一、(1). 求 $f(x)=e^{-a|x|}$ 的 Fourier 变换.

$$\mathcal{F}[f](\lambda)=\int_{\mathbb{R}} e^{-i\lambda x}e^{-a|x|}dx=\int_{-\infty}^{0} e^{(-i\lambda+a)x}dx+\int_{0}^{+\infty} e^{(-i\lambda-a)x}dx$$

$$=\frac{1}{a-i\lambda}e^{(-i\lambda+a)x}\Big|_{-\infty}^{0}-\frac{1}{a+i\lambda}e^{-(i\lambda+a)x}\Big|_{0}^{+\infty}=\frac{1}{a-i\lambda}+\frac{1}{a+i\lambda}=\frac{2a}{a^2+\lambda^2}$$

故 $\mathcal{F}[e^{-a|x|}](\lambda)=\frac{2a}{a^2+\lambda^2}$

(2). 求 $\begin{cases} u_t+au_x+cu=0 & (t,x)\in(0,+\infty)\times\mathbb{R} \\ u(0,x)=\varphi(x). & x\in\mathbb{R}. \end{cases}$

对上述方程两侧做 Fourier 变换. 记 $\tilde{u}=\mathcal{F}[u]$. $\tilde{\varphi}=\mathcal{F}[\varphi]$. (对 x 作) $\boxed{u(t,\infty)=0}$

则有 $\mathcal{F}[u_t]=\tilde{u}_t$ $\mathcal{F}[u_x]=i\lambda\tilde{u}$

故有 $\begin{cases} \tilde{u}_t+(i\lambda a+c)\tilde{u}=0 \\ \tilde{u}(0,\lambda)=\tilde{\varphi}(\lambda). \end{cases} \Rightarrow \tilde{u}(t,\lambda)=\tilde{\varphi}(\lambda)e^{-(i\lambda a+c)t}$

由 $f_1 * f_2=\mathcal{F}[f_1]\cdot\mathcal{F}[f_2]$. $\Rightarrow u(t,x)=\varphi(x)*\mathcal{F}^{-1}[e^{-(i\lambda a+c)t}]$

$\mathcal{F}^{-1}[e^{-(i\lambda a+c)t}]=e^{-ct}\mathcal{F}[e^{-i\lambda at}]=e^{-ct}\delta(x-at)$

故 $u(t,x)=\int_{\mathbb{R}} e^{-ct}\varphi(s)\delta(x-s-at)\,ds=\varphi(x-at)e^{-ct}$

$u(t,x)=\varphi(x-at)e^{-ct}$

2. (1): $G|_{\partial\Omega}=0$, $\int_{\partial\Omega}\frac{\partial G}{\partial n}dS=-1$.

(2): G 在 $\Omega\setminus B(x_0,r)$ 上调和. (r 充分小). $G|_{\partial B(x_0,r)}>0$. (与 $\frac{1}{4\pi r}$ 同阶).

故由极值原理 $G|_{\Omega\setminus B(x_0,r)}\in(0,M_r)$ $(\exists M_r)$ 令 $r\to0$. 则 $G|_{\Omega\setminus\{x_0\}}>0$.

(3): $G(P_0,P)=\frac{1}{4\pi}\left(\frac{1}{r_{P_0P}}-\frac{R}{r_0}\frac{1}{r_{P_1P}}\right)$. 则 $R=1=r_0r_1$ $|P_1P|\cdot|OP_0|=\frac{1}{|P_0P|}$ 且 $R=1$.

则 $G(P_0,P)=0$

3. $\begin{cases} u_t-u_{xx}=0 \\ u(0,x)=x^2(l-x)^2 & (t,x)\in(0,T)\times(0,l) \\ u_x(t,0)=u_x(t,l)=0. \end{cases}$

令 $u(t,x)=X(x)T(t) \Rightarrow \frac{X''(x)}{X(x)}=\frac{T'(t)}{T(t)}=-\lambda$.

若 $\lambda\le0$ 时. 有 0 解. 故 $\lambda>0$.

则 $X(x)=C_1\sin\sqrt{\lambda}x+C_2\cos\sqrt{\lambda}x$.

