UNIT-2 **MATRICES**

Rank of the matrix:

Let A be any matrix then rank of A is the order of highest non-vanishing minor of matrix. The rank of the matrix is denoted by $\rho(A)$ and if $\rho(A) = r$ then we say that there is at least one minor of order r which is non-vanishing and others of order greater than r are zero.

Note: (i) $\rho(A) = \rho(A')$

(ii) $\rho(A \cdot B) \leq \rho(A)$ or $\rho(B)$

(iii) $\rho(\text{null matrix}) = 0$

Eg: ① $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ As $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\rho(A) = 2$

② $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2-1 = 1 \neq 0$ $\rho(A) = 2$

$|A| = 0$

$\Rightarrow \rho(A) < 3$

③ $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

$|A| = 0$

$\Rightarrow \rho(A) < 3$

$\rho(A) \neq 2$ because all are zero

Hence $\rho(A) = 1$

Elementary Operations:

The following operation are said to be elementary operations →

(i) Interchange of 2 rows/columns.

$R_{ij} \leftrightarrow R_{ji}$ $(C_{ij} \leftrightarrow C_{ji})$

(ii) Scalar multiplication by constant to any row/column.

$R_i \rightarrow kR_i$ $(C_i \rightarrow kC_i)$

(iii) Addition of k-times any row (column) to other row (column)

$$R_i \to kR_j + R_i \quad (C_i \to kC_j + C_i)$$

Equivalent Matrix: Two matrix A and B are said to be equivalent if they can be obtained from each other by elementary operations
And we write $A \sim B$.

Echleon Form: A matrix A is said to be in Echleon form if
(a) First element of first row should be ~~non~~ unity (1)
(b) The no. of zero before non-zero element in the 2nd, 3rd, higher rows should be in increasing order.

If there are 'r' non-zero rows then rank of matrix is r
$\boxed{\rho(A) = r.}$

Here, we will use e-row operation to reduce a matrix in echleon form (upper triangular matrix)

Eg: ① $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ Echleon form ✓ $\rho(A) = 2$

② $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Echleon ✓ $\rho(A) = 3$

Normal form of a matrix: A matrix A is said to be in normal form if this is of the type.

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

When I_r is identity matrix of order 'r'. As we have r non-zero rows, $\rho(A) = r$.

<u>Note</u>: If A is any matrix then there exists 2 non-singular matrix P & Q such that

$$PAQ = \text{Normal Form}$$

If A is a square matrix of order r

then $PAQ = I_{r \times r}$

$AQ = P^{-1}$

$A = P^{-1}Q^{-1}$

$A^{-1} = (P^{-1}Q^{-1})^{-1}$

$\boxed{A^{-1} = QP}$

Gauss Jordan Method: (Inverse of matrix)

Let A be any square matrix then

$A = AI$

Now, after e-row operation

$I = AB$

$\boxed{A^{-1} = B}$

Q. Find rank of –

① $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

③ $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Ans: ① Transform to Echleon form

$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_3 \to R_3 - 3R_1$

$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_4 \to R_4 - R_1$

$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - R_2$

$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$

(2) $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$\cancel{R_3 \to R_3 - 2R_3}$

$\cancel{A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 4 & 5 & -1 \\ 6 & 8 & 7 & 5 \end{bmatrix}}$

$R_3 \to R_3 - 3R_1$

$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_4 \to R_4 - 6R_1$

$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$

$R_4 \to R_4 - R_2$

$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -8 & 3 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \end{bmatrix}$

$R_4 \to R_2 - R_4$

$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\rho(A) = 3$

(3) $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

$R_2 \to \frac{1}{3} R_2$

$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 1 & 3 & 4 & 1 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$R_2 \to R_1 - R_2$

$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\rho(A) = 2$

Q. Reduce the following matrix in normal form & hence find its rank.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

Ans: $$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$R_2 \to R_2 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - 2C_1$

$C_3 \to C_3 - 3C_1$

$C_2 \to C_2 - 2C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$C_4 \to C_4 - 3C_2$

$R_2 \to (-1)R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$$

$\rho(A) = 2$

Q

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

~~$R_1 \to R_1 - R_2$~~

$$A = \begin{bmatrix} 1 & 4 & 1 & 3 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_3 \to R_2 + R_3$

$R_4 \to R_4 - R_3$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_3 - R_2$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & -1 & 6 & 3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_1 + C_2$

$C_3 \to C_3 + 2C_1$

$C_4 \to C_4 + 4C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 6 & 3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 6 & 3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - 2C_4$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 4 & -11 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 + 3C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & -11 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + C_4$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & 1 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to (-1)C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\cancel{C_4 \to C_4 + 3C_2}$

