30/09/22

Unit - I

Differential Equation

Linear Differential Equation with constant coefficient

A differential equatn of the form is known as non-homogenous diff eqn with constant coeff.

$$a_n\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots a_0 y = R(x)$$

$$\left(a_n\frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \dots a_0\right)y = R(x)$$

If $D = \frac{d}{dx}$

$$(a_n D^n + a_{n-1} D^{n-1} + \dots a_0)y = R(x)$$

$$f(D)y = R(x) \quad \text{--- (1)}$$

Solution of (1) is given by

$$y = C.F + P.I$$

where C.F is complimentary functn.
PI is Particular Integral

Complimentary Function

C.F is the solution of the homogenous diff equation.

$$f(D).y = 0$$

$$(a_n D^n + a_{n-1} D^{n-1} + \dots a_0)y = 0 \quad \text{--- (2)}$$

Let $y = e^{mx}$ be sol^n of ②

$$(a_n m^n e^{mx} + a_{n-1} m^{n-1} e^{mx} + \dots a_0 e^{mx}) = 0$$

$$(a_n m^n + a_{n-1} m^{n-1} + \dots a_0) e^{mx} = 0$$

Then,

$$\boxed{a_n m^n + a_{n-1} m^{n-1} + \dots a_0 = 0} \quad \text{--- ③}$$

- Auxiliary Eq^n.

Case 1 :- Let ③ has distinct and real roots say,

$$m = m_1, m_2 \dots m_n$$

then sol^n of ② is

$$y = C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots C_n e^{m_n x}$$

Case 2 :- Let $m_1 = m_2 = m_3 = m$ & other roots are distinct & real, then

$$y = C.F = (C_1 + C_2 x + C_3 x^2) e^{mx} + C_4 e^{m_4 x} + \dots C_n e^{m_n x}$$

Case 3 :- Let $m = \alpha \pm i\beta$

$$C.F = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

$$= e^{\alpha x} (C_1 e^{i\beta x} + C_2 e^{-i\beta x})$$

$$= e^{\alpha x} \left\{ C_1 (\cos\beta x + i\sin\beta x) + C_2 (\cos\beta x - i\sin\beta x) \right]$$

$$= e^{\alpha x} [A\cos\beta x + B\sin\beta x]$$

$[\because A = C_1 + C_2,\ B = iC_1 - iC_2]$

↳ all constant,

OR

$$y = C.F = Ae^{\alpha x} \sin(\beta x + B)$$

OR

$$y = C.F = Ae^{\alpha x} \cos(\beta x + B)$$

If $m = \alpha \pm \sqrt{\beta}$

$$C.F. = e^{\alpha x}[c_1 \cosh\sqrt{\beta} + c_2 \sinh\sqrt{\beta}]$$

If $m = \alpha \pm i\beta$ (twice)

then

$$y = C.F. = e^{\alpha x}\left[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x\right]$$

Particular Integral

$$P.I = \frac{1}{f(D)} \times R(x)$$

Note:- $\frac{1}{(D-\alpha)}[f(x)] = e^{\alpha x}\int e^{-\alpha x} f(x)\,dx$

$$\frac{1}{D-\alpha}[f(x)] = y$$

$$(D-\alpha)y = f(x)$$

$$\frac{dy}{dx} - \alpha y = f(x)$$

$$y \times e^{-\alpha x} = \int f(x) \cdot e^{-\alpha x} dx$$

$$y = e^{\alpha x}\int f(x)\, e^{-\alpha x} dx$$

$$\frac{1}{(D-\alpha)}[f(x)] = e^{\alpha x}\int f(x)\, e^{-\alpha x} dx$$

If $\alpha = 0$

$$\frac{1}{D}[f(x)] = \int f(x)\,dx$$

$$\frac{1}{D^2}[f(x)] = \iint [f(x)\,dx]\,dx$$

Note :-

$$\frac{1}{D-\alpha}[e^{\alpha x}] = e^{\alpha x}\int e^{\alpha x} \times e^{-\alpha x}\,dx$$

$$= \frac{x}{1} e^{\alpha x}$$

Formula

① $$\frac{1}{f(D)}[e^{ax}] = \frac{1}{f(a)} e^{ax}, \quad f(a) \neq 0$$

$$= \frac{x}{f'(a)} e^{ax}, \quad f'(a) \neq 0,\ f(a) = 0$$

$$= \frac{x^2}{f''(a)} e^{ax}, \quad f''(a) \neq 0,\ f'(a) = 0,\ f(a) = 0$$

② $$\frac{1}{f(D)}[x^m] = \frac{1}{\phi(1+D)}[x^m]$$

$$= \{\phi(1+D)\}^{-1} x^m$$

then we take D common from lowest degree of $f(D)$ term & then apply binomial expansion in $\{\phi(1+D)\}^{-1}$.

(3) $y = \frac{1}{f(D^2)}(\sin ax \text{ or } \cos ax)$

$D^2 = -a^2$

$= \frac{1}{f(-a^2)}(\sin ax \text{ or } \cos ax) = \frac{x^2}{f''(-a^2)}(\sin ax \text{ or } \cos ax)$

↳ if $f'(-a^2) = 0$.

$= \frac{x}{f'(-a^2)}(\sin ax \text{ or } \cos ax)$ if $f(-a^2) = 0$

Formula to remember.

(1) $\cosh x = \frac{e^x + e^{-x}}{2}$

(2) $\sinh x = \frac{e^x - e^{-x}}{2}$

(3) $\cosh^2 x - \sinh^2 x = 1$

(4) $\sinh(-x) = -\sinh x$

(5) $\cos i\theta = \cosh\theta$

(6) $\sin i\theta = i\sinh\theta$

Questn to Solve

(1) $\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 5y = -2\cosh x$

(2) $(D^3 - D^2 - D + 1)y = 1 + x^2$

(3) $(D^3 + 3D^2 + 2D)y = x^2 + 1$

(4) $(D^2 - 3D + 2)y = 2\cos(2x+3) + 2e^x$

(5) $(D^2 - 4D + 3)y = \sin 3x \cos 2x$

Soln

(1) $(D^2 + 4D + 5)y = -2\cosh x$

$(D^2 + 4D + 5)y = -2 \times \left(\frac{e^x + e^{-x}}{2}\right)$

$(D^2 + 4D + 5)y = -(e^x + e^{-x})$

$y = -\left[\frac{1}{(D^2+4D+5)} e^x + \frac{1}{D^2+4D+5} e^{-x}\right]$

$= -\left[\frac{1}{10}e^x + \frac{1}{2}e^{-x}\right]$

$y = \left(\frac{e^x + 5e^{-x}}{-10}\right) = \text{P.I}$

② For P.I

$$y = \frac{1}{(D^3-D^2-D+1)} e^{0x} + \frac{1}{(D^3-D^2-D+1)} x^2$$

$$= \frac{1}{1} e^{0x} + \{1+(D^3-D^2-D)\}^{-1} x^2$$

$$= 1 + \{1 - D^3 + D^2 + D + D^2\} x^2$$

$$= 1 + \{x^2 - 0 + 2 + 2x + 2\}$$

$$= x^2 + 2x + 5$$

③ For P.I.

