Primal -dual problem

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Every linear programming problem has an associated with it another linear programming problem. So, the original problem is called primal and its associated problem is called its dual.

If the optimal solution to one problem is known, then the optimal solution of the other is available. The dual of a dual is primal.

Symmetric primal-dual problem

Let the primal problem is

Maximize $f_x = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ Subject to

$$
\begin{aligned}
&a_{11}x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n \le b_1,\\
&a_{21}x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n \le b_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad\qquad\qquad\qquad (1)\\
&a_{m1}x_1 + a_{m2}\,x_2 + \cdots + a_{mn}\,x_n \le b_m
\end{aligned}
$$

$x_1, x_2, \ldots, x_n \ge 0.$

This is the linear programming problem with m constants and n variables.

Or this can be written as

$$
\begin{aligned}
\text{Max } f_x = CX \quad &\text{subject to}\\
\text{A X} \ge \text{B}, \qquad &\text{X} \ge 0 \qquad (2)
\end{aligned}
$$

Where, A is the coefficient matrix.

So, transpose of the above problem is obtained by

(1) transposing the coefficient matrix.
(2) interchanging the role of constant terms and the coefficients of the objective function.
(3) reverting the inequalities
(4) minimizing the objective function in place of maximizing it.

So, dual of above problem (1) is

Find $w_1, w_2, \ldots, w_m$, which

Minimize $f_w = b_1w_1 + b_2w_2 + \cdots + b_m\,w_m$ Subject to

$a_{11}w_1 + a_{21}\,w_2 + \cdots + a_{m1}\,w_m \geq c_1,$
$a_{12}w_1 + a_{22}\,w_2 + \cdots + a_{m2}\,w_m \geq c_2,$
………………………………………………..
……………………………………………….
$a_{1n}w_1 + a_{2n}\,w_2 + \cdots + a_{mn}\,w_m \geq c_n$, and

$w_1, w_2, \ldots, w_m \geq 0.$

Rules for converting any primal into its dual

Step1- if objective function is minimizing, convert to maximize.

Step2- If a constraints have inequality of type $AX \geq B$, concert to inequality type $AX \leq B$.

Step3- If a constraint has an equality sign, then it is replaced by two constraints involving the inequalities going in opposite directions.

For example, if the constraint is
$x_1 - x_2 + 3\,x_3 = 4,$

Then, it can be convert as

$x_1 - x_2 + 3\,x_3 \leq 4,$
$x_1 - x_2 + 3\,x_3 \geq 4,$

Step4- every unrestricted variable is replaced by the difference of two non - negative variables

There are some theorem related to primal dual problems

Theorem 1: if the k-th constraint of the primal is an equality sign, the dual variable w_k is unrestricted in sign.

Theorem 2: If the p-th variable of the primal is unrestricted in sign, the p-th constraint of the dual is an equality sign.

Theorem 3: If either of the problem has unbounded optimum solution, then the other problem has no feasible solution at all.

Theorem 4: Both problems may be infeasible, i.e., may not have any solution.

Theorem 5: If x is any feasible solution to the primal problem and w be the any feasible solution to the dual problem, then $C\,x \le b^T\,w$, i.e., $z_x \le z_w$.

Theorem 6: Basic duality theorem:- If $x_0(w_0)$ *is* an optimum solution to the primal (dual), then there exists a feasible solution $w_0(x_0)$ to the dual (primal), such that

$$c\,x_0 = b^T w_0.$$

For Example:-

Find the dual of the problem:

1. Min $Z_x = 2x_2 + 5x_3$ Subject to

$$x_1 + x_2 \ge 2$$
$$2x_1 + x_2 + 6x_3 \le 6,$$
$$x_1 - x_2 + 3\,x_3 = 4, \qquad x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$$

