CALEY - HAMILTON THEOREM :-

Every square matrix satisfies its own characteristic equation. i.e.

~~Proof~~: $|A - \lambda I| = ((-1)^n \lambda^n + K_1 \lambda^{n-1} + K_2 \lambda^{n-2} + \dots + K_n) = 0$

then

$$(-1)^n A^n + K_1 A^{n-1} + K_2 A^{n-2} + \dots + K_n = 0$$

Proof:- Let the adjoint of the matrix $A - \lambda I$ be P. Clearly, the element of P will be polynomials of $(n-1)$ degree in λ, for the cofactors of the elements in $|A - \lambda I|$ will be such polynomials.

$\therefore$ P can be split up into a number of matrices containing terms with the same power of λ, such that

$$P = P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \dots + P_{n-1} \lambda + P_n$$

where $P_1, P_2, \dots, P_n$ are square matrices of order 'n'. whose elements are functions of the elements of A.

Since the product of a matrix by its adjoint = determinant of the matrix x determinant

$$[A - \lambda I] P = |A - \lambda I| . I$$

$$(A - \lambda I)((-1)^n P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \dots + P_{n-1} \lambda + P_n)$$
$$= (-1)^n \lambda^n + K_1 \lambda^{n-1} + K_2 \lambda^{n-2} + \dots + K_n$$

Equating the coefficient of various powers of λ, we get

$$-P_1 = (-1)^n I$$

$$AP_1 - P_2 = K_1 I$$
$$AP_2 - P_3 = K_2 I$$
$$\vdots$$
$$AP_{n-1} - P_n = K_{n-1} I$$
$$AP_n = K_n I$$

Now pre-multiplying equations by $A^n, A^{n-1}, A^{n-2}, \ldots, A, I$ respectively and adding, we get

$$(-1)^n A^n + K_1 A^{n-1} + K_2 A^{n-2} + \cdots + K_n I = 0$$

This proves the theorem.

Another method for finding the inverse :-

Characteristic equation for any matrix A is

$$(-1)^n \lambda^n + K_1 \lambda^{n-1} + K_2 \lambda^{n-2} + \cdots + K_{n-1}\lambda + K_n = 0 \quad -(1)$$

by Caley-Hamilton theorem

$$(-1)^n A^n + K_1 A^{n-1} + K_2 A^{n-2} + \cdots + K_{n-1} A + K_n I = 0 \quad -(2)$$

multiplying equation (2) by A^{-1}, we have

$$(-1)^n A^{n-1} + K_1 A^{n-2} + K_2 A^{n-3} + \cdots + K_{n-1} I + K_n A^{-1} = 0$$

$$\Rightarrow \boxed{A^{-1} = -\frac{1}{K_n}\left((-1)^n A^{n-1} + K_1 A^{n-2} + \cdots + K_{n-1} I\right)}$$

Q

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Verify caley-Hamilton theorem for this matrix and find its inverse

Soln

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

or, $(1-\lambda)(3-\lambda) - 8 = 0$

$3 - 3\lambda - \lambda + \lambda^2 - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0 \qquad -(i)$

By Caley-Hamilton theorem, A must satisfy its characteristic equation (i), so that

$$A^2 - 4A - 5I = 0 \qquad -(ii)$$

L.H.S

$A^2 - 4A - 5I$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 16-16-0 \\ 8-8-0 & 17-12-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O = R.H.S.$$

$\Rightarrow$ A satisfies its characteristic equation multiplying eqn (2) by A^{-1} we have

$$A - 4I - 5A^{-1} = 0$$

$$\Rightarrow \quad 5A^{-1} = A - 4I$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}\left[A - 4I\right]$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -3/5 & 4/5 \\ 2/5 & -1/5 \end{bmatrix}$$

Q. Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

and hence find its inverse.

Soln The characteristic eqn of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left\{(3-\lambda)(-4-\lambda) - 12\right\} - 1\left\{1\cdot(-4-\lambda) - 6\right\}$$
$$+ 3\left\{1\cdot(-4) - (3-\lambda)(-2)\right\} = 0$$

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Simplifying it, we have

$$\lambda^3 - 20\lambda + 8 = 0 \quad \text{— (i)}$$

By Caley-Hamilton theorem.

$$A^3 - 20A + 8I = 0 \quad \text{—— (ii)}$$

multiply eqn (ii) with A^{-1}, we have

$$A^2 - 20I + 8A^{-1} = 0$$

$$\Rightarrow 8A^{-1} = -A^2 + 20I$$

$$= -\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= -\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} + \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 4+20 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{8}\begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

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