多项式插值 2

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Theorem 定理:: 对任意的 x, 若 $x \neq x_i, i=0,1,2,\cdots,n$, 则

$$
\begin{aligned}
f(x)= & f\left(x_0\right)+f\left[x_0, x_1\right]\left(x-x_0\right) \\
& +f\left[x_0, x_1, x_2\right]\left(x-x_0\right)\left(x-x_1\right)+\cdots \\
& +f\left[x_0, x_1, \cdots, x_n\right]\left(x-x_0\right)\left(x-x_1\right) \cdots\left(x-x_{n-1}\right) \\
& +f\left[x_0, x_1, \cdots, x_n, x\right]\left(x-x_0\right)\left(x-x_1\right) \cdots\left(x-x_{n-1}\right)\left(x-x_n\right)
\end{aligned}
$$

Hermite 插值

问题: 给定函数 $f(x)$ 在节点 $x_0, x_1, \cdots, x_n$ 处的函数值及一阶导数值, 求 $2n+1$ 次多项式 $H_{2n+1}(x)$ 满足条件

$$
H_{2n+1}\left(x_i\right)=f\left(x_i\right), \quad H_{2n+1}^{\prime}\left(x_i\right)=f^{\prime}\left(x_i\right), \quad i=0,1,\cdots,n.
$$

求解:$H_{2n+1}(x)=\sum_{i=0}^n y_i A_i(x)+\sum_{i=0}^n y_i^{\prime} B_i(x)$(其中 A,B 是待定的次数不超过 2n+1 次的多项式)

$$
A_i\left(x_j\right)=\delta_{ij}, \quad A_i^{\prime}\left(x_j\right)=0, \quad i,j=0,1,\cdots,n
$$

并且有 $B_i\left(x_j\right)=0, \quad B_i^{\prime}\left(x_j\right)=\delta_{ij}, \quad i,j=0,1,\cdots,n.$

解得:

$$B_i(x)=\frac{(x-x_0)^2(x-x_1)^2\cdots(x-x_{i-1})^2(x-x_i)(x-x_{i+1})^2\cdots(x-x_n)^2}{(x_i-x_0)^2(x_i-x_1)^2\cdots(x_i-x_{i-1})^2(x_i-x_{i+1})^2\cdots(x_i-x_n)^2}$$

$$=\frac{\omega_n^2(x)}{(x-x_i)\left[\omega_n'(x_i)\right]^2}=(x-x_i)\,l_i^2(x).$$

$$A_i(x)=\left(1-2(x-x_i)\,l_i'(x_i)\right)\frac{\omega_n^2(x)}{(x-x_i)^2\left[\omega_n'(x_i)\right]^2}$$

$$=\left(1-2(x-x_i)\,l_i'(x_i)\right)l_i^2(x).$$

最后得到:

$$H_{2n+1}(x)=\sum_{i=0}^{n}\left(y_i\left(1-2(x-x_i)\sum_{\substack{j=0\\j\neq i}}^{n}\frac{1}{x_i-x_j}\right)+y_i'(x-x_i)\right)\prod_{\substack{j=0\\j\neq i}}^{n}\left(\frac{x-x_j}{x_i-x_j}\right)^2$$

$$=\sum_{i=0}^{n}\left(y_i+(x_i-x)\left(2y_i\sum_{\substack{j=0\\j\neq i}}^{n}\frac{1}{x_i-x_j}-y_i'\right)\right)\left(\prod_{\substack{j=0\\j\neq i}}^{n}\frac{x-x_j}{x_i-x_j}\right)^2$$

误差估计: 定理: 设 $x_0,x_1,\cdots,x_n$ 是区间 $[a,b]$ 上的 $n+1$ 个互不相同的点, $f(x)\in C^{2n+2}[a,b]$, 且 $f(x_i)=y_i$, $f'(x_i)=y_i'(i=0,1,\cdots,n)$, $H_{2n+1}(x)$ 是 Hermite 插值多项式. 则对每个 $x\in[a,b]$, 存在 $\xi\in(a,b)$, 使得

$$f(x)-H_{2n+1}(x)=\frac{f^{(2n+2)}(\xi)}{(2n+2)!}\omega_n^2(x).$$

$$\omega_n(x)=(x-x_0)(x-x_1)\cdots(x-x_n).$$