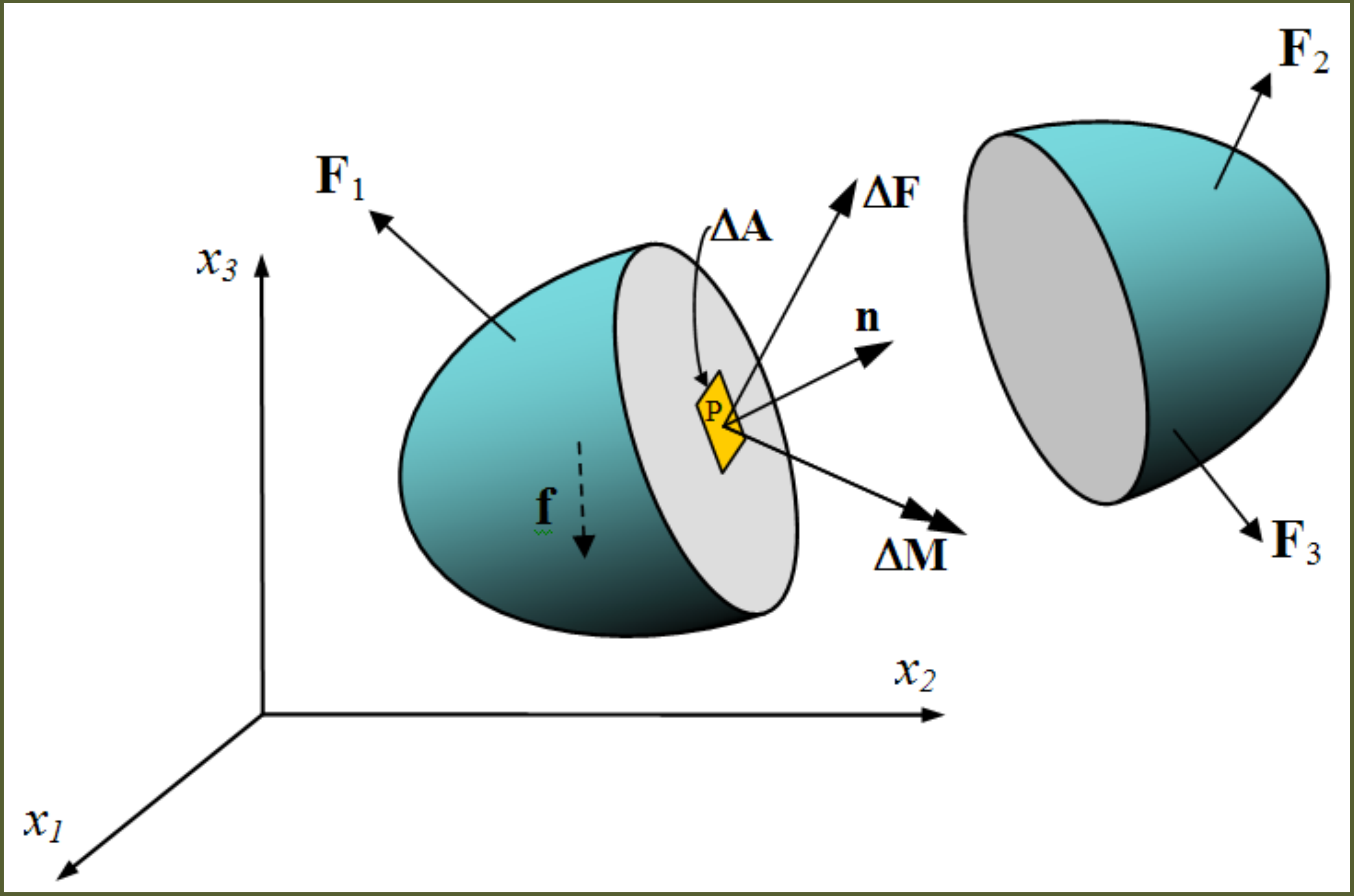
Why Mechanical Engineers
have to major in Mathematics

