1. $\begin{cases} u_t + \sum_{i=1}^{n} b_i u_{x_i} = f(t,x) & (t,x)\in(0,+\infty)\times R^n \\ u(0,x)=g(x) & \forall x\in R^n \end{cases}$

构造特征方程组.

$\frac{dx_i}{dt}=b_j \Rightarrow x_j = x_j^0 + b_j t$ 则 $\frac{du(t,x(t))}{dt} = u_t + \sum_{i=1}^{n} u_{x_i}(x_i)_t = f(t,x(t))$

$\Rightarrow u(t,x) = u(0,x(0)) \int_0^t f(t, x_1^0+b_1 t, \dots, x_n^0+b_n t)\,dt$.

故 $u(t,x)=g(x)\int_0^t f(\tau, x_0+B\tau)\,d\tau$.

2. $\begin{cases} u_{tt}-u_{xx}=0 & (t,x)\in(0,+\infty)\times R \\ u(0,x)=\varphi(x), \quad u_t(0,x)=\psi(x). & x\in R. \end{cases}$

由达朗贝尔公式: $u(t,x)=\frac{1}{2}[\varphi(x-t)+\varphi(x+t)]+\frac{1}{2}\int_{x-t}^{x+t}\psi(s)\,ds$

3. $\Delta u=0$ 形如 $u(x)=v(r)$ 的解.

$\frac{\partial u}{\partial x_i}=\frac{x_i}{r}\frac{\partial u}{\partial r} \Rightarrow \frac{\partial^2 u}{\partial x_i^2} = \frac{1}{r}\frac{\partial u}{\partial r} - \frac{x_i^2}{r^3}\frac{\partial u}{\partial r} + \frac{x_i^2}{r^2}\frac{\partial^2 u}{\partial r^2}$

$\Rightarrow \Delta = \frac{n}{r}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}$

① $n=2$时 $\Delta u = \frac{1}{r}u_r + u_{rr}=0$

$\Rightarrow u(r)=C_1\ln r + C_2$

② $n\geq 3$时 $\Delta u=\frac{n-1}{r}u_r+u_{rr}=0$

$\Rightarrow u(r)=C_1 r^{2-n}+C_2$

4. 5. ?

6. $\begin{cases} u_t-\Delta u=0 & (t,x)\in(0,+\infty)\times\Omega \\ u(0,x)=f(x) \\ u|_{\partial\Omega}=g(t). \end{cases}$

① 设 u_1,u_2 为上述方程之解. 记 $u=u_2-u_1$. 则 u 是 $\begin{cases} u_t-\Delta u=0 \\ u(0,x)=0 \\ u|_{\partial\Omega}=0 \end{cases}$ 的解.

由极值原理 $u|_{\Omega}=0$. 故 $u_1=u_2$.

...

7.

1. $\begin{cases} u_{tt} - u_{xx} = 0, & (t,x) \in [0,+\infty) \times R \\ u(0,x) = f(x), \quad u_t(0,x) = g(x), & x \in R^+ \\ u(t,0) = 0. \end{cases}$

令 $F(x) = \begin{cases} f(x) & x > 0 \\ 0 & x = 0 \\ -f(-x) & x < 0. \end{cases}$ $\quad G(x) = \begin{cases} g(x) & x > 0 \\ 0 & x = 0 \\ -g(-x) & x < 0 \end{cases}$

则有 $u(t,x) = \frac{1}{2}[F(x-t) + F(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} G(s)\,ds$. 可以验证. 此为上述方程的解.