Environmental Remote Sensing
GEOG0027

Lecture 7

Mathematical Modelling in Geography II

Last time….

- Different types of model
- statistical, empirical, physically-based, combinations..
- This time some examples
- Simple population growth model
- Require:
  - model of population Q as a function of time t i.e. $Q(t)$
- Theory:
  - in a ‘closed’ community, population change given by:
    - increase due to births
    - decrease due to deaths
  - over some given time period δt

‘Physically-based’ models

- population change given by:
  - Old population + increase due to births - decrease due to deaths

$$Q(t+\delta t) = Q(t) + births(t) - deaths(t) \quad [1]$$

‘Physically-based’ models

- For period δt
  - rate of births per head of population is *B*
  - rate of deaths per head of population is *D*
- So…

$$births(t) = B * Q(t) * \delta t$$

$$deaths(t) = D * Q(t) * \delta t \qquad [2]$$

  - Implicit assumption that *B,D* are constant over time δt
  - So, IF our assumptions are correct, and we can ignore other things (big IF??)

$$\delta Q(t) = Q(t+\delta t) - Q(t)$$

so :

$$\frac{\delta Q}{\delta t} = \frac{Q(t+\delta t) - Q(t)}{\delta t}$$

$$= \frac{births(t) - deaths(t)}{\delta t}$$

$$= (B - D)Q(t) \qquad [3]$$

‘Physically-based’ models

- As time period considered decreases, can write eqn [3] as a differential equation:

$$\frac{dQ}{dt} = (B - D)Q$$

- i.e. rate of change of population with time equal to birth-rate minus death-rate multiplied by current population
- Solution is ...

