$$=\iiint \frac{a^l b^m c^n}{p\,q\,r} u_1^{(l/p)-1} u_2^{(m/q)-1} u_3^{(n/r)-1}\, du_1\, du_2\, du_3,$$

where $u_1 + u_2 + u_3 \le 1$

$$= \frac{a^l b^m c^n}{p\,q\,r} \cdot \frac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma\{1 + (l/p) + (m/q) + (n/r)\}}.$$

Example 3. Prove that $\displaystyle\iint\ldots\int \frac{dx_1\, dx_2 \ldots dx_n}{\sqrt{\{1 - x_1^2 - x_2^2 - \ldots - x_n^2\}}}$

$= \dfrac{\pi^{(n+1)/2}}{2^n\,\Gamma\{\frac{1}{2}(n+1)\}}$ **the integral being extended to all positive values of the variables for which the expression is real.**

Solution. The expression is real if $x_1^2 + x_2^2 + \ldots + x_n^2 < 1$. **(Note)**

Putting $x_1^2 = u_1, x_2^2 = u_2, \ldots, x_n^2 = u_n$, we get

$$x_1 = \sqrt{u_1}, x_2 = \sqrt{u_2}, \ldots x_n = \sqrt{u_n}$$

$$\therefore \quad dx_1 = \tfrac{1}{2} u_1^{\frac{1}{2}-1} du_1;\ dx_2 = \tfrac{1}{2} u_2^{\frac{1}{2}-1} du_2, \ldots, dx_n = \tfrac{1}{2} u_n^{\frac{1}{2}-1} du_n.$$

$\therefore$ The given integral

$$= \iint\ldots\int \frac{\left(\frac{1}{2}\right)^n u_1^{\frac{1}{2}-1} u_2^{\frac{1}{2}-1} \ldots u_n^{\frac{1}{2}-1}\, du_1\, du_2 \ldots du_n}{\sqrt{(1 - u_1 - u_2 - \ldots u_n)}}$$

where $u_1 + u_2 + \ldots u_n < 1$

$$= \left(\tfrac{1}{2}\right)^n \cdot \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}) \ldots \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2})} \int_0^1 \frac{h^{\frac{n}{2}-1}}{\sqrt{(1-h)}}\, dh$$

$$= \left(\tfrac{1}{2}\right)^n \cdot \frac{\{\Gamma(\frac{1}{2})\}^n}{\Gamma(n/2)} \int_0^{\pi/2} \frac{(\sin^2\theta)^{(n/2)-1} \cdot 2\sin\theta\cos\theta\, d\theta}{\sqrt{(1 - \sin^2\theta)}},$$

where $h = \sin^2\theta$

$$= \frac{\left(\frac{1}{2}\right)^n \{\Gamma(\frac{1}{2})\}^n}{\Gamma(\frac{1}{2}n)} \int_0^{\pi/2} 2\sin^{n-1}\theta\, d\theta$$

$$= \frac{(\sqrt{\pi})^n}{2^n\,\Gamma(\frac{1}{2}n)} 2 \cdot \frac{\Gamma(\frac{1}{2}n)\,\Gamma(\frac{1}{2})}{2\Gamma\{\frac{1}{2}(n - 1 + 0 + 2)\}} \qquad \textbf{(Note)}$$

$$= \frac{\pi^{\frac{n}{2}}\sqrt{\pi}}{2^n\,\Gamma\{\frac{1}{2}(n+1)} = \frac{(\pi)^{\frac{(n+1)}{2}}}{2^n\,\Gamma\{\frac{1}{2}(n+1)\}}.$$

Example. 4. Evaluate $\iiint xyz \sin(x+y+z)\, dx\, dy\, dz$ **the integr being extened to all positive values of the variables subject to the conditi** $x+y+z \le \frac{1}{2}\pi$.

