Unit-III

$\beta(m,n)$

Beta & Gamma function

Beta function

Let $m>0, n>0$ be positive numbers then

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

~~Beta~~ Gamma function

Let $n>0$ then

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$$

$\Gamma(0) =$ undefined

Properties of Beta funcⁿ

① $B(m,n) = B(n,m)$

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

let $x = 1-y \Rightarrow dx = -dy$

$$= -\int_1^0 (1-y)^{m-1} (1-1+y)^{n-1} (-dy)$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1} dy$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1} dx$$

$$= B(n,m)$$

(2) $B(m,n) = \int_0^\infty \frac{x^{n-1}\,dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Proof:

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}$

Put $x = \frac{1}{1+y}$ $\left(\Rightarrow 1+y = \frac{1}{x},\ y = \frac{1}{x} - 1\right)$

$\Rightarrow dx = \frac{-1}{(1+y)^2}\,dy$

$= \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1 - \frac{1}{1+y}\right)^{n-1} \times \frac{-1}{(1+y)^2}\,dy$

$= \int_0^\infty \frac{1}{(1+y)^{m-1}} \times \frac{y^{n-1}}{(1+y)^{n-1}} \times \frac{1}{(1+y)^2}\,dy$

$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

(3) $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$\cancel{\int_0^{\frac{\pi}{2}} \sin^{p}\theta \cos^{q}\theta \, d\theta}$

Pf:

$\because \quad B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

$$B(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \times 2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2n-2}\theta \times 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$$

Properties of Gamma function

(i) $\Gamma(n+1) = n!$ if n is +ve integer

$= n\Gamma(n)$ otherwise

$$\lim_{x\to\infty} e^{-x} x^n \quad (0\times\infty)$$

$$= \lim_{x\to\infty} \frac{x^n}{e^x} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{n!}{e^x} = \frac{n!}{e^\infty} = 0$$

Pf: $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$ — (1)

$$\Rightarrow \Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx$$

$$= \left\{-e^{-x} x^n\right\}_0^\infty + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$= 0 - 0 + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$= n\int_0^\infty e^{-x} x^{n-1}\, dx \quad \text{— (ii)}$$

$$= n\Gamma(n)$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx$$

$$\left[-e^{-x}\right]_0^\infty = 0+1$$
$$= 1$$

$$= n \int_0^\infty e^{-x} x^{n-1} \, dx$$

$$= n(n-1) \int_0^\infty e^{-x} x^{n-2} \, dx$$

$$= n(n-1)(n-2) \int_0^\infty e^{-x} x^{n-3} \, dx$$

$$= n(n-1)(n-2) \cdots (n-(n-1)) \int_0^\infty e^{-x} x^{n-n} \, dx$$

$$= n!$$

$$\therefore \quad \Gamma(n+1) = n\Gamma(n)$$
$$= n!$$

$\Gamma 1 = 0! = 1$

$\Gamma(2) = 1!$

$\Gamma(6) = 5!$

$\Gamma(11) = 10!$ and so on.

As we know that

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(\tfrac{1}{2}+1\right)$$

$$= \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \Gamma\left(\tfrac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\tfrac{3}{2}\right)$$

$$= \frac{3}{4}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{7}{2}\right) = \Gamma\left(\tfrac{5}{2}+1\right)$$

$$= \frac{5}{2}\Gamma(5/2) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{15}{8}\sqrt{\pi}.$$

(2) $\frac{\Gamma(n)}{k^n} = \int_0^\infty e^{-kx} x^{n-1} dx$

$\therefore \int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$

Pf: $\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

put $x = ky \Rightarrow dx = k\,dy$

$$\Gamma(n) = \int_0^\infty e^{-ky} (ky)^{n-1} (k\,dy)$$

$$= \int_0^\infty e^{-ky} k^{n-1} y^{n-1} kx\, dy$$

$$= k^n \int_0^\infty e^{-ky} y^{n-1} dy$$

$$= k^n \int_0^\infty e^{-kx} x^{n-1} dx$$

(3) $\boxed{\int_0^\infty e^{-x^{\frac{1}{n}}}\,dx = n\Gamma(n)}$

$\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$

put $x = y^{\frac{1}{n}}$

$dx = \frac{1}{n} y^{\frac{1}{n}-1}\,dy$

$$= \int_0^\infty e^{-y^{\frac{1}{n}}} \left(y^{\frac{1}{n}}\right)^{n-1} \frac{1}{n} y^{\frac{1}{n}-1}\,dy$$

$$= \int_0^\infty e^{-y^{\frac{1}{n}}} y^{1-\frac{1}{n}} \times \frac{1}{n} y^{\frac{1}{n}-1}\,dy$$

$$= \frac{\int_0^\infty e^{-y^{\frac{1}{n}}}\,dy}{n}$$

$$= \frac{1}{n}\int_0^\infty e^{-x^{\frac{1}{n}}}\,dx$$

$$\Rightarrow \boxed{\int_0^\infty e^{-x^{\frac{1}{n}}}\,dx = n\Gamma(n)}$$

If $n = \frac{1}{2}$

$$\boxed{\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}}$$

Relationship b/w Beta & Gamma funcⁿ

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Proof: $\Gamma(m) = \int_0^\infty e^{-x} x^{m-1}\,dx$, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$

