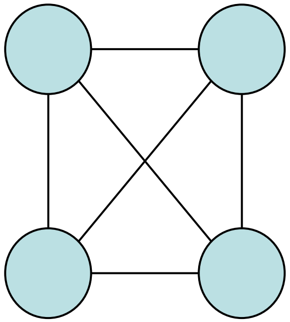
Data Structures

25. Minimum Spanning Tree (MST)

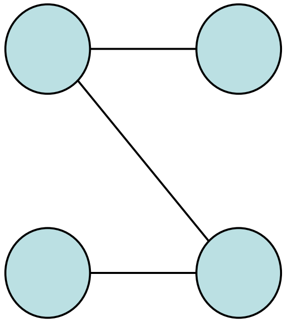
Spanning Trees

- A spanning tree of a graph is just a subgraph that contains all the vertices and is a tree

- Formal definition
  - Given a connected graph with $|V| = n$ vertices
  - A spanning tree is defined a collection of $n - 1$ edges which connect all n vertices
  - The n vertices and $n - 1$ edges define a connected sub-graph



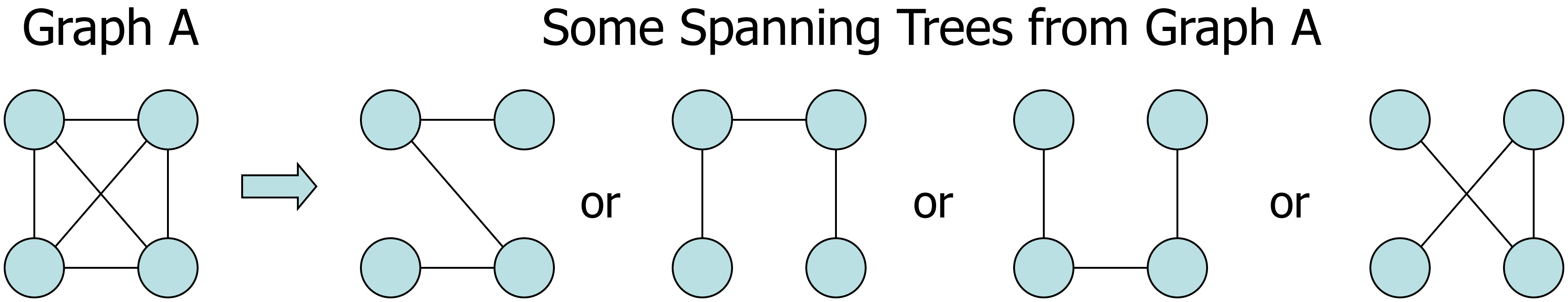
Graph



Spanning Tree

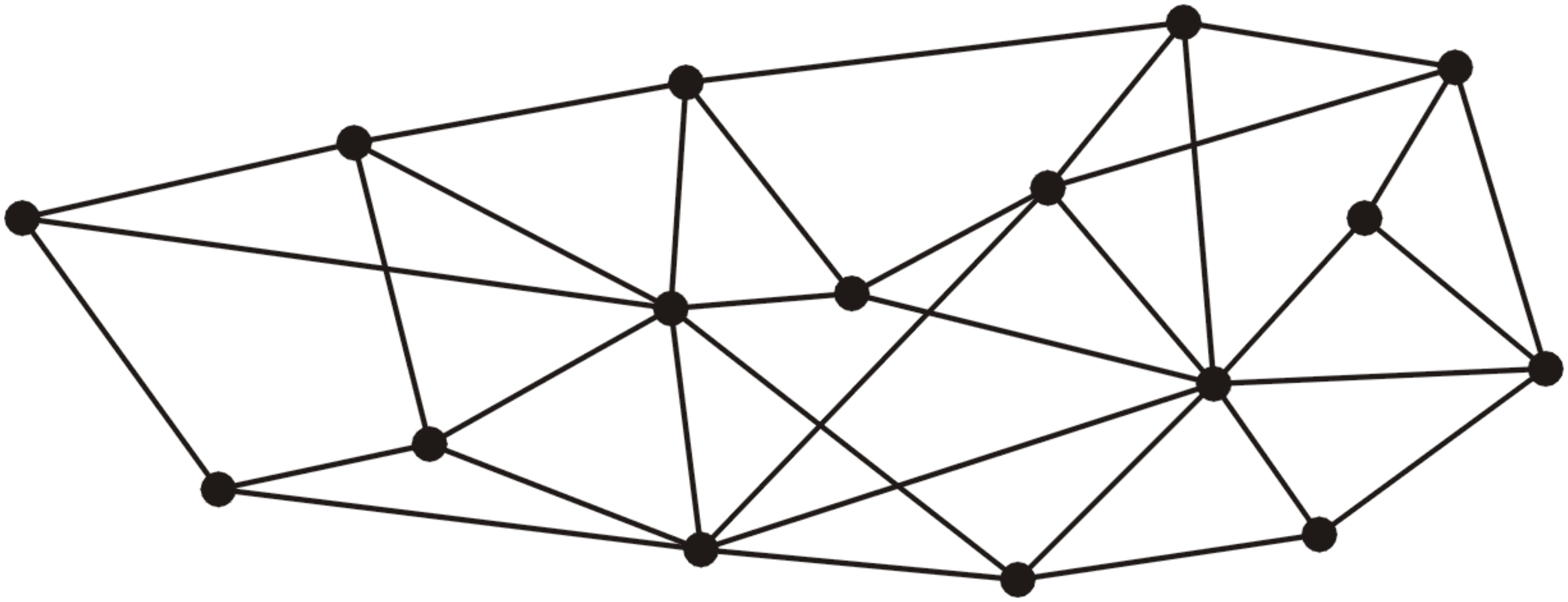
Spanning Trees

- A spanning tree is not necessarily unique



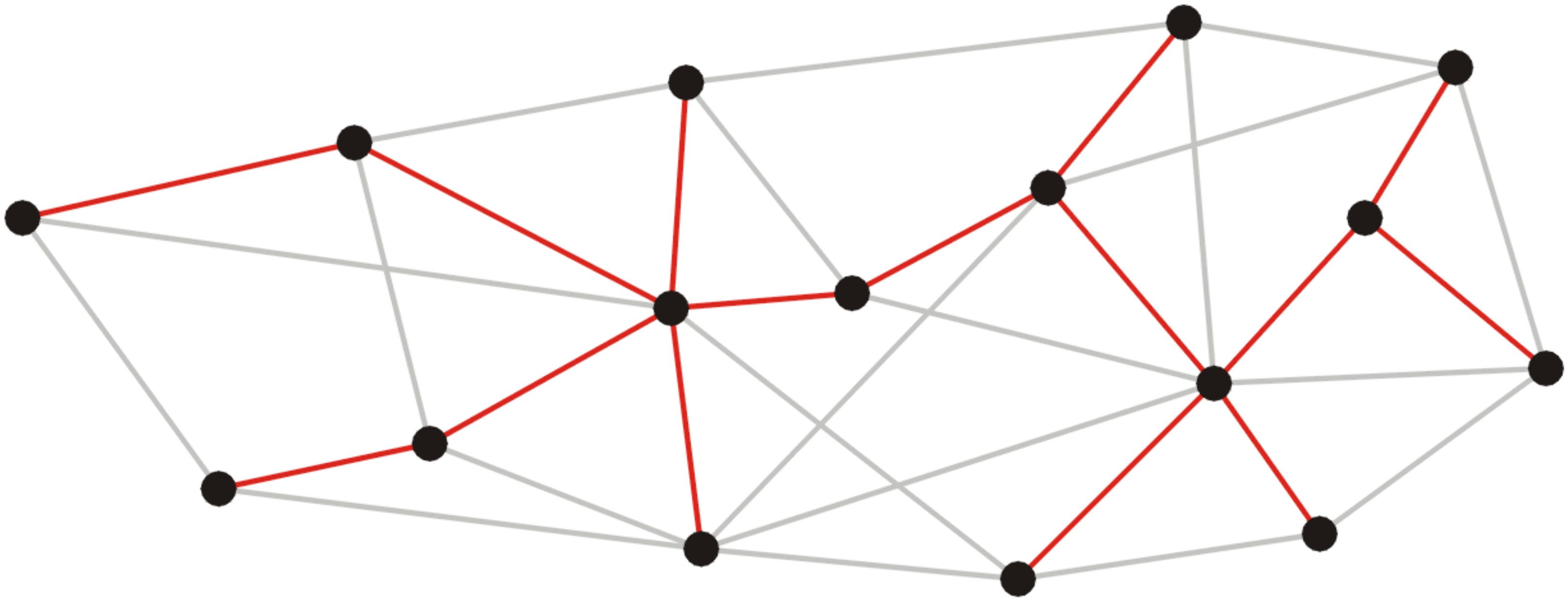
Spanning Trees – Example

- This graph has 16 vertices and 35 edges



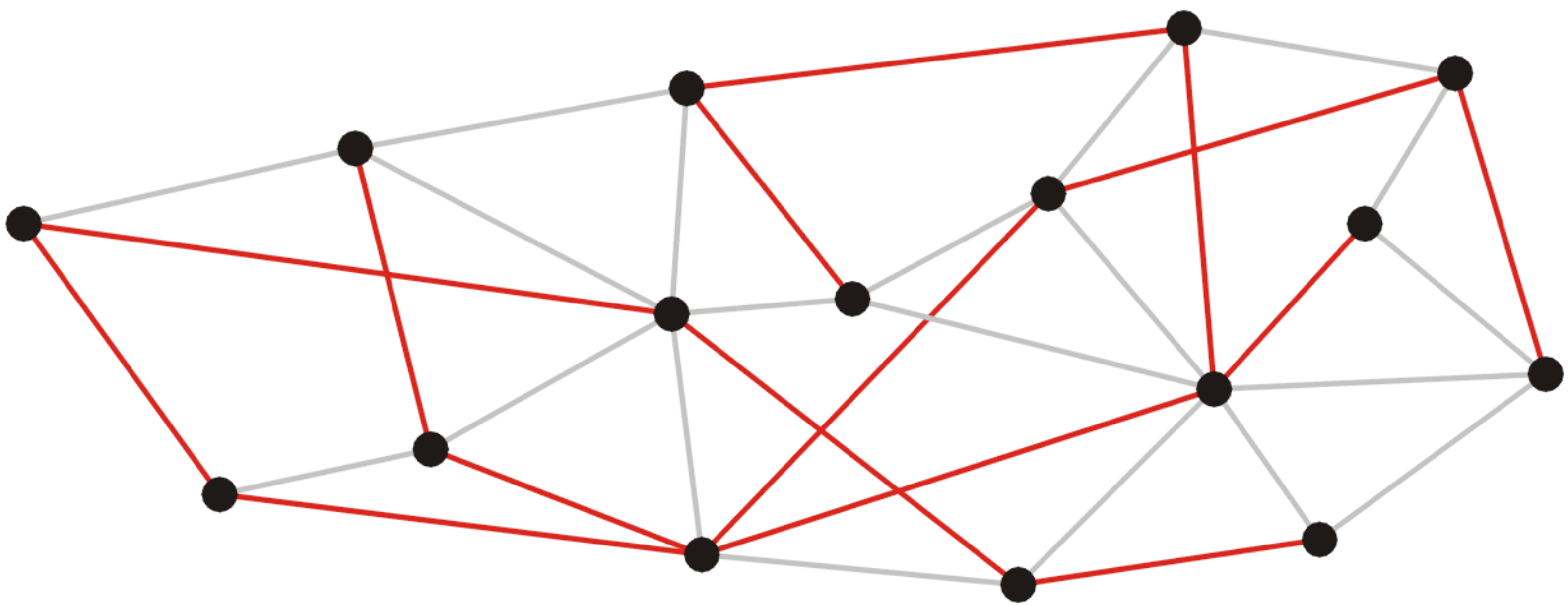
Spanning Trees – Example

- These 15 edges form a minimum spanning tree

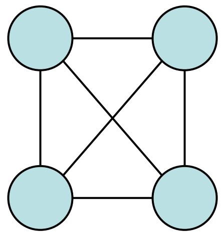


Spanning Trees – Example

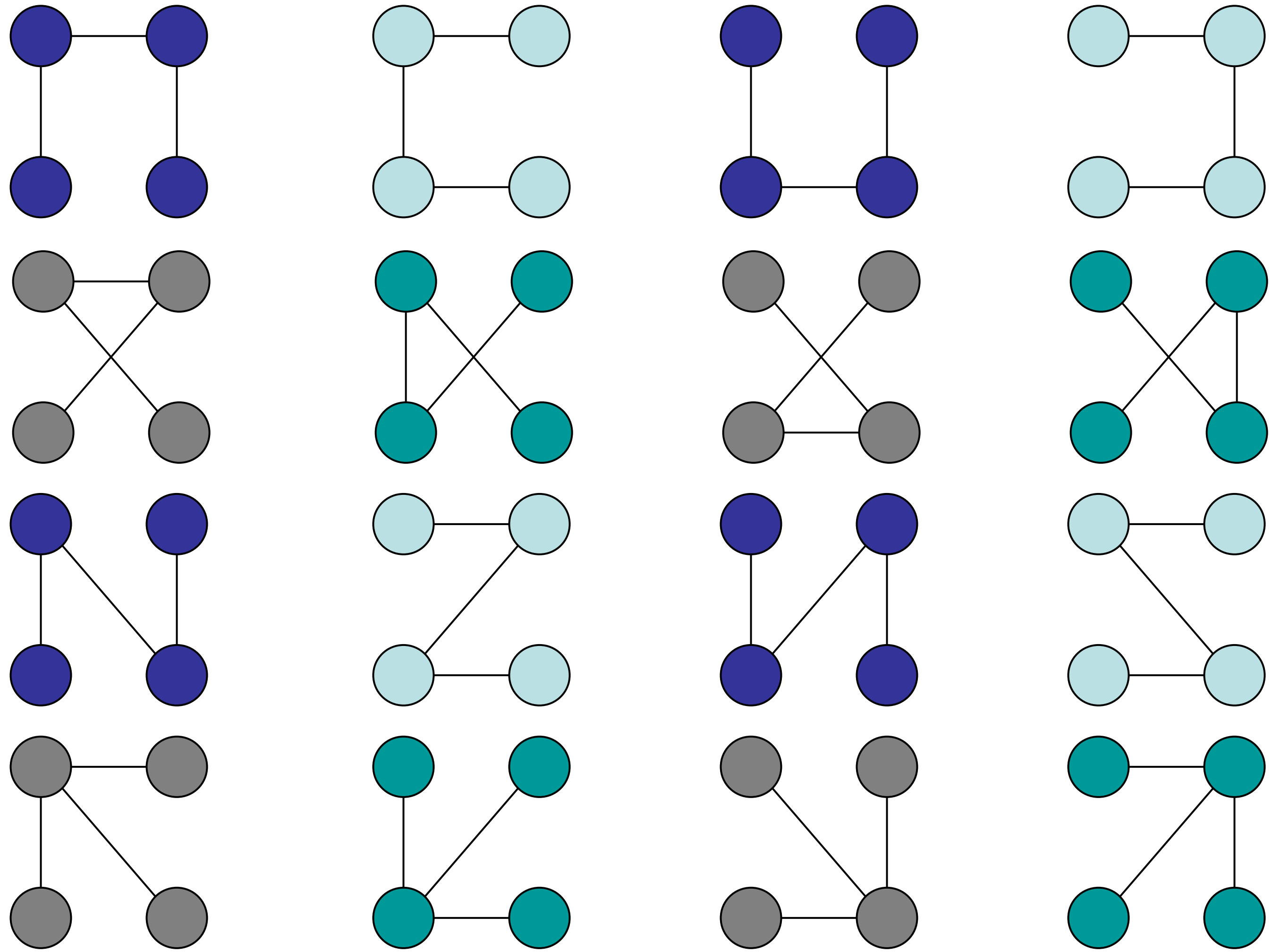
- As do these 15 edges



Spanning Trees – Example



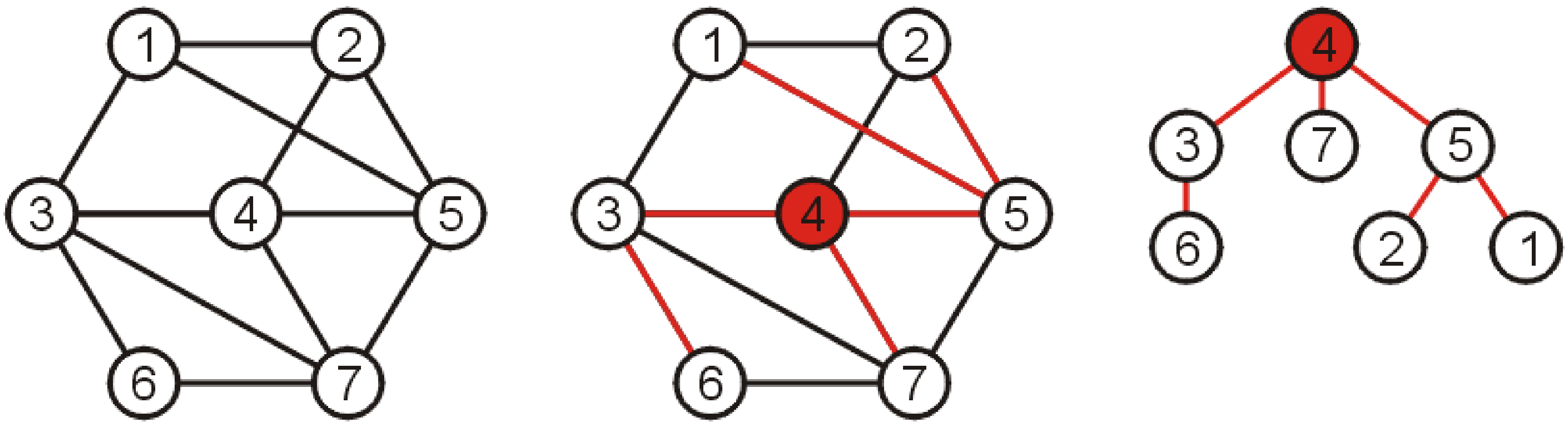
Graph



All 16 of its Spanning Trees

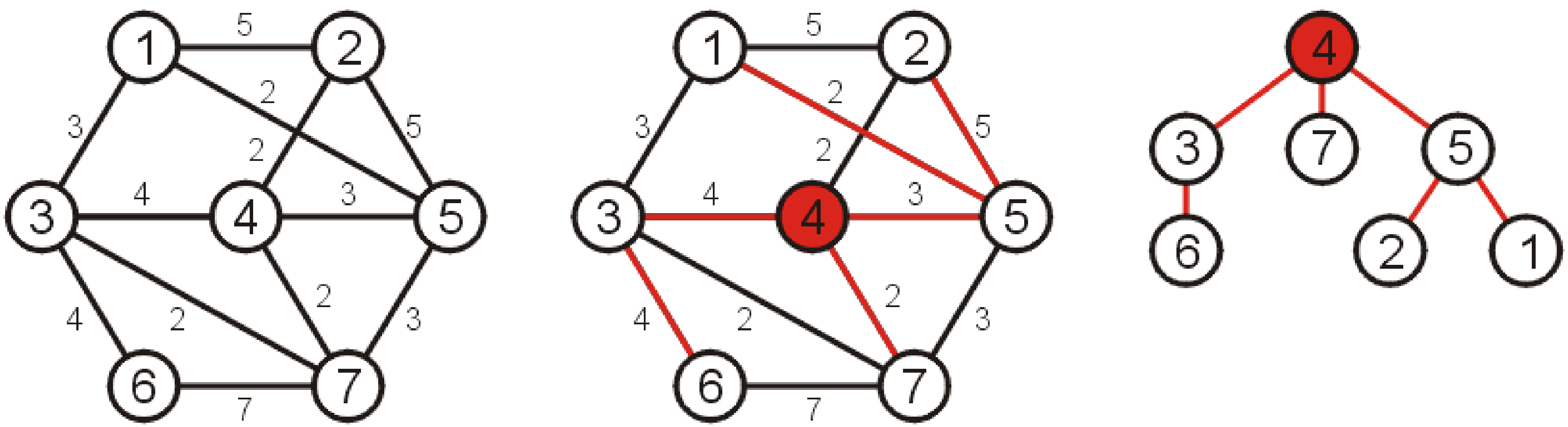
Spanning Trees

- Why such a collection of `|V|-1` edges is called a tree?
  - If any vertex is taken to be the root, we form a tree by treating the adjacent vertices as children, and so on...



