P189.

$2x^4-3x^2$

1\. (1) $f(x,y)=x^4+y^4-x^2-2xy-y^2$

$f_x(x,y)=4x^3-2x-2y$ $f_y(x,y)=4y^3-2x-2y$ $f_{xy}(x,y)=-2$

$f_{xx}(x,y)=12x^2-2$ $f_{yy}(x,y)=12y^2-2$

$$\begin{cases} f_x(x,y)=0 \\ f_y(x,y)=0 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=1 \end{cases} \begin{cases} x=-1 \\ y=-1 \end{cases} \begin{cases} x=0 \\ y=0. \end{cases}$$

$$H(x,y)=\begin{vmatrix} 12x^2-2 & -2 \\ -2 & 12y^2-2 \end{vmatrix} \quad \det H=4[(6x^2-1)(6y^2-1)-1]$$

$\det H(0,0)=0$ 当 $(x,y)\to 0$. $f(x,x)=x^2(2x^2-3)$ $f(x,-x)=2x^4$

故 $(0,0)$ 不是极值点.

$\det H(1,1)=\cancel{4\cdot 2}\ 4>0$ $(12x^2-2)|_{(1,1)}=10>0$. 故 $(1,1)$ 是极小值点.

$f_{极小}(x,y)=f(1,1)=-2$

$\det H(-1,-1)=\cancel{4\cdot 2}\ 4>0$ $(12x^2-2)|_{(-1,-1)}=10>0$ 故 $(-1,-1)$ 是极小值点.

$f_{极小}(x,y)=f(-1,-1)=-2$

综上. f 极小值 $=-2$

(6) $f_x=1-\frac{y}{x^2}$ $f_y=\frac{1}{x}-\frac{z}{y^2}$ $f_z=\frac{1}{y}-\frac{2}{z^2}$

$f_{xx}=\frac{2y}{x^3}$ $f_{xy}=-\frac{1}{x^2}$ $f_{xz}=0$ $f_{yy}=\frac{2z}{y^3}$ $f_{yz}=-\frac{1}{y^2}$ $f_{zz}=\frac{4}{z^3}$

$$\det H(x,y,z)=\det\begin{pmatrix} \frac{2y}{x^3} & -\frac{1}{x^2} & 0 \\ -\frac{1}{x^2} & \frac{2z}{y^3} & -\frac{z}{y^2} \\ 0 & -\frac{z}{y^2} & \frac{4}{z^3} \end{pmatrix}=\frac{2}{x^3}\left(\frac{8}{y^2z^2}-\frac{z^2}{y^3}-\frac{2}{xz^3}\right)$$

$$\begin{cases} f_x=0 \\ f_y=0 \\ f_z=0 \end{cases} \Rightarrow \begin{cases} x=2^{\frac{1}{4}} \\ y=2^{\frac{1}{2}} \\ z=2^{\frac{3}{4}} \end{cases}$$

$\det H(2^{\frac{1}{4}},2^{\frac{1}{2}},2^{\frac{3}{4}})=2^{\frac{1}{4}}(\sqrt{2}-1-\frac{1}{2^{\frac{3}{4}}})>0$ $\frac{2y}{x^3}|_P>0$

f极小值 $= f(2^{\frac{1}{4}}, 2^{\frac{1}{2}}, 2^{\frac{3}{4}}) = 5\sqrt[4]{2}$

8. 令 $y = f(x)$

有：$x^2 + 2xf(x) + 2f^2(x) = 1$. 同时取 x 导数，

有 $2x + 2f(x) + 2xf'(x) + 4f(x)f'(x) = 0$

$\Rightarrow f'(x) = -\frac{x+f(x)}{x+2f(x)} = -1 + \frac{f(x)}{x+2f(x)}$　　（当 $x+2f(x) \neq 0$）.

①当 $x + 2f(x) = 0$ 则 $x + f(x) = 0 \Rightarrow x = y = 0$. 不符合.

② 令 $f'(x) = 0 \Rightarrow x + y = 0$

$\Rightarrow x = \pm 1$.

当 $x = -1$ 时 y极大 $= 1$　　$x = 1$ 时 y极小 $= -1$

13.

