-JE2-jXSaw
(7) https://youtu.be/0Fv2iMLL77A
(8) https://youtu.be/4ZyN99gNCOo
(9) https://youtu.be/Il-X_8BeFog
(10) https://youtu.be/FHWaoaPX4rk
(11) https://youtu.be/B2xwuEQRJxs
(12) https://youtu.be/jBukOH3HxhU