

UNIT-3 Combinators

Counting Techniques

Basic counting Techniques

- (1) Permutation $n P_r = n! / (n-r)!$
(2) combination $n C_r = n! / r! (n-r)!$

Theorem:

$$n P_n = \frac{1}{m} = n!$$

Theorem:

$$10 = 1$$

Proof We know $n P_n = \frac{n!}{(n-n)!}$

Put $n=10$

$$n P_n = \frac{1}{10} \rightarrow 10 = \frac{1}{10}$$
$$10 = \frac{1}{10}$$
$$10 = 1.$$

$$12 P_9 = ?$$

$$\frac{(12)!}{(12-9)!} = \frac{12!}{8!} = \frac{12 \times 11 \times 10 \times 9}{8!} = \underline{\underline{11880}}$$

$$20 P_9 = 6840 \quad 9!=?$$

$$\frac{20!}{(20-9)!} = 6840$$
$$12! \quad 9!=\underline{\underline{3}}$$

$$\text{m} P_{g_1} = 20 \times \text{m} P_2 \quad m = ?$$

$$\frac{n!}{(n-g_1)!} = 20 \quad \frac{n!}{(n-2)!}$$

$$\frac{(n-2)!}{(n-g_1)!} = 20$$

$$\frac{(n-2)!}{(n-4)!} = 20$$

$$\frac{(n-2)(n-3)(n-4)!}{(n-4)!} = 20$$

$$n^2 - 2n - 3n + 6 - 20 = 0$$

$$n^2 - 5n - 14 = 0$$

$$n^2 - 7n + 2n - 14 = 0$$

From $(n-7)(n+2) = 0$

$$\underline{\underline{n=7}}$$

Prove that $\text{m} P_{g_1} = m-1 P_{g_1} + g_1 \cdot m-1 P_{g_1-1}$

$$\frac{(n-1)!}{(n-1-g_1)!} + g_1 \times \frac{(n-1)!}{(n-g_1-1)!}$$

$$\frac{(n-1)!}{(n-g_1-2)!} \left(\frac{1}{(n-g_1-1)!} + g_1 \right)$$



Date / /
Page No.

Permutation with Repetition

→ Out of 'n' objects 'p' objects are of one kind 'q' objects are of second types and remaining objects are of third kind and number of permutation of n objects taken all at a time $\frac{(p!)^q \cdot q! \cdot n!}{n!}$

→ the number of permutation of 'n' different objects taken 'q' at a time when each objects may be repeated then, the no. of permutation is equal to n^q

Ex: ways to arrange

ways to arrange the letters in

'A L L A H A B A D':
 $n=9$; $p=4$; $q=2$

$$\text{ways} = \frac{9!}{4!2!} = 7560$$

find the no. of different words that can be represented by sequences of 4 digits and three dots.

$$n=7; p=4; q=3$$

$$\frac{7!}{4!3!} = \frac{7 \times 6 \times 5}{3 \times 2} = \underline{\underline{35}}$$

Teacher's Signature

Circular Permutation

The number of circular permutations of n different objects is $(n-1)!$

Theorem: There are $n \cdot (n-1)!$ ways in which n different books can be arranged to form a shelf.

$$\text{arrangement} = \frac{n!}{2}$$

Proof: Fixed the position of one book in $(n-1)!$ ways

These are no restrictions for clockwise anti-clockwise so it is allowed to rotate ways are $\frac{(n-1)!}{2}$.

Color

Color

Such a arrangement

is called vertical coloring.

Next

$$i(1-k) i(1+k-n)$$

$$= \frac{i(n)}{i(k) i(k-n)} = \text{Shif}$$

$$\frac{1-k}{n} = \frac{n}{n-k}$$

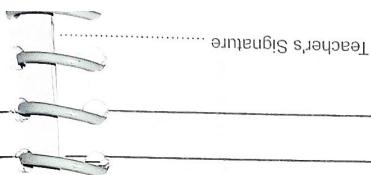
$n \leq k \leq n$: If

$$n = 1$$

$$\frac{\cancel{n}}{n} = \frac{\cancel{n}}{1}$$

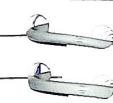
$$\frac{(1+k-n)}{1+n} = \frac{k}{1}$$

$$\frac{k(1+k-n)}{(1+n)} = \frac{i(k-n) n}{1}$$



Teacher's Signature
is formed but due to association it is considered as semicommutation permutation that only a group and word does not matter in combination study a group is made