由 $u_x(t,0)=0 \Rightarrow C_1=0$

$u_x(t,l)=0 \Rightarrow \sin\sqrt{\lambda}l=0$ 记 $\lambda=\left(\frac{k\pi}{l}\right)^2$ $k\geq1$

$u_x(t,l)=0 \Rightarrow \sin\sqrt{\lambda}\,l=0$ i.e. $\lambda=\left(\frac{k\pi}{l}\right)^2$ $k\geq 1$

不妨设 $X_k(x)=\sin\frac{k\pi}{l}x$. $T_k(t)=C_k e^{-\left(\frac{k\pi}{l}\right)^2 t}$

故 $u(t,x)=\sum_{k=1}^{\infty} C_k e^{-\left(\frac{k\pi}{l}\right)^2 t}\sin\frac{k\pi}{l}x$

由 $u(0,x)=x^2(l-x)^2 \Rightarrow \sum_{k=1}^{\infty} C_k\sin\frac{k\pi}{l}x=x^2(l-x)^2$

$\Rightarrow C_k=\frac{2}{l}\int_0^l x^2(l-x)^2\sin\frac{k\pi}{l}x\,dx$ 令 $I=\int_0^l x^2(l-x)^2\sin\frac{k\pi}{l}x\,dx$.

$I=-\left(\frac{l}{k\pi}\right)x^2(l-x)^2\cos\frac{k\pi}{l}x\Big|_0^l+\frac{l}{k\pi}\int_0^l(4x^3-6lx^2+2l^2x)\cos\frac{k\pi}{l}x\,dx$

$=\left(\frac{l}{k\pi}\right)^2(4x^3-6lx^2+2l^2x)\sin\frac{k\pi}{l}x\Big|_0^l-\left(\frac{l}{k\pi}\right)^2\int_0^l(12x^2-12lx+2l^2)\sin\frac{k\pi}{l}x\,dx$

$=\left(\frac{l}{k\pi}\right)^3(12x^2-12lx+2l^2)\cos\frac{k\pi}{l}x\Big|_0^l-\left(\frac{l}{k\pi}\right)^3\int_0^l(24x-12l)\cos\frac{k\pi}{l}x\,dx$

$=\left(\frac{l}{k\pi}\right)^3(2l^2)[(-1)^k-1]-\left(\frac{l}{k\pi}\right)^4(24x-12l)\sin\frac{k\pi}{l}x\Big|_0^l+\left(\frac{l}{k\pi}\right)^4\int_0^l 24\sin\frac{k\pi}{l}x\,dx$

$=2[(-1)^k-1]l^2\left(\frac{l}{k\pi}\right)^3-24\left(\frac{l}{k\pi}\right)^5[(-1)^k-1]$

显然 当 $k\equiv 0 \mod 2$ 时. $C_k=0$. 故令 $C_{2k-1}=\frac{96l^4}{(2k-1)^5\pi^5}-\frac{8l^4}{(2k-1)^3\pi^3}$

故 $u(t,x)=\sum_{k=1}^{\infty}\left(\frac{96l^4}{(2k-1)^5\pi^5}-\frac{8l^4}{(2k-1)^3\pi^3}\right)\exp\left\{-\frac{(2k-1)^2\pi^2}{l^2}t\right\}\sin\frac{2k-1}{l}\pi x$

4. $\begin{cases} u_t+u_{xx}=2u \\ u|_\Gamma\leq M \end{cases}$

令 $u(t,x)=X(x)T(t)$. 则有 $T'X+TX''=2TX$ i.e. $\frac{T'}{T}+\frac{X''}{X}=2$

令 $\frac{T'}{T}=v(t,x)$ 则 $\frac{X''}{X}=2-v(t,x)$ $\Rightarrow T(t)=T_0 e^{\int_0^t v(\tau,x)d\tau}$ $T_0=T(0)$

$T(t)X(x)\big|_\Gamma\leq M$. 故 $\int_0^\infty v(\tau,x)d\tau<\infty$, i.e. $v(\infty,x)=0$.

