Chapter2-Cauchy Theorem and it's application

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Goursat's theorem

Theorem 1.1:*if Ω is an open set in C and $T \subset \Omega$ a triangle whose interior is also contained in Ω .then*

$$\int_T f(z)dz = 0$$

whenever f is holomorphic in Ω

Corollary 1.2:*If f is holomorphic in a open set Ω that contains a rectangle R and its interior .then*

$$\int_R f(z)dz = 0$$

Local existence of primitives and Cauchy's Theorem in a disc

Theorem 2.1:*A holomorphic function f in an open disc has a primitive in that disc*

Theorem 2.2 Cauchy's theorem for a disc:*if f is holomorphic in a disc then*

$$\int_\gamma f(z)dz = 0$$

for any closed curve γ in that disc

Corollary 2.3:*suppose f is holomorphic in an open set containing the circle C and its interior ,then*

$$\int_C f(z)dz = 0$$

Evaluation of some integrals

there are some common toy contours needed to be discuss

Cauchy's integral formula

Theorem 4.1:*suppose f is holomorphic in an open set that contains the closure of a disc D,if Cdenotes the boundary of D with the positive orientation, then*

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}d\zeta$$

for any point $z \in D$

f is holomorphic on $\overline{D}$!, $z \in D$!: To understand it, let's look at the proof (p46). In the proof, we calculate the integral $F(\zeta) = \frac{f(\zeta)}{\zeta - z}$ on the disc D and divide it into two parts. The first part is $\int_{F_{\delta,\epsilon}} F(\zeta)\,d\zeta = 0$ since $F_{\delta,\epsilon} \subset \Omega$ and F is holomorphic on Ω (except at z). The second part is $\int_{C_\epsilon} F(\zeta)\,d\zeta$, which evaluates to $-2\pi i f(z)$ since F is not holomorphic at z. We need to calculate this part because $C = F_{\delta,\epsilon} \cup C_\epsilon$, where $\delta, \epsilon \to 0$. Therefore, we have $0 = \oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta - 2\pi i f(z)$, which is what we need to prove.

In fact, the circle C can be replaced by any closed curve γ that contains z in Ω. Also, if we choose $z \in \Omega - \overline{D}$, then the integral will be zero since F is holomorphic on those points. That's why we choose z in D; otherwise, the conclusion is trivial.

Theorem 4.2: *If f is hol on an open set* Ω *, then f has infinitely many complex dirivatives in* Ω *, and for any circle* $C \subset \Omega$ *whose interior is also in* ω*, we have:*

$$f^n(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

for all z in the interior of C

Corollary 4.3 Cauchy's inequalities: *If f is hol in an open set that contains the closure of disc of D centered at z of radius of R, then*

$$|f^n(z)| \leq \frac{n!||f||_C}{R^n}$$

whenever $||f||_C = \sup_{z\in C}|f(z)|$ *denotes the supremum of* $|f|$ *on the boundary circle C*

Theorem 4.4: *Suppose f is holomorphic in an open set* Ω*. If D is a disc centered at* z_0 *and whose closure is contained in* Ω*, then f has a power series expansion at* z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$*, and the coefficients are given by*

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \text{for all } n \geq 0.$$

In fact there are not many restiction on D

Theorem 4.5 Lilouvile's theorem: *if f is entire and bounded, then f is constant*

Corollary 4.6: *Every non-constant polynomial* $P(z) = a_n z^n + \cdots + a_0$ *with the complex coefficients has a root in C*

Corollary 4.7: *Every polynomial* $P(z) = a_n z^n + \cdots + a_0$ *of degree* $n \geq 1$ *has precisely n roots in* $\mathbb{C}$*. If these roots are denoted by* $w_1, \ldots, w_n$*, then P can be factored as*

$$P(z) = a_n (z - w_1)(z - w_2) \cdots (z - w_n)$$

Theorem 4.8: *Suppose f is a holomorphic function in a region Ω that vanishes on a sequence of distinct points with a limit point in Ω. Then f is identically 0*

it's easy to underststand since that a function f is hol on Ω means f is continuous in Ω , then using the continuity to get the conclusion. In the proof we just need to consider the neighborhood of the point

Corollary 4.9: *Suppose f and g are holomorphic in a region Ω and f(z) = g(z) for all z in some non-empty open subset of Ω (or more generally for z in some sequence of distinct points with limit point in Ω). Then f(z) = g(z) throughout Ω*

Futher application

Theorem 5.1 Morera's theorem: *Suppose f is a continuous function in the open disc D such that for any triangle T contained in D*

$$\int_T f(z)dz = 0$$

then f is holomorphic

In fact if true, for any γ the integration is zero, that the inverse proposition of Cauchy' theorem