$C_2 \to C_2 + \frac{1}{3}C_4$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - 12C_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{c|c} I_3 & 0 \\ \hline 0 & 0 \end{array}\right]$$

$\boxed{\rho(A) = 3}$

Q. For $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

find two non-singular matrix P & Q such that

PAQ = normal form.

and if A is non-singular, verify that $A^{-1} = QP$

Q. Find inverse of matrix using Gauss – Jordan method:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$A = A I$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = A \cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 6 & -3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

$R_3 \to \frac{1}{2} R_3$

$R_1 \to R_1 + R_3$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$I = A B$$

$$A^{-1} = B =$$

Q. Find inverse of matrix using Gauss-Jordan Method.

$$A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 6 \end{bmatrix}$$

$$A = AI$$

$$\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 6 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 1 & 3 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 6 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 3 & 8 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & -2 & -5 \\ 0 & 2 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 3 & 8 \end{bmatrix} = A \begin{bmatrix} 0 & 4 & -3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 3 & 8 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 8 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

$R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & -3 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 0 & 4 & -3 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -2 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 0 & 4 & -3 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -2 & 2 & 0 \\ 1 & 1 & -2 & 1 \end{bmatrix}$$

$R_1 \to R_1 + R_4$

$$\begin{bmatrix} 1 & 5 & -5 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & -2 & 2 & 0 \\ 1 & 1 & -2 & 1 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$\begin{bmatrix} 1 & 5 & -5 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & -2 & 2 & 0 \\ 3 & 5 & -6 & 1 \end{bmatrix}$$

$R_4 \to (-1) R_4$

$R_4 \to R_4 - 2R_3$
$R_4 \to (-1) R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 5 & -5 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & -2 & 2 & 0 \\ -3 & -5 & 6 & -1 \end{bmatrix}$$

Q. Show that $(x_1, y_1)(x_2, y_2)(x_3, y_3)$ are coplanar if $\rho(A) < 3$ where

$$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_1 & 1 \end{bmatrix}$$

A. As $\rho(A) \neq 3$

$|A| = 0$

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

$$\frac{1}{2}\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

Hence, area of triangle is 0 where vertices are $(x_1, y_1)(x_2, y_2)(x_3, y_3)$

Hence, the points are coplan.

Q. Find the value of x in the following matrix which $\rho(A) = 3$ where

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & x \end{bmatrix}$$

Ans As $\rho(A) = 3$ $\boxed{x \neq 1}$

$|A| \neq 0$

$$\begin{bmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & x \end{bmatrix} \neq 0$$

$2(x) - 4(2x-2) + 2(-1) \neq 0$

$2x - 8x + 8 - 2 \neq 0$

$-6x + 6 \neq 0$

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Solution of system of linear eq^n.

Consider the non-homogeneous system of equation —

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots a_{2n}x_n = b_2 \\ \text{— — — — — — —} \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots a_{mn}x_n = b_m \end{array}\right\} \quad (1)$$

Eq^n (1) can be written as

$$\boxed{AX = B}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

If A is non-singular

$$\boxed{X = A^{-1}B}$$

Consider the augmented matrix

$$[A, B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdot & - & - & - & \cdots \\ \cdot & - & - & - & \cdots \\ \cdot & - & - & - & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

System of eq^n (1) is said to be consistent if $\rho(A, B) = \rho(A)$

(i) If $\rho(A, B) = \rho(A) =$ No. of unknowns then we will have a unique sol^n.

(ii) If $\rho(A, B) = \rho(A) = r <$ No. of variables (n) then we will have infinite sol^n and $(n - r)$ variables have to assign a constant value.

Consider the homogeneous system of eq^n –

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = 0 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = 0 \end{array}\right\} (2)$$

Equation (2) can be written as –

$$\boxed{AX = B}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Consider the coeffecient matrix

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Now, we have a

(i) unique sol^n (zero sol^n or trivial sol^n)
if $\rho(A)$ = Number of variables.

(ii) infinite sol^n (non-trivial sol^n) if
$\rho(A) = r <$ the number of variables (n)
and we will assign $(n-r)$ variables a constant value.

Q. Examine the consistency of the following system of eq^n –

$$x_1 + 2x_2 - x_3 = 3$$
$$3x_1 - x_2 + 2x_2 = 1$$
$$x_1 - x_2 + x_3 = -1$$

Sol^n This system can be written as –

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

Augmented matrix is

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -3 & 2 & -4 \end{array}\right]$$

$R_3 \to 7R_3 - 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & -1 & -4 \end{array}\right]$$

$\therefore \rho(A, B) = \rho(A) = 3 =$ number of unknowns

Hence, We have a unique solⁿ.

