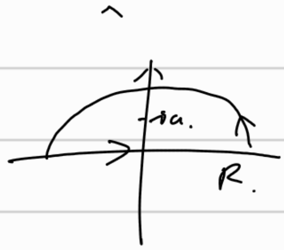
1. (1) $\mathcal{F}[f] = \mathcal{F}\left[\frac{1}{a^2+x^2}\right] = \int_R f(x) e^{-i\lambda x} dx = \int_R \frac{1}{a^2+x^2} e^{-i\lambda x} dx$

$= \int_R \frac{\cos\lambda x}{a^2+x^2} dx$



设 $g(z) = \frac{e^{i\lambda z}}{a^2+z^2}$. $\quad \underset{ia}{\text{Res}}\, g(z) = \lim_{z\to ia} (z-ia) g(z) = \frac{e^{-\lambda a}}{2ai}$

$$\int_R \frac{e^{i\lambda x}}{a^2+x^2} dx + \int_0^\pi \frac{iRe^{i\theta} e^{i\lambda Re^{i\theta}}}{a^2+R^2e^{2i\theta}} d\theta = 2\pi i \underset{ia}{\text{Res}}\, g(z)$$

$$LHS = \int_R \frac{\cos\lambda x}{a^2+x^2} dx + \int_0^\pi \frac{iRe^{i\lambda R\cos\theta - \lambda R\sin\theta + i\theta}}{a^2+R^2e^{2i\theta}} d\theta.$$

其中 $\left|\int_0^\pi \frac{iRe^{i\lambda R\cos\theta - \lambda R\sin\theta + i\theta}}{a^2+R^2e^{2i\theta}} d\theta\right| \le \int_0^\pi \left|\frac{Re^{-\lambda R\sin\theta}}{a^2+R^2e^{2i\theta}}\right| d\theta = 0. \quad R\to\infty.$

$RHS = \frac{\pi}{a} e^{-\lambda a}$

故 $\mathcal{F}[f](\lambda) = \frac{\pi}{a} e^{-\lambda a}.$

(2): $\begin{cases} u_t + 3u_x = 0 \\ u(0,x) = x^2 \end{cases}$ $\quad \frac{dZ(t)}{dt} = 3 \quad$ 令 $U(t) = u(t, Z(t))$

$Z(t) = 3t + Z(0) = 3t + \alpha$

则 $\frac{dU(t)}{dt} = u_t + 3u(t, Z(t)) = 0$. 故 $U(t) = U(0) = u(0, Z(0)) = (Z(0))^2$

令 $\alpha = Z(0)$. 则 $\alpha = x - 3t$ 为特征线. $U(t) = \varphi(\alpha) = u(t,x)$.

故 $u(t,x) = (x-3t)^2$

2. $\begin{cases} -\Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases}$

(1): $G(P_0, P) = \frac{1}{4\pi r_{P_0P}} - f(P_0, P)$

其中 G 在 Ω 中调和, $G|_{\partial\Omega} = 0$.

令 $G(P_0, P) = \frac{1}{4\pi r_{P_0P}} - \frac{R}{r_0}\frac{1}{4\pi r_{P_1P}}$. 其中 P_1 为 P 的反演点. $r_0 r_1 = R^2$. 且 P_1 在 $\overline{OP_0}$ 上.

(2). $G(P_0P) = \frac{1}{4\pi}\left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right]$

(3). $u(z) = -\iint \left(f \frac{\partial G}{\partial n}\right) dS_P$

$u(x_0, y_0, z_0) = \frac{z_0}{2\pi}\int_R\int_R \frac{f(x,y)}{((x-x_0)^2+(y-y_0)^2+z_0^2)^{\frac{3}{2}}} dx dy.$

3. (1) $\begin{cases} u_{tt} - u_{xx} = 0 \quad (t,x) \in (0,+\infty)\times(0,\pi) \\ u(0,x) = \varphi(x).\ u_t(0,x) = \psi(x). \quad 0 \le x \le \pi. \\ u(t,0) = u(t,\pi) = 0 \quad t \ge 0 \end{cases}$

$u(0,t) = u(\pi,t) = 0$

设 $u(t,x) = X(x)T(t)$. $\Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{T''(t)} = -\lambda$.

① 当 $\lambda \le 0$ 时. 只有 0 解.

② 当 $\lambda > 0$ 时. $X(x) = C_1 \sin\sqrt{\lambda}x + C_2\cos\sqrt{\lambda}x$. 由 $u(0,t) = 0 \Rightarrow C_2 = 0$.

