Week 1: Potential Outcomes

PLSC 30600 - Causal Inference

Welcome!

Course Overview

- Instructor: Anton Strezhnev
- TA: Cindy Wang
- Logistics:
  - Lectures T/Th - 2:00pm - 3:20pm; Sections F - 2:30pm - 3:20pm
  - 3 Problem Sets (~ 2 weeks)
  - Take-home midterm (~ 1 week)
  - In-class final (exact day/time TBA)
  - My office hours: Tuesdays 4pm-6pm (Pick 328)
  - Cindy's office hours: TBA
- What is this course about?
  - *Defining* causal effects w/ a structured statistical framework (potential outcomes)
  - *Outlining* assumptions necessary to identify causal effects from data
  - *Estimating* and conducting *inference* on causal effects
- Goals for the course
  - Give you the tools you need to develop your own causal research designs and comment on others' designs.
  - Equip you with an understanding of the fundamentals of causal inference to enable you to learn new methods.
  - Professionalization -- how did causal inference emerge as a field and how do different disciplines approach the topic.

Course workflow

- Lectures
  - Topics organized by week
  - Tuesday lecture: Introduction + higher-level overview
  - Thursday lecture: More in-depth details + applications
  - You should do the readings prior to each week, but definitely be sure to do them before the Thursday.
- Readings
  - Mix of textbooks and papers - theory + application.
  - Organized on the syllabus by priority. Try to do all of them, but if you need to prioritize, start with the first ones on the list for the week.
  - All readings available digitally on Canvas

Course workflow

- Problem sets (25% of your grade)
  - Main day-to-day component of the class -- meant to get you working with your colleagues and thinking hard about the material.
  - Collaboration is **strongly encouraged** -- you should ask and answer questions on our class Ed Discussion board
  - Grading is on a **plus/check/minus** scale.
  - Conversion to grade point is somewhat hollistic.
  - Majority plusses is an A, Majority checks is an A-, Majority minus is a B
  - Solutions will be posted after the due-date.

Course workflow

- Midterm Exam (30% of your grade)

  - The midterm will be structured like the problem sets with two main differences:
    1. You have about 1 week to complete them instead of 2
    2. You may **not** collaborate with one another on the exams.

- Final Exam (35% of your grade)

  - The final exam will be in-person and written.
  - Combination of theoretical questions and code/results analysis.
- Participation (10% of your grade)
  - It is important that you actively engage with lecture and section -- ask and answer questions.
  - Do the reading!
  - Participating on Ed counts towards this as well.

Class Requirements

- **Overall**: An interest in learning and willingness to ask questions.
- Generally assume some background in probability theory and statistics (e.g. an intro course taught in most departments)
  - Regression a plus but not required -- if you know what $(X'X)^{-1}X'Y$ is, then that is great, but not strictly necessary.
- Main concepts to be familiar with:
  - properties of random variables
  - estimands and estimators
  - bias, variance, consistency
  - central limit theorem
  - confidence intervals; hypothesis testing

A brief overview

- **Week 1:** Introduction to potential outcomes
  - Defining causal estimands
- **Week 2-3:** Experiments
  - Why they work, design, inference
- **Week 4-5:** Selection-on-observables
  - Key assumptions, graphical models, how to think about confounding
  - Estimation via regression, weighting, matching + combinations
- **Week 6-8:** Designs for dealing with unobserved confounding
  - Instrumental variables
  - Differences-in-differences
  - Regression discontinuity
- **Week 9:** Causal mediation; Sensitivity analyses

Random variables and estimators

Random variables

- Understanding the behavior and properties of random variables is at the core of statistical theory.
- (Simply put) a random variable X is a mapping from a **sample space** to the real number line - Random variables have a *distribution* (which we may or may not assume we know) defined by the cumulative distribution function (CDF)

$$F(x) = Pr(X \leq x)$$

Random variables

- Discrete random variables take on a countable number of values (e.g. Bernoulli r.v. can take on 0 or 1) and have a probability mass function (PMF)

$$p(x) = Pr(X = x)$$

- Continuous random variables take on an uncountable number of values (e.g. the Normal distribution on $(-\infty, \infty)$). No PMF, but have a *density* function (PDF) that integrates to a probability

$$Pr(X \in \mathcal{A}) = \int_{\mathcal{A}} f(x) dx$$

- Remember: PMFs (and PDFs) sum (integrate) to 1 over the support of the random variable.

