Classical optimization:-

Maxima and minima for one variable:-

$f = f(x)$

$\frac{df}{dx} = 0 \Rightarrow x^*$

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{x=x^*}$ — $-ve$: maxima; $+ve$: minima; zero

$\downarrow$

$\left(\frac{d^3 f}{dx^3}\right)_{x^*}$ — $\neq 0 \rightarrow$ Neither max. nor min.; 0

$\downarrow$

$\left(\frac{d^4 f}{dx^4}\right)_{x=x^*}$ — $+ve$ - minima; $-ve$ - maxima; zero

Ques1. Find the maxima and minima of the function

$f(x) = x^5 - 5x^4 + 5x^3 - 1$

Soln:- $\frac{df}{dx} = 5x^4 - 20x^3 + 15x^2 = 0$

$5x^2 = 0$

$\cancel{5x^2}\; x^2 - 4x + 3 = 0$

$\Rightarrow \cancel{x=2}\; (x-1)(x-3) = 0$

$x = \cancel{0, 1, 3}\; 0, 0, 1, 3.$

$\frac{d^2 f}{dx^2} = 20x^3 - 60x^2 + 30x$

Case 1: At $x=0$, $\frac{d^2 f}{dx^2} = 0$

~~At~~

~~Case 2: At x=1, 20-60+30 = -10.~~

~~Case 3 At~~ $\frac{d^3 f}{dx^3} = 60x^2 - 120x + 30$

putting $x=0$ $= 30$

$=$ Neither maxima nor minima

Case 2: $x = 1$

$$\frac{d^2y}{dx^2} = -10 < 0$$

Maxima

Case 3: $x = 3$

$$\frac{d^2y}{dx^2} = 20\times 27 - 60\times 9 + 30\times 3$$

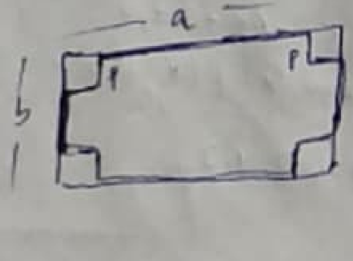
$$= 540 - 540 + 90$$

$$= (90) > 0$$

~~Maxima~~ Minima

Ques 2: A rectangular sheet of metal of sides a and b has four equal square portion removed at the corners and the sides are then turn up so as to form an open rectangular box. Find the depth of the box when the vol. of box is minimum

Sol^n :-



$$\text{Volume} = (a-2p)\times(b-2p)\times p$$

$$= (ab - 2ap - 2bp + 4p^2)\,p$$

$$V = abp - 2ap^2 - 2bp^2 + 4p^3$$

$$\frac{dV}{dp} = 0 \qquad = ab - 4ap - 4bp + 12p^2 = 0$$

$$\Rightarrow \cancel{4p(3p-b-a) = -ab}$$

$$12p^2 - 4p(a+b) + ab = 0$$

$$p = \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 4 \times 12 \times ab}}{24}$$

$$= \frac{4(a+b) \pm \sqrt{16a^2 + 16b^2 + 32ab - 48ab}}{24}$$

$$\Rightarrow \frac{4(a+b) \pm 4\sqrt{a^2+b^2-ab}}{24}$$

$$\Rightarrow \frac{(a+b) \pm \sqrt{a^2+b^2-ab}}{6}$$

$$\frac{d^2y}{dx^2} = -4a - 4b + 24p$$

case 1: put $p = \frac{(a+b) \pm \sqrt{a^2+b^2-ab}}{6}$ in $\frac{d^2y}{dx^2}$

$$= -4a - 4b + 4\left[(a+b) + \sqrt{a^2+b^2-ab}\right]$$

$$\Rightarrow 4\sqrt{a^2+b^2-ab} > 0$$

minima. Ans

case 2: put $p = \frac{(a+b) - \sqrt{a^2+b^2-ab}}{6}$

$$\Rightarrow -4\sqrt{a^2+b^2-ab} < 0$$

maxima.

multivariable optimization with no constraints.

- definite and indefinite matrix.

↓

(1)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

minors -

$$A_1 = |a_{11}|, \quad A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdots$$

$$A_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

case(i) if all $A_i > 0$ then the matrix is called positive definite.

case(ii) if sign of A_i is in the form $(-1)^i$, ~~then the matrix is~~ called $A_1 = -ve$, $A_2 = +ve$, $A_3 = -ve$, then the matrix is called negative definite.

- Any case other than these two are considered ~~negative~~ indefinite.

2. Maxima and minima for two or more variable :-

$f = f(x_1, x_2)$

- to find the points of maxima and minima for two variables :-

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} &= 0 \\ \frac{\partial f}{\partial x_2} &= 0 \end{aligned}\right\} \rightarrow (x_1^*, x_2^*)$$

~~Hasing~~ Hasian matrix :

$$J = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix}_{2\times 2}$$

If $(J)_{(x_1^*, x_2^*)}$ $\rightarrow$ case (i) $J_1 > 0$, $J_2 > 0$, then point is minimum

case(ii) $J_1 < 0$, $J_2 > 0$, then the point is maximum.

other than these two cases, the point is neither maximum nor minimum.

