1. $\begin{cases} u_t - u_x + au = 0 & x \in \mathbb{R}. \\ u(0,x) = x^2 \end{cases}$

设 $x(t) = -t + \alpha$. $\alpha = x(0)$.

则 $\dfrac{du(t,x(t))}{dt} = u_t - u_x = -au(t,x(t))$

$\Rightarrow u(t,x(t)) = u(0,\alpha) e^{-at} = \alpha^2 e^{-at}$.

故 $u(t,x) = (x+t)^2 e^{-at}$

$\mathcal{F}\left[\dfrac{x}{4+x^2}\right](\lambda) = \int_{\mathbb{R}} \dfrac{x}{4+x^2} e^{-i\lambda x} dx = \int_{\mathbb{R}}$

$f(z) = \dfrac{z}{4+z^2} e^{-i\lambda z}$ 考虑 $f(z)$ 在

上积分

$\operatorname{Res}_{2i} f(z) = \int_{\mathbb{R}} \dfrac{x}{4+x^2} e^{-i\lambda x} dx + \int_0^\pi \dfrac{Re^{i\theta}}{4+R^2e^{2i\theta}} iRe^{i\theta} e^{-i\lambda Re^{i\theta}} d\theta$.

$\left|\int_0^\pi \dfrac{Re^{i\theta} iRe^{i\theta}}{4+R^2e^{2i\theta}} e^{-i\lambda Re^{i\theta}} d\theta\right| \le \int_0^\pi \left|\dfrac{R^2 e^{-\lambda R}}{4+R^2}\right| d\theta \to 0$.

$\operatorname{Res}_{2i} f(z) = \lim_{z\to 2i} (z-2i)\dfrac{z}{4+z^2} e^{-i\lambda z} = \lim_{z\to 2i} \dfrac{z}{z+2i} e^{-i\lambda z} = \dfrac{1}{2} e^{2\lambda}$

故 $\mathcal{F}\left[\dfrac{x}{4+x^2}\right](\lambda) = \dfrac{1}{2} e^{2\lambda}$

2. $\begin{cases} u_{tt} - u_{xx} = \sin wt. & (t,x) \in (0,+\infty) \times (0,1) \\ u(0,x) = \sin \pi x \quad u_t(0,x) = 0 & x \in (0,1) \\ u(t,0) = u(t,1) = 0. \end{cases}$

$u(t,x) = \sin \dfrac{k\pi}{L} x \sum_{k=1}^{\infty} \int_0^t B_k(\tau) \sin \dfrac{k\pi c}{L}(t-\tau) d\tau$ 其中. $L = c = 1$.

$B_k(\tau) = \dfrac{2}{k\pi c} \int_0^L f(\tau,\xi) \sin \dfrac{k\pi}{L} \xi \, d\xi$

$\Rightarrow B_k(\tau) = \dfrac{2}{k\pi} \int_0^1 \sin w\tau \sin k\pi\xi \, d\xi = \dfrac{2\sin w\tau}{(k\pi)^2} \cdot [1 - (-1)^k]$

$\Rightarrow u(t,x) = \sin k\pi x \sum_{k=1}^{\infty} \int_0^t \dfrac{2(1-(-1)^k)}{(k\pi)^2} \sin w\tau \sin k\pi(t-\tau) d\tau$

其中. $I_k = \int_0^t \sin w\tau \sin k\pi(t-\tau) d\tau = -\dfrac{1}{w} \cos w\tau \sin k\pi(t-\tau)\Big|_0^t - \dfrac{k\pi}{w} \int_0^t \cos w\tau \cos k\pi(t-\tau) d\tau$

$= \dfrac{1}{w} \sin k\pi t - \dfrac{k\pi}{w^2} \sin w\tau \cos k\pi(t-\tau)\Big|_0^t + \left(\dfrac{k\pi}{w}\right)^2 \int_0^t \sin w\tau \sin k\pi(t-\tau) d\tau$

$= \dfrac{1}{w} \sin k\pi t - \dfrac{k\pi}{w^2} \sin wt + \left(\dfrac{k\pi}{w}\right)^2 I_k$.

