Unit - II

(Matrix - Algebra)

$A =$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

$$= (a_{ij})_{3\times 3}$$

Symmetric matrix

$A = (a_{ij})_{m\times n}$ is said to be symmetric if

$$A' = A$$

$$\boxed{a_{ji} = a_{ij}}$$

e.g: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}$, $A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

$\Rightarrow A' = A \Rightarrow A$ is symmetric.

skew symmetric matrix

A matrix $A = (a_{ij})_{m\times n}$ is said to be skew symmetric if $A' = -A$

$$\boxed{a_{ji} = -a_{ij}}$$

$$\left(\Rightarrow a_{21} = -a_{12} \quad a_{31} = -a_{13}\right)$$

$$\begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{bmatrix}$$

if $j = i$, then

$$a_{ii} = -a_{ii}$$

$$\Rightarrow 2a_{ii} = 0$$

$$\Rightarrow \boxed{a_{ii} = 0}$$

$\Rightarrow$ diagonals of the skew symmetric matrix are 0.

e.g. $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix}$, $A' = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}$

$$\Rightarrow A' = -\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix} = -A$$

Hermitian & skew Hermitian matrix

A matrix A is said to Hermitian if

$$(\bar{A})' = A \quad \text{OR} \quad A^{\theta} = A, \quad A^{\theta} = (\bar{A})'$$

OR $\boxed{\bar{a}_{ji} = a_{ij}}$ $\Rightarrow$ $a_{12} = \bar{a}_{21}$ ✓

$a_{13} = \bar{a}_{31}$ ✓

If $j = i$,

$$\bar{a}_{ii} = a_{ii}$$

If $a_{ii} = x + iy$, then

$$\cancel{x} - iy = \cancel{x} + iy$$

$$\Rightarrow 2iy = 0$$

$$y = 0$$

$$\therefore \boxed{a_{ii} = x}$$

$A = \begin{bmatrix} 1-i & i \\ 1 & -i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 1+i & -i \\ 1 & i \end{bmatrix}$

$A^{\theta} = (\bar{A})' = \begin{bmatrix} 1+i & 1 \\ -i & i \end{bmatrix}$

A is said to be skew hermitian if

$$(\bar{A})' = -A \text{ or } A^{\theta} = -A \quad \text{where } A^{\theta} = (\bar{A})'$$

$$\Rightarrow \bar{a}_{ji} = -a_{ij}$$

If $j = i$,

$$\bar{a}_{ii} = -a_{ii}$$

If $a_{ii} = x + iy$, then

$$x - iy = -(x + iy)$$

$$\Rightarrow x - \cancel{iy} = -x - \cancel{iy}$$

$$\Rightarrow 2x = 0$$

$$x = 0$$

$a_{ij} = \cancel{x +} iy$

$\therefore \boxed{a_{ii} = iy}$ or $a_{ii} = 0$ (if $y = 0$)

$\therefore$ Diagonals are either 0 or pure imaginary.

Idempotent matrix $\rightarrow$ $A^2 = A$

Nilpotent matrix $\rightarrow$ $A^k = 0$, k is known as index of nilpotent matrix.

Involutory matrix $\rightarrow$ $A^2 = I$

Singular matrix $\rightarrow$ $|A| = 0$ otherwise non-singular. $(|A| \neq 0)$

Q Every square matrix can be written as a sum of two matrices one symmetric (or Hermitian) and the other skew symmetric (or skew Hermitian)

$A = X + Y$

$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

$X = \frac{1}{2}(A + A')$

$Y = \frac{1}{2}(A - A')$

Sol: (i) Let $A = \frac{1}{2}\underbrace{(A + A')}_{} + \frac{1}{2}(A - A')$

$= X + Y$

where $X = \frac{1}{2}(A + A')$, $Y = \frac{1}{2}(A - A')$

$X' = \frac{1}{2}(A' + A'')$, $Y' = \frac{1}{2}(A' - A'')$

$= \frac{1}{2}(A' + A)$, $Y' = \frac{1}{2}(A' - A)$

$= -\frac{1}{2}(A - A') = -Y$

$= X$,

$\Rightarrow X' = X$, $Y' = -Y$

$\Rightarrow$ X is symmetric and Y is skew symmetric.

