1. (1) $\begin{cases} y > x \\ 1-x^2-y^2 > 0 \end{cases} \Rightarrow (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \Rightarrow y > x \in (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

(2) $x \in (0,+\infty)$. $y \in (-\infty,+\infty)$. $z \in (0,+\infty)$

(3) $r \le x^2+y^2+z^2 \le R$

(4) $\begin{cases} z \le x^2+y^2 \\ x^2+y^2 \ne 0 \end{cases}$

2. $f(\frac{y}{x}) = \frac{(\frac{x}{y})^3}{[(\frac{x}{y})^2+1]^{\frac{3}{2}}} \Rightarrow f(x) = \frac{\frac{1}{x^3}}{(\frac{1}{x^2}+1)^{\frac{3}{2}}} = \frac{1}{(x^2+1)^{\frac{3}{2}}}$ 即 $f(x) = (x^2+1)^{-\frac{3}{2}}$

3. $z(x,4) = 2 + f(\sqrt{x}-1) = x+1$

$\Rightarrow f(\sqrt{x}-1) = x-1 = \sqrt{x}^2-1 = (\sqrt{x}-1)(\sqrt{x}-1+2) \Rightarrow f(x) = x(x+2) = x^2+2x$

$\Rightarrow z(x,y) = \sqrt{y} + (\sqrt{x}-1)(\sqrt{x}+1) = x+\sqrt{y}-1$

即 $f(x) = x^2+2x$ $\quad z(x,y) = x+\sqrt{y}-1$

4. (1): $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0}\frac{x}{x} = 1 \ne \lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y) = \frac{-y}{y} = -1$ 不∃

(2) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2} = \lim\limits_{t\to 1}\frac{1}{1+t^2} = \frac{1}{2}$

(3) $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y) = 0 \ne \lim\limits_{x\to 0}\lim\limits_{y\to 0} = 1$ ，不存在

(4) $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim\limits_{t\to 0}\frac{t^6}{t^4+t^8} = \lim\limits_{t\to 0}\frac{t^2}{1+t^4} \to 0$

5.

6.

7.

(1) $\lim\limits_{(x,y)\to(0,1)} \dfrac{1-xy}{x^2+y^2} = 1$

(2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{1+x^2+y^2}{x^2+y^2} = \lim\limits_{t\to 0} \dfrac{1+t}{t}$ 不存在.

(3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{1+xy}-1}{xy} = \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{xy(\sqrt{1+xy}+1)} = \dfrac{1}{2}$

(4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{1+x^2+y^2}-1} = \lim\limits_{(x,y)\to(0,0)} \dfrac{(x^2+y^2)(\sqrt{1+x^2+y^2}+1)}{x^2+y^2} = 2$

(5) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\ln(x^2+e^{y^2})}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \dfrac{(x^2+y^2+1)}{x^2+y^2} = 1$

(6) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^3+y^3)}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2} = 0$

(7) $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2} = 2\lim\limits_{(x,y)\to(0,0)} \dfrac{\left[\dfrac{x^2+y^2}{2}\right]^2}{(x^2+y^2)x^2y^2} = \dfrac{1}{2}\lim\limits_{(x,y)\to(0,0)} \dfrac{(x^2+y^2)}{x^2y^2} = 0$

(8) $\lim\limits_{\substack{x\to+\infty\\ y\to+\infty}} \dfrac{x^2+y^2}{e^{x+y}} = 0,$

8. (1) $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0} \dfrac{2x^2}{2x^2+2} = 0 = \lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$

令 $x=r\cos\theta$, $y=r\sin\theta$, $r\to 0^+$ $\theta\in[0,2\pi)$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2y^2+(x-y)^2} = \lim\limits_{r\to 0} \dfrac{r^4\sin^2\theta\cos^2\theta}{r^4\sin^2\theta\cos^2\theta+r^2(\sin\theta-\cos\theta)^2} = \lim\limits_{r\to 0} \dfrac{r^2\sin^2 2\theta}{r^2\sin^2 2\theta+4-4\sin 2\theta}$

$= \lim\limits_{\varepsilon\to 0} \dfrac{\varepsilon}{4-4\sin 2\theta}$ 不存在. 二重不存在. 二次相同为 0

(2) $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0} (1+x^2) = 1 \neq \lim\limits_{y\to 0}\lim\limits_{x\to 0} -(1+y^2) = -1$

令 $x=r\cos\theta$, $y=r\sin\theta$, $r\to 0$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{r\to 0} (r^2+1)\cos 2\theta = \cos 2\theta$ 不存在.

