一、

(1) $\exists \varepsilon>0$, s.t. $\forall N>0$, $\exists n>N$. s.t. $P(A_n)>\varepsilon$.

$\forall m>0$, 取一 A_{i_1} s.t. $P(A_{i_1})>\varepsilon$. 再取 A_{i_2}, $i_2>i_1$ s.t. $P(A_{i_2})>\varepsilon$.

…… 得到 $\{A_{i_n}\}_{n=1}^{\infty}$ s.t. $P(A_{i_n})>\varepsilon$. 则 $P(\bigcup_{n=m}^{\infty} A_n)$

二、$S_n=\sum_{k=1}^{n}\sin^2 kU$, $U\sim U(0,2\pi)$.

(1) $ES_n=\int_0^{2\pi}\frac{1}{2\pi}\sum_{k=1}^{n}\sin^2 kx\,dx=\frac{1}{2\pi}\sum_{k=1}^{n}\int_0^{2\pi}\sin^2 kx\,dx=\frac{1}{2}\sum_{k=1}^{n}1=\frac{n}{2}$

(2): 令 $\xi_k=\sin^2 kU$, $U\sim U(0,2\pi)$. 则 $E\xi_k=\int_0^{2\pi}\frac{1}{2\pi}\sin^2 kx\,dx=\frac{1}{2}<\infty$

故由辛钦大数定律 $\frac{S_n}{n}\xrightarrow{P}\frac{1}{2}$. $c=\frac{1}{2}$.

三、$X,Y\sim N(0,0,1,1,r)$.

(1): 只需令 $Cov(Y,X-aY)=0$. i.e. $EY(X-aY)=EY\,E(X-aY)$.

$LHS=EY(X-aY)=EXY-aEY^2$

其中 由 $Cov(X,Y)=r=EXY-EXEY=EXY$ 知 $EXY=r$. 而 $EY^2=E(Y-EY)^2=VarY=1$

故 $LHS=r-a$ 而 $RHS=0$.

故令 $a=r$. 则 Y, $X-aY$ 不相关

(2): $r_{X^2,Y^2}=\frac{Cov(X^2,Y^2)}{\sqrt{VarX^2}\sqrt{VarY^2}}$

$VarX^2=E(X^2-EX^2)^2=E(X^2-1)^2=EX^4-2EX^2+1=EX^4-1$.

其中 $EX^4=\int_R\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cdot x^4dx=3$ 故 $VarX^2=2$. 同理 $VarY^2=2$.

$Cov(X^2,Y^2)=EX^2Y^2-EX^2EY^2=E(XY)^2-1$

$p(x,y)=\frac{1}{2\pi\sqrt{1-r^2}}\exp\{-\frac{1}{2(1-r^2)}\times(x^2-2rxy+y^2)\}$

$EX^2Y^2=\iint_{R^2}p(x,y)x^2y^2dxdy=\frac{1}{2\pi\sqrt{1-r^2}}\int_R\int_R x^2y^2e^{-\frac{x^2-2rxy+y^2}{2(1-r^2)}}dxdy$

其中 $I=\int_R\int_R x^2y^2e^{-\frac{x^2-2rxy+y^2}{2(1-r^2)}}dxdy=\int_R y^2e^{-\frac{1}{2}y^2}\left(\int_R x^2e^{-\frac{(x-ry)^2}{2(1-r^2)}}dx\right)dy$

$=\int_R y^2e^{-\frac{1}{2}y^2}(1-r^2+r^2y^2)\sqrt{2(1-r^2)\pi}\,dx$

$=\sqrt{2(1-r^2)\pi}\,(1+2r^2)\sqrt{2\pi}$

故 $EX^2Y^2=1+2r^2$ 从而 $Cov(X^2,Y^2)=2r^2$

故 $r_{X^2,Y^2}=\frac{Cov(X^2,Y^2)}{\sqrt{VarX^2\,VarY^2}}=r^2$

四: $p(x,y)=C(x-y)^2e^{-\frac{1}{2}(x^2+y^2)}$

$1=\iint p(x,y)dxdy=C\int_0^{2\pi}\int_0^{+\infty}r^3(\cos\theta-\sin\theta)^2e^{-\frac{1}{2}r^2}dr\,d\theta$

(1): $1=\iint_{R^2} p(x,y)\,dxdy = c\int_0^{2\pi}\int_0^{+\infty} r(\cos\theta-\sin\theta)^2 e^{-\frac{1}{2}r^2}\,drd\theta$

$2\times\frac{1}{2} r^2 e^{-\frac{1}{2}r^2} d\frac{1}{2}r^2$

$= c\int_0^{2\pi}(1-\sin 2\theta)\,d\theta\int_0^{+\infty} 2te^{-t}\,dt = 4\pi c.$

故 $c=\frac{1}{4\pi}$

(2): $p_X(x)=\int_R p(x,y)\,dy = \frac{1}{4\pi}e^{-\frac{1}{2}x^2}\left(x^2\int_R e^{-\frac{1}{2}y^2}dy - 2x\int_R ye^{-\frac{1}{2}y^2}dy + \int_R y^2 e^{-\frac{1}{2}y^2}dy\right)$

