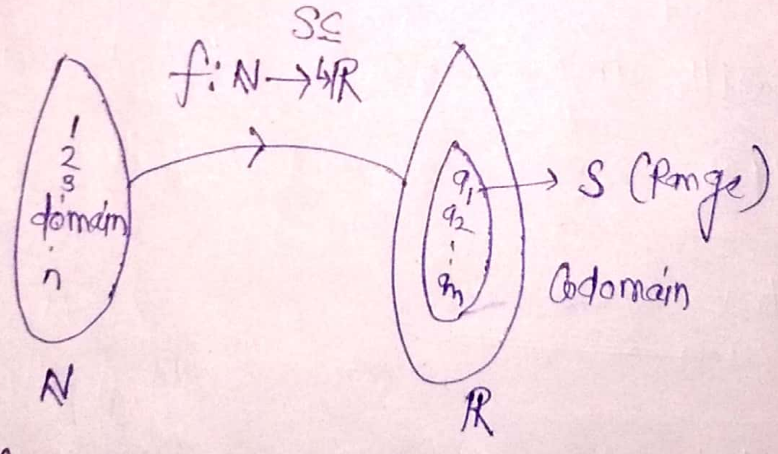
Sequence and Series :-

Sequence :- A sequence is a function whose domain is the set of Natural number whereas range may be any set, i.e., for each $n \in N$ there exists $a_n \in S$.

Real sequence :-

A sequence (real sequence) is a function whose domain is the set of Natural number N and range a subset of real number.



for each $n \in N$, $\exists\ a_n \in S \subseteq R$

Symbolically $f: N \rightarrow R$ is a real sequence.

⊛ The terms occurring at different positions are treated as distinct terms even if they have same value.

Range of sequence –:

The set of all distinct terms of a sequence is called its range.

e.g. If $x_n = \{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, -1, 1, \cdots\}$

The range of sequence $\{x_n\} = \{1, -1\}$ which is a finite set.

Constant sequence –:

A sequence $\{x_n\}_{n=1}^{\infty} = c \in \mathbb{R} \quad \forall n \in \mathbb{N}$ is called a constant sequence.

Eg. $\{x_n\}_{n=1}^{\infty} = \{c, c, c, c, \cdots\}$ is a constant sequence with range $= \{c\}$.

Bounded sequence –:

Bounded above sequence –:

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be bounded above if $\exists$ a real number K such that $a_n \leq K \quad \forall n \in \mathbb{N}$.

Bounded below sequence –:

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be bounded below if $\exists$ a real number k such that $k \leq a_n \quad \forall n \in \mathbb{N}$

Bounded Sequence—:

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be bounded when it is bounded above as well as bounded below

$\Rightarrow$ A sequence $\{a_n\}_{n=1}^{\infty}$ is bounded if $\exists$ two real numbers k & K $(k \leq K)$ such that

$$k \leq a_n \leq K \quad \forall\, n \in \mathbb{N}$$

Choose $M = \max(|k|, |K|)$, we also can define a sequence $\{a_n\}_{n=1}^{\infty}$ to be bounded if

$$|a_n| \leq M \quad \forall\, n \in \mathbb{N}$$

Unbounded Sequence—:

If $\exists$ no real number M such that $|a_n| \leq M \;\; \forall\, n \in \mathbb{N}$, then sequence $\{a_n\}_{n=1}^{\infty}$ is said to be unbounded.

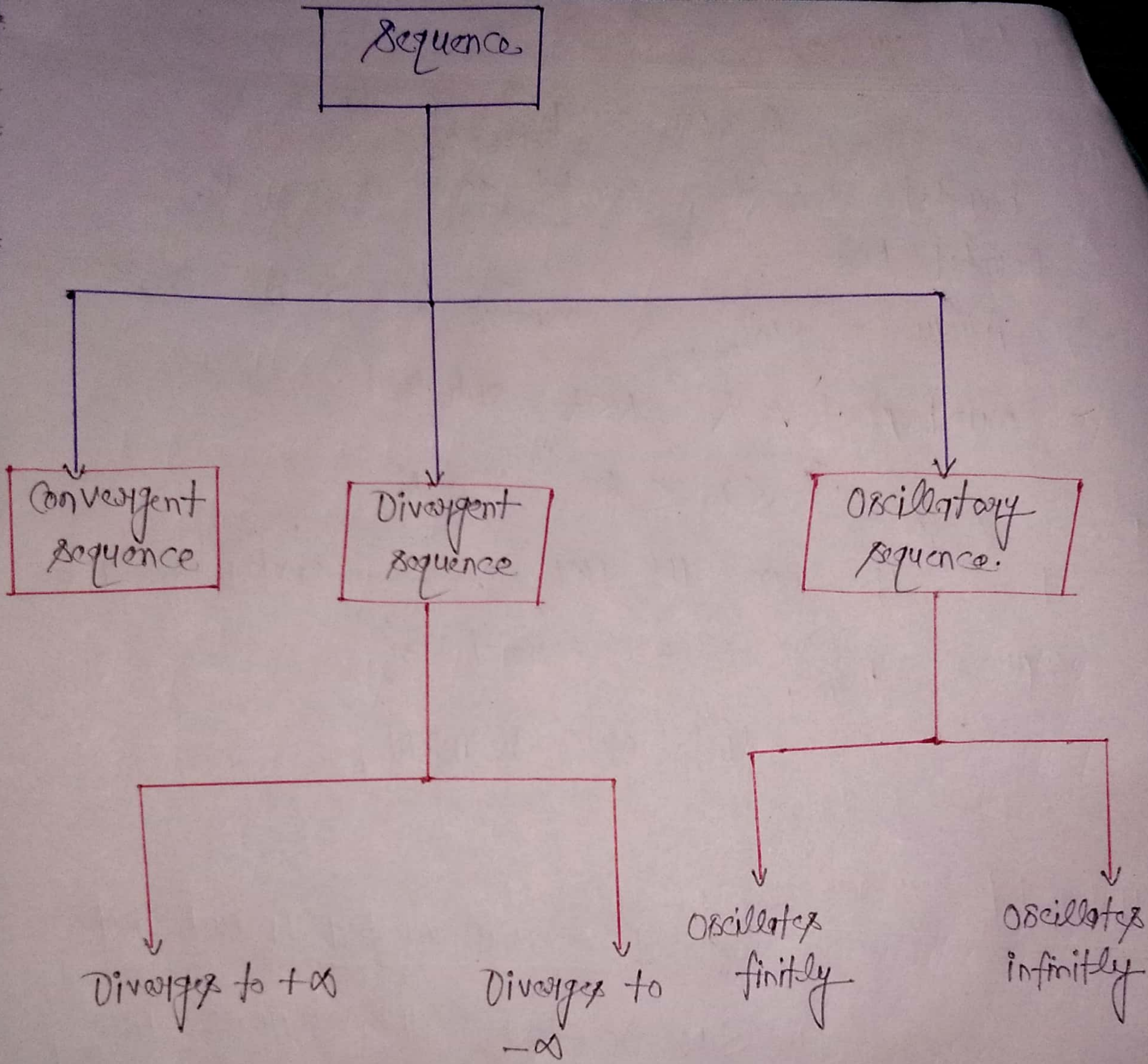
Example—: (1) The sequence $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$

$\because \; 0 \leq a_n \leq 1 \quad \forall\, n \in \mathbb{N}$

$\Rightarrow \{a_n\}_{n=1}^{\infty}$ is bounded sequence.

(2) The sequence $\{a_n\}_{n=1}^{\infty} = \{2^{n-1}\}_{n=1}^{\infty} = \{1, 2, 2^2, 2^3, \dots\}$

$1 \leq a_n \;\; \forall\, n \in \mathbb{N}$, $\nexists$ a real number K s.t. $a_n \leq K \;\; \forall\, n \in \mathbb{N} \Rightarrow$ The sequence is unbounded.



Convergent sequence :-

A sequence $\{x_n\}_{n=1}^{\infty}$ is said to tends to number α if corresponding to an arbitrary chosen positive number ϵ there exists a positive integer N such that

$$|x_n - \alpha| < \epsilon \quad \forall \; n \geq N$$

We write $\boxed{\lim_{n \to \infty} x_n = \alpha}$

* Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of real number

$$|\alpha_n - \alpha| < \epsilon \quad \forall n \geq N$$

| $|x| < \alpha$ |
| $x < \alpha$ |
| $-x < \alpha$ |

$$\Rightarrow (\alpha_n - \alpha) < \epsilon$$
$$-(\alpha_n - \alpha) < \epsilon$$

$$\Rightarrow \alpha - \epsilon < \alpha_n < \alpha + \epsilon \quad \forall n \geq N$$

$$\Rightarrow \alpha_n \in (\alpha - \epsilon, \alpha + \epsilon) \quad \forall n \geq N$$

Divergent sequence :-

If corresponding to any chosen positive A there exists a positive number N such that

$$\alpha_n \geq A \quad \forall n \geq N$$

then we say that the sequence $\{\alpha_n\}_{n=1}^{\infty}$ of real number diverges to $+\infty$.

Then we write $\lim_{n \to \infty} \alpha_n = +\infty$

The sequence $\{\alpha_n\}_{n=1}^{\infty}$ of real numbers diverges to $-\infty$ if corresponding to any chosen positive number A, there correspond a positive integer N s.t.

$$\alpha_n < -A \quad \forall n \geq N$$

In this case, we write

$$\boxed{\lim_{n \to \infty} (\alpha_n) = -\infty}$$

Oscillatory sequence -:

If a sequence $\{x_n\}_{n=1}^{\infty}$ neither converges to a finite number α nor diverges to $+\infty$ or $-\infty$ then we say that the sequence $\{x_n\}_{n=1}^{\infty}$ oscillates.

For an oscillatory sequence if there exists a finite positive number A such that

$$|x_n| \leq A \quad \forall \; n \geq 1$$

then we say that $\{x_n\}_{n=1}^{\infty}$ oscillates finitly

If there is no such number A, then we say that the sequence $\{x_n\}_{n=1}^{\infty}$ oscillates infinitly.

Example (i) $\{x_n\}_{n=1}^{\infty} = \{(-1)^{n-1}\}_{n=1}^{\infty}$ is oscillates finitly

(ii) $\{x_n\}_{n=1}^{\infty} = \{n(-1)^{n-1}\}_{n=1}^{\infty}$ is oscillates infinitly.

upper bound and lower bound of a sequence -:

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real number then either:

(i) There exists a positive number K such that every member of the sequence is less than or equal to K, this number K is called a rough upper bound of the sequence $\{x_n\}_{n=1}^{\infty}$.

