Series

→ Let $\langle a_n \rangle$ be a sequence of real no. then the expression $a_1+a_2+\dots+a_n+\dots$ defines as a series of real numbers. and denoted by $\sum_{n=1}^{\infty} a_n = a_1+\dots+a_n+\dots\infty$
or Σa_n

→ If all the terms are +ve the series is called series of +ve terms. If terms are alternatively +ve and -ve then series is called alternating series.

Ex (i) $\Sigma (-1)^{n+1} a_n$ or $\Sigma (-1)^n a_n$ where $a_n > 0 \ \forall n$

(ii) $\Sigma (-1)^n \frac{1}{n}$ or $\Sigma \frac{(-1)^{n+1}}{n}$

Sequence of partial sums (SOPS):-
If $\langle a_n \rangle$ be a sequence of real number and Σa_n be a series then the SOPS of series Σa_n is denoted by S_n, where nth term of this sequence is given by

$$S_n = a_1 + a_2 + \dots + a_n$$

i.e.

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + \dots a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + \dots + a_n$$
$$\vdots$$

$$\langle S_n \rangle = \langle S_1, S_2, \dots \rangle$$

i.e. $\lim_{n\to\infty} S_n = \Sigma a_n$

Behaviour of series :-

Let Σa_n be series of real number and $\langle S_n \rangle$ be sequence of partial sum of Σa_n then

(i) $\langle S_n \rangle$ is convergent iff Σa_n is convergent

(ii) $\langle S_n \rangle$ is divergent iff Σa_n is divergent

(iii) If $\langle S_n \rangle$ oscillate finite then Σa_n oscilates finite.

(iv) If $\langle S_n \rangle$ oscillate infinite the Σa_n oscilates infinite.

(v) i.e. $\lim\limits_{n \to \infty} S_n = l$ iff $\Sigma a_n = l$.

e.g. $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots$

$$\Sigma a_n = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

Let $\langle S_n \rangle$ be its sops.

$$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}$$

$$= \frac{1(1-r^n)}{(1-r)} = \frac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}$$

$$S_n = 2\left[1 - \frac{1}{2^n}\right]$$

$$\lim_{n \to \infty} S_n = 2.$$

$\Rightarrow \Sigma a_n = 2$ ✓

e.g. $\Sigma (-1)^n$

Let $\langle S_n \rangle$ be sops

$S_n = -1 + 1 - 1 + 1 + \dots (-1)^n$

$= \begin{cases} 0, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$

$\langle S_n \rangle = \{-1, 0, -1, 0, \dots\}$

$\langle S_n \rangle$ is bounded and not convergent

$\Rightarrow \langle S_n \rangle$ is oscillate finitely

then $\lim_{n \to \infty} S_n \begin{smallmatrix} \nearrow 0 \\ \searrow -1 \end{smallmatrix}$

$\Rightarrow \Sigma a_n \begin{smallmatrix} \nearrow 0 \\ \searrow -1 \end{smallmatrix}$

$\Rightarrow$ series oscillate finitely.

eg. $\Sigma (-1)^n \cdot n$

$\langle S_n \rangle = \{-1, 1, -2, 2, -3, 3, \dots\}$

$\lim_{n \to \infty} S_n \begin{smallmatrix} \nearrow 0 \\ \searrow -\infty \end{smallmatrix}$.

$\Rightarrow \Sigma a_n \begin{smallmatrix} \nearrow 0 \\ \searrow -\infty \end{smallmatrix}$

$\Rightarrow$ Series oscillates infinitely.

Geometric series:-

$\Sigma a_n = 1 + x + x^2 + \dots + x^n + \dots$

Let $\langle S_n \rangle$ be its sops and

$S_n = 1 + x + \dots + x^n$

$S_n = \dfrac{1[1 - x^n]}{1 - x}$, $|x| < 1$ & $x \neq 1$;

or $S_n = \frac{1\cdot[x^n - 1]}{x-1}$, ~~[illegible]~~ $x > 1$

case(I) If $|x| < 1$

$\lim_{n\to\infty} x^n = 0$

$\Rightarrow \lim_{n\to\infty} S_n = \frac{1}{1-x}$ finite.

