Cauchy'Theorem and Its Application

luojunxun

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Goursat'theorem

Theorem 1.1: *If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)dz = 0$$

whenever f is holomorphic in Ω.

Corollary 1.2: *If f is holomorphic in an open set Ω that contains a rectangle R and its interior, then*

$$\int_R f(z)dz = 0$$

Local existence of primitives and Cauchy's theorem ina disc

Theorem 2.1: *A holomorphic function in an open disc has a primitive in that disc.*

Corollary 2.2: *A holomorphic function f in an open disc Ω ,$\gamma \subset \Omega$ is closed,then:* $\int_\gamma f(z)dz = 0$

Theorem 2.2(Cauchy's theorem for a disc): *If f is holomorphic in a disc, then*

$$\int_\gamma f(z)dz = 0$$

for any closed curve γ in that disc.

. *Proof. Since f has a primitive, we can apply Corollary 3.3 of Chapter 1 .*

Corollary 2.3:*Suppose f is holomorphic in an open set containing the circle C and its interior. Then*

$$\int_C f(z)dz = 0$$

Evaluation of some integrals

1.$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx; \xi = 0 \Rightarrow 1 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx$

2.$\int_0^\infty \frac{1-\cos x}{x^2} dx = \frac{\pi}{2}$

Cauchy's integral formulas

Theorem 4.1:*Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then:*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta \quad \textit{for any point } z \in D$$

4.1:while $\zeta \in \Omega - \overline{D} : \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = 0$

Corollary 4.2: *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta$$

for all z in the interior of C.

Corollary 4.3(Cauchy inequalities): *If f is holomorphic in an open set that contains the* *closure of a disc D centered at z_0 and of radius R, then*

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!\|f\|_C}{R^n}$$

where $\|f\|_C = \sup_{z\in C}|f(z)|$ denotes the supremum of $|f|$ on the boundary circle C.

Theorem 4.4: *Suppose f is holomorphic in an open set Ω. If D is a disc centered at z_0 and whose closure is contained in Ω, then f has a power series expansion at z_0*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$, and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \textit{for all } n \geq 0.$$

Corollary 4.5 Liouville's theorem: *If f is entire and bounded, then f is constant.*