Expectations

- One important property of a random variable is its expectation $E[X]$. We'll often make assumptions about the expectation of an R.V. while remaining agnostic about its true distribution.
- The expectation is a *weighted average*. For a discrete r.v. X, we sum over the support of the random variable $\mathcal{X}$.

$$E[X] = \sum_{x \in \mathcal{X}} x Pr(X = x)$$

- For continuous r.v. we have an integral

$$E[X] = \int_{x\mathcal{X}} x f(x) dx$$

- Fun fact: we can get the expectation of any function of $g(X)$ just by plugging it into the integral

$$E[g(X)] = \int_{x\mathcal{X}} g(x) f(x) dx$$

Expectations

- You'll probably not have to do any integration in this class. Rather, we'll derive expectations of functions of random variables via known properties

- Most important. **Linearity**. For any two random variables X and Y and constants a and b

$$E[aX + bY] = aE[X] + bE[Y]$$

- Note that for any generic function $g()$, $E[g(X)] \neq g(E[X])$. If $g()$ is convex, by Jensen's inequality $E[g(X)] \geq g(E[X])$

- For a binary r.v. $X \in \{0, 1\}$, it's helpful to remember the "fundamental bridge" between expectations and probability

$$E[X] = Pr(X = 1)$$

Variance

- We also care about the *spread* of a random variable -- how far is the average draw of X from its mean $E[X]$. One measure of this is the variance.

$$Var(X) = E[(X - E[X])^2]$$

- Also written as

$$Var(X) = E[X^2] - E[X]^2$$

- Note that the square is a convex function. Which means that by Jensen's inequality $E[X^2] \geq E[X]^2$. Variances *cannot* be negative!

- We also can define a *covariance* between two variables (does X take high values when Y takes high values?)

$$Cov(X, Y) = E\left[(X - E[X])(Y - E[Y])\right] = E[XY] - E[X]E[Y]$$

Variance

- Again, you won't have to do the integral. Variances have some useful properties.
- For a constant a

$$Var(aX) = a^2 Var(X)$$

- For any two random variables X and Y

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

$$Var(X-Y) = Var(X) + Var(Y) - 2Cov(X,Y)$$

- For independent random variables X and Y

$$Var(X+Y) = Var(X) + Var(Y)$$

$$Var(X-Y) = Var(X) + Var(Y)$$

Conditional probabilities

- We will also spend *a lot of time* with conditional distributions and conditional expectations of random variables.
  - What's the probability that an individual enrolls in a job training program given their income?
  - We represent the conditioning set using a vertical bar with the right-hand side denoting what is being conditioned on.
    - For example: $Pr(D_i = 1|X_i = x)$

Conditional probabilities

- **Key concept** - Dependence and independence. If two variables are *independent*, the distribution of one does not change conditional on the other. We'll write this using the $\perp\!\!\!\perp$ notation.

- $Y_i \perp\!\!\!\perp D_i$ implies

$$Pr(Y_i = 1|D_i = 1) = Pr(Y_i = 1|D_i = 0) = Pr(Y_i = 1)$$

- Otherwise, the two variables are dependent: $Y_i \perp\!\!\!\perp D_i$.

$$Pr(Y_i = 1|D_i = 1) \neq Pr(Y_i = 1|D_i = 0)$$

- Two variables can be *conditionally independent* in that they are independent only when conditioning on a third variable. For example, we can have $Y_i \perp\!\!\!\perp D_i$ but $Y_i \perp\!\!\!\perp D_i|X_i$. This implies

$$Pr(Y_i = 1|D_i = 1, X_i = x) = Pr(Y_i = 1|D_i = 0, X_i = x) = Pr(Y_i = 1|X_i = x)$$

- **Remember**: Conditional independence *does not* imply independence or vice-versa!