Spanning Tree on Weighted Graphs

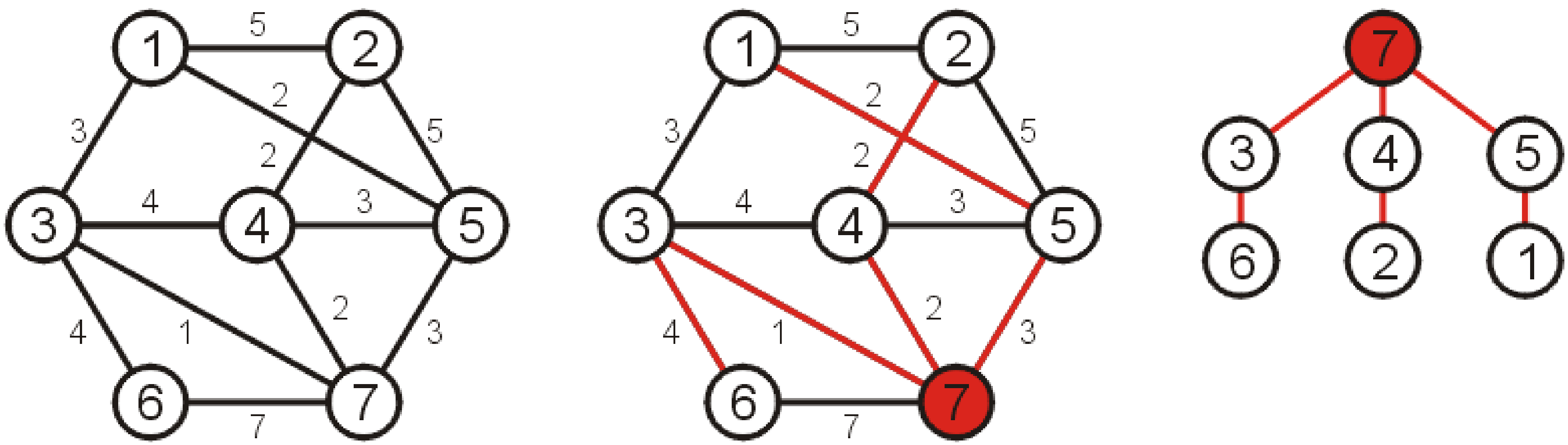
- Weight of a spanning tree
  - Sum of the weights on all the edges which comprise the spanning tree



- The weight of this spanning tree is 20

Minimum Spanning Tree (MST)

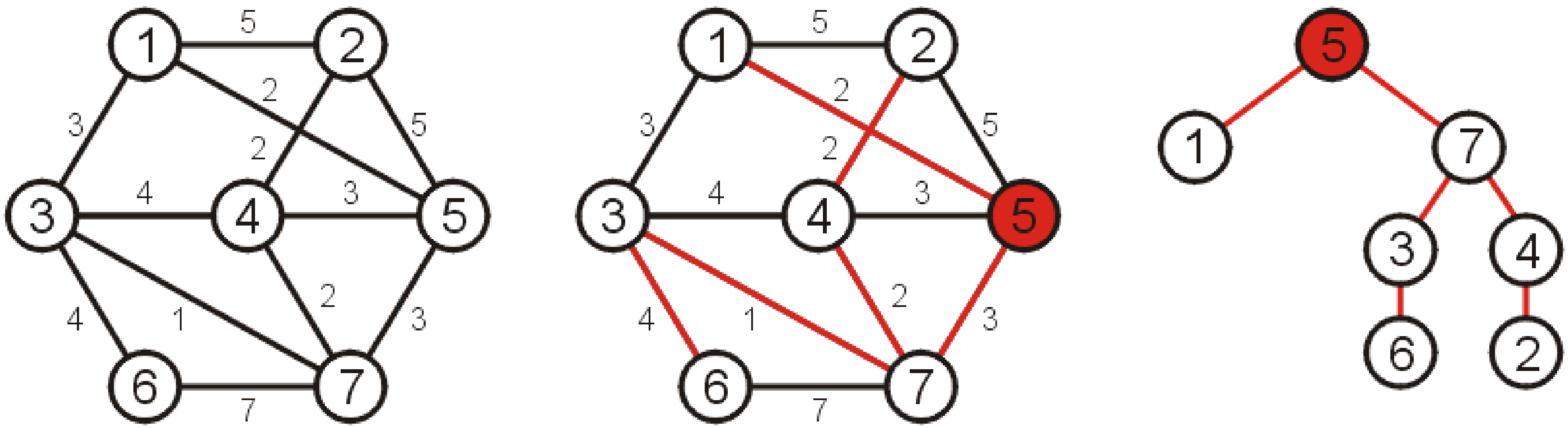
- Spanning tree that minimizes the weight
  - Such a tree is termed a minimum spanning tree



- The weight of this spanning tree is 14

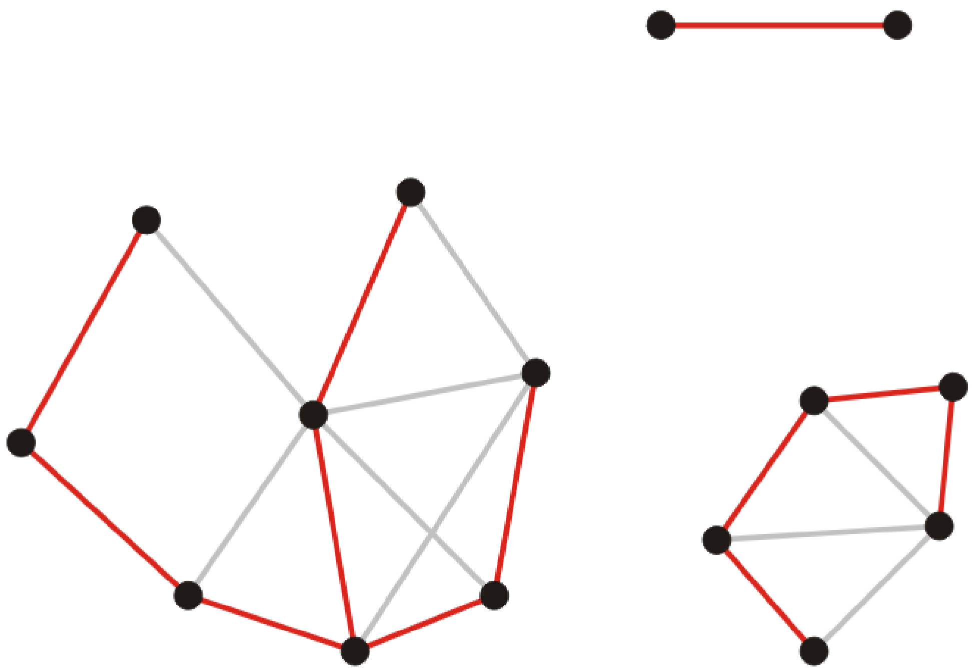
Minimum Spanning Tree (MST)

- If a different vertex is used as the root
  - A different tree is obtained
  - However, this is simply the result of one or more rotations



Spanning Forest

- Suppose that a graph is composed of N connected sub-graphs
- A spanning forest is a collection of N spanning trees
  - One for each connected sub-graph



- A minimum spanning forest
  - A collection of N minimum spanning trees
  - One for each connected vertex-induced sub-graph

Algorithms For Obtaining MST

- Kruskal's Algorithm
- Prim's Algorithm
- Boruvka's Algorithm

Kruskal’s Algorithm

Kruskal’s Algorithm

- Kruskal’s algorithm creates a forest of trees
- Initially forest consists of N single node trees (and no edges)
- Sorts the edges by weight and goes through the edges from least weight to greatest weight
- At each step one edge (with least weight) is added so that it joins two trees together
  - As long as the addition does not create a cycle

Kruskal's Algorithm

The halting conditions are as follows:

1. When $|V| - 1$ edges have been added
   - In this case we have a minimum spanning tree

2. We have gone through all edges
   - A forest of minimum spanning trees on all connected sub-graphs

Kruskal's Algorithm

1. The forest is constructed – with each node in a separate tree

2. The edges are placed in a priority queue

3. Until we've added `n-1` edges (assumption: connected graph)
    1. Extract the cheapest edge from the queue

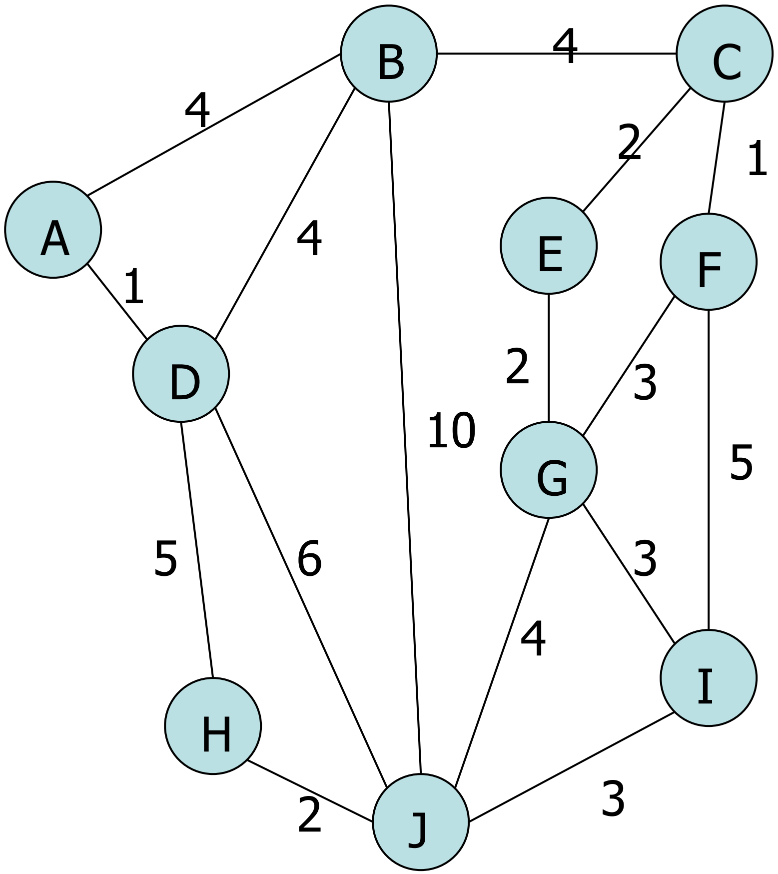
    2. If it forms a cycle, reject it

    3. Else add it to the forest. Adding it to the forest will join two trees

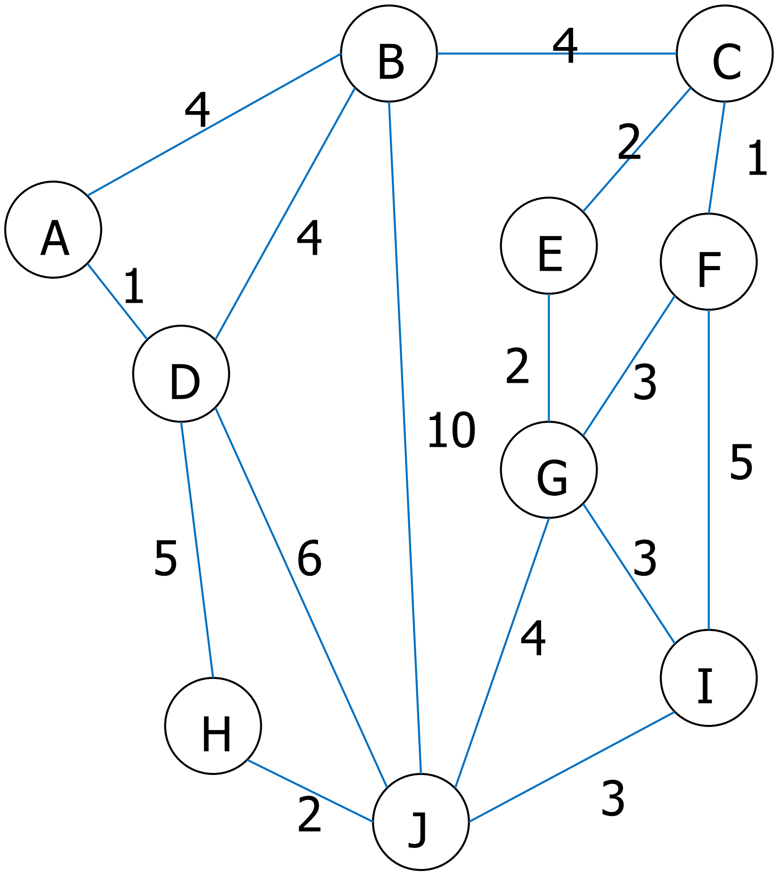
- Every step will have joined two trees in the forest together, so that at the end, there will only be one tree

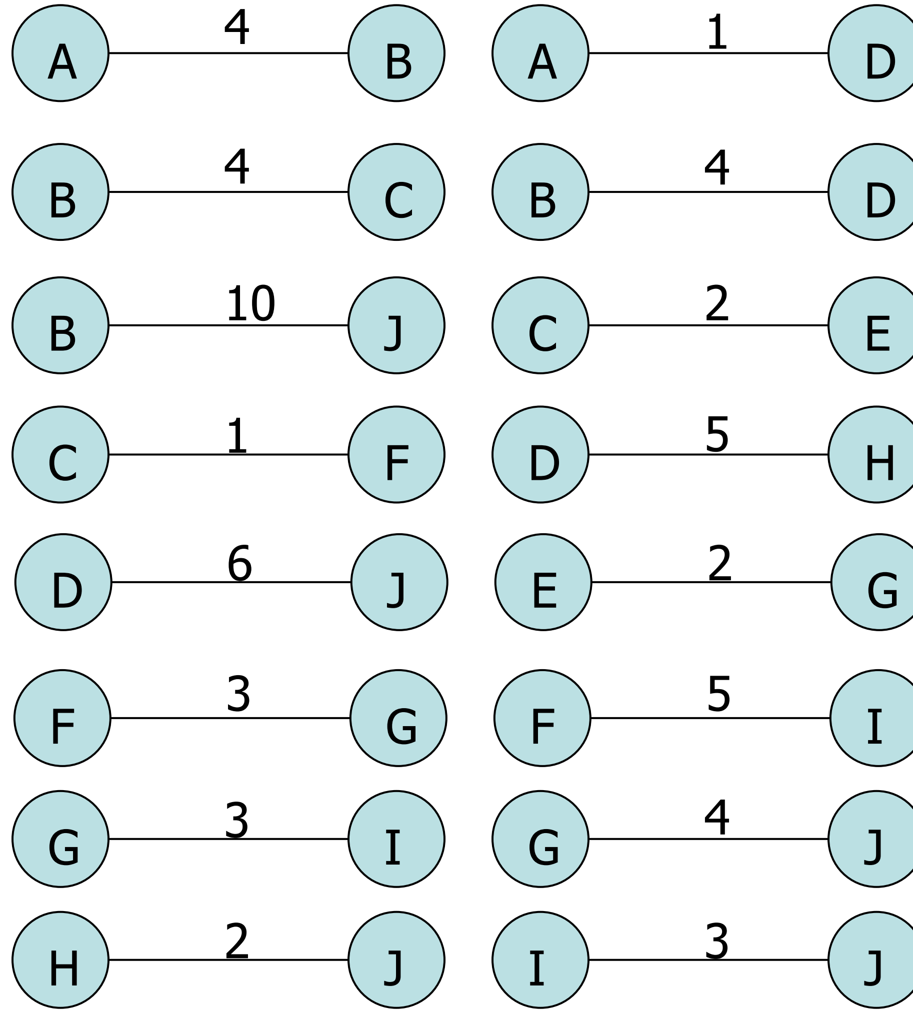
Kruskal’s Algorithm – Example

- Complete Graph



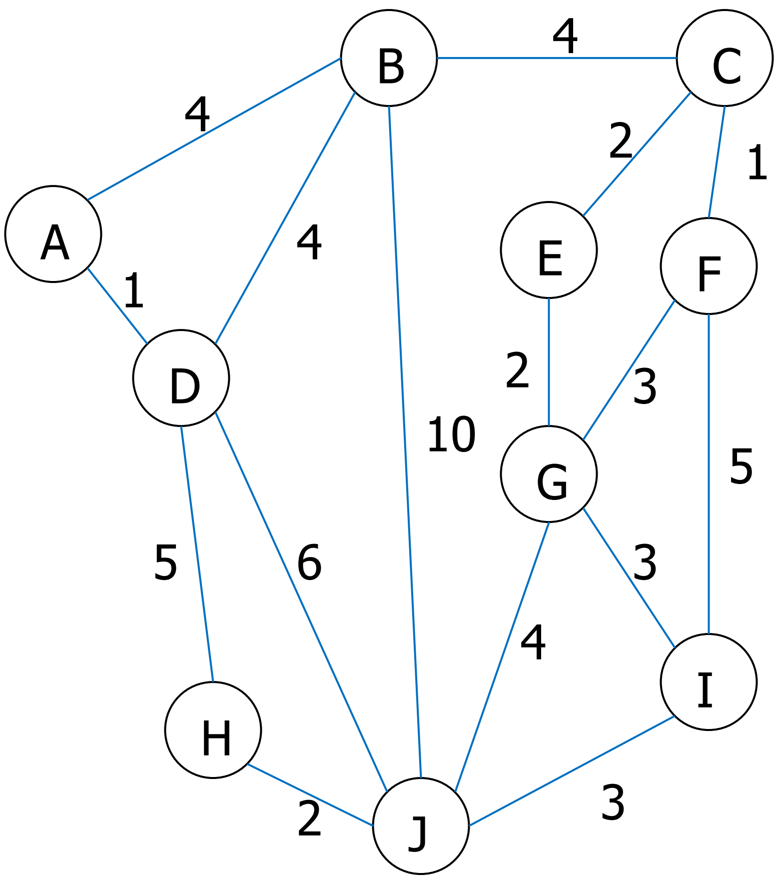
Kruskal's Algorithm – Example

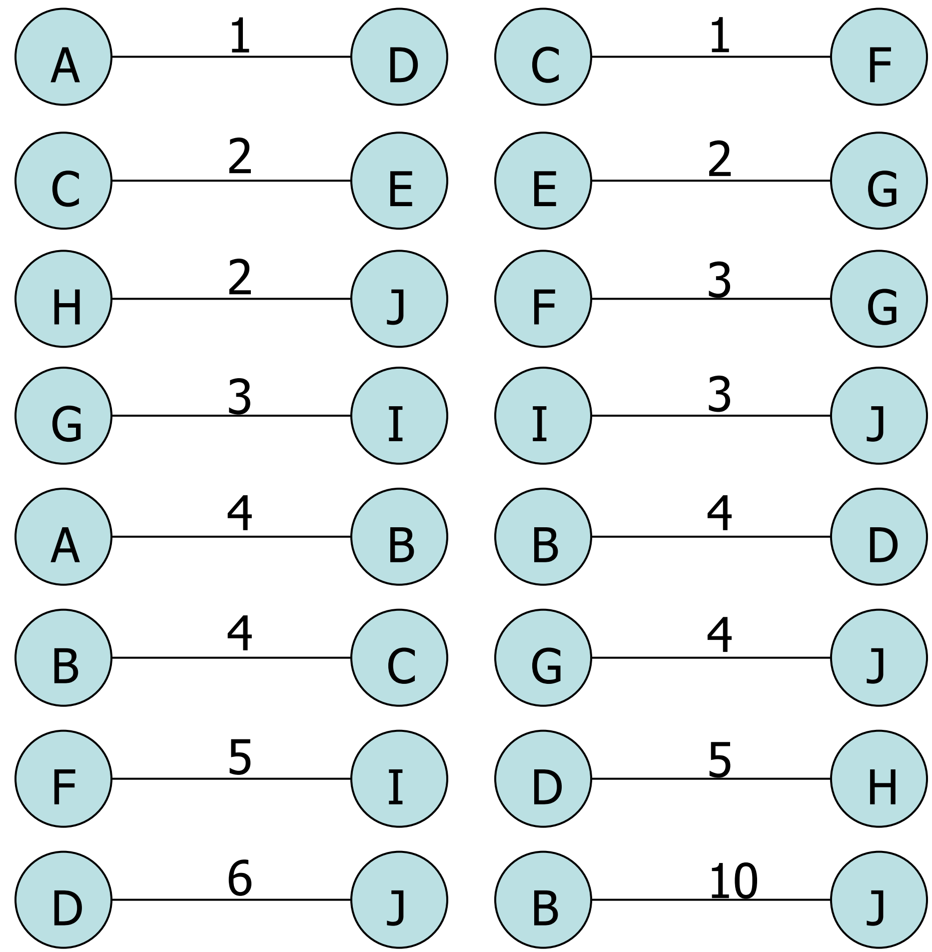




Kruskal’s Algorithm – Example

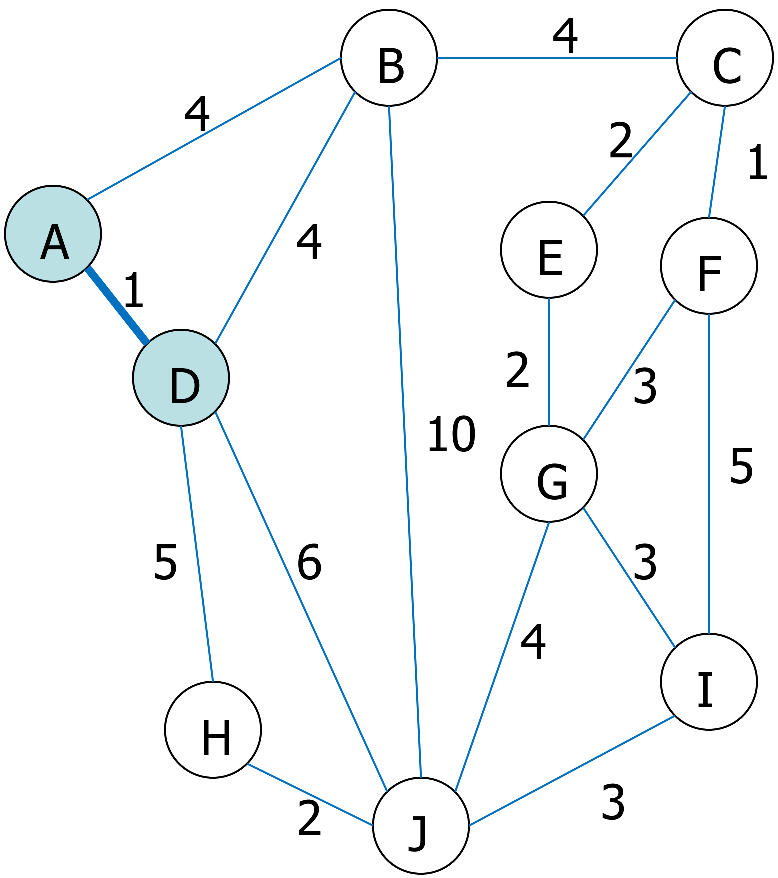
- Sort Edges: In reality priority queue is used