Differences b/w P & C



defining sum of all is called combination solution for which can be formed by

$$\frac{i(k) i(k-n)}{i(n)}$$

$$\binom{n}{k} n^k$$

COMBINATION

$$\binom{n}{k}$$

Theorem: Let $0 \leq r_1 \leq n$ Prove that $nC_{r_1} = nC_{r_1 - n}$

Theorem:

$$nC_{r_1} + nC_{r_1 - 1} = n+1C_{r_1}$$

$$nC_{r_1} = n+1C_{n-r_1+1} - nC_{n-n+1}$$

$$nC_{r_1} =$$

$$\frac{n!}{r_1!(n-r_1)!} = \frac{(n+1)!}{(n-r_1+1)r_1!} - \frac{n!}{(n-r_1+1)(r_1-1)!}$$

$$1 = \frac{(n+1)}{r_1(r_1+1)} - \frac{1}{(n-r_1+1)!}$$

$$\frac{1}{r_1} = \frac{n+1}{(n-r_1+1)r_1} - \frac{1}{(n-r_1+1)}$$

$$\frac{1}{r_1} = \frac{n+1}{r_1(n-r_1+1)}$$

$$1 = \frac{n+1-r_1}{n+1-r_1} = 1$$

Proved

Theorem: If $1 \leq r_1 \leq n$ then

$$\frac{nC_{r_1}}{nC_{r_1-1}} = \frac{n-r_1+1}{n}$$

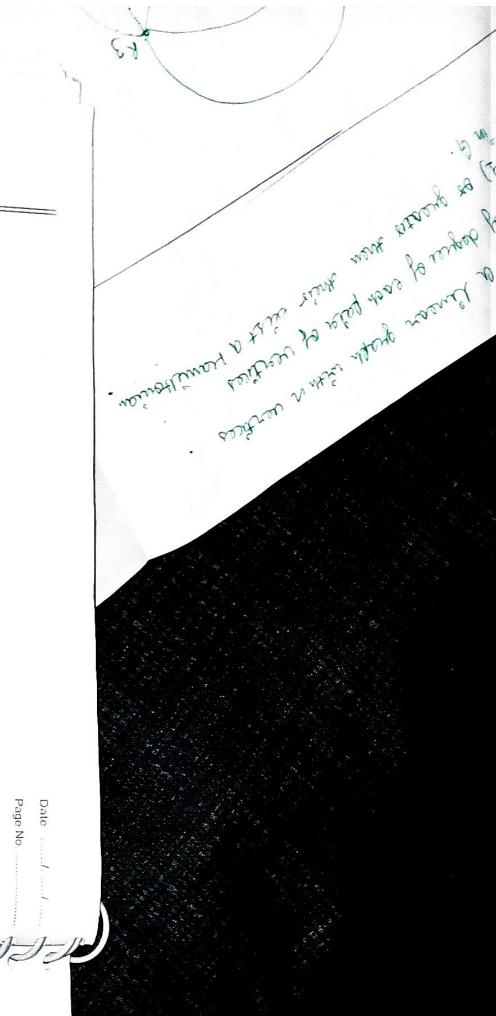
$$LHS = \frac{\frac{n!}{(n-r_1)!r_1!}}{r_1!} =$$

$$= \frac{(n-r_1+1)!}{(n-r_1+1)r_1!}$$

$$LHS = RHS$$

Teacher's Signature

2) If you want to make
any part of each path
then use a boundary
of a larger path within which



Date / /

Page No.

$$\text{Theorem: } {}^nC_q + {}^nC_{q+1} = {}^{n+1}C_{q+1}$$

$$\text{LHS} = \frac{(n-q)!}{(n+1)!} + \frac{1}{n!} = \frac{(n+1)!}{(n+1)(n+1)}$$

$$\text{LHS} = \frac{(q+1+n-q)!}{(n+1)(n+1)} = \frac{(n+1)!}{(n+1)(n+1)}$$

LHS = RHS Proved

$$\text{theorem } {}^nC_p = {}^nC_q \Rightarrow p=q \text{ or } p+q=n$$

$$\begin{aligned} {}^nC_p &= \frac{n!}{p!(n-p)!} \\ &= \frac{n!}{q!(n-q)!} \\ &= \frac{(n-p)!p!}{(n-q)!q!} \end{aligned}$$

$$P=Q$$

for $\boxed{P=Q}$ being equal

$$n-q=p$$

$n-p=p$

Teacher's Signature

Ex

$${}^n P_9 = 720 ; \quad {}^n C_9 = 120 \quad \text{then } n = ?$$

$${}^n P_9 = 9_! \quad {}^n C_9$$

$$720 = 9_! \cdot 120$$

$$9_! = \frac{720}{12} = 6$$

$$9_! = 3_!$$

$$\underline{\underline{9_! = 3_!}}$$

In how many ways can a committee of five members been selected from 6 men and 5 women consisting of 3 men and 2 women

