Method of proof

Part 2

Definition

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r is rational $\Leftrightarrow$ $\exists$ integers a and b such that $r = \frac{a}{b}$ and $b \neq 0$.

Example:

1. Is 10/3 a rational number?
2. Is − 5 a rational number?
3. Is 0.281 a rational number?
4. Is 7 a rational number?
5. Is 2/0 a rational number?
6. Is 2/0 an irrational number?
7. Is 0.12121212 . . . a rational number (where the digits 12 are assumed to repeat forever)?

Example:

Is 0.12121212 . . . a rational number (where the digits 12 are assumed to repeat forever)?

Yes!

$$x = 0.12121212\ldots$$

Then,

$$100x = 12.12121212\ldots$$
$$100x - x = 12.121212\ldots - 0.121212\ldots$$
$$99x = 12$$
$$x = \frac{12}{99}$$

Zero Product Property

If neither of two real numbers is zero, then their product is also not zero.

Theorem:

Every integer is a rational number.

$$\forall\, r \in \mathbb{R}, if\ r\ is\ an\ integer\ then\ r\ is\ rational$$

$$\forall\, r \in \mathbb{Z}, r\ is\ rational$$

Proof:

Let z be any integer. Since $z = \frac{z}{1}$ where both $z, 1 \in \mathbb{Z}$ and $1 \neq 0$, by definition of rational numbers z is a rational number. ■

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Let z be any integer. Since $z = \frac{z}{1}$ where both $z, 1 \in \mathbb{Z}$ and $1 \neq 0$, by definition of rational numbers z is a rational number. ■

Theorem: The sum of any two rational numbers is rational.

Proof:

Suppose r and s are rational numbers. [We must show that $r + s$ is rational.] Then, by definition of rational,$r = a/b$ and $s = c/d$ for some integers a, b, c, and d with $b \neq 0$ and $d \neq 0$. Thus

$$r + s = \frac{a}{b} + \frac{c}{d} \qquad \text{by substitution}$$

$$= \frac{ad + bc}{bd} \qquad \text{by basic algebra.}$$

Let $p = ad + bc$ and $q = bd$. Then p and q are integers because products and sums of integers are integers and because a, b, c, and d are all integers. Also $q \neq 0$ by the zero-product property. Thus

$$r + s = \frac{p}{q},$$

where p and q are integers and $q \neq 0$.

Therefore, $r + s$ is rational by definition of a rational number. ■

Exercise:

1. The product of any two rational numbers is a rational number.
2. If r and s are any two rational numbers, then $\frac{r+s}{2}$ is rational.
3. For all real numbers a and b, if $a < b$ then $a < \frac{a+b}{2} < b$.
4. given any two rational numbers *r* and *s* with *r* , *s*, there is another rational number between *r* and *s*.
5. if a is any even integer and b is any odd integer, then $\frac{a^2+b^2+1}{2}$ is an integer.

Example:

Suppose a, b, and c are integers and x, y, and z are nonzero real numbers that satisfy the following equations:

$$\frac{xy}{x+y} = a, \frac{xz}{x+z} = b \ and \ \frac{yz}{y+z} = c$$

Is x rational? If so, express it as ratio of two integers.

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$$\frac{1}{a} = \frac{1}{y} + \frac{1}{x}, \qquad \frac{1}{b} = \frac{1}{x} + \frac{1}{z}, \qquad \frac{1}{c} = \frac{1}{y} + \frac{1}{z}$$

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$$\frac{1}{b} - \frac{1}{c} + \frac{1}{a} = ?$$