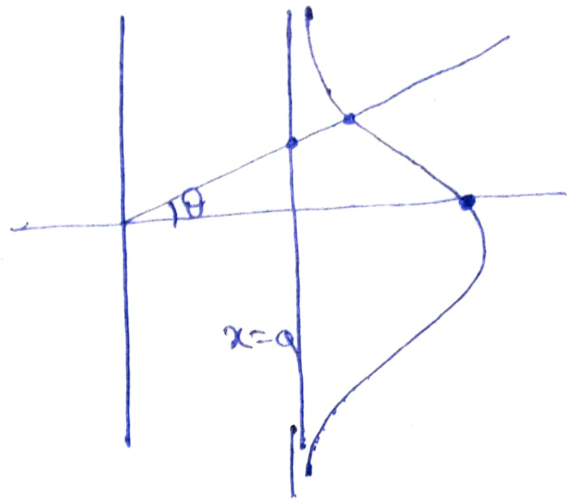
Q Calculate the area included b/w the curve $r = a(\sec\theta + \cos\theta)$ and its assymptote.



at $\theta = 0$

$r = 2a$

at $\theta = \pi/2$

$r \to \infty$

$\because r = a(\sec\theta + \cos\theta)$

$r\cos\theta = a(1+\cos^2\theta)$

$\theta \to \pi/2$, $\quad r\cos\theta \to a$

$\Rightarrow$ $x = a$ is verticle assymptote

$f(\theta) = f(-\theta)$ i.e. the curve is symmetric about x-axis

$$\text{Area} = 2\int_0^{\pi/2}\int_{r=a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} = a^2\int_0^{\pi/2}(2+\cos^2\theta)\,d\theta = 5\pi a^2/4$$

Triple integral:-

Ex. Evaluate $\int_{-1}^{1}\int_0^{x}\int_{y=x-z}^{x+z}(x+y+z)\,dy\,dz\,dx$

$$I = \int_{-1}^{1}\int_0^{x}\left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+z} dz\,dx$$

$$= \int_{-1}^{1}\int_{0}^{x} (2xz+2z^2+2xz)\,dz\,dx$$

$$= \int_{-1}^{1} \left(xz^2 + \frac{2z^3}{3} + xz^2\right)_0^x dx$$

$$= \int_{-1}^{1} (x^3 + \tfrac{2}{3}x^3 + x^3)\,dx = \frac{8}{3}\int_{-1}^{1} x^3\,dx = \frac{8}{3}\times\left[\frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{2}{3}\left[1^4 - (-1)^4\right] = 0 \quad \text{Ans}$$

<u>Volume of solids:-</u>

<u>volume as double integral:-</u>

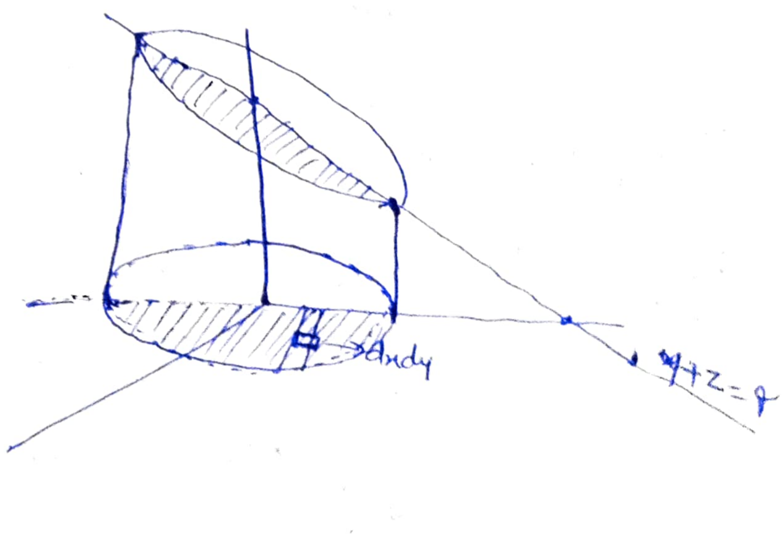
$$V = \iint z\,dx\,dy \qquad \text{(for cartesian)}$$