Solⁿ is given by $\boxed{AX = B}$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ -4 \end{bmatrix}$$

$x_1 + 2x_2 - x_3 = 3$

$-7x_2 + 5x_3 = -8$

$-x_3 = -4$

$\boxed{x_3 = 4}$

$-7x_2 + 20 = -8$

$\boxed{x_2 = 4}$

$\cancel{4 + 8 =}$

$x_1 + 8 - 4 = 3$

$\boxed{x_1 = -1}$

$$\boxed{\begin{aligned} x_1 &= -1 \\ x_2 &= 4 \\ x_3 &= 4 \end{aligned}}$$

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Q. Examine the consistency of following system of linear eq^n.

$$2x + 6y = -11$$
$$6x + 20y - 6z = -3$$
$$6y - 18z = -1$$

$\boxed{AX = B}$

This system of eq^n can be written as —

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$$

Augmented Matrix —

$$[A, B] = \left[\begin{array}{ccc|c} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{array}\right]$$

$$R_2 \to R_2 - 3R_1$$

$$\left[\begin{array}{ccc|c} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 6 & -18 & -1 \end{array}\right]$$

$$R_3 \to R_3 - 3R_2$$

$$\left[\begin{array}{ccc|c} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 0 & 0 & -91 \end{array}\right]$$

$$\cancel{R_3 \to R_3 - 3R_2}$$

$$\cancel{\left[\begin{array}{ccc|c} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & -6 & 18 & \end{array}\right]}$$

$\therefore \rho(A) = 2 \neq \rho(A, B) = 3 \Rightarrow$ system of eq^n is inconsistent

Hence, we don't have any sol^n

Q. For what value of λ & μ the system of eq^n

$$x+y+z=\lambda$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

has a (i) unique sol^n
(ii) No sol^n
(iii) ∞ sol^n.

Sol^n The system of eq^n can be written as –

$AX=B$ where

$$A=\begin{bmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad B=\begin{bmatrix}\lambda\\10\\\mu\end{bmatrix}$$

Augmented matrix –

$$[A,B]=\left[\begin{array}{ccc|c}1&1&1&\lambda\\1&2&3&10\\1&2&\lambda&\mu\end{array}\right]$$

$R_2\to R_2-R_1$
$R_3\to R_3-R_1$

$$\left[\begin{array}{ccc|c}1&1&1&\lambda\\0&1&2&10-\lambda\\0&1&\lambda-1&\mu-\lambda\end{array}\right]$$

$R_3\to R_3-R_2$

$$\left[\begin{array}{ccc|c}1&1&1&\lambda\\0&1&2&10-\lambda\\0&0&\lambda-3&\mu-10\end{array}\right]$$

Case ①: (unique sol^n)

If $\lambda\neq3,\ \mu\in R$

then $\rho(A,B)=\rho(A)=3$

Case ②: (No sol^n) $\lambda=3,\ \mu\neq10$

$\rho(A)=2\neq\rho(A,B)$

Case ③:

If $\lambda=3,\ \mu=10$

then $\rho(A)=\rho(A,B)=2$

< No. of variables

$\Rightarrow$ Infinite sol^n.

Q. Solve

$$2x + 3y + 4z = 0$$
$$x + y + z = 0$$
$$x - y + z = 0$$

Solⁿ System of eqⁿ can be written as —

$$AX = B$$

where

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Coefficient matrix →

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & -1 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$
$$R_3 \rightarrow R_3 - R_1$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 2R_2$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

As $\rho(A) = 3$ = No. of variables
we have unique (zero solⁿ)

Solⁿ is given by —

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y + z = 0$$
$$y + 2z = 0$$
$$4z = 0$$

$\boxed{z = 0}$ $\boxed{y = 0}$ $\boxed{x = 0}$

Q. Show that only real value of λ for which the following system of equations have non-trivial sol^n is $\lambda = 6$ and solve it for $\lambda = 6$.

$$x + 2y + 3z = \lambda x$$
$$2x + y + 2z = \lambda y$$
$$2x + 3y + z = \lambda z$$

$\Rightarrow$ ~~$(\lambda-1)x + 2y + 3z = 0$~~

~~$2x + (\lambda-1)$~~

Sol^n.

$$(1-\lambda)x + 2y + 3z = 0$$
$$2x + (1-\lambda)y + 2z = 0$$
$$2x + 3y + (1-\lambda)z = 0.$$

The following system of eq^n can be written as –

$AX = B$ where

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

coefficient matrix –

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 2 & 1-\lambda & 2 \\ 1-\lambda & 2 & 3 \\ 2 & 3 & 1-\lambda \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$A = \begin{bmatrix} 2 & 1-\lambda & 2 \\ 1-\lambda & 2 & 3 \\ 0 & 2+\lambda & -1-\lambda \end{bmatrix}$$

