Predicates and Quantifiers

Predicate

A ***predicate*** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The ***domain*** of a predicate variable is the set of all values that may be substituted in place of the variable.

Example: Let $P(n)$ be the predicate "$n^2 \leq 30$."

Write $P(2)$: $2^2 \leq 30$, $P(-2)$: $(-2)^2 \leq 30$, $P(7)$, and $P(-7)$, and indicate which of these statements are true and which are false?

Definition:

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}.$$

Example: Let A(x) denote the statement “Computer x is under attack by an intruder.” Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of A(CS1), A(CS2), and A(MATH1)?

Solution: We obtain the statement A(CS1) by setting x = CS1 in the statement “Computer x is under attack by an intruder.” Because CS1 is not on the list of computers currently under attack, we conclude that A(CS1) is false. Similarly, because CS2 and MATH1 are on the list of computers under attack, we know that A(CS2) and A(MATH1) are true.

Exercise:

Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Example:

Finding the Truth Set of a Predicate

Let $Q(n)$ be the predicate "n is a factor of 8." Find the truth set of $Q(n)$ if

a. the domain of n is the set $\mathbb{Z}^{+}$ of all positive integers
b. the domain of n is the set $\mathbb{Z}$ of all integers.

Solution

a. The truth set is $\{1,2,4,8\}$ because these are exactly the positive integers that divide 8 evenly.
b. The truth set is $\{1,2,4,8,-1,-2,-4,-8\}$ because the negative integers $-1,-2,-4$, and -8 also divide into 8 without leaving a remainder.

Quantifiers

Another way to obtain statements from predicates is to add ***quantifiers***. Quantifiers are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true. There are mainly two types of quantifiers:

1. Universal quantifier
2. Existential quantifier

The Universal Quantifier: ∀

Let $Q(x)$ be a predicate and D the domain of x. A universal statement is a statement of the form

$$\text{“}\forall x \in D, Q(x).\text{”}$$

- It is defined to be true if, and only if, $Q(x)$ is true for every x in D.
- It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D.
- A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Example:

“All human beings are mortal” is to write

$\forall$ human beings x, x is mortal.

Example: Truth and Falsity of Universal Statements

Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Show that this statement is true.

Solution

Check that "$x^2 \geq x$" is true for each individual x in D.
$1^2 \geq 1,\ 2^2 \geq 2,\ 3^2 \geq 3,\ 4^2 \geq 4,\ 5^2 \geq 5.$
Hence "$\forall x \in D, x^2 \geq x$" is true. This technique used to show the truth of the universal statement is called the ***method of exhaustion***.

Example: Truth and Falsity of Universal Statements

Consider the statement

$$\forall x \in \mathbb{R}, x^2 \geq x.$$

Find a counterexample to show that this statement is false.

Solution

Counterexample: Take $x = 1/2$. Then x is in $\mathbb{R}$ (since $1/2$ is a real number) and

$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \ngeq \frac{1}{2}$.Hence "$\forall x \in \mathbb{R}, x^2 \geq x$" is false.

The Existential Quantifier: ∃

The symbol ∃ denotes "there exists" and is called the *existential quantifier*. For example, the sentence "There is a student in Math 140" can be written as,

∃ a person p such that p is a student in Math 140,

or, more formally,

$\exists p \in P$ such that p is a student in Math 140,

Definition:

Let Q(x) be a predicate and D the domain of x. An existential statement is a statement of the form

$$\text{“}\exists x \in D \text{ such that } Q(x)\text{.”}$$

- It is defined to be true if, and only if, Q(x) is true for at least one x in D.
- It is false if, and only if, Q(x) is false for all x in D.

Example: Truth and Falsity of Existential Statements

1. Consider the statement
$$\exists m \in \mathbb{Z}^+ \text{ such that } m^2 = m.$$
Show that this statement is true.

2. Let E = {5, 6, 7, 8} and consider the statement
$$\exists m \in E \text{ such that } m^2 = m.$$
Show that this statement is false.

Solution

1. Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m. Hence "$\exists m \in \mathbb{Z}$ such that $m^2 = m$" is true.
2. Note that $m^2 = m$ is not true for any integers m from 5 through 8:
$5^2 = 25 \neq 5, \qquad 6^2 = 36 \neq 6, \qquad 7^2 = 49 \neq 7, \qquad 8^2 = 64$
Thus "∃m ∈ E such that $m^2 = m$" is false.

Comparison between universal and existential statements:

Universal Statement

Existential Statement

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- $\forall x \in D, Q(x)$.

Existential Statement

- $\exists x \in D$ such that $Q(x)$.

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Existential Statement

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- To show it is true, we must find x in D where $Q(x)$ is true.

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Universal Statement

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- It is true if, and only if, $Q(x)$ is true for every x in D.
- It is false if, and only if, $Q(x)$ is false for at least one x in D.
- To show it is true, we must check the truth of $Q(x)$ for every x in D.
- To show that it is false, we must give a counter example.

Existential Statement

- $\exists x \in D$ such that $Q(x)$.
- It is true if, and only if, $Q(x)$ is true for at least one x in D.
- It is false if, and only if, $Q(x)$ is false for all x in D.
- To show it is true, we must find x in D where $Q(x)$ is true.
- To show that it is false, we must check the falsity of $Q(x)$ for every x in D.

Formal vs. Informal Language

Example:

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol ∀ or ∃.

1. $\forall x \in \boldsymbol{R}, x^2 \geq 0$.

   Every real number has a nonnegative square.

2. $\forall x \in \boldsymbol{R}, x^2 \neq -1$.

   All real numbers have squares that do not equal -1
   Or: No real numbers have squares equal to -1.
   (The words ***none are*** or ***no ... are*** are equivalent to the words ***all are not***.)

3. $\exists m \in \boldsymbol{Z}^{+} such\ that\ m^2 = m$.

   There is a positive integer whose square is equal to itself.

Example: Translating from Informal to Formal Language

Rewrite each of the following statements formally. Use quantifiers and variables.

1. All triangles have three sides.
2. No dogs have wings.
3. Some programs are structured.

Solution

1. $\forall$ triangles t, t has three sides.
   Or: $\forall t \in T$, t has three sides (where T is the set of all triangles).
2. $\forall$ dogs d, d does not have wings.
   Or: $\forall d \in D$, d does not have wings (where D is the set of all dogs).
3. $\exists$ a program p such that p is structured.
   Or: $\exists p \in P$ such that p is structured (where P is the set of all programs).

Universal Conditional Statements

A reasonable argument can be made that the most important form of statement in mathematics is the universal conditional statement:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Familiarity with statements of this form is essential if you are to learn to speak mathematics.

Example:

Rewrite the following statement informally, without quantifiers or variables.

$$\forall x \in \mathbb{R}, \text{if } x > 2 \text{ then } x^2 > 4.$$

Solution:

If a real number is greater than 2 then its square is greater than 4.

Or: Whenever a real number is greater than 2, its square is greater than 4.

Or: The square of any real number greater than 2 is greater than 4.

Or: The squares of all real numbers greater than 2 are greater than 4.

Example:

Rewrite each of the following statements in the form

$$\forall\_\_\_\_\_\_\_\_\ ,\ \text{if}\_\_\_\_\_\_\_\_\_\ \text{then}\_\_\_\_\_\_\_\_\_\_\_\_\ .$$

1. If a real number is an integer, then it is a rational number.
2. All bytes have eight bits.
3. No fire trucks are green.

Solution

1. ∀ real number x, if x is an integer, then x is a rational number.
Or: $\forall\ x \in \mathbb{R}$, if $x \in \mathbb{Z}$ then $x \in \mathbb{Q}$.

Equivalent Forms of Universal and Existential Statements

Observe that the two statements

$$\text{“}\forall \text{ real number } x, \text{ if } x \text{ is an integer then } x \text{ is rational”}$$

and

$$\text{“}\forall \text{ integer } x, x \text{ is rational”}$$

mean the same thing because the set of integers is a subset of the set of real numbers. Both have informal translations “All integers are rational.”

In fact, a statement of the form

$$\forall\, x \in U, \text{ if } P(x) \text{ then } Q(x)$$

can always be rewritten in the form

$$\forall\, x \in D, Q(x)$$

by narrowing U to be the subset D consisting of all values of the variable x that make $P(x)$ true.

Conversely, a statement of the form

$$\forall\, x \in D, Q(x)$$

can be rewritten as

$$\forall\, x, \text{ if } x \text{ is in } D \text{ then } Q(x).$$

Example:

All squares are rectangles.

can also be written as

$\forall\ x$, if x is a square then x is a rectangle.

Or

$\forall$ square x, x is a rectangle.

Equivalent Forms for Existential Statements

"There is an integer that is both prime and even."

Equivalent Forms for Existential Statements

"There is an integer that is both prime and even."

Can be written formally as

$\exists$ *n* such that Prime(*n*) $\wedge$ Even(*n*).

Or

Two answers: $\exists$ a prime number *n* such that Even(*n*).

$\exists$ an even number *n* such that Prime(*n*).

Negations of Quantified Statements

Definition:

The negation of a statement of the form

$$\forall x \in D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \; in \; D \text{ such that} \sim Q(x).$$

Symbolically,

$$\sim (\forall x \;\in\; D, Q(x)) \equiv \exists x \in D \; such \; that \;\; \sim Q(x).$$

Example:

Consider the statement “All mathematicians wear glasses.”

The negation is

“There is at least one mathematician who does not wear glasses.”

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$$\sim (\exists x\ \in\ D\ such\ that\ Q(x))\ \equiv\ \forall x\ \in\ D, \sim Q(x).$$

Example:

Now consider the statement “Some snowflakes are the same.”

The negation will be

“No snowflakes are the same,” or “All snowflakes are different.”

Exercise: Write formal negations for the following statements:

∀ primes p, p is odd.

∃ prime p such that p is even.

∃ a triangle T such that the sum of the angles of T equals 200◦.

∀ triangle T, the sum of the angles of T is not 200 ◦.

All computer programs are finite.

∃ computer program p such that p is infinite.

Some computer hackers are over 40.

∀ computer hacker h such that h is 40 or younger.

The number 1,357 is divisible by some integer between 1 and 37.

∀ integers x between 1 and 37, x does not divide 1,357.

Example:

Which of the following is a negation for "All dogs are loyal"? More than one answer may be correct.

a) All dogs are disloyal.
b) No dogs are loyal.
c) Some dogs are disloyal.
d) Some dogs are loyal.
e) There is a disloyal animal that is not a dog.
f) There is a dog that is disloyal.
g) No animals that are not dogs are loyal.
h) Some animals that are not dogs are loyal.

Negations of Universal Conditional Statements

By definition of the negation of a for all statement,

$$\begin{aligned} \sim(\forall x, P(x) \to Q(x)) &\equiv \exists x \; such\ that \sim(P(x) \to Q(x)). \\ &\equiv \exists x \; such\ that\ P(x) \wedge \sim Q(x). \end{aligned}$$

Example: Negating Universal Conditional Statements

Write a formal negation for statement (a) and an informal negation for statement (b).

a. ∀ people p, if p is blonde then p has blue eyes.
b. If a computer program has more than 100,000 lines, then it contains a bug.

Solution

a. ∃ a person p such that p is blonde, and p does not have blue eyes.
b. There is at least one computer program that has more than 100,000 lines and does not contain a bug.

Variants of Universal Conditional Statements

Recall that a conditional statement has a contrapositive, a converse, and an inverse. The definitions of these terms can be extended to universal conditional statements.

Definition:

Consider a statement of the form: $\forall x \in D, if\ P(x)\ then\ Q(x)$.

1. Its contrapositive is the statement:
   $$\forall x \in D, if \sim Q(x) \text{ then } \sim P(x).$$
2. Its converse is the statement:
   $$\forall x \in D, if\ Q(x) \text{ then } P(x).$$
3. Its inverse is the statement:
   $$\forall x \in D, if \sim P(x) \text{ then } \sim Q(x).$$

Example:

Write a formal and an informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

The formal version of this statement is

$$\forall x \in \mathbb{R}, if\ x > 2\ then\ x^2 > 4.$$

Contrapositive:

$$\forall x \in \mathbb{R}, if\ x^2 \leq 4 \text{ then } x \leq 2.$$

Or: If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.

Converse:

$$\forall x \in \mathbb{R}, if x^2 > 4\ then\ x > 2.$$

Or: If the square of a real number is greater than 4, then the number is greater than 2.