$\Rightarrow \Sigma x^n$ is convergent if $|x| < 1$

case II if $x = 1$

$\sum_{n=0}^{\infty} x^n = 1 + 1 + \cdots = \infty$

case III :- If $x = -1$

$\sum_{n=0}^{\infty} x^n$ oscilates finite.

case IV :- If $x > 1$

$x^n \to \infty \Rightarrow \Sigma x_n \to \infty$.

convergence test :-

Comparison test :- Let Σa_n and Σb_n are +ve term series such that $a_n \le b_n \ \forall \ n \ge 1$ then If

(i) If Σb_n convergent then Σa_n is convergent.

(ii) If Σa_n is divergent then Σb_n is divergent

ex $\sum_{n=0}^{\infty} \frac{1}{3^n + n}$, $\frac{1}{3^n+n} < \frac{1}{3^n}$ & $\Sigma \frac{1}{3^n} = 3/2 \Rightarrow \Sigma \frac{1}{3^n+n}$ is conv.

Necessary cond for convergent :-

If a positive term series Σa_n is convergent then $\lim_{n\to\infty} a_n = 0$.

Harmonic series – diverges

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

$$\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$$= 1 + \cancel{\frac{1}{2} + + +}$$

$$= \infty$$

$\Rightarrow$ b_n diverges

$\Rightarrow$ $\sum \frac{1}{n}$ diverges.

eg: $\sum \tan^{-1}(1/n)$

expansion of $\tan^{-1}(x)$

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots$

$$\frac{\tan^{-1}(1/n)}{1/n} = \frac{\frac{1}{n}\left[1 - \frac{(1/n)^2}{3} + \frac{(1/n)^4}{5} \cdots\right]}{1/n} \to 1 \text{ as } n \to \infty$$

since $\sum \frac{1}{n}$ diverge.

eg: $\sum \frac{1}{n^2}$ is convergent.

$$\frac{1}{n^2} \leq \frac{1}{n(n-1)}$$

$$= \frac{1}{n-1} - \frac{1}{n}$$

$$\sum \frac{1}{n^2} \leq \sum \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$\sum a_n \qquad \sum b_n$$

sops of b_n

$$S_M = \sum_{n=2}^{M} b_n$$

$$= \left(1-\cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \cdots + \left(\cancel{\tfrac{1}{M-1}} - \frac{1}{M}\right)$$

$$= 1 - \frac{1}{M}$$

as $\lim_{M \to \infty} S_M = 1$

$\Rightarrow \sum b_n = 1 \Rightarrow$ convergent

. using comparison test $\sum a_n$ also converge.

eg: (i) $\sum \cos\left(\frac{1}{n^2}\right)$

$\because \frac{1}{n^2} \longrightarrow 0$ as $n \to \infty$

$\cos\left(\frac{1}{n^2}\right) \longrightarrow \cos(0) = 1$ as $n \to \infty$

$\neq 0$

$\Rightarrow \sum \cos\left(\frac{1}{n^2}\right)$ is not a convergent series.

(ii) $\sum \frac{1}{n}$ is not a convergent series.

Limit form test :-

If two positive term series $\sum a_n$ and $\sum b_n$ be such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l \; (\neq 0 \text{ finite}) \text{ then}$$

$\sum u_n$ and $\sum v_n$ converge or diverge together.

ex $\sum \sin(1/n)$

Let $1/n = x$

as $n \to \infty$, $\frac{1}{n} \to 0 \Rightarrow x \to 0$

and we know that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 \neq 0$$

$\because \sum \frac{1}{n}$ is divergent sequence

$\Rightarrow \sum \sin\left(\frac{1}{n}\right)$ is a divergent sequence.

Ex $u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{n(2-1/n)}{n^3(n+1}$

$$= \frac{1}{n^2}\left(\frac{(2-1/n)}{(1+1/n)(1+2/n)}\right)$$

$v_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = 2 \Rightarrow \Sigma u_n$ converges.

ex $u_n = \frac{n^n}{(n+1)^{n+1}} = \left(\frac{1}{n+1}\right)\cdot\left(\frac{n}{n+1}\right)^n$

$v_n = \frac{1}{n}$

$\frac{u_n}{v_n} = \left(\frac{n}{n+1}\right)\cdot\left(\frac{n}{n+1}\right)^n$

$\frac{u_n}{v_n} = \frac{1}{(1+1/n)}\cdot\left(\frac{1}{1+1/n}\right)^n$

$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = 1\cdot\frac{1}{e} \neq 0$