$R_2 \to 2R_2 - R_1$

$$A = \begin{bmatrix} 2 & 1-\lambda & 2 \\ -2\lambda & 3+\lambda & 4 \\ 0 & 2+\lambda & -1-\lambda \end{bmatrix}$$

A

For non-toivial solⁿ $\rho(A) < 3$

$|A| = 0$

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$|A| = \begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix}$$

$$|A| = (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix}$$

$(6-\lambda)[(1+\lambda^2-2\lambda-6) - 1(3-3\lambda-4) + 9-2+2\lambda] = 0$

$(6-\lambda)[\lambda^2 - \cancel{2\lambda} - 5 + 1 + 3\lambda + 7 + \cancel{2\lambda}] = 0$

$(6-\lambda)[\underbrace{\lambda^2 + 3\lambda + 3}_{\text{Imaginary roots}}] = 0$

$\boxed{\lambda = 6}$

$$A = \begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 3 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to 2R_2 - 3R_1$

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 <$ No. of variables

solⁿ is given by -

$AX = B$

$$\begin{bmatrix} 2 & 3 & -5 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x + 3y - 5z = 0$

$-19y + 19z = 0$

$y = z$

$2x - 2y = 0$

$x = y$

Q. Find the value of λ for which

$$2x + y + 2z = 0$$
$$x + y + 3z = 0$$
$$4x + 3y + \lambda z = 0$$

have a non-trivial sol^n and solve it for this value of λ.

Sol^n The system of eq^n can be written as -

$$AX = B$$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & \lambda \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For non-trivial sol^n

$$\rho(A) < 3$$
$$\Rightarrow |A| = 0$$

$$|A| = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & \lambda \end{vmatrix} \Rightarrow 2(\lambda - 9) - 1(\lambda - 12) + 2(3 - 4) = 0$$

$$\Rightarrow 2\lambda - 18 + 12 - \lambda - 2 = 0$$

$$\lambda - 8 = 0$$

$$\boxed{\lambda = 8}$$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & 8 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 8 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 4R_1$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_2 + R_1$

Sol^n is given by -

$$Ax = B$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y + 3z = 0$$

$$-y - 4z = 0$$

$$\boxed{y = -4z}$$

$$x - 4z + 3z = 0$$

$$\boxed{x = z}$$

Let $x = z = k$.

$x = k$

$z = k$

$y = -4k$

---

d. Solve -

$$x_1 + 2x_2 + x_3 = 2$$

$$3x_1 + x_2 - 2x_3 = 1$$

$$4x_1 - 3x_2 - x_3 = 3$$

$$2x_1 + 4x_2 + 2x_3 = 4$$

Sol^n: The system of eq^n can be written as - $AX = B$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

Augmented matrix is -

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 4R_1$

$R_4 \to R_4 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & 0 & 30 & 30 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \to 5R_3 - 11R_2$

$\therefore \rho(A, B) = \rho(A) = 3 =$ No. of variables.

we have a unique sol^n

Sol^n is given by –

$$AX = B$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -5 \\ 0 & 0 & 30 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 2$$
$$-5x_2 - 5x_3 = 1$$
$$30x_3 = 4$$
$$\boxed{x_3 = \frac{4}{30}}$$

$$5x_2 = \frac{-5x_3 - 1}{5}$$
$$= -\frac{2}{3} - 1$$
$$5x_2 = \frac{-5}{3}$$
$$\boxed{x_2 = -\frac{1}{3}}$$

$$x_1 = \frac{60}{30} - \frac{4}{30} - \frac{20}{30}$$
$$x_1 = \frac{36}{30}$$
$$\boxed{x_1 = 6/5}$$

Q. Show that

$$-2x + y + z = a$$
$$x - 2y + z = b$$
$$x + y - 2z = c$$

have no sol^n unless

$$a + b + c = 0$$

By which case they have infinite sol^n? Solve for $a = 1$, $b = 1$, $c = -2$

Sol^n The system of eq^n can be written as —

$$AX = B$$

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Augmented matrix →

$$[A,B] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$[A,B] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \\ 1 & 1 & -2 & c \end{array}\right]$$

$R_3 \to R_1 + R_2 + R_3$

$$[A,B] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \\ 0 & 0 & 0 & a+b+c \end{array}\right]$$

$R_2 \to R_2 + 2R_1$

$$[A,B] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & b \\ 0 & -3 & 3 & a+2b \\ 0 & 0 & 0 & a+b+c \end{array}\right]$$

$\rho(A,B) = 2$ We have no sol^n if $(a+b+c) \ne 0$

or

unless $a+b+c = 0$.

