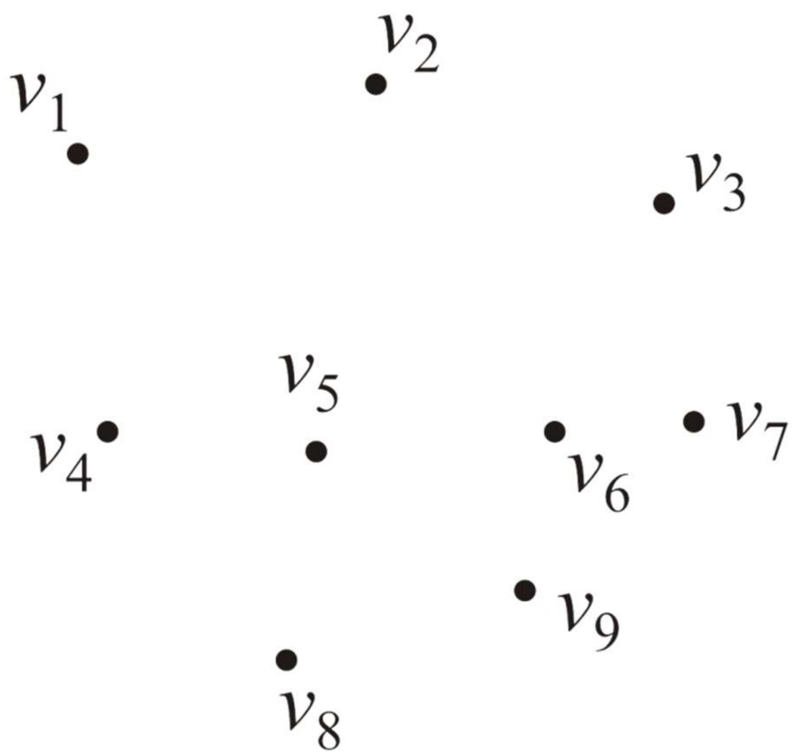
Data Structures

19. Graphs

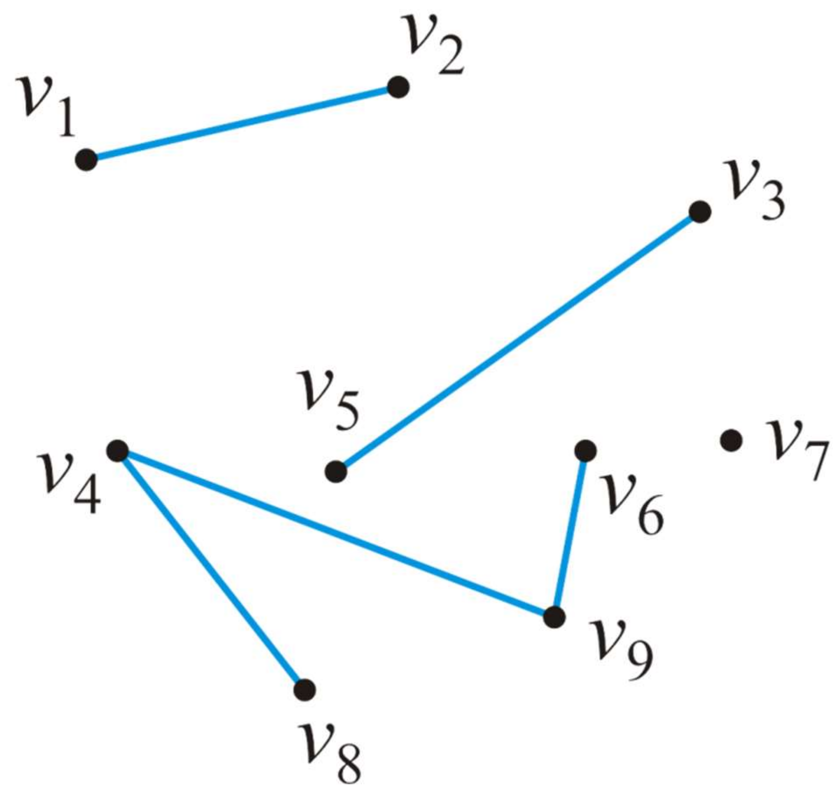
Graphs

- Consider this collection of vertices
  - `V = {`v_1`, `v_2`, ..., `v_9`}`
  - Where `|V| = n`



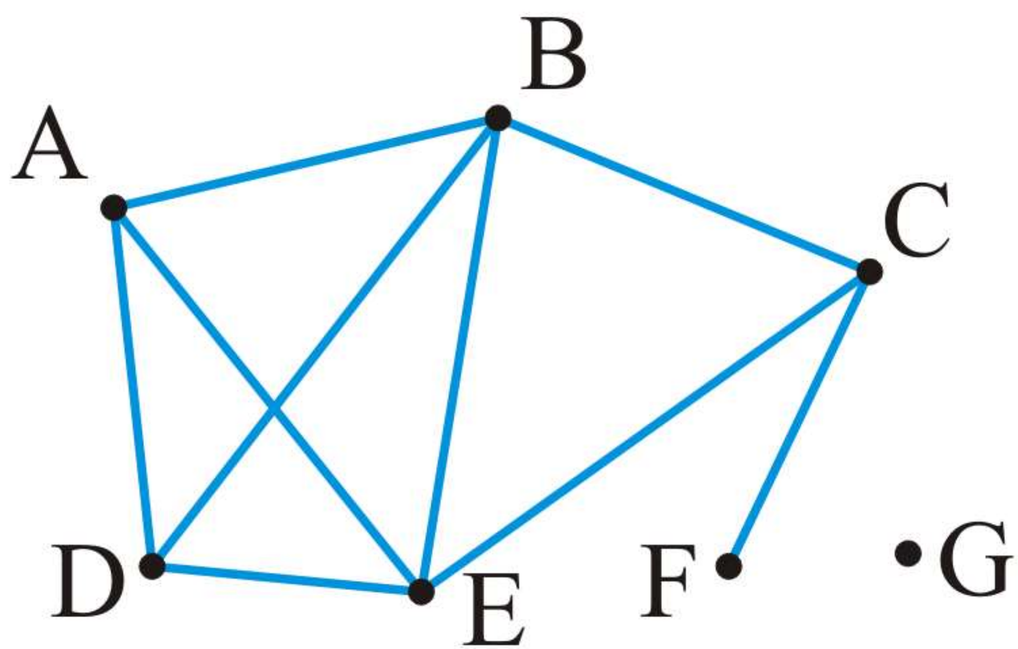
Undirected Graphs

- Associated with these vertices are $|E| = 5$ edges
  - $E = \{\{v_1, v_2\}, \{v_3, v_5\}, \{v_4, v_8\}, \{v_4, v_9\}, \{v_6, v_9\}\}$
- Pair $\{v_j, v_k\}$ indicates following relations
  - Vertex v_j is adjacent to vertex v_k
  - Vertex v_k is adjacent to vertex v_j



Undirected Graphs – Example

- Given `|V| = 7` vertices and `|E| = 9` edges
  - `V = {A, B, C, D, E, F, G}`
  - `E = {{A, B}, {A, D}, {A, E}, {B, C}, {B, D}, {B, E}, {C, E}, {C, F}, {D, E}}`



Applications Of Graphs

- Driving Map
  - Vertex = Intersection, destinations
  - Edge = Road

- Airline Traffic
  - Vertex = Cities serviced by the airline
  - Edge = Flight exists between two cities

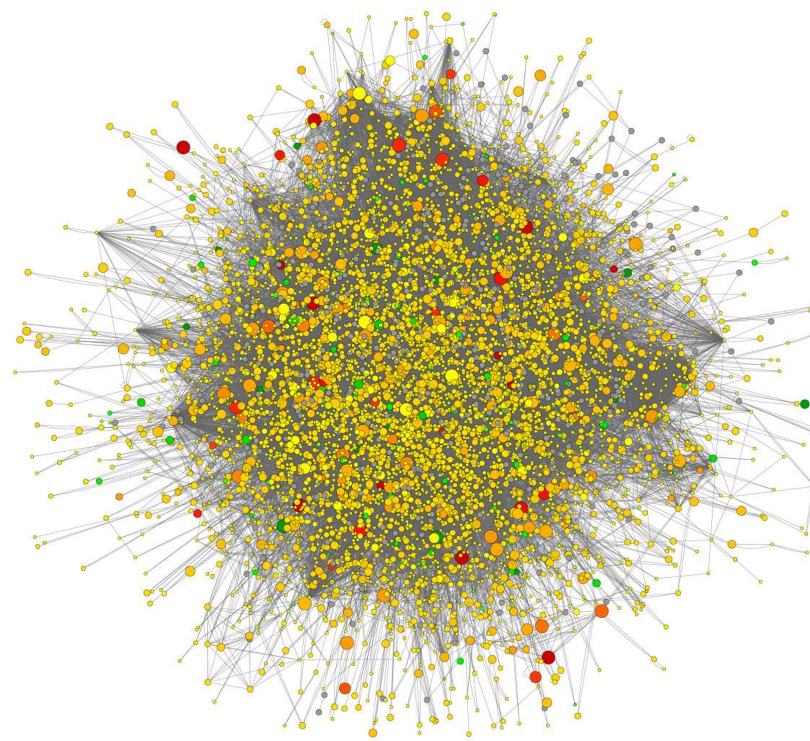
- Computer networks
  - Vertex = Server nodes, end devices, routers
  - Edge = Data link

Applications Of Graphs

- Many real-world applications concern large graphs

- Web document graph - 1 trillion webpages
  - Vertex = Webpage
  - Edge = Hyperlink

- Social networks - 1.3 billion users
  - Vertex = Users
  - Edge = Friendship relation



Undirected Graphs – Definition

- An undirected Graph is defined as `G=(V,E)` consisting of
  - Set `V` of vertices: $V = \{v_1, v_2, \ldots, v_n\}$
    - Number of vertices is denoted by $|V| = n$
  - Set `E` of unordered pairs $\{v_i, v_j\}$ termed edges
    - Edges connect the vertices

- Maximum number of edges in an undirected graph is $O(|V|^2)$

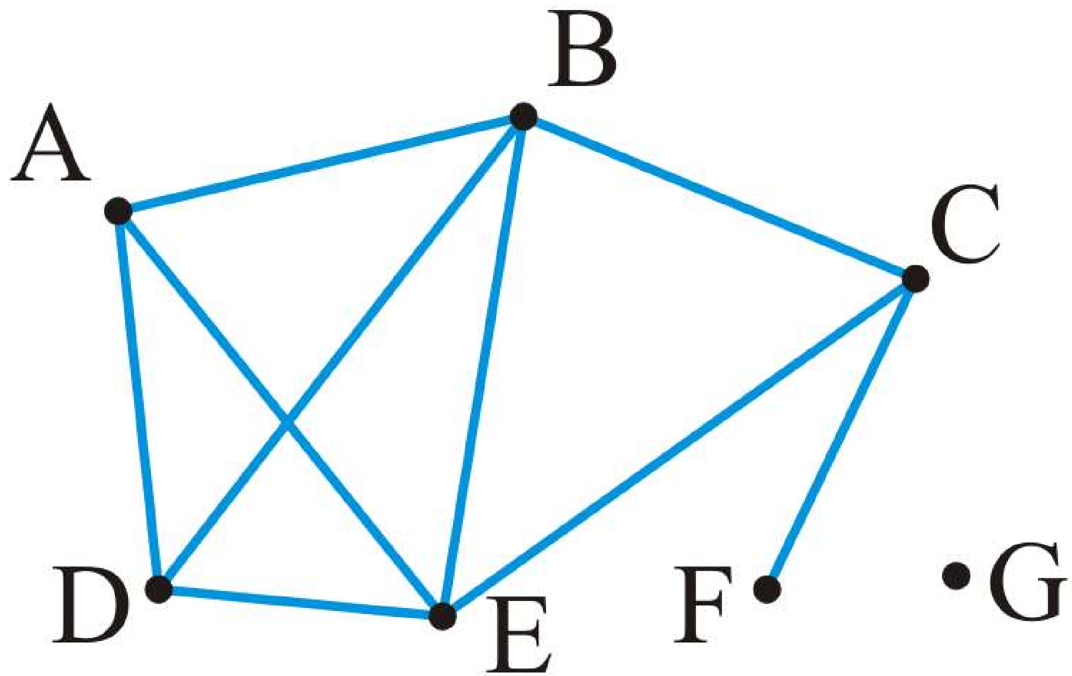
$$|E| \le \binom{|V|}{2} = \frac{|V|\left(|V|-1\right)}{2} = O\left(|V|^2\right)$$