$\Rightarrow \Sigma u_n$ & Σv_n diverges together as $\Sigma v_n = \Sigma \frac{1}{n}$ diverges

(limit form test)

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u}$$

ex (1) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{(n+1)}}$

$$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} \times \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}} = \sqrt{n+1} - \sqrt{n}$$

$$= \sqrt{n}\left((1+1/n)^{1/2} - 1 \right)$$

$$= \sqrt{n}\left\{ \left(1+\frac{1}{2n}-\frac{1}{8n^2}+\cdots\right) - 1 \right\}$$

$$= \sqrt{n}\left\{ \frac{1}{2n} - \frac{1}{8n^2} + \cdots \right\}$$

$$= \frac{1}{\sqrt{n}}\left\{ \frac{1}{2} - \frac{1}{8n^2} + \cdots \right\}, \quad v_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2} \neq 0$$

$\Rightarrow \sum u_n$ diverges.

ex $\boxed{1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \cdots + \frac{n^2}{n!}, \cdots \quad u_n = \frac{n^2}{n!} = \frac{n \cdot n}{n\cdot(n-1)\cdots 1}}$

eg

v_n !

$$= [\log n]_2^{\infty}$$

$$= \infty$$

Ex $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots$

$$\sum a_n = \sum_{n=1}^{\infty} \frac{n^2}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{\cancel{n} \cdot n}{\cancel{n} \cdot (n-1)!}$$

$$= \sum_{n=2}^{\infty} \frac{n-1+1}{(n-1)!}$$

$$= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!}$$

$$= e + e$$

$$= 2e$$

ex $\sum \frac{\sqrt{n}}{n^2+1}$

$$u_n = \frac{1}{n^{3/2}(1 + 1/n^2)}$$

$$v_n = \frac{1}{n^{3/2}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1 \neq 0$$

$\Rightarrow \sum u_n$ converges as $\sum v_n$ converges.

eg $\sum \frac{1}{\sqrt{n}} \sin \frac{1}{n}$

$$u_n = \frac{1}{\sqrt{n}} \sin \frac{1}{n}$$

$$v_n = \frac{1}{n^{3/2}}$$

$$\frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{(1/n)}$$

$$= 1 \neq 0$$

$\Rightarrow \sum u_n$ converges.

ex Let $f(n) = \sum \frac{1}{n(\log n)^p}$

$$\Rightarrow f(x) = \frac{(\log x)^{-p}}{x}$$

$$f'(x) = -p\frac{(\log x)^{-p-1}}{x^2} - \frac{(\log x)^{-p}}{x^2}$$

$$= \frac{-1}{x^2}\left(p(\log x)^{p-1} + (\log x)^{-p}\right)$$

< 0 as $x = n \geq 1$

i.e. $f(x)$ is decreasing funcn

$$\int_2^{\infty} \frac{(\log x)^{-p}}{x} dx = \left| \frac{(\log x)^{-p+1}}{-p+1} \right|_2^{\infty}$$

If $p > 1$, $p-1 = k > 0$

$$= \left| \frac{(\log x)^{-k}}{-k} \right|_2^{\infty}$$

$= 0 + \frac{1}{k}(\log 2)^{-k}$ finite.

$\Rightarrow \sum f(n)$ is convergent

If $p<1$ $\Rightarrow 1-p \geq k > 0$

$$= \left| \frac{(\log x)^k}{k} \right|_0^\infty$$

$$= \cancel{\log \infty}\ \infty \quad \text{not finite}$$

$\Rightarrow \sum f(n)$ diverges.

If $p=1$

$$\int_2^\infty \frac{1}{x \log x} = \left[\log(\log x)\right]_2^\infty$$

$$= \infty$$

Comparison of ratios:-

If $\sum u_n$ and $\sum v_n$ be two positive term series, then $\sum u_n$ converges if (i) $\sum v_n$ converges and (ii) from and after some particular term

$$\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$$

D'ALEMBERT'S RATIO TEST:-

In a positive term series $\sum u_n$, if

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$, then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$.

$\boxed{\text{But fails for } \lambda = 1}$

Another form of ratio test

If $\sum u_n$ is positive term series then

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = k$$

(i) $k>1$ convergent

(ii) $k<1$ divergent.

(iii) $k=1$ test fails.