Now, we have a sol^n if $(a+b+c) = 0$
and $\rho(A) = \rho(A,B) = 2 <$ No. of variables.
⇒ we have infinite sol^n.

$$AX = B$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$x - 2y + z = 1$
$-3y + 3z = 3$
$\boxed{z - y = 1}$
$\boxed{z = 1 + y}$

$x - 2y + 1 + y = 1$
$x - y = 0$
$\boxed{x = y}$

$$\boxed{\begin{array}{l} \text{Let } x = K \\ y = K \\ z = 1 + K \end{array}}$$

Linear dependent and linear independent

The vectors $x_1, x_2, \ldots x_n$ are said to be linearly dependent (l·d) if there exists scalars $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ (not all of them are 0) such that -

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots \lambda_n x_n = 0$$

If ~~no~~ such λ exists i.e if all the $\lambda_i = 0 \ (i = 1, 2, \ldots n)$. then we say that -

$x_1, x_2, x_3 \ldots x_n$ are linearly independent.

Q. Show that -

$x_1 (1, -1, 1)$
$x_2 (2, 1, 1)$
$x_3 (3, 0, 2)$

are l·d and hence find a relationship in x_1, x_2 & x_3

Let $\lambda_1, \lambda_2, \lambda_3$ are scalars such that -

~~then~~ $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$

$\lambda_1 (1, -1, 1) + \lambda_2 (2, 1, 1) + \lambda_3 (3, 0, 2) = (0, 0, 0)$

$\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$
$-\lambda_1 + \lambda_2 = 0$
$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$

$\boxed{\lambda_1 = \lambda_2}$

$2\lambda_1 = -2\lambda_3$

$\boxed{\lambda_1 = -\lambda_3 = \lambda_2}$

coefficient matrix -

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$R_2 \to R_1 + R_2$
$R_3 \to R_1 - R_3$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_2 - 3R_3$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

As $\rho(A) < 3$

$\Rightarrow x_1, x_2, x_3$ are l.d.

If $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$.

then $AX = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$$

$$3\lambda_2 + 3\lambda_3 = 0$$

$$\boxed{\lambda_2 = -\lambda_3 = \lambda_1}$$

$$\lambda_1 x_1 + \lambda_1 x_2 - \lambda_1 x_3 = 0$$

$$\boxed{x_1 + x_2 - x_3 = 0}$$

Eigen Values & Eigen vectors:

Let A be any square matrix of order n
λ be any scalar, I be the identity matrix of order n
then the matrix -

$[A - \lambda I]$ is known as characteristic matrix.
and the eq^n

$|A - \lambda I| = 0$ is known as characteristic eq^n.

Roots of $|A - \lambda I| = 0$ are known as eigen values or characteristic roots or latent roots.

The ~~space~~ set of all eigen values is known as spectrum.

Eigen Vector: Let ~~A~~ X be any vector and A be square matrix of order n then the linear transformation

AX which carries each vectors of X into λX i.e.

(*) — $AX = \lambda X$ gives vector X and is known as characteristic vector or eigen vector.

Now by (*)

$$AX - \lambda X = 0$$

$$\boxed{(A - \lambda I)X = 0}$$

The collection of all eigen vectors is known as eigen space.

Properties :

(i) Eigen values of A and A' are same.

Proof : Now

$$|(A-\lambda I)'| = |A-\lambda I|$$

$$|A'-\lambda I| = |A-\lambda I|$$

If ~~A-λI~~ $|A-\lambda I| = 0$

then

$$|A'-\lambda I| = 0$$

$$\therefore |A'-\lambda I| = |A-\lambda I| = 0$$

$\Rightarrow$ A and A' have same eigen values.

(ii) Eigen values of triangular matrix ~~of~~ or diagonal matrix are diagonal of the matrix.

Eg: $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$

$$[A-\lambda I] = \begin{bmatrix} a-\lambda & c \\ 0 & b-\lambda \end{bmatrix}$$

$$[A-\lambda I] = \begin{bmatrix} a-\lambda & c \\ 0 & b-\lambda \end{bmatrix} = 0$$

$$(a-\lambda)(b-\lambda) = 0$$

$$\lambda = a, \lambda = b$$

(iii) Sum of all eigen values is equal to sum of diagonal elements i.e trace of the matrix.

and

the product of all eigen values = ~~the~~ determinant of the matrix.