Also

$$\frac{\Gamma(n)}{z^n} = \int_0^\infty e^{-zx} x^{n-1}\,dx$$

$$\Rightarrow \Gamma(n) = \int_0^\infty e^{-zx} z^n x^{n-1}\,dx$$

multiplying both sides by $z^{m-1} e^{-z}$ and integrating w.r.t 'z' from 0 to ∞, we get

$$\Gamma(m)\Gamma(n) = \int_0^\infty e^{-z} z^{m-1} \left[\int_0^\infty e^{-zx} z^n x^{n-1}\,dx\right] dz$$

$$= \int_0^\infty \int_0^\infty e^{-z-zx} z^{m+n-1} x^{n-1}\,dx\,dz$$

$$= \int_0^\infty \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{n-1}\,dx\,dz$$

let $z(1+x) = t$

$$\Rightarrow dz = \frac{dt}{1+x}$$

$$= \int_0^\infty \int_0^\infty e^{-t} \left(\frac{t}{1+x}\right)^{m+n-1} x^{n-1} \times dx \times \frac{dt}{1+x}$$

$$= \int_0^\infty \left\{ \int_0^\infty e^{-t}\, t^{m+n-1}\, dt \right\} \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$= \Gamma(m+n) \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$= \Gamma(m+n)\, B(m,n)$$

$$\therefore \quad \Gamma(m)\,\Gamma(n) = \Gamma(m+n)\, B(m,n)$$

$$\therefore \quad \boxed{B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$

Deduction

① let $m+n=1$ is

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\Rightarrow \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = \frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)}$$

$$\Rightarrow \int_0^\infty \frac{x^{n-1}\, dx}{1+x} = \Gamma(n)\,\Gamma(1-n)$$

As $\displaystyle\int_0^\infty \frac{x^{n-1}}{1+x}\, dx = \frac{\pi}{\sin n\pi}$

$$\therefore \quad \boxed{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$$

Put $n = \frac{1}{2}$

$$\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}}$$

$$\Rightarrow \left\{\Gamma\left(\tfrac{1}{2}\right)\right\}^2 = \pi$$

$$\Rightarrow \boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

If $n = 1/3$

$$\Gamma(1/3)\Gamma(2/3) = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{\pi}{\sqrt{3}/2}$$

$$\Rightarrow \boxed{\Gamma\left(\tfrac{1}{3}\right)\Gamma(2/3) = \frac{2\pi}{\sqrt{3}}}$$

If $n = \frac{1}{4}$

$$\Gamma\left(\tfrac{1}{4}\right)\Gamma(3/4) = \frac{\pi}{\sin\frac{\pi}{4}} = \pi\sqrt{2}$$

$$\therefore \boxed{\Gamma\left(\tfrac{1}{4}\right)\Gamma(3/4) = \pi\sqrt{2}}$$

If $n = \frac{1}{6}$

$$\Gamma(1/6)\Gamma(5/6) = \frac{\pi}{\sin\frac{\pi}{6}} = 2\pi$$

$$\Gamma(1/6)\Gamma(5/6) = 2\pi$$

(2) we know that

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\Rightarrow \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} = \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

let $2m-1 = p$, $2n-1 = q$

$$\Rightarrow m = \frac{p+1}{2}, \; \& \; n = \frac{q+1}{2}$$

$$\therefore \boxed{\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\frac{\pi}{2}} \sin^{p}\theta \cos^{q}\theta \, d\theta}$$

If $q=0$, $p=\frac{1}{2}$

$$\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \, d\theta$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(5/4)}$$

$$= \frac{\Gamma(3/4)\sqrt{\pi}}{2\Gamma\left(\frac{1}{4}+1\right)}$$

$$= \frac{\Gamma(3/4)\sqrt{\pi}}{2\times\frac{1}{4}\times\Gamma(1/4)}$$

$$= \frac{2\Gamma(3/4)\sqrt{\pi}}{\Gamma(1/4)}$$

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Legendre's duplication formula

$B(m,n) = B(n,m)$

$$\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}} \times \Gamma(2n)$$

Pf: we know that

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \quad \text{--- (1)}$$

put $m = \frac{1}{2}$ in (1)

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \, d\theta = \frac{\Gamma(1/2)\,\Gamma(n)}{2\,\Gamma(n+1/2)} \quad \text{--- (2)}$$

If $m = n$ in (1)

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(n)\,\Gamma(n)}{2\,\Gamma(2n)}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2n-1}\, d\theta = \frac{1}{2}\frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{2^{2n-1}} \sin^{2n-1} 2\theta \, d\theta = \frac{1}{2}\frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

let $2\theta = t$

$\Rightarrow d\theta = \frac{dt}{2}$

$$\Rightarrow \int_0^{\pi} \frac{1}{2^{2n-1}} \sin^{2n-1} t \,\frac{dt}{2} = \frac{1}{2}\frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

$$\Rightarrow \int_0^{2\frac{\pi}{2}} \frac{1}{2^{2n-1}} \sin^{2n-1} t \,\frac{dt}{2} = \frac{1}{2}\frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

$\left\{ \because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(2a-x)\,dx \right\}$

$$\Rightarrow \cancel{2}\int_0^{\frac{\pi}{2}} \frac{1}{2^{2n-1}} \sin^{2n-1}\theta \,\frac{d\theta}{\cancel{2}}$$

$$= \frac{1}{2}\frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \, d\theta$$

$$= \frac{2^{2n-1}}{2}\frac{\{\Gamma(n)\}^2}{\Gamma(2n)} \quad \text{— (3)}$$

By (2) & (3)

$$\frac{1}{\cancel{2}}\frac{\cancel{\Gamma(n)}\,\Gamma(1/2)}{\Gamma(n+1/2)} = \frac{2^{2n-1}}{\cancel{2}}\frac{\{\Gamma(n)\}^{\cancel{2}}}{\Gamma(2n)}$$

$$\Rightarrow \boxed{\Gamma(n)\,\Gamma(n+1/2) = \frac{\Gamma(2n) \times \sqrt{\pi}}{2^{2n-1}}}$$

