14.1 Counting Techniques

14.1.1 Permutation

The different arrangement which can be made out of a given number of things by taking some o
time, are called ***Permutations***.

For example, consider arranging the digits 1, 2 and 3. The possible arrangements as follows:

123, 132, 231, 213, 312, 321.

These are 6 permutations. Here the same three digits 1, 2 and 3 have been used but the number
when the order of digits is changed.

Thus, forming numbers with given digits means arranging the digits and hence it is the pr
permutation.

➢ Notations

Let r and n be positive integers such that $1 \le r \le n$.

Then, the number of all permutations of n things taking r at a time is denoted by $P(n, r)$ or ^{n}

Theorem 1: *Let $1 \le r \le n$. Then the number of all permutations of n dissimilar things taken r at a time is*

$$^{n}P_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{\lfloor\underline{n}}{\lfloor\underline{n-r}}.$$

Theorem 2: *The number of all permutations of n different things taken all at a time is given by* ^{n}

Proof: We have, $^{n}P_r = \dfrac{\lfloor\underline{n}}{\lfloor\underline{n-r}}$ Put $r = n$, we find $^{n}P_n = \dfrac{\lfloor\underline{n}}{\lfloor\underline{n-n}} = \dfrac{\lfloor\underline{n}}{\lfloor\underline{0}} = \dfrac{\lfloor\underline{n}}{1} = \lfloor\underline{n}$.

Theorem 3: *Prove that $\lfloor\underline{0} = 1$.*

Proof: We have,

$$^{n}P_r = \frac{\lfloor\underline{n}}{\lfloor\underline{n-r}}$$

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$$\Rightarrow \quad ^{n}P_n = \frac{\lfloor\underline{n}}{\lfloor\underline{0}}$$

$$\Rightarrow \quad \lfloor\underline{n} = \frac{\lfloor\underline{n}}{\lfloor\underline{0}} \qquad \Rightarrow \quad \lfloor\underline{0} = \frac{\lfloor\underline{n}}{\lfloor\underline{n}} = 1.$$

Example 5: *If* $^nP_4 = 2 \times {}^5P_3$*, find n.*

Solution:

$$^nP_4 = 2 \times {}^5P_3$$

$\Rightarrow$ $$n(n-1)(n-2)(n-3) = 2 \times 5 \times 4 \times 3$$

$\Rightarrow$ $$n(n-1)(n-2)(n-3) = 120$$

$\Rightarrow$ $$(n^2 - 3n)(n^2 - 3n + 2) = 120$$

$\Rightarrow$ $m(m+2) = 120$ where $m = n^2 - 3n$

$\Rightarrow$ $$m^2 + 2m - 120 = 0$$

$\Rightarrow$ $$(m+12)(m-10) = 0$$

$\Rightarrow$ $$m = -12, \text{ or } m = 10$$

$\Rightarrow$ $$n^2 - 3n = -12 \quad \text{or} \quad n^2 - 3n = 10$$

$\Rightarrow$ $$n^2 - 3n + 12 = 0 \quad \text{or} \quad n^2 - 3n - 10 = 0$$

$\Rightarrow$ $$n = \frac{3 \pm \sqrt{9 - 48}}{2} \quad \text{or} \quad (n-5)(n+2) = 0$$

$\Rightarrow$ $$n = \frac{3 \pm i\sqrt{39}}{2} \quad \text{or} \quad n = 5 \quad \text{or } n = -2$$

$\Rightarrow$ $n = 5$ [neglecting negative and imaginary

Example 6: *Find n if* $^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_n$.

Solution: We have, $^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_n$

$\Rightarrow$ $$\frac{\underline{|9}}{\underline{|9-5}} + 5 \cdot \frac{\underline{|9}}{\underline{|9-4}} = \frac{\underline{|10}}{\underline{|10-n}}$$

$\Rightarrow$ $$\frac{\underline{|9}}{\underline{|4}} + 5 \cdot \frac{\underline{|9}}{\underline{|5}} = \frac{\underline{|10}}{\underline{|10-n}}$$

$\Rightarrow$ $$\frac{\underline{|9}}{\underline{|4}} + \frac{\underline{|9}}{\underline{|4}} = \frac{\underline{|10}}{\underline{|10-n}}$$

$\Rightarrow$ $$2 \times \frac{\underline{|9}}{\underline{|4}} = \frac{10 \times \underline{|9}}{\underline{|(10-n)}}$$