Conditional expectations

- A central object of interest in statistics is the **conditional expectation** function (CEF) $E[Y|X]$.
  - Given a particular value of X, what is the expectation of Y?
  - The CEF is a function of X.
- All the usual properties of expectations apply to conditional expectations.

- We also will often make use of the *law of total expectation*

$$E[Y] = E[E[Y|X]]$$

- Easiest to think about this in terms of discrete r.v.s

$$E[Y] = \sum_{x \in \mathcal{X}} E[Y|X = x] Pr(X = x)$$

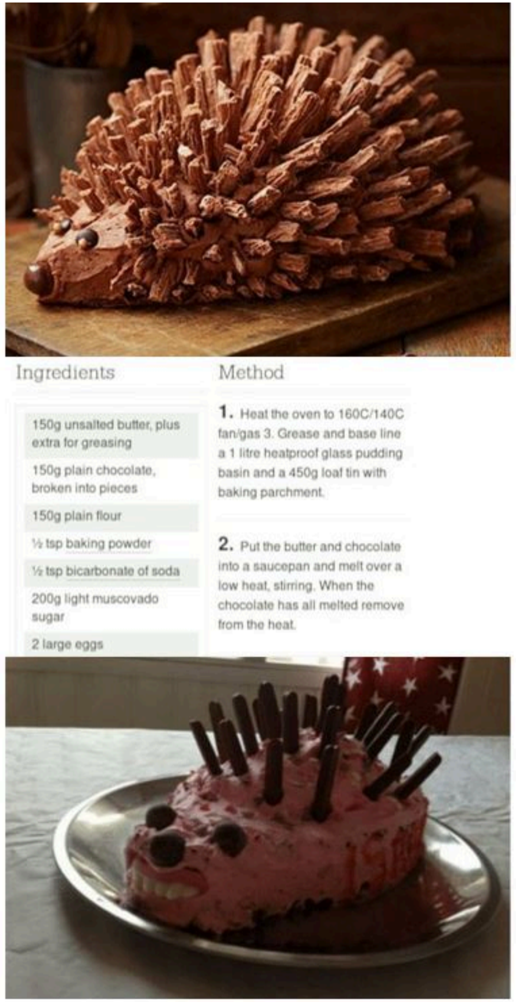
Estimation

- One critical use of statistical theory is understanding how to learn about things we don't observe using things that we do observe. We call this estimation.
  - e.g. What is the share of voters in Wisconsin who will turn out in the 2022 election?
  - What is the share of voters who turn out among those assigned to receive a GOTV phone call?
- Estimand: The **unobserved** quantity that we want to learn about. Often denoted via a greek letter (e.g. μ, π)
  - Often a "population" characteristic that we want to learn about via a sample.
    - But you'll learn another reason why we sometimes can't observe a quantity of interest even in a sample!
  - Important to define your estimand well. (Lundberg, Johnson and Stewart, 2022)

Estimation

- Estimator: The **function** of random variables that we will use to try to estimate the quantity of interest. Often denoted with a hat on the parameter of interest (e.g. $\hat{\mu}$, $\hat{\pi}$)
  - Why are the variables random?
    - Classic inference: We have a random sample from the population -- if we took another sample, we would obtain a different realization of our estimator.
    - Randomization inference: We have a randomly assigned treatment -- if we were to re-run the experiment, we would observe a different treatment/control allocation.
- Estimate: A single realization of our estimator (e.g. 0.3, 9.535)
  - We often report both point estimates ("best guess") and interval estimates (e.g. confidence intervals).
  - Careful not to confuse properties of estimators with properties of the estimates themselves.

