Universidad Complutense de Madrid

Sistemas dinámicos y realimentación

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Capítulo 2

Modelado de sistemas dinámicos

2.1. Sistemas en el espacio de estados

Nos vamos a centrar en sistemas que puedan ser descritos por características cuantificables. A estas características las vamos a llamar **estados**, como por ejemplo, una temperatura, una velocidad, o un voltaje. Si estos estados dependen del tiempo, entonces, llamamos **señal** a la sucesión de valores de los estados en el tiempo. Uno podría interaccionar con el sistema a través de una **entrada** cuantificable, y a su vez medir información del sistema a través de una **salida** cuantificable.

Vamos a definir $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$ y $u(t) \in \mathbb{R}^k$ como el vector apilado de estados, la salida, y la entrada a un sistema Σ respectivamente. En particular, son señales, e.g., $x : [0, \infty) \to \mathbb{R}^n$.

El sistema Σ es un modelo que predice el valor de los estados y la salida a lo largo del tiempo. Esta predicción incorpora la interacción de la entrada en los estados y la salida. En particular, vamos a emplear ecuaciones diferenciales como herramienta para predecir la evolución en el tiempo de los estados del sistema Σ como se muestra a continuación

$$\Sigma := \begin{cases} \dot{x}(t) = & f(x(t), u(t)) \\ y(t) = & g(x(t), u(t)) \end{cases}, \tag{2.1}$$

en donde $\dot{x} := \frac{\mathrm{d}}{\mathrm{dt}}(x(t))$ es la notación para la derivada total con respecto del tiempo, y $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ y $g : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ son funciones.

Podemos representar el sistema Σ como un bloque con puertos de entrada y de salida como se muestra en la figura 2.1.

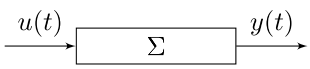

Figura 2.1: Diagrama de bloque entrada/salida del sistema Σ.

2.2. Ejemplos

2.2.1. Péndulo invertido

Vamos a derivar las funciones f y g en (2.1) para el sistema del péndulo invertido.

Primero, vamos a hallar la ecuación diferencial que describe la dinámica de una masa $m \in \mathbb{R}_+$ en el extremo de un péndulo de longitud $l \in \mathbb{R}_+$ tal y como se muestra en la figura 2.2. Vamos a considerar que podemos interactuar con el sistema por medio de un torque $T \in \mathbb{R}$ en el otro extremo del péndulo, que la masa sufre un rozamiento proporcional $b \in \mathbb{R}_+$ a su celeridad, y que podemos medir el ángulo $\theta \in \mathbb{R}$ que forma el péndulo con la vertical.

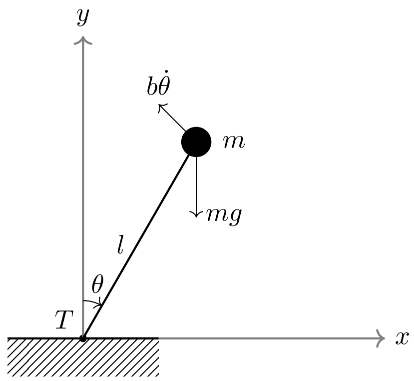

Figura 2.2: Inverted pendulum

Define $g = 9{,}8$ y $I \in \mathbb{R}_+$ como la aceleración gravitatoria y el momento de inercia del péndulo respectivamente. Es sencillo comprobar que $I = ml^2$, y explotaremos que $I\ddot{\theta} =$ suma de torques. De hecho, tenemos que considerar tres torques: 1. Torque T ejercido por nosotros en la base del péndulo; 2. Torque $-bl\dot{\theta}$ ejercido por la fricción en la masa; 3. Torque $mgl\sin\theta$ ejercido por la atracción gravitatoria. Por lo que la ecuación diferencial que modela el comportamiento del péndulo invertido es

$$\ddot{\theta} = \frac{1}{ml^2}\left(mgl\sin\theta - bl\dot{\theta} + T\right). \tag{2.2}$$

Parece razonable escoger θ como uno de los estados para construir $x(t)$ en (2.1). De hecho, la ecuación (2.2) es de segundo orden en θ, por lo que es conveniente escoger $\dot{\theta}$ como un estado también. Por lo tanto, definamos nuestro vector de estados como

$$x := \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}, \tag{2.3}$$

y como el torque T es como interactuamos con el sistema, escogemos como entrada $u(t) = T(t)$.

Ahora estamos listos para construir las funciones f and g en (2.1) para el péndulo invertido. Atención a que f y g solo toma como argumentos los vectores de estados x y de entradas u. En el lado izquierdo de (2.1) tenemos la derivada temporal de $x(t)$, por lo que

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}\right) = f(x(t), u(t)) = \begin{bmatrix}f_1(x(t), u(t)) \\ f_2(x(t), u(t))\end{bmatrix}, \tag{2.4}$$

donde automáticamente obtenemos que $f_1 = \dot{\theta}$. Fijarse que la primera fila de f en (2.4), a la izquierda tenemos que $\frac{\mathrm{d}}{\mathrm{dt}}\theta$, y a la derecha tenemos que $f_1 = \dot{\theta}$ porque $\dot{\theta}$ es un estado o elemento de x. Desafortunadamente, no podemos decir que $f_2 = \ddot{\theta}(t)$ porque $\ddot{\theta}$ no es un estado o elemento de x. No obstante, tenemos que la segunda fila f_2 viene dada por la ecuación (2.2). Por lo que podemos escribir f como

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}\right) = f(x(t), u(t)) = \begin{bmatrix}\dot{\theta} \\ \frac{1}{ml^2}\left(mgl\sin\theta - bl\dot{\theta} + T\right)\end{bmatrix}. \tag{2.5}$$

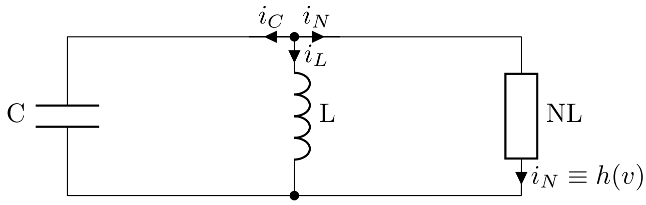

Figura 2.3: Circuito eléctrico no lineal

El cálculo de g es más sencillo en este caso. Hemos establecido al comienzo que solo podemos medir el ángulo θ. Por lo que $y(t) = \theta(t)$, i.e.,

$$g(x(t), u(t)) = \theta(t). \tag{2.6}$$

2.2.2. El oscilador de Van der Pol

Se trata de un oscilador, propuesto por primera vez por Balthasar Van der Pol, cuando trabajaba en Philips, para explicar las oscilaciones observadas en tubos de vacío. Podemos obtener la ecuación del oscilador, empleando el circuito de la figura 2.3.

Donde el elemento no lineal NL, presenta una relación entre voltaje e intensidad caracterizada por la función $h(v)$.

El voltaje v en los tres componentes del circuito debe ser igual; además,

$$v = L\frac{di_L}{dt} \tag{2.7}$$

$$i_C = C\frac{dv}{dt} \tag{2.8}$$

Si aplicamos la primera ley de Kirchohff al nodo superior del circuito,

$$i_C + i_L + i_N = 0 \tag{2.9}$$

$$C\frac{dv}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(s)ds + h(v) = 0 \tag{2.10}$$

Si derivamos (2.10) con respecto al tiempo, dividimos por C y reordenamos,

$$\frac{d^2v}{dt^2} + \frac{1}{C}\frac{dh(v)}{dv}\frac{dv}{dt} + \frac{1}{LC}\cdot v = 0 \tag{2.11}$$

Se trata de un caso particular de la ecuación de Liénard,

$$\ddot{v} + f(v)\dot{v} + g(v) = 0 \tag{2.12}$$

Si definimos ahora, $h(v)$,

$$h(v) = m(\frac{1}{3}v^3 - 1) \tag{2.13}$$

$$\frac{dh}{dv} = m(v^2 - 1) \tag{2.14}$$

y sustituyendo en (2.11),

$$\ddot{v} + m\frac{1}{C}(v^2 - 1)\dot{v} + \frac{1}{LC}v = 0 \tag{2.15}$$

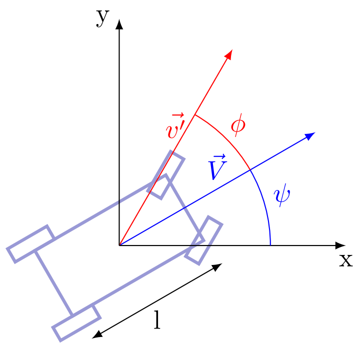

Figura 2.4: Esquema de un vehículo terrestre de 4 ruedas

Podemos representarla finalmente en variables de estado, tomando $x_1 = v$ y $x_2 = \dot{v}$,

$$\dot{x}_1 = x_2 \tag{2.16}$$

$$\dot{x}_2 = -\frac{1}{LC}x_1 - m\frac{1}{C}(x_1^2 - 1)x_2 \tag{2.17}$$

Podemos ahora hacer un primer análisis cualitativo. El primer término a la derecha del igual en la ecuación (2.17) representa una fuerza recuperadora proporcional al desplazamiento, el segundo término, crecerá con la velocidad para $x_1 < 1$, alejando así al sistema del origen y representará un término disipativo para $x_1 > 1$ acercándolo por tanto de nuevo al origen. Es por tanto esperable, que se alcance algún tipo de situación de equilibrio. Más adelante definiremos esta situación rigurosamente como un ciclo límite.

