Chapter1

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Complex number and complex plane

Theorem 1.1: *C, the complex numbers is complete*

a set is complete is to say $X \overset{closed}{\subset} Y$, (Y, ρ) is Metric space .for any point $x \in X$,x is limit point

Theorem 1.2: *The set $\Omega \subset C$ is compact iff every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point z_0 in Ω*

the most important is $z_0 \in \Omega$, for example for $\Omega = (0,1] \times [0,1], z_n = 1/n + i/2$, it's obviously $z_n \to i/2$ but $i/2 \notin \Omega$

Theorem 1.3: *A set Ω is compact iff every open covering has a finite subcovering that covers Ω*

proposition 1.4: *if $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset \cdots$ is a sequence of non-empty compact set in C with the property that*

$$diam(\Omega_n) \to 0 \; as \; n \to \infty$$

then there exists a unique point $\omega \in C$ s.t. $\omega \in \Omega_n$ for all n

Function of the complex plane

Theorem 2.1: *A continuous function f on a compace set Ω is abounded and attains a maximun and minimun in Ω*

Holomophic Function:f hol at z iff $\lim\limits_{h\to 0}\frac{f(z+h)-f(z)}{h}$ converges to a limit

f is said to be hol on Ω iff f hol at every point of it

for example $f(z)=\frac{1}{z}$ isn't hol at (0,0);any polynomial hol in C,$f(z)=\overline{z}$ isn't hol since $\lim\limits_{h\to 0}\frac{f(z+h)-f(z)}{h}=\frac{\overline{h}}{h}$ has no limit

proposition 2.2: *f,g hol on Ω:(1):f+g hol, (f+g)'=f'+g' ; (2):fg hol, (fg)'=f'g+g'f ; (3):(f/g) hol where g novanish $(f/g)'=(f'g-g'f)/g^2$ and (4): $\Omega\xrightarrow{f}U\xrightarrow{g}C, g(f(z))$ hol on Ω*

proposition 2.3: *if f is hol at z_0 then :$\frac{\partial f}{\partial \overline{z}}f(z_0)=0, f'(z_0)=\frac{\partial f}{\partial z}(z_0)=2\frac{\partial u}{\partial z}(z_0)$*

proposition 2.4: *suppose f=u+iv is a complex-value function defined on Ω ,if u,v sre continuesly differentiable on Ω and satisify the Cauchy-Riemann Equations on Ω, then f is hol on Ω and $f'(z)=\frac{\partial f}{\partial z}$*

Theorem 2.5: *Given a power seties $\sum\limits_{n=0}^{\infty}a_nz^n$ there exists $0\le R\le\infty$ s.t.*

(1):if |z|<R,the series converges absolutely

(2):if |z|>R,the series diverges

while |z|=R the situation need to discuss

$\frac{1}{R}=\limsup|a_n|^{\frac{1}{n}}$,we call R "radius of convergence",|z|<R "the disc of convergence"

Theorem 2.6: *the power series $f(z)=\sum\limits_{n=0}^{\infty}a_nz^n$ defines a hol function in it's disc of convergence.the derivative of f is also a power serier for f that's*

$$f'(z)=\sum_{n=0}^{\infty}na_nz^{n-1}$$

moverover f' has the same radius of convergence as f

Corollary 2.7: *A power series is infinitely complex differentiable in it's disc of convergence and the higher derivatives are also seties obtained by termwise differentiation*

f is Holomophic iff f is Analytic iff f has a power series expansion

integration along curves

proposition 3.1:*integration of continuous functions over cuvers satisfies the followling properties: (i) It is linear, that is, if* $\alpha, \beta \in \mathbb{C}$, *then*

$$\int_{\gamma}(\alpha f(z)+\beta g(z))dz = \alpha\int_{\gamma} f(z)dz + \beta\int_{\gamma} g(z)dz$$

(ii) If γ^{-} *is* γ *with the reverse orientation, then*

$$\int_{\gamma} f(z)dz = -\int_{\gamma^{-}} f(z)dz$$

(iii) One has the inequality

$$\left|\int_{\gamma} f(z)dz\right| \leq \sup_{z\in\gamma}|f(z)| \cdot \textit{ length } (\gamma)$$

Theorem 3.2:*if a function f has a primitive* F *in* Ω *and* γ *is a curve in it that begins* w_1 *and ends* w_2 *then:*

$$\int_{\gamma} f(z)dz = F(w_2) - F(w_1)$$

it's same like Newton-Leibniz Formula

Corollary 3.3: *If* γ *is a closed curve in an open set* Ω, *and* f *is continuous and has a primitive in* Ω, *then*

$$\int_{\gamma} f(z)dz = 0.$$

This is immediate since the end-points of a closed curve coincide.