UNIT-2

Algebraic Structure

Let G be a non-empty set & $*$ be the binary operation, then the ordered pair $(G, *)$ is c/a algebraic structure

Group :- An algebraic structure $(G, *)$ where $G \rightarrow$ non empty set and $*$ is binary operation, then algebraic structure is called group if satisfy following property

1) Closure property :- $\forall\ a, b \in G$, $a * b \in G$

2) Associative property :- $a * (b * c) = (a * b) * c$ $\forall\ a, b, c \in G$

3) Existance of Identity :- $\forall\ a \in G$, $\exists\ e \in G$ s.t. $a * e = e * a = a$

4) Existence of inverse :- $\forall\ a \in G$ $\exists\ a' \in G$ s.t. $a * a' = a' * a = e$

G SM Grp abelian

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Groupoid :- $(G,*)$ is groupoid if
Closure $\forall a,b \in G$, $a*b \in G$

Semigroup :- $(G,*)$ is semigroup if
1) Closure $\forall a,b \in G$, $a*b \in G$
2) Associative $\forall a,b,c \in G$
$a*(b*c) = (a*b)*c$

Monoid :- $(G,*)$ is monoid if
1) Closure $\forall a,b \in G$, $a*b \in G$
2) Associative $\forall a,b,c \in G$
$a*(b*c) = (a*b)*c$
3) Existance of identity
$\forall a \in G$ $\exists e \in G$ s.t
$a*e = e*a = a$

Abelian / commutative Grp :-
1) Closure
2) Associative
3) Existance of identity
4) Existance of inverse
5) Commutative $\forall a,b \in G$
$a*b = b*c$

Q $(Z,+)$
Closure ✓
Associative ✓
Existance of Identity ✓
Existence of inverse ✓
Comm ✓

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Q $(Z, \cdot) \rightarrow$ Monoid.

Q $(C, +) \rightarrow$ Abelian

Closure :- $(a_1 + ib_1) + (a_2 + ib_2)$

Associative :-

Q $(Z^-, \cdot)$

Addition modulo m $(+_m)$

$G = \{1, 2, 3, \ldots m-1\}$

Q. ~~Show~~ check the set $\{1,2,3,4,5\}$ is group or what under addition & multiplication modulo 6.

$+_6$	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

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$\times_6$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Not a group.

Q $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

Is multiplicative grp?

$\Rightarrow$ Properties of grps :-

Theorem :- Cancellation law

If $(G, *)$ is grp & $a, b, c \in G$ then

① $a * b = a * c \Rightarrow b = c$ (Left cancellation law)

② $b * a = c * a \Rightarrow b = c$ (Right cancellation law)

Proof :-

① $\because a \in G \quad \exists \; a^{-1} \in G$

$a * b = a * c$

$\Rightarrow a^{-1} * (a * b) = a^{-1} * (a * c)$

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$\Rightarrow (a^{-1} * a) * b = (a^{-1} * a) * c$

$\Rightarrow e * b = e * c$

$\Rightarrow b = c$

(2) $\because a \in G \quad \exists\, a^{-1} \in G$

$b * a = c * a$

$(b * a) * a^{-1} = (c * a) * a^{-1}$

$b * (a * a^{-1}) = c * (a * a^{-1})$

$b * e = c * e$

$b = c$

Theorem :- Left identity = Right identity

i.e.

$e * a = a * e = a$

<u>Proof</u> :-

If a^{-1} is inverse of a then

$a^{-1} * (a * e) = (a^{-1} * a) * e$

$= e * e$

$= e = a^{-1} * a$

$\Rightarrow a^{-1} * (a * e) = a^{-1} * a$

$a * e = a$ — (1) [By L.C.L]

Similarly, $e * a = a$ — (2) [By R.C.L]

From (1) & (2)

$a * e = e * a = a$

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Theorem :- Left inverse = Right inverse
i.e. $a^{-1} * a = a * a^{-1} = e$

Proof :-

$$a^{-1} * (a * a^{-1}) = (a^{-1} * a) * a^{-1} \quad \text{(Ass.)}$$
$$= e * a^{-1} \quad \text{(Identity)}$$
$$= a^{-1} * e \quad \text{[Identity]}$$

(2)

$\Rightarrow a^{-1} * (a * a^{-1}) = a^{-1} * e$

$\Rightarrow a * a^{-1} = e$ —— (1) [By L.C.L]

Similarly,
$a^{-1} * a = e$ —— (2) [By R.C.L.]

From eqn (1) & (2)
$a * a^{-1} = a^{-1} * a = e$

Theorem :- In a grp $(G, *)$
(1) $a * x = b$ has unique soln $x = a^{-1} * b$
(2) $y * a = b$ has unique soln $y = b * a^{-1}$

Proof :- Let $a * x = b$ has soln x & x'
then $a * x = b$ & $a * x' = b$
$\Rightarrow a * x = a * x'$
$\Rightarrow \boxed{x = x'}$ [L.C.L]

$\therefore$ $a * x = b$ has unique soln
$a * x = a * (a^{-1} * b)$ (Given)
$= (a * a^{-1}) * b$ (Ass.)

$= e * b$ (identity)
$= b$

$\Rightarrow x = a^{-1} * b$ satisfy the eq^n

(2) Let $y * a = b$ has unique soln y & y'
then

$y * a = b$ & $y' * a = b$.
$\Rightarrow y * a = y' * a$
$\Rightarrow y = y'$ [By R.C.L]

$\therefore y * a = b$ has unique soln
$y * a = y * (y^{-1} * b)$
$= (y * y^{-1}) * b$
$= e * b$
$= b$.

