1. $\Gamma^{-1}(s)=e^{\gamma s}s\prod_{n=1}^{\infty}(1+\frac{s}{n})e^{-\frac{s}{n}}$ 其中 $\gamma=\lim_{n\to\infty}(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n)$.

故 $\Gamma(s)=\lim_{n\to\infty}e^{-(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n)s}\frac{1}{s}\prod_{k=1}^{n}\frac{k}{s+k}e^{\frac{s}{k}}$

$=\lim_{n\to\infty}\frac{n!}{s(s+1)\cdots(s+n)}e^{-(1+\frac{1}{2}+\cdots+\frac{1}{n})s+s+\frac{s}{2}+\cdots+\frac{s}{n}+s\ln n}$

$=\lim_{n\to\infty}\frac{n^s n!}{s(s+1)\cdots(s+n)}$

2. 由 1. $\Gamma(s)=\lim_{n\to\infty}\frac{n^s n!}{s(s+1)\cdots(s+n)}=\lim_{n\to\infty}\frac{1}{s}\cdot n^s\prod_{k=1}^{n}\frac{k}{(s+k)}$

则 $\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}=\lim_{n\to\infty}\frac{a+b+1}{(a+1)(b+1)}\cdot\prod_{k=2}^{n}\frac{k}{k-1}\cdot\prod_{k=1}^{n}\frac{k}{a+1+k}\cdot\frac{k}{b+1+k}\cdot\frac{a+b+1+k}{k}$

$=\lim_{n\to\infty}\prod_{k=2}^{n}\frac{(a+b+k)k}{(a+k)(b+k)}\cdot\frac{n(a+b+n+1)}{(a+n+1)(b+n+1)}\cdot\frac{b+a+1}{(a+1)(b+1)}$

$=\lim_{n\to\infty}\prod_{k=1}^{n+1}\frac{(a+b+k)k}{(a+k)(b+k)}=\prod_{n=1}^{\infty}\frac{n(a+b+n)}{(a+n)(b+n)}$.

故 $\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}=\prod_{n=1}^{\infty}\frac{n(a+b+n)}{(a+n)(b+n)}$.

3. $\Gamma(\frac{1}{2})=\sqrt{\pi}$. 故 $\sqrt{\frac{\pi}{2}}=\Gamma(\frac{1}{2})/\sqrt{2}$. 由(1).

$\sqrt{\frac{\pi}{2}}=\lim_{n\to\infty}\frac{n^{\frac{1}{2}}\cdot n!}{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n+1}{2}}\frac{1}{\sqrt{2}}=\lim_{n\to\infty}=\lim_{n\to\infty}\frac{n^{\frac{1}{2}}\,n!\,2^{n+1}\cdot(2n)!!}{(2n+1)!!\,(2n)!!}=\lim_{n\to\infty}\frac{\sqrt{2n}\,(n!)^2\,2^{2n}}{(2n+1)!}$

$=\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}\sqrt{2n+1}$ 往证 $\Gamma(s)\Gamma(s+\frac{1}{2})=\sqrt{\pi}\,2^{1-2s}\Gamma(2s)$.

$\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+\frac{1}{2})}=\lim\frac{n^{2s}n!}{2s(2s+1)\cdots(2s+n)}\lim\frac{s(s+1)\cdots(s+n)\cdot(s+\frac{1}{2})(s+\frac{3}{2})\cdots(s+\frac{1}{2}+n)}{n^s n!\;n^{\frac{1}{2}+s}n!}$

$=\lim\frac{(2n+1)^{2s}(2n+1)!}{2s\cdots(2s+2n+1)}\frac{s(s+1)\cdots(s+n)\cdot(s+\frac{1}{2})\cdots(s+\frac{1}{2}+n)}{n^s n!\;n^{\frac{1}{2}+s}n!}$

$=\lim\frac{2s\cdots(2s+2n+1)}{2^{2n+2}}\cdot\frac{(2n+1)^{2s}(2n+1)!}{n^{2s+1/2}(n!)^2\,2\,s(2s+1)\cdots(2s+2n+1)}=\lim\left(\frac{2n+1}{n}\right)^{2s}\sqrt{\frac{2n+1}{n}}\frac{(2n+1)!}{2^{2n+2}(n!)^2\sqrt{2n+1}}$

$=2^{2s}\frac{\sqrt{2}}{4}\sqrt{\frac{\pi}{2}}=2^{2s-1}\sqrt{\pi}$.