$$A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta}), \quad A^{\theta} = (\bar{A})' \quad \cancel{(\bar{\bar{A}})'' = A}$$

$$= X + Y$$

$$X^{\theta} = \frac{1}{2}(A^{\theta} + A^{\theta\theta}), \quad Y^{\theta} = \frac{1}{2}(A^{\theta} - A^{\theta\theta})$$

$$= \frac{1}{2}(A^{\theta} + A), \quad Y^{\theta} = \frac{1}{2}(A^{\theta} - A) = -\frac{1}{2}(A - A^{\theta}) = -Y$$

$$= X$$

$$\Rightarrow X^{\theta} = X, \quad Y^{\theta} = -Y$$

$\Rightarrow X$ is Hermitian and Y is skew Hermitian.

$\begin{bmatrix} 1 & 2 \\ 3k & 4k \end{bmatrix}$

① $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

Elementary operations

The following operations are known as Elementary operations

$|A| \neq 0$

$A^{-1} = \frac{\text{Adj} A}{|A|}$

$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \sim R_2 \to 2R_1 + R_2$

(i) Interchange of two rows or column

$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

(ii) multiplication of ith row/column by a constant

$R_i \to kR_i$ or $C_i \to kC_i$

$x+y+z=1$
$x-y+z=2$
$2x+3y+4z=5$

(iii) Addition of k times of ith row (column) in jth row (column)

$R_j \to kR_i + R_j$ or $C_j \to kC_i + C_j$

$AX = B$

$X = A^{-1}B$

$|A-\lambda I|=0$

$\downarrow$

$\boxed{\lambda = 1, 2, 3}$

$\boxed{A = AI}$

$AB = I$

$\Rightarrow \bar{A} = B$

$\Rightarrow A$

{ Gauss-Jordan Method }

Inverse of the matrix using E-row operations

$\boxed{A \sim I}$

$\downarrow$

$[\;] \sim [\;]$

$A = A I$

Applying E-row operations we get

$I = A B$

$$\Rightarrow \boxed{A^{-1} = B}$$

$\cancel{AA} = I$

Q Ex: Find the inverse of the following matrix using E-row operations

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$A = A I$

$\downarrow \qquad \downarrow$

$I = A B$

$$\boxed{A^{-1} = B}$$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$A = AI$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 2R_3 - 5R_1$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 2 \end{bmatrix}$$

$R_3 \to 2R_3 + R_2$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -10 & 1 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ -10 & 2 & 4 \\ -10 & 1 & 4 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 11 & -1 & -4 \\ -10 & 2 & 4 \\ -10 & 1 & 4 \end{bmatrix}$$

$R_1 \to 2R_1 - R_2$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 32 & -4 & -12 \\ -10 & 2 & 4 \\ -10 & 1 & 4 \end{bmatrix}$$

$$I = A \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

$-6+5$

$A = AI$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 2R_3 - 5R_1$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 2 \end{bmatrix}$$

$R_3 \to 2R_3 + R_2$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -10 & 1 & 4 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -10 & 1 & 4 \end{bmatrix}$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 1 & 0 \\ -10 & 2 & 4 \\ -10 & 1 & 4 \end{bmatrix}$$

$R_1 \to 2R_1 - 3R_2$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 32 & -4 & -12 \\ -10 & 2 & 4 \\ -10 & 1 & 4 \end{bmatrix}$$

$$I = A \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 15 & -1 & -4 \end{bmatrix} = AB$$

$$\boxed{B = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 15 & -1 & -4 \end{bmatrix}}$$