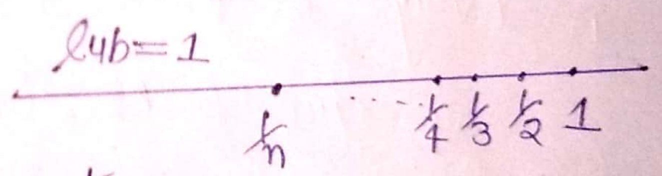
If $K_1 > K$ then each member of the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is less than or equal to K_1

If $K' < K$, then also possible through not certainly, every member of the sequence is less than or equal to K'

or

(ii) There is no number K for which (i) holds.

$$\{\alpha_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$$



$K = 2,\ K_1 = 3,\ K_2 = 4,\ K_3 = 5 \ldots$

$K, K_1, K_2, K_3 \ldots$ rough upper bound

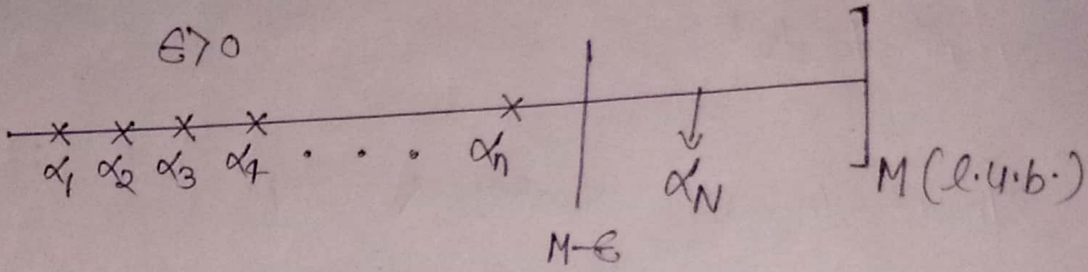
least upper bound

$$\{\beta_n\}_{n=1}^{\infty} = \left\{1 + \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{2, 1+\frac{1}{2}, 1+\frac{1}{3}, \ldots\right\}$$

$2, 3, 4, 5, 6, \ldots \infty$ rough upper bound

$M = 2 \longrightarrow$ least upper bound.

(*) It is known that a set of rough upper bounds of a sequence has a least member. This least member is known as least upper bound (l.u.b) or supremum of the sequence.



Let M be the supremum of the sequence $\{\alpha_n\}_{n=1}^{\infty}$

(a) Every member of the sequence is less than or equal to M, i.e.,

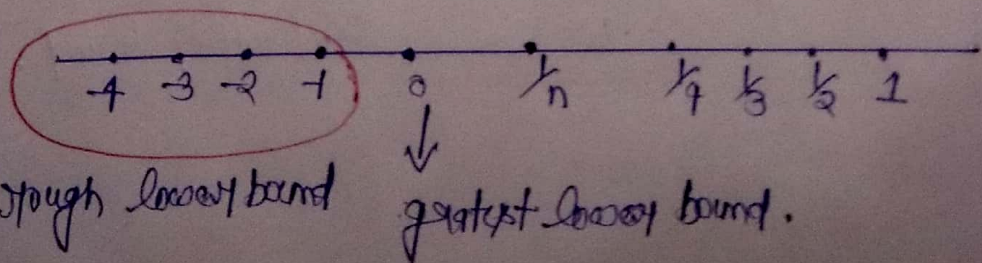
$$\alpha_n \le M \quad \forall\ n \ge 1$$

(b) However small positive number ϵ may be, there is a number α_N of the sequence such that

$$\alpha_N > M - \epsilon$$

In case of lower bound of the sequence $\{\alpha_n\}_{n=1}^{\infty}$

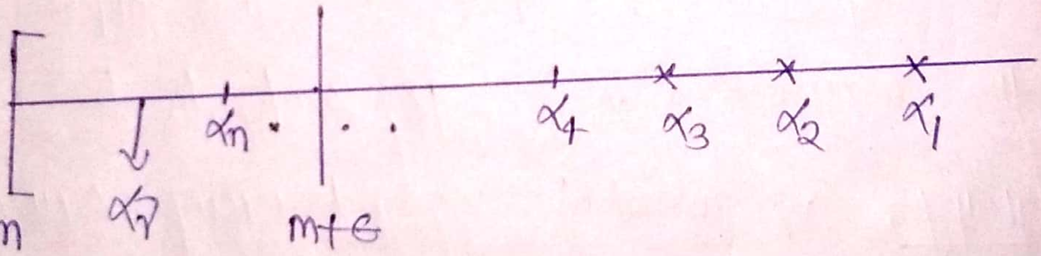
e.g. $\{\alpha_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$



There exists a number k such that every member of the sequence is greater than or equal to k this number k is called a rough lower bound.

It is known that a set of ~~of~~ rough lower bounds of a sequence has a greatest member. This greatest member is known as greatest lower bound (g.l.b.) or infimum of the sequence.

Let m be the infimum of the sequence.



(a) Every member of the sequence is greater than or equal to m i.e.,

$$m \leq \alpha_n \quad \forall\, n \geq 1$$

(b) However small positive number ϵ may be there, there exists a number α_p of the sequence such ~~that~~

$$\alpha_p < m+\epsilon$$

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Proposition :- Every convergent sequence of real number is bounded.

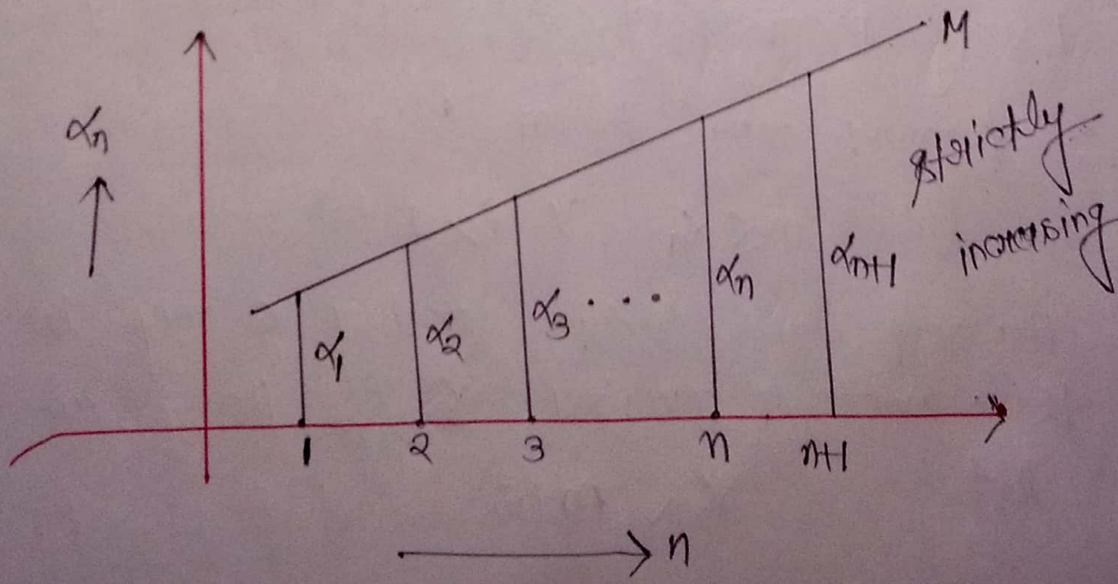
Remark :- The converse of above proposition is not necessarily true.

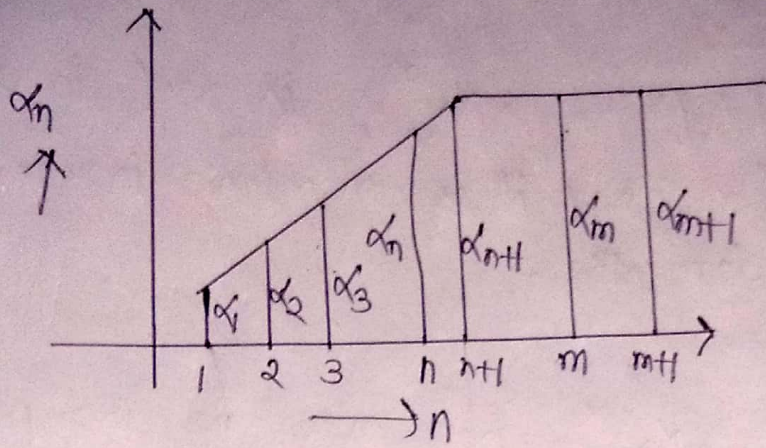
Eg. Let $\{\alpha_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$ is a bounded sequence but it is not convergent.

Monotonic increasing sequence :- Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of real numbers

$$\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \cdots \alpha_n \leq \alpha_{n+1} \leq \cdots$$

then the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is said to be monotonic increasing or monotonic non-decreasing sequence.





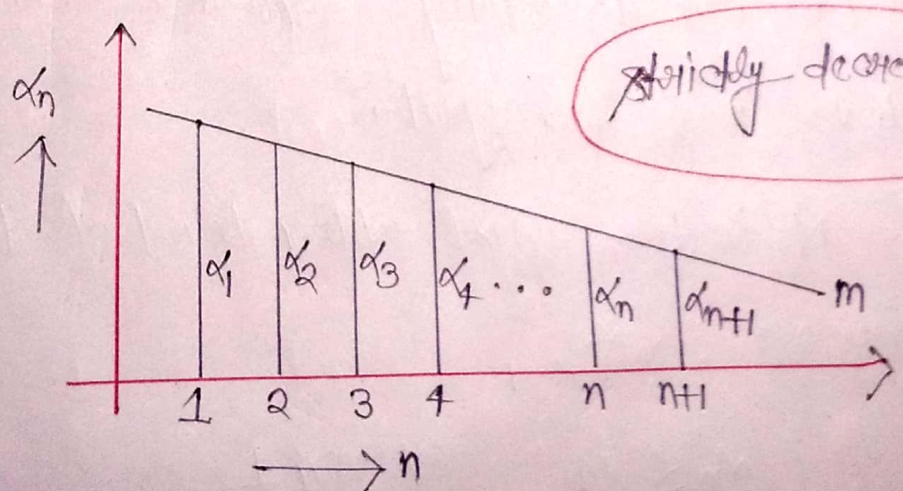
Monotonicaly increasing.

