Sequence

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A **sequence** is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer. each individual element a_k (read "a sub k") is called a **term.** The k in a_k is called a **subscript** or **index,** *m* (which may be any integer) is the subscript of the **initial term,** and *n* (which must be greater than or equal to *m*) is the subscript of the **final term.**

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$$a_m, a_{m+1}, a_{m+2}, \ldots$$

denotes an **infinite sequence.** An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k.

Example:

Find an explicit formula for a sequence that has the following initial terms:

$$a_1 = 1, a_2 = -\frac{1}{4}, a_3 = \frac{1}{9}, a_4 = -\frac{1}{16}, a_5 = \frac{1}{25}, \ldots$$

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$$a_k = \frac{(-1)^{k+1}}{k^2}$$

Example:

Define sequences $a_1, a_2, a_3, \ldots$ and $b_2, b_3, b_4, \ldots$ by the following explicit formulas:

1. $a_k = \frac{k}{k+1}$ for all integers $k \geq 1$,

2. $b_j = \frac{j-1}{j}$ for all integers $j \geq 2$.

Definition:

If *m* and *n* are integers and *m* ≤ *n*, the symbol $\sum_{k=m}^{n} a_k$, read the **summation from *k* equals *m* to *n* of *a*-sub-*k***, is the sum of all the terms $a_m, a_{m+1}, a_{m+2}, \dots, a_n$. We say that $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$ is the **expanded form** of the sum, and we write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

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Solution:

$$k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

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Express the following using summation notation:

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$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}.$$

Solution:

$$\sum_{k=0}^{n} \frac{k+1}{n+k} \quad \text{or} \quad \sum_{k=1}^{n+1} \frac{k}{n+k-1}.$$

Example: A Telescoping Sum

Some sums can be transformed into telescoping sums, which then can be rewritten as a simple expression. For instance, observe that

$$\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}.$$

Use this identity to find a simple expression for $\sum_{k=1}^{n} \frac{1}{k(k+1)}$

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Solution

$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

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$$= \left(\frac{1}{1} - \frac{1}{2}\right)$$

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Solution

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k(k+1)} &= \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)\\ &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)\end{aligned}$$

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Solution

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$
$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

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Use this identity to find a simple expression for $\sum_{k=1}^{n} \frac{1}{k(k+1)}$

Solution

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{(n+1)}\right) \\ &= 1 - \frac{1}{n+1}.\end{aligned}$$

Product Notation:

If m and n are integers and m ≤ n, the symbol $\prod_{k=m}^{n} a_k$, read the product from k equals m to n of a-sub-k, is the product of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$.

We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

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A recursive definition for the product notation is the following: If m is any integer, then

$$\prod_{k=m}^{m} a_k = a_m \quad \text{and} \quad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right). a_n$$

Example:

Compute the following products:

1. $\prod_{k=1}^{5} k$.
2. $\prod_{k=1}^{1} \frac{1}{k+1}$.

Theorem:

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$
2. $c \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c . a_k$
3. $(\prod_{k=m}^{n} a_k)(\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n} (a_k . b_k)$.

Example:

Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expressions as a single summation or product:

1. $\sum_{k=m}^{n} a_k + 2\sum_{k=m}^{n} b_k$,
2. $\left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=m}^{n} b_k\right)$.

Factorial

For each positive integer n, the quantity n factorial denoted $n!$, is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n - 1) \cdots 3 \cdot 2 \cdot 1.$$

Zero factorial, denoted 0!, is defined to be 1: 0! = 1

Or,

$$n! = \begin{cases} 1, & n = 0 \\ n(n-1)!, & n \geq 1 \end{cases}.$$

Example:

What are the values of the following factorials.

1. $\frac{(n+1)!}{n!}$

2. $\frac{n!}{(n-3)!}$.

n choose r

Definition:

Let n and r be integers with $0 \le r \le n$. The symbol,

$$\binom{n}{r}$$

is read "n choose r" and represents the number of subsets of size r that can be chosen from a set with n elements.

Formula for computing $\binom{n}{r}$ is

For all integers n and r with $0 \;\le\; r \;\le\; n$,

$$\binom{n+1}{n} = \frac{(n+1)!}{n!\,(n+1-n)!} = n+1.$$

Example:

Use the formula for computing $\binom{n}{r}$ to evaluate the following expressions:

1. $\binom{8}{4}$
2. $\binom{4}{0}$
3. $\binom{n+1}{n}$.