2.2.3. Un vehículo de cuatro ruedas.

La figura 2.4 muestra un esquema de un vehículo terrestre de cuatro ruedas, visto desde arriba. Si consideramos que se mueve en el plano x, y, y que su velocidad instántanea $\vec{V}$ está siempre orientada en la dirección de avance del vehículo *psi*–asumimos que no derrapa, ni se mueve lateralmente–, podemos entoces definir el velocidad, en el sistema de referencia x, y como,

$$\dot{x} = V_x(t) = V\cos(\psi(t)) \tag{2.18}$$

$$\dot{y} = V_y(t) = V\sin(\psi(t)) \tag{2.19}$$

$$\tag{2.20}$$

Además el vehículo girará, siempre que las ruedas delanteras no estén alineadas con las ruedas traseras, cambiando así su dirección de avance . Podemos relacionar la velocidad de giro del vehículo $\dot{\psi}$ con el ángulo de orientación de las ruedas delanteras ϕ, y la velocidad a la que avanzan $\vec{v'}$. podemos obtener las componetes de dicha velocidad en ejes cuerpo (paralela y perpendicular a la dirección de avance del vehículo),

$$v_P = v'\cos(\phi(t)) \tag{2.21}$$

$$v_T = v'\sin(\phi(t)) \tag{2.22}$$

Pero la rueda esta unida al vehículo así que su velocidad en la direccíon de avance debe ser la misma que la del vehículo: $v_P \equiv V$. A partir de esta relación podemos obtener la velocidad tangencial de las ruedas como,

$$v_T = V\frac{\sin(\phi)}{\cos(\phi)} = V\tan(\phi) \tag{2.23}$$

Si tomamos como centro de giro del vehículo el centro de su eje trasero, y la batalla (distancia entre ejes) es l, obtenemos una expresión para su velocidad de giro,

$$\dot{\psi} = \frac{V}{l}\tan(\phi) \tag{2.24}$$

En resumen, podemos describir el sistema mediante tres ecuaciones de estado $x_1 \equiv x, x_2 \equiv y, x_3 \equiv \psi$:

$$\begin{cases} \dot{x}_1 & = V\cos(x_3) \\ \dot{x}_2 & = V\sin(x_3) \\ \dot{x}_3 & = \frac{V}{l}\tan(\phi) \end{cases} \tag{2.25}$$

Si consideramos $V = cte$, la única entrada al sistema sería el ángulo de giro de las ruedas $u(t) = \phi(t)$, controlando su valor, podemos hacer girar al vehículo en la dirección deseada.

2.3. Simulación o soluciones numéricas del sistema Σ

Dado un punto inicial $x(0)$, podemos predecir o calcular numéricamente $x(t)$. El método de *integración de Euler* es un método numérico sencillo que puede darnos información sobre la evolución temporal de los estados y salidas de Σ. El siguiente algoritmo describe la integración numérica por Euler:

Algorithm 1.

1. *Define el paso de tiempo ΔT*
2. *Define $x = x(0)$*
3. *Define $y = g(x, u)$*
4. *Registra x and y, para poder procesarlos después si fuera necesario*
5. *Define $t = 0$*
6. *Define un tiempo final t^**
7. *Mientras $t \leq t^*$ entonces:*
   a) $x_{nuevo} = x_{viejo} + f(x_{viejo}, u)\Delta T$
   b) $y_{nuevo} = g(x_{nuevo}, u)$
   c) *Representa x gráficamente*
   d) *Registra x e y, para poder procesarlos más adelante si fuera necesario*
   e) $t = t + \Delta t$
8. *Representa x e y a lo largo de t*

Este algoritmo rinde bien cuando ΔT es suficientemente pequeño en función de como de rápido varíe f en el tiempo. Por ahora hemos considerado $u = 0$, es decir, no hay control o interacción alguna con el sistema.

Capítulo 3

Comportamiento dinámico y estabilidad

El concepto de estabilidad y su análisis constituye unos de los aspectos claves para el estudio de los sistemas dinámicos. La estabilidad de un sistema esta estrechamente relacionada con su comportamiento dinámico y puede definirse de diversas maneras. En este capítulo nos centraremos en el análisis de los llamados puntos de equilibrio de un sistema y los estudiaremos de acuerdo con el concepto de estabilidad de Lyapunov. Además, incidiremos en otros aspectos de la estabilidad de los sistemas tales como la existencia de ciclos límite o el movimiento nominal. Empecemos pues por el estudio de los puntos de equlibrio.

3.1. Sistemas autónomos

3.1.1. Puntos de equilibrio

Definition 1 (Punto de equlibrio)**.** *Dados un sistema dinámico general definido por las ecuaciones,*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{3.1}$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}) \tag{3.2}$$

donde $\mathbf{x} \in \mathbb{R}^n$*,* $\mathbf{u} \in \mathbb{R}^m$ *e* $\mathbf{y} \in \mathbb{R}^l$*. Se definen como puntos de equilibrio o puntos estacionarios los valores del vector de estados* $\mathbf{x_e}$ *y del vector de entradas* $\mathbf{u_e}$ *para los cuales el estado y, consecuentemente, la salida del sistema permanecen constantes,*

$$\dot{\mathbf{x}}_\mathbf{e} \equiv 0 = \mathbf{f}(\mathbf{x_e}, \mathbf{u_e}) \tag{3.3}$$
$$\mathbf{y_e} = \mathbf{g}(\mathbf{x_e}, \mathbf{u_e}) \tag{3.4}$$

Si el vector de estados no cambia, su derivada será cero. Por tanto, mientras no se altere el valor de la entrada, el sistema permanecerá en el mismo estado y el valor del vector de salidas permanecerá tambien constante

Podemos, a partir de esta definición, obtener algunas propiedades importantes de los puntos de equilibrio:

1. Una vez que un sistema alcanza un punto de equilibrio, permance en él indifinidamente. (Todas las derivadas temporales de las componentes del vector de estado son cero.

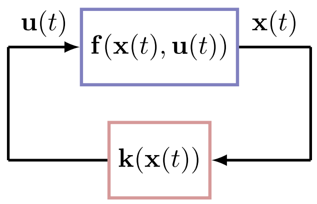

Figura 3.1: Esquema general de un sistema realimentado

2. Desde el punto de vista del control de sistemas, los puntos de equilibrio juegan un papel importante ya que representan condiciones de operación constante.

3. Un sistema dinámico puede tener uno o más puntos de equilibrio, o no tener ninguno.

Sistemas autónomos. Para simplificar el estudio de la estabilidad, podemos empezar por considerar el casos de sistemas que tienen entrada nula $\mathbf{u} = \mathbf{0}$, $\forall t$. en los que la entrada es una función directa de los estados del sistema. Hablaremos entonces de un *sistema autónomo*; la salida evoluciona a partir de un estado inicial $\mathbf{x_0} \equiv \mathbf{x(t_0)}$,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{3.5}$$

$$\mathbf{y} = \mathbf{g}(\mathbf{x}) \tag{3.6}$$

Sistemas realimentados. Del mismo modo, podemos considerar sistemas en los que la entrada es una función directa del valor de los estados; $\mathbf{u} = \mathbf{c(x)}$. Hablaremos entonces de un *sistema realimentado*. A efectos de análisis de la estabilidad del sistema, no hay diferencia entre un sistema realimentado y un sistema autónomo,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{c}(\mathbf{x})) = \bar{f}(\mathbf{x}) \tag{3.7}$$

$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{c}(\mathbf{x})) = \bar{\mathbf{g}}(\mathbf{x}) \tag{3.8}$$

El nombre de sistema realimentado, proviene de considerar que se están *realimentando* los estados en la entrada del sistema. El concepto de realimentación constituye uno de los pilares de los sistemas de control. La figura 3.1, muestra esquemáticamente este concepto.

Tanto para un sistema autónomo como para uno realimentado los puntos de equilibrio deberán satisfacer la condición de que las derivadas temporales de los estados sean todas nulas,

$$\mathbf{0} = \mathbf{f}(\mathbf{x_e}) \tag{3.9}$$

3.2. Lyapunov

3.3. Principio de invarianza de LaSalle

Capítulo 4

Diseño de controladores para sistemas en el espacio de estados

4.1. Seguidor de trayectorias deseadas

Considera el sistema Σ en (2.1). Dada una señal deseada o trayectoria $x^*(t) \in \mathbb{R}^n$, nuestro objetivo es diseñar $u(t) \in \mathbb{R}^k$ tal que $x(t) \to x^*(t)$ para $t \to \infty$.

