一、(a): $P(A|B)=P(A|\bar{B}) \Leftrightarrow P(AB)/P(B) = P(A\bar{B})/P(\bar{B}) \Leftrightarrow P(AB)(1-P(B))=P(B)(P(A)-P(AB))$

$\Leftrightarrow P(AB)=P(A)P(B) \Leftrightarrow$ A.B独立.

(b):

$\frac{9}{16}=P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(ABC)$

$= 3P(A) - 3P^2(A)$

$\Rightarrow P(A) = \frac{1}{4}$ 或 $\frac{3}{4}$(舍).

故 $P(A)=\frac{1}{4}$

2. (a): $P(S_5=1) = C_5^3(\frac{1}{2})^3(\frac{1}{2})^2 = \frac{5}{16}$

(b): $P(S_5=1 | S_2=0) = P(S_5=1, S_2=0)/P(S_2=0) = \frac{C_2^1(\frac{1}{2})(\frac{1}{2})\cdot C_3^2(\frac{1}{2})^2(\frac{1}{2})}{C_2^1(\frac{1}{2})(\frac{1}{2})} = \frac{3}{8}$

(c): $P(S_2=0|S_5=1)P(S_5=1) = P(S_5=1|S_2=0)P(S_2=0)$.

i.e. $P(S_2=0|S_5=1) = P(S_2=0)\frac{P(S_5=1|S_2=0)}{P(S_5=1)} = \frac{1}{2}\times\frac{\frac{3}{8}}{\frac{5}{16}} = \frac{3}{5}$

3. $f(x,y)=\frac{e^{-y}}{y}e^{-\frac{x}{y}} \quad (x,y)\in(0,+\infty)^2$

(a): $E(X|Y=y) = \int_0^{+\infty} x f(x,y)dx = \int_0^{+\infty}\frac{e^{-y}}{y}\cdot xe^{-\frac{x}{y}}dx = ye^{-y}$

(b): $P_Y(y)=\int_0^{+\infty} f(x,y)\,dx = e^{-y}$

故 $EX=\int E(X|Y=y)P_Y(y)dy = \int_0^{+\infty} ye^{-2y}dy = \frac{1}{4}$

4. $U=\min\{X,Y\}$. $V=X-Y$.

(a): $P(U=k) = P(X=k, Y>k) + P(X>k, Y=k) + P(X=Y=k)$

$= 2pq^{k-1}\sum_{j=k+1}^{\infty} pq^{j-1} + p^2q^{2k-2} = p^2q^{2k-2}\frac{1+q}{1-q}$

$P(V=k) = \sum_j P(Y=j, X=j+k)$

$k\geq0$时 $P(V=k) = \sum_{j=1}^{\infty}P(Y=j, X=j+k) = \sum_{j=1}^{\infty} pq^{j-1}\cdot pq^{j+k-1} = \frac{p^2q^k}{1-q^2} = \frac{pq^k}{1+q}$

故 $P(V=k) = \begin{cases} \frac{pq^k}{1+q} & k\geq 0 \\ \frac{pq^{-k}}{1+q} & k<0. \end{cases}$

(b) 显然不独立. 若 $V>0$ 则 $U=Y$ 若 $V<0$ 则 $U=X$. 故不独立.

5. $\mathrm{Var}X=16$. $\mathrm{Var}Y=9$ $\mathrm{Cov}(X,Y)=-6$.

$\rho = \frac{\mathrm{Cov}(3X+2, 2Y-1)}{}$

$$\rho_{3X+2,2Y-1} = \frac{\quad}{\sqrt{Var(3X+2)}\sqrt{Var(2Y-1)}}$$

$Var(3X+2) = 9Var X = 144 \qquad Var(2Y-1) = 4Var Y = 36.$

$-6 = Cov(X,Y) = EXY - EXEY$

$Cov(3X+2, 2Y-1) = E(3X+2)(2Y-1) - E(3X+2)E(2Y-1)$

$= 6EXY - 3EX + 4EY - 2 - 6EXEY + 3EX - 4EY + 2$

$= 6(EXY - EXEY) = -36$

故 $\rho_{3X+2,2Y-1} = \frac{-36}{12\times 6} = -\frac{1}{2}$

6. 令数 $\xi \sim B(600, \frac{1}{6})$. 即求 $P(|\frac{\xi}{600} - \frac{1}{6}| \le 0.02) = P(88 \le \xi \le 112)$. $E\xi = 100$

(1) 令 $\eta = \frac{\xi}{600}$. 则 $E\eta = \frac{1}{6}$. $Var\eta = \frac{1}{600^2}\cdot 600\times\frac{1}{6}\times\frac{5}{6} = \frac{1}{4320}$

故 $P(|\eta - E\eta| > 0.02) < \frac{Var\eta}{0.02^2} = \frac{125}{216}$

故 $P(|\eta - E\eta| \le 0.02) \ge \frac{91}{216}$

(2). $P = P(\frac{112-100}{\sqrt{250/3}}) - P(\frac{88-100}{\sqrt{250/2}}) \approx \Phi(1.31) - \Phi(-1.31) \approx 0.905 - (1-0.905) = 0.81$

7. $\xi_1 \sim U(-1,1)$. $\xi_n \sim N(0, 2^{n-1})$ $n \ge 2$.

$ES_n = E\sum_{k=1}^{n}\xi_k = \sum_{k=1}^{n}E\xi_k = 0$. $Var\xi_1 = \int_{-1}^{1}\frac{1}{2}x^2dx = \frac{1}{3}$

$Var S_n = \sum_{k=1}^{n} Var\xi_k = \frac{1}{3} + \sum_{k=2}^{n} 2^{k-1} = 2^n - \frac{5}{3} \qquad X_n = \frac{S_n}{\sqrt{2^n - \frac{5}{3}}}$

$f_{\xi_1}(t) = Ee^{it\xi_1} = \int_{-1}^{1}\frac{1}{2}e^{itx}dx = \frac{e^{it} - e^{-it}}{2it} = \frac{\sin t}{t}$

$f_{\xi_k}(t) = Ee^{it\xi_k} = e^{-2^{k-2}t^2}$

故 $f_{X_n}(t) = \prod_{k=1}^{n} f_{\xi_k}(\frac{t}{\sqrt{2^n - \frac{5}{3}}}) = \frac{\sin t/\sqrt{B_n}}{t/\sqrt{B_n}}\cdot e^{-\sum_{k=0}^{n-2}2^k\frac{t^2}{2^n-\frac{5}{3}}} = \frac{\sqrt{B_n}}{t}\sin\frac{t}{\sqrt{B_n}}\cdot \exp\{-\frac{2^{n-1}-1}{2^n-\frac{5}{3}}t^2\}$

当 $n\to\infty$. $\sqrt{B_n}\to\infty$. 故 $\sin\frac{t}{\sqrt{B_n}} / (t/\sqrt{B_n}) = 1$. $\frac{2^{n-1}-1}{2^n-\frac{5}{3}} = \frac{1}{2}$.

故 $f_{X_n}(t) \to \exp\{-\frac{1}{2}t^2\}$ 就正是 $N(0,1)$ 的特征函数.

故 $X_n \xrightarrow{d} N(0,1)$ $n\to\infty$.