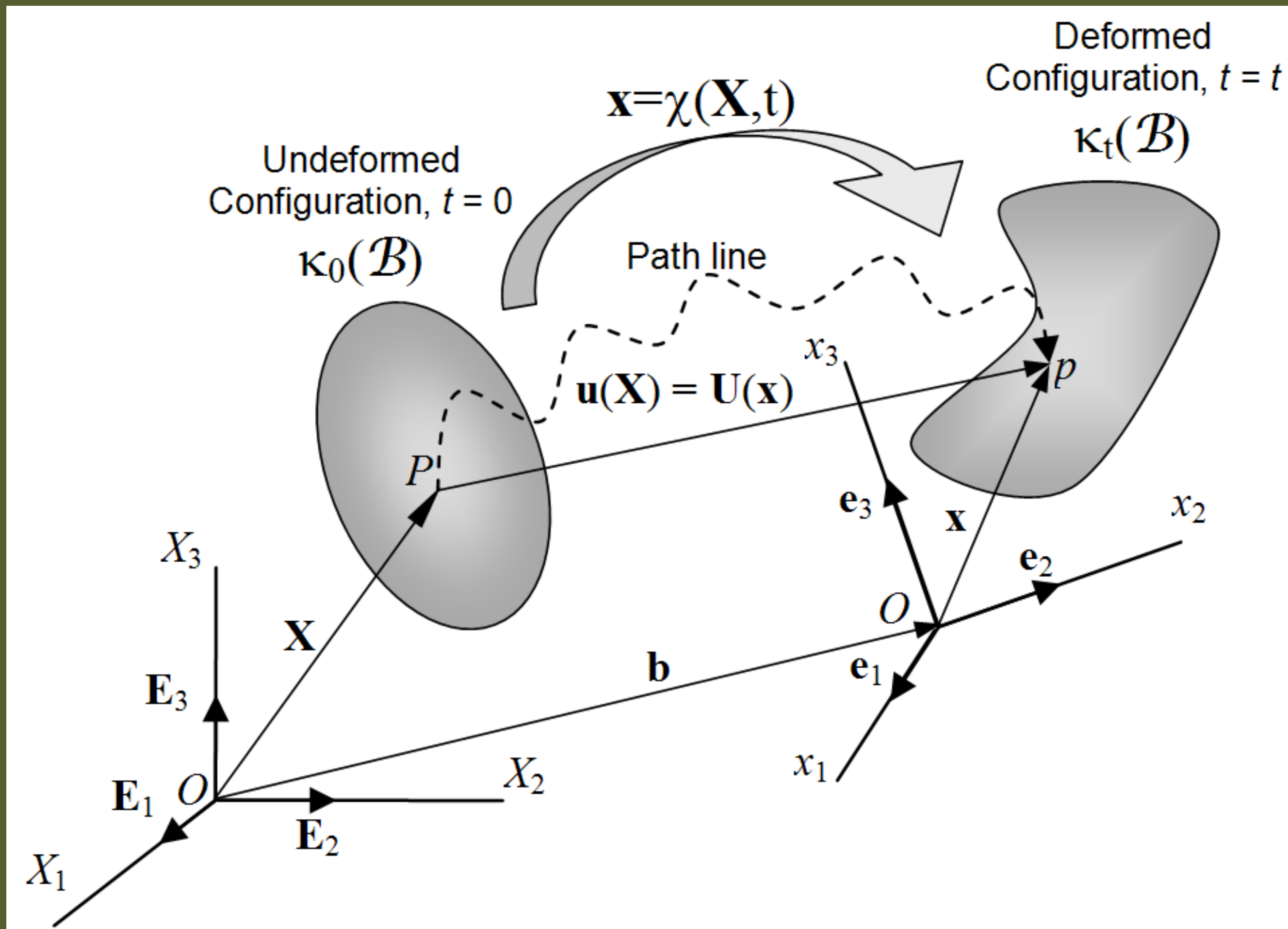
Tuesday, November 18, Math Club
Misha Shvartsman

Answer: Mechanical engineers
need to know HOW things break

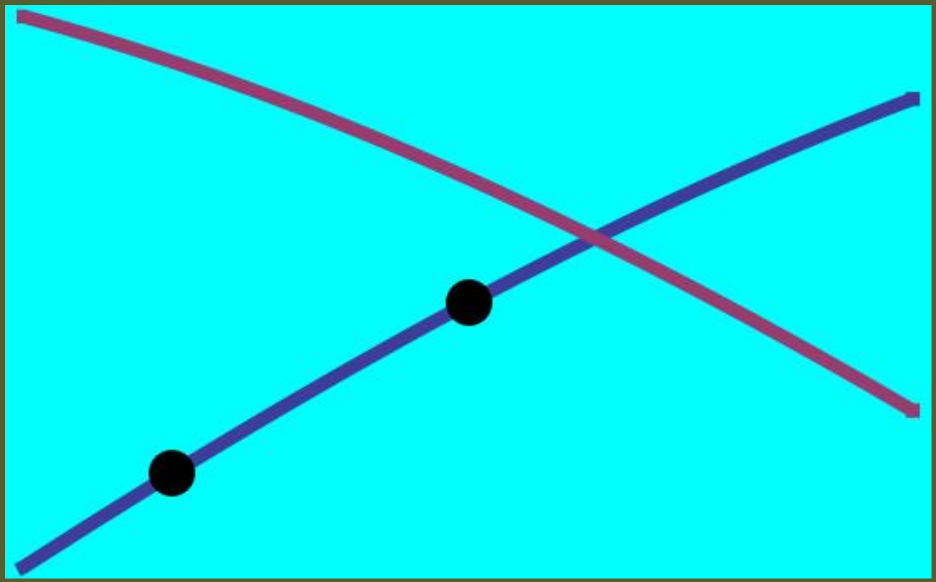
Mechanical engineers are trained to understand WHY things break