Sol:-

$A = AI$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 2R_3 - 5R_1$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 2 \end{bmatrix}$$

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} -4 & 0 & 2 \\ 0 & 1 & 0 \\ -5 & 0 & 2 \end{bmatrix}$$

$4-5$

$-6+5$

$R_1 \to R_1 - R_3$

$R_1 \to R_1 - 2R_3$

$R_1 \to R_2 + R_3$

R_2

$R_3 \to 2R_3 + R_2$

R_3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A = I$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} -4 & 0 & 2 \\ -5 & 1 & 2 \\ -5 & 0 & 2 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} -4 & 0 & 2 \\ -5 & 1 & 2 \\ -10 & 1 & 4 \end{bmatrix}$$

$R_1 \to R_1 - 2R_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 16 & -2 & -6 \\ -5 & 1 & 2 \\ -10 & 1 & 4 \end{bmatrix}$$

$R_1 \to \frac{1}{2} R_1 \;,\; R_3 \to R_3 \times -1$

$A A^{-1} = I$ ✓

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

$\therefore \quad I = AB$

$$\Rightarrow A^{-1} = B = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} \quad \text{Ans.}$$

$$\left\{ \because A A^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$ ✓

Q Find the inverse of the matrix using E-row operation

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol:-

$A = AI$

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} 4 & 3 & 1 \\ 4 & 8 & 16 \\ \hline 0 & -5 & -15 \end{array}$$

$R_1 \rightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} = A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -15 \\ 0 & -1 & -4 \end{bmatrix} = A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & -2 \end{bmatrix}$$

-20

$R_3 \to 5R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -15 \\ 0 & 0 & -5 \end{bmatrix} = A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 5 & -1 & -6 \end{bmatrix}$$

$$\begin{array}{r} -10 \\ 4 \\ \hline \end{array}$$

$5 - 24$

$R_1 \to 5R_1 + 4R_3$

$$\begin{bmatrix} 5 & 10 & 0 \\ 0 & -5 & -15 \\ 0 & 0 & -5 \end{bmatrix} = A \begin{bmatrix} 20 & -4 & -19 \\ 0 & 1 & -4 \\ 5 & -1 & -6 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_3$$

$$\begin{bmatrix} 5 & 10 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} = A \begin{bmatrix} 20 & -4 & -19 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

$$\begin{array}{r} 28 \\ -19 \\ \hline \end{array}$$

$$R_1 \to R_1 + 2R_2$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} = A \begin{bmatrix} -10 & 4 & 9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

$$R_1 \to \frac{1}{5}R_1, \quad R_2 \to \frac{-1}{5}R_2, \quad R_3 \to \frac{-1}{5}R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \times \frac{1}{5} \begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$$

$I = AB$

$$\Rightarrow \boxed{A^{-1} = B = \frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}}$$

$\{ A A^{-1} = I \}$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$ 4×5

$\downarrow$

$\boxed{|A| \neq 0}$ ✓

✓ $|A| = 0 \rightarrow$

$\rho(A) < 4$

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $\rho(A) < 2$

$|A| = 2 - 2 = 0$ ✓

$\boxed{A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$ ✓ $\rightarrow \rho(A) = 2$

$|A| = 4 - 6 = -2 \neq 0$

$$\left[\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} 4 \\ 1 \\ 1 \end{matrix}\right]_{3\times 4}$$

Rank of the matrix

Rank of a matrix A is the order of the highest non-zero minor of the matrix.

i.e., A matrix 'A' is said to have rank 'r' if $\exists$ a minor of A whose order is r and determinant is non-zero and all other minors of order (r+1) are zero

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 6 \end{bmatrix}, \quad \boxed{|A| = 1} \neq 0$$

$\rho(A) = 3$

Echleon from

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$|A| \neq 0$ ✓

(0 0 0)

Rank of the matrix is denoted by $\rho(A)$.

