Isomorphisms of Graphs

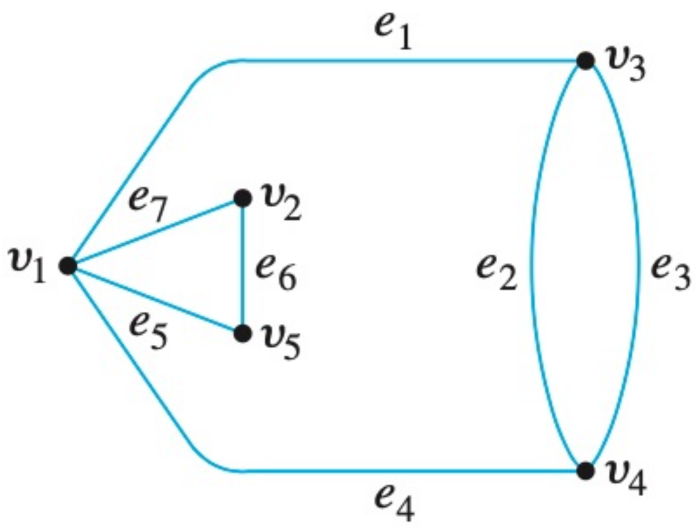
Definition:

Let G and G' be graphs with vertex sets $V(G)$ and $V(G')$ and edge sets $E(G)$ and $E(G')$, respectively. ***G* is isomorphic to *G'*** if, and only if, there exist one-to-one correspondences $g: V(G) \to V(G')$ and $h: E(G) \to E(G')$ that preserve the edge-endpoint functions of G and G' in the sense that for each $v \in V(G)$ and $e \in E(G)$,

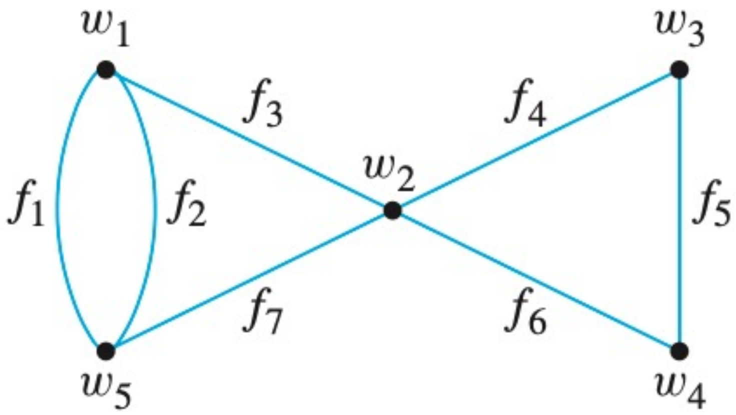
$$v \text{ is an endpoint of } e \Leftrightarrow g(v) \text{ is an endpoint of } h(e).$$

Example:

Show that the following two graphs are isomorphic.



G



G'

