Using Trains to Model Recurrence Relations

Benjamin Sinwell

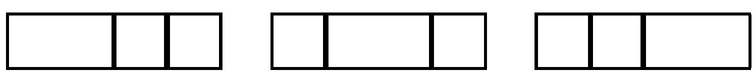
Northwood High School

Summary

This project gives students a hands-on approach to experiment with recurrence relations. By building trains using cars of different lengths students are given a concrete model with which to represent and better understand the concept of recursion.

Notes for the instructor

A brief introduction to building trains of different lengths using cars of different lengths is all that is necessary for students to complete this assignment. For example, a train of length four can be made from two cars of length one and one car of length two in three different ways.



Cuisenaire rods or colored paper (cut to match the different train sizes) could be used, but are not necessary, for students to complete this project. Students can easily record the trains on graph paper. That said, the use of manipulatives forces students to build the trains rather than simply represent the trains symbolically. This project could be given as a homework assignment or be used as a group activity during class. The first six problems should take approximately an hour for students to work through. The problems could be worked during class time or outside of class. Providing a forum (i.e., class discussion or an online discussion) for students to share their thinking on how they went about solving the problems would provide an additional and valuable learning opportunity.

Bibliography

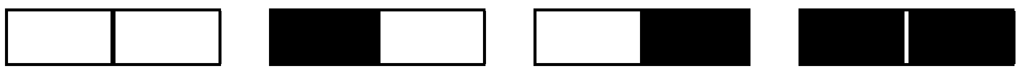
[1] Benjamin, A.T. and J.J. Quinn, *Proofs That Really Count: The Art of Combinatorial Proof*, Dolciani Mathematical Expositions, Volume 27, Mathematical Association of America, 2003.

Worksheet on Using Trains to Model Recurrence Relations

Each of the following questions will involve building trains of different lengths using cars of different lengths. The following shows how a train of length 4 can be made from two cars of length 1 and one car of length 2 in three different ways.

When asked how many trains of a given length you can make, simply count the number of *different* trains you can make.

1. Using only length 1 cars and length 2 cars, build and record all the trains that you can make for each of the following lengths: 1, 2, 3, 4, 5, and 6. Predict how many length 10 trains you can make. Explain how you determined your answer for length 10 trains. How do you know that you have counted all the trains?
2. Using only length 1 cars and now two different colors of length 2 cars, build and record all the trains that you can make for each of the following lengths: 1, 2, 3, 4, and 5. Predict how many length 6 trains you can make. Explain how you obtained your answer for the number of length 6 trains. The following are different trains.



3. Using only length 1 cars and two different colors of length 3 trains, build and record all the trains that you can make for each of the following lengths: 1, 2, 3, 4, and 5. Predict how many length 6 trains you can build. Explain your prediction using only the drawings of the different length 3 trains and length 5 trains.
4. For each of #1, #2, and #3, write a rule that would tell you how many ways there are to make a length n train if you already knew the number of ways to make trains of length less than n.
5. Predict how many trains of length 1, 2, 3, 4, and 5 you can make using only length 2 cars and two different colors of length 1 cars. Verify your predictions. How can you determine the number of length 6 trains?
6. What types of cars (give the length and/or number of colors) would you need to use in order to create a pattern where the number of length n trains is equal to three times the number of length $n-1$ trains plus two times the number of length $n-2$ trains?

Extensions:

7. Using only cars of lengths 1, 2, and 3, how many trains can you make of each of the following lengths: 1, 2, 3, 4, 5, and 6? Explain how many ways there are there to make a length 7 train using what you know about the number of ways to make trains of length less than 7.
8. Using cars of any length except 1, how many trains can you make of each of the following lengths: 1, 2, 3, 4, 5, and 6? How many ways are there to make a length 7 train? Have you seen these numbers before? Compare your results to #1. What is happening?
9. Using cars of any length except 2, how many trains can you make of the following lengths: 1, 2, 3, 4, 5, and 6? Explain how to find how many ways there are to make a length 7 train.
10. What types of cars would you need to produce the following table?

