Q If $u = \log_e \sin\left(\dfrac{\pi (2x^2+y^2+xz)^{\frac{1}{2}}}{2(x^2+xy+2yz+z^2)^{1/3}}\right)$

then show that $xu_x + yu_y + zu_z = \dfrac{\pi}{12}$ at $x=0,\ y=1,\ z=2$.

Sol:

$$\Rightarrow \sin^{-1}(e^u) = \frac{\pi (2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}$$

$$\Rightarrow v = \frac{\pi (2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}, \quad v = \sin^{-1}(e^u)$$

here u is homogeneous of degree $1/3$.

$$\Rightarrow x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = \frac{1}{3}v$$

$$\Rightarrow x\frac{\partial}{\partial x}\left(\sin^{-1}(e^u)\right) + y\frac{\partial}{\partial y}\left(\sin^{-1}(e^u)\right) + z\frac{\partial}{\partial z}\left(\sin^{-1}(e^u)\right) = \frac{1}{3}\sin^{-1}(e^u)$$

$$\Rightarrow x\times\frac{1}{\sqrt{1-e^{2u}}}\times e^u\times\frac{\partial u}{\partial x} + y\times\frac{1}{\sqrt{1-e^{2u}}}\times e^u\times\frac{\partial u}{\partial y} + z\times\frac{1}{\sqrt{1-e^{2u}}}\times e^u\times\frac{\partial u}{\partial z}$$

$$= \frac{1}{3}\sin^{-1}(e^u)$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{\sqrt{1-e^{2u}}}{e^u}\times\frac{1}{3}\sin^{-1}(e^u) \quad \text{--- } (1)$$

At $x=0$, $y=1$, $z=2$, we have

$$u = \log_e \sin\left(\frac{\pi}{2}\frac{(2x^2+y^2+xz)^{1/2}}{(x^2+xy+2yz+z^2)^{1/3}}\right)$$

$$= \log_e \sin\left(\frac{\pi}{2}\times\frac{1}{2}\right)$$

$$= \log_e \sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \boxed{e^u = \frac{1}{\sqrt{2}}}$$

Hence by (1), $xu_x + yu_y + zu_z = \dfrac{\sqrt{1-\frac{1}{2}}}{\frac{1}{\sqrt{2}}} \times \dfrac{1}{3}\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$

$$= \frac{\pi}{12} \#.$$

Differentiation of composite functions

Let $u = f(x,y)$ and $x = \phi(t)$, $y = \psi(t)$

Then the total derivative of u w.r.t 't' is defined as

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt}$$

$\rightarrow$ If $f(x,y) = c$, then

$$\frac{dc}{dx} = \frac{\partial f}{\partial x} \times \frac{dx}{dx} + \frac{\partial f}{\partial y} \times \frac{dy}{dx}$$

$$\Rightarrow \quad 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}}$$

Hence if $f(x,y) = c$

then $\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$

$u = \cancel{x^2 + y^2}$

$x = t^2 \sin t$

$f = x^3 y + y^{1/2} x^{t^3} + xy^{2} = 1$

→ Chain rule of partial differentiation

let $z = f(x, y)$, $x = \phi(u, v)$, $y = \psi(u, v)$

then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}$$

Q If $u = f(y-z, z-x, x-y)$, then show

that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol! Let $u = f(r, s, t)$ where $r = y-z$, $s = z-x$, $t = x-y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \times \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \times \frac{\partial t}{\partial x}$$

$$= -u_s + u_t$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \times \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \times \frac{\partial t}{\partial y}$$

$$= u_r - u_t$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \times \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \times \frac{\partial t}{\partial z} = -u_r + u_s$$

$\therefore$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Q If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$

then show that

$$\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = 0$$

Q If $u = f(x^2+2yz,\ y^2+2zx)$, show that

$$(y^2-zx)u_x + (x^2-yz)u_y + (z^2-xy)u_z = 0$$

Q If $f(x,y,z)=0$, then show that $\left(\frac{dy}{dz}\right)_x \left(\frac{dz}{dx}\right)_y \left(\frac{dx}{dy}\right)_z = -1$

Sol

let x is constant, then

$$f(y,z) = x = \text{constant}$$

$$\frac{dy}{dz} = \frac{-f_z}{f_y}$$

If $f(x,y) = z = \text{constant}$

$$\frac{dx}{dy} = -\frac{f_y}{f_x}$$

then $f(z,x) = y = \text{constant}$

$$\frac{dz}{dx} = \frac{-f_x}{f_z}$$

so

$$\left(\frac{dy}{dz}\right)_x \left(\frac{dx}{dy}\right)_z \left(\frac{dz}{dx}\right)_y$$

$$= \frac{-\cancel{f_z}}{\cancel{f_y}} \times \frac{-\cancel{f_y}}{\cancel{f_x}} \times \frac{-\cancel{f_x}}{\cancel{f_z}}$$

$$= -1.$$

Ans.