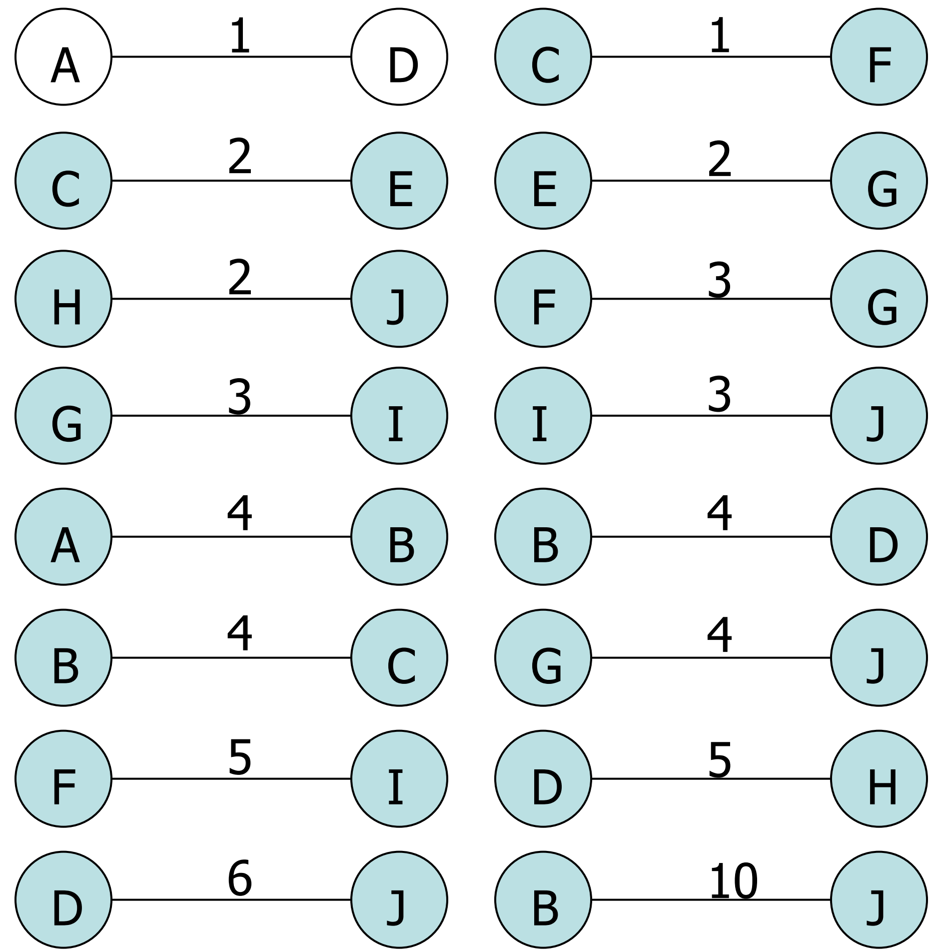




Kruskal's Algorithm – Example

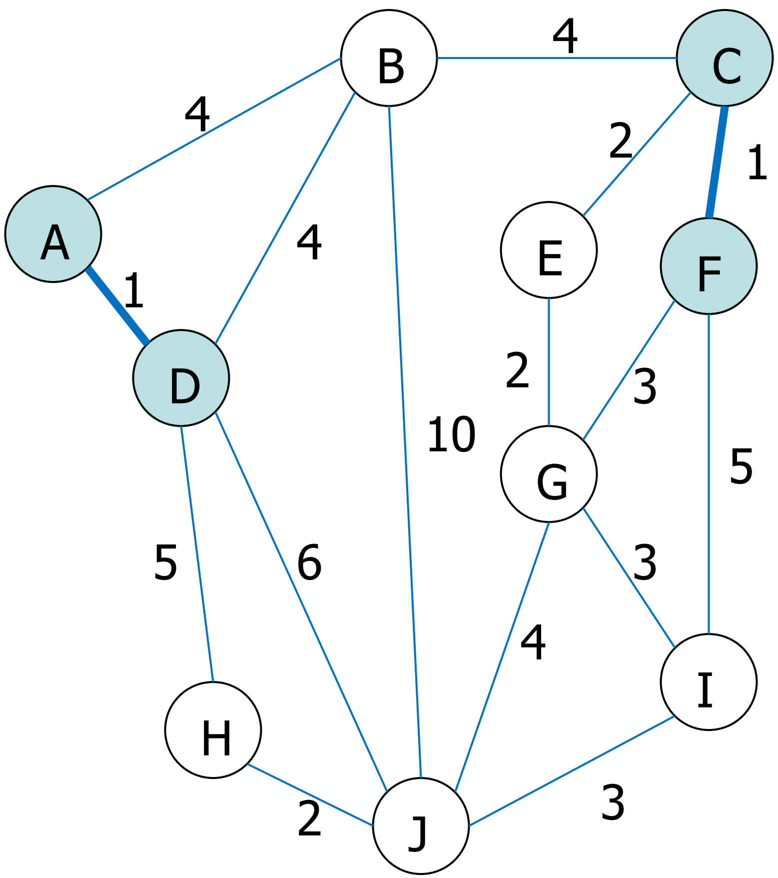
- Add Edge

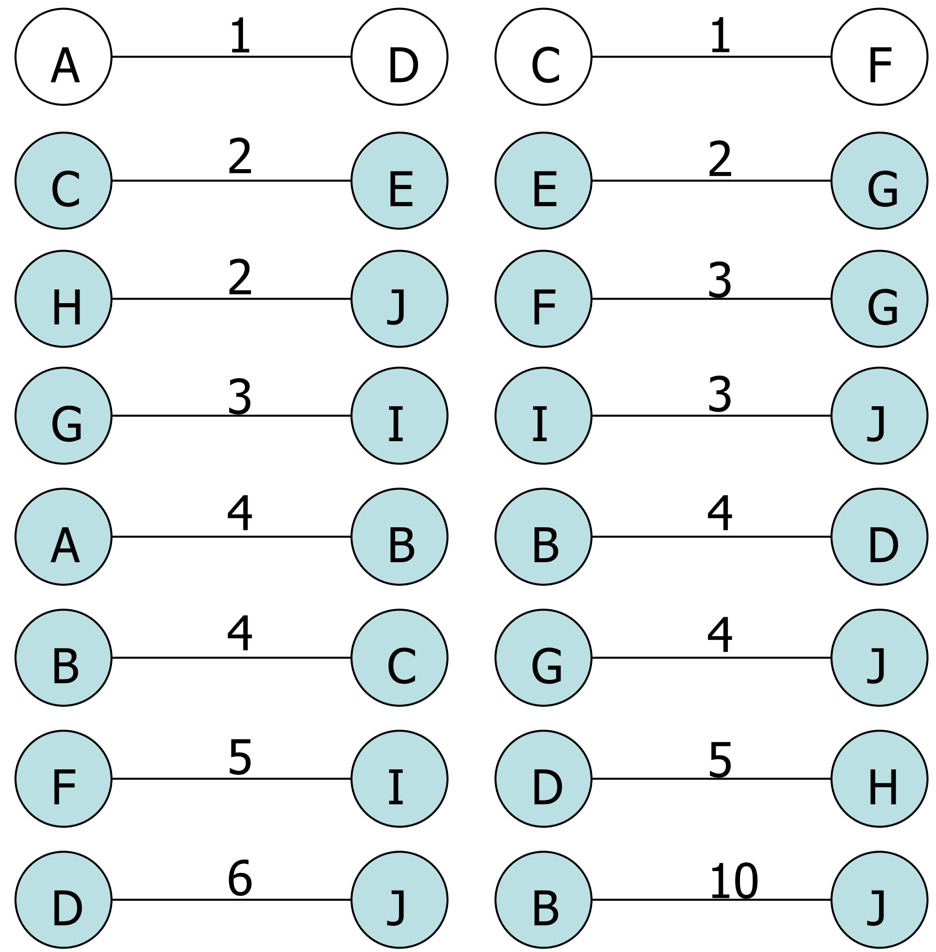




Kruskal’s Algorithm – Example

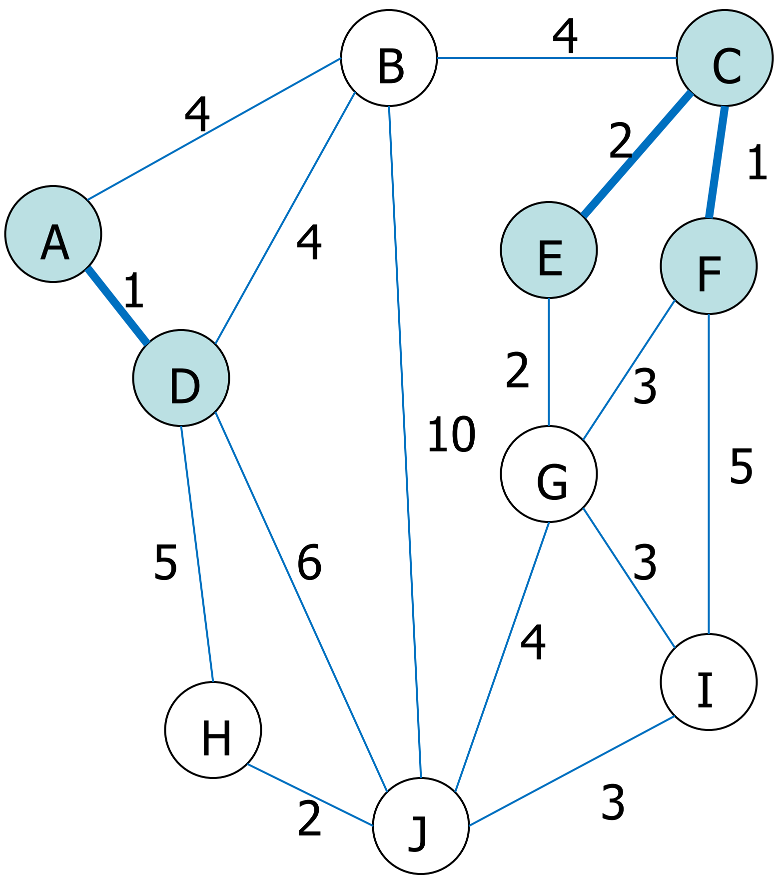
- Add Edge

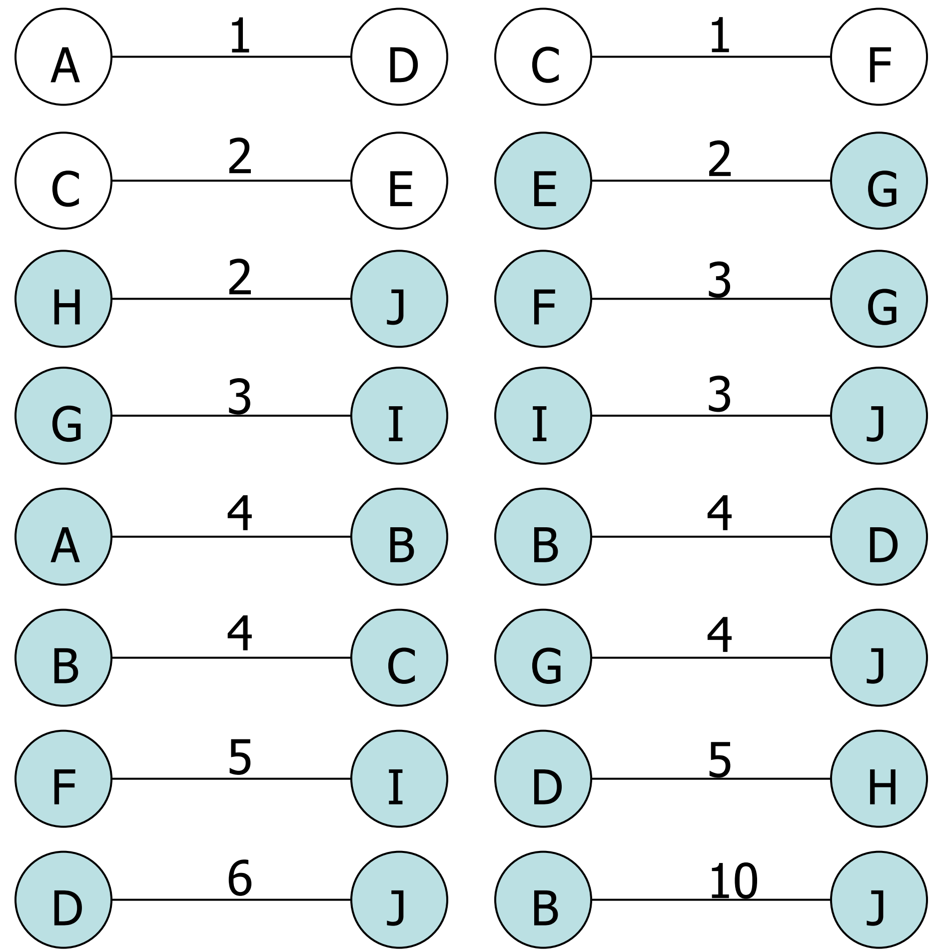




Kruskal's Algorithm – Example

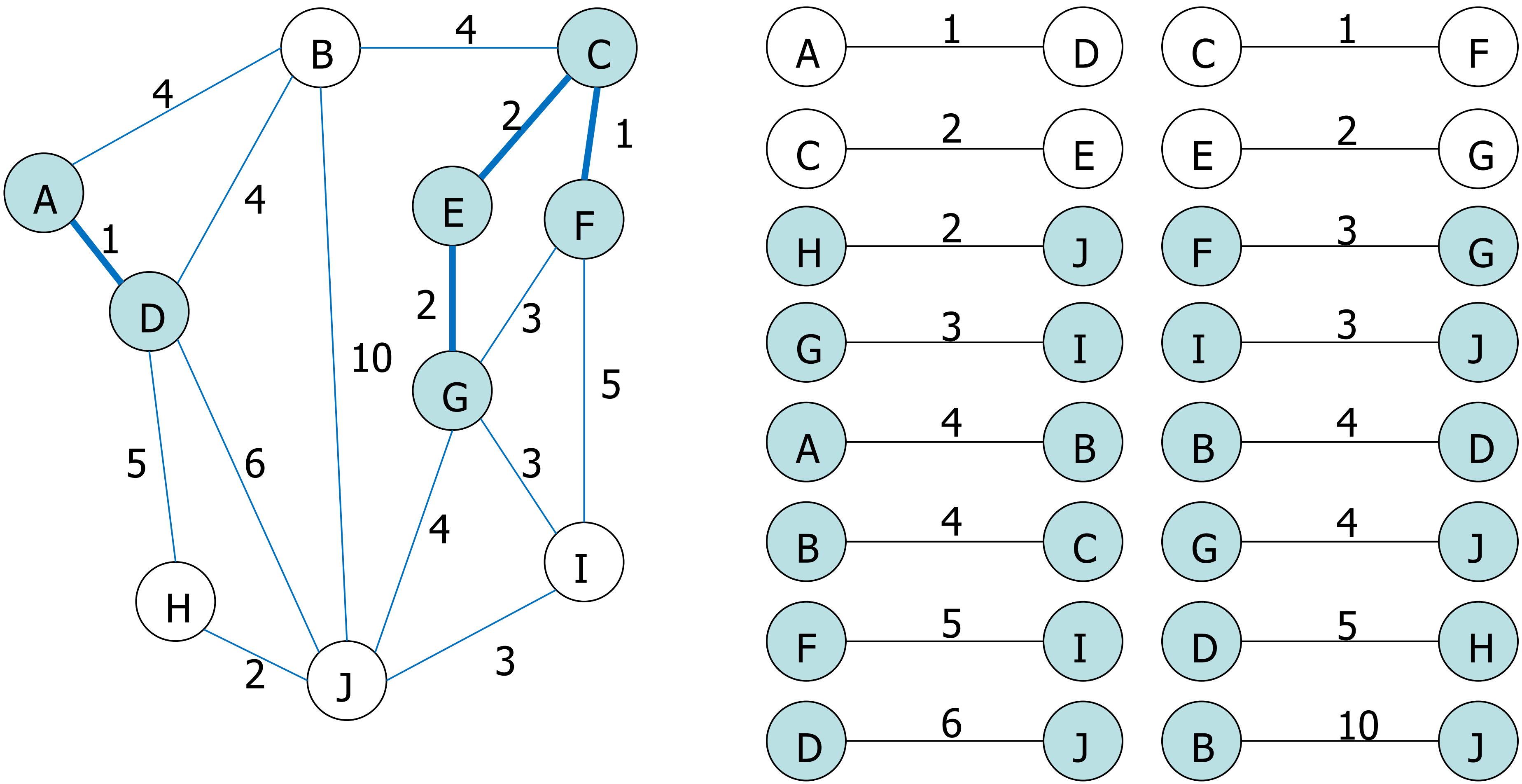
- Add Edge





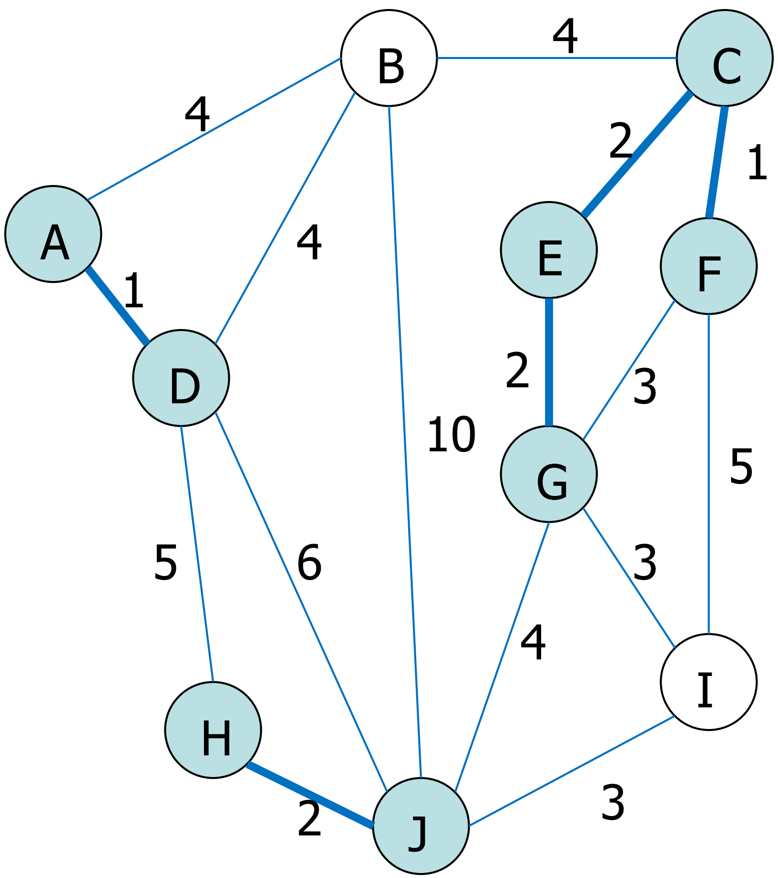
Kruskal's Algorithm – Example

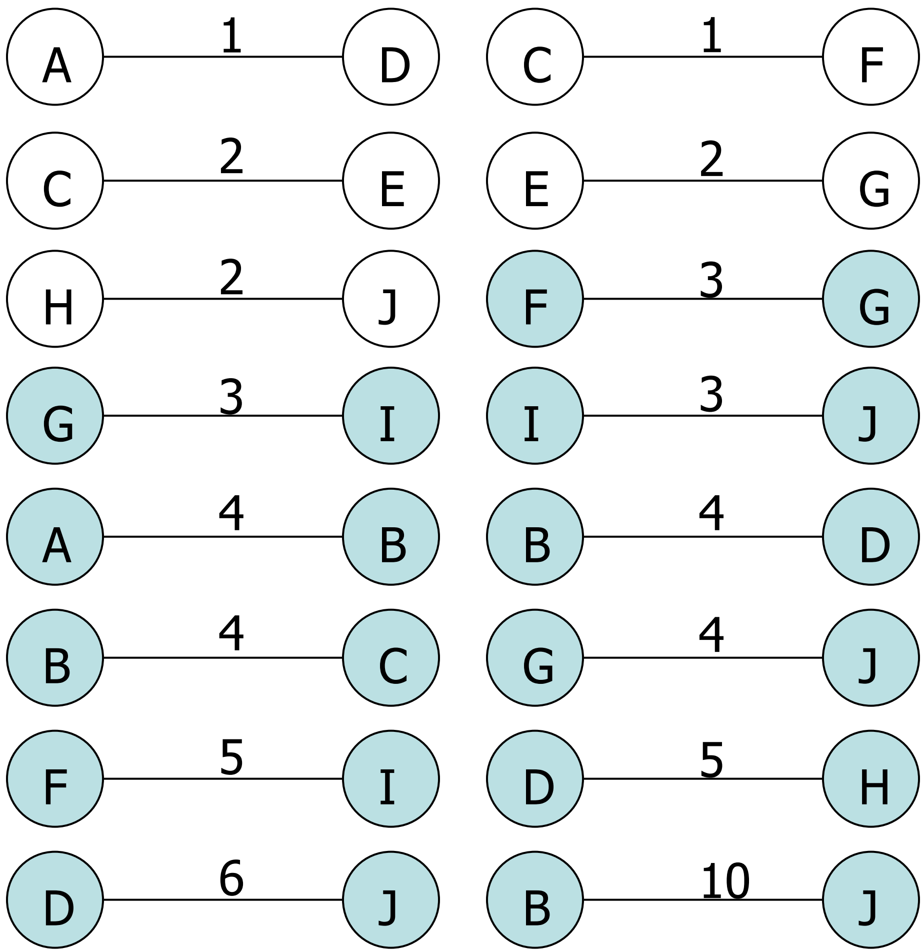
- Add Edge



Kruskal’s Algorithm – Example

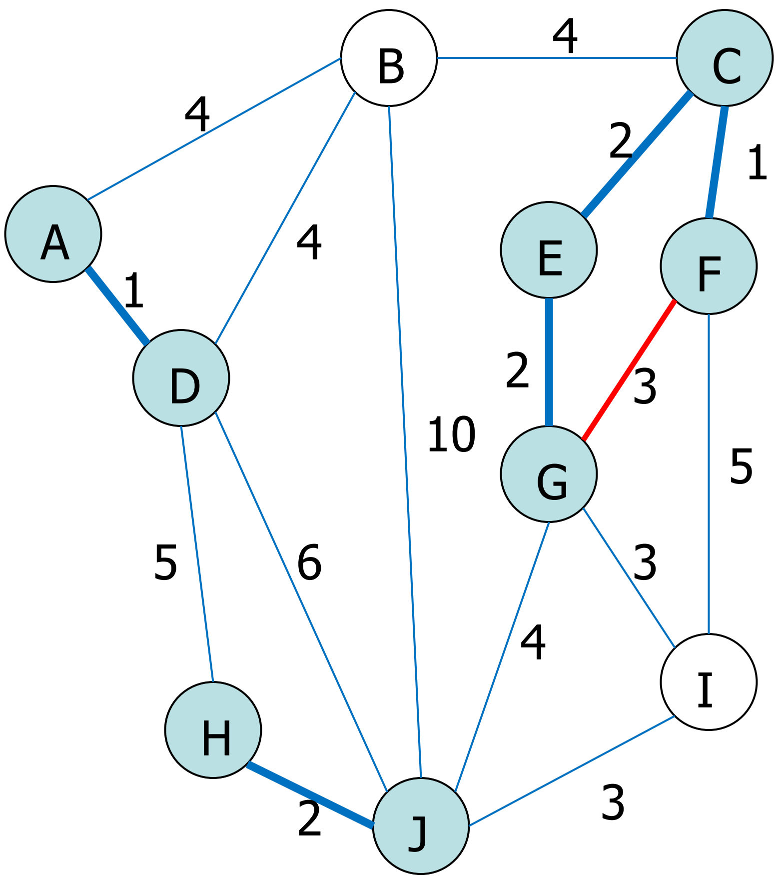
- Add Edge