$$y = \frac{1}{(D^3+3D^2+2D)} e^{0x} + \frac{1}{(D^3+3D^2+2D)} x^2$$

$$= \frac{x}{(3D^2+6D+2)} e^{0x} + \frac{1}{2D}\left\{1+\frac{1}{2}(D^2+3D)\right\}^{-1} x^2$$

$$= \frac{x}{2} e^{0x} + \frac{1}{2D}\left\{1 - \frac{1}{2}(D^2+3D) + \frac{1}{4}(D^2+3D)^2\right\} x^2$$

$$= \frac{x}{2} + \frac{1}{2D}\left\{x^2 - 1 - \frac{3x}{2} + \frac{9}{2}\right\}$$

$$\Rightarrow \frac{x}{2} + \frac{1}{2}\left\{\frac{x^3}{3} - x - \frac{3x^2}{2} + \frac{9x}{2}\right\} = \frac{x^3}{6} - \frac{3x^2}{4} + \frac{9x}{4}$$

④ For P.I

$$y \Rightarrow 2\left\{\frac{2}{(D^2-3D+2)} \cos(2x+3) + \frac{1}{(D^2-3D+2)} e^x\right\}$$

$$\Rightarrow 2\left\{\frac{x(\cos 3)}{(2D-3)} \cos 2x - \frac{x(\sin 3)}{(2D-3)} \sin 2x + \frac{x}{(2D-3)} e^x\right\}$$

$$\Rightarrow 2\left\{x(\cos 3 \cos 2x - \sin 3 \sin 2x) - xe^x\right\}$$

$$\Rightarrow 2x\left\{\cos(2x-3) - e^x\right\}$$

Solve

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 1 + x^2 + 3^x + \log 2$$

Solⁿ: $(D^3 - D^2 + 6D)y = 1 + x^2 + 3^x + \log 2.$

For C.F.

$y(D^3 - D^2 + 6D) = 0$

$m^3 - m^2 + 6m = 0$

$m(m^2 - m + 6) = 0$

$$m = \frac{+1 \pm \sqrt{1-24}}{2}$$

$$= \frac{+1 \pm \sqrt{23}i}{2}$$

$$m = 0, \ \frac{+1 + \sqrt{23}i}{2}, \ \frac{+1 - \sqrt{23}i}{2}.$$

$$C.F = C_1 + e^{x/2}\left[C_2 \cos\frac{\sqrt{23}}{2}x + C_3 \sin\frac{\sqrt{23}}{2}x\right]$$

For P.I

$$y = \frac{1}{D^3 - D^2 + 6D}(1 + x^2 + 3^x + \log 2)$$

$$= \frac{1}{D^3 - D^2 + 6D}(e^{0x}) + \frac{1}{D^3 - D^2 + 6D}(x^2) + \frac{1 \cdot 3^x}{D^3 - D^2 + 6D} + \frac{1 \times \log 2}{D^3 - D^2 + 6D}(e^{0x})$$

$$\Rightarrow \frac{x}{6} + \frac{1}{6D}\left[1 + \frac{1}{6}(D^2 - D)\right]^{-1}(x^2) + \frac{1}{D^3 - D^2 + 6D}(e^{x\log 3}) + \frac{x}{6}\log 2$$

$$= \frac{x}{6} + \frac{1}{6D}\left(1 - \frac{1}{6}(D^2-D) + \frac{1}{36}(D^2-D)^2 + \dots\right)x^2$$

$$+ \frac{1}{(\log 3)^3 - (\log 3)^2 + 6(\log 3)} 3^x + \frac{x}{6}\log 2$$

$$= \frac{x}{6} + \frac{1}{6D}\left(x^2 - \frac{1}{6}\left((2-2x) + \frac{1}{18}\right]\right) + \text{"} + \text{"}$$

$$= \frac{x}{6} + \frac{1}{6}\left(\frac{x^3}{3} - \frac{\cancel{2}}{6}\left(2x - \frac{2x^2}{2}\right) + \frac{2x}{18}\right) + \text{"} \quad \text{"}$$

$$= \frac{x}{6} + \frac{1}{6}\left[\frac{x^3}{3} - \frac{1}{3}\left(x - \frac{x^2}{2}\right) + \frac{1}{18}x\right] + \text{"} \quad \text{"}$$

$$y = C.F + P.I$$

Solve :-

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = 2\sin 3x + e^{2x}$$

$$y = 1, \ \frac{dy}{dx} = 0, \ \text{at } x = 0.$$

$$(D^2 - 7D + 6)y = 2\sin 3x + e^{2x}$$

For C.F.

$$D^2 - 7D + 6 = 0$$

$$D = \frac{7 \pm \sqrt{49 - 24}}{2}$$

$$= \frac{7 \pm \sqrt{25}}{2}$$

$$= \frac{7 \pm 5}{2} = 6, 1$$

$C.F = c_1 e^{x} + c_2 e^{6x}$

For PI.

$$y = 2\frac{1}{(D^2-7D+6)}(2\sin 3x) + \frac{1}{D^2-7D+6}e^{2x}$$

$$\Rightarrow \because D^2 = -9$$

$$= \frac{1}{-9-7D+6}(2\sin 3x) + \frac{1}{4-14+6}e^{2x}$$

$$= \frac{1}{-3-7D}2\sin 3x + \frac{1}{-4}e^{2x}$$

$$= -\left[\frac{2}{3+7D}(\sin 3x) + \frac{1}{4}e^{2x}\right]$$

$$= -\left[\frac{2(7D-3)}{(7D-3)(7D+3)}(\sin 3x) + \frac{1}{4}e^{2x}\right]$$

$$= -\left[\frac{2(7D-3)(\sin 3x)}{-(49\times 9)-9} + \frac{1}{4}e^{2x}\right]$$

$$\Rightarrow -\left[\frac{2(7D)(\sin 3x)}{-9\times 50} - \frac{6}{9\times 50}\sin 3x + \frac{1}{4}e^{2x}\right]$$

$$= -\left[\frac{3\times 14\cos 3x - 6\sin 3x}{-9\times 50} + \frac{1}{4}e^{2x}\right]$$

$$\Rightarrow -\left[\frac{42\cos 3x - 6\sin 3x}{-450} + \frac{1}{4}e^{2x}\right]$$

$$\Rightarrow -\left[\frac{7\cos 3x - \sin 3x}{-75} + \frac{1}{4}e^{2x}\right]$$

$y = C.F + P.I$

$\Rightarrow C_1 e^{x} + C_2 e^{6x} + \cancel{\sin 3x} \dfrac{(7\cos 3x - \sin 3x)}{75} - \dfrac{1}{4} e^{2x}$

$\dfrac{dy}{dx} \Rightarrow c_1 e^{x} + 6 c_2 e^{6x} \cancel{21\cos} \dfrac{-21\sin 3x - 3\cos 3x}{75} - \dfrac{2}{4} e^{2x}$

If $x = 0$, $y = 1$ & $\dfrac{dy}{dx} = 0$

$0 = C_1 + 6C_2 - \dfrac{3}{75} - \dfrac{1}{2}$ —— ①

$1 = C_1 + C_2 + \dfrac{7}{75} - \dfrac{1}{4}$ —— ②

$C_1 + 6C_2 = \dfrac{3}{75} + \dfrac{1}{2}$

$C_1 + 6C_2 = \dfrac{81}{150}$ 28

$C_1 + C_2 = \dfrac{5}{4} - \dfrac{7}{75} = \dfrac{347}{300}$

$C_1 = \dfrac{347}{300} - C_2$

$\dfrac{347}{300} + 5C_2 = \dfrac{162}{300}$

$C_2 = -\dfrac{185}{300 \times 5} = \dfrac{-37}{300}$

$C_1 = \dfrac{384}{300}$

Formula :—

④ $$\frac{1}{f(D)}\left[e^{ax} \times v(x)\right] = e^{ax}\frac{1}{f(D+a)}\left[v(x)\right]$$

⑤ $$\frac{1}{f(D)}\left[x\sin ax\right] = x\frac{1}{f(D)}\left[\sin ax\right] - \frac{f'(D)}{\left[f(D)\right]^2}(\sin ax)$$

or

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta = Re\,(e^{i\theta})$$

$$\sin\theta = Im\,(e^{i\theta})$$

$$\therefore \frac{1}{f(D)} \times \left[x\sin ax\right]$$

$$= Im\left[\frac{1}{f(D)}\left[x.e^{iax}\right]\right]$$

Now use ④.