Train length	1	2	3	4	5	6	7
number of trains	0	2	0	4	0	8	0

Solutions

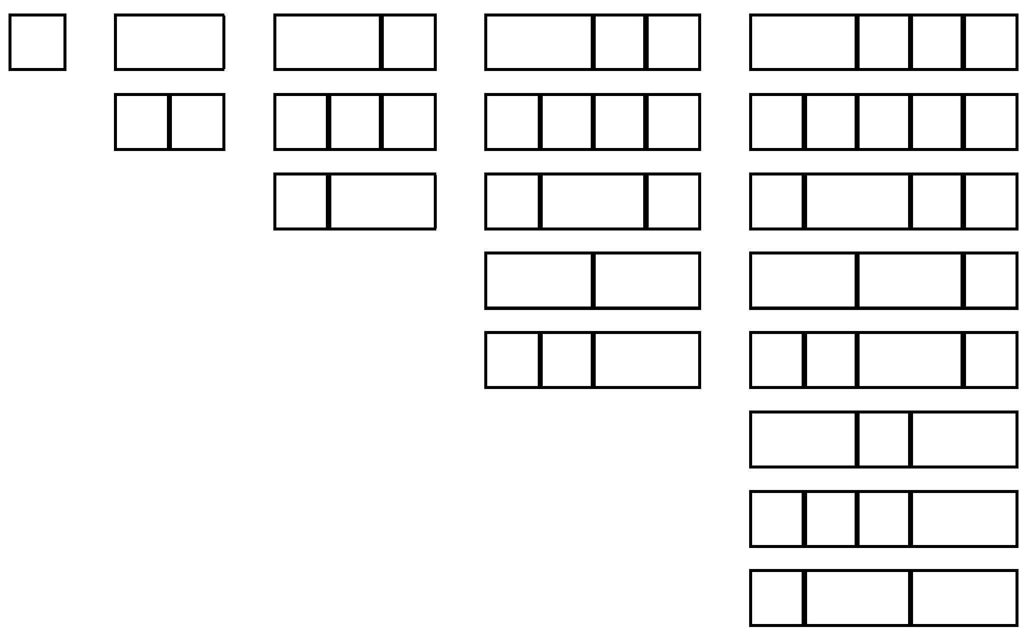
1. Using cars of lengths 1 and 2, the number of ways to make each of the first six train lengths is 1, 2, 3, 5, 8, and 13, respectively. There are 89 trains of length 10. This is the sum of the number of all length 9 trains and the number of all length 8 trains. To make a train of length n, add a length 1 car to the end of each length $n-1$ train and add a length 2 car to the end of each length $n-2$ train. If the number of trains one can make of length 1 and 2 are correct, it follows that one has counted all of the trains.

   Let t_n be the number of length n trains. We have the following recurrence relation:

   $$t_n = t_{n-1} + t_{n-2} \text{ with initial conditions } t_1 = 1, t_2 = 2.$$

   This is the same recurrence relation as the Fibonacci numbers, but with different starting values.

   The diagram below shows the trains 1, 2, 3, 4, and 5 that can be made from length 1 and 2 cars. Notice the order of the trains in columns for $n = 3, 4, 5$: first are the trains whose rightmost car is length 1, added to the trains of length $n-1$, and below those are the trains whose rightmost car is length 2, added to the trains of length $n-2$.

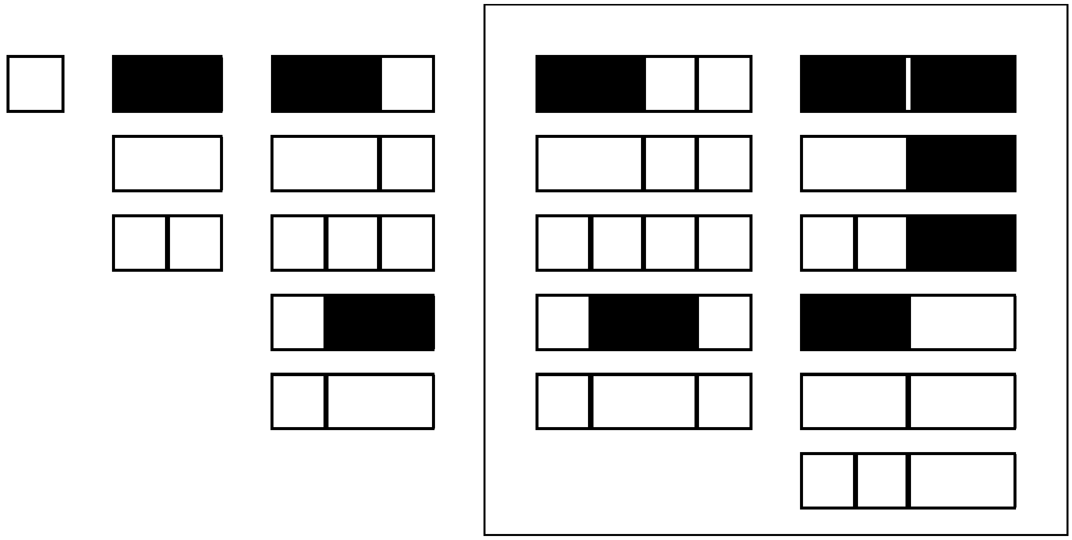


2. Using cars of length 1 and two types of cars of length 2, the number of ways to make each of the first five train lengths is 1, 3, 5, 11, and 21, respectively. There are 43 trains of length 6. To build them, add a length 1 car to each of the length 5 trains and add one of each color of length 2 cars to the length 4 trains.

   These satisfy the following recurrence relation:

   $$t_n = t_{n-1} + 2t_{n-2} \text{ with initial conditions } t_1 = 1, t_2 = 3.$$

   The diagram below shows all the trains that can be made of length 1, 2, 3, and 4.



The trains of length 4 are boxed in the last two columns. They can be built by adding a car of length 1 to each train of length 3 (left hand column), a black car of length 2 to each train of length 2, and a white car of length 2 to each train of length 2 (right hand column).