- to find points of max and min. for three variables. $\left(f = f(x_1, x_2, x_3)\right)$

$$= \left.\begin{aligned} \frac{\partial f}{\partial x_1} &= 0 \\ \frac{\partial f}{\partial x_2} &= 0 \\ \frac{\partial f}{\partial x_3} &= 0 \end{aligned}\right\} (x_1^*, x_2^*, x_3^*)$$

$$J = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{vmatrix}$$ 3x3 matrix.

case (i) $J_1 > 0$, $J_2 > 0$, $J_3 > 0$, then the point is minima.

case (ii) $J_1 < 0$, $J_2 > 0$, $J_3 < 0$, then the point is maxima

other than the above two cases, the case will be of neither maxima nor minima.

Ques 1: Find the extreme point of function $f(x_1, x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$.

Soln: differentiate w.r.t x_1 and x_2 :-

$$\frac{\partial f}{\partial x_1} = 3x_1^2 + 4x_1 = 0 \Rightarrow x_1 = 0, -\frac{4}{3}$$

$$\frac{\partial f}{\partial x_2} = 3x_2^2 + 8x_2 = 0 \Rightarrow x_2 = 0, -8/3$$

The combination of points are:

$A(0,0)$, $B(0, -8/3)$, $C(-4/3, 0)$, $D(-\frac{4}{3}, -\frac{8}{3})$.

$$J = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} : \begin{vmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{vmatrix}$$

Now, check positive definite or negative definite for each of the points.

Case (i) $A(0,0)$

$$J = \begin{vmatrix} 4 & 0 \\ 0 & 8 \end{vmatrix} \Rightarrow \quad J_1 = |4| = 4 > 0$$

$$J_2 = \begin{vmatrix} 4 & 0 \\ 0 & 8 \end{vmatrix} = 32 > 0$$

Hence the point A is $(0,0)$ is a minima (+ definite)

Case (ii) $B(0, -8/3)$

minor

$$J = \begin{vmatrix} 4 & 0 \\ 0 & -8 \end{vmatrix} \quad J_1 = |4| = 4 > 0$$

$$J_2 = \begin{vmatrix} 4 & 0 \\ 0 & -8 \end{vmatrix} = -32 < 0$$

Hence J is indefinite, therefore the point $B(0, -8/3)$ is saddlepoint. (neither maximum nor minimum)

Case (iii) $C(-\frac{4}{3}, 0)$

minor

$$J = \begin{vmatrix} -4 & 0 \\ 0 & 8 \end{vmatrix} \quad J_1 = |-4| = -4 < 0$$

$$J_2 = \begin{vmatrix} -4 & 0 \\ 0 & 8 \end{vmatrix} = -32 < 0$$

Hence the point is saddle point (indefinite).

Case (iv) $D(-\frac{4}{3}, -\frac{8}{3})$

$$J = \begin{vmatrix} -4 & 0 \\ 0 & -8 \end{vmatrix} \quad J_1 = |-4| = -4 < 0$$

$$J_2 = \begin{vmatrix} -4 & 0 \\ 0 & -8 \end{vmatrix} = 32 > 0$$

J is negative definite,
therefore the point is maxima.

Classification optimization

multivariable optimization with constraints (inequality constraints) :-

* **Kuhn- Tucker conditions:-**

(i) Min $f(x_1, x_2, x_3)$

subject to condition $g_j(x_1, x_2, x_3) \geq 0$, $j = 1, 2, 3, 4$.

Langrange's function

$$L(x, \lambda) = f + \sum_{j=1}^{4} \lambda_j g_j(x)$$

cond^n:

1. $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial x_3} = 0$

2. $\lambda_j g_j(x) = 0$, $j = 1, 2, 3, 4$

3. $g_j(x) \geq 0$, $j = 1, 2, 3, 4$ → subject cond^n (given)

4. $\lambda_j \leq 0$, $j = 1, 2, 3, 4$ → sign will be opposite of 3.

↓
Langrange's multiplier

[if $g_j(x) \leq 0$, $\lambda_j(x) \geq 0$ in case of minimize operation].

(ii) Max $f(x_1, x_2, x_3)$.

All same except:-

The sign of 3rd and 4th condition will be same.

Ques: Use Kuhn-Tucker condition to solve

$$\min f(x,y,z) = x^2+y^2+z^2+20x+10y$$

subject to

$$\left.\begin{aligned} x &\geq 40 & (g_1(x)) \\ x+y &\geq 80 & = g_2(x) \\ x+y+z &\geq 120. & = g_3(x) \end{aligned}\right]$$ sign of inequality must be same.