$A_{3\times 2}$

$\boxed{\rho(A) \leq 2}$

✓ (i) I_n is the identity matrix of order $n \times n$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ 0 & 0 & 0 & 1 & \cdots \end{bmatrix}$$

$$\rho(I_n) = n.$$

✓ (ii) If A is $m \times n$ matrix then $\rho(A) \leq$ minimum of m and n

$A = [\;]_{3\times 4}$

Ex: (1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$|A| = 1(0-1) - 2(0-0) + 3(1-0)$$

$$= -1 + 3 = 2 \neq 0 \Rightarrow \boxed{\rho(A) = 3}$$

(2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$\cancel{\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}}$ $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

$B = \cancel{\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}}$

$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$|A| = 0 \Rightarrow \boxed{\rho(A) < 3}$

$\because$ All the submatrix of order 2×2 have determinant 0, hence $\rho(A) < 2$

$\Rightarrow \boxed{\rho(A) = 1}$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_1$, $R_2 \to R_2 - R_1$

$\boxed{\rho(A) = 1}$ Ans

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 6 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Echleon form :-

A matrix A is said to be in Echleon form if

(i) First element of the first row is non-zero and is 1

(ii) The number of zeros before the non-zero element in the 2^{nd}, 3^{rd}, 4^{rth} row should be in increasing order.

Rank of the matrix in this case is the number of non-zero rows. (Here we will apply only row or column)

Ex:- (1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow \rho(A) = 3$

(2) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 3$

Normal form of a matrix

A matrix A is said to be in normal form if

$$A = \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right]$$

where I_r is the identity matrix of order $r \times r$ and O is the zero matrix.

e.g.

(i) $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \rho(A) = 2$ (Here I can apply both operations)

(ii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \rho(A) = 3$

(iii) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 3$

Q2 $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$ Find $\rho(A)$ reducing it in Echleon form

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & -2 & -5 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\Rightarrow \rho(A) = 3$$

Q Reduce the following matrix in both Echelon & Normal form.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad \begin{matrix} R_4 \to R_4 - R_1 \\ R_3 \to R_3 - 3R_1 \\ R_2 \to R_2 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -11 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4 \to 2R_4 + R_3$$

$$\Rightarrow \rho(A) = 3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 5 & -11 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} C_4 \to C_4 - 3C_1 \\ C_3 \to C_3 + C_1 \\ C_2 \to C_2 - 2C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -7 & 28 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} C_4 \to 7C_4 - 11C_2 \\ C_3 \to 7C_3 + 5C_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_4 \to C_4 + 4C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} C_2 \to -\frac{1}{7} C_2 \\ C_3 \to \frac{1}{7} C_3 \end{array}$$

$$\Rightarrow \underline{\rho(A) = 3}$$

Q Find non-singular matrices P, Q such that $PAQ = \underline{\underline{\text{Normal form}}}$

where $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}_{3\times 4}$

Sol: $\quad A = I_{3\times 3}\, A_{3\times 4}\, I_{4\times 4}$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \underset{R}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} A \underset{C}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1, \quad R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 6R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & 0 & -28 & -56 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 6 & -1 & -9 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to \frac{-1}{28} R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ \frac{-3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 - 2C_1,\ C_3 \to C_3 - C_1,\ C_2 \to C_2 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ -3/14 & 1/28 & 9/28 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 - 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \downarrow \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 3C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} ,, \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to \frac{-1}{6} C_2, \quad C_3 \to \frac{1}{3} C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ -3/14 & 1/28 & 9/28 \end{bmatrix} A \begin{bmatrix} 1 & 1/6 & -2/3 & 0 \\ 0 & -1/6 & -1/3 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Normal form $= P \; A \; Q$

Hence $PAQ =$ Normal form where

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ -3/14 & 1/28 & 9/28 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1/6 & -2/3 & 0 \\ 0 & -1/6 & -1/3 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q Find the rank of the matrix by reducing it in normal form