Para ello vamos a construir la función de error

$$e(t) := x(t) - x^*(t), \tag{4.1}$$

junto con la siguiente función candidata de Lyapunov

$$V(t) = \frac{1}{2}||e(t)||^2, \tag{4.2}$$

cuya derivada temporal satisface

$$\frac{\mathrm{d}V}{\mathrm{dt}} = e^T \dot{e} = e^T\big(\dot{x}(t) - \dot{x}^*(t)\big) = e^T f(x,u) - e^T \dot{x}^*(t). \tag{4.3}$$

Podemos garantizar que $e(t) \to 0$ cuando $t \to \infty$ si para un conjunto compacto $\mathcal{B} := \{e : ||e|| \leq \rho,\ \rho \in \mathbb{R}_+\}$ tenemos que $\frac{\mathrm{d}V}{\mathrm{dt}} < 0$ si $e \in \mathcal{B} \setminus \{0\}$ y $\frac{\mathrm{d}V}{\mathrm{dt}} = 0$ si $e = 0$.

4.1.1. Ejemplo con un sistema cinemático de segundo orden

Considera el siguiente sistema cinemático de segundo orden

$$\ddot{p}(t) = u, \tag{4.4}$$

donde $p, u \in \mathbb{R}^n$ son las *posiciones* y las *aceleraciones* respectivamente en un espacio $l \in \mathbb{N}$-dimensional. Vamos a particularizar para $l = 2$. Apilamos las posiciones y velocidades $p_x, p_y, \dot{p}_x$ y $\dot{p}_y$ en $x \in \mathbb{R}^4$, consideramos que podemos medir x, y apilamos las aceleraciones $\ddot{p}_x$ y $\ddot{p}_y$ en $u \in \mathbb{R}^2$. Por lo que las funciones f y g del sistema Σ (2.1) son

$$\begin{cases} f &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \\ g &= x. \end{cases} \tag{4.5}$$

Definimos como trayectorias deseadas para seguir

$$p^*(t) = f(t), \quad \dot{p}^*(t) = f'(t), \tag{4.6}$$

donde $f(t) \in C^2$. Vamos a construir las siguientes señales de error

$$e_1(t) = p(t) - p^*(t), \quad e_2(t) = \dot{p}(t) - \dot{p}^*(t), \tag{4.7}$$

para definir la siguiente función candidata de Lyapunov al estilo de (4.2).

$$V(e(t)) = \frac{1}{2}||e(t)||^2, \tag{4.8}$$

en donde $e(t) \in \mathbb{R}^4$ hemos apilado e_1 y e_2. La derivada temporal de (4,8) satisface

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{dt}} &= e^T\dot{e} = e_1^T\dot{e}_1 + e_2^T\dot{e}_2 = e_1^T(\dot{p} - f'(t)) + e_2^T(u - f''(t)) \\ &= e_1^Te_2 + e_2^T(u - f''(t)) \end{aligned} \tag{4.9}$$

por lo que si uno escoge

$$u = f''(t) - e_1 - e_2 = f''(t) - p(t) + f(t) - \dot{p}(t) + f'(t), \tag{4.10}$$

nos lleva a

$$\frac{\mathrm{d}V}{\mathrm{dt}} = -||e_2||^2 \leq 0. \tag{4.11}$$

De hecho, la derivada temporal de $V(t)$ es cero sí y solo sí $e_2 = 0$. Por lo que para invocar al principio de invariance de LaSalle, debemos comprobar cual es el conjunto invariante más grande del sistema (autónomo) error $\dot{e}$ cuando $e_2 = 0$, esto es,

$$\begin{cases} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= -e_1 - e_2 \end{cases}, \tag{4.12}$$

donde se puede ver que cuando $e_2 = 0$, el conjunto invariante más grande es $e_1 = e_2 = 0$. Consecuentemente, podemos concluir que $e(t) \to 0$ cuando $t \to \infty$ para cualquier $e(0)$ empezando en $\mathcal{B}$ con un ρ arbitrario, esto es, tenemos convergencia global.

4.2. Diseño de ciclos límites para sistemas planos

Consideremos de nuevo el sistema (4.4) para $l = 2$, es decir, seguimos en 2D, o un sistema en el plano. En vez de querer seguir una trayectoria $p^*(t)$, queremos que la posición converja a un *camino* cerrado que además sea un ciclo límite. El camino cerrado lo podemos definir como una curva $\phi(p_x, p_y) = 0$, es decir, todos los puntos con curva de nivel igual a cero definen *el camino*. Por ejemplo, un círculo de radio $r \in \mathbb{R}_+$ satisface

$$\phi(p) := p_x^2 + p_y^2 - r^2 = 0. \tag{4.13}$$

Atención a que podemos considerar $\phi(p)$ como una señal de error, i.e., $e(t) := \phi(p(t))$. Consecuentemente, podemos definir la siguiente función candidata de Lyapunov.

$$V(e(t)) = \frac{1}{2}e(t)^2 + \frac{1}{2}||\dot{p}(t) - \dot{p}^*(t)||^2, \tag{4.14}$$

cuya derivada temporal satisface (por ahora considera $\dot{p}^*(t) = 0$).

$$\frac{\mathrm{d}V}{\mathrm{dt}} = e\dot{e} + \dot{p}^T\ddot{p} = e\nabla\phi(p)\dot{p} + \dot{p}^T u, \tag{4.15}$$

entonces, si uno escoge $u = -e\nabla\phi(p)^T - \dot{p}$, entonces tenemos que

$$\frac{\mathrm{d}V}{\mathrm{dt}} = -||\dot{p}||^2 \leq 0, \tag{4.16}$$

y, similarmente a como hemos hecho en la sección anterior, por el principio de invarianza de LaSalle's concluimos que $e(t) \to 0$ cuando $t \to \infty$.

4.2.1. Ejercicio

Para que $\phi(p)$ sea un ciclo límite, la posición no solo ha de converger a ese camino, sino que ha de evolucionar en él en el tiempo. Para ello considera $\dot{p}^*(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla\phi(p)^T$, y concluye que ahora $\phi(p)$ sí es un ciclo límite.

Capítulo 5

Sistemas lineales

5.1. Linear maps

In this chapter, we focus on a particular class of state-space systems called *state-space linear systems*. First, we need the notion of *linear map*.

Definition 2. *Consider the mapping $H : V \to W$. If H preserves the operations of addition and scalar multiplication, i.e.,*

$$H(v_1 + v_2) = H(v_1) + H(v_2), \quad v_1, v_2 \in \mathbb{V}$$
$$H(\alpha v_1) = \alpha H(v_1), \quad \alpha \in \mathbb{K},$$

then H is a linear map.

5.1.1. Exercise: Check whether the following maps are linear or not

1. $H_1(v) := Av, A \in \mathbb{R}^{n\times n}, \quad v \in \mathbb{R}^n$
2. $H_2(v) := \frac{\mathrm{d}}{\mathrm{dt}}(v(t)), \quad v \in \mathcal{C}^1$
3. $H_3(v) := \int_0^T v(t)dt, \quad v \in \mathcal{C}^1, T \in \mathbb{R}_{\geq 0}$
4. $H_4(v) := D(v) := v(t-T), \quad v \in \mathcal{C}^1, T \in \mathbb{R}_{\geq 0}$
5. $H_5(v) := Av + b, \quad A \in \mathbb{R}^{n\times n}, v, b \in \mathbb{R}^n$

5.2. Continuous state-space linear systems

The following system defines a continuous state-space linear system

$$\Sigma := \begin{cases} \dot{x}(t) & = A(t)x(t) + B(t)u(t), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^k \\ \dot{y}(t) & = C(t)x(t) + D(t)u(t), \quad y \in \mathbb{R}^m \end{cases} \tag{5.1}$$

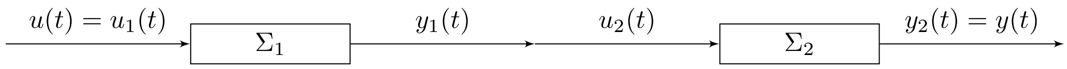

Figura 5.1: Example of series interconnection.

5.2.1. Exercise: Write as a block diagram the continuous state-space linear system and check that consists only of linear maps

5.2.2. Exercise: Interconnections of continuous state-space linear systems

Rewrite as a single system, i.e., as in (5.1):

1. the series (or cascade) interconnection of two continuous state-space linear systems, i.e., $y_1(t) = u_2(t)$.

2. the parallel interconnection of two continuous state-space linear systems, i.e., $y(t) = y_1(t) + y_2(t)$.

3. the feedback interconnection, i.e., $u_1(t) = u(t) - y(t)$, assuming $u, y, \in \mathbb{R}^k$.

