3. (1) $l=\int_0^4\sqrt{1+\frac{9}{4}x}\,dx=\frac{8}{27}(1+\frac{9}{4}x)^{\frac{3}{2}}\Big|_0^4=\frac{8}{27}(10^{\frac{3}{2}}-1)$

(4) $l=\int_0^{2\pi}3a|\sin t\cos t|dt=\frac{3a}{4}\int_0^{2\pi}|\sin 2t|d2t=6a$

(8) $l=a\int_0^{3\pi}\sqrt{\sin^6\frac{\theta}{3}+\sin^4\frac{\theta}{3}\cos^2\frac{\theta}{3}}\,d\theta=\frac{a}{2}\int_0^{3\pi}(1-\cos\frac{2}{3}\theta)d\theta=\frac{3}{2}\pi a^2$

5. (2).

$S(z)=\frac{ab}{c^2}(c^2-z^2)\pi$

$V=2\int_0^c S(z)dz=2\int_0^c\frac{ab}{c^2}\pi(c^2-z^2)dz=\frac{2ab\pi}{c^2}(c^3-\frac{1}{3}c^3)=\frac{4}{3}\pi abc$

$V=\frac{4}{3}\pi abc$

7. (1) $V=2\int_0^a\pi(b^2-\frac{b^2}{a^2}x^2)dx=\frac{4}{3}ab^2\pi$

(2)

i: $V=\pi\int_0^\pi\sin^2x\,dx=\pi\int_0^\pi(\frac{1}{2}-\frac{1}{2}\cos 2x)dx=\frac{1}{2}\pi^2$

ii: $V=\int_0^\pi 2\pi x\sin x\,dx=-2\pi x\cos x\Big|_0^\pi+2\pi\int_0^\pi\cos x\,dx=2\pi^2$

(3) $V=\int_0^\pi a^2\sin^6t\cdot a\,3\cos^2t\sin t\,dt=3a^3\int_0^\pi\sin^7t\cos^2t\,dt=3a^3\int_0^\pi(\sin^7t-\sin^9t)dt$

$=6a^3\left(\frac{6!!}{7!!}-\frac{8!!}{9!!}\right)=\frac{32}{105}a^3$

13. (2) $V=\int_0^\pi\pi\sin^2x\,dx=\frac{1}{2}\pi^2$

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3. (3) $\int_{-\infty}^{+\infty}\frac{1}{x^2+x+1}dx=\int_{-\infty}^{+\infty}\frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}d(x+\frac{1}{2})=\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}\Big|_{-\infty}^{+\infty}=\frac{2}{\sqrt{3}}\pi$

(8) $\int_0^{+\infty}\frac{dx}{(e^x+e^{-x})^2}=\int_0^{+\infty}\frac{e^x}{(e^{2x}+1)^2}de^x=\int_0^{+\infty}\frac{t\,dt}{(t^2+1)^2}=-\frac{1}{2(t^2+1)}\Big|_0^{+\infty}=\frac{1}{2}$

(9) $\int_0^{+\infty}\frac{1}{x^4+1}dx=\int_0^{+\infty}\frac{dx}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}=\int_0^{+\infty}\left(\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}-\frac{\frac{\sqrt{2}}{4}x-\frac{1}{2}}{x^2-\sqrt{2}x+1}\right)dx$

$$\int \frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}dx = \frac{\sqrt{2}}{8}\int \frac{d[(x+\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2]}{(x+\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2} + \frac{1}{4}\int \frac{d(x+\frac{\sqrt{2}}{2})}{(x+\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2} = \frac{\sqrt{2}}{8}\ln|x^2+\sqrt{2}x+1| + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}x+1)$$

同理 $\int \frac{\frac{\sqrt{2}}{4}x-\frac{1}{2}}{x^2-\sqrt{2}x+1}dx = \frac{\sqrt{2}}{8}\ln|x^2-\sqrt{2}x+1| - \frac{\sqrt{2}}{4}\arctan(\sqrt{2}x-1)$

$$\Rightarrow \int_0^{+\infty}\frac{dx}{x^4+1} = \frac{\sqrt{2}}{8}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\Big|_0^{+\infty} + \frac{\sqrt{2}}{4}[\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)]\Big|_0^{+\infty}$$

$$= 0 + \frac{\sqrt{2}}{4}\pi = \frac{\pi}{2\sqrt{2}}.$$

故 $\int_0^{+\infty}\frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$

(10). $\int_0^{+\infty}\frac{\ln x}{1+x^2}dx = \int_0^1\frac{\ln x}{1+x^2}dx + \int_1^{+\infty}\frac{\ln x}{1+x^2}dx = \int_0^1\frac{\ln x}{1+x^2}dx + \int_1^0\frac{\ln x\,dx}{1+x^2} = 0$

4.(2) $\int_1^e \frac{1}{x\sqrt{1-\ln^2 x}}dx = \int_1^e\frac{d\ln x}{\sqrt{1-\ln^2 x}} = \arcsin\ln x\Big|_1^e = \arcsin 1$

