习题 6.1.

2. $\begin{cases} x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' \\ y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \end{cases}$ 代入. $\Rightarrow 2\cdot\frac{\sqrt{2}}{2}(x-y)\frac{\sqrt{2}}{2}(x+y)=a$

$\Rightarrow x^2-y^2=a$

4.

(1). 设P为σ的不动点. 考虑以P为圆心. 半径为r的圆上的点A

$d(A,P) = d(\sigma(A),\sigma(P)) = d(\sigma(A),P) = r$. 故 $r(A)$ 仍在圆上.

由A. r的任意性知σ是旋转

(2). 若P_1,P_2是σ的不动点. 考虑 $\overrightarrow{P_2P_3}\perp\overrightarrow{P_1P_2}$ 且 $|\overrightarrow{P_2P_3}|=r$

$d(P_2,P_3)=d(\sigma(P_2),\sigma(P_3)) = d(P_2,\sigma(P_3))$. 由$P_1P_2$在$l$上. $|\overrightarrow{P_2P_3}|=r$知与$l$相距r

的直线l'经σ后变为 $\sigma(l')$且间距未变. 又σ不是恒等变换. 则$\sigma(l')$只能是原l'

关于l的对称直线. 则σ为反射.

5.

在复平面上考虑. z是任意一点.

$r_1(z)-O_1 = (z-O_1)e^{i\theta_1}$ $\Rightarrow r_1(z)=O_1+(z-O_1)e^{i\theta_1}$.

同理 $r_2(z)=O_2+(z-O_2)e^{i\theta_2}$

$r_2\circ r_1(z)=r_2(r_1(z)) = O_2+(r_1(z)-O_2)e^{i\theta_2}$

$= z e^{i(\theta_1+\theta_2)} + (1-e^{i\theta_1})e^{i\theta_2}O_1 + (1-e^{i\theta_2})O_2$

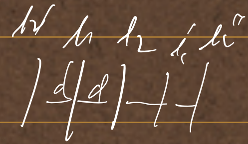
(1). $\theta_1+\theta_2=2k\pi$ 时

$r_2\circ r_1(z)=z+(1-e^{i\theta_2})\overrightarrow{O_1O_2}$

(2) $r_2(r_1(z))-x=(z-x)e^{i(\theta_1+\theta_2)}$

$\Rightarrow x=\dfrac{r_2(r_1(z))-ze^{i(\theta_1+\theta_2)}}{1-e^{i(\theta_1+\theta_2)}}$ 是旋转中心. $\theta_1+\theta_2$是转角.

6.



① 当 $l_1 \parallel l_2$ 时是平移变换.

$\sigma_1(l_1)=l_1$ $\sigma_1(l_2)=l_2'$ 且 $d(l_2',l_1)=d(l_1,l_2)$

$\sigma_2(l_1)=l_1'$ $\sigma_2(l_2')=l_2''$ 且 $d(l_2'',l_1')=d$. $d(l_1',l_2)=d$.

故 $\sigma_2\circ\sigma_1$ 是平移变换. l_1', l_2' 相对 l_1, l_2 平移 d 单位 ($l_1\to l_2$ 方向)

② 当 l_1 与 l_2 相交时. 设 $l_1\cap l_2=P_0$. 是旋转变换.

设 $l_1: x\cos\alpha - y\sin\alpha + p_1=0$ $\quad l_2: x\cos\beta - y\sin\beta + p_2=0$ 任取 $P(x,y)$.

$$\begin{cases} x_1 = -x\cos2\alpha + y\sin2\alpha - 2p_1\cos\alpha \\ y_1 = x\sin2\alpha + y\cos2\alpha + 2p_1\sin\alpha \end{cases} \text{且} \begin{cases} x_2 = -x_1\cos2\beta + y_1\sin2\beta - 2p_2\cos\beta \\ y_2 = x_1\sin2\beta + y_1\cos2\beta + 2p_2\sin\beta \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = x\cos(2\alpha-2\beta) - y\sin(2\alpha-2\beta) + 2p_1\cos(2\beta-\alpha) - 2p_2\cos\beta \\ y_2 = x\sin(2\alpha-2\beta) + y\cos(2\alpha-2\beta) - 2p_1\sin(2\beta-\alpha) + 2p_2\sin\beta \end{cases}$$

即 $X_2 = MX + X_0$. 其中 $M=\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$ 是正交矩阵 $\theta=2\alpha-2\beta$.

故是一个第一类等距变化. 又 P_0 不动.

故 $\sigma_2\circ\sigma_1$ 是旋转.

习题6.2

1. 不妨设位似中心是 $O(0,0)$. 否则可重新建立坐标系使之成立.

位似变化 τ: $\begin{cases} x' = \lambda x \\ y' = \lambda y \end{cases}$ $(\lambda>0)$. 则令 $\sigma_1: x\mapsto \lambda x$ $\quad \sigma_2: y\mapsto\lambda y$

则 $\sigma_1\sigma_2=\tau$. 且 它方向与它方向垂直

3.

(1) 反设 AB上 P 不是不动点. 且 P 点.. 且 $\lambda\vec{PA}=\vec{PB}$

由 σ 映射保持简单比不变 $\sigma(\vec{PB})=\lambda\,\sigma(\vec{PA})$

即 $\overrightarrow{\sigma(p)B} = \lambda\overrightarrow{\sigma(p)A}$. 故 $\sigma(p)=p$. 矛盾.