  - Assumption: A vertex is never adjacent to itself
  - For example, $\{v_1, v_1\}$ will not define an edge

- Many data structures can implement abstract undirected graphs
  - Adjacency matrices, Adjacency lists

Degree

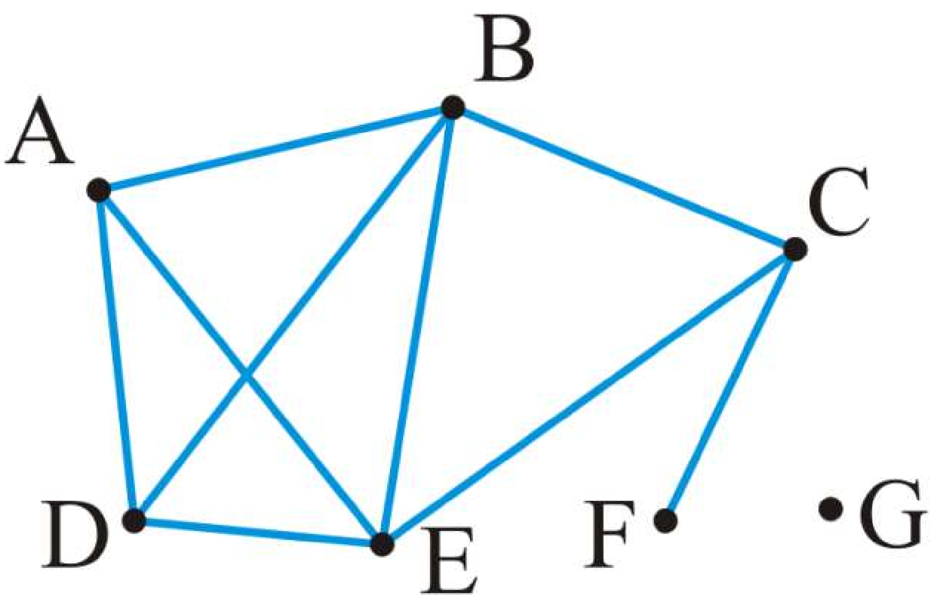
- Degree of a vertex is defined as the number of adjacent vertices
  - degree(A) = degree(D) = degree(C) = 3
  - degree(B) = degree(E) = 4
  - degree(F) = 1
  - degree(G) = 0

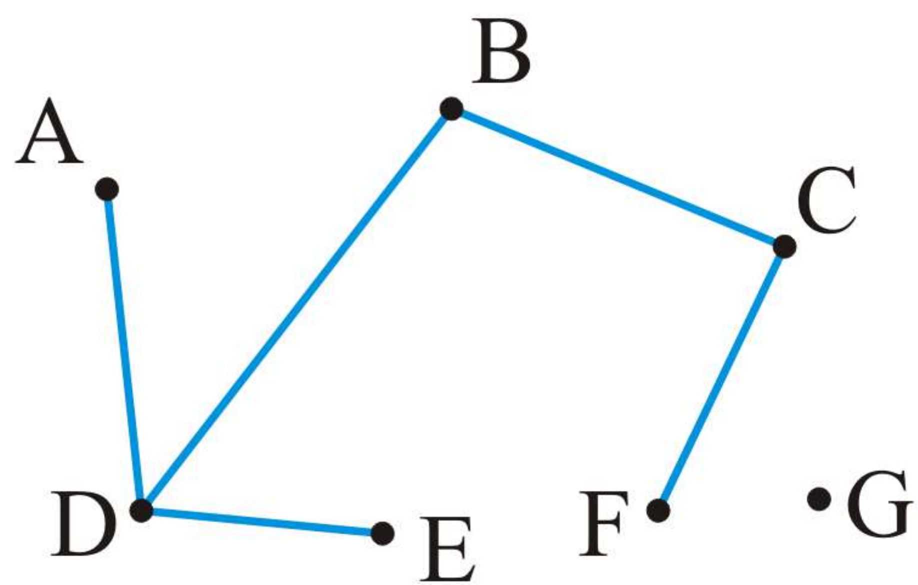


- Vertices adjacent to a given vertex are its neighbors

Subgraph

- A sub-graph of a graph G is defined by
  - Subset of the vertices
  - Subset of the edges that connected the subset of vertices in the original graph
- Every graph is a subgraph of itself





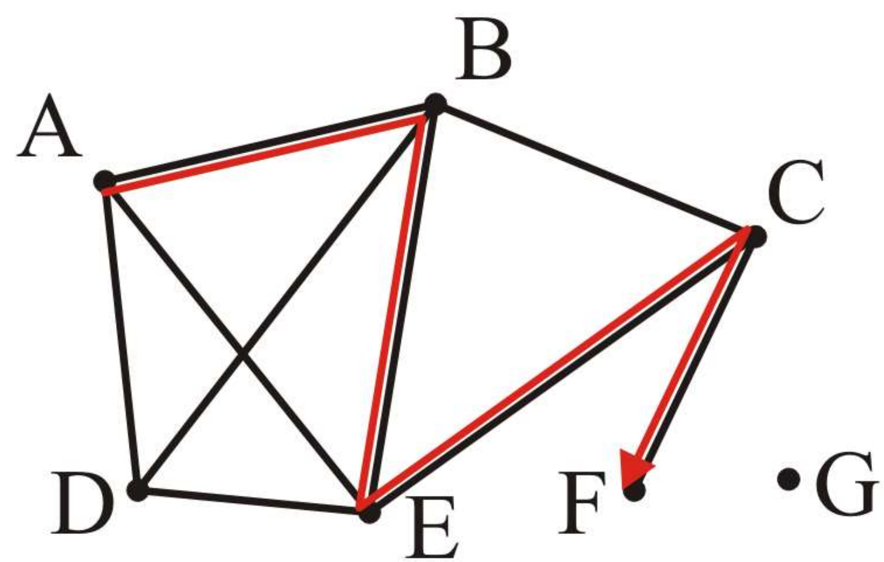
Path

- Path in an undirected graph is an ordered sequence of vertices
  - Consecutive vertices are connected through edges

- Path from vertex `0` to vertex `k` is $(v_0, v_1, v_2, ..., v_k)$
  - where $\{v_j - 1, v_j\}$ is an edge for $j = 1, ..., k$

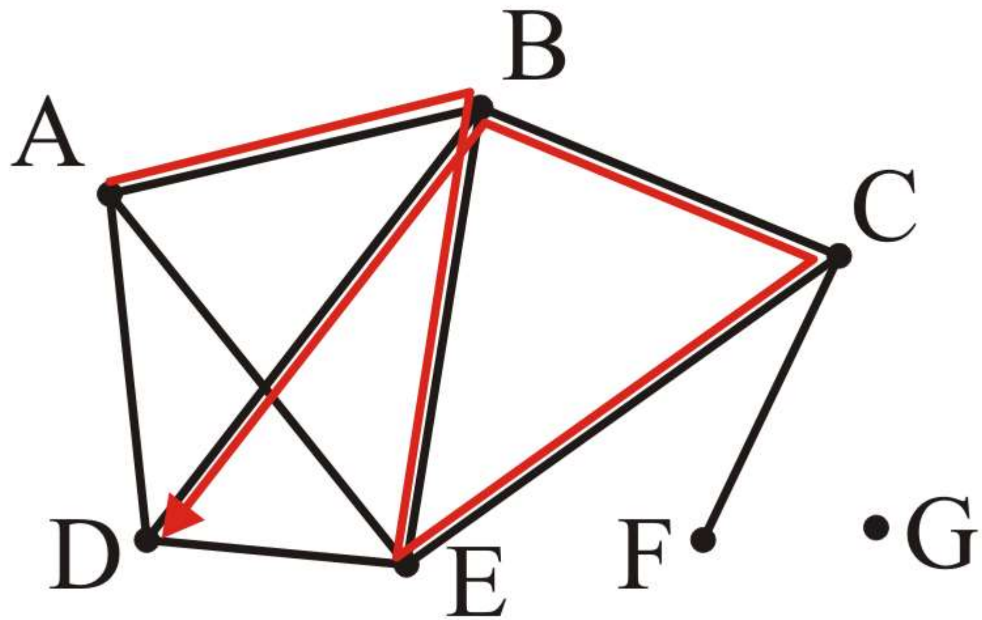
- Length of a path is equal to the number of edges

- Example: Path from A to F
  - Path: `(A, B, E, C, F)`
  - Length of the path is 4