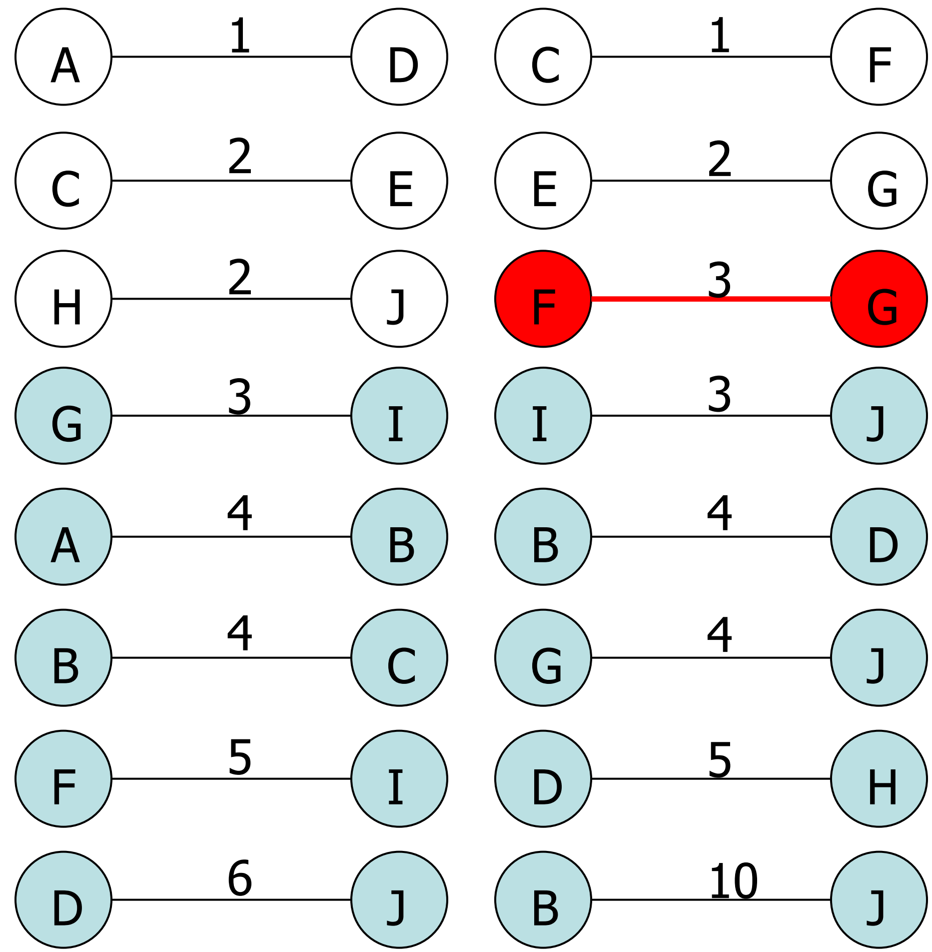




Kruskal's Algorithm – Example

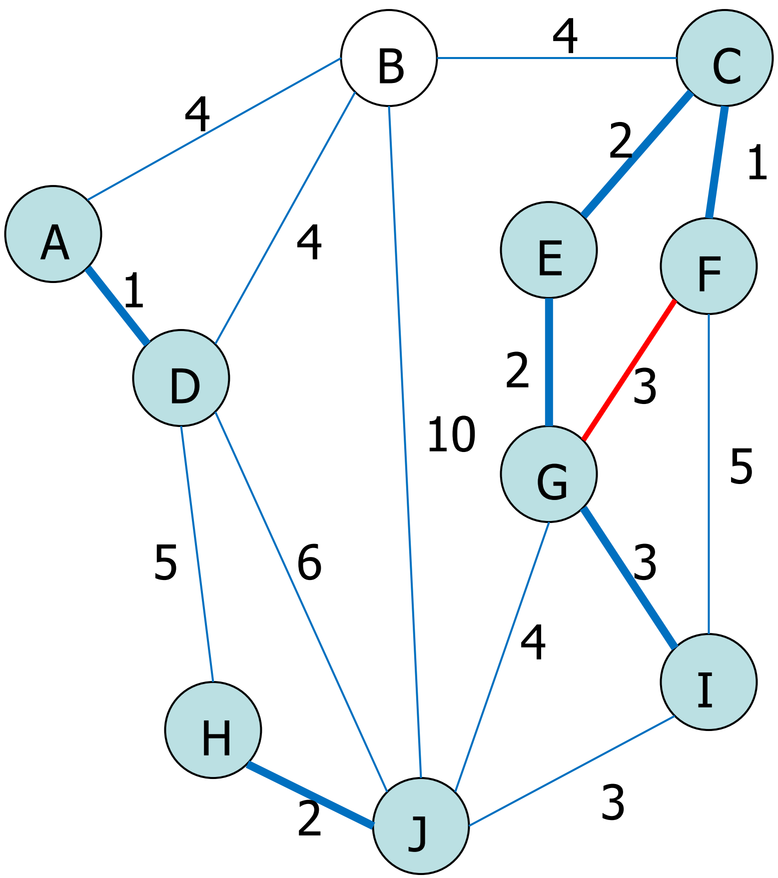
- Add Edge

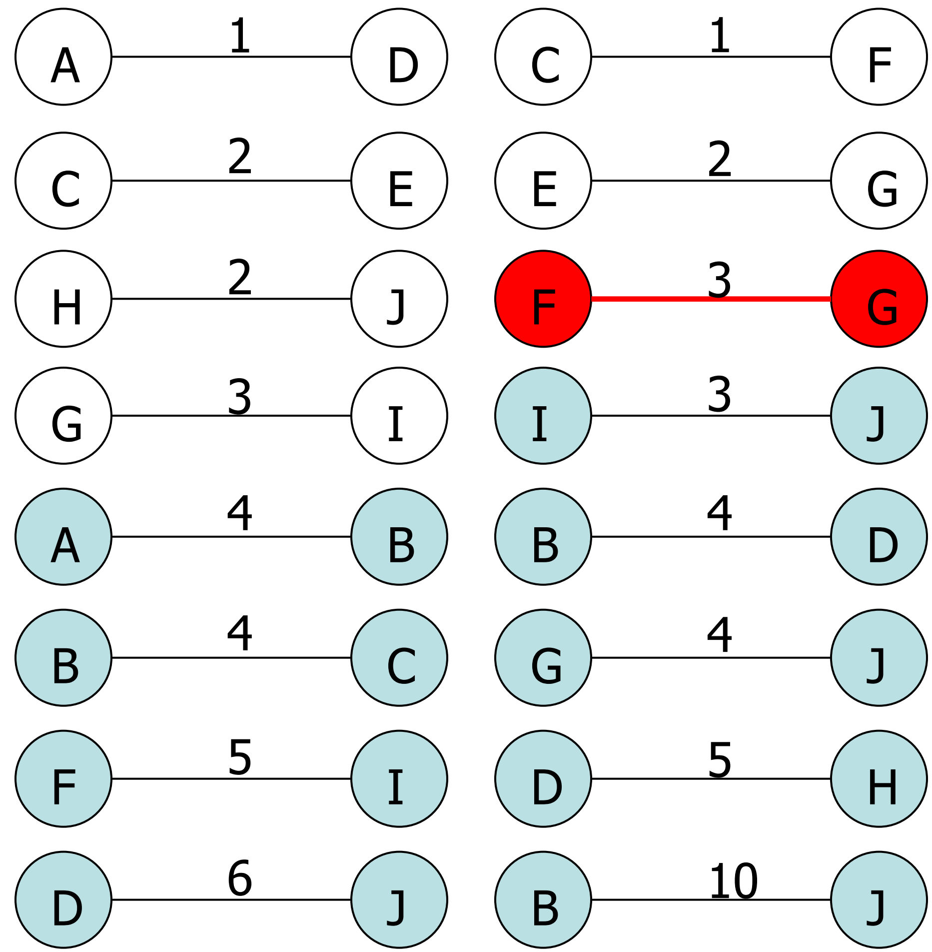




Kruskal's Algorithm – Example

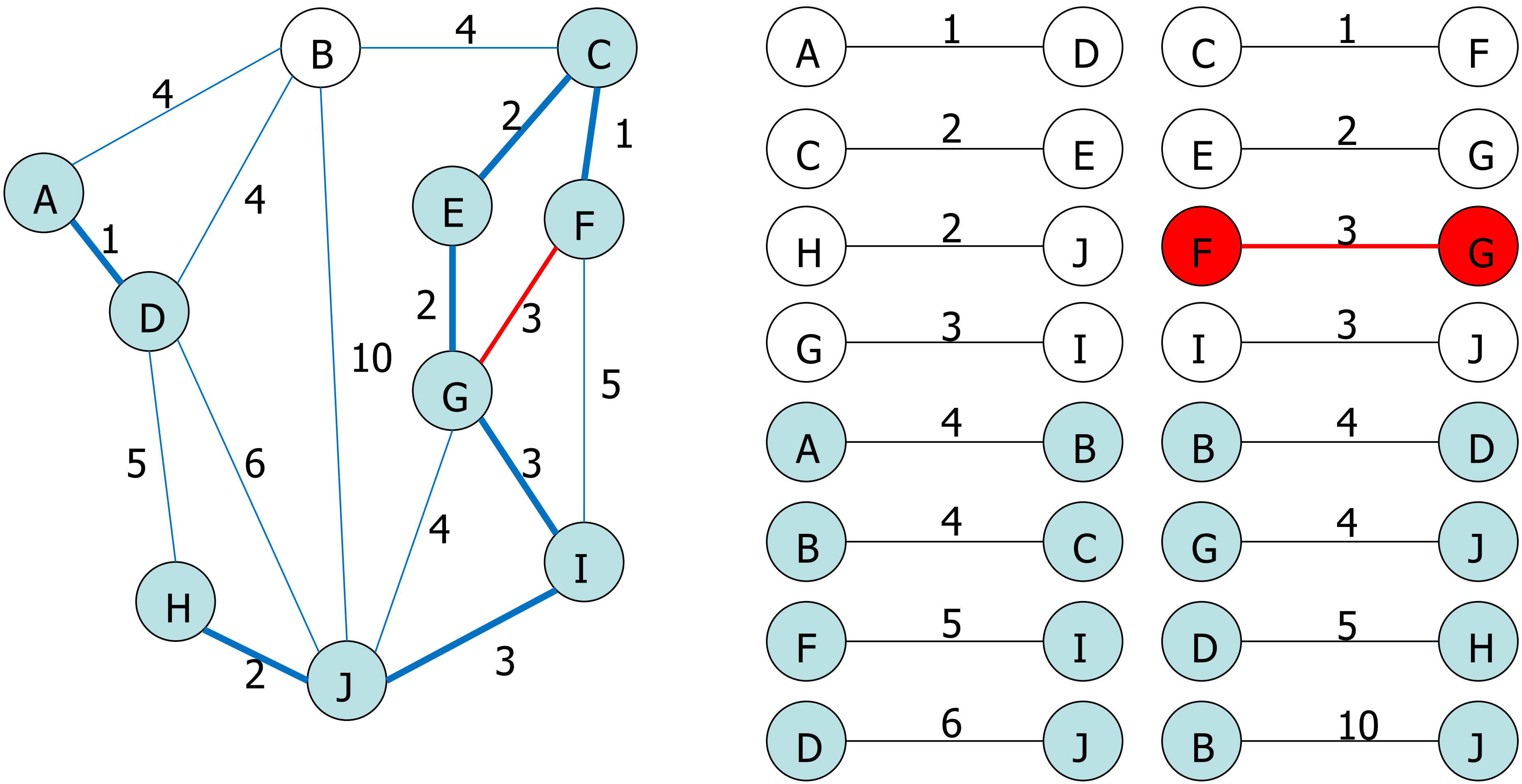
- Add Edge





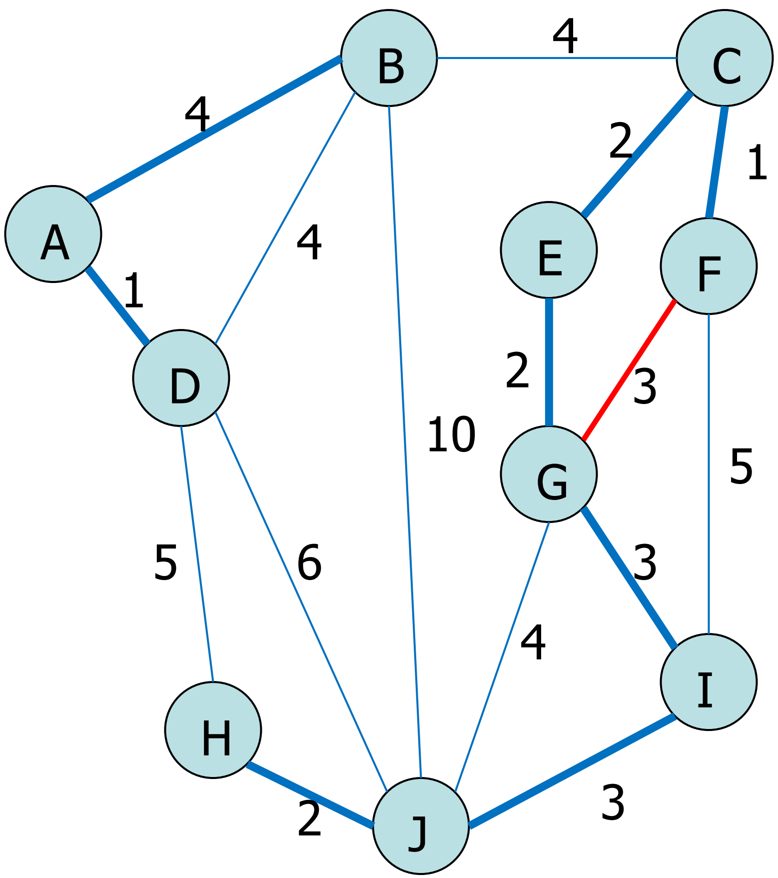
Kruskal's Algorithm – Example

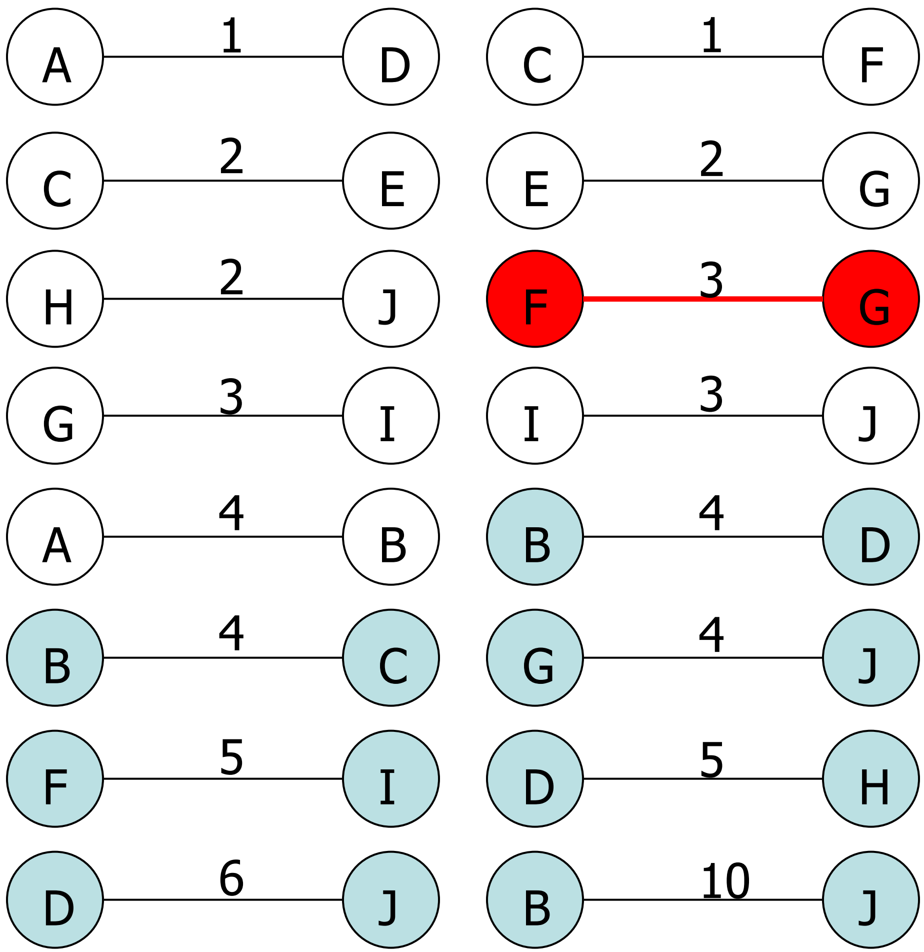
- Add Edge



Kruskal's Algorithm – Example

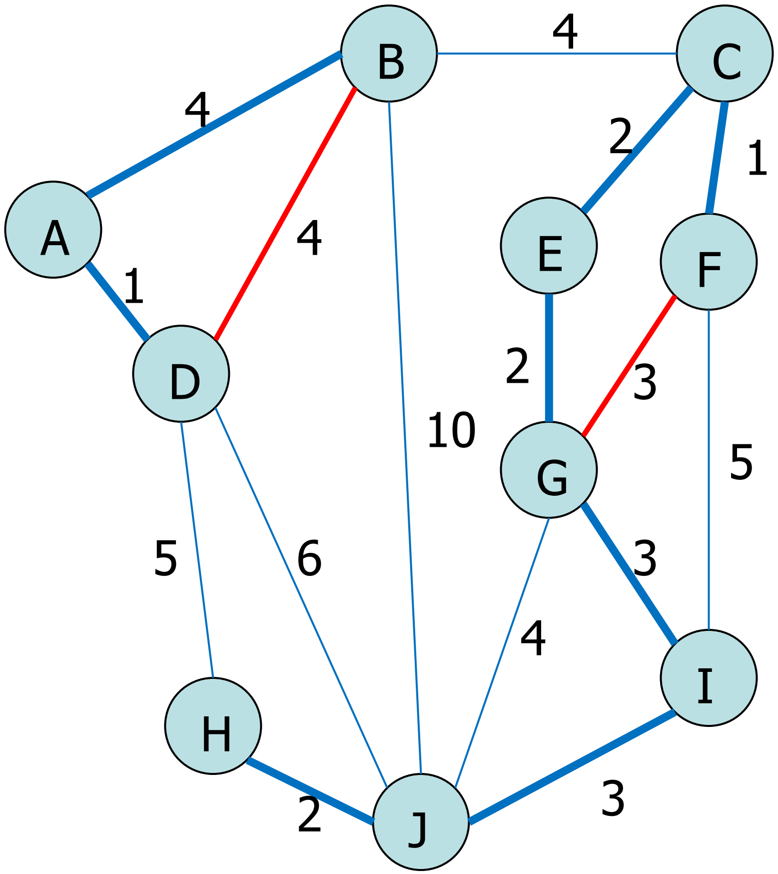
- Add Edge

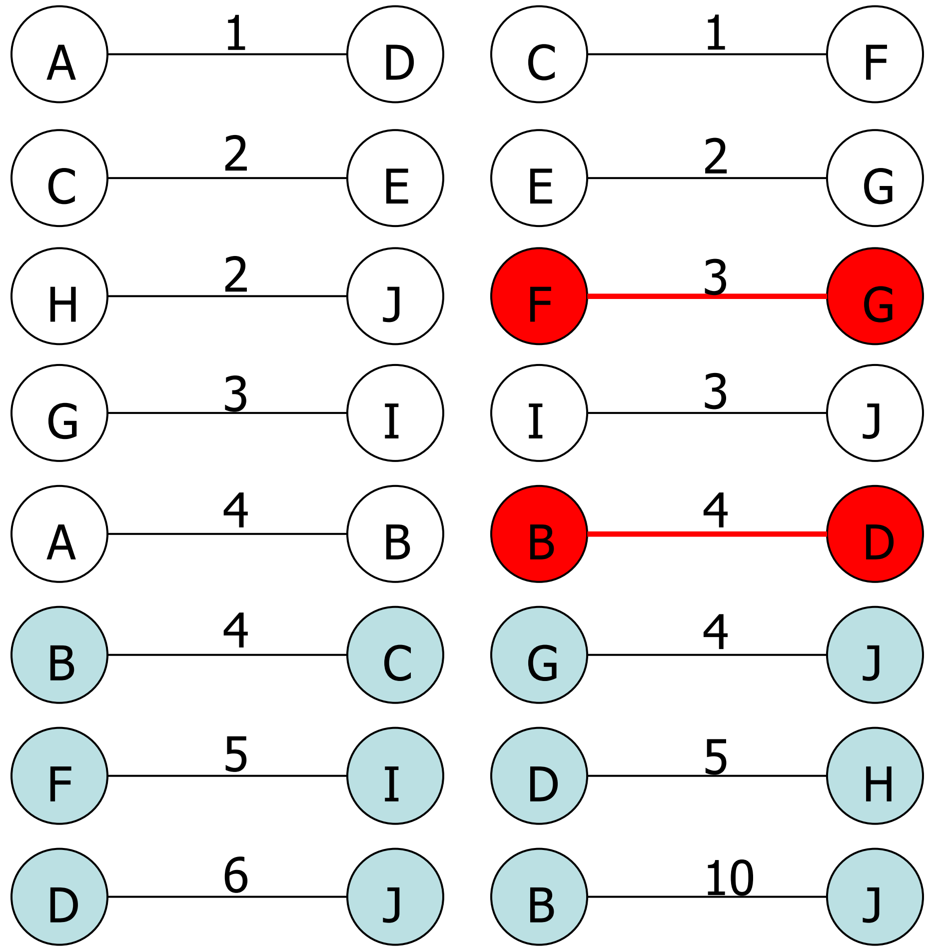




Kruskal's Algorithm – Example

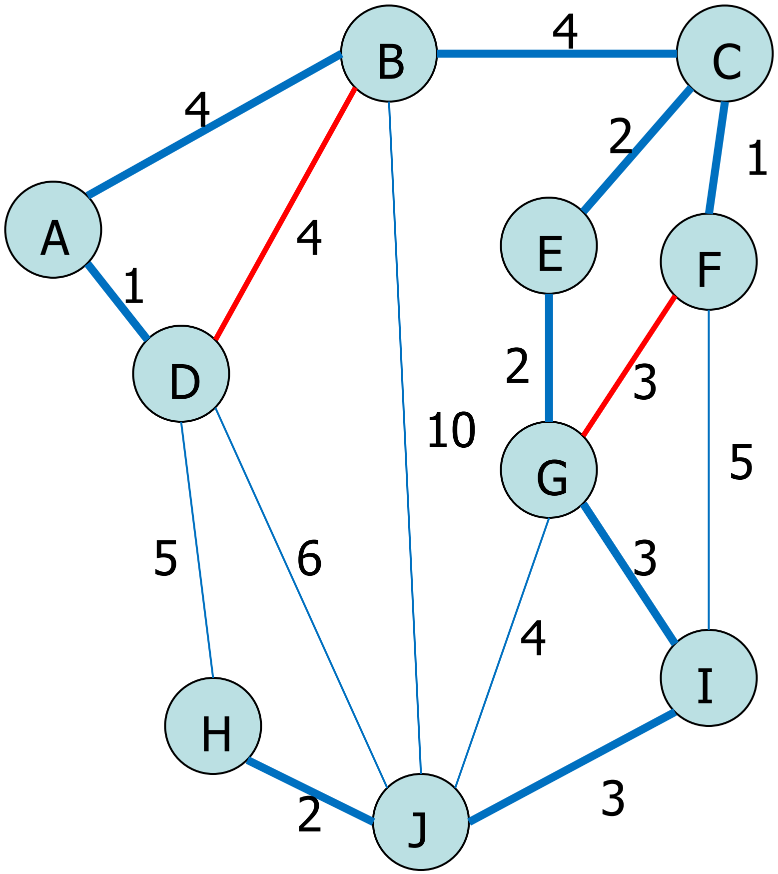
- Add Edge