Sol^n: $L = f + \sum_{j=1}^{3} \lambda_j g_j$

$= x^2+y^2+z^2+20x+10y+\lambda_1(x-40)+\lambda_2(x+y-80)+\lambda_3(x+y+z-120)$

cond^{ns}:

(1) $\frac{\partial L}{\partial x} = 0 \Rightarrow 2x+20+\lambda_1+\lambda_2+\lambda_3 = 0$

$\frac{\partial L}{\partial y} = 0 \Rightarrow 2y+10+\lambda_2+\lambda_3 = 0$

$\frac{\partial L}{\partial z} = 0 \Rightarrow 2z+\lambda_3 = 0$

(2) $\lambda_1(x-40) = 0$

$\lambda_2(x+y-80) = 0$

$\lambda_3(x+y+z-120) = 0$

(3) $g_j(x) \geq 0,\ j=1,2,3$] sign opposite for minimize

(4) $\lambda_j \leq 0;\ j=1,2,3$

Using cond^n 2:

Let $\lambda_1, \lambda_2, \lambda_3 \neq 0$

then $x - 40 = 0$
$x + y - 80 = 0$
$x + y + z - 120 = 0$

~~$x, y, z = 40$~~
$x = 40$
$y = 40$
$z = 40$

put the value in cond^n ①

$\frac{\partial L}{\partial x} = 80 + 20 + \lambda_1 + \lambda_2 + \lambda_3 = 0$

$\frac{\partial L}{\partial y} = 80 + 10 + \lambda_2 + \lambda_3 = 0$

$\frac{\partial L}{\partial z} = 80 + \lambda_3 = 0$

$\lambda_3 = -80$
~~$\lambda_2 = -10$~~ $\lambda_2 = -10$
$\lambda_1 = -10$

∴ all $\lambda_j \leq 0$, Hence all cond are satisfied.

So final $x = 40$, $y = 40$, $z = 40$

$f_{min} = 6000$.

Ans:- Consider the following optimization problem:

Max $f = -x_1 - x_2$, subject to

$x_1^2 + x_2 \geq 2$

$x_1 + 3x_2 \geq 4$

$x_1 + x_2^4 \leq 30$

Find whether the vector $x^* \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ satisfies the Kuhn-Tucker cond^ns and what are the values of Langrange's multipliers at the given vector?

Sol^n:- $f = -x_1 - x_2$ ~~$+ \lambda_1 t x_1$~~

$g_1(x) = x_1^2 + x_2 - 2 \geq 0$

$g_2(x) = x_1 + 3x_2 - 4 \geq 0$

$g_3(x) = -x_1 - x_2^4 + 30 \geq 0$

$L(x, \lambda) = f + \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3$

$= -x_1 - x_2 + \lambda_1(x_1^2 + x_2 - 2) + \lambda_2(x_1 + 3x_2 - 4) + \lambda_3(-x_1 - x_2^4 - 30)$

K.T. conditions:

① $\frac{\partial L}{\partial x_1} = -1 + 2x_1\lambda_1 + \lambda_2 - \lambda_3 = 0$

$\frac{\partial L}{\partial x_2} = -1 + \lambda_1 + 3\lambda_2 - 4\lambda_3 x_2^3 = 0$

② $\lambda_j g_j(x) = 0$

$\lambda_1(x_1^2 + x_2 - 2) = 0$

$\lambda_2(x_1 + 3x_2 - 4) = 0$

$\lambda_3(-x_1 - x_2^4 + 30) = 0$

③ $g_j(x) \geq 0$ → sign of inequality same as that in given in ques

$j = 1, 2, 3$

④ $\lambda_j \geq 0$ (sign of inequality will be same as that of subject to cond^n in case of maximize).

$j = 1, 2, 3$

As the solution is given

$x_1 = 1$

$x_2 = 1$

start from ②,

$\cdot g_1(x) = x_1^2 + x_2 - 2 = 0$

$= 1 + 1 - 2 = 0 \Rightarrow \lambda_1 \neq 0$ (λ_1 is not necessary to be 0).

$g_2(x) = 1 + 3 - 4 = 0 \Rightarrow \lambda_2 \neq 0$

$g_3(x) = -1 - 1 + 30 = 28 \Rightarrow \lambda_3 = 0.$

Now use (1).

$$\left.\begin{aligned} -1 + 2\lambda_1 + \lambda_2 &= 0 \\ -1 + \lambda_1 + 3\lambda_2 &= 0 \end{aligned}\right] \quad \begin{aligned} \lambda_1 &= \frac{2}{5} \\ \lambda_2 &= \frac{1}{5} \end{aligned}$$

Now all the values of λ ~~ar~~ satisfy the (a) condⁿ.

So the given KT condⁿ are satisfied.

