(In this matrix, vertices appear in the order as they do in the directed Hamiltonian path $e_2\ e_3\ e_4\ e_{11}\ e_{12}\ e_{14}\ e_{15}$.)

The cofactor of any term in this matrix is 16, and therefore $\sigma = 16$ in Theorem 9-13. Since $d^-(v_i) = 2$ for each v_i in Fig. 9-10,

$$\prod_{i=1}^{8} [d^-(v_i) - 1]! = 1.$$

Therefore, the number of Euler lines in Fig. 9-10 is 16.

However, for a regular Euler digraph, such as the one in Fig. 9-10, it is often easier to compute the number of Euler lines by other methods (Problem 9-18).

9-10. PAIRED COMPARISONS AND TOURNAMENTS

In many experiments, specially in the social sciences, one is required to rank a number of given objects by comparing only two at a time. This is called the *method of paired comparisons*, and is used in situations where a numerical measurement is difficult, for example, individual preference for pieces of music. The items are presented two at a time to a subject and he is asked to state his preference. After having noted the results of all possible $n(n-1)/2$ paired comparisons of the n objects, the experimenter ranks the n objects in order of preference.

A digraph is a natural way of representing the results of a paired-comparison experiment. The results of a classic experiment of Kendall [9-5] are shown in Fig. 9-21. Six different dog foods $\{1, 2, \ldots, 6\}$ were to be ranked. Each day two of the six delicacies were served to a dog, and the dog established preference for one food over the other according to which plate he finished first. The experiment was conducted for 15 days, so that all possible pairs could be tried. In the graph representation, an edge is drawn from the preferred dish to the less preferred. For example, 1 was preferred to 2 in Fig. 9-21. Such a graph is called a *preference graph*.

Establishing a rank from a given preference graph is, in general, not easy. In Fig. 9-21, for example, due to some canine inconsistency, the dog preferred food 1 over 2, 2 over 4, and then 4 over 1. So which of the three is the best?

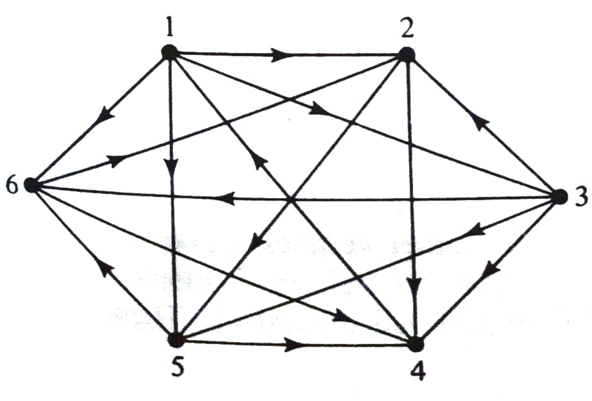


Fig. 9-21 Results of a paired-comparison experiment.

On Tournaments: A similar situation is encountered in tournaments. The results of a round-robin tournament in which every player has played against every other may also be represented by a digraph in which an edge directed from vertex a to b represents the victory of player a over player b. This is why a complete asymmetric digraph was called a tournament or a complete tournament in Section 9-2. The digraph in Fig. 9-21 can also be viewed as the result of a six-player tournament. The problem of ranking players in a tournament is identical to that of ranking in a paired-comparison experiment.

Ranking by Score: A straightforward method of ranking, and the one that has been traditionally used in round-robin tournaments, is to rank each player by his *score*. The score is the number of games the player has won. In terms of the dog food, the number of times the particular dish was preferred is its score. The score of a player in a tournament equals the out-degree of the corresponding vertex in the digraph.

Thus if we use the scores for ranking, we would rank the six dog foods as

$$(1, 3), \quad (2, 5, 6), \quad \text{and} \quad 4.$$

That is, foods 1 and 3 are tied for the first rank; there is a three-way tie for the second rank; and food 4 is the least preferred.

Ranking the vertices according to their out-degrees is not always a satisfactory method, although it is the easiest. In particular, this method loses significance if the tournament is incomplete (that is, the players do not compete in the same number of games).

Ranking by Hamiltonian Path: Another method sometimes used is to rank the players in a directed Hamiltonian path, such that each player has defeated his successor. One such ranking in Fig. 9-21 is 1 3 2 5 6 4. In this context, let us prove the following result regarding Hamiltonian paths in a tournament.

THEOREM 9-14

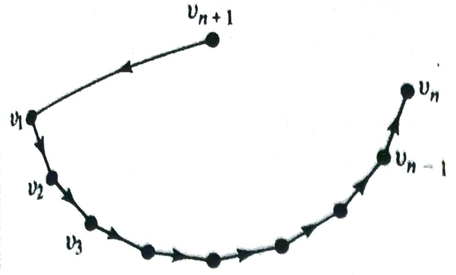
Every complete tournament has a directed Hamiltonian path.

Proof: The theorem will be proved by induction on the number of vertices. By actual sketching, the theorem can be shown to hold for all complete tournaments of 1, 2, 3, and 4 vertices. Let us make the inductive assumption that the theorem is true for all complete tournaments of n vertices, and then prove that it also holds for all tournaments of $n + 1$ vertices.