5.3. Solution to Linear State-Space systems

The solution to an *ordinary differential equation* (ODE) is given by the addition of two terms: the solution to the homogeneous part, and a particular solution to the non-homogeneous.

$$\dot{x}(t) = \underbrace{A(t)x(t)}_{\text{homogeneous}} + \underbrace{B(t)u(t)}_{\text{non-homogeneous}} \tag{5.2}$$

Theorem 1. *Peano-Barker series The unique solution to the homogeneous $\dot{x} = Ax$ is given by*

$$x(t) = \Phi(t, t_0)x(t_0), \quad x(t_0) \in \mathbb{R}^n, t \geq 0, \tag{5.3}$$

where

$$\begin{aligned}\Phi(t, t_0) := I + \int_{t_0}^{t} A(s_1)ds_1 + \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2)ds_2ds_1 \\ + \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2) \int_{t_0}^{s_2} A(s_3)ds_3ds_2ds_1 + \dots .\end{aligned} \tag{5.4}$$

Sketch of the proof: First we calculate the following time derivative

$$\begin{aligned}\frac{d}{dt}\Phi(t, t_0) &= A(t) + A(t) \int_{t_0}^{t} A(s_2)ds_2 \\ &+ A(t) \int_{t_0}^{t} A(s_2) \int_{t_0}^{s_2} A(s_3)ds_3ds_2 + \dots \\ &= A(t)\Phi(t, t_0).\end{aligned} \tag{5.5}$$

We claim that the solution to the homogenenous part of (5.2) is $x(t) = \Phi(t,t_0)x_0$ (x_0 is the short notation for $x(t_0)$), whose time derivative is given by

$$\begin{aligned} \frac{d}{dt}x &= \frac{d}{dt}\Phi(t,t_0)x_0 \\ &= A(t)\Phi(t,t_0)x_0 \\ &= A(t)x(t), \end{aligned} \tag{5.6}$$

which is proving the identity $\dot{x} = A(t)x(t)$ given that $x(t) = \Phi(t,t_0)x_0$. In order to make this proof complete, we would need to prove that the series (5.4) converges for $t \geq t_0$. That material should be covered in a standard course on differential equations.

The matrix $\Phi(t,t_0)$ is called the ***state transition matrix***. Given an initial condition x_0, we can predict $x(t)$ in (5.2) by *iterating* over and over with $\Phi(t,t_0)$ given that we do not interact with the system, i.e., $u(t) = 0, t \geq t_0$.

5.3.1. Exercise

Check that

$$\begin{aligned} x(t) &= \Phi(t,t_0)x_0 + \int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)d\tau \\ y(t) &= C(t)\phi(t,t_0)x_0 + \int_{t_0}^{t}C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t) \end{aligned}$$

are the solutions to

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ \dot{y}(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

5.4. Solution to Linear Time Invariant Systems

The matrix $\Phi(t,t_0)$ can be calculated analytically when A is a matrix with constant coefficients. If A is constant, we can take it out from the integrals in (5.4)

$$\begin{aligned} \Phi(t,t_0) := I + A\int_{t_0}^{t}ds_1 + A^2\int_{t_0}^{t}\int_{t_0}^{s_1}ds_2ds_1 \\ +A^3\int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}ds_3ds_2ds_1 + \dots, \end{aligned} \tag{5.7}$$

and noting that the following integrals can be easily solved

$$\begin{aligned} \int_{t_0}^{t}ds_1 &= (t-t_0) \\ \int_{t_0}^{t}\int_{t_0}^{s_1}ds_2ds_1 &= \frac{(t-t_0)^2}{2} \\ &\vdots \\ \int_{t_0}^{t}\int_{t_0}^{s_1}\cdots\int_{t_0}^{s_{k-2}}\int_{t_0}^{s_{k-1}}ds_kds_{k-1}\cdots ds_2ds_1 &= \frac{(t-t_0)^k}{k!}, \end{aligned}$$

then we have that (5.7) can be calculated by

$$\Phi(t,t_0) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} A^k, \tag{5.8}$$

which resembles to the power series of the scalar exponential function, i.e., $e^x := \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$. In fact, the definition of the *exponential of a matrix* is

$$exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots \tag{5.9}$$

Let us set $t_0 = 0$ for the sake of convinience, then

$$\begin{aligned} \Phi(t,0) &= I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \dots \\ &= exp(At), \end{aligned} \tag{5.10}$$

therefore the solution to the homogeneous (5.2) with A constant and setting $t_0 = 0$ is

$$x(t) = exp(At)x_0, \quad t \geq 0. \tag{5.11}$$

To continue further, we need the following result from Linear Algebra.

Theorem 2. ***Jordan Form***. *For every square matrix $A \in \mathbb{C}^{n\times n}$, there exists a non-singular change of basis matrix $P \in \mathbb{C}^{n\times n}$ that transform A into*

$$J = PAP^{-1} = \begin{bmatrix} J_1 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ 0 & 0 & J_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \cdots & J_l \end{bmatrix}, \tag{5.12}$$

where each J_i is a Jordan block of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ 0 & 0 & \lambda_i & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}_{n_i \times n_i}, \tag{5.13}$$

where each λ_i is an eigenvalue of A, and the number l of Jordan blocks is equal to the total number of independent eigenvectors of A. The matrix J is unique up to a reordering of the Jordan blocks and is called the ***Jordan normal form*** *of A.*

Note that $A = P^{-1}JP$ as well, and we leave as an exercise to prove that

$$A^k = P^{-1}J^kP, \tag{5.14}$$

so we can calculate

$$
\begin{aligned}
exp(At) &= P^{-1}\left(\sum_{k=1}^{\infty}\frac{t^k}{k!}\begin{bmatrix} J_1^k & 0 & \cdots & 0 \\ 0 & J_2^k & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & J_l^k \end{bmatrix}\right)P \\
&= P^{-1}\begin{bmatrix} exp(J_1 t) & 0 & \cdots & 0 \\ 0 & exp(J_2 t) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & exp(J_l t) \end{bmatrix}P
\end{aligned} \tag{5.15}
$$

Therefore if J is just a diagonal matrix with the eigenvalues of A, i.e., $J_l = \lambda_l \in \mathbb{C}$, then $exp(J_l t) = e^{\lambda_l t} \in \mathbb{C}$ is a trivial calculation.

Now, let us check the consequencues on the following two conditions

1. J is diagonal.

2. All the eigenvalues of A have negative real part.

Knowing that $\lim_{t\to\infty} e^{\lambda t} \to 0$ if $\lambda \in \mathbb{R}_{<0}$, then we will have that $exp(At) \to 0$ as $t \to \infty$ if the previous two conditions are satisfied! So if we take a look at (5.11), we can conclude that

$$
\lim_{t\to\infty} x(t) \to 0, \tag{5.16}
$$

therefore we can make a prediction on the evolution of $x(t)$ by just checking the eigenvalues of A. If J is not diagonal we can also conclude similar results, but we will not cover them here. We will talk about stability in the next lecture, and how to design a controller such that we can guarantee (5.16).

5.5. Linearization of state-space systems

Unfortunately, it is really (really) hard to calculate the analytic solution of $x(t)$ and $y(t)$ for a generic system Σ. Nevertheless, we will see that we can find the analytic solution for a state-space linear system.

The question then is whether we can relate a generic Σ to a state-space linear system.

If $f(x,t)$ and $g(x,t)$ are real analytic around a specific point (x^*, u^*), then we can approximate them around (x^*, u^*) by a Taylor series expansion. This approximation is what we call *linearization* if we stop at order one in the Taylor series

$$
\Sigma := \begin{cases} \dot{x}(t) = & f(x(t), u(t)) \\ y(t) = & g(x(t), u(t)) \end{cases}\Bigg|_{x\approx x^*, u\approx u^*} \approx \begin{cases} x(t) & = x^* + \delta x(t) \\ u(t) & = u^* + \delta u(t) \\ \delta\dot{x}(t) & = A(t)\delta x(t) + B(t)\delta u(t) \\ \delta y(t) & = C(t)\delta x(t) + D(t)\delta u(t) \end{cases},
$$

where

$$A(t) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{|x=x^*, u=u^*} \qquad B(t) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_k} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_k}{\partial u_1} & \cdots & \frac{\partial f_k}{\partial u_k} \end{bmatrix}_{|x=x^*, u=u^*}$$

$$C(t) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}_{|x=x^*, u=u^*} \qquad D(t) = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_k} \\ \vdots & \vdots & \vdots \\ \frac{\partial g_m}{\partial u_1} & \cdots & \frac{\partial g_m}{\partial u_k} \end{bmatrix}_{|x=x^*, u=u^*} .$$

Roughly speaking, we calculate the sensitivity (up to first order) of f and g when we make a small variation on x and u around (x^*, u^*). How close (x, u) must be to (x^*, u^*) depends on the particular system Σ. Later in the course, we will provide bounds for δx and δu such that we can apply with guarantees our control algorithms.

5.5.1. Linearization of the inverted pendulum

We will see that, with the linearization, we can design controllers $u(t)$, i.e., a signal that our torque T must follow, to drive the state of the pendulum where we wish. Let us define this point of interest as $x^* = \begin{bmatrix}\theta^* \\ 0\end{bmatrix}$, i.e., a fixed angle with (obviously) zero velocity. Indeed, this is an equilibrium point for the angle θ. In order to have an equilibrium, we need to find a $u(t)$ in (2.5) such that $\frac{\mathrm{d}}{\mathrm{dt}}\left(\begin{bmatrix}\theta \\ \dot\theta\end{bmatrix}\right) = \begin{bmatrix}0 \\ 0\end{bmatrix}$. A quick inspection to the dynamics (2.2) we have that

$$u^* = T^* = -\frac{g}{l}\sin\theta^*, \tag{5.17}$$

for example, for the vertical position of the pendulum corresponding to $\theta^* = 0$ we have that $T^* = 0$, i.e., $x^* = \begin{bmatrix}0 \\ 0\end{bmatrix}$ and $u^* = 0$.