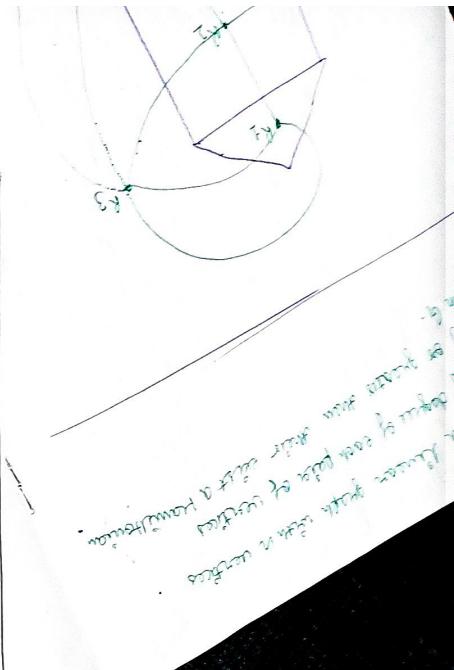
$$\begin{aligned} {}^6 C_3 \times {}^5 C_2 &= \frac{6!}{3!3!} \times \frac{5!}{2!2!} \\ &= \frac{2 \times 6 \times 5 \times 4}{3 \times 2} \times \frac{5 \times 4}{2} \\ &= 10 \times 20 \\ &= \underline{\underline{200}} \end{aligned}$$

Pigeonhole Principle / Principe des tiroirs

↳ Pigeonhole Principle

If n pigeons are assigned to m pigeon holes and $n > m$ then atleast one pigeon holes contains two or more pigeons

Teacher's Signature



Date / /

Page No.

Proof: Let $H_1, H_2, H_3, \dots, H_m$ and pigeons holes and $P_1, P_2, P_3, \dots, P_n$ are in pigeon holes as follows.

Let P_i is assigned to H_1 .

P_1
 P_2
 P_3
 \vdots
 P_m

then there are $(n-m)$ pigeons are left.

At least one pigeonholes will be assigned to a second pigeon.

Ex Show that among 13 people there are at least 2 people who were born in same months.

We consider 13 peoples as pigeons and 12 months as pigeonholes (Jan, feb - dec).

There no of pigeons > no of pigeon holes

By pigeonholes principle.

Teacher's Signature

Principle: We can prove this theorem by contradiction.

$$\text{Pigeons} = \left\lceil \frac{n}{d} \right\rceil \quad (\text{largest integer}) \leq \frac{n}{d} \text{ EA}$$

must contain at least $(m+1)$ pigeons, thus one of the pigeon holes

Theorem: If n pigeons are assigned to m

Extremal / Generalized Pigeon Hole Principle

Assignment Date _____

13

3

than $\frac{n-m}{m}$ pigeons

then, there are almost $m(\frac{n-1}{m})$ pigeons

$$\text{in all but } \frac{n-1}{m} \leq \frac{(n-1)m}{m} \Rightarrow \frac{n-1}{m} < n-1$$

i.e. there are $(n-1)$ regions in all which is contradiction.

Hence one of the pigeon holes must contain atleast $\left(\frac{n-1}{m} + 1\right)$ or $\left(\frac{n+1}{m}\right)$ pigeons.

Show that if any 26 peoples are selected then we may choose a subset of four so that all four were born on the same day of a week.

Soln We assign each person to the ~~26~~ day of the week on which he/she was born. The number of pigeons are to be assigned to seven pigeon holes with $n=26$ and $m=7$ atleast $\frac{n-1}{m} + 1$ or $\frac{n-1}{m}$

$$\begin{aligned} \frac{n-1}{m} + 1 &= \frac{26-1}{7} + 1 = \frac{25+7}{7} \\ &= \frac{32}{7} \\ &= 4.57 \end{aligned}$$

b4

Ex:

If 9 books are to be kept in 4 shelves their must be atleast ~~atleast~~ 1 shelf which contains atleast 3 books

9 books can be considered as pigeon
4 shelves as pigeonholes.

$$n=9$$

$$m=4$$

By generalized pigeonhole principle

$$\left(\frac{m-1}{m} + 1\right) = \frac{9-1}{4} + 1 = 3$$

Atleast 1 shelf will contain 3 books.