Proof: Consider

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$[A - \lambda I] = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} \quad \text{--- (A)}$$

$$\Rightarrow (a_{11}-\lambda)\left[(a_{22}-\lambda)(a_{33}-\lambda) - a_{32}a_{23}\right]$$
$$- a_{12}\left[a_{21}(a_{33}-\lambda) - a_{31}a_{23}\right]$$
$$+ a_{13}\left[a_{21}a_{32} - a_{31}(a_{22}-\lambda)\right]$$

$$[A-\lambda I] \Rightarrow (-1)^3\lambda^3 + \lambda^2(a_{11}+a_{22}+a_{33}) - \lambda(\ \ ) + (\ \ ) \quad \text{--- (1)}$$

If $\lambda_1, \lambda_2, \lambda_3$ are roots of $|A-\lambda I| = 0$, then we have

$$|A - \lambda I| = (-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) \quad \text{--- (2)}$$
$$= (-1)^3\lambda^3 + \lambda^2(\lambda_1+\lambda_2+\lambda_3) + \ldots$$

~~on comparing~~ From (1) & (2)

Thus, $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$

If $\lambda = 0$ in (2) & (A), then -

$$\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{vmatrix}$$

(iv) Eigen values of an idempotent matrix are either 0 or unity.

Proof: A matrix 'A' is said to be idempotent if

$$A^2 = A$$

Let λ is eigen value of A or A^2 then -

$$AX = \lambda X \quad \text{--- (1)}$$
$$A(AX) = A(\lambda X)$$
$$A^2X = \lambda(AX)$$
$$AX = \lambda^2 X \quad (\text{by (1)})$$
$$\lambda X = \lambda^2 X$$
$$(\lambda - \lambda^2)X = 0$$

$$\lambda - \lambda^2 = 0$$
$$\lambda(1-\lambda) = 0$$
$$\lambda = 0 \text{ or } \lambda = 1$$

(v) If $\lambda_1, \lambda_2, \cdots \lambda_n$ are eigen values of A then $\lambda_1^m, \lambda_2^m, \cdots \lambda_n^m$ are eigen values of A^m.

Proof: $\because$ λ_i are eigen values of A so we have

$$AX = \lambda_i X \quad —(1)$$
$$A(AX) = A(\lambda_i X)$$
$$A^2 X = \lambda_i (AX)$$
$$A^2 X = \lambda_i^2 X \quad (\text{by } (1))$$

In general, we have—

$$A^m X = \lambda_i^m X$$

$\Rightarrow$ Eigen values of A^m are $\lambda_1^m, \lambda_2^m, \cdots \lambda_n^m$

If $m = -1$ then eigen values of A^{-1} are $\lambda_1^{-1}, \lambda_2^{-1} \cdots \lambda_n^{-1}$.

(vi) Eigen values of a Hermitian matrix are real.

Proof: Let A is Hermitian matrix then

$$(\bar{A})^t = A$$

or

$$A^\theta = A$$

If λ is eigen value of A then

$$AX = \lambda X$$
$$\Rightarrow X^\theta A X = X^\theta \lambda X$$
$$\Rightarrow (X^\theta A X)^\theta = (X^\theta \lambda X)^\theta$$
$$\Rightarrow X^\theta A^\theta X^{\theta\theta} = (\lambda^\theta)(X^\theta X^{\theta\theta})$$
$$\Rightarrow X^\theta A^\theta X = \bar{\lambda} X^\theta X \quad — (1)$$
$$\Rightarrow X^\theta A X = \bar{\lambda} X^\theta X$$
$$\Rightarrow X^\theta \lambda X = \bar{\lambda} X^\theta X$$
$$(\lambda - \bar{\lambda}) X^\theta X = 0$$
$$\lambda - \bar{\lambda} = 0$$
$$\Rightarrow \lambda = \bar{\lambda}$$

If $\lambda = a+ib$
then $a+ib = a-ib$
$b = 0$

$\therefore \boxed{\lambda = a}$ $\Rightarrow$ eigen value is real.

(vii) Eigen values of skew Hermitian matrix are either zero or pure imaginary.

Proof: As A is skew Hermitian, so-

$$A^{\theta} = -A$$

Hence, by ①

$$-X^{\theta}AX = \bar{\lambda}X^{\theta}X$$
$$-X^{\theta}\lambda X = \bar{\lambda}X^{\theta}X$$
$$(\lambda+\bar{\lambda})X^{\theta}X = 0$$
$$(\lambda+\bar{\lambda}) = 0$$

If $\lambda = a+ib$

then

$$a+ib+a-ib = 0$$
$$\Rightarrow a = 0$$

$\therefore \lambda = ib$

If $b=0$, $\lambda = 0$

else, λ is pure imaginary.

(viii) Eigen values of a ~~the~~ unitary matrix are of unit modulus.