The calculation of the matrices A, B, C, and D are the corresponding Jacobians for (2,5) and (2,6), i.e.,

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= 0 \\
\frac{\partial f_1}{\partial x_2} &= 1 \\
\frac{\partial f_2}{\partial x_1} &= \frac{g}{l}\cos\theta \\
\frac{\partial f_2}{\partial x_2} &= -\frac{b}{ml^2} \\
\frac{\partial f_1}{\partial u_1} &= 0 \\
\frac{\partial f_2}{\partial u_1} &= 1 \\
\frac{\partial g_1}{\partial x_1} &= 1 \\
\frac{\partial g_1}{\partial x_2} &= 0 \\
\frac{\partial g_1}{\partial u_1} &= 0,
\end{aligned}$$

therefore we can arrive at

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\left(\begin{bmatrix}\delta\theta \\ \dot{\delta\theta}\end{bmatrix}\right) &= \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos\theta & -\frac{b}{ml^2}\end{bmatrix}_{|\theta=\theta^*} \begin{bmatrix}\delta\theta \\ \dot{\delta\theta}\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}\delta T \\
\delta y &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\delta\theta \\ \dot{\delta\theta}\end{bmatrix} + 0\,\delta T,
\end{aligned} \tag{5.18}$$

to model the dynamics of $x(t)$ and the output $y(t)$ around the points x^* and u^*.

Finally, we would like to hightlight that the Jacobians can have time-varying elements, and still have a linear system. For example, we can consider that the lenght l depends on the time explicitly, e.g., $l(t) = l + \sin(t)$. In such a case, we would have a $A(t)$.

5.6. (Internal or Lyapunov) Stability

We say that the linear system (**??**) *in the sense of Lyapunov*

1. is *(marginally) stable* if for every initial condition x_0, then $x(t) = \Phi(t, t_0)x_0$ is uniformily bounded for all $t > t_0$.

2. is *asymptotically stable* if, in addition, $x(t) \to 0$ as $t \to \infty$.

3. is *exponentially stable* if, in addition, $||x(t)|| \leq ce^{\lambda(t-t_0)}||x(t_0)||$ for some constants $c, \lambda > 0$.

4. is *unstable* if it is not marginally stable.

Let us now check the particular case when A is constant, i.e., $\Phi(t, t_0) = e^{A(t-t_0)}$. Then, we can stablish a clear relation between the eigenvalues of A and the stability definitions in the sense of Lyapunov by inspecting the solution to $\dot{x}(t) = Ax(t)$ given by (**??**).

The system $\dot{x}(t) = Ax(t)$

1. is marginally stable if and only if all the eigenvalues of A have negative real part or zero real parts (with all the Jordan blocks being 1×1).
2. is asymptotically (and equivalently exponentially) stable if and only if all the eigenvalues of A have strictly negative real parts.
3. is unstable if and only if at least one eigenvalues of A has a positive real part or zero real part (with the corresponding Jordan block being larger than 1×1).

Checking the solution(s) to a linear system (**??**)-(**??**) we can say that if A is constant and $\dot{x} = Ax$ is asymptotically stable, then $x(t) \to \int_{t_0}^{t} e^{A(t-\tau)}B(\tau)u(\tau)d\tau$ as $t \to \infty$.

5.6.1. Stability of locally linearized systems

The following equation is also equivalent to have asymptotic (exponential) stability for $\dot{x}(t) = Ax(t)$. There exists a unique solution P for the following *Lyapunov equation*

$$A^T P + PA = -Q, \quad \forall Q \succ 0. \tag{5.19}$$

You can prove (5.19) by considering

$$P := \int_0^\infty e^{A^T t} Q e^{At} dt. \tag{5.20}$$

Hint: First, substitute P in (5.19), and then check the calculation $\frac{\mathrm{d}}{\mathrm{dt}}\left(e^{A^T t} Q e^{At}\right)$. Since P is unique, then P must be positive definite according to its definition (5.20).

Let us consider an autonomous continuous-time nonlinear system

$$\dot{x}(t) = f(x(t)), \quad x \in \mathbb{R}^n, \tag{5.21}$$

with an equilibrium point $x^* \in \mathbb{R}^n$, i.e., $f(x^*) = 0$. The dynamics of $x(t)$ around x^* can be approximated by considering $x(t) = x^* + \delta x(t)$ where

$$\dot{\delta x}(t) = A\,\delta x(t), \quad A := \frac{\partial f(x)}{\partial x}. \tag{5.22}$$

What is this approximation good for?

Theorem 3. *Assume that $f(x)$ is twice differentiable. If (5.22) is exponentially stable, then there exists a neighborhood $\mathcal{B}$ around x^* and constants $c, \lambda > 0$ such that for every solution $x(t)$ to the nonlinear system (5.21) that starts at $x(t_0) \in \mathcal{B}$, we have*

$$||x(t) - x^*|| \leq c e^{\lambda(t-t_0)} ||x(t_0) - x^*||, \quad \forall t \geq t_0. \tag{5.23}$$

How big is $\mathcal{B}$? Can we estimate it? Sketch of the proof of Theorem 3

Since f is twice differentiable, from its Taylor's series we have that

$$r(x) := f(x) - (f(x^*) + A(x - x^*)) = f(x) - A\,\delta x = O(||\delta x||^2), \tag{5.24}$$

which means that there exist a constant c and a ball $\bar{B}$ around x^* such that

$$||r(x)|| \leq c||\delta x||^2, \quad x \in \bar{B}. \tag{5.25}$$

If the linearized system is exponentially stable, we have that

$$A^T P + PA = -I. \tag{5.26}$$

Now consider the following scalar signal

$$v(t) := (\delta x)^T P \delta x, \quad \forall t \geq 0. \tag{5.27}$$

Noting that $\delta x(t) = x(t) - x^*$, then $\dot{\delta x}(t) = \dot{x}(t) = f(t)$. Therefore, the time derivative of $v(t)$ satisfies

$$\begin{aligned}
\dot{v} &= f(x)^T P \delta x + (\delta x)^T P f(x) \\
&= (A\delta x + r(x))^T P \delta x + (\delta x)^T P (A\delta x + r(x)) \\
&= (\delta x)^T (A^T P + PA)\delta x + 2(\delta x)^T P r(x) \\
&= -||\delta x||^2 + 2(\delta x)^T P r(x) \\
&\leq -||\delta x||^2 + 2||P||\, ||\delta x||\, ||r(x)||.
\end{aligned} \tag{5.28}$$

We have that $v(t)$ is positive excepting when $\delta x = 0$. If we can guarantee that $\dot{v}(t) < 0$ and $\dot{v}(t) = 0$ only when $\delta x = 0$, then $v(t) \to 0$ as $t \to \infty$, which means that $\delta x(t) \to 0$ as $t \to \infty$.

Now, if $x \in \bar{\mathcal{B}}$, then

$$\dot{v} \leq -\Big(1 - 2c\,||P||\,||\delta x||\Big)||\delta x||^2, \tag{5.29}$$

Thus, if the deviation δx is small enough, i.e.,

$$||\delta x|| < \frac{1}{2c||P||}, \tag{5.30}$$

then $\dot{v}(t) < 0$ if $\delta x(0) \neq 0$ and $\delta x(0) < \frac{1}{2c||P||}$.

We can conclude that an estimation of $\mathcal{B}$ is

$$\mathcal{B} := \{\delta x : ||\delta x|| < \frac{1}{2c||P||}\}. \tag{5.31}$$

5.7. Controllability

5.7.1. Reachable and Controllable subspaces

We recall that when we apply an input $u(\cdot)$ to (**??**), we transfer the system from a state $x(t_0) := x_0$ to a state $x(t_1) := x_1$, and it is calculated from (**??**) as follows

$$x_1 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau, \tag{5.32}$$

where $\Phi(\cdot)$ is the system's state transition matrix.

Questions:

1. Which states can I reach from x_0?
2. Is there always an input $u(\cdot)$ that transfers the system from an arbirtrary state x_0 to another arbitrary state x_1?

These two questions lead to the definition of the reachable and controllable subspaces.

Definition 3 (Reachable subspace)**.** *Given two times $t_1 > t_0 \geq 0$, the reachable or controllable-from-the-origin on $[t_0, t_1]$ subspace $\mathcal{R}[t_0, t_1]$ consists of all the states x_1 for wichi there exists an input $u : [t_0, t_1] \to \mathbb{R}^k$ that transfers the state from $x_0 = 0$ to $x_1 \in \mathbb{R}^n$; i.e.,*

$$\mathcal{R}[t_0, t_1] := \Big\{x_1 \in \mathbb{R}^n : \exists u(\cdot),\, x_1 = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau\Big\}. \tag{5.33}$$

Definition 4 (Controllable subspace)**.** *Given two times $t_1 > t_0 \geq 0$, the controllable or controllable-to-the-origin on $[t_0, t_1]$ subspace $\mathcal{C}[t_0, t_1]$ consists of all the states x_1 for wichi there exists an input $u : [t_0, t_1] \to \mathbb{R}^k$ that transfers the state from $x_0 \in \mathbb{R}^n$ to $x_1 = 0$; i.e.,*

$$\mathcal{C}[t_0, t_1] := \Big\{x_0 \in \mathbb{R}^n : \exists u(\cdot),\, 0 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau\Big\}. \tag{5.34}$$

How to calculate the $\mathcal{R}[t_0, t_1]$ and $\mathcal{C}[t_0, t_1]$ subspaces? We will exploit the following two matrices called *Gramians.*

Definition 5 (Reachability and Controllability Gramians)**.**

$$W_R(t_0, t_1) := \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B(\tau)^T\Phi(t_1, \tau)^T d\tau \tag{5.35}$$

$$W_C(t_0, t_1) := \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B(\tau)^T\Phi(t_0, \tau)^T d\tau \tag{5.36}$$

Theorem 4. *Given two times $t_1 > t_0 \geq 0$,*

$$\mathcal{R}[t_0, t_1] = Im\{W_R(t_0, t_1)\} \tag{5.37}$$

$$\mathcal{C}[t_0, t_1] = Im\{W_C(t_0, t_1)\}, \tag{5.38}$$

where $Im\{A\} := \Big\{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n, y = Ax\Big\}$ for a matrix $A \in \mathbb{R}^{m\times n}$.

