1.

(3) $\left(\int_0^x \sin^2 t\,dt\right)' = \sin^2 x \qquad \int_0^x \sin^2 t\,dt = \int_0^x \frac{1-\cos 2t}{2}\,dt = \frac{1}{2}x - \frac{1}{4}\sin 2x$

$$f'(x) = \sin^2 x \cdot \frac{1}{1+\left(\int_0^x \sin^2 t\,dt\right)^2} = \frac{4\sin^2 x}{4+(x-\sin x\cos x)^2}$$

2.

(2) $\lim\limits_{x\to 0}\frac{x^2}{\int_{\cos x}^1 e^{-w^2}\,dw} = -\lim\limits_{x\to 0}\frac{x^2}{\int_1^{\cos x} e^{-w^2}\,dw} = \lim\limits_{x\to 0}\frac{2x}{e^{-\cos^2 x}\sin x} = \lim\limits_{x\to 0} 2e^{\cos^2 x} = 2e$

4. $f'(x) = (x-1)(x-2)^2$ 故 $x=1$ 是极值点(小)

$f(x)_{极小} = f(1) = \int_0^1 (t-1)(t-2)^2\,dt = \left(\frac{1}{4}t^4 - \frac{5}{3}t^3 + 4t^2 - 4t\right)\Big|_0^1 = -\frac{17}{12}$

故 $f(x)$ 有极小值 $f(1) = -\frac{17}{12}$

6.

(5) $\int_{-1}^1 \frac{(x+1)\,dx}{(x^2+2x+5)^2} = \int_{-1}^1 \frac{(x+1)\,d(x+1)}{[(x+1)^2+4]^2} \xlongequal{t=x+1} \int_0^2 \frac{t\,dt}{(t^2+4)^2} \xlongequal{t=2\tan u} \int_0^{\frac{\pi}{4}} \frac{2\tan u\, d2\tan u}{16\sec^4 u}$

$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \sin u\cos u\,du = \frac{1}{8}\int_0^{\frac{\pi}{4}} \sin 2u\,du = \frac{1}{8}\left(-\frac{1}{2}\cos 2u\right)\Big|_0^{\frac{\pi}{4}} = \frac{1}{16}$

(6) $\int_0^1 \arcsin x\,dx = x\arcsin x\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} + \sqrt{1-x^2}\Big|_0^1 = \frac{\pi}{2} - 1$

(10) $\int_1^e \sin\ln x\,dx = \int_1^e x\sin\ln x\,d\ln x = -x\cos\ln x\Big|_1^e + \int_1^e \cos\ln x\,dx$

其中 $\int_1^e \cos\ln x\,dx = \int_1^e x\,d\sin\ln x = x\sin\ln x\Big|_1^e - \int_1^e \sin\ln x\,dx$

$\Rightarrow \int_1^e \sin\ln x\,dx = \frac{1}{2}\left(x\sin\ln x\Big|_1^e - x\cos\ln x\Big|_1^e\right) = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2}$

(11) $\int_0^1 x^2\arctan x\,dx = \frac{1}{3}x^3\arctan x\Big|_0^1 - \int_0^1 \frac{1}{3}\frac{x^3}{1+x^2}\,dx$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0 \frac{\quad}{1+x^2}dx = \frac{\pi}{12} - \frac{1}{6}\left[x^2 - \ln(1+x^2)\right]\Big|_0 = \frac{\pi}{12} - \frac{1}{6} + \frac{\quad}{6}$$

即 $\int_0^1 x^2 \arctan x\,dx = \frac{1}{12}(\pi + 2\ln 2 - 2)$

(15) $\int_0^1 \frac{dx}{\sqrt{1+e^{x}}} \xlongequal{e^x=\tan t} \int_{\frac{\pi}{4}}^{\arctan e} \frac{d\ln\tan t}{\sec t} = \int_{\frac{\pi}{4}}^{\arctan e} \csc t\,dt = \ln|\csc t - \cot t|\Big|_{\frac{\pi}{4}}^{\arctan e}$

$$= \ln(\sqrt{e^2+1} - 1) - \ln(\sqrt{2}-1) - 1$$

即 $\int_0^1 \frac{dx}{\sqrt{1+e^x}} = \ln(\sqrt{e^2+1} - 1) - \ln(\sqrt{2}-1) - 1$