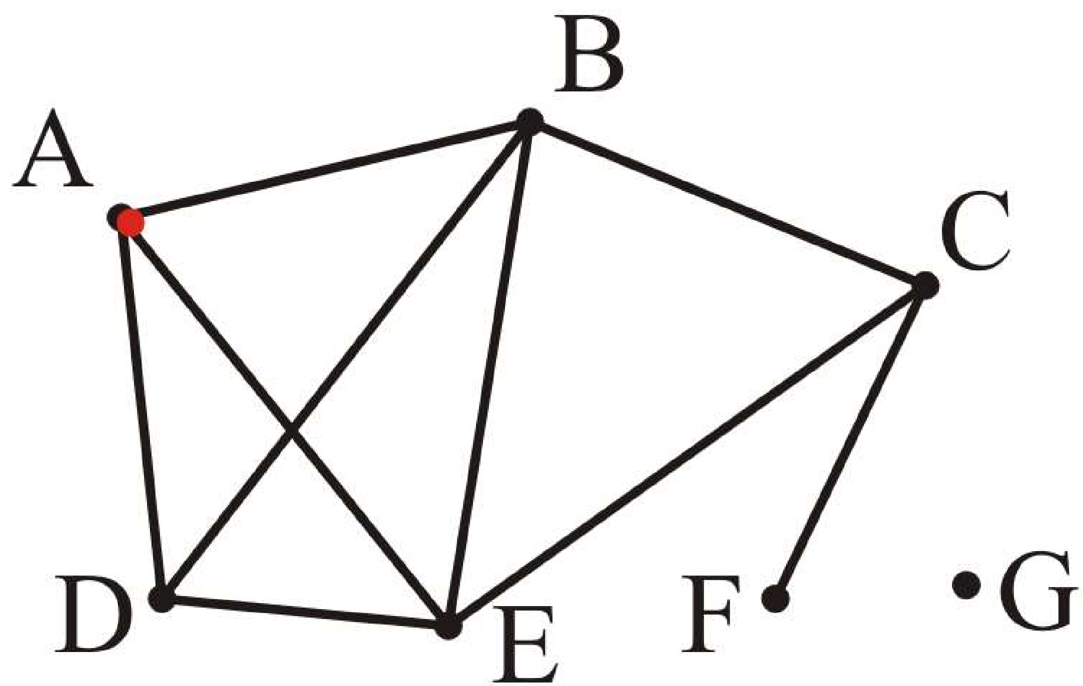
Path – Example

- Path of length 5: `(A, B, E, C, B, D)`
  - Repitition of vertex B



Path – Example

- A trivial path of length 0: `(A)`

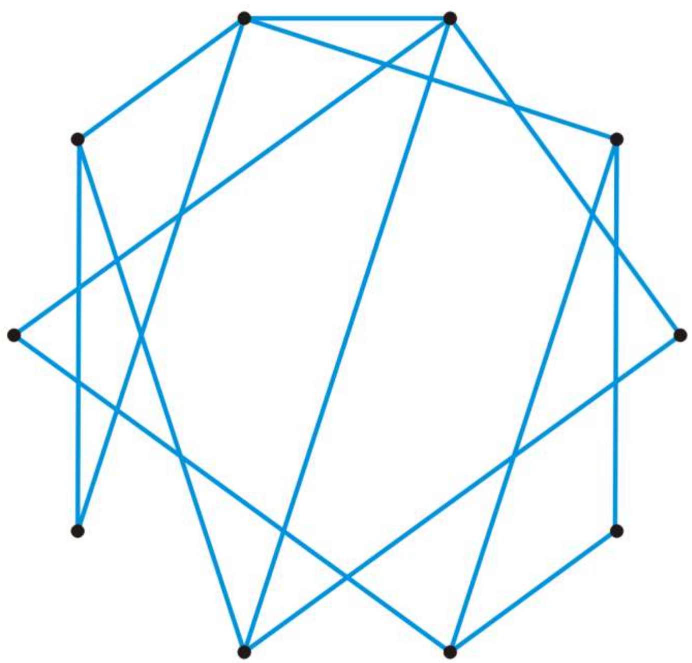


Simple Path

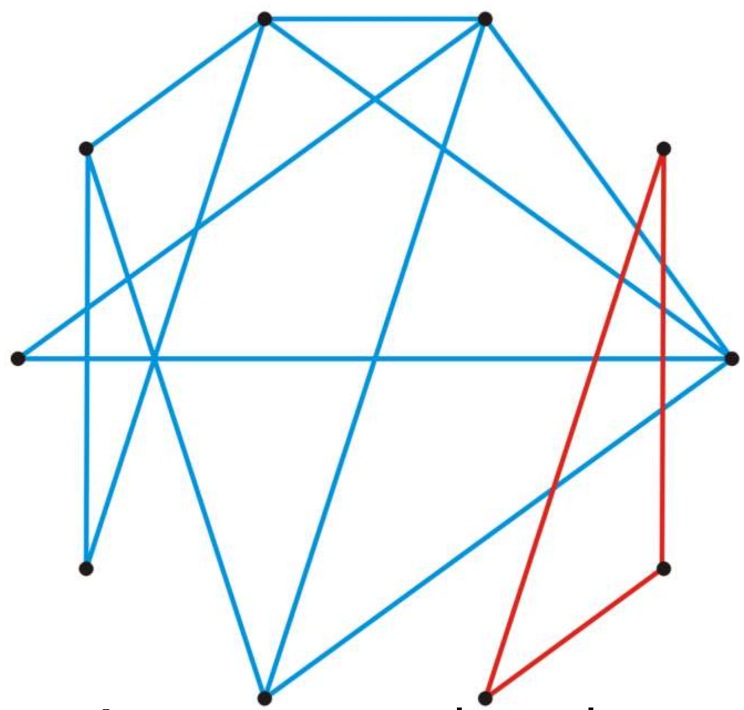
- A simple path has no repetitions other than perhaps the first and last vertices
- A simple cycle is a simple path of at least two vertices with the first and last vertices equal
  - Note: these definitions are not universal
- A loop is an edge from a vertex onto itself

Connectedness

- Two vertices v_i, v_j are said to be connected if there exists a path from v_i to v_j
- A graph is connected if there exists a path from every vertex to every other vertex



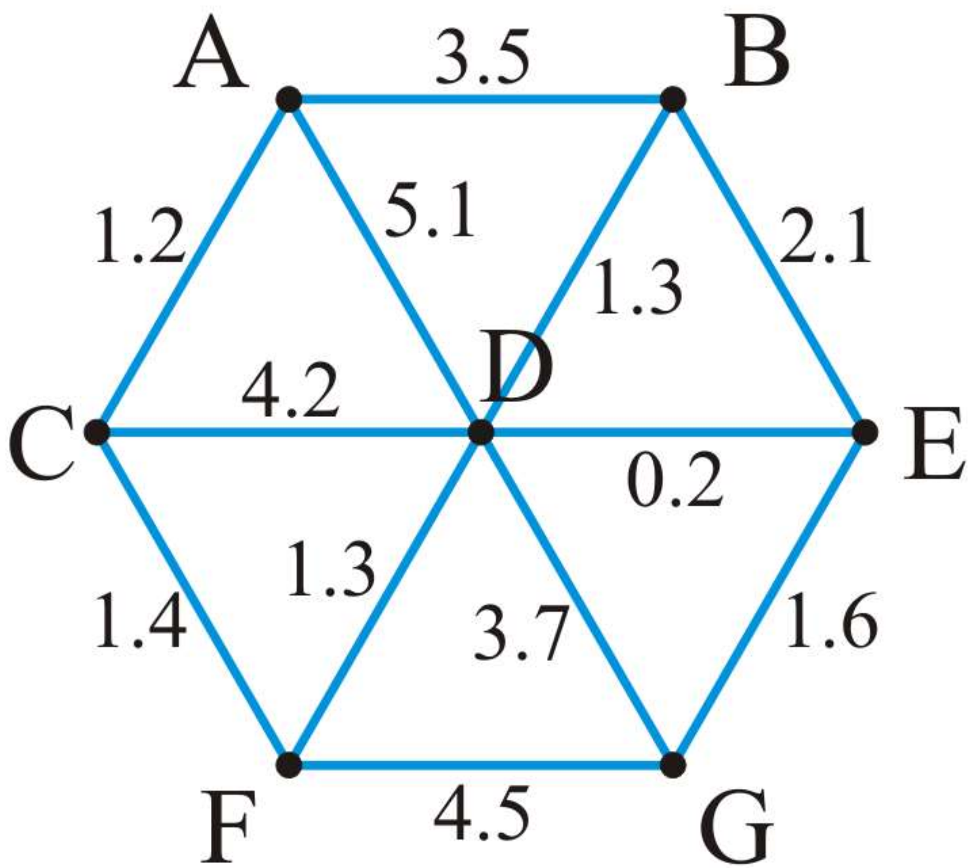
A connected graph



An unconnected graph

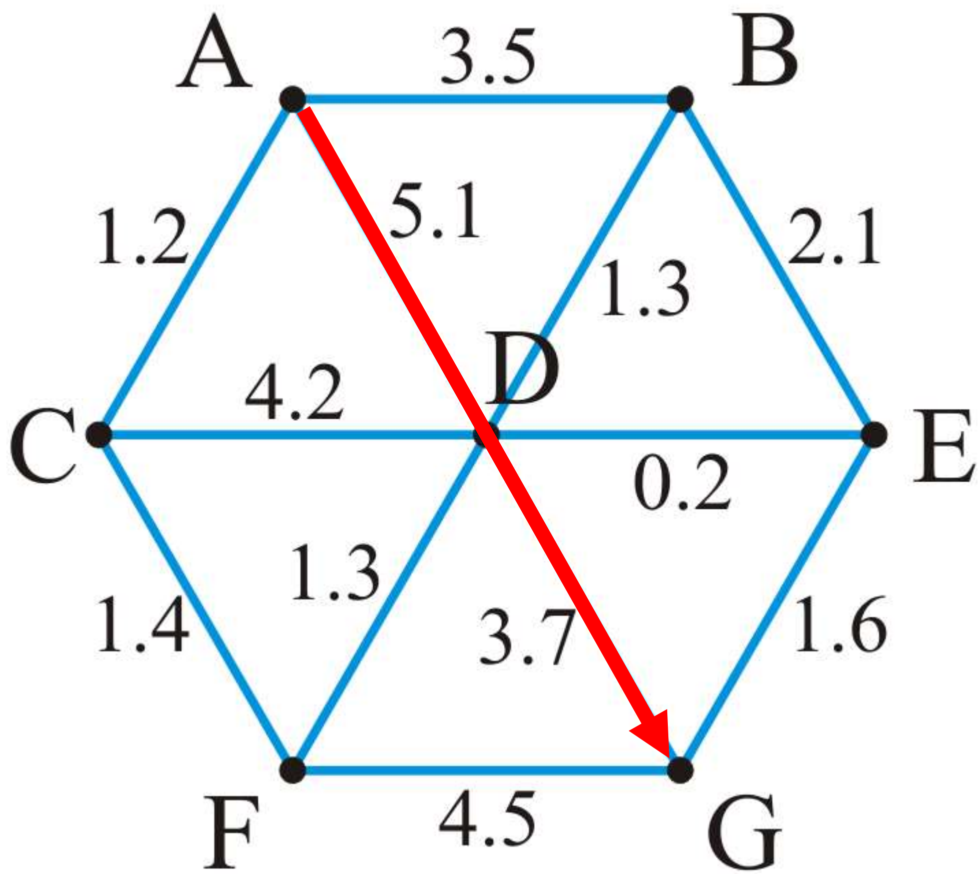
Weighted Graphs

- A weight may be associated with each edge in a graph
  - This could represent distance, energy consumption, cost, etc.
  - Such a graph is called a weighted graph
- Pictorially, we will represent weights by numbers next to the edges



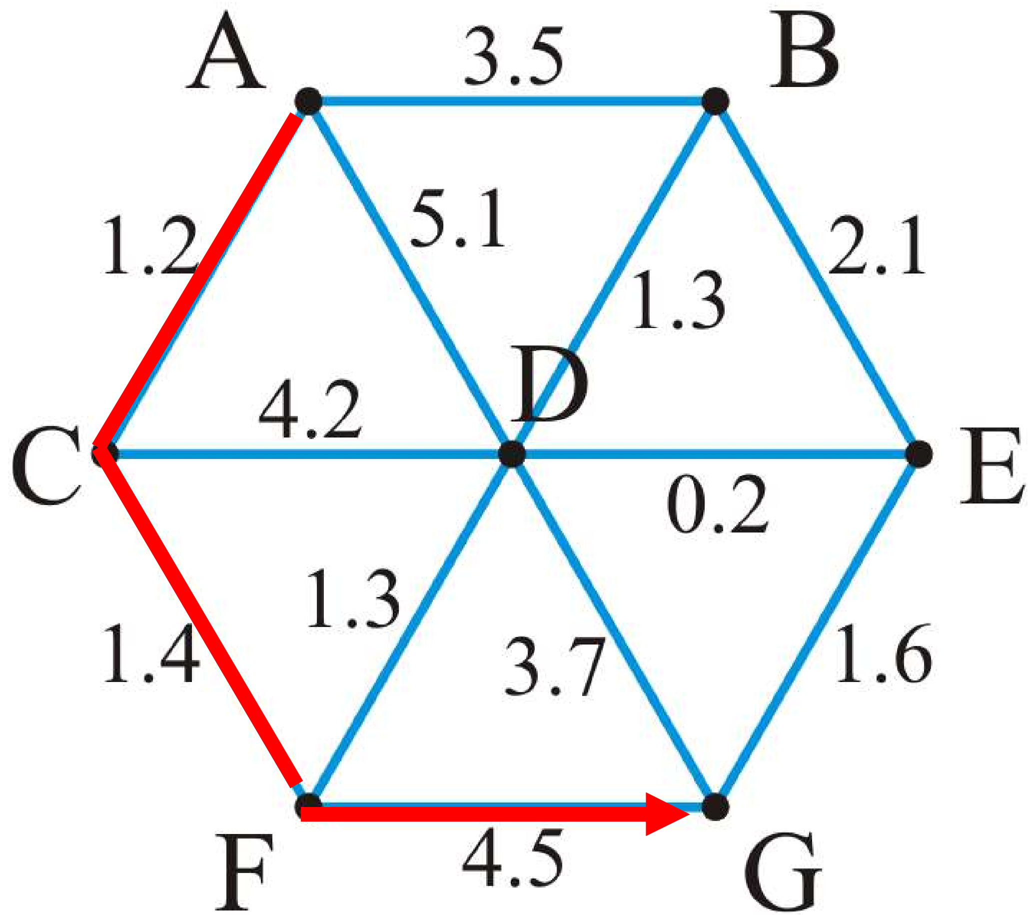
Weighted Graphs

- Length of a path within a weighted graph is the sum of all of the edges which make up the path
- The length of the path `(A, D, G)` in the following graph is `8.8`



