exPIPG

Extrapolated Proportional Integral Projected Gradient Method

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Random system regulation

Consider an linear time-invariant system

$$x_{t+1} = Ax_t + Bu_t,$$

where $x_t \in \mathbb{R}^n,\ u_t \in \mathbb{R}^m$ for $t \in \mathbb{Z}_+$, and $A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}$. The initial condition is z_0 and the maximum absolute value of the states and control inputs is bounded by $x_{\max}$ and $u_{\max}$, respectively, at all time instants. Entries of A, B and z_0 are sampled from the standard normal distribution. Matrix A is normalized by the spectral radius to ensure that the system is neutrally stable, and entries of z_0 are saturated to lie in the interval $[-x_{\max}, x_{\max}]$.

```
A = randn(n);
% Scale by spectral radius to create a neutrally stable system
A = A/max(abs(eig(A)));

B = randn(n,m);
```

Optimal control problem

$$\begin{aligned}
\underset{\substack{u_t,\ t=0,\ldots,N-1\\ x_t,\ t=0,\ldots,N}}{\text{minimize}}\quad & x_N^\top Q x_N + \sum_{t=0}^{N-1} x_t^\top Q x_t + u_t^\top R u_t \\
\text{subject to}\quad & Bu_t + Ax_t - x_{t+1} = 0, && t = 0,\ldots,N-1, \\
& \|x_t\|_\infty \le x_{\max}, && t = 1,\ldots,N-1, \\
& \|u_t\|_\infty \le u_{\max}, && t = 0,\ldots,N-1, \\
& x_0 = z_0, \\
& x_N = \mathbf{0}_n.
\end{aligned}$$

Conversion to a quadratic program

$$\begin{aligned}
\text{minimize}\ & \xi^\top P \xi \\
\text{subject to}\ & H\xi - g = \mathbf{0}_{nN}, \\
& \xi_{\min} \le \xi \le \xi_{\max},
\end{aligned}$$

where

$$
\begin{aligned}
\xi &= \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \\ x_0 \\ \vdots \\ x_N \end{bmatrix}, \\
P &= \begin{bmatrix} I_N \otimes R & \mathbf{0}_{mN\times n(N+1)} \\ \mathbf{0}_{n(N+1)\times mN} & I_{N+1}\otimes Q \end{bmatrix}, \\
H &= \begin{bmatrix} H_u & H_x \end{bmatrix}, \quad H_u = I_N \otimes B, \quad H_x = \begin{bmatrix} I_N \otimes A & \mathbf{0}_{nN\times n} \end{bmatrix} - \begin{bmatrix} \mathbf{0}_{nN\times n} & I_{nN} \end{bmatrix}, \\
g &= \mathbf{0}_{nN}, \\
\xi_{\min} &= \begin{bmatrix} -\mathbf{1}_{mN}\otimes u_{\max} \\ z_0 \\ -\mathbf{1}_{nN-n}\otimes x_{\max} \\ \mathbf{0}_n \end{bmatrix}, \quad \xi_{\max} = \begin{bmatrix} \mathbf{1}_{mN}\otimes u_{\max} \\ z_0 \\ \mathbf{1}_{nN-n}\otimes x_{\max} \\ \mathbf{0}_n \end{bmatrix}.
\end{aligned}
$$

Infeasibility detection in exPIPG

Yue Yu, 04/30/2022

We wish to obtain a infeasibility certificate $\overline{\eta} \in \mathbb{R}^m$ such that

$$\inf_{\xi\in\mathbb{D}} \langle H\xi - g, \overline{\eta}\rangle > \sup_{\eta\in\mathbb{K}} \langle \eta, \overline{\eta}\rangle.$$