Let G be any complete tournament of $n + 1$ vertices. Let g be an n-vertex complete subtournament of G. By inductive assumption, g has a directed Hamiltonian path. Let that path be $v_1 v_2 \ldots v_n$. Let the vertex present in G but not in g be called v_{n+1}.

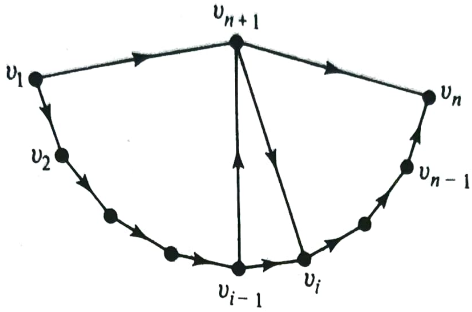
Since G is a complete tournament of $n + 1$ vertices, the vertex v_{n+1} in G has a directed edge either to or from each of the other vertices $v_1, v_2, \ldots, v_n$. The following three cases are possible.

Case 1: The edge between v_{n+1} and v_1 is directed toward v_1. Then we have



(a)
Case 1

(b)
Case 2



(c)
Case 3

Fig. 9-22 Three cases of Theorem 9-14.

a Hamiltonian path $v_{n+1}\ v_1\ v_2 \ldots v_n$ in G, and the proof is complete [Fig. 9-22(a)].

Case 2: There is an edge directed from v_n to v_{n+1}. Then also we have a Hamiltonian path in G, which is $v_1\ v_2 \ldots v_n\ v_{n+1}$, and the proof is complete [Fig. 9-22(b)].

Case 3: Instead, both these edges are directed from v_1 to v_{n+1} and from v_{n+1} to v_n. In this case, as we move from v_1 to v_n, we encounter a reversal of direction in the edges incident on v_{n+1}. This reversal must occur because edge (v_1, v_{n+1}) is directed toward v_{n+1}, but edge (v_n, v_{n+1}) is directed away from v_{n+1}. Call the vertex at which the first such reversal occurs v_i (v_i may be v_n itself). Then edge (v_{i-1}, v_{n+1}) must be directed toward v_{n+1}. See Fig. 9-22(c). In this case we have a directed Hamiltonian path $v_1\ v_2 \ldots v_{i-1}\ v_{n+1}\ v_i\ v_{i+1} \ldots v_n$ in G. Therefore, the theorem. ■

Coming back to the original problem of ranking the vertices, we now know that if the digraph is a complete tournament, at least one Hamiltonian ranking is always possible.

However, this method of ranking also suffers from some drawbacks. For one, there may be discrepancies between such a ranking and the scores of the players. Second, a tournament may have more than one directed Hamiltonian path, and therefore several different rankings are possible. In Fig. 9-21, for instance, 1 3 2 5 6 4 and 1 3 5 6 2 4 are two different Hamiltonian rankings.

Ranking with Minimum Violations: For a given ranking of the n vertices in any tournament (complete or incomplete), a *violation* is defined as an edge directed from v_i to v_j if v_j precedes v_i in the ranking. For example, in Fig.

9-21 the order 1 3 2 5 6 4 has the following two violations—edges 4 to 1 and 6 to 2. The order 3 2 5 6 4 1 has five violations, edges 1 to 3, 1 to 2, 6 to 2, 1 to 5, and 1 to 6.

Ranking with the minimum number of violations represents the fewest possible upsets for a given tournament. It can be shown that a ranking with minimum violations automatically includes the ranking according to scores, as well as a Hamiltonian ranking. Moreover, a minimum-violation ranking is also meaningful for incomplete tournaments. Thus this may be considered the best method of ranking.

However, out of all $n!$ possible orders of n vertices, to find one with minimum violations is computationally difficult. A method using dynamic programming has been used and is the best available so far, but it is computationally slow and cumbersome.

A minimum number of violations among all $n!$ rankings represents a smallest set of edges whose removal from the digraph will eliminate all directed circuits, that is, make the digraph acyclic. Acyclic digraphs are discussed in the next section.

9-11. ACYCLIC DIGRAPHS AND DECYCLIZATION

In many situations semicircuits are of no significance, and one is concerned only with whether or not a given digraph has a directed circuit. A digraph that has no directed circuit is called *acyclic*. Let us make the following observations about acyclic digraphs:

1. Every tree (with directed edges) is an acyclic digraph, but the converse is not true. For example, the digraph in Fig. 9-4 is acyclic, but it is not a tree.
2. An acyclic digraph cannot be condensed. That is, the condensation G_c of an acyclic digraph G is G itself. The converse is also true, because if $G_c = G$, obviously G has no directed circuit.
3. An acyclic digraph represents an irreflexive, asymmetric relation. But the digraph of an irreflexive, asymmetric relation is not necessarily acyclic. (Why?)
4. A digraph G is acyclic if and only if every directed walk in G is also a directed path.
5. Observation 4 has a significant implication: If a digraph is acyclic, the (i, j)th entry in X^k gives the number of distinct directed paths of length k from the ith vertex to the jth vertex.

THEOREM 9-15

Every acyclic digraph