Solution. The given integral

$$= \iiint [\sin(x+y+z)]\, x^{2-1} y^{2-1} z^{2-1}\, dx\, dy\, dz$$

(Note

$$= \frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2)} \int_0^{\pi/2} h^{2+2+2-1} \sin h\, dh$$

$$= \frac{[\Gamma(2)]^3}{\Gamma(6)} \int_0^{\pi/2} h^5 \sin h\, dh = \frac{1}{5!} \int_0^{\pi/2} h^5 \sin h\, dh \qquad \text{...(i}$$

Now $\int_0^{\pi/2} h^n \sin h\, dh$

$$= \left[h^n(-\cos h)\right]_0^{\pi/2} - \int_0^{\pi/2} nh^{n-1}(-\cos h)\, dh$$

$$= n \int_0^{\pi/2} h^{n-1} \cos h\, dh$$

$$= n \left[\left\{h^{n-1} \sin h\right\}_0^{\pi/2} - \int_0^{\pi/2} (n-1)\, h^{n-2} \sin h\, dh \right]$$

or
$$\int_0^{\pi/2} h^n \sin h\, dh = n\left(\tfrac{1}{2}\pi\right)^{n-1} - n(n-1) \int_0^{\pi/2} h^{n-1} \sin h\, dh \qquad \text{...(ii)}$$

Putting $n = 5$ in (ii) we get

$$\int_0^{\pi/2} h^5 \sin h\, dh = 5\left(\tfrac{1}{2}\pi\right)^4 - 5\,(4) \int_0^{\pi/2} h^3 \sin h\, dh$$

$$= \frac{5}{16}\pi^4 - 20 \int_0^{\pi/2} h^3 \sin h\, dh \qquad \text{...(iii)}$$

Putting $n = 3$ in (ii) we get $\int_0^{\pi/2} h^3 \sin h\, dh$

$$= 3\left(\tfrac{1}{2}\pi\right)^2 - 3\,(2) \int_0^{\pi/2} h \,\text{din}\, h\, dh$$

$$= \frac{3}{4}\pi^2 - 6\left[\left(-h\cos h\right)_0^{\pi/2} + \int_0^{\pi/2} \cos h \, dh\right]$$

$$= \frac{3}{4}\pi^2 - 6\left(\sin h\right)_0^{\pi/2} = \frac{3}{4}\pi^2 - 6 \qquad \text{...(iv)}$$

$\therefore$ From (iii) and (iv) we get

$$\int_0^{\pi/2} h^2 \sin h \, h \, dh = (5/16)\pi^4 - 20\left[\tfrac{3}{4}\pi^2 - 6\right]$$

$$= (5/16)\pi^4 - 15\pi^2 + 120$$

$\therefore$ From (i), the given integral$= \frac{1}{5!}\left[(5/16)\pi^4 - 15\pi^2 + 120\right]$ **Ans.**

Example 5. Show that $\iiint \frac{dx\,dy\,dz}{\sqrt{[1-(x^2+y^2+z^2)]}} = \frac{1}{8}\pi^2$, the integral being extended to all +ve values of the variables for which the expression is real.

Solution. Putting $x^2 = u, y^2 v, z^2 = w$, we get

$$x = \sqrt{u},\ y = \sqrt{v},\ z = \sqrt{w},\ dx = \tfrac{1}{2}u^{\frac{1}{2}-1}\,du \text{ etc.}$$

Also the expression is real if $x^2 + y^2 + z^2 < 1$ *i.e.* $u + v + w < 1$

$\therefore$ The given integral

$$= \iiint \frac{u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1}\,du\,dv\,dw}{\sqrt{[1-(u+v+w)]}},$$

where $u + v + w < 1$

$$= \left(\tfrac{1}{2}\right)^3 \iiint \frac{u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1}}{\sqrt{[1-(u+v+w)]}}\,du\,dv\,dw$$

$$= \frac{1}{8}\cdot\frac{\Gamma(\frac{1}{2})\cdot\Gamma(\frac{1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})}\int_0^1 \frac{h^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}}{\sqrt{(1-h)}}\,dh,$$