Monotonic decreasing sequence —:

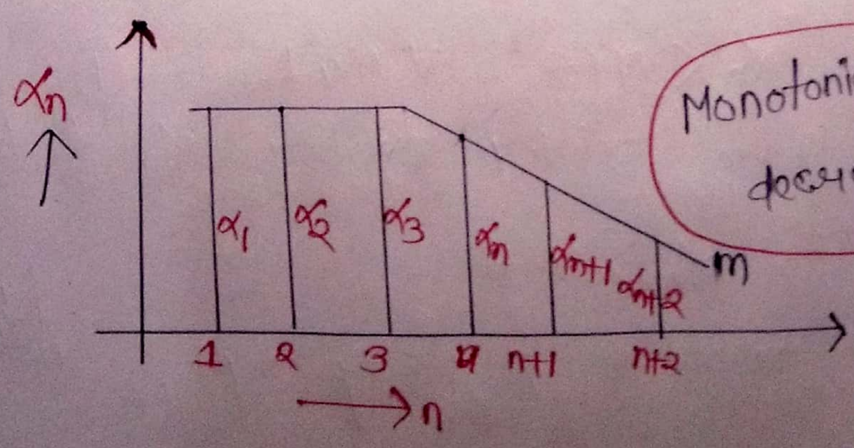
If $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of real number s.t.

$$\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \cdots \geq \alpha_n \geq \alpha_{n+1} \geq \cdots$$

then we say that sequence $\{\alpha_n\}_{n=1}^{\infty}$ is monotonic decreasing or monotonic non-increasing.



strictly decreasing



Monotonicalley decreasing

Monotonic sequence –:

A sequence $\{\alpha_n\}_{n=1}^{\infty}$ is said to be monotonic if either it is monotonic increasing or monotonic decreasing.

Proposition –: Let $\{\alpha_n\}_{n=1}^{\infty}$ be a monotonic increasing sequence. If M is the least upper bound of $\{\alpha_n\}_{n=1}^{\infty}$ then

$$\lim_{n\to\infty} \alpha_n = M$$

If the upper bound of $\{\alpha_n\}_{n=1}^{\infty}$ is $+\infty$ then

$$\lim_{n\to\infty} \alpha_n = +\infty$$

or

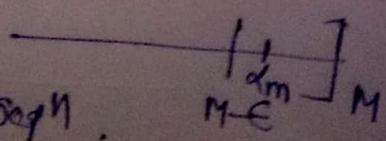
Every monotonically increasing sequence which is bounded above is convergent.

Proof –: Since M is the least upper bound of the sequence $\{\alpha_n\}_{n=1}^{\infty}$

$$\therefore \quad \alpha_n \leq M \quad \forall\ n \geq 1$$

for $\epsilon > 0$ $\quad \alpha_n < M + \epsilon \quad \forall\ n \geq 1$ —(1)

Given $\epsilon > 0$, there exists a number α_m of the sequence s.t. $\alpha_m > M - \epsilon$



$\because$ $\{\alpha_n\}_{n=1}^{\infty}$ is monotonic increasing seq^n.

$\therefore \quad \alpha_n \geq \alpha_m \quad \forall\ n \geq m$

$\alpha_n > M - \epsilon \quad \forall\ n \geq m$ — (ii)

(i) & (ii) $\Rightarrow$

$M - \epsilon < \alpha_n < M + \epsilon \quad \forall\ n \geq m$

$\Rightarrow \quad |\alpha_n - M| < \epsilon \quad \forall\ n \geq m$

$\therefore \quad \boxed{\lim_{n\to\infty} (\alpha_n) = M}$

$\Rightarrow$ The sequence $\{\alpha_n\}_{n=1}^{\infty}$ is convergent and converges to its l.u.b. M

Proved

Moreover, suppose the upper bound of the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is ∞.

Thus for every arbitrary chosen positive number $A > 0$ there exists a ~~seq~~ number α_m s.t

$\alpha_m > A$

Since $\{\alpha_n\}_{n=1}^{\infty}$ is increasing sequence

$\alpha_n \geq \alpha_m \quad \forall\ n \geq m$

$\alpha_n \geq A \quad \forall\ n \geq m$

$\therefore \quad \boxed{\lim_{n\to\infty} (\alpha_n) = \infty}$

Proved

Proposition [B] Let $\{\alpha_n\}_{n=1}^{\infty}$ be a monotonic decreasing sequence of real number. If m is the glb. of the sequence $\{\alpha_n\}_{n=1}^{\infty}$ then

$$\lim_{n\to\infty} (\alpha_n) = m,$$

Moreover, if the lower bounds of sequence is $-\infty$ then

$$\lim_{n\to\infty} (\alpha_n) = -\infty .$$

or

Every monotonicaly decreasing sequence which is bounded below is convergent and converges to its glb. (m).

Proof: Since m is the glb of the sequence $\{\alpha_n\}_{n=1}^{\infty}$

$$m \leq \alpha_n \quad \forall\ n \geq 1$$

for $\epsilon > 0$

$$m - \epsilon \leq \alpha_n \quad \forall\ n \geq 1$$

$$m - \epsilon < \alpha_n \quad \forall\ n \geq 1 \quad \text{— (I)}$$

for $\epsilon > 0$, there exists a number α_p of the sequence $\{\alpha_n\}$ s.t

$$\alpha_p < m + \epsilon$$

$\because$ $\{\alpha_n\}_{n=1}^{\infty}$ is monotonic decreasing

$$\alpha_n \leq \alpha_p \quad \forall\ n \geq p$$

$$\Rightarrow \quad \alpha_n < m + \epsilon \quad \forall\ n \geq p \quad \text{— (II)} .$$

(i) and (ii) $\Rightarrow$

$$m-\epsilon < \alpha_n < m+\epsilon \quad \forall\, n \geq \nu$$

$$\therefore \quad |\alpha_n - m| < \epsilon \quad \forall\, n \geq \nu$$

$$\therefore \quad \boxed{\lim_{n\to\infty} (\alpha_n) = m} \qquad \underline{\text{Proved}}$$

$\Rightarrow$ The seq^n $\{\alpha_n\}_{n=1}^{\infty}$ is ~~mono~~ Convergent and Converges to its g.l.b.

Moreover, suppose the lower bound of the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is $-\infty$, i.e., for a chosen positive number $A > 0$, there exists a number α_m s.t.

$$\alpha_m < -A$$

$\because$ $\{\alpha_n\}_{n=1}^{\infty}$ is monotonically decreasing

$$\alpha_n \leq \alpha_m \quad \forall\, n \geq m$$

$$\Rightarrow \quad \alpha_n \leq -A \quad \forall\, n \geq m$$

$$\therefore \quad \boxed{\lim_{n\to\infty} (\alpha_n) = -\infty} \qquad \underline{\text{Proved}}$$

Remark -: Thus, it is concluded that Every monotonic and bounded sequence is convergent. #

Convergent, divergent and oscillating sequence -:

Convergent sequence -:

A sequence $\{x_n\}_{n=1}^{\infty}$ is said to be Convergent if $\lim_{n\to\infty}(x_n)$ is finite.

Eg. Consider the sequence

$$\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{2n}\right\}_{n=1}^{\infty}$$

Here $\lim_{n\to\infty}(x_n) = 0$, which is finite

$\Rightarrow$ The sequence $\{x_n\}_{n=1}^{\infty}$ is Convergent. #

Divergent sequence -:

A sequence $\{x_n\}_{n=1}^{\infty}$ is said to be divergent if $\lim_{n\to\infty}(x_n)$ is not finite, i.e., if

$$\lim_{n\to\infty}(x_n) = +\infty \text{ or } -\infty.$$

Eg. Consider the seqn

$$\{x_n\}_{n=1}^{\infty} = \{n^2\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty}(x_n) = +\infty$$

$\Rightarrow$ The sequence $\{n^2\}_{n=1}^{\infty}$ is divergent.

Oscillatory Sequence -:

If a sequence $\{x_n\}_{n=1}^{\infty}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an oscillatory sequence.

Oscillatory sequences are of two types;

(i) A bounded sequence which does not converge is said to be oscillate finitely.

Eg. Consider the sequence

$$\{x_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$$

Here, it is a bounded sequence

$$\& \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} (-1)^{2n} = 1$$

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} (-1)^{2n+1} = -1$$

Thus $\lim_{n\to\infty} x_n$ does not exists

$\Rightarrow$ The sequence does not converge.

Hence this sequence oscillates finitely.

(ii) An unbounded sequence, which does not diverge is said to be oscillate infinitely.

Eg. Consider the sequence

$$\{\alpha_n\}_{n=1}^{\infty} = \{(-1)^n n\}_{n=1}^{\infty}$$

Here, $\{\alpha_n\}_{n=1}^{\infty}$ is an unbounded sequence.

$$\lim_{n\to\infty} \alpha_{2n} = \lim_{n\to\infty} \left((-1)^{2n} \cdot 2n\right) = +\infty$$

$$\lim_{n\to\infty} \alpha_{2n+1} = \lim_{n\to\infty} \left((-1)^{2n+1}(2n+1)\right) = -\infty.$$

Thus, sequence does not diverge

Hence, this sequence oscillates infinitly.

Note that -:

(i) When we say the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is said to be convergent

i.e., $\lim_{n\to\infty} (\alpha_n) = l$

it means $\boxed{\lim_{n\to\infty} \alpha_{2n} = \lim_{n\to\infty} \alpha_{2n+1} = l}$

(ii) When $\lim_{n\to\infty} \alpha_n = +\infty$

means $\lim_{n\to\infty} \alpha_{2n} = \lim_{n\to\infty} \alpha_{2n+1} = +\infty$

