Chapter3-Meromorphic Functions and the Logarithm

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Zeros and poles

Theorem 1.1: *Suppose that f is holomorphic in a connected open set Ω,has a zero at a point* $z_0 \in \Omega$*, and does not vanish identically in Ω. Then there exists a neighborhood* $U \subset \Omega$ *of* z_0*, a non-vanishing holomorphic function g on U, and a unique positive integer n such that;*

$$f(z) = (z - z_0)^n g(z)$$

for all $z \in U$

we say f has zero of order n at z_0(n=1,say it' simple)

Theorem 1.2: *If f has a pole at* $z_0 \in \Omega$*, then in a neighborhood of that point there exist a non-vanishing holomorphic function h and a unique positive integer n such that:*

$$f(z) = (z - z_0)^{-n} h(z)$$

The integer n is called the order of the pole(n=1,say it' simple)

Theorem 1.3: *if f has a pole of order n at* z_0*,then*

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z - z_0)} + G(z)$$

where G is a holomorphic function in a neighborhood of z_0

the sum $P(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0}$ is called the Principal part of f at z_0. the coefficient $a_{-1}$is the Residue of f at that pole ($res_{z_0} f = a_{-1}$)

In fact we can write $f(z) = P(z) + G(z)$,G is at most an infinite term series.Consider the integral along $C = C(z_0)$,we have

$$\int_C f(z)dz = \int_C P(z)dz + \int_C G(z)dz$$

for the second part,its value is obviously zero.for the first part,the value is $2\pi i a_{-1}$.

Theorem 1.4:*if f has a pole of order n at z_0,then*

$$res_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} (\frac{d}{dz})^{n-1} (z - z_0)^n f(z)$$

The residue formula

Theorem 2.1:*Suppose that f is holomorphic in an open set containing a circle C and its interior, except for a pole at z_0 inside C. Then*

$$\int_C f(z)dz = 2\pi i res_{z_0} f.$$

Corollary 2.2:*Suppose that f is holomorphic in an open set containing a circle C and its interior, except for poles at the points $z_1, ..., z_N$ inside C. Then:*

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{N} res_{z_k} f.$$

Corollary 2.3(Residue formula):*Suppose that f is holomorphic in an open set containing a toy contour γ and its interior, except for poles at the points $z_1, ..., z_N$ inside γ. Then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} res_{z_k} f.$$

Singularities and Meromorphic functions

Theorem 3.1 Riemann's theorem on removable singularities: *Suppose that f is holomorphic in an open set Ω except possibly at a point z_0 in Ω. If f is bounded on $\Omega - \{z_0\}$, then z_0 is a removable singularity.*

Proof of 3.1 use the Taylor series of function $F(z) = (z - z_0)^2 f(z)$

Corollary 3.2: *Suppose that f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $|f(z)| \to \infty, z \to z_0$.*

Proof:[3.2]The necessity is obvious.Conversely,$1/f$ bounded on the neighborhood of z_0 and vanishs at z_0.Suppose the order of zero is n,that's $1/f = \sum\limits_{k=n} a_k(z - z_0)^k$,so we have

$$f(z) = \frac{1}{(z - z_0)^n \times \sum\limits_{k=n}^{\infty} a_k(z - z_0)^{k-n}} = \frac{1}{(z - z_0)^n} \frac{1}{a_n + a_{n+1}(z - z_0) + \cdots} = \frac{1}{(z - z_0)^n} g(z)$$

where $g(z) \neq 0$ on the neighborhood.that's what we need

Isolated singularities belong to one of three categories:

1.Removeable singularity(f is bounded near z_0)

2.Pole singularity($|f| \to \infty \ as \ z \to z_0$)

3.Essential singularity(Singularities those are not Pole or Removeable singularities)

Theorem 3.3 Casorati-Weistrass: *Suppose f is holomorphic in the punctured disc $D_r(z_0) - \{z_0\}$ and has an essential singularity at z_0. Then, the image of $D_r(z_0) - \{z_0\}$ under f is dense in the complex plane.*

f is dense is to say:$\forall w \in C, \forall \epsilon, \delta > 0, \exists z, |z - z_0| < \delta \ s.t. |f(z) - w| < \epsilon$

In fact, Picard proved a much stronger result. He showed that under the hypothesis of the above theorem, the function f takes on every complex value infinitely many times with at most one exception

*(Meromorphic function):*We now turn to functions with only isolated singularities that are poles. A function f on an open set Ω is meromorphic if there exists a sequence of points

$\{z_0, z_1, z_2, \ldots\}$ that has no limit points in Ω, and such that (i) the function f is holomorphic in $\Omega - \{z_0, z_1, z_2, \ldots\}$, and (ii) f has poles at the points $\{z_0, z_1, z_2, \ldots\}$.