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Sol:-

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to 2R_4 + R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} C_4 \to C_4 - 3C_1 \\ C_3 \to C_3 + C_1 \\ C_2 \to C_2 - 2C_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -14 & 28 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} C_4 &\to 7C_4 - 11C_2 \\ C_3 &\to 7C_3 + 6C_2 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -14 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_4 \to C_4 + 2C_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} C_2 &\to -\frac{1}{7}C_2 \\ C_3 &\to -\frac{1}{14}C_3 \end{aligned}$$

$$\sim \left[\begin{array}{c|c} I_3 & O \\ \hline O & O \end{array}\right] \Rightarrow \boxed{\rho(A) = 3}$$

$$\cancel{A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \ \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4/3 & 1 \\ & & \end{bmatrix}}$$

Q Find x such that

$$A = \begin{bmatrix} 3 & x & x \\ x & 3 & x \\ x & x & 3 \end{bmatrix}, \quad \rho(A) = 1.$$

$$\sim \begin{bmatrix} 3+2x & 3+2x & 3+2x \\ x & 3 & x \\ x & x & 3 \end{bmatrix} \quad R_1 \to R_1 + R_2 + R_3$$

$$\sim \cancel{(3+2x)} \begin{bmatrix} 1 & 1 & 1 \\ x & 3 & x \\ x & x & 3 \end{bmatrix} \quad R_1 \to \frac{1}{3+2x} R_1$$

$$\sim \cancel{(3+2x)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3-x & 0 \\ 0 & 0 & 3-x \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - xR_1 \\ R_3 \to R_3 - xR_1 \end{array}$$

$\Rightarrow \rho(A) = 1$ iff $3 - x = 0 \Rightarrow \boxed{x = 3}$ #

$$\boxed{|A| = 0}$$

$\rho(A) < 3$

$$\begin{vmatrix} 3 & x \\ x & 3 \end{vmatrix} = 0$$

$9 - x^2 = 0$

$x = 3, -3$ ✓

$\boxed{\cancel{3, -3, 0}}$

$\boxed{\cancel{x = 3}}$

$$\begin{vmatrix} x & x \\ 3 & x \end{vmatrix}$$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0$ or $x = 3$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times$$

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - R_1 \\ R_2 \to R_2 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \boxed{\rho(A) = 1}$$

Q Find two non-singular matrices P and Q such that PAQ is in normal form. Again if $PAQ = I$, then find A^{-1}.

when $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$A_{3\times 2} = I_{3\times 3} A_{3\times 2} I_{2\times 2}$

$\cancel{A} = \cancel{I}A$

$I = BA$

$\cancel{A^{-1}} = B$

$PAQ = [\;]$

Sol:-

Canonical form of the matrix

$\because A = IAI$

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \underset{R}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} A \underset{C}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$R_3 \to R_3 - 3R_1$, $R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + C_1, \quad C_2 \to C_2 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to \frac{1}{2} C_2$

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$PAQ = I$

$AQ = P^{-1}$

$A = P^{-1}Q^{-1}$

$A^{-1} = (P^{-1}Q^{-1})^{-1}$

$= QP$

$$\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right] = \underline{\underline{P}}\,\underline{\underline{A}}\,\underline{\underline{Q}}$$

$\Rightarrow PAQ =$ Normal form where

$$P = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & -2 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution of system of linear equations

Minor-Test

Q1 → 10 (I)
Q2 → 20 (I+II)
Q3 — III (10)
Q4 — IV (10)

~~TAQ~~ ~~10 (AT)~~ 50
~~+10 (Quiz)~~

Case i (Non-Homogeneous system of linear equations)

consider the system of equations

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{aligned}\right\} \text{ — } (1)$$

we can write it as

$AX = B$ where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

$[A:B]$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\rho(A) = \rho(A, B)$
$= 2 < 3$

Augmented matrix

$$[A ; B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

(a) If $\rho(A) \neq \rho(A ; B)$, then the system of equations (1) is said to be inconsistent and we have no solution.