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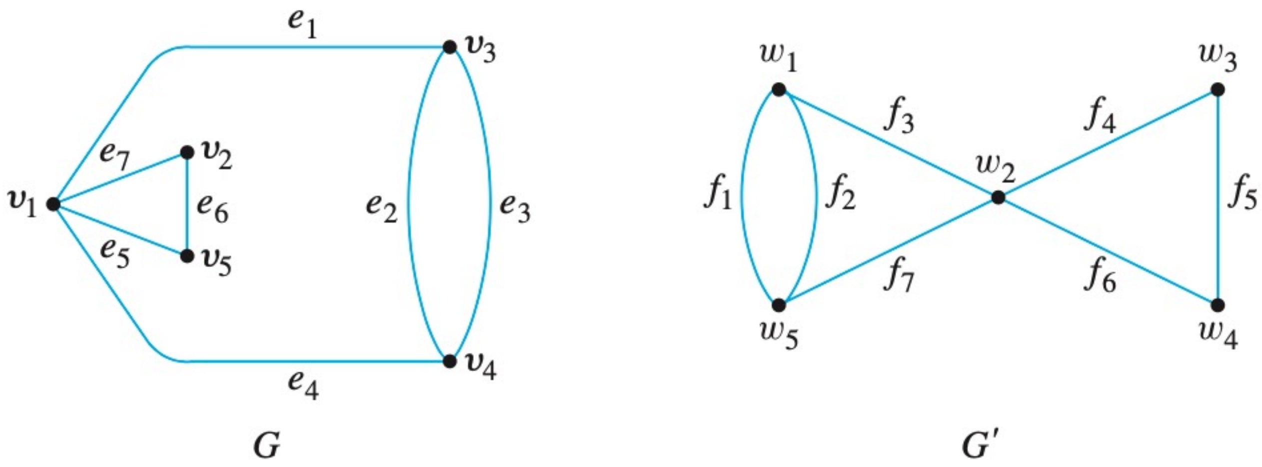
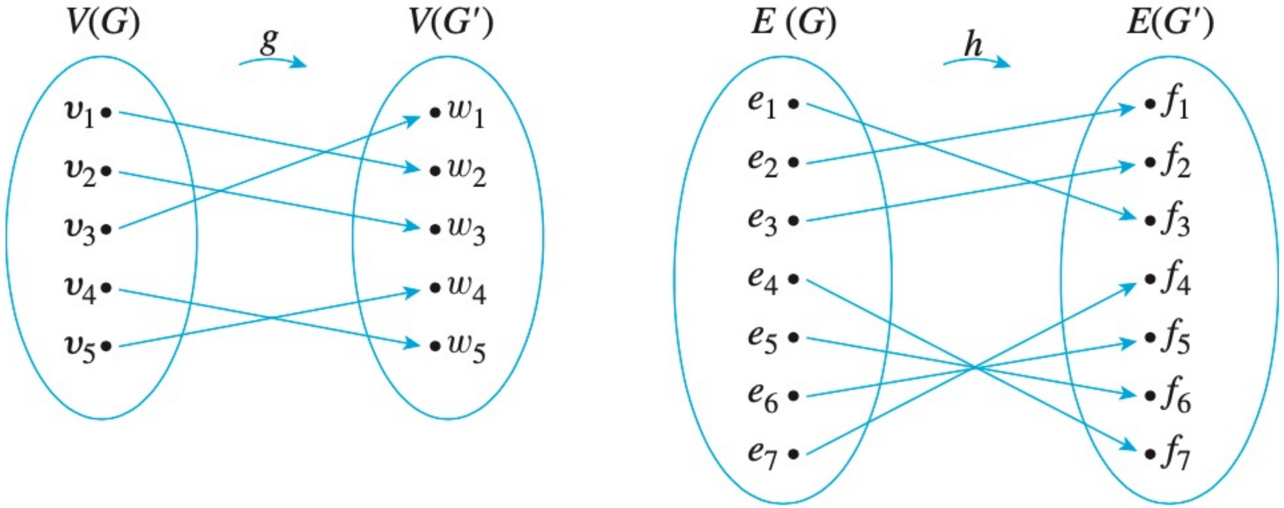
Show that the following two graphs are isomorphic.





Example:

Find all non-isomorphic graphs that have two vertices and two edges. In other words, find a collection of representative graphs with two vertices and two edges such that every graph with two vertices and two edges is isomorphic to one in the collection.

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Definition:

A property P is called an **invariant for graph isomorphism** if, and only if, given any graphs G and G', if G has property P and G' is isomorphic to G, then G' has property P.