从而 p是不动点. 由附性质性, 知 AB上每点都是σ的不动点.

(2) 反射 p是平面π上一点. $\sigma(p)\neq p$

$\exists\ u,\lambda$ s.t. $\overrightarrow{Ap}=\lambda\overrightarrow{AB}+u\overrightarrow{AC}$ 则 $\overrightarrow{Ap}$, $\lambda\overrightarrow{AB}$, $u\overrightarrow{AC}$构成三角形

由σ保持图形仿射性质

$\sigma(\overrightarrow{Ap})=\overrightarrow{A\sigma(p)}=\lambda\overrightarrow{AB}+u\overrightarrow{AC}$ 由向量表示的唯一性. $\sigma(p)=p$.

故 π是每点都是σ的不动点

5.

(1) 设 $\begin{cases} x' = a_{11}x + a_{12}y + a_{13} \\ y' = a_{21}x + a_{22}y + a_{23} \end{cases}$

$\begin{cases} 3 = a_{11} + 0 + a_{13} \\ 0 = a_{21} + 0 + a_{23} \end{cases}$ $\begin{cases} -2 = -a_{12} + a_{13} \\ 1 = -a_{22} + a_{23} \end{cases}$ $\begin{cases} 0 = -2a_{11} + a_{12} + a_{13} \\ -5 = -2a_{21} + a_{22} + a_{23} \end{cases}$

$\Rightarrow a_{11}=2 \quad a_{12}=3 \quad a_{13}=1 \quad a_{21}=1 \quad a_{22}=-2 \quad a_{23}=-1$

故 $\begin{cases} x' = 2x+3y+1 \\ y' = x-2y-1 \end{cases}$

(2) $\begin{cases} x'+y'-3 = \lambda(3x+2y+1) \\ 2x'+3y'-3 = u(8x+3y+10) \end{cases}$ 代入 $(1,1)$. $(3,6)$

$\lambda=1 \quad u=1$

$\Rightarrow \begin{cases} x' = x+3y-1 \\ y' = 2x-y+5 \end{cases}$

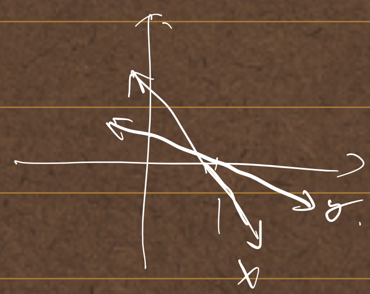
(3). $\begin{cases} 3x'+2y'-1 = x(3x+2y-1) \\ x'+2y'+1 = u(x+2y+1) \end{cases}$ 代入 $(0,0)\to(1,1)$

$\Rightarrow x = -4 \quad u = 4$

$$\Rightarrow \begin{cases} x' = 4x - 3 \\ y' = 4y + 3 \end{cases}$$

6. (1). 令 $x' = x$ $y' = y$. 则 $x + 4y - 1 = 0$ $3x + 2y - 3 = 0$ 是不变直线.

$\det \begin{vmatrix} 2 & 4 \\ 3 & 3 \end{vmatrix} = -6$. 故变换系数为 6



(2) $\vec{e}_1^* = \left(\frac{-1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right)$ $\vec{e}_2^* = \left(\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\right)$

$(\vec{e}_1^*, \vec{e}_2^*) = (\vec{e}_1, \vec{e}_2) M$ 其中. $M = \begin{pmatrix} \frac{-1}{\sqrt{17}} & \frac{2}{\sqrt{13}} \\ \frac{4}{\sqrt{17}} & \frac{-3}{\sqrt{13}} \end{pmatrix}$

在基 $(\vec{e}_1, \vec{e}_2)$ 下坐标为 X. 在 $(\vec{e}_1^*, \vec{e}_2^*)$ 下坐标为 Y.

则 $X = MY$.

$\Rightarrow MZ_1' = AMZ + M^{-1}Z_0$ $\quad M^{-1} = \begin{pmatrix} \frac{3}{5}\sqrt{17} & \frac{2}{5}\sqrt{17} \\ \frac{4}{5}\sqrt{13} & \frac{1}{5}\sqrt{13} \end{pmatrix}$

$$\Rightarrow \begin{aligned} -\tfrac{1}{\sqrt{17}}x' + \tfrac{2}{\sqrt{13}}y' &= \tfrac{4}{\sqrt{17}}x - \tfrac{8}{\sqrt{13}}y - \tfrac{3}{5}\sqrt{17} \\ \tfrac{4}{\sqrt{17}}x' - \tfrac{3}{\sqrt{13}}y' &= \tfrac{2}{\sqrt{17}}x - \tfrac{3}{\sqrt{13}}y - \tfrac{7}{5}\sqrt{13} \end{aligned}$$

$$\Rightarrow \begin{cases} x' = 12x - 6\sqrt{\frac{17}{13}}\,y - \frac{459}{25} - \frac{14}{25}\sqrt{221} \\ y' = 13\sqrt{\frac{13}{17}}\,x - 7y - \frac{36}{25}\sqrt{221} - \frac{91}{25} \end{cases}$$