RECURRANCE RELATION

A formulae which defines any term in terms of its previous term is called recurrence relation.

n^{th} term a_n of sequence 3, 8, 13, 18-
is $a_n = a_{n-1} + 5$ for $n \geq 2$ and $a_1 = 3$

Differential eqn

Sometimes recurrence relation is called differential eqn as the beginning info a_1 is boundary cond.

The recurrence relation can be written as the even

$$f(x+3h) + 3f(x+2h) + 4f(x+h) + 0 + f(x) = 0$$

$$a_{n+3} + 3a_{n+2} + 4a_{n+1} + 0 = 0$$

$$y_{n+3} + 3y_{n+2} + 4y_{n+1} + 0 = 0$$

$$u_{x+3} + 3u_{x+2} + u_{x+1} + 0 = 0$$

Order of recurrence relation

the difference b/w highest and lowest subscript.

Ex 1 $a_n + 3a_{n-1}$, order = 1

Ex 2 $f(x) + f(x+1) + 5f(x+2) = k(n)$. order = 2

Degree of recurrence relation

Highest power of $f(x)$ or a_n or y_n

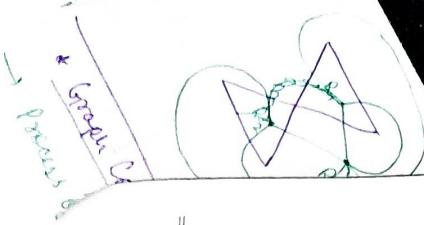
Ex 3 $y_{n+3}^3 + 2y_{n+2}^2 + 2y_{n+1} = 0$ degree = ?

degree = 3

Ex 4 $y_{n+4} + y_{n+1} + y_n = 0$

degree = 1

with vertices
and vertical
vector of a Hamiltonian.
This is a Hamiltonian.



Date:

Page No.

The recurrence relation is called

if it is of degree '1'.
quadratic if it is of degree '2'.

The recurrence relation is called homogeneous
if it contains no terms and depends only on ~~on~~ variables.

Homogeneous

$$a_n = a_{n-2}^2 \quad \text{Homogeneous}$$

order = 2 & degree = 2

$$a_n = a_{n-1} + n^2$$

Non-homogeneous, order = 1,
degree = 1

Linear recurrence relation with const. coefficient

Recurrence relation with degree '1'

general form:-

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = f(n)$$

Teacher's Signature

Linear Homogeneous Recurrence Relation
with const. coeff

If and only if $f(n) = 0$

solⁿ of Homogeneous Recurrence Relation

$$\text{consider } C_0 a_n + C_1 a_{n-1} + \dots + C_K a_{n-K} = 0 \quad (1)$$

solⁿ of (1) is of the form $A\alpha^n$

where α_i is root of characteristic eqⁿ

$$\text{substituting } a_n = A\alpha^n, a_{n-1} = A\alpha^{n-1}$$

$$C_0 A\alpha^{n-1} + C_1 A\alpha^{n-2} + C_2 A\alpha^{n-3} + \dots + C_K A\alpha^{n-K} = 0$$

$$A\alpha^{n-K} (C_0 \alpha^K + C_1 \alpha^{K-1} + C_2 \alpha^{K-2} + \dots + C_K)$$

$$C_0 \alpha^K + C_1 \alpha^{K-1} + C_2 \alpha^{K-2} + \dots + C_K = 0$$

which is the characteristic eqⁿ of recurrence relation

∴ If α_i is a root then the characteristic eqⁿ of Kth degree will give Kth character

CHARACTERISTIC EQUATION

Distinct Roots

If $r_1, r_2, r_3, \dots, r_k$ are distinct roots
then general solⁿ = a_n

$$a_n = b_1 r_1^n + b_2 r_2^n + b_3 r_3^n + \dots + b_k r_k^n$$

where $b_1, b_2, b_3, \dots, b_k$ are constants

Solve $a_n = a_{n-1} + 2a_{n-2}$, $n \geq 2$ with
 $a_0 = 0$ and $a_1 = 1$

$$\text{Given } a_n - a_{n-1} - 2a_{n-2} = 0 \quad \text{--- (1)}$$

A second order homogeneous recurrence solⁿ with constant coeff.