Monotonic Sequences :-

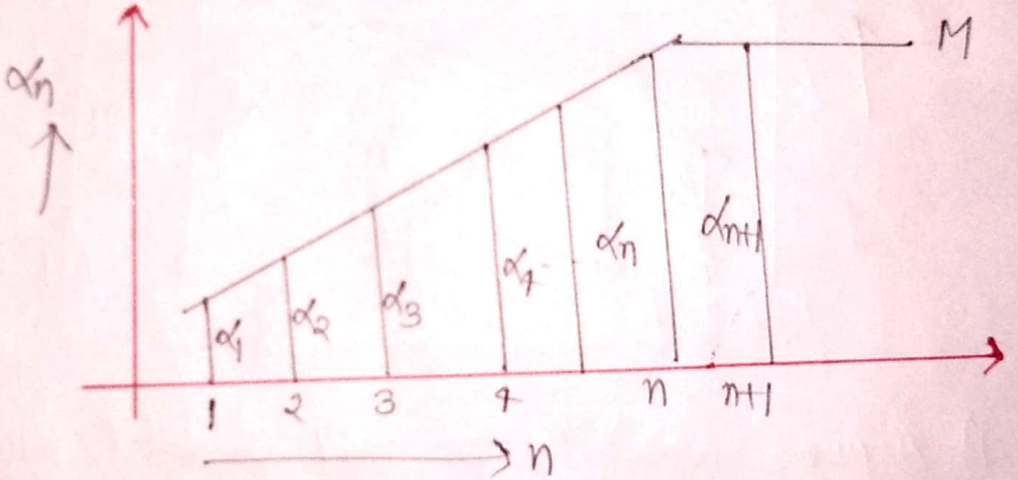
(i) A sequence $\{\alpha_n\}$ is said to be monotonically increasing if $\alpha_{n+1} \geq \alpha_n \quad \forall\, n \in \mathbb{N}$

i.e., if

$$\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \cdots\cdots \leq \alpha_n \leq \alpha_{n+1} \leq \cdots$$

i.e., $\forall\, n \geq m$ & $\{\alpha_n\} \uparrow$

$\Rightarrow \alpha_n \geq \alpha_m$



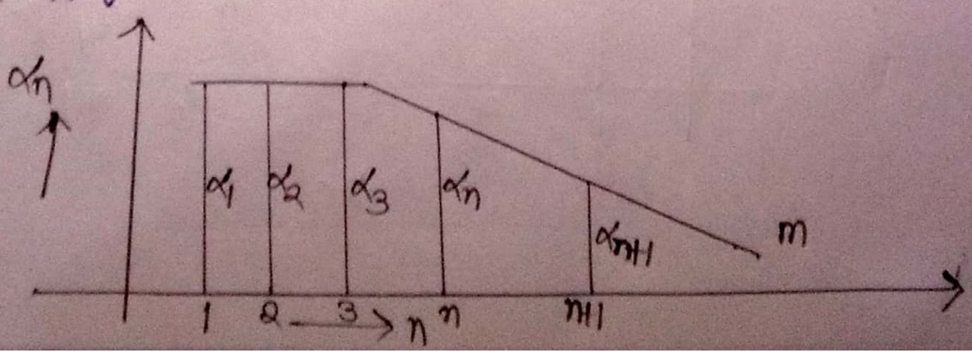
(ii) A sequence $\{\alpha_n\}_{n=1}^{\infty}$ is said to be monotonically decreasing if $\alpha_{n+1} \leq \alpha_n \quad \forall\, n \in \mathbb{N}$

i.e., if

$$\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \cdots\cdots \geq \alpha_n \geq \alpha_{n+1} \geq \cdots$$

for $n \geq m$

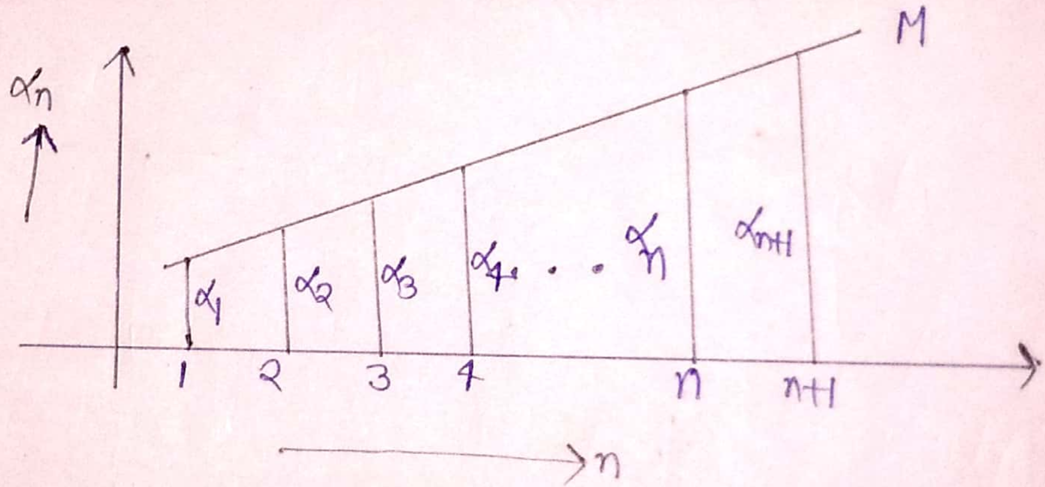
$\{\alpha_n\} \downarrow \Rightarrow \alpha_n \leq \alpha_m$



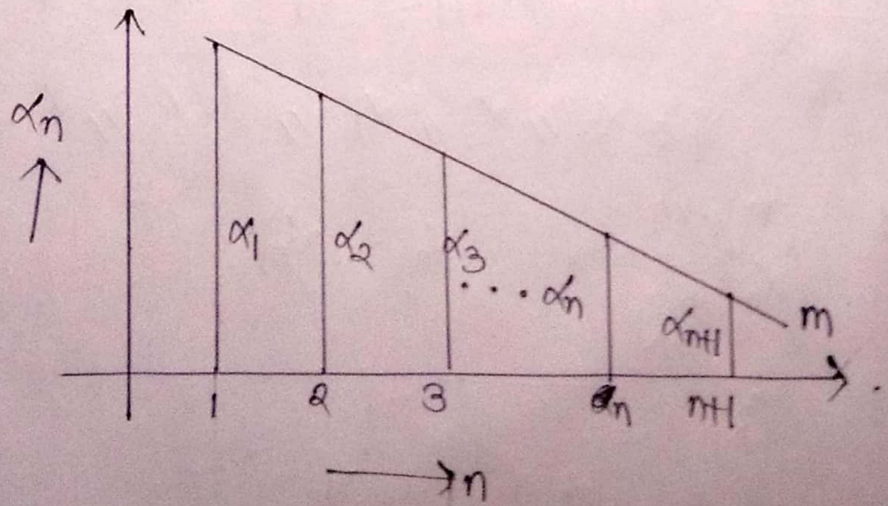
(iii) A sequence $\{\alpha_n\}_{n=1}^{\infty}$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

(iv) A sequence $\{\alpha_n\}_{n=1}^{\infty}$ is said to be strictly increasing if

$$\alpha_n < \alpha_{n+1} \quad \forall\, n \in \mathbb{N}$$



(v) A sequence $\{\alpha_n\}_{n=1}^{\infty}$ is said to be strictly decreasing if $\alpha_{n+1} < \alpha_n \quad \forall\, n \in \mathbb{N}$



(vi) A sequence $\{\alpha_n\}_{n=1}^{\infty}$ is said to be strictly monotonic if it is either strictly increasing or strictly decreasing. (11)

Limit of the sequence –:

A sequence $\{\alpha_n\}_{n=1}^{\infty}$ is said to approach the limit l (say) when $n \to \infty$ if for each $\epsilon > 0$, $\exists$ a +ive integer m (depending upon ϵ) such that $|\alpha_n - l| < \epsilon \quad \forall\, n \geq m$

In this case, we write

$$\lim_{n \to \infty} (\alpha_n) = l.$$

Note that –:

$$|\alpha_n - l| < \epsilon \quad \forall\, n \geq m$$

$$\Rightarrow \quad l - \epsilon < \alpha_n < l + \epsilon \quad \text{for } n = m, m+1, m+2, \ldots$$

Theorem –: Every convergent sequence is bounded. Moreover converse of above is not necessarily true.

Proof: Let the sequence $\{\alpha_n\}_{n=1}^{\infty}$ be convergent and converges to the limit l, i.e., for given $\epsilon > 0$, there exists a positive integer m, such that

$$|\alpha_n - l| < \epsilon \quad \forall\ n \geq m$$

$$\Rightarrow \quad l - \epsilon < \alpha_n < l + \epsilon \quad \forall\ n \geq m$$

let $k = \min\{\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_{m-1}, l+\epsilon, l-\epsilon\}$

$K = \max\{\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_{m-1}, l+\epsilon, l-\epsilon\}$

$$\Rightarrow \quad k \leq \alpha_n \leq K \quad \forall\ n \in \mathbb{N}$$

$\Rightarrow$ The sequence $\{\alpha_n\}_{n=1}^{\infty}$ is bounded.

To Converse may not always true, i.e., a sequence may be bounded, yet it may not be Convergent

let $\{\alpha_n\} = \{(-1)^n\}$ is bounded sequence but does not Convergent.

i.e. $\lim\limits_{n\to\infty} \alpha_{2n} = \lim\limits_{n\to\infty} (-1)^{2n} = 1$

$\lim\limits_{n\to\infty} \alpha_{2n+1} = \lim\limits_{n\to\infty} (-1)^{2n+1} = -1$

oscillates finity

$\Downarrow$

Not Convergent.

Convergence of monotonic sequence :

Theorem : The necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.

(i) A monotonic increasing sequence which is bounded above converges.

(ii) A monotonic decreasing sequence which is bounded below converges.

Theorem : If a monotonic increasing sequence is not bounded above, it diverges to $+\infty$.

Theorem : If a monotonic decreasing sequence is not bounded below, it diverges to $-\infty$.

Theorem : (i) If $\{\alpha_n\}$ and $\{\beta_n\}$ are two convergent sequences, then sequence $\{\alpha_n \pm \beta_n\}$ is also convergent.

or

If $\lim_{n\to\infty} \alpha_n = \alpha$ & $\lim_{n\to\infty} (\beta_n) = \beta$,

then $\lim_{n\to\infty} (\alpha_n \pm \beta_n) = \alpha \pm \beta$.

(ii) If $\{\alpha_n\}$ and $\{\beta_n\}$ are two convergent sequences, such that $\lim_{n\to\infty}(\alpha_n) = \alpha$ & $\lim_{n\to\infty}(\beta_n) = \beta$, then

(i) sequence $\{\alpha_n \beta_n\}$ is also convergent and converges to $\alpha\beta$.

(ii) sequence $\left\{\frac{\alpha_n}{\beta_n}\right\}$ is also convergent and converges to $\left\{\frac{\alpha}{\beta}\right\}$ $(\beta \neq 0)$ $\beta_n \neq 0\ \forall n \in N$.

Theorem—:

The sequence $\{|\alpha_n|\}$ converges to zero if and only if the sequence $\{\alpha_n\}$ converges to zero.

Theorem—: (i) If a sequence $\{\alpha_n\}$ converges to α and $\alpha_n \geq 0\ \forall n \in N$, then $\alpha \geq 0$.

(ii) If $\alpha_n \to \alpha$, $\beta_n \to \beta$ and $\alpha_n \leq \beta_n\ \forall n$, then $\alpha \leq \beta$.