Note:

① $\Gamma(1/n)\,\Gamma(2/n) \times \cdots \Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$

② $\int_0^\infty e^{-ax}\, x^{n-1} \cos bx \, dx = \frac{\Gamma(n) \cos n\theta}{(a^2+b^2)^{n/2}}$

$\int_0^\infty e^{-ax}\, x^{n-1} \sin bx \, dx = \frac{\Gamma(n) \sin n\theta}{(a^2+b^2)^{n/2}}$ when $\theta = \tan^{-1}(b/a)$

Q Evaluate

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$$

$\boxed{x^n = \sin^2\theta}$ $\boxed{\int_0^1 t^{n-1}(1-t)^{m-1} = B(n,m)}$

$\int_0^\infty$

sol: let $x^n = t$

$\Rightarrow x = t^{1/n}$

$\Rightarrow dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

$$I = \int_0^1 \frac{1}{n} \frac{t^{\frac{1}{n}-1} dt}{(1-t)^{1/2}}$$

$$= \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1}(1-t)^{\frac{1}{2}-1} dt$$

$$= \frac{1}{n} B(1/n, 1/2) = \frac{1}{n} \frac{\Gamma(1/n)\Gamma(1/2)}{\Gamma(\frac{1}{n}+1/2)} = \frac{\sqrt{\pi}\,\Gamma(1/n)}{n\,\Gamma(1/n+1/2)} \text{ Ans.}$$

$$\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}$$

Q. Evaluate

(1) $I=\int_0^1 \left(1-x^{\frac{1}{n}}\right)^m dx$ $\{x^{\frac{1}{n}}=t\}$

(2) $I=\int_0^1 x^m(1-x^n)^p\,dx$ $\{x^n=t\}$

(3) $I=\int_0^1 \frac{dx}{(1-x^n)^{1/n}}$ $\{x^n=t\}$

(4) $I=\int_0^{\pi/2} \tan^p\theta\,d\theta=\frac{\pi}{2}\sec\frac{p\pi}{2}$

(5) $I=\int_0^1 \frac{dx}{\sqrt{-\log x}}$

(6) $I=\int_0^\infty e^{-h^2x^2}\,dx$ $(h^2x^2=t)$

(7) $I=\int_0^\infty x^4 e^{-x^2}\,dx$ $(x^2=t)$

(5) sol:

$$I = \int_0^1 \frac{dx}{\sqrt{-\lg x}}$$

let $-\lg x = t$

$\Rightarrow x = e^{-t}$

$dx = -e^{-t}\,dt$

$$\therefore \quad I = \int_\infty^0 \frac{-e^{-t}\,dt}{\sqrt{t}}$$

$$= \int_0^\infty t^{\frac{1}{2}-1} e^{-t}\,dt$$

$$= \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\int_0^\infty e^{-kx} x^{n-1}\,dx = \frac{\Gamma(n)}{k^n}$$

(4)

$$I = \int_0^{\frac{\pi}{2}} \tan^p\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^p\theta \cos^{-p}\theta\,d\theta$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{-p+1}{2}\right)}{2\Gamma\left(\frac{2}{2}\right)}$$

$$= \frac{1}{2}\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(1-\left(\frac{p+1}{2}\right)\right)$$

$$= \frac{1}{2} \times \frac{\pi}{\sin\left(\frac{p+1}{2}\right)\pi} = \frac{1}{2}\,\frac{\pi}{\sin\left(\frac{\pi}{2}+\frac{p\pi}{2}\right)}$$

$$= \frac{\pi}{2}\sec\frac{p\pi}{2} \quad \#$$

Q Evaluate $\int_0^\infty \cos x^2 \, dx$

Sol: $I = \int_0^\infty \cos x^2 \, dx$

let $x^2 = t$

$\Rightarrow x = t^{\frac{1}{2}}$

$dx = \frac{1}{2} t^{\frac{1}{2}-1} dt$

$\therefore I = \frac{1}{2}\int_0^\infty t^{\frac{1}{2}-1} \cos t \, dt$

$= \frac{1}{2} \frac{\Gamma(\frac{1}{2}) \cos n\theta}{(0+1)^{n/2}}$ where $\theta = \tan^{-1}(b/a) = \frac{\pi}{2}$, $n = \frac{1}{2}$

$\left\{ \because \int_0^\infty e^{-ax} x^{n-1} \cos bx \, dx = \frac{\Gamma(n) \cos n\theta}{(a^2+b^2)^{n/2}}, \; \theta = \tan^{-1}(b/a) \right\}$

$= \frac{1}{2}\sqrt{\pi} \cos\frac{\pi}{4} = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ #

Q Evaluate

$I = \int_0^1 \log \Gamma(x) \, dx$

Sol:

$I = \int_0^1 \log \Gamma(x) \, dx$ — (1)

OR

$I = \int_0^1 \log \Gamma(1-x) \, dx$ — (2)

$\left\{ \because \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \right\}$