$$= \frac{1}{8}\cdot\frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(\frac{3}{2})}\int_0^1 \frac{h^{1/2}}{\sqrt{(1-h)}}\,dh$$

$$= \frac{1}{8}\cdot\frac{[\Gamma(\frac{1}{2})]^3}{\frac{1}{2}\Gamma(\frac{1}{2})}\cdot\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}\cdot 2\sin\theta\cos\theta\,d\theta.$$

puting $h = \sin^2\theta$

$$= \frac{1}{2}[\Gamma(\tfrac{1}{2})]^2 \int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{1}{2}\pi \cdot \frac{1}{2} \cdot \frac{1}{2}\pi = \frac{1}{8}\pi^2$$

Example 6. Prove that $\iiint \dfrac{dx\,dy\,dz}{\sqrt{(a^2 - x^2 - y^2 - z^2)}} = \dfrac{\pi^2 a^2}{8}$, **the integral being extended to all positive values of the variables for which the expression is real.**

Solution. The expressin is real provided $x^2 + y^2 + z^2 < a^2$

or $$(x/a)^2 + (y/a)^2 + (z/a)^2 < 1$$

$\therefore$ Putting $(x/a)^2 = u$, $(y/a)^2 = v, (z/a)^2 = w$, we get

$$x = au^{\frac{1}{2}}, y = av^{\frac{1}{2}}, z = aw^{\frac{1}{2}}, dx = \frac{1}{2}au^{\frac{1}{2}-1}\, du \text{ etc.}$$

$\therefore$ The given integral

$$= \iiint \frac{(\frac{1}{2}au^{\frac{1}{2}-1}\,du)\,(\frac{1}{2}av^{\frac{1}{2}-1}\,dv)\,(\frac{1}{2}aw^{\frac{1}{2}-1}\,dw)}{\sqrt{(a^2 - a^2u - a^2v - a^2w)}}$$

where $u + v + w < 1$

$$= \frac{1}{8}a^2 \iiint \frac{u^{(\frac{1}{2})-1}\,v^{(\frac{1}{2})-1}\,w^{(\frac{1}{2})-1}\,du\,du\,dw}{\sqrt{[1-(u+v+w)]}}$$

$$= \frac{1}{8}a^2 \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(\frac{3}{2})} \cdot \int_0^1 \frac{h^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}}{\sqrt{(1-h)}}\, dh$$

by Liouville's Theorem

$$= \frac{1}{8} \frac{a^2\,[\Gamma(\frac{1}{2})]^3}{\frac{1}{2}\Gamma(\frac{1}{2})} \int_0^1 \frac{h^{1/2}}{\sqrt{(1-h)}}\, dh$$

$$= \frac{1}{4}a^2\,[\Gamma(\tfrac{1}{2})]^2 \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}\, 2\sin\theta\cos\theta\, d\theta,$$

where $h = \sin^2\theta$

$$= \frac{1}{2}a^2\,(\sqrt{\pi})^2 \int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{1}{2}\pi a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2 a^2}{8}.$$

Hence proved.

Example 7. Prove that the value of $\iiiint dx\,dy\,dz\,dw$, **for all values of the variables for which** $x^2 + y^2 + z^2 + w^2$ **is net less than** a^2 **and not greater than** b^2 **is** $(\pi^2/32)\,(b^4 - a^4)$.

Solution. The given condition is $a^2 < x^2 + y^2 + z^2 + w^2 < b^2$.

Putting $x^2 = u_1, y^2 = u_2, z^2 = u_3, w^2 = u_4$, we get

$$x = \sqrt{u_1}, y = \sqrt{u_2}, z = \sqrt{u_3}, w = \sqrt{u_4}$$

or $dx = \frac{1}{2}u_1^{(1/2)-1}\, du_1, dy = \frac{1}{2}u_2^{(1/2)-1}\, du_2$, etc.