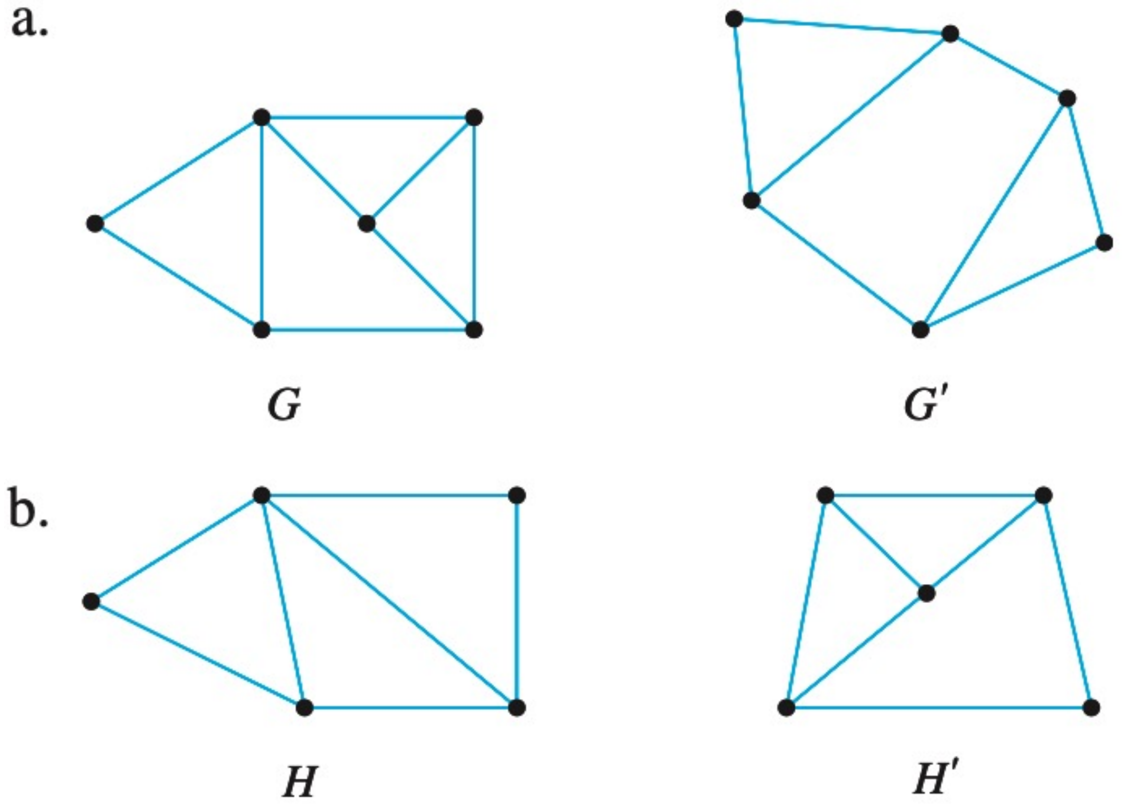
Theorem

Each of the following properties is an invariant for graph isomorphism, where *n*, *m*, and *k* are all nonnegative integers:

1. has *n* vertices
2. has *m* edges
3. has a vertex of degree *k*
4. has *m* vertices of degree *k*
5. has a circuit of length *k*
6. has a simple circuit of length *k*
7. has *m* simple circuits of length *k*
8. is connected
9. has an Euler circuit
10. has a Hamiltonian circuit.

Example:

Show that the following pairs of graphs are not isomorphic by finding an isomorphic invariant that they do not share.

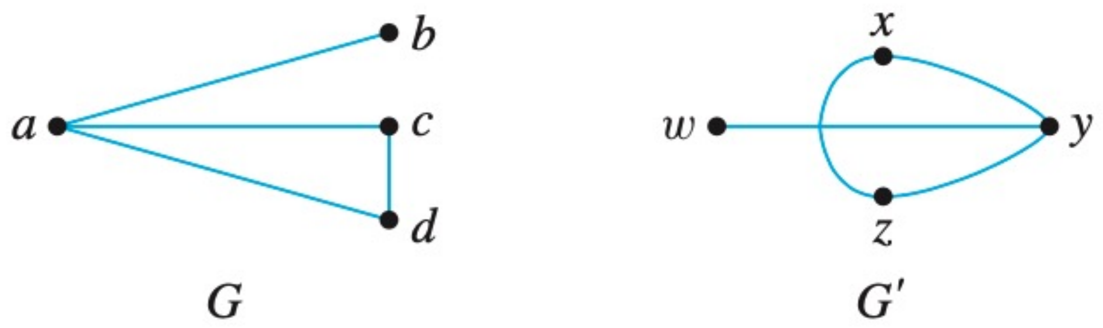


Definition:

