Discrete Mathematics Notes

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Computer Science Notes

Unit 1: Sets, Relation & Func.

A set is a collection of distinct and well defined objects.

$$S = \{1, 2, 3, 4, 5\}$$

$\{a,b\} \cup \{a,c\} = \{a,b,c\}$

$\{a,b\} \cap \{a,c\} = \{a\}$

$\{a,b,c\} - \{a,d\} = \{b,c\}$

$\{a,b\} \oplus \{a,c\} = \{b,c\}$

$\{a,b\} \oplus \{a,b\} = \emptyset$

Power Set of a set A, P(A), is the set that contains exactly all subsets of A.

$$P(\{a,b\}) = \{\{\}, \{a\}, \{b\}, \{a,b\}\}$$

$$P\{\{\}\} = \{\{\}\}$$

$|P \cup Q| \leq |P| + |Q|$

$|P \cap Q| \leq \min(|P|, |Q|)$

$|P \oplus Q| = |P| + |Q| - 2|P \cap Q|$

$|P - Q| = |P| - |Q|$

Principle of Inclusion and Exclusion

The number of elements in union of 2 sets is sum of no. of elements in two sets A_1 and A_2 minus no. of elements in intersection of A_1 and A_2.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

★ Multiset

A collection of objects that are not necessarily distinct.

Thus, $\{a, a, a, b, b, c\}$, $\{a, a, a\}$, $\{a, b, c\}$ and $\{\}$ are all examples of multiset.

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★ Multiplicity of an element in a multiset is the number of times the element appears in the multiset.
Thus multiplicity of element a in multiset {a, a, a, b, c, c} is 3, multiplicity of b is 1 and multiplicity of c is 2.

★ Principle of mathematical Induction
For a given statement involving a natural no. 'n'.
If we can show that:
1. The statement is true for $n = n_0$
2. The statement is true for $n = K+1$, assuming that statement is true for $n = K$.

★ Principle of strong mathematical induction
For a given statement involving a natural no. 'n'.
If we can show that:
1. The statement is true for $n = n_0$
2. The statement is true for $n = K+1$, assuming that statement is true for $n_0 \leq n \leq K$.

Ex= Show that $1^2+2^2+3^2+\text{----}+n^2 = \frac{n(n+1)(2n+1)}{6}$, $n \geq 1$ by mathematical induction.

Sol= Basis of induction:
For $n=1$,
$$1^2 = \frac{1(1+1)(2+1)}{6} = 1$$

Induction Step:
Assume that: $1^2+2^2+\text{----}+K^2 = \frac{K(K+1)(2K+1)}{6}$

we have, $1^2+2^2+\text{----}+K^2+(K+1)^2 = \frac{K(K+1)(2K+1)}{6} + (K+1)^2$

$$= \frac{(K+1)[K(2K+1)+6(K+1)]}{6}$$

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$$= \frac{(K+1)(2K^2+7K+6)}{6}$$

$$= \frac{(K+1)\ (K+2)\ (2K+3)}{6}$$

Ex= Show that $n! \geq 2^{n-1}$ by PMI

Sol= Basis of induction,

For $n=1$,

$$1! \geq 2^{1-1} = 1$$

Induction Step:

Assume that: $K! \geq 2^{K-1}$

we have, $(K+1)K! \geq (K+1)\ 2^{K-1}$

$$(K+1)\ K! \geq (K+1)\ \frac{2^K}{2}$$

$$(K+1)\ K! \geq 2^K\ \frac{(K+1)}{2} \qquad \left[\because \frac{K+1}{2} \geq 1\right]$$

$$(K+1)\ K! \geq 2^K$$

$$(K+1)! \geq 2^K \qquad \left[\because (K+1)\ K! = (K+1)!\right]$$

Ex= Show that $2^n > n^3$ $\forall\ n \geq 10$

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Ex= Show that any +ve int ≥ 2 is either a prime no. or can be presented as product of prime.

Sol= Basis of Induction:
For $n=2$, since 2 is a prime, the statement is true.

Induction step:
Assume that statement is true for any integer n, $2 \leq n \leq k$.

for $k+1$, if $k+1$ is prime then statement is true.
if $k+1$ is not prime, then $k+1$ can be written as pq, where $p \leq k$ & $q \leq k$.

Acc. to PMI, p is either a prime or a product of primes.
Also, q is either a prime or a product of primes.

Consequently, pq is a product of primes.

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Relation & func.

Cartesian product

The cartesian product of A & B, denoted as $A \times B$, is set of all ordered pairs of form (a, b) where $a \in A$ & $b \in B$.

Ex= $\{a, b\} \times \{a, c, d\} = \{(a,a), (a,c), (a,d), (b,a), (b,c), (b,d)\}$

A binary relation from A to B is subset of $A \times B$.

Ex= Let $A = \{a, b, c\}$ and $B = \{\alpha, \beta, \gamma\}$

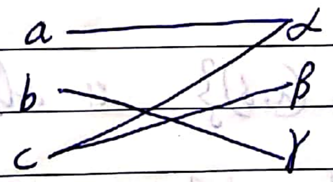
Let $R = \{(a, \alpha), (b, \gamma) (c, \alpha), (c, \beta)\}$ be a binary relation from A to B.

In tabular form, R,
$$\begin{array}{c|ccc} & \alpha & \beta & \gamma \\ \hline a & 1 & 0 & 0 \\ b & 0 & 0 & 1 \\ c & 1 & 1 & 0 \end{array}$$

We can place a 1 in ith row & jth column if element $a_i \in A$ is related to element $b_j \in B$, otherwise a 0 is placed. The matrix for a relation R is termed as <u>Relation Matrix.</u> $[M_R]$.

$$M_R = [m_{ij}] \quad \text{where } m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

Graphical representation



A ternary relation among 3 sets A, B, C is defined as a subset of cartesian product of 2 sets $A \times B$ and C, denoted $(A \times B) \times C$.

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Note that $(A \times B) \times C$ is set of all ordered triples of form $((a,b),c)$ where $(a,b) \in A \times B$ and $c \in C$.

Ex= $A = \{a,b\}$, $B = \{\alpha, \beta\}$, $C = \{1,2\}$

$((A \times B) \times C) = \{((a,\alpha),1), ((a,\alpha),2), ((a,\beta),1), ((a,\beta),2), ((b,\alpha),1), ((b,\alpha),2), ((b,\beta),1), ((b,\beta),2)\}$.

1) Reflexive Relation
A relation R on set A is reflexive if (a,a) is related to a for every $a \in A$.

2. Irreflexive Relation
A relation R on set A is Irreflexive if (a,a) does not related to R for every $a \in A$.

3. Transitive Relation
A relation R on set A is transitive if whenever A is related to B & B is related to C, then A is related to C. Thus, R is not transitive if there exists $(a,b,c) \in A$, such that A, B

Ex= $A = (a,b,c)$ $B = (a,b)$ $C = (b,c)$
$X = \{(a,a)\ (a,b)\ (a,c)\ (b,c)\}$ X is transitive

$Y = \{(a,b)\}$ is also Transitive rel.

4. Symmetric Relation
A relation R is said to be symmetric if (a,b) in R implies that (b,a) is also in R.

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Ex: Let A be a set of +ve integers & T be a binary relation on A such that (a,b) is in T if and only if $a \geq b$.
for instance, (10,9) is in T but (9,10) is not in T, T is not symmetric but T is antisymmetric.

5. Antisymmetric Relation
~~If whenever~~ a relation R is said to be antisymmetric relation if (a,b) in R implies that (b,a) is not in R unless a=b.
In other words, if both (a,b) and (b,a) are in R then it must be case that a=b.
Ex: Let $A = \{a,b,c\}$
$S = \{(a,a),(b,b)\}$
$N = \{(a,b),(a,c),(c,a)\}$
S is both symmetric & antisymmetric.
N is neither symmetric nor antisymmetric.

6. Asymmetric Relation
If $(a,b) \in R$, then $(b,a) \notin R$ it means R is not asymmetric if for some a & b from A both (a,b) & $(b,a) \in R$.

	Symmetric	Antisymmetric
$<$	✗	✓
$\leq$	✗	✗
null	✓	✓
$=$	✓	✓

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Ex= Let R be relation on integers defined as $R=\{(x,y): x,y \in$ $(x-y)$ is div. by 6.

Sol= Reflexive: Let $x \in Z$ such that xRx like
$$x-x = 0/6 = 0$$

Symmetric: $x, y \in Z$
$(x-y)$ is div. by 6
$-(x-y)$ is div. by 6
hence, $(y-x)$ is div. by 6

Transitive: xRy yRz we have to prove xRz
$$(x-y)+(y-z) = (x-z)$$

Hence, R is an equivalence relation

Ex= Prove that $\geq$ relation is partial order rel. on set of integers.

Sol= Ref. $a \geq a$ $\forall a \in Z$

AntiSym. $a \geq b$ but $b \geq a$ unless $a=b$

Trans. $a \geq b$ & $b \geq c$ then $a \geq c$

hence, $\geq$ is partial order relation.

Ex= Prove that binary relation R on set of +ve int. A is partial order rel. where R is def. as (a,b) will be in R if a/b.

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Sol= Ref. $a/a = 1 \quad \forall a \in Z^+$

Anti Sym. a/b but b/a unless $a=b$

Transitive a/b & b/c then a/c

Th. If A is poset & B is poset, then $(A \times B, \leq)$ is poset with partial order $\leq$ is poset def. by $(a,b) \leq (a',b')$ if $a \leq a'$ in poset A & $b \leq b'$ in poset B.

$A = \{1,2\}$ Poset $A = \{(1,1), (2,2), (1,2)\}$
$B = \{2,3\}$ Poset $B = \{(2,2), (3,3), (2,3)\}$

$R = \{(1,2)(1,2), (2,2)(2,2), (1,2)(2,2)\ (1,3)(1,3), (2,3)(2,3)\ (1,3)(2,3),$
$(1,2)(2,3), (1,3)(1,2), (2,2)(2,3)\}$

Proof To prove that it is partial ordered we need to prove for antisymmetric, reflexivity & transitivity.

Ref: If $(a,b) \in A \times B$ then
$(a,b) \leq (a,b)$ since $a \leq a'$ in A, $b \leq b'$ in B

Antisymmetric: Suppose that $(a,b) \leq (a',b')$ & $(a',b') \leq (a,b)$
where $a, a' \in A$ & $b, b' \in B$ then
$a \leq a'$ & $a' \leq a$ in A
$b \leq b'$ & $b' \leq b$ in B
$\Rightarrow a = a'$ & $b = b'$

Transitive: Suppose that $(a,b) \leq (a',b')$ & $(a',b') \leq (a'',b'')$ where
$a, a', a'' \in A$ & $b, b', b'' \in B$ then

$a \leq a'$ & $a' \leq a''$ in A since A is Poset

Then $a \leq a''$

$b \leq b'$ & $b' \leq b''$ in B

Since B is Poset then $b \leq b''$

$\Rightarrow (a,b) \leq (a'',b'')$

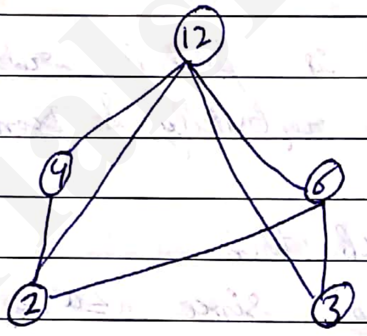
☆ Diagraph

Diagraph of partial order has no cycle of >1 i.e. it has only cycles of length 1

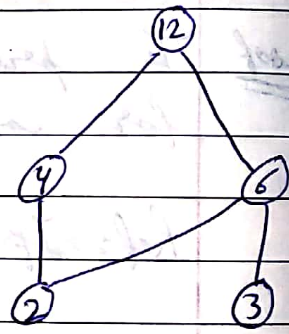
Ex = $A = \{2, 3, 4, 6, 12\}$

R = partial divisibility a divides b

$R = \{(2,2), (3,3), (4,4), (6,6), (12,12), (2,4), (2,6), (2,12), (3,6), (3,12), (4,12), (6,12)$



Diagraph



Hasse diagraph

☆ Closure

The closure of a relation is the smallest extension of relation that has certain specific properties such as reflexivity, symmetricity & transitivity.

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1. Reflexive Closure [$r(R)$]
A relation R' is reflexive closure of a relation R if and only if,
(i) R' is reflexive
(ii) $R \subseteq R'$
(iii) for any relation R'', if $R \subseteq R''$ & R'' is reflexive then $R' \subseteq R''$, i.e. R' is smallest relation that satisfy (a) & (b).

Ex: if $A = \{1,2,3,4\}$
$R = \{(1,1),(2,2)\}$
$r(R) = \{(1,1),(2,2),(3,3),(4,4)\}$

2. Symmetric closure [$S(R)$]
A relation R' is symmetric closure of a relation R if and only if.
(a) R' is symmetric
(b) $R \subseteq R'$
(c) for any relation R'', if $R \subseteq R''$ & R'' is symmetric, then $R' \subseteq R''$, i.e. R' is smallest relation that satisfies (a) & (b).