(b) If $\boxed{\rho(A) = \rho(A ; B)}$ = No. of variables, then we will have a unique solution.

(c) If $\rho(A) = \rho(A ; B) = n <$ no. of variables $(= r)$, then we will have infinite solution and "n-r" variables will have ~~...~~ arbitrary value.

Case 2:- (Homogeneous system of equations)

consider the system of equations

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0$$

and solution is

$$AX = B$$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\exists \rightarrow$ there exists

If (i) $\rho(A) =$ The number of variables

($i.e.$ $|A| \neq 0$)

then we will have a trivial (or zero) solution

($x = y = z = 0$)

(ii) $\rho(A) <$ The number of variables

($|A| = 0$)

then we will have a (non-zero) solution for infinite (or non-trivial.) solution.

Q Find the values of λ & μ for which

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

have (i) no solution
(ii) unique solution
(iii) Infinite solution.

Sol: solution is given by

$$AX=B$$

$$A=\begin{bmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$B=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$$

Augmented matrix

$$[A,B]=\left[\begin{array}{ccc|c}1&1&1&6\\1&2&3&10\\1&2&\lambda&\mu\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1&1&1&6\\0&1&2&4\\0&1&\lambda-1&\mu-6\end{array}\right]\qquad \begin{array}{l}R_3\to R_3-R_2\\R_2\to R_2-R_1\end{array}$$

$$\sim\left[\begin{array}{ccc|c}1&1&1&6\\0&1&2&4\\0&0&\lambda-3&\mu-10\end{array}\right]\qquad R_3\to R_3-R_2$$

(i) No solution $\lambda=3,\ \mu\neq 10$

(then $\rho(A)=2\neq\rho(A,B)$)

(ii) unique solution $\lambda\neq 3,\ \mu\in R$

(iii) Infinite solution $\lambda=3,\ \mu=10$

Q Discuss the consistency of

$$2x+3y+4z=11$$
$$x+5y+7z=15$$
$$3x+11y+13z=25$$

Sol: solution is given by

$$AX = B$$

where $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

$B = \begin{bmatrix} 11 \\ 15 \\ 25 \end{bmatrix}$

Augmented matrix $[A, B] = \left[\begin{array}{ccc|c} 2 & 3 & 4 & 11 \\ 1 & 5 & 7 & 15 \\ 3 & 11 & 13 & 25 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 5 & 7 & 15 \\ 2 & 3 & 4 & 11 \\ 3 & 11 & 13 & 25 \end{array}\right]$ $R_2 \Leftrightarrow R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & -4 & -8 & -20 \end{array}\right]$ $R_3 \to R_3 - 3R_1$, $R_2 \to R_2 - 2R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & 1 & 2 & 5 \end{array}\right]$ $R_3 \to -\frac{1}{4} R_3$

$\sim \left[\begin{array}{ccc|c} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & 0 & 4 & 16 \end{array}\right]$ $R_3 \to 7R_3 + R_2$

$\Rightarrow \rho(A) = \rho(A, B) = 3 =$ no. of variables

we have a unique solution

$\therefore AX = B$

$$\begin{bmatrix} 1 & 5 & 7 \\ 0 & -7 & -10 \\ 5 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ -19 \\ 11 \end{bmatrix}$$

$\therefore$
$x = 2$
$y = -3$
$z = 4.$

$x + 5y + 7z = 15$
$-7y - 10z = -19$
$4z = 16$
$\Rightarrow z = 4$

$-7y - 40 = -19$
$-7y = 40 - 19$
$= 21$

$\Rightarrow y = -3$

$x = 15 - 5y - 7z$
$= 15 + 15 - 28 = 2$

Q For what values of λ the system of equations

$$3x + y - \lambda z = 0$$

$$4x - 2y - 3z = 0$$

$$2\lambda x + 4y + \lambda z = 0$$

have a non-trivial solution? obtain the solution.