(iii) If $\alpha_n \to l$, $\beta_n \to l$ and $\alpha_n \leq c_n \leq \beta_n\ \forall n \in N$, then $c_n \to l$ (Sandwitch Theorem).

Question :- Give example of a monotonic increasing sequence which is (i) Convergent (ii) divergent.

Soln. Consider the sequence

$$\{x_n\} = \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\right\}$$

$$x_{n+1} - x_n = \left(\frac{n+1}{n+2}\right) - \left(\frac{n}{n+1}\right) = \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)}$$

$$= \frac{(\cancel{n^2}+\cancel{2n}+1) - \cancel{n^2} - \cancel{2n}}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0$$

$\Rightarrow \quad x_n \leq x_{n+1} \quad \forall\, n \in \mathbb{N}$

$\Rightarrow \{x_n\}_{n=1}^{\infty}$ is a monotonically increasing sequence

and $\quad |x_n| \leq 1 \quad \forall\, n \in \mathbb{N}$

$\Rightarrow \{x_n\}$ is monotonically increasing and bounded above therefore it is convergent and converges to it l.u.b $= 1$

$$\lim_{n\to\infty} (x_n) = \lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right) = 1.$$

$\therefore$ The sequence is convergent. #

(ii) Consider the sequence $\{x_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$

$$= \{1, 2, 3, 4, \cdots\}$$

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Since $\{\alpha_n\}$ is monotonically increasing not bounded above $\Rightarrow$ The sequence $\{\alpha_n\}$ is diverges to $+\infty$.

$$\boxed{\lim_{n\to\infty} (\alpha_n) = +\infty}$$

$\Rightarrow$ The sequence diverges to $+\infty$.

Question :- Give an example of a monotonic decreasing sequence which (i) Convergent (ii) divergent.

(i) Consider the sequence

$$\{\alpha_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$$

is a monotonic decreasing sequence and

$$0 \leq \alpha_n \leq 1 \quad \forall\, n \in \mathbb{N} \quad (\text{bound below by } 0)$$

$\Rightarrow$ The sequence $\{\alpha_n\}_{n=1}^{\infty}$ is Convergent and Convergent to glb 0.

$$\lim_{n\to\infty} (\alpha_n) = \lim_{n\to\infty} (\alpha_n) = 0$$

$\therefore$ The sequence Converges to 0.

(ii) Consider the sequence $\{\alpha_n\}_{n=1}^{\infty} = \{-n\}_{n=1}^{\infty} = \{-1, -2, -3, \ldots\}$ is motonic decreasing sequence and not bounded below

$$\lim_{n\to\infty} (\alpha_n) = -\infty \Rightarrow \text{The seq}^n \text{ diverges to } -\infty.$$

(14)

Question: Discuss the convergence of the sequence $\{x_n\}_{n=1}^{\infty}$

(i) $\{x_n\} = \left(\frac{n+1}{n}\right)$ (ii) $\{x_n\} = \left\{\frac{n}{n^2+1}\right\}$

(iii) $\{x_n\} = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n}$

Soln (i) We have given

$$x_n = \frac{n+1}{n} = \left(1 + \frac{1}{n}\right)$$

$$x_{n+1} - x_n = \frac{(n+2)}{n+1} - \frac{n+1}{n}$$

$$= \frac{n(n+2) - (n+1)^2}{n(n+1)} = \frac{n^2+2n-(n^2+2n+1)}{n(n+1)}$$

$$\Rightarrow x_{n+1} - x_n = \frac{-1}{n(n+1)} < 0 \quad \forall\, n \in \mathbb{N}$$

$$\Rightarrow x_n > x_{n+1} \quad \forall\, n$$

$\Rightarrow$ $\{x_n\}$ is a decreasing sequence

Also $x_n = \frac{n+1}{n} = 1 + \frac{1}{n} > 1 \quad \forall\, n$

$\because$ $\{x_n\}_{n=1}^{\infty}$ is a decreasing and bounded below, therefore it is convergent and converges to glb (1)

$$\lim_{n\to\infty} (x_n) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$$

Ans

(ii) We have given

$$\{\alpha_n\}_{n=1}^{\infty} = \frac{n}{n^2+1}.$$

Now $$\alpha_{n+1} - \alpha_n = \frac{(n+1)}{(n+1)^2+1} - \left(\frac{n}{n^2+1}\right)$$

$$= \frac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+1)(n^2+2n+2)}$$

$$= \frac{\cancel{n^3}+n+n^2+1-\cancel{n^3}-2n^2-2n}{(n^2+1)(n^2+2n+2)}$$

$$= \frac{-n^2-n+1}{(n^2+1)(n^2+2n+2)} = \frac{-(n^2+n-1)}{(n^2+1)(n^2+2n+2)} < 0 \quad \forall n \in \mathbb{N}$$

$\Rightarrow$ $\alpha_n > \alpha_{n+1}$ $\forall n \in \mathbb{N}$

$\Rightarrow$ $\{\alpha_n\}$ is a decreasing sequence.

and $\alpha_n = \left(\frac{n}{n^2+1}\right) > 0$ $\forall n$

$\Rightarrow$ $\{\alpha_n\}$ is bounded below by 0.

$\because$ $\{\alpha_n\}$ is decreasing and bounded below, therefore it is convergent, converges to glb $= 0$.

$$\lim_{n\to\infty}(\alpha_n) = \lim_{n\to\infty}\left(\frac{n}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1+1/n^2}\right) = 0$$

Ans

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(iii) We have given

$$\{\alpha_n\}_{n=1}^{\infty} = 1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^n}$$

$$= \frac{1\left(1-\frac{1}{3^{n+1}}\right)}{1-\frac{1}{3}} = \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right)$$

Now $\alpha_{n+1} = \underline{1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}}+\frac{1}{3^{n+1}}$

$$\alpha_{n+1}-\alpha_n = \frac{1}{3^{n+1}} > 0 \quad \forall\, n \in \mathbb{N}$$

$$\Rightarrow \quad \alpha_n < \alpha_{n+1} \quad \forall\, n$$

$\Rightarrow$ $\{\alpha_n\}_{n=1}^{\infty}$ is a increasing sequence.

Also, $\alpha_n = \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right) < \frac{3}{2} \quad \forall\, n$

$\Rightarrow$ $\{\alpha_n\}$ is bounded above by $3/2$.

$\because$ $\{\alpha_n\}_{n=1}^{\infty}$ is increasing and bounded above, therefore it is convergent.

$$\lim_{n\to\infty}(\alpha_n) = \lim_{n\to\infty}\frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right) = 3/2$$

Ans

Infinite series

Series :- let $\{u_n\}_{n=1}^{\infty} = \{u_1, u_2, u_3, u_4, \ldots\ldots u_n, \ldots\}$

be a sequence.

Then $$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is an infinite series.

Write,

$$s_1 = u_1$$
$$s_2 = u_1 + u_2$$
$$s_3 = u_1 + u_2 + u_3$$
$$\vdots \quad \vdots \quad \vdots$$
$$s_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum_{i=1}^{n} u_i$$

Called the nth partial sum of $\sum_{n=1}^{\infty} u_n$.

$$\lim_{n\to\infty} (u_1 + u_2 + u_3 + \cdots + u_n) = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

or

$$\lim_{n\to\infty} (s_n) = u_1 + u_2 + \cdots + u_n + \cdots$$

If $\lim_{n\to\infty} (s_n)$ = a finite number say s, then we say that $\sum_{n=1}^{\infty} u_n$ is Convergent and its sum is s.

If $\lim_{n\to\infty}(s_n) = +\infty$ (or $-\infty$) then we say that $\sum_{n=1}^{\infty} u_n$ diverges to $+\infty$ (or $-\infty$).

If $\lim_{n\to\infty}(s_n)$ oscillates, then $\sum_{n=1}^{\infty} u_n$ oscillates.

Positive term series -:

The series

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is said to be positive term series if every term of series is positive.

Alternating series -:

If the terms of the series are alternatively positive and negative beginning with the first terms, then series is said to be alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n+1} u_n + \cdots$$

is an alternating series.

Some important limits :

ⓐ $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

ⓑ $\lim_{n\to\infty} \left(\frac{\log n}{n}\right) = 0$

ⓒ $\lim_{n\to\infty} n x^n = 0$ for $0<x<1$

ⓓ $\lim_{n\to\infty} \left(\frac{a_1 n^p + a_2 n^{p-1} + \cdots + a_{p+1}}{b_1 n^q + b_2 n^{q-1} + \cdots + b_{q+1}}\right)$

$$= \begin{cases} 0 & \text{if } p<q \\ \frac{a_1}{b_1} & \text{if } p=q \\ \infty & \text{if } a_1 \text{ and } b_1 \text{ are same sign } \& \ p>q \\ -\infty & \text{if } a_1 \text{ and } b_1 \text{ are opposite sign } \& \ p>q \end{cases}$$

ⓔ $\lim_{n\to\infty} (x^n) = 0 \quad x<1.$

Convergence and divergence of a series :

Let $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ be an infinite series. Then the nth partial sum

$$S_n = \sum_{i=1}^{n} u_i = u_1 + u_2 + u_3 + \cdots + u_n.$$

(i) If $\lim_{n\to\infty}(S_n) = S$, where S is finite and unique quantity, $\sum_{n=1}^{\infty} u_n$ is Convergent. Converges to S.

(ii) If $\lim_{n\to\infty}(S_n) = +\infty$ or $-\infty$, then $\sum_{n=1}^{\infty} u_n$ is divergent.

(iii) If $\lim_{n\to\infty}(S_n)$ is neither a finite quantity nor $-\infty$ or ∞, then the series $\sum_{n=1}^{\infty} u_n$ is said to be oscillatory.

A series can be oscillates finitly or infinitly.

(a) The series $\sum_{n=1}^{\infty} u_n$ is said to oscillates finitly if $\lim_{n\to\infty}(S_n)$ is a finite but not a unique quantity i.e., $\lim_{n\to\infty}(S_n)$ fluctuates between finite limits

(b) The series $\sum_{n=1}^{\infty} u_n$ is said to be oscillates infinitly if $\lim_{n\to\infty}(S_n)$ fluctuates between $-\infty$ and $+\infty$.

Question (i) Consider the series

$$\sum_{n=1}^{\infty} u_n = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$$

Here, $S_n = 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$

$$S_n = \frac{1\left(1-\frac{1}{2^n}\right)}{\left(1-\frac{1}{2}\right)} = 2\left(1-\frac{1}{2^n}\right)$$

Clearly $\lim_{n\to\infty}(S_n) = 2$, which is finite unique quantity, hence the series is convergent.