(1) + (2) $\Rightarrow$

$2I = \int_0^1 \log \Gamma(x) \, dx + \int_0^1 \log \Gamma(1-x) \, dx$

$$= \int_0^1 \log\{\Gamma(x)\Gamma(1-x)\}\,dx$$

$$= \int_0^1 \log\left\{\frac{\pi}{\sin x\pi}\right\}dx \qquad \left\{\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right\}$$

$$= \int_0^1 \log\pi\,dx - \int_0^1 \log\sin x\pi\,dx$$

$$= \log\pi - \int_0^1 \log\sin\pi x\,dx$$

let $\pi x = t$

$$dx = \frac{dt}{\pi}$$

$$= \log\pi - \frac{1}{\pi}\int_0^{\pi} \log\sin t\,dt$$

$$= \log\pi - \frac{1}{\pi}\int_0^{2\times\frac{\pi}{2}} \log\sin t\,dt$$

$$= \log\pi - \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \log\sin t\,dt$$

$$\left\{\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(2a-x)\,dx\right\}$$

$$= \log\pi - \frac{2}{\pi}\left(-\frac{\pi}{2}\log 2\right)$$

$$\left\{\because \int_0^{\frac{\pi}{2}} \log\sin t\,dt = -\frac{\pi}{2}\log 2\right\}$$

$$= \log\pi + \log 2$$

$$2I = \log 2\pi$$

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$$\Rightarrow \boxed{I = \frac{1}{2}\log 2\pi}$$

Q Evaluate $\int_0^\infty \frac{x^c}{c^x}\,dx$, $(c>1)$

Sol:

$$I = \int_0^\infty x^c\, c^{-x}\, dx$$

$$= \int_0^\infty x^c\, e^{\log c^{-x}}\, dx \quad \left\{\because a = e^{\log a}\right\}$$

$$= \int_0^\infty x^c\, e^{-x \log c}\, dx$$

$$= \int_0^\infty e^{-(\log c)x}\, x^{c+1-1}\, dx$$

$$\left\{\because \frac{\Gamma(n)}{k^n} = \int_0^\infty e^{-kx}\, x^{n-1}\, dx\right\}$$

$$= \frac{\Gamma(c+1)}{(\log c)^{c+1}}$$

#

Q Evaluate

$$I = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$$

Sol: we know that

$$I_1 = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx + \int_1^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$= A + B \quad \text{--- (1)}$$

where $B = \int_1^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$ --- (2)

let $x = \frac{1}{t}$ in (2)

$$\Rightarrow dx = -\frac{1}{t^2}\, dt$$

$$B = \int_1^0 \frac{\left(\frac{1}{t}\right)^{n-1}}{\left(1+\frac{1}{t}\right)^{m+n}} \left(\frac{-1}{t^2}\right) dt$$

$$= \int_0^1 \frac{dt \times t^{m+n}}{t^{n-1}(1+t)^{m+n}} \times \frac{1}{t^2}$$

$$= \int_0^1 \frac{t^{m-1}}{(1+t)^{m+n}}\, dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

on putting this value in (1), we get

$$I_1 = A + B$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{n-1} + x^{m-1}}{(1+x)^{m+n}}\,dx$$

Q. Show that $\beta(m,m) = 2^{1-2m}\,\beta(m,\tfrac{1}{2})$

Sol: $R.H.S = 2^{1-2m}\,\beta(m,\tfrac{1}{2})$

$$= 2^{1-2m}\,\frac{\Gamma(m)\,\Gamma(\tfrac{1}{2})}{\Gamma(m+\tfrac{1}{2})}$$

$$= 2^{1-2m}\,\frac{\Gamma(m)\,\Gamma(\tfrac{1}{2})}{\left\{\dfrac{\sqrt{\pi}\,\Gamma(2m)}{\Gamma(m)\,2^{-1+2m}}\right\}}$$

$$\left\{\because\ \Gamma(m)\,\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(2m)}{2^{-1+2m}}\right\}$$

$$= \frac{\cancel{2^{1-2m}} \times \Gamma(m)\,\cancel{2^{-1+2m}}\,\Gamma(m)\,\cancel{\Gamma(\tfrac{1}{2})}}{\cancel{\sqrt{\pi}}\,\Gamma(2m)}$$

$$= \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)}$$

$$= \beta(m,m) = L.H.S$$

(1) Prove that $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{\beta(m,n)}{a^n(a+b)^m}$

(2) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{(a\cos^4 x + b\sin^4 x)^{\frac{1}{2}}} = \frac{\{\Gamma(\frac{1}{4})\}^2}{4(ab)^{\frac{1}{4}}\sqrt{\pi}}$

Sol: (2)

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{(a\cos^4 x + b\sin^4 x)^{\frac{1}{2}}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{(a+b\tan^4 x)^{\frac{1}{2}}}\,dx$$

$\tan x = t$

$\sec^2 x\, dx = dt$

$$I = \int_0^{\infty} \frac{1+t^2}{a+bt^4}\,dt$$

$bt^4 = ay$

$\Rightarrow t = \left(\frac{a}{b}\right)^{\frac{1}{4}} y^{\frac{1}{4}}$

$\Rightarrow dt = \left(\frac{a}{b}\right)^{\frac{1}{4}} \frac{1}{4} y^{\frac{1}{4}-1}\,dy$

$$= \int_0^\infty \frac{dt}{(a+bt^4)^{\frac{1}{2}}}$$

let $bt^4 = ay$

$$\Rightarrow t = \left(\frac{a}{b}\right)^{\frac{1}{4}} y^{\frac{1}{4}}$$

$$dt = \left(\frac{a}{b}\right)^{\frac{1}{4}} \frac{1}{4} y^{\frac{1}{4}-1} dy$$

$$\therefore I = \int_0^\infty \frac{\left(\frac{a}{b}\right)^{\frac{1}{4}} \frac{1}{4} y^{\frac{1}{4}-1} dy}{(a+ay)^{\frac{1}{2}}}$$