Estimation



estimand

estimator

estimate

Estimation

- The classic estimation problem in statistics is to estimate some unknown population mean μ from an i.i.d. sample of n observations $Y_1, Y_2, \ldots, Y_n$.
  - We assume that each Y_i is a draw from the target population with mean μ. (identically distributed) -- therefore $E[Y_i] = \mu$
  - We'll also assume that conditioning on Y_i tells us nothing about any other Y_j $Y_i \perp\!\!\!\perp Y_j$ (independently distributed) -- this implies $Cov(Y_i, Y_j) = 0$
- Our estimand: μ
- Our estimator: The sample mean $\hat{\mu} = \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$
- Our estimate: A particular realization of that estimator based on our observed sample (e.g. 0.4)
- Note that our estimator is a *random variable* -- it's a function of Y_is which are random variables.
  - Therefore it has an expectation $E[\hat{\mu}]$ (assuming Y_i has an expectation)
  - It has a variance $Var(\hat{\mu})$ (again, under regularity conditions)
  - It has a distribution (which we may or may not know).

Estimation

- How do we know if we've picked a good estimator? Will it be close to the truth? Will it be systematically higher or lower than the target?
- We want to derive some of its properties
    - Bias: $E[\hat{\mu}] - \mu$
    - Variance: $Var(\hat{\mu})$
    - Consistency: Does $\hat{\mu}$ converge in probability to μ as n goes to infinity?
    - Asymptotic distribution: Is the sampling distribution of $\hat{\mu}$ well approximated by a known distribution?

Unbiasedness

- Is the expectation of $\hat{\mu}$ equal to μ?
- First we pull out the constant.

$$E[\hat{\mu}] = E\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \frac{1}{n}E\left[\sum_{i=1}^{n} Y_i\right]$$

- Next we use linearity of expectations

$$\frac{1}{n}E\left[\sum_{i=1}^{n} Y_i\right] = \frac{1}{n}\sum_{i=1}^{n} E\left[Y_i\right]$$

- Finally, under our i.i.d. assumption

$$\frac{1}{n}\sum_{i=1}^{n} E\left[Y_i\right] = \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{n\mu}{n} = \mu$$

- Therefore, the bias, $\text{Bias}(\hat{\mu}) = E[\hat{\mu}] - \mu = 0$

Variance

- What is the variance of $\hat{\mu}$? Again, start by pulling out the constant.

$$Var(\hat{\mu}) = Var\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \frac{1}{n^2} Var\left[\sum_{i=1}^{n} Y_i\right]$$

- We can further simplify by using our i.i.d. assumption. The variance of a sum of i.i.d. random variables is the sum of the variances.

$$\frac{1}{n^2} Var\left[\sum_{i=1}^{n} Y_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} Var\left[Y_i\right]$$

- "identically distributed"

$$\frac{1}{n^2}\sum_{i=1}^{n} Var\left[Y_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

- Therefore, the variance is $\frac{\sigma^2}{n}$

Asymptotic behavior

- As n gets large, what can we say about the estimator $\hat{\mu}$.
- First, we can show that it is **consistent** -- it converges in probability to the true parameter μ
    - Unbiasedness + Variance that goes to 0 as n gets large.
    - Some estimators may be biased but have bias terms that go to 0 -- if variance also goes to 0 these are still consistent.
- Second, we can say something about the distribution of $\hat{\mu}$.
    - Remember, we've only made assumptions about $E[Y_i]$ and $Var(Y_i)$ (that they exist). We have made no assumptions on the distribution of Y_i. Y_i can be normal, poisson, bernoulli, or whatever!
    - However, we know something about sums and means of random variables -- they are well-approximated by a normal distribution. The Central Limit Theorem!
    - So in large samples, the **sampling distribution** of $\hat{\mu}$ is close to normal. This lets us construct confidence intervals and do inference with this approximation and be confident that we won't be far off!

The potential outcomes model

Thinking about causal effects

- Two types of causal questions (Gelman and Rubin, 2013)
- **Causes of effects**
  - What are the factors that cause some outcome Y?
  - "Why?" questions: Why do states go to war? Why do politicians get re-elected?
- **Effects of causes**
  - If X were to change, what might happen to Y?
  - "What if?" questions: What if a pair of states were democratic, would that change their chances of going to war? What if a politician were an incumbent, would that affect their re-election probability?
- We'll spend more time on the **effects of causes**
  - Why? Because we can connect them to well-defined statistical quantities of interest (e.g. an "average treatment effect")
  - "Causes of effects" are still important questions, and theoretical inquiry can lead you towards possible causes to evaluate.