Ex Test for convergence of the series

(i) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \dots \infty$

we have

$$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$u_{n+1} = \frac{x^{2(n+1)-2}}{(n+2)\sqrt{n+1}}$$

$$= \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \times \frac{(n+2)\sqrt{n+1}}{x^{2n}}$$

$$= \frac{1}{x^2} \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}}$$

$$= \frac{1}{x^2} \frac{(n+2)}{\sqrt{n+1}\sqrt{n}}$$

$$= \frac{1}{x^2} \cdot \frac{\cancel{n}(1+2/n)}{n\sqrt{1+1/n}\cdot\sqrt{1}}$$

$$= \frac{1}{x^2}$$

$\Rightarrow$ converges for

$$\frac{1}{x^2} > 1$$

$$x^2 < 1$$

$$\Rightarrow -1 < x < 1$$

diverges for

$$\frac{1}{x^2} < 1$$

$$\Rightarrow x^2 > 1$$

i.e. $x \in (-\infty, -1] \cup [1, \infty)$

for $1/x^2 = 1$

$$u_n = \frac{1}{(n+1)\sqrt{n}}$$

$$v_n = \frac{1}{n^{3/2}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\frac{1}{\cancel{n^{3/2}}(1+1/n)}}{\frac{1}{\cancel{n^{3/2}}}} = 1 \neq 0$$

$\Rightarrow \Sigma u_n$ converges as Σv_n converges

$\Rightarrow \Sigma u_n$ converges for $x^2 \leq 1$

and diverges for $x^2 > 1$.

ex $\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!}$

$$u_n = \frac{n^2 (n+1)^2}{n!}$$

$$u_{n+1} = \frac{(n+1)^2 (n+2)^2}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = (n+1) \cdot \left(\frac{n}{n+2}\right)^2$$

$$\lim_{n\to\infty} = \lim_{n} (n+1) \cdot \lim_{n\to\infty} \left(\frac{1}{1+2/n}\right)^2$$

$$= \infty \cdot 1$$

$$= \infty.$$

Integral test :- A positive term series $f(1)+f(2)+\cdots+f(n)+\cdots$, where $f(n)$ decreases as n increases, converges or diverges according as the integral

$$\int_1^{\infty} f(x)\,dx$$

is finite or infinite.

i.e. if (i) $\int_1^{\infty} f(x)\,dx$ is finite $\Rightarrow \sum f(n)$ converges

(ii) $\int_1^{\infty} f(x)\,dx$ is infinite $\Rightarrow \sum f(n)$ diverges.

eg: Test for convergence of $\sum \frac{1}{n^p}$, $p>0$

$$f(x) = \frac{1}{x^p}$$

$$f'(x) = \frac{-p}{x^{p-1}}, \quad p>0$$

$\Rightarrow f'(x) < 0$

$\Rightarrow f(x)$ is a decreasing function

$$I = \int_1^{\infty} \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^{\infty}$$

$$= \left(\frac{1}{-p+1}\right) \lim_{m\to\infty} \left(m^{-p+1} - 1\right)$$

for $p<1$, $\Rightarrow -p > -1 \Rightarrow -p+1 > 0$

$\Rightarrow I = \infty \qquad \Rightarrow \sum \frac{1}{n^p}$ diverges

for $p>1 \Rightarrow -p < -1 \Rightarrow -p+1 < 0$

$\Rightarrow I = \left(\frac{1}{-p+1}\right)(0-1) = \frac{1}{p-1}$ finite

$\Rightarrow \sum \frac{1}{n^p}$ converges.

$a_n = \sqrt[3]{n^3+1} - n$ Let $\sqrt[3]{n^3+1} = x$ & $n = y$

$\Rightarrow \sqrt[3]{n^3+1} - n = x - y$

$= \dfrac{x^3 - y^3}{x^2+xy+y^2} = \dfrac{n^3+1-n^3}{x^2+xy+y^2}$

$= \dfrac{1}{x^2+xy+y^2}$

$\because x \geq y$

$\Rightarrow x^2+xy+y^2 \geq y^2+y^2+y^2 = 3y^2$

$\Rightarrow \dfrac{1}{x^2+xy+y^2} \leq \dfrac{1}{3y^2}$

$$\leq \frac{1}{3n^2}$$

$$\Rightarrow \quad \sqrt[3]{n^3+1} - n \leq \frac{1}{3n^2}$$

$\because \sum \frac{1}{3n^2}$ is convergent

$\Rightarrow \sum \sqrt[3]{n^3+1} - n$ is also convergent