In conclusion, f is meromorphic is to say instead of poles f has no other singularities(removable singularity is allowed since we can replace it with a complex num)

Theorem 3.4: *the meromorphic function f ($f : \overline{C} \to D$) in the extended complex plane ($\overline{C} = C \cup \{\infty\}$) are the rational functions*

theorem 3.4

The argument principle and applications

Theorem 4.1 Argument principle: *Suppose f is meromorphic in an open set containing a circle C and its interior. If f has no poles and never vanishes on C, then:*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = (number\ of\ zeros\ of\ f\ inside\ C)\ minus\ (number\ of\ poles\ of\ f\ insideC)$$

where the zeros and poles are counted with their multiplicities.

Corollary 4.2: *The above theorem holds for toy contours*

Theorem 4.3 Rouche's theorem: *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If*

$$|f(z)| > |g(z)|\ forall\ z \in C,$$

then f and f + g have the same number of zeros inside the circle C.

here |f|>|g| at C instead of at Int(C)!

Theorem 4.4 Open mapping theorem: *If f is holomorphic and non-constant in a region Ω, then f is open.*

A mapping is said to be open if it maps open sets to open sets.

Theorem 4.5 Maximum modulus principle: *If f is a non-constant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.*

Corollary 4.6: *Suppose that Ω is a region with compact (so bounded) closure $\overline{\Omega}$. If f is holomorphic on Ω and continuous on $\overline{\Omega}$. then*

$$\sup_{z\in\Omega}|f(z)| \leq \sup_{z\in\bar{\Omega}-\Omega}|f(z)|.$$

In fact, since $f(z)$ is continuous on the compact set $\bar{\Omega}$, then $|f(z)|$ attains its maximum in $\bar{\Omega}$; but this cannot be in Ω if f is non-constant. If f is constant, the conclusion is trivial.

Homotopies and simply connected domains

Loosely speaking, two curves are homotopic if one curve can be deformed into the other by a continuous transformation without ever leaving Ω

Theorem 5.1: *If f is holomorphic in Ω, then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

whenever the two curves γ_0 and γ_1 are homotopic in Ω.

A region Ω in the complex plane is simply connected if any two pair of curves in Ω with the same end-points are homotopic.

Theorem 5.2: *Any holomorphic function in a simply connected domain has a primitive.*

Corollary 5.3: *If f is holomorphic in the simply connected region Ω, then*

$$\int_{\gamma} f(z)dz = 0$$

for any closed curve γ in Ω.

The complex logarithm

logarithm

Theorem 6.1: *Suppose that Ω is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_{\Omega}(z)$ so that* *(i) F is holomorphic in Ω,*

(ii) $e^{F(z)} = z$ for all $z \in \Omega$,

(iii) $F(r) = \log r$ whenever r is a real number and near 1 .

In other words, each branch $\log_{\Omega}(z)$ is an extension of the standard logarithm defined for positive numbers.

Theorem 6.2: *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that*

$$f(z) = e^{g(z)}$$

The function g(z) in the theorem can be denoted by log f(z), and determines a "branch" of that logarithm.

Fourier series and harmonic functions

Theorem 7.1: *The coefficients of the power series expansion of f are given by*

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) e^{-in\theta} d\theta$$

for all $n \geq 0$ and $0 < r < R$. Moreover,

$$0 = \frac{1}{2\pi r^n} \int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) e^{-in\theta} d\theta$$

whenever $n < 0$.

Corollary 7.2: *(Mean-value property) If f is holomorphic in a disc $D_R(z_0)$, then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta, \quad \text{for any } 0 < r < R.$$

Corollary 7.3: *If f is holomorphic in a disc $D_R(z_0)$, and $u = \mathrm{Re}(f)$, then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) d\theta, \quad \text{for any } 0 < r < R.$$

every harmonic function in a disc is the real part of a holomorphic function in that disc.(Exercise 12 in Chapter 2)