Defining a causal effect

- Historically, causality was seen as a *deterministic* process.
    - Hume (1740): Causes are regularities in events of "constant conjunctions"
    - Mill (1843): Method of difference
- This became problematic -- empirical observation alone does not demonstrate causality.
    - Russell (1913): Scientists aren't interested in causality!
- How do we talk about causation that both incorporates *uncertainty* in measurement and clearly defines what we mean by a "causal effect"?
- Rubin (1974) - formalizes a framework for understanding causation from a statistical perspective. Inspired by earlier Neyman (1923) and Fisher (1935) on randomized experiments.
- We'll spend most of our time with this approach, often called the Rubin Causal Model or potential outcomes framework.

Causality and interventions

- Causal effects are effects of interventions
  - What happens to an outcome when a treatment is changed.
- It's very difficult to learn about vague causal statements:
  - What is the effect of growth on democracy or conflict?
  - What is the effect of exercise on health?
- The potential outcomes framework clarifies:
  1. **What** action is doing the causing?
  2. Compared to **what** alternative action?
  3. On **what** outcome metric?
  4. How would we learn about the effect from data?

Statistical setup.

- Population of units
  - Finite population or infinite super-population
- Sample of N units from the population indexed by i - Observed outcome Y_i
- Binary treatment indicator D_i.
  - Units receiving "treatment": $D_i = 1$
  - Units receiving "control": $D_i = 0$
- Covariates (observed prior to treatment) X_i

Potential outcomes

- Let D_i be the value of a treatment assigned to each individual.
- $Y_i(d)$ is the value that the outcome would take if D_i were set to d.
  - For binary D_i: $Y_i(1)$ is the value we would observe if unit i were *treated*.
  - $Y_i(0)$ is the value we would observe if unit i were under *control*
- We model potential outcomes as fixed attributes of the units.
- Notation alert! -- Sometimes you'll see potential outcomes written as:
  - Y_i^1, Y_i^0 or $Y_i^{d=1}$, $Y_i^{d=0}$
  - Y_{i0}, Y_{i1}
  - $Y_1(i)$, $Y_0(i)$
- Causal effects are contrasts in potential outcomes.
  - Individual treatment effect: $\tau_i = Y_i(1) - Y_i(0)$
  - Can consider ratios or other transformations (e.g. $\frac{Y_i(1)}{Y_i(0)}$)

Consistency/SUTVA

- How do we link the potential outcomes to observed ones?
- Consistency/Stable Unit Treatment Value (SUTVA) assumption

$$Y_i(d) = Y_i \text{ if } D_i = d$$

- Sometimes you'll see this w/ binary D_i (often in econometrics)

$$Y_i = Y_i(1)D_i + Y_i(0)(1 - D_i)$$

- Implications

  1. No interference -- other units' treatments don't affect i's potential outcomes.
  2. Single version of treatment
  3. D is in principle manipulable -- a "well-defined intervention"
  4. The means by which treatment is assigned is irrelevant (a version of 2)

Positivity/Overlap

- We also need some assumptions on the treatment assignment mechanism D_i.
- In order to be able to observe *some units'* values of $Y_i(1)$ or $Y_i(0)$ treatment can't be deterministic. For all i:

$$0 < Pr(D_i = 1) < 1$$

- If no units could ever receive treatment or control it would be impossible to learn about $E[Y_i|D_i = 1]$ or $E[Y_i|D_i = 0]$
- This is sometimes called a **positivity** or overlap assumption.
    - Pretty trivial in a randomized experiment, but can be tricky in observational studies when D_i is perfectly determined by some covariates X_i

A missing data problem

- It's useful to think of the causal inference problem in terms of *missingness* in the complete table of potential outcomes.

