P233.

1. (1) $\iint_D e^{-(x^2+y^2)}\,dxdy = \iint_{D^*} e^{-r^2} r\,drd\theta$ $\quad D^* = \{(r,\theta) \mid 0\le r\le R,\ 0\le\theta\le 2\pi\}$

$I = \frac{1}{2}\int_0^{2\pi} d\theta \int_0^R e^{-r^2} dr^2 = \pi(1-e^{-R^2})$

(2) $\iint_D \sqrt{x}\,dxdy = \iint_{D^*} \sqrt{r}\,r\sqrt{\cos\theta}\,drd\theta$ $\quad D^* = \{(r,\theta) \mid 0\le r\le \cos\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\}$

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos\theta}\,d\theta \int_0^{\cos\theta} r^{\frac{3}{2}} dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{5}\cos\theta\,d\theta - \frac{2}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\,d\sin\theta$

$= \frac{4}{5} - \frac{2}{5}\times\frac{1}{3}\times 2 = \frac{8}{15}$

(3) $\iint_D (x+y)\,dxdy = \iint_{D^*} r^2(\cos\theta+\sin\theta)\,drd\theta = \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} (\cos\theta+\sin\theta)\,d\theta \int_0^{\cos\theta+\sin\theta} r^2 dr$

$= \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \left(\frac{1}{2} + \frac{2}{3}\sin 2\theta - \frac{1}{8}\cos 4\theta\right) d\theta = \frac{\pi}{2}$

(4) $\iint_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dxdy = \iint_{D^*} \sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr = \frac{\pi^2}{8} - \frac{\pi}{4}$

2. (2) $\iint_D dxdy \xlongequal[v=y/x]{u=\frac{y^2}{x}} \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \iint_{D^*} \frac{u}{v^4}\,dudv = \int_m^n u\,du \int_\alpha^\beta \frac{1}{v^4}\,dv$

$= \frac{1}{6}(n^2-m^2)\left(\frac{1}{\alpha^3} - \frac{1}{\beta^3}\right)$

4. (1) $\iint (\sqrt{x}+\sqrt{y})\,dxdy \xlongequal[v=\sqrt{y}]{u=\sqrt{x}} \iint_{D^*} 4(u+v)uv\,dudv = 4\int_0^1 du \int_0^{1-u} (u+v)uv\,dv$

$= \frac{2}{3}\int_0^1 (u-1)^2(u^2+2u)\,du = 0.3$

(2) i: $\iint_D \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) dxdy \xlongequal[y=br\sin\theta]{x=ar\cos\theta} \iint_{D^*} r^2\,ab\,r\,drd\theta = ab\int_0^{2\pi} d\theta \int_0^1 r^3 dr = \frac{\pi}{2}ab$

ii: $\iint_D \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) dxdy \xlongequal[y=r\sin\theta]{x=r\cos\theta} \iint_D \left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right) r^3\,drd\theta = \int_0^{2\pi} \left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right) d\theta \int_0^R r^3 dr$

$= \frac{\pi}{4}R^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$

(3) $\iint_D y\,dx\,dy = \int_{-2}^{0} dx\int_0^2 y\,dy - \int_0^2 dy\int_{-\sqrt{2y-y^2}}^{0} y\,dx = 4 - \int_0^2 y\sqrt{2y-y^2}\,dy$

$= 4 - \frac{\pi}{2}$

(4) $\iint_D e^{\frac{x-y}{x+y}}\,dx\,dy \xlongequal[v=x-y]{u=x+y} \iint_{D^*} e^{\frac{v}{u}} \frac{1}{2}\,du\,dv = \frac{1}{2}\int_0^2 du\int_{-u}^{u} e^{\frac{v}{u}}\,dv = e - \frac{1}{e}$

(5) $\iint_D \frac{(x+y)^2}{1+(x-y)^2}\,dx\,dy \xlongequal[v=x+y]{u=x-y} \iint_{D^*} \frac{v^2}{1+u^2}\frac{1}{2}\,du\,dv = \frac{1}{2}\int_{-1}^{1}\frac{du}{1+u^2}\int_{-1}^{1} v^2\,dv = \frac{\pi}{6}$

(6) $\iint_D \sqrt{\frac{x^2+y^2}{4a^2-x^2-y^2}}\,dx\,dy \xlongequal[y=r\sin\theta]{x=r\cos\theta} \iint_{D^*} \sqrt{\frac{r^2}{4a^2-r^2}}\cdot r\,dr\,d\theta = \int_{-\frac{\pi}{4}}^{0} d\theta\int_0^{-2a\sin\theta} \frac{r^2}{\sqrt{4a^2-r^2}}\,dr$

其中: $\int_0^{-2a\sin\theta} -r\cdot\left(\sqrt{4a^2-r^2}\right)'dr = -r\sqrt{4a^2-r^2}\Big|_0^{-2a\sin\theta} + \int_0^{-2a\sin\theta}\sqrt{4a^2-r^2}\,dr$

$= 2a^2\sin 2\theta + \left(\frac{1}{2}r\sqrt{4a^2-r^2} + \frac{4a^2}{2}\arcsin\frac{r}{2a}\right)_0^{-2a\sin\theta}$

$= a^2\sin 2\theta - 2a^2\theta$

故 $I = a^2\int_{-\frac{\pi}{4}}^{0}(\sin 2\theta - \theta)\,d\theta = a^2\left(-\frac{1}{2}\cos 2\theta - \theta^2\right)\Big|_{-\frac{\pi}{4}}^{0} = \frac{\pi^2-8}{16}a^2$

14. $\iint_{|x|+|y|\le 1} f(x+y)\,dx\,dy \xlongequal[v=x+y]{u=x-y} \frac{1}{2}\iint_{D^*} f(v)\,du\,dv$ 其中 $D^* = [-1,1]\times[-1,1]$

故 $I = \frac{1}{2}\iint_{D^*} f(v)\,du\,dv = \frac{1}{2}\int_{-1}^{1} du\int_{-1}^{1} f(v)\,dv = \int_{-1}^{1} f(v)\,dv = \int_{-1}^{1} f(u)\,du$. #.