Ex= $A = \{1,2,3,4\}$ $R = \{(1,1)\ (2,2)\ (1,2)\ (3,4)\}$
$R^{-1} = \{(1,1),(2,2),(2,1),(4,3)\}$

$S(R) = R \cup R^{-1} = \{(1,1),(2,2),(1,2),(2,1),(3,4),(4,3)\}$

3. Transitive Closure [$t(R)$]
A relation R' is transitive closure of a relation R if and only if
(a) R' is transitive (b) $R \subseteq R'$
(c) for any relation R'', if $R \subseteq R''$ & R'' is transitive then $R' \subseteq R''$, i.e. R' is smallest relation that satisfies (a) & (b).

Ex = $A = \{a, b, c, d\}$

$R = \{(a,b), (b,c), (c,b), (c,d)\}$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^2 = R_1 = R o R = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_1 = R o R_1^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$t(R) = R \cup \underbrace{R_1}_{RoR} \cup \underbrace{R_2}_{RoR_1} \cup \text{-------} \cup \underbrace{R_n}_{RoR_{n-1}}$$

$$R_2 = R o R_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$t(R) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

* Let A be a set of cordinality n & R be a relation on A,

$$R^T = R \cup R_1 \cup R_2 \text{-----} \cup R_n$$

i.e. power of R greater than n need not considered to R^T.

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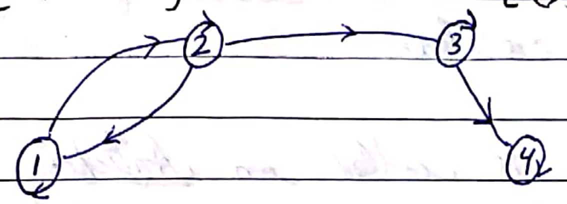
Q= $A=\{1,2,3,4\}$ $R=\{(1,2)\ (2,3)\ (3,4)\ (2,1)\}$



vertex 1: 1,2,3,4

vertex 2: 1,2,3,4

vertex 3: 4

$R=\{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,1)\ (2,2)\ (2,3)\ (2,4)\ (3,3)\ (3,4)\ (4,4)\}$

$$R=\begin{bmatrix}1&1&1&1\\1&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix} \qquad R_1=R\circ R=\begin{bmatrix}1&1&1&1\\1&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}$$

$$R^T=R\circ R_1=\begin{bmatrix}1&1&1&1\\1&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}$$

☆ Antichain

If no 2 distinct elements are related in a subset of A, This subset is called an antichain.

If A is poset & every 2 elements of A are comparable, then A is chain, A is called totally ordered chain.

Length of chain = 1 less than no. of elements in chain

Max. chain is a chain that is not a subset of a larger chain.

A set with an ordering rel is well ordered if every non empty subset of a set has a least element.

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* A subset of A is called a <u>chain</u> if every 2 elements in subset are related.

* A subset of A is called an <u>Antichain</u> if no 2 distinct elements in subset are related.

Ex = In partially ordered set
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

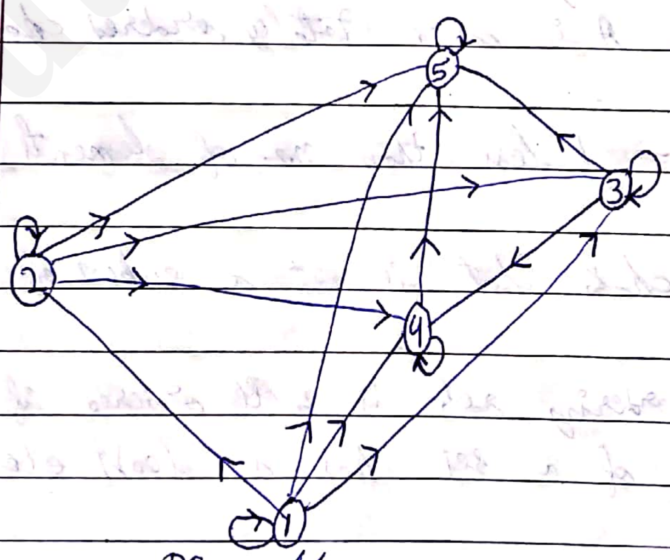
Ans = Chain : $\{a, b, c, e\}$, $\{a, b, c\}$
$\{a, d, e\}$ & $\{a\}$

Antichain : $\{b, d\}$, $\{c, d\}$, $\{a\}$

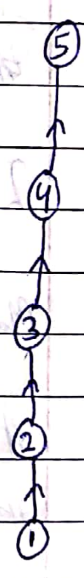
Q = Draw Hasse diagram for relation R on $A = \{1, 2, 3, 4, 5\}$,

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Sol = $R = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 5,5\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 1,5\rangle,$
$\langle 2,3\rangle, \langle 2,4\rangle, \langle 2,5\rangle, \langle 3,4\rangle, \langle 3,5\rangle, \langle 4,5\rangle\}$



Digraph



Hasse Digraph

★ Function

If f is a func. from A to B, then A is domain of f & B is codomain of f.

If $f(a) = b$, then b is image of a & a is preimage of b.

The range of f is set of all images of element of A.

*** Equal func.** ⇒ 2 func. are equal when they have same domain, same codomain & map elements of their common domain to same element in their common codomain.

1. $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

2. $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$

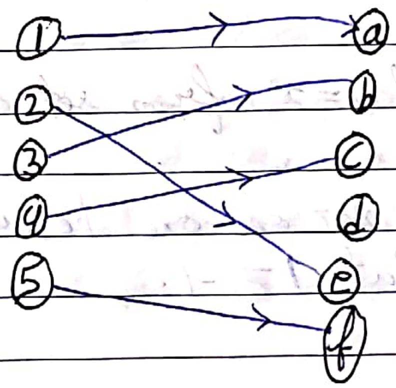
Ex = Let f_1 & f_2 be func. from R to R such that $f_1(x) = x^2$ & $f_2(x) = x - x^2$.

Sol = $f_1(x) + f_2(x) = x^2 + (x - x^2) = x$

$f_1(x) \cdot f_2(x) = x^2(x - x^2) = x^3 - x^4$

★ One to one function (Injection)

A func. from A to B is said to be a one-to-one func. if no 2 elements of A have same image.



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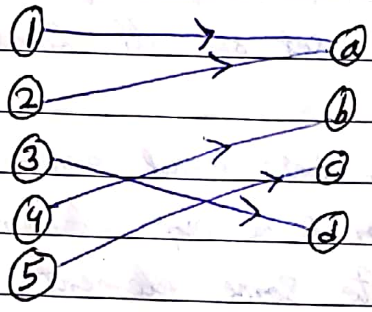
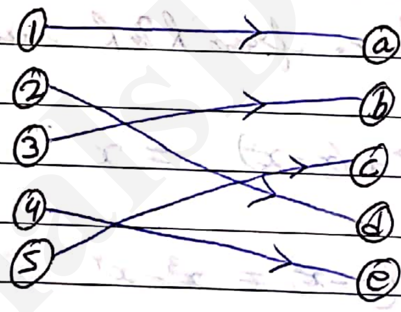
☆ Onto function (Surjection)

A func. from A to B is said to be an onto func. if every element of B image of one or more element of A.



☆ One to one onto function (Bijection)

A func. from A to B is said to be a one to one onto func. if it is both an onto & a one to one func.



Ex= Determine whether func. f from {1,2,3,4,5} to {a,b,c,d,e,f} with $f(1) = a$, $f(2) = e$, $f(3) = b$, $f(4) = c$, $f(5) = f$ is one-one?

Sol= The func. f is one-one because f takes on diff values at 5 elements of its domain.

Ex= Determine whether func. $f(x) = x^2$ from set of int to set of int is one to one.

Sol= The func. $f(x) = x^2$ is not one-one because, for instance $f(1) = f(-1) = 1$, But $1 \neq -1$.

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Ex= Determine whether func. $f(x) = x+1$ from set of real no. to itself is one to one.

Sol= The func $f(x) = x+1$ is a one to one func, to demonstrate this note that $x+1 \neq y+1$ when $x \neq y$.

Ex= Let f be fn from $\{1,2,3,4,5\}$ to $\{a,b,c,d\}$ defined by $f(1) = a$, $f(2) = a$, $f(3) = d$, $f(4) = b$, $f(5) = c$.

Sol= Because all four elements of codomain are images of elements in domain, we see that f is onto.
If codomain is $\{a,b,c,d,e\}$, then f would not be onto.

Ex= Is the fn $f(x) = x^2$ from set of int to set of int. onto?

Sol= The fn. f is not onto because there is no int x with $x^2 = -1$, for instance.

Ex= Is func $f(x) = x+1$ from set of int to set of int onto?

Sol= The func is onto because for every int y there is an int x such that $f(x) = y$.
$f(x) = y$ if & only if $x+1 = y$ which holds if & only if $x = y-1$.

Ex= Let f be func from $\{a,b,c,d\}$ to $\{1,2,3,4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$ & $f(d) = 3$. Is f a bijection?

Sol= The func f is one-one & onto.
It is one-one because no 2 values in domain are assigned same func value.
It is onto because all 4 elements of codomain are images of element in domain.
Hence, f is bijective func.

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★ Inverse function

Only one-one func. is Invertible

Ex= let f be func. from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, $f(c) = 1$. Is f invertible?

Sol= Func. f is invertible because it is a one-one func.
The inverse func. f^{-1} reverses correspondance given by f,
so $f^{-1}(1) = c$, $f^{-1}(2) = a$, $f^{-1}(3) = b$.

Ex= Let $f: Z \to Z$ be such that $f(x) = x+1$. Is f invertible?

Sol= The func. f is invertible because it is a one-one func.
Suppose y is image of x, so that $y = x+1$.
Then, $x = y-1$.
This means that $y-1$ is unique element of Z that is sent to y by f.

$$f^{-1}(y) = y-1$$

Ex= Let f be func. from R to R with $f(x) = x^2$. Is f invertible?

Sol= Because $f(-2) = f(2) = 4$, f is not one-~~on~~ one.
If an inverse func. were defined, it would have to assign 2 elements to 4.
Hence, f is not Invertible.

★ Composition of function

Let g be a func. from set A to set B & let f be a func. from set B to set C.
The composition of func. f & g denoted by $f \circ g$,

$$(f \circ g)(a) = f(g(a))$$

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* The composition $f \circ g$ cannot be defined unless range of g is a subset of domain of f.

Ex= Let g be func from set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, $g(c) = a$. Let f be func from set $\{a, b, c\}$ to set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$ & $f(c) = 1$.
What is composition of $f \circ g$ & $g \circ f$?

Sol= The composition $(f \circ g)(a) = f(g(a))$
$= f(b)$
$= 2$

$(f \circ g)(b) = f(g(b))$
$= f(c)$
$= 1$

$(f \circ g)(c) = f(g(c))$
$= f(a)$
$= 3$

$g \circ f$ is not defined because range of f is not a subset of domain of g.

Ex= Let f & g be func from set of int. to set of int. defined by $f(x) = 2x+3$ & $g(x) = 3x+2$. What is $f \circ g$?

Sol= Both compositions $f \circ g$ & $g \circ f$ are defined

$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)+3 = 6x+7$

$(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3)+2 = 6x+11$

* Pigeonhole Principle

It states that if there are more pigeons than no. of pigeonholes then, there must be one pigeon hole with atleast 2 pigeons in it.

* If K is +ve int. & K+1 or more objects are placed into K boxes then there must be one box containing 2 or more boxes.

* Extended pigeonhole principle

If there are x pigeons & y pigeonholes then one pigeonhole must be occupied by at least $\left[\frac{x-1}{y}\right]+1$ pigeons

Ex= If 30 magazines contain a total of 61,324 pages, then one of magazine must be atleast 2045 pages.

Sol= Let the pages as pigeons & magazines as pigeonholes.

By extended pigeonhole principle, one magazine must contain at least $\frac{(61,324-1)}{30}+1 = 2045$ pages

Ex= Show that if seven no.s from 1 to 12 are chosen, then 2 of them will add upto 13.

Sol= Let $S_1=(1,12)$, $S_2=(2,11)$, $S_3=(3,10)$, $S_4=(4,9)$, $S_5=(5,8)$, $S_6=(6,7)$

Each of 7 numbers chosen must belong to one of these sets, since there are only 6 sets.

Acc. to pigeonhole principle, 2 of the chosen numbers will belong to same set. These no.s add up to 13.

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Ex= Let T be an equilateral triangle whose sides are of length 1 unit. Show that if any 5 points are chosen lying on or inside the Δ, then 2 of them must ~~not~~ be no more than 1/2 unit apart.

Sol= Let T be equilateral ΔABC.
given that $AB = BC = CA = 1$ unit.
Let D, E, F be mid points of AB, BC, CA respectively.
Join DE, EF, FD.
Now, $AD = DB = BE = EC = AF = FC = DF = DE = EF = 1/2$ units.
The triangle ABC is divided into 4 equilateral triangle ADE, DBE, EFC, DEF, each have side of length 1/2 units.
Each of 5 points (pigeons) chosen must belong to one of these triangles (pigeonholes).
Since, there are only 4 triangles & any 2 points chosen from a triangle out of these 4 are not more than 1/2 unit apart. Then,
Acc. to pigeonhole principle, if 5 points are chosen, then 2 of them must be no more than 1/2 unit apart. because 2 points must belong to a single triangle.

