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# Unit-1

## *) Sequence

- In map, set of natural numbers (N) to any set A are define as a sequence.
- If the set A is subset of real numbers then the sequence is called sequence of real numbers.
- If the set A is subset of complex numbers then the sequence is called sequence of complex number.
- Generally, the sequence is denoted by $\langle a_n \rangle$ and defined as $\langle a_1, a_2, \dots, a_n, \dots \rangle$

## *) Range of a sequence

The range of a sequence is the set consisting of all distinct elements of a sequence and without regards to the position term.
Thus the range of a sequence may be finite or infinite.

## *) Some Important Sequences

(i) $a_n = \langle n \rangle$

(ii) $a_n = \langle -n \rangle$

(iii) $a_n = \langle (-1)^n n \rangle$

(iv) $a_n = (-1)^n \rightarrow$ (finite range)

(v) $a_n = \frac{1}{n}$

(vi) $a_n = 1 - \frac{(-1)^n}{n}$

(vii) $a_n = \begin{cases} 2 & n=1 \text{ or prime} \\ n & \text{else} \end{cases}$

(viii) $a_n = \begin{cases} -1+\frac{1}{n}, & n \text{ is even} \\ 1+\frac{1}{n}, & n \text{ is odd} \end{cases}$

(ix) $a_{n+1} = \sqrt{2+a_n}$, where $a_1 = 0$

(x) $a_{n+2} = a_n + a_{n+1}$, where $a_1 = 1$ & $a_2 = 1$.

## *) Bounded sequence

A sequence $\langle a_n \rangle$ is defined as bounded if its range set is bounded. Hence, the sequence $\langle a_n \rangle$ is bounded if there exist real number $k$ and $k'$ such that $a_n$ lies b/w $k$ and $k'$ for every $n \geq 1$

$$k \leq a_n \leq k' \quad \forall n \geq 1$$

E.g. ⇒ ① $\langle a_n \rangle = (-1)^n$

② $\langle a_n \rangle = \frac{1}{n}$

③ $\langle a_{n+1} \rangle = \sqrt{2+a_n}$, $a_1 = 0$

## *) Monotonic Sequence

Let $a_n$ be a sequence of real numbers then it is monotonic if -

① $a_{n+1} \geq a_n \quad \forall n$ (monotonic inc. seq. or non-decreasing seq.)

② $a_{n+1} \leq a_n \quad \forall n$ (monotonic dec. seq. or non-inc. seq.)

③ $a_{n+1} > a_n \quad \forall n$ (monotonic strictly inc. seq.)

④ $a_{n+1} < a_n \quad \forall n$ (monotonic strictly dec. seq.)

## *) Limit point of a sequence

A real number $p$ is said to be a limit point or a clusted point of a seq. if every neighbourhood of $p$ contains an infinite no. of elements of the seq. $\langle a_n \rangle$. In other words, a real no. $p$ is a limit point of a sequence $\langle a_n \rangle$ if for any $\varepsilon > 0$ $a_n \in (p-\varepsilon, p+\varepsilon)$ for infinite values of $n$.

Note: $p$ may be or may not be an element of $n$.

E.g. ⇒ ① $a_n = (-1)^n$ l.p. $\to -1$, $\to 1$

② $a_n = \frac{1}{n} \to 0$

$p = 0$ $\qquad \varepsilon = \frac{1}{10^2}$

$$\left| \frac{1}{n} - 0 \right| < \varepsilon$$

$$-\varepsilon < \frac{1}{n} < \varepsilon$$

$$0 - \varepsilon < \frac{1}{n} < 0 + \varepsilon$$

For infinite value of $n$

$$\left| \frac{1}{n} \right| < \frac{1}{10^2}$$

$$\frac{1}{n} < \frac{1}{100}$$

$$n > 100$$

$$a_{101}, a_{102}, \ldots\ldots$$

## *) Limit of a sequence

Let $\langle a_n \rangle$ be a sequence of real no. and $l$ be a real number then we say $l$ is limit of a sequence $\langle a_n \rangle$ if for any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$|a_n - l| < \varepsilon \quad \forall\, n \geq m$$

$$l - \varepsilon < a_n < l + \varepsilon$$

E.g. $\Rightarrow a_n = \frac{1}{n}$ $\quad \varepsilon = \frac{1}{10^2}$

$$\left|\frac{1}{n} - 0\right| < \frac{1}{100}$$

i.e. $\quad n > 100 = m$

## *) Convergent sequence

A seq. $\langle a_n \rangle$ is said to be convergent if it has a finite limit. Then, if $\lim_{n\to\infty} a_n = l$.