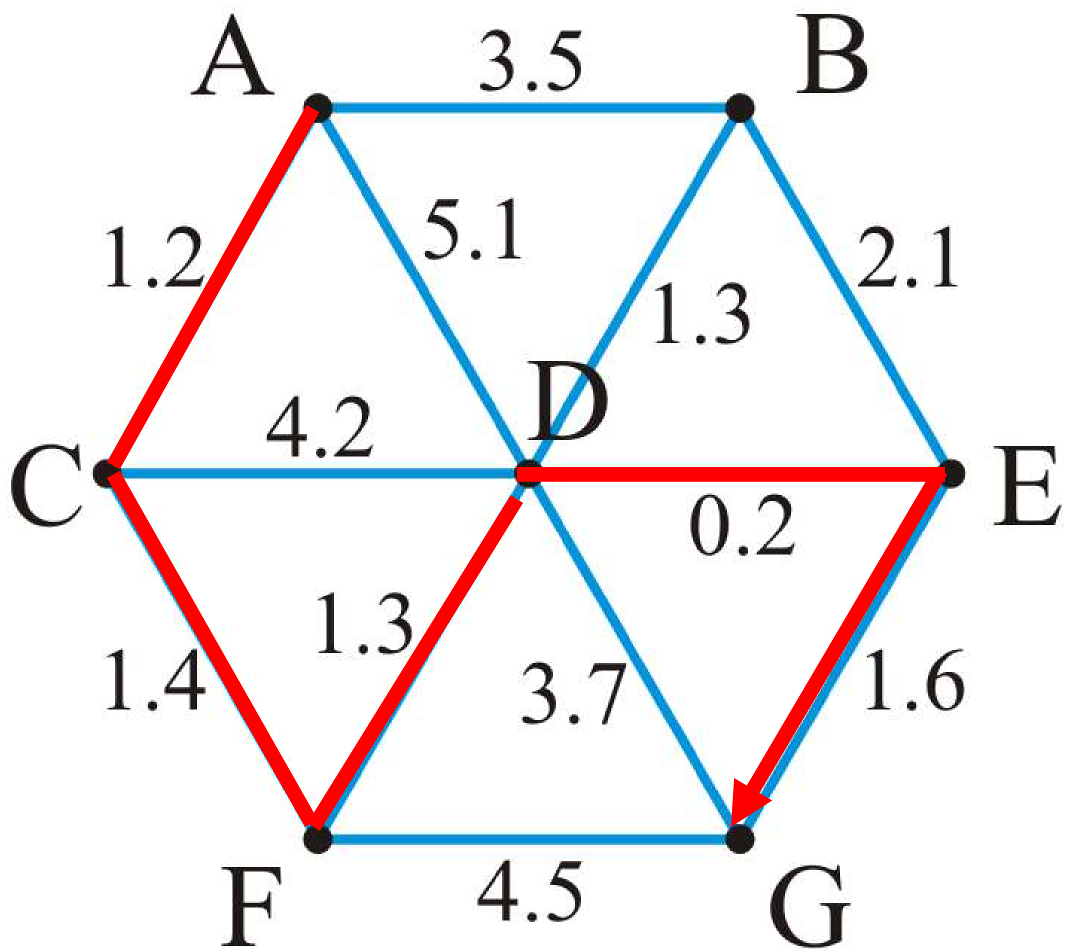
Weighted Graphs – Example

- Different paths may have different weights
  - Another path is `(A, C, F, G)` with length `1.2 + 1.4 + 4.5 = 7.1`



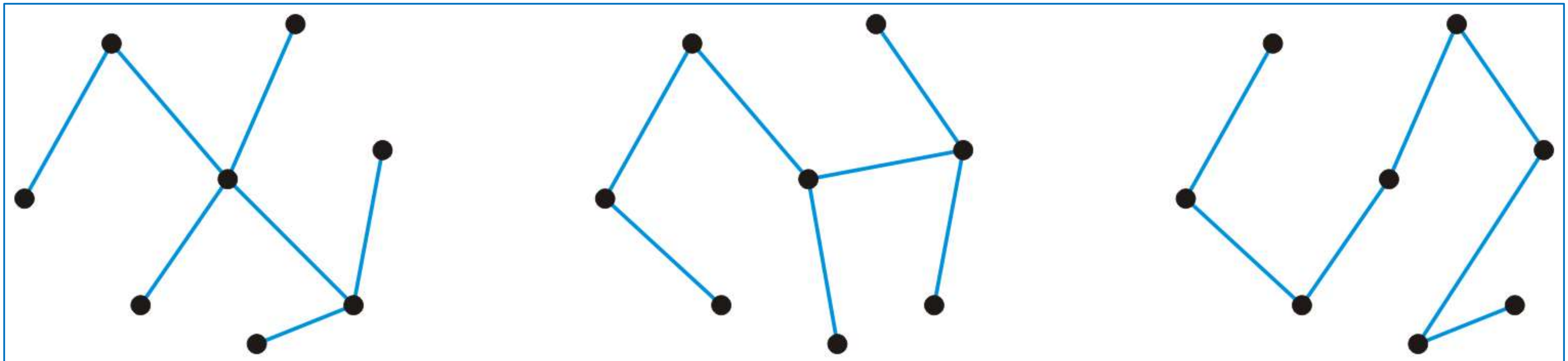
Weighted Graphs – Example

- Find the shortest path between two vertices A and G
- Shortest path is `(A, C, F, D, E, G)` with length `5.7`



Trees

- A graph is a tree if it satisfies the following two conditions
  - Graph is connected
  - There is a unique path between any two vertices

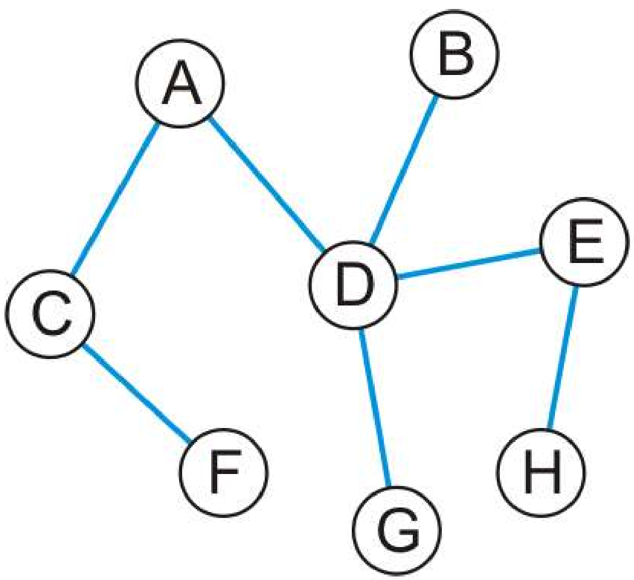


Three trees on same 8 vertices

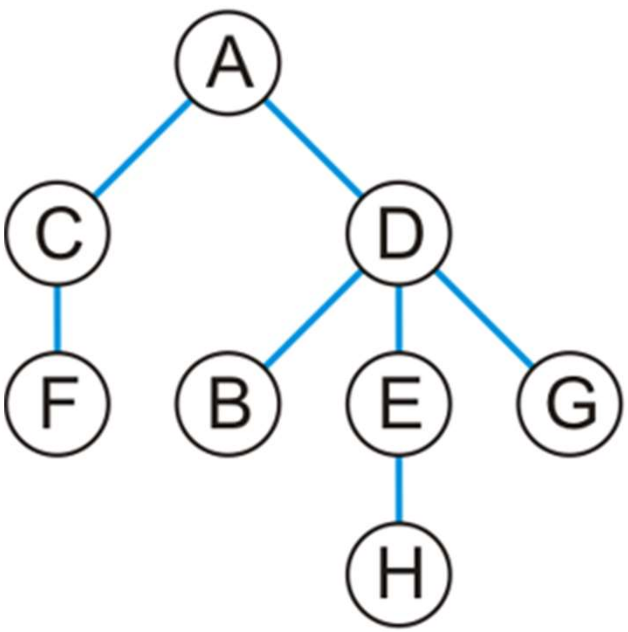
- Consequences
  - The number of edges is `|E| = |V| - 1`
  - The graph is acyclic, that is, it does not contain any cycles
  - Adding one more edge must create a cycle
  - Removing any one edge creates two disjoint non-empty sub-graphs

Trees

- Any tree can be converted into a rooted tree by
  - Choosing any vertex to be the root
  - Defining its neighboring vertices as its children
- Recursively defining
  - All neighboring vertices other than that one designated as parent to be that vertex children