Unit i	Treatment D_i	$Y_i(1)$	$Y_i(0)$	Observed Y_i
1	1	5	$?$	5
2	0	$?$	-3	-3
3	1	9	$?$	9
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	0	$?$	8	8

- If we could observe both $Y_i(1)$ and $Y_i(0)$ for each unit, then this would be easy!
- But we can't - we only observe what we're given by D_i
- Holland (1986) calls this "The Fundamental Problem of Causal Inference"

Causal Estimands

- All causal inference starts with a definition of the estimand.
- The individual causal effect: τ_i

$$\tau_i = Y_i(1) - Y_i(0)$$

  - **Problem**: Can't identify this without extremely strong assumptions!
  - "The Fundamental Problem of Causal Inference"

- The **average treatment effect** (ATE): τ

$$\tau = E[Y_i(1) - Y_i(0)]$$

  - What is the *expected* effect of the treatment?

Causal Estimands

- The **conditional average treatment effect** (CATE): $\tau(x)$

$$\tau(x) = E[Y_i(1) - Y_i(0) | X_i = x]$$

  - The average treatment effect for the sub-population with a particular set of covariate values $X_i = x$

- The **average treatment effect on the treated** (ATT): τ_{ATT}

$$\tau_{\text{ATT}} = E[Y_i(1) - Y_i(0) | D_i = 1]$$

  - The average treatment effect among those units that were *assigned* treatment.
  - Why do we care about this?
    - Sometimes more policy-relevant.
    - Often easier to identify in selection-on-observables designs.

Causal Estimands

- The **risk ratio** (RR) for binary outcomes:

$$RR = \frac{Pr(Y_i(1) = 1)}{Pr(Y_i(0) = 1)}$$

- Political science and social sciences rarely use these.
- Medicine *loves* risk ratios (e.g. Vaccine effectiveness = 1 - RR)

Finite Population Estimands

- For these estimands we have two sources of uncertainty
  - Sampling from a population.
  - Random assignment of treatment (unobserved P.O.s)

- What if we're just interested in doing inference on the treatment effect in the sample, or there is no population from which we sample from?

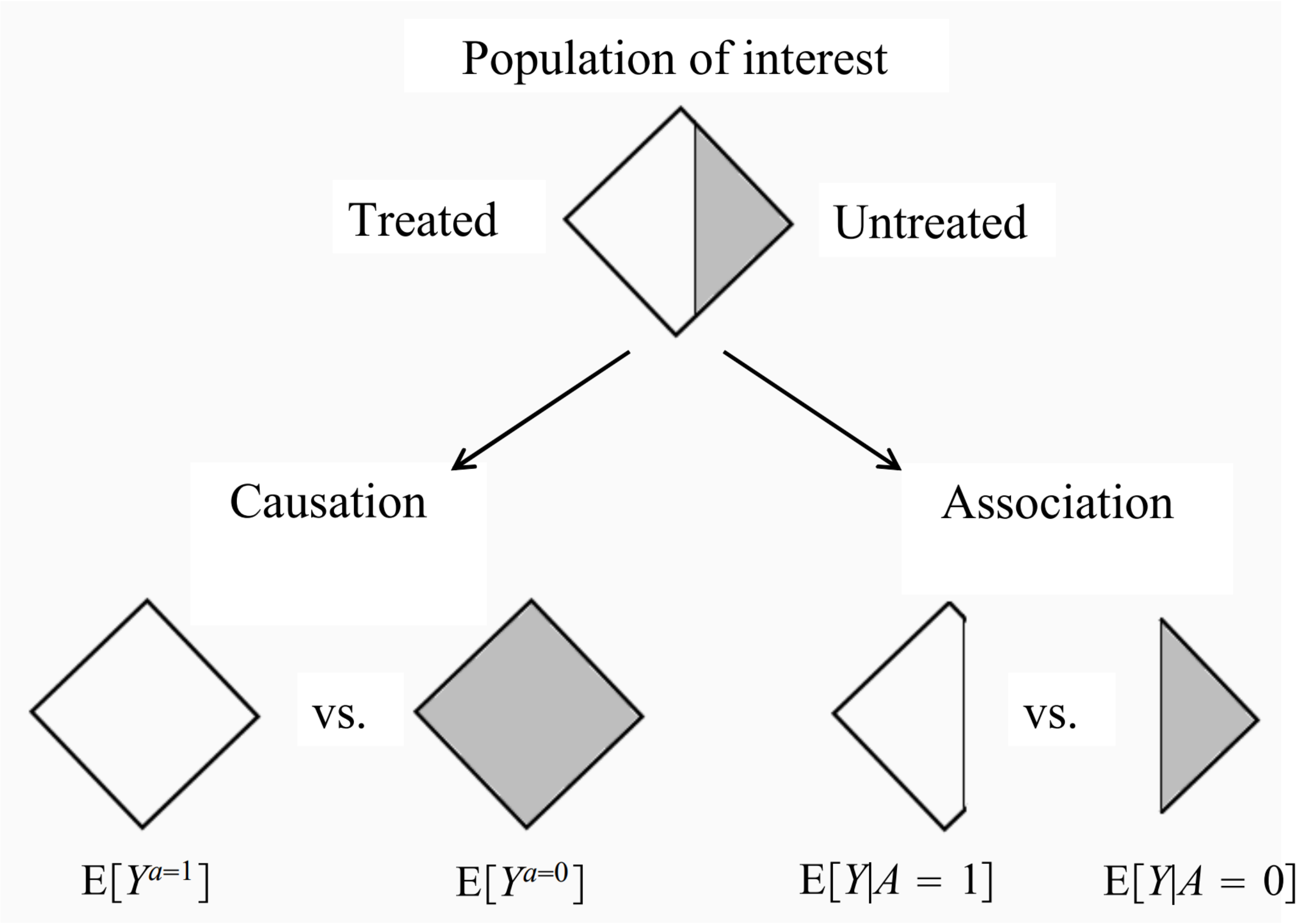
  - We can define analogues of the ATE for the sample -- the *sample average treatment effect*