Solve :—

① $(D^2 - 2D + 1)y = e^x \sin x$

② $(D^2 - 2D + 1)y = x e^x \sin x$

③ $(D^2 - 2D + 2)y = x e^x \cos x$

Solⁿ :—

② $(D^2 - 2D + 1)y = xe^x \sin x$

Auxiliary Eqⁿ

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1$$

$$C.F = (C_1 + C_2 x)e^x$$

P.I

$$y = \cancel{\text{cos}} \frac{1}{(D-1)^2} (e^x \sin x)$$

$$= e^x \frac{1}{(D+1-1)^2} [x \sin x]$$

$$= e^x \frac{1}{D^2} [x \sin x]$$

Formula :-

$$[uv]_1 = uv_1 - u'v_2 + u''v_3 + \ldots\ldots$$

where $v_1 = \int v$, $u' = du$

$$y = e^x \frac{1}{D} [x(-\cos x) - (1)(-\sin x)]$$

$$= e^x \frac{1}{D} [\sin x - x\cos x]$$

$$= e^x [-\cos x - (x \sin x - (1)(\cos x)]$$

$$= e^x [-x\sin x - 2\cos x]$$

$$y = C.F + P.I$$

$$= e^x [-x\sin x - 2\cos x] + (C_1 + C_2 x) e^x$$

(3) $(D^2-2D+2)y = xe^x\cos x$

C.F

$m^2-2D+2=0$

$m = \dfrac{2\pm\sqrt{4-8}}{2} = 1\pm i\times 1.$

C.F $= e^x(c_1\cos x + c_2\sin x)$

P.I

$y = \dfrac{1}{(D^2-2D+2)}[xe^x\cos x]$

$\Rightarrow e^x \dfrac{1}{[(D-1)^2+1]}[x\cos x]$

$\Rightarrow e^x \dfrac{1}{D^2+1}[x\cos x]$

$= e^x\left\{x\dfrac{1}{(D^2+1)}[\cos x] - \dfrac{2D}{(D^2+1)^2}[\cos x]\right\}$

$= e^x\left\{x\times\dfrac{x}{2D}[\cos x] + \dfrac{2}{(D^2+1)^2}(\sin x)\right\}$

$= e^x\left\{\dfrac{x^2}{2}\sin x + \dfrac{2x}{2(D^2+1)(2D)}\sin x\right\}$

$= e^x\left\{\dfrac{x^2}{2}\sin x + \dfrac{x(\cos x)}{2(D^2+1)}\right\}$

$= e^x\left\{\dfrac{x^2}{2}\sin x + \left(-\dfrac{x^2}{4D}\cos x\right)\right\} = e^x\left\{\dfrac{x^2}{2}\sin x - \dfrac{x^2}{4}\sin x\right\}$

$= \dfrac{x^2}{4}e^x\sin x$

$y = C.F + P.I$

ive :-

1. $(D^2+1)y = \tan x$
2. $(D^2-2D+2)y = e^x \tan x$
3. $(D^2+3D+2)y = e^{e^x}$
4. $(D^2-3D+2)y = \sin e^x$

Soln :-

1. $(D^2+1)y = \tan x$

Auxiliary eqn

$$m^2+1=0$$

$$m = \pm i$$

$$C.F = (c_1 \cos x + c_2 \sin x)$$

P.I.

$$= \frac{1}{(D^2+1)}[\tan x]$$

$$= \frac{1}{(D+i)(D-i)}[\tan x]$$

$$= \frac{1}{-2i}\left[\frac{1}{D+i} - \frac{1}{D-i}\right][\tan x]$$

$$= \frac{1}{2i}\left\{\frac{1}{(D-i)}[\tan x] - \frac{1}{(D+i)}[\tan x]\right\}$$

$$= \frac{1}{2i}\{I_1 - I_2\}$$

$$I = \frac{1}{(D-i)}[\tan x]$$

$$= e^{ix}\int e^{-ix}\tan x\,dx$$

$$\Rightarrow e^{ix}\int (\cos x - i\sin x)\frac{\sin x}{\cos x}\,dx$$

$$= e^{ix}\int \sin x\,dx - i\int \frac{\sin^2 x}{\cos x}\,dx$$

$$= e^{ix}\left[\cos x - i\int \cancel{\tan x \sin x\,dx}\ \frac{\sin^2 x}{\cos x}\,dx\right]$$

$$= e^{ix}\left[\cos x - i\left(\int (\sec x - \cos x)\,dx\right)\right]$$

$$= e^{ix}\left[-\cos x - i\left[\ln|\sec x + \tan x| - \sin x\right]\right]$$

$$= e^{ix}\left[-\cos x + i\sin x - i\ln|\sec x + \tan x|\right]$$

$$I_2 \Rightarrow e^{-ix}\left[-\cos x + i\log|\sec x + \tan x| - i\sin x\right]$$

$$\text{P.I.} = \frac{1}{2i}\Big[-\cos x\,(e^{ix} - e^{-ix}) - i\log(\sec x + \tan x)(e^{ix} + e^{-ix}) + i\sin x\,(e^{ix} + e^{-ix})\Big]$$

$$= \frac{1}{2i}\Big(-2i\cos x\sin x - 2i\cos x\log(\sec x + \tan x) + 2i\cos x\sin x\Big)$$

$$= -\cos x\log(\sec x + \tan x)$$

$\cos x = t$

(2) $(D^2+3D+2)y = e^{e^x}$ ★ (quiz)

Auxiliary eq^n :-

$$m^2+3m+2 = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

$$C.F = c_1 e^{-x} + c_2 e^{-2x}$$

P.I $= \dfrac{1}{(D^2+3D+2)}[e^{e^x}]$

$= \dfrac{1}{(D+1)(D+2)}[e^{e^x}]$

$= -\dfrac{1}{D+2}[e^{e^x}] + \dfrac{1}{D+1}[e^{e^x}]$

$\Rightarrow$ As $\dfrac{1}{D-\alpha}[f(x)] = e^{\alpha x}\int e^{-\alpha x} f(x)\,dx$