‘Physically-based’ models

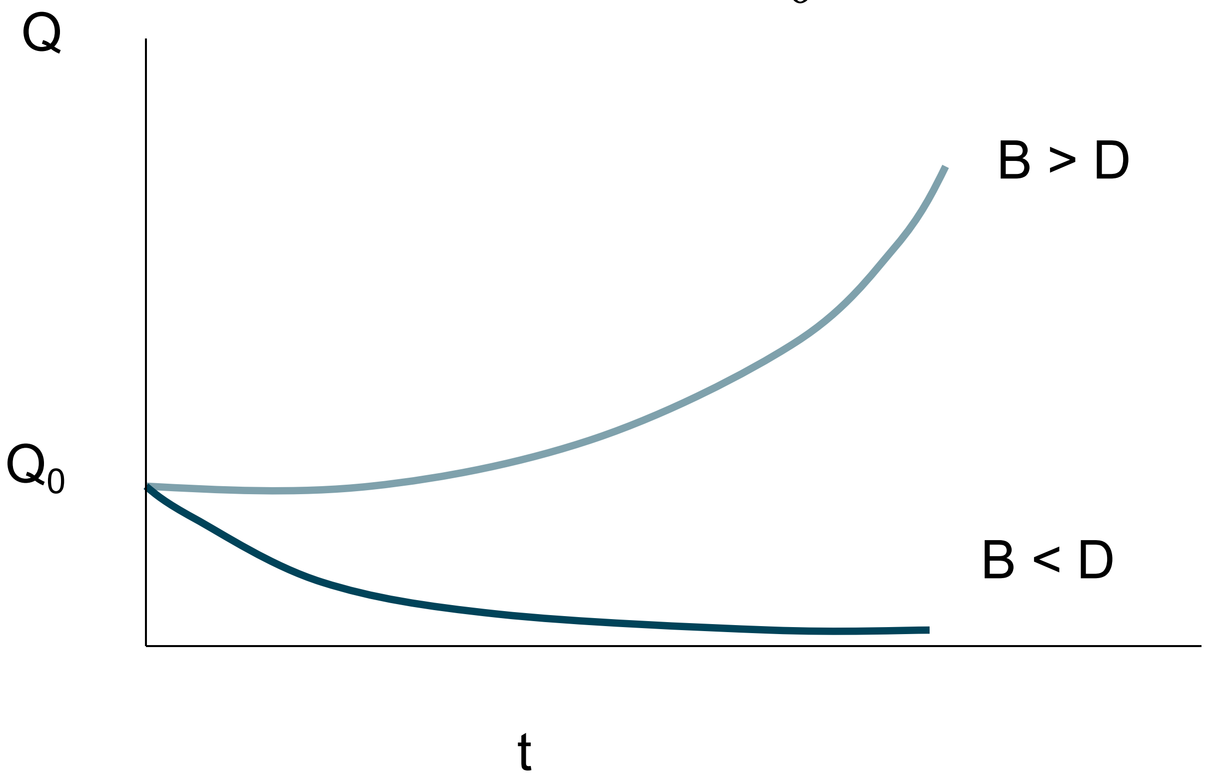
$$Q(t) = Q_0 e^{(B-D)t}$$

- Consider the following:
  - what does Q_0 mean?
  - does the model work if the population is small?
  - What happens if B>D (and *vice-versa*)?
  - How might you ‘calibrate’ the model parameters?

[hint - think logarithmically]

'Physically-based' models

$$Q(t) = Q_0 e^{(B-D)t}$$



- So B, D and Q_0 all have 'physical' meanings in this system

Hydrological (catchment) models

- Want to know water volume in/out of catchment
- Simplest e.g. time-area method not dependent on space (1D) (lumped model)
- next level of complexity - semi-distributed models e.g. sub-basin division
- spatially distributed i.e. need 2D representation of catchment area
- More complex still, use full 3D representation of catchment area, totpography, soil types etc.

Simplest catchment models: time-area hydrograph

- Schematic model
  - very simplified, no physics
    - Empirical/statistical
  - predicts discharge, Q (m^3s^{-1}), based on rainfall intensity, i (mm hr^{-1}), and catchment area, A (m^2)i.e. Q = ciA
    - c is (empirical) runoff coefficient i.e. fraction of rainfall which becomes runoff
    - c is particular to a given catchment (limitation of model)
  - more than one area? Divide drainage basins into isochrones (lines of equal travel time along channel), and add up….
  - $Q(t) = c_1A_1i_{(t-1)} + c_2A_2i_{(t-2)} + \ldots.. + c_nA_ni_{(t-n)}$
  - e.g. TIMAREA model in Kirkby et al.