$= \frac{1}{4\pi}e^{-\frac{1}{2}x^2}(\sqrt{2\pi}x^2+\sqrt{2\pi}) = \frac{x^2+1}{2\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$

$EX=\int_R x\,p_X(x)\,dx = \frac{1}{2\sqrt{2\pi}}\int_R (x^3e^{-\frac{1}{2}x^2}+xe^{-\frac{1}{2}x^2})\,dx = 0.$ 同理 $EY=0.$

$\mathrm{Var}X=\int_R x^2 p_X(x)\,dx = \frac{1}{2\sqrt{2\pi}}\int_R (x^4e^{-\frac{1}{2}x^2}+x^2e^{-\frac{1}{2}x^2})\,dx = 2$ 同理 $\mathrm{Var}Y=2$

故 $EX=EY=0.$ $\mathrm{Var}X=\mathrm{Var}Y=2$

(3): 只需证 $E(X-Y)(X+Y)=E(X-Y)E(X+Y).$

$LHS=E(X^2-Y^2)=EX^2-EY^2=0.$

$RHS=E(X-Y)E(X+Y)=0\times 0=0.$ 故 $LHS=RHS.$

即 $X-Y$ 与 $X+Y$ 不相关.

(4): 作坐标变换 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$ 则 $p_{r,\theta}(r,\theta)=\frac{1}{4\pi}r^3(1-\sin 2\theta)e^{-\frac{1}{2}r^2}$

故 $p_r(r)=\int_0^{2\pi}p_{r,\theta}\,d\theta = \frac{1}{2}r^3e^{-\frac{1}{2}r^2}$

故 $p_{X^2+Y^2}(x)=\frac{1}{2}x^{\frac{3}{2}}e^{-\frac{1}{2}x}$

五: 设 $P(X=x_i)=p_i^x$ $\quad P(Y=y_i)=p_i^y.$

则 $EX=\sum_{i=1}^{\infty}p_i^x x_i.$ $EY=\sum_{i=1}^{\infty}p_i^y y_i.$ 而 $EXY=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}p_i^x x_i p_j^y y_j.$

显然有 $EXEY=EXY.$

六、(1): $P(\eta_n=\frac{1}{n}\sum_{k=1}^{s}\xi_k)=\frac{n^s}{s!}e^{-n}$

$f_{\xi_k}(t)=Ee^{it\xi_k}=\int_R \lambda e^{-\lambda x}e^{itx}\,dx=\frac{\lambda}{\lambda-it}$

故 $f_{\eta_n}(t)=\sum_{s=0}^{\infty}\frac{n^s}{s!}e^{-n}Ee^{it\frac{1}{n}\sum_{k=1}^{s}\xi_k} = \sum_{s=0}^{\infty}\frac{n^s}{s!}e^{-n}\prod_{k=1}^{s}f_{\xi_k}(\frac{t}{n})$

$=\sum_{s=0}^{\infty}\frac{n^s}{s!}e^{-n}\left(\frac{\lambda n}{\lambda n-it}\right)^s = e^{-n}\sum_{s=0}^{\infty}\frac{\left(\frac{\lambda n^2}{\lambda n-it}\right)^s}{s!} = e^{-n}e^{\frac{\lambda n^2}{\lambda n-it}} = e^{\frac{itn}{\lambda n-it}}$

故 $f_{\eta_n}(t)=\exp\{\frac{itn}{\lambda n-it}\}$ η_n为随机变量.

(2). $E\eta_n=-if'(0)=\frac{1}{\lambda}$ $E\eta_n^2=-f''(0)=(1+\frac{2}{n})\frac{1}{\lambda^2}$ 故 $Var\eta_n=\frac{2}{n\lambda^2}$

令 $\xi\sim N(0,b)$. 则 $f_\xi(t)=e^{-\frac{b}{2}t^2}$ $f_{\eta_n}(t)=e^{\frac{itn}{\lambda n-it}}$

$$Ee^{it\sqrt{n}(\eta_n-a)}=e^{-ita}Ee^{it\sqrt{n}\eta_n}=e^{-ita}\exp\{\frac{it\sqrt{n}n}{\lambda n-it\sqrt{n}}\}$$

$$=e^{-ita\sqrt{n}}\exp\{\frac{itn}{\lambda\sqrt{n}-it}\}=\exp\{\frac{itn(1-a\lambda)-a\sqrt{n}t^2}{\lambda\sqrt{n}-it}\}$$

当 $a=\frac{1}{\lambda}$ 时 有极限 $\exp\{-\frac{a}{\lambda}t^2\}$.

故 $a=\frac{1}{\lambda}$. $b=\frac{2}{\lambda^2}$

设 $f(\xi)$ 与 分布F时. 直接证 $Ee^{itf(\xi)}$ 收敛到 F

若. $\eta\sim N(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,r)$

$$\Rightarrow p(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}\exp\{-\frac{1}{2(1-r^2)}(\frac{(x-\mu_1)^2}{\sigma_1^2}-2r\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2})\}.$$