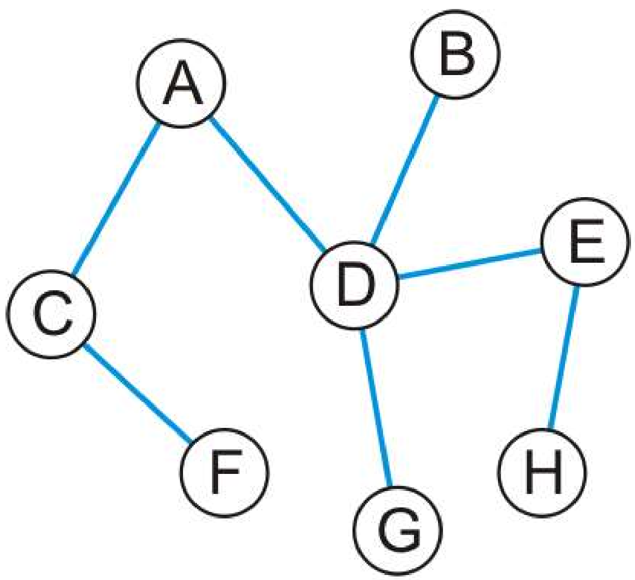


Unrooted Tree

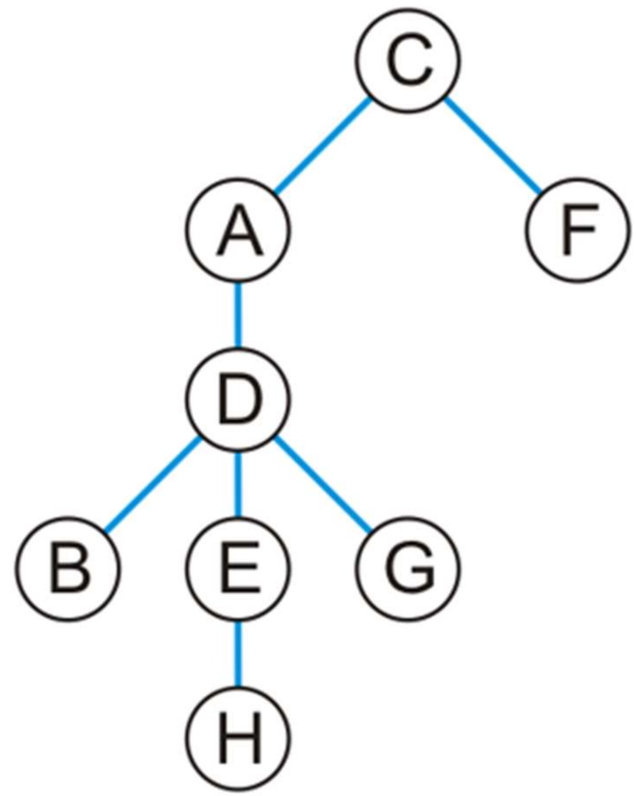


Tree rooted at A

Trees – Example

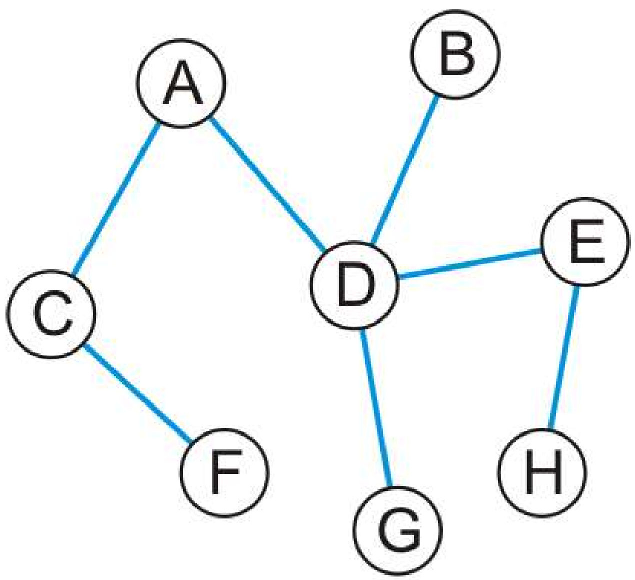


Unrooted Tree

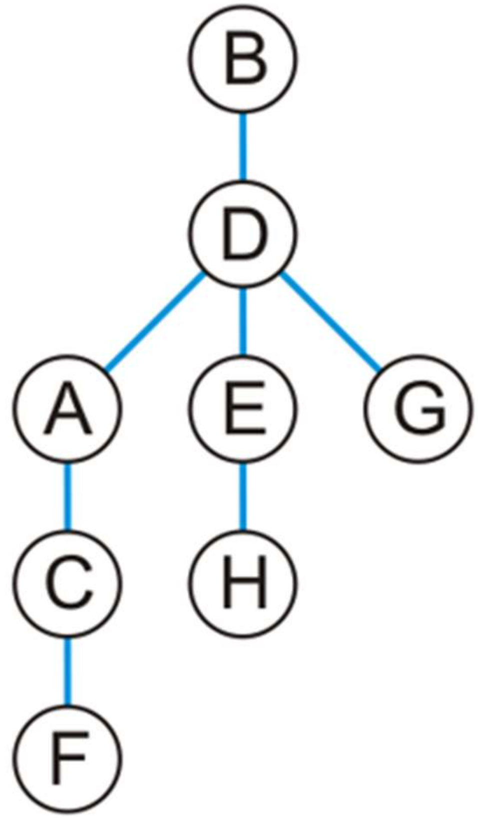


Tree rooted at C

Trees – Example



Unrooted Tree

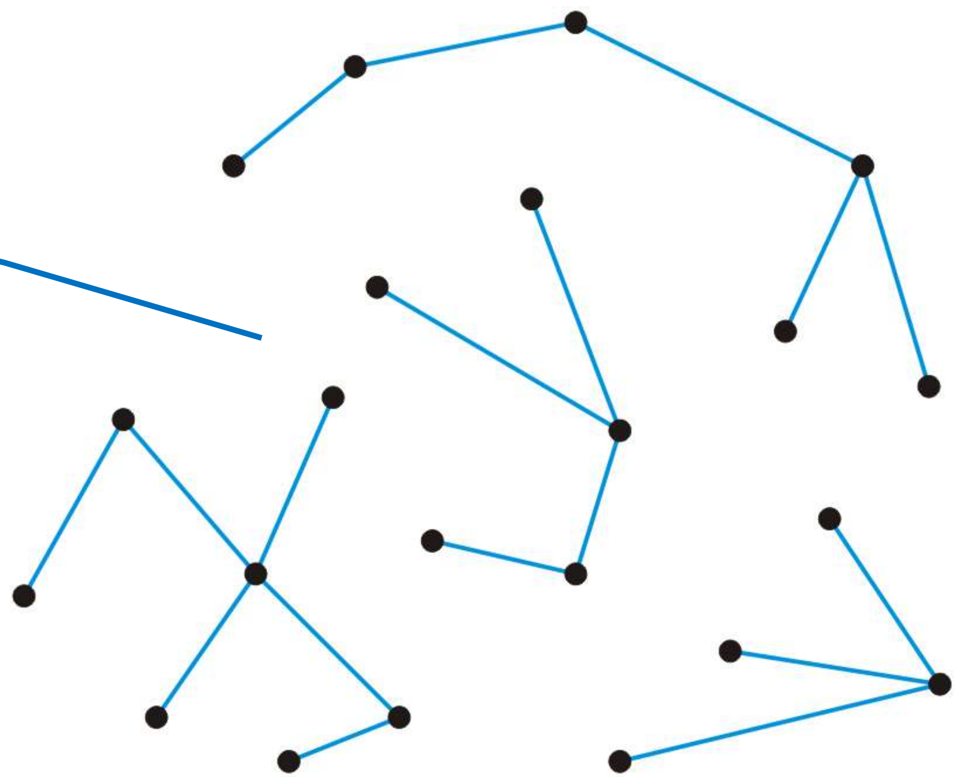


Tree rooted at B

Forest

- A forest is any graph that has no cycles
- Consequences
  - The number of edges is `|E| < |V|`
  - The number of trees is `|V| - |E|`
  - Removing any one edge adds one more tree to the forest

- Forest with 22 vertices and 18 edges
- Four trees



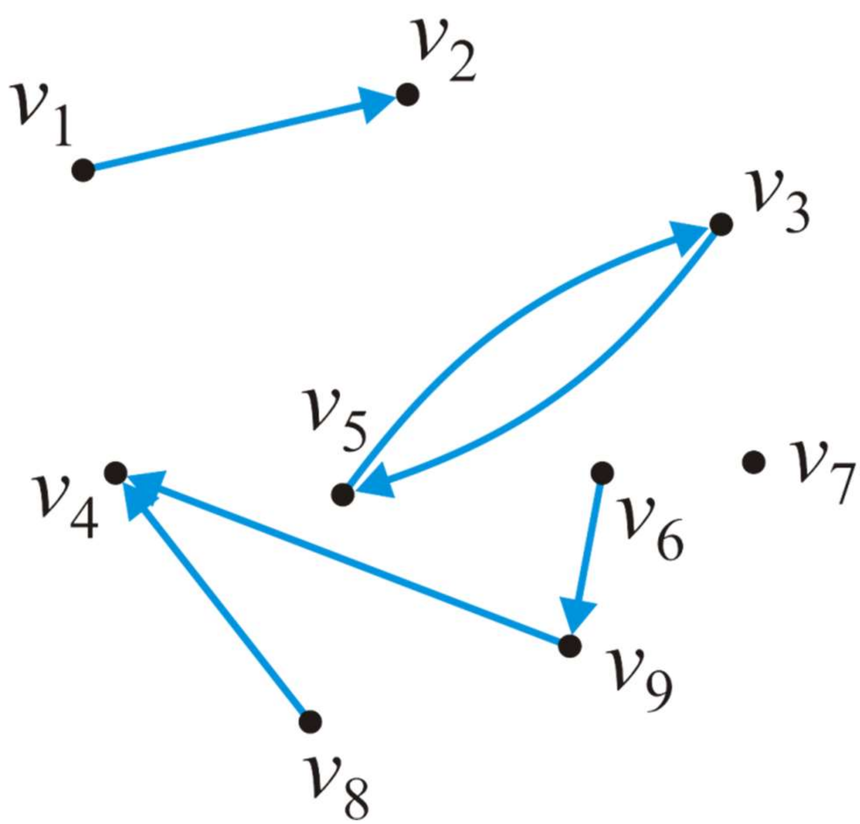
Directed Graphs

- In a directed graph, the edges on a graph are associated with a direction
  - Edges are ordered pairs (v_j, v_k) denoting a connection from v_j to v_k
  - The edge (v_j, v_k) is different from the edge (v_k, v_j)

- Streets are directed graphs
  - In most cases, you can go two ways unless it is a one-way street

Directed Graphs

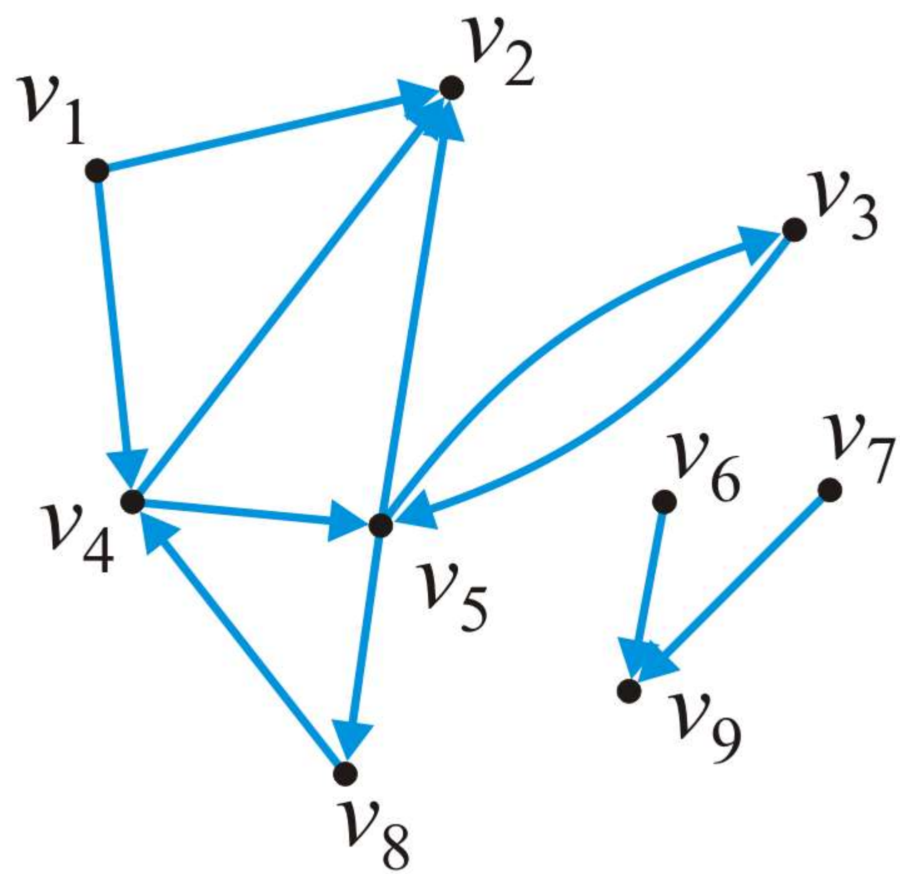
- Given our graph of nine vertices $V = \{v_1, v_2, \ldots v_9\}$
  - These six pairs (v_j, v_k) are directed edges
  - $E = \{(v_1, v_2), (v_3, v_5), (v_5, v_3), (v_6, v_9), (v_8, v_4), (v_9, v_4)\}$



In and Out Degree

- Degree of a vertex must be modified to consider both cases:
  - Out-degree of a vertex is the number of vertices which are adjacent to the given vertex
    - Number of outgoing edges
  - In-degree of a vertex is the number of vertices which this vertex is adjacent to
    - Number of incoming edges

- In this graph:
  - In-degree(v_1) = 0 out-degree(v_1) = 2
  - In-degree(v_5) = 2 out-degree(v_5) = 3

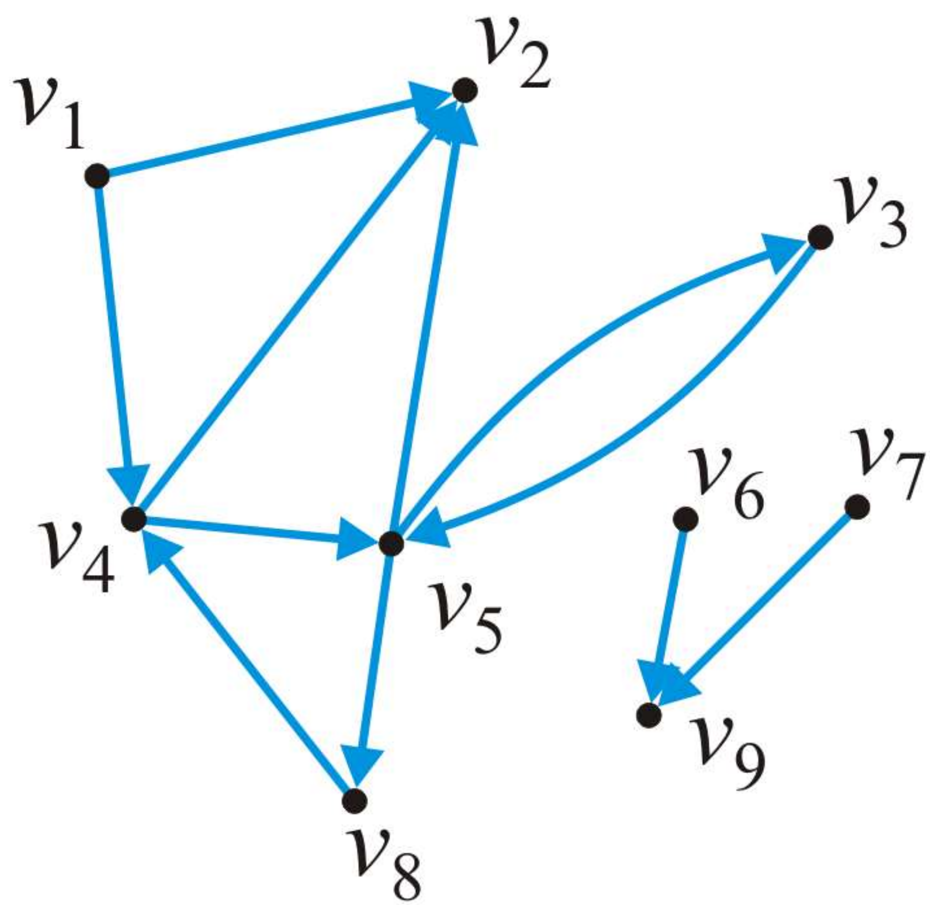


Path

- A path in a directed graph is an ordered sequence of vertices
  - $(v_0, v_1, v_2, \ldots, v_k)$
  - where $(v_j - 1, v_j)$ is an edge for $j = 1, \ldots, k$