Proof: Let A is unitary then

$$A^{\theta}A = I$$

If λ is eigen value of A then

$$AX = \lambda X \quad \text{— ①}$$
$$\Rightarrow (AX)^{\theta} = (\lambda X)^{\theta} \quad \text{— ⑪}$$

By ① & ⑪

$$(AX)^{\theta}AX = (\lambda X)^{\theta}\lambda X$$
$$\Rightarrow X^{\theta}A^{\theta}AX = \bar{\lambda}X^{\theta}\lambda X$$
$$\Rightarrow X^{\theta}X = \bar{\lambda}\lambda X^{\theta}X$$
$$(1-\lambda\bar{\lambda})XX^{\theta} = 0$$
$$(1-\lambda\bar{\lambda}) = 0$$

$$|\lambda|^2 = 1$$
$$\boxed{|\lambda| = 1}$$

Similarly,

① for orthogonal matrix, eigen value is 1.

② for real symmetric matrix, eigen value are real.

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Cayley Hamilton Theorem :

Every square matrix satisfies its characteristic eq^n.

i.e if $A = [A_{ij}]_{n \times n}$

then characteristic eq^n →

$$|A - \lambda I| = (-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \dots k_n = 0$$

and we have –

$$(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \dots k_n = 0$$

Proof: Let A is the square matrix
then characteristic eq^n →

$$[A - \lambda I] = (-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + k_n = 0$$

$$[A - \lambda I] = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & - & - & - \\ \vdots & - & - & - \\ a_{n1} & - & & a_{nn} - \lambda \end{vmatrix}$$

Let adjoint of $[A - \lambda I]$ is the matrix P
Then P is a polynomial of degree $(n-1)$ and hence, we can write P as →

$P = P_0 \lambda^{n-1} + P_1 \lambda^{n-2} + \dots P_{n-1}$ where $P_0, P_1, \dots P_{n-1}$ are square matrix of order 'n'.

~~Contd~~ Contd.

Q: Find eigen values & eigen vectors of $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Sol^n Characteristic eq^n is →

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5 - \lambda & 4 \\ 1 & 2 - \lambda \end{vmatrix} = 0$$

$$(5 - \lambda)(2 - \lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0 \quad \boxed{\lambda = 1, 6}$$

If $\lambda=1$, then the eigen vectors $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is given by

$$(A-\lambda I)X_1 = 0$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - 4R_1$$

$$\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 4x_1 + 4x_2 = 0$$

$$\boxed{x_1 = -x_2}$$

Let $x_1 = a$

$$x_2 = -x_1$$

$$\Rightarrow \boxed{x_2 = -a}$$

eigen vector is $X_1 = \begin{bmatrix} a \\ -a \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

If $\lambda = 6$, then the eigen vectors $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is given by.

$$(A-\lambda I)X_2 = 0$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_1 + R_2$$

$$\begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 4x_2 = 0$$

$$x_1 = 4x_2$$

Let $x_2 = a$

$x_1 = 4a$

eigen vector is $X_2 = \begin{bmatrix} 4a \\ a \end{bmatrix} = a\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Q. Find eigen values & eigen vectors for the matrix →

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Solⁿ Characteristic eqⁿ →

$$(A-\lambda I) = 0 \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[(5-\lambda)(1-\lambda)-1\right]-1\left[1-\lambda-3\right]+3\left[1-15+3\lambda\right]=0$

$\Rightarrow (1-\lambda)\left[\lambda^2-6\lambda+4\right]+2+\lambda+9\lambda-42=0$

$= \lambda^2-6\cancel{\lambda}+4-\lambda^3+6\lambda^2-4\cancel{\lambda}+10\cancel{\lambda}-40=0$

$\Rightarrow -\lambda^3+7\lambda^2-36=0$

$\Rightarrow \lambda^3-7\lambda^2+36=0$

$\Rightarrow \lambda^3+2\lambda^2-9\lambda^2+36=0$

$\Rightarrow \lambda^2(\lambda+2)-9(\lambda^2-4)=0$

$\Rightarrow (\lambda+2)(\lambda^2-9\lambda+18)=0$

$\Rightarrow (\lambda+2)(\lambda-3)(\lambda-6)=0$

$\lambda=-2, 3, 6$

If $\lambda=-2$, eigen vector $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is given by

$(A-\lambda I)X=0$

$$\begin{bmatrix} 4 & 1 & 3 \\ 1 & 8 & 1 \\ 3 & 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$R_2 \rightarrow 4R_2 - R_1$

$R_3 \rightarrow 4R_2 - 3R_1$

$$\begin{bmatrix} 4 & 1 & 3 \\ 0 & 31 & 1 \\ 0 & 1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