The value of Langrange's multiplier is :-

$\lambda_1 = \frac{2}{5}$

$\lambda_2 = \frac{1}{5}$

$\lambda_3 = 0$

• Method of Lagrange's multipliers:

If $f(x_1, x_2, x_3)$ subject to:

$$g_1(x_1, x_2, x_3) = 0$$

$$g_2(x_1, x_2, x_3) = 0$$

$\rightarrow$ $L(x, \lambda) = f + \lambda_1 g_1 + \lambda_2 g_2$ —(1)

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial x_3} = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0, \quad \frac{\partial L}{\partial \lambda_2} = 0$$

$X^* = (x_1^*, x_2^*, x_3^*), \lambda_1^*, \lambda_2^*$ $\rightarrow$ (all the required values)

$$M = \begin{vmatrix} \frac{\partial^2 L}{\partial x_1^2} - k & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} - k & \frac{\partial^2 L}{\partial x_2 \partial x_3} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} - k & \frac{\partial g_1}{\partial x_3} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} & 0 & 0 \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} & 0 & 0 \end{vmatrix}$$

case(i) ~~if [att]~~ if all $k > 0$, then the point is ~~minimum~~ minima

case(ii) if all $k < 0$, then the point is maxima

Ques 1 : Solve $\min f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ By Langrange's method.

Subject to $g_1(x) \equiv x_1 - x_2 = 0$
$g_2(x) \equiv x_1 + x_2 + x_3 - 1 = 0$.

Soln: $L(x,\lambda) = f + \lambda_1 g_1 + \lambda_2 g_2$

$$L = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \lambda_1(x_1 - x_2) + \lambda_2(x_1 + x_2 + x_3 - 1) \quad -①$$

diff w.r.t. $x_1, x_2, x_3, \lambda_1, \lambda_2$.

$\Rightarrow \frac{\partial L}{\partial x_1} : x_1 + \lambda_1 + \lambda_2 = 0$

$\frac{\partial L}{\partial x_2} = x_2 - \lambda_1 + \lambda_2 = 0$

$+ \quad x_1 + x_2 + 2\lambda_2 = 0$
$2x_1 + 2\lambda_2 = 0$
$\Rightarrow x_1 = -\lambda_2$

$\frac{\partial L}{\partial x_3} = x_3 + \lambda_2 = 0 \quad ; \quad x_3 = -\lambda_2$

$x_1 = x_3$

$\frac{\partial L}{\partial \lambda_1} = x_1 - x_2 = 0 \quad ; \quad x_1 = x_2$

$\frac{\partial L}{\partial \lambda_2} = x_1 + x_2 + x_3 - 1 = 0 \quad -ⓐ$

$\Rightarrow x_1 = x_3 = x_2 = -\lambda_2$

$\Rightarrow$ From ⓐ

$3x_1 = 1 \Rightarrow x_1 = \frac{1}{3}$.

$\cancel{\lambda_2} = x_1 = x_2 = x_3 = \frac{1}{3}$

$\lambda_2 = -\frac{1}{3}$.

$\frac{1}{3} - \lambda_1 + -\frac{1}{3} = 0$

$\lambda_1 = 0$.

$$H = \begin{vmatrix} \frac{\partial^2 L}{\partial x_1^2} - k & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} - k & \frac{\partial^2 L}{\partial x_2 \partial x_3} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} - k & \frac{\partial g_1}{\partial x_3} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} & 0 & 0 \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} & 0 & 0 \end{vmatrix} = 0$$

$$= x_2 \quad = \begin{array}{c} + \quad - \quad + \quad - \quad + \\ \begin{vmatrix} 1-k & 0 & 0 & 1 & 1 \\ 0 & 1-k & 0 & -1 & 1 \\ 0 & 0 & 1-k & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{vmatrix}_{5\times 5} \end{array} = 0$$

opening by 4 column

$$(-1)\begin{vmatrix} 0 & 1-k & 0 & 1 \\ 0 & 0 & 1-k & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 1-k & 0 & 0 & 1 \\ 0 & 0 & 1-k & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$$= -1\left\{ -(1-k)\begin{vmatrix} 0 & 1-k & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 & 1-k \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \right\} +$$

$$(-1)\left\{ (1-k)\begin{vmatrix} 0 & 1-k & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 0 & 1-k \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \right\} = 0$$

$(1-k)(1-0) + (1-k)(2) - \{(1-k)(-1-0) + (1-k)3\} = 0$

$\Rightarrow \quad 6(1-k) = 0$

$\underline{k = 1 > 0}$

hence the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is minima

Multivariable optimization with equality constraints :-

* method of constrained variation :

Min/max $f(x_1, x_2, x_3, x_4, x_5)$

Subject to $g_1(x_1, x_2, x_3, x_4, x_5) = 0$ — (1)

$g_2(x_1, x_2, x_3, x_4, x_5) = 0$ — (2)

→ gives only extreme point but cannot tell whether the point gives maxima or minima. If ques is maximise, then the obtained point maximises the function. If the ques says ~~maxim~~ minimise, the obtained point minimises the function.