$\frac{1}{2}-\frac{1}{4}$

$$= \left(\frac{a}{b}\right)^{\frac{1}{4}} \frac{1}{a^{\frac{1}{2}}} \frac{1}{4} \int_0^\infty \frac{y^{\frac{1}{4}-1} dy}{(1+y)^{\frac{1}{4}+\frac{1}{4}}}$$

$$= \frac{1}{4(ab)^{\frac{1}{4}}} B\left(\frac{1}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{4(ab)^{\frac{1}{4}}} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{4}\right)}$$

$$= \frac{1}{4(ab)^{\frac{1}{4}}\sqrt{\pi}} \times \left\{\Gamma\left(\frac{1}{4}\right)\right\}^2$$

#

(1)

let $\frac{x}{a+bx} = \frac{z}{(a+b)}$

$$\Rightarrow \frac{\{(a+bx) - xb\}}{(a+bx)^2} dx = \frac{dz}{(a+b)}$$

$$\Rightarrow dx = \frac{1}{a} \frac{(a+bx)^2}{(a+b)} dz$$

At $x=0$, $z=0$

At $x=1$, $z=1$

Q Evaluate

(a) $\displaystyle\int_0^\infty \frac{x\,dx}{1+x^6}$ (c) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^n}}$

(b) $\displaystyle\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx$ (d) $\displaystyle\int_0^3 \frac{dx}{\sqrt{3x-x^2}}$

$$\boxed{B(m,n) = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx}$$

$$\rightarrow \quad = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$$

Sol:

(a) $$I = \int_0^\infty \frac{x\,dx}{1+x^6}$$

let $x^6 = t$

$x = t^{\frac{1}{6}}$

$dx = \frac{1}{6} t^{\frac{1}{6}-1}\,dt$

$$\therefore I = \int_0^\infty \frac{t^{\frac{1}{6}} \, \frac{1}{6} t^{\frac{1}{6}-1}\,dt}{1+t} = \frac{1}{6}\int_0^\infty \frac{t^{\frac{1}{3}-1}}{(1+t)^1}\,dt$$

$$= \frac{1}{6}\int_0^\infty \frac{t^{\frac{1}{3}-1}}{(1+t)^{\frac{1}{3}+\frac{2}{3}}}\,dt$$

$n = \frac{1}{3}$

$m+n = 1$

$m = 1-\frac{1}{3}$

$= \frac{2}{3}$

$$= \frac{1}{6} B\left(\frac{1}{3}, \frac{2}{3}\right)$$

$$= \frac{1}{6} \frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3}+\frac{2}{3})}$$

$$= \frac{1}{6} \Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$$

$$= \frac{1}{6} \frac{\pi}{\sin\frac{\pi}{3}} = \frac{1}{6} \times \frac{2\pi}{\sqrt{3}}$$

$$= \frac{\pi}{3\sqrt{3}} \text{ Ans.}$$

(ii)

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$$

let $x^n = t$

$\Rightarrow x = t^{\frac{1}{n}}$

$dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

$$\therefore I = \int_0^1 \frac{\frac{1}{n} t^{\frac{1}{n}-1} dt}{(1-t)^{\frac{1}{2}}}$$

$$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{\frac{1}{2}-1} dt$$

$$= \frac{1}{n} B\left(\frac{1}{n}, \frac{1}{2}\right)$$

$$= \frac{1}{n} \frac{\Gamma(\frac{1}{n})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{n}+\frac{1}{2})} \#$$

$$\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = B(m,n)$$

Evaluate

Q

(i) $I = \int_0^2 x\,(8-x^3)^{\frac{1}{3}}\,dx$

(ii) $I = \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx$

Sol:-

(i) $I = \int_0^2 x\,(8-x^3)^{\frac{1}{3}}\,dx$

let $x^3 = 8t$

$\Rightarrow x = (8t)^{\frac{1}{3}}$

$= 2t^{\frac{1}{3}}$

$dx = \frac{2}{3} t^{\frac{1}{3}-1}\,dt$

$\therefore$

$$I = \int_0^1 2t^{\frac{1}{3}}\,(8-8t)^{\frac{1}{3}}\,\frac{2}{3}t^{\frac{1}{3}-1}\,dt$$

$$= \frac{4}{3}\times 2\int_0^1 t^{\frac{2}{3}-1}(1-t)^{\frac{4}{3}-1}\,dt$$

$$= \frac{8}{3}\,B\left(\frac{2}{3},\frac{4}{3}\right)$$

$$= \frac{8}{3}\,\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{6}{3}\right)}$$

$$= \frac{8}{3}\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)$$

$$= \frac{8}{3}\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(1+\frac{1}{3}\right)$$

$$= \frac{8}{9}\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)$$

$$\int_0^\infty \cancel{\frac{du\; u^{m-1}}{(1+u)^{m+n}}}$$

Hence by ①

$$2I = \log\pi + \log 2$$

$$= \log(2\pi)$$

$$x-a=t$$

$$(b-(a+t))^n$$

$$(b-a-t)$$

$$t=(b-a)u$$

$$\boxed{I = \frac{1}{2}\log 2\pi}$$

Show that

(a) $\int_a^b (x-a)^m (b-x)^n\,dx$

$$= (b-a)^{m+n+1}\beta(m+1, n+1)$$

(b) $\int_0^{\pi/2} \frac{d\theta}{(a\cos^4\theta + b\sin^4\theta)^{1/2}} = \frac{\{\Gamma(\frac{1}{4})\}^2}{4(ab)^{\frac{1}{4}}\sqrt{\pi}}$

$$\int \frac{\sec^2\theta}{(a+b\tan^4\theta)^{\frac{1}{2}}} \rightarrow b\tan^4\theta = au$$