P.I $\Rightarrow e^{-x}\int e^x \cdot e^{e^x}dx - e^{-2x}\int e^{2x} e^{e^x} dx$

$\Rightarrow$ Let $e^x = t$

$e^x dx = dt$

$\Rightarrow e^{-x}\int e^t dt - e^{-2x}\int t \cdot e^t dt$

$\Rightarrow e^{-x} e^{e^x} - e^{-2x}(te^t - e^t]$

$\Rightarrow e^{-2x} e^{e^x}$

$y = C.F + P.I$

(4) $(D^2+3D+2)y = \sin e^x$

Auxiliary Equation

$$m^2+3m+2=0$$

$$m = \frac{3 \pm \sqrt{9-8}}{2}$$

$$= \frac{3\pm 1}{2} = -1, -2$$

$$C.F = C_1 e^{-x} + C_2 e^{-2x}$$

P.I $\Rightarrow$ $y = \frac{1}{(D^2+3D+2)}[\sin e^x]$

$$= \frac{1}{(D+1)(D+2)}[\sin e^x]$$

$$= \left[\frac{1}{(D+1)} - \frac{1}{(D+2)}\right][\sin e^x]$$

$$= \frac{1}{(D+1)}[\sin e^x] - \frac{1}{(D+2)}[\sin e^x]$$

$$= I_1 - I_2$$

$$\left[\because \frac{1}{(D-\alpha)}[f(x)] = e^{\alpha x}\int e^{-\alpha x} f(x)\,dx\right]$$

$$I_2 = \frac{1}{(D+2)}[\sin e^x]$$

$$= e^{-2x}\int e^{2x}\sin e^x\,dx$$

$$e^x = t$$
$$e^x dx = dt$$

$$= t^{-2}\int t^2 \sin t\,dt$$

$$= t^{-2}[-t^2\cos t - 2t\sin t + 2\cos t]$$

$$= -t^{-2}[t^2\cos t + 2t\sin t - 2\cos t]$$

$$= \frac{-1}{e^{2x}}[e^{2x}\cos e^x + 2e^x \sin e^x - 2\cos e^x]$$

$$I_1 = \frac{1}{(D+1)}[\sin e^x]$$

$$= e^{-x}\int e^{x}\sin e^x\,dx$$

$$e^x = t$$

$$= t^{-1}\int \sin t\,dt$$

$$= t^{-1}(-\cos t)$$

$$= \frac{\cos e^x}{e^x}$$

$$\cdot \frac{\cos e^x}{e^x}$$

(9) $(D^2-2D+2)y = e^x \tan x$

For C.F

$$m^2-2m+2=0$$

$$m = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= 1 \pm i$$

$$C.F. = e^x[c_1\cos x + c_2 \sin x]$$

For P.I.

$$y = \frac{1}{(D^2-2D+2)}[e^x \tan x]$$

$$= e^x \frac{1}{((D+1)^2-2(D+1)+2)}[\tan x]$$

$$= e^x \frac{1}{(D^2+1 \cancel{+2D} \cancel{-2D} \cancel{-2} \cancel{+2})}[\tan x]$$

$$= e^x \frac{1}{(D^2+1)}[\tan x]$$

$$= e^x \frac{1}{(D+i)(D-i)}[\tan x]$$

$$= -\cos x \log(\sec x + \tan x)$$ [Refer to questn no (1)]

$$y = e^x[c_1\cos x + c_2\sin x] - \cos x \log(\sec x+\tan x).$$

Homogeneous Linear Differential Eq^n.

① Cauchy - Euler diff equation

The differential Equation

$$(a_n x^n D^n + a_{n-1} x^{n-1} D^{n-1} + \dots a_1 x D + a) y = R(x)$$

Let $x = e^z$ $\qquad D = \frac{d}{dx}$

$z = \log x$

$\frac{dz}{dx} = \frac{1}{x}$

$\therefore Dy = \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \frac{dy}{dz} \times \frac{1}{x}$

$x Dy = \frac{dy}{dz}$

$xD = \frac{d}{dz}$ or $xD = \mathcal{D}$ $\qquad \boxed{\mathcal{D} = \frac{d}{dz}}$

$$D^2 y = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$D^2 y = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$D^2 y = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2 y}{dz^2}$$

$$x^2 D^2 y = -\frac{dy}{dz} + \frac{d^2 y}{dz^2}$$

$$x^2 D^2 = -\mathcal{D} + \mathcal{D}^2$$

$$\boxed{x^2 D^2 = \mathcal{D}(\mathcal{D}-1)}$$

Similarly

$$x^3 D^3 = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2) \text{ \& so on.}$$

(2) Legendre's Diff Equation

The diff. eqⁿ

$$(a_n (ax+b)^n D^n + a_{n-1} (ax+b)^{n-1} D^{n-1} + \ldots \ldots \; a_0) y = R(x)$$

is said to be Legendre's diff eqⁿ.

If $(ax+b) = e^z$

then $(ax+b) D = a \mathcal{D}$ $\qquad \mathcal{D} = \frac{d}{dz}$

$(ax+b)^2 D^2 = a^2 \mathcal{D}(\mathcal{D}-1)$ & so on.

Solve :-

① $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$

$(x^2 D^2 + 4xD + 2) y = e^x$

$x = e^z \qquad z = \log x$

$xD = \mathcal{D}$

$x^2 D^2 = \mathcal{D}(\mathcal{D}-1) \qquad \mathcal{D} = \frac{d}{dz}$

$(\mathcal{D}(\mathcal{D}-1) + 4\mathcal{D} + 2) y = e^{e^z}$

$(\mathcal{D}^2 + 3\mathcal{D} + 2) y = e^{e^z}$ (Refer to quiz qⁿ)

P.I $= e^{-2z} e^{e^z}$

C.F. $= c_1 e^{-z} + c_2 e^{-2z}$

$y = \frac{c_1}{x} + \frac{c_2}{x^2} + x^{-2} e^x$

(2) Solve :-

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = (\log x)\sin(\log x)$$

Sol^n

$$(x^2D^2 + xD + 1)y = \log x \sin(\log x)$$

$x = e^z$

$z = \log x$

$xD = \mathcal{D}$

$x^2D^2 = \mathcal{D}(\mathcal{D}-1)$

$$(\mathcal{D}^2 - \cancel{\mathcal{D}} + \cancel{\mathcal{D}} + 1)y = z\sin z$$

$$(\mathcal{D}^2+1)y = z\sin z.$$

$$(\mathcal{D}^2+1)y = z\sin z$$

m ~~C.F~~ $= \pm i$

$$C.F = (C_1\cos x + C_2 \sin x)$$

P.I.

$$y = \frac{1}{\mathcal{D}^2+1}[z\sin z]$$

$$\boxed{= \cancel{\frac{1}{\mathcal{D}^2-i^2}[z\sin z]} \quad = \frac{1}{(D-i)}[z\sin z] \quad \cancel{\frac{1}{D+i}[z\sin z]}}$$

$$\Rightarrow z\frac{1}{D^2+1}[\sin z] - \frac{2\mathcal{D}}{(\mathcal{D}^2+1)^2}(\sin x)$$

$$\Rightarrow z\frac{z}{2D}[\sin z] - \frac{z}{2(\mathcal{D}^2+1)\mathcal{D}}(\cos x)$$

$$= -\frac{z^2}{2}\cos z - \frac{z}{2(\mathcal{D}^2+1)}\sin z$$

$$= -\frac{z^2}{2}\cos z - \frac{z^2}{4\mathcal{D}}(\sin z)$$

$$= -\frac{z^2}{2}\cos z + \frac{z^2}{4}\cos z = -\frac{1}{4}z^2\cos z$$

$$= -\frac{1}{4}(\log x)^2\cos(\log x)$$

Solve :-

$$(2x+3)^2\frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} + 2y = 6x$$

$$((2x+3)^2D^2 - 2(2x+3)D + 2)y = 6x$$

$$2x+3 = e^z$$

$$x = \frac{e^z-3}{2}$$

$$(2x+3)D = 2\mathcal{D}, \quad \mathcal{D} = \frac{d}{dz}$$

$$(2x+3)^2D^2 = 4\mathcal{D}(\mathcal{D}-1)$$

$$\therefore [4\mathcal{D}(\mathcal{D}-1) - 2(2D) - 12]y = 3(e^2-3)$$

$$(4\mathcal{D}^2 - 8\mathcal{D} - 12)y = 3e^2 - 9$$

Simultaneous Linear Differential Equation.