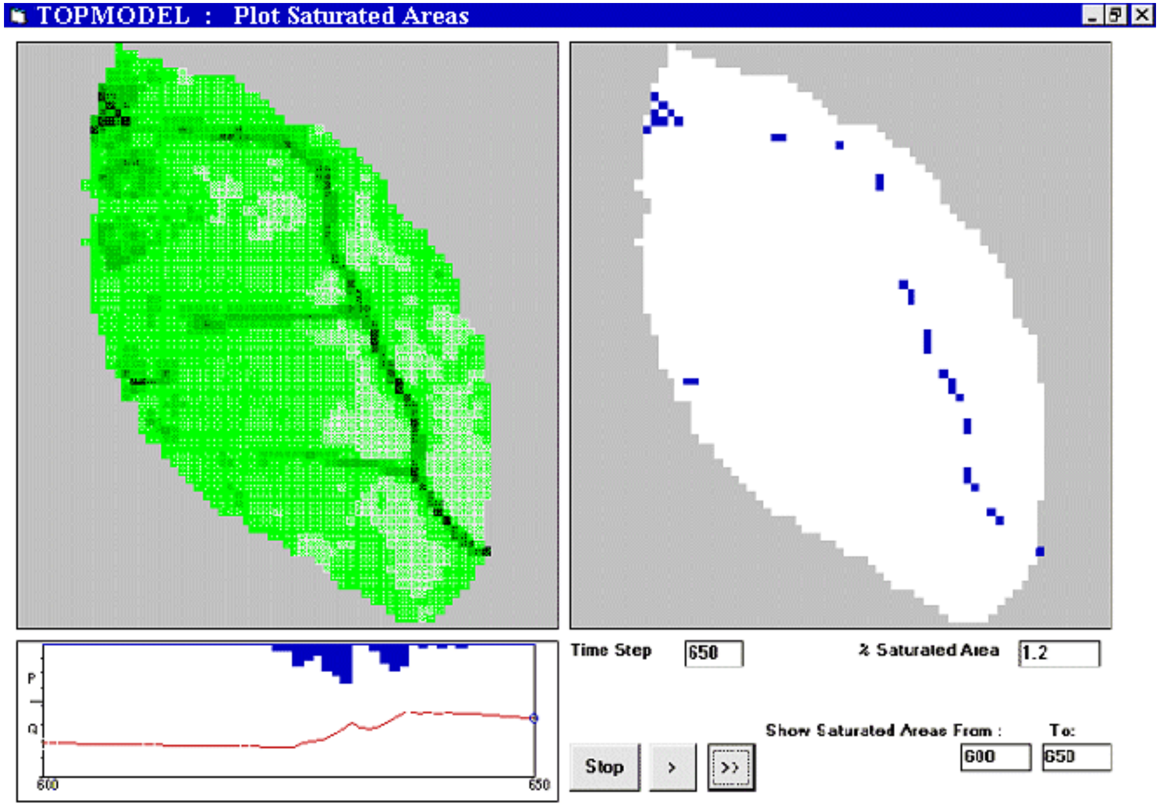
Process-type catchment models

- More complex – some physics
  - precipitation, evapotranspiration, infiltration
  - soil moisture conditions (saturation, interflow, groundwater flow, throughflow, overland flow, runoff etc.)
- From conservation of "stuff" - water balance equation
  - $dS/dt = R - E - Q$
  - i.e. rate of change of storage of moisture in the catchment system, S, with time t, is equal to inflow (rainfall, R), minus outflow (runoff, Q, plus evapotranspiration, E)
  - E.g. STORFLO model (in Kirkby et al.)