Let $a_n = r_1^n$ is a solⁿ of (1) then

$$(r_1^n - r_1^{n-1} - 2r_1^{n-2} = 0) r_1^{2-n}$$

$$r_1^2 - r_1 - 2 = 0 \Rightarrow (r_1 - 2)(r_1 + 1) = 0$$

$$r_1 = 2, -1$$

General solⁿ is $a_n = b_1 2^n + b_2 (-1)^n$

$$\text{for } n=0, a_0 = b_1 + b_2 \Rightarrow b_1 + b_2 = 0$$

$$\text{for } n=1, a_1 = b_1(2) + b_2(-1) \Rightarrow 2b_1 - b_2 = 1$$

Solving we get $k_1 = \frac{1}{3}$ $k_2 = \frac{1}{3}$

so the solⁿ is

$$a_n = \frac{1}{3} 2^n - \frac{1}{3} (-1)^n$$

Teacher's Signature

Multiple root

If γ_1 is a root of multiplicity m of m^{th} order of a recurrence relation with multiplicity m , then general solⁿ is

$$a_n = (b_1 + nb_2 + n^2 b_3 + \dots + n^{m-1} b_m) \gamma_1^n$$

where $b_1, b_2, b_3, \dots, b_m$ are const.

Ques Solve $a_n = 4(a_{n-1} - a_{n-2})$ with initial condⁿ
 $a_0 = a_1 = 1$

Given $a_n - 4a_{n-1} + 4a_{n-2} = 0$
 Let $a_n = \gamma_1^n$ is a root solⁿ then

$$(\gamma_1^n - 4\gamma_1^{n-1} + 4\gamma_1^{n-2} = 0) \gamma_1^{2-n}$$

$$\gamma_1^2 - 4\gamma_1 + 4 = 0 \Rightarrow \gamma_1 = 2, 2$$

general solⁿ is $a_n = (b_1 + nb_2) 2^n$
 for $n=0$, $a_0 = (b_1 + 0) 2^0 \Rightarrow b_1 = 1$
 $n=1$ $a_1 = (b_1 + b_2) 2 \Rightarrow b_1 + 2b_2 = 1$

$\therefore b_2 = -1$
 Now solⁿ: $a_n = \left(1 - \frac{n}{2}\right) 2^n$

Ans

Non-Homogeneous Recurrence Relation

A second order non-linear homogeneous linear recurrence relation with const coeff is $a_{n-2} + b a_{n-1} + c a_n = f(n)$

The solⁿ of non-homogeneous recurrence relation is

→ sum of particular solⁿ and solⁿ of the associated linear homogeneous recurrence relation

there is no general method of finding particular solⁿ of recurrence relation of function f(x)

this method for finding particular solⁿ is called trial sequence method.

$f(n)$	Trial form
b^n (If b is not a root of characteristic eq ⁿ) ⇒ Polynomial P(n) of degree m	Ab^n $A_0 + A_1 n + A_2 n^2 + \dots + A_m n^m$.
$c^n P(n)$ (If c is not a root of characteristic eq ⁿ)	$c^n (A_0 + A_1 n + A_2 n^2 + \dots + A_m n^m)$

Teacher's Signature

a^n (if b is not a root of characteristic eq a^n of multiplicity)

$A_0 n^m b^n$

$c^n P(n) \cdot$ (if c is a root of characteristic eq a^n of multiplicity)

$n! (A_1 + A_2 n + A_3 n^2 + \dots)$

If funcⁿ of n is const that is polynomial is zero then trial solⁿ is taken as (A)

If funcⁿ of n is linear combination then trial solⁿ is taken as sum of corresponding trial funcⁿ with different unknown constants that are to be determined

Case I:

$$f(n) = \text{const.}$$

(1) $a_n = p$

(2) $a_n = np$ (if ① fails)

(3) $a_n = n^2 p$ (if ② fails)

Case II:

$f(n)$ is polynomial

$$\text{i.e. } f(n) = c_0 + c_1 n + c_2 n^2 + \dots + c_m n^m$$

(1) Put $a_n = d_0 + d_1 n + d_2 n^2 + \dots + d_m n^m$

(2) Find $d_0, d_1, d_2, \dots, d_m$ by comparing coefficients

Case III

$$\text{When } f(x) = P a^n$$

$$a_n = d a^{n-1}$$

Ex solve $a_{n+2} - 5a_{n+1} + 6a_n = 2$ with initial condition $a_0 = 1, a_1 = -1$