① $\frac{dx}{dt} + 7x - y = 0$, $\frac{dy}{dt} + 2x + 5y = 0$

Soln:

$$D = \frac{d}{dt}$$

$Dx + 7x - y = 0$, $Dy + 2x + 5y = 0$

$(D+7)x - y = 0$ $(D+5)y + 2x = 0$

$2x + (D+5)y = 0$

$2x + (D+5)(D+7)x = 0$

$x(D^2 + 12D + 37) = 0$

C.F

$m^2 + 12m + 37 = 0$

$$m = \frac{-12 \pm \sqrt{144 - 148}}{2}$$

$$= \frac{-12 \pm 2i}{2} = -6 \pm i$$

C.F $= e^{-6t}(c_1 \cos t + c_2 \sin t)$

$$\boxed{x = e^{-6t}(c_1 \cos t + c_2 \sin t).}$$

Now

$$y = \frac{dx}{dt} + 7x$$

$$= -6e^{-6t}(c_1 \cos t + c_2 \sin t) + e^{-6t}(-c_1 \sin t + c_2 \cos t) + 7e^{-6t}(c_1 \cos t + c_2 \sin t)$$

$$\boxed{y = e^{-6t}(A \cos t + B \sin t)}$$

Solve

$$\frac{dx}{dt} + 2x - 3y = t, \quad \frac{dy}{dt} - 3x + 2y = e^{2t}$$

$$\frac{dx}{dt} + 2x - 3y = t \quad ,$$

$$(D+2)x - 3y = t, \quad (Dy + 2y) - 3x = e^{2t}$$

$$(D+2)x - 3y = t \qquad (D+2)y - 3x = e^{2t}$$

$$3(D+2)x - 9y = 3t$$

$$+ \quad (D+2)^2 y - 3(D+2)x = (D+2)[e^{2t}]$$

$$[(D+2)^2 - 9] \quad y = (D+2)[e^{2t}] + 3t$$

$$(D^2 + 4D - 5)y = 2e^{2t} + 2e^{2t} + 3t$$

$$[D^2 + 4D - 5]y = 4e^{2t} + 3t$$

C.F

$$m^2 + 4m - 5 = 0$$

$$m = \frac{-4 \pm \sqrt{16+20}}{2}$$

$$= \frac{-4 \pm 6}{2}$$

$$= -5, 1.$$

$$\therefore \text{C.F} \Rightarrow c_1 e^{-5t} + c_2 e^{t}$$

P.I

$$y \Rightarrow \frac{4}{(D^2+4D-5)}[e^{2t}] + \frac{3}{(D^2+4D-5)}[t]$$

$$\Rightarrow \frac{4}{(4+8-5)} e^{2t} + \frac{3}{-5}\left[1-\frac{1}{5}(D^2+4D)\right]^{-1} t$$

$$= \frac{4}{7} e^{2t} + \frac{3}{-5}\left\{1-\frac{1}{5}(D^2+4D)+\frac{1}{25}(D^2+4D)^2+\dots\right\}$$

$$= \frac{4}{7}e^{2t} + \frac{3}{-5}\left\{t-\frac{4}{5}\right\}$$

$$= \frac{4}{7}e^{2t} - \frac{3t}{5} + \frac{12}{25}.$$

$$y = C.F + P.I$$

$$\Rightarrow C_1e^{-5t} + C_2e^{t} + \frac{4}{7}e^{2t} - \frac{3t}{5} + \frac{12}{25}.$$

Solve :-

$$\frac{d^2x}{dt^2} + 4x + 5y = t^2 \quad , \quad \frac{d^2y}{dt^2} + 5x + 4y = t + 1$$

$$(D^2+4)x + 5y = t^2 \quad , \quad (D^2+4)y + 5x = t+1.$$

$$5(D^2+4)x + 25(y) = 5t^2 \quad — ① \quad , \quad (D^2+4)^2 y + 5(D^2+4)x = (t+1)(D^2+4) \quad — ②$$

② − ①

$$(D^2+4)^2 y - 25y = (D^2+4)[t+1] - 5t^2$$

$$(D^2+4-5)(D^2+4+5)y = 4t + 4 - 5t^2$$

$$(D^2-1)(D^2+9)y = -5t^2 + 4 + 4t$$

Let $D^2 = k$

$$(k-1)(k+9)y = -5t^2 + 4 + 4t$$

$$(k^2 - 9 + 8k)y = -5t^2 + 4t + 4.$$

C.F.

$$k^2 + 8k - 9 = 0$$

$$k = \frac{-8 \pm \sqrt{64+36}}{2}$$

$$= \frac{-8 \pm \sqrt{100}}{2}$$

$$= \frac{-8 \pm 10}{2}$$

$$= -9, 1.$$

$D^2 = -9$, $D^2 = 1, 1$

$D = \pm 3i$

C.F $y = (C_1 + C_2 x)e^x + C_1 \cos 3x + C_2 \sin 3x$

For. P.I.

$$y = \frac{-5}{D^4+8D^2-9}[t^2] + \frac{4}{D^4+8D^2-9}[t] + \frac{4}{D^4+8D^2-9}[e^{0x}]$$

$$\Rightarrow \frac{-5}{-9}\left\{1-\frac{1}{9}(D^4+8D^2)\right\}^{-1}[t^2] + \frac{4}{-9}\left\{1-\frac{1}{9}(D^4+8D^2-9\right\}[t$$

$$+\frac{4}{-9}$$

$$= \frac{5}{9}\left\{1+\frac{1}{9}(D^4+8D^2)-\frac{1}{81}(D^4+8D^2)^2\right\}[t^2]$$

$$-\frac{4}{9}\left\{1+\frac{1}{9}(D^4+8D^2)-\frac{1}{81}(D^4+8D^2)^2\right\}[t]-\frac{4}{9}$$

$$= \frac{5}{9}\left\{t^2+\frac{8}{9}\right\}-\frac{4}{9}\{t\}-\frac{4}{9}$$

$$= \frac{1}{9}\left(5t^2+\frac{40}{9}-4t-4\right\}$$

$$= \frac{1}{9}\left(5t^2-4t+\frac{4}{9}\right\}$$

$$= \left\{\frac{5}{9}t^2-\frac{4}{9}t+\frac{4}{81}\right\}$$

Solve :-

$$\frac{dx}{dt} + \frac{2}{t}(x-y) = 1 \quad \& \quad \frac{dy}{dt} + \frac{1}{t}(x+5y) = t$$

$$t\frac{dx}{dt} + 2(x-y) = t \quad \& \quad t\frac{dy}{dt} + (x+5y) = t^2$$

$$(tD+2)x - 2y = t \quad , \quad (tD+5)y + x = t^2$$

$$t = e^z$$

$$tD = \mathcal{D} \qquad \frac{d}{dz} = \mathcal{D}$$

$$(\mathcal{D}+2)x - 2y = e^z \;-①\;, \quad (\mathcal{D}+5)y + x = e^{2z}$$

$$(\mathcal{D}+5)(\mathcal{D}+2)y + (\mathcal{D}+2)x = (\mathcal{D}+2)e^{2z} \;-②$$