Demostración. We will only prove (5.37) since (5.38) has a similar proof. We need to show both ways: firsty, if $x_1 \in \text{Im}\{W_R(t_0, t_1)$, then $x_1 \in \mathcal{R}[t_0, t_1]$; secondly, if $x_1 \in \mathcal{R}[t_0, t_1]$ then $x_1 \in \text{Im}\{W_R(t_0, t_1)$.
When $x_1 \in \text{Im}\{W_R(t_0, t_1)$, there exists a vector $\mu_1 \in \mathbb{R}^n$ such that

$$x_1 = W_R(t_0, t_1)\eta_1. \tag{5.39}$$

Choose $u(\tau) = B(\tau)^T\Phi(t_1, \tau)^T\eta_1$, and plug it into (5.33), then we have that

$$x_1 = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B(\tau)^T\Phi(t_1, \tau)^T\eta_1 d\tau = W_R(t_0, t_1)\eta_1. \tag{5.40}$$

When $x_1 \in \mathcal{R}[t_0, t_1]$, there exists an input $u(\cdot)$ for which

$$x_1 = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau. \tag{5.41}$$

If (5.41) is in $\text{Im}\{W_R(t_0, t_1)\}$, then $x_1^T\mu = 0$, $\mu \in \text{Ker}\{W_R(t_0, t_1)\}$[1]. Let us calculate

$$x_1^T\mu = \int_{t_0}^{t_1} u(\tau)^T B(\tau)^T\Phi(t_1, \tau)^T\mu\, d\tau. \tag{5.42}$$

[1]If $x \in \text{Ker}\{A^T\}$, then $A^Tx = 0$. If $y \in \text{Im}\{A\}$, then $y = A\eta$. Thus, $x^Ty = x^TA\eta = \eta^TA^Tx = \eta \cdot 0 = 0$. Note that $W_R^T = W_R$ by definition.

And noting that

$$
\begin{aligned}
\mu \in \operatorname{Ker}\{W_R(t_0,t_1)\} \implies \mu^T W_R(t_0,t_1)\mu &= 0 \\
&= \int_{t_0}^{t_1} \mu^T \Phi(t_1,\tau)B(\tau)B(\tau)^T\Phi(t_1,\tau)^T \mu \, d\tau \\
&= \int_{t_0}^{t_1} ||B(\tau)^T\Phi(t_1,\tau)^T\mu||^2 d\tau,
\end{aligned} \tag{5.43}
$$

give us that $B(\tau)^T\Phi(t_1,\tau)^T\mu = 0$, leading to (5.42) equals zero. □

Remark 1. *Note that we have proven that $u(\tau) = B(\tau)^T\Phi(t_1,\tau)^T\eta_1$ can be used as a control input to transfer the system from $x_0 = 0$ to $x_1 \in \mathbb{R}^n$ in the finite time $(t_1 - t_0)$. In fact, this is the* open-loop minimun energy control.

To see this fact, consider another control input $\bar{u}(t)$ so that

$$
x_1 = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)\bar{u}(\tau)d\tau. \tag{5.44}
$$

For this to hold, we must have

$$
\int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)v(\tau)\, d\tau = 0, \tag{5.45}
$$

where $v(\tau) = u(\tau) - \bar{u}(\tau)$. Let us see the energy of $\bar{u}$

$$
\begin{aligned}
\int_{t_0}^{t_1} ||\bar{u}(\tau)||^2 d\tau &= \int_{t_0}^{t_1} ||B(\tau)^T\Phi(t_1,\tau)^T\eta_1 + v(\tau)||^2 d\tau \\
&= \eta_1^T W_R(t_0,t_1)\eta_1 + \int_{t_0}^{t_1} ||v(\tau)||^2 d\tau + 2\eta_1^T \int_{t_0}^{t_1} B(\tau)\Phi(t_1,\tau)v(\tau)d\tau
\end{aligned} \tag{5.46}
$$

where the third term is zero because of (5.45). Hence, if $\bar{u}(t)$ difers $v(t)$ from $u(t)$, it will expend $\int_{t_0}^{t_1} ||v(\tau)||^2 d\tau$ more energy than $u(t)$.

5.7.2. Controllability matrix for A, B being constant

We have that the Cayley-Hamilton theorem allows us to write

$$
e^{At} = \sum_{i=0}^{n-1} \alpha_i(t)A^i, \quad \forall t \in \mathbb{R}, \tag{5.47}
$$

for some appropriate scalar functions $\alpha_i(t)$. We also have that if A and B are constant then

$$
\begin{aligned}
x_1 &= \int_0^{t_0 - t_1} e^{At}Bu(t)dt \\
&= \sum_{i=0}^{n-1} A^iB\Big(\int_0^{t_1-t_0} \alpha_i(t)u(t)dt\Big) \\
&= \mathcal{C}\begin{bmatrix} \int_0^{t_1-t_0} \alpha_0(t)u(t)dt \\ \vdots \\ \int_0^{t_1-t_0} \alpha_{n-1}(t)u(t)dt \end{bmatrix},
\end{aligned} \tag{5.48}
$$

where $\mathcal{C} := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}_{n\times(kn)}$ [2]. Note that we just have proven that the image of $\mathcal{C}$ is the same as the image of $W_R(t_0,t_1)$. The following more general result can be proven as well:

Theorem 5. *For any two times t_0, t_1, with $t_1 > t_0$, we have*

$$\mathcal{R}[t_0,t_1] = Im\{W_R(t_0,t_1)\} = Im\{\mathcal{C}\} = Im\{W_C(t_0,t_1)\} = \mathcal{C}[t_0,t_1], \tag{5.49}$$

Two important consequences can be extracted from Theorem 5 for systems with A and B constant. Note that $\mathrm{Im}\{\mathcal{C}\}$ does not depend on any time variable!

1. *Time reversibility*: If you can reach x_1 from zero, then you can reach zero from x_1, i.e, the controllability and reachibility subspaces are the same.

2. *Time scaling*: The notions of reachable and controllable subspaces do not depend on the considered time interval. If you can transfer the state from x_0 to x_1 in t seconds, then you can also do it in $\bar{t}$ seconds.

Definition 6. *Given two times $t_1 > t_0 \geq 0$, the pair (A,B) from system (**??**) is reachable on $[t_0,t_1]$ if $\mathcal{R}[t_0,t_1] = \mathbb{R}^n$, i.e., if we can drive the state of the system from the origin to any arbitrary state in finite time.*

Definition 7. *Given two times $t_1 > t_0 \geq 0$, the pair (A,B) from system (**??**) is controllable on $[t_0,t_1]$ if $\mathcal{C}[t_0,t_1] = \mathbb{R}^n$, i.e., if every state can be driven to the origin in finite time.*

5.7.3. Controllability tests

The following theorem is the combination of the Definition 7 and Theorem 5:

Theorem 6. *The (constant) pair (A,B) is controllable if and only if the rank of $\mathcal{C}$ is n.*

The following theorem is an easy check numerically.

Theorem 7. *The (constant) pair (A,B) is controllable if and only if there is no eigenvector of A^T in the kernel of B^T.*

The following theorem is a restatement of the previous one.

Theorem 8. *The (constant) pair (A,B) is controllable if and only if the rank of $\begin{bmatrix} A-\lambda I & B \end{bmatrix}$ is n.*

Remark 2. *Note that having an asymptotically stable system does not imply to have a controllable system (pair (A,B) controllable). Let us also illustrate Theorem 7 with the following system:*

$$\dot{x} = \begin{bmatrix} -3 & 0 \\ 0 & -7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x \in \mathbb{R}^2, u \in \mathbb{R}. \tag{5.50}$$

*The kernel of B^T is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which is an eigenvector of $A = A^T$. Therefore, the system is not controllable according to Theorem 7. Note that we do not have any control over x_1 via u. However, it is asympotically stable. One can see that $x_{\{1,2\}}(t) \to 0$ as $t \to \infty$ when $u = 0$. Remember that controllability is about going from a nonzero state x^* to 0 in **finite time**.*

[2] Please, note that $\mathcal{C}[t_0,t_1]$ and $\mathcal{C}$ are different objects. The former is a space, the latter is a matrix.