(18) $\int_0^1 \frac{x^2+1}{x^4+1}dx = \int_0^1 \frac{x^2+1}{x^2}\cdot\frac{x^2}{(x^2)^2+1}dx = \int_0^1 \frac{x^2}{x^4+1}d\left(x-\frac{1}{x}\right) = \int_0^1 \frac{1}{\left(x-\frac{1}{x}\right)^2+2}d\left(x-\frac{1}{x}\right)^2$

$$= \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x}\Big|_0^1 = \frac{\sqrt{2}}{4}\pi \quad (\text{取极限})$$

7. (3) $\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n-1}{n}\pi\right) = \frac{1}{\pi}\lim_{n\to\infty}\frac{\pi}{n}\sum_{i=0}^{n-1}\sin\frac{i\pi}{n} = \frac{1}{\pi}\int_0^{\pi}\sin x\,dx = \frac{2}{\pi}$

8. (5) $\int_0^1 x^n \ln^m x\,dx = \frac{x^{n+1}}{n+1}\ln^m x\Big|_0^1 - \int_0^1 \frac{m}{n+1}x^n \ln^{m-1}x\,dx = -\frac{m}{n+1}\int_0^1 x^n \ln^{m-1}x\,dx$

$\Rightarrow \int_0^1 x^n \ln^m x\,dx = (-1)^m\frac{m!}{(n+1)^m}\int_0^1 x^n dx = (-1)^m\frac{m!}{(n+1)^{m+1}}$

即 $\int_0^1 x^n \ln^m x\,dx = (-1)^m\frac{m!}{(n+1)^{m+1}}$

9. (1) $\int_0^{\frac{\pi}{2}} f(\cos x)dx \xlongequal{t=x+\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} f(\sin t)dt \xlongequal{\text{对称性}} \int_0^{\frac{\pi}{2}} f(\sin t)dt = \int_0^{\frac{\pi}{2}} f(\sin x)dx$.

(2) $F(x) = \int_0^x t f(\sin t)dt - \frac{\pi}{2}\int_0^x f(\sin t)dt$. $x\in[0,\pi]$

$F'(x) = x f(\sin x) - \frac{\pi}{2}f(\sin x) = \left(x-\frac{\pi}{2}\right)f(\sin x)$

有 $F'(x)$ 关于 $(\frac{\pi}{2},0)$ 中心对称，故 $F(x)$ 关于 $(\frac{\pi}{2},0)$ 中心对称.

而 $F(0)=0$. 故 $F(\pi)=0$.

从而 $\int_0^{\pi} x f(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

10. (3) $\int_0^{\pi}\frac{x}{1+\sin^2x}dx=\frac{\pi}{2}\int_0^{\pi}\frac{1}{1+\sin^2x}dx=\pi\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin^2x}dx=\pi\int_0^{\frac{\pi}{2}}\frac{d\sqrt{2}\tan t}{1+(\sqrt{2}\tan t)^2}\cdot\frac{1}{\sqrt{2}}$

$=\frac{\pi}{\sqrt{2}}\arctan\sqrt{2}\tan t\Big|_0^{\frac{\pi}{2}}=\frac{\sqrt{2}}{4}\pi^2$

11. (3) $\int_0^1 x|x-a|dx=\begin{cases}\int_0^1 x(x-a)\,dx=\frac{1}{3}-\frac{1}{2}a & a\le 0\\ \int_0^a x(a-x)dx+\int_a^1 x(x-a)\,dx=\frac{1}{3}a^3-\frac{a}{2}+\frac{1}{3} & 0<a<1\\ \int_0^1 x(a-x)\,dx=-\frac{1}{3}+\frac{a}{2} & a\ge 1\end{cases}$ $\quad -\frac{1}{3}x^3+\frac{1}{2}ax^2$

13.

