Properties of Eigen values

1. Any square matrix A and its transpose A' have the same eigen values

We have

$$(A-\lambda I)' = A' - \lambda I' = (A' - \lambda I)$$

$$\Rightarrow \quad |(A-\lambda I)'| = |A' - \lambda I|$$

$$\Rightarrow \quad |A-\lambda I| = |A' - \lambda I|$$

$$\Rightarrow \therefore \quad |A-\lambda I| = 0 \text{ iff } |A' - \lambda I| = 0$$

λ is eigen value of A $\Rightarrow$ λ is eigen value of A'.

2. The eigen values of a triangular matrix are just the diagonal elements of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) = 0$$

$$\Rightarrow \quad \lambda = a_{11}, a_{22}, \cdots a_{nn}$$

3. The eigen values of Idempotent matrices are either zero or one.

A matrix A is said to be idempont if $A^2 = A$.

Let λ be an eigen value of A, then there exist a non zero vector X such that

$$AX = \lambda X$$

$$\therefore A(AX) = A(\lambda X)$$

$$\Rightarrow A^2X = \lambda AX$$

$$\Rightarrow AX = \lambda \cdot \lambda x$$

$$\Rightarrow AX = \lambda^2 X$$

$$\Rightarrow \lambda^2 X = \lambda X$$

$$\Rightarrow (\lambda^2 - \lambda)X = 0$$

$$\lambda(\lambda - 1) = 0$$

$$\lambda = 0, 1$$

Hence the result.

4. The sum of eigen values of a matrix is the sum of the elements at principal diagonal.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A - \lambda I| = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} = 0$$

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$\Rightarrow (a_{11}-\lambda)\{(a_{22}-\lambda)(a_{33}-\lambda)-a_{23}a_{32}\} - a_{12}\{a_{21}(a_{33}-\lambda) - a_{23}a_{31}\}$
$+ a_{13}\{a_{21}a_{32} - a_{31}(a_{22}-\lambda)\} = 0$

$\Rightarrow \lambda^3 - (a_{11}+a_{22}+a_{33})\lambda^2 + \dots = 0$

Let λ_1, λ_2 & λ_3 are roots

$\because$ sum of roots $= -b/a = \frac{-(-(a_{11}+a_{22}+a_{33}))}{1}$

$= a_{11}+a_{22}+a_{33}$

$\Rightarrow$ sum of eigen values $= a_{11}+a_{22}+a_{33}$ = Trace of A $= Tr(A)$

(V) The product of eigen values of a matrix A is equal to its determinant.

Let $\lambda_1, \lambda_2, \lambda_3$ be eigen values of $A_{3\times 3}$

then

$$|A-\lambda I| = (-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$

$$\Rightarrow |A-\lambda I| = -1\{\lambda^3 - (\lambda_1+\lambda_2+\lambda_3)\lambda^2 + (\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\lambda - \lambda_1\lambda_2\lambda_3\}$$

$$= -\lambda^3 + (\lambda_1+\lambda_2+\lambda_3)\lambda^2 - (\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\lambda + \lambda_1\lambda_2\lambda_3$$

Put $\lambda = 0$ in above, we have

$$\boxed{\lambda_1\lambda_2\lambda_3 = |A|}$$

(6) If λ is an eigen value of matrix A, then $\frac{1}{\lambda}$ is the eigen value of A^{-1}. (A is nonsingular)

Let λ be an eigen value of matrix A, there there exist a non zero vector X such that

$$AX = \lambda X$$

$$\Rightarrow A^{-1}(AX) = A^{-1}(\lambda X)$$

$$\Rightarrow IX = \lambda A^{-1}X$$

$$\Rightarrow X = \lambda (A^{-1}X)$$

$$\Rightarrow A^{-1}X = \frac{1}{\lambda}X$$

$\Rightarrow \frac{1}{\lambda}$ is eigen value of A^{-1}.

(7) If λ is an eigen value of an orthogonal matrix, then $1/\lambda$ is also its eigen value

$$\because AA^{T} = I.$$

$$\Rightarrow A^{-1} = A^{T}$$

If λ is an eigen value of A, we have $\frac{1}{\lambda}$ is an eigen value of A^{-1}

$\Rightarrow \frac{1}{\lambda}$ is eigen value of A'

$\Rightarrow \frac{1}{\lambda}$ is eigen value of A.

Hence the result.

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(8) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen value of $A_{n\times n}$ then $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ are eigen value of A^m

Let X_i be eigen vector for λ_i of A

$\Rightarrow \quad AX_i = \lambda_i X_i$

$\Rightarrow \quad A(AX_i) = A(\lambda_i X_i)$

$\Rightarrow \quad A^2 X_i = \lambda_i (AX_i)$

$\Rightarrow \quad A^2 X_i = \lambda_i(\lambda_i X_i) = \lambda_i^2 X_i$

$\vdots$

$A^m X_i = \lambda_i^m X_i$

hence, the result.