5.8. Feedback stabilization based on the Lyapunov test

5.8.1. Lyapunov test for stabilization

Definition 8. *The system (??) is stabilizable if there exists an input $u(t)$ for every $x(0)$ such that $x(t) \to 0$ as $t \to \infty$.*

This is somehow a *controllable* version of the system in infinite time instead of in finite time. In the next theorem, we will see that *stabilizable* is less restrictive than *controllable*.

Theorem 9. *The system (??) is stabilizable if and only if every eigen-vector of A^T corresponding to an eigenvalue with a positive or zero real part is not in the kernel of B^T.*

The components of $x(t)$ corresponding to the eigenvectors with associated eigenvalues with negative real part go to zero as the time goes to infinity without any *assistance* from an input. Therefore, the input $u(t)$ must counterreact the components corresponding to the eigenvectors with associated eigenvalues with non-negative real part. For example:

$$\dot{x} = \begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x \in \mathbb{R}^2, u \in \mathbb{R}, \tag{5.51}$$

while we do not have any *authority* over $x_1(t)$, we can employ $u(t)$ to drive $x_2(t)$ to zero so that $x(t) \to 0$ as $t \to \infty$.

Theorem 10. *The system (??) is stabilizable if and only if there is a positive-definite solution P to the following Lyapunov matrix inequality*

$$AP + PA^T - BB^T \prec 0 \tag{5.52}$$

Demostración. We will see only one direction of the proof. Do not confuse (5.52) with the Lyapunov equation[3] $PA + A^T P \prec -I$ in (5.19).

Consider x being an eigenvector associated to an eigenvalue λ with non-negative real part of A^T. Then

$$x^*(AP + PA^T)x < x^* BB^T x = ||B^T x||^2, \tag{5.53}$$

where x^* is the complex conjugate transpose of x. But the left-hand side of (5.53) equals

$$(A^T(x^*)^T)^T Px + x^* PA^T x = \lambda^* x^* Px + \lambda x^* Px = 2\text{Real}\{\lambda\} x^* Px. \tag{5.54}$$

Since P is positive-definite and $\text{Real}\{\lambda\} \geq 0$, we can conclude that

$$0 \leq 2\text{Real}\{\lambda\} x^* Px < ||B^T x||^2, \tag{5.55}$$

and therefore x must not belong to the kernel of B, otherwise we will have

$$0 \leq 2\text{Real}\{\lambda\} x^* Px < 0, \tag{5.56}$$

which cannot be possible. □

[3]Note the order of the transposed matrices, and also note the opposite sign of $-BB^T$ and $+I$ for the two equations.

5.8.2. State-feedback controller

Define the *control matrix gain*

$$K := \frac{1}{2}B^T P^{-1}, \tag{5.57}$$

where P is calculated from (5.52) if and only if the system (**??**) is stabilizable. Therefore, we can rewrite (5.52) as

$$(A - \frac{1}{2}BB^T P^{-1})P + P(A - \frac{1}{2}BB^T P^{-1})^T = (A - BK)P + P(A - BK)^T \prec 0, \tag{5.58}$$

and by multiplying on the left and right by $Q := P^{-1}$ we have that

$$Q(A - BK) + (A - BK)^T Q \prec 0, \tag{5.59}$$

which is the Lyapunov equation. Thus, we can conclude that $(A - BK)$ has all its eigenvalues with negative real part, i.e., if one chooses the input

$$u = -Kx = -\frac{1}{2}B^T P^{-1}x, \tag{5.60}$$

in the stabilizable (**??**), then $x(t) \to 0$ as $t \to \infty$ exponentially fast.

5.9. Observability for linear time invariant (lti) systems

In this section we will consider only the following linear system

$$\Sigma_{\text{lti}} := \begin{cases} \dot{x}(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) + Du(t) \end{cases}. \tag{5.61}$$

5.9.1. Unobservable subspace and the observability Gramian

Definition 9. *The* unobservable *subspace $\mathcal{UO}$ of system (5.61) consists of all the states $x_0 \in \mathbb{R}^n$ such that*

$$Ce^A x_0 = 0. \tag{5.62}$$

This definition is motivated by the following facts. Recall that from (**??**) we have that

$$\begin{aligned} y(t) &= Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \\ \tilde{y}(t) := y(t) - \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau - Du(t) &= Ce^{At}x_0. \end{aligned} \tag{5.63}$$

On the left hand side of (5.63) we have a pair input/output, and on the right hand side of (5.63) we have the initial state x_0. From (5.63) we can observe two interesting properties:

1. When a particular x_0 is compatible with an input/output pair, then every initial state of the form $x_0 + x_u$, $x_u \in \mathcal{UO}$, is also compatible with the same input/output pair.

2. When $\mathcal{UO}$ contains only the zero vector, then there exists at most one initial state x_0 that is compatible with the input/output pair.

Definition 10. *The system (5.61) is observable if its $\mathcal{UO}$ contains only the zero vector.*

Definition 11. *Given two times $t_1 > t_0 \geq 0$, the observability Gramian is defined by*

$$W_O(t_0, t_1) := \int_{t_0}^{t_1} e^{A^T(\tau - t_0)} C^T C e^{A(\tau - t_0)} d\tau \tag{5.64}$$

There is difference w.r.t. the controllability Gramian, the transposes appear on the left for the observability Gramian, whereas the transposes appear on the right for the controllability Gramian. Starting with (5.62), it can be shown that

$$\operatorname{Ker} W_O(t_0, t_1) = \mathcal{UO}. \tag{5.65}$$

5.9.2. Observability tests

The following can be proven formally by applying the theorem of Cayley-Hamilton[4]. Let us now check the sensibility of y and its time derivatives with respect to x when[5] $u(t) = 0$

$$\begin{aligned} y(t) = Cx(t) \implies \dot{y}(t) = C\dot{x}(t) = CAx(t) \implies \ddot{y}(t) = CA^2x(t) \quad \ldots \\ \implies \frac{\mathrm{d}^{n-1}y}{\mathrm{d}\mathrm{t}^{n-1}} = CA^{n-1}x(t), \end{aligned}$$

If we do not want to loose any information of the signal $x(t)$, then the matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{(kn)\times n}, \tag{5.66}$$

must be full column rank. We, then, introduce the following equivalent results.

Theorem 11. *The system (5.61) is observable if and only if the rank of $\mathcal{O}$ equals n.*

Theorem 12. *The system (5.61) is observable if and only if no eigenvector of A is in the kernel of C.*

Note that from (5.66) we can derive a *controllability* test

$$\mathcal{O}^T = \begin{bmatrix} C^T & A^T C^T & \cdots & (A^{n-1})^T C^T \end{bmatrix}_{n\times(kn)}, \tag{5.67}$$

from which we can derive from the following *dual* system constructed from (5.61)

$$\Sigma_{\text{dual}} := \begin{cases} \dot{\bar{x}}(t) & = A^T \bar{x}(t) + C^T \bar{u}(t) \\ \bar{y}(t) & = B^T \bar{x}(t) + D^T \bar{u}(t) \end{cases}, \tag{5.68}$$

then, we have the following result

Theorem 13. *The system (5.61) is controllable if and only if (5.68) is observable.*

Therefore, we only need to study the controllability of the dual system (5.61) to conclude about its observability.

---

[4] To see why we stop at $(n-1)$.

[5] Note that B and D do not play any role for the observability, only A and C.

5.10. State estimation for linear time invariant systems

The simplest estimator consists of a copy of the system (5.61)

$$\Sigma_{\text{estimator}} := \begin{cases} \dot{\hat{x}}(t) & = A\hat{x}(t) + Bu(t) \\ \hat{y}(t) & = C\hat{x}(t) + Du(t) \end{cases}, \tag{5.69}$$

and let us define the error signal $e(t) := \hat{x}(t) - x(t)$. Now, let us check the dynamics of the error signal

$$\dot{e}(t) = \dot{\hat{x}}(t) - \dot{x}(t) = A\hat{x} + Bu - Ax - Bu = Ae(t), \tag{5.70}$$

therefore $e(t) \to 0$ as $t \to \infty$ exponentially fast if A is a stability matrix for every input signal $u(t)$.

What if A is not a stability matrix? Then, consider the following system

$$\Sigma_{\text{estimator2}} := \begin{cases} \dot{\hat{x}}(t) & = A\hat{x}(t) + Bu(t) - L(\hat{y}(t) - y(t)) \\ \hat{y}(t) & = C\hat{x}(t) + Du(t) \end{cases}, \tag{5.71}$$

where we have *two inputs* u and $(\hat{y} - y)$, and $L \in \mathbb{R}^{n \times m}$ is a matrix gain. Note that the signal y comes from (5.61). Now, let us see the dynamics of the error signal

$$\dot{e} = A\hat{x} + Bu - L(\hat{y} - y) - (Ax + Bu) = (A - LC)e, \tag{5.72}$$

thus, if we can make $(A - LC)$ a stability matrix, then $e(t) \to 0$ as $t \to \infty$ exponentially fast for every input signal $u(t)$.

Sometimes, it is not possible to find such a L because of the structure of C (something similar happened for K and B for the stabilizability and controllability of (**??**)). Then we have a similar result about *detectatbility*.

Theorem 14. *The system (5.61) is* detectable *if and only if every eigenvector of A corresponding to a non stable eigenvalue is not in the kernel of C.*

Remember, a system is detectable if its dual is stabilizable.