More complex?

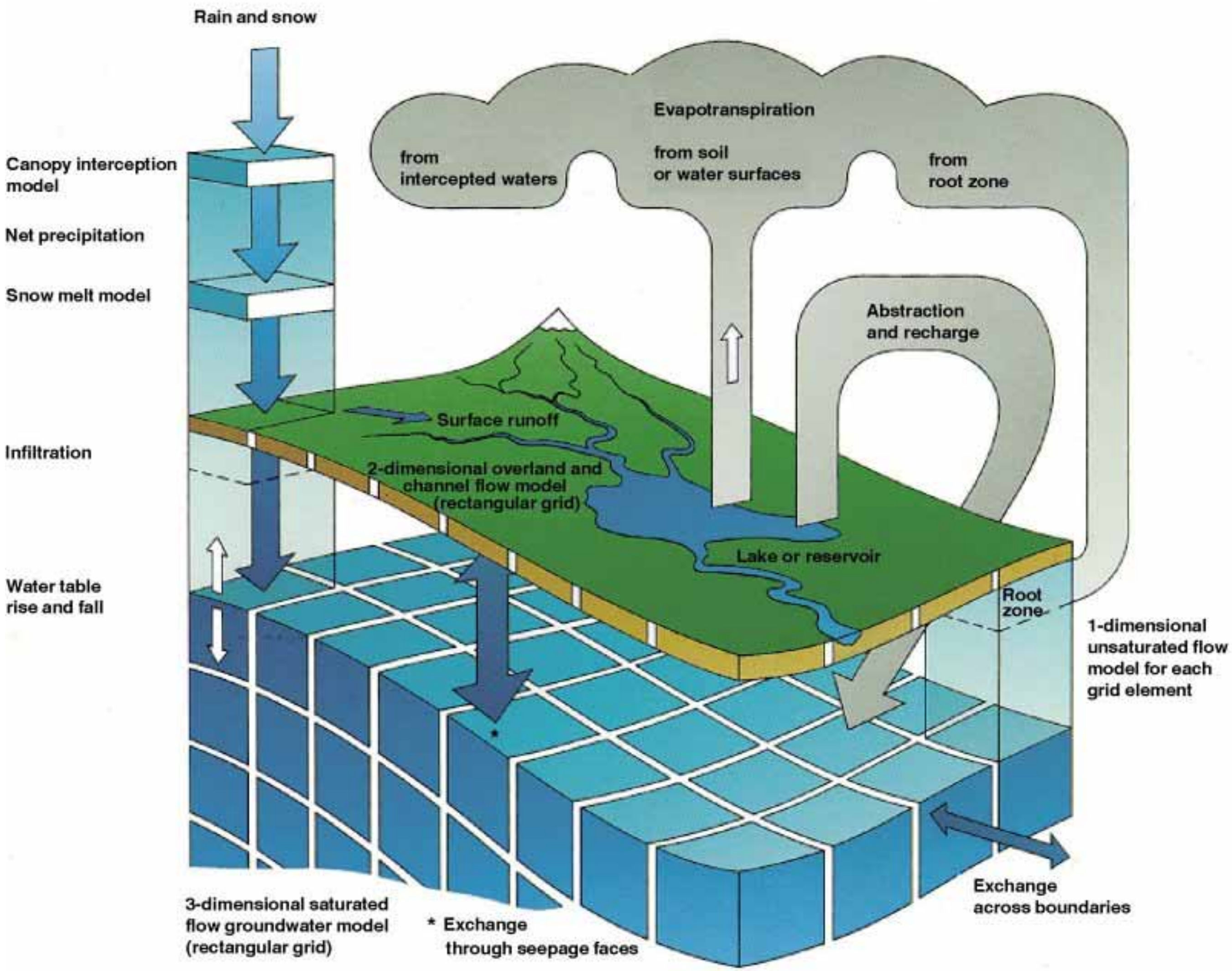
- Consider basin morphometry (shape) on runoff
  - Slope, area, shape, density of drainage networks
  - Consider 2D/3D elements, soil types and hydraulic properties
- How to divide catchment area?
  - Lumped models
    - Consider all flow at once... Over whole area
  - Semi-distributed
    - isochrone division, sub-basin division
  - Distributed models
    - finite difference grid mesh, finite element (regular, irregular)
  - Use GIS to represent - vector overlay of network?
  - Time and space representation?