$\Rightarrow$ $$5 \times \underline{|4} = \underline{|10-n}$$

$\Rightarrow$ $$\underline{|10-n} = \underline{|5} \quad \Rightarrow \quad 10 - n = 5 \quad \Rightarrow \quad n =$$

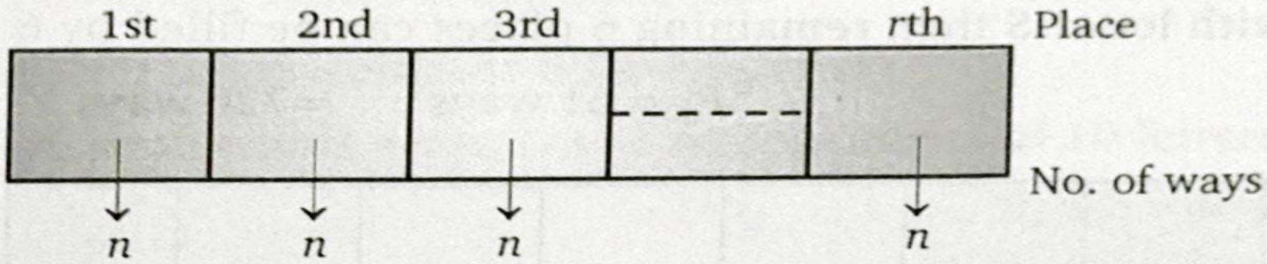
14.1.2 Permutation with Repetition

If out of n objects in a set, p objects are exactly alike of one kind, q objects exactly alike of se
objects exactly alike of third kind and remaining objects are all different then the number
of n objects taken all at a time is

$$= \frac{\lfloor n}{\lfloor p \, \lfloor q \, \lfloor r}$$

Theorem 4: *The number of permutations of n different objects taken r at a time when each object replaced any number of times in each arrangement is n^r.*

Proof: The number of permutations of n objects taken r at a time is the same as the number of filling r places with n different objects.



Number of ways of filling first place $= n$,

Number of ways of filling second place $= n$, (Since the object used in filling the first place repeated)

Number of way of filling third place $= n$,

....

....

....

Number of way of filling rth place $= n$

Therefore, by the fundamental principle of counting, the required number of permutations

$$= n \cdot n \cdot n \ldots\ldots r \text{ (times) } = n^r$$

Example 36: *In a shipment, there are 40 floppy disks of which 5 are defective, determine:*

(i) *In how many ways can we select five floppy disks?*

(ii) *In how many ways can we select five non-defective floppy disks?*

(iii) *In how many ways can we select five floppy disks containing exactly three defective flop*

(iv) *In now many ways can we select five disks containing at least 1 defective floppy disk?*

[U.P.T.U.

Solution: (i) The required number of ways

$$= 40C_5 = \frac{\lfloor 40}{\lfloor 5 \lfloor 35} = \frac{40\times 39\times 38\times 37\times 36\times \lfloor 35}{5\times 4\times 3\times 2\times \lfloor 35} = 39\times 38\times 37\times 12 = 65$$

(ii) The non-defective floppy are 35

The number of ways to select five non-defective floppy $= 35C_5 = \dfrac{35}{\lfloor 5 \lfloor 30}$

$$= \frac{35\times 34\times 33\times 32\times 31\times \lfloor 30}{5\times 4\times 3\times 2\times 1\times \lfloor 30} = 324632$$

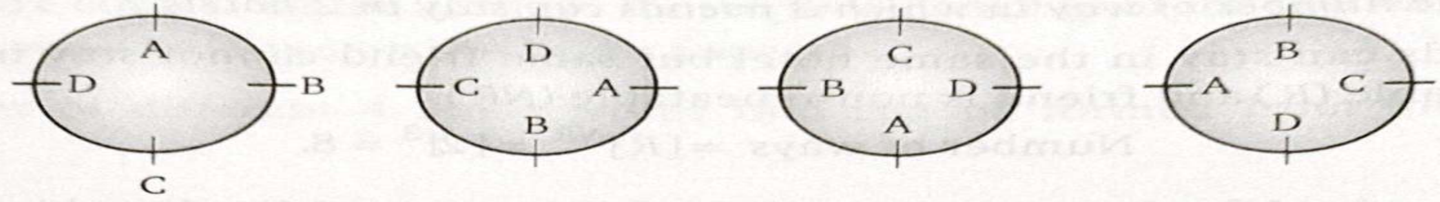
(iii) The number of ways to select five floppy contain exactly 3 defective floppy disks =

(iv) The required selection $= \dfrac{35C_4\times 5C_1 + 35C_3\times 5C_2 + 35C_2\times 5C_3 + 35C_1\times 5C_4}{40C_5}$

14.1.3 Circular Permutations

So far we have been considering the arrangements of objects in a line. Such permutation are linear permutations. In circular permutations, what really matters is the position of an object the others. If we consider the linear permutations ABCD, BCDA, CDAB and DABC. Then, clear distinct.