Ex= 10 people come forward to volunteer for a 3 person committe. Every possible committe of 3 that can be formed from these 10 names is written on a slip of paper, one slip for each possible committe & slips are put in 10 hats. Show that at least 1 hat contains 12 or more slips of paper.

Sol= A committe of 3 can be chosen from 10 names in
$C(10,3) = \frac{10!}{7!\,3!} = 120$ ways
So there are 120 slips (pigeons) & 10 hats (pigeonholes).

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By extended pigeonhole principle, one hat must contain at least $\frac{(120-1)}{10}+1 = 12$ or more slips of paper.

Ex= Show that there must be at least 90 ways to chose 6 numbers from 1 to 15 so that all choices have same sum.

Sol= 6 numbers out of 1 to 15 can be chosen in $C(15,6) = 5005$ ways.
The sum varies from 21 to 75 as $[1+2+3+4+5+6] = 21$ & $[10+11+12+13+14+15] = 75$.
The no. of sums (pigeonholes) that can be generated
$= 75-21+1 = 55$.
So, by extended pigeonhole principle, there are
$$= \frac{(5005-1)}{55}+1 = 91 \text{ ways.}$$

Hence, there must be at least 90 ways to chose 6 no. from 1 to 15 so that all choices have same sum.

Ex= Prove that it is not possible to arrange no. 1,2,3......10. in a circle so that every triplet of consecutively placed no. has sum less than 15.

Sol= In any arrangement of 1,2,3,.......,10 in a circle, there are 10 triplets of consecutively placed no.
Each no. appears in 3 of these triplets.
If sum of each triplet were less than 15, then total sum of all triplets would be less than 10 times 15 i.e. 150.
But, 1+2+3+.....+10 is 55 and since each no. appears in 3 triplets, total sum should be 3 times 55, i.e. 165.
This is a contradiction.
So, not all triplets can have a sum less than 15.

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Ex= Show that among 6 persons either there are 3 persons who are mutual friends or there are 3 persons who are complete strangers to each other.

Sol= Let A be a person in the group.
Acc. to pigeonhole principle, either there are 3 (or more) persons who are friends of A or there are 3 (or more) persons who are strangers to A.
Let B, C, D denote friends of A. If any 2 of B, C, D know each other, then these 2, together with A, form a friendly threesome.
If no. 2 of B, C, D know each other then they are 3 persons who are complete strangers to each other.

Ex= How many students must be in a class to guarantee that at least 2 students receive same score on final exam if exam is graded on a scale from 0 to 100 points?

Sol= There are 101 possible scores on final.
Pigeonhole principle shows that among any 102 students there must be atleast 2 students with same ~~course~~ score

* Among any group of 367 people, there must be atleast 2 with same birthday because there are only 366 possible birthdays.

* Among 100 people there are atleast $\lceil 100/12 \rceil = 9$ who were born in same month.

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PSC

★ **Rule of product:** If one exp. has 'm' possible outcomes & another exp. has 'n' possible outcomes then there are $m \times n$ possible outcomes when both of these exp. takes place.

★ **Rule of Sum:** If one exp. has m possible outcomes & another exp. has n possible outcomes then there are $m+n$ possible outcomes when exactly one of these exp. takes place.

Ex= If there are 52 ways to select a representative for junior class & 49 ways to select a representative for senior class then,
Acc. to rule of product, there will be 52×49 ways to select the representative for both junior & senior classes.
Acc. to rule of sum, there will be $52+49$ ways to select a representative for either junior or senior class.

★ **Permutation**

$$P(n,r) = \frac{n!}{(n-r)!}$$

Ex= In how many ways can 3 examination be scheduled within a 5 day period of so that no 2 exams are scheduled on same day?

Sol= Considering 3 examinations as distinctly colored balls & 5 days as distinctly no. boxes, we obtain the result,

$$5 \times 4 \times 3 = 60$$

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Ex= If we have 7 rooms & want to assign 4 of them to 4 programmers as offices & use the remaining 3 rooms for comp. terminals.

Sol= The assignment can be made in $7 \times 6 \times 5 \times 4 = 840$ diff. ways we can view the problem as that of placing 4 distinct balls (programmers) into 7 distinct boxes (rooms), with 3 boxes that are left empty being rooms for computer terminals.

Ex= Determine no. of 4-digit decimal no. that contain no repeated digits?

Sol= Since this is a problem of arranging 4 of 10 digits $0, 1, 2, \ldots, 9$ the answer is $P(10,4) = 5040$

Among these 5040 no., $9 \times 8 \times 7 = 504$ of them have a leading 0.

So, $5040 - 504 = 4536$ of them do not have a leading 0.

~~Result can be~~.

Ex= In how many ways, we can schedule 3 exams within 5-day period with no restriction on no. of exams scheduled each day?

Sol= The total no. of ways is $5 \times 5 \times 5 = 125$.

* There are $P(n,r)$ ways to place r distinctly colored ~~boxes~~ balls in 'n' numbered boxes, total no. of ways to place r colored boxes in 'n' numbered boxes, where q_1 of these balls are of ~~1~~ one colour, q_2 of them are of second color, ... and q_t of them are of t_{th} color, is

$$\boxed{\frac{P(n,r)}{q_1!\, q_2!\, q_t}}$$

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Ex= How many no. of ways to paint 12 offices so that 3 of them will be green, 2 of them pink, 2 of them yellow & remaining ones white?

Sol= The no. of ways is $\frac{12!}{3!2!2!5!} = 166,320$

Ex= How many diff. messages can be represented by sequence of 3 dashes & 2 dots?

Sol= The no. of ways is $\frac{5!}{3!2!} = 10$

★ Combination

The no. of ways of placing 'r' balls of same color in 'n' numbered boxes is:

$$\boxed{C(n,r) = \frac{n!}{r!(n-r)!}}$$

No. of ways to select r objects from 'n' distinct objects, allowing repeated selection is,

$$\boxed{C(n+r-1,r) = \frac{(n+r-1)!}{r!(n-1)!}}$$

* No. of ways to choose 3 out of 7 days (with repetition allowed) is $C(7+3-1,3) = C(9,3) = 84$

* No. of ways to choose 7 out of 3 days (with repetition necessarily allowed) is

$$C(3+7-1,7) = C(9,7) = 36$$

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Ex= When 3 dice are rolled, what are the diff. outcomes.

Sol= The no. of diff. outcomes is $C(6+3-1,3) = C(8,3) = 56$ because rolling 3 dice is equivalent to selecting 3 numbers from 6 numbers 1, 2, 3, 4, 5, 6 with repetition allowed.

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Logic

★ Proposition

* Tautology: A proposition that is true under all circumstances.
* Contradiction: A proposition that is false under all circumstances.

* Compound proposition: A proposition obtained from combination of other proposition.

* Atomic proposition: A proposition that is not a combination of other propositions.

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

Ex = p and q are 2 atomic propositions
$p \wedge q$, $\bar{p}$, $p \vee (\bar{q})$ are compound proposition

* A proposition "if p then q" denoted by $p \to q$, which is true if both p and q are true ~~or~~ if p is false and is false if p is true and q is false.
* It is not mandatory that there should be any kind of relation b/w p and q in order to construct $p \to q$.

★ Conditional statement / Implications
Compound proposition "if p then q" (also reads p implies q).

Ex = If you try then you will succeed.
Clearly, if you try and succeed, statement is true.
If you try and fail then statement is false.
If you did not try then statement is true.

Ex = Consider the order from security officer of a company that every visitor must wear a badge.

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The order can be rephrased as a proposition "If one is a visitor then one wears a badge.
If she is a visitor, we can determine whether the order has been enforced by observing whether she is wearing a badge.
On other hand, if she is not a visitor then there is no way we can possibly conclude that the order has been enforced and hence statement is true.

For proposition $p \to q$, the proposition $q \to p$ is converse.
Proposition $\bar{q} \to \bar{p}$ is contrapositive of $p \to q$.
Proposition $\bar{p} \to \bar{q}$ is inverse of $p \to q$.

Contrapositive of an implication $p \to q$ has same value of $p \to q$.

Ex= Give converse, contrapositive & inverse of following implication.
"If it rains today, I will go to college tomorrow."

Ans: Converse: If I will go to college tomorrow, then it would have rained today.
Contrapositive: If I do not go to college tomorrow, then it will not have rained today.
Inverse: If it does not rain today, then I will go to college tomorrow.

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* Let p and q be 2 propositions.
we define a proposition, 'p if and only if q', $p \leftrightarrow q$ which is true if both p and q are true or if both p and q are false. and which which is false if p is true while q is false and if p is false while q is ~~false~~ true.

The compound statement $p \leftrightarrow q$ is also interpreted as "p is necessary & sufficient for q" & called Biconditional statement.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

☆ Compound propositions which have same truth value in all possible cases are called Logically equivalent.

Ex= There are 2 restaurants next to each other. One has a sign that says "Good food is not cheap" & other has a sign that says "Cheap food is not good". Are signs saying the same thing?

Ans= Let g denote the proposition that food is good & c denote the proposition that food is cheap.
The first sign can then be written as $g \to \bar{c}$ & second sign be written as $c \to \bar{g}$

g	c	$\bar{g}$	$\bar{c}$	$g \to \bar{c}$	$c \to \bar{g}$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

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☆ Rule of Inference

Rule P: A premise can be inserted at any point in derivation.

Rule T: If formula q is tautologically implied by any one or more of previous formula in a derivation, then q can be inserted in derivation.

Rule CP: If we can derive a formula q from p & a set of premises then we can derive $p \to q$ from set of premises alone.

* Rule CP, is used in cases, where conclusion is of form $p \to q$. In these cases, p is taken as an extra premise and q is derived from given premise & from p.

* Tautology $(p \wedge (p \to q)) \to q$ is basis of rule of inference, called modus ponens & law of detachment.

Ex= Let implication, "If it rains today then I shall carry an umbrella" and its hypothesis, "It is raining today" are True.

Sol= Then by modus ponens, it is clear that conclusion of implication, "I shall carry umbrella" is True.

Ex= Show that q is a valid inference from premise $p \to q$, $p \vee q$ and $\bar{q}$.

Sol=

{1}	(1)	$p \to q$	Rule P
{2}	(2)	$\bar{q}$	Rule P
{1,2}	(3)	$\bar{p}$	Rule T, (1),(2) & modus tollens
{4}	(4)	$p \vee q$	Rule P
{1,2,4}	(5)	q	Rule T, (3),(4) & disjunctive syllogism

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EQ_1	$\bar{\bar{p}} \equiv p$	(double negation low)
EQ_2	$p \vee q \equiv q \vee p$	(commutative low)
EQ_3	$p \wedge q \equiv q \wedge p$	(commutative low)
EQ_4	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	(associative low)
EQ_5	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	(associative low)
EQ_6	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (q \vee r)$	(distributive low)
EQ_7	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (q \wedge r)$	(distributive low)
EQ_8	$\overline{p \vee q} \equiv \bar{p} \wedge \bar{q}$	(De Morgan's low)
EQ_9	$\overline{p \wedge q} \equiv \bar{p} \vee \bar{q}$	(De Morgan's low)
EQ_{10}	$p \vee p \equiv p$	(idempotent low)
EQ_{11}	$p \wedge p \equiv p$	(idempotent low)
EQ_{12}	$r \vee (p \wedge \bar{p}) \equiv r$	
EQ_{13}	$r \wedge (p \vee \bar{p}) \equiv r$	
EQ_{14}	$r \vee (p \vee \bar{p}) \equiv T$	
EQ_{15}	$r \wedge (p \wedge \bar{p}) \equiv F$	
EQ_{16}	$p \rightarrow q \equiv \bar{p} \vee q$	
EQ_{17}	$\overline{p \rightarrow q} \equiv p \wedge \bar{q}$	
EQ_{18}	$p \rightarrow q \equiv \bar{q} \rightarrow \bar{p}$	
EQ_{19}	$p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$	
EQ_{20}	$\overline{p \leftrightarrow q} \equiv p \leftrightarrow \bar{q}$	
EQ_{21}	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	
EQ_{22}	$(p \leftrightarrow q) \equiv (p \wedge q) \vee (\bar{p} \wedge \bar{q})$	
EQ_{23}	$p \vee (p \wedge q) \equiv p$	(absorption low)
EQ_{24}	$p \wedge (p \vee q) \equiv p$	(absorption low)
EQ_{25}	$p \vee \bar{p} \equiv T$	(negation low)
EQ_{26}	$p \wedge \bar{p} \equiv F$	(negation low)

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Rule No.	Tautology	
IM_1	$p \wedge q \rightarrow p$	(simplification)
IM_2	$p \wedge q \rightarrow q$	(simplification)
IM_3	$p \rightarrow (p \vee q)$	(addition)
IM_4	$q \rightarrow (p \vee q)$	(addition)
IM_5	$\bar{p} \rightarrow (p \rightarrow q)$	
IM_6	$q \rightarrow (p \rightarrow q)$	
IM_7	$(\overline{p \rightarrow q}) \rightarrow p$	
IM_8	$(\overline{p \rightarrow q}) \rightarrow \bar{q}$	
IM_9	$p, q \rightarrow p \wedge q$	(conjunction)
IM_{10}	$[\bar{p}, (p \vee q)] \rightarrow q$	(disjunctive syllogism)
IM_{11}	$[p, (p \rightarrow q)] \rightarrow q$	(modus ponens)
IM_{12}	$[\bar{q}, (p \rightarrow q)] \rightarrow \bar{p}$	(modus tollens)
IM_{13}	$[(p \rightarrow q), (q \rightarrow r)] \rightarrow (p \rightarrow r)$	(hypothetical syllogism)
IM_{14}	$[p \vee q, p \rightarrow r, q \rightarrow r] \rightarrow r$	(dilemma)
IM_{15}	$[p \vee q, \bar{p} \vee r] \rightarrow (q \vee r)$	(resolution)

Ex = Show that $\bar{p}$ is tautologically implied by $(\overline{p \wedge \bar{q}})$, $\bar{q} \vee r$, $\bar{r}$.