Theorem: (1) $(a^{-1})^{-1} = a$
(2) $(ab)^{-1} = b^{-1}a^{-1}$

Proof: Let e be identity in G
We have $a * a^{-1} = e$ [$\forall a \in G \; \exists a^{-1} \in G$
$= (a^{-1})^{-1} * a^{-1}$

$\Rightarrow a * a^{-1} = (a^{-1})^{-1} * a^{-1}$
$a = (a^{-1})^{-1}$ [R.C.L]

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(2) $a * b \in G$ (Closure)

$\Rightarrow (a*b)^{-1} * (a*b) = e$ [inverse] (1)

$\forall\ a, b \in G\ \exists\ a^{-1}, b^{-1} \in G$

$\therefore\ (b^{-1} * a^{-1}) * (a*b)$

$= b^{-1} * (a^{-1} * a) * b$

$= b^{-1} * e * b$

$= b^{-1} * b$

$= e$ —— (2)

From (1) & (2)

$(a*b)^{-1} * (a*b) = (b^{-1} * a^{-1}) * (a*b)$

$(a*b)^{-1} = b^{-1} * a^{-1}$

[RCL]

Order of grp :-

Order of element:- $g \in G$ is the smallest positive integer n such that $g^n = e$ if no such integer exist then g has infinite order, denoted by $O(g)$

Ex:- Let $G = \{1, -1, i, -i\}$ be multiplicative grp. Find order of each element.

Soln:- The multiplicative identity is 1

$O(1) = 1^1 = 1$

$(-1)^2 = 1 \Rightarrow O(-1) = 2$

$(i)^4 = 1 \Rightarrow O(i) = 4$

$(-i)^4 = 1 \Rightarrow O(-i) = 4$

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Subgrp :- Let $(G,*)$ be a grp & H is subset of G s.t. $(H,*)$ itself is a grp then $(H,*)$ is subgrp of $(G,*)$

Normal subgrps :- A subgrp H of a grp G is said to be normal if for every $x \in G$ and for every $h \in H$, $xhx^{-1} \in H$

Coset :- $\forall\ a \in (G,+)$ & $h \in (H,+)$

$(a+h)$ is left coset of G w.r.t. addition

$(h+a)$ is right " " " "

$a \in (G,\cdot)$ & $h \in (H,\cdot)$

ah is left coset w.r.t multiplication

ha is right " " " "

Theorem :- A subgrp H of grp G is normal iff $xHx^{-1} = H \ \forall\ x \in G$

Soln

Let $xHx^{-1} = H \ \forall\ x \in G$

$xHx^{-1} \subseteq H \ \forall\ x \in G$

$\Rightarrow$ H is normal subgrp of G

Conversely, H is normal subgrp of G, then

$xHx^{-1} \subseteq H \ \forall\ x \in G$ —— (1)

Then $x \in G$, $x^{-1} \in G$

$x^{-1}H(x^{-1})^{-1} \in H \ \forall\ x \in G$

$x^{-1}Hx \in H \ \forall\ x \in G$

$x(x^{-1}Hx)x^{-1} \subseteq xHx^{-1} \ \forall\ x \in G$

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$\Rightarrow H \subseteq xHx^{-1}$ —— (2)

From (1) & (2)

$H = xHx^{-1}$

$\Rightarrow$ Theorem :- A subgrp H of grp G is normal if left coset of H in G is right coset of H in G

i.e. H is normal subgrp of G
$\iff xH = Hx \quad \forall x \in G$

Proof :-

Let H be normal subgrp of G
then $xHx^{-1} = H$
$\Rightarrow (xHx^{-1})x = Hx$
$\Rightarrow xH = Hx$
$\Rightarrow$ left coset = right coset

Conversely

Let $xH = Hx \quad \forall x \in G$

then

$xHx^{-1} = Hxx^{-1}$
$\Rightarrow xHx^{-1} = H$

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Q Let $(\{a,b\}, *)$ be semigrp where $a * a = b$
show $a * b = b * a$

Proof: $a * b = a * (a * a)$
$= (a * a) * a$
$= b * a$

(2) $b * b = b$

Qu Let G be a grp with binary operation multiplication & H be a subgrp.
Let $a \in G$ then
$Ha = \{ ha : h \in H \}$
is right coset of H in G generated by a
$aH = \{ ah : h \in H \}$
is left coset of H on G generated by a

If operation is addition

$H + a = \{ h + a : h \in H \}$ is right coset
& $a + H = \{ a + h : h \in H \}$ is left coset

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Ex. If G is an additive grp of integers and H is additive subgrp of all even integers of G, then find all cosets of H in G.

Sol We have, $G = \{0, \pm 1, \pm 2, \pm 3 \ldots\}$

& $H = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$

Let $0, 1, 2 \ldots \in G$ then

$H + 0 = \{0+0, \pm 2+0, \pm 4+0 \ldots\}$

$= \{0, \pm 2, \pm 4, \ldots\} = H$

$H+1 = \{0+1, \pm 2+1, \pm 4+1, \ldots\}$

$= \{1, 3, -1, 5, -3, \ldots\}$

$= \{\pm 1, \pm 3, \pm 5 \ldots\}$

$H+2 = \{0+2, \pm 2+2, \pm 4+2, \ldots\}$

$= \{2, 4, 0, 6, -2, \ldots\}$

$= \{0, \pm 2, \pm 4, \ldots\}$

H & $H+1$ are different cosets