Q

If $x = r\cos\theta$, $y = r\sin\theta$

then show that

a) $\dfrac{\partial^2 r}{\partial x^2} \cdot \dfrac{\partial^2 r}{\partial y^2} = \left(\dfrac{\partial^2 r}{\partial x \partial y}\right)^2$

b) $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right]$

__Sol__

As $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x^2 + y^2 = r^2$, $\tan\theta = \dfrac{y}{x}$

$\Rightarrow r^2 = x^2 + y^2$

$\Rightarrow 2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ and similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

$\because \quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \frac{1 \times r - x \times \frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - x \times \frac{x}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r^3}$$

Similarly

$$\frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3}$$

Also $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

$$\Rightarrow \frac{\partial^2 r}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{y}{r}\right)$$

$$= y \frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$= y \times \frac{-1}{r^2} \frac{\partial r}{\partial x}$$

$$= \frac{-xy}{r^3}$$

(a)

L.H.S

$$= \frac{\partial^2 r}{\partial x^2} \times \frac{\partial^2 r}{\partial y^2}$$

$$= \frac{r^2 - x^2}{r^3} \times \frac{r^2 - y^2}{r^3}$$

$$= \frac{y^2}{r^3} \times \frac{x^2}{r^3}$$

$$= \frac{x^2 y^2}{r^6}$$

$$\text{R.H.S} = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2 = \left(\frac{-xy}{r^3}\right)^2 = \frac{x^2 y^2}{r^6}$$

$$\therefore \quad \frac{\partial^2 r}{\partial x^2} \times \frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2$$

(ii)

L.H.S

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}$$

$$= \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3}$$

$$= \frac{2r^2 - (x^2 + y^2)}{r^3}$$

$$= \frac{r^2}{r^3} = \frac{1}{r}$$

$$\text{R.H.S} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

$$= \frac{1}{r}\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right] = \frac{1}{r}\left[\frac{x^2+y^2}{r^2}\right] = \frac{1}{r}$$

$$\therefore \frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$$

Q If $u = f(r)$ then show that

$$u_{xx}+u_{yy} = f''(r)+\frac{1}{r}f'(r)$$

Sol: $u = f(r)$, $r^2 = x^2+y^2$ $\left(\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}\right)$

$$\Rightarrow \frac{\partial u}{\partial x} = f'(r)\times\frac{\partial r}{\partial x}$$

$$= f'(r)\times\frac{x}{r}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{f'(r)\,x}{r}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\left\{f''(r)\times\frac{\partial r}{\partial x}\times x + f'(r)\right\}r - f'(r)\times x\times\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{\left\{f''(r)\times\frac{x^2}{r} + f'(r)\right\}r - f'(r)\times x\times\frac{x}{r}}{r^2}$$

$$= \frac{f''(r)\times x^2}{r^2} + \frac{f'(r)}{r} - \frac{f'(r)x^2}{r^3}$$

$$\boxed{u_{xx} = \frac{f''(r)\times x^2}{r^2} + \frac{f'(r)\left\{r^2 - x^2\right\}}{r^3}}$$

$r^2 = x^2 + y^2 + z^2$

Similarly $u_{yy} = \frac{f''(r)\, y^2}{r^2} + \frac{f'(r)}{r^3}\left[r^2 - y^2\right]$.

on addition we set

$$u_{xx} + u_{yy} = \frac{f''(r)}{r^2}\{x^2 + y^2\} + \frac{f'(r)}{r^3}\{r^2 - x^2 + r^2 - y^2\}$$

$$= f''(r) + \frac{f'(r)}{r^3} \times r^2$$

$$= f''(r) + \frac{f'(r)}{r} \quad \#1$$

Q If $f(x,y)=0$, $\phi(y,z)=0$, then

show that

$$\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z}\frac{dz}{dx}=\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}$$

Sol' As $\frac{dz}{dx}=\left(\frac{dz}{dy}\right)\left(\frac{dy}{dx}\right)$

Now $f(x,y)=0 \Rightarrow \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}$

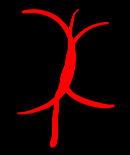
and $\phi(y,z)=0 \Rightarrow \frac{dz}{dy}=-\frac{\partial \phi/\partial y}{\partial \phi/\partial z}$

$\therefore$ $\frac{dz}{dx}=\frac{-\partial \phi/\partial y}{\partial \phi/\partial z}\times\frac{-\partial f/\partial x}{\partial f/\partial y}$

$\Rightarrow \frac{\partial \phi}{\partial z}\times\frac{\partial f}{\partial y}\frac{dz}{dx}=\frac{\partial \phi}{\partial y}\times\frac{\partial f}{\partial x}$

#.