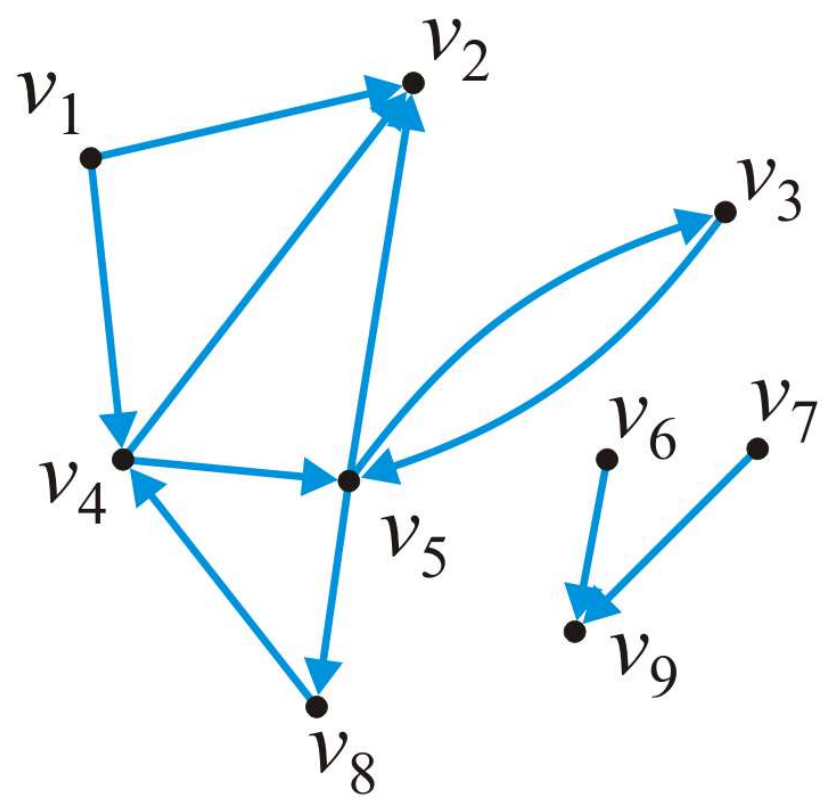
- A path of length 5 in this graph is
  - $(v_1, v_4, v_5, v_3, v_5, v_2)$

- A simple cycle of length 3 is
  - (v_8, v_4, v_5, v_8)



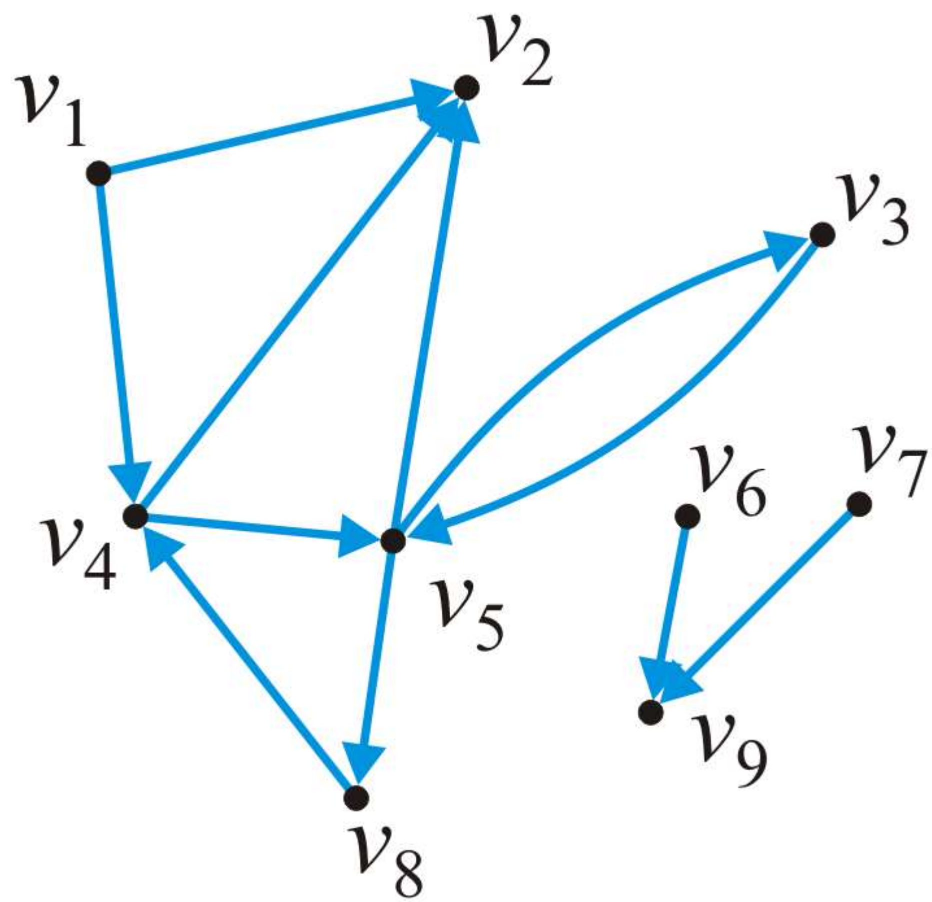
Connectedness

- Two vertices v_j, v_k are said to be connected if there exists a path from v_j to v_k
  - A graph is strongly connected if there exists a directed path between any two vertices
  - A graph is weakly connected if there exists a path between any two vertices that ignores the direction



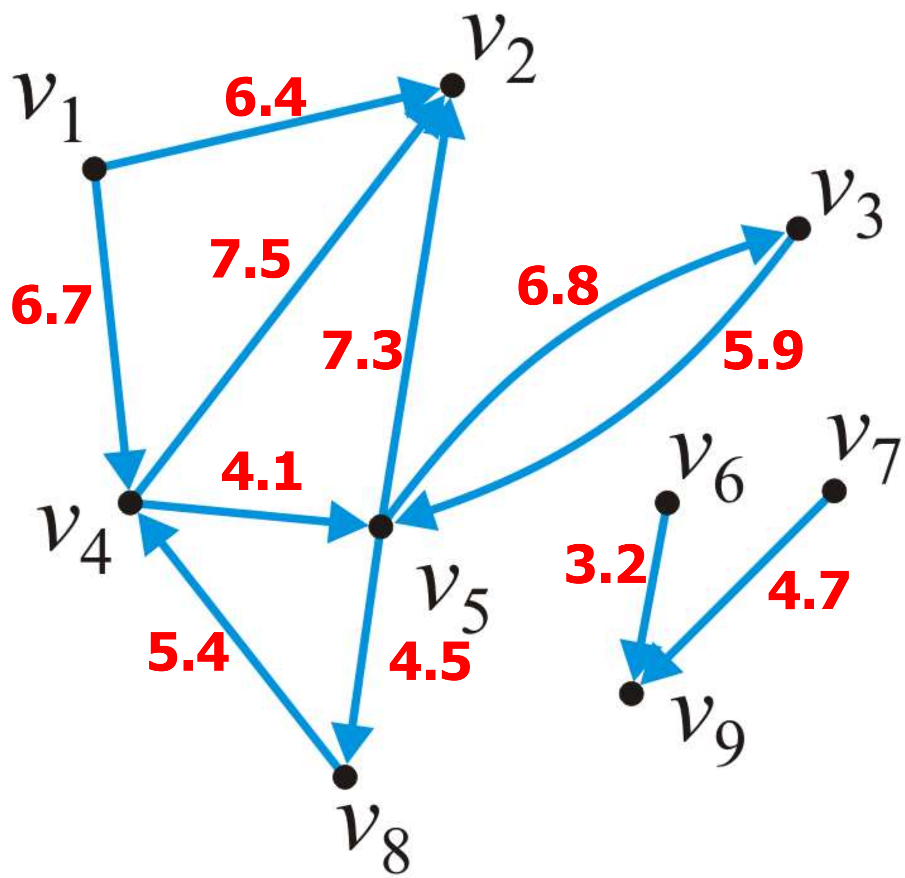
Connectedness – Example

- The sub-graph $\{v_3, v_4, v_5, v_8\}$ is strongly connected
- The sub-graph $\{v_1, v_2, v_3, v_4, v_5, v_8\}$ is weakly connected



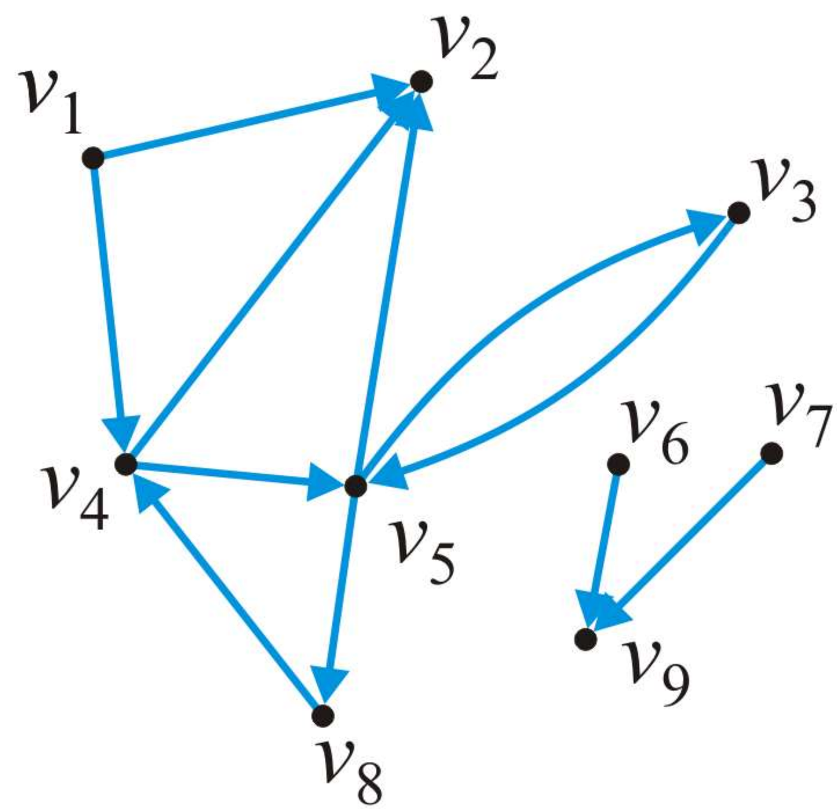
Weighted Directed Graphs

- Each edge is associated with a value
- If both `(vj, vk)` and `(vj, vk)` are edges
  - It is not required that they have the same weight



Representation

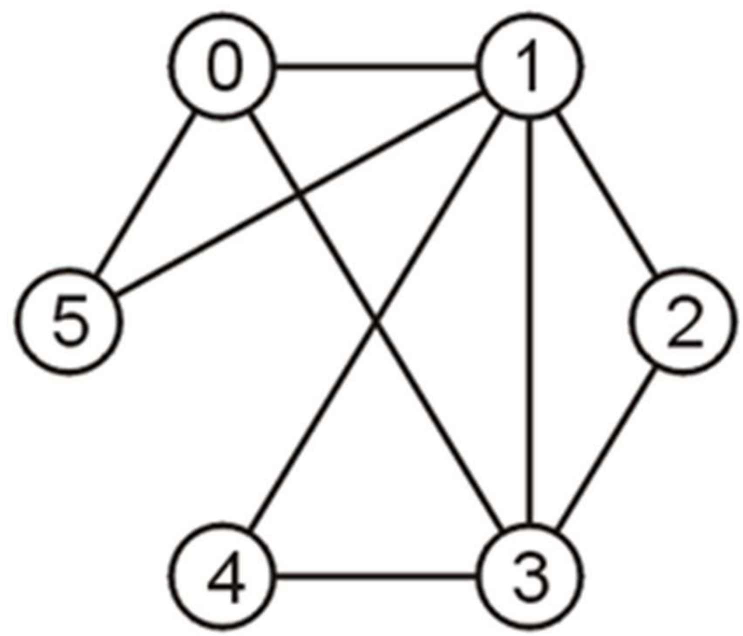
- How do we store the adjacency relations?
  - Adjacency matrix
  - Adjacency list



Adjacency Matrix

- Two dimensional matrix of size `n x n` where `n = |V|`
- `a[i, j] = 0 (F)` if there is no edge between vertices v_i and v_j
- `a[i, j] = 1 (T)` if there is an edge between vertices v_i and v_j
- Adjacency matrix of undirected graphs is symmetric
  - `a[i, j] = a[j, i]`

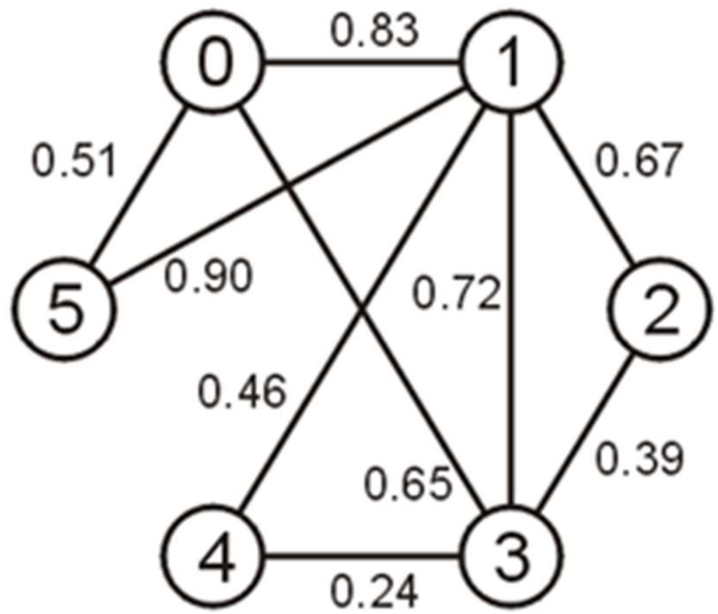
	0	1	2	3	4	5
0	F	T	F	T	F	T
1	T	F	T	T	T	T
2	F	T	F	T	F	F
3	T	T	T	F	T	F
4	F	T	F	T	F	F
5	T	T	F	F	F	F



Adjacency Matrix – Weighted Graph

- The matrix entry [j, k] is set to the weight of the edge (v_j, v_k)
- How to indicate absence of an edge in the graph?

