1. 解： $EX_n = E(A\cos\lambda n + B\sin\lambda n) = \cos\lambda n \cdot EA + \sin\lambda n \cdot EB = 0+0=0$;

$$Cov(X_n, X_m) = E[(X_n - EX_n)(X_m - EX_m)] = E(X_n X_m)$$

$$= E[A^2\cos\lambda n\cdot\cos\lambda m + AB(\cos\lambda n\cdot\sin\lambda m + \sin\lambda n\cdot\cos\lambda m) + B^2\sin\lambda n\cdot\sin\lambda m]$$

$$= E(A^2)\cdot\cos\lambda n\cdot\cos\lambda m + E(AB)(\cos\lambda n\cdot\sin\lambda m + \sin\lambda n\cdot\cos\lambda m) + E(B^2)\cdot\sin\lambda n\cdot\sin\lambda m$$

$$= \cos\lambda n\cdot\cos\lambda m + \sin\lambda n\cdot\sin\lambda m$$

$$= \cos(\lambda n - \lambda m).$$

2. 解： $EX_n = 0$;

$$Cov(X_n, X_m) = E[(X_n - EX_n)(X_m - EX_m)] = E(X_n X_m)$$

$$= \sum_{i=0}^{r}\sum_{j=0}^{r}\alpha_i\alpha_j E(Z_{n-i}Z_{m-j}) = \begin{cases} \sum\limits_{i=0}^{r-|n-m|}\alpha_i\alpha_{i+|n-m|}, 若|n-m|<r+1; \\ 0, 其它. \end{cases}$$

3.证明：
$$\begin{aligned} X_n &= \alpha X_{n-1} + Z_n \\ &= \alpha(\alpha X_{n-2} + Z_{n-1}) + Z_n \\ &= \alpha^2 X_{n-2} + \alpha Z_{n-1} + Z_n \\ &= \cdots \\ &= \sum_{i=0}^{\infty}\alpha^i Z_{n-i} \end{aligned}$$

不妨先假设$m \ge 0$，$Cov(X_n, X_{n+m})$

$$= Cov(\sum_{i=0}^{\infty}\alpha^i Z_{n-i}, \sum_{j=0}^{\infty}\alpha^j Z_{m+n-j})$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} Cov(Z_{n-i}, Z_{n+m-j})\cdot\alpha^i\cdot\alpha^j$$

令$n-i=n+m-j$， $\Rightarrow\ j=m+i\ge 0$

所以$Cov(X_n, X_{n+m}) = \sum_{i=0}^{\infty}\alpha^i\alpha^{m+i}\cdot 1 = \alpha^m\sum_{i=0}^{\infty}(\alpha^2)^i = \frac{\alpha^m}{1-\alpha^2}$.

若$m \le 0$，则$Cov(X_n, X_{n+m}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} Cov(Z_{n-i}, Z_{n+m-j})\cdot\alpha^i\cdot\alpha^j$

令$n-i=n+m-j$， $\Rightarrow i = j-m \ge 0$

所以$Cov(X_n, X_{n+m}) = \sum_{j=0}^{\infty}\alpha^{j-m}\alpha^j\cdot 1 = \alpha^{-m}\sum_{j=0}^{\infty}(\alpha^2)^j = \frac{\alpha^{-m}}{1-\alpha^2}$

综上，$Cov(X_n, X_{n+m}) = \frac{\alpha^{|m|}}{1-\alpha^2}$.

4. 解：**引理**：随机向量服从多元正太分布的**充要条件**是它的各分量的任意线性组合服从正太分布.

任给$t_1,t_2,\cdots,t_n \in R$，由上述引理可知$(X(t_1),X(t_2),\cdots,X(t_n))$服从多元正太分布.

下面计算上述正太分布的均值向量μ及协方差矩阵Σ：

由于$EX(t)=E(Y\cos\theta t+Z\sin\theta t)=EY\cdot\cos\theta t+EZ\cdot\sin\theta t=0$,

$$Cov(X(t_i),X(t_j))=\cos((t_i-t_j)\theta),$$

可得：$\mu=[0,0,\cdots 0]^T$，$\Sigma=[\cos((t_i-t_j)\theta)]_{ij},1\le i,j\le k$.

进而，随机向量$(X(t_1),X(t_2),\cdots,X(t_n))$的密度函数为：

$$f_{(X(t_1),X(t_2),\cdots,X(t_n))}(x_1,x_2,\cdots,x_n)=\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}e^{-\frac{1}{2}X^T\Sigma^{-1}X},\text{其中}X=(x_1,x_2,\cdots,x_n).$$