Inverse:

$$\forall x \in \mathbb{R}, if x \leq 2\ then\ x^2 \leq 4.$$

Or: If a real number is less than or equal to 2, then the square of the number is less than or equal to 4.

Definition:

1. "$\forall x, r(x)$ is a sufficient condition for $s(x)$" means
   "$\forall x, if\ r(x)\ then\ s(x).$"

2. "$\forall x, r(x)$ is a necessary condition for $s(x)$" means
   "$\forall x, if\ \sim r(x)\ then\ \sim s(x)$" or, equivalently,
   "$\forall x, if\ s(x)\ then\ r(x)$."

3. "$\forall x, r(x)$ only if $s(x)$" means
   "$\forall x, if \sim s(x)\ then \sim r(x)$" or, equivalently,
   "$\forall x, if\ r(x)\ then\ s(x)$."

Example:

Rewrite the following statements as quantified conditional statements. Do not use the word necessary or sufficient.

Squareness is a sufficient condition for rectangularity.

Solution

A formal version of the statement is

$\forall$x, if x is a square, then x is a rectangle.

Or, in informal language:

If a figure is a square, then it is a rectangle.

Example:

Rewrite the following statements as quantified conditional statements. Do not use the word necessary or sufficient.

Being at least 35 years old is a necessary condition for being President of the United States.

Solution

Using formal language, you could write the answer as

∀ people x, if x is younger than 35, then x cannot be President of the United States.

Or, by the equivalence between a statement and its contrapositive:

∀ people x, if x is President of the United States, then x is at least 35 years old.

Example:

Rewrite the following as a universal conditional statement:

A product of two numbers is 0 only if one of the numbers is 0.

Solution: Using informal language, you could write the answer as

If neither of two numbers is 0, then the product of the numbers is not 0.

Or, by the equivalence between a statement and its contrapositive,

If a product of two numbers is 0, then one of the numbers is 0.

Statements with Multiple Quantifiers

Interpreting Statements with Two Different Quantifiers

If you want to establish the truth of a statement of the form

$$\forall x \in D, \exists y \in E \text{ such that } P(x, y),$$

- your challenge is to allow someone else to pick whatever element x in D they wish
- and then you must find an element y in E that “works” for that particular x.

For every x there is a y to make $\mathrm{P}(x, y)$ true!

Statements with Multiple Quantifiers

Interpreting Statements with Two Different Quantifiers

If you want to establish the truth of a statement of the form

$$\exists x \in D \text{ such that } \forall y \in E, P(x, y),$$

- your job is to find one particular x in D that will "work"
- no matter what y in E anyone might choose to challenge you with.

There is an x that works for every y to make $\mathrm{P}(x, y)$ true!

Statements with Multiple Quantifiers

$\forall x \in D, \exists y \in E$ such that $P(x, y)$,

- For any x from the set D
- We must find a y from the set E
- Such that $P(x, y)$ is true.

$\exists x \in D$ such that $\forall y \in E, P(x, y)$,

- There must be at least one x in D such that
- every y from E makes
- $P(x, y)$ is true.

Example:

The reciprocal of a real number a is a real number b such that ab = 1. The following two statements are true. Rewrite them formally using quantifiers and variables:

a. Every nonzero real number has a reciprocal.
b. There is a real number with no reciprocal.

Solution

a. ∀ nonzero real numbers u, ∃ a real number v such that $uv = 1$.

b. ∃ a real number c such that ∀ real numbers d,$cd \neq 1$.

Example:

Recall that every integer is a real number and that real numbers are of three types: positive, negative, and zero (zero being neither positive nor negative). Consider the statement

“There is a smallest positive integer.”

Write this statement formally using both symbols $\exists$ and $\forall$.

Example:

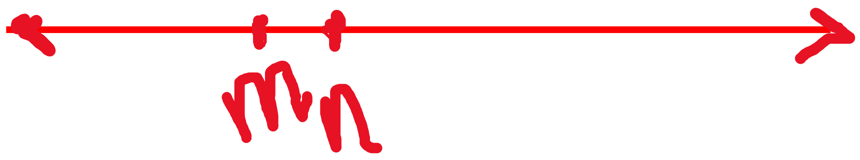
Recall that every integer is a real number and that real numbers are of three types: positive, negative, and zero (zero being neither positive nor negative). Consider the statement

"There is a smallest positive integer."

Write this statement formally using both symbols $\exists$ and $\forall$.

Solution

To say that there is a smallest positive integer means that there is a positive integer m with the property that no matter what positive integer n a person might pick, m will be less than or equal to n:



Example:

Recall that every integer is a real number and that real numbers are of three types: positive, negative, and zero (zero being neither positive nor negative). Consider the statement

"There is a smallest positive integer."

Write this statement formally using both symbols ∃ and ∀.

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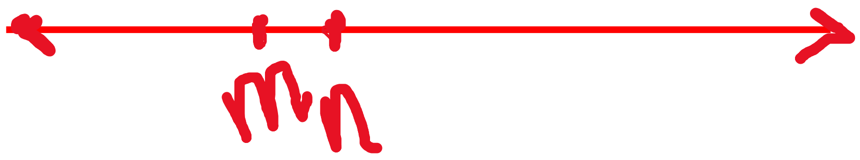

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Write this statement formally using both symbols $\exists$ and $\forall$.

Solution

To say that there is a smallest positive integer means that there is a positive integer m with the property that no matter what positive integer n a person might pick, m will be less than or equal to n:

$$\exists \text{ a positive integer } m \text{ such that } \forall \text{ positive integers } n,\ m \leq n.$$

Example:

"There is no smallest positive real number."

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"There is no smallest positive real number."

Write this statement formally using both symbols ∀ and ∃.

Solution

This means no matter how small a positive real number we take, there is always a smaller positive real number.

Example:

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Write this statement formally using both symbols ∀ and ∃.
Solution

This means no matter how small a positive real number we take, there is always a smaller positive real number.

∀ positive real numbers x,∃ a positive real number y such that $y < x$.

Statement	When True?	When False?
$\forall x \forall y P\,(x, y)$ $\forall y \forall x P\,(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P\,(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P\,(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P\,(x, y)$ $\exists y \exists x P\,(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P\,(x, y)$ is false for every pair x, y.

Exercise:

Express the statement as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

a. Everyone trusts someone.

b. Someone does not trust anyone.

c. Someone trusts everyone.

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Express the statement as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

a. Everyone trusts someone.

$$\forall\ people\ x, \exists\ person\ y\ such\ that\ x\ trusts\ y.$$

b. Someone does not trust anyone.

$$\exists\ person\ x, such\ that \forall\ people\ y, x\ do\ not\ trust\ y.$$

c. Someone trusts everyone.

$$\exists\ person\ x, such\ that \forall\ people\ y, x\ trusts\ y.$$

Exercise:

Express the statement as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

a. Everyone has exactly one best friend. [Hint: To say that x has exactly one best friend means that there is a person y who is the best friend of x, and furthermore, that for every person z, if person z is not person y, then z is not the best friend of x.]

b. Someone has no best friend.

Example:

A college cafeteria line has four stations: salads, main courses, desserts, and beverages.

The salad station: green salad or fruit salad;

main course station: spaghetti or fish;

dessert station: pie or cake; and

beverage station: milk, soda, or coffee.

Three students, Uta, Tim, and Yuen, go through the line and make the following choices:

Uta: green salad, spaghetti, pie, milk

Tim: fruit salad, fish, pie, cake, milk, coffee

Yuen: spaghetti, fish, pie, soda

Write each of following statements informally and find its truth value.

1. $\exists$ an item I such that $\forall$ students S, S chose I.
2. $\exists$a student S such that $\forall$ items I, S chose I.
3. $\exists$ a student S such that $\forall$ stations Z, $\exists$ an item I in Z such that S chose I.
4. $\forall$ students S and $\forall$ stations Z, $\exists$ an item I in Z such that S chose I.

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Similarly

$$\sim (\exists x \in D \text{ such that } \forall y \in E, P(x, y))$$

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$$\equiv\ \exists x \in D \text{ such that } \forall y \in E, \sim P(x, y).$$

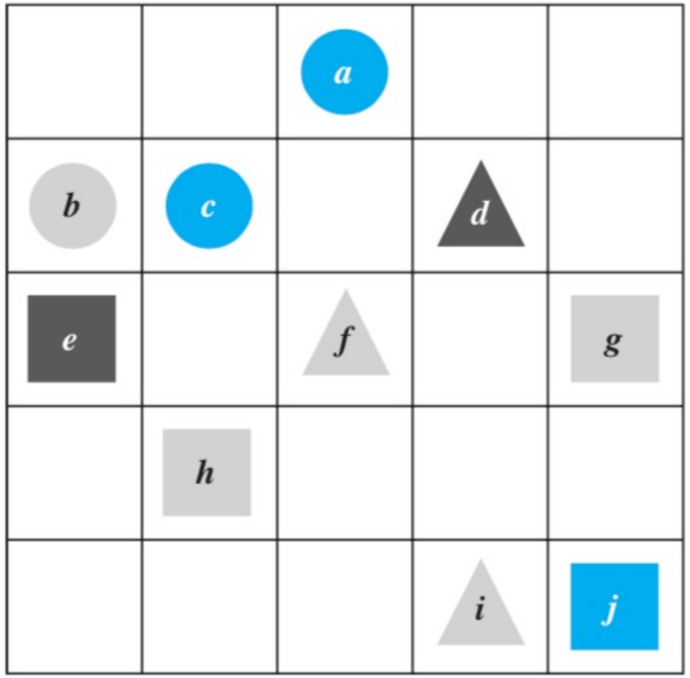
Similarly

$$\sim (\exists x \in D \text{ such that } \forall y \in E, P(x, y))$$
$$\equiv\ \forall x \in D, \sim (\forall y \in E, P(x, y))$$
$$\equiv\ \forall x \in D, \exists y \in E \text{ such that} \sim P(x, y).$$

Example: Refer to the following Tarski world.

Write a negation for each of the following statements, and determine which is true, the given statement or its negation.

1. For all squares x, there is a circle y such that x and y have the same color.
2. There is a triangle x such that for all squares y, x is to the right of y.



Solution

1. $\exists$ a square x such that $\forall$ circles y, x and y do not have the same color.
   The negation is true. Square e is black and no circle is black, so there is a square that does not have the same color as any circle.
2. $\forall$ triangles x, $\exists$ a square y such that x is not to the right of y.
   The negation is true because no matter what triangle is chosen, it is not to the right of square g (or square j).

Recall

$$\text{“}\forall x \in D, P(x)\text{”}$$

can be written as

$$\text{“}\forall x\ (x\ in\ D \rightarrow P(x)),\text{”}$$

Recall

$$\text{“}\forall x \in D, P(x)\text{”}$$

can be written as

$$\text{“}\forall x\ (x\ in\ D \rightarrow P(x)),\text{”}$$

and

$$\text{“}\exists\ x \in D\ such\ that\ P(x)\text{”}$$

can be written as

$$\text{“}\exists\ x\ (x\ in\ D \wedge P(x)).\text{”}$$

Example:

Let Triangle(x), Circle(x), and Square(x) mean

"x is a triangle," "x is a circle," and "x is a square";

let Blue(x), Gray(x), and Black(x) mean

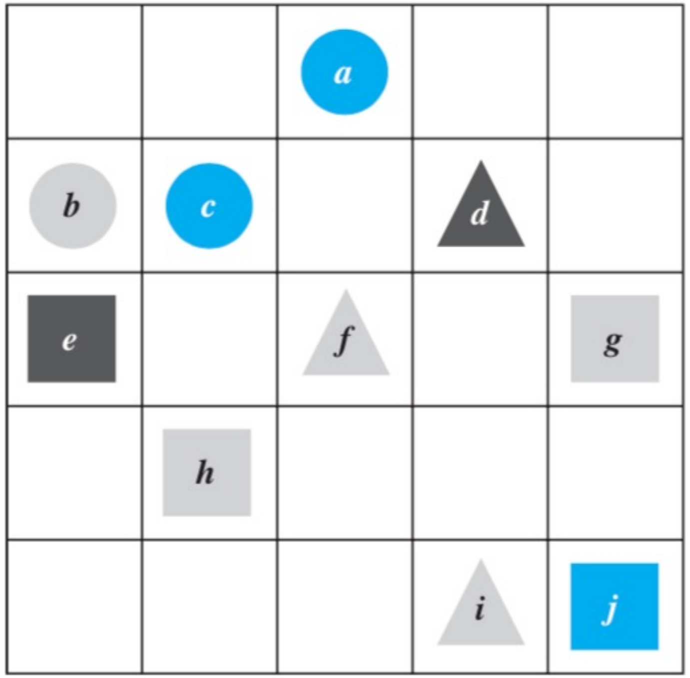
"x is blue," "x is gray," and "x is black";

let RightOf(x, y), Above(x, y), and SameColorAs(x, y) mean

"x is to the right of y," "x is above y," and "x has the same color as y";

and use the notation $x = y$ to denote the predicate "x is equal to y".

Let the common domain D of all variables be the set of all the objects in the Tarski world. Use formal, logical notation to write each of the following statements, and write a formal negation for each statement.



Let

Triangle(x), "x is a triangle"

Circle(x), "x is a circle"

Square(x), "x is a square"

Blue(x), "x is blue"

Gray(x), "x is gray"

Black(x) "x is black"

RightOf(x, y), "x is to the right of y"

Above(x, y), "x is above y"

SameColorAs(x, y), "x has the same color as y"

$x = y$, "x is equal to y".

For all circles x, x is above f.

$$\forall\ circle\ x, x\ is\ above\ f$$

Statement: $\forall x$(Circle(x) $\rightarrow$ Above(x, f)).

Negation: $\sim$ ($\forall x$(Circle(x) $\rightarrow$ Above(x, f)))

$\equiv$ $\exists x \sim$ (Circle(x) $\rightarrow$ Above(x, f)).

$\equiv$ $\exists x$(Circle(x) $\wedge \sim$Above(x, f))

Let
Triangle(x), "x is a triangle"
Circle(x), "x is a circle"
Square(x), "x is a square"
Blue(x), "x is blue"
Gray(x), "x is gray"
Black(x) "x is black"
RightOf(x, y), "x is to the right of y"
Above(x, y), "x is above y"
SameColorAs(x, y), "x has the same color as y"
$x = y$, "x is equal to y".

There is a square x such that x is black.

Statement: $\exists x(\text{Square}(x) \wedge \text{Black}(x))$.

Negation: $\sim (\exists x(\text{Square}(x) \wedge \text{Black}(x)))$
$\equiv \forall x \sim (\text{Square}(x) \wedge \text{Black}(x))$
$\equiv \forall x(\sim\text{Square}(x) \vee \sim\text{Black}(x))$

Let

Triangle(x), "x is a triangle"

Circle(x), "x is a circle"

Square(x), "x is a square"

Blue(x), "x is blue"

Gray(x), "x is gray"

Black(x) "x is black"

RightOf(x, y), "x is to the right of y"

Above(x, y), "x is above y"

SameColorAs(x, y), "x has the same color as y"

$x = y$, "x is equal to y".

For all circles x, there is a square y such that x and y have the same color.

Statement:

$\forall x(\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$.

Negation:

$\sim (\forall x(\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

$\equiv \exists x \sim(\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$

$\equiv \exists x(\text{Circle}(x) \wedge \sim (\exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

$\equiv \exists x(\text{Circle}(x) \wedge \forall y(\sim(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

$\equiv \exists x(\text{Circle}(x) \wedge \forall y(\sim\text{Square}(y) \vee \sim\text{SameColor}(x, y)))$

Let

Triangle(x), "x is a triangle"

Circle(x), "x is a circle"

Square(x), "x is a square"

Blue(x), "x is blue"

Gray(x), "x is gray"

Black(x) "x is black"

RightOf(x, y), "x is to the right of y"

Above(x, y), "x is above y"

SameColorAs(x, y), "x has the same color as y"

$x = y$, "x is equal to y".

There is a square x such that for all triangles y, x is to right of y.

Statement:

$\exists x(\text{Square}(x) \wedge \forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y)))$.

Negation:

$\sim (\exists x(\text{Square}(x) \wedge \forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))))$
$\equiv \forall x \sim(\text{Square}(x) \wedge \forall y(\text{Triangle}(x) \rightarrow \text{RightOf}(x, y)))$
$\equiv \forall x(\sim\text{Square}(x) \vee\sim (\forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))))$
$\equiv \forall x(\sim\text{Square}(x) \vee \exists y(\sim(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))))$
$\equiv \forall x(\sim\text{Square}(x) \vee \exists y(\text{Triangle}(y) \wedge \sim\text{RightOf}(x, y)))$

Prolog

The programming language Prolog (short for programming in logic) was developed in France in the 1970s by A. Colmerauer and P. Roussel to help programmers working in the field of artificial intelligence.

A simple Prolog program consists of a set of statements describing some situation together with questions about the situation. Built into the language are search and inference techniques needed to answer the questions by deriving the answers from the given statements. This frees the programmer from the necessity of having to write separate programs to answer each type of question.

Prolog programs include a set of declarations consisting of two types of statements,

Prolog facts and *Prolog rules*.

Prolog facts define predicates by specifying the elements that satisfy these predicates. Prolog rules are used to define new predicates using those already defined by Prolog facts.

Example: Consider a Prolog program given facts telling it the instructor of each class and in which classes students are enrolled. The program uses these facts to answer queries concerning the professors who teach particular students. The predicates

instructor(p, c): professor p is the instructor of course c

and

enrolled(s, c): student s is enrolled in course c.

The Prolog facts in such a program might include:

- instructor(chan, math273)
- instructor(patel, ee222)
- instructor(grossman, cs301)
- enrolled(kevin,math273)
- enrolled(juana,ee222)
- enrolled(juana,cs301)
- enrolled(kiko,math273)
- enrolled(kiko,cs301)

instructor(p, c): professor p is the instructor of course c

and

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The Prolog facts in such a program might include:

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- enrolled(juana,cs301)
- enrolled(kiko,math273)
- enrolled(kiko,cs301)

A new predicate teaches(p, s), representing that professor p teaches student s, can be defined using the Prolog rule

teaches(P, S): instructor(P, C), enrolled(S, C)

instructor(p, c): professor p is the instructor of course c

and

enrolled(s, c): student s is enrolled in course c.

The Prolog facts in such a program might include:

- instructor(chan, math273)
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- enrolled(juana,cs301)
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- enrolled(kiko,cs301)

Note that a comma is used to represent a conjunction of predicates in Prolog. Similarly, a semicolon is used to represent a disjunction of predicates.

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- enrolled(juana,ee222)
- enrolled(juana,cs301)
- enrolled(kiko,math273)
- enrolled(kiko,cs301)

Prolog rule

teaches(P, S):

instructor(P, C), enrolled(S, C)

Prolog answers queries using the facts and rules it is given.

For example,

using the facts and rules listed, the query

?enrolled(kevin,math273)

produces the response

yes

because the fact enrolled(kevin, math273) was provided as input.

instructor(p, c): professor p is the instructor of course c

and

enrolled(s, c): student s is enrolled in course c.

Prolog facts:

- instructor(chan, math273)
- instructor(patel, ee222)
- instructor(grossman, cs301)
- enrolled(kevin,math273)
- enrolled(juana,ee222)
- enrolled(juana,cs301)
- enrolled(kiko,math273)
- enrolled(kiko,cs301)

Prolog rule

teaches(P, S):

instructor(P, C), enrolled(S, C)

The query

?enrolled(X ,math273)

produces the response

kevin
kiko

instructor(p, c): professor p is the instructor of course c

and

enrolled(s, c): student s is enrolled in course c.

Prolog facts:

- instructor(chan, math273)
- instructor(patel, ee222)
- instructor(grossman, cs301)
- enrolled(kevin,math273)
- enrolled(juana,ee222)
- enrolled(juana,cs301)
- enrolled(kiko,math273)
- enrolled(kiko,cs301)

Prolog rule

teaches(P, S):

instructor(P, C), enrolled(S, C)

?teaches(X,juana)

This query returns

patel

grossman

instructor(p, c): professor p is the instructor of course c

and

enrolled(s, c): student s is enrolled in course c.

Prolog facts:

- instructor(chan, math273)
- instructor(patel, ee222)
- instructor(grossman, cs301)
- enrolled(kevin,math273)
- enrolled(juana,ee222)
- enrolled(juana,cs301)
- enrolled(kiko,math273)
- enrolled(kiko,cs301)

Prolog rule

teaches(P, S):

instructor(P, C), enrolled(S, C)

Define a new rule for classfellow(u,v)?

Map Coloring

A famous problem in mathematics concerns coloring adjacent planar regions. It is required that whatever colors are used, no two adjacent regions may not have the same color. Two regions are considered adjacent provided they share some boundary line segment. Consider the following map.

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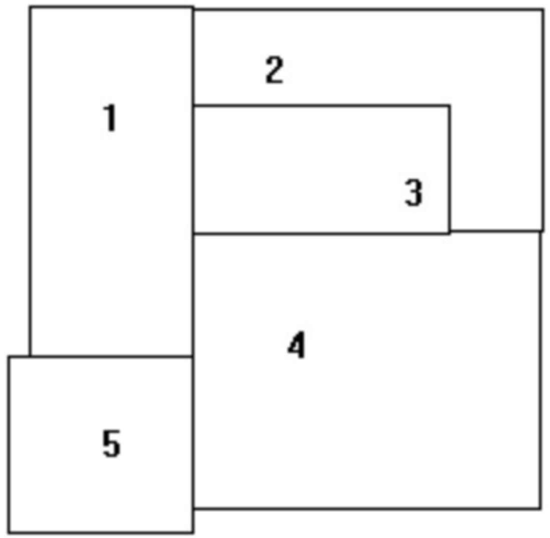

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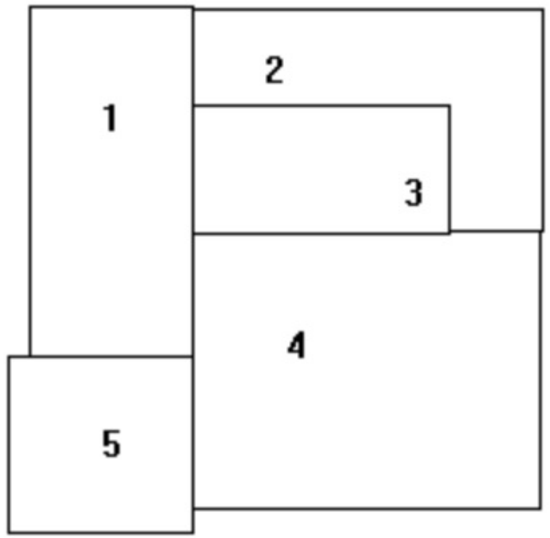
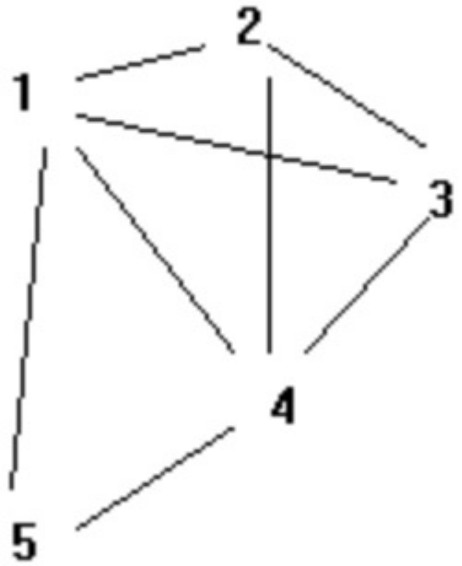
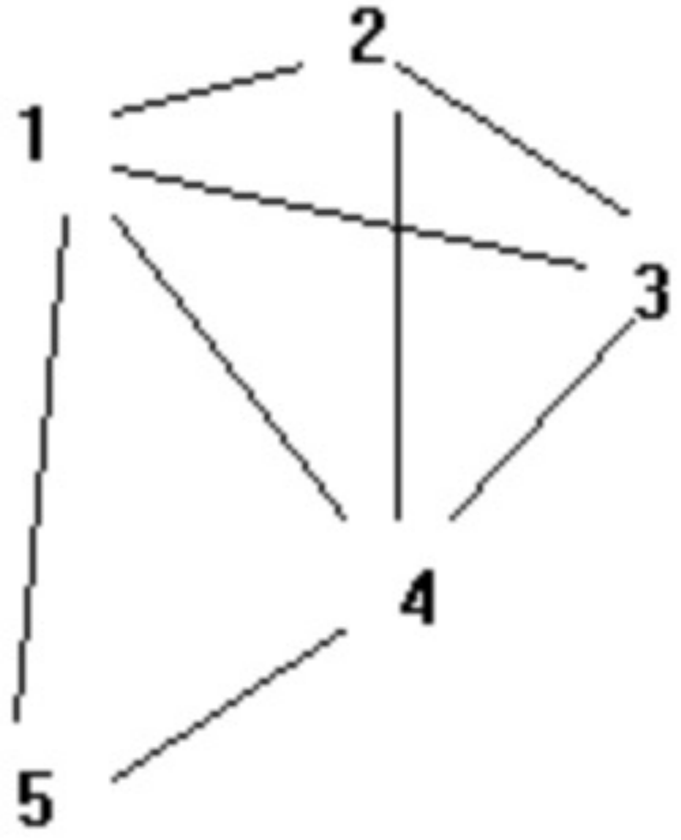






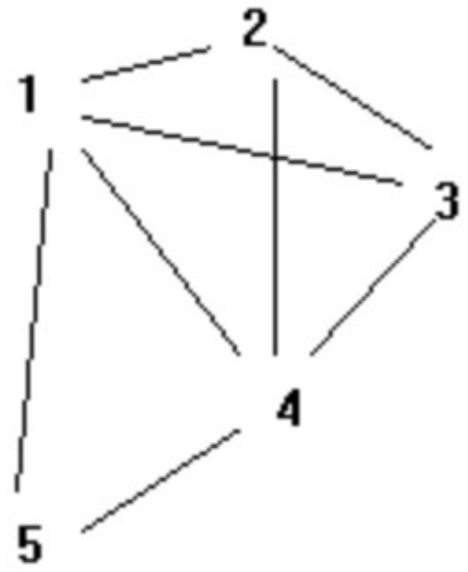
A Prolog representation for the adjacency information could be represented by the following *unit* clauses, or facts.

```
adjacent(1,2).
adjacent(1,3).
adjacent(1,4).
adjacent(1,5).
adjacent(2,3).
adjacent(2,4).
adjacent(3,4).
adjacent(4,5).
```

```
adjacent(2,1).
adjacent(3,1).
adjacent(4,1).
adjacent(5,1).
adjacent(3,2).
adjacent(4,2).
adjacent(4,3).
adjacent(5,4).
```

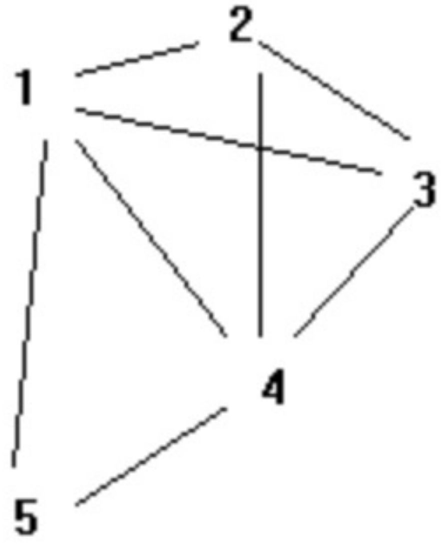
If these clauses were loaded into Prolog, we could observe the following behavior for some goals.

```
?- adjacent(2,3).
yes
?- adjacent(5,3).
no
?- adjacent(3,R).
R = 1 ;
R = 2 ;
R = 4 ;
no
```



One could declare colorings for the regions in Prolog also using unit clauses.

```
color(1,red,a).    color(1,red,b).
color(2,blue,a).   color(2,blue,b).
color(3,green,a).  color(3,green,b).
color(4,yellow,a). color(4,blue,b).
color(5,blue,a).   color(5,green,b).
```



One could declare colorings for the regions in Prolog also using unit clauses.

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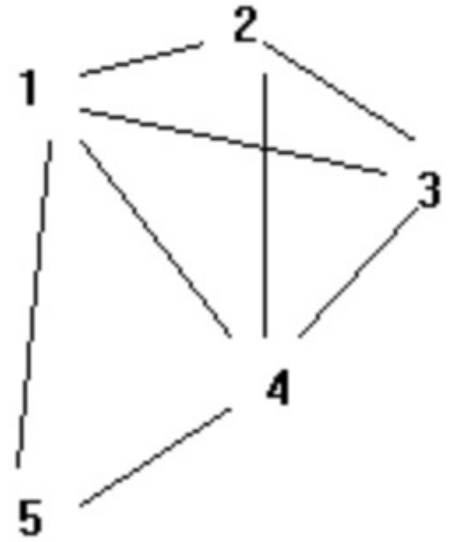
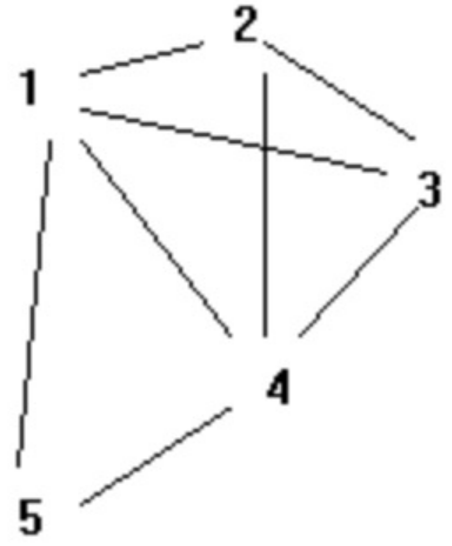
color(1,red,a).    color(1,red,b).
color(2,blue,a).   color(2,blue,b).
color(3,green,a).  color(3,green,b).
color(4,yellow,a). color(4,blue,b).
color(5,blue,a).   color(5,green,b).
```



Here we have encoded 'a' and 'b' colorings. We want to write a Prolog definition of a conflictive coloring, meaning that two adjacent regions have the same color. For example, here is a Prolog clause, or rule to that effect.

One could declare colorings for the regions in Prolog also using unit clauses.

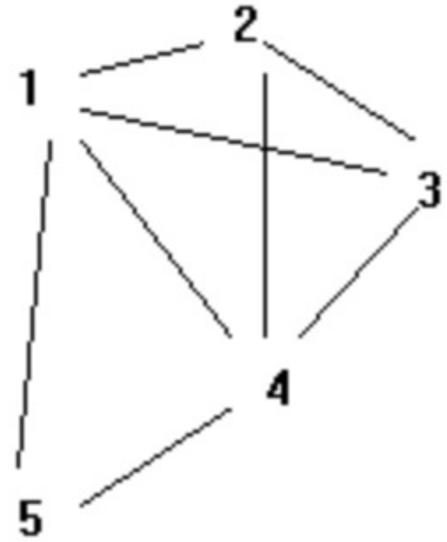
```
color(1,red,a).    color(1,red,b).
color(2,blue,a).   color(2,blue,b).
color(3,green,a).  color(3,green,b).
color(4,yellow,a). color(4,blue,b).
color(5,blue,a).   color(5,green,b).
```



Here we have encoded 'a' and 'b' colorings. We want to write a Prolog definition of a conflictive coloring, meaning that two adjacent regions have the same color. For example, here is a Prolog clause, or rule to that effect.

```
conflict(Coloring) :-
   adjacent(X,Y),
   color(X,Color,Coloring),
   color(Y,Color,Coloring).
```

One could declare colorings for the regions in Prolog also using unit clauses.




```
color(1,red,a).     color(1,red,b).
color(2,blue,a).    color(2,blue,b).
color(3,green,a).   color(3,green,b).
color(4,yellow,a).  color(4,blue,b).
color(5,blue,a).    color(5,green,b).
```

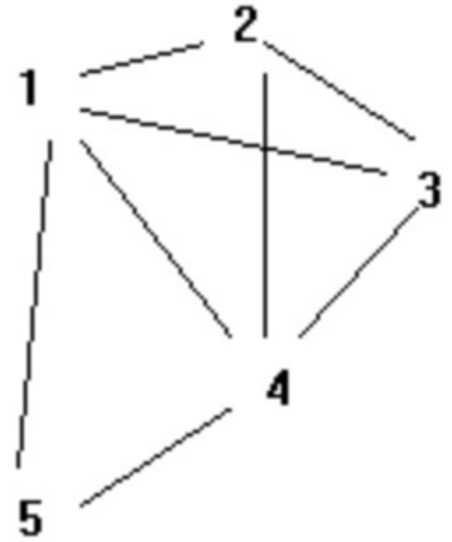
Here we have encoded 'a' and 'b' colorings. We want to write a Prolog definition of a conflictive coloring, meaning that two adjacent regions have the same color. For example, here is a Prolog clause, or rule to that effect.

```
conflict(Coloring) :-
   adjacent(X,Y),
   color(X,Color,Coloring),
   color(Y,Color,Coloring).
```

```
?- conflict(a).
no
?- conflict(b).
yes
?- conflict(Which).
Which = b
```

One could declare colorings for the regions in Prolog also using unit clauses.

```
color(1,red,a).    color(1,red,b).
color(2,blue,a).   color(2,blue,b).
color(3,green,a).  color(3,green,b).
color(4,yellow,a). color(4,blue,b).
color(5,blue,a).   color(5,green,b).
```

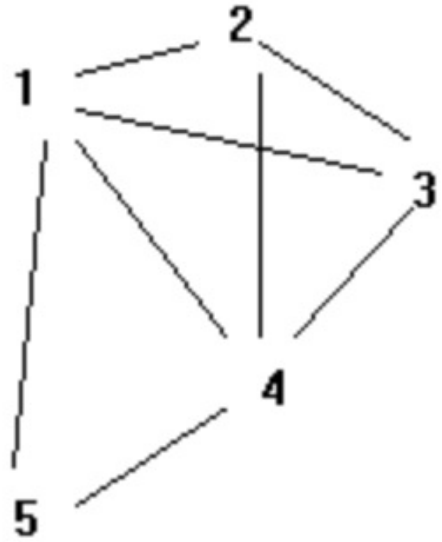


Here is another version of 'conflict' that has more logical parameters.

```
conflict(R1,R2,Coloring) :-
   adjacent(R1,R2),
   color(R1,Color,Coloring),
   color(R2,Color,Coloring).
```

One could declare colorings for the regions in Prolog also using unit clauses.

```
color(1,red,a).     color(1,red,b).
color(2,blue,a).    color(2,blue,b).
color(3,green,a).   color(3,green,b).
color(4,yellow,a).  color(4,blue,b).
color(5,blue,a).    color(5,green,b).
```



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```
conflict(R1,R2,Coloring) :-
   adjacent(R1,R2),
   color(R1,Color,Coloring),
   color(R2,Color,Coloring).
```

```
?- conflict(R1,R2,b).
R1 = 2   R2 = 4
?- conflict(R1,R2,b),color(R1,C,b).
R1 = 2   R2 = 4   C = blue
```

Arguments with Quantified Statements

Universal Modus Ponens

The rule of universal instantiation can be combined with modus ponens to obtain the valid form of argument called universal modus ponens.

Definition:

If x makes $P(x)$ true, then x makes $Q(x)$ true.

a makes $P(x)$ true.

$\therefore$ a makes $Q(x)$ true.

Example:

Rewrite the following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

If an integer is even, then its square is even.

k is a particular integer that is even.

$\therefore$ k^2 is even.

Solution

The major premise of this argument can be rewritten as

$\forall x$, if x is an even integer then x^2 is even.

Let $E(x)$ be "x is an even integer," let $S(x)$ be "x^2 is even," and let k stand for a particular integer that is even. Then the argument has the following form:

$\forall x$, if $E(x)$ then $S(x)$.

$E(k)$, for a particular k.

$\therefore$ $S(k)$.

This argument has the form of universal modus ponens and is therefore valid.

Universal Modus Tollens

Another crucially important rule of inference is universal modus tollens. Its validity results from combining universal instantiation with modus tollens. Universal modus tollens is the heart of proof of contradiction, which is one of the most important methods of mathematical argument.

If x makes P(x) true, then x makes Q(x) true.

a does not make Q(x) true.

∴ a does not make P(x) true.

Example:

Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All human beings are mortal.

Zeus is not mortal.

∴ Zeus is not human.

Solution The major premise can be rewritten as
∀x,if x is human then x is mortal.

Let H(x) be “x is human,” let M(x) be “x is mortal,” and let Z stand for Zeus. The argument becomes

$\forall x$, if $H(x)$ then $M(x)$

$\sim M(Z)$

$\therefore \sim H(Z)$.

Proving Validity of Arguments with Quantified Statements

To say that an argument form is valid means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true. An argument is called valid if, and only if, its form is valid.