ON GOOD *ABC* TRIPLES

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Abstract. We define an *abc* triple as (a, b, c), where a, b, and c are relatively prime positive integers such that $a + b = c$. We say that an *abc* triple (a, b, c) is good if $\text{rad}(abc) < c$. In this article, we seek to extend previously established literature pertaining to *abc* triples where $a = 1$ to include cases where $a > 1$. Our work is motivated by the *abc* conjecture and utilizes data from the abc@home project. We discuss generalized equivalence statements that allow us to recover sequences of good *abc* triples with given conditions.

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1. Introduction

To understand the concept of good *abc* triples, we must first introduce two important ideas.

Definition 1.1. For a positive integer $n > 1$, the *radical of* n, denoted $\text{rad}(n)$, is the product of all the distinct prime factors of n. We set $\text{rad}(1) = 1$.

For example:

$$\text{rad}(16) = 2$$

The prime factorization of 16 is 2^4. The radical ignores the exponent, as it pertains only to the distinct prime factors. Similarly, since $72 = 2^3 \cdot 3^2$, we deduce that $\text{rad}(72) = 2 \cdot 3 = 6$.

Definition 1.2. Two integers a and b are said to be *relatively prime* (or *coprime*) if their greatest common divisor is 1. In this case, we write

$$\gcd(a, b) = 1$$

In other words, a and b have no common positive integer factors other than 1.

Now that these two concepts have been introduced, we can define what (good) *abc* triples are.

Definition 1.3. We say (a, b, c) is an *abc triple* if a, b, and c are relatively prime positive integers such that $a + b = c$. We say that an *abc* triple (a, b, c) is *good* if $\text{rad}(abc) < c$.

For instance, $(9, 16, 25)$ is an abc triple, but it is not a good abc triple, since $\mathrm{rad}(9 \cdot 16 \cdot 25) = 30 > 25$. However, $1 + 8 = 9$ satisfies $\mathrm{rad}(1 \cdot 8 \cdot 9) = 6 < 9$. Thus, $(1, 8, 9)$ is a good abc triple. The abc conjecture asserts that good abc triples are very rare