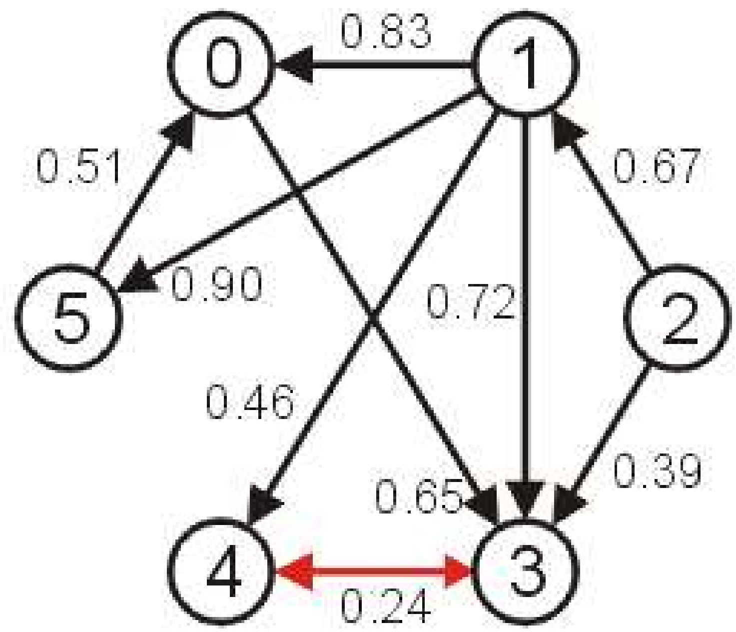
	0	1	2	3	4	5
0		0.83		0.65		0.51
1	0.83		0.67	0.72	0.46	0.90
2		0.67		0.39		
3	0.65	0.72	0.39		0.24	
4		0.46		0.24		
5	0.51	0.90				



Adjacency Matrix – Directed Graph

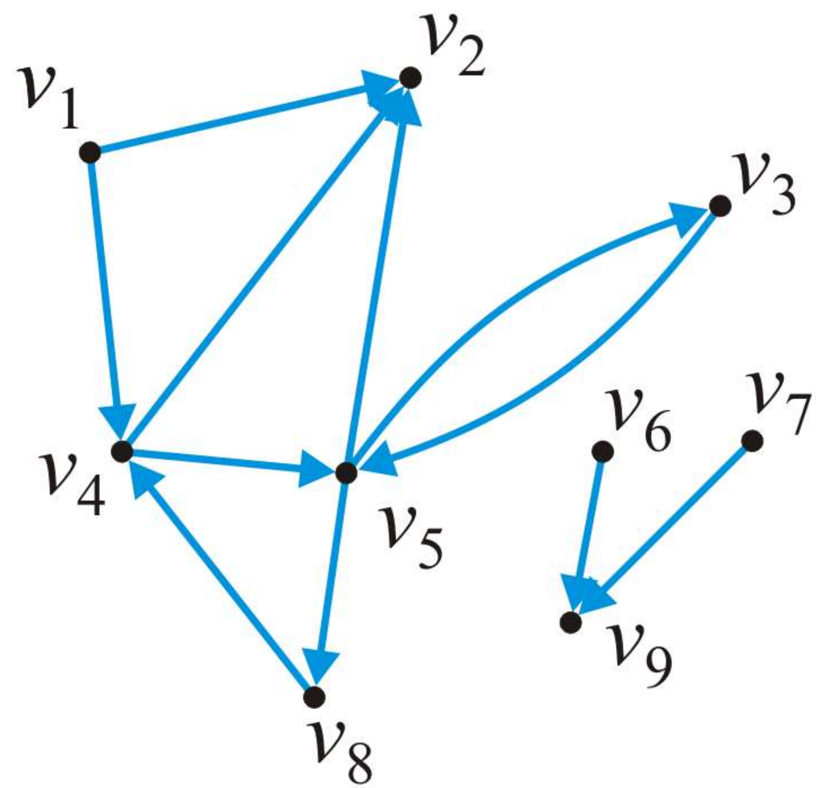
- For directed graph the matrix would not necessarily be symmetric

	0	1	2	3	4	5
0				0.65		
1	0.83			0.72	0.46	0.90
2		0.67		0.39		
3					0.24	
4				0.24		
5	0.51					



Adjacency Matrix – Analysis

	1	2	3	4	5	6	7	8	9
1		1		1					
2									
3					1				
4		1			1				
5		1	1					1	
6									1
7									1
8				1					
9									

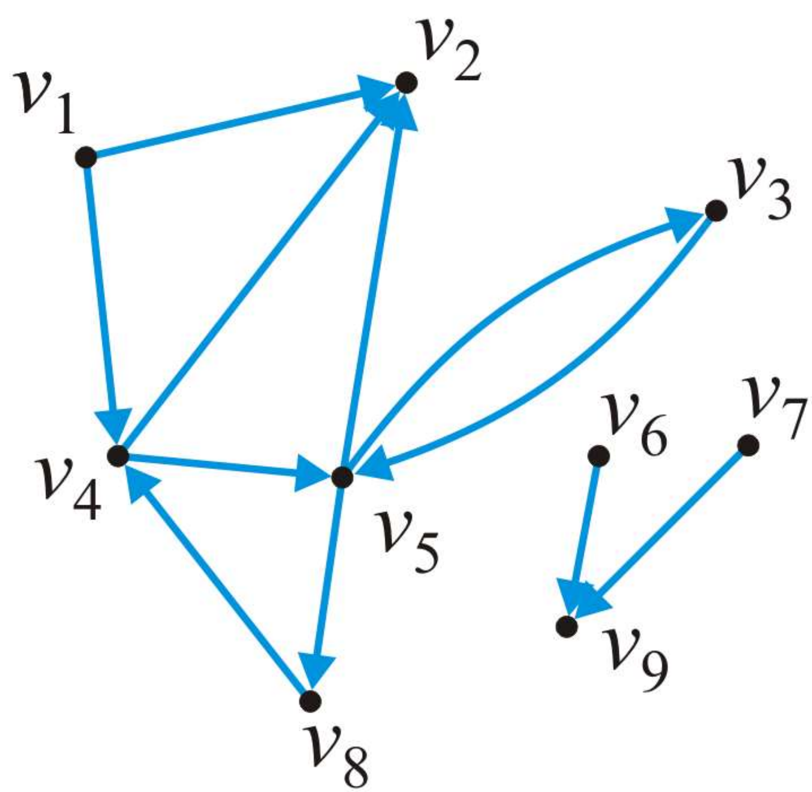


- Requires memory : $O(|V|^2)$
- Determining if v_j is adjacent to v_k : $O(1)$
- Finding all neighbors of v_j : $O(|V|)$

Adjacency Matrix – Problem

- Very sparsely populated
  - Out of 81 cells only 11 are 1 (or T)

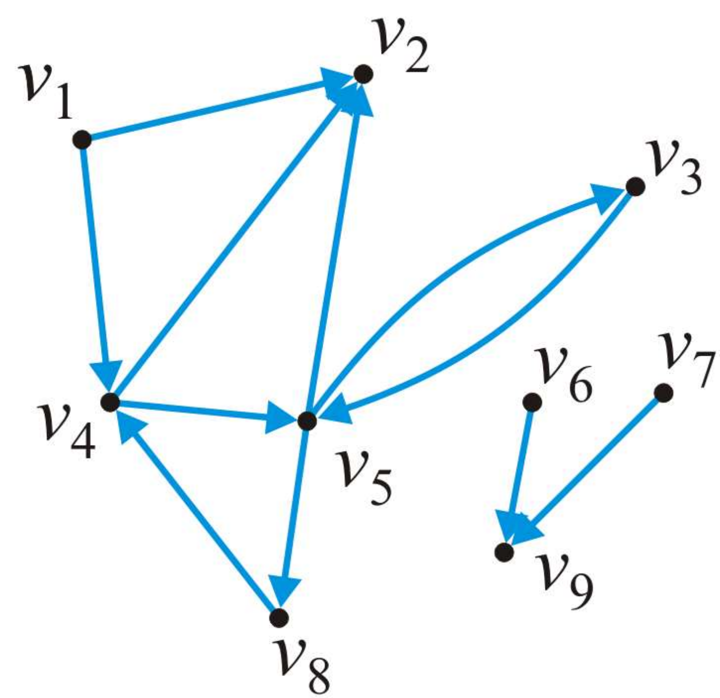
	1	2	3	4	5	6	7	8	9
1		1		1					
2									
3					1				
4		1			1				
5		1	1					1	
6									1
7									1
8				1					
9									



Adjacency List

- Each vertex is associated with a list of its neighbors
  - A vertex `w` is inserted in the list for vertex `v` if edge `(v, w)` exists

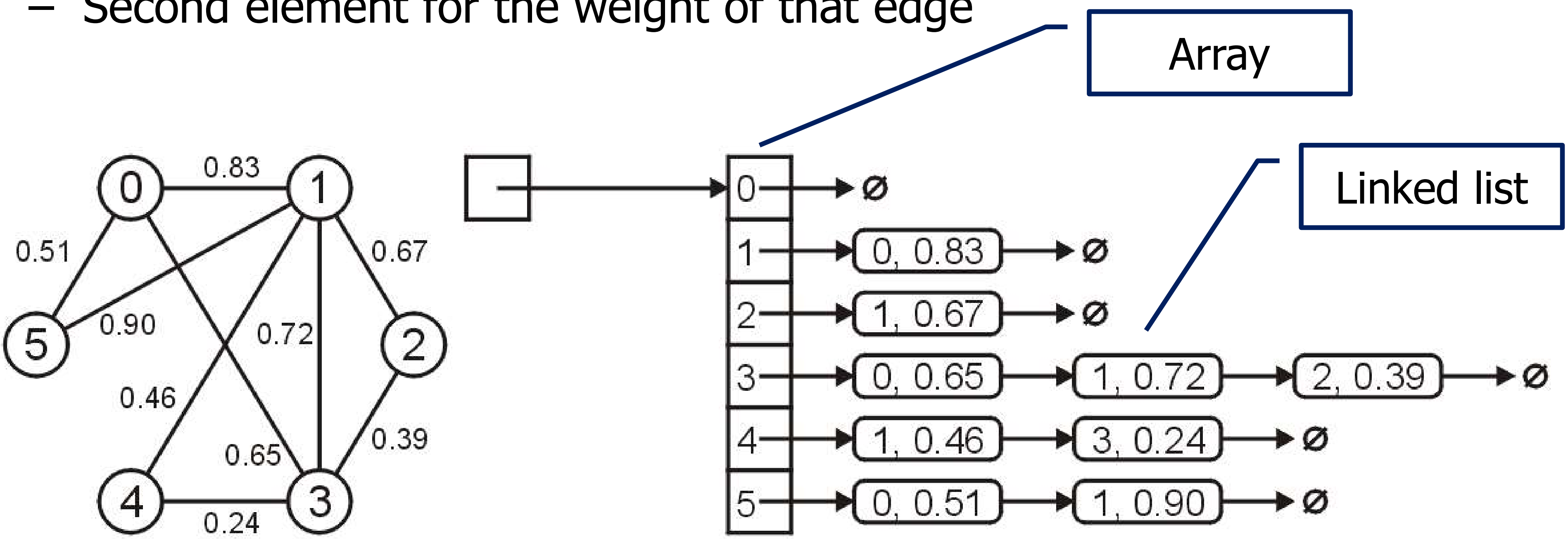
1	• → 2 → 4
2	•
3	• → 5
4	• → 2 → 5
5	• → 2 → 3 → 8
6	• → 9
7	• → 9
8	• → 4
9	•