5. 令 $s=\frac{1}{2}+it$. $\Gamma(\frac{1}{2}+it)\Gamma(\frac{1}{2}-it)=\frac{\pi}{\sin\pi(\frac{1}{2}+it)}$

故 $|\Gamma(\frac{1}{2}+it)|^2=\left|\frac{\pi}{\sin\pi(\frac{1}{2}+it)}\right|$ $\quad \sin\pi z=\frac{e^{i\pi z}-e^{-i\pi z}}{2i}$

故 $\left|\frac{\pi}{\sin\pi(\frac{1}{2}+it)}\right|=\left|\frac{2\pi i}{e^{i\pi(\frac{1}{2}+it)}-e^{-i\pi(\frac{1}{2}+it)}}\right|=\left|\frac{2\pi}{e^{-\pi t}+e^{\pi t}}\right|=\frac{2\pi}{e^{-\pi t}+e^{\pi t}}$

因为 $e^{-\pi t+\frac{\pi}{2}i}-e^{\pi t-\frac{\pi}{2}i}=e^{-\pi t}\cos\frac{\pi}{2}+e^{-\pi t}\sin\frac{\pi}{2}-e^{\pi t}\cos\frac{\pi}{2}+e^{\pi t}\sin\frac{\pi}{2}=e^{-\pi t}+e^{\pi t}$.

故 $|\Gamma(\frac{1}{2}+it)|=\sqrt{\frac{2\pi}{e^{-\pi t}+e^{\pi t}}}$.

7. (a) $\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt$

$\Gamma(\alpha)\Gamma(\beta)=\int_0^\infty e^{-x}x^{\alpha-1}dx\int_0^\infty e^{-y}y^{\beta-1}dt=\int_0^\infty\int_0^\infty e^{-(x+y)}x^{\alpha-1}y^{\beta-1}dydx$

令 $u=x+y$. $v=\frac{x}{x+y}$. 则 $\begin{cases}x=uv\\ y=u-uv\end{cases}$

$\Gamma(\alpha)\Gamma(\beta)=\int_0^\infty du\int_0^1 e^{-u}\cdot u\cdot(uv)^{\alpha-1}u^{\beta-1}(1-v)^{\beta-1}dv$

$=\int_0^\infty u^{\alpha+\beta-1}e^{-u}du\int_0^1 v^{\alpha-1}(1-v)^{\beta-1}dv=\Gamma(\alpha+\beta)B(\alpha,\beta)$

(b). $B(\alpha,\beta)=\int_0^1(1-t)^{\alpha-1}t^{\beta-1}dt=\int_\infty^0\left(\frac{u}{u+1}\right)^{\alpha-1}\left(\frac{1}{u+1}\right)^{\beta-1}\frac{-1}{(u+1)^2}du$

$=\int_0^\infty\frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du$.

8.

9. $\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}=\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt$

$=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left(1+\sum_{n=1}^{\infty}\frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-n+1)}{n!}(-zt)^n\right)dt$

$=1+\sum_{n=1}^{\infty}\frac{\Gamma(\gamma)\cdot\beta\cdot(\beta+1)\cdots(\beta+n-1)\Gamma(\beta)}{\Gamma(\beta)\cdot\gamma\cdot(\gamma+1)\cdots(\gamma+n-1)\cdot\Gamma(\gamma)}\frac{\alpha\cdot(\alpha+1)\cdots(\alpha+n-1)}{n!}z^n$

$$= 1 + \sum_{n=1}^{\infty} \frac{\Gamma(\gamma) \cdot \beta \cdot (\beta+1) \cdots (\beta+n-1)\Gamma(\beta)}{\Gamma(\beta) \gamma \cdot (\gamma+1) \cdots (\gamma+n-1) \cdot \Gamma(\gamma)} \quad \frac{\alpha \cdot (\alpha+1) \cdots (\alpha+n-1)}{n!} z^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1) \cdots (\alpha+n-1) \cdot \beta(\beta+1) \cdots (\beta+n-1)}{n! \gamma(\gamma+1) \cdots (\gamma+n-1)} z^n = F(\alpha, \beta, \gamma, z)$$