∴ The given integral

$$= \frac{1}{16}\iiiint u_1^{(1/2)-1} u_2^{(1/2)-1} u_3^{(1/2)-1} u_4^{(1/2)-1}\, du_1\, du_2\, du_3\, du_4,$$

where $a^2 < u_1 + u_2 + u_3 + u_4 < b^2$

$$= \frac{1}{16}\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2})}\int_{a^2}^{b^2} h^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}\, dh$$

$$= \frac{1}{16}\frac{[\Gamma(\frac{1}{2})]^4}{\Gamma(2)}\int_{a^2}^{b^2} h\, dh = \frac{1}{16}\cdot\frac{(\sqrt{\pi})^4}{1}\left(\frac{1}{2}h^2\right)_{a^2}^{b^2} = \frac{\pi^2}{32}(b^4 - a^4)$$

Hence proved.

Example 8. Find the volume of $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ by using Dirichlet's Integral. **(Rohilkhand 91)**

Solution. Putting $(x/a)^2 = u, (y/b)^2 = v, (z/c)^2 = w$, we get $x = au^{1/2}$, $y = bv^{1/2}$, $z = cw^{1/2}$ where $u + v + w < 1$, since $(x/a)^2 + (y/b)^2 + (z/c)^2 < 1$ for the rigion within the given surface.

Here $dx = \frac{1}{2}a\, u^{\frac{1}{2}-1}\, du, dy = \frac{1}{2}b\, v^{\frac{1}{2}-1}\, dv, dz = \frac{1}{2}c\, w^{\frac{1}{2}-1}\, dw.$

∴ The required volume

$$= 8\ [\text{Volume in positive octant}] = 8\iiint dx\, dy\, dz$$

$$= 8\iiint \left(\frac{1}{2}\right)^3 abc\, u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1}\, du\, dv\, dw$$

$$= 8\cdot\frac{1}{8}abc\iiint u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1}\, du\, dv\, dw$$

where $u + v + w < 1$

$$= abc\,\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})}\int_0^1 h^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}\, dh,$$

by Dirichlet's Theorem

$$= abc \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(3/2)} \int_0^1 h^{1/2}\, dh = abc \frac{[\Gamma(\frac{1}{2})]^3}{\frac{1}{2}\Gamma(1/2)} \left[\frac{2}{3} h^{3/2}\right]_0^1$$

$$= abc \frac{(\sqrt{\pi})^2}{(1/2)} \cdot \frac{2}{3}, \qquad \because \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$= (4/3)\, \pi\, abc$$

Example 9. Apply Dirichlet's Theorem to find the volume of the solid $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1$.

Solution. Putting $(x/a)^{2/3} = u, (y/b)^{2/3} = v\ (z/c)^{2/3} = w$

or $\qquad x = au^{3/2}, y = bv^{3/2}, z = cw^{3/2}$,

we get $\quad dx = \frac{3}{2} au^{1/2}\, du, dy = \frac{3}{2} bv^{1/2}\, dv, dz = \frac{3}{2} cw^{1/2}\, dw$

Also the solid exists for all positive and negative values of x subject to the condition

$$(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} < 1 \text{ i.e. } u + v + w < 1$$

$\therefore$ The required volume = 8 (volume in positive octant)

$$= 8 \iiint dx\, dy\, dz = 8 \iiint \left(\frac{3}{2}\right)^3 abc\, u^{1/2} v^{1/2} w^{1/2}\, du\, dv\, dw$$

$$= 27\, abc \iiint u^{(3/2)-1} v^{(3/2)-1} w^{(3/2)-1}\, du\, du\, dw$$

$$= 27\, abc \frac{\Gamma(3/2)\,\Gamma(3/2)\,\Gamma(3/2)}{\Gamma\left[\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)\right]} \int_0^1 h^{(3/2)+(3/2)+(3/2)-1}\, dh,$$