② - ①

$$\{(\mathcal{D}+5)(\mathcal{D}+2) + 2\}y = (\mathcal{D}+2)[e^{2z}] - e^z$$

$$= 2e^{2z} + 2e^{2z} - e^z$$

$$(\mathcal{D}^2 + 7\mathcal{D} + 12)y = 4e^{2z} - e^z$$

For C.F

$$(\mathcal{D}^2 + 7\mathcal{D} + 12)y = 0$$

$$m^2 + 7m + 12 = 0$$

$$m = \frac{-7 \pm \sqrt{49-48}}{2}$$

$$= \frac{-7 \pm 1}{2}$$

$$= -3, -4$$

$$C.F \Rightarrow c_1 e^{-3x} + c_2 e^{-4x}$$

For P.I

$$y = \frac{4}{(D^2+7D+12)}[e^{2z}] - \frac{1}{(D^2+7D+12)} e^{z}$$

$$= \frac{4(e^{2z})}{4+14+12} - \frac{1}{1+7+12}(e^{z})$$

$$= \frac{4e^{2z}}{30} - \frac{e^{z}}{20}$$

$$= \frac{2e^{2z}}{15} - \frac{e^{z}}{20}$$

$$= \frac{2t^2}{15} - \frac{t}{20}$$

$$y = C.F + P.I$$

$$= c_1 e^{-3x} + c_2 e^{-4x} + \frac{2t^2}{15} - \frac{t}{20}.$$

(*) Solution of differential Equation when one part of C.F is known.

Consider a diff. eq^n

$$y'' + Py' + Qy = R \quad \text{—} \; (1)$$

where u is one part of C.F i.e. u solution

$$u'' + Pu' + Qu = 0 \quad \text{—} \; (2)$$

Let $y = uv$ is the complete solution of (1)

$$y' = u'v + uv'$$

$$y'' = u''v + 2u'v' + uv''$$

By eq (1)

$$(u''v + 2u'v' + uv'') + P(u'v + uv') + Quv = R$$

$$= v(u'' + Pu' + Qu) + uv'' + v'(2u' + Pu) = R$$

$$= uv'' + v'(2u' + Pu) = R$$

$$v'' + v'\left(\frac{2u'}{u} + P\right) = \frac{R}{u}$$

Let $v' = z$, $v'' = \frac{dz}{dx}$

$$\boxed{\frac{dz}{dx} + \left(P + \frac{2u'}{u}\right) z = \frac{R}{u}} \quad \text{—} \; (3)$$

From (3) one can get v.

when u is not given

consider

$$y'' + Py' + Qy = 0 \quad \text{— (4)}$$

Let $u = e^{mx}$ is the solution of (4)

then

$$m^2 e^{mx} + mPe^{mx} + Qe^{mx} = 0$$

$$(m^2 + mP + Q) = 0$$

$\Rightarrow$ e^x is one of C.F if

$$1 + P + Q = 0$$

$\Rightarrow$ e^{-x} is one part of C.F. if

$$1 - P + Q = 0$$

$\Rightarrow$ e^{2x} is one part of C.F. if

$$4 + 2P + Q = 0$$

Let $u = x^m$ is one part of C.F.

then, $u'' + Pu' + Qu = 0$

$$m(m-1)x^{m-2} + Pmx^{m-1} + Qx^m = 0$$

$$(m(m-1) + Pmx + Qx^3)x^{m-2} = 0$$

$$m(m-1) + mPx + Qx^2 = 0$$

$u = x$ is one part of C.F. if

$$Px + Qx^2 = 0$$

$$\text{or} \quad P + Qx = 0$$

$u = x^{-1}$ is one part of C.F

$$2 - Px + Qx^2 = 0$$

solve :

$$(x+2)y'' - (2x+5)y' + 2y = (x+1)e^x$$

$$y'' - \left(\frac{2x+5}{x+2}\right)y' + \left(\frac{2}{x+2}\right)y = \left(\frac{x+1}{x+2}\right)e^x$$

comparing it with $y'' + ~~Px~~y' + Qy = R$

we get $P = -\left(\frac{2x+5}{x+2}\right)$, $Q = \frac{2}{x+2}$, $R = \left(\frac{x+1}{x+2}\right)e^x$.

. as $4 + 2P + Q = 4 - \frac{2(2x+5)}{x+2} + \frac{2}{x+5}$

$= 0$

$u = e^{2x}$ is ~~also~~ one part of C.F.

Let $y = uv$ is the complete solⁿ of ① then.

$$\frac{dz}{dx} + \left(P + \frac{2u'}{u}\right)z = \frac{R}{u}$$

$$= \frac{dz}{dx} + \left(\frac{-2x-5}{x+2} + 2 \times \frac{2e^{2x}}{e^{2x}}\right)z = \frac{(x+1)}{(x+2)} \frac{e^x}{e^{2x}}$$

$$\frac{dz}{dx} + \left(\frac{2x+3}{x+2}\right)z = \frac{(x+1)}{(x+2)}e^{-x}$$

$$\text{I.F.} = e^{\int \frac{2x+3}{x+2} dx}$$

$$= e^{\int \frac{2(x+2) - 1}{x+2} dx}$$

$$= e^{2x} e^{-\log(x+2)}$$

$$= \frac{e^{2x}}{x+2}$$

$$z \times I.F = \int \frac{(x+1)}{(x+2)} e^{-x} \times \frac{e^{2x}}{x+2}\, dx$$

$$z \times \frac{e^{2x}}{x+2} = \int \frac{(x+1)e^{x}}{(x+2)^2}\, dx + c_1$$

$$= \int \left[\frac{e^x}{x+2} - \frac{e^x}{(x+2)^2} \right] dx + c_1$$

$$z \times \frac{e^{2x}}{x+2} = \int \left(\frac{e^x}{x+2} \right) + c_1$$

$$z = e^{-x} + c_1 (x+2) e^{-2x}$$

$$\frac{dz}{dx} + \left(\frac{-2x-5}{x+2} + 2 \times \frac{2e^{2x}}{e^{2x}} \right) z = \frac{(x+1)e^x}{(x+2)e^{2x}}$$

$$\frac{dz}{dx} + \left(\frac{2x+3}{x+2} \right) z = \frac{(x+1)e^{-x}}{x+2}$$

$$I.F = e^{\int \frac{2x+3}{x+2} dx} = \frac{e^{2x}}{x+2}$$

$$\therefore v' = e^{-x} + c_1 (x+2) e^{-2x}$$

On Integrating,

$$v = -e^{-x} + c_1 \left\{ (x+2)\left(-\frac{e^{-2x}}{2}\right) - \left(\frac{e^{-2x}}{4}\right) \right\} + c_2$$

$$y = uv$$

$$= e^{2x} \left[-e^{-x} - \frac{c_1}{2}(x+2)e^{-2x} - \frac{c_1}{4} e^{-2x} \right] + c_2 \Big]$$