(i) If $l$ is finite, then $\langle a_n \rangle$ is convergent.
E.g. $\Rightarrow a_n = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

(ii) If $l$ is infinite, then $\langle a_n \rangle$ is divergent.
E.g. $\Rightarrow$ $a_n = n \xrightarrow[n\to\infty]{} \infty$

(iii) If $l$ is not unique, then $\langle a_n \rangle$ is oscillatory.
E.g. $\Rightarrow$ $a_n = (-1)^n \rightleftarrows {}^{-1}_{1}$

## *) Cauchy Sequence

Let $\langle a_n \rangle$ be a sequence, then this sequence $\langle a_n \rangle$ is said to be cauchy if for any $\varepsilon > 0$ there exist a natural no. $M$ such that

$$|a_n - a_m| < \varepsilon \quad , n, m > M$$

E.g., ① $a_n = \frac{1}{n}$

$$|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right|$$

$$= \left|\frac{1}{n} + \left(\frac{-1}{m}\right)\right|$$

$$\leq \frac{1}{n} + \frac{1}{m}$$

Let us choose, $\varepsilon > 0$

then

there exist $M > \frac{2}{\varepsilon}$

and for $n, m > M$

$$|a_n - a_m| < \frac{1}{n} + \frac{1}{m}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$|a_n - a_m| < \varepsilon$ for $n, m > M$

② $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n}$

$$|a_n - a_m| = \frac{1}{m+1} + \frac{1}{m+2} + \dots \frac{1}{2m}$$ $n = 2m > m$

$$> \frac{1}{2m} + \frac{1}{2m} + \dots \frac{1}{2m}$$ ($m$ terms)

$$\geq \frac{m}{2m}$$

$$|a_n - a_m| \geq \frac{1}{2}$$

Hence, $\langle a_n \rangle$ is not a cauchy sequence

## *) Some Important Theorems

### ① Cauchy first Theorem

Let $\langle a_n \rangle$ be a sequence which converges to limit $l$ ~~that~~ i.e.

$$\lim_{n\to\infty} a_n = l$$

Then,

$$\lim_{n\to\infty} \left( \frac{a_1 + a_2 + \cdots a_n}{n} \right) = l$$

E.g. $$\lim_{n\to\infty} \left( \frac{1}{n} + \frac{1}{2n} + \cdots \frac{1}{n^2} \right)$$

$$\Rightarrow \lim_{n\to\infty} \left( \frac{1}{n} \left( 1 + \frac{1}{2} + \cdots \frac{1}{n} \right) \right)$$

$\Rightarrow$ Since $\lim_{n\to\infty} \frac{1}{n} = 0$

Using cauchy first theorem

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \left( 1 + \frac{1}{2} + \cdots \frac{1}{n} \right) = 0$$

### ② Cauchy Second Theorem

Let $\langle a_n \rangle$ be a convergent sequence of a positive terms i.e.,

$$\lim_{n\to\infty} a_n = l \quad (l \neq 0)$$

Then, $$\lim_{n\to\infty} (a_1 . a_2 . \cdots a_n)^{1/n} = l$$

E.g. $\lim_{n\to\infty}\left(\frac{2}{1}\cdot\left(\frac{3}{2}\right)^2\cdot\left(\frac{4}{3}\right)^3\cdots\cdots\left(\frac{n+1}{n}\right)^n\right)^{1/n}$

$a_n = \left(\frac{n+1}{n}\right)^n$

$\lim_{n\to\infty} a_n = e$

Using cauchy second theorem,

$\lim_{n\to\infty}\left(\frac{2}{1}\cdot\left(\frac{3}{2}\right)^2\cdot\left(\frac{4}{3}\right)^3\cdots\cdots\left(\frac{n+1}{n}\right)^n\right)^{1/n} = e$

③ Sandwitch Theorem

Let $\langle a_n\rangle$, $\langle b_n\rangle$, $\langle C_n\rangle$ are sequences of ~~three~~ real numbers such that

$a_n \leq b_n \leq C_n \quad \forall\ n \geq 1$

Then,

if $\lim_{n\to\infty} a_n = l = \lim_{n\to\infty} c_n$

$\lim_{n\to\infty} b_n = l$

E.g. $\lim_{n\to\infty} \frac{\sin n}{n}$

$-1 \leq \sin(n) \leq 1$

$\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$

$\because \lim_{n\to\infty}\left(\frac{-1}{n}\right) = 0 = \lim_{n\to\infty}\left(\frac{1}{n}\right)$

Using sandwitch theorem,

$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$