(ii) Again consider the series

$1+2+3+4+\cdots$

Here $S_n = \frac{n(n+1)}{2} \Rightarrow \lim_{n\to\infty}(S_n) = +\infty$.

Hence series is divergent.

Further: Consider the series

$-1-2-3-4-5-\cdots$

Here, $S_n = -\frac{(n)(n+1)}{2}$

clearly, $\lim_{n\to\infty}(S_n) = -\infty$

Hence series is divergent.

(iii) Now consider the series

$1-1+1-1+1-1+1-1-\cdots$

Here,

$$s_n = \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd} \end{cases}$$

Then s_n oscillates (fluctuates) between 0 & 1, which are the finite limits. So the series oscillates finitely.

* Consider the series

$$1-2+3-4+5-6 \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} n$$

$$s_n = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Hence $\lim_{n\to\infty} (s_n) = -\infty$ if n is even

$\lim_{n\to\infty} (s_n) = \infty$ if n is odd

Therefore, series oscillates infinitely between $-\infty$ to ∞.

Note that-: (i) If all the terms of a series are the same sign (either positive or negative) then this series can be convergent or divergent but never oscillates.

Geometric series :-

A infinite geometric series

$$1+r+r^2+r^3+\cdots+r^{n-1}+\cdots$$

with common ratio r is

(i) Convergent if $-1<r<1$ i.e., $|r|<1$

(ii) divergent if $r>1$

(iii) finitly oscillating if $r=-1$

(iv) infinitly oscillating if $r<-1$.

proof :- Consider the infinite geometric series

$$1+r+r^2+r^3+\cdots+r^{n-1}+\cdots$$

with common ratio is r.

Case-I When $-1<r<1$, i.e., $|r|<1$

The nth partial sum

$$S_n = 1+r+r^2+\cdots+r^{n-1} = \frac{1(1-r^n)}{(1-r)}$$

as $|r|<1$, $r^n \to 0$ as $n\to\infty$

$$\lim_{n\to\infty}(S_n) = \left(\frac{1}{1-r}\right),$$ which is finite

unique quantity,

Hence the series is Convergent for $|r|<1$.

Case-II When $r>1$

$$S_n = \frac{(r^n-1)}{(r-1)} \quad \left(\because \; r>1 \; r^n \to \infty \text{ as } n\to\infty\right)$$

$\therefore \lim_{n\to\infty}(S_n) = \infty$

Hence the series is divergent

Also when $r=1$, the series becomes

$1+1+1+1+1+1+\cdots$

Then $S_n = n$; $\lim_{n\to\infty}(S_n) = \infty$.

Hence, the series is divergent.

$\Rightarrow$ The series $\sum_{n=1}^{\infty} r^{n-1}$ is divergent $r \geq 1$.

Case-III When $r=-1$, the series becomes

$1-1+1-1+1-1+1-1 \cdots$

$$S_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Here, the limit are finite but not unique.

Hence series is finitly oscillating series.

Case-IV When $r<-1$

then $-r>1$

Let $x=(-r)$; $x>1 \Rightarrow x^n \to \infty$ as $n\to\infty$

Now $S_n = 1+r+r^2+r^3+\cdots+r^{n-1} = \frac{1-r^n}{1-r} = \frac{1-(-x)^n}{1+x}$

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$$= \frac{1+x^n}{1+x} \text{ or } \frac{1-x^n}{1+x}$$

$$\lim_{n\to\infty} (S_n) = \infty \text{ or } -\infty.$$

Hence, the series is infinitly oscillating. # proved

Fundamental properties -:

(i) The nature of series remains unaltered if signs of all the terms are altogether changed.

(ii) The convergence or divergence of a series remains uneffected if finite number of terms are added or neglected (omitted)

(iii) If each term of a given series is multiplied or devided by some fixed quantity other than zero, then the new series so obtained will remains convergent or divergent according as it was originally convergent or divergent.

(iv) If two infinite series are given, then series formed by their sum will be;

(a) Convergent if both the given series are convergent

(b) divergent if any of the given series is divergent.

(4) If $a_n \leq b_n \quad \forall\, n \in N$

if $\{b_n\}$ is convergent

$\Rightarrow \{a_n\}$ is convergent

If $\{a_n\}$ is divergent $\Rightarrow \{b_n\}$ is divergent.

(VI) If a series $\sum_{n=1}^{\infty} u_n$ converges to a fixed finite quantity S, then series obtained by grouping the terms in bracket without affecting the order of terms also converges to S.

Limit u_n test -:

For a series $\sum_{n=1}^{\infty} u_n$ to be convergent, it is necessary but not sufficient that

$$\lim_{n \to \infty} (u_n) = 0$$

A test for divergence of a series -:

Statement -: If all the terms of an infinite series are positive and each term is greater than some fixed finite quantity however small, then series will be divergent.

proof: let $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + u_4 \cdots + u_n + \cdots$

Then the nth partial sum S_n is given by

$$S_n = \sum_{i=1}^{n} u_i = u_1 + u_2 + u_3 + \cdots + u_n$$

let $u_i > k$ for some finite k

$\forall\ i = 1, 2, 3, \cdots n$

$\Rightarrow \quad S_n > k + k + \cdots + k$ (n times)

$> nk$

Hence $S_n \to \infty$ as $n \to \infty$

Hence, $\sum_{n=1}^{\infty} u_n$ is divergent. #

Leibnitz's Test or Alternating series test -:

The infinite $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ is convergent if

$u_n > 0$, $u_n < u_{n+1}$ and $\lim_{n\to\infty} (u_n) = 0$.

or

A infinite series in which, the terms are alternatively positive and negative is convergent if if each terms is numerically less than the previous term and if $\lim_{n\to\infty} u_n = 0$.

Proof: Let the alternating series be

$$u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \cdots$$

where $u_1 > u_2 > u_3 > u_4 > \cdots$

Let s_{2n} and s_{2n+1} represent the sum of first $2n$ & $2n+1$ terms

Then $s_{2n} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) \cdots + (u_{2n-1} - u_{2n}) > 0$

$=$ a positive quantity > 0 $\because u_i > u_{i+1} \ \forall i$ — (1)

and $s_{2n+1} = u_1 - [(u_2 - u_3) + (u_4 - u_5) \cdots + (u_{2n} - u_{2n+1})]$
$= u_1 -$ a positive quantity
$< u_1$

$\Rightarrow s_{2n+1} < u_1$ — (2)

Also $s_{2n+1} = s_{2n} + u_{2n+1}$

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} (u_{2n+1})$$

$$\Rightarrow \lim_{n\to\infty}(s_{2n+1}) = \lim_{n\to\infty}(s_{2n}) \quad \because \lim_{n\to\infty}(u_n) = 0$$

Hence, the sum of odd terms is same as sum of even terms

Hence the series can not be oscillating

Now ① & ② $\Rightarrow$ The common limit lies between

0 and u_1

$\because$ u_1 is finite and definite

Hence the series must be convergent.

Note that :- let $\sum_{n=1}^{\infty} u_n$ be a series of positive term

s.t $\lim_{n\to\infty} (u_n) \neq 0$

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is not convergent.

Question :- Examine the convergence of the series

$$\frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \cdots = \sum_{n=1}^{\infty} u_n$$

Here $u_n > 1 \quad \forall n$

$$s_n = 2 + \frac{3}{2} + \frac{4}{3} + \cdots + \left(\frac{n+1}{n}\right) > 1+1+1+\cdots 1$$
$$= n$$

$\lim_{n\to\infty} (s_n) = +\infty$

Hence, the series is divergent;

Question :- Test the convergence of the series

$$\sum_{n=1}^{\infty} u_n = \sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \sqrt{\frac{4}{5}} + \cdots$$

Here $u_n \geq \sqrt{\frac{1}{2}} \quad \forall\, n$

$\therefore \quad s_n \geq \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}} - \cdots + \sqrt{\frac{1}{2}}$ (n times)

$\Rightarrow \quad s_n > n\sqrt{\frac{1}{2}}$

$\Rightarrow \quad \lim_{n\to\infty}(s_n) = \infty$

Hence, the series is divergent. #

Question-: Test whether the series

$$\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \cdots$$

is Convergent or divergent.

Soln We have given

$$\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \cdots$$

$$\frac{1}{1+\frac{1}{2}} + \frac{2}{1+\frac{1}{2^2}} + \frac{3}{1+\frac{1}{2^3}} + \cdots$$

$$\sum_{n=1}^{\infty} u_n = \frac{2}{3} + \frac{8}{5} + \frac{24}{9} + \cdots$$

Here $u_n > \frac{1}{2} \quad \forall\, n$

Therefore nth partial sum

$s_n > \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$ (n time)

$\Rightarrow \quad s_n > \frac{n}{2} \quad \forall\, n$

$\Rightarrow \quad \boxed{\lim_{n\to\infty}(s_n) = \infty}$

Hence, the given series is divergent. #

Question :- Test the Convergence of the series

$$\sqrt{\tfrac{1}{2}} + \frac{2}{\sqrt{5}} + \frac{4}{\sqrt{17}} + \frac{8}{\sqrt{65}} + \cdots + \frac{2^n}{\sqrt{4^n+1}}$$

Soln After leaving the first term, we have

$$u_n = \frac{2^n}{\sqrt{4^n+1}} = \frac{\cancel{2^n}}{\cancel{2^n}\sqrt{1+4^{-n}}}$$

$$\lim_{n\to\infty}(u_n) = 1 \neq 0$$

Hence, the given series is not Convergent. #

Question :- Prove that the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Here $u_n > 0 \ \forall n$ $\quad u_n = 1/n$

$u_{n+1} < u_n \quad \forall n \geq 1$

and $\lim_{n\to\infty}(u_n) = 0$

Thus, all the three Conditions of Leibnitz's test are satisfied. Hence, the given series is Convergent.

Question :- Test the Convergence of the series

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{n+1}{n}\right)$$

Here $u_n = \left(\frac{n+1}{n}\right) > 0 \quad \forall n$

and $u_{n+1} - u_n = \left(\frac{n+2}{n+1}\right) - \left(\frac{n+1}{n}\right) = \frac{n(n+2)-(n+1)^2}{n(n+1)}$

$u_{n+1} - u_n = \frac{-1}{n(n+1)} < 0$

$\Rightarrow u_n > u_{n+1} \quad (u_n)\downarrow$

and $\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{n+1}{n}\right) = 1 \neq 0 \Rightarrow$ Not Convergent

Sequence and Series Contd ...