If $\mathbb{K} = \{0_m\}$ and $\mathbb{D} = \{\xi \,|\, \xi_{\min} \leq \xi \leq \xi_{\max}\} \subset \mathbb{R}^n$, we have

$$\sup_{\eta\in\mathbb{K}} \langle \eta, \overline{\eta}\rangle = 0.$$

and

$$
\begin{aligned}
&\inf_{\xi\in\mathbb{D}} \langle H\xi - g, \overline{\eta}\rangle \\
&= -\langle g, \overline{\eta}\rangle + \inf_{\xi\in\mathbb{D}} \langle H^\top \overline{\eta}, \xi\rangle \\
&= -\langle g, \overline{\eta}\rangle + \langle H^\top \overline{\eta}, \overline{\xi}\rangle
\end{aligned}
$$

where

$$\left[\overline{\xi}\right]_i = \begin{cases} [\xi_{\min}]_i, & \left[H^\top\overline{\eta}\right]_i \geq 0, \\ [\xi_{\max}]_i, & \left[H^\top\overline{\eta}\right]_i < 0, \end{cases}$$

for $i = 1, \ldots, n$.

The intuition is that, when minimizing a scalar-to-scalar linear function over an interval, the minimum is attained at either left or right boundary of the interval, depends on whether the linear function has positive or negative slope.

We can define $\overline{\xi}$ compactly as follows:

$$\overline{\xi} = (H^\top\overline{\eta} \geq 0) \odot \xi_{\min} + (H^\top\overline{\eta} < 0) \odot \xi_{\max}$$

where $\odot$ denotes element-wise product.

An *ϵ-approximate infeasibility certificate* is hence given by $\overline{w}$ such that

$$\inf_{\xi\in\mathbb{D}} \langle H\xi - g, \overline{\eta}\rangle > -\epsilon,$$

which is equivalent to the following

$$\langle H\overline{\xi} - g, \overline{\eta}\rangle > -\epsilon$$

In exPIPG, we let $\overline{\eta} = \eta^{k+1} - \eta^k$ and test if the above condition holds; if so, we terminate and declare the optimization infeasible.

Devectorization

$$\overline{x}_t = \begin{cases} z_0, & t = 0, \\ \big(\overline{w}_{t-1} - A^\top \overline{w}_t \geq 0\big) \odot x_{\min} + \big(\overline{w}_{t-1} - A^\top \overline{w}_t < 0\big) \odot x_{\max}, & t = 1, \dots, N-1, \\ z_N, & t = N, \end{cases}$$
$$\overline{u}_t = \big(-B^\top \overline{w}_t \geq 0\big) \odot u_{\min} + \big(-B^\top \overline{w}_t < 0\big) \odot u_{\max}, \qquad t = 0, \dots, N-1.$$

ϵ-approximate infeasibility certificate

$$\sum_{t=0}^{N-1} \overline{w}_t^\top \left(\overline{x}_{t+1} - A\overline{x}_t - B\overline{u}_t\right) > -\epsilon$$

Implementation notes

`Primal error:`	$\lVert \xi^{k+1} - \xi^k \rVert_\infty$
`Dual error:`	$\lVert \eta^{k+1} - \eta^k \rVert_\infty$
`Infeas. cert.:`	$\langle H\overline{\xi} - g, \overline{\eta} \rangle$

Infeasibility detection criteria based on ratio dual variable error

$$\left| \frac{\lVert w^{k+1+M} - w^{k+M} \rVert_\infty}{\lVert w^{k+1} - w^k \rVert_\infty} - 1 \right| \leq \epsilon,$$

where M is the frequency at which the above criteria is tested. This criteria is denoted by `old_infeas` and is implemented in

- `expipg_vec.m`
- `expipg_vec_mex.mexmaci64`
- `expipg_dvec.m`
- `expipg_dvec_mex.mexmaci64`

The new infeasibility detection criteria developed by Yue Yu on 04/30/2022 (described in section above)

$$\langle H\overline{\xi} - g, \overline{\eta} \rangle \geq \epsilon$$

This criteria is denoted by `new_infeas` and is implemented in

- `expipg_vec_v2.m`
- `expipg_vec_v2_mex.mexmaci64`
- `expipg_dvec_v2.m`
- `expipg_dvec_v2_mex.mexmaci64`