TOPMODEL: Rainfall runoff



From: http://www.es.lancs.ac.uk/hfdg/topmodel.html

Very complex: MIKE-SHE



- Name
- Combination of physical, empirical and black-box…
- Can "simulate all major processes in land phase of hydrological cycle" !!

From: http://www.dhisoftware.com/mikeshe/Key_features/

Other distinctions

- Analytical
  - resolution of statement of model as combination of mathematical variables and ‘analytical functions’
  - i.e. “something we can actually write down”
  - Very handy & (usually) very unlikely.
  - e.g. biomass = a + b*NDVI
  - e.g.

$$\frac{dQ}{dt} = aQ \longrightarrow Q = Q_0 e^{at}$$

Other Distinctions

- Numerical
  - solution to model statement found e.g. by calculating various model components over discrete intervals
    - e.g. for integration / differentiation

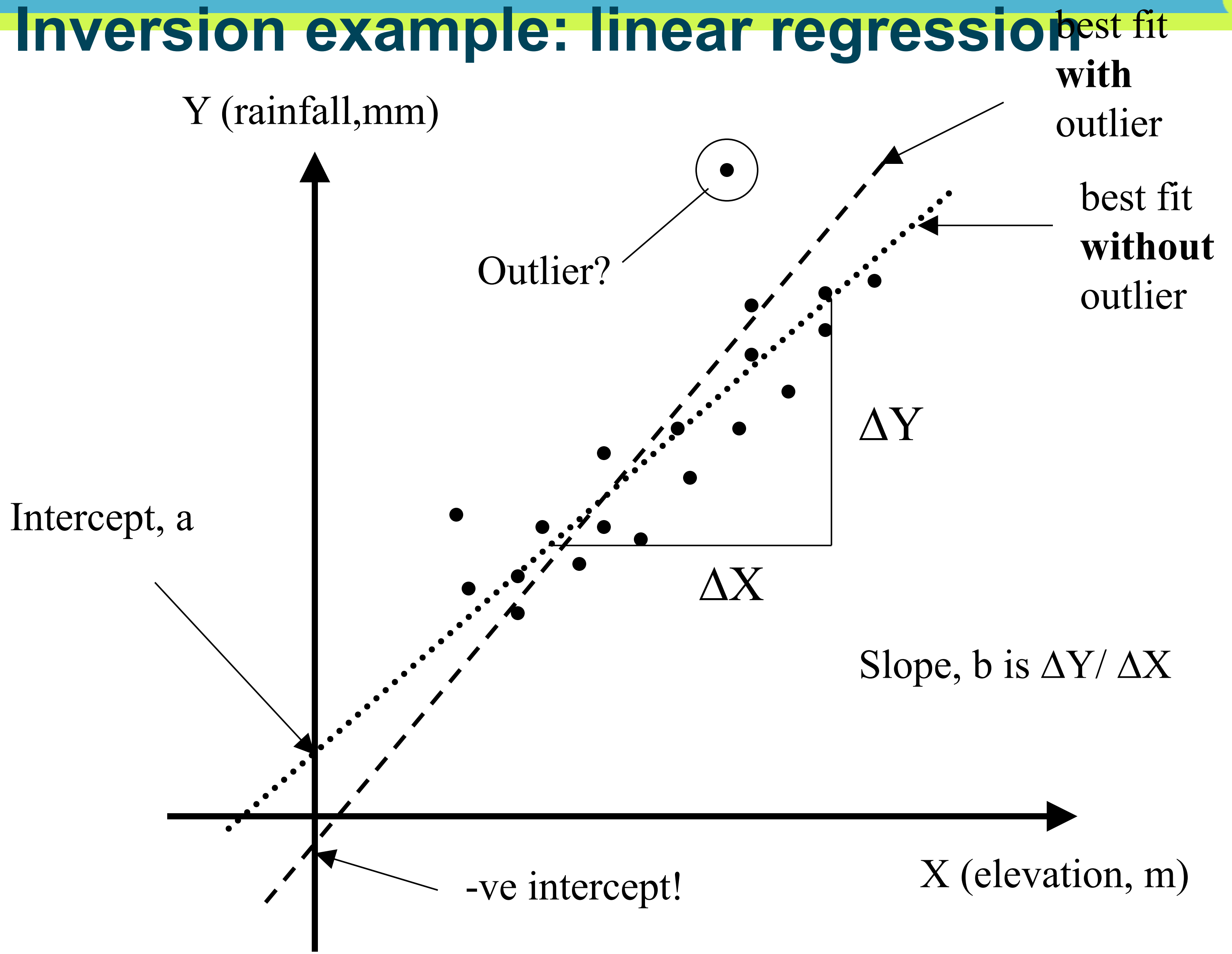
Inversion

- Remember - turn model around i.e. use model to explain observations (rather than make predictions)
  - Estimate value of model parameters
  - E.g. physical model: canopy reflectance, ρ_{canopy} as a function of leaf area index, LAI
    - ρ_{canopy} = f(LAI, …..)
  - Measure reflectance and use to estimate LAI
    - LAI = f^{-1}(measured ρ_{canopy})

Inversion example: linear regression

- E.g. model of rainfall with altitude
  - $Y = a + bX + \varepsilon$
  - Y is predicted rainfall, X is elevation, a and b are constants of regression (slope and intercept), ε is residual error
    - ε arises because our observations contain error
    - (and also if our model does not explain observed data perfectly e.g. there may be dependence on time of day, say….)
  - fit line to measured rainfall data, correlate with elevation

