P151.

1. $f_x(x,y) = \cos x\cos y$ $f_y(x,y) = -\sin x\sin y$

$f(x+\Delta x, y+\Delta y) = f(x,y) + (\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y})f(x,y) = f(x,y) + f_x(x+\theta\Delta x, y+\theta\Delta y)\cdot\Delta x$
$+ f_y(x+\theta\Delta x, y+\theta\Delta y)\Delta y$

令 $x=y=0$. $\Delta x = \frac{\pi}{3}$ $\Delta y = \frac{\pi}{6}$

$\Rightarrow f(\frac{\pi}{3}, \frac{\pi}{6}) = \frac{3}{4} = \frac{\pi}{3}\cos\frac{\pi}{3}\theta\cos\frac{\pi}{6}\theta - \frac{\pi}{6}\sin\frac{\pi}{3}\theta\sin\frac{\pi}{6}\theta$. 证毕.

2. 由f是多项式函数, 其展开为其自身

$f(x,y) = 3[(x-1)+1]^3 + [(y-2)+2]^3 - 2[(x-1)^2+1][(y-2)+1] - 2[(x-1)+1][(y-2)+2] - 8[(x-1)+1]$
$- 8[(y-2)+2] + 9$

$= 3(x-1)^3 + (y-2)^3 - 2(x-1)^2(y-2) - 2(x-1)(y-2)^2 - 12(x-1)(y-2) + 5(x-1)^2 + 4(y-2)^2$
$- 13(x-1) - 6(y-2) - 14$

5. $f_x(1,0) = f_x(x,y)|_{(1,0)} = -\frac{\cos y}{x^2}|_{(1,0)} = -1$

(1) $f_y(1,0) = f_y(x,y)|_{(1,0)} = -\frac{\sin y}{x}|_{(1,0)} = 0$

$f_{xx}(1,0) = f_{xx}(x,y)|_{(1,0)} = \frac{2\cos y}{x^3}|_{(1,0)} = 2$

$f_{yy}(1,0) = f_{yy}(x,y)|_{(1,0)} = -\frac{\cos y}{x}|_{(1,0)} = -1$

$f_{xy}(1,0) = f_{xy}(x,y)|_{(1,0)} = \frac{\sin y}{x^2}|_{(1,0)} = 0$

$f_{xxx}(x,y) = \frac{-6\cos y}{x^4}$ $f_{yyy}(x,y) = \frac{\sin y}{x}$ $f_{xxy}(x,y) = \frac{-2\sin y}{x^3}$ $f_{xyy}(x,y) = \frac{\cos y}{x^2}$

$f(x,y) = 1 - (x-1) + 2(x-1)^2 - y^2 + R_2$

$R_2 = \frac{1}{3!}(\frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial y}\Delta y)f(1+\theta(x-1), \theta y)$

$= \frac{1}{6}(f_{xxx}(1+\theta(x-1), \theta y)(x-1)^3 + f_{yyy}(1+\theta(x-1), \theta y)\cdot y^3 + 3f_{xxy}(1+\theta(x-1), \theta y)\cdot(x-1)^2 y$
$+ 3f_{xyy}(1+\theta(x-1), \theta y)\cdot(x-1)y^2)$

$= \frac{1}{6}(\frac{-6\cos\lambda}{u^4}\cdot(x-1)^3 + \frac{\sin\lambda}{u}y^3 - \frac{3\sin\lambda}{u^3}(x-1)^2 y + 3\frac{\cos\lambda}{u^2}(x-1)y^2)$

$R_2 = -\frac{\cos\lambda}{u^4}(x-1)^3 + \frac{\sin\lambda}{6u}y^3 - \frac{\sin\lambda}{2u^3}(x-1)^2 y + \frac{\cos\lambda}{2u^2}(x-1)y^2$.

$\lambda = 1+\theta(x-1)$ $u = \theta y$

(2).

$$f(x,y)=\sum_{i=1}^{k}\frac{1}{i!}\left(\frac{\partial}{\partial x}(x-1)+\frac{\partial}{\partial y}\right)^{i}f\Big|_{(1,0)}+R_k$$

$$=\sum_{i=1}^{k}\frac{1}{i!}\sum\frac{\partial^{l}f}{\partial x^{l}}(1,0)\cdot(x-1)^{l}\frac{\partial^{i-l}f}{\partial y^{i-l}}(1,0)y^{i-l}+R_k.$$

$$R_k=\frac{1}{(k+1)!}\left(\frac{\partial}{\partial x}(x-1)+\frac{\partial}{\partial y}y\right)^{k+1}f(1+\theta(x-1),\theta y)$$

$$\lim_{(x,y)\to(1,0)}\frac{(\cos y)^n}{|x|^n}<\lim_{(x,y)\to(1,0)}\frac{1}{|x|^n}=1.$$ 即 f 的偏导的绝对值小于1（在 $(1,0)$ 处）.

$\forall\varepsilon>0$. $\exists\delta=\sqrt[k+1]{\varepsilon(k+1)!}$ $\quad R_k<\frac{1}{(k+1)!}\cdot\delta^{k+1}\cdot\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{k+1}f(\lambda,\mu)<\varepsilon$.

故当 $k\to\infty$ 时. $R_k\to0$.

7. 设 $f(0,0)=C$. 再任取 $P_0(x_0,y_0)\in\mathbb{R}^2$. 设 $\vec{OP}=\vec{l}$

由于 f 在 $\mathbb{R}^2$ 上可微. 且 $\vec{e_1}$ 与 $\vec{e_2}$ 不共线.

$$\operatorname{grad}f=\frac{\partial f}{\partial x}(x,y)\vec{i}+\frac{\partial f}{\partial y}(x,y)\vec{j}=\left(\frac{\partial f}{\partial l_1}(x,y),\ \frac{\partial f}{\partial l_2}(x,y)\right)\begin{pmatrix}e_1\\e_2\end{pmatrix}=0.$$

故 f 在 $O((0,0),\delta)$ 邻域内增长率为0. 故 $\forall P\in O((0,0),\delta)$. $f(P)\equiv f(0,0)=C$.

取 $P_1=O((0,0),\delta_1)\cap l$ $\quad P_2=[O(P_1,\delta)\setminus O((0,0),\delta_2)]\cap l\cdots$ 取 P_i 为原点.

依次下去. 并重复上述过程. 得到 $f(x_0,y_0)=f(0,0)=C$.

由于 P_0 是任意的. 故 $f(x,y)\equiv C$ 为常数.