习题 4.2

1.(1) $\mathscr{A}(\varepsilon_0)=1-1=0$

$\mathscr{A}(\varepsilon_1)=(x+1)-x=1=\varepsilon_0$

$\mathscr{A}(\varepsilon_2)=\frac{x(x+1)}{2}-\frac{x(x-1)}{2}=x=\varepsilon_1$

$\vdots$

$\mathscr{A}(\varepsilon_{n-1})=\varepsilon_{n-2}$

则有 $(0,\varepsilon_0,\varepsilon_1,\cdots,\varepsilon_{n-2})=(\varepsilon_0,\varepsilon_1,\cdots,\varepsilon_{n-1})A$

$$\Rightarrow A=\begin{pmatrix}0&1&0&\cdots&0&0\\0&0&1&\cdots&0&\vdots\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&0&1\\0&0&0&\cdots&&0\end{pmatrix}$$

(4). $E_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $E_{12}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $E_{21}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ $E_{22}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$\mathscr{A}E_{11}=\begin{pmatrix}1&1\\0&0\end{pmatrix}=E_{11}+E_{12}$ 故 $\mathscr{A}$ 在 $E_{11}.E_{12}.E_{21}.E_{22}$ 下矩阵为

$\mathscr{A}E_{12}=\begin{pmatrix}1&-1\\0&0\end{pmatrix}=E_{11}-E_{12}$

$\mathscr{A}E_{21}=\begin{pmatrix}0&0\\1&1\end{pmatrix}=E_{21}+E_{22}$

$\mathscr{A}E_{22}=\begin{pmatrix}0&0\\1&-1\end{pmatrix}=E_{21}-E_{22}$

$$A=\begin{pmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&-1\end{pmatrix}$$

$\mathscr{B}E_{11}=\begin{pmatrix}1&0\\-2&0\end{pmatrix}=E_{11}-2E_{21}$

$\mathscr{B}E_{12}=\begin{pmatrix}0&1\\0&-2\end{pmatrix}=E_{12}-2E_{22}$

$\mathscr{B}E_{21}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

$\mathscr{B}E_{22}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$.

$$B=\begin{pmatrix}1&0&0&0\\0&1&0&0\\-2&0&0&0\\0&-2&0&0\end{pmatrix}$$

故 $\mathscr{AB}$, $\mathscr{A}+\mathscr{B}$ 在 $E_{11}.E_{12}.E_{21}.E_{22}$ 下矩阵为 AB 与 $A+B$、

即 $\begin{pmatrix}1&1&0&0\\1&-1&0&0\\-2&-2&0&0\\-2&2&0&0\end{pmatrix}$ 与 $\begin{pmatrix}2&1&0&0\\1&0&0&0\\-2&0&1&1\\0&-2&1&-1\end{pmatrix}$

2.(1) 只需证 l 是线性的.(已经是 $P^n\to P^m$ 的映射了)

$(\lambda l_A+\mu l_B)X=(\lambda A+\mu B)X=\lambda(AX)+\mu(BX)=\lambda l_A(X)+\mu l_B(X)$.

故 l 是线性的. 从而 $l_A\in Hom_P(P^n,P^m)$

(2). $X_1=(a_1,a_2\cdots a_n)\in P^n$ $X_2=(b_1,b_2\cdots b_m)\in P^m$

则有 $\varphi: \begin{cases} b_1 = k_{11}a_1 + k_{21}a_2 + \cdots + k_{n1}a_n \\ \vdots \\ b_m = k_{1m}a_1 + k_{2m}a_2 + \cdots + k_{nm}a_n \end{cases}$ 即 $\Sigma_2 = (k_{ij})_{m\times n} \cdot \Sigma_1$.

故 $\exists A = (k_{ij})_{m\times n} \in P^{m\times n}$ s.t. $\varphi = \mathcal{A}$.

习题 5.1

1、给定 $P[x]_n$ 中一组基.

(1) $\varepsilon_1 = 1 \quad \varepsilon_2 = x \quad \varepsilon_3 = x^2 \quad \cdots \varepsilon_n = x^{n-1}$

$\tau(\varepsilon_1) = -1 \quad \tau(\varepsilon_2) = 0 \quad \tau(\varepsilon_3) = x^2 = \varepsilon_3 \quad \tau(\varepsilon_4) = 2x^3 \cdots \quad \tau(\varepsilon_n) = (n-2)x^{n-1}$

令 $\tau(f(x)) = xf'(x) - f(x) = 0 \Rightarrow f(x) = kx.\ k \in P.$

故 $\ker\tau = \{f(x) \mid f(x) = kx,\ k \in P\}$.