Sol =

{1}	(1)	$(\overline{p \wedge \bar{q}})$	Rule P
{1}	(2)	$\bar{p} \vee q$	Rule T, (1) & Demorgan's law
{1}	(3)	$p \rightarrow q$	Rule T, (2) & EQ_{16} $(p \rightarrow q \equiv \bar{p} \vee q)$
{4}	(4)	$\bar{q} \vee r$	Rule P
{4}	(5)	$q \rightarrow r$	Rule T, (4) & EQ_{16} $(p \rightarrow q \equiv \bar{p} \vee q)$
{1,4}	(6)	$p \rightarrow r$	Rule T, (3), (5) & hypothetical syllogism
{7}	(7)	$\bar{r}$	Rule P
{1,4,7}	(8)	$\bar{p}$	Rule T, (6), (7) & modus tollens

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Ex= Derive the following using CP rule:

$$(\bar{P} \vee q,\ \bar{q} \vee r,\ r \to s) \to (P \to S)$$

Sol= Acc. to CP rule, we will include P as an additional premise & show S first.

{1}	(1)	$\bar{P} \vee q$	Rule P
{1}	(2)	$P \to q$	Rule T, (1) & EQ_{16} $(P \to q \equiv \bar{P} \vee q)$
{3}	(3)	P	P (assumed premise)
{1,3}	(4)	q	Rule T, (2), (3) & modus ponens
{5}	(5)	$\bar{q} \vee r$	Rule P
{1,3,5}	(6)	r	Rule T, (4), (5) & disjunctive syllogism
{7}	(7)	$r \to S$	Rule P
{1,3,5,7}	(8)	S	Rule T, (6), (7) & modus ponens
{1,5,7}	(9)	$P \to S$	Rule CP.

Ex= Show that the following system is inconsistent:

$$P \to q,\ P \to r,\ q \to \bar{r},\ P$$

Sol=

{1}	(1)	$P \to q$	Rule P
{2}	(2)	P	Rule P
{1,2}	(3)	q	Rule T, (1), (2) and modus ponens
{4}	(4)	$P \to r$	Rule P
{2,4}	(5)	r	Rule T, (2), (4) & modus ponens
{1,2,4}	(6)	$q \wedge r$	Rule T, (3), (5) & conjunction
{7}	(7)	$q \to \bar{r}$	Rule P
{7}	(8)	$\bar{q} \vee \bar{r}$	Rule T, (7) & EQ_{16} $(P \to q \equiv \bar{P} \vee q)$
{7}	(9)	$\overline{q \wedge r}$	Rule T, (8) & (De Morgan's law)
{1,2,4,7}	(10)	$(q \wedge r) \wedge (\overline{q \wedge r})$	Rule T, (6), (9) & conjunction

Since, we obtain $(q \wedge r) \wedge (\overline{q \wedge r})$ which is a contradiction, so we conclude that given system is inconsistent.

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Ex= Using Indirect method, show that
$(r \to \bar{q},\ r \vee s,\ s \to \bar{q},\ p \to q) \to \bar{p}$

Sol= Following the indirect method, we introduce p as an additional premise & show that this additional premise leads to a contradiction.

{1}	(1)	$p \to q$	Rule P
{2}	(2)	p	Rule P (assumed)
{1,2}	(3)	q	Rule T, (1),(2) & modus ponens
{4}	(4)	$s \to \bar{q}$	Rule P
{1,2,4}	(5)	$\bar{s}$	Rule T, (3),(4) & modus tollens
{6}	(6)	$r \vee s$	Rule P
{1,2,4,6}	(7)	r	Rule T, (5),(6) & disjunctive syllogism
{8}	(8)	$r \to \bar{q}$	Rule P
{8}	(9)	$\bar{r} \vee \bar{q}$	Rule T, (8) & EQ_{16} $(p \to q \equiv \bar{p} \vee q)$
{8}	(10)	$\overline{r \wedge q}$	Rule T, (8) & De Morgan's law
{1,2,4,6}	(11)	$r \wedge q$	Rule T, (7),(3) & conjunction
{1,2,4,6,8}	(12)	$r \wedge q \wedge \overline{r \wedge q}$	Rule T, (10),(11) & conjunction

Hence it leads to contradiction

★ Predicate Calculus

We use capital letters to represent predicates & lower case letters to represent names of individuals or objects.

A statement can be written symbolically in terms of predicate letters followed by name of object to which predicate is applied.

1. Dog is an animal
2. Cat is an animal

The part, " is an animal" is called a predicate.

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Suppose, the predicate, "is an animal" is represented by predicate letter A, "Dog by d and "Cat" by c. Then the statements (1) & (2) can be written as A(d) & A(c).

* In general, any statement of type "x is S" where S is predicate and x is subject can be written as S(x).

*5 All dogs are animals
6 Every rose is red

we can also write above statements in following way.
5' For all x, if x is a dog then x is an animal.
6' For all x, if x is a rose then x is red.

The symbols (x) or $\forall x$ are called universal quantifier & represent "for all x", "Every x", "for any x".

The symbol $(\exists x)$ represent "there exist some" or "there is at least one" & called essential quantifier.

$D(x)$: x is a dog
$A(x)$: x is an animal
$R(x)$: x is red
$C(x)$: x is rose

5' & 6' can be written as
5". $\forall x \, (D(x) \rightarrow A(x))$
6". $\forall x \, (C(x) \rightarrow R(x))$

GRAPH

A graph $G = (V, E)$ consists of V, a non empty set of vertices and E, a set of edges.
Each edge has either one or 2 vertices.

Self loops: Defined on some vertex. 

Pseudographs: Graphs that may include loops and possibly multiple edges connecting some pair of vertices.

Order of graph (n): No. of finite vertices.

Size of graph: (m) No. of finite edges

$$(n, m) = E(u, v)$$

Undirected graph G consists of set V no. of vertices & E no. of edges such that $e \in E$ is associated with an unordered pair of vertices.

Directed graph (diagraph) (V, E) consists of non empty set of vertices V and a set of directed edges E. Each directed edge is associated with an ordered pair (u, v) is said to start at u and end at v.

Simple directed graph
When a directed graph has no loops and has no multiple directed edges.
Because a simple directed graph has at most one edge associated to each ordered pair of vertices (u, v), we call (u, v) an edge if there is an edge associated to it in graph

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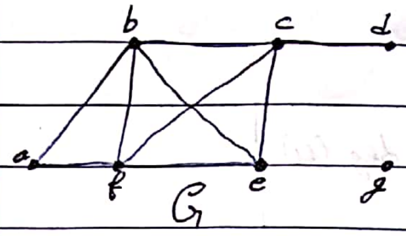
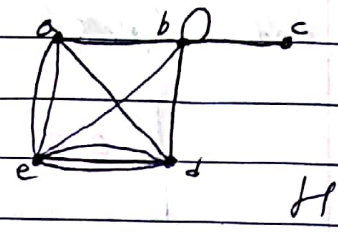
Ex= What are the degrees of vertices in graph G & H?





deg (a) = 2	deg (a) = 4
deg (b) = 4	deg (b) = 6
deg (c) = 4	deg (c) = 1
deg (d) = 1	deg (d) = 5
deg (e) = 3	deg (e) = 6
deg (f) = 4	
deg (g) = 0	

* A vertex of degree 0 is called isolated.
* A vertex is pedant if and only if it has degree one.

☆ Hand Shaking Theorem
Let $G = (V, E)$ be an undirected graph with e edges.

Sum of degree of vertex of an undirected graph is Even.

$$\boxed{2e = \sum_{v \in V} \deg(v)}$$

Ex= How many edges are there in a graph with 10 vertices each of degree 6?

Sol= Because sum of degrees of vertices is $6 \cdot 10 = 60$, it follows that, $2e = 60$.
Therefore $e = 30$.

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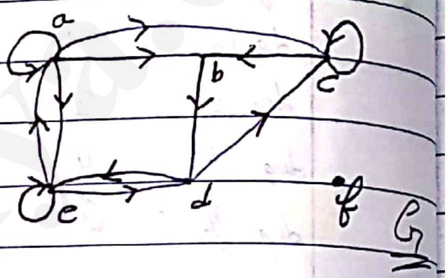
Th. An undirected graph has an even number of vertices of odd degree.

$$\boxed{2e = \sum_{u \in V_1} \deg(u) + \sum_{u \in V_2} \deg(u)}$$

Ex= Find in-degree & out degree of each vertex in graph G with directed edges.

Sol= In degree, $\deg^-(a) = 2$, $\deg^-(b) = 2$, $\deg^-(c) = 3$, $\deg^-(d) = 2$, $\deg^-(e) = 3$, $\deg^-(f) = 0$

Out degree, $\deg^+(a) = 4$, $\deg^+(b) = 1$, $\deg^+(c) = 2$, $\deg^+(d) = 2$, $\deg^+(e) = 3$, $\deg^+(f) = 0$



Th. let G(V,E) be a graph with directed edges then

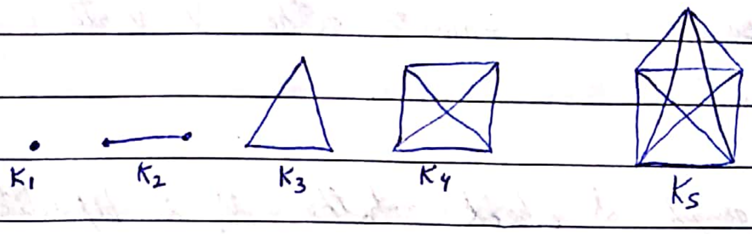
$$\text{No. of edges } |E| = \sum_{v \in V} \deg^-(v) = \sum_{u \in V} \deg^+(u)$$

* Underlying undirected graph
The undirected graph that results from ignoring direction of edges.

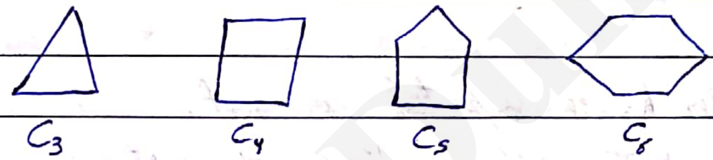
A graph with directed edges & its underlying undirected graph have same no. of edges.

★ Some Special Graphs

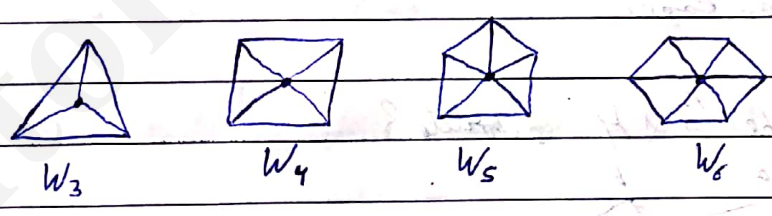
1. **Complete graph** (K_n): Complete graph on n vertices is simple graph that contains exactly one edge b/w each pair of distinct vertices.



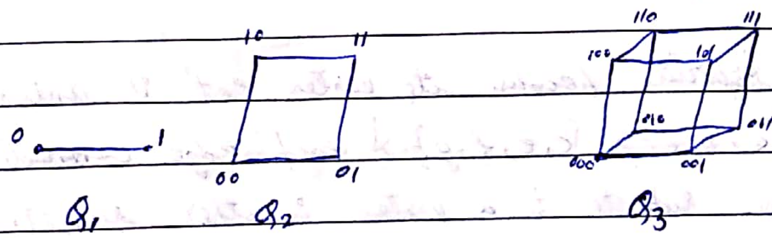
2. **Cycles** (C_n): Cycles ($n \geq 3$) consists of n vertices $1, 2, \ldots, n$ and edges $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}$ & $\{n, 1\}$.



3. **Wheels** (W_n): When we add an additional vertex to cycle, C_n, for $n \geq 3$ and connect this new vertex to each of n vertices in C_n by new edges.



4. **N-dimensional hypercube, or n-cube** (Q_n)
Graph that has vertices representing 2^n bit string of length n.

