Vectors:- A n-tuple is a set of n similar things. If the place of every members of a set is fixed then it is called an ordered set. An ordered n-tuple of numbers is called n-vector. Thus the co-ordinates of a point in space is called 3-vector (x, y, z). The members of a set are called the components of a vector so x, y, z in a 3-vector are called components.

→ Each row and each columns of a matrix is also a vector.

Linear dependense and independence of vectors :—

Vectors $X_1, X_2, \cdots, X_n$ are said to be dependent if

(i) all the vectors (row or column matrices) are of the same order.

(ii) n scalars $\lambda_1, \lambda_2, \cdots, \lambda_n$ (not all zero) exist such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n = 0$$

Otherwise they are linearly independent.

Note:- If in a set of vectors, any vector of the set is the combination of the remaining vectors, then the vectors are called dependent.

eg: Examine the following vectors for linear dependence and find the relation if it exist.

$x_1 = (1,2,4)$, $x_2 = (2,-1,3)$, $x_3 = (0,1,2)$, $x_4 = (-3,7,2)$

Sol^n Consider the matrix

$$\lambda_1(1,2,4) + \lambda_2(2,-1,3) + \lambda_3(0,1,2) + \lambda_4(-3,7,2) = 0$$

$$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$$

$$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$$

$$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0 \quad \text{— (i)}$$

$$-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0 \quad \text{— (ii)}$$

$$\lambda_3 + \lambda_4 = 0 \quad \text{— (iii)}$$

Let $\lambda_3 = t$

from (iii)

$\lambda_4 = -t$

from (ii)

$-5\lambda_2 + t - 13t = 0$

$\lambda_2 = -\frac{12}{5}t$

from (i)

$\lambda_1 + 2 \times \left(-\frac{12}{5}t\right) - 3(-t) = 0$

$\lambda_1 - \frac{24}{5}t + 3t = 0$

$\lambda_1 + \left(\frac{-24+15}{5}\right)t = 0$

$\lambda_1 = \frac{9}{5}t$

$t\left(\frac{9}{5}X_1 - \frac{12}{5}X_2 + X_3 - X_4\right) = 0$

$\Rightarrow \boxed{\frac{9}{5}X_1 - \frac{12}{5}X_2 + X_3 = X_4}$

$\Rightarrow$ given vectors are linearly dependent.

Ex Show that row vectors of the matrix $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ are linearly independent.

Sol^n $X_1 = (1, 2, -2)$, $X_2 = (-1, 3, 0)$, $X_3 = (0, -2, 1)$

Let $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0$

$\lambda_1(1, 2, -2) + \lambda_2(-1, 3, 0) + \lambda_3(0, -2, 1) = 0$

$\lambda_1 - \lambda_2 = 0, \quad 2\lambda_1 + 3\lambda_2 - 2\lambda_3 = 0, \quad -2\lambda_1 + \lambda_3 = 0$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + \frac{2}{5} R_2$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & -2 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_1 - \lambda_2 = 0$ — (i)

$4\lambda_2 - 2\lambda_3 = 0$ — (ii)

$\frac{1}{5}\lambda_3 = 0$ — (iii) $\Rightarrow \lambda_3 = 0$

from (ii)

$\lambda_2 = 0$

from (i)

$\lambda_1 = 0$

$\lambda_1 = \lambda_2 = \lambda_3 = 0$

$\Rightarrow x_1, x_2 \;\&\; x_3$ are L.I.

solution of linear system of equation

consider the equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

this can be represent in matrix form

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\boxed{AX = b}$$

(i) Cramer's Rule:- $(\Delta \neq 0)$

If Δ denote determinant of coefficient matrix

i.e

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$x\Delta = \begin{vmatrix} xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \\ xa_3 & b_3 & c_3 \end{vmatrix}$$

$$C_1 \rightarrow C_1 + yC_2 + zC_3$$

$$x\Delta = \begin{vmatrix} xa_1 + yb_1 + zc_1 & b_1 & c_1 \\ xa_2 + yb_2 + zc_2 & b_2 & c_2 \\ xa_3 + yb_3 + zc_3 & b_3 & c_3 \end{vmatrix}$$

$$x\Delta = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Similarly

$$y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and

$$z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(ii) Matrix inversion Method:-

$\because \quad AX = b$

Let A is non singular i.e. $|A| \neq 0$

then

$$\boxed{X = A^{-1} b}$$

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Consistency of linear system of equations:

~~consider~~

Consider the system of m linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

matrix form

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

here we have two matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

$$[A:b] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right]$$

Procedure to test the consistency

(i) If $\rho(A) = \rho(A:b)$, then system is consistent

(ii) If $\rho(A) \neq \rho(A:b)$, then system is not consistent

(iii) If $\rho(A) = \rho(A:b) = n$ (equal to the no. of variables)

$\Rightarrow$ unique solⁿ

(iv) If $\rho(A) = \rho(A:b) = r < (n)$ (no. of variable)

(infinite solⁿ)