→ ~~Select no of dependent variable on the basis~~ No. of dependent variable = No. of given conditions (variables)

Sol: $n = 5$, $m = 2$ (equations)

So we have two select 2 dependent and 3 independent variables.

$J\left(\frac{g_1, g_2}{x_1, x_2}\right) \neq 0$ J - Jacobian

If the eqn get satisfied then for independent variable

$$J\left(\frac{f, g_1, g_2}{x_3, x_1, x_2}\right) = 0 \quad - (3)$$

$$J\left(\frac{f, g_1, g_2}{x_4, x_1, x_2}\right) = 0 \quad - (4)$$

$$J\left(\frac{f, g_1, g_2}{x_5, x_1, x_2}\right) = 0 \quad - (5)$$

Solving (1), (2), (3), (4), (5), get the point.

$$J\left(\frac{g_1, g_2}{x_1, x_2}\right) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$

$$J\left(\frac{f, g_1, g_2}{x_3, x_1, x_2}\right) = \begin{vmatrix} \frac{\partial f}{\partial x_3} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g_1}{\partial x_3} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \\ \frac{\partial g_2}{\partial x_3} & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$

Ques: minimize $f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$

s.t. $g_1(x) = x_1 - x_2 = 0$ — (1)

$g_2(x) = x_1 + x_2 + x_3 - 1 = 0$ — (2)

Soln: $n = 3$, $m = 2$.

Select 2 dependent and 1 independent variable let x_1, x_2 be dependent.

$$J\left(\frac{g_1, g_2}{x_1, x_2}\right) \neq 0$$

$$J\left(\frac{g_1, g_2}{x_1, x_2}\right) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$

$$\therefore \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1-(-1) = 2 \neq 0$$

so, x_1 and x_2 may be selected as dependent variable

so x_3 is independent variable

for independent variable,

$$J\left(\frac{f, g_1, g_2}{x_3, x_1, x_2}\right) = 0$$

$$= \begin{vmatrix} \frac{\partial f}{\partial x_3} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g_1}{\partial x_3} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_3} & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} \therefore \begin{vmatrix} x_3 & x_1 & x_2 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x_3(1+1) - x_1(1) + x_2(-1) = 0$$

$$2x_3 - x_1 - x_2 = 0 \quad \text{--- (3)}$$

From (1), (2), (3)

$$x_1 = x_2 \text{ — from (1)}$$

$$x_1 + x_2 + x_3 = 1 \quad \text{— (2)}$$

from ③, $x_3 = \frac{x_1 + x_2}{2}$

$x_3 = \frac{2x_1}{2} \Rightarrow x_3 = x_1$

$x_3 = x_1 = x_2$

$3x_1 = 1$

$x_1 = \frac{1}{3} = x_2 = x_3$

$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ - This must give minimum value.

→ Single - variable optimization left topics :-

• Necessary condition :- If $f(x)$ be a single variable function defined in the interval $a \le x \le b$ and has a stationary point i.e., relative minimum or maximum or point of inflection at $x = x^*$ where $a \le x^* \le b$ and if the first order derivative $\frac{df(x)}{d(x)}$ ~~exists~~ $= f'(x)$ exists as a finite number at $x = x^*$ then $f'(x) = 0$.

• Proof :- Let x^* be a point of relative minimum where $f'(x^*)$ exists.

$$f'(x^*) = \lim_{h \to 0} \frac{f(x^*+h) - f(x^*)}{h}$$

we need to prove $f'(x^*) = 0$.

x^* be the point of relative minimum

$$\Rightarrow f(x^*) \le f(x^*+h)$$

$$\Rightarrow \frac{f(x^*+h) - f(x^*)}{h} \ge 0 \text{ for } h > 0$$

and

$$\frac{f(x^*+h) - f(x^*)}{h} \le 0 \text{ for } h < 0$$

thus since $f'(x^*)$ exists for $h \to 0$,

$$f'(x^*) = 0.$$

Sufficient condition for relative minimum and maximum of a single variable function:-

→ $f(x)$ on $[a,b]$. and $x = x^*$ is the point of relative minimum or maximum..

$$\Rightarrow a \leq x^* \leq b$$

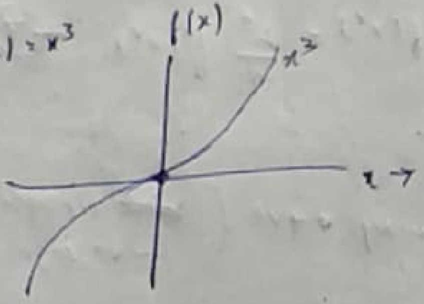
$$f'(x^*) = 0$$

~~(i) We cannot say the~~

The drawbacks:- (condn not satisfied):-

1) We cannot say the nature of x^* at the end points.
2) If $f'(x)$ is undefined or does not exist at $x = x^*$, then we cannot conclude the relative minimum or relative max. of $f(x)$ at x^*.
3) We cannot conclude the rela. min. and rel. max of $x = x^*$ for some functions given $f'(x^*) = 0$.

eg:- $f(x) = x^3$



$f'(x) = 3x^2 = 0$

$\underline{x = 0}$.