Q solve : ① $xy'' - y' + (1-x)y = x^2 e^{-x}$

$y = e^x$ is one part of C.F

② $x^2 y'' - 2x(1+x)y' + 2(1+x)y = x^3$. [$\because y = x$ is C.F].

solution :-

① $xy'' - y' + (1-x)y = x^2 e^{-x}$

$\boxed{y = e^x}$ is one of the solⁿ. $(u = e^x)$

$$y'' - \frac{1}{x}y' + \frac{(1-x)}{x}y = \frac{x^2 e^{-x}}{x} = xe^{-x}$$

$$P = -\frac{1}{x}, \quad Q = \frac{(1-x)}{x}, \quad R = xe^{-x}$$

Let $y = uv$ is complete solⁿ of diff. eqⁿ

$$\frac{dz}{dx} + \left(P + \frac{2u'}{u}\right)z = \frac{R}{u}$$

$$\frac{dz}{dx} + \left(-\frac{1}{x} + 2\right)z = xe^{-2x}$$

$$I.F = e^{\int (2 - 1/x)\,dx}$$

$$= \cancel{e^{(2x + 1/x^2)}} \; e^{2x - \ln x} = \frac{e^{2x}}{x}$$

$$z \frac{e^{(2x)}}{x} = \int xe^{-2x} \cdot \frac{e^{2x}}{x} \, dx$$

$$= \int x \cdot e^{1/x^2} dx$$

$$= \int \frac{x^4 e^{1/x^2}}{x^3} dx$$

$$= -\frac{1}{2}\int \frac{1}{t^2} e^t \, dt$$

① ②

$$= -\frac{1}{2}\left[\frac{1}{t^2}e^t + \frac{2}{t^3}e^t + \frac{-6}{t^4}e^t \ldots\right.$$

$\frac{1}{x^2} = t$

$x^2 = \frac{1}{t}$

$\frac{1}{x^3}dx = \frac{dt}{-2}$

$$z e^{2x} = x^2 + C_1$$

$$z = x^2 e^{-2x} + C_2 e^{-2x}$$

$$v' = x^2 e^{-2x} + C_2 e^{-2x}$$

$$v = x^2 \frac{e^{-2x}}{-2} - 2x\frac{e^{-2x}}{4} + 2\frac{e^{-2x}}{-8} + \frac{C_2 e^{-2x}}{-2} + C_3$$

$$= \frac{x^2 e^{-2x}}{-2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + \frac{C_1 e^{-2x}}{-2} + C_2$$

$y = uv$

$$= e^{x}\left[\frac{x^2 e^{-2x}}{-2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} - \frac{c_1 e^{-2x}}{2} + c_2\right]$$

$$= \frac{x^2 e^{-x}}{-2} - xe^{-x} - \frac{e^{-2x}}{4} - \frac{c_1 e^{-2x}}{4} + c_2.$$

② $x^2 y'' - 2x(1+x)y' + 2(1+x)y = x^3$

$$y'' - \frac{2(1+x)}{x}y' + \frac{2(1+x)}{x^2}y = x$$

$$P = -\frac{2(1+x)}{x}, \quad Q = \frac{2(1+x)}{x^2}, \quad R = x$$

Let $u = x$ is one of the solution,

$P + Qx = 0$

Let $y = uv$ is complete solⁿ of diff. eqⁿ.

$$\frac{dz}{dx} + \left(P + \frac{2u'}{u}\right)z = \frac{R}{u}$$

$$\frac{dz}{dx} + \left(-\frac{2(1+x)}{x} + \frac{2}{x}\right)z = \frac{x}{x}$$

$$= \frac{dz}{dx} + \left(\frac{-2x}{x}\right)z = 1$$

$$= \frac{dz}{dx} - 2z = 1$$

$$= \frac{dz}{dx} = 1 + 2z$$

$$= \int \frac{dz}{1+2z} = \int dx$$

$$= \frac{\ln(1+2z)}{2} = x + c.$$

$$= 1 + 2z = e^{2x+c} \Rightarrow z = \frac{e^{2x} - 1}{2} + c = \frac{e^{2x}}{2} + c$$

$$v' = \frac{e^{2x}}{2} + c$$

$$v = \frac{e^{2x}}{4} + cx$$

$$y = uv$$

$$= x\left[\frac{e^{2x}}{4} + cx\right]$$

$$= \frac{xe^{2x}}{4} + \frac{cx^2}{2}$$

(2) Variation of Parameters

consider the differential eq^n

$$y'' + Py' + Qy = R$$

and let

$$C.F. = C_1 u + C_2 v$$

complete sol^n is

$$y = Au + Bv$$

where

$$A = -\int \frac{Rv}{w} dx, \quad B = \int \frac{Ru}{w} dx$$

and

$$w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

$= w$ kian of u & v

Solve :-

1) $y'' + y = \tan x$

2) $y'' + 3y' + 2y = e^{e^x}$

3) $y'' - y = \dfrac{2}{1+e^x}$

4) $x^2 y'' + xy' - y = x^3 e^x$

5) $y'' + y = x - \cot x$

6) $y'' - 2y' = e^x \sin x$

7) $y'' + (1 - \cot x) y' - y \cot x = \sin^2 x$

Sol^n :

1) $y'' + y = \tan x$

$(D^2+1)y = \tan x$

Auxiliary eq^n.

$m^2 + 1 = 0$

$m = \pm i$

$C.F = C_1 \cos x + C_2 \sin x$

Let $u = \cos x$, $v = \sin x$

complete sol^n is

$y = Au + Bv$

$$w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$w = 1$

$$A = -\int \tan x \times \sin x \, dx$$

$$= -\int \frac{\sin^2 x}{\cos x} dx$$

$$= \cancel{\int \frac{\sin x \sqrt{1-\cos^2 x}}{\cos x} dx}$$

$$= -\int \frac{1-\cos^2 x}{\cos x} dx$$

$$= -\int \sec x \, dx + \int \cos x \, dx$$

$$= -\ln|\sec x + \tan x| + \sin x + C_1$$

$$B = \int \tan x \times \cos x \, dx$$

$$= \int \sin x \, dx$$

$$= -\cos x + C_2$$

$$y = Au + Bv$$

$$= (-\ln|\sec x + \tan x| + \sin x + C_1)\cos x + (-\cos x + C_2)\sin x$$

$$= -\ln|\sec x + \tan x|\cos x + \sin x \cancel{\cos x} + C_1 \cos x - \cancel{\sin x}\cos x + C_2 \sin x)$$

$$y'' - y = \frac{2}{1+e^x}$$

$$(D^2 - 1)y = \frac{2}{1+e^x}$$

C.F. $\quad m^2 - 1 = 0$

$$m^2 = 1$$

$$m = \pm 1$$

$$y = c_1 e^x + c_2 e^{-x}$$

Complete sol^n.