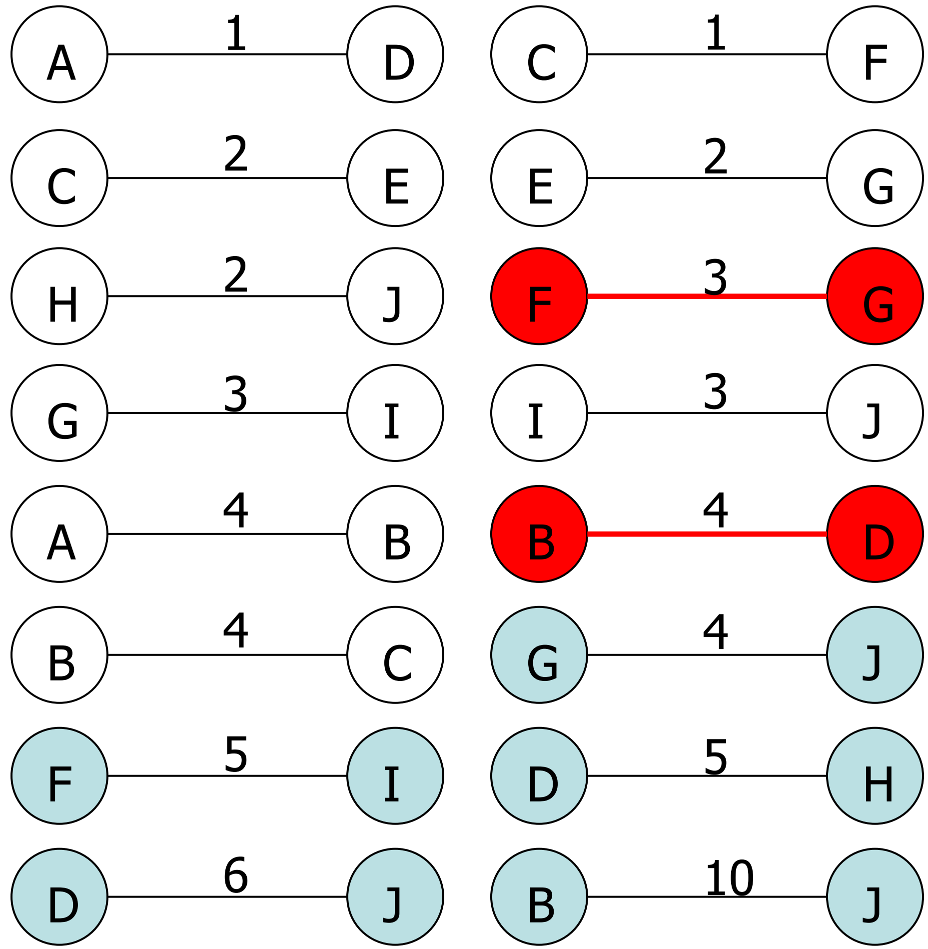




Kruskal's Algorithm – Example

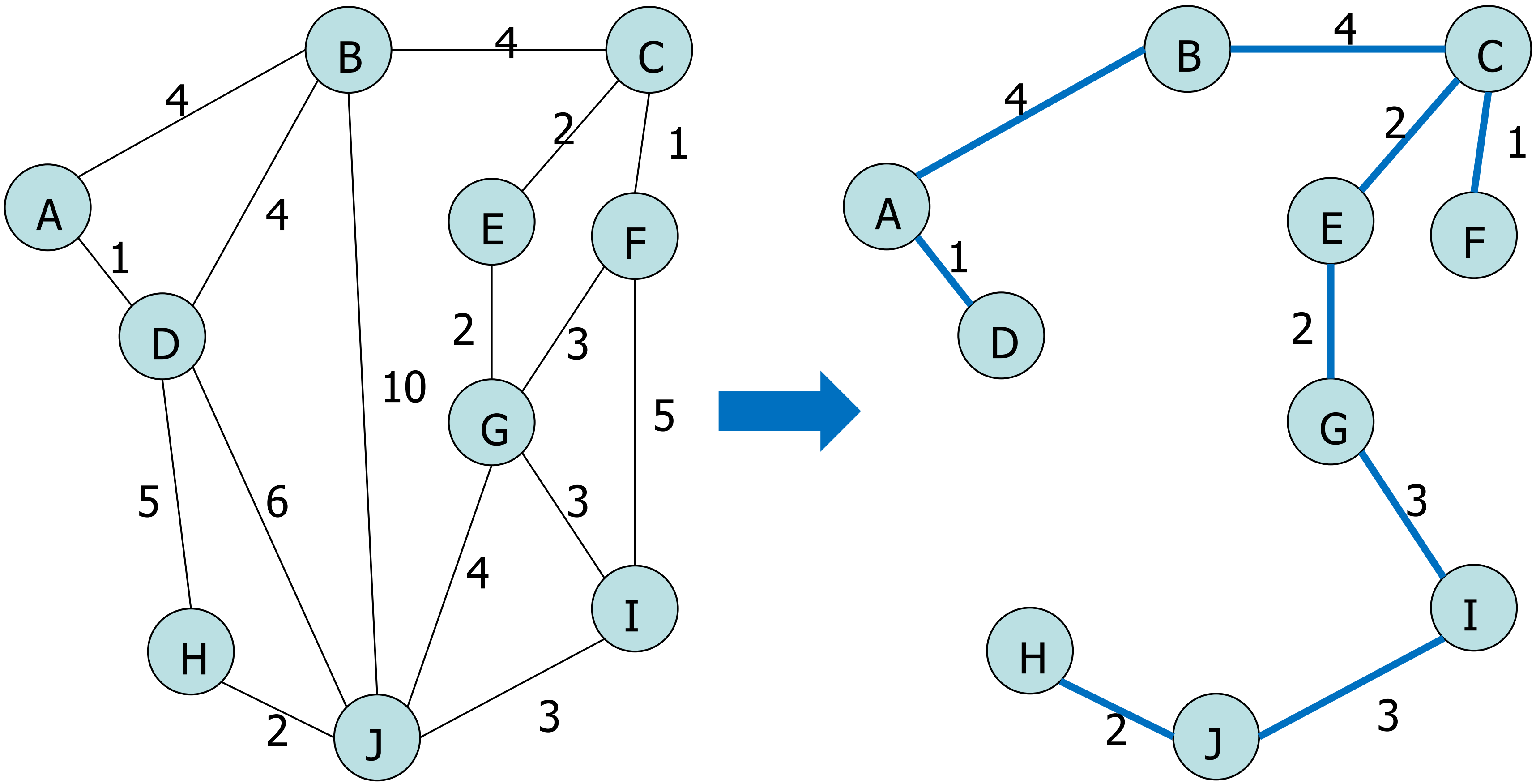
- Add Edge





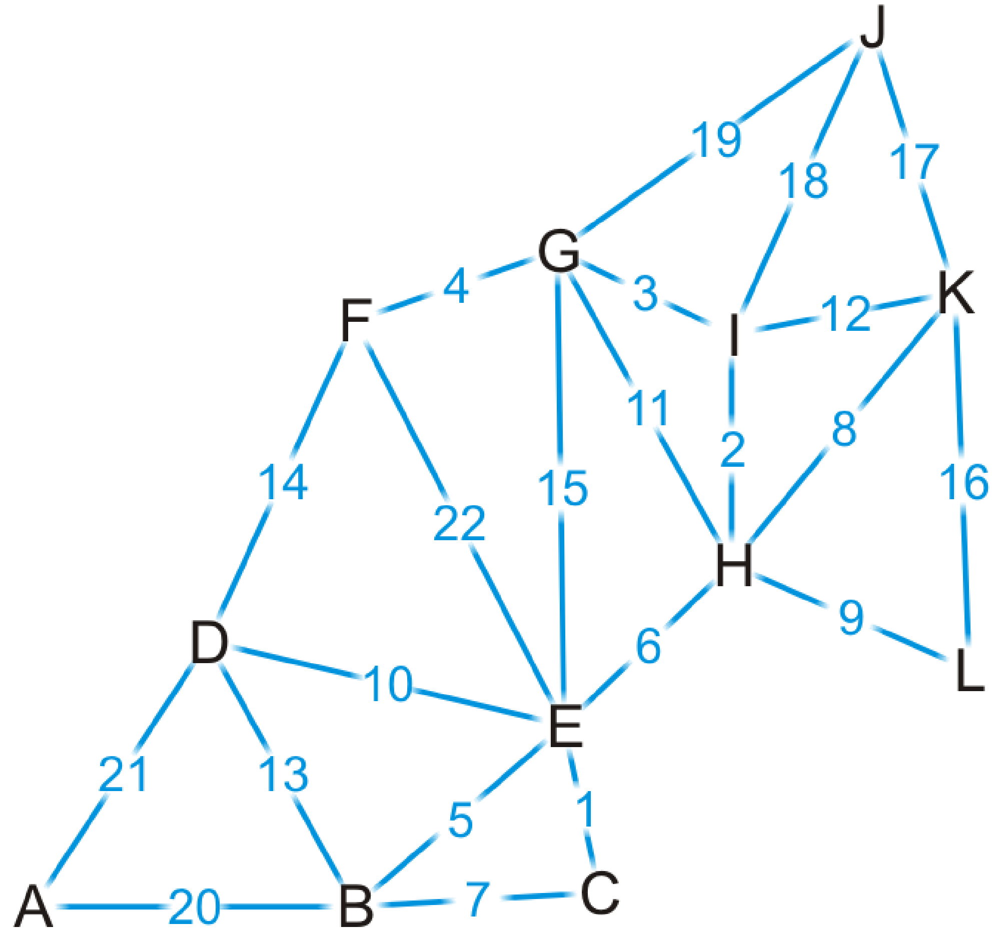
Kruskal's Algorithm – Example

- Minimum spanning tree



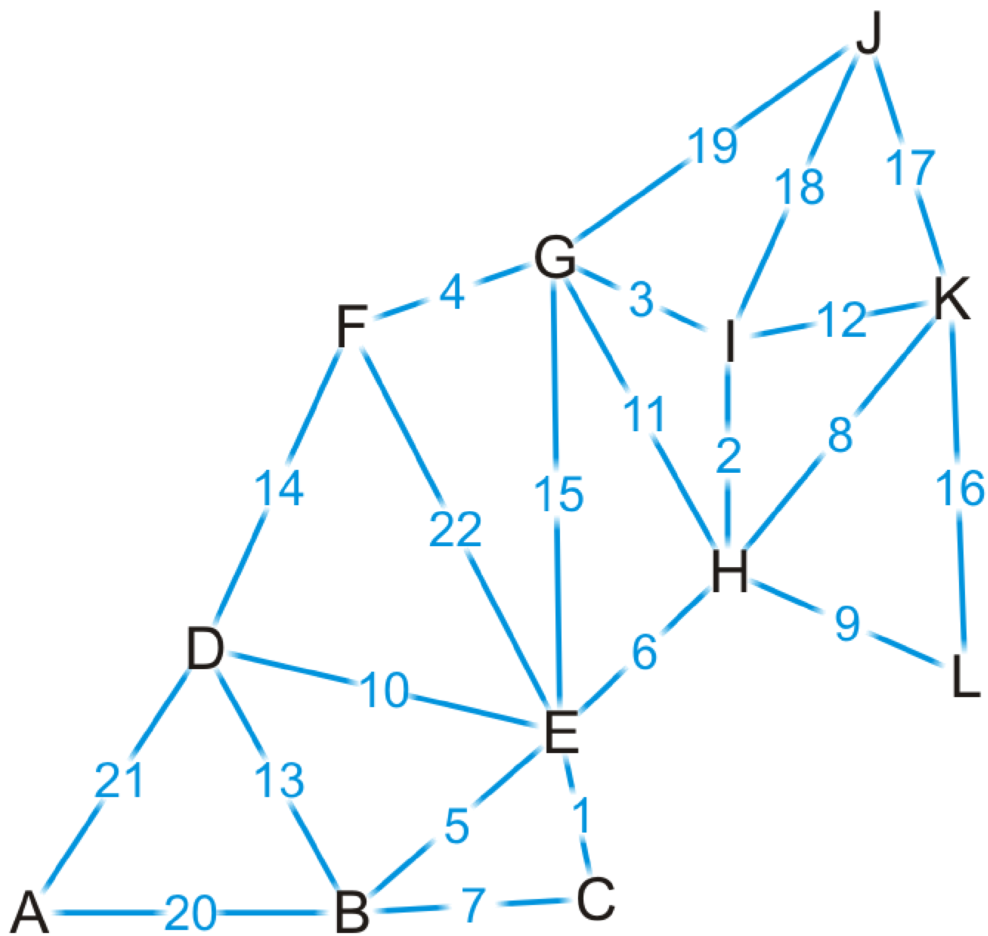
Kruskal’s Algorithm – Example

- Complete graph



Kruskal's Algorithm – Example

- Sort edges based on weight



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
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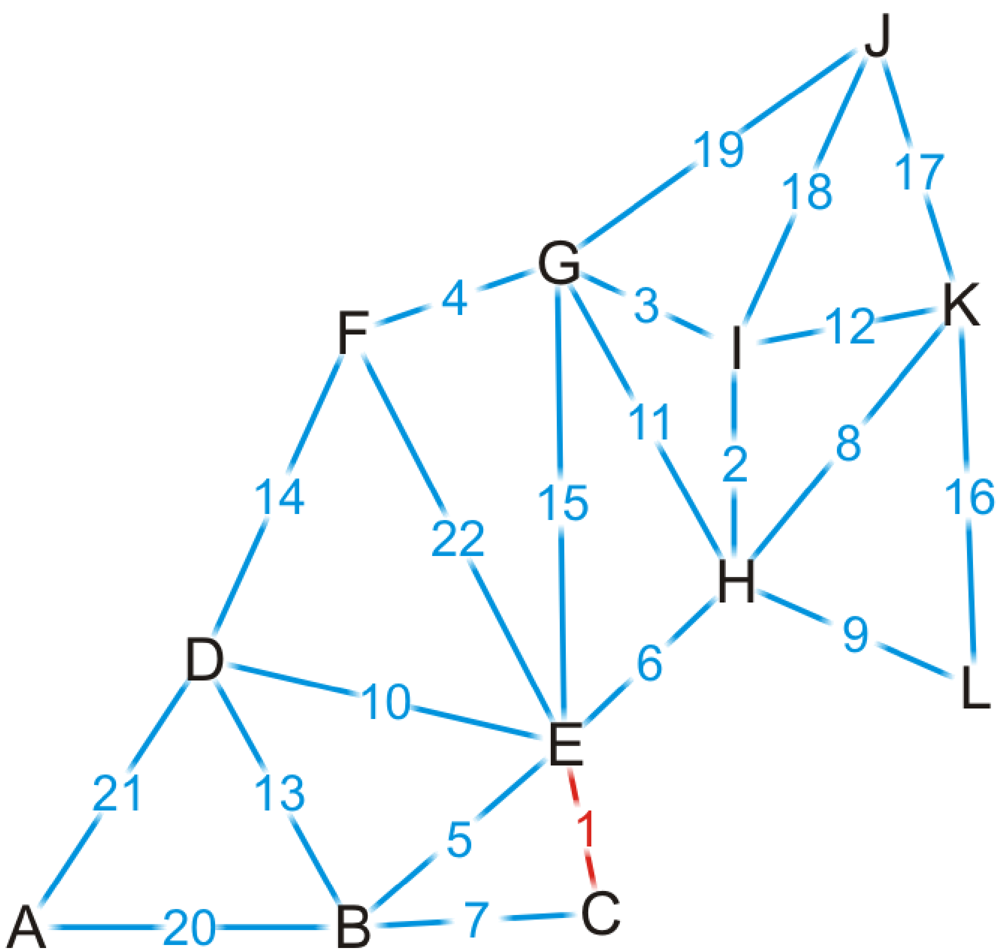
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal’s Algorithm – Example