- Requires memory : `O(|V| + |E|)`

Adjacency List – Weighted Graphs

- An adjacency list for a weighted graph contains two elements
  - First element for the vertex
  - Second element for the weight of that edge



- When the vertices are identified by a name (i.e., string)
  - Hash-table of lists is used to implement the adjacency list

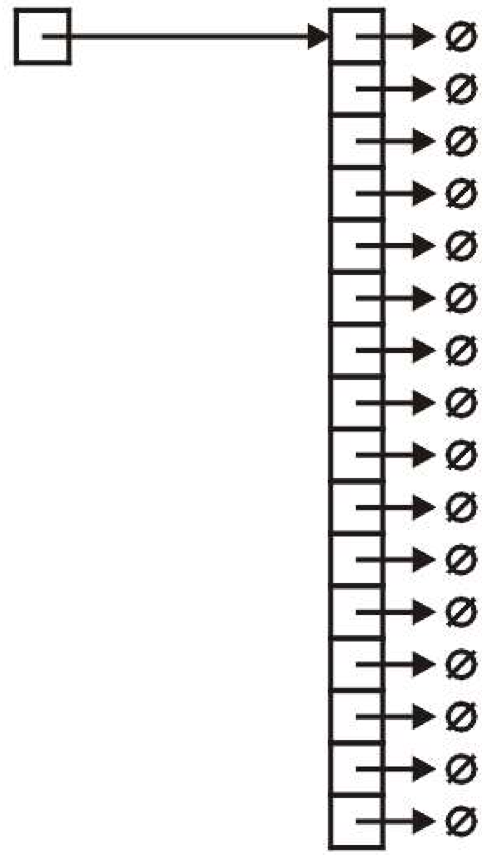
Adjacency List – Implementation

- Node to store adjacent vertex and weight of the edge

```
class SingleNode {
   private:
      int adjacent_vertex;
      double edge_weight;
      SingleNode * next_node;
};
```

- Define and create Array

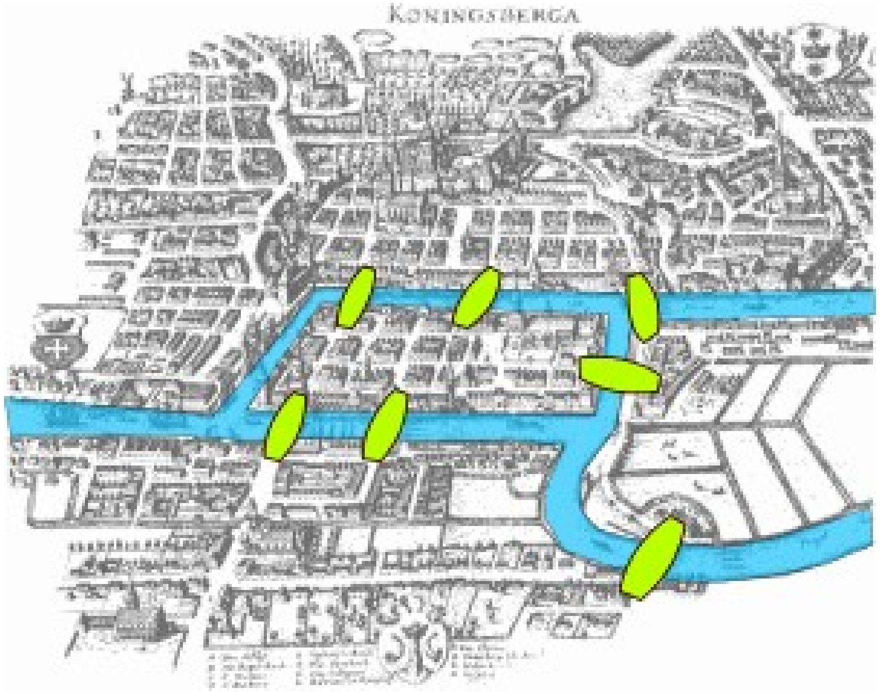
```
SingleNode * array;
array = new SingleNode[16];
```



Graph Problems – Euler Tour

- A sequence of vertices that traverse all edges in the graph exactly once
  - Leonhard Euler in 1736
  - Laid the foundations of graph theory

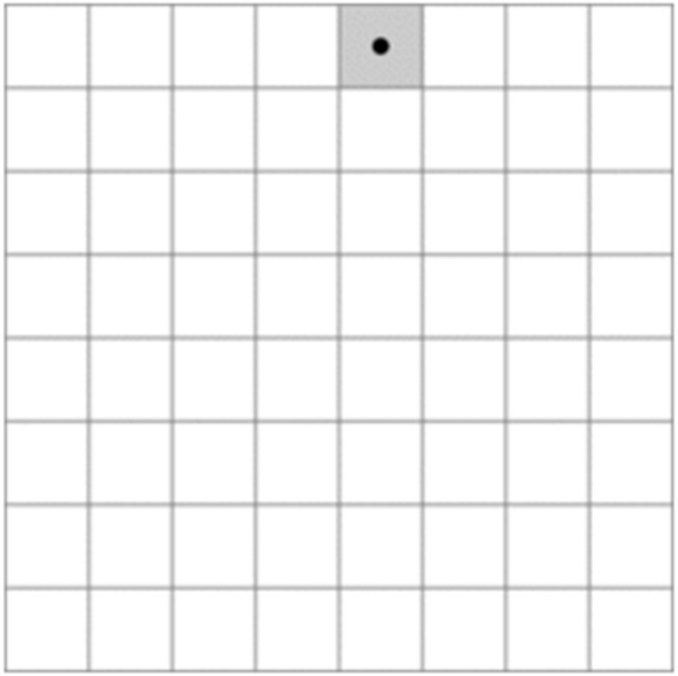
- Problem: To devise a walk through the city that would cross each of the seven bridges of Königsberg once and only once
  - Euler proved problem has no solution



Graph Problems – Hamiltonian Cycle

- Is there a simple cycle that connects all vertices in the graph
  - NP-Complete

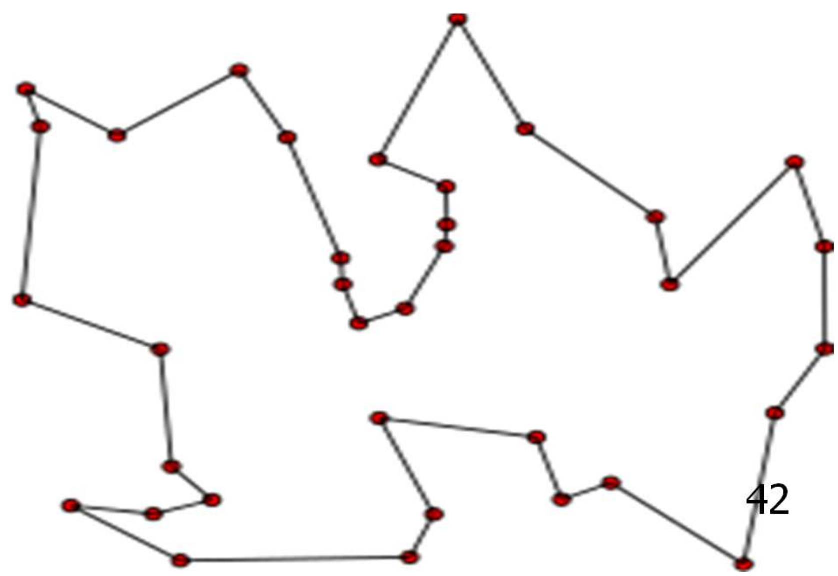
- Knight's Tour of chessboard
  - A sequence of knight's moves
  - Visit every square of a chessboard precisely once
  - Returns to its initial square



Graph Problems – Traveling Salesman

- A salesman wishes to
  - Visit a number of towns, and then
  - Return to his starting town
- Given the travelling times between towns, how should the travel be planned, so that:
  - He visits each town exactly once, and
  - He travels in as short time as possible

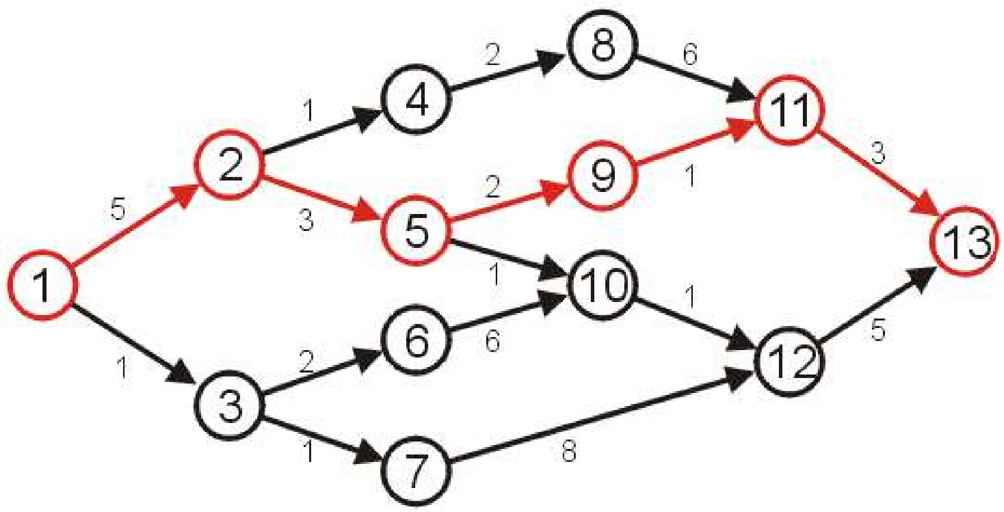
- Problem: Given a weighted graph G, provide shortest cycle that contains all vertices in G
  - NP-Hard problem

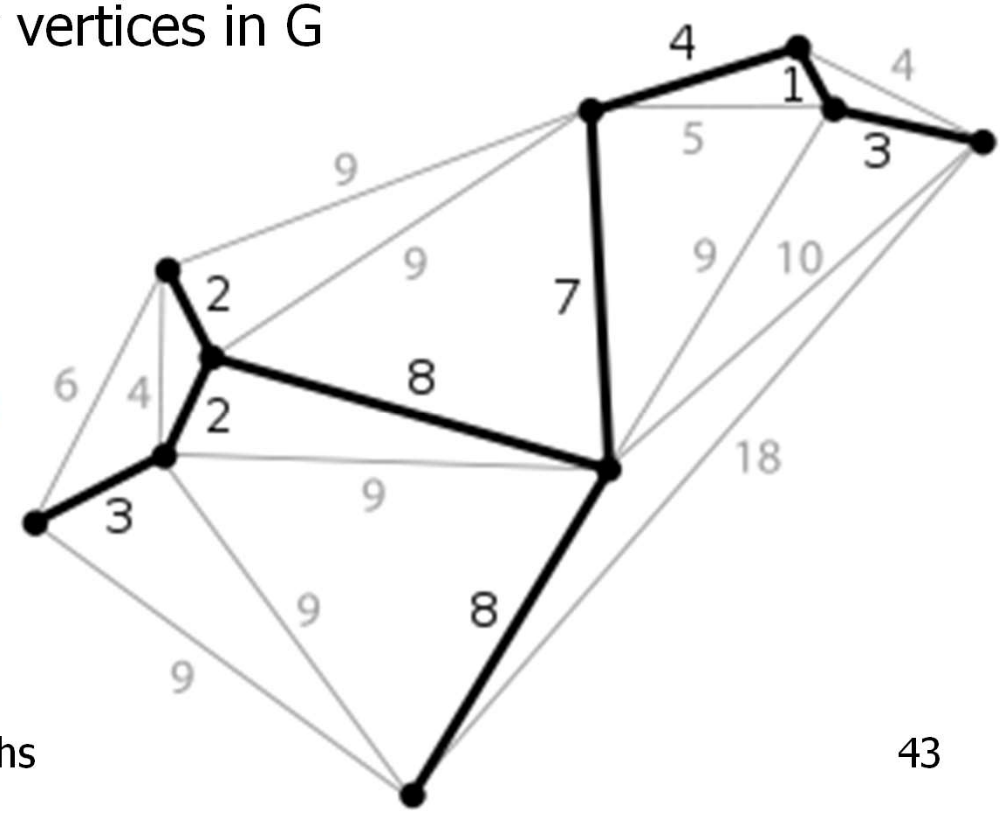


Graph Problems – Others

- Minimum-cost spanning tree
  - Given a weighted graph G, determine a spanning tree with minimum total edge cost

- Single-source shortest path
  - Given a weighted graph G and a source vertex v in G, determine the shortest paths from v to all other vertices in G





Any Question So Far?