由 $u(\pi,t) = 0 \Rightarrow \sin\sqrt{\lambda}\pi = 0 \Rightarrow \sqrt{\lambda}\pi = k\pi \Rightarrow \lambda = k^2 \quad k = 1,2\cdots$

不妨令 $X_k(x) = \sin kx$. 则 $T_k(t) = A_k \sin kt + B_k \cos kt$.

$u(0,x) = \varphi(x) \Rightarrow \varphi(x) = \sum_{k=1}^{\infty} B_k \sin kx$ 故 $B_k = \frac{2}{\pi}\int_0^\pi \varphi(s)\sin ks\,ds$

$u_t(0,x) = \psi(x) \Rightarrow \psi(x) = \sum_{k=1}^{\infty} A_k k \sin kx$ 故 $A_k = \frac{2}{k\pi}\int_0^\pi \psi(s)\sin ks\,ds$

故 $u(t,x) = \sum_{k=1}^{\infty}\left(\frac{2\sin kt}{k\pi}\int_0^\pi \psi(s)\sin ks\,ds + \frac{2\cos kt}{\pi}\int_0^\pi \varphi(s)\sin ks\,ds\right)\sin kx$.

(2). $u(t,x) = \sum_{k=1}^{\infty}(A_k \sin kt + B_k \cos kt)\sin kx$.

$u_t^2 + u_x^2 = k^2\left(\sum_{k=1}^{\infty}(A_k \cos kt - B_k \sin kt)\sin kx\right)^2 + k^2\left(\sum_{k=1}^{\infty}(A_k \sin kt + B_k \cos kt)\cos kx\right)^2$

$\int_0^\pi (u_t^2 + u_x^2)\,dx = k^2\int_0^\pi \sum_{k=1}^{\infty}\left[(A_k \cos kt - B_k \sin kx)^2 \sin^2 kx + (A_k \sin kt + B_k \cos kt)^2 \cos^2 kx\right]dx$.

因为 $\int_0^\pi \sin kx \sin lx\,dx = 0$. $k \ne l$. $\int_0^\pi \sin^2 kx\,dx = \int_0^\pi \cos^2 kx\,dx = \frac{\pi}{2}$

$\Rightarrow \int_0^\pi (u_t^2 + u_x^2)\,dx = \frac{\pi}{2}k^2\sum_{k=1}^{\infty}(A_k^2 + B_k^2)$

而 $\int_0^\pi (\psi^2 + \varphi_x^2)\,dx = \int_0^\pi\left[\left(\sum_{k=1}^{\infty} A_k k \sin kx\right)^2 + \left(\sum_{k=1}^{\infty} k B_k \cos kx\right)^2\right]dx = \frac{\pi}{2}k^2\sum_{k=1}^{\infty}(A_k^2 + B_k^2)$

故有 $\int_0^\pi (|u_t|^2 + |u_x|^2)\,dx = \int_0^\pi (|\varphi_x|^2 + \psi^2)\,dx$.

4. $\begin{cases} -\Delta u = 1 & x \in \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$ $\quad u(x_0) = \frac{1}{4\pi R^2}\iint_{B_R(x_0)} u\,ds$

$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial r}\cdot\frac{x_i}{r} \Rightarrow \frac{\partial^2}{\partial x_i^2} = \frac{x_i^2}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{x_i^2}{r^3}\frac{\partial}{\partial r}$ 故 $\Delta = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}$.

i.e. $u_{rr} + \frac{n-1}{r}u_r + 1 = 0$

① $n=1$ 时. 不妨取 $\Omega = (-l, l)$. 则 $u(x) = \frac{l^2}{2} - \frac{1}{2}x^2$ 显然 $\min|x-x_0|^2 = 0$ $\max|x-x_0|^2 = l^2$

故 $\min|x-x_0|^2 \le 2u(x_0) \le \max|x-x_0|^2$ $\quad u = \frac{|R|^2}{2n} - \frac{1}{2n}r^2$

② $n \ge 2$ 时. $u_r = -\frac{1}{n}r \Rightarrow u = -\frac{1}{2n}r^2 + C_0$.

由 $u|_{\partial\Omega} = 0 \Rightarrow C_0 = \frac{R^2}{2n}$. Ω 为球对称. 则如①中 $\min|x-x_0|^2 \le 2n\,u(x_0) \le \max|x-x_0|^2$

5. ? x

6.