二重极限不存在 二次极限为 1 与 -1.

(3). $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ 不存在. 同理 $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ 不存在

$|f(x,y)| < |x| + |y| \to 0$ 故二重极限为 0. 二次极限不存在.

9. $\lim\limits_{\substack{x\to 0\\ y=x^2\to 0}} f(x,y) = 1$ $\lim\limits_{\substack{x\to 0\\ y=\frac{1}{2}x^2\to 0}} f(x,y) = +\infty$. $\lim\limits_{\substack{x\to 0\\ y=2x^2\to 0}} f(x,y) = +\infty$. 故 $f(x,y)$ 在 $(0,0)$ 不连续

在 $\{(x,y) \mid x>0 \wedge \frac{1}{2}x^2 < y < 2x^2\}$ 显然连续.
$\wedge x>0$
考虑点集 $\{(x,y) \mid y=\frac{1}{2}x^2$ 或 $y=x^2$ 或 $y=2x^2\}$ 下的连续性.

① 当 $y=\frac{1}{2}x^2$ $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = 0$ 符合.

② 同上 $y=2x^2$ 时符合.

③ 当 $y=x^2$ 时

$$\lim_{(x,x^2)} f_1(x,y) = 1 = \lim_{(x,x^2)} f_2(x,y) = 1 \text{ 符合.}$$

故 f 在除 $(0,0)$ 以外的点连续.

10. 显然 f 在 $x\neq 0, y\neq 0$ 连续.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{x\to 0\\ y=kx\to 0}} \frac{k^2x^3}{(k^2+1)x^2} = \lim_{x\to 0} \frac{k^2}{k^2+1} x = 0$$

故 f 在 $R\times R$ 上连续

11. 令 $x = c + r\cos\theta$ $y = c + r\sin\theta$. $r\to 0$.

则 $f'(c) = \dfrac{f(c+r\cos\theta) - f(c+r\sin\theta)}{r\cos\theta - r\sin\theta}$ $r\to 0$.

故 $\lim\limits_{(x,y)\to(c,c)} f(x,y) = \dfrac{f'(c)\cdot r(\cos\theta - \sin\theta)}{r(\cos\theta - \sin\theta)} = f'(c) = F(c,c)$

故 $\forall c \in (a,b)$ 成立. $\lim\limits_{(x,y)\to(c,c)} F(x,y) = f'(c)$

12. 对给定的 x_0，$\forall \varepsilon>0\ \exists \delta=\frac{\varepsilon}{L}>0$

s.t. $\forall y_1, y_2\ |y_1-y_2|<\delta$，有 $|f(x_0,y_1)-f(x_0,y_2)|\le L\cdot\frac{\varepsilon}{L}=\varepsilon$.

故 f 对 y 一致连续，故 f 对 y 连续，又 f 对 x 连续，由 x_0 的任意性，

f 在 D 上连续.

13. 只需证明对 $f: R\to R$，$g: R\to R$，$f+g$，$f\cdot g$ 在 f, g 连续条件下连续即可.

f, g 连续，则 $\forall \varepsilon>0\ \exists\delta>0\ \forall |x_1-x_2|<\delta$ $|f(x_1)-f(x_2)|<\frac{\varepsilon}{2}$，$|g(x_1)-g(x_2)|<\frac{\varepsilon}{2}$

① $|f(x_1)+g(x_1)-[f(x_2)+g(x_2)]|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon$. 符合 $(f+g)$

② $|f(x_1)g(x_1)-f(x_2)g(x_2)|=|f(x_1)g(x_1)-f(x_2)(g(x_2)+g(x_1)-g(x_1))|$

$=|(f(x_1)-f(x_2))g(x_1)-f(x_2)(g(x_2)-g(x_1))|\le\frac{\varepsilon}{2}|g(x_1)|+\frac{\varepsilon}{2}|f(x_2)|<\varepsilon_0$

故 $f+g$，$f\cdot g$ 一致连续，故连续.

故 $f+g$ 于 $\langle f, g\rangle$ 连续.

14. $f: A\to B$ $g: C\to D$ $B\subseteq C$.

$\forall x_0\in A$ $f(x_0)\in B\subset C$ $\forall\varepsilon>0\ \exists\delta_1, \delta_2>0$ s.t. $\forall x\in O(x_0,\delta_1)$

$f|_{O(x_0,\delta_1)}\mapsto O(f(x_0),\delta_2)$ $g|_{O(f(x_0),\delta_2)}\mapsto O(g(f(x_0)),\varepsilon)$

故 $f\cdot g$ 连续.