But $x = 0$ is neither ~~for~~ relative max nor rel. minimum and it is called point of inflection.

Sufficient condⁿ for relative minimum/max. :-

→ If $f'(x^*) = f''(x^*) = \dots = f^{n-1}(x^*) = 0$ — ① and $f^n(x^*) \neq 0$

then $x = x^*$ is said to be :-

(i) relative minimum if $f^n(x^*) > 0$ and n is even

(ii) relative maximum if $f^n(x^*) < 0$ and n is even

(iii) inflection point if n is odd (neither min nor max.)

Proof:- $f(x^*+h) = f(x^*) + h f'(x^*) + \frac{h^2}{2!} f''(x^*) + \dots$

$\frac{h^n}{n!} f^n(x^*) + \dots$ ②

~~$f(x^*+h)$~~ putting ① in ②,

$f(x^*+h) = f(x^*) + \frac{h^n}{n!} f^n(x^*)$

$\Rightarrow f(x^*+h) - f(x^*) = \frac{h^n}{n!} f^n(x^*)$ — ③

If n is even,

$\frac{h^n}{n!}$ is always +ve (irrespective of +ve or -ve sign of h)

from this, -

① If $f^n(x^*)$ is positive, $f^n(x^*) > 0$

then $f(x^*+h) - f(x^*) > 0 \Rightarrow f(x^*) < f(x^*+h)$

→ x^* is relative minimum point.

② If $f^n(x^*)$ is -ve, $f^n(x^*) < 0$

then $f(x^*+h) - f(x^*) < 0 \Rightarrow f(x^*) > f(x^*+h)$.

→ x^* is relative max. point.

If n is odd,

For negative h, $\Rightarrow \frac{h^n}{n!} < 0$

For +ve h, $\frac{h^n}{n!} > 0$

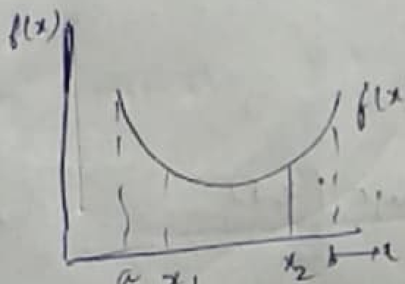
Fibonacci Search method :-

Algorithm:

(1) ~~choose a lower bound and an upper bound (a and b respectively)~~

~~set $L = b-a$~~

Region elimination Rule :-



$f(x)$ is a unimodal function defined on $[a, b]$ for minimum

(1) If $f(x_1) > f(x_2)$ then we have to delete the region (a, x_1)

(2) If $f(x_1) < f(x_2)$ then we have to delete the region (x_2, b)

(3) If $f(x_1) = f(x_2)$, then we have to delete the region (a, x_1) and (x_2, b)

n' Algo of fibonacci method :-

Step 1: choose — lower bound "a"
upper bound "b"

Set $L = b - a$; Assume the desired no. of function evaluation to be "n", set $k = 2$

Step 2: Compute $L_k^* = \left(\frac{F_{n-2}}{F_n}\right) \times L$

$F_0 = 1$
$F_1 = 1$
$F_2 = 2.$

Set $x_1 = a + L_k^*$
$x_2 = b - L_k^*$

Step 3:- Compute $f(x_1)$ or $f(x_2)$, which was not evaluated earlier. Use fundamental region elimination rule to eliminate a region. set new "a"; "b"

Step 4 Is $k = n$? If no, set $k = k+1$ and go to step 2.

Ans: minimize the function :.

$f(x) = x^2 + \frac{54}{x}$. in range $(0, 5)$

$n = 3$.

Soln: Iteration 1

$n = 3.$

$a = 0 \quad b = 5$

$L = 5$

$L_2^* = \left(\frac{F_{n-k+1}}{F_{n+1}}\right) \times L$

$= \frac{F_{3-2+1}}{F_4} \times 5 = \frac{F_2}{F_4} \times 5 = \frac{2}{5} \times 5 = 2$

