测度的平移不变性

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(E 关于 y 的平移):$E \subset R, y \in R : E_y = \{x + y | x \in E\}$ 称为 E 关于 y 的平移

Lemma 2.4.1:$E, F \subset R, \forall y \in R \Rightarrow$

$(i) : E \cap F_y = (E_{-y} \cap F)_y$

$(ii) : (E^c)_y = (E_y)^c$

$(iii) : m^*(E) = m^*(E_y)$

Proof:[2.4.1]$1 : z \in (E_{-y} \cap F)_y \iff z - y \in E_{-y}\ \&\ z \in F_y \iff (z - y) - (-y) = z \in E\ \&\ z - y \in F \iff z \in E \cap E_y$

$2.z \in (E^c)_y \iff z - y \in E^c \iff z - y \notin E \iff z \notin E_y \iff z \in (E_y)^c$

$3.\forall I\text{开区间} : l(I) = l(I_y) \Rightarrow E \subset \bigcup\limits_n I_n \Rightarrow E_y \subset \bigcup\limits_n (I_n)_y \Rightarrow m^*(E_y) \le \sum\limits_{n=1}^{\infty} l((I_n)_y) = \sum\limits_{n=1}^{\infty} l(I_n) = m^*(E) \Rightarrow m^*(E) = m^*((E_y)_{-y}) \le m^*(E_y) \Rightarrow m^*(E) = m^*(E_y)$

Theorem 2.4.1 测度平移不变性:E 可测, 那么 $\forall y \in R, E_y$ 可测并且有 $m(E) = m(E_y)$

Proof:[2.4.1]$\forall A \subset R : m^*(A) = m^*(A_{-y}) \overset{E\text{可测}}{\ge} m^*(A_{-y} \cap E) + m^*(A_{-y} \cap E^c) = m^*((A \cap E_y)_{-y}) + m^*((A \cap E_y^c)_{-y}) = m^*(A \cap E_y) + m^*(A \cap (E_y)^c)$ 因此 E_y 可测

不可测集案例

$\forall x \in [0,1] : E(x) = \{y \in [0,1] : y - x \in Q\}$

$(i) : [0,1] = \bigcup\{E(x) : x \in [0,1]\}\ \&\ (ii) : x_1 - x_2 \in Q \iff E(x_1) = E(x_2)\ \&\ (iii) : \forall x_1, x_2 \in [0,1] : E(x_1) = E(x_2)$或者$E(x_1) \cap E(x_2) = \emptyset\ \&\ (iv) : \exists F \subset [0,1] s.t. \forall x_1, x_2 \in F : x_1 \neq x_2 \iff E(x_1) \cap E(x_2) = \emptyset$

下面证明 F 不可测

$let\ \{r_n\}_{n=1}^{\infty} = [-1,1] \cap Q\ \&\ F_n = F_{r_n} = \{x + r_n : x \in F\} \rightleftarrows$

$(1) : \forall m \neq n, F_m \cap F_n = \emptyset \quad since\ if\ \exists z \in F_m \cap F_n \Rightarrow \exists x_m, x_n \in F\ s.t. x_m + r_m = x_n + r_n \Rightarrow x_m - x_n = r_n - r_m \in Q \Rightarrow E(x_m) = E(x_n)$矛盾 $i.e. \{F_n\}_{n\geq 1}$互不相交

$(2) : [0,1] \subset \bigcup\limits_n F_n \subset [-1,2]$ 后者是显然的, 对于前者, 任取 $y \in [0,1], \exists x \in F\ s.t. y \in E(x) \Rightarrow y - x \in Q, let\ r_k = y - x \Rightarrow y \in F_k\ i.e. [0,1] \subset \bigcup\limits_n F_n$

假设 F 可测, 由 $thm2.4.1 : F_n$可测并且$m(F_n) = m(F)$, 由可数可加性 :

$1 = m([0,1]) \leq m(\bigcup\limits_n F_n) = \sum\limits_{n=1}^{\infty} m(F_n) \leq m([-1,2]) = 3\ i.e. \leq 1 \leq \sum\limits_{n=1}^{\infty} m(F) \leq 3$ 若 $m(F) = 0$ 则和为零, 否则 $m(F)$ 严格大于零, 从而级数发散到无穷大. 两者都矛盾

思考题:R^n 中不可测集的构造和不可测集的构造机理

下一章节的: 用开集和闭集刻画可测集

Theorem 2.5.1:E可测 $\iff$

$\forall \epsilon > 0, \exists G \overset{open}{\supset} E\ s.t. m^*(G - E) < \epsilon \iff \forall \epsilon > 0, \exists F \overset{closed}{\subset} E\ s.t. m^*(E - F) < \epsilon$