Q Evaluate

(a) $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

(b) $\frac{\Gamma(\frac{1}{3})\Gamma(5/6)}{\Gamma(\frac{2}{3})}$

$$2^{2n-1}\Gamma(n)\Gamma(n+\tfrac{1}{2})$$

$$= \sqrt{\pi}\,\Gamma(2n)$$

(c) $\int_0^1 \frac{dx}{\sqrt{-\log x}} \rightarrow (\log x = t)$

(d) $\Gamma(\frac{3}{2}-p)\Gamma(\frac{3}{2}+p)$

(e) $\int_0^1 x^m (\log x)^n\,dx \rightarrow (\log x = -t)$

(f) $\int_0^\infty x^n e^{-a^2x^2}\,dx$

(a)

$$I = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

$$= I_1 \times I_2$$

where $I_1 = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}}$

let $x^4 = t$

$\Rightarrow x = t^{\frac{1}{4}}$

$dx = \frac{1}{4} t^{\frac{1}{4}-1} dt$

$$\therefore I_1 = \int_0^1 \frac{t^{\frac{1}{2}} \frac{1}{4} t^{\frac{1}{4}-1} dt}{\sqrt{1-t}}$$

$$\therefore I_1 = \frac{1}{4} \int_0^1 t^{\frac{3}{4}-1} (1-t)^{\frac{1}{2}-1} dt$$

$$= \frac{1}{4} B\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \frac{\Gamma(3/4)\Gamma(\frac{1}{2})}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}$$

$$= \frac{1}{4} \frac{\Gamma(3/4)\sqrt{\pi}}{\Gamma(5/4)}$$

$$= \frac{1}{4} \frac{\Gamma(3/4) \times \sqrt{\pi}}{\Gamma\left(1+\frac{1}{4}\right)}$$

$$= \cancel{\frac{1}{4}} \frac{\Gamma(3/4)\sqrt{\pi}}{\cancel{\frac{1}{4}}\Gamma(1/4)} = \frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(\frac{1}{4})}$$

Now

$$I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

let $x^2 = \tan\theta$

$\Rightarrow x = \tan^{\frac{1}{2}}\theta$

$$dx = \frac{1}{2}\tan^{-\frac{1}{2}}\theta \sec^2\theta \, d\theta$$

$$\therefore I_2 = \int_0^{\frac{\pi}{4}} \frac{\frac{1}{2}\tan^{-\frac{1}{2}}\theta \sec^2\theta \, d\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\tan^{-\frac{1}{2}}\theta \cancel{\sec^2\theta}}{\cancel{\sec\theta}} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\cos^{\frac{1}{2}}\theta}{\sin^{\frac{1}{2}}\theta} \times \frac{1}{\cos\theta}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\frac{\sin 2\theta}{2}}}$$

let $2\theta = t$

$$d\theta = \frac{dt}{2}$$

$$\therefore I_2 = \frac{\sqrt{2}}{2\times 2}\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{\sin t}}$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}} t \cos^0 t \, dt$$

$$= \frac{1}{2\sqrt{2}} \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\times\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)}$$

$$= \frac{1}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}$$

$\therefore$

$$I = I_1 \times I_2$$

$$= \frac{\pi}{4\sqrt{2}}$$

(e) $I = \int_0^1 x^m (\log x)^n \, dx$

let $\log x = -t$

$\Rightarrow x = e^{-t}$

$dx = -e^{-t} dt$

$\therefore I = \int_\infty^0 (e^{-t})^m (-t)^n \, -e^{-t} dt$

$= \int_0^\infty e^{-mt} e^{-t} (-1)^n t^n dt$

$\therefore I = (-1)^n \int_0^\infty e^{-(1+m)t} t^{n+1-1} dt$

$= (-1)^n \dfrac{\Gamma(n+1)}{(1+m)^{n+1}}$ Ans.

$\left\{ \because \int_0^\infty e^{-kx} x^{n-1} dx = \dfrac{\Gamma(n)}{k^n} \right\}$