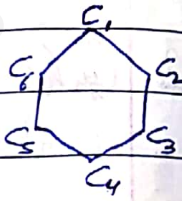


☆ Bipartite

A simple graph G is called bipartite if its vertex set V can be partitioned into 2 disjoint sets V_1 & V_2 such that every edge in graph connects a vertex in V_1 and a vertex V_2, so that no edge in G connects either 2 vertices in V_1 or 2 vertices in V_2.

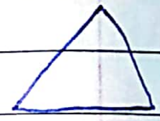
Ex= Draw a C_6 graph & check whether it is bipartite or not.

Ans= C_6 is bipartite because its vertex set can be partitioned into 2 sets $V_1 = \{1,5,3\}$ & $V_2 = \{2,4,6\}$ & every edge of C_6 connects a vertex V_1 & a vertex in V_2.



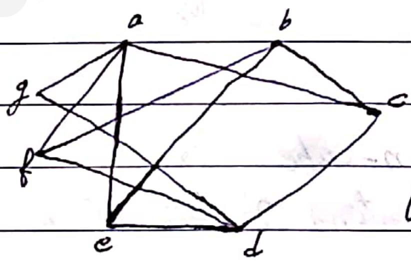
Ex= Check whether K_3 is bipartite or not.

Ans= K_3 is not bipartite, if we divide the vertex set of K_3 into 2 disjoint sets, one of 2 sets must contain 2 vertices. If graph were bipartite, these 2 vertices could not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge.



K_3

Ex= Are graphs G & H bipartite?



G

Graph G is bipartite because its vertex set is union of 2 disjoint sets {a, b, d} & {c, e, f, g} & each edge connects a vertex in one of these subsets to a vertex in other subset.

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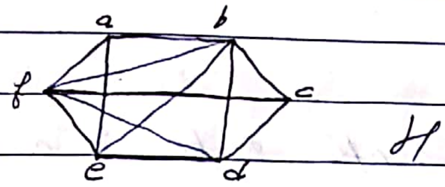


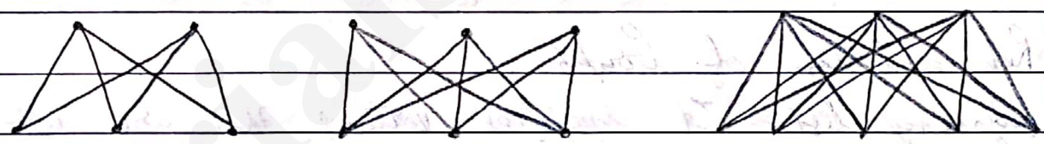
Graph H is not bipartite because its vertex set cannot be partitioned into 2 subsets so that edges do not connect 2 vertices from same subset.

Th. A simple graph is bipartite if and only if it is possible to assign one of 2 diff. colors to each vertex of graph so that no 2 adjacent vertices are assigned same color.

☆ Complete Bipartite Graph ($K_{m,n}$)

Graph that has its vertex set partitioned into 2 subsets of m and n vertices, respectively.

There is an edge b/w 2 vertices if and only if one vertex is in first subset & other vertex is in 2nd subset.

Ex= 

$K_{2,3}$ $K_{3,3}$ $K_{3,5}$

✱ Regular graph: Graph in which all vertices are at regular degree.

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☆ Matching

A matching in a simple graph is a set of edges without common vertex.

* It may also be an entire graph consisting of edges without common vertices.

* A maximum matching is a matching m of graph G with property that if any edge not on 'm' is added to m is no longer a matching.

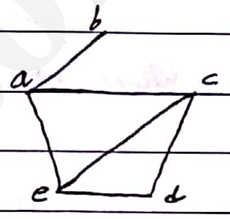
* A maximal matching is a matching with largest no. of edges.

* Every maximum matching is maximal but vice versa is not true.

☆ Representation of Graph

Adjacency list ⇒ It specifies vertices that are adjacent to each vertex of graph and has no multiple edges.

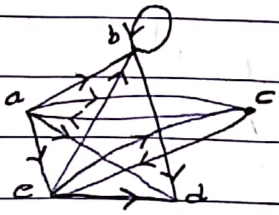
Ex ⇒



vertex	adjacent vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Adjacency list

Ex= Represent the directed graph by listing all vertices that are terminal vertices of edges starting at each vertex of graph.



Initial vertex	Terminal vertices
a	b, c, d, e
b	b, d
c	a, c, e
d	—
e	b, c, d

☆ Adjacency Matrix

Adjacency matrix $A = [a_{ij}]$

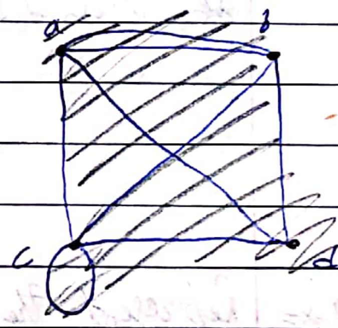
$$a_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

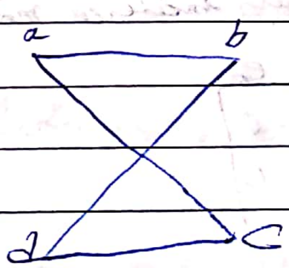
Ex= Draw a graph with adjacency matrix w.r.t. ordering of vertices a, b, c, d.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Sol=

vertex	adjacency vertices
a	b, c
b	a, d
c	a~~b~~, d
d	b, ~~d~~c





adjacency Matrix

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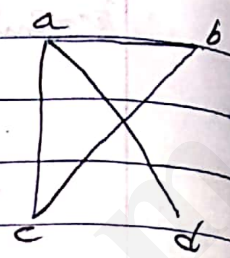
Ex= Use an adjacency matrix to represent the graph.

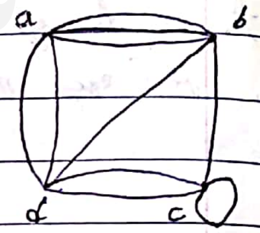


Sol=

vertex	adjacent vertices
a	b, d, c
b	a, c
c	a, b
d	a

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Ex= Use an adjacency matrix to represent pseudograph.



Pseudograph

Sol=

vertex	adjacent vertices
a	b, b, b, d, d
b	a, a, a, c, d
c	b, c, d, d
d	a, a, b, c, c

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

☆ Incidence Matrices

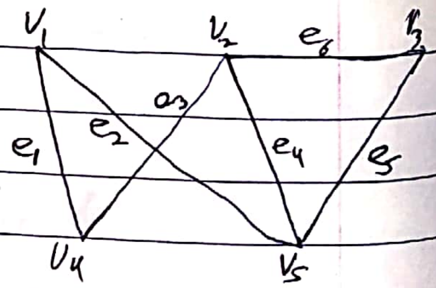
Suppose that $1, 2, \ldots, n$ are vertices & $e_1, e_2, \ldots, e_m$ are edges of G. Then incidence matrix with respect to this ordering of V and E is $n \times m$ matrix $M = [m_{ij}]$ where

$$m_{ij} = \begin{cases} 1 & \text{where edge } e_j \text{ is incident with } i \\ 0 & \text{otherwise} \end{cases}$$

Ex= Represent the graph with incidence matrix.

Sol=

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$



undirected graph

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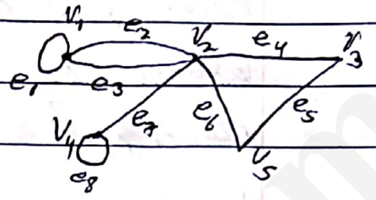
Ex= Represent the pseudograph using an incidence matrix.

Sol=

$$\begin{array}{c|cccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}$$

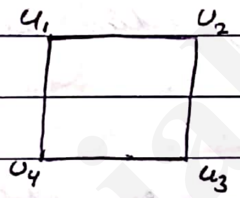


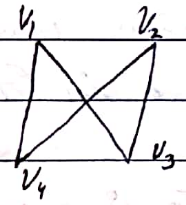
Pseudograph

☆ Graph Isomerism

Simple graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ are isomorphic if there is a one-one and onto function f from V_1 to V_2 with property that a and b are adjacent in G_1 if and only if $f(a)$ & $f(b)$ are adjacent in G_2, for all a and b in V_1.

Ex= Show that graphs G & H are isomorphic.

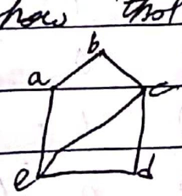
G


H


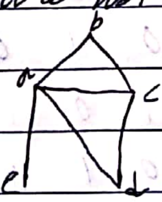
Sol= Let $f(u_1) = v_3$, $f(u_2) = v_2$, $f(u_3) = v_4$, $f(u_4) = v_1$

$$\begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline u_1 & 0 & 1 & 0 & 1 \\ u_2 & 1 & 0 & 1 & 0 \\ u_3 & 0 & 1 & 0 & 1 \\ u_4 & 1 & 0 & 1 & 0 \end{array} \qquad \begin{array}{c|cccc} & v_3 & v_2 & v_4 & v_1 \\ \hline v_3 & 0 & 1 & 0 & 1 \\ v_2 & 1 & 0 & 1 & 0 \\ v_4 & 0 & 1 & 0 & 1 \\ v_1 & 1 & 0 & 1 & 0 \end{array}$$

Ex= Show that following graphs are not isomorphic.



G



H

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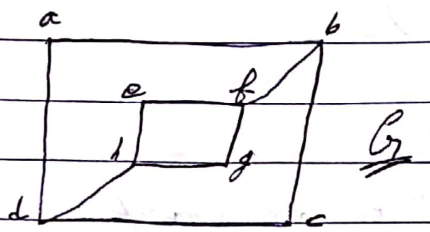
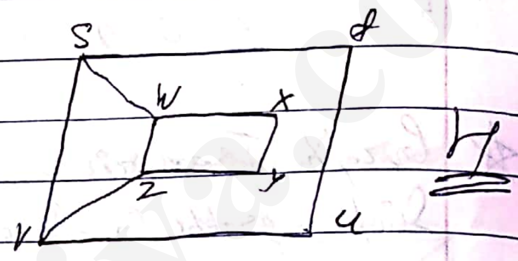
Sol= Both graphs G & H have 5 vertices & 6 edges.
However, H has a vertex of degree 1, e, whereas G has no vertices of degree 1.
Hence, G & H are not ~~isop~~ isomorphic.

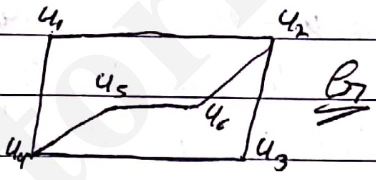
Ex= Determine whether graphs are isomorphic.

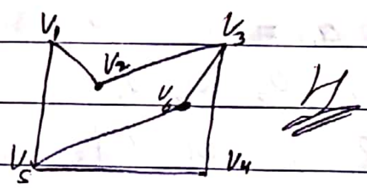




Sol= In G, $\deg(a) = 2$, so a must correspond to either t, u, x or y in H, because these are vertices of degree 2 in H.
However, each of these 4 vertices in H is adjacent to another vertex of degree 2 in H, which is not true for a in G.

Ex= Determine whether graphs G & H are isomorphic.





Sol= Let $f(u_1) = v_6$, $f(u_2) = v_3$, $f(u_3) = v_4$, $f(u_4) = v_5$, $f(u_5) = v_1$, $f(u_6) = v_2$

$$
\begin{array}{c}
 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6
\end{array}
\begin{array}{c}
\begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
\qquad
\begin{array}{c}
 \\ 6 \\ 3 \\ 4 \\ 5 \\ 1 \\ 2
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 6 & 3 & 4 & 5 & 1 & 2 \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

G Isomorphic H

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☆ Connectivity

Path is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of graph.

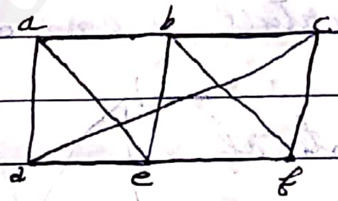
* A path is a circuit if it begins and ends at same vertex and has length greater than zero.

* A circuit in a graph is also called as ~~cyl~~ cycle in a graph.

* A path or circuit is simple if it does not contain same edge more than once.

The term walk is used instead of path where a walk is defined to be an alternating sequence of vertices & edges of a graph.
Closed walk is used instead of circuit to indicate a walk that begins and ends at same vertex.
Trail [simple path] is used to denote a walk that has no repeated edge.
Path is often used for a trail with no repeated vertices.

Ex =



Simple graph

Sol = a, d, c, f, e is a simple path of length 4 because {a, d}, {d, c}, {c, f} and {f, e} are all edges.
⇒ d, e, c, a is not a path because {e, c} is not an edge.
→ b, c, f, e, b is a circuit of length 4 because {b, c} {c, f} {f, e} & {e, b} are edges and this path begins and ends at b.

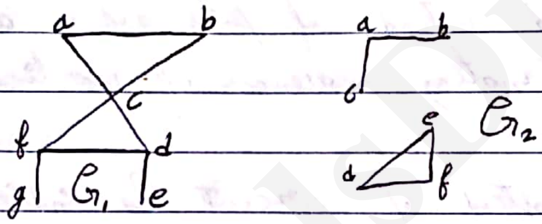
→ Path a, b, c, d, a, b, which is of length 5, is not simple because it contains edge {a, b} twice.