$$= \frac{27\, abc\, [\Gamma(3/2)]^3}{\Gamma(9/2)} \int_0^1 h^{7/2}\, dh$$

$$= \frac{27\, abc\, [\frac{1}{2}\Gamma(\frac{1}{2})]^3}{(7/2)\,.\,(5/2)\,.\,(3/2)\,.\,\frac{1}{2}\Gamma(\frac{1}{2})} \left[\frac{2}{9} h^{9/2}\right]_0^1, \text{ where } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$= \frac{18\, abc\, \pi}{35} \cdot \frac{2}{9} = \frac{4\pi}{35} abc$$

Ans

Example 10. Evaluate $\iiint \sqrt{\left[\frac{1 - x^2 - y^2 - z^2}{1 + x^2 + y^2 + z^2}\right]}\, dx\, dy\, dz$, integral being taken over all positive values of x, y, z such that $x^2 + y^2 + z^2 \le 1$.

Solution. Putting $x^2 = u, y^2 = v, z^2 = w$

or $\qquad x = \sqrt{u}, y = \sqrt{v}, z = \sqrt{w}$

we get $\qquad dx = \frac{1}{2} u^{(1/2)-1}\, du, dy = \frac{1}{2} v^{(1/2)-1}\, dv, dz = \frac{1}{2} w^{(1/2)-1}\, dw.$

$\therefore$ The given integral

$$= \iiint \sqrt{\left[\frac{1-(u+v+w)}{1+(u+v+w)}\right]} \left(\frac{1}{2}\right)^3$$

$$u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1} \, du \, dv \, dw$$

$$= \frac{1}{8} \iiint \left[\frac{1-(u+v+w)}{1+(u+v+w)}\right] u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1} \, du \, dv \, dw$$

$$= \frac{1}{8} \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})} \int_0^1 \sqrt{\left(\frac{1-h}{1+h}\right)} h^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}$$

$$= \frac{1}{8} \frac{[\Gamma(\frac{1}{2})]^3}{\frac{1}{2}\Gamma(\frac{1}{2})} \int_0^1 \frac{(1-h)}{\sqrt{[(1+h)(1-h)]}} h^{1/2} \, dh$$

$$= \frac{1}{4}\pi \int_0^1 \frac{(1-h)\, h^{1/2} \, dh}{\sqrt{(1-h^2)}}$$

$$= \frac{1}{4}\pi \left[\int_0^1 h^{1/2} (1-h^2)^{-1/2} \, dh - \int_0^1 h^{3/2} (1-h^2)^{-1/2} \, dh\right]$$

$$= \frac{1}{4}\pi \left[\int_0^1 X^{1/4} (1-X)^{-1/2} \cdot \frac{1}{2} X^{(-1/2)} \, dX\right.$$

$$\left. - \int X^{3/4} (1-X)^{-1/2} \frac{1}{2} X^{-1/2} dX\right], \text{ putting } h^2 = X \text{ or } h = X^{1/2}$$

$$= \frac{1}{8}\pi \left[\int_0^1 X^{(3/4)-1} (1-X)^{(1/2)-1} \, dX - \int_0^1 X^{(5/4)-1}\right.$$

$$\left. (1-X)^{(1/2)-1} \, dX\right]$$

$$= \frac{\pi}{8}\left[B\left(\frac{3}{4}, \frac{1}{2}\right) - B\left(\frac{5}{4}, \frac{1}{2}\right)\right].$$ **Ans.**

EXERCISE 10.1

1. Evaluate $\iiint x^2 \, dx \, dy \, dz$,

 where $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \le 1$. *(Gorakhpur 2002)*

2. Show that $\iiint_V x^{l-1} y^{m-1} z^{n-1} \, dx \, dy \, dz = \dfrac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$,

 where V is the region given by $x \ge 0, y \ge 0, z \ge 0, x + y + z \le 1$.