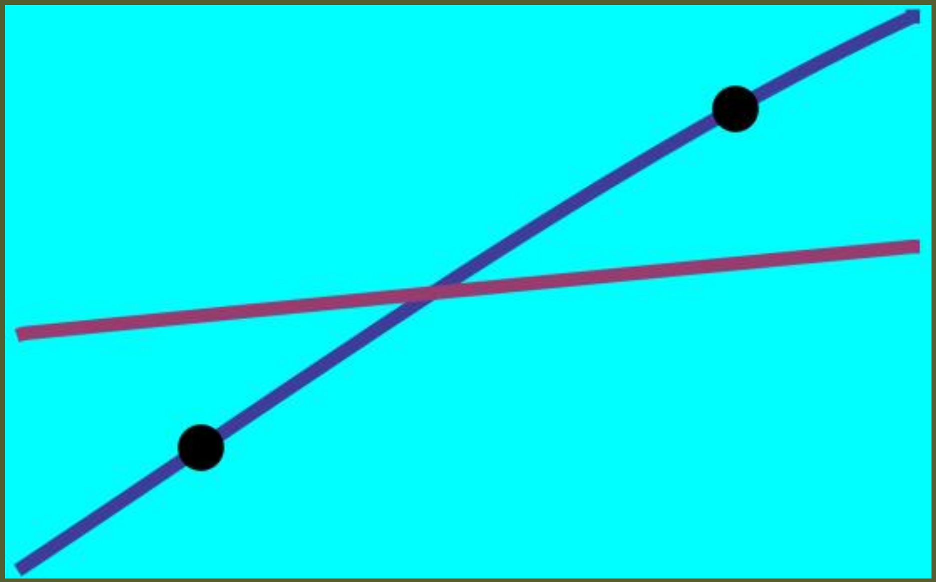
Deformation Vector



Rigid Motions



Conformal
Mappings



Volume-
Preserving
Mappings

Mechanical Engineers know how to connect ∇u (**deformations**) and forces (**stresses**)

How does one find ∇u ???

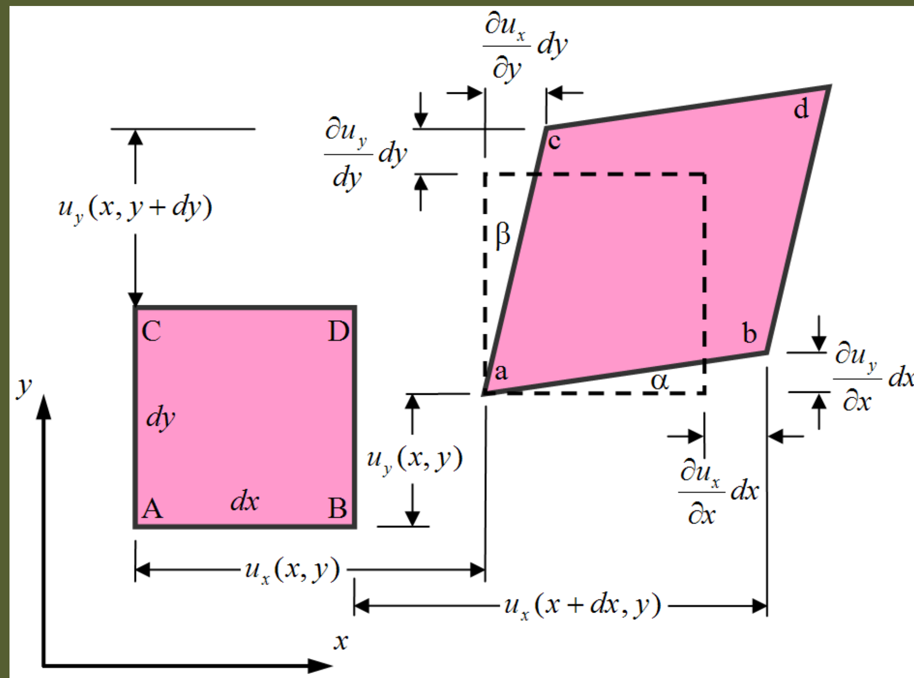
u $\in C$?? u $\in C^1$??

