Optimization Techniques
Paper Code – BMS-09
Lecture – 03(Unit -1)
Topic-Multiple Variables Optimization - Lagrange Method

Dr. Ram Keval

Mathematics and Scientific Computing Deptt.

M. M. M. University of Technology Gorakhpur

Unit-01

Classical Optimization Techniques: Single variable optimization, Multi-variable with no constraints. Non-linear programming: One Dimensional Minimization methods. Elimination methods: Fibonacci method, Golden Section method

Unit-02

Unit-02

Linear Programming: Constrained Optimization Techniques: Simplex method, Solution of System of Linear Simultaneous equations, Revised Simplex method, Transportation problems, Karmarkar’s method, Duality Theorems, Dual Simplex method, Decomposition principle.

MULTIVARIABLE OPTIMIZATION WITH EQUALITY CONSTRAINTS

consider the optimization of continuous functions subjected to equality constraints:

$$\text{Minimize } f = f(X) \text{ subject to constraints}$$

$$g_j(X) = 0, \quad j = 1, 2, 3, \dots, m, \quad \text{where} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

for solving such problem, we will use **the Method of Lagrange Multipliers**

$f(x) = x_1^2 + 2x_1x_2 + x_2$ 1/to

$g_1(x) \cong x_1 - 5 = 0$

$g_2(x) \cong x + x_2 = 6$

(i) $x^3 + 3x^2 + 6x + 5$

(ii) $x_1^2 + 2x_1x_2 + x_2^2 + 5$

(iii) $x_1^2 + 3x_1x_2 + 4x_3 + 5x_2^2 + \dots$

Solution by the Method of Lagrange Multipliers – Here, Lagrange multiplier method is given for two variables with one constraint as given below-

Consider the problem

Minimize $$f(x_1, x_2)$$

subject to (1)

$$g(x_1, x_2) = 0$$

So, for the necessary conditions, first, constructing a function L, known as the Lagrange function, as

$$L = f(x_1, x_2) + \lambda\, g(x_1, x_2) \tag{2}$$

$g_1(x),\ g_2(x_2) \quad L = f(x_1, x_2) + \lambda_1 g_1(x) + \lambda_2 g_2(x) + \cdots$

By treating L as a function of three variables, x_1, x_2 and λ the necessary conditions for its extreme are given by

$$\begin{aligned}
\frac{\partial L}{\partial x_1}(x_1, x_2, \lambda) &= \frac{\partial f}{\partial x_1}(x_1, x_2) + \lambda \frac{\partial g}{\partial x_1}(x_1, x_2) = 0 \\
\frac{\partial L}{\partial x_2}(x_1, x_2, \lambda) &= \frac{\partial f}{\partial x_2}(x_1, x_2) + \lambda \frac{\partial g}{\partial x_2}(x_1, x_2) = 0 \\
\frac{\partial L}{\partial \lambda}(x_1, x_2, \lambda) &= g(x_1, x_2) = 0
\end{aligned} \tag{3}$$

After solving eqn. (3), get extreme point.

Necessary Conditions for a General Problem

Suppose a general problem with n variables and m equality constraints as:

Minimize $f = f(X)$ subject to constraints

$$g_j(X) = 0, \quad j = 1, 2, 3, \dots, m, \quad \text{where} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

so, the Lagrange function, L, in this case is defined by as

$$L(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m) = f(\mathbf{X}) + \lambda_1 g_1(\mathbf{X}) + \lambda_2 g_2(\mathbf{X}) + \cdots + \lambda_m g_m(\mathbf{X}) \tag{4}$$

Were, L is the function of *n + m* unknowns, $x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m$.

The necessary conditions for the extremum of L, which are the solution of the equations as given below-

✓ n

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0, \qquad i = 1, 2, \ldots, n \tag{5}$$

$$\frac{\partial L}{\partial \lambda_j} = g_j(\mathbf{X}) = 0, \qquad j = 1, 2, \ldots, m \tag{6}$$

m eqn ✓

Equations (5) and (6) give the solutions as

$$\mathbf{X}^* = \begin{Bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{Bmatrix} \quad \text{and} \quad \lambda^* = \begin{Bmatrix} \lambda_1^* \\ \lambda_2^* \\ \vdots \\ \lambda_m^* \end{Bmatrix}$$

Sufficiency Conditions for a General Problem

A sufficient condiⁿ for $f(x)$ to have a relative minimum at x^* is that the quadratic Q, defined as

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 L}{\partial x_i \partial x_j} dx_i dx_j \quad — (1) \quad \text{evaluated}$$

at $X = X^* = (x_1^*, x_2^*, \ldots x_n^*, \lambda_1^*, \lambda_2^*, \ldots \lambda_n^*)$ must be +ve definite for all values of dx for which the constraints satisfied.