- We start by adding edge {C, E}



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{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
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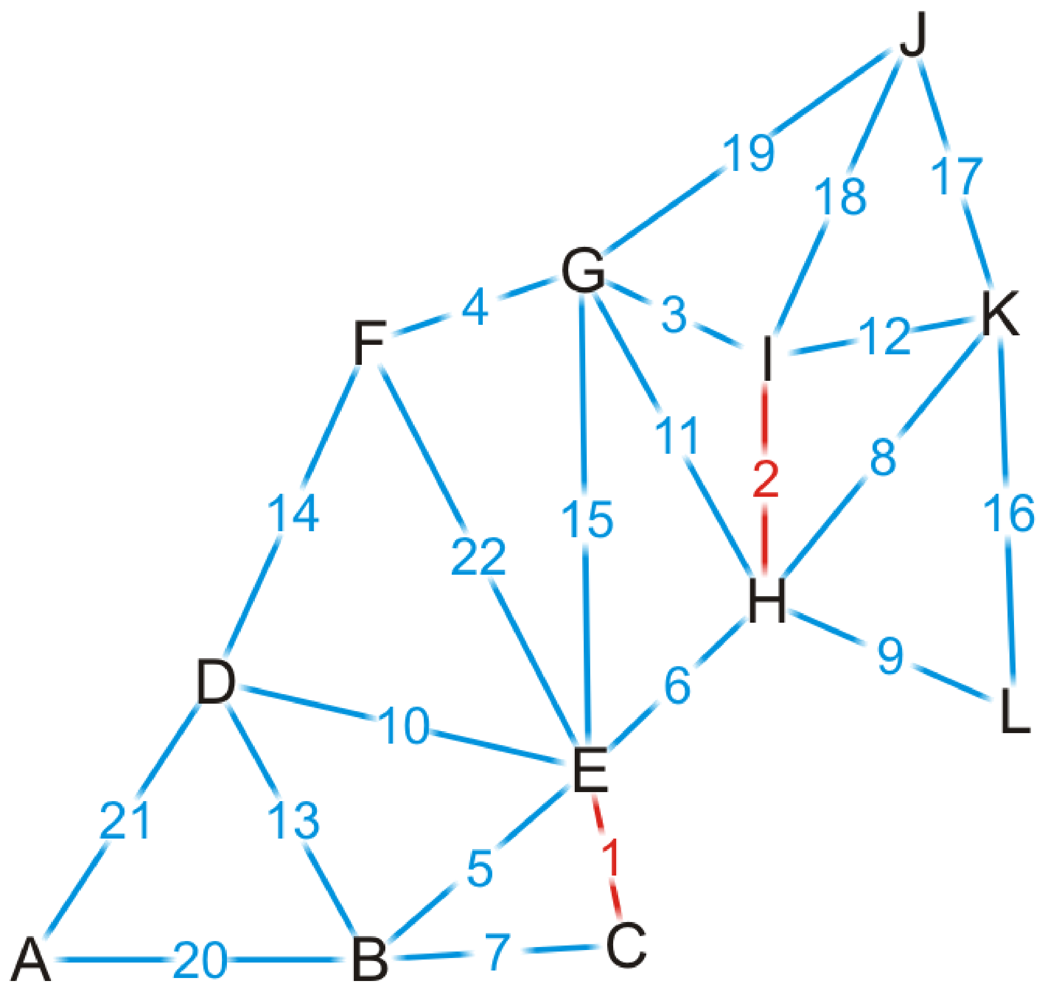
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

- We add edge {H, I}



{C, E}
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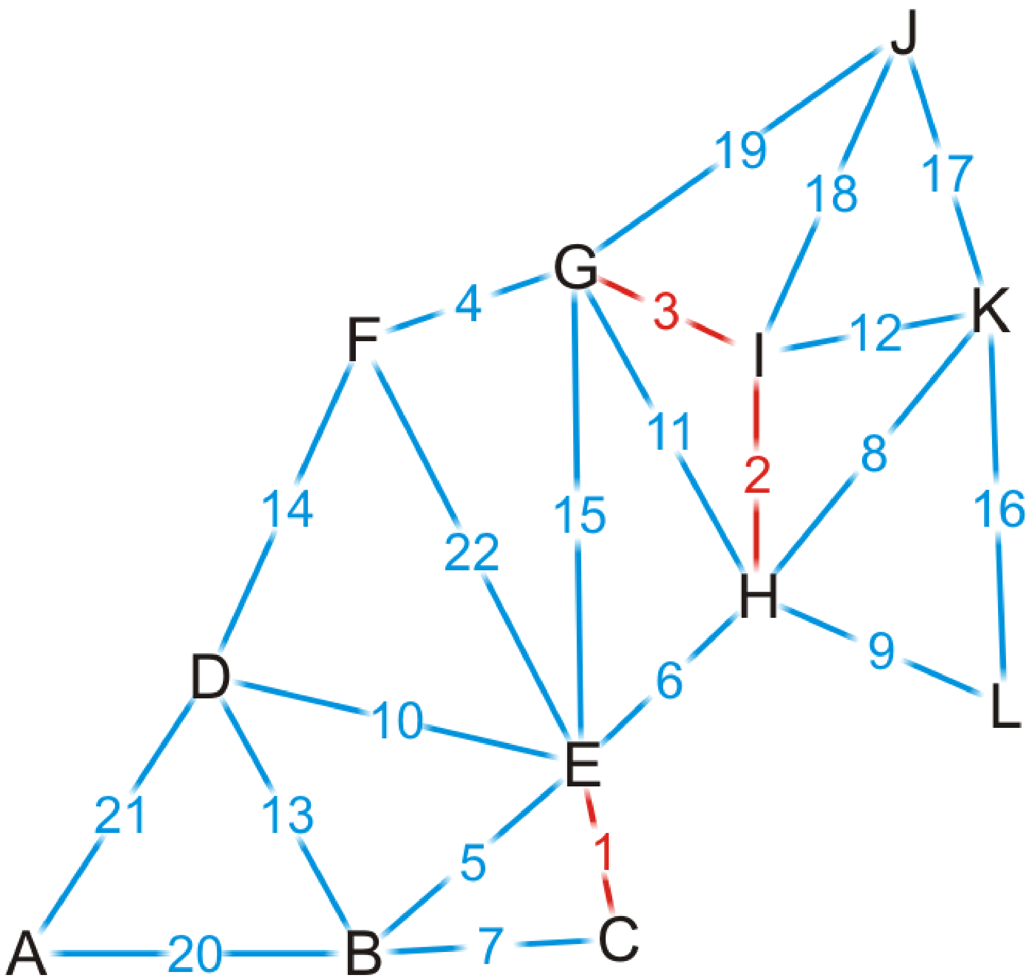
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal’s Algorithm – Example

- We add edge {G, I}



{C, E}
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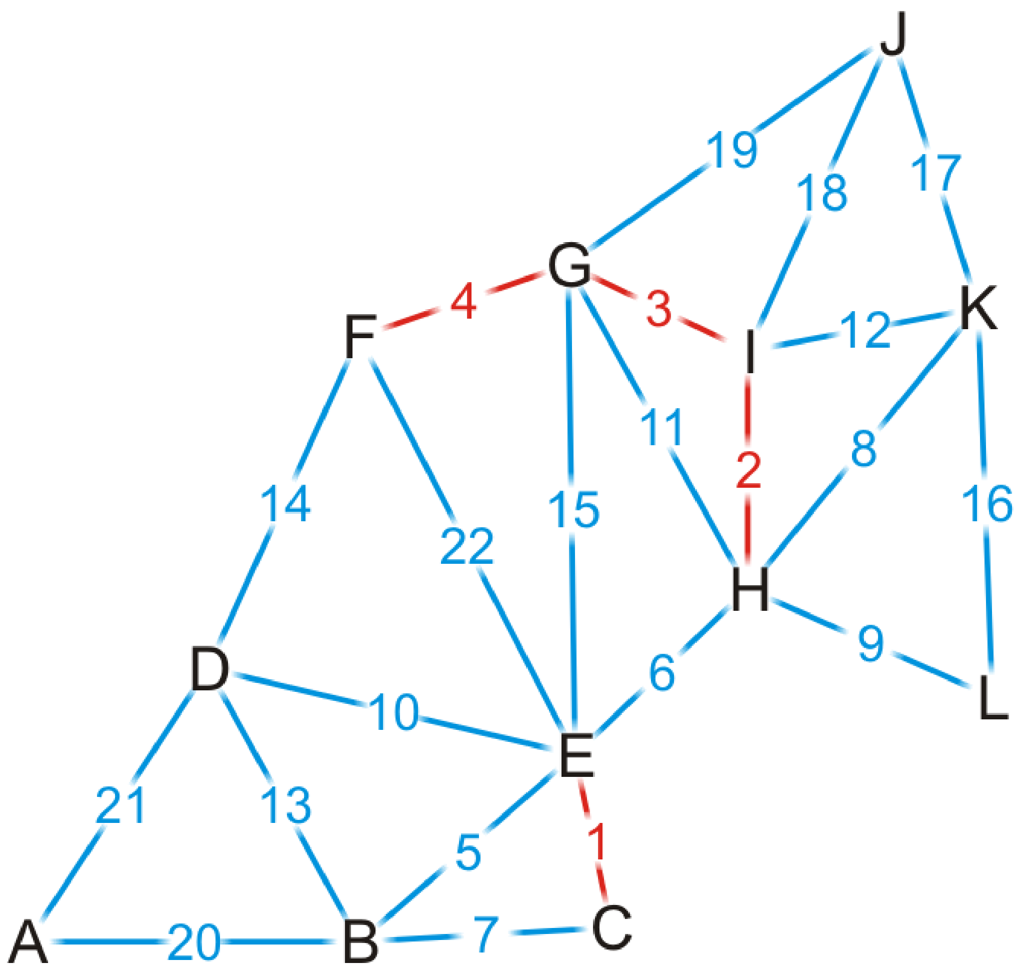
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal’s Algorithm – Example

- We add edge {F, G}



{C, E}
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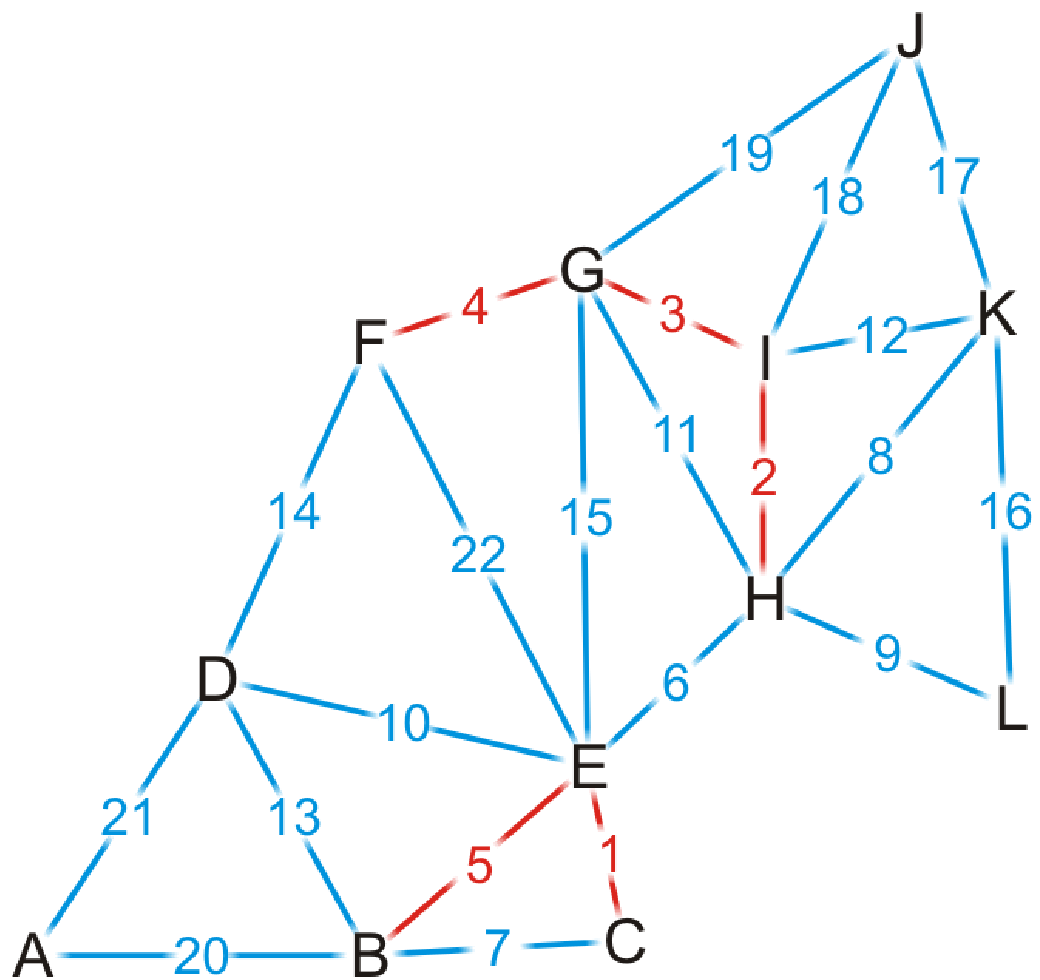
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

- We add edge {B, E}



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{J, I}
{J, G}
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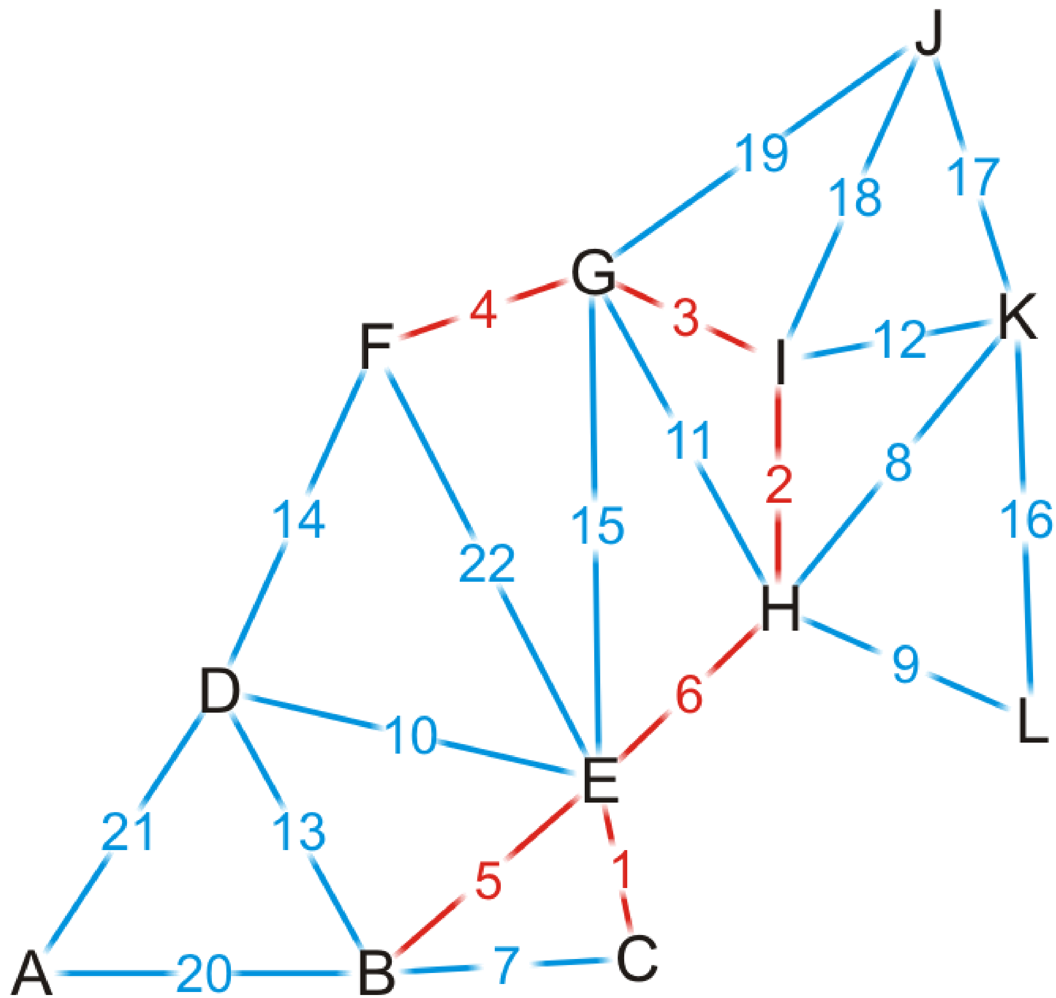
{A, D}
{E, F}

Kruskal's Algorithm – Example

- We add edge {E, H}
  - This coalesces the two spanning sub-trees into one



{C, E}
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{J, K}
{J, I}
{J, G}
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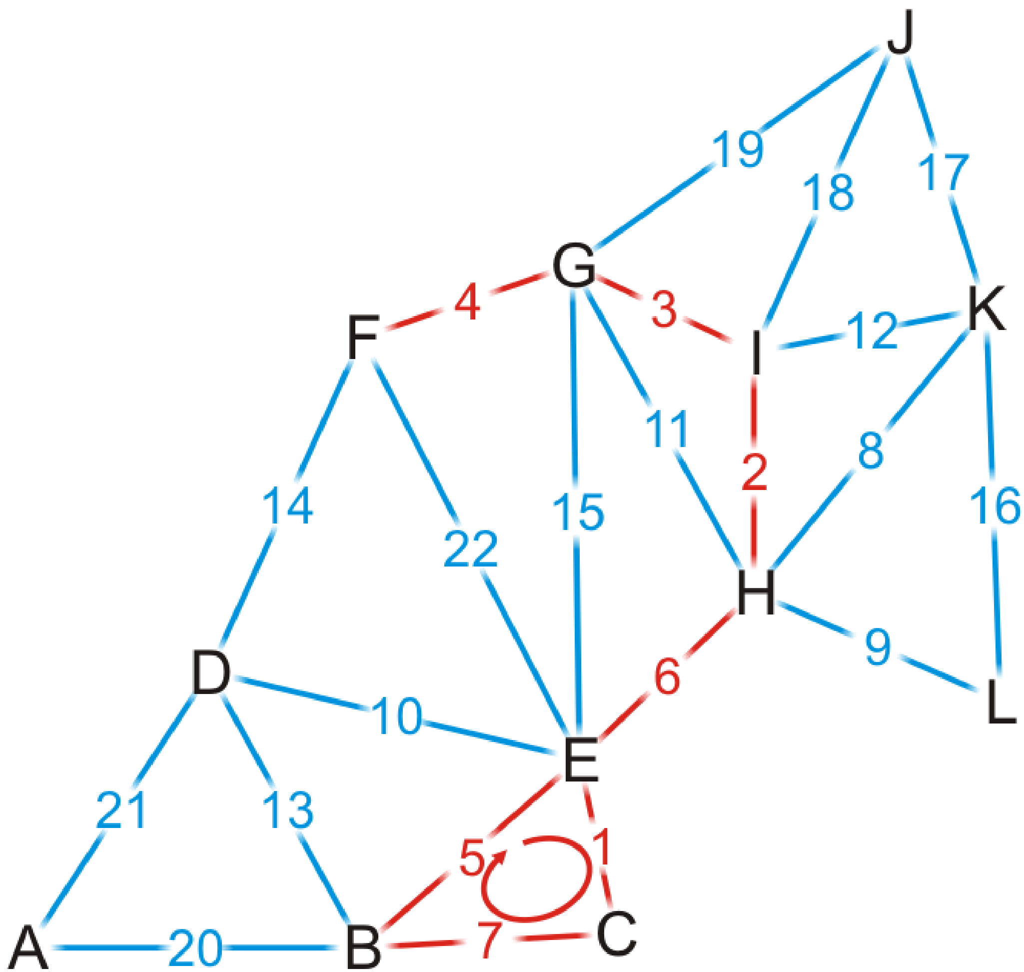
{A, D}
{E, F}