Q If the curve $f(x,y)=0$, $\phi(x,y)=0$ touch each other



then show that $\dfrac{\partial f}{\partial x}\dfrac{\partial \phi}{\partial y}-\dfrac{\partial f}{\partial y}\dfrac{\partial \phi}{\partial x}=0.$

Sol: As $f(x,y)=0$, $\phi(x,y)=0$ touch each other

$\Rightarrow \dfrac{dy}{dx}=\dfrac{-\partial f/\partial x}{\partial f/\partial y}$ and $\dfrac{dy}{dx}=\dfrac{-\partial \phi/\partial x}{\partial \phi/\partial y}$ must be equal

$$\therefore \quad \frac{-\partial f/\partial x}{\partial f/\partial y}=\frac{-\partial \phi/\partial x}{\partial \phi/\partial y}$$

$$\Rightarrow \frac{\partial f}{\partial x}\times\frac{\partial \phi}{\partial y}=\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x} \quad \text{hs.}$$

Q ① If $z = f(x, y)$

$x = e^u \cos v, \quad y = e^u \sin v$

show that $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right]$

② If $x = e^r \cos\theta, \quad y = e^r \sin\theta$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\right\}$$

where $u = f(x, y)$

$z = f(x, y)$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial u}$$

$u = f(x, y)$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

Sol:

(2) $x = e^r \cos\theta, \quad y = e^r \sin\theta$

$u = f(x, y)$

$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial r}$

$= \frac{\partial u}{\partial x} \times e^r \cos\theta + \frac{\partial u}{\partial y} \times e^r \sin\theta$

$= x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$

$\Rightarrow \boxed{\frac{\partial}{\partial r} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}}$

Now $\frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right) = \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)$

$= x \frac{\partial}{\partial x}\left(x \frac{\partial}{\partial x}\right) + x \frac{\partial}{\partial x}\left(y \frac{\partial}{\partial y}\right) + y \frac{\partial}{\partial y}\left(x \frac{\partial}{\partial x}\right) + y \frac{\partial}{\partial y}\left(y \frac{\partial}{\partial y}\right)$

$= x\left\{\frac{\partial}{\partial x} + x \times \frac{\partial^2}{\partial x^2}\right\} + 2xy \frac{\partial^2}{\partial x \partial y} + y\left\{\frac{\partial}{\partial y} + y \frac{\partial^2}{\partial y^2}\right\}$

$$= x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$\therefore \frac{\partial^2 u}{\partial r^2} = x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \quad \text{— (1)}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\times\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\times\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}\times(-e^r\sin\theta) + \frac{\partial u}{\partial y}\times(e^r\cos\theta)$$

$$= -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial \theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$\boxed{-y\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right) = -y\left\{\frac{\partial}{\partial y} + x\frac{\partial^2}{\partial x\partial y}\right\}}$$

$$\frac{\partial^2}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial \theta}\right)$$

$$= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)$$

$$= \cancel{y^2\frac{\partial^2}{\partial x^2} - 2xy\frac{\partial^2}{\partial x\partial y} + x^2\frac{\partial^2}{\partial y^2}}$$

$$= y^2\frac{\partial^2}{\partial x^2} - y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x} - 2xy\frac{\partial^2}{\partial x\partial y} + x^2\frac{\partial^2}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial\theta^2} = y^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x\partial y} + x^2\frac{\partial^2 u}{\partial y^2} - x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} \quad \text{—} (2)$$

(1) + (11) $\Rightarrow$

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial\theta^2} = (x^2+y^2)\frac{\partial^2 u}{\partial x^2} + (x^2+y^2)\frac{\partial^2 u}{\partial y^2}$$

$$\cancel{+\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)}$$

$x = e^r\cos\theta,\quad y = e^r\sin\theta$

$\Rightarrow x^2+y^2 = e^{2r}$

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial\theta^2} = e^{2r}\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial\theta^2}\right\}$$

pro.

