Introduction to Logic

Class code:b7xpd4s

Logic

The definition of logic is a science that studies the principles of correct reasoning.

Statement

- A ***statement (or proposition)*** is a declarative sentence that is either true or false but not both.
- The ***truth value*** of a statements refers to the truth or falsity of the statement.

Examples:

1. π is an irrational number.
2. 2+3=5.
3. $\forall n \in \mathbb{N}, n^2 + n + 41$ is a positive integer.
4. Get up and do your work.
5. The number x is an integer.

Examples 1, 2 and 3 are statements however 4 and 5 are not statements as the truth values of these statements cannot be determined.

Examples:

1. π is a rational number.
2. 2+3=6.
3. $\forall n \in \mathbb{N}, n^2 + n - 41$ is a positive integer.
4. Don’t get up and do your work.
5. The number x is not an integer.

Compound Statements

- In logic, the letters $p, q, r, \ldots$ denote ***propositional variables (statement variables)***; that is
- variables that can be replaced by statements.
- Statements or propositional variables can be combined by **logical connectives** (negation $\sim$, conjunction $\wedge$, disjunction $\vee$, implication $\rightarrow$ etc.) to obtain ***compound statements***.

Compound Statements

- **Negation of $\boldsymbol{p}$**: For a given statement p, '$\sim p$' is read 'not p' or 'it is not the case that p'.
- **Conjunction of $\boldsymbol{p}$ and $\boldsymbol{q}$**: '$p \wedge q$'read as 'p and q' is called conjunction. Note that 'but' is also translated as a conjunction.
- **Disjunction of $\boldsymbol{p}$ and $\boldsymbol{q}$**: '$p \vee q$'read as 'p or q' is called disjunction.

Example:

p: It is snowing $\qquad$ q: I am cold

Express the following statement forms into sentences:

1. $p \wedge q$
   It is snowing and I am cold
2. $p \vee q$
   Either it is snowing, or I am cold
3. $p \wedge \sim q$
   It is snowing but I am not cold
4. $\sim p \wedge \sim q$
   It is neither snowing nor I am cold

Example:

Let p and q be the propositions

p : You drive over 65 miles per hour. q : You get a speeding ticket.

Write these propositions using p and q and logical connectives (including negations).

a) You do not drive over 65 miles per hour

$$\sim p$$

b) You drive over 65 miles per hour, but you do not get a speeding ticket.

$$p \land \sim q$$

c) You get a speeding ticket, but you do not drive over 65 miles per hour.

$$\mathrm{q} \land \sim p$$

Example:

1. $x \leq a$ means

$$x < a \text{ or } x = a$$

2. $a \leq x \leq b$ means

$$a \leq x \text{ and } x \leq b$$

3. $\sim(x \leq a)$ is $x > a$

Statement form

A ***statement form* (or *propositional form)*** is an expression made up of statement variables (such as p, q, and r) and logical connectives (such as $\sim, \wedge$, and $\vee$) that becomes a statement when actual statements are substituted for the component statement variables.

Truth Table

The ***truth table*** for a given statement form displays the truth values that correspond to all possible combination of the truth values for its component statement variables.

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If p is a statement variable, the ***negation*** of p is "not p" or "It is not the case that p" and is denoted $\sim p$. It has opposite truth value from p: if p is true, $\sim p$ is false; if p is false, $\sim p$ is true.

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p	$\sim p$
T	F
F	T

If p and q are statement variables, the ***conjunction*** of p and q is "p and q," denoted $p \ \wedge \ q$. It is true when, and only when, both p and q are true. If either p or q is false, or if both are false, $p \ \wedge \ q$ is false.

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p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

If p and q are statement variables, the ***disjunction*** of p and q is "p or q," denoted $p \vee q$. It is true when either p is true, or q is true, or both p and q are true; it is false only when both p and q are false.

If p and q are statement variables, the ***disjunction*** of p and q is "p or q," denoted $p \lor q$. It is true when either p is true, or q is true, or both p and q are true; it is false only when both p and q are false.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \lor q}$
T	T	T
T	F	T
F	T	T
F	F	F

Exercise: Construct a truth table for the statement form $(p \wedge q) \vee \sim r$.

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

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p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T	T	F	
T	T	F	T	T	
T	F	T	F	F	
T	F	F	F	T	
F	T	T	F	F	
F	T	F	F	T	
F	F	T	F	F	
F	F	F	F	T	

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p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

Example: (Exclusive Or)

Denoted by $p \oplus q$ or $p\ XOR\ q$ means ‘p or q but not both’ or ‘p or q and not both’.

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$$(p \vee q) \wedge \sim(p \wedge q)$$

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Denoted by $p \oplus q$ or $p\ XOR\ q$ means ‘p or q but not both’ or ‘p or q and not both’.

$$(p \vee q) \wedge \sim(p \wedge q)$$

Find the truth table of the above expression.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \oplus q}$
T	T	
T	F	
F	T	
F	F	

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Denoted by $p \oplus q$ or $p\ XOR\ q$ means 'p or q but not both' or 'p or q and not both'.

$$(p \vee q) \wedge \sim(p \wedge q)$$

Find the truth table of the above expression.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

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Denoted by $p \oplus q$ or $p\ XOR\ q$ means 'p or q but not both' or 'p or q and not both'.

$$(p \vee q) \wedge \sim(p \wedge q)$$

Find the truth table of the above expression.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \oplus q}$	$\boldsymbol{(p \vee q) \wedge \sim(p \wedge q)}$
T	T	F	
T	F	T	
F	T	T	
F	F	F	

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Denoted by $p \oplus q$ or $p\ XOR\ q$ means 'p or q but not both' or 'p or q and not both'.

$$(p \vee q) \wedge \sim(p \wedge q)$$

Find the truth table of the above expression.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \oplus q}$	$\boldsymbol{(p \vee q) \wedge \sim(p \wedge q)}$	$\boldsymbol{(\sim p \wedge q) \vee (p \wedge \sim q)}$
T	T	F		
T	F	T		
F	T	T		
F	F	F		

Logical Equivalence

Two statement forms are called ***logically equivalent*** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted by writing $P \equiv Q$.

Two statements are called ***logically equivalent*** if, and only if, they have logically equivalent forms when identical component statement variables are used to replace identical component statements.

Example: Double Negative Property:

$$\sim(\sim p) \equiv p$$

Construct a truth table to show that the negation of the negation of a statement is logically equivalent to the statement, annotating the table with a sentence of explanation.

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p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

Example: Showing Nonequivalence

Show that the statement forms

$$\sim (p \ \wedge \ q) \text{ and } \sim p \wedge \sim q$$

are not logically equivalent.

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p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

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Show that the statement forms

$$\sim (p \ \wedge \ q) \text{ and } \sim p \wedge \sim q$$

are not logically equivalent.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \wedge \sim q$	$\sim p \vee \sim q$
T	T	F	F	T	F	F	F
T	F	F	T	F	T	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	T	T	T

De Morgan's Law:

The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

and

The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

$$\sim (p \wedge q) \equiv \sim p \vee \sim q.$$

Example: Applying De Morgan's Laws

Write negations for each of the following statements:

1. Ana is 5 feet tall and she weighs at least 50 kg.

   Ana is not 5 feet tall or she weighs less than 50 kg.

2. Norma is doing her homework and Karen is practicing her piano lessons.

   Either Norma is not doing her homework or Karen is not practicing her piano lessons.

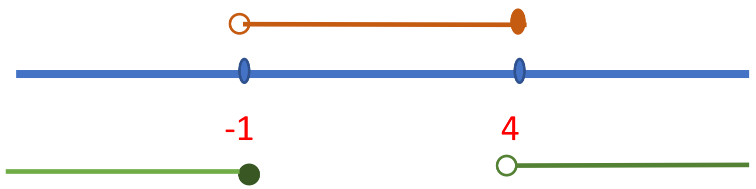
Example:

Use De Morgan's laws to write the negation of $-1 < x \leq 4$.

Solution

The given statement is equivalent to

$$-1 < x \text{ and } x \leq 4.$$



By De Morgan's laws, the negation is

$$\sim(-1 < x \text{ and } x \leq 4) \equiv \sim(-1 < x) \text{ or } \sim(x \leq 4).$$

$$-1 \geq x \text{ or } x > 4.$$

Tautologies and Contradictions

A ***tautology*** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a tautological statement.

A ***contradiction*** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradiction is a contradictory statement.

Example:

Show that the statement form $p \vee \sim p$ is a tautology and that the statement form $p \wedge \sim p$ is a contradiction.

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F		
F	T		

Example:

Show that the statement form $p \lor \sim p$ is a tautology and that the statement form $p \land \sim p$ is a contradiction.

p	$\sim p$	$p \lor \sim p$	$p \land \sim p$
T	F	T	F
F	T	T	F

Example:

If t is a tautology and c is a contradiction, show that

$p \wedge t \equiv p$ and $p \wedge c \equiv c$.

Example:

If t is a tautology and c is a contradiction, show that

$$p \wedge \ t \equiv \ p \text{ and } p \wedge c \equiv c.$$

p	$\boldsymbol{t}$	$p \wedge \boldsymbol{t}$
T	T	T
F	T	F

p	$\boldsymbol{c}$	$p \wedge \boldsymbol{c}$
T	F	F
F	F	F

Summary of Logical Equivalences

Commutative laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative laws:	$(p \wedge q) \wedge r \ \equiv\ p \wedge (q \wedge r)$	$(p \vee q) \vee r \ \equiv\ p \vee (q \vee r)$
Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws:	$p \wedge t \equiv\ p$	$p \vee c \equiv\ p$
Negation laws:	$p\ \vee \sim p\ \equiv\ t$	$p\ \wedge \sim p\ \equiv\ c$
Double negative law:	$\sim (\sim p)\ \equiv\ p$	
idempotent laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal bound laws	$p\ \vee\ t\ \equiv\ t$	$p \wedge c \equiv c$
DeMorgan's laws:	$\sim (p \wedge q) \equiv \sim p \vee \sim q$	$\sim (p \vee q) \equiv \sim p \wedge \sim q$
Absorption laws:	$p \vee (p \wedge q) \equiv\ p$	$p \wedge (p \vee q) \equiv\ p$
Negations of t and c:	$\sim t \equiv c$	$\sim c \equiv t$

Example:

Simplify statements form

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$$

L.H.S:

$$\sim(\sim p \wedge q) \wedge (p \vee q)$$

$$\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q) \qquad \text{De Morgan's Law}$$

$$\equiv (p \vee \sim q) \wedge (p \vee q) \qquad \text{Double negation}$$

$$\equiv p \vee (\sim q \wedge q) \qquad \text{Distributive Law}$$

$$\equiv p \vee (\boldsymbol{c}) \qquad \text{Negation Law}$$

$$\equiv p \qquad \text{Identity Law}$$

Conditional Statements

If p and q are statement variables, the *conditional* of q by p is "If p then q" or "p implies q" and is denoted $p \to q$. We call p the *hypothesis* (or *antecedent*) of the conditional and q the *conclusion* (or *consequent*).

- We don't want a true statement to lead us into believing something that is false.

- So, the **only** way an implication ($p \to q$, p implies q) is false is when $\boldsymbol{p}$ **is not implying** $\boldsymbol{q}$**.**

- It is false when $\boldsymbol{p}$ **is true and** $\boldsymbol{q}$ **is false**; otherwise it is true.

Conditional Statements

It is false when p is true and q is false; otherwise it is true.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example:

'If 2+3=5 then 2+4=6'

is a true statement, however

'If 2+3=5 then 2+4=7'

is false.

On the other hand, whenever the hypothesis is false, we consider the conditional statement to be true irrespective of the truth value of the conclusion.

So,

'If 2+3=7 then 2+4=6' and 'If 2+3=7 then 2+4=7'

both are considered to be true statements.

Example:

"If Juan has a smartphone, then 2 + 3 = 5"

is true from the definition of a conditional statement, because its conclusion is true (The truth value of the hypothesis does not matter then).

The conditional statement

"If Juan has a smartphone, then 2 + 3 = 6"

is true if Juan does not have a smartphone, even though 2 + 3 = 6 is false.

The notation $p \rightarrow q$ indicates that $\rightarrow$ is a connective, like $\wedge$ and $\vee$, that can be used to join statements to create new statements.

In expressions that include $\rightarrow$ as well as $\wedge, \vee$ and $\sim$ the ***order of operations*** is that $\rightarrow$ is performed last. That is $\sim$ is performed first, then $\wedge$ and $\vee$ and finally $\rightarrow$.

Exercise:

Using truth table, show that

$$p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r).$$

Representation of ‘If-Then’ as ‘or’

Another definition of a conditional statement $p \to q$ is that it is true only when either p is false or q is true.

$$p \to q \equiv \sim p \vee q.$$

Exercise: Prove the above equivalence.

Exercise:

Using truth table, show that

$$p \vee q \to r \equiv (p \to r) \wedge (q \to r).$$

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Exercise: Prove the above equivalence.

The Negation of a Conditional Statement

The negation of "if p then q" is logically equivalent to "p and not q." This can be restated symbolically as follows:

$$\sim (p \to q) \equiv p \wedge \sim q$$

Because,

$$\begin{aligned} \sim (p \to q) &\equiv \sim (\sim p \vee q) & \\ &\equiv \sim (\sim p) \wedge (\sim q) & \text{by De Morgan's Law} \\ &\equiv p \wedge \sim q & \text{by Double negation} \end{aligned}$$

Example:

a) If the decimal expansion of *r* is terminating, then *r* is rational.
b) If *n* is divisible by 6, then *n* is divisible by 2 and *n* is divisible by 3.

$$\sim (p \to q) \equiv p \wedge \sim q$$

Answer:

a) The decimal expansion of *r* is terminating but *r* is not rational.
b) *n* is divisible by 6 but either *n* is not divisible by 2 or *n* is not divisible by 3.

Exercise:

Suppose that p and q are statements so that $p \rightarrow q$ is false. Find the truth values of each of the following:

$a)\ \sim p \rightarrow q$

$b)\ p \vee q$

$c)\ q \rightarrow p$

The Contrapositive of a Conditional Statement

Definition:
The contrapositive of a conditional statement of the form "If p then q" is If $\sim q$ then $\sim p$.
That is the contrapositive of $p \rightarrow q$ is

$$\sim q \rightarrow \sim p.$$

$$\begin{aligned} \sim q \rightarrow \sim p &\equiv \sim(\sim q) \vee \sim p \\ &\equiv q \vee \sim p \\ &\equiv \sim p \vee q \\ &\equiv p \rightarrow q. \end{aligned}$$

This tells us that a conditional statement is logically equivalent to its contrapositive.

The Contrapositive of a Conditional Statement

Definition:
The contrapositive of a conditional statement $p \to q$ is

$$\sim q \to \sim p.$$

Example:

If 4,686 is divisible by 6, then 4,686 is divisible by 3.

Contrapositive:

If 4,686 is not divisible by 3, then it is not divisible by 6.

The Converse and Inverse of a Conditional Statement

The fact that a conditional statement and its contrapositive are logically equivalent is very important and has wide application. Two other variants of a conditional statement are not logically equivalent to the statement.

Suppose a conditional statement of the form "If p then q" is given.

The **converse** is "If q then p".

The **inverse** is "If $\sim p$ then $\sim q$".

Symbolically,

The converse of $p \rightarrow q$ is $q \rightarrow p$, and the inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

Question:

- Is the converse logically equivalent to the original conditional statement?
- Is the inverse logically equivalent to the original conditional statement?
- What is the relation between converse and inverse of a conditional statement?

Example:

What is the contrapositive, the converse, and the inverse of the conditional statement?

"The home team wins whenever it is raining".

Conditional Statement: If it is raining then the home team wins.

Contrapositive: If the home team does not win, then it is not raining.

Converse: If the home team wins, then it is raining.

Inverse: If it is not raining, then the home team does not win.

Biconditional

Given statement variables p and q, the biconditional of p and q is

“p if, and only if, q”

and is denoted $p \leftrightarrow q$.

It is true if both p and q have the same truth values and is false if p and q have opposite truth values. The words if and only if are sometimes abbreviated iff.

The biconditional has the following truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Exercise:

Construct the truth table for the following statement form:

$$(p \to q) \wedge (q \to p)$$

Therefore,

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Necessary and Sufficient Conditions

The phrases necessary condition and sufficient condition, as used in formal English, correspond exactly to their definitions in logic.

Definition
If r and s are statements:

- r is a **sufficient** condition for s means "if r then s."

$$r \rightarrow s$$

- r is a **necessary** condition for s means "if not r then not s."

$$\sim \mathrm{r} \rightarrow \sim s \equiv s \rightarrow r$$

- r is a **necessary and sufficient** condition for s means "r if, and only if, s."

$$\mathrm{r} \leftrightarrow s$$

Example: Rewrite the following in "if-then" form

1. Pia's birth on U.S soil is a sufficient condition for her to be a U.S. citizen.

   If Pia was born on U.S. soil, then she is a U.S. citizen.

2. John is at least 18 years old is *necessary* for the condition John to be eligible to vote.

   If John is not at least 18 years old, then John is not eligible to vote

   If John is eligible to vote, then he is at least 18 years old.

Other expressions equivalent to a conditional statement

The conditional connective $p \to q$ represents the following English constructs:

- if p then q
- p only if q
- q follows from p
- p is a sufficient condition for q
- q is a necessary condition for p
- q unless $\sim p$
- q if p
- p implies q
- q whenever p
- a sufficient condition for q is p
- a necessary condition for p is q

Other expressions equivalent to a biconditional statement

The biconditional connective $p \leftrightarrow q$ represents the following English constructs:

- p if and only if q (often written p iff q)
- p and q imply each other
- p is a necessary and sufficient condition for q
- p and q are equivalent

$$p \equiv q \text{ also means } p \leftrightarrow q$$