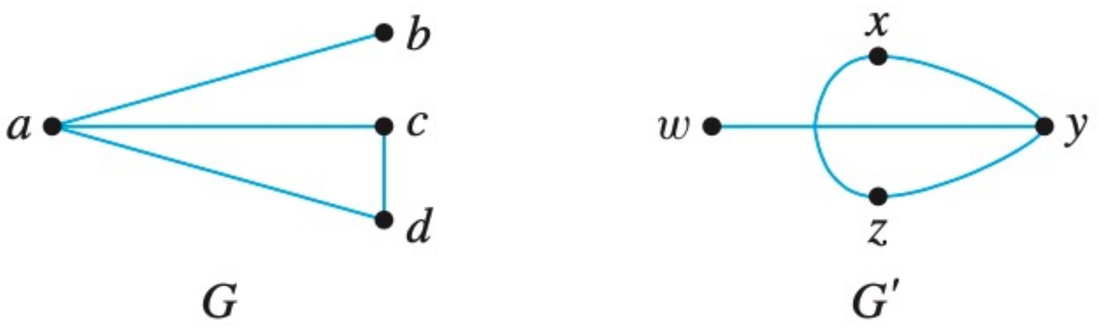
If G and G' are simple graphs, then **G is isomorphic to G'** if, and only if, there exists a one-to-one correspondence g from the vertex set $V(G)$ of G to the vertex set $V(G')$ of G' that preserves the edge-endpoint functions of G and G' in the sense that for all vertices u and v of G,

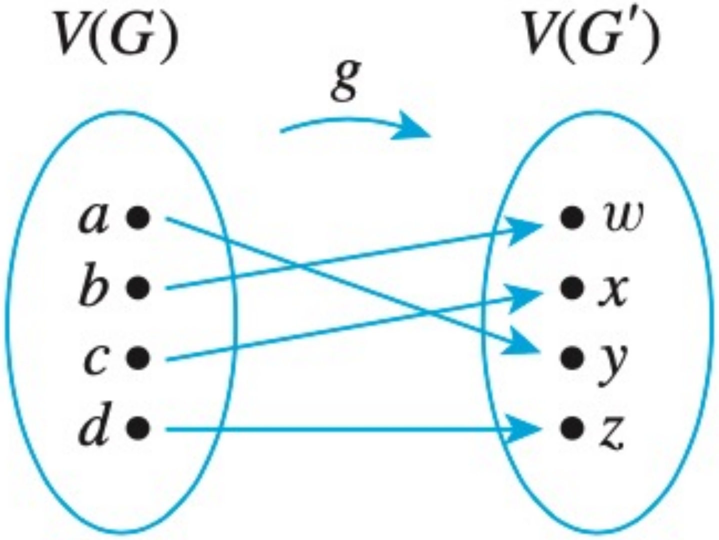
$$\{u, v\} \text{ is an edge in } G \Leftrightarrow \{g(u), g(v)\} \text{ is an edge in } G'.$$

Example: Are the two graphs shown below isomorphic? If so, define an isomorphism.



Example: Are the two graphs shown below isomorphic? If so, define an isomorphism.





Edges of G	Edges of G'
$\{a, b\}$	$\{y, w\} = \{g(a), g(b)\}$
$\{a, c\}$	$\{y, x\} = \{g(a), g(c)\}$
$\{a, d\}$	$\{y, z\} = \{g(a), g(d)\}$
$\{c, d\}$	$\{x, z\} = \{g(c), g(d)\}$