Preliminaries to Complex Analysis

luojunxun

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(C): $C \sim R^2 : algebra + geometry + analysis + topology$

there are multiply,addition,conjugation and mudule in C ;

$$z = x + iy = re^{i\theta} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

algebra

the relationship between vector , complex and linear transformation:

$$z = x + iy \to \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : \quad i(x+iy) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} = -y + ix$$

$$r \to \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \quad e^{i\theta} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Generraly:$f : C \to Gl_2(R) \cup O; s.t. f(x+iy) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ C operate on itself.

topology

$$D_r(z_0) = \{z||z - z_0| < r\}; \overline{D_r}(z_0) = \{z||z - z_0| \leq r\}; C_r(z_0) = \{z||z - z_0| = r\}$$

$\Omega \subset C : diam(\Omega) = \sup\limits_{z,w \in \Omega} |z - w|$

1.interior point:$\exists r > 0, s.t.D_r(z_0) \subset \Omega$

2.open set:every point is interior point.

3.closed set:$C - \Omega$ is open.

4.limit point:$\exists \{z_n\}_{n=1}^{\infty}, z_n \neq z_O, \lim\limits_{n \to \infty} z_n = z_0$

5.closure of Ω is the union of Ω and its limit points

6.the boundry $\partial\Omega = \{z|\forall r > 0, D_r(z) \cap \Omega \neq \emptyset; D_r(z) \cap (C - \Omega) \neq \emptyset\}$

7.Ω closed $\iff \Omega = \overline{\Omega}$

8.compact = closed + boundry:$\begin{cases} 1.for\ every\ sequence\{z_n\}, there\ exists\ a\ subsequence\{z_{n_k}\}\ that \\ \ converges\ to\ a\ point\ in\ \Omega \\ 2.every\ open\ covering\ has\ a\ finite\ subcovering \end{cases}$

9.$\Omega_1 \subset \Omega_2 \cdots \Omega_n \subset \cdots, diam(\Omega_n) \to 0; \to \exists! z \in \omega_n, \forall n$

10.connected:it is impossiable to find two disjoint non-empty open set Ω_1 and Ω_2 $s.t.\Omega_1 \cup \Omega_2 = \Omega$

11.path connected:for any two points in Ω,there is a path connected them.

12.a connected open set is called region.

analysis

Continues Function f:$\Omega \to C$;

Complex Differentiable Function = Holompic Function

$f\ is\ Holompic \iff \lim\limits_{h \to 0} \frac{f(z+h)-f(z)}{h} coverges, called\ f'(z)$

for complex function , the First-order Differentiablility $\iff$ n-order Differentiable.

Theorem 1.1: *f is Holompic iff f is real-Differentiable and maintain C-R-Eq*

(Def): $F = f : \Omega \subset C \to C, f(z) = f(x+iy) = F(x,y) = u(x,y)+iv(x,y) = (u(x,y), v(x,y))^T$

$P_0(x_0, y_0), H = (h_1, h_2) = h_1 + ih_2$

then F is said to be differrentiable at P_0 if there exists a linear transformation

$J : R^2 \to R^2;$

$s.t. \lim\limits_{|H|\to 0} \frac{|F(P_0+H)-F(P_0)-JH|}{|H|\to 0} = 0 \iff F(P_0+H) - F(P_0) = JH + H\psi(H); \psi(H) = o(H)$

then J is called the derivative of F at P_0 ,if F is differrentiable , the partial derivative of u and c exists , the J is the Jacobian matrix:$J = J_F = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$

Chuchy-Riemann-Equationa

$$\text{assume f=F is Holompic ,and then } \frac{\partial f}{\partial z} \xlongequal{h=h_1+h_2} \begin{cases} \lim\limits_{h_1\to 0} \frac{f(x_0+h_1,Y_0)-f(x_0,y_0)}{h_1} & h_2 = 0 \\ \lim\limits_{h_2\to 0} \frac{f(x_0,Y_0+h_2)-f(x_0,y_0)}{h_2} & h_1 = 0 \end{cases}$$

$$\text{thus } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; and \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$define : \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

$$\text{If f is Holompic at } P_0 \Rightarrow \frac{\partial f}{\partial \bar{z}} = 0 \; and \; f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z} \; and \; \det J_f(P_0) = |f'(z_0)|^2$$

$$\text{with C-R-Eq , } J_F(x_0, y_0) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f_x(x_0, y_0)$$