$$\mathrm{u} = \langle\, u_1,\ u_2,\ u_3 \,\rangle$$

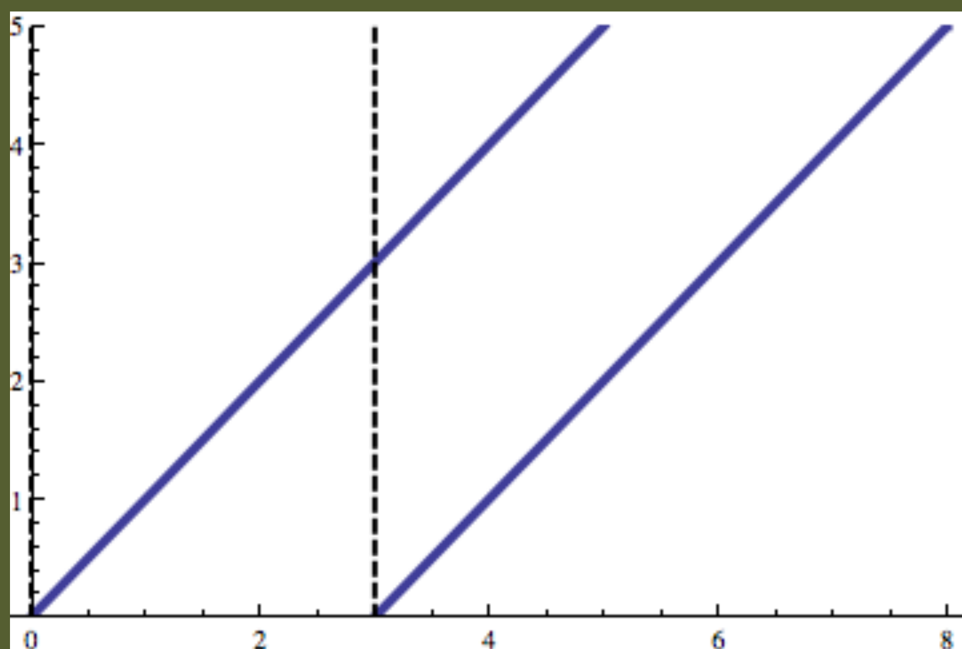
$$\nabla \mathrm{u} = \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{pmatrix}$$

What matrices are allowed?

∇u (x) = ???



$\frac{\partial u_x}{\partial y} dy$
$\frac{\partial u_y}{dy} dy$
$u_y(x, y+dy)$
d
c
β
C
D
b
a
α
$\frac{\partial u_y}{\partial x} dx$
y
dy
$u_y(x, y)$
$\frac{\partial u_x}{\partial x} dx$
A
dx
B
$u_x(x, y)$
$u_x(x+dx, y)$
x



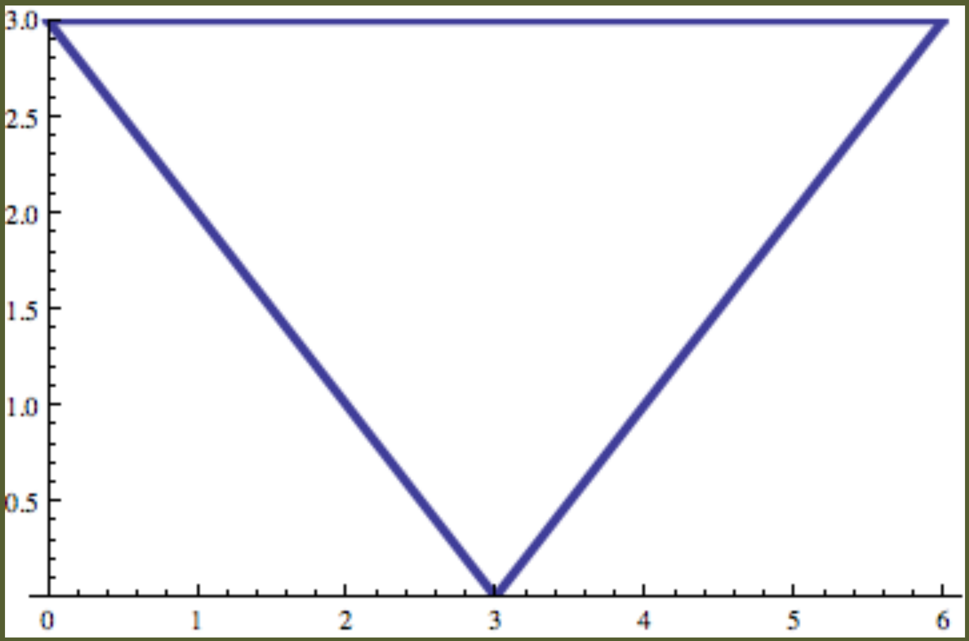
5
4
3
2
1
0
2
4
6
8

$u \in C^1$

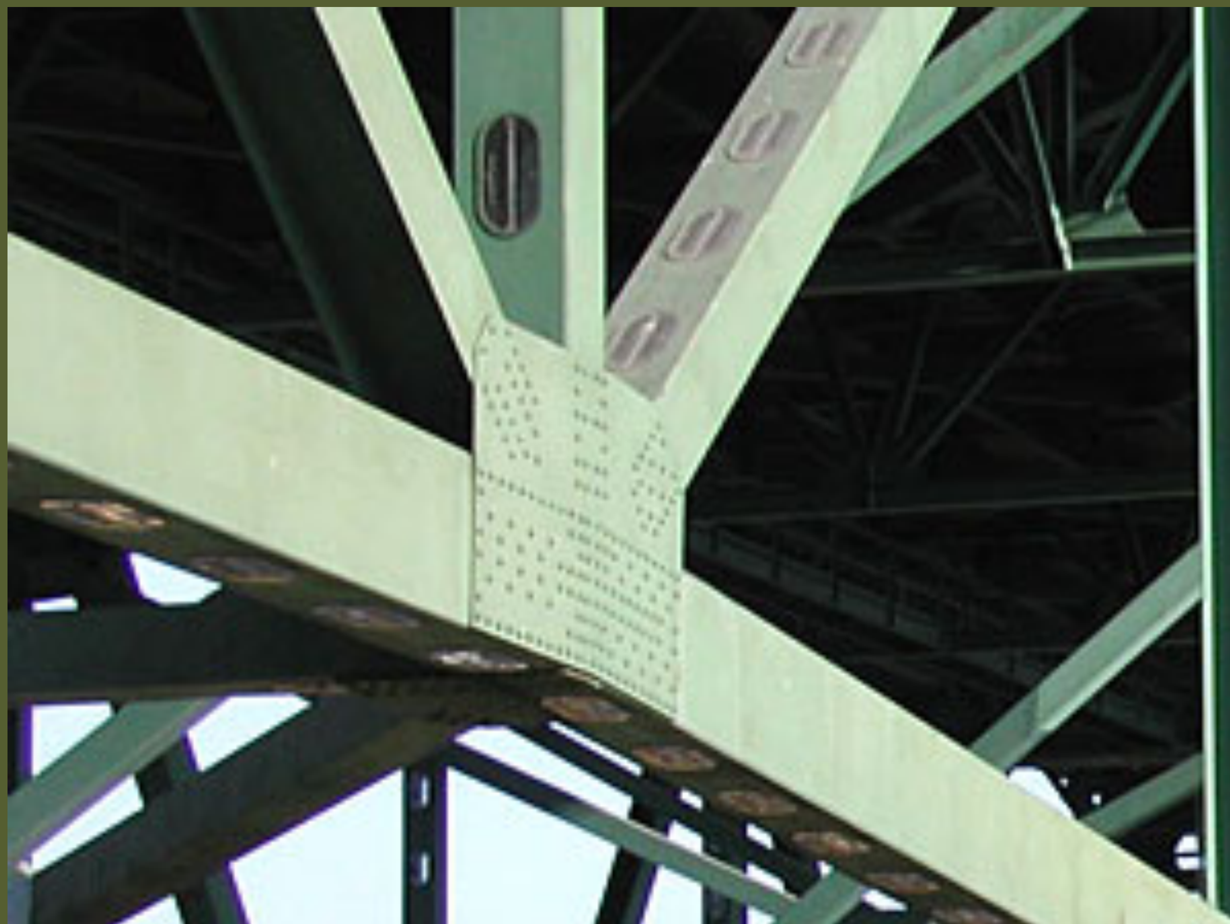
$$\nabla u = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

