Thermodynamic Balance in Tornado Theory

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We address the energy balance in a thunderstorm, in particular, how energy is redistributed on a local level inside a tornadic flow.

Outline

1. Motivation
2. Views from Branches of Thermodynamics
3. Rational Vs Mesoscopic Thermodynamics
4. Evolution with Macroscopic Fluctuation
5. Thermodynamic Fluxes
6. Vertical vs Horizontal Scales
7. Summary

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Motivation

- (Rotunno, [2015]): ...there is a strong nexus with **thermodynamics**, because these thunderstorms are driven by the phase change of water vapor. There are lots and lots of things other than pure fluid dynamics in this field. It is a very rich subject ...

- (Doswell et al., [2006]): ...forecasters and researchers are seeking a "magic bullet" when they offer up yet another combined variable or index for consideration ...

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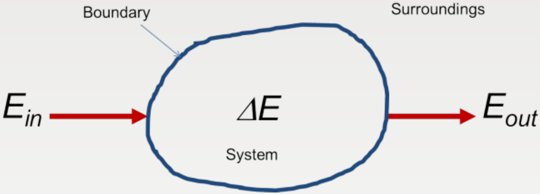
Global Reversible Thermodynamics



$S = S(A_1, A_2, \ldots A_n)$, S is *entropy*, A_i are extensive state variables, F_i are conjugate intensive variables

$$dS = \sum F_i \, dA_i$$

Example: Ideal gas, $dS = (1/T)dU + (P/T)dV$

Local Irreversible Dynamics

Local Equilibrium Hypothesis:

$$\Delta s(\mathbf{x},t) \approx \sum F_i(\mathbf{x},t)\, \Delta a_i(\mathbf{x},t),$$

s is specific entropy (per unit mass), and a_i are specific variables (per unit mass). Dividing by Δt and sending it to zero, we obtain a relation for material derivatives:

$$\frac{Ds(\mathbf{x},t)}{Dt} = \sum F_i(\mathbf{x},t)\, \frac{Da_i(\mathbf{x},t)}{Dt},$$

$$D/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla$$

$\mathbf{u}$: velocity field

Extended Irreversible Thermodynamics

Non-equilibrium variables: thermodynamic fluxes $b_j(\mathbf{x},t)$

$$ds(\mathbf{x},t) = \sum F_i(\mathbf{x},t)\,da_i(\mathbf{x},t) + \sum G_j(\mathbf{x},t)\,db_j(\mathbf{x},t),$$

$b_j(\mathbf{x},t)$ satisfy transport equations.

heat flux $\mathbf{q}(\mathbf{x},t)$, Cattaneo equation:

$$\tau\frac{\partial \mathbf{q}}{\partial t} = -(\mathbf{q} + \lambda\nabla T)$$

where λ is Fourier's Law constant and τ is relaxation time.

$$ds = \frac{\partial s}{\partial u}du + \frac{\partial s}{\partial \mathbf{q}}\cdot d\mathbf{q}$$

where $\boldsymbol{\theta} = \frac{\partial s}{\partial \mathbf{q}}$ is non-equilibrium temperature.

Internal Variables Thermodynamics

Internal Variables $c_k(\mathbf{x}, t)$: compensate for lack of information (no corresponding conjugate computable forces).

$$ds(\mathbf{x}, t) = \sum F_i(\mathbf{x}, t)\, da_i(\mathbf{x}, t) + \sum dc_k(\mathbf{x}, t)$$

No *a priori* constitutive equation. Variables depend on the system non-locally (space and time). Conservation laws plus Clausius–Duhem's apply:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$
$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\textbf{stress}) = 0,$$
$$\rho \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = 0,$$
$$\rho \frac{\partial s}{\partial t} + \nabla \cdot \frac{\mathbf{q}}{T} - \frac{\rho r}{T} \geq 0.$$

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Rational Vs Mesoscopic Thermodynamics

Axiom of Local Action: An air parcel is only influenced by its immediate neighborhood in space and time, so higher-order space and time derivatives are excluded from the constitutive relations. Its validity is controversial as it ignores "memory".

Mesoscopic approach appreciates limitations of the local equilibrium hypothesis and takes into consideration fluctuations of thermodynamic variables. Statistical entropy $S = k \ln W$, W is the number of microstates corresponding to a macrostate with the specific value of S. The probability of such a macrostate is (Einstein, [1910])

$$W \approx \exp(S/k).$$

Rational Vs Mesoscopic Thermodynamics

If **fluctuation** is associated with entropy change ΔS we can write

$$\text{Probability of Fluctuation} \sim \exp(\Delta S/k).$$

Einstein's formula underwent a range of generalizations (Bertini et al., [2015]). Thermodynamic flux: thermodynamic variable $\rho(\mathbf{x}, t)$, associated current $\mathbf{j}(\mathbf{x}, t)$, *mobility* matrix χ

$$\text{Probability of Fluctuation} \approx \exp\left(-\frac{B}{kT}\right),$$

where

$$B = \int dt \int dx(\mathbf{j} - \mathbf{J}(\rho)) \cdot \chi(\rho)^{-1}(\mathbf{j} - \mathbf{J}(\rho))$$

and where $\mathbf{J}(\rho)$ is a *hydrodynamic* flux of ρ.