(c)

$$I = \int_0^1 \frac{dx}{\sqrt{-\log x}}$$

Let $-\log x = t$

$$\Rightarrow x = e^{-t}$$

$$\Rightarrow dx = -e^{-t}\,dt$$

$$\therefore I = \int_\infty^0 \frac{-e^{-t}\,dt}{\sqrt{t}}$$

$$= \int_0^\infty e^{-t}\, t^{\frac{1}{2}-1}\,dt$$

$$= \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\frac{3}{2} - p > 0$$

$$3 > 2p$$

$$-3 < 2p < 3$$

$$-1 < 2p < 1$$

(d)

$$I = \Gamma\left(\tfrac{3}{2} - p\right)\Gamma\left(\tfrac{3}{2} + p\right)$$

$$= \Gamma\left(1 + \tfrac{1}{2} - p\right)\Gamma\left(1 + \tfrac{1}{2} + p\right)$$

$$= \left(\tfrac{1}{2} - p\right)\Gamma\left(\tfrac{1}{2} - p\right)\left(\tfrac{1}{2} + p\right)\Gamma\left(\tfrac{1}{2} + p\right)$$

$$= \left(\tfrac{1}{4} - p^2\right)\Gamma\left(\tfrac{1}{2} - p\right)\Gamma\left(\tfrac{1}{2} + p\right)$$

$$= \left(\tfrac{1}{4} - p^2\right)\Gamma\left(\tfrac{1}{2} - p\right)\Gamma\left(1 - \left(\tfrac{1}{2} - p\right)\right)$$

$$= \left(\tfrac{1}{4} - p^2\right)\frac{\pi}{\sin\left(\tfrac{1}{2} - p\right)\pi}$$

$$\left\{ \because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \right\}$$

$$= \left(\frac{1}{4}-p^2\right)\frac{\pi}{\cos p\pi}$$

$$= \left(\frac{1}{4}-p^2\right)\pi \sec p\pi$$

Multiple integral

$\int_a^b f(x)\,dx$ | Change of order

Double integral

The integral $\iint_R f(x,y)\,dx\,dy$ is known as the double integral of $f(x,y)$ over the region R.

case i :- $\int_a^b \int_{x=\phi(y)}^{\psi(y)} f(x,y)\,dx\,dy$

To evaluate this integral we first integral w.r.t 'x' and then w.r.t 'y' i.e.,

$$I = \int_a^b \left[\int_{\phi(y)}^{\psi(x)} f(x,y)\,dx \right] dy$$

Case 2:- $I = \int_a^b \int_{y=\phi(x)}^{\psi(x)} f(x,y)\, dy\, dx$

then we first integrate w.r.t 'y' and then w.r.t 'x'

$$I = \int_a^b \left[\int_{\phi(x)}^{\psi(x)} f(x,y)\, dy \right] dx$$

Case 3:- $I = \int_{x=a}^{b} \int_c^d f(x,y)\, dy\, dx$

In this case we can integrate first w.r.t 'x' then w.r.t 'y' or vice-versa.

Q. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

$\int \frac{dx}{\cancel{\sec\theta}}\ \sec^2\theta\, d\theta$

Sol:- $I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

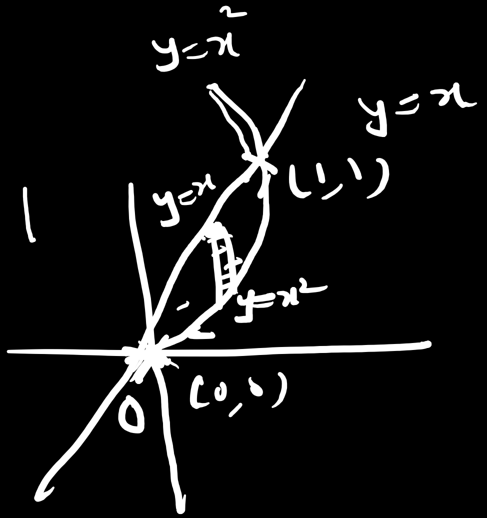
$= \int_0^1 \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{(\sqrt{1+x^2})^2+y^2} \right] dx$

$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}} \right) \right]_{y=0}^{\sqrt{1+x^2}} dx$

$= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}} = \frac{\pi}{4} \left[\log|x+\sqrt{1+x^2}| \right]_0^1$

$= \frac{\pi}{4} \log(1+\sqrt{2})$ Ans.

Q Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ and $y=x$.



Sol.

$$I = \iint xy(x+y)\,dx\,dy$$

$$= \int_{x=0}^{1}\left[\int_{y=x^2}^{y=x} xy(x+y)\,dy\right]dx$$

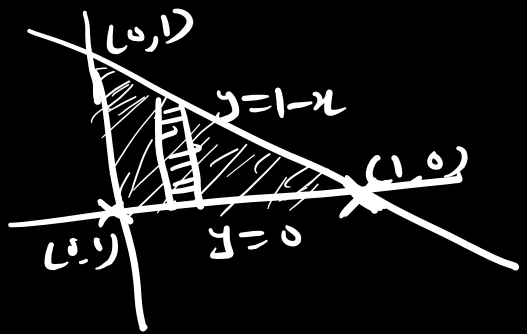
$$= \int_{x=0}^{1}\left[\int_{y=x^2}^{y=x} (x^2y+xy^2)\,dy\right]dx$$

$$= \int_{x=0}^{1}\left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_{y=x^2}^{y=x} dx = \int_0^1 \left[\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right]dx$$

$$= \frac{1}{10}+\frac{1}{15}-\frac{1}{14}-\frac{1}{24}$$

$$= \frac{3}{56}$$

Q evaluate $\iint xy\,dx\,dy$ over the +ve quadrant of $x+y\le 1$



Sol:

$$I = \iint xy\,dx\,dy$$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{1-x} xy\,dy\right]dx$$