$$u = e^x, \quad v = e^{-x}$$

$$y = Au + Bv$$

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= -2e^x e^{-x}$$

$$= -2$$

$$B = \int \frac{Ru}{W}\,dx$$

$$= \int \frac{\cancel{2}\, e^x\, dx}{-\cancel{2}(1+e^x)}$$

$$= -\int \frac{e^x}{1+e^x}\,dx \qquad 1+e^x = t$$

$$= -\int \frac{dt}{t}$$

$$= -\ln|1+e^x| + c_1$$

$$A = -\int \frac{Rv}{W}\,dx$$

$$= +\int \frac{e^{-x}}{1+e^x}\,dx$$

$$= \int \frac{1}{e^x(1+e^x)}\,dx$$

$$= \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)dx$$

$$= \cancel{\ln e^x \quad \ln e^x}$$

$$= \int e^{-x}\,dx - \frac{(1+e^x)}{1+e^x} + \frac{e^x}{(1+e^x)}\,dx$$

$$= -e^{-x} - x + \ln|1+e^x| + c_2$$

$y = Au + Bv$

$= e^x(-e^{-x} - x + \ln|1+e^x| + c_2) + (-\ln|1+e^x| + c_1)$

$= -1 - xe^x + e^x \ln|1+e^x| + c_2 e^x - \ln|1+e^x| e^{-x}$

$+ c_1 e^{-x}.$

4) $x^2 y'' + xy' - y = x^3 e^x$

$x = e^z$

$xD = \theta$

$x^2 D^2 = \theta(\theta - 1)$

$(x^2 D^2 + xD - 1)y = e^{3z} e^{e^z}$

$(\theta(\theta-1) + \theta - 1)y = e^{3z} e^{e^z}$

$(\theta^2 - 1)y = e^{3z} e^{e^z}$

<u>For C.F</u>

$m^2 - 1 = 0$

$m = \pm 1$

$y = c_1 e^z + c_2 e^{-z}$

$y = c_1 x + \frac{c_2}{x}$

$u = x, \quad v = \frac{1}{x}.$

$$w = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix}$$

$= \frac{-1}{x} - \frac{1}{x} = -\frac{2}{x}$

Let $y = Au + Bv$ is the complete solution of

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = xe^x$$

$$R = xe^x$$

$$A = -\int \frac{Rv}{\omega}\,dx \qquad B = \int \frac{Ru}{\omega}\,dx$$

$$= -\int \frac{xe^x \cdot \frac{1}{x}}{\frac{-2}{x}}\,dx \qquad = \int \frac{xe^x \cdot x}{-2/x}$$

$$= \frac{1}{2}\int xe^x\,dx \qquad = \frac{-1}{2}\int x^3 e^x\,dx$$

$$= \frac{1}{2}\left[xe^x - e^x\right] + c_1 \qquad = \frac{-1}{2}\left[x^3e^x - 3x^2e^x + 6xe^x - 6e^x\right] + c_2$$

$$y = \frac{1}{2}\left[\frac{x^2e^x}{2} - \frac{xe^x}{2} + \frac{c_1 x}{2} - \frac{x^2e^x}{2} + \frac{3xe^x}{2} - 3e^x + \frac{3e^x}{x} - \frac{c_2 x}{2}\right]$$

(3) Change of Independent variable

consider $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ —— ①

Let us consider z such that

$$\left(\frac{dz}{dx}\right)^2 = Q$$

where Q is perfect square & should not have any (−) negative sign or square root.

After simplifying ①, we get

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

where

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

Solve :-

$$\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} - y\sin^2 x = \cos x - \cos^3 x$$

Sol^n

Let us consider such that $Q = \sin^2 x$, $P = -\cot x$,

$R = \cos x - \cos^3 x$

$$\left(\frac{dz}{dx}\right)^2 = Q = \sin^2 x$$

$$\frac{dz}{dx} = \sin x$$

$$\therefore \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$\Rightarrow P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x - \cot x \sin x}{\left(\frac{dz}{dx}\right)^2} = 0.$$

$Q_1 = -1$

$$R = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x - \cos^3 x}{\sin^2 x}$$

$$= \cos x = -z$$

$$\frac{d^2 y}{dz^2} - y = -z$$

$$(D^2 - 1)y = -z$$

$$C.F. = C_1 e^{z} + C_2 e^{-z} = C_1 e^{-\cos x} + C_2 e^{\cos x}$$

P.I.

$$\frac{1}{(D^2-1)}(-z)$$

$$= (1-D^2)^{-1} z$$

$$= (1 + D^2 -)z$$

$$= z = -\cos x$$

$$y = C.F + P.I.$$

$$= C_1 e^{-\cos x} + C_2 e^{\cos x} - \cos x$$

Solve :-

① $\frac{d^2y}{dx^2} + (3\sin x - \cot x)\frac{dy}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x.$

② $\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$

(4) Reduction to Normal form

Consider the differential equation

$$y'' + Py' + Qy = R \quad \text{--- (1)}$$

Let $y = uv$ be the complete solution of (1)

the (1) can be written as

$$\frac{d^2v}{dx^2} + Iv = S$$

where $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$

$$S = \frac{R}{u}, \quad u = e^{-\frac{1}{2}\int P dx}$$

Q Solve -

(1) $y'' - 4xy' + (4x^2-1)y = -3e^{x^2} \sin 2x$

(2) $y'' - 2xy' + (x^2+2)y \cancel{} = e^{\frac{1}{2}(x^2+2x)}$

(3) $y'' - 4xy' + (4x^2-3)y = e^{x^2}$.

Solⁿ

(1) $y'' - 4xy' + (4x^2-1)y = -3e^{x^2} \sin 2x$

$P = -4x, \quad Q = 4x^2-1, \quad R = -3e^{x^2} \sin 2x$.

$$I = 4x^2 - 1 + \frac{4}{2} - \frac{16x^2}{4}$$

$$= 4x^2 - 1 + 2 - 4x^2$$

$$= 1$$

$$u = e^{-\frac{1}{2}\int -4x\,dx} = e^{2\left(\frac{x^2}{2}\right)} \cdot e^{x^2}$$

$$S = -3\sin 2x$$

$$\frac{d^2v}{dx^2} + v = -3\sin 2x$$

C.F

$m^2+1=0$

$m=\pm i.$

$C.F = (c_1\cos x + c_2\sin x)$

P.I.

$(D^2+1)y = -3\sin 2x$

$y = \frac{-3}{(D^2+1)}[\sin 2x]$

$= \frac{-3}{-3}\sin 2x$

$= \sin 2x$

$v = c_1\cos x + c_2\sin x + \sin 2x.$

uv is solⁿ of y

$y = e^{x^2}(c_1\cos x + c_2\sin x + \sin 2x)$

② $y'' - 2xy' + (x^2+2)y = e^{1/2}(x^2+2x)$

$P = -2x,\ Q = (x^2+2),\ R = e^{1/2}(x^2+2x)$

$I = x^2+2 - \frac{1}{\cancel{2}}x - \cancel{x} - \frac{4x^2}{4}$

$= x^2+2+1-x^2$

$= 3.$

$u = e^{-\frac{1}{2}\int -2x\,dx} = e^{x^2/2}$

$S = e^{x}$

$\frac{d^2v}{dx^2} + 3v = e^{x}$

Auxiliary Eq^n.

$$(m^2+3) = 0$$

$$m = \pm\sqrt{3}\,i$$

$$v = \cos\sqrt{3}x + \sin\sqrt{3}x.$$

P.I.

$$v = \frac{1}{(D^2+3)}[e^x]$$

$$= \frac{e^x}{4}$$

$$v = \cos\sqrt{3}x + \sin\sqrt{3}x + \frac{e^x}{4}$$

$uv = y$

$$y = e^{x^2/2}\left[\cos\sqrt{3}x + \sin\sqrt{3}x + \frac{e^x}{4}\right)$$