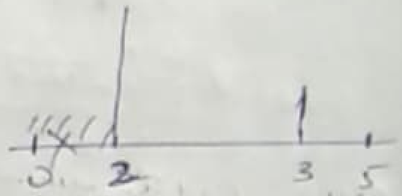
$x_1 = a + L_2^* = 2$

$x_2 = b - L_2^* = 3.$

$f(x_1) = 4 + \frac{54}{2} = 4 + 27 = 31$

$f(x_2) = 9 + 18 = 27$

so $f(x_1) > f(x_2)$



So new range is $(2, 5)$

Now, $k \neq n$

$2 \neq 3$

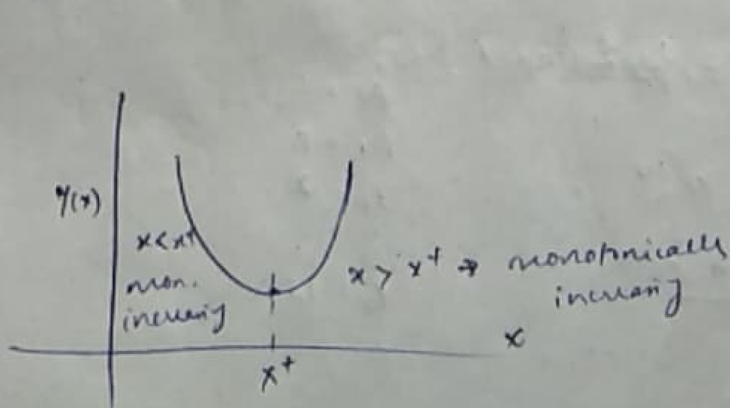
set $k = k+1 = 3$

Iteration 2

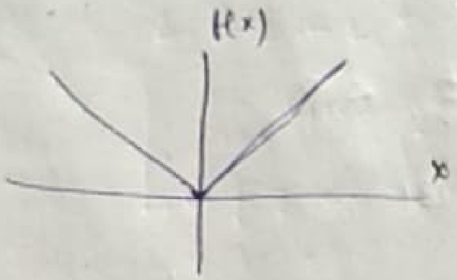
$L_3^* = \left(\frac{F_{n-k+1}}{F_{n+1}}\right) L$

$= \frac{F_1}{F_4} \times$

Unimodal function:- A function $f(x)$ is unimodal in $a \leq x \leq b$ iff it is monotonic on either side of x^*, where x^* is the single optimal point.



Hence a unimodal function.



$y = |x|$

= unimodal function

Que: Find the minimum value of $f(x) = x^2 + 2x$ within the interval $[-3, 4]$ using the fibonacci method and obtain the optimal value within 5% of exact value.

$f(x)$ is unimodal function

Sol: No. of experiments is not given but we ~~~~ can deduce it from given accuracy.

$$\frac{\text{Length of final interval of uncertainity}}{2 \times \text{length of initial interval of uncertainity}} \le \frac{5}{100}$$

$$\Rightarrow \frac{L_n}{2} \le \frac{1}{20} \times L_0$$

$$L_n \le \frac{L_0}{10}$$

~~Measure of effi~~ Length of interval of uncertainity after j^{th} experiment :-

$$L_j = \frac{F_{n-(j-1)}}{F_n} L_0$$

$$\text{Measure of efficiency} = \frac{L_n}{L_0} = \frac{1}{F_n}$$

$$\Rightarrow \frac{L_n}{L_0} = \frac{1}{F_n} \le \frac{1}{10}$$

$F_n \geq 10$

$n \geq 6.$

$F_0 = 1$

$F_1 = 1$

$F_2 = 2$

$\vdots$

$F_6 = 13.$

Step 1: $L_0 = [-3, 4]$, $n = 6$

Step 2: To obtain x_1 and x_2,

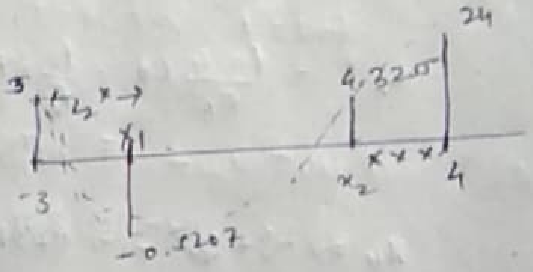
$$L_2^* = \frac{F_{n-2}}{F_n} \times L_0 = \frac{F_4}{F_6} \times L_0 = \frac{5}{13} \times 7 = 2.6923.$$

$x_1 = -3 + 2.6923 = -0.3077$

$x_2 = 4 - 2.6923 = 1.3077$

$f(x_1) = -0.5207$

$f(x_2) = 4.3255$



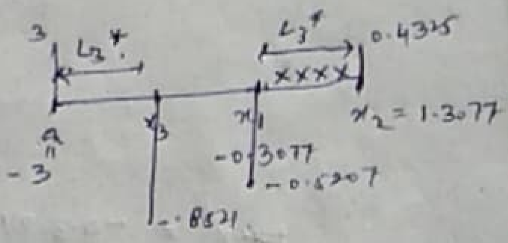
$\Rightarrow f(x_1) < f(x_2)$

Discard $(x_2, 4]$. obtain $L_2 = [-3, x_2]$

Step 3: $L_2 = [-3, x_2] = [-3, 1.3077]$

$L_2 = L_0 - L_2^* = 7 - 2.6923 = 4.3077.$

can be thought as and the values match.