★ **Connected undirected Graph**
If there is a path b/w every pair of distinct vertices of graph.

Theorem 1. There is a simple path b/w every pair of distinct vertices of a connected undirected graph.

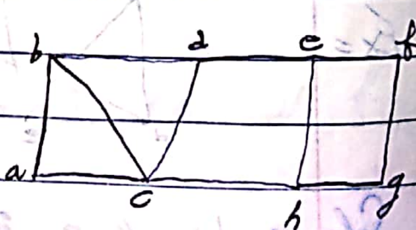
★ **Connected Component** of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G.

Ex =

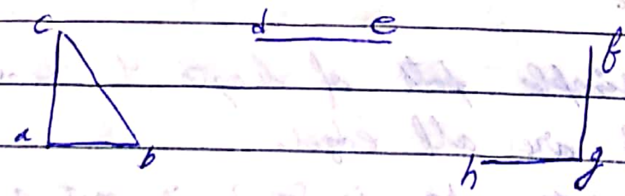


Sol = The graph G_1 is connected because every pair of distinct vertices there is a path b/w them. Graph G_2 is not connected because there is no path in G_2 b/w vertices a and d.

Ex = What are connected components of



Sol =



These 3 subgraphs are connected components.

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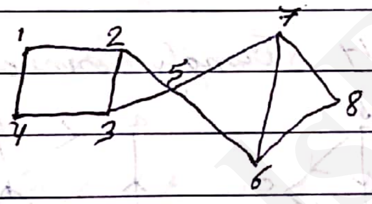
Cut Vertices: Removal of a vertex & all edges incident with it produces a subgraph with more connected components than is original graph.

Cut Edge [bridge]: Edge whose removal produces a graph with more connected components than in original graph.

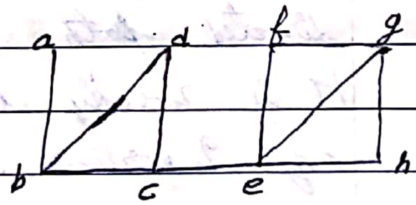
Cut Set [cocycle]: Set of edges whose removal from G leaves G disconnected, provided removal of no proper subset of these edges disconnects G.

Ex =



Sol = {(2,5),(3,5)} is a cut set.
{(1,2), (2,3), (3,5)} is also a cut set.
{(1,2), (2,3), (3,5), (2,5)} is not a cut-set because a proper subset of this is a cut-set.
Cut-set {(1,2),(2,3),(3,5)} has 3 edges whereas cut-set {(2,5),(3,5)} has 2 edges. {(5,6),(5,7)} is also a cut-set with 2 edges.
These are called minimal cut-sets or simple cut-sets.

Ex = Find cut vertices & cut edges in graph



Sol = Cut vertices: b, c, e
Cut edges: {a, b} and {c, e}

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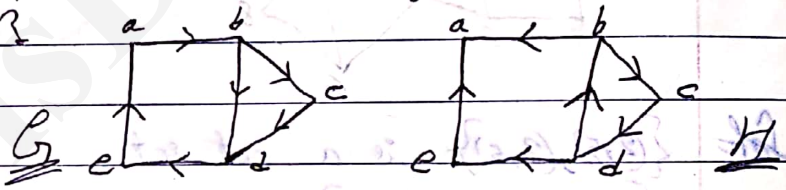
☆ Connected Directed Graph

A directed graph is strongly connected if there is a path from a to b and from b to a whenever a & b are vertices in the graph.

A directed graph is weakly connected if there is a path b/w every 2 vertices in underlying undirected graph.

The subgraphs of a directed graph G that are strongly connected but not contained in larger strongly connected subgraphs, i.e., maximal strongly connected subgraphs are called Strongly Connected components.

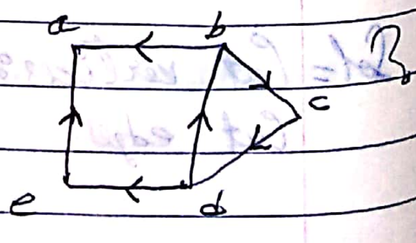
Ex= Are directed graphs G & H strongly connected or weakly connected?



Sol:- G is strongly connected because there is a path b/w any 2 vertices in this directed graph.
Hence, G is also weakly connected.

H is not strongly connected because there is no directed path from a to b in this graph.
H is weakly connected because there is a path b/w any 2 vertices in underlying undirected graph of H.

Ex= What are strongly connected components of



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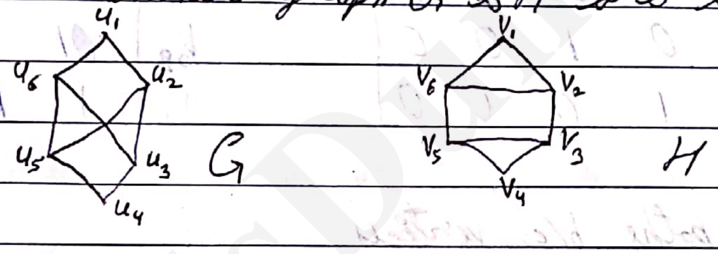
Sol= The graph H has 3 strongly connected components, consisting of vertex a, vertex e and graph consisting of vertices b, c, d and edges (b,c) (c,d) (d,b).

★ Path in Graph Isomerism

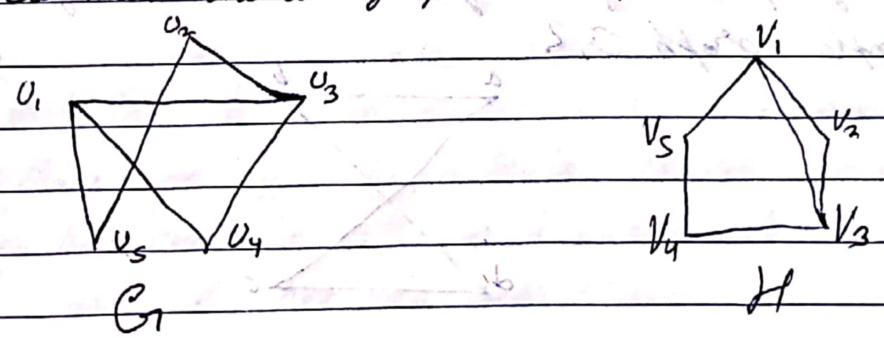
* Graph isomorphic invariant
① Same no. of vertices
② Same no. of edges
③ Same degree of vertices

Ex= Determine whether graph G & H are isomorphic.



Sol= Both G and H have 6 vertices & 8 edges. Each has 4 vertices of degree 3 and 2 vertices of degree 2.
So, all 3 invariants agree for 2 graphs.
However, H has a simple circuit of length 3, v_1, v_2, v_6, v_1
whereas G has no simple circuit of length 3.
Because existence of simple circuit of length 3 is an isomorphic invariant, G & H are not isomorphic.

Ex= Determine whether graphs G & H are isomorphic or not?



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Sol:- Both G & H have 5 vertices and 6 edges, both have 2 vertices of degree 3 and 3 vertices of degree 2 & both have a simple circuit of length 3, a simple circuit of length 4 and a simple circuit of length 5. Because all these isomorphic invariants agree, G & H may be isomorphic.

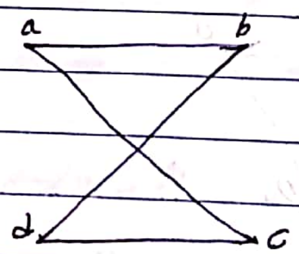
$f(u_1) = V_3$, $f(u_2) = V_2$, $f(u_3) = V_1$, $f(u_4) = V_5$, $f(u_5) = V_4$

$$\begin{array}{c|ccccc} & u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline u_1 & 0 & 0 & 1 & 1 & 1 \\ u_2 & 0 & 0 & 1 & 0 & 1 \\ u_3 & 1 & 1 & 0 & 1 & 0 \\ u_4 & 1 & 0 & 1 & 0 & 0 \\ u_5 & 1 & 1 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|ccccc} & V_3 & V_5 & V_1 & V_2 & V_4 \\ \hline V_3 & 0 & 0 & 1 & 1 & 1 \\ V_5 & 0 & 0 & 1 & 0 & 1 \\ V_1 & 1 & 1 & 0 & 1 & 0 \\ V_2 & 1 & 0 & 1 & 0 & 0 \\ V_4 & 1 & 1 & 0 & 0 & 0 \end{array}$$

☆ Counting paths b/w vertices
No. of paths b/w 2 vertices in a graph can be determined using its adjacency matrix.

Theorem 2: Let G be a graph with adjacency matrix A w.r.t. ordering 1, 2, ..., n. The no. of diff. paths of length r from i to j, where r is a positive integer, equals (i,j)th entry of A^r.

Ex:- How many paths of length 4 are there from a to d in simple graph G?



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Sol= Adjacency matrix of G is

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Hence, no. of paths of length 4 from a to d is (1, 4)th entry of A^4.

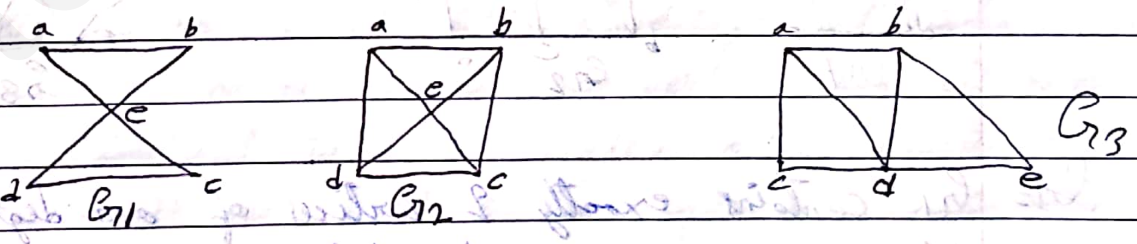
$$A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

There are exactly 8 paths of length 4 from a to d.

Euler circuit in a graph G is a simple circuit containing every edge of G.

Euler path in G is a simple path containing every edge of G.

Ex= Which of undirected graph have an euler circuit?



Sol= Graph G_1 has an euler circuit, a, e, c, d, e, b, a.
Neither of graphs G_2 or G_3 has euler circuit.
G_3 has euler path a, c, d, e, b, d, a, b.
G_2 does not have an euler path.

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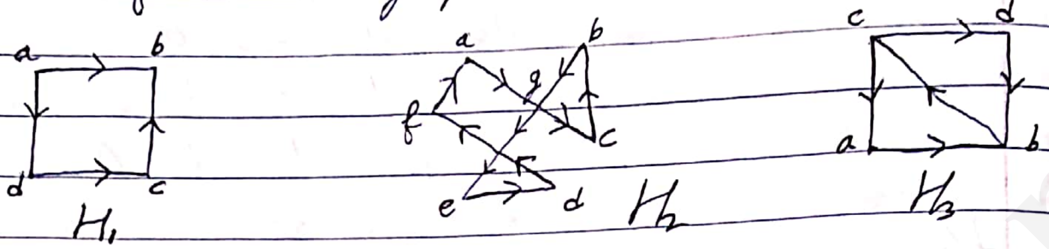
Ex= Which of directed graphs have euler circuit?

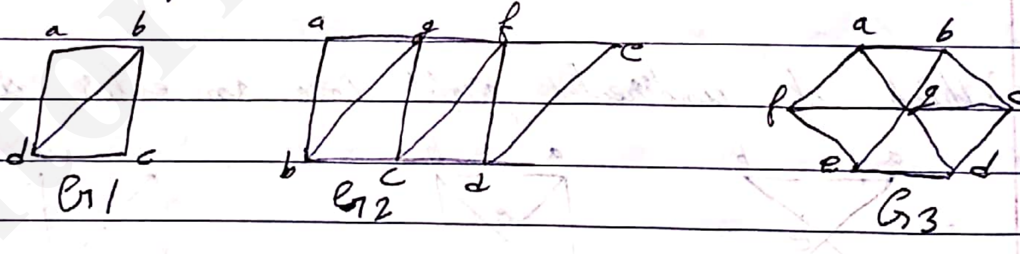


Sol= H_2 has euler circuit a, g, c, b, g, e, d, f, a.
Neither H_1 nor H_3 has euler circuit.
H_3 has euler path c, a, b, c, d, b. but H_1 does not.

Theorem 1 A connected multigraph with at least 2 vertices has a euler circuit if & only if each of its vertices has even degree.

Theorem 2 A connected multigraph has an euler path but not an euler circuit if & only if it has exactly 2 vertices of odd degree.

Ex= Which graphs have an euler path?



Sol= G1 contains exactly 2 vertices of odd degree, d & b. Hence it has an euler path that must have b & d as its endpoints.
G2 has exactly 2 vertices of odd degree, b & d. So it has an euler path that must have b & d as endpoints.
G3 has no euler ~~circuit~~ path because it has 6 vertices of odd degree.