$I=\int_1^4 f(x-2)dx=\int_{-1}^2 f(x)dx=\int_{-1}^0 f(x)dx+\int_0^2 f(x)dx$

$=\int_{-1}^0\frac{1}{1+e^x}dx+\int_0^2 xe^{-x^2}dx=\int_0^{-1}\frac{de^{-x}}{1+e^{-x}}+(-\frac{1}{2})\int_0^2 d\,e^{-x^2}$

$=\ln(1+e^{-x})\Big|_0^{-1}-\frac{1}{2}e^{-x^2}\Big|_0^2=\ln(1+e)-\ln 2-\frac{1}{2e^4}+\frac{1}{2}$

$=\ln(1+e)-\ln 2-\frac{1}{2e^4}+\frac{1}{2}$

16. $\int_0^1 t f(2x-t)\,dt=\int_{2x}^{2x-1}f(u)(2x-u)\,d(2x-u)=\int_{2x}^{2x-1}(u-2x)f(u)\,du$

$=\int_{2x}^{2x-1}uf(u)\,du-2x\int_{2x}^{2x-1}f(u)\,du=\frac{1}{2}\arctan x^2$

同时对左右求导. $xf(x)-2\int_{2x}^{2x-1}f(u)\,du-2xf(x)=\frac{x}{1+x^4}$

$\Rightarrow\int_{2x-1}^{2x}f(u)\,du=\frac{1}{2}\frac{x}{1+x^4}+\frac{1}{2}xf(x)$. 令 $x=1$

$\Rightarrow\int_1^2 f(x)dx=\frac{1}{4}+\frac{1}{2}f(1)=\frac{5}{4}$

21. $f(\xi)\triangleq\max\limits_{a\le x\le b}|f(x)|\quad f(\eta)\triangleq\min\limits_{a\le x\le b}|f(x)|$

$\max|f(x)|-\min|f(x)|\le\int_\eta^\xi|f'(x)|dx\le\int_a^b|f'(x)|\,dx$

由积分第一中值定理, $\exists v\in(a,b)$ s.t. $\frac{1}{b-a}\int_a^b f(x)dx=f(v)\ge f(\eta)$

故 $\max|f(x)|\le\int_a^b|f'(x)|dx+\min|f(x)|\le\int_a^b|f'(x)|dx+f(v)\le\int_a^b|f'(x)|dx+\left|\frac{1}{b-a}\int_a^b f(x)dx\right|$

即 $\max\limits_{a\le x\le b}|f(x)|\le\left|\frac{1}{b-a}\int_a^b f(x)dx\right|+\int_a^b|f'(x)|dx.$

23. $f(x)$是下凸函数.

由Jensen不等式 $\frac{1}{a}\int_0^a f(x)dx \geq f\left(\frac{a+0}{2}\right) = f\left(\frac{a}{2}\right)$.

得到 $\int_0^a f(x)dx \geq a f\left(\frac{a}{2}\right)$

25: ① 当$f(0) \geq 0$. $\int_0^{2\pi} f(x)\sin nx\,dx = f(0)\int_0^{\xi} \sin nx\,dx = f(0)\int_0^{\eta} \sin x\,dx$. 其中 $\eta = \xi n - \left[\frac{\xi n}{2\pi}\right]\cdot 2\pi$

则 $\int_0^{\eta} \sin x\,dx = \int_0^{\pi\pm\theta} \sin x\,dx$ 其中 $\theta \in (-\pi,\pi)$. 则由η任意 $\int_0^{\eta} \sin x\,dx \geq 0$

故 $\int_0^{2\pi} f(x)\sin nx\,dx \geq 0$

② 当$f(0) < 0$时. $-f(x)$单增

$\int_0^{2\pi} f(x)\sin nx\,dx = -\int_0^{2\pi}(-f(x))\sin nx\,dx = -(-f(2\pi))\int_{\xi}^{2\pi} \sin nx\,dx$ *

同理 *式等于 $f(2\pi)\int_{\eta}^{2\pi} \sin x\,dx = f(2\pi)\int_{2\pi-\theta}^{2\pi} \sin x\,dx$ $\theta \in (0,2\pi)$.

而 $\int_{2\pi-\theta}^{2\pi} \sin x\,dx < 0$ $f(2\pi) < 0$. 故 $\int_0^{2\pi} f(x)\sin nx\,dx > 0$.

综上: $\int_0^{2\pi} f(x)\sin nx\,dx \geq 0$