Kruskal's Algorithm – Example

- We try adding {B, C}, but it creates a cycle



{C, E}
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→ {B, C}
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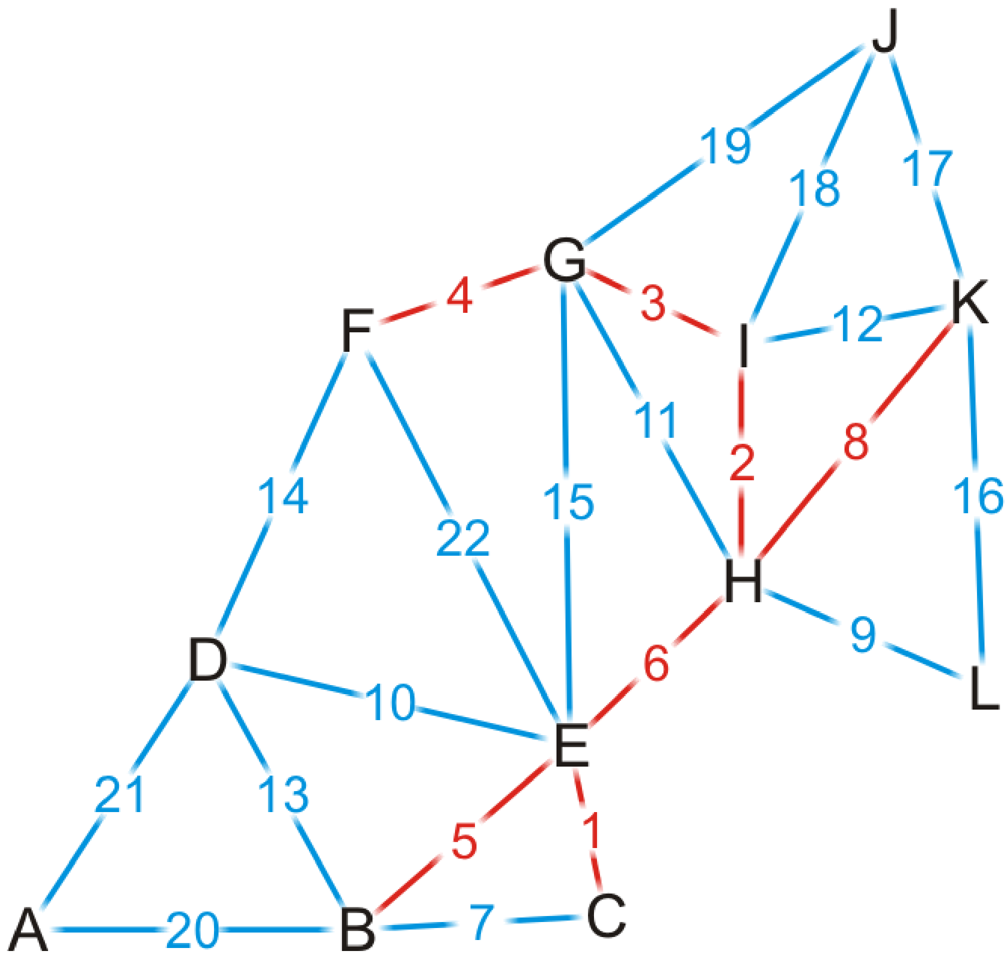
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal’s Algorithm – Example

- We add edge {H, K}



{C, E}
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{G, I}
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{E, H}
{B, C}
→ {H, K}
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{D, E}
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{B, D}
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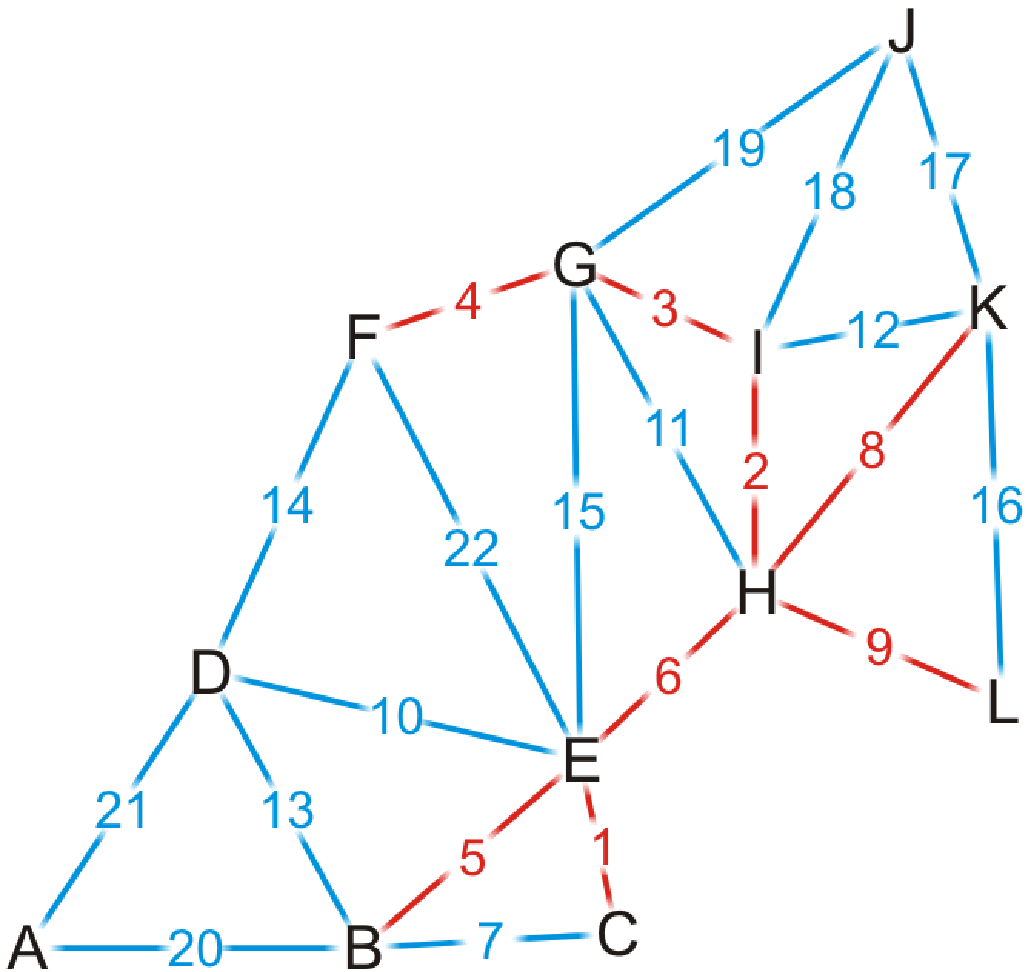
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

- We add edge {H, L}



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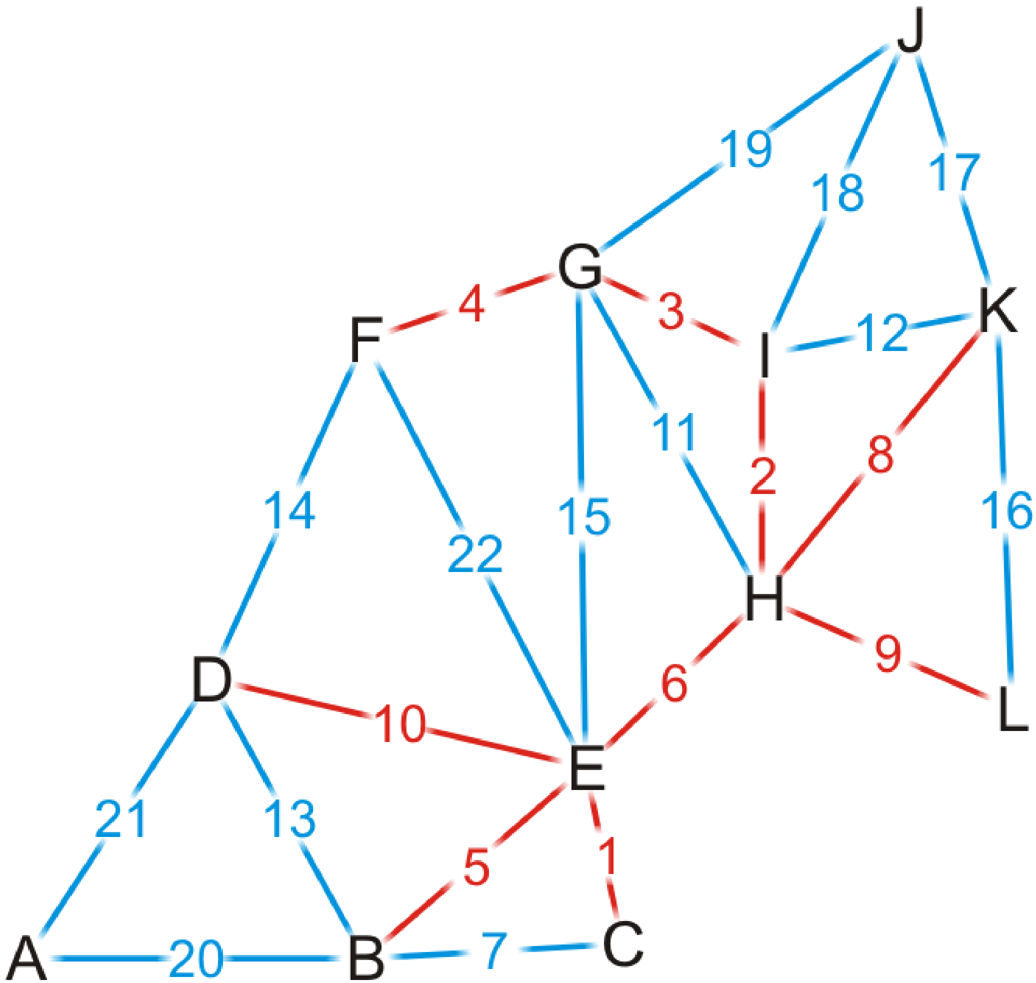
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

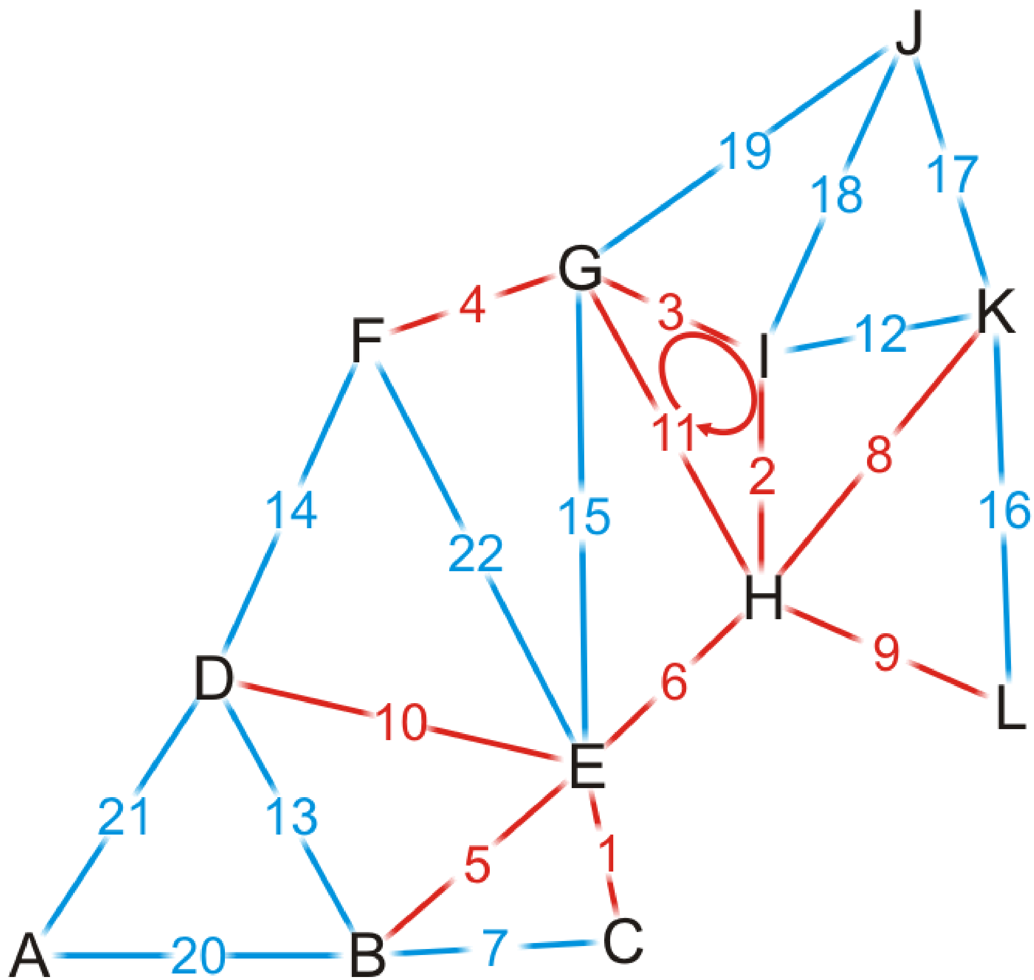
- We add edge {D, E}



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

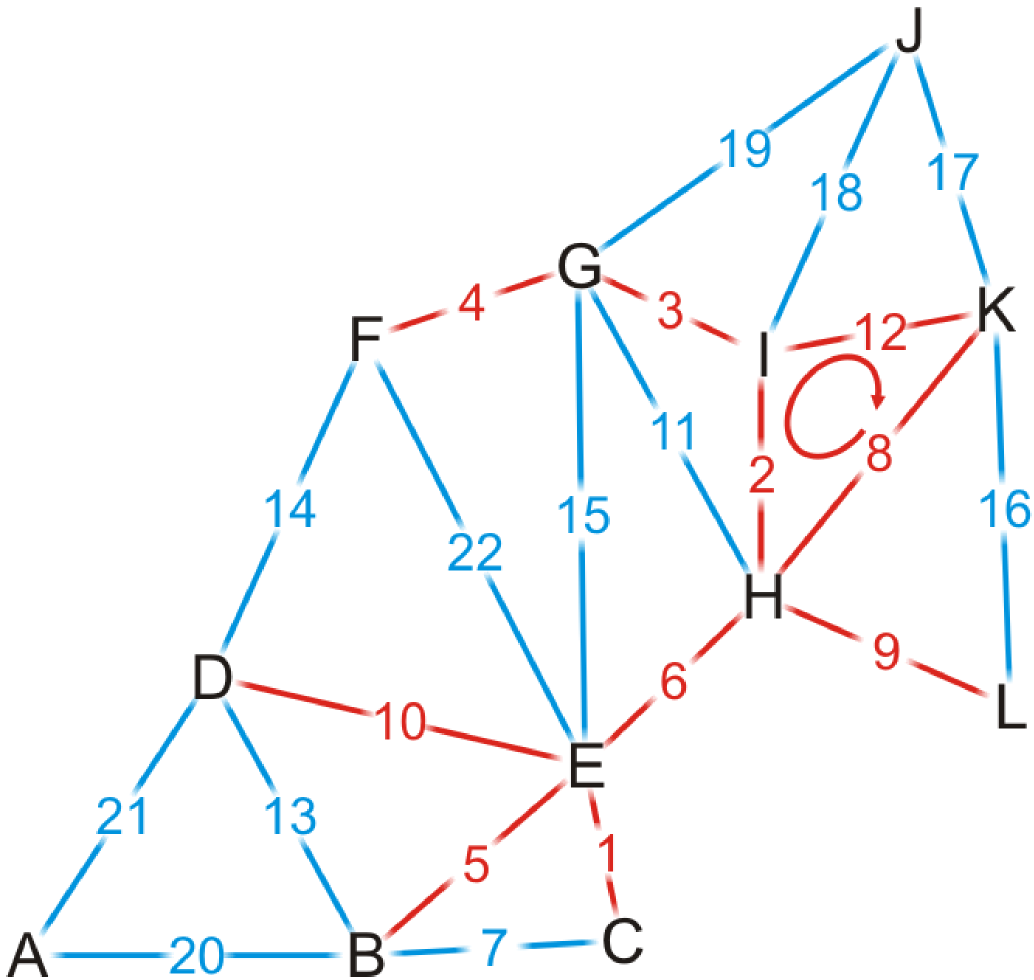
- We try adding {G, H}, but it creates a cycle



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

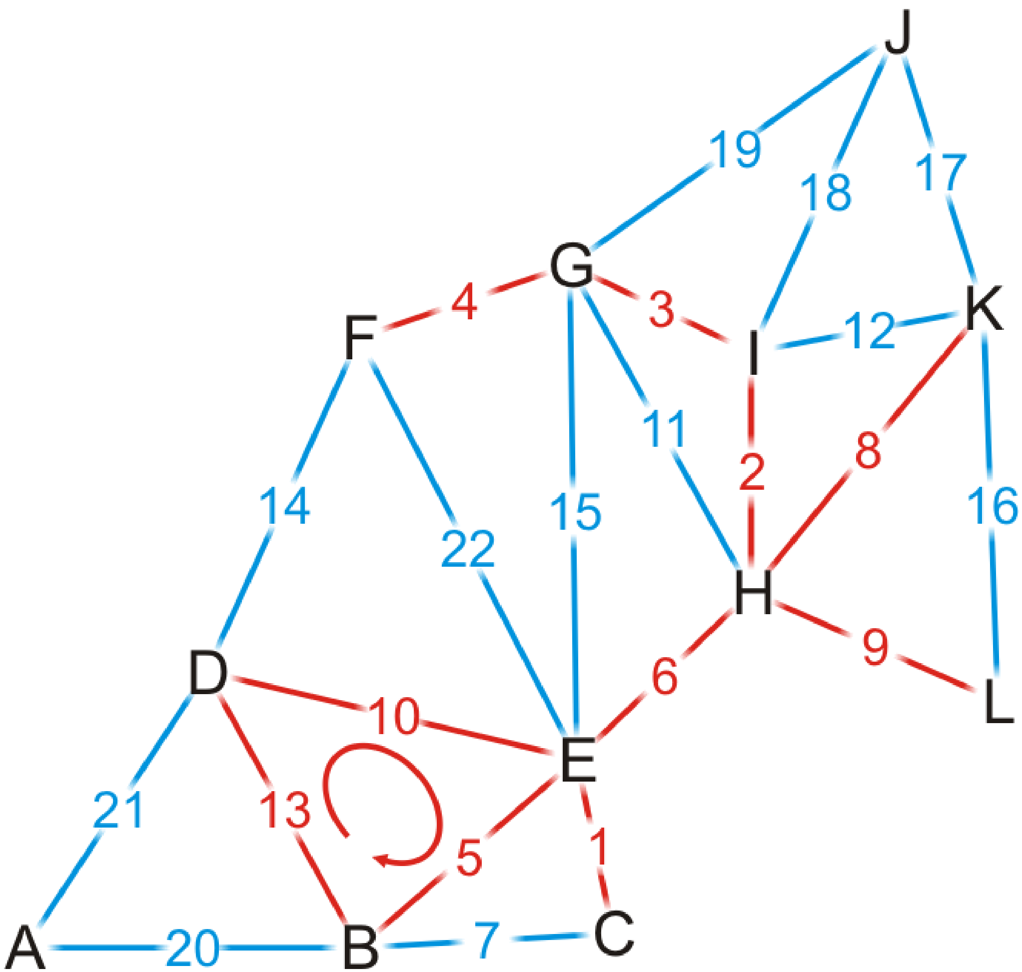
- We try adding {I, K}, but it creates a cycle



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

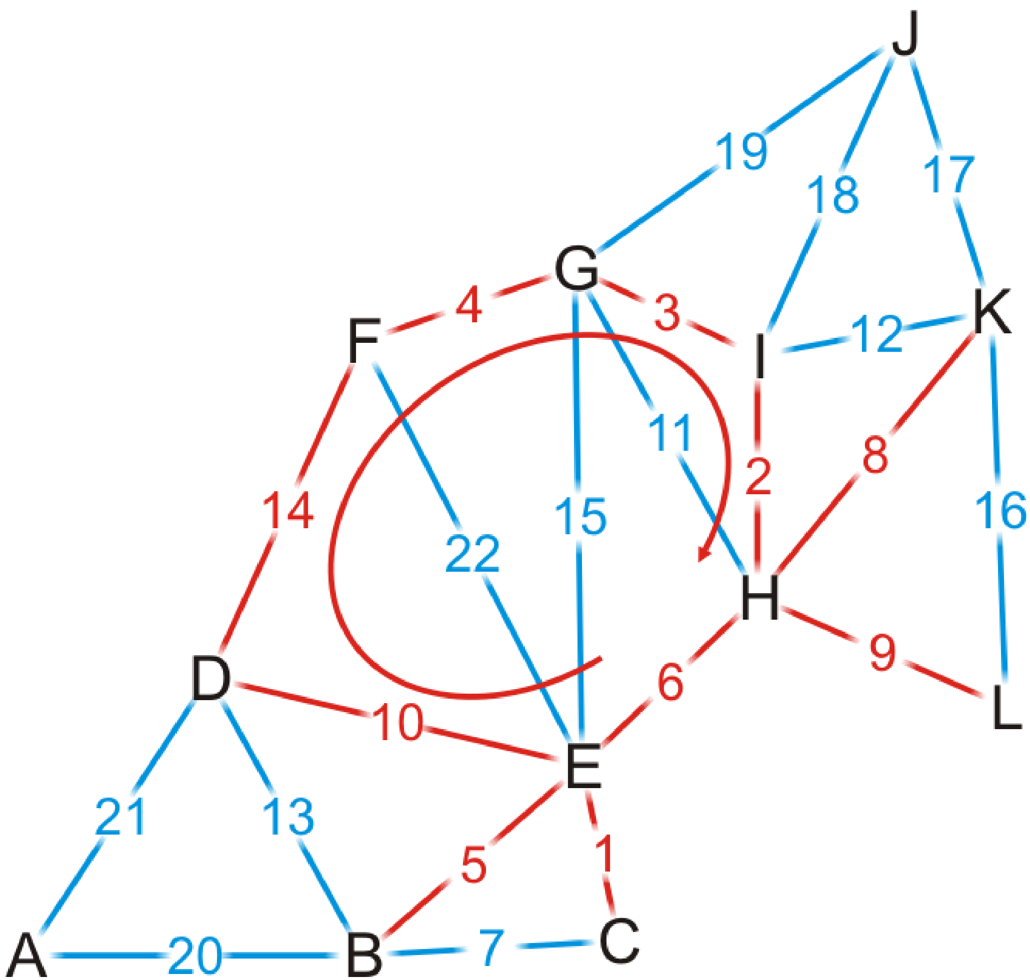
- We try adding {B, D}, but it creates a cycle



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
→ {B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