故 $X''/X\leq 2$ 即 $X(x)\leq C_1e^{\sqrt{2}x}+C_2e^{-\sqrt{2}x}$ 故 $X(x)\big|_{\overline{Q_T}}\leq A\in\mathbb{R}$ $(\exists A)$

则 $u(t,x)\leq Ae^{\int_0^t v(\tau,x)d\tau}$

由 $u(t,x)\big|_\Gamma\leq M$. 取 $t=0$. 则 $A\geq M$. 又有 $v(t,x)\leq 2$ 恒成立.

故 $u(t,x)\leq Me^{2t}$

5. $\begin{cases} u_{tt}-u_{xx}=0 & (t,x)\in[0,T)\times(0,l). \\ u(0,x)=\varphi(x).\ u_t(0,x)=\psi(x) & \\ u(t,0)=0,\ u_x(t,l)+au(t,l)=0. & a\geq 0. \end{cases}$

令 $F(t)=\int_0^l(|u_t|^2+|u_x|^2)dx$

由能量守恒. 则 $\frac{dF(t)}{dt}=0$. 故 $\int_0^l|u_t|^2+|u_x|^2\,dx=A\in\mathbb{R}$. 故 $\exists F_0$. s.t. $\int_0^l|u_t|^2+|u_x|^2dx\leq F_0$

若不唯一, 不妨设 u_1,u_2 为对应的两解. 考虑 $u=u_1-u_2$.

则 $\begin{cases} u_{tt}-u_{xx}=0 \\ u(0,x)=0,\ u_t(0,x)=0 \\ u(t,0)=0.\ (u_x+au)\big|_{(t,l)}=0 \end{cases}$ $u_t=0$. 则 $u_{tt}=0\Rightarrow u_{xx}=0\Rightarrow u_x=c\in\mathbb{R}$.

$\Rightarrow u=cx+b$. 又 $u(t,0)=0\Rightarrow b=0$.

且 $(u_x+au)|_{(t,l)}=0 \Rightarrow a=0.$

故 $u\equiv 0$. i.e. $u_1=u_2$. 矛盾.

故解唯一.

6. $\begin{cases} u_{tt}-a^2u_{xx}=0 \\ u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x) \end{cases}$

$\Rightarrow u(t,x)=\frac{1}{2}(\varphi(x+at)+\varphi(x-at))+\frac{1}{2a}\int_{x-at}^{x+at}\psi(s)ds$

$e(t)=\frac{1}{2}\int_{x_0-a(t_0-t)}^{+\infty}(u_t^2+a^2u_x^2)dx \Rightarrow e'(t)=-\frac{a}{2}(u_t^2+a^2u_x^2)+\frac{1}{2}\int_{x_0-a(t_0-t)}^{+\infty}(2u_tu_{tt}+2a^2u_xu_{xt})dx$

其中 $\int_{C_t}u_xu_{xt}dx = u_xu_t|_{C_t}-\int u_tu_{xx}dx$

则有: $\int_{C_t}(2u_tu_{tt}+2a^2u_xu_{xt})dx=\int_{C_t}2u_t(u_{tt}-a^2u_{xx})dx+2a^2u_xu_t|_{C_t}=2a^2u_xu_t|_{C_t}$

从而: $e'(t)=-\frac{a}{2}(u_t^2+a^2u_x^2)+a^2u_xu_t=-\frac{a}{2}(u_t-au_x)^2\leq 0$

若 $\varphi=\psi=0$. 则 $u_t=\psi=0$ 则 $u(t,x)=f(x)$

又 $u_{xx}=\frac{1}{a^2}u_{tt}=0$. 则 $u_x=g(t)$ 故 $u(t,x)=g(t)\cdot x+B$ 故 $g(t)=A\in\mathbb{R}$.

从而 $u(t,x)=Ax+B$ 而 $u(0,x)=\varphi(x)=0$. 故 $Ax+B=0$, i.e. $u=0$.

故 $u\equiv 0$.

7. $\int_\Omega u^2dxdy\leq C(d)\int_\Omega(u_x^2+u_y^2)dxdy$

故 u 在 P 处取正极大. 则 $\nabla u|_P=0$. 而 $u^2>0$. 故不成立.

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