$$= \iint z r\,d\theta\,dr \qquad \text{(for polar co-ordinate)}$$

Q Find the volume of the cylinder $x^2+y^2=4$ and the planes $y+z=4$ & $z=0$

$\therefore y+z=4$

$z = 4-y$



volume $= 2\int_{y=-2}^{2}\int_{0}^{\sqrt{4-y^2}} z\,dx\,dy$

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} (4-y)\,dx\,dy$$

$$= 2\int_{-2}^{2} (4-y)\,[x]_0^{\sqrt{4-y^2}}\,dy$$

$$= 2\int_{-2}^{2} 4\sqrt{4-y^2} - 2\int_{-2}^{2} y\sqrt{4-y^2}\,dy = 0$$ (as f y is odd fun

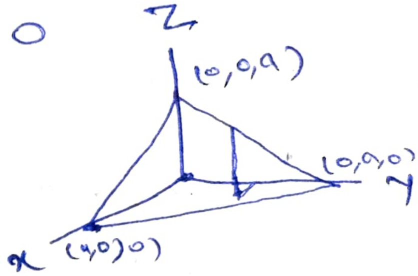
$$= 8\left[\frac{y}{2}\sqrt{4-y^2}+\frac{4}{2}\sin^{-1}\frac{y}{2}\right]_{-2}^{2}$$

$$= 16\pi$$ ans

<u>Volume as triple integral:-</u>

$$V = \iiint_R dx\,dy\,dz$$

Q calculate the volume of the solid bounded by curve Plane $x=0, y=0, x+y+z=a$ & $z=0$



Sol

$$V = \int_0^a\int_0^{a-x}\int_0^{a-x-y} dz\,dy\,dx$$

$$= \int_0^a\int_0^{a-x} [z]_0^{a-x-y}\,dy\,dx$$

$$= \int_0^a\int_0^{a-x} (a-x-y)\,dy\,dx$$

$$= \int_0^a \left(ay - xy - \frac{y^2}{2}\right)_0^{a-x} dx = \int_0^a \left[a(a-x) - x(a-x) - \frac{(a-x)^2}{2}\right] dx$$

$$= \frac{1}{2}\int_0^a 2(a^2 - ax - ax + x^2) - (a^2 + x^2 - 2ax)$$

$$= \frac{1}{2}\int_0^a (a^2 + x^2 - 2ax)\,dx = \frac{1}{2}\left[a^2x + \frac{x^3}{3} - 2a\frac{x^2}{2}\right]_0^a = \frac{a^3}{6}$$

basic of beta and Gamma func^n

(1) $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \qquad n > 0$ (Gamma func^n)

for n a natural no, $\boxed{\Gamma n = (n-1)\Gamma n-1}$

$\boxed{\Gamma n = \lfloor n-1 = (n-1)!}$. $\boxed{\Gamma \tfrac{1}{2} = \sqrt{\pi}}$

(2) $\beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx \quad \begin{cases} m > 0 \\ n > 0 \end{cases}$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

e.g: $\beta(2,2) = \frac{\Gamma 2 \, \Gamma 2}{\Gamma 2+2} = \frac{1! \cdot 1!}{3!} = \frac{1}{6}$

Dirichlet Theorem:-

Dirichlet theorem for three variables:-

If l, m, n are all positive, then the triple integral

$$\iiint x^{l-1} y^{m-1} z^{n-1} dx dy dz = \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma l+m+n+1}$$

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x+y+z \leq 1$.

n-variables:-

$$\iiint \cdots \int x_1^{\lambda_1 - 1} x_2^{\lambda_2 - 1} \cdots x_n^{\lambda_n - 1} dx_1 dx_2 \cdots dx_n = \frac{\Gamma \lambda_1 \, \Gamma \lambda_2 \cdots \Gamma \lambda_n}{\Gamma 1+\lambda_1+\lambda_2+\cdots \lambda_n}$$