Q. If $w = \sqrt{x^2+y^2+z^2}$ $\left\{ w = f(x,y,z) \right.$

$x = u\cos v$, $y = u\sin v$, $z = uv$

then $u\dfrac{\partial w}{\partial u} - v\dfrac{\partial w}{\partial v} = \dfrac{u}{\sqrt{1+v^2}}$

Sol: $\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x} \times \dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y} \times \dfrac{\partial y}{\partial u} + \dfrac{\partial w}{\partial z} \times \dfrac{\partial z}{\partial u}$

$= \dfrac{x}{w} \times 0 + \dfrac{y}{w} \times \sin v + \dfrac{z}{w} \times v$

$= \dfrac{y\sin v}{w} + \dfrac{zv}{w}$

$\Rightarrow u\dfrac{\partial w}{\partial u} = \dfrac{uy\sin v}{w} + \dfrac{zuv}{w}$

Taylor's & Maclaurin's series

let $f(x)$ is the function of x having derivative of different order then

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

This is known as Taylor's series.

If $h \to x-a$, $x \to a$

then $$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) \cdots$$

This is the Taylor's series in terms of "$x-a$" or we call it as Taylor series at $x=a$.

If $a=0$, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$$

or $$y = y_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots$$

is known as Maclaurin's series in powers of 'x'.

Q Find the expansion of

(a) $f(x) = \sin x$

(b) $f(x) = \cos x$

(c) $f(x) = \tan^{-1}(x)$

(d) $f(x) = \log(1+x)$

(e) $f(x) = e^{a\sin^{-1}x}$ including its general term.

$(y_n)_0 = -((n-2)^2+a^2)(y_{n-2})_0$

$y = e^{a\sin^{-1}x}$

$y_1 = \frac{a}{\sqrt{1-x^2}}\, y$

(c) let $y = \tan^{-1}x$ — (i) $(y)_0 = 0$

$y_1 = \frac{1}{1+x^2}$ — (ii)

$\Rightarrow (1+x^2)y_1 = 1 \rightarrow (y_1)_0 = 1$

$(1+x^2)y_2 + 2xy_1 = 0$ — (iii)

$\Rightarrow (y_2)_0 = 0$

$(1+x^2)y_3 + 2xy_2 + 2y_1 + 2xy_2 = 0$ — (iv)

$(y_3)_0 = -2$ and so on.

$y = (y_0) + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \cdots$

$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ Ans

Q. Expand $\sin x$ in powers of $x - \frac{\pi}{4}$

Sol: $f(x) = \sin x \Rightarrow f(\pi/4) = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x \Rightarrow f'(\pi/4) = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x \Rightarrow f''(\pi/4) = -1/\sqrt{2}$

$f'''(x) = -\cos x \Rightarrow f'''(\pi/4) = -1/\sqrt{2}$ and so on.

Now we have

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) - - -$$

$$\sin x = \frac{1}{\sqrt{2}} + \left(x - \frac{\pi}{4}\right) \times \frac{1}{\sqrt{2}} + \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 \times -\frac{1}{\sqrt{2}} + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 \times \left(\frac{-1}{\sqrt{2}}\right) \cdots\cdots$$

Q. expand $\tan(x+h)$ in powers of 'x'.

sol: As

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) - - - -$$

$$\Rightarrow \tan(x+h) = f(x+h)$$

$$\Rightarrow f(x) = \tan x$$

$$\Rightarrow f'(x) = \sec^2 x$$

$$f''(x) = 2\sec^2 x \tan x$$ and so on.

$$\therefore f(x+h) = \tan h + x(\sec^2 h) + \frac{x^2}{2!}(2\sec^2 h \tan h) - - - -$$

or

$$\tan(x+h) = \tan h + x(\sec^2 h) + x^2(\sec^2 h \tan h) - - - - -$$

Taylor's series for function of two variable

Let $z = f(x, y)$ be the function of two variable and suppose that z have continuous partial derivatives w.r.t 'x' and 'y' then

$$f(x+h, y+k) = f(x, y) + \left\{ h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right\} + \frac{1}{2!}\left\{ h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + h^2\frac{\partial^2 f}{\partial y^2} \right\}$$

$$+ \cdots\cdots$$

or

$$f(x+h, y+h) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right) f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^2 f$$

$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^3 f + \cdots$$

This series is know as Taylor series of $f(x,y)$ at (x,y).