~~$4x_1 + x_2 + 3x_3$~~

$R_1 \rightarrow R_1 - R_3$

$$\begin{bmatrix} 4 & 0 & -4 \\ 0 & 31 & 1 \\ 0 & 1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$4x_1 - 4x_3 = 0$ $\qquad \boxed{x_1 = x_3}$

$31x_2 + x_3 = 0$ $\qquad 31(-7x_3)+x_3=0$

$x_2 + 7x_3 = 0$ $\qquad -217x_3+x_3=0$

$x_2 = -7x_3$

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\qquad x_3 = 0 \quad x_1 = 0$

$x_2 = 0$

If $\lambda = 3$, eigen vector X_2 is given by $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$(A - \lambda I) X_2 = 0$.

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$R_2 \to R_1 + 2R_2$

$R_3 \to 2R_3 + 3R_1$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & 4 & 7 \\ 0 & 5 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$R_2 \to R_3 - R_2$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 5 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$-2y_1 + y_2 + 3y_3 = 0$

$y_2 - 2y_3 = 0$ $\boxed{y_2 = 2y_3}$ $y_2 = y_3 = 0$

$5y_2 + 5y_3 = 0$ $\boxed{y_2 = -y_3}$

Q. Show that the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -3 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

satisfies its char. eq^n and evaluate the value of $A^6 + 4A^5 + A^4 + A^3 + 4A^2 - A + 12$

Also calculate A^{-1}.

Sol^n:

Q. Show that the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Solⁿ Characteristic eqⁿ is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(2-\lambda)^2 - 1] + 1[-2+\lambda+1] + 1[1-2+\lambda] = 0$$

$$\Rightarrow (2-\lambda)[4+\lambda^2-4\lambda-1] + \lambda - 1 + \lambda - 1 = 0$$

$$\Rightarrow (2-\lambda)[\lambda^2 - 4\lambda + 3] + 2\lambda - 2 = 0$$

$$\Rightarrow 2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda + 2\lambda - 2 = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0.$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

Now, we show that

$$A^3 - 6A^2 + 9A - 4 = 0 \quad -(2)$$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 . A$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$22-36+18-4$

$-21+30-9$

$5-6$

$-5+6$

$6-12+9$

By (2)

$$A^2 - 6A + 9 - 4A^{-1} = 0$$

$$A^{-1} = \frac{1}{4}\left(A^2 - 6A + 9\right)$$

$$= \frac{1}{4}\left(\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{4}\left(\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\right)$$

$6-12+9$

$-5+6$

Diagonalization:

Let A is a matrix with linearly independent eigen vectors then there exists a non-singular matrix P such that

$$P^{-1}AP = \text{Diagonal Matrix}.$$

where diagonal of the matrix D are eigen values of A.

Again $P^{-1}AP = D$ $\qquad$ $P \rightarrow$ modal matrix.

$$\boxed{A = PDP^{-1}}$$

$$A^2 = (PDP^{-1})(PDP^{-1})$$

$$A^2 = PD^2P^{-1}$$

In general

$$\boxed{A^n = PD^nP^{-1}}$$

Similar Matrices:

Two matrices A & B are said to be similar if there exists a non singular matrix P such that—

$$B = P^{-1} A P.$$

Q. Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$ and hence evaluate A^4.

Sol^n: Characteristic eq^n →

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(5-\lambda)(1-\lambda) - 1\right] - 1\left[1-\lambda-3\right] + 3\left[1-15+3\lambda\right]$$

$$\Rightarrow (1-\lambda)\left[\lambda^2 - 6\lambda + 4\right] + \lambda + 2 + 9\lambda - 42 = 0$$

$$\Rightarrow \lambda^2 - \cancel{6\lambda} + 4 - \lambda^3 + 6\lambda^2 - \cancel{4\lambda} + \cancel{10\lambda} - 40 = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 36 = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$$

$$\lambda = -2, 3, 6.$$

$$\boxed{\lambda^3 - (\mathrm{tr}(A))\lambda^2 + (\overset{4}{A_{11}} + \overset{-8}{A_{22}} + \overset{4}{A_{33}})\lambda - \det A = 0}$$

Eigen vectors for $\lambda = -2, 3, 6$ are →

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

P = modal matrix $= \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$\det P = (-1)(-3) - 1(-2) + 1(1)$
$= 6$

$$Adj(P) = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 2 \\ 3 & 2 & 1 \end{bmatrix}'$$

$$= \begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{Adj(P)}{|P|}$$

$$= \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$D = P^{-1}AP = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = D$$

$P^{-1}AP = D$
$\Rightarrow A = PDP^{-1}$
$\Rightarrow A^4 = PD^4P^{-1}$

End