(6) $\int_0^{\frac{\pi}{2}}\sqrt{\cot x}\,dx = \int_0^{+\infty}\frac{1}{t}d\,\mathrm{arccot}\,t^2 = 2\int_0^{+\infty}\frac{dt}{1+t^4} = \frac{\pi}{\sqrt{2}}$ (由3题(8))

7.(1) (cpv) $\int_{-\infty}^{+\infty}\frac{1+x}{1+x^2}dx = \int_{-\infty}^{+\infty}\frac{1}{1+x^2}dx + \frac{1}{2}\int_{-\infty}^{+\infty}\frac{dx^2}{1+x^2} = \arctan x\Big|_{-\infty}^{+\infty} + \frac{1}{2}\ln|1+x^2|\Big|_{-\infty}^{+\infty}$

$= \pi$

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3.(2) $\lim\limits_{x\to+\infty}\frac{\frac{\arctan x}{1+x^3}}{\frac{1}{x^3}} = \frac{\pi}{2}\lim\limits_{x\to\infty}\frac{x^3}{1+x^3} = \frac{\pi}{2}$. 即 $\int_1^{+\infty}\frac{\arctan x}{1+x^3} \le \frac{\pi}{2}\cdot\frac{1}{1+x^3}$. 收敛.

5.(2) 当 $p\ge1$. $\int_1^{+\infty}\left|\frac{\sin x}{x^p}\right|dx < \int_1^{+\infty}\frac{1}{x^p}dx$ 收敛. 故此时 $\int_1^{+\infty}\frac{\sin x}{x^p}dx$ 绝对收敛.

当 $0<p<1$ $\int_1^{+\infty}\frac{\sin x}{x^p}dx < \int_1^{+\infty}\frac{1}{x^p}dx$. 收敛.

但 $\int_1^{+\infty}\frac{|\sin x|}{x^p}dx > \int_1^{+\infty}\frac{\sin^2 x}{x^p}dx = \int_1^{+\infty}\frac{1-\cos 2x}{2x^p} = \int_1^{+\infty}\frac{1}{2x^p}dx - \int_1^{+\infty}\frac{\cos 2x}{2x^p}$

其中 $\int_1^{+\infty}\frac{1}{2x^p}=\frac{1}{1-p}\cdot x^{1-p}\Big|_1^{+\infty}$ 发散. $\int_1^{+\infty}\frac{\cos 2x}{2x^p}dx$ 收敛

故 $p\in(0,1)$ 时. 条件收敛. $p\in[1,+\infty)$ 时. 绝对收敛.

7.(6) $\int_0^1 x^{p-1}(1-x)^{q-1}dx=\int_0^1\frac{x^{p-1}}{(1-x)^{q-1}}dx=\int_0^1 f(x)dx$

当 $x\to 0$. $f(x)\to\frac{1}{x^{1-p}}$ 当 $x\to 1$. $f(x)\to\frac{1}{(1-x)^{1-q}}$

当 $p>0$ 且 $q>0$ 时. $f(x)$ 有界. 收敛.

其余情况. $f(x)$ 无界. 发散.

p-积分 $p<1$ 时收敛.
$p>1$ 时发散.

8.(6) $\int_0^{+\infty}x^{p-1}e^{-x}dx=\int_0^{+\infty}f(x)dx$. $x\to 0^+$ 时 $f(x)\to\frac{1}{x^{1-p}}$ $x\to+\infty$ 时 $f(x)\to e^{-\infty}$.

当 $p>0$ $\int_0^{+\infty}x^{p-1}e^{-x}dx$ 收敛.

当 $p<0$ 时 $\int_0^{+\infty}x^{p-1}e^{-x}dx$ 发散.

12. $\int_0^1 f(x)dx$ 收敛 $\Leftrightarrow$ $\forall\varepsilon>0$ $\exists\delta>0$ $\forall x_1,x_2\in(0,\delta)$, $\int_{x_1}^{x_2}f(x)dx<\varepsilon$

取 $x_1=0$. 即 $\forall\varepsilon>0$ $\exists\delta>0$ $\forall x\in(0,\delta)$ $\int_0^x f(x)dx<\varepsilon$.

任取 $t\in(0,x)$. 则 $t\cdot f(t)$ 是 $f(x)$ 在 $(0,x)$ 黎曼和的一部分. ($\exists P$)

故 $t\cdot f(t)<\int_0^x f(x)dx$ $\|P\|\to 0$ 时.

即 $\lim_{x\to 0}xf(x)<\int_0^x f(x)dx<\varepsilon$. ($\forall\varepsilon>0$. $\exists\delta$ 成立)

即 $\lim_{x\to 0}xf(x)=0$.

14. 令 $h(x)=\frac{f(x)}{e^{-x}}$ 则 $\lim_{x\to+\infty}h(x)=\lim_{x\to+\infty}\frac{f(x)}{e^{-x}}=\lim_{x\to+\infty}\frac{e^{-x}(f'(x)+f(x))}{e^{-2x}}=\lim_{x\to+\infty}\frac{f'(x)+f(x)}{e^{-x}}$.

即 $\lim_{x\to+\infty}f(x)=0$. 而 $\sin^2x<1$ 有界.

由狄利克雷判别法知 $\int_0^{+\infty}f(x)\sin^2x\,dx$ 收敛.