① $\rho(A) =$ No. of variables
$\Rightarrow$ trivial solution ($|A| \neq 0$)

② $\rho(A) <$ No. of variables
Non-trivial solution
($|A| = 0$)

Sol:- solution of the system of equation is given by

$$AX = B$$

where $A = \begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

For non-trivial solution

$$|A| = 0 \quad (\Rightarrow \rho(A) < 3)$$

$$\Rightarrow \begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow 3(-2\lambda + 12) - 1(4\lambda + 6\lambda) - \lambda(16 + 4\lambda) = 0$$

$$\Rightarrow -6\lambda + 36 - 10\lambda - 16\lambda - 4\lambda^2 = 0$$

$$\Rightarrow -4\lambda^2 - 32\lambda + 36 = 0$$

$$\Rightarrow \lambda^2 + 8\lambda - 9 = 0$$

$$\Rightarrow \lambda^2 + 9\lambda - \lambda - 9 = 0$$

$$(\lambda + 9)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 1, -9$$

$Ax=0$

Case i :- $(\lambda=1)$

$$A=\begin{bmatrix}3 & 1 & -1\\ 4 & -2 & -3\\ 2 & 4 & 1\end{bmatrix}$$

$-9+4$

12

$A^T \neq B^T$

$$\sim\begin{bmatrix}3 & 1 & -1\\ 0 & -10 & -5\\ 0 & 10 & 5\end{bmatrix} \quad \begin{matrix}R_2\to 3R_2-4R_1\\ R_3\to 3R_3-2R_1\end{matrix}$$

$$\sim\begin{bmatrix}3 & 1 & -1\\ 0 & -10 & -5\\ 0 & 0 & 0\end{bmatrix} \quad R_3\to R_3+R_2$$

solution is

$$Ax=B$$

$$\begin{bmatrix}3 & 1 & -1\\ 0 & -10 & -5\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$3x+y-z=0$

$-10y-5z=0$

$\boxed{z=-2y}$

Let $y=k$

$\Rightarrow z=-2k$

$3x=z-y=-2k-k$

$\Rightarrow x=-k$

$$\therefore X=\begin{bmatrix}-k\\ k\\ -2k\end{bmatrix}=k\begin{bmatrix}-1 & 1 & -2\end{bmatrix}'$$

Case 2:- $x = -9$

$-9-36$

Here $A = \begin{bmatrix} 3 & 1 & 9 \\ 4 & -2 & -3 \\ -18 & 4 & -9 \end{bmatrix}$

$-9+54$

$\sim \begin{bmatrix} 3 & 1 & 9 \\ 0 & -10 & -45 \\ 0 & 10 & 45 \end{bmatrix}$ $R_3 \to R_3 + 6R_1$

$R_2 \to 3R_2 - 4R_1$

$\sim \begin{bmatrix} 3 & 1 & 9 \\ 0 & -10 & -45 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 + R_2$

solve 16 $AX = B$

$$\begin{bmatrix} 3 & 1 & 9 \\ 0 & -10 & -45 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 3x + y + 9z = 0$$

$$-10y - 45z = 0$$

$$y = \frac{-45}{10} z = \frac{-9}{2} z$$

let $\underset{\sim\sim\sim}{z = 2k}$, $y = -9k$

$$3x = -9z - y$$

$$= -18k + 9k = -9k$$

$$\Rightarrow x = -3k$$

$\therefore$ soln is

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} -3k \\ -9k \\ 2k \end{bmatrix} = k \begin{bmatrix} -3 & -9 & 2 \end{bmatrix}'$$

Unit II(Matrix Theory)

- L1: https://youtu.be/jBukOH3HxhU
- L2: https://youtu.be/J-QMiJwXT9Q
- L3: https://youtu.be/XNLWHDhGDiA
- L4: https://youtu.be/OCXWrlMWK6g
- L5: https://youtu.be/Y58bx36-z2c