Now, we arrange A, B, C, D along the circumference of a circle as shown



If we consider the position of an object retative to others then we find that the above four arrang the same.

***Theorem* 5:** *The number of circular permutations of n different objects, is* $\lfloor\underline{n-1}$.

***Proof*:** Fixing the position of an object can be done in n ways, as the position of anyone of the fixed. Thus, each circular permutation corresponding to n linear permutations depending u from we start.

Since, there are $\lfloor\underline{n}$ linear permutations, it follows that there are $\frac{\lfloor\underline{n}}{n}$ *i. e.*, $\lfloor\underline{(n-1)}$ circular permu

***Theorem* 6:** *The number of ways in which n persons can be seated round a table is* $\lfloor\underline{n-1}$.

***Proof*:** Let us fix the position of one person and then arrange the remaining $(n-1)$ persons in a ways. Clearly, this can be done in $\lfloor\underline{(n-1)}$ ways. Hence, the required number of ways $= \lfloor\underline{n-1}$.

***Theorem* 7:** *Show that the number of ways in which n different beads can be arranged to form a* $\frac{1}{2}\lfloor\underline{(n-1)}$.

***Proof*:** Fixed the position of one bead, the remaining $(n-1)$ beads can be arranged in $\lfloor\underline{n-1}$ w

In case of arranging the beads, there is no distinction between the clockwise and anti arrangements. So, the required number of ways $= \frac{1}{2}\lfloor\underline{(n-1)}$.

Example 40: *A telegraph has 5 arms and each arm is capable of 4 distinct positions, includin of rest. Find the total signals that can be made ?*

Solution: Two arms may have the same position, but same arms cannot have two positio
Hence, position is repeatable (R), but arm is non-repeatable (*NR*)

$$\text{Number of ways } = [R]^{NR} = 4^5 = 1{,}024$$

But, in one case, when all the 5 arms will be in rest position, no signal will be made.

Hence, the required number of signals = 1,024 – 1 = 1,023.

Example 41: *Find the number of way in which 3 friends can stay in 2 hotels ?*

Solution: Two friends can stay in the same hotel but same friend cannot stay in two ho
Hence, hotel is repeatable (*R*) and friend is non-repeatable (*NR*).

$$\text{Number of ways } = [R]^{NR} = [2]^3 = 8.$$

14.2 Combinations

Combinations: Each of the different groups or selections which can be formed by taking some number of objects, irrespective of their arrangements, is called a combination. The total combinations of n distinct objects taking r $(1 \leq r \leq n)$ at a time is denoted by $C(n, r)$ or ${}^nC_r$ or $\binom{n}{r}$

where, ${}^nC_r$ is defined only when n and r integers such that $n \geq r$ and $n > 0$, $r \geq 0$.

Illustration: *The combinations of 4 objects a, b, c, d taking 2 at a time are* ab, ac, ad, bc, bd, cd

14.2.1 Difference between a Permutation and a Combination

In combination, only a group is made and the order in which the objects are arranged is imm permutation, not only a group is formed, but also an arrangement in definite order is consider

Note: We used the word 'arrangement' for permutation and selections for combinations.

Theorem 8: *The number of all combinations of n distinct objects, taken r at a time, is given by*

$${}^nC_r = \frac{\lfloor n}{\lfloor r \, \lfloor n-r}$$

Proof: Let the number of all combinations of n objects, taken r at a time, be x. Then, ${}^nC_r =$

Now, each combination contains r objects, which may be arranged amongst themselves is $\lfloor r$ w

Thus, each combination gives rise to $\lfloor r$ permutations.

$\therefore$ x combinations will give rise to $x \times \lfloor r$ permutations.