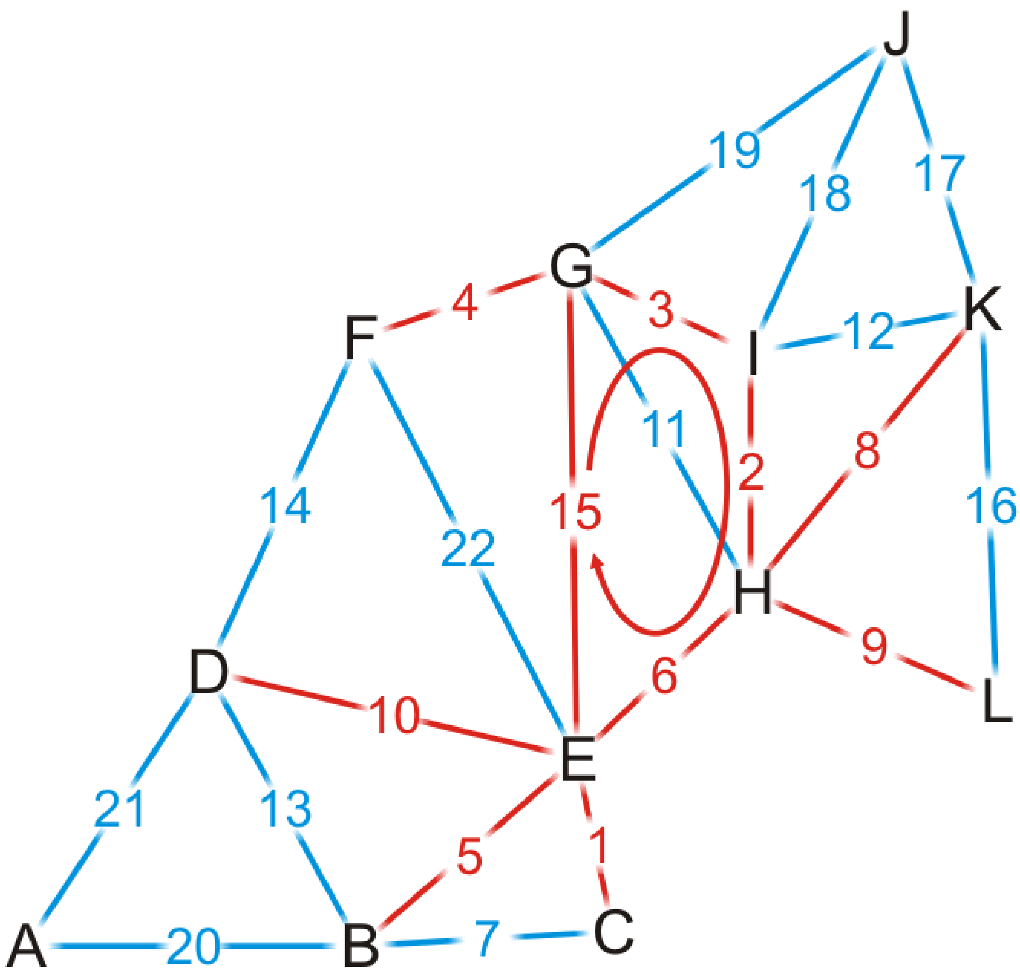
- We try adding {D, F}, but it creates a cycle



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
→ {D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

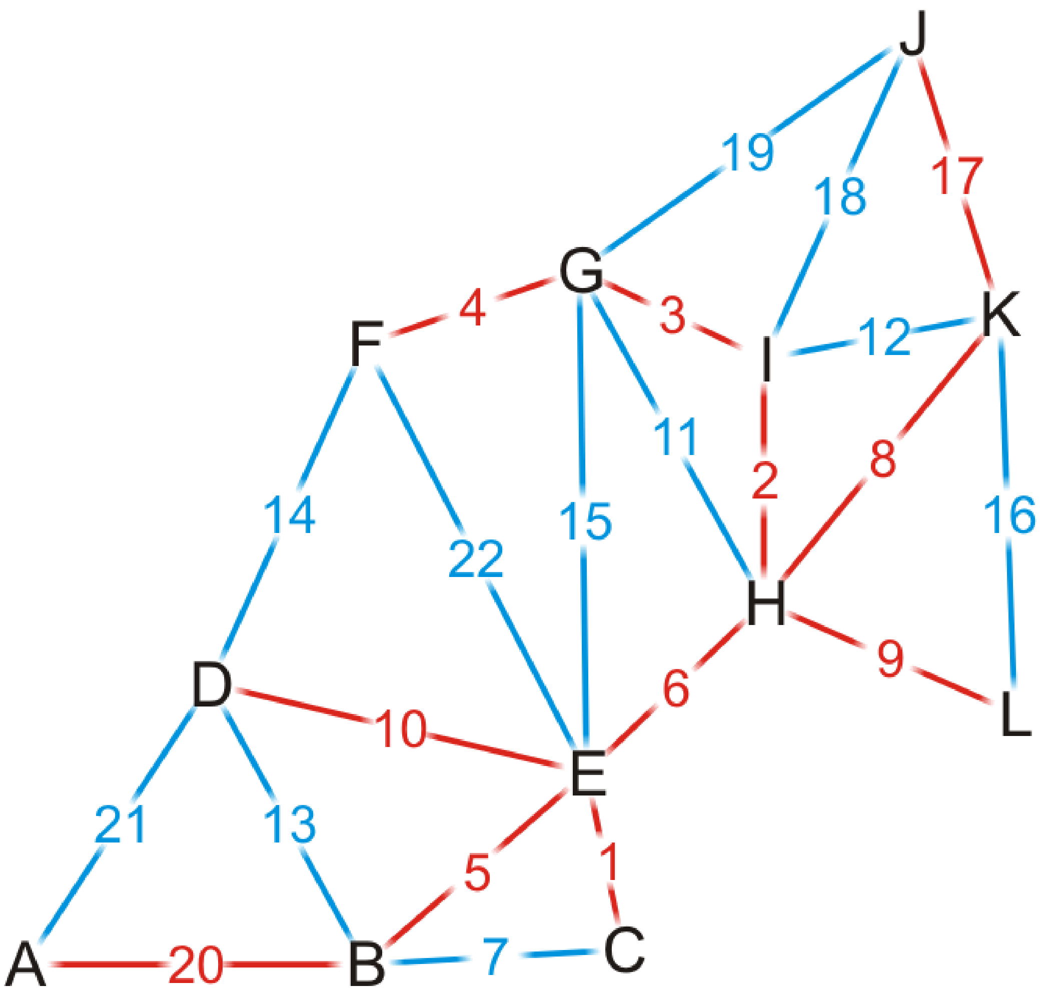
- We try adding {E, G}, but it creates a cycle



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
→ {E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

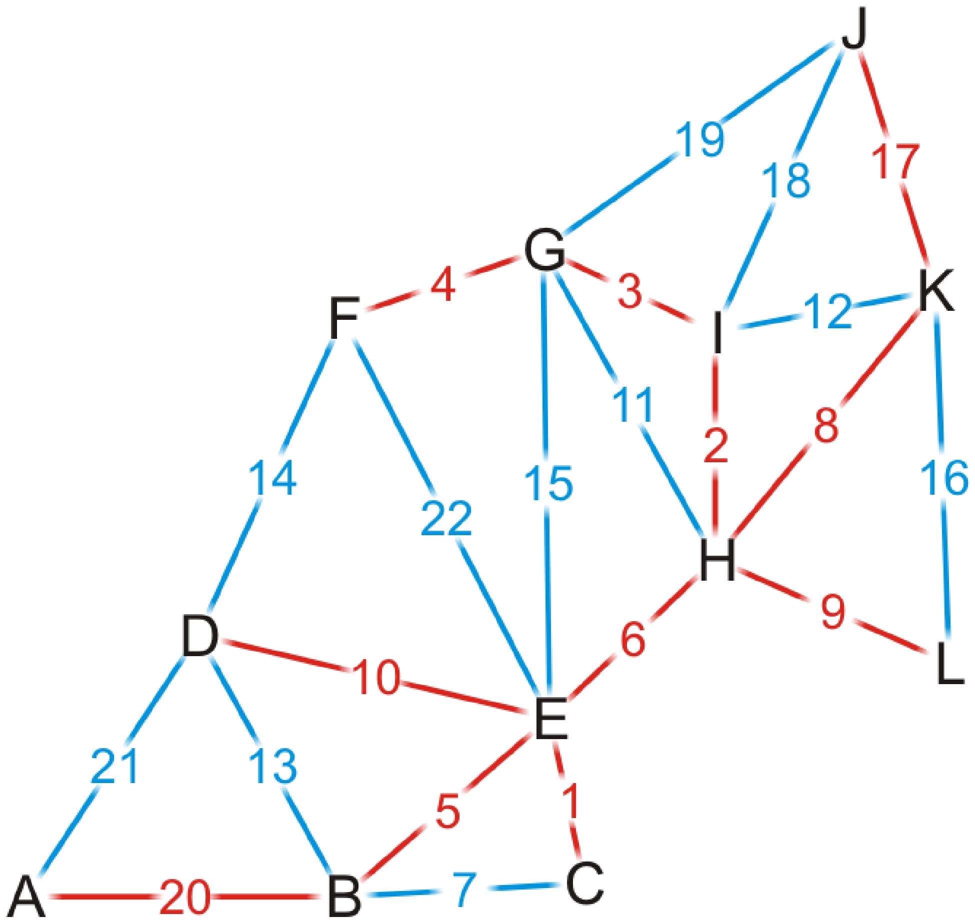
- By observation, we can still add edges {J, K} and {A, B}



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Kruskal's Algorithm – Example

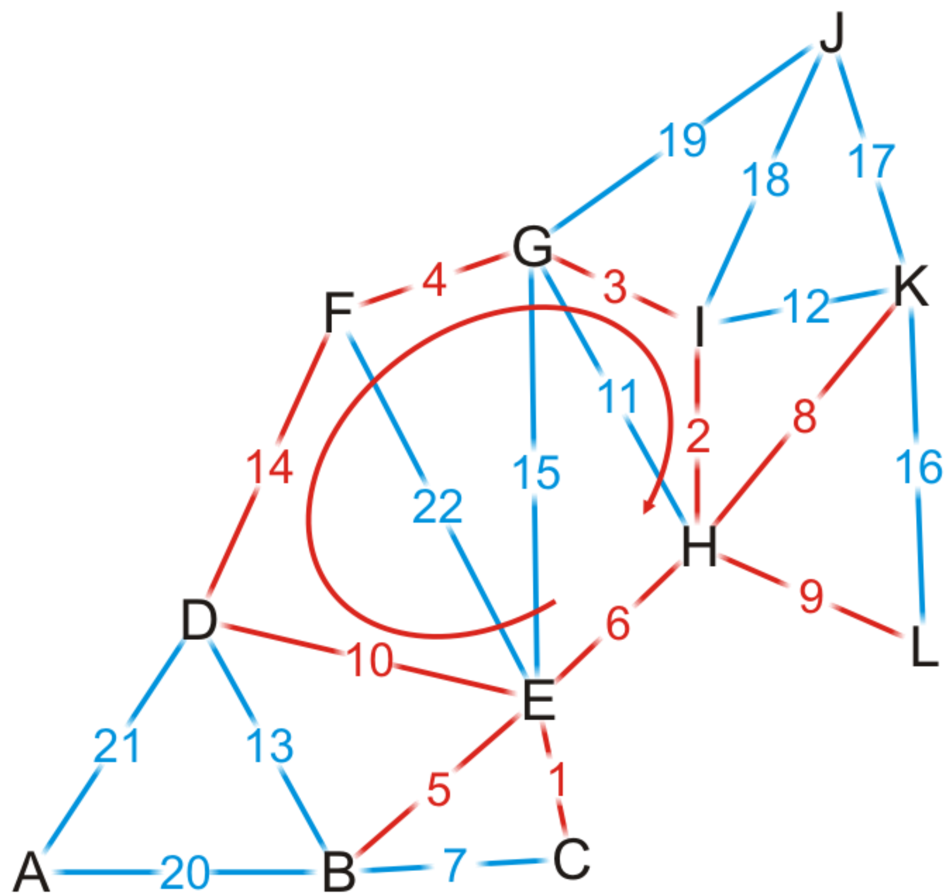
- Having added {A, B}, we now have 11 edges
  - We terminate the loop
  - We have our minimum spanning tree



{C, E}
{H, I}
{G, I}
{F, G}
{B, E}
{E, H}
{B, C}
{H, K}
{H, L}
{D, E}
{G, H}
{I, K}
{B, D}
{D, F}
{E, G}
{K, L}
{J, K}
{J, I}
{J, G}
{A, B}
{A, D}
{E, F}

Detecting a Cycle

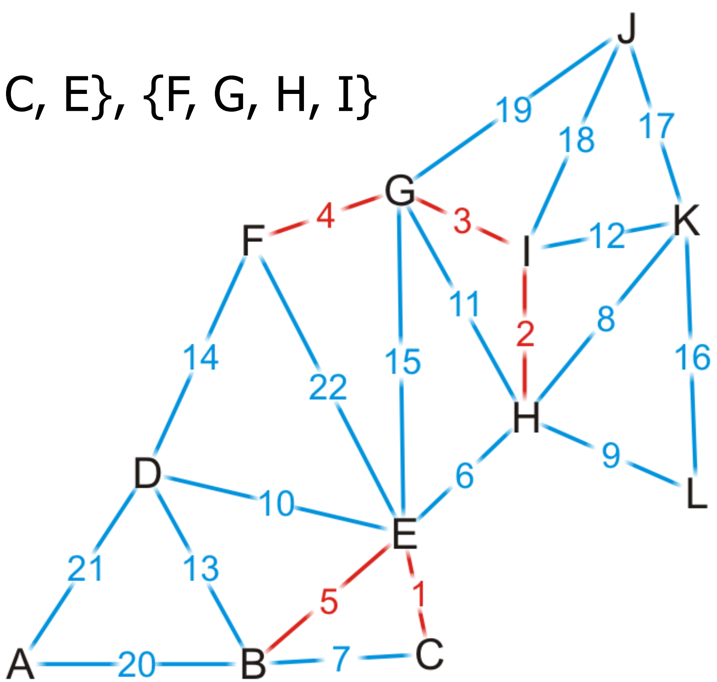
- To determine if a cycle is created, we could perform a traversal
  - A run-time of O(|V|)



Detecting a Cycle – Disjoint Sets

- Consider edges in the same connected sub-graph as forming a set

{B, C, E}, {F, G, H, I}

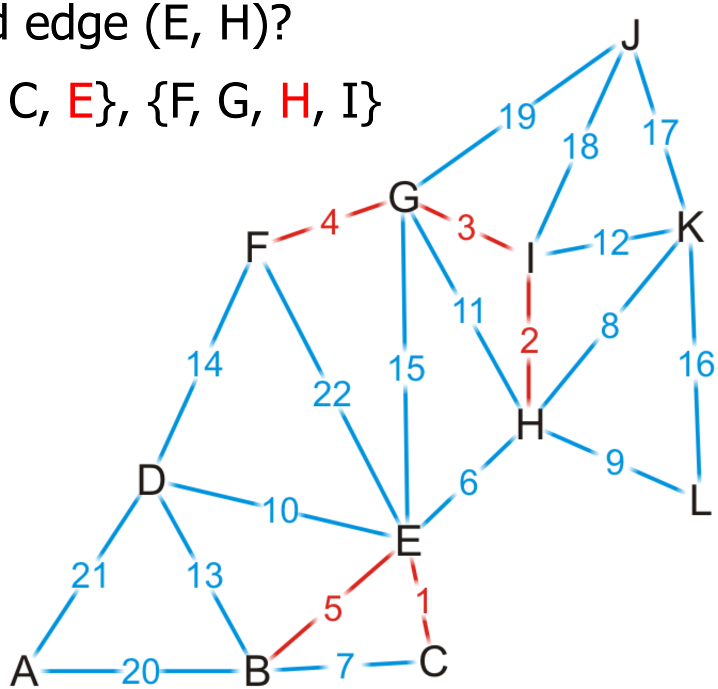


Detecting a Cycle – Disjoint Sets

- Consider edges in the same connected sub-graph as forming a set
- If the vertices of the next edge are in different sets
  - Take the union of the two sets

Add edge (E, H)?

{B, C, E}, {F, G, H, I}



Detecting a Cycle – Disjoint Sets

- Consider edges in the same connected sub-graph as forming a set
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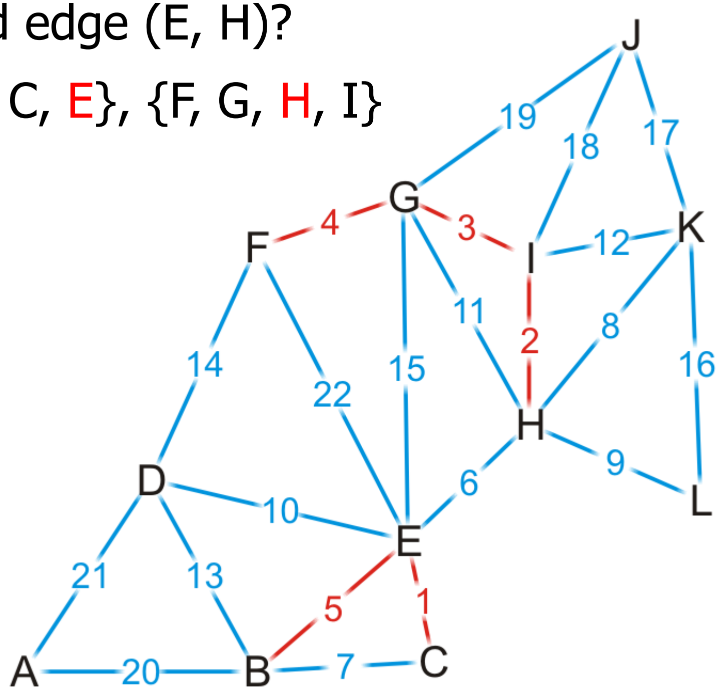
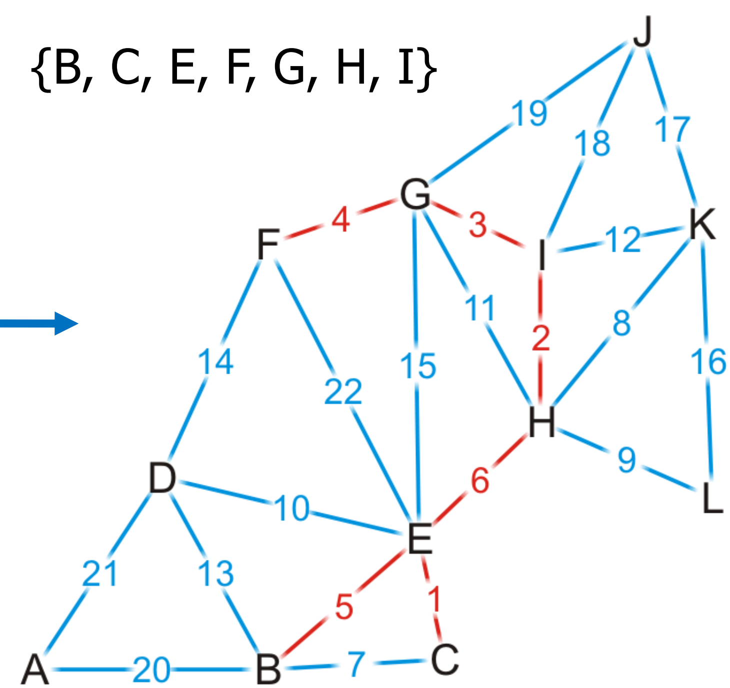
Add edge (E, H)?

{B, C, E}, {F, G, H, I}



{B, C, E, F, G, H, I}

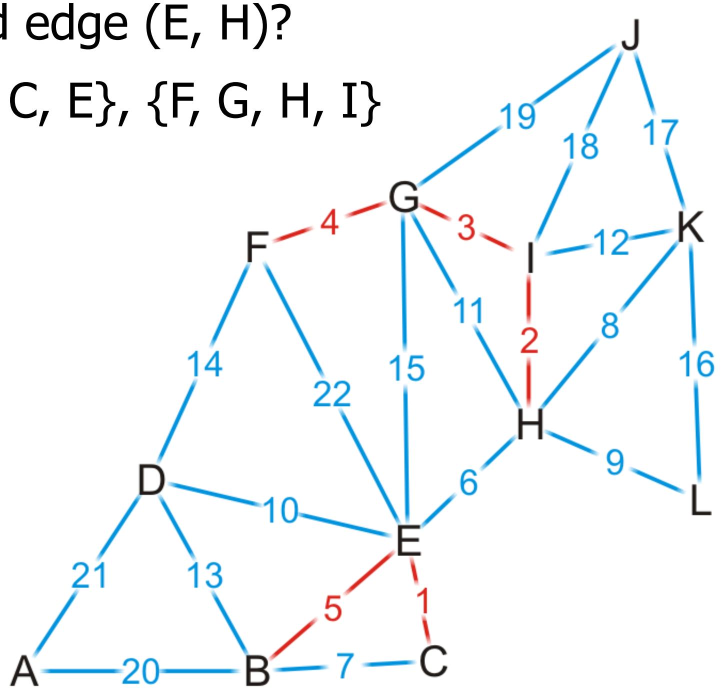


Detecting a Cycle – Disjoint Sets

- Consider edges in the same connected sub-graph as forming a set
- If the vertices of the next edge are in different sets
  - Take the union of the two sets
- Do not add an edge if both vertices are in the same set

Add edge (E, H)?

{B, C, E}, {F, G, H, I}



{B, C, E, F, G, H, I}