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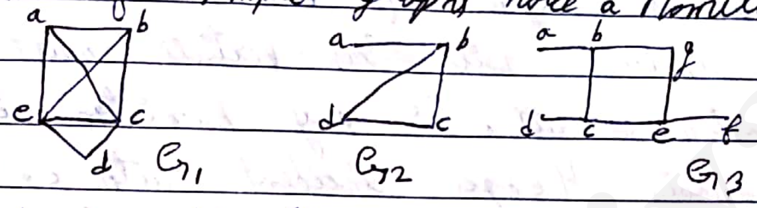
Hamilton Path: A simple path in a graph G that passes through every vertex exactly once.

Hamilton circuit: A simple circuit in a graph G that passes through every vertex exactly once.

Ex= Which of simple graphs have a Hamilton circuit.

Sol=



G_1 has a Hamilton circuit a, b, c, d, e, a.

There is no Hamilton circuit in G_2 but G_2 does have a Hamilton path a, b, c, d.

G_3 has neither a Hamilton circuit nor a Hamilton path because any path containing all vertices must contain one of edges {a, b}, {e, f} and {c, d} more than once.

* A graph with a vertex of degree one cannot have a Hamilton circuit because in a Hamilton circuit, each vertex is incident with 2 edges in circuit.

* When a hamilton circuit is being constructed & this circuit has passed through a vertex then all remaining edges incident with this vertex, other than 2 used in circuit, can be removed from consideration.

* Hamilton circuit cannot contain a smaller circuit within it.

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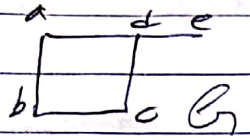
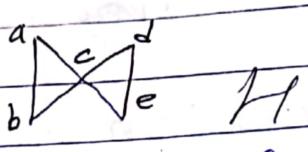
Ex= Show that neither graph has Hamilton circuit?





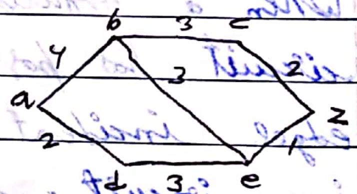
Sol= There is no Hamilton circuit in G because G has a vertex of degree one, e.
In H, because degree of vertices a, b, d & e are all 2, every edge incident with these vertices must be part of any Hamilton circuit. For any Hamilton circuit would have to contain 4 edges incident with c, which is impossible.

DIRAC's Th. If G is a simple graph with n vertices with $n \geq 3$ such that degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

ORE'S Th. If G is a simple graph with n vertices with $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u & v in G then G has Hamilton circuit.

Ex= What is the length of a shortest path b/w a and z in weighted graph.

Sol= The length of a shortest path from 'a' to successive vertices, until z is reached is a, d, e, z of length 6.



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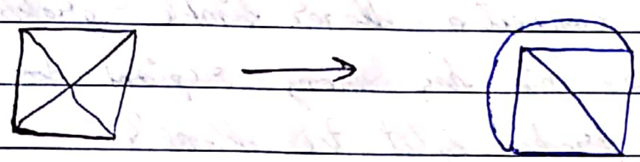
Planar Graphs

A graph is called planar if it can be drawn in plane without any edges crossing.

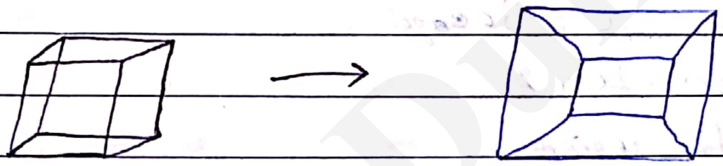
Ex= Is K_4 planar?

Sol: K_4 is planar because it can be drawn without crossings.

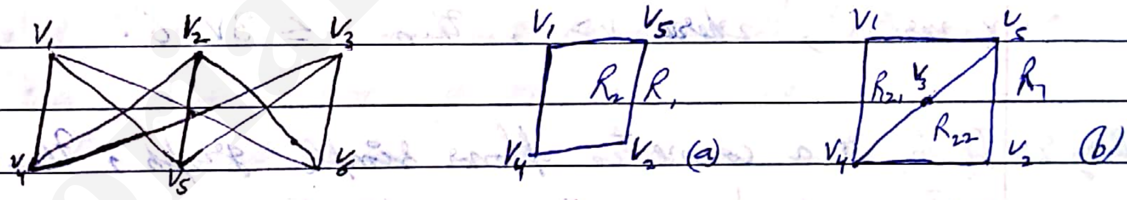


Ex= Is Q_3 planar?

Sol: Q_3 is planar because it can be drawn without crossing.



Ex= Is $K_{3,3}$ planar?



In any planar representation of $K_{3,3}$, vertices V_1 & V_2 must be connected to both V_4 & V_5. These 4 edges form a closed curve that splits the plane into 2 regions R_1, R_2. The vertex V_3 is in either R_1 or R_2. When V_3 is in R_2, inside of closed curve, edges b/w V_3 & V_4 & b/w V_3 and V_5 separate R_2 into 2 subregions, R_{21} and R_{22}.

Now, there is no way to place find vertex V_6 without forcing a crossing.

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* Euler's Formula

Let G be a connected planar simple graph with e edges & v vertices. Let r be no. of regions in a planar representation of G.

$$\boxed{r = e - v + 2}$$

Ex= Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Sol= This graph has 20 vertices.

$$3v = 2e$$

$$3 \times 20 = 2 \times e$$

$$e = 30 \text{ edges}$$

$$r = 30 - 20 + 2$$

$$r = 12 \text{ regions}$$

Corollary 1 If G is a connected planar simple graph with e edges & v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Corollary 2 If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

* Degree of a region is number of edges on boundary of region. When an edge occurs twice on boundary, it contributes 2 to degree.

Ex= Show that K_5 is non planar using Corollary 1.

Sol= The graph K_5 has 5 vertices & 10 edges. However the inequality $e \leq 3v - 6$ is not satisfied for this graph because $e = 10$, $3v - 6 = 9$.

Therefore, K_5 is not planar.

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Corollary 3: If a connected planar simple graph has e edges and v vertices with $v \geq 3$ & no circuits of length 3, then $e \leq 2v-4$.

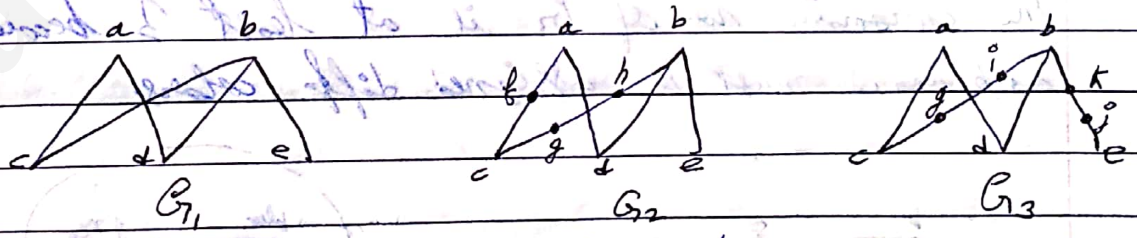
Ex= Use Corollary 3 to show that $K_{3,3}$ is non planar.
Sol= Because $K_{3,3}$ has no circuit of length 3.
$K_{3,3}$ has 6 vertices & 9 edges. Because $e=9$ & $2v-4=8$, hence $K_{3,3}$ is non planar.

★ Kuratowski's Theorem
A graph is non planar if & only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

If a graph is planar, so will be any graph obtained by removing an edge $\{u,v\}$ & adding a new vertex w together with edges $\{u,w\}$ & $\{w,v\}$. Such an operation is called an elementary subdivision.

* Graphs G_1 & G_2 are homeomorphic if they can be obtained from some graph by a sequence of elementary subdivision.

Ex=1: Show that graphs G_1 & G_2 & G_3 are all homeomorphic.



Sol= All 3 graphs are homeomorphic because all 3 can be obtained from G_1 by elementary subdivision.
G_1 can be obtained from itself by an empty sequence of elementary subdivision.

To obtain G_2 from G_1 we can use this sequence of elementary subdivisions; (i) remove edge {a, c}, add vertex f, add edges {a, f} & {f, c}; (ii) remove edge {b, c} add vertex g, add add edges {b, g} & {g, c}; (iii) Remove edge {b, g}, add vertex h and add edges {g, h} & {h, b}.

★ Graph Coloring

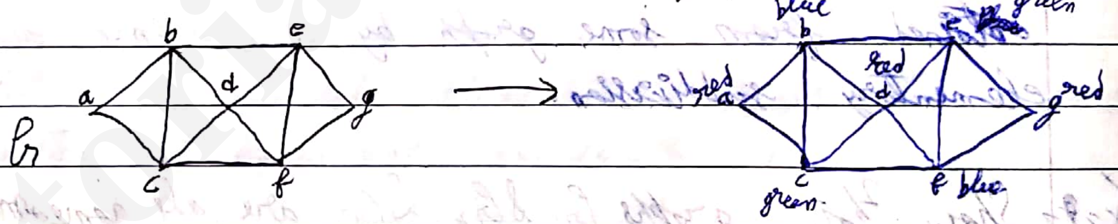
A coloring of simple graph is assignment of a color to each vertex of graph so that no 2 adjacent vertices are assigned same color.

The Chromatic Number $\chi(G)$ of a graph G is least no. of colors needed for a coloring of this graph.

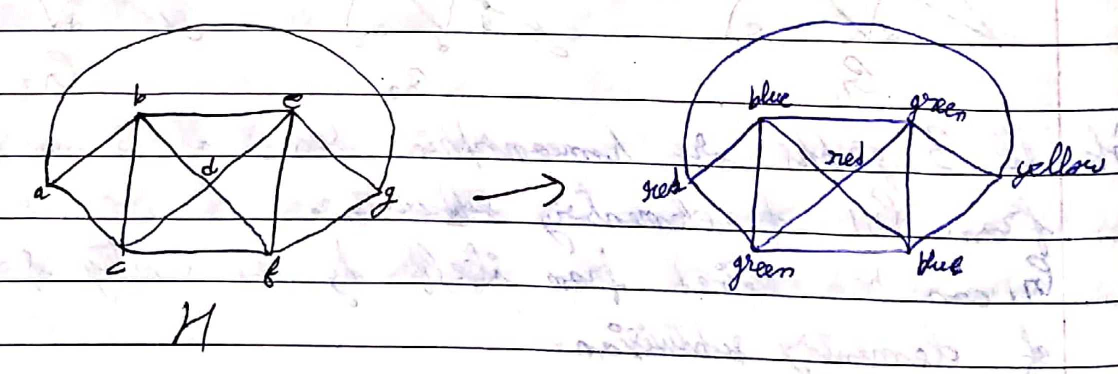
* Four color Theorem

Chromatic no. of a planar graph is no greater than 4.

Ex = What are chromatic no. of graphs G & H?

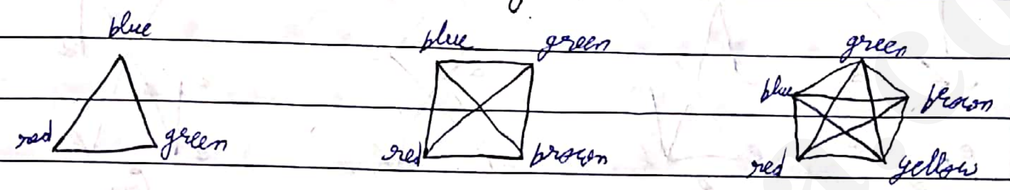


The chromatic no. of G is at least 3 because vertices a, b and c must be assigned diff. colors.



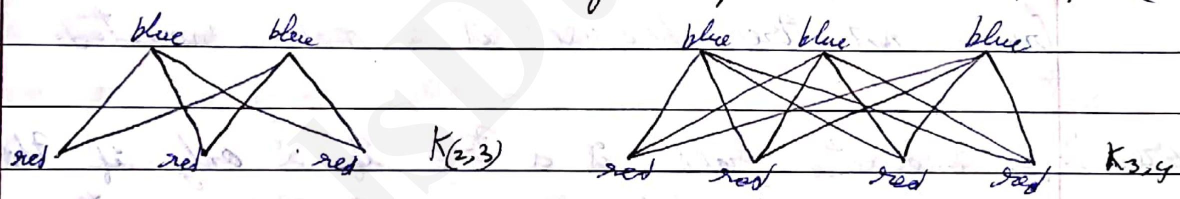
Any attempt to color H using 3 colors must follow same reasoning as that used to color G_1, except at last stage, when all vertices other than g have been colored. Hence, H has a chromatic no equal to 4.

Ex:- What is chromatic no. of K_n?



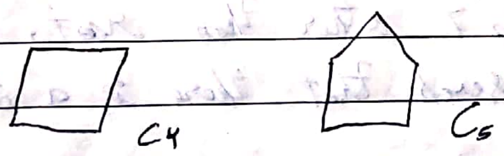
Sol:- A coloring of K_n can be constructed using n colors by assigning a diff. color to each vertex.

Ex:- What is chromatic no. of complete bipartite graph $(K_{m,n})$?