So, the number of permutations of n things, taken r at a time is $x \times \lfloor r$

Hence, $${}^nP_r = x \times \lfloor r = {}^nC_r \times \lfloor r$$

$\therefore$ $${}^nC_r = \frac{{}^nP_r}{\lfloor r} = \frac{\lfloor n}{\lfloor r \, \lfloor n-r} \qquad \left[\because {}^nP \right.$$

Note: We can write $${}^nC_r = \frac{n(n-1)(n-2)\ldots.\ n \text{ factors}}{\lfloor r}.$$

Theorem 9: *Let $0 \le r \le n$. Prove that* $\quad {}^nC_r = {}^nC_{n-r}$.

proof: We have

$$ {}^nC_{n-r} = \frac{\lfloor\underline{n}}{\lfloor\underline{n-r}\,\lfloor\underline{n-(n-r)}} = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}} = {}^nC_r. $$

Theorem 10: *To prove that* ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$.

proof: We have,

$$ {}^nC_r + {}^nC_{r-1} = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}} + \frac{\lfloor\underline{n}}{\lfloor\underline{r-1}\,\lfloor\underline{n-(r-1)}} = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}} + \frac{\lfloor\underline{n}}{\lfloor\underline{r-1}\,\lfloor\underline{n-r+1}} = \frac{\lfloor\underline{n}\cdot(n-r+1)}{\lfloor\underline{r}\cdot\lfloor\underline{(n-r+1)}} + $$

$$ = \left\{\frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r+1}}\right\}\cdot\{n-r+1+r\} = \frac{(n+1)\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r+1}} = \frac{\lfloor\underline{n+1}}{\lfloor\underline{r}\,\lfloor\underline{n+1-r}} = {}^{n+1}C_r $$

Theorem 11: *If $1 \le r \le n$, prove that* $n \times {}^{(n-1)}C_{r-1} = (n-r+1) \times {}^nC_{r-1}$.

Proof: We have $\quad n \times {}^{(n-1)}C_{r-1} = n \times \dfrac{\lfloor\underline{n-1}}{\lfloor\underline{r-1} \times \lfloor\underline{(n-1)-(r-1)}} = \dfrac{n \times \lfloor\underline{n-1}}{\lfloor\underline{r-1}\,\lfloor\underline{n-r}}$

$$ = \frac{\lfloor\underline{n}\,(n-r+1)}{\lfloor\underline{r-1} \times \lfloor\underline{n-r} \times (n-r+1)} = (n-r+1) \times \frac{\lfloor\underline{n}}{\lfloor\underline{r-1} \times \lfloor\underline{n-r+1}} = (n-r+1) \times {}^nC_{r-1} $$

Theorem 12: *If n and r are positive integers such that $1 \le r \le n$ then* $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$.

Proof : We know that $\quad {}^nC_r = \dfrac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}}$ and ${}^nC_{r-1} = \dfrac{\lfloor\underline{n}}{\lfloor\underline{r-1} \times \lfloor\underline{n-r+1}}$

$$ \therefore \quad \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}} \times \frac{\lfloor\underline{r-1}\,\lfloor\underline{n-r+1}}{\lfloor\underline{n}} = \frac{\lfloor\underline{r-1} \times (n-r+1)\,\lfloor\underline{n-r}}{r \times \lfloor\underline{(r-1)}\,\lfloor\underline{n-r}} = \left(\frac{n-r+1}{r}\right) $$

Theorem 13: *If $1 \le r \le n$, prove that* ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$.

Proof:

$$ {}^nC_r + {}^nC_{r+1} = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}} + \frac{\lfloor\underline{n}}{\lfloor\underline{(r+1)}\,\lfloor\underline{n-r-1}} $$

$$ = \frac{\lfloor\underline{n}}{\lfloor\underline{r} \times (n-r)\,\lfloor\underline{n-r-1}} + \frac{\lfloor\underline{n}}{(r+1)\lfloor\underline{r} \times \lfloor\underline{n-r-1}} $$

$$ = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r-1}}\left(\frac{1}{n-r} + \frac{1}{r+1}\right) $$

$$ = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r-1}} \times \frac{(n+1)}{(n-r)(r+1)} = \frac{(n+1)\lfloor\underline{n}}{(r+1)\lfloor\underline{r} \times (n-r) \times \lfloor\underline{n-r-1}} $$

$$ = \frac{\lfloor\underline{n+1}}{\lfloor\underline{r+1}\,\lfloor\underline{n-r}} = \frac{\lfloor\underline{n+1}}{\lfloor\underline{r+1}\,\lfloor\underline{(n+1)-(r+1)}} = {}^{n+1}C_{r+1} $$

Theorem 14: *Prove that* ${}^nC_p = {}^nC_q \Rightarrow p = q$ *or* $p + q = n.$

Proof: We have, ${}^nC_p = {}^nC_q = {}^nC_{n-q}$

$$\Rightarrow \qquad p = q \quad \text{or} \quad p = n - q \quad \text{or} \quad p + q = n.$$