Liouville's extension of Dirichlet's theorem:-

If x, y, z are all +ve such that
$$h_1 \le x+y+z \le h_2, \text{ then}$$

$$\iiint F(x+y+z) \cdot x^{l-1} y^{m-1} z^{n-1} \, dx\,dy\,dz$$
$$= \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma l+m+n} \int_{h_1}^{h_2} F(h)\, h^{l+m+n-1} \, dh$$

$$\boxed{h = x+y+z}$$

Ques Evaluate $\iiint x^{-1/2} y^{-1/2} z^{-1/2} (1-x-y-z)^{1/2} \, dx\,dy\,dz$ extended to all positive values of the variables subject to the condition $x+y+z < 1$.

Sol^n The given integral

$$\iiint x^{-1/2} y^{-1/2} z^{-1/2} \left(1 - (x+y+z)\right)^{\frac{1}{2}} dx\,dy\,dz$$

$$= \frac{\Gamma \tfrac12 \, \Gamma \tfrac12 \, \Gamma \tfrac12}{\Gamma \tfrac12+\tfrac12+\tfrac12 + \cancel{1}} \int_0^1 h^{\frac12+\frac12+\frac12-1} (1-h)^{1/2} \, dh$$

$$= \frac{(\Gamma \tfrac12)^3}{\Gamma \tfrac32} \int_0^1 h^{\frac12} (1-h)^{1/2} \, dh = (\Gamma \tfrac12)^2 \int_0^1 h^{\frac32 - 1} (1-h)^{\frac32 - 1} \, dh$$

$$= 2(\Gamma \tfrac12)^2 \, \beta\left(\tfrac32, \tfrac32\right)$$

$$= 2(\sqrt{\pi})^2 \, \frac{\Gamma \tfrac32 \, \Gamma \tfrac32}{\Gamma \tfrac32 + \tfrac32} \quad \frac{\cancel{2} \times \pi \times \frac12 \times \frac12 (\sqrt{\pi})^2}{2}$$

$$= \frac{\pi^2}{8} \times 2 = \pi^2/4$$

Q Find the value of $\iiint x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz$, where x, y, z are always +ve but $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \le 1$

Soln

$$\left(\frac{x}{a}\right)^p = u_1, \quad \left(\frac{y}{b}\right)^q = u_2, \quad \left(\frac{z}{c}\right)^r = u_3$$

$$x = a u_1^{1/p}, \quad y = b u_2^{1/q}, \quad z = c u_3^{1/r}$$

$$dx = \frac{a}{p} u_1^{\frac{1}{p}-1} du_1, \quad dy = \frac{b}{q} u_2^{\frac{1}{q}-1} du_2, \quad dz = \frac{c}{r} u_3^{\frac{1}{r}-1} du_3$$

$$\iiint x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz = \iiint \left(a u_1^{1/p}\right)^{l-1} \left(b u_2^{1/q}\right)^{m-1} \left(c u_3^{1/r}\right)^{n-1} \left(\frac{a}{p}\right) u_1^{\frac{1}{p}-1} \cdot \left(\frac{b}{q}\right) u_2^{\frac{1}{q}-1} \left(\frac{c}{r}\right) u_3^{\frac{1}{r}-1} du_1\,du_2\,du_3$$

$$= \frac{a^l b^m c^n}{pqr} \iiint u_1^{\frac{l}{p}-1} u_2^{\frac{m}{q}-1} u_3^{n/r-1} du_1\,du_2\,du_3$$

where $u_1 + u_2 + u_3 \le 1$

$$= \frac{a^l b^m c^n}{pqr} \frac{\Gamma l/p \,\Gamma m/q\, \Gamma n/r}{\Gamma \left(\frac{l}{p} + \frac{m}{q} + \frac{n}{r} + 1\right)}$$

(using dirichlet theorem).