Cauchy root test:- In a positive term series $\sum a_n$ if $\lim_{n\to\infty} (a_n)^{1/n} = \lambda$, then the series converges for $\lambda < 1$ & diverges for $\lambda > 1$. test fails when $\lambda = 1$.

Example $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1$$

$\Rightarrow \sum a_n$ converges.

example Discuss the nature of the series.

(i) $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \dots \infty$, $(x>0)$

$$a_{n+1} = \left(\frac{n+1}{n+2}\right)^{n} x^{n}$$

$$(a_{n+1})^{1/n} = \left(\frac{n+1}{n+2}\right) \cdot x$$

$$\lim_{n\to\infty} (a_{n+1})^{1/n} = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1+2/n}\right) \cdot x = x$$

By cauchy root test this series converges for $x<1$ diverges for $x>1$

when $x=1$,

$$a_{n+1} = \left(\frac{n+1}{n+2}\right)^n = \frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}} \cdot \left(1+\frac{1}{n+1}\right)$$

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{e} \neq 0$$

$\Rightarrow \sum a_n$ diverges.

(iii) $\sum \frac{(n+1)^n x^n}{n^{n+1}}$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{(n+1)x}{n^{1+1/n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\cdot\left(\frac{1}{n^{1/n}}\right)\cdot x$$

$$= 1\cdot 1\cdot x$$
$$= x$$

$\Rightarrow \sum a_n$ converges for $x<1$, diverges for $x>1$

at $x=1$

$$\sum a_n = \sum \frac{(n+1)^n}{n^{n+1}}$$

$$a_n = \left(\frac{n+1}{n}\right)^n \cdot \frac{1}{n}$$

$$b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$$

$\Rightarrow \sum a_n$ diverges as $\sum b_n$ diverges.

(iii) Alternating series:- A series in which the terms are alternatively +ve and -ve is called an alternating series.

Leibnitz's series:- An alternating series $a_1 - a_2 + a_3 - a_4 + \dots$ converges if (i) each term is numerically less than its preceding term and (ii) $\lim_{n\to\infty} a_n = 0$

If $\lim_{n\to\infty} a_n \neq 0$ then series is oscillating.

ex (ii) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

Since

$$a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{n-n-1}{n(n+1)} = \frac{-1}{n(n+1)} < 0$$

$\Rightarrow a_{n+1} < a_n \quad \forall n$

& $\lim_{n\to\infty} \frac{1}{n} = 0$

$\Rightarrow \sum \frac{1}{n}(-1)^{n-1}$ is convergent. (using Leibnitz series)

Absolute convergence and conditional convergence

Absolute convergente series :- Let $\sum_{n=1}^{\infty}$ be a series of real numbers, then $\sum_{n=1}^{\infty} u_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |u_n|$ converges.

Conditional convergent series :- If $\sum_{n=0}^{\infty} u_n$ be a series of real numbers, then $\sum_{n=1}^{\infty} u_n$ is said to conditional convergent if the series $\sum u_n$ converges, but $\sum_{n=1}^{\infty} |u_n|$ diverges.

ex (i) $\sum (-1)^{n-1} \cdot \frac{1}{n} \rightarrow$ conditional convergent

(ii) $\sum (-1)^{n-1} \cdot \frac{1}{n^2} \longrightarrow$ Absolutely convergent.

ex Examine the convergance of the series.

$\sum \frac{(-1)^{n-1} x^n}{n(n-1)}$, $0<x<1$

$$a_n - a_{n-1} = \frac{x^n}{n(n-1)} - \frac{x^{n-1}}{(n-1)(n-2)} = \frac{x^{n-1}[x(n-2) - n]}{n(n-1)(n-2)}$$

$< 0 \quad \forall n \geq 2 \quad (\because 0<x<1$

& $\lim_{n\to\infty} \frac{x^n}{n(n-1)} = 0$

$\Rightarrow \Sigma a_n$ is convergent.

Ex.

$$\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \cdots$$

$$a_n = (-1)^{n-1} \cdot \frac{1+2+3+\cdots+n}{(n+1)^3} = (-1)^{n-1} \cdot \frac{n(n+1)}{2(n+1)^3}$$

$$= (-1)^{n-1} \cdot \frac{n}{2(n+1)^2}$$

$$a_n - a_{n+1} = \frac{n}{2(n+1)^2} - \frac{(n+1)}{2(n+2)^2}$$

$$= \frac{1}{2}\left[\frac{n(n+2)^2 - (n+1)^3}{(n+1)^2(n+2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{n(n^2+4n+4) - (n^3+3n^2+3n+1)}{(n+1)^2(n+2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{n^2+n-1}{(n+1)^2(n+2)^2}\right] > 0$$

$\Rightarrow a_{n+1} < a_n$

$$\lim_{n\to\infty} \frac{n}{2(n+1)^2} = 0$$

$\Rightarrow \Sigma a_n$ converges.

Now $\Sigma |a_n| = \Sigma \left(\frac{1+2+3+\cdots+n}{(n+1)^3}\right) = \Sigma \frac{n(n+1)}{2(n+1)^3} = \Sigma \frac{n}{2(n+1)^2}$

$|a_n| = \frac{n}{2(n+1)^2}$, $\quad \frac{|a_n|}{|b_n|} = \frac{n^2}{2(n+1)^2} = \frac{1}{2} \neq 0$

$b_n = \frac{1}{n}$, $\quad \Rightarrow \Sigma |a_n|$ is diverg

$\Rightarrow \Sigma a_n$ is conditionaly convergent