Step 4: To obtain x_3,

$$L_3^* = \frac{F_{n-3}}{F_n} L_0 = \frac{F_3}{F_6} \times 7 = 1.6154.$$

$x_3 = -3 + 1.6154 = -1.3846$

$f(x_3) = -0.8521.$

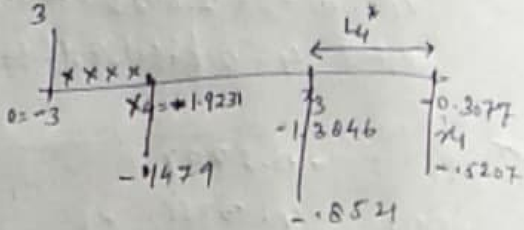
$f(x_3) < f(x_1) \Rightarrow$ discard $[x_1, x_2]$

Step 5:- $L_3 = L_2 - L_3^*$

$= 4.3077 - 1.6154 = 2.6923$

$\rightarrow [-3, -0.3077]$

Step 6:- ~~To~~ To obtain x_4,



$L_4^* = \frac{F_{n-4}}{F_n} L_0 = \frac{F_2}{F_6} \times 7$

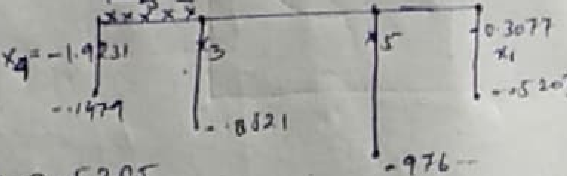
$= 1.0769$

$x_4 = -3 + L_4^* = -1.9231$

$\Rightarrow f(x_3) < f(x_4) \Rightarrow$ discard $[-3, x_4)$

$L_4 = [-1.9231, -0.3077]$

Step 7 $L_5^* = \frac{L_{n-5}}{F_n} \times L_0 = \frac{1}{13} \times 7 = 0.5385$



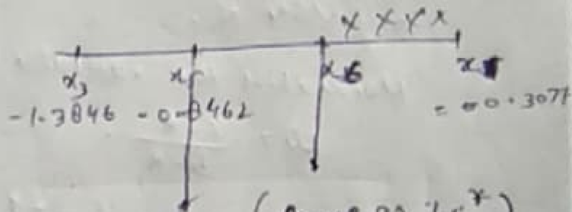
$\Rightarrow x_5 = -0.3077 - 0.5385$

$= -0.8462$

$f(x_5) = -0.97634556$

$f(x_5) < f(x_3) \Rightarrow$ discard $[x_4, x_3)$

Step 8: $L_5 = [-1.3846, -0.3077]$



$$L_6^* = \frac{F_{n-6}}{F_n} L_0 = \frac{1}{13} \times L_0 = .5385 \text{ (same as } L_5^*)$$

$$x_6 = x_3 + L_6^* = -0.8461$$

$$f(x_6) = -0.97631479$$

$$f(x_6) > f(x_5)$$

Final

Final interval of uncertainity is

$$L_6 = [-1.3846, -0.8461]$$

Final ans: $\frac{L_6}{2} = \frac{-1.3846 - 0.8461}{2}$

Algorithm:-

(1) choose a lower bound a and an upper bound b, a small number ϵ.

Normalise the variable x by using the equation

$$\omega = \frac{x-a}{b-a}.$$

Thus, $a_\omega = 0$, $b_\omega = 1$ and

$L_\omega = 1$, set $k=1$

(2) Set $\omega_1 = a_\omega + (0.618)L_\omega$ and

$\omega_2 = b_\omega - (0.618)L_\omega$.

compute $f(\omega_1)$ and $f(\omega_2)$. Use region elimination rule and set new a_ω and b_ω.

(3) stop when $|L_\omega| < \epsilon$

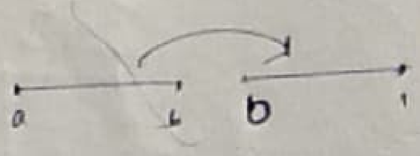
$$\omega = \frac{x-a}{b-a},$$

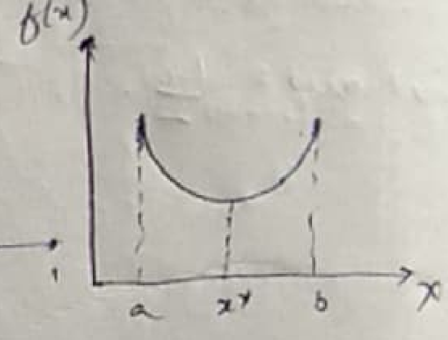
If $x=a$

$\omega=0$

If $x=b$

$\omega=1$





$L_\omega = b_\omega - a_\omega = 1$

$\omega_1 = a_\omega + (0.618)L_\omega$, $\omega_2 = b_\omega - (0.618)L_\omega$.

If $f(\omega_1) > f(\omega_2)$, $[0, \omega_1)$ will be eliminated

and

the similar rules of fibonacci ~~...~~ method

Stop. when $|h(x)| < \varepsilon$

Ans: