Defining Sequences recursively and Solving Recursive Sequences by Iterations

Defining Sequences Recursively

A **recurrence relation** for a sequence $a_0, a_1, a_2, \ldots$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where i is an integer with $k - i \geq 0$.

The **initial conditions** for such a recurrence relation specify the values of $a_0, a_1, a_2, \ldots, a_{i-1}$, if i is a fixed integer, or $a_0, a_1, \ldots, a_m$, where m is an integer with $m \geq 0$, if i depends on k.

Example:

Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ satisfy the recurrence relation that the kth term equals 3 times the $(k-1)$st term for every integer $k \geq 2$:

$$a_k = 3a_{k-1} \text{ and } b_k = 3b_{k-1}.$$

But suppose that the initial conditions for the sequences are different:

$$a_1 = 2 \text{ and } b_1 = 1$$

$a_1, a_2, a_3, \ldots$ begins 2, 6, 18, 54, ... and
$b_1, b_2, b_3, \ldots$ begins 1, 3, 9, 27, ...

Example: Showing That a Sequence Given by an Explicit Formula Satisfies a Certain Recurrence Relation

The sequence of Catalan numbers arises in a remarkable variety of different contexts in discrete mathematics. It can be defined as follows: For each integer n ≥ 1,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

1. Find C_1, C_2, and C_3.
2. Show that this sequence satisfies the recurrence relation

$$C_k = \frac{4k-2}{k+1}\,C_{k-1}$$

   for all integers $k \geq 2$

Example: If $C_n = \frac{1}{n+1}\binom{2n}{n}$

$$C_1 = \frac{1}{1+1}\binom{2}{1} = \frac{1}{2}\frac{2!}{1!\,(2-1)!} = \frac{1}{2}\frac{2}{1} = 1\,,$$

$$C_2 = \frac{1}{2+1}\binom{4}{2} = \frac{1}{3}\frac{4!}{2!\,(4-2)!} = \frac{1}{3}\frac{24}{(2)(2)} = 2,$$

and

$$C_3 = \frac{1}{3+1}\binom{6}{3} = \frac{1}{4}\frac{6!}{3!\,(6-3)!} = \frac{1}{4}\frac{720}{(6)(6)} = 5.$$

Example: If $C_n = \frac{1}{n+1}\binom{2n}{n}$ show that $C_k = \frac{4k-2}{k+1}\ C_{k-1}, \forall\ int\ k \geq 2$

L.H.S=

$$C_k = \frac{1}{k+1}\binom{2k}{k} = \frac{1}{k+1}.\frac{(2k)!}{k!\,(2k-k)!} = \frac{1}{k+1}\frac{(2k)!}{k!\,k!} = \frac{(2k)!}{k!\,(k+1)!}\ ,$$

R.H.S=

$$\frac{4k-2}{k+1}C_{k-1} = \frac{4k-2}{k+1}.\frac{1}{(k-1)+1}\binom{2(k-1)}{k-1}$$

$$= \frac{2(2k-1)}{k+1}.\frac{1}{k}.\frac{(2k-2)!}{(k-1)!\,(2k-2-k+1)!}$$

$$= \frac{2(2k-1)(2k-2)!}{(k+1).k.(k-1)!.(k-1)!} = \frac{2(2k-1)!}{(k+1)!.(k-1)!}$$

$$= \frac{2(2k-1)!}{(k+1)!.(k-1)!}.\frac{k}{k} = \frac{2k(2k-1)!}{(k+1)!.k.(k-1)!} = \frac{(2k)!}{(k+1)!\,k!} = L.H.S$$

Example:

Let $c_0, c_1, c_2, \ldots$ be defined by the formula

$$c_n = 2^n - 1$$

for every integer $n \geq 0$.

Show that this sequence satisfies the recurrence relation

$$c_k = 2c_{k-1} + 1$$

for every integer $k \geq 1$.

$$\begin{aligned} R.H.S &= 2c_{k-1} + 1 \\ &= 2\left(2^{k-1} - 1\right) + 1 \\ &= 2^{k-1+1} - 2 + 1 \\ &= 2^k - 1 = c_k = L.H.S \end{aligned}$$

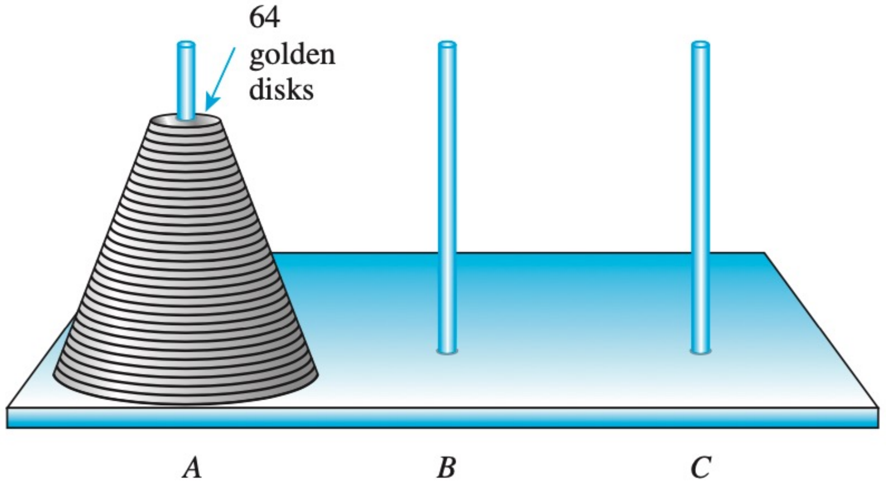
The Tower of Hanoi

In 1883 a French mathematician, Édouard Lucas, invented a puzzle that he called the Tower of Hanoi (La Tour D'Hanoï).

The puzzle consisted of eight disks of wood with holes in their centers, which were piled in order of decreasing size on one pole in a row of three.

Those who played the game were supposed to

- move all the disks one by one from one pole to another,
- never placing a larger disk on top of a smaller one.

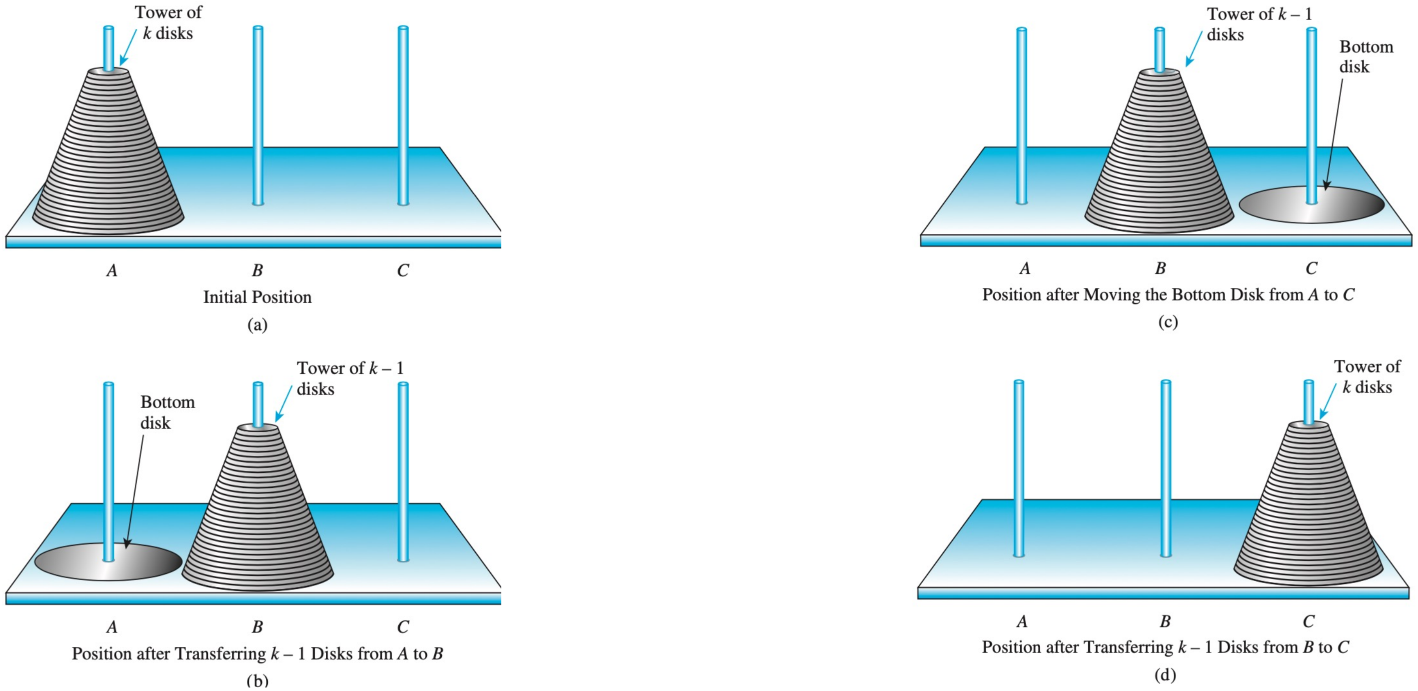


The Tower of Hanoi

Suppose that you, somehow or other, have found the most efficient way possible to transfer a tower of k-1 disks one by one from one pole to another, obeying the restriction that you never place a larger disk on top of a smaller one. What is the most efficient way to transfer a tower of k disks from one pole to another?

The Tower of Hanoi

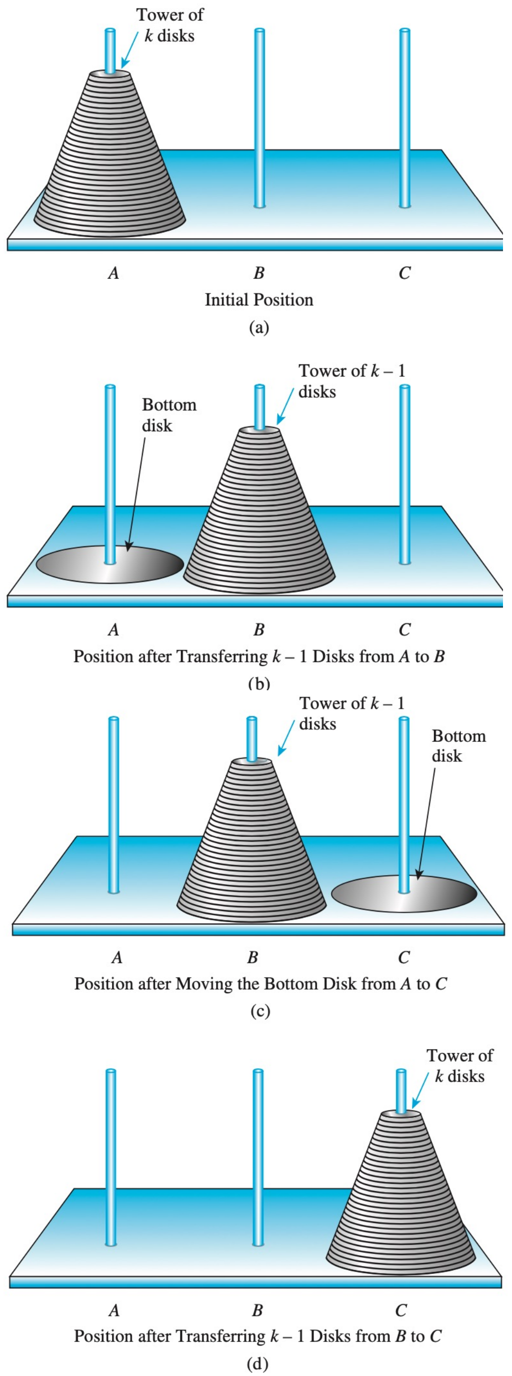
Suppose that you, somehow or other, have found the most efficient way possible to transfer a tower of *k*-1 disks one by one from one pole to another, obeying the restriction that you never place a larger disk on top of a smaller one. What is the most efficient way to transfer a tower of *k* disks from one pole to another?



The Tower of Hanoi

For each integer $k \geq 1$, let

$$m_k = \begin{bmatrix} \text{the minimum number} \\ \text{of moves needed} \\ \text{to transfer a tower} \\ \text{of k disks from} \\ \text{one pole to another} \end{bmatrix} \qquad m_{k-1} = \begin{bmatrix} \text{the minimum number} \\ \text{of moves needed} \\ \text{to transfer a tower} \\ \text{of k−1 disks from} \\ \text{one pole to another} \end{bmatrix}$$

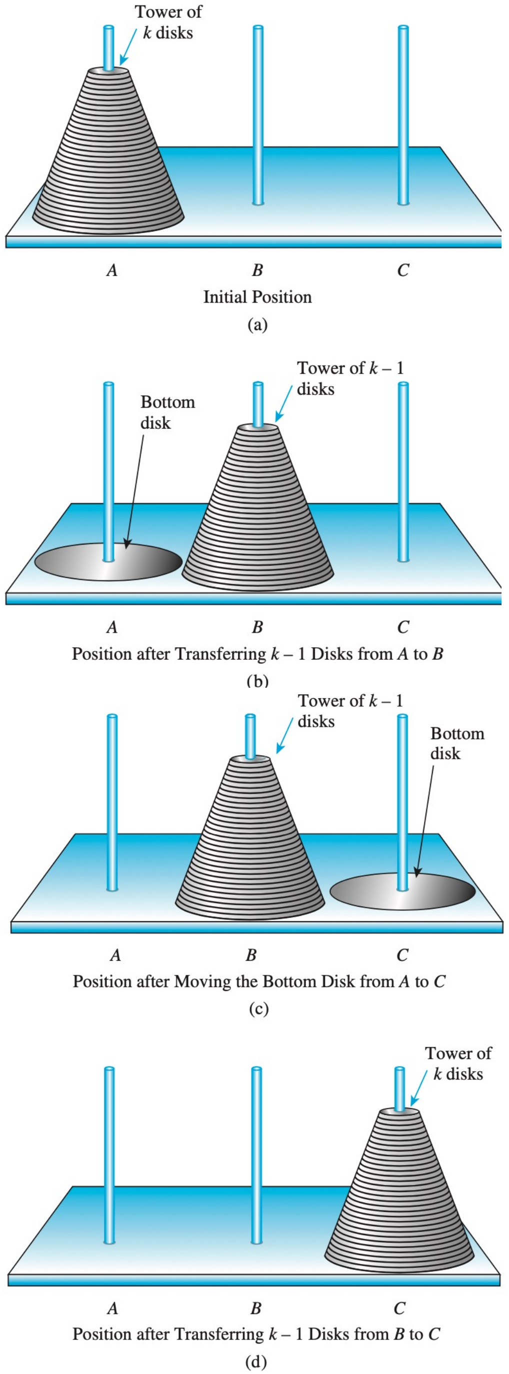


The Tower of Hanoi

For each integer $k \geq 1$, let

$$m_k = \begin{bmatrix} \text{the minimum number} \\ \text{of moves needed} \\ \text{to transfer a tower} \\ \text{of k disks from} \\ \text{one pole to another} \end{bmatrix}$$

$$= \begin{bmatrix} \text{the minimum number} \\ \text{of moves needed} \\ \text{to go from} \\ \text{position (a)} \\ \text{to position (b)} \end{bmatrix} + \begin{bmatrix} \text{the minimum number} \\ \text{of moves needed} \\ \text{to go from} \\ \text{position (b)} \\ \text{to position (c)} \end{bmatrix} + \begin{bmatrix} \text{the minimum number} \\ \text{of moves needed} \\ \text{to go from} \\ \text{position (c)} \\ \text{to position (d)} \end{bmatrix}$$



The Tower of Hanoi

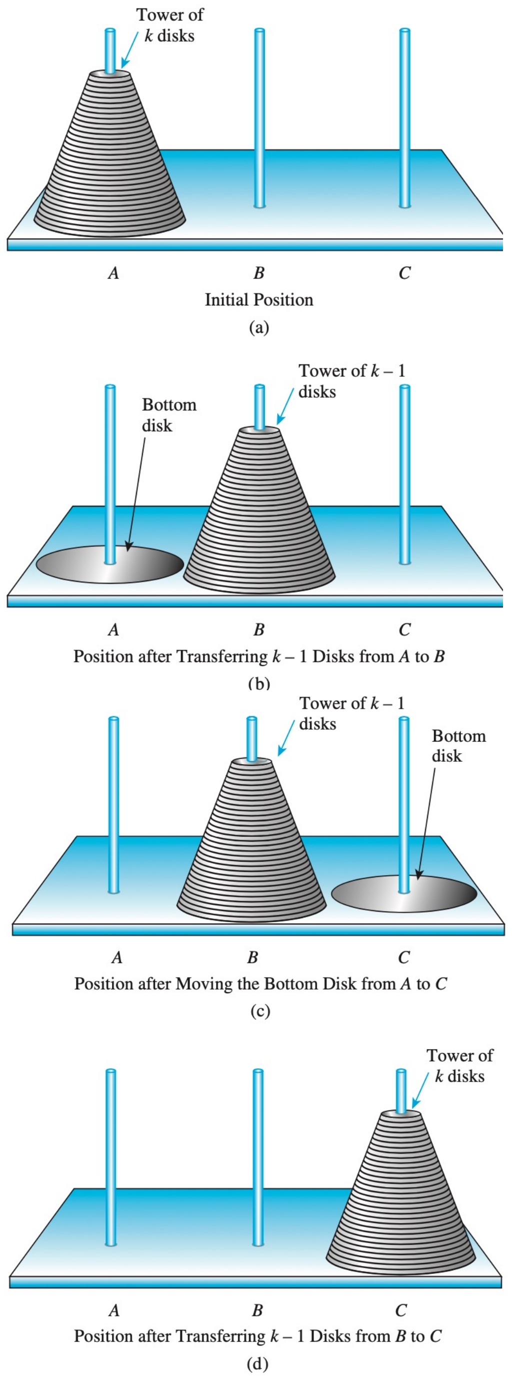
For each integer $k \geq 1$, let

$$m_k = \left[\begin{array}{c}\text{the minimum number}\\ \text{of moves needed}\\ \text{to transfer a tower}\\ \text{of k disks from}\\ \text{one pole to another}\end{array}\right]$$

$$= \left[\begin{array}{c}\text{the minimum number}\\ \text{of moves needed}\\ \text{to go from}\\ \text{position (a)}\\ \text{to position (b)}\end{array}\right] + \left[\begin{array}{c}\text{the minimum number}\\ \text{of moves needed}\\ \text{to go from}\\ \text{position (b)}\\ \text{to position (c)}\end{array}\right] + \left[\begin{array}{c}\text{the minimum number}\\ \text{of moves needed}\\ \text{to go from}\\ \text{position (c)}\\ \text{to position (d)}\end{array}\right]$$

Therefore,

$$m_k = m_{k-1} + 1 + m_{k-1}$$

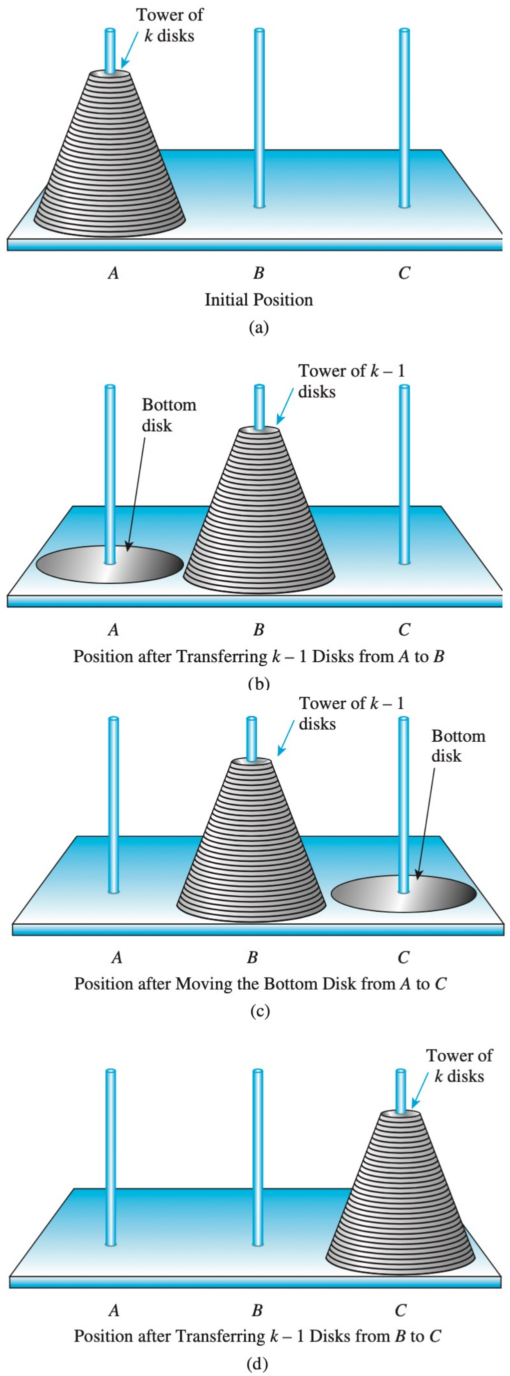


The Tower of Hanoi

For each integer $k \geq 1$, let

$$m_k = \left[\begin{array}{c}\text{the minimum number}\\ \text{of moves needed}\\ \text{to transfer a tower}\\ \text{of k disks from}\\ \text{one pole to another}\end{array}\right]$$

$$= \left[\begin{array}{c}\text{the minimum number}\\ \text{of moves needed}\\ \text{to go from}\\ \text{position (a)}\\ \text{to position (b)}\end{array}\right] + \left[\begin{array}{c}\text{the minimum number}\\ \text{of moves needed}\\ \text{to go from}\\ \text{position (b)}\\ \text{to position (c)}\end{array}\right] + \left[\begin{array}{c}\text{the minimum number}\\ \text{of moves needed}\\ \text{to go from}\\ \text{position (c)}\\ \text{to position (d)}\end{array}\right]$$



Therefore,

$$m_k = m_{k-1} + 1 + m_{k-1} = 2m_{k-1} + 1$$

$$m_1 = 1, m_2 = 2(1) + 1 = 3, m_3 = 2(3) + 1 = 7,$$
$$m_4 = 2(7) + 1 = 15$$

The Fibonacci Numbers

In 1202 Fibonacci posed the following problem:

A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions:

1. Rabbit pairs are not fertile during their first month of life but thereafter give birth to one new male/female pair at the end of every month.
2. No rabbits die.

How many rabbits will there be at the end of the year?

The Fibonacci Numbers

$$\begin{bmatrix}\textbf{the number of}\\ \textbf{rabbit pairs alive}\\ \textbf{at the end of month } \boldsymbol{k}\end{bmatrix} = \begin{bmatrix}\textbf{the number of}\\ \textbf{rabbit pairs alive}\\ \textbf{at the end of month k−1}\end{bmatrix} + \begin{bmatrix}\textbf{the number of}\\ \textbf{rabbit pairs born}\\ \textbf{at the end of month } \boldsymbol{k}\end{bmatrix}$$

The Fibonacci Numbers

$$\begin{bmatrix}\textbf{the number of}\\ \textbf{rabbit pairs alive}\\ \textbf{at the end of month } \boldsymbol{k}\end{bmatrix} = \begin{bmatrix}\textbf{the number of}\\ \textbf{rabbit pairs alive}\\ \textbf{at the end of month k}-\textbf{1}\end{bmatrix} + \begin{bmatrix}\textbf{the number of}\\ \textbf{rabbit pairs alive}\\ \textbf{at the end of month } \boldsymbol{k-2}\end{bmatrix}$$

The Fibonacci Numbers

$$\begin{bmatrix} \textbf{the number of} \\ \textbf{rabbit pairs alive} \\ \textbf{at the end of month } \boldsymbol{k} \end{bmatrix} = \begin{bmatrix} \textbf{the number of} \\ \textbf{rabbit pairs alive} \\ \textbf{at the end of month k−1} \end{bmatrix} + \begin{bmatrix} \textbf{the number of} \\ \textbf{rabbit pairs born} \\ \textbf{at the end of month } \boldsymbol{k} \end{bmatrix}$$

Hence the complete specification of the Fibonacci sequence is as follows: For every integer $k \geq 2$,

$F_k = F_{k-1} + F_{k-2}$ recurrence relation

$F_0 = 1, F_1 = 1$ initial conditions.

Example:

Prove that $F_k = 3F_{k-3} + 2F_{k-4}$ for every integer $k \geq 4$.

Example:

Prove that $F_k = 3F_{k-3} + 2F_{k-4}$ for every integer $k \geq 4$.

Solution:

$$3F_{k-3} + 2F_{k-4}$$

Example:

Prove that $F_k = 3F_{k-3} + 2F_{k-4}$ for every integer $k \geq 4$.

Solution:

$$3F_{k-3} + 2F_{k-4}$$
$$= 2(F_{k-3} + F_{k-4}) + F_{k-3}$$

Example:

Prove that $F_k = 3F_{k-3} + 2F_{k-4}$ for every integer $k \geq 4$.

Solution:

$$3F_{k-3} + 2F_{k-4}$$
$$= 2(F_{k-3} + F_{k-4}) + F_{k-3}$$
$$= 2F_{k-2} + F_{k-3}$$

Example:

Prove that $F_k = 3F_{k-3} + 2F_{k-4}$ for every integer $k \geq 4$.

Solution:

$$3F_{k-3} + 2F_{k-4}$$
$$= 2(F_{k-3} + F_{k-4}) + F_{k-3}$$
$$= 2F_{k-2} + F_{k-3}$$
$$= F_{k-2} + F_{k-1}$$

Example:

Prove that $F_k = 3F_{k-3} + 2F_{k-4}$ for every integer $k \geq 4$.

Solution:

$$\begin{gathered} 3F_{k-3} + 2F_{k-4} \\ = 2(F_{k-3} + F_{k-4}) + F_{k-3} \\ = 2F_{k-2} + F_{k-3} \\ = F_{k-2} + F_{k-1} \\ = F_k. \end{gathered}$$

Example:

On your twenty-first birthday you get a letter informing you that on the day you were born an eccentric rich aunt deposited $100,000 in a bank account earning 4% interest compounded annually and she now intends to turn the account over to you, provided you can figure out how much it is worth. What is the amount currently in the account?

Example:

On your twenty-first birthday you get a letter informing you that on the day you were born an eccentric rich aunt deposited \$100,000 in a bank account earning 4% interest compounded annually and she now intends to turn the account over to you, provided you can figure out how much it is worth. What is the amount currently in the account?

$$A_k = A_{k-1} + 0.04A_{k-1} = 1.04\, A_{k-1},$$

Solving Recurrence Relations by Iteration

Example: Let a_0 , a_1 , a_2 , . . . be the sequence defined recursively as follows:

For all integers $k \geq 1$,

$$a_k = a_{k-1} + 2 \text{ recurrence relation and}$$

$$a_0 = 1 \text{ initial condition.}$$

Use iteration to guess an explicit formula for the sequence.

Example: For all integers $k \geq 1$, $a_k = a_{k-1} + 2$ recurrence relation and $a_0 = 1$ initial condition.

Solution:

$$a_0 = 1$$

Example: For all integers $k \geq 1$, $a_k = a_{k-1} + 2$ recurrence relation and $a_0 = 1$ initial condition.

Solution:

$$a_0 = 1$$
$$a_1 = a_0 + 2 = 1 + 2$$

Example: For all integers $k \geq 1$, $a_k = a_{k-1} + 2$ recurrence relation and $a_0 = 1$ initial condition.

Solution:

$$a_0 = 1$$
$$a_1 = a_0 + 2 = 1 + 2$$
$$a_2 = a_1 + 2 = 1 + 2 + 2 = 1 + 2(2)$$

Example: For all integers $k \geq 1$, $a_k = a_{k-1} + 2$ recurrence relation and $a_0 = 1$ initial condition.

Solution:

$$a_0 = 1$$
$$a_1 = a_0 + 2 = 1 + 2$$
$$a_2 = a_1 + 2 = 1 + 2 + 2 = 1 + 2(2)$$
$$a_3 = a_2 + 2 = 1 + 2 + 2 + 2 = 1 + 3(2)$$

Example: For all integers $k \geq 1$, $a_k = a_{k-1} + 2$ recurrence relation and $a_0 = 1$ initial condition.

Solution:

$$a_0 = 1$$
$$a_1 = a_0 + 2 = 1 + 2$$
$$a_2 = a_1 + 2 = 1 + 2 + 2 = 1 + 2(2)$$
$$a_3 = a_2 + 2 = 1 + 2 + 2 + 2 = 1 + 3(2)$$
$$a_4 = a_3 + 2 = 1 + 2 + 2 + 2 + 2 = 1 + 4(2)$$

Example: For all integers $k \geq 1$, $a_k = a_{k-1} + 2$ recurrence relation and $a_0 = 1$ initial condition.

Solution:

$$a_0 = 1$$
$$a_1 = a_0 + 2 = 1 + 2$$
$$a_2 = a_1 + 2 = 1 + 2 + 2 = 1 + 2(2)$$
$$a_3 = a_2 + 2 = 1 + 2 + 2 + 2 = 1 + 3(2)$$
$$a_4 = a_3 + 2 = 1 + 2 + 2 + 2 + 2 = 1 + 4(2)$$

Then,

$$a_n = 1 + n(2).$$

Definition:

A sequence $a_0, a_1, a_2, \ldots$ is called an ***arithmetic sequence*** if, and only if, there is a constant d such that

$$a_k = a_{k-1} + d \text{ for all integers } k \geq 1$$

It follows that,

$$a_n = a_0 + dn \text{ for all integers } n \geq 0.$$

Example:

Let r be a fixed nonzero constant, and suppose a sequence $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

$$a_k = \ ra_{k-1} \text{ for all integers } k \geq 1, \ a_0 = 1.$$

Use iteration to guess an explicit formula for this sequence.

Example:

Let r be a fixed nonzero constant, and suppose a sequence $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

$$a_k = \ r a_{k-1} \text{ for all integers } k \geq 1, \ a_0 = 1.$$

Use iteration to guess an explicit formula for this sequence.

$$a_0 = 1$$

Example:

Let r be a fixed nonzero constant, and suppose a sequence $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

$$a_k = \ ra_{k-1} \text{ for all integers } k \geq 1, \ a_0 = 1.$$

Use iteration to guess an explicit formula for this sequence.

$$a_0 = 1$$
$$a_1 = ra_0 = r.1 = r$$

Example:

Let r be a fixed nonzero constant, and suppose a sequence $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

$$a_k = \; r a_{k-1} \text{ for all integers } k \geq 1, \;\; a_0 = 1.$$

Use iteration to guess an explicit formula for this sequence.

$$a_0 = 1$$
$$a_1 = r a_0 = r.1 = r$$
$$a_2 = r a_1 = r.r = r^2$$

Example:

Let r be a fixed nonzero constant, and suppose a sequence $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

$$a_k = \ ra_{k-1} \text{ for all integers } k \geq 1, \ a_0 = 1.$$

Use iteration to guess an explicit formula for this sequence.

$$a_0 = 1$$
$$a_1 = ra_0 = r.1 = r$$
$$a_2 = ra_1 = r.r = r^2$$
$$a_3 = ra_2 = r.r^2 = r^3$$

Example:

Let r be a fixed nonzero constant, and suppose a sequence $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

$$a_k = \ r a_{k-1} \text{ for all integers } k \geq 1,\ \ a_0 = 1.$$

Use iteration to guess an explicit formula for this sequence.

$$a_0 = 1$$
$$a_1 = ra_0 = r.1 = r$$
$$a_2 = ra_1 = r.r = r^2$$
$$a_3 = ra_2 = r.r^2 = r^3$$
$$\vdots$$
$$a_n = r^n$$

Definition:

A sequence $a_0, a_1, a_2, \ldots$ is called a *geometric sequence* if, and only if, there is a constant r such that

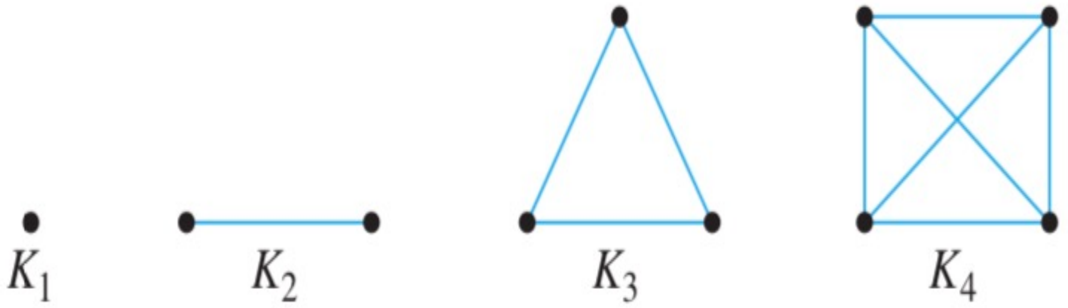
$$a_k = \ ra_{k-1} \text{ for all integers } k \geq 1.$$

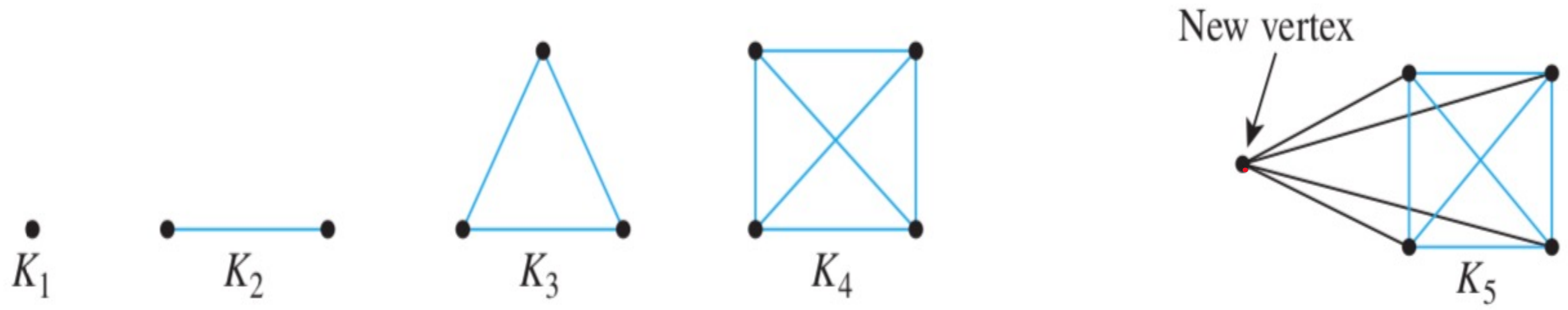
It follows that,

$$a_n = \ a_0 r^n \text{ for all integers } n \geq 0.$$

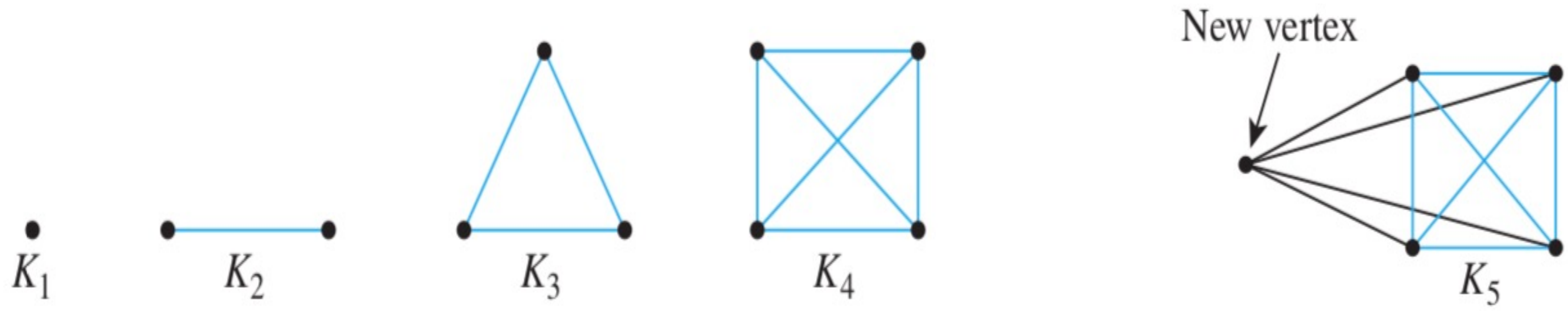
Example:

Let K_n be the picture obtained by drawing n dots (which we call vertices) and joining each pair of vertices by a line segment (which we call an edge). Then K_1, K_2, K_3, and K_4 are as follows:

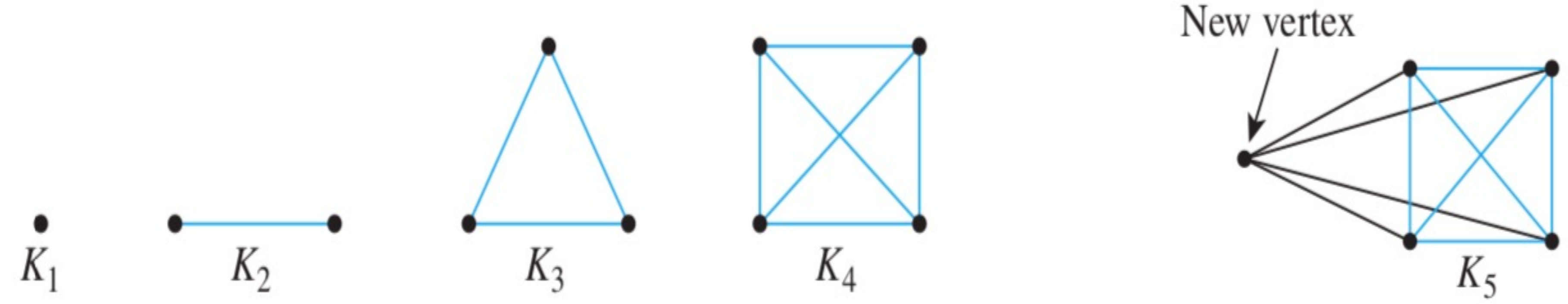




Observe that K_5 may be obtained from K_4 by adding one vertex and drawing edges between this new vertex and all the vertices of K_4 (the old vertices).



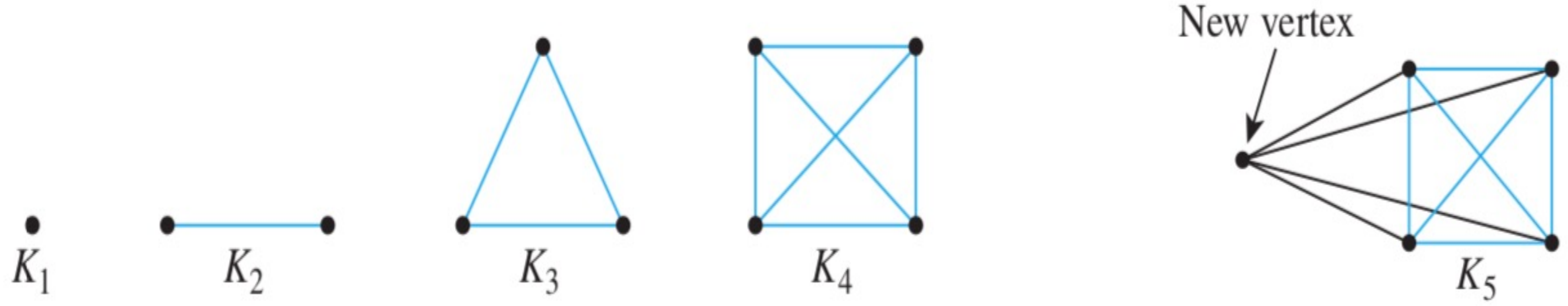
Observe that K_5 may be obtained from K_4 by adding one vertex and drawing edges between this new vertex and all the vertices of K_4 (the old vertices). The reason this procedure gives the correct result is that each pair of old vertices is already joined by an edge, and adding the new edges joins each pair of vertices consisting of an old one and the new one.



Observe that K_5 may be obtained from K_4 by adding one vertex and drawing edges between this new vertex and all the vertices of K_4 (the old vertices). The reason this procedure gives the correct result is that each pair of old vertices is already joined by an edge, and adding the new edges joins each pair of vertices consisting of an old one and the new one.

Thus

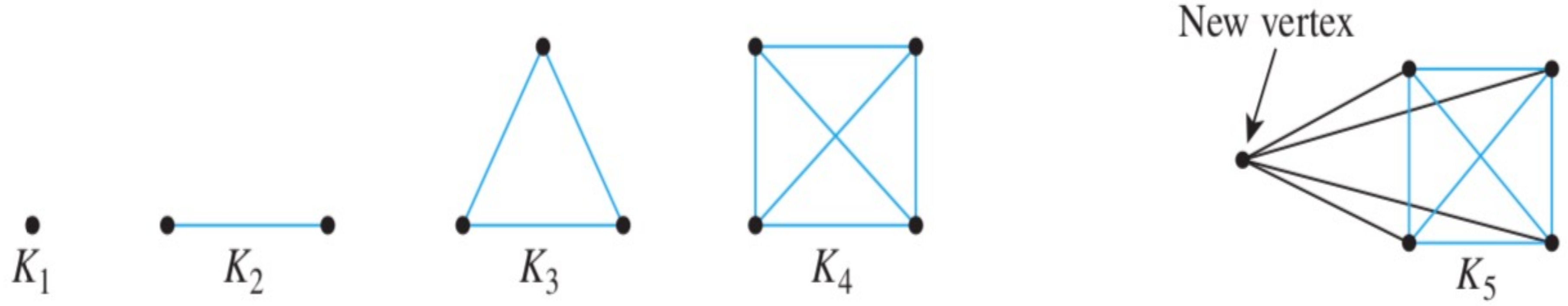
the number of edges of K_5 = 4 + the number of edges of K_4.



Observe that K_5 may be obtained from K_4 by adding one vertex and drawing edges between this new vertex and all the vertices of K_4 (the old vertices). The reason this procedure gives the correct result is that each pair of old vertices is already joined by an edge, and adding the new edges joins each pair of vertices consisting of an old one and the new one.
Thus

the number of edges of K_5 = 4 + the number of edges of K_4.

By the same reasoning, for all integers $k \geq 2$, the number of edges of K_k is $k - 1$ more than the number of edges of K_{k-1}.

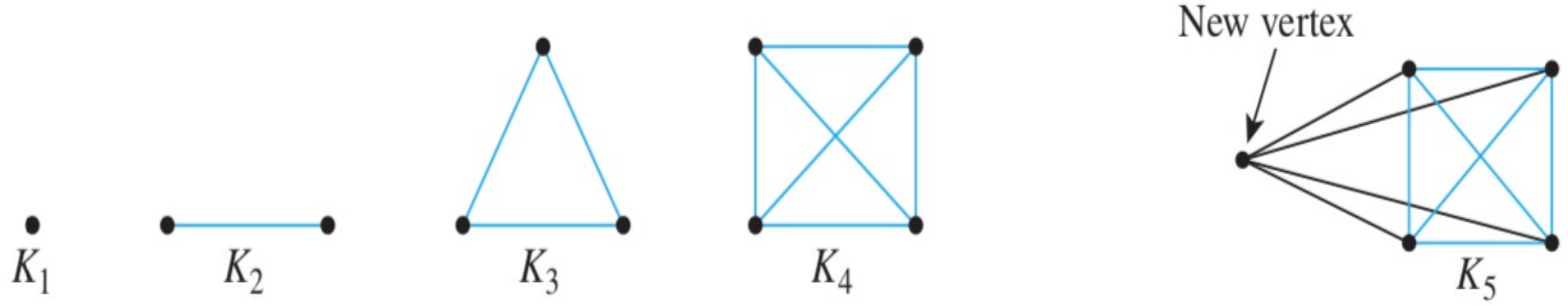


Observe that K_5 may be obtained from K_4 by adding one vertex and drawing edges between this new vertex and all the vertices of K_4 (the old vertices). The reason this procedure gives the correct result is that each pair of old vertices is already joined by an edge, and adding the new edges joins each pair of vertices consisting of an old one and the new one.

Thus

the number of edges of K_5 = 4 + the number of edges of K_4.

By the same reasoning, for all integers $k \geq 2$, the number of edges of K_k is $k - 1$ more than the number of edges of K_{k-1}. That is, if for each integer $n \geq 1$ and if $s_n =$ number of edges of K_n, then



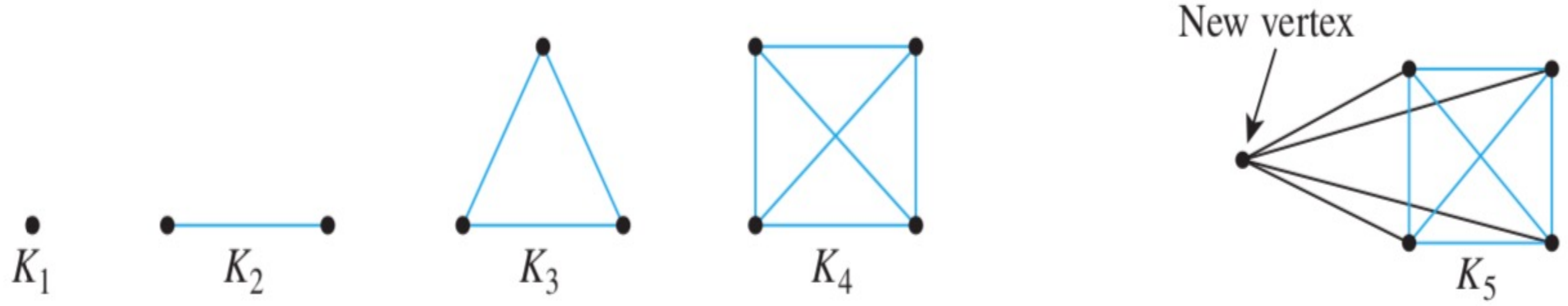
Observe that K_5 may be obtained from K_4 by adding one vertex and drawing edges between this new vertex and all the vertices of K_4 (the old vertices). The reason this procedure gives the correct result is that each pair of old vertices is already joined by an edge, and adding the new edges joins each pair of vertices consisting of an old one and the new one.

Thus

the number of edges of K_5 = 4 + the number of edges of K_4.

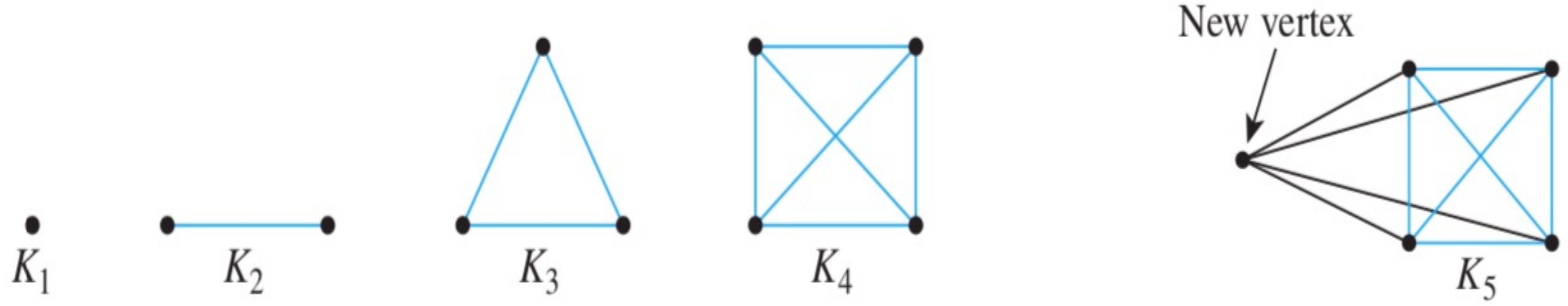
By the same reasoning, for all integers $k \ \geq \ 2$, the number of edges of K_k is $k - 1$ more than the number of edges of K_{k-1}. That is, if for each integer $n \geq 1$ and if $s_n =$ number of edges of K_n, then

$$s_n = s_{(n-1)} + (n - 1).$$



$$s_n = s_{(n-1)} + (n-1).$$

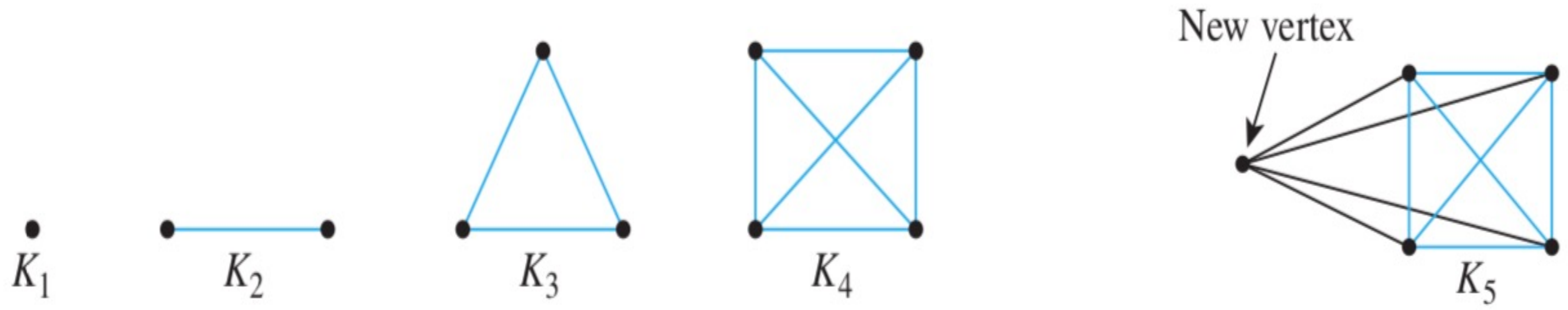
To find an explicit formula for this, we use iterations:



$$s_n = s_{(n-1)} + (n-1).$$

To find an explicit formula for this, we use iterations:

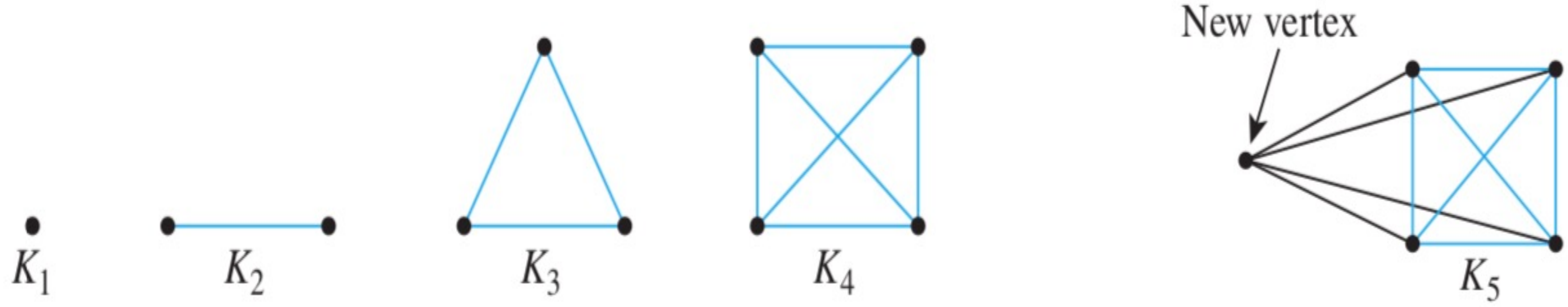
$$s_1 = 0$$



$$s_n = s_{(n-1)} + (n-1).$$

To find an explicit formula for this, we use iterations:

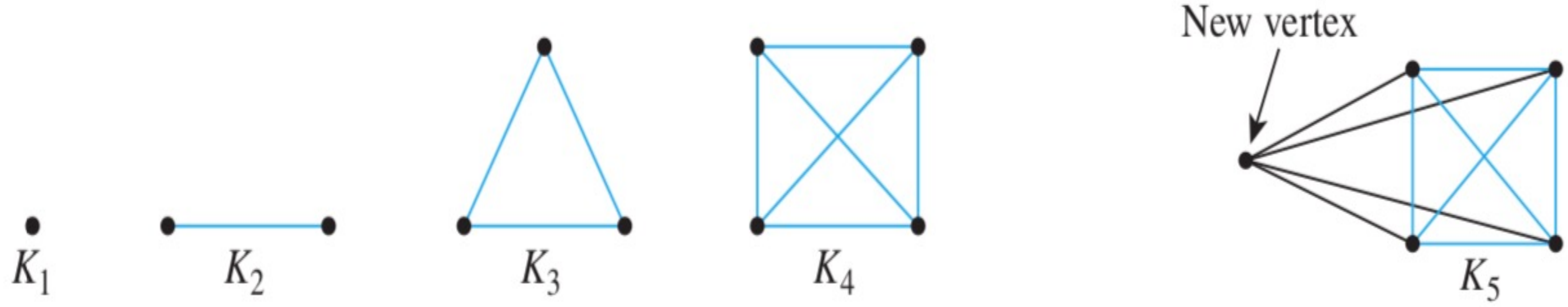
$$s_1 = 0$$
$$s_2 = s_1 + 1 = 0 + 1$$



$$s_n = s_{(n-1)} + (n-1).$$

To find an explicit formula for this, we use iterations:

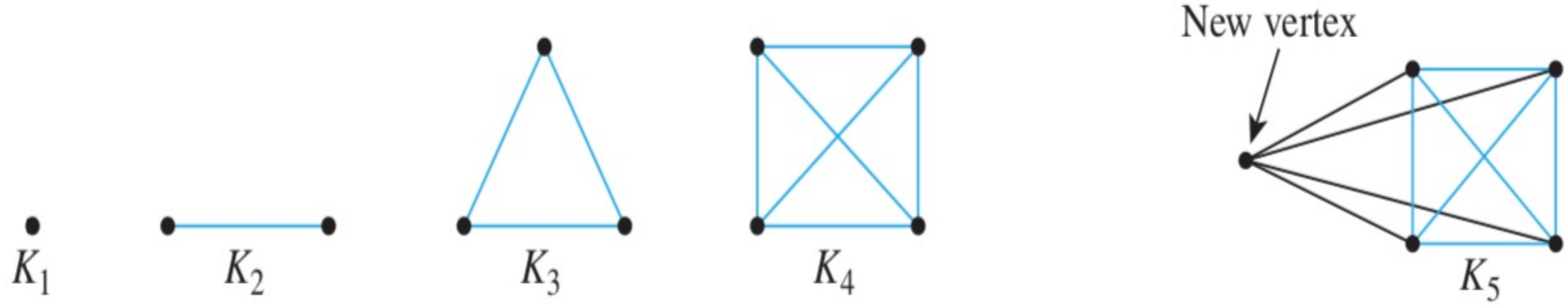
$$s_1 = 0$$
$$s_2 = s_1 + 1 = 0 + 1$$
$$s_3 = s_2 + 2 = 0 + 1 + 2$$



$$s_n = s_{(n-1)} + (n-1).$$

To find an explicit formula for this, we use iterations:

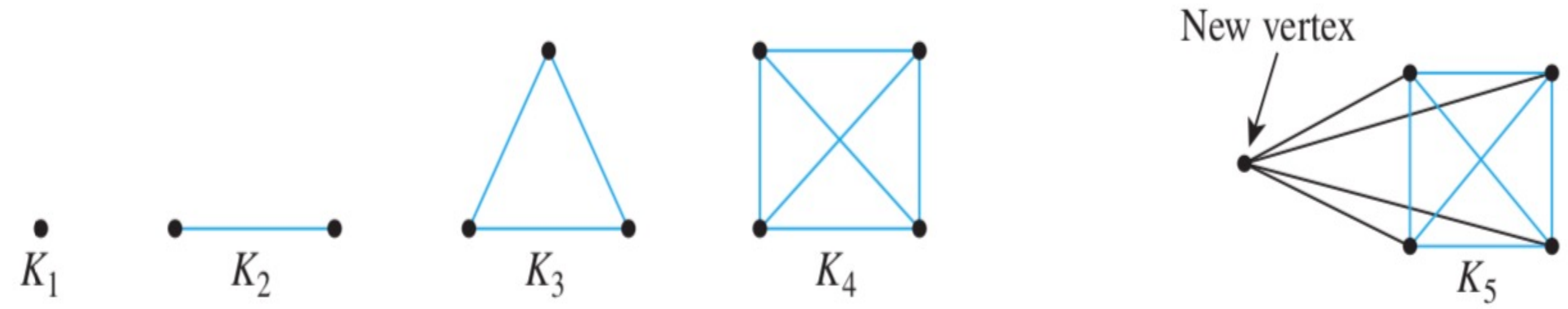
$$s_1 = 0$$
$$s_2 = s_1 + 1 = 0 + 1$$
$$s_3 = s_2 + 2 = 0 + 1 + 2$$
$$s_4 = s_3 + 3 = 0 + 1 + 2 + 3$$



$$s_n = s_{(n-1)} + (n-1).$$

To find an explicit formula for this, we use iterations:

$$s_1 = 0$$
$$s_2 = s_1 + 1 = 0 + 1$$
$$s_3 = s_2 + 2 = 0 + 1 + 2$$
$$s_4 = s_3 + 3 = 0 + 1 + 2 + 3$$
$$\vdots$$
$$s_n = 0 + 1 + 2 + \cdots + (n-1)$$



$$s_n = s_{(n-1)} + (n-1).$$

To find an explicit formula for this, we use iterations:

$$s_1 = 0$$
$$s_2 = s_1 + 1 = 0 + 1$$
$$s_3 = s_2 + 2 = 0 + 1 + 2$$
$$s_4 = s_3 + 3 = 0 + 1 + 2 + 3$$
$$\vdots$$
$$s_n = 0 + 1 + 2 + \cdots + (n-1) = \frac{(n-1)n}{2}.$$

Example:

The sequence $s_1, s_2, \ldots$ is defined

$$s_n = s_{(n-1)} + (n-1) \quad and \; s_1 = 0$$

For all int $n \geq 2$.

Show that

$$s_k = \frac{k(k-1)}{2},$$

For all int $k \geq 1$.

Example:

The sequence $s_1, s_2, \ldots$ is defined

$$s_n = s_{(n-1)} + (n-1) \quad and\ s_1 = 0$$

For all int $n \geq 2$.

Show that

$$s_k = \frac{k(k-1)}{2},$$

For all int $k \geq 1$.

$$P(n): s_n = \frac{n(n-1)}{2}$$

Step1: $P(1)$ true

Step2: Suppose P(k) is true, that is

$$s_k = \frac{k(k-1)}{2}$$

By the definition:

$$s_{k+1} = s_k + k = \frac{k(k-1)}{2} + k$$

$$= k\left(\frac{k-1}{2} + 1\right) = \frac{k(k+1)}{2}$$

P(k+1) is true.

Example:

Recall that the Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ satisfies the recurrence relation

$$m_k = 2m_{k-1} + 1$$

and has the initial condition $m_1 = 1$ for each integer $k \geq 2$.

Use iteration to guess an explicit formula for this sequence in a closed form.

Example: Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ satisfies the recurrence relation $m_k = 2m_{k-1} + 1$
and has the initial condition $m_1 = 1$ for each integer $k \geq 2$.

$$m_1 = 1$$

Example: Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ satisfies the recurrence relation $m_k = 2m_{k-1} + 1$
and has the initial condition $m_1 = 1$ for each integer $k \geq 2$.

$$m_1 = 1$$
$$m_2 = 2m_1 + 1 = 2(1) + 1 = 2 + 1$$

Example: Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ satisfies the recurrence relation $m_k = 2m_{k-1} + 1$
and has the initial condition $m_1 = 1$ for each integer $k \geq 2$.

$$m_1 = 1$$
$$m_2 = 2m_1 + 1 = 2(1) + 1 = 2 + 1$$
$$m_3 = 2m_2 + 1 = 2(2 + 1) + 1 = 2^2 + 2 + 1$$

Example: Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ satisfies the recurrence relation $m_k = 2m_{k-1} + 1$
and has the initial condition $m_1 = 1$ for each integer $k \geq 2$.

$$m_1 = 1$$
$$m_2 = 2m_1 + 1 = 2(1) + 1 = 2 + 1$$
$$m_3 = 2m_2 + 1 = 2(2 + 1) + 1 = 2^2 + 2 + 1$$
$$m_4 = 2m_3 + 1 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1$$

Example: Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ satisfies the recurrence relation $m_k = 2m_{k-1} + 1$
and has the initial condition $m_1 = 1$ for each integer $k \geq 2$.

$$m_1 = 1$$
$$m_2 = 2m_1 + 1 = 2(1) + 1 = 2 + 1$$
$$m_3 = 2m_2 + 1 = 2(2 + 1) + 1 = 2^2 + 2 + 1$$
$$m_4 = 2m_3 + 1 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1$$
$$m_5 = 2m_4 + 1 = 2^4 + 2^3 + 2^2 + 2 + 1$$

Example: Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ satisfies the recurrence relation $m_k = 2m_{k-1} + 1$
and has the initial condition $m_1 = 1$ for each integer $k \geq 2$.

$$m_1 = 1$$
$$m_2 = 2m_1 + 1 = 2(1) + 1 = 2 + 1$$
$$m_3 = 2m_2 + 1 = 2(2 + 1) + 1 = 2^2 + 2 + 1$$
$$m_4 = 2m_3 + 1 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1$$
$$m_5 = 2m_4 + 1 = 2^4 + 2^3 + 2^2 + 2 + 1$$

$$r^n + r^{n-1} + \cdots + r^2 + r + 1 = \frac{r^{n+1} - 1}{r - 1}$$

Example: Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ satisfies the recurrence relation $m_k = 2m_{k-1} + 1$
and has the initial condition $m_1 = 1$ for each integer $k \geq 2$.

$$m_1 = 1$$
$$m_2 = 2m_1 + 1 = 2(1) + 1 = 2 + 1$$
$$m_3 = 2m_2 + 1 = 2(2 + 1) + 1 = 2^2 + 2 + 1$$
$$m_4 = 2m_3 + 1 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1$$
$$m_5 = 2m_4 + 1 = 2^4 + 2^3 + 2^2 + 2 + 1$$
$$\vdots$$
$$m_k = 2^{k-1} + 2^{k-2} + \cdots + 2^2 + 2 + 1 = \frac{2^k - 1}{2 - 1} = 2^k - 1$$

Example: Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ satisfies the recurrence relation $m_k = 2m_{k-1} + 1$
and has the initial condition $m_1 = 1$ for each integer $k \geq 2$.

$$m_1 = 1$$
$$m_2 = 2m_1 + 1 = 2(1) + 1 = 2 + 1$$
$$m_3 = 2m_2 + 1 = 2(2+1) + 1 = 2^2 + 2 + 1$$
$$m_4 = 2m_3 + 1 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1$$
$$m_5 = 2m_4 + 1 = 2^4 + 2^3 + 2^2 + 2 + 1$$
$$\vdots$$
$$m_k = 2^{k-1} + 2^{k-2} + \cdots + 2^2 + 2 + 1 = \frac{2^k - 1}{2 - 1} = 2^k - 1$$

Therefore,

$$m_k = 2^k - 1 \text{ for all } k \geq 1$$

Proof of the correctness of the formula

The sequence $m_1, m_2, m_3, \ldots$ satisfies the recurrence relation

$$m_k = 2m_{k-1} + 1$$

and has the initial condition $m_1 = 1$ for each integer $k \geq 2$.

Show that

$$m_n = 2^n - 1 \text{ for all } n \geq 1$$

Proof of the correctness of the formula

The sequence $m_1, m_2, m_3, \ldots$
satisfies the recurrence relation

$$m_k = 2m_{k-1} + 1$$

and has the initial condition
$m_1 = 1$ for each integer $k \geq 2$.

Show that

$$m_n = 2^n - 1 \text{ for all } n \geq 1$$

$P(n)\colon\ m_n = 2^n - 1$

Example:

$$p_k = p_{k-1} + 2 \cdot 3^k, \qquad p_1 = 2$$

For all integers $k \geq 2$

Example:

$$p_k = p_{k-1} + 2 \cdot 3^k, \qquad p_1 = 2$$

$$p_2 = p_1 + 2.3^2 = 2 + 2.3^2$$

Example:

$$p_k = \ p_{k-1} + 2 \cdot 3^k, \qquad p_1 = \ 2$$

$$p_2 = p_1 + 2.3^2 = 2 + 2.3^2$$

$$p_3 = p_2 + 2.3^3 = 2 + 2.3^2 + 2.3^3$$

Example:

$$p_k = p_{k-1} + 2 \cdot 3^k, \qquad p_1 = 2$$
$$p_2 = p_1 + 2.3^2 = 2 + 2.3^2$$
$$p_3 = p_2 + 2.3^3 = 2 + 2.3^2 + 2.3^3$$
$$p_4 = p_3 + 2.3^4 = 2 + 2.3^2 + 2.3^3 + 2.3^4$$

Example:

$$p_k = p_{k-1} + 2 \cdot 3^k, \qquad p_1 = 2$$

$$p_2 = p_1 + 2.3^2 = 2 + 2.3^2$$

$$p_3 = p_2 + 2.3^3 = 2 + 2.3^2 + 2.3^3$$

$$p_4 = p_3 + 2.3^4 = 2 + 2.3^2 + 2.3^3 + 2.3^4$$

$$p_5 = p_4 + 2.3^5 = 2 + 2.3^2 + 2.3^3 + 2.3^4 + 2.3^5$$

Example:

$$p_k = p_{k-1} + 2 \cdot 3^k, \qquad p_1 = 2$$

$$p_2 = p_1 + 2.3^2 = 2 + 2.3^2$$

$$p_3 = p_2 + 2.3^3 = 2 + 2.3^2 + 2.3^3$$

$$p_4 = p_3 + 2.3^4 = 2 + 2.3^2 + 2.3^3 + 2.3^4$$

$$p_5 = p_4 + 2.3^5 = 2 + 2.3^2 + 2.3^3 + 2.3^4 + 2.3^5$$

$$\vdots$$

Example:

$$p_k = p_{k-1} + 2 \cdot 3^k, \qquad p_1 = 2$$

$$p_2 = p_1 + 2.3^2 = 2 + 2.3^2$$

$$p_3 = p_2 + 2.3^3 = 2 + 2.3^2 + 2.3^3$$

$$p_4 = p_3 + 2.3^4 = 2 + 2.3^2 + 2.3^3 + 2.3^4$$

$$p_5 = p_4 + 2.3^5 = 2 + 2.3^2 + 2.3^3 + 2.3^4 + 2.3^5$$

$$\vdots$$

$$p_n = 2 + 2.3^2 + 2.3^3 + 2.3^4 + 2.3^5 + \cdots + 2.3^n$$

Example:

$$p_k = p_{k-1} + 2 \cdot 3^k, \qquad p_1 = 2$$
$$p_2 = p_1 + 2.3^2 = 2 + 2.3^2$$
$$p_3 = p_2 + 2.3^3 = 2 + 2.3^2 + 2.3^3$$
$$p_4 = p_3 + 2.3^4 = 2 + 2.3^2 + 2.3^3 + 2.3^4$$
$$p_5 = p_4 + 2.3^5 = 2 + 2.3^2 + 2.3^3 + 2.3^4 + 2.3^5$$
$$\vdots$$
$$p_n = 2 + 2.3^2 + 2.3^3 + 2.3^4 + 2.3^5 + \cdots + 2.3^n$$
$$= 2(1 + 3^2 + 3^3 + \cdots + 3^n)$$

Example:

$$p_k = p_{k-1} + 2 \cdot 3^k, \qquad p_1 = 2$$

$$p_2 = p_1 + 2.3^2 = 2 + 2.3^2$$

$$p_3 = p_2 + 2.3^3 = 2 + 2.3^2 + 2.3^3$$

$$p_4 = p_3 + 2.3^4 = 2 + 2.3^2 + 2.3^3 + 2.3^4$$

$$p_5 = p_4 + 2.3^5 = 2 + 2.3^2 + 2.3^3 + 2.3^4 + 2.3^5$$

$$\vdots$$

$$p_n = 2 + 2.3^2 + 2.3^3 + 2.3^4 + 2.3^5 + \cdots + 2.3^n$$

$$= 2(1 + 3^2 + 3^3 + \cdots + 3^n + 3 - 3)$$

Example:

$$p_k = p_{k-1} + 2 \cdot 3^k, \qquad p_1 = 2$$

$$p_2 = p_1 + 2.3^2 = 2 + 2.3^2$$

$$p_3 = p_2 + 2.3^3 = 2 + 2.3^2 + 2.3^3$$

$$p_4 = p_3 + 2.3^4 = 2 + 2.3^2 + 2.3^3 + 2.3^4$$

$$p_5 = p_4 + 2.3^5 = 2 + 2.3^2 + 2.3^3 + 2.3^4 + 2.3^5$$

$$\vdots$$

$$p_n = 2 + 2.3^2 + 2.3^3 + 2.3^4 + 2.3^5 + \cdots + 2.3^n$$

$$= 2(1 + 3^2 + 3^3 + \cdots + 3^n + 3 - 3)$$

$$2\left(\frac{3^{n+1} - 1}{3 - 1} - 3\right)$$

Example:

$$p_k = p_{k-1} + 2 \cdot 3^k, \qquad p_1 = 2$$

$$p_2 = p_1 + 2.3^2 = 2 + 2.3^2$$

$$p_3 = p_2 + 2.3^3 = 2 + 2.3^2 + 2.3^3$$

$$p_4 = p_3 + 2.3^4 = 2 + 2.3^2 + 2.3^3 + 2.3^4$$

$$p_5 = p_4 + 2.3^5 = 2 + 2.3^2 + 2.3^3 + 2.3^4 + 2.3^5$$

$$\vdots$$

$$p_n = 2 + 2.3^2 + 2.3^3 + 2.3^4 + 2.3^5 + \cdots + 2.3^n$$

$$= 2(1 + 3^2 + 3^3 + \cdots + 3^n)$$

$$= 2\left(\frac{3^{n+1} - 1}{3 - 1} - 3\right) = 2\left(\frac{3^{n+1} - 1 - 6}{2}\right) = 3^{n+1} - 7.$$

Example: A worker is promised a bonus if he can increase his productivity by 2 units a day every day for a period of 30 days. If on day 0 he produces 170 units, how many units must he produce on day 30 to qualify for the bonus?

$$p_0 = 170,$$
$$p_1 = p_0 + 2 = 170 + 2,$$
$$p_2 = p_1 + 2 = 170 + 2 + 2$$
$$\vdots$$
$$p_n = 170 + 2n.$$

Therefore, at the 30th day, he must produce $p_{30} = 170 + 2(30) =$ 230units.

Example: Using Verification by Mathematical Induction to Find a Mistake

Let $c_0, c_1, c_2, \ldots$ be the sequence defined as follows:

$$c_k = 2c_{k-1} + k \text{ for all integers } k \geq 1, \text{ and } c_0 = 1.$$

Suppose your calculations suggest that $c_0, c_1, c_2, \ldots$ satisfies the following explicit formula:

$$c_n = 2^n + n \text{ for all integers } n \geq 0.$$

Is this formula correct?

Example: Using Verification by Mathematical Induction to Find a Mistake

Let $c_0, c_1, c_2, \ldots$ be the sequence defined as follows:

$$c_k = 2c_{k-1} + k \text{ for all integers } k \geq 1, \text{ and } c_0 = 1.$$

Suppose your calculations suggest that $c_0, c_1, c_2, \ldots$ satisfies the following explicit formula:

$$c_n = 2^n + n \text{ for all integers } n \geq 0.$$

Is this formula correct?

$$c_k = 2^k + k$$

$$c_{k+1} = 2c_k + k = 2\left(2^k + k\right) + k$$

$$= 2^{k+1} + 2k + k$$

$$= 2^{k+1} + 3k \neq R.H.S\ of\ P(k+1)$$