Evaluate

Example 47: *If* ${}^nP_r = 720$ *and* ${}^nC_r = 120$, *find r.*

Solution: We know that $\qquad {}^nC_r = \dfrac{{}^nP_r}{\lfloor r}$

$$\therefore \qquad 120 = \frac{720}{\lfloor r} \Rightarrow \lfloor r = \frac{720}{120} = \lfloor 3$$

Hence, $\qquad r = 3.$

Example 48: *Prove that* ${}^{2n}C_n = \dfrac{2^n \times [1 \cdot 3 \cdot 5 \ldots . (2n-1)]}{\lfloor n}.$

Solution: ${}^{2n}C_n = \dfrac{\lfloor 2n}{\lfloor n \lfloor 2n - n} = \dfrac{\lfloor 2n}{(\lfloor n)^2}$

$$= \frac{(2n)(2n-1)(2n-2) \ldots \ldots 4 \cdot 3 \cdot 2 \cdot 1}{(\lfloor n)^2}$$

$$= \frac{[2n(2n-2)(2n-4) \ldots . 4 \cdot 2] \times [(2n-1)(2n-3) \ldots . 5 \cdot 3 \cdot 1]}{(\lfloor n)^2}$$

$$= \frac{2^n [n(n-1)(n-2) \ldots . 2 \cdot 1][(2n-1)(2n-3) \ldots . 5 \cdot 3 \cdot 1]}{(\lfloor n)^2}$$

$$= \frac{2^n \lfloor n [1 \cdot 3 \cdot 5 \ldots . (2n-3)(2n-1)]}{(\lfloor n)^2}$$

$$= \frac{2^n \times [1 \cdot 3 \cdot 5 \ldots . (2n-3)(2n-1)]}{\lfloor n}.$$

14.4 Permutations and Combinations with Unlimited Repetitions

Let $U(n, r)$ denote-permutation of n-objects with unlimited repetitions, and $V(n, r)$ denote the r-combinations with unlimited repetitions, then

$U(n, r) = n^r$ and $V(n, r) = (n-1+r, n-1)$ consider the set $\{\infty, a_1, \infty, a_2, \ldots \infty, a_n\}$ where $a_1, a_2, \ldots$ distinct. Any r-combination is of the form $\{x_1, a_1, x_2, a_2 \ldots x_n, a_n\}$ where each x_i is non-negative in

$$x_1 + x_2 + \ldots + x_n = r$$

The number $x_1 + x_2 + \ldots + x_n$ are called ***Repetition Numbers***. Conversely any sequence of no integers $x_1 + x_2 + \ldots + x_n$, where $\sum_{i=1}^{n} x_i = r$

Corresponding to $a - r$ combination $\{x_1, a_1, x_2, a_2, \ldots x_n, a_n\}$. The following results are made.

The number of r-combinations of $\{\infty, a_1, \infty, a_2, \ldots \infty, a_n\}$

= The number of non-negative integers solution of $x_1 + x_2 + x_3 + \ldots + x_n = r$

= The number of ways of placing r-indistinguishable balls in n numbered boxes.

= The number of binary number with $n-1$ one r and r-zeros $= C(n-1+r, r) = C(n-1+r,$

Example 70: *Enumerate the number of non-negative integral to the inequality* $x_1 + x_2 + x_3 + x$

[R.G.P.V. (B.E.) Raipur 2005, 2009; Pu

Solution: We can express the given inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 19$$

As

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0$$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1$$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 19$$

The number of non-negative integral solution of $x_1 + x_2 + x_3 + x_4 + x_5 = 0$ is $C(5-1+0, 0)$

The number of non-negative integral solution of $x_1 + x_2 + x_3 + x_4 + x_5 = 1$ is $C(5-1+1, 1)$

....

....

The number of non-negative integral solution of $x_1 + x_2 + x_3 + x_4 + x_5 = 19$ is $C(5-1+19, 1$

$\therefore$ The number of non-negative integral solution of

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 19$$

is $C(5-1+0, 0) + C(5-1+1, 1) + \ldots + C(5-1+19, 19)$

Example 71: *Find the number of 3-combinations of* $\{\infty, a_1, \infty, a_2, \infty, a_3, \infty, a_4\}$ [Osmar

Solution: We have $n = 4$, $r = 3$

$\therefore$ The number of 3-combination of the given set is $C(4-1+3, 3) = C(6, 3) = 6C_3 = \frac{6}{}$