1. $\forall \alpha \in V$. $\exists ! \alpha_1 \in V_1$, $\alpha_2 \in V_2$ $\alpha = \alpha_1 + \alpha_2$ $\exists ! \alpha_{11} \in V_{11}$ $\alpha_{12} \in V_{12}$ s.t. $\alpha_1 = \alpha_{11} + \alpha_{12}$

故 $\forall \alpha \in V$. $\exists ! \alpha_{11}, \alpha_{12}, \alpha_2$ s.t. $\alpha = \alpha_{11} + \alpha_{12} + \alpha_2$.

故 $V = V_{11} \oplus V_{12} \oplus V_2$

2. 设V是n维线性空间. $\varepsilon_1 \cdots \varepsilon_n$是其一组正交基

令 $V_i = L(\varepsilon_i)$. 则 $\dim V_i = 1$. 显然 $V_1 + V_2 + \cdots + V_n = V_1 \oplus V_2 \oplus \cdots \oplus V_n$.

即 $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$

3. (1) 证明:

① $(A, A) = tr(AA) = tr(AA^T)$ $(AA^T)_{ii} = x_i^2 \geq 0$. $(AA^T)_{ii} = 0 \Leftrightarrow A = 0$.

② $(A, B) = tr(AB) = tr((BA)^T) = tr(BA) = (B, A)$.

③ $(kA, B) = tr(kAB) = k\,tr(AB) = k(A, B)$.

④ $(A, B+C) = tr(A(B+C)) = tr(AB) + tr(AC) = (A, B) + (A, C)$.

故是欧氏空间

(2) 形如 $C = \begin{pmatrix} & & a_{1m} \\ & & a_{2m} \\ & & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}$ 的n和矩阵各构成n个V的子空间 V_i. ($i = 1, 2, \ldots n$).

而 $\dim V_i = i$ 且 $V_i \cap V_j = \phi$. $i \neq j$.

故 $\dim V = \dim (V_1 + V_2 + \cdots + V_n) = \sum_{i=1}^{n} \dim V_i - \dim \sum_{i=1}^{n} \cap V_i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(3). $\dim W = \dim V - n = \frac{n^2 - n}{2}$

(4). E_{ii} 是其一个基. $i = 1, 2, \ldots n$

4. x^3. x^2. x. 1 是 $R[x]_4$ 的一组基.

$\eta_1 = 1$ 是R的基.

$$\eta_2 = x - \frac{(x, \eta_1)}{(\eta_1, \eta_1)} \cdot \eta_1 = x - \frac{\int_0^1 x\,dx}{1} = x - \frac{1}{2}$$

$$\eta_3 = x^2 - \frac{(x^2, \eta_2)}{(\eta_2, \eta_2)} \eta_2 - \frac{(x^2, \eta_1)}{(\eta_1, \eta_1)} \cdot \eta_1 = x^2 - \frac{\int_0^1 (x^3 - \frac{1}{2}x^2)\,dx}{\int_0^1 (x^2 - x + \frac{1}{4})\,dx} (x - \frac{1}{2}) - \frac{\int_0^1 x^2\,dx}{1} \cdot 1$$

$= x^2 - (x-\frac{1}{2}) - \frac{1}{3} = x^2 - x + \frac{1}{6}$

$\eta_4 = x^3 - \frac{(x^3, \eta_3)}{(\eta_3, \eta_3)} \cdot \eta_3 - \frac{(x^3, \eta_2)}{(\eta_2, \eta_2)} \cdot \eta_2 - \frac{(x^3, \eta_1)}{(\eta_1, \eta_1)} \cdot \eta_1$

$= x^3 - \frac{3}{2}(x^2 - x + \frac{1}{6}) - \frac{2}{5}(x - \frac{1}{2}) - \frac{1}{4} = x^3 - \frac{3}{2}x^2 + \frac{11}{10}x - \frac{3}{10}$

故 $R^{\perp} = L(x^3 - \frac{3}{2}x^2 + \frac{11}{10}x - \frac{3}{10}, x^2 - x + \frac{1}{6}, x - \frac{1}{2})$

η_2, η_3, η_4 是其一个基.