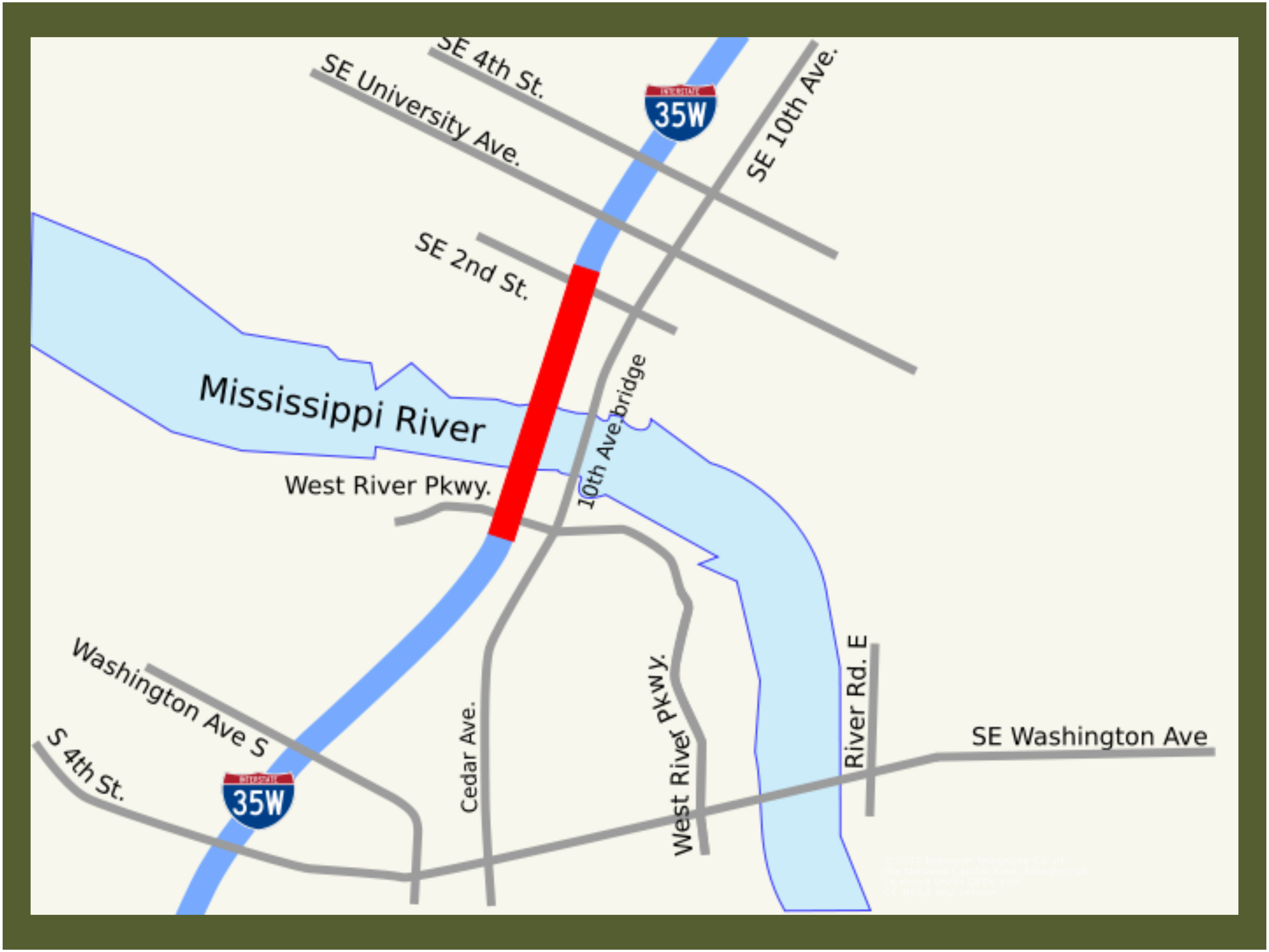
$\det(\nabla u) = 1$

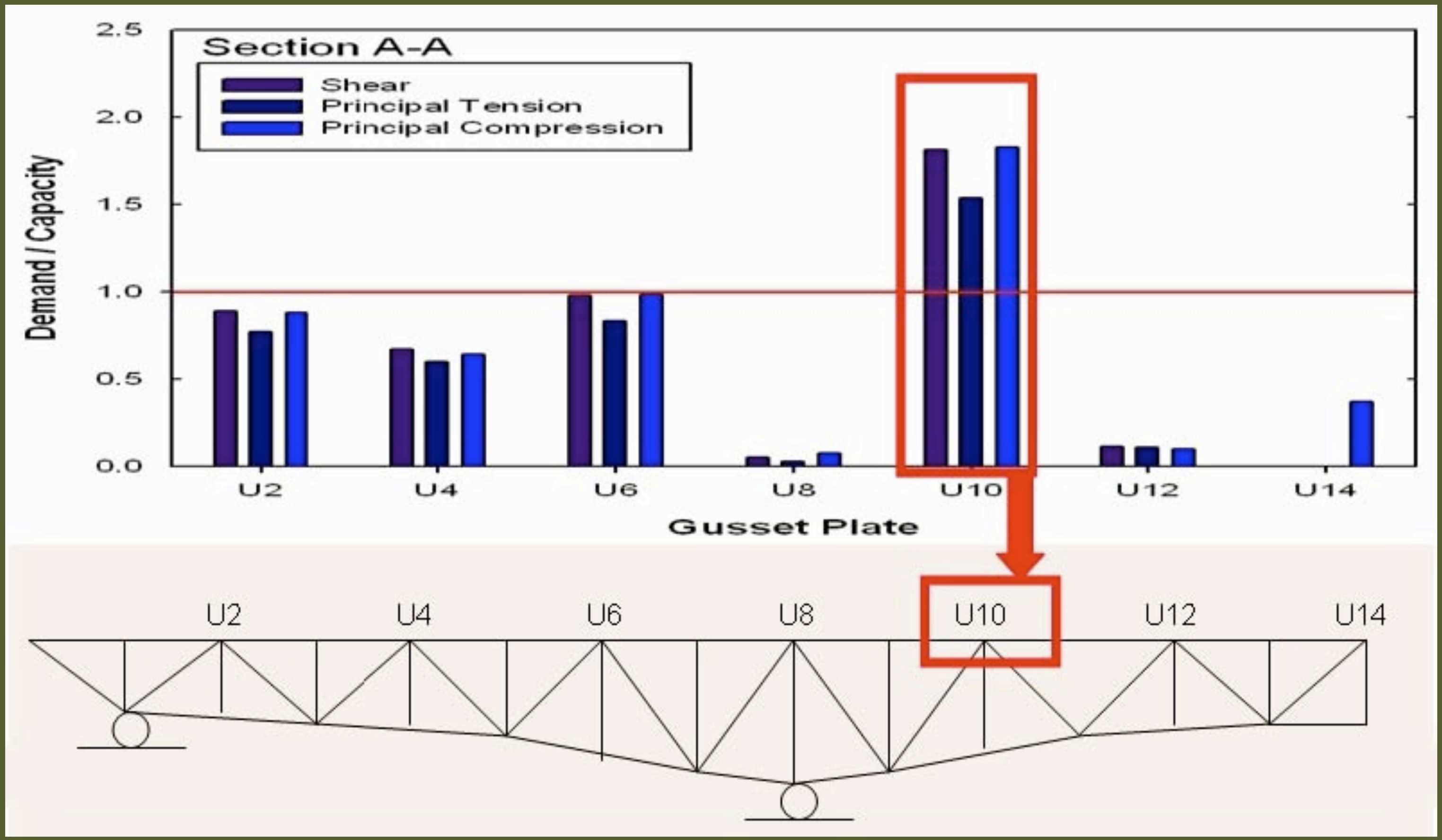
Rigidity Theorems



Peter Olver & Cheri Shakiban, Applied Linear Algebra,
Prentice Hall, 2005




SE 4th St.
SE University Ave.
35W
SE 10th Ave.
SE 2nd St.
Mississippi River
10th Ave. bridge
West River Pkwy.
Washington Ave S
S 4th St.
35W
Cedar Ave.
West River Pkwy.
River Rd. E
SE Washington Ave


Section A-A
Shear
Principal Tension
Principal Compression
Demand / Capacity
2.5
2.0
1.5
1.0
0.5
0.0
U2
U4
U6
U8
U10
U12
U14
Gusset Plate
U2
U4
U6
U8
U10
U12
U14

NTSB, 11/14/2008

The National Transportation Safety Board determines that the probable cause of the collapse of the I-35W bridge in Minneapolis, Minnesota, was the inadequate load capacity, due to a design error by Sverdrup & Parcel and Associates, Inc., of the **gusset plates** at the U10 nodes, which failed under a combination of

(1) Substantial increases in the weight of the bridge, which resulted from previous bridge modifications, and

(2) The traffic and concentrated construction loads on the bridge on the day of the collapse.