We have given

$$2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\cdots$$

It can be written as —

$$(1+1)-\left(1+\tfrac{1}{2}\right)+\left(1+\tfrac{1}{3}\right)-\left(1+\tfrac{1}{4}\right)+\left(1+\tfrac{1}{5}\right)-\cdots$$

i.e., $(1-1+1-1+1-1\cdots)+\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\right)$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$S_n=\begin{cases}0 & \text{if } n \text{ even}\\ 1 & \text{if } n \text{ odd}\end{cases} \qquad \frac{1}{2}\log\left(\frac{4m}{n}\right)$$

$$\Rightarrow \sum u_n = \frac{1}{2}\log 4\left(\frac{4}{1}\right)$$

$$= \log 2.$$

$\Rightarrow \sum u_n$ is either $1+\log 2$ or $\log 2$.

Hence the given series is finite oscillatory.

Comparison test for positive term series —:

If $\sum u_n$ & $\sum v_n$ be two series of positive terms such that $\lim\limits_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is fixed finite non-zero quantity, then both the series will converge or diverge simultaneously, i.e., the two series are either both convergent or both divergent.

Note that: v_n can be obtain by taking common highest degree term from Nr^r and Dr^r of u_n.

p-series test or Hyper-Harmonic Test :

The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

is convergent if $p > 1$ and divergent if $p \leq 1$.

Question: Test the convergence of the series;

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$$

Solution We have given

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n-1} + \cdots$$

Here, $u_n = \frac{1}{2n-1} = \frac{1}{n}\left(\frac{1}{2-1/n}\right)$

$\therefore \quad v_n = 1/n$

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{2-1/n}\right) = \frac{1}{2} \neq 0 \text{ finite.}$$

Hence by Comparison test $\sum u_n$ & $\sum v_n$ converges or divergent together.

Now by p-test $\sum v_n = \sum \frac{1}{n}$ is divergent $(p=1)$

$$\Rightarrow \sum u_n = \sum_{n=1}^{\infty} \frac{1}{2n-1} \text{ is divergent. } \#$$

Question:- Test the convergence of the series;

$$\frac{14}{1^3}+\frac{24}{2^3}+\frac{34}{3^3}+\cdots+\frac{10n+4}{n^3}+\cdots$$

Here, $u_n=\dfrac{10n+4}{n^3}=\dfrac{1}{n^2}\left(10+\dfrac{4}{n}\right)$

Let $v_n=\dfrac{1}{n^2}$

$\therefore \lim\limits_{n\to\infty}\left(\dfrac{u_n}{v_n}\right)=\lim\limits_{n\to\infty}\left(10+\dfrac{4}{n}\right)=10\neq 0$ finite

Then, by comparison test $\sum u_n$ & $\sum v_n$ converges or diverges together.

Now by p-test $\sum v_n=\sum \frac{1}{n^2}$ is convergent $(p=2>1)$

$$\Rightarrow \sum_{n=1}^{\infty} u_n=\sum_{n=1}^{\infty}\frac{10n+4}{n^3} \text{ is } \underline{\text{convergent}} \ \#$$

Question:- Test the convergence of the series

$$\frac{\sqrt{1}}{1+\sqrt{1}}+\frac{\sqrt{2}}{2+\sqrt{2}}+\frac{\sqrt{3}}{3+\sqrt{3}}+\cdots\cdots$$

Soln We have given

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+\sqrt{n}}=\frac{\sqrt{1}}{1+\sqrt{1}}+\frac{\sqrt{2}}{2+\sqrt{2}}+\frac{\sqrt{3}}{3+\sqrt{3}}+\cdots$$

Here, $u_n=\dfrac{\sqrt{n}}{n+\sqrt{n}}=\dfrac{1}{\sqrt{n}}\left(\dfrac{1}{1+\frac{1}{\sqrt{n}}}\right)$

Let $v_n=\dfrac{1}{\sqrt{n}}$

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$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{\sqrt{n}}}\right) = 1 \neq 0 \text{ finite.}$$

Then, by Comparison test, $\sum u_n$ & $\sum v_n$ Converges or diverges together.

Now by p-test $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is divergent $\left(p = \frac{1}{2} < 1\right)$

$$\Rightarrow \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+\sqrt{n}} \text{ is divergent } \#$$

Question-: Test the Convergence of the series

$$1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots$$

Soln After leaving the first term, we have

$$\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{n+1}} = \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots$$

Here, $$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{n^{n+1}} \frac{1}{(1+\frac{1}{n})^{n+1}}$$

$$= \frac{1}{n} \frac{1}{(1+\frac{1}{n})^{n+1}}$$

let $v_n = \frac{1}{n}$

$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{(1+\frac{1}{n})^{n+1}}\right) = \frac{1}{e^{\lim_{n\to\infty}\left(\frac{n+1}{n}\right)}}$$

$$= \frac{1}{e} \neq 0 \text{ finite.}$$

Then by Comparison test $\sum u_n$ & $\sum v_n$ Converges or diverges together. #

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Now by p-test, $\sum u_n = \sum \frac{1}{n}$ is divergent. (3)

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{n+1}} \text{ is divergent.} \quad \#$$

Question :- Test the convergence of the series;

$$\frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \frac{5^p}{4^q} + \cdots$$

Sol^n We have given

$$\sum_{n=1}^{\infty} \frac{(n+1)^p}{n^q} = \frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \frac{5^p}{4^q} + \cdots$$

Here, $u_n = \frac{(n+1)^p}{n^q} = \frac{1}{n^{q-p}} \left(1+\frac{1}{n}\right)^p$

Let $v_n = \frac{1}{n^{q-p}}$

$$\therefore \lim_{n\to\infty} \left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^p\right) = 1 \neq 0 \text{ (finite)}$$

Hence by Comparison test $\sum u_n$ & $\sum v_n$ converges or diverges together.

Now by p-test $\sum v_n = \sum \frac{1}{n^{q-p}}$ is cgt if $q-p > 1$
dgt if $q-p \leq 1$

$$\Rightarrow \sum u_n = \sum \frac{(n+1)^p}{n^q} \text{ is convergent if } q > p+1$$
divergent if $q \leq p+1$.

\#

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Question:- Test the convergence of the series

$$\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\cdots$$

Solution: We have given

$$\sum_{n=1}^{\infty}\frac{(2n-1)}{n(n+1)(n+2)}=\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\cdots$$

Here, $u_n=\frac{2n-1}{n(n+1)(n+2)}=\frac{1}{n^2}\frac{(2-1/n)}{(1+1/n)(1+2/n)}$

let $v_n=\frac{1}{n^2}$

$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\left(\frac{(2-1/n)}{(1+1/n)(1+2/n)}\right)=2\neq0 \text{ (finite)}$$

Hence by Comparison test, $\sum u_n$ & $\sum v_n$ converges or diverges together.

Now by p-test $\sum v_n=\sum\frac{1}{n^2}$ is Convergent

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(2n-1)}{n(n+1)(n+2)} \text{ is } \underline{\text{Convergent}} \quad \#$$

Question:- Test the convergence of the series

$$\sum_{n=1}^{\infty}\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$$

Solution:- We have given

$$\sum_{n=1}^{\infty}\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$$

Here, $u_n=\sqrt{n^3+1}-\sqrt{n^3}=\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$

$$=\frac{1}{n^{3/2}}\frac{1}{\left(\sqrt{1+1/n^3}+1\right)}$$

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$$v_n = \frac{1}{n^{3/2}}$$

$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{\sqrt{1+1/n^3}+1}\right) = 1 \neq 0 \text{ (finite)}$$

Then by Comparison test, $\sum u_n$ & $\sum v_n$ Converges or diverges together.

Now, by p-test $\sum v_n = \sum \frac{1}{n^{3/2}}$ is convergent $\left(p = \frac{3}{2} > 1\right)$

$$\Rightarrow \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\left(\sqrt{n^3+1} - \sqrt{n^3}\right) \text{ is convergent. } \#$$

Cauchy's Root test :-

An infinite series $\sum u_n$ of positive terms is convergent if $\lim_{n\to\infty} (u_n)^{1/n} < 1$,

divergent if $\lim_{n\to\infty} (u_n)^{1/n} > 1$,

The test fails if $\lim_{n\to\infty} (u_n)^{1/n} = 1$

Question :- Test the Convergence of the series

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$$

Solⁿ Here $u_n = (1-\frac{1}{n})^{n^2}$

$$(u_n)^{1/n} = (1-\frac{1}{n})^n$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} (1-\frac{1}{n})^n = e^{-1} = \frac{1}{e} < 1$$

Hence by Cauchy's root test, $\sum u_n$ is convergent.

\#

Question: Test the convergence of the series

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$$

Solⁿ We have given

$$\sum_{n=1}^{\infty} \left(\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right)^{-n}$$

$$u_n = \left((1+\frac{1}{n})^{n+1} - (1+\frac{1}{n})\right)^{-n}$$

$$(u_n)^{1/n} = \left((1+\frac{1}{n})^{n+1} - (1+\frac{1}{n})\right)^{-1}$$

$$= (1+\frac{1}{n})^{-1}\left((1+\frac{1}{n})^n - 1\right)^{-1}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \left(\frac{1}{e-1}\right) < 1$$

Hence, by Cauchy's test $\sum u_n$ is convergent.

\#

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Question: Test the convergence of the series

$$\frac{1}{2} + \left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots$$

Sol^n Neglecting the I^{st} term, now the given series

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n x^n = \left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots$$

Here, $u_n = \left(\frac{n+1}{n+2}\right)^n x^n$

$$\therefore (u_n)^{1/n} = \left(\frac{n+1}{n+2}\right) x$$

$$\therefore \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1+1/n}{1+2/n}\right) x = x$$

Hence, by Cauchy Root test, $\sum u_n$ is convergent if $x < 1$ and divergent if $x > 1$.

When $x = 1$

$$u_n = \left(\frac{n+1}{n+2}\right)^n$$

$$\lim_{n\to\infty} (u_n) = \lim_{n\to\infty} \frac{(1+1/n)^n}{(1+2/n)^n} = \frac{1}{e} \neq 0$$

Hence $\sum u_n$ is divergent. #
for $x = 1$

Therefore, $\sum u_n$ is convergent if $x < 1$ and divergent if $x \geq 1$.