Inversion example: linear regression



Inversion

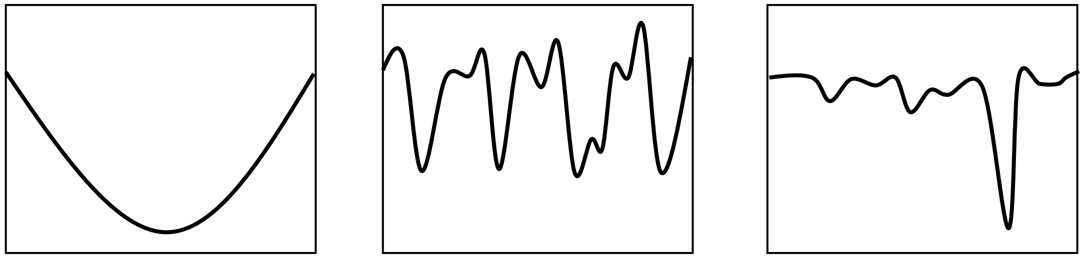
- Seek to find "best fit" of model to observations somehow
  - most basic is linear least-squares (regression)
  - "Best fit" - find parameter values which minimise some error function e.g. RMSE (root mean square error)

$$RMSE = \sqrt{\sum_{n=1}^{n=N} \frac{(measured - modelled)^2}{N-1}}$$

- Easy if we can write inverse model down (analytical)
- If we can't …… ?

Inversion

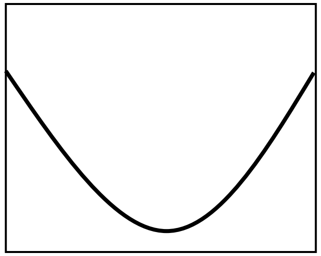
- Have to solve numerically i.e. using sophisticated trial and error methods
  - Generally more than 1 parameter, mostly non-linear
  - Have an error surface (in several dimensions) and want to find lowest points (minima)
  - Ideally want global minimum (very lowest) but can be problematic if problem has many near equivalent minima



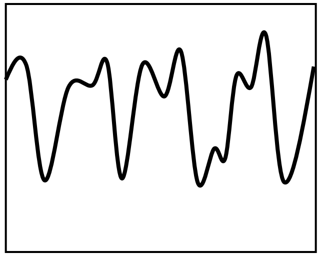
'Easy' Hard 'Easy'

Inversion

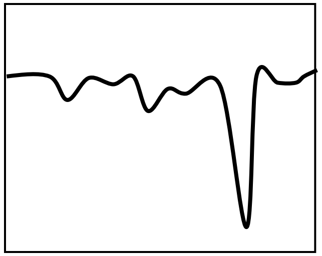
- E.g. of aircraft over airport and organising them to land in the right place at the right time
  - Many parameters (each aicraft location, velocity, time, runway availability, fuel loads etc.)
  - Few aircraft? 1 solution i.e. global minimum & easy to find
  - Many aircraft? Many nearly equivalent solutions
  - Somewhere in middle? Many solutions but only one good one



‘Easy’



‘Easy’



‘Hard’

Which type of model to use?

- Statistical
  - **advantages**
    - simple to formulate & (generally) quick to calculate
    - require little / no knowledge of underlying (e.g. physical) principles
    - (often) easy to invert as have simple analytical formulation
  - **disadvantages**
    - may only be appropriate to limited range of parameter
    - may only be applicable under limited observation conditions
    - validity in extrapolation difficult to justify
    - does not improve general understanding of process

Which type of model to use?

- Physical/Theoretical/Mechanistic
  - **advantages**
    - if based on fundamental principles, more widely applicable
    - may help to understand processes e.g. examine role of different assumptions
  - **disadvantages**
    - more complex models require more time to calculate
    - Need to know about all important processes and variables AND write mathematical equations for processes
    - often difficult to obtain analytical solution & tricky to invert

Summary

- Empirical (regression) vs theoretical (understanding)
- uncertainties
- validation
  - *Computerised Environmetal Modelling: A Practical Introduction Using Excel*, Jack Hardisty, D. M. Taylor, S. E. Metcalfe, 1993 (Wiley)
  - *Computer Simulation in Physical Geography*, M. J. Kirkby, P. S. Naden, T. P. Burt, D. P. Butcher, 1993 (Wiley)
  - http://www.sportsci.org/resource/stats/models.html

Summary

- No one trusts a model except the person who wrote it
- Everyone trusts an observation except the person who made it