令 $f(x,y) = \ln y + y\ln x + \ln(1-x)$　　$0 < x < 1$, $0 < y < +\infty$

$f_x = \frac{y}{x} - \frac{1}{1-x}$　　$f_y = \frac{1}{y} + \ln x$　　$f_{xy} = \frac{1}{x}$

$f_{xx} = -\frac{y}{x^2} - \frac{1}{(1-x)^2}$　　$f_{yy} = -\frac{1}{y^2}$

$$\det H(x,y) = \det \begin{vmatrix} -\frac{y}{x^2} - \frac{1}{(1-x)^2} & \frac{1}{x} \\ \frac{1}{x} & -\frac{1}{y^2} \end{vmatrix} = \frac{1}{x^2 y} + \frac{1}{(1-x)^2 y^2} - \frac{1}{x^2}$$

$(x,y) \to (1, +\infty)$　$\det H(x,y) \to \frac{1}{x^2 y} \to 0^+$　而 $-\frac{y}{x^2} - \frac{1}{(1-x)^2} \to -\infty$

故 当 $x \to 1$, $y \to +\infty$, f 取最大值 且 $y = \frac{x}{1-x}$

$f < f|_{(1,+\infty)} = \lim\limits_{x \to 1} \frac{\ln x}{1-x} = -1$.

故 $f < -1$. 即 $y x^y (1-x) < e^{-1}$.

P201.

1.(2). $L(x,y,z,\lambda) = x - 2y + 2z + \lambda(x^2 + y^2 + z^2 - 1)$

$L_x = 1 + 2x\lambda = 0$　　$\lambda = \frac{3}{2}$　　$\lambda = -\frac{3}{2}$

$$\begin{cases} L_y = -2+2\lambda y = 0 \\ L_z = 2+2\lambda z = 0 \end{cases}$$ 且 $x^2+y^2+z^2=1 \Rightarrow \begin{cases} x=-\frac{1}{3} \\ y=\frac{2}{3} \\ z=-\frac{2}{3} \end{cases}$ 或 $\begin{cases} x=\frac{1}{3} \\ y=-\frac{2}{3} \\ z=\frac{2}{3} \end{cases}$

$f(-\frac{1}{3},\frac{2}{3},-\frac{2}{3})=-3 \quad f(\frac{1}{3},-\frac{2}{3},\frac{2}{3})=3$

故 $f_{极大}=3 \quad f_{极小}=-3$

6. $d(x.y.z)=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$

$d|_\pi = L(x.y.z,\lambda)=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}+\lambda(Ax+By+Cz-D)$

$$\begin{cases} L_x = \frac{x-a}{\rho}+\lambda A = 0 \\ L_y = \frac{y-b}{\rho}+\lambda B = 0 \\ L_z = \frac{z-c}{\rho}+\lambda C = 0 \end{cases}$$ 且 $Ax+By+Cz=D$

$\Rightarrow \lambda = \frac{Aa+Bb+Cc-D}{\rho(A^2+B^2+C^2)}$

$\Rightarrow d_{min}(x.y.z)=|\lambda\rho\sqrt{A^2+B^2+C^2}|=\frac{|Aa+Bb+Cc-D|}{\sqrt{A^2+B^2+C^2}}$

9. $L(x.y.z.\lambda)=\ln x+2\ln y+3\ln z+\lambda(x^2+y^2+z^2-6R^2)$

$$\begin{cases} L_x = \frac{1}{x}+2\lambda x = 0 \\ L_y = \frac{2}{y}+2\lambda y = 0 \\ L_z = \frac{3}{z}+2\lambda z = 0 \end{cases}$$ 且 $x^2+y^2+z^2=6R^2$

$\Rightarrow \lambda = -\frac{1}{2R^2} \Rightarrow f_{max}=\ln 6\sqrt{3}R^6$

有 $ab^2c^3 \le 6\sqrt{3}R^6 = 6\sqrt{3}(\frac{a^2+b^2+c^2}{6})^3$

$\Rightarrow \sqrt{a}\,b\,c^{\frac{3}{2}} \le 6\sqrt{3}(\frac{a+b+c}{6})^3$

$\Rightarrow ab^2c^3 \le 108(\frac{a+b+c}{6})^6.$

11. 相切. $\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \\ (x-1)^2+y^2=1 \end{cases}$

$\Rightarrow \left(\frac{1}{a^2}-\frac{1}{b^2}\right)x^2+\frac{2}{b^2}\cdot x-1=0$

$\Rightarrow \delta=\frac{4}{b^4}+\frac{4}{a^2}-\frac{4}{b^2}=0 \Rightarrow \frac{1}{a^2}=\frac{1}{b^2}-\frac{1}{b^4} \Rightarrow a^2+b^4=a^2b^2$

$f(a,b,\lambda)=\pi ab-\lambda(a^2+b^4-a^2b^2)$

$\begin{cases} f_a=\pi b-(2a-2ab^2)\lambda=0 \\ f_b=\pi a-(4b^3-2a^2b)\lambda=0 \end{cases} \Rightarrow a=\sqrt{2}b^2 \Rightarrow \begin{cases} a=\frac{3}{2}\sqrt{2} \\ b=\frac{\sqrt{6}}{2} \end{cases}$ 时. f最小

$a=\frac{3}{2}\sqrt{2} \quad b=\frac{\sqrt{6}}{2}$