Theorem 5.2: *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.*

the key point is that the uniform-convergence in a compact set make the integral and limit can be exchanged for order:let's look at the impact of different convergent property(suppose f>0,$f_n \nearrow f_n$ converge to f in $D \overset{closed}{\subset} \Omega$):that's

$$\forall \epsilon > 0, \forall z \in \Omega, \exists N_z > 0, \forall n \geq N_z : \int_T (f(z) - f_n(z))dz < \epsilon; here : T \overset{triangle}{\subset} \Omega$$

then:

$$\int_T (f(z) - f_{N_z}(z))dz < \int_T \epsilon dz$$

there is nomore information we can get unless $f_n \Rightarrow f$,then $\exists N, \forall z \in D, N_z < N$ then

$$\lim_{n\to\infty} \int_T (f(z) - f_n(z))dz \le \int_T \epsilon dz < \epsilon_0$$

that's what we need(According to the arbitrariness of T and theorem 5.1 : f is hol)

Theorem 5.3: *Under the hypotheses of the previous theorem, the sequence of derivatives* $\{f_n\}$ *converges uniformly to f on every compact subset of Ω.*

this two theorem show that if every f_n is hol on Ω ,and the sum of these functions converge uniformly to F on Ω.then this series defines a holomorphic function F on Ω,which is $F(z) = \sum_{n=1}^{\infty} f_n(z)$

Theorem 5.4: *Let F(z, s) be defined for* $(z, s) \in \Omega \times [0, 1]$ *where Ω is an open set in C. Suppose F satisfies the following properties:*

(i):F(z, s) is holomorphic in z for each s. (ii):F is continuous on $\Omega \times [0, 1]$*.*

Then the function f defined on Ω by:

$$f(z) = \int_0^1 F(z, s)dz$$

is holomorphic

In fact, the properties(ii) means to show F(z,s) is continuous about s

proof of 5.4:In the proof we make a function sequence $f_n(z) = \frac{1}{n} \sum_{k=1}^{n} F(z, k/n)$,then $f_n \Rightarrow f, on\ D \overset{compact}{\subset} \Omega$,then use Thm5.2 ,f is hol

so if we need to proof a function f is hol on a set Ω,we can make a function sequence f_n which converges uniformly to f on any compact subset of Ω,then we can say f is hol on Ω

Theorem 5.5 Symmetry principle: *If* f^+ *and* f^+ *are holomorphic functions in* Ω^+ *and* Ω^-*respectively, that extend continuously to I and*

$$f^+(x) = f^+(x), \forall x \in I$$

then the function f defined on Ω by

$$f(z) = \begin{cases} f^+(z) & z \in \Omega^+ \\ f^-(z) & z \in \Omega^- \\ f^+(z) = f^-(z) & z \in I \end{cases}$$

is holomorphic on all of Ω

Theorem 5.6 Schwarz reflection principle:*Suppose that f is a holomorphic function in* Ω^+ *that extends continuously to I and such that f is real-valued on I. Then there exists a function F holomorphic in all of Ω such that* $F = f$ *on* Ω^+*.*

Proof:[Schwarz reflection principle]let $F(z) = \overline{f(\overline{z})}$ *on* Ω^- , which is holomorphic,then use the Symmetry principle

Theorem 5.7:*Any function holomorphic in a neighborhood of a compact set K can be approximated uniformly on K by rational functions whose singularities are in* K^c

If K^c *is connected, any function holomorphic in a neighborhood of K can be approximated uniformly on K by polynomials*

wikipedia about Runge's approximation theorem

Lemma 5.8:Suppose f **is holomorphic in an open set** Ω**, and** $K \subset \Omega$ **is compact. Then, there exists finitely many segments** $\gamma_1, \dots, \gamma_N$ **in** $\Omega - K$ **such that**

$$f(z) = \sum_{n=1}^{N} \frac{1}{2\pi i} \int_{\gamma_n} \frac{f(\zeta)}{\zeta - z} d\zeta \quad \textbf{for all } z \in K$$

Lemma 5.9:For any line segment γ **entirely contained in** $\Omega - K$**, there exists a sequence of rational functions with singularities on** γ **that approximate the integral** $\int_\gamma f(\zeta)/(\zeta - z) d\zeta$ **uniformly on** K**.**

Lemma 5.10:If K^c **is connected and** $z_0 \notin K$**, then the function** $1/(z - z_0)$ **can be approximated uniformly on K by polynomials.**

if $K \overset{compact}{\subset} \Omega$,there are rational function which can be used to approximated uniformly.if K^c is connected,then the rational function can be polynomials

In fact I didn't understand the proof clearly...