Sol:- Only 2 colors are needed to color $K_{m,n}$ graph because $K_{m,n}$ is a bipartite graph.
We can color the set of m vertices with one color & set of n vertices with a second color.

Ex:- What is chromatic no. of graph C_n, where $n \geq 3$?



Sol:- In general, 2 colors are needed to color C_n when n is even. When n is odd & $n>1$, chromatic number is 3.

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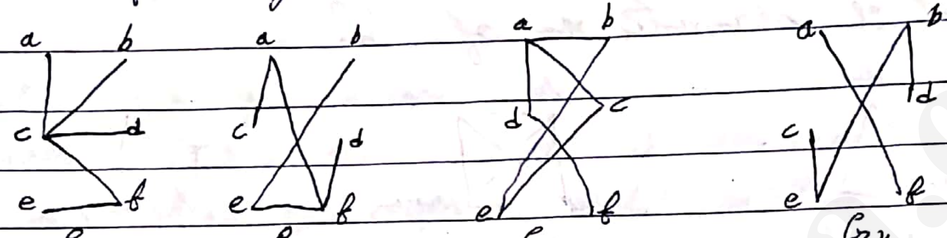
☆ TREES

A tree is a connected undirected graph with no simple circuit.

Ex= Which of the graphs are trees?



G_1 G_2 G_3 G_4

Sol= G_1 & G_2 are trees because both are connected graphs with no simple circuit.
G_3 is not a tree because e, b, a, d, e is a simple circuit in this graph.
G_4 is not a tree because it is not connected.

Theorem 1: An undirected graph is a tree if & only if there is a unique simple path b/w any 2 of its vertices.

* Rooted Tree

A tree in which one vertex has been designated as root vertex.

Suppose that T is a rooted tree.
If v is a vertex in T other than root, parent of v is unique vertex u such that there is a directed edge from u to v.
When u is parent of v, v is called child of u.
Vertices with same parents are called siblings.

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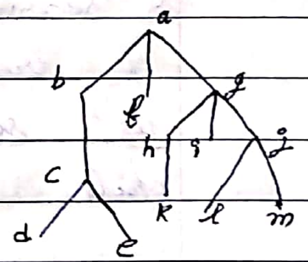
Ancestors of a vertex other than root are vertices in path from root to this vertex, excluding itself & including root.
Descendants of a vertex v are those vertices that have v as an ancestor.
A vertex of a tree is called leaf if it has no children.
Vertices that have children are called internal vertices.
If a is a vertex in a tree, subtree of a as its root is subgraph of tree consisting of a and its descendants and all edges incident to these descendants.

* m-ary tree

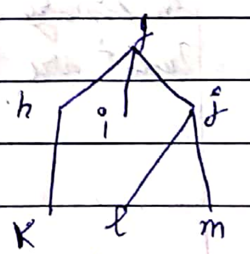
A rooted tree is called an m-ary tree if every internal vertex has no more than m children.
The tree is called a full m-ary tree if every internal vertex has exactly m children.
An m-ary tree with m=2 is called a binary tree.

Ex= In a rooted tree T (with root a), find parent of c, children of g, sibling of h, all ancestors of e, all descendents of b, all internal vertices & all leaves. What is a subtree rooted at g?



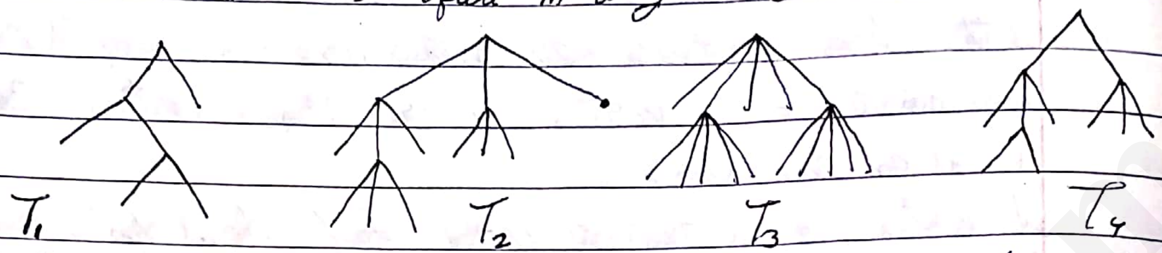
rooted tree T

Sol= Parent of c is b
children of g are h, i, j.
Siblings of h are i & j.
Ancestors of e are c, b & a.
Descendants of b are c, d & e.
Internal vertices are a, b, c, g, h & j
Leaves are d, e, f, i, k, l & m



Subtree rooted at g

Ex= Are rooted trees full m-ary tree?

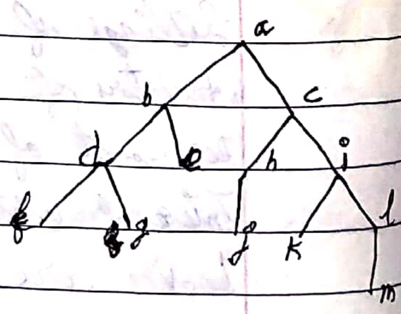


Sol= T_1 is full binary tree because each of its internal vertices has 2 children.
T_2 is full 3-ary tree because each of its internal vertices has 3 children.
In T_3 each internal vertex has 5 children, so T_3 is a full 5-ary tree.
T_4 is not a full m-ary tree for any m because some of its internal vertices have 2 children & others have 3 children.

* Ordered rooted tree

An ordered rooted tree is a rooted tree where children of each internal vertex are ordered.
In an ordered binary tree if an internal vertex has 2 children, first child is called left child. & second is called Right child.
If tree rooted at left child of a vertex is called left subtree of this vertex & tree rooted at right child of a vertex is called right subtree of vertex.

Ex= What are left & right children of d in binary tree T?
What are left & right subtree of c?



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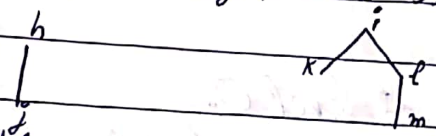
Sol: Left child of d is f & right child is g.



~~left~~ left subtree right subtree

☆ Properties of Trees

1. A Tree with n vertices has $n-1$ edges.
2. A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.
3. A full m-ary tree with
   (i) n vertices has $i = (n-1)/m$ internal vertices & $l = [(m-1)n+1]/m$ leaves.
   (ii) i internal vertices has $n = mi+1$ vertices & $l = (m-1)i + 1$ leaves.
   (iii) l leaves has $n = (ml-1)/(m-1)$ vertices & $i = (l-1)/(m-1)$ internal vertices.
4. There are atmost m^h leaves in an m-ary tree of height h.

* A rooted m-ary tree of height h is Balanced if all leaves are at level h or $h-1$.

Corollary 1: If an m-ary tree of height h has l leaves, then $h \geq \lceil \log_m l \rceil$.
If m-ary tree is full & balanced, then $h = \lceil \log_m l \rceil$.

* Level of a vertex in a rooted tree is length of unique path from root to this vertex.
Level of root is defined to be 0.

* Height of rooted tree is max. levels of vertices.

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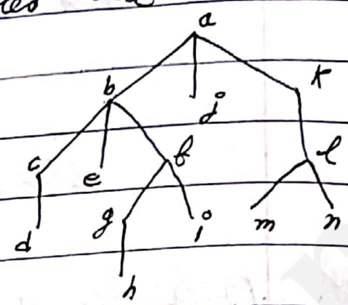
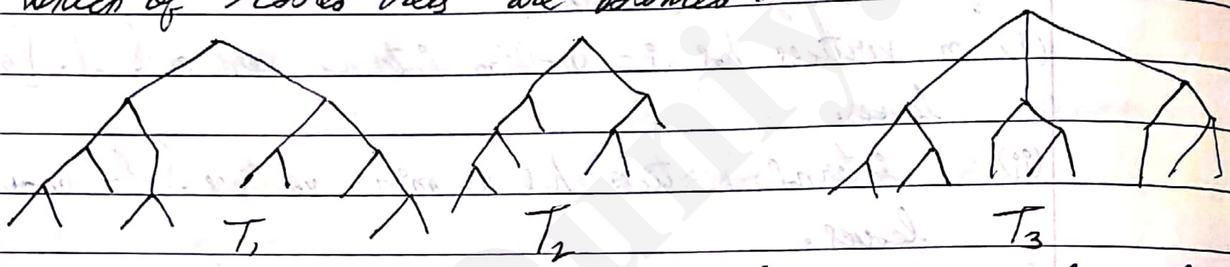
Ex= find level of each vertex in rooted tree?



Sol= Root a is at level 0.
Vertices b, j & k are at level 1.
Vertices c, e, f & l are at level 2.
Vertices d, g, i, m & n are at level 3.
Vertex h is at level 4.
Because largest level of vertex is 4, height of tree is 4.

Ex= Which of rooted trees are balanced?



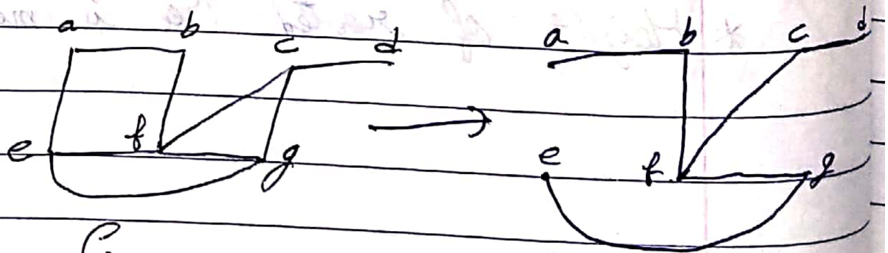
Sol= T_1 is balanced because all its leaves are at level 3 & 4.
T_2 is not balanced because it has leaves at levels 2, 3 & 4.
T_3 is balanced because all its vertices are at level 3.

☆ Spanning Tree

Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.

Theorem: A simple graph is connected if & only if it has a spanning tree.

Ex= Find a spanning tree of simple graph G.



G

Spanning Tree

Sol= Graph G is connected but it is not a tree because it contains simple circuits. Remove the edge {a, e}. This eliminates one simple circuit & resulting subgraph is still connected & still connects every vertex of G.

Next, remove edge {e, f} to eliminate a second simple circuit.

Finally, remove edge {c, g} to produce a simple graph with no simple circuit.

This subgraph is a spanning tree because it is a tree that contains every vertex of G.

* Depth First ~~Algorith~~ Search [DFS]

Algorithm: procedure DFS(G: connected graph with vertices $v_1, v_2, \dots, v_n$)
T: = Tree consisting only of vertex v_1.
visit (v_1)
procedure visit (: vertex of G)
for each vertex w adjacent to & not yet in T
begin add vertex w & edge {v, w} to T
visit (w)
end

Depth first Search is also called Backtracking because algorithm returns to vertices previously visited to add paths.

Tree edges: edges selected by depth first search of graph.

Back edges: All other edges of graph must connect a vertex to an ancestor or descendant of this vertex in tree.

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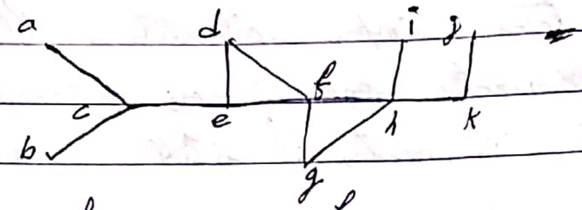
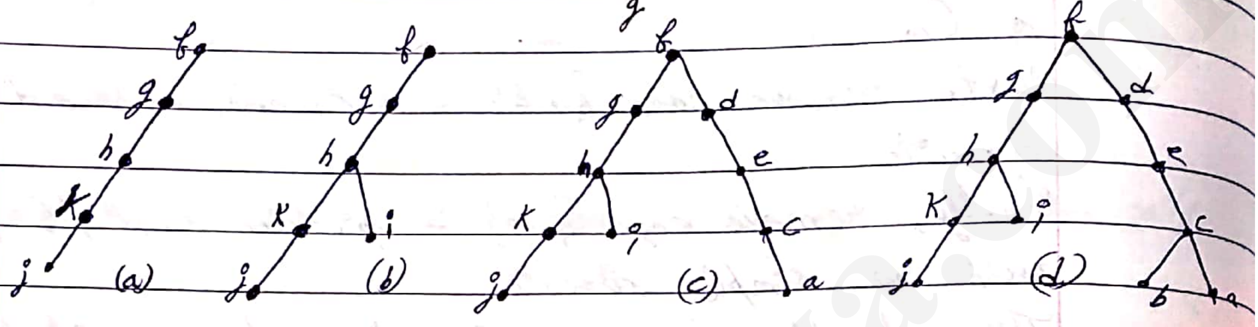
Ex= Use depth first search to find a spanning Tree for graph G.



Sol=



The steps used by DFS to produce a spanning Tree of G.
We started with vertex f. A path is built by successively adding edges incident with vertices not already in path. This produces a path f,g,h,k,j.
Next, backtrack to k. There is no path beginning at k containing vertices not already visited.
So we backtrack to h, form path h,i. Then backtrack to h & then to f.
From f build the path f,d,e,c,a. Then backtrack to c & form path c,b. This produce spanning tree.