If $x = a$, $h \to x-a$, $y \to b$, $k \to y-b$, then Taylor series at (a,b) is given by

$$\boxed{f(x,y) = e^x \cos y}$$

$$f(x,y) = f(a,b) + \left\{ (x-a)\left(\frac{\partial f}{\partial x}\right)_{(a,b)} + (y-b)\left(\frac{\partial f}{\partial y}\right)_{(a,b)} \right\} + \frac{1}{2!}\left\{ (x-a)^2 \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)} + 2(x-a)(y-b)\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)} + (y-b)^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} \right\} + \cdots$$

If $(a,b) = (0,0)$ then the above series is known as Maclaurin's series

$$f(x,y) = f(0,0) + \left\{ x\left(\frac{\partial f}{\partial x}\right)_{(0,0)} + y\left(\frac{\partial f}{\partial y}\right)_{(0,0)} \right\} + \frac{1}{2!}\left\{ x^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} + 2xy\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} \right\} + \cdots$$

Q Expand $e^x \log(1+y)$ in Taylor series in the neighbourhood of $(0,0)$.

Sol:
$$f(x,y) = e^x \log(1+y) \Rightarrow f(0,0) = 0$$
$$\Rightarrow f_x = e^x \log(1+y) \Rightarrow f_x(0,0) = 0$$
$$\Rightarrow f_{xx} = e^x \log(1+y) \Rightarrow (f_{xx})_{(0,0)} = 0$$
$$\Rightarrow f_{xy} = \frac{e^x}{1+y} \Rightarrow (f_{xy})_{(0,0)} = 1$$
$$f_y = \frac{e^x}{1+y} \Rightarrow (f_y)_{(0,0)} = 1$$
$$f_{yy} = \frac{-e^x}{(1+y)^2} \Rightarrow (f_{yy})_{(0,0)} = -1 \quad \text{and so on.}$$

$$\therefore f(x,y) = f(0,0) + \left\{ x(f_x)_{(0,0)} + y(f_y)_{(0,0)} \right\} + \frac{1}{2!}\left\{ x^2 (f_{xx})_{(0,0)} + 2xy(f_{xy})_{(0,0)} + y^2 (f_{yy})_{(0,0)} \right\} + \cdots$$

$$\boxed{e^x \log(1+y) = y + xy - \frac{y^2}{2} \cdots\cdots}$$

Expand y^x about $(1,1)$ and hence evaluate $(1.02)^{(1.03)}$

Sol: $f(x,y) = y^x \Rightarrow f(1,1) = 1$

$f_x = y^x \log y \Rightarrow f_x(1,1) = 0$

$f_{xx} = y^x (\log y)^2 \Rightarrow f_{xx}(1,1) = 0$

$f_{xy} = x y^{x-1} \log y + y^{x-1} \Rightarrow (f_{xy})_{(1,1)} = 1$

$f_y = x y^{x-1} \Rightarrow (f_y)_{(1,1)} = 1$

$f_{yy} = x(x-1) y^{x-2} \Rightarrow (f_{yy})_{(1,1)} = 0$ and so on.

$$f(x,y) = f(1,1) + \{(x-1)(f_x)_{(1,1)} + (y-1)(f_y)_{(1,1)}\} + \frac{1}{2!}\{(x-1)^2 (f_{xx})_{(1,1)} + 2(x-1)(y-1)(f_{xy})_{(1,1)} + (y-1)^2 (f_{yy})_{(1,1)}\} + \cdots$$

$y^x = 1 + (y-1) + (x-1)(y-1) \cdots\cdots$

let $y = 1.02$, $x = 1.03$

$(1.02)^{(1.03)} \simeq 1 + .02 + 0.0006$

$\simeq 1.0206$

Jacobian

Defn: Let u and v are functions of two independent variables x and y then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is known as Jacobian of u and v w.r.t x and y and we write it as

$$\boxed{J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}}$$

If $u = f(x,y,z)$, $v = \phi(x,y,z)$, $w = \psi(x,y,z)$ then

$$J(u,v,w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties

① If u, v are functions of x, y and x, y are functions of r, s then

$$\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,s)}$$

$u = f(x,y)$, $x \to \phi(r,s)$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

Pf: R.H.S $= \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,s)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix}$$

$$= \frac{\partial(u,v)}{\partial(r,s)} = \text{L.H.S}$$

(2) If u, v is function of x, y and $J = \frac{\partial(u,v)}{\partial(x,y)}$

$J' = \frac{\partial(x,y)}{\partial(u,v)}$, then

$JJ' = 1$

sol: $JJ' = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$