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Evolution with Macroscopic Fluctuations

(Eyink et al., [1990]):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}(t) = 0, \quad \mathbf{j}(t) = \mathbf{J}(t, \rho(t))$$

$$\mathbf{J}(t, \rho) = -D(\rho)\nabla\rho + \chi(\rho)E(t)$$

$D(\rho)$: diffusion matrix, $E(t)$: external field

So

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \chi(\rho)E(t) = \nabla \cdot (D(\rho)\nabla\rho)$$

Local Equilibrium:

$$D(\rho) = \chi(\rho) f''(\rho)$$

f is the (local) Helmholtz free energy

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Thermodynamic Fluxes

Free energy density (Bystrai et al., 2011):

$$\hat{f}(t) = f(\theta(t, \boldsymbol{\xi}(t)), v(t), t)$$

θ: non-equilibrium temperature, $\boldsymbol{\xi}(t)$: parameters

$$\frac{df}{dt} = -s\frac{d\theta}{dt} - p\frac{dv}{dt} - s\frac{\partial\theta}{\partial\boldsymbol{\xi}}\frac{\partial\boldsymbol{\xi}}{\partial t} \qquad (*)$$

(*) are compatible with Navier - Stokes equations and with Kuramoto–Tsuzuki equation for complex velocity Φ

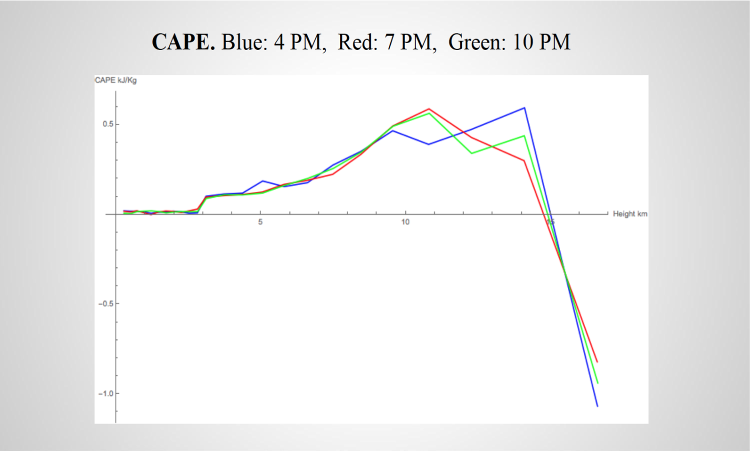
$$\frac{d\Phi}{dt} = \nu_1(1 + ic_1)\Delta\Phi + q\Phi - (\alpha_1(1 + ic_2)|\Phi|^2\Phi$$

that has a plane wave solution

$$\Phi(x, y, t) = R(x, y)\exp(i\omega t + ia(x, y))$$

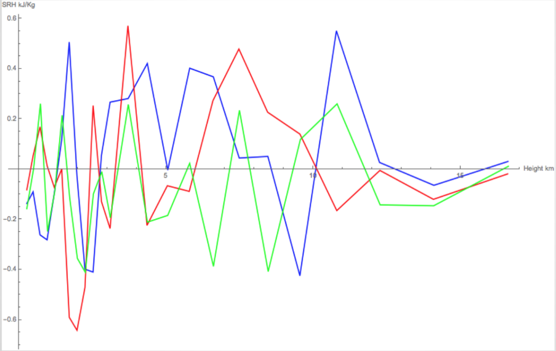
or a *spiral-wave* solution.

CAPE


CAPE. Blue: 4 PM, Red: 7 PM, Green: 10 PM
CAPE kJ/Kg
0.5
−0.5
−1.0
5
10
15
Height km

SRH

SRH. Blue: 4 PM, Red: 7 PM, Green: 10 PM



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Vertical vs Horizontal Scales and Energy Evolution

Upper portion of the flow: pressure drop due to condensation (Makarieva et al, 2011). Clausius-Clapeyron relation

$$\Delta P = \frac{L}{T\Delta v}\Delta T$$

where L is the *latent heat* of condensation.

In the cooler upper portion of the flow the vertical speed plays important role and so does the pressure drop, with an additional term free energy evolution:

$$\frac{df}{dt} = -s\frac{d\theta}{dt} - p\frac{dv}{dt} - s\frac{\partial\theta}{\partial\boldsymbol{\xi}}\frac{\partial\boldsymbol{\xi}}{\partial t} - \frac{d}{dt}\left(vdp\right)$$

where we still can neglect the volume changes due to condensation.

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Summary

We described the integrated parameters CAPE and SRH as thermodynamic fluxes associated with a non-equilibrium air parcel in a thunderstorm and their influence on free energy density formula. It is not yet clear if this formula incorporates all macroscopic fluctuations associated with a non-equilibrium state. We also considered a thermodynamic flux associated with the condensation process in the cooler portion of the flow. Our long-term goal is to find a free energy density formula.

Thank You!!

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