$\nabla u\,(x) \in SO(n) \;\Rightarrow\; \det(\nabla u) = 1$

Liouville (1847)

$u \in C^1 \quad \nabla u\,(x) \in SO(3) \;\Rightarrow\; \nabla u\,(x) = \text{const}$

Reshetnyak (1967, 1989, 1997)

$u \in W^{1,2}(U, \mathbb{R}^n)$ and $\nabla u\,(x) \in SO(n)$ a.e. $\Rightarrow\; \nabla u\,(x) = \text{const}$

Friesecke, James, Müller (2002)

$u \in W^{1,2}(U, \mathbb{R}^n) \;\Rightarrow$ there is a rotation

$R \in SO(n)$ and $C > 0$:

$\|\nabla u - R\|_{L^2(U)} \le C(U)\, \|\mathrm{dist}(\nabla u, SO(n))\|_{L^2(U)}$

JERRARD, LORENT (2008)
ON MULTIWELL LIOUVILLE THEOREMS IN HIGHER DIMENSIONS

$i = \sqrt{-1}, \quad z = x + i\,y = r\,e^{it}$

$|z| = r, \qquad e^{it} = \cos t + i \sin t$

$f : \mathbb{C} \to \mathbb{C}$ is holomorphic if

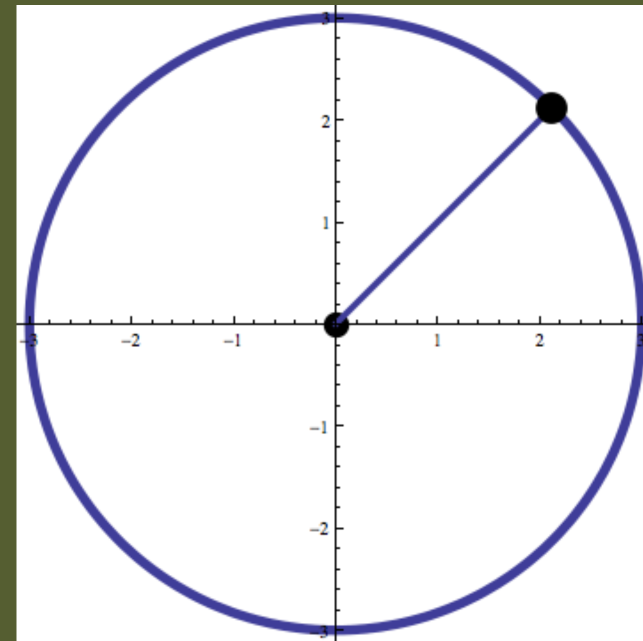
$$f'(z) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

$D = \{ z : |z - a| \leq r \}$

$$\oint_{\partial D} f(z)\, dz = F(z_2) - F(z_1) = 0$$

$$\oint_{\partial D} \frac{1}{z-a}\, dz = \int_0^{2\pi} \frac{ir\,e^{it}}{re^{it}}\, dt = 2\pi i$$

$z = a + re^{it} \qquad dz = ir\,e^{it}\, dt$



Circle of radius r
centered at a

Cauchy's Integral Formula

$$f(a) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(z)}{z-a}\, dz, \quad f^{(k)}(a) = \frac{k!}{2\pi i} \oint_{\partial D} \frac{f(z)}{(z-a)^{k+1}}\, dz$$

$$\left| \frac{1}{2\pi i} \oint_{\partial D} \frac{f(z)}{z-a}\, dz - f(a) \right| = \left| \frac{1}{2\pi i} \oint_{\partial D} \frac{f(z)-f(a)}{z-a}\, dz \right|$$

$$z = a + r e^{it} \qquad dz = i r e^{it}\, dt$$

$$\le \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(a + r e^{it}\right) - f(a) \right| dt \to 0 \text{ as } r \to 0$$

Liouville's Theorem. Every bounded holomorphic function must be constant.

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(z)}{z^{k+1}} \, dz$$

$$|a_k| \le \frac{1}{2\pi} \oint_{\partial D} \frac{|f(z)|}{|z^{k+1}|} \, dz \le \frac{1}{2\pi} \oint_{\partial D} \frac{M}{r^{k+1}} \, dz$$

$$\le \frac{M}{r^k}, \; r \to \infty. \qquad f(z) = a_0$$

The fundamental theorem of algebra: Every non-constant single-variable polynomial with complex coefficients has at least one complex root.

Proof. Let $p: \mathbb{C} \to \mathbb{C}$ be a polynomial
Suppose that $p(z) \neq 0$ all $z \in \mathbb{C}$. Then
$f(z) = \dfrac{1}{p(z)}$ is holomorphic on $\mathbb{C}$
$|\,p(z)\,| \to \infty$ as $|\,z\,| \to \infty \;\Rightarrow\; |\,f(z)\,| \to 0$

So $|\,f(z)\,| < 1$, $|\,z\,| > M$, some $M > 0$

$|\,f(z)\,| \leq K$ for some $K > 0$ on $\{\, z : |\,z\,| \leq M\}$

Liouville's $\Rightarrow$ $f(z) = \text{constant}$