Ans Max $z'_x = -2x_2 - 5x_3$, $z'_x = -z_x$, subject to

$$-x_1 - x_2 - 0x_3 \le -2$$
$$2x_1 + x_2 + 6x_3 \le 6$$
$$x_1 - x_2 + 6x_3 \le 4$$
$$-x_1 + x_2 - 6x_3 \le -4 \quad , x_1 \ge 0, \ x_2 \ge 0, \ x_3 \ge 0.$$

So, we can see that original problem, now, becomes the standard primal problem. So, write as

$$\begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 6 \\ 1 & -1 & 6 \\ -1 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} -2 \\ 6 \\ 4 \\ -4 \end{bmatrix}$$

now, dual problem is $\rightarrow$ min $z'_y = -2y_1 + 6y_2 + 4(y_3 - y_4)$ subject to now, dual problem is $\rightarrow$ $\min Z'_y = -2y_1 + 6y_2 + 4(y_3 - y_4)$ subject to

$$\begin{bmatrix} -1 & 2 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 0 & 6 & 6 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \geq \begin{bmatrix} 0 \\ -2 \\ -5 \end{bmatrix}$$

or dual problem is

$\text{Min } Z'_y = -2y_1 + 6y_2 + 4(y_3 - y_4)$ subject to

$-y_1 + 2y_2 + (y_3 - y_4) \geq 0$

$-y_1 + y_2 - (y_3 - y_4) \geq -2$

$0y_1 + 6y_2 + (y_3 - y_4) \geq -5$, $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$, $y_4 \geq 0$.

or $\min Z'_y = -2y_1 + 6y_2 + 4y_5$ subject to

$-y_1 + 2y_2 + y_5 \geq 0$

$-y_1 + y_2 - y_5 \geq -2$

$6y_2 + y_5 \geq -5$, $y_1 \geq 0$, $y_2 \geq 0$,

y_5 is unrestricted. Ans.

2. Min $Z_x = 2x_1 + 3x_2 + 4x_3$ Subject to

$2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 = 3,$

$x_1 + 4x_2 + 6x_3 \leq 5,$ $x_1 \geq 0, x_2 \geq 0, x_3$ *is unrestricted.*

Ans $\max Z'_x = -2x_1 - 3x_2 - 4(x_3' - x_3'')$, $Z'_x = -Z_x$, subject to

$-2x_1 - 3x_2 - 5(x_3' - x_3'') \leq -2 \rightarrow y_1$

$3x_1 + x_2 + 7(x_3' - x_3'') \leq 3 \rightarrow y_2'$

$-3x_1 - x_2 - 7(x_3' - x_3'') \leq -3 \rightarrow y_2'$

$x_1 + 4x_2 + 6(x_3' - x_3'') \leq 5 \rightarrow y_3$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3' \geq 0$, $x_3'' \geq 0$.

$x_1 \geq 0,\ x_2 \geq 0,\ x_3' \geq 0,\ x_3'' \geq 0.$

dual of the problem is

$$\text{Min } z_y' = -2y_1 + 3(y_2' - y_2'') + 5y_3 \quad \text{subject to}$$

$$-2y_1 + 3(y_2' - y_2'') + y_3 \geq -2$$

$$-3y_1 + (y_2' - y_2'') + 4y_3 \geq -3$$

$$-5y_1 + 7(y_2' - y_2'') + 6y_3 \geq -4$$

$$5y_1 - 7(y_2' - y_2'') - 6y_3 \geq 4, \text{ with}$$

$$y_1 \geq 0,\ y_2' \geq 0,\ y_2'' \geq 0,\ y_3 \geq 0$$

or,

$$\text{min } z_y' = -2y_1 + 3y_2 + 5y_3 \quad \text{subject to}$$

$$-2y_1 + 3y_2 + y_3 \geq -2$$

$$-3y_1 + y_2 + 4y_3 \geq -3$$

$$5y_1 - 7y_2 - 6y_3 = 4 \quad \text{with}$$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3$ is unrestricted. Ans.