ON THERMODYNAMIC EQUATIONS IN LAYERED TORNADO MODEL

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Introduction

We explore the energy balance in a thunderstorm, in particular, how energy is redistributed on a local level inside a tornado-like flow. The notions of non-equilibrium thermodynamics are used to describe the problem. We show that fluctuations on a macroscopic level play an especially important role in this model.

The Main Goal

To develop a set of thermodynamic criteria to adequately describe the behavior of an air parcel in a tornado-like flow

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Motivation

- **Rotunno** [2015], *Weatherwise*, May/June 2015, 56–57: “ *. . . there is a strong nexus with **thermodynamics**, because these thunderstorms are driven by the phase change of water vapor. There are lots and lots of things other than pure fluid dynamics in this field. It is a very rich subject.* ”

Thermodynamic System

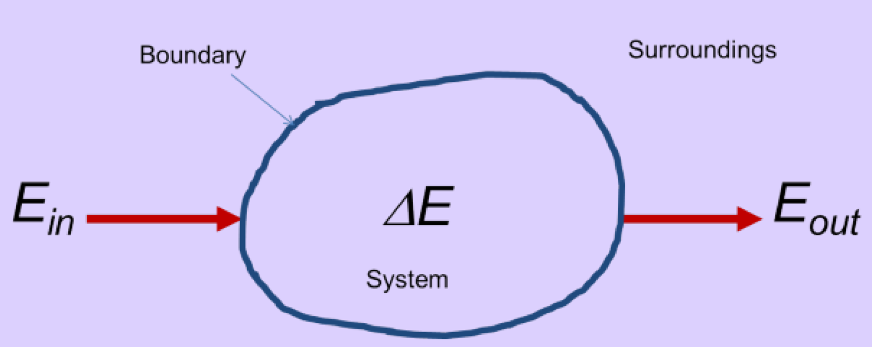

Figure 1: Balance Diagram

Global Reversible Thermodynamics

Equation of state: $S = S(A_1, A_2, \ldots A_n)$, S is *entropy*, A_i are the independent extensive variables, that fully determine the state of the system. If the change happens infinitely slowly, then system moves to a different equilibrium state according to

$$dS = \sum F_i \, dA_i$$

F_i are the corresponding conjugate intensive variables (forces), d stands for the (material) change in variables in the system under consideration.

Example: $S = S(U, V)$ where U is the internal energy and V is the specific volume. For the ideal gas, $dS = (1/T)dU + (P/T)dV$ where T is the temperature and P is pressure.

Local Irreversible Thermodynamics

We employ the **Local Equilibrium Hypothesis** where state of air parcel depends on position $\mathbf{x}$ and time t (cf. [1])

$$ds(\mathbf{x}, t) = \sum F_i(\mathbf{x}, t) \, da_i(\mathbf{x}, t),$$

s is specific S (per unit mass), and a_i is specific A_i (per unit mass), d is **material** change in variables in the air parcel.

Local Dynamics

Differentiating along the path of the parcel that moves with local velocity $\mathbf{u}$:

$$\frac{Ds(\mathbf{x}, t)}{Dt} = \sum F_i(\mathbf{x}, t) \frac{Da_i(\mathbf{x}, t)}{Dt},$$
$$D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$$

Extended Irreversible Thermodynamics

The state of the system may include non-equilibrium variables, more precisely, thermodynamic fluxes $b_j(\mathbf{x}, t)$. Then

$$ds(\mathbf{x}, t) = \sum F_i(\mathbf{x}, t) \, da_i(\mathbf{x}, t) + \sum G_j(\mathbf{x}, t) \, db_j(\mathbf{x}, t),$$

where $b_j(\mathbf{x}, t)$ have to satisfy the appropriate transport equations. For example, the heat flux $\mathbf{q}(\mathbf{x}, t)$ may behave according to the Cattaneo equation

$$\tau \frac{\partial \mathbf{q}}{\partial t} = -(\mathbf{q} + \lambda \nabla T)$$

where λ is Fourier's Law constant and τ is relaxation time

Example:

$$ds = \frac{\partial s}{\partial u} du + \frac{\partial s}{\partial \mathbf{q}} \cdot d\mathbf{q} \tag{1}$$

where $\boldsymbol{\theta} = \frac{\partial s}{\partial \mathbf{q}}$ is non-equilibrium temperature.

Internal Variables Thermodynamics

Internal Variables $c_k(\mathbf{x}, t)$ are introduced to compensate for lack of knowing the behavior of the system. It does not have the corresponding conjugate forces that can be directly calculated. Then

$$ds(\mathbf{x}, t) = \sum F_i(\mathbf{x}, t) \, da_i(\mathbf{x}, t) + \sum dc_k(\mathbf{x}, t),$$

Rational Thermodynamics

In this case we do not assume an *a priori* constitutive equation. Instead, the thermodynamic variables depend on the system non-locally in space and time. Conservation of energy and linear momentum for each control volume remain to be valid. Then the balance equations and Clausius–Duhem's inequality:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$
$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{stress}) = 0,$$
$$\rho \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = 0,$$
$$\rho \frac{\partial s}{\partial t} + \nabla \cdot \frac{\mathbf{q}}{T} - \frac{\rho r}{T} \geq 0,$$

where r is the internal heat source.

Axiom of Local Action: An air parcel is only influenced by its immediate neighborhood in space and time, so higher-order space and time derivatives are excluded from the constitutive relations. Its validity is controversial as it ignores “memory”.