D'Alembert's Ratio Test -:

An infinite series $\sum_{n=1}^{\infty} u_n$ of positive terms is convergent if

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = k < 1$$

and divergent if

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = k > 1$$

Remark -: In practic, ratio test is applied in the following form;

The series $\sum_{n=1}^{\infty} u_n$ is of positive terms is,

convergent if $\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) > 1$

divergent if $\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) < 1$

and test fails if $\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = 1$

Question -: Test the convergence of the series;

$$\frac{2}{1^2+1} + \frac{2^2}{2^2+1} + \frac{2^3}{3^2+1} + \cdots$$

Soln We have given

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2+1} = \frac{2}{1^2+1} + \frac{2^2}{2^2+1} + \frac{2^3}{3^2+1} + \cdots$$

Here,

$$u_n = \frac{2^n}{n^2+1}$$

$$u_{n+1} = \frac{2^{n+1}}{(n+1)^2+1}.$$

$$\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\left(\frac{2^n}{n^2+1} \cdot \frac{(n^2+2n+2)}{2^{n+1}}\right)$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(\frac{1+2/n+2/n^2}{1+1/n^2}\right)$$

$$= \frac{1}{2} < 1$$

Hence by Ratio test, $\sum u_n$ is divergent.

Question-: Test the convergence of the series;

$$1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots+\frac{x^n}{n^2+1}+\cdots$$

Soln After leaving the first term,

$$\sum_{n=1}^{\infty}\frac{x^n}{n^2+1} = \frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots$$

Here, $u_n = \dfrac{x^n}{n^2+1}$; $u_{n+1} = \dfrac{x^{n+1}}{(n+1)^2+1}$

$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\left(\frac{x^n}{n^2+1} \cdot \frac{n^2+2n+2}{x^{n+1}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n^2+2n+2}{n^2+1}\right)\left(\frac{1}{x}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1+2/n+2/n^2}{1+1/n^2}\right)\left(\frac{1}{x}\right).$$

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$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \frac{1}{x}$$

Hence, by ratio test, $\sum u_n$ is

Convergent if $\frac{1}{x} > 1$, i.e., $x < 1$

divergent if $\frac{1}{x} < 1$, i.e., $x > 1$.

and test fails if $\frac{1}{x} = 1$, i.e., $x = 1$

When $x = 1$ $\quad u_n = \frac{1}{n^2+1}$

$v_n = 1/n^2$

$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{n^2}{n^2+1}\right) = 1 \neq 0$$

finite.

Then by Comparison test $\sum u_n$ & $\sum v_n$ Converges or diverges together.

Now by p-test $\sum v_n = \sum \frac{1}{n^2}$ is Convergent.

$\Rightarrow \sum u_n$ is Convergent when $x = 1$

Hence, $\sum u_n$ is Convergent if $x \leq 1$

and divergent if $x > 1$.

Ans

Question:- Test the Convergence of series;

$1+(n-1)\times 3$ $\qquad \frac{1}{1\cdot2\cdot3} + \frac{x}{4\cdot5\cdot6} + \frac{x^2}{7\cdot8\cdot9} + \cdots$

Sol^n We have given

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{(3n-2)(3n-1)3n} = \frac{1}{1\cdot2\cdot3} + \frac{x}{4\cdot5\cdot6} + \frac{x^2}{7\cdot8\cdot9} + \cdots$$

Here, $$u_n = \frac{x^{n-1}}{(3n-2)(3n-1)(3n)}$$

$$u_{n+1} = \frac{x^n}{(3n+1)(3n+2)(3n+3)}$$

$$\frac{u_n}{u_{n+1}} = \frac{(3n+1)(3n+2)(3n+3)}{(3n-2)(3n-1)(3n)} \left(\frac{1}{x}\right)$$

$$\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\left(\frac{(3+\frac{1}{n})(3+\frac{2}{n})(3+\frac{3}{n})}{(3-\frac{2}{n})(3-\frac{1}{n})3}\right)\frac{1}{x}$$

$$= \frac{3\cdot3\cdot3}{3\cdot3\cdot3}\frac{1}{x} = \frac{1}{x}$$

Hence, by Ratio test, $\sum u_n$ is

Convergent if $\frac{1}{x} > 1$ i.e., $x < 1$

divergent if $\frac{1}{x} < 1$, i.e., $x > 1$

and test fails if $\frac{1}{x} = 1$, i.e., $x = 1$

when $x = 1$

$$u_n = \frac{1}{(3n-2)(3n-1)(3n)}$$

$$u_n = \frac{1}{n^3}\left[\frac{1}{(3-\frac{2}{n})(3-\frac{1}{n})(3)}\right]$$

Let $v_n = \frac{1}{n^3}$

$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty} \frac{1}{3(3-2/n)(3-1/n)} = \frac{1}{27} \neq 0$$
(finite)

Hence, by Comparison test, $\sum u_n$ & $\sum v_n$ converges or diverges together.

Now by p-test $\sum v_n = \sum \frac{1}{n^3}$ is Convergent ($p = 3 > 1$)

$$\Rightarrow \sum u_n = \sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n-1)(3n)}$$ is Convergent for $x = 1$.

Hence $\sum_{n=1}^{\infty} u_n$ is Convergent if $x \leq 1$

divergent if $x > 1$ Ans

An important Comparison test —:

If $\sum u_n$ and $\sum v_n$ are series of positive terms and if form and after some fixed terms,

(a) $\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$ for all values of n

then $\sum u_n$ is Convergent if $\sum v_n$ is Convergent.

(b) $\frac{u_{n+1}}{u_n} > \frac{v_{n+1}}{v_n}$ for all values of n

then $\sum u_n$ will be divergent if

$\sum v_n$ is divergent.

Note that -:

(a) If $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$ and $\sum v_n$ is convergent

then $\sum u_n$ is also convergent.

(b) If $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$ and $\sum v_n$ is divergent

then $\sum u_n$ is also divergent.

Raabe's Test -:

The series $\sum u_n$ of positive terms is convergent if

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1$$

divergent if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1$

This test fails if

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1 .$$

Logarithmic test-:

The series $\sum u_n$ of positive terms is convergent if

$$\lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right) > 1.$$

divergent if $\lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right) < 1.$

This test fails if $\lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right) = 1$

Question-: Test the Convergence of the series;

$$1 + \frac{2}{3}\cdot\frac{1}{4} + \frac{2\cdot 4}{3\cdot 5}\,\frac{1}{6} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\,\frac{1}{8} + \cdots$$

Soln Leaving the first term

$$u_n = \frac{2\cdot 4\cdot 6\cdot 8 \cdots 2n}{3\cdot 5\cdot 7 \cdots (2n+1)}\left(\frac{1}{2n+2}\right)$$

$$u_{n+1} = \frac{2\cdot 4\cdot 6\cdot 8 \cdots 2n\,(2n+2)}{3\cdot 5\cdot 7 \cdots (2n+1)(2n+3)}\,\frac{1}{2n+4}$$

$$\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\left[\left(\frac{2n+3}{2n+2}\right)\frac{(2n+4)}{(2n+2)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(2+3/n)(2+4/n)}{(2+2/n)(2+2/n)}\right] = 1$$

$$\therefore \lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = 1.$$

Hence, Ratio test fails;

Now
$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+3)(2n+4)}{(2n+2)^2}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{4n^2}+\cancel{8n}+6n+12 - (\cancel{4n^2}+\cancel{8n}+4)}{(2n+2)^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{n(6n+8)}{(2n+2)^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(6+8/n)}{(2+2/n)^2}\right] = \frac{3}{2} > 1$$

Hence, by Rabee's test, $\sum u_n$ is Convergent. #

Question-: Test for the Convergence of the series;

$$1+\frac{x}{2}+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\cdots$$

Solution-: Neglating the first terms of $\sum u_n$

$$u_n = \frac{1\cdot 3\cdot 5\cdot 7 --(2n-1)}{2\cdot 4\cdot 6\cdot 8 -- 2n}x^n$$

$$u_{n+1} = \frac{1\cdot 3\cdot 5\cdot 7 --(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdot 8 -- 2n(2n+2)}x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)}{(2n+1)}\left(\frac{1}{x}\right)$$

$$\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\left(\frac{2+2/n}{2+1/n}\right)\frac{1}{x} = \frac{1}{x}$$

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Hence, by Ratio test, $\sum u_n$ is

Convergent if $\frac{1}{x} > 1$, i.e., $x < 1$

divergent if $\frac{1}{x} < 1$, i.e., $x > 1$

and test fails if $\frac{1}{x} = 1$, i.e., $x = 1$.

When $x = 1$,

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(2n+2)}{(2n+1)} - 1 = \frac{1}{(2n+1)}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty}\left(\frac{n}{2n+1}\right) = \frac{1}{2} + 0 < 1$$

Hence by Rabees test $\sum u_n$ is divergent when $x = 1$.

Therefore $\sum u_n$ is Convergent if $x < 1$

and divergent if $x \geq 1$. Ans

Question-: Test the Convergence of the series;

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \frac{5^5x^5}{5!} + \cdots$$

Solution-: we have given

$$\sum_{n=1}^{\infty} \frac{n^n x^n}{n!} = x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots$$

Here, $u_n = \frac{n^n x^n}{n!}$

$$u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^n}{n!}\,\frac{(n+1)!}{(n+1)^{n+1}}\,\frac{1}{x}$$

$$= \frac{n^{n}}{(n+1)^{n}}\,\frac{1}{x} = \frac{1}{(1+1/n)^n}\,\frac{1}{x}$$

$$\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\frac{1}{(1+1/n)^n}\left(\frac{1}{x}\right) = \left(\frac{1}{ex}\right)$$

Hence, by Ratio test, $\sum u_n$ is Convergent

if $\frac{1}{ex} > 1$ i.e., $x < 1/e$

divergent if $\frac{1}{ex} < 1$, i.e., $x > \frac{1}{e}$

and test fails if $\frac{1}{ex} = 1$, i.e., $x = 1/e$.

When $x = 1/e$

$$\frac{u_n}{u_{n+1}} = \frac{e}{(1+1/n)^n}$$

Applying Logarithmic test

$$\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty} n\log\left(\frac{e}{(1+1/n)^n}\right)$$

$$= \lim_{n\to\infty} n\left[\log e - \log(1+1/n)^n\right]$$

$$= \lim_{n\to\infty} n\left[1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots\right)\right]$$

$$= \lim_{n\to\infty} n\left[1 - \left(1 - \frac{1}{2n} + \frac{1}{3n^2} + \cdots\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \cdots\right] = \frac{1}{2} < 1$$

Thus, by Logarthemic test, $\sum u_n$ is divergent when $x = 1/e$.

Hence $\sum u_n$ is convergent if $x < 1/e$

divergent if $x \geq 1/e$ <u>Ans</u>

Done