$$= \int_{x=0}^{1}\left[\frac{xy^2}{2}\right]_{y=0}^{1-x} dx$$

$$= \frac{1}{2}\int_{x=0}^{1} x(1-x)^2\,dx$$

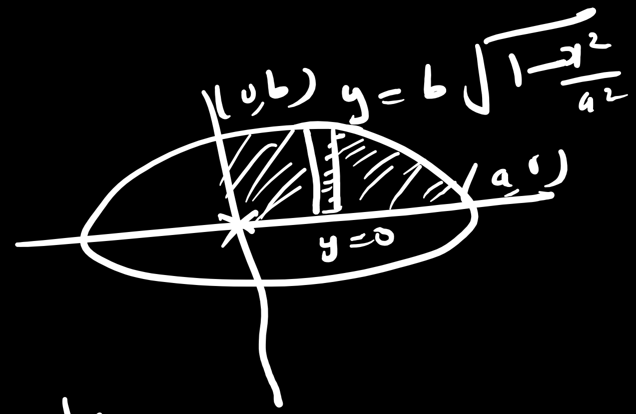
$$= \frac{1}{2}\int_{x=0}^{1} x(1+x^2-2x)\,dx = \frac{1}{2}\int_{x=0}^{1}(x+x^3-2x^2)\,dx$$

$$\frac{6+3-8}{12}$$

$$= \frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right) = \frac{1}{24}$$

Ans.

Q Evaluate $\iint_R \left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) dx\, dy$ over the +ve quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Sol:

$$I = \iint_R \left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) dx\, dy$$

$$= \int_0^a \left[\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) dy\right] dx$$

$$= \frac{\pi ab}{4}$$

Q Evaluate $\iint y\, dx\, dy$ over the part of the plane bounded by the lines $y = x$ and the parabola $y = 4x - x^2$

Sol: Intersection parts are given by

$$x = 4x - x^2$$

$$\Rightarrow 3x - x^2 = 0$$

$$\Rightarrow x(3 - x) = 0$$

$$\Rightarrow x = 0, 3$$

$$\therefore y = 0, 3$$

$\therefore$ (0,0) & (3,3) are two intersection pts.

$\because y = 4x - x^2$

$$x^2 - 4x = -y$$

$$x^2 - 4x + 4 = -(y - 4)$$

$$(x - 2)^2 = -(y - 4)$$

$$\Rightarrow \boxed{X^2 = -4AY}$$

vertex $(X, Y) = (0, 0)$

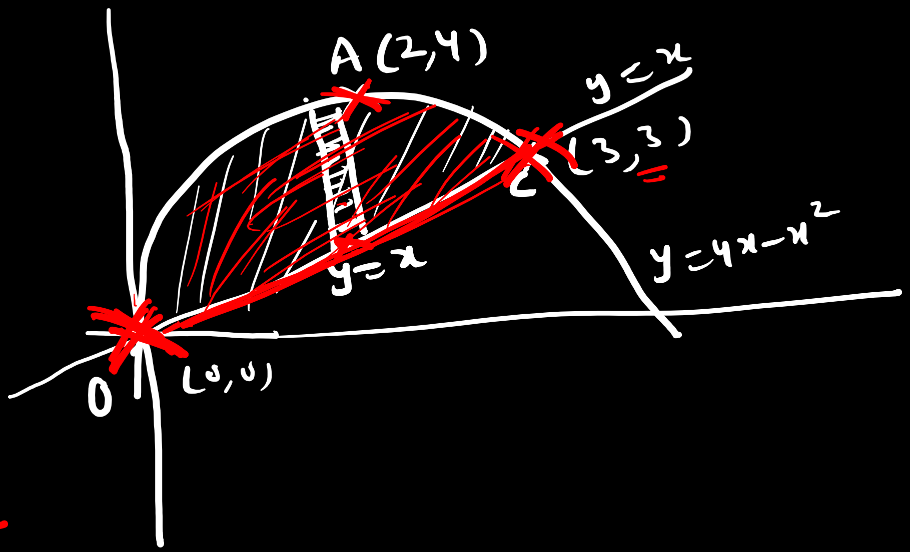
$\Rightarrow x = 2,\ y = 4$

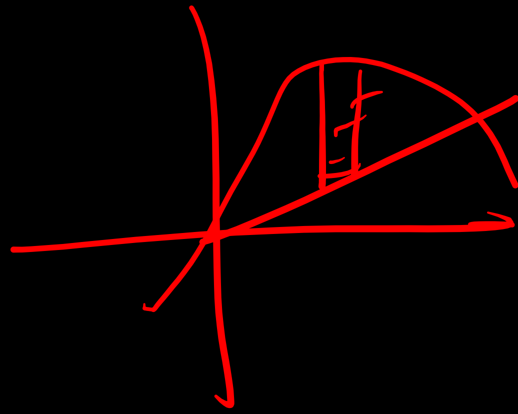
$\Rightarrow (2, 4)$ is the vertex.

$\boxed{\iint dx\,dy}$

$\therefore\ I = \iint y\,dx\,dy$

$$= \int_{x=0}^{3}\left[\int_{y=x}^{4x-x^2} y\,dy\right]dx$$





$$= \frac{1}{2}\int_{x=0}^{3}\left[(4x-x^2)^2 - x^2\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{3}\left[16x^2 + x^4 - 8x^3 - x^2\right]dx$$

$$= \frac{1}{2}\left[15\times\frac{9}{3} + \frac{1}{5}\times\cancel{22} - \frac{8}{4}\times\cancel{16}\right] = \frac{54}{5}\ \text{Ans}$$

- You Tube Link:
- 1. https://youtu.be/EUV1kpKS24c
- 2. https://youtu.be/zY9yf1N5Vbs
- 3. https://youtu.be/McJFWZVvBvw