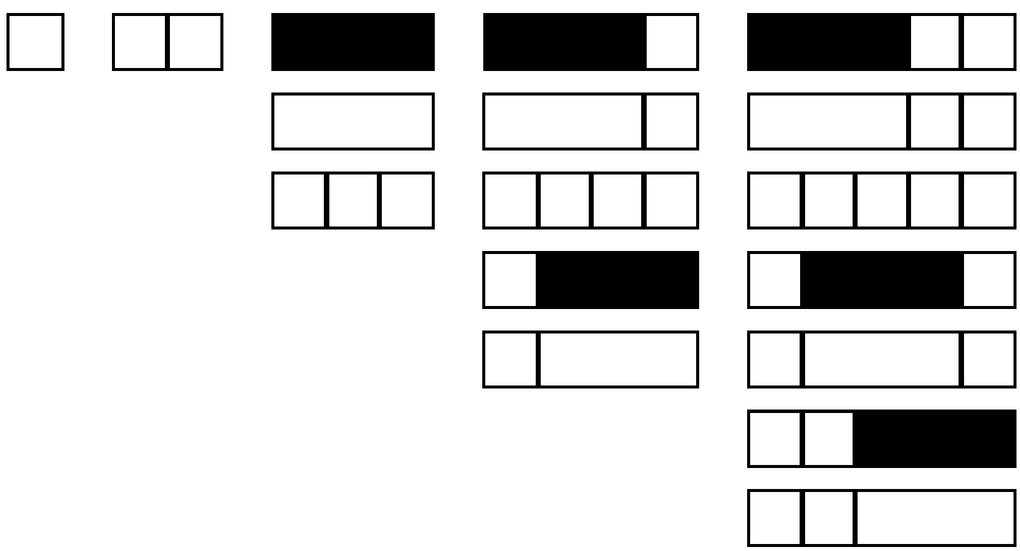
3. Using cars of length 1 and two types of cars of length 2, the number of ways to make each of the first five train lengths is 1, 1, 3, 5, and 7, respectively. There are 13 trains of length 6. To build them, add a length 1 car to each of the length 5 trains and add one of each color of length 3 cars to the length 3 trains.

   The diagram below illustrates how this can be done using the drawings of trains of length 5 and length 3.

   These satisfy the following recurrence relation:

$$t_n = t_{n-1} + 2t_{n-3} \text{ with initial conditions } t_1 = 1, t_2 = 1, t_3 = 3.$$

   The diagram below shows all the trains that can be made of length 1, 2, 3, 4, and 5.



4. (Recurrence relations and initial conditions are given above.)

5. Using cars of length 2 and two types of cars of length 1, the number of ways to make each of the first five train lengths is 2, 5, 12, 29, and 70, respectively. There are 13 trains of length 6. To build them, add a length 1 car to each of the length 5 trains and add one of each color of length 3 cars to the length 3 trains.

The number of ways to make each of the first five train lengths is: 2, 5, 12, 29, and 70. To make all length 6 trains, double the number of length 5 trains (because one can add two different cars of length 1 to each of them) and add that to the number of length 4 trains (because one can add a car of length 2 to each of those, giving $2 \cdot 70 + 29 = 169$ length 6 trains.

These satisfy the following recurrence relation:

$$t_n = 2t_{n-1} + t_{n-2} \text{ with initial conditions } t_1 = 2, t_2 = 5.$$

6. Using three different colors of length 1 cars and two different colors of length 2 cars will produce such a pattern.

7. Using cars of lengths 1, 2, and 3, the number of ways to make each of the first six train lengths is 1, 2, 4, 7, 13, and 24, respectively. There are 44 length 7 trains: to make them all, add a length 1 car to all length 6 trains, a length 2 car to all length 5 trains, and a length 3 car to all length 4 trains.

   These satisfy the following recurrence relation:

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} \text{ with initial conditions } t_1 = 1, t_2 = 2, t_3 = 4.$$

8. Using cars of any length except 1, the number of ways to make each of the first six train lengths is 0, 1, 1, 2, 3, and 5, respectively. There are 8 length 7 trains: to make them all, extend the last car in each length 6 train by one unit and add a length 2 car to each length 5 train. These are the same numbers as problem #1, just shifted over two terms. They follow the same recurrence relation as those in problem #1, $t_n = t_{n-1} + t_{n-2}$, but have different initial values, $t_1 = 0$ and $t_2 = 1$.

9. Using cars of any length except 2, the number of ways to make each of the first six train lengths is 1, 1, 2, 4, 7, and 12, respectively. There are 21 length 7 trains: to make them all, add a length 1 car to each length 6 train, extend the last car in each length 5 train by two units, and add a length 4 car to each length 3 train. (Since none of the length 5 train can end in a length 2 car, none of their extended versions end in a length 4 car.)

   These satisfy the following recurrence relation:

$$t_n = t_{n-1} + t_{n-2} + t_{n-4} \text{ with initial conditions } t_1 = 1, t_2 = 1, t_3 = 2, t_4 = 4.$$

10. Using just two different colors of length 2 cars produces the pattern. An explicit formula for the number of length n trains in this setting is

$$\frac{1}{2}\left[(\sqrt{2})^n + (\sqrt{-2})^n\right].$$

   (Determining the number of certain types of length n trains could be used to motivate finding explicit formulas from recurrence relations.)