Mesoscopic Thermodynamics

Mesoscopic approach appreciates limitations of the local equilibrium hypothesis and takes into consideration fluctuations of thermodynamic variables. Statistical entropy $S = k \ln W$, W is the number of microstates corresponding to a macrostate with the specific value of S. The probability of such a macrostate is (**Einstein**, [2])

$$W \approx \exp(S/k).$$

If **fluctuation** is associated with entropy change ΔS we can write

$$\text{Probability of Fluctuation} \approx \exp(\Delta S/k).$$

Einstein's formula underwent a range of generalizations, in particular, for the thermodynamic fluxes to be included it requires for any thermodynamic variable $\rho(\mathbf{x}, t)$ with the associated current $\mathbf{j}(\mathbf{x}, t)$ in the *mobility* matrix χ to be

$$\text{Probability of Fluctuation} \approx \exp\left(-\frac{B}{kT}\right),$$

where

$$B = \int dt \int dx (\mathbf{j} - \mathbf{J}(\rho)) \cdot \chi(\rho)^{-1} (\mathbf{j} - \mathbf{J}(\rho))$$

and where $\mathbf{J}(\rho)$ is a *hydrodynamic* flux of ρ. The derivation and justification with the references is given in [3]. The evolution of a system subject to macroscopic fluctuations has to satisfy

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}(t) = 0, \quad \mathbf{j}(t) = \mathbf{J}(t, \rho(t)),$$
$$\mathbf{J}(t, \rho) = -D(\rho) \nabla \rho + \chi(\rho) E(t).$$

So

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \chi(\rho) E(t) = \nabla \cdot (D(\rho) \nabla \rho).$$

where $D(\rho)$ is a diffusion matrix and $E(t)$ is the external field. Local Equilibrium implies that

$$D(\rho) = \chi(\rho) f''(\rho)$$

where f is the (local) Helmholtz free energy per unit volume. These equations can be justified using microscopic stochastic dynamics [4].

Thermodynamic Fluxes

According to [5] in a thin horizontal layer of tornado-like flow the Helmholtz free energy density of an air parcel is defined by

$$\hat{f}(t) = f(\theta(t, \xi_e(t), \boldsymbol{\xi}_i(t)), v(t), t)$$

where θ is non-equilibrium temperature defined in (1), $\boldsymbol{\xi}_i(t)$ is a vector of internal parameters, and $\xi_e(t)$ is an external parameter. Then, neglecting explicit dependence on time t and external parameter $\xi_e(t)$,

$$\frac{df}{dt} = -s \frac{d\theta}{dt} - p \frac{dv}{dt} - S \frac{\partial \theta}{\partial \boldsymbol{\xi}_i} \frac{\partial \boldsymbol{\xi}_i}{\partial t} \tag{2}$$

It can be proven that if $\boldsymbol{\xi}_i$ are linear functions of $\nabla \mathbf{u}$, then (2) is compatible with Navier-Stokes governing equations [5]. Solutions of (2) also are particular solutions of the Kuramoto–Tsuzuki system of equations of motion in a plane layer [5]:

$$\frac{du}{dt} = \nu_1 \Delta u - \nu_2 \Delta v + qu - (\alpha_1 |\mathbf{u}|^2 u - \alpha_2 |\mathbf{u}|^2 v) \tag{3}$$
$$\frac{dv}{dt} = \nu_1 \Delta v + \nu_2 \Delta u + qv - (\alpha_1 |\mathbf{u}|^2 v + \alpha_2 |\mathbf{u}|^2 u) \tag{4}$$

where Δ is the Laplacian in 2-D, and ν_1, ν_2, q, α_1, and α_2 are the parameters describing the air parcel at the altitude h.

Thermodynamic Potential and Instability

(3) and (4) can be combined in a vector equation for a complex velocity $\Phi = u + iv$

$$\frac{d\Phi}{dt} = \nu_1 (1 + ic_1) \Delta \Phi + q\Phi - (\alpha_1 (1 + ic_2)) |\Phi|^2 \Phi$$

that has a plane wave solution (cf. [5])

$$\Phi(x, y, t) = R(x, y) \exp(i\omega t + ia(x, y))$$

or a *spiral-wave* solution (cf. [5]).

Vertical vs Horizontal Scales

Non-equilibrium thermodynamics of condensation in the upper portion of the flow would have to take into consideration the pressure drop due to condensation (see [6]) as equations of motion do not explain the downdraft inside the vortex core ([6]). In case of infinitely slow simplified model we can use Clausius-Clapeyron relation

$$\Delta P = \frac{L}{T \Delta v} \Delta T$$

where L is the *latent heat* of condensation.

Evolution of the Helmholtz Energy

In the cooler upper portion of the flow the vertical speed plays dominating role as radial velocity vanishes at certain height, so the pressure drop is significant, and an additional term has to be added to the internal flux system in (2), more precisely

$$\frac{df}{dt} = -s \frac{d\theta}{dt} - p \frac{dv}{dt} - S \frac{\partial \theta}{\partial \boldsymbol{\xi}_i} \frac{\partial \boldsymbol{\xi}_i}{\partial t} - \frac{d}{dt}(v \Delta p)$$

where we still can neglect the volume changes due to condensation.

Summary

We investigated the thermodynamic equation for a tornado-like vortex using the *Gibbs Relations* in various contexts (equilibrium and non-equilibrium case), as well as the theory when the Gibbs Relations are abandoned and *Rational Thermodynamics* is used to form a constitutive behavior of moist air subject to tornado-like behavior of pressure, temperature, and velocity profiles. We also discussed the thermodynamic flux associated with the faster condensation process in the cooler portion of the flow. Our long-term goal is to model the evolution of the Helmholtz free energy density f in the tornado-like vortex consistent with the available soundings.

References

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