$\operatorname{Im}\tau = \{f(x) \mid f(x) = a_{n-1}x^{n-1} + \cdots + a_2x^2 + b,\ a_i \in P\ i=2,3\cdots n-1,\ b \in P\}$

(2) $\ker\tau \cap \operatorname{Im}\tau = \{f(x) \mid f(x) = 0\}$. $\dim\ker\tau = 1$ $\dim\operatorname{Im}\tau = n-1$.

有 $\dim\ker\tau + \dim\operatorname{Im}\tau = n = \dim P[x]_n$.

即 $P[x]_n = \ker\tau \oplus \operatorname{Im}\tau$.

2、设 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 令 $\tau(A) = AM - MA = O$

$\Rightarrow \begin{cases} a = a + 2c \\ 2a + 3b = b + 2d \\ c = 3c \\ 2c + 3d = 3d \end{cases} \Rightarrow \begin{cases} a + b = d \\ c = 0 \end{cases}$ 令 $\begin{cases} a = 1 \\ b = 0 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 0 \\ c = 0 \\ d = 1 \end{cases}$ 令 $\begin{cases} a = 0 \\ b = 1 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 1 \\ c = 0 \\ d = 1. \end{cases}$

则 $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ $\ker\tau = L(A_1, A_2)$

$$\tau(A) = \begin{pmatrix} -2c & 2a+2b-2d \\ -2c & 2c \end{pmatrix} = -2\begin{pmatrix} c & d-a-b \\ c & c \end{pmatrix}$$

令 $c=1$. $d-a-b=0 \Rightarrow A_3=\begin{pmatrix}1&0\\1&1\end{pmatrix}$ $c=0$ $d-a-b=1 \Rightarrow A_4=\begin{pmatrix}0&1\\0&0\end{pmatrix}$

则 $\mathrm{Im}\tau=L(A_3,A_4)$.

故设 $A_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $A_2=\begin{pmatrix}0&1\\0&1\end{pmatrix}$ $A_3=\begin{pmatrix}1&0\\1&1\end{pmatrix}$ $A_4=\begin{pmatrix}0&1\\0&0\end{pmatrix}$

$\mathrm{Ker}\tau=L(A_1,A_2)$ $\mathrm{Im}\tau=L(A_3,A_4)$

3. $\sigma: V\mapsto V$ $\tau: V\mapsto V$ $X_G P$

由 $\sigma\in \mathrm{Hom}_P(V,V)$ 知. $\exists A$. s.t. $\sigma P_V = P_V A$

故不妨设 σ 对应的矩阵为 A. τ 对应的矩阵为 B.

同时有: $X A^2=XA$ $XB^2=XB$. A 是

(1). σ 与 τ 有相同像集 即 $\forall X$. $\exists Y$. s.t $AX=BY$. 或 $\exists C$. s.t. $X=CY$

$\Rightarrow$ 故 $\exists C$. s.t. $B=AC$

故 $AB=A^2C=AC=B$ $BA=B^2C^{-1}=BC^{-1}=A$.

即 $\sigma\tau=\tau$. $\tau\sigma=\sigma$.

$\Leftarrow$ $\sigma\tau=\tau$. 知 $\mathrm{Im}\tau\subset\mathrm{Im}\sigma$. 同理 $\mathrm{Im}\sigma\subset\mathrm{Im}\tau$. $AB=P_1Q_1P_2Q_2=P_1Q_1=A$

故 σ 与 τ 有相同像集.

(2) $A-E$ 是 A 的解空间子空间. $B-E$ 是 B 解空间子空间. 而 $\mathrm{Ker}\tau=\mathrm{Ker}\sigma$.

$\Rightarrow$

故 $B(A-E)=0=A(B-E)$. 则 $BA=B$ $AB=A$.

$\Leftarrow$. $AB=A=A^2$ $A(B-A)=0$. $B-A$ 是 A 的解空间. $A-B$ 是 B 的解空间.

$L(B-A)=L(A-B)$. 故. σ 与 τ 有相同的核.

5. $\varphi: R\mapsto R^+$

$\varphi(a+b)=\varphi(a)\varphi(b)$ $\varphi(ka)=\varphi^k(a)$

令 $\varphi(x)=e^x$ 显然 φ 为是双射. 则符合以上条件.

故 $R\cong R^+$. $\varphi(x)=e^x$.