Theorem 15. *When a pair (A, C) is detectable, it is always possible to find L such that $(A - LC)$ is a stability matrix.*

Theorem 16. *When a pair (A, C) is observable, it is always possible to find L such that $(A - LC)$ is a stability matrix with an arbitrary set of stable eigenvalues.*

In such a case, we can always compute L as we have computed K as in (5.57) for controllable systems by analyzing the dual system of (5.61). Note that in such a computation, since we are dealing with the dual system, the matrix gain K will be L^T.

5.11. Stabilization of linear time invariant systems through output feedback

If C is not invertible, then we do not have *direct access* to the states x in (5.61); therefore we cannot apply the control input $u = -Kx$ as in (5.60).

What if (5.61) is observable, or at least detectable? Then, we are going to show that we can employ

$$u = -K\hat{x}, \tag{5.73}$$

where $\hat{x}$ are the states of our estimator (5.71), as a control action to stabilize (5.61) around the origin. Let us apply (5.73) in (5.61), therefore we have that

$$\dot{x} = Ax - BK\hat{x} = Ax - BK(e + x) = (A - BK)x - BKe, \tag{5.74}$$

that together with (5.72) give us the following autonomous system

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}, \tag{5.75}$$

which is exponentially stable since $(A - BK)$ and $(A - LC)$ are stability matrices (if our system (5.61) is at least stabilizable and detectable) and the triangular structure of the state-transition matrix.

Capítulo 6

Control por realimentación de estados

6.1. Guideline to design a linear controller for the inverted pendulum

1.

Given the model Σ of a dynamical system

$$\Sigma := \begin{cases} \dot{x}(t) = & f(x(t), u(t)) \\ y(t) = & g(x(t), u(t)) \end{cases},$$

choose an operational/equilibrium point x^* of interest.

For the pendulum, let us choose when it is in vertical position and at rest, i.e., $\theta^* = 0$, $\dot{\theta}^* = 0$. So you we have that $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

2.

Since x^* is an equilibrium, find out which u^* makes $\dot{x}(t) = 0$, i.e., $f(x^*, u^*) = 0$.
We have that $u = T$, and that $\ddot{\theta} = \frac{1}{ml^2}\left(mgl\sin\theta - b\dot{\theta} + T\right)$. To keep x^* fixed, we need to set $\ddot{\theta} = 0$, therefore $u^* = T^* = 0$. Note that for another x^*, we would have different T^*.

3.

Now we want the system around x^* and u^* to be an stable equilibrium, i.e., for a small deviation/disturbance δx, we need to calculate the necessary δu to keep the system at x^*.

We calculate the dynamics of δx and δu, i.e., we linearize Σ around x^* and u^*.

$$\Sigma := \begin{cases} \dot{x}(t) = & f(x(t), u(t)) \\ y(t) = & g(x(t), u(t)) \end{cases} \approx \begin{cases} \dot{\delta x}(t) & = A(t)\delta x(t) + B(t)\delta u(t) \\ \dot{\delta y}(t) & = C(t)\delta x(t) + D(t)\delta u(t) \end{cases} \quad \text{if } x \approx x^* + \delta x, u \approx u^* + \delta u,$$

where the matrices $A(t), B(t), C(t)$ and $D(t)$ are the Jacobians from Week 14. Note that it is usual to set the origin of the coordinates x the system at x^*, this is why you will find in many places

(including Week 14) $\delta x = x$ for the linearized version of Σ.
The Jacobians for the inverted pendulum are

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ \frac{1}{ml^2}(mgl\cos\theta) & -\frac{b}{ml^2} \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ D &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{6.1}$$

Note that A has to be evaluated at x^* (check the notes from Week 14). Therefore, for $\theta = 0$, we have that $A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix}$. We assume that $C = I$, i.e., we can measure all the elements from the state vector x.

4.

Let us calculate the linear controller

$$\delta u = K\delta y, \tag{6.2}$$

such that x^* is stable. We substitute (6.2) in the linearized Σ resulting in

$$\begin{aligned} \dot{\delta x}(t) &= A(t)\delta x(t) + B(t)K\delta y \\ &= A(t)\delta x(t) + B(t)KC(t)\delta x(t) + KD(t)\delta u(t) \\ &= (A(t) + B(t)KC(t))\, x(t) + KD(t)u(t) \end{aligned} \tag{6.3}$$

Consider that $D(t) = 0$, and $A(t)$ and $B(t)$ are constant matrices, i.e., their elements do not depend on time. Then, we can write (6.3) as

$$\dot{\delta x}(t) = (A + BKC)\, x(t) \tag{6.4}$$

$$= Mx(t). \tag{6.5}$$

Then, the linearized system Σ is stable around x^* under small disturbances if and only if M has all its eigenvalues with negative real part (Week 15). To have the addition $A + BKC$, we need K with the appropriate dimensions. For $C = I$ we have that $K = \begin{bmatrix} k_{11} & k_{12} \end{bmatrix}$, so we have that

$$M = A + KBC = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} + \frac{k_{11}}{ml^2} & -\frac{b}{ml^2} + \frac{k_{12}}{ml^2} \end{bmatrix} \tag{6.6}$$

5.

A matrix $M \in \mathbb{R}^{n\times n}$ has n eigenvalues. The eigenvalues of M can be calculated from the following determinant

$$\det\{M - \lambda_i I\} = 0, \quad i \in \{1, \dots, n\}. \tag{6.7}$$

For example, for a 2×2 matrix we have that $\det\{A\} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$. Therefore we have that (6.7) is

$$(m_{11} - \lambda_i)(m_{22} - \lambda_i) - m_{12}m_{21} = 0, \quad i \in \{1, 2\}. \tag{6.8}$$

The values for the elements of K are calculated by setting an arbitrary $\lambda_i < 0$. This solution is guaranteed for $C = I$ and $D = 0$.
We have that (6.8) from M in (6.6) is

$$\lambda^2 + \lambda\left(\frac{1}{ml^2}(b - k_{12})\right) - \frac{g}{l} - \frac{k_{11}}{ml^2},$$

whose solution is given by

$$\lambda_{1,2} = \frac{-\frac{1}{ml^2}(b - k_{12}) \pm \sqrt{\frac{(b-k_{12})^2}{m^2l^4} + 4(\frac{g}{l} + \frac{k_{11}}{ml^2})}}{2}. \tag{6.9}$$

Let us find some conditions for k_{11} and k_{12} such that we can guarantee that λ_1 and λ_2 are two real negative numbers. For example, if force

$$k_{12} < b, \tag{6.10}$$

then $-\frac{1}{ml^2}(b-k_{12})$ in (6.9) is a negative number. Note that b is a coefficient friction in the pendulum equation, therefore if $b = 1$, we can have $-\infty < k_{12} < 1$, i.e., the gain k_{12} can be even positive as long as it is smaller than b. Now we turn our attention at the square root in (6.9). Assume that we take the positive solution $r > 0$ of the square root in (6.9). Now we need to add or subtract r to $-\frac{1}{ml^2}(b - k_{12})$. We note that for λ_1 if $-\frac{1}{ml^2}(b - k_{12})$ is negative then $-\frac{1}{ml^2}(b - k_{12}) - r$ is still negative, so $\lambda_1 < 0$. For λ_2 we need to calculate k_{11} such that $-\frac{1}{ml^2}(b - k_{12}) + r < 0$. If we set $k_{11} < -gml$ then $\sqrt{\frac{(b-k_{12})^2}{m^2l^4} + 4(\frac{g}{l} + \frac{k_{11}}{ml^2})} < \sqrt{\frac{(b-k_{12})^2}{m^2l^4}} = \frac{1}{ml^2}(b - k_{12})$. Therefore we guarantee that $r < \frac{1}{ml^2}(b - k_{12})$, thus $\lambda_2 < 0$. Note that k_{11} not only needs to be negative but *negative enough*. Check in the Python script the consequences of playing around these limits for k_{11} and k_{12}.

6.2. Controller for the inverted pendulum

Design a controller for

$$x_1^* = \begin{cases} \theta^* & = 0 \\ \dot{\theta}^* & = 0 \end{cases}$$

$$x_2^* = \begin{cases} \theta^* & = \frac{\pi}{4} \\ \dot{\theta}^* & = 0 \end{cases}.$$

We will first assume that we can measure θ and $\dot{\theta}$, i.e., $C = I$. Note that for $M = (A - BKC)$ with $C = I$ the dimensions of K must be 1×2, i.e., $K = \begin{bmatrix} k_{11} & k_{12} \end{bmatrix}$. Note that in this case we have that $K\delta y = k_{11}\delta\theta + k_{12}\delta\dot{\theta}$. Remember that the input $u = u^* + \delta u$, and that for the pendulum $u = T$, i.e., the applied torque.

The exercise asks to find the values k_{11} and k_{12} for two arbitrary negative real eigenvalues λ_1 and λ_2 in (6.8).

Simulate your designed controller in the Python script. Check that your x^* are stable if you start close to them, and you can further check the robustness by applying small disturbances, e.g., add a small random number to x at every iteration.

Is it possible to design a stable controller with $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$? and for $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$? Note that for this cases K will have different dimensions than for $C = I$ since C has different dimensions as well.