$\Rightarrow I_k = \dfrac{w}{w^2 - k^2\pi^2} \sin k\pi t - \dfrac{k\pi}{w^2 - k^2\pi^2} \sin wt$

故 $u(t,x) = \sin k\pi x \sum_{k=1}^{\infty} \dfrac{2[1-(-1)^k]}{w^2 - k^2\pi^2} (w \sin k\pi t - k\pi \sin wt)$

3. $\begin{cases} u_{tt} - \Delta_x u = f(t,x) \\ u(0,x) = \varphi(x). \quad u_t(0,x) = \psi(x). \\ u|_{\partial\Omega} = 0. \end{cases}$

令 $E(t) = \int_\Omega |u_t|^2 + |\nabla u|^2 dx.$

$\frac{dE(t)}{dt} = \int_\Omega |2u_t u_{tt}| + 2\sum_{i=1}^n |u_{x_i t} u_{x_i}| dx \le \int_\Omega |u_t|^2 dx + \int_\Omega f^2 dx$

$\le E(t) + \int_\Omega f^2 dx$

$\Rightarrow \frac{d}{dt}[e^{-t}E(t)] \le e^{-t}\int_\Omega f^2 dx \Rightarrow E(t) \le e^T[E(0) + \int_0^t e^{-s}\int_\Omega f^2 ds dx]$

$\Rightarrow E(t) \le e^T[E(0) + \int_0^T \int_\Omega f^2 dx d\tau]$

$E(0) = \int_\Omega |\psi|^2 + |\nabla\varphi|^2 dx$

故 $E(t) \le e^T\{\int_\Omega |\psi|^2 + |\nabla\varphi|^2 dx + \int_0^T\int_\Omega |f|^2 dx d\tau\}$. 同取 sup 有.

$\sup_{0\le t\le T} \int_\Omega |u_t|^2 + |\nabla u|^2 dx \le \sup_{0\le t\le T} e^T\{\int_\Omega |\psi|^2 + |\nabla\varphi|^2 dx + \int_0^T \int |f|^2 dx d\tau\}.$

4. ① 极值原理

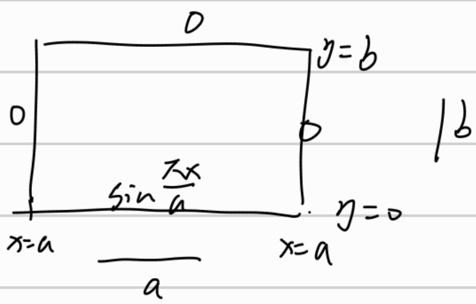
② $u = u_1 - u_2$ 由 $E(t)|_{\Omega_t} \le E|_{\Omega_0} = 0.$

因为 $g(t,x) = f(t) = 0$ $\quad u_t = 0$ 故唯一.

5. ✓ $\boxed{\Delta = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}}$

6. ✓

7. $\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u(a,y) = 0 \\ u(x,0) = \sin\frac{\pi x}{a}, \ u(x,b) = 0 \end{cases}$



令 $u(x,y) = X(x)Y(y) \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$. 当 $\lambda \le 0$ 时. 只有0解. 故 $\lambda > 0$.

$X = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$

由 $u(0,y) = 0 \Rightarrow B = 0$

由 $u(a,y) = 0 \Rightarrow \sin\sqrt{\lambda}a = 0 \Rightarrow \lambda = (\frac{k\pi}{a})^2 \quad k \ge 1.$

本征方程 $X_k(x) = \sin\frac{k\pi}{a}x$. $\Rightarrow$ $y_k(y) = C_k e^{\frac{k\pi}{a}y} + D_k e^{-\frac{k\pi}{a}y}$

$\Rightarrow u(x,y) = \sum_{k=1}^{\infty} \sin\frac{k\pi}{a}x \cdot \left(C_k e^{\frac{k\pi}{a}y} + D_k e^{-\frac{k\pi}{a}y} \right)$

$u(x,0) = \sin\frac{\pi x}{a} \Rightarrow \sum_{k=1}^{\infty} \sin\frac{k\pi}{a}x \, (C_k + D_k) = \sin\frac{\pi x}{a}$

$\Rightarrow C_1 + D_1 = 1 \qquad C_k, D_k = 0. \ k \geq 2.$

$u(x,b) = \sin\frac{\pi x}{a}\left(C_1 e^{\frac{\pi b}{a}} + D_1 e^{-\frac{\pi b}{a}} \right) = 0$

$\Rightarrow C_1 e^{\frac{2\pi b}{a}} + D_1 = 0$

$\Rightarrow C_1 = 1/\left(1 - e^{\frac{2\pi b}{a}}\right) \quad D_1 = \dfrac{-e^{\frac{2\pi b}{a}}}{1 - e^{\frac{2\pi b}{a}}}$

$$\Rightarrow u(x,y) = \frac{\sin\frac{\pi x}{a}}{1 - e^{\frac{2\pi b}{a}}} e^{\frac{\pi y}{a}} - \frac{e^{\frac{2\pi b}{a}}}{1 - e^{\frac{2\pi b}{a}}} \sin\frac{\pi x}{a} \, e^{-\frac{\pi y}{a}}$$