$$\tau_{\text{SATE}} = \frac{1}{N}\sum_{i=1}^{N} Y_i(1) - Y_i(0)$$

Finite Population Estimands

- There's *still* uncertainty -- we don't observe all of the potential outcomes. But now we don't have to justify generalizing to the population.
  - Under random sampling, the SATE is unbiased for the (population) ATE or PATE.
  - Large literature on when and how we can generalize absent random sampling ("transportability")
- Inference justified on the basis of *randomization* of the treatment.
  - A lot of the same tools for inference in classical super-population sampling transfer over nicely.

Causal vs. Associational Estimands



Identification

- In statistics, we often talk about whether a parameter is *identifiable*.
    - A parameter is "point identified" if, having access to infinite data, the parameter could only take on a single value.
    - In other words, no other value of the parameter could generate the same observable data.
- In statistical models, non-identifiability arises when different values of the parameter could give rise to the same observable data
    - In classical regression: More parameters than observations.
    - Or perfectly collinear regressors

Causal Identification

- **Causal identification**: Can we learn about the value of a causal effect from the observed data.
  - Can we express the causal estimand (e.g. $\tau = E[Y_i(1) - Y_i(0)]$) entirely in terms of *observable* quantities
- Causal identification comes prior to questions of estimation
  - It doesn't matter whether you're using regression, weighting, matching, doubly-robust estimation, double-LASSO, etc...
  - If you can't answer the question "What's your identification strategy?" then no amount of fancy stats will solve your problems.
- Identification requires *assumptions* about the connection between the observed data Y_i, D_i and the unobserved counterfactuals $Y_i(d)$
  - Under what assumptions will the observed difference-in-means *identify* the average treatment effect?

Conclusion

- Causal effects are contrasts in counterfactuals
- The potential outcomes framework gives us a tool for defining statistical *estimands* in terms of functions of individual effects of treatment.
  - The potential outcome $Y_i(d)$ denotes the outcome Y_i we would observe for unit i if they were assigned treatment d
  - Implicitly making a consistency/SUTVA assumption to link *observed* to *counterfactual*
- Fundamental problem of causal inference: We're going to need more assumptions -- don't get to directly observe individual treatment effects.
- Consistency and positivity only get us part of the way there.
- Next week: What additional assumptions do we need to **identify** causal effects from the observed data. Why randomized experiments satisfy those assumptions.