Have recurrence relation is

$$a_{n+2} - 5a_{n+1} + 6a_n = 0$$

$$\text{Let } a_n = q_i^n \quad (q_i^{n+2} - 5q_i^{n+1} + 6q_i^n = 0) \quad q_i^n$$
$$\Rightarrow q_i^2 - 5q_i + 6 = 0$$
$$q_i = 3, 2$$

$$\text{sol}^n \text{ is } \stackrel{(P)}{a_n} = C_1 3^n + C_2 2^n$$

for particular solⁿ $a_n^{(P)} = P$ gives the eqⁿ

$$P - 5P + 6 = 2$$

$$\therefore P = 1 \Rightarrow a_n^{(P)}$$

Hence general solⁿ is

$$a_n = C_1 3^n + C_2 2^n + 1$$
$$n=0 \quad a_0 = C_1 3^0 + C_2 2^0 + 1 = C_1 + C_2 + 1 = 1$$
$$C_1 + C_2 = 0$$
$$n=1 \quad a_1 = C_1 3 + C_2 2 + 1 = 3C_1 + 2C_2 + 1 = 1$$
$$3C_1 + 2C_2 = -2$$

$$C_1 = -2 \quad C_2 = 2$$

$$a_n = -2 \cdot 3^n + 2 \cdot 2^n + 1$$

Teacher's Signature

Generating func^n

The generating func^n for the sequence $a_0, a_1, a_2, \dots, a_k, \dots$ of real no is infinite series $G(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k + \dots$

$$= \sum_{k=0}^{\infty} a_k x^k$$

Ex Sequence

$$a_k = 2$$

generating function

$$\sum_{k=0}^{\infty} 2x^k$$

$$a_k = 3^k$$

$$\sum_{k=0}^{\infty} 3^k x^k$$

$$a_k = k+1$$

$$\sum_{k=0}^{\infty} (k+1) (x^k)$$

Some special Cases in generating func^n

General Term of a_k

$$\frac{G(x)}{1-x}$$

(1) Sequence = 1, 1, 1, ...

$$G(x) = 1 + x + x^2 + \dots + (1-x)^{-1} = \frac{1}{1-x}$$

(2) Sequence = 1, 2, 3, ...

$$G(x) = 1 + 2x + 3x^2 + \dots + (k+1)x^k + \dots = \frac{1}{(1-x)^2}$$

$$G(x) = 1 + 2x + 3x^2 + \dots + (k+1)x^k + \dots = \frac{1}{(1-x)^2}$$

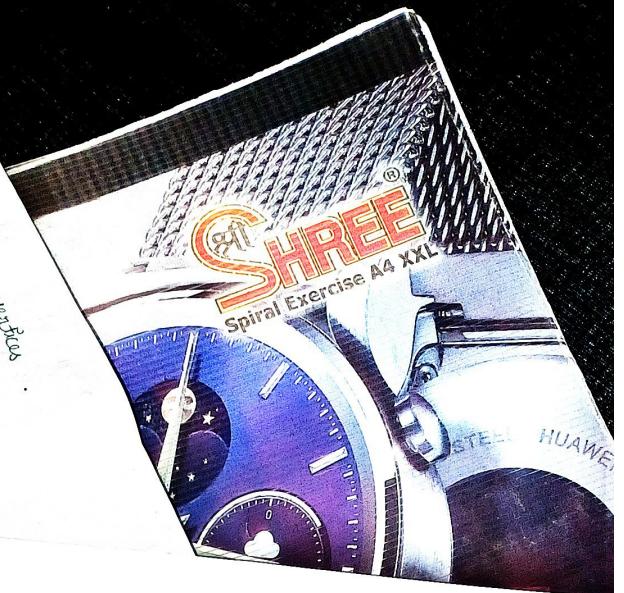
(3) K

$$\frac{x}{(1-x)^2}$$

(4) $K(K+1)$

$$\frac{2x}{(1-x)^3}$$

Teacher's Signature



→
 1. Known quality within surfaces
 2. Known quality but vertical
 3. The ~~size~~ or degree of each portion of a known surface
 4. Known by degrees known
 5. By qualities
 6. By size
 7. By nature
 8. By nature & size

Date / /
Page No.

$$(5) \quad (k+1)(k+2)$$

$$\frac{2}{(1-x)^3}$$

we have
had numbers
and numbers.
~~and~~ numbers
and numbers.

Teacher's Signature