For +ve definite of (1), we have to find the each root of the polynomial z_i defined by the following determinantal equation, be +ve (−ve)

$$
\begin{vmatrix}
L_{11}-z & L_{12} & \cdots & L_{1n} & g_{11} & g_{21} & \cdots & g_{m1} \\
L_{21} & L_{22}-z & \cdots & L_{2n} & g_{12} & g_{22} & \cdots & g_{m2} \\
\vdots & & & & & & & \\
L_{n1} & L_{n2} & \cdots & L_{nn}-z & g_{1n} & g_{2n} & \cdots & g_{mn} \\
g_{11} & g_{12} & \cdots & g_{1n} & 0 & 0 & & 0 \\
g_{21} & g_{22} & \cdots & g_{2n} & 0 & 0 & & 0 \\
\vdots & & & & \vdots & \vdots & & \vdots \\
g_{m1} & g_{m2} & \cdots & g_{mn} & 0 & 0 & & 0
\end{vmatrix} = 0
$$

where,

$$L_{ij} = \frac{\partial^2 L}{\partial x_i \partial x_j}(X^*, \lambda^*)$$

$$g_{ij} = \frac{\partial g_i(X^*}{\partial x_j}$$

* If some of the roots of this polynomial are +ve while the others are negative, the point X^* is not an extreme point.

$$L_{11} = \frac{\partial^2 L}{\partial x_1^2}, \quad L_{12} = \frac{\partial^2 L}{\partial x_1 \partial x_2}, \quad L_{13} = \frac{\partial^2 L}{\partial x_1 \partial x_3}$$

$$g_{11} = \frac{\partial g_1}{\partial x_1}, \quad g_{12} = \frac{\partial g_1}{\partial x_2}$$

Q Find the maximum value of the following :

Maximize $f(x_1, x_2) = \pi x_1^2 x_2$ s/to

Suppose $A_0 = 24\pi$

Constraints

$$2\pi x_1^2 + 2\pi x_1 x_2 = A_0 \quad \to g_1$$

Ans Construct a Lang. function $L = f(x_1, x_2) + \lambda g_1(x_1, x_2)$

$$L = \pi x_1^2 x_2 + \lambda(2\pi x_1^2 + 2\pi x_1 x_2 - A_0)$$

the necessary condition for maxima of f,

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2\pi x_1 x_2 + 4\pi\lambda x_1 + 2\pi x_2 = 0 \quad \text{—(I)}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow \pi x_1^2 + 2\pi x_1 \lambda = 0 \quad \text{—(II)}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 2\pi x_1^2 + 2\pi x_1 x_2 = A_0 \quad \text{—(III)}$$

$x_1^* = \sqrt{\frac{A_0}{6\pi}}$, $x_2^* = \sqrt{\frac{2A_0}{3\pi}}$ and $d^* = -\sqrt{\frac{A_0}{24\pi}}$

$\Rightarrow x_1^* = \sqrt{\frac{24\pi}{6}} = 2$

$x_2^* = \sqrt{\frac{48\pi}{3\pi}} = 4$

$d^* = -\sqrt{\frac{24\pi}{24\pi}} = -1$

extreme point (x_1^*, x_2^*, d^*)

here no. of variables $= 2$ (x_1, x_2)

no. of constraint $= 1$,

Sufficient condi is the polynomial in term of z should be +ve or −ve, provided, the determinantal eqn.

$$\begin{vmatrix} L_{11}-z & L_{12} & g_{11} \\ L_{21} & L_{22}-z & g_{12} \\ g_{11} & g_{12} & 0 \end{vmatrix} = 0 \quad \text{— (IV)}$$

$$L_{11} = \frac{\partial^2 L}{\partial x_1^2} = 2\pi x_2 + 4\pi\lambda =$$

$$L_{12} = \frac{\partial^2 L}{\partial x_1 \partial x_2} = 2\pi x_1 + 2\pi\lambda = L_{21}$$

$$L_{22} = \frac{\partial^2 L}{\partial x_2^2} = 0$$

$$2\pi x_1^2 + 2\pi x_1 x_2 = A_0 = 24\pi$$

$$2\pi x_1^2 + 2\pi x_1 x_2 = 24\pi$$

$$g_{11} = \frac{\partial g_1}{\partial x_1} = 4\pi x_1 + 2\pi x_2$$

$$g_{12} = \frac{\partial g_1}{\partial x_2} = 2\pi x_1$$

(4) becomes

$$\Rightarrow \begin{vmatrix} 2\pi x_1^* + 4\pi\lambda^* - z & 2\pi x_1^* + 2\pi\lambda^* & 4\pi x_1^* + 2\pi x_2^* \\ 2\pi x_1^* + 2\pi\lambda^* & 0 - z & 2\pi x_1^* \\ 4\pi x_1^* + 2\pi x_2^* & 2\pi x_1^* & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 8\pi - 4\pi - z & 4\pi - 2\pi & 8\pi + 8\pi \\ 4\pi - 2\pi & -z & 4\pi \\ 8\pi + 8\pi & 4\pi & 0 \end{vmatrix} = 0$$

as
$x_1^* = 2$
$x_2^* = 4$
$\lambda^* = -1$

$$\Rightarrow \begin{vmatrix} 4\pi - z & 2\pi & 16\pi \\ 2\pi & -z & 4\pi \\ 6\pi & 4\pi & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad 272\pi^2 z + 192\pi^3 = 0$

$$\Rightarrow \quad z = -\frac{12}{17}\pi$$

as the value of z is negative, the point $(x_1^*, x_2^*) = (2$ corresponds to the maximum and maximum value is

$f(x^*) = \pi (2)^2 \cdot 4 = 16\pi \Rightarrow f^* = 16\pi$ Ans

Q2- Minimize $z = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$

subject to constraints

$$x_1 + x_2 + x_3 = 11, \text{ and } x_1, x_2, x_3 \geq 0$$

Ans Langrange function $L = z + \lambda g_1$, where

$$g_1 \cong x_1 + x_2 + x_3 - 11 = 0$$

$$\Rightarrow L = (2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200) + \lambda(x_1 + x_2 + x_3 - 11)$$

now, for maxima and minima -

$$\frac{\partial L}{\partial x_1} = 4x_1 - 24 + \lambda \quad \Rightarrow 4x_1 - 24 + \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - 8 + \lambda \quad \Rightarrow 4x_2 - 8 + \lambda = 0$$

$$\frac{\partial L}{\partial x_3} = 4x_3 - 12 + \lambda = 4x_3 - 12 + \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 11 \Rightarrow x_1 + x_2 + x_3 = 11$$

after solving these equations, get

$x_1^* = 6, \quad x_2^* = 2, \quad x_3^* = 3, \quad \lambda^* = 0$

sufficient condition for max or mini

$$\begin{vmatrix} L_{11} - P & L_{12} & L_{13} & g_{11} \\ L_{21} & L_{22} - P & L_{23} & g_{12} \\ L_{31} & L_{32} & L_{33} - P & g_{13} \\ g_{11} & g_{12} & g_{13} & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 4-p & 0 & 0 & 1 \\ 0 & 4-p & 0 & 1 \\ 0 & 0 & 4-p & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (4-p)\begin{vmatrix} 4-p & 0 & 1 \\ 0 & 4-p & 1 \\ 1 & 1 & 0 \end{vmatrix} + 0 + 0 - 1\begin{vmatrix} 0 & 4-p & 0 \\ 0 & 0 & 4-p \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (4-p)\left[(4-p)(0-1) + 0 - 1(0-4+p)\right] - 1\left((4-p)^2\right) = 0$$

$$\Rightarrow (4-p)\left[\cancel{-(4-p)} + \cancel{(4-p)}\right] - (4-p)^2 = 0$$

$$\Rightarrow (4-p)^2 = 0$$

$P = $ 4, 4

we can say here as the value of P is +ve

$\Rightarrow$ at $x^* = (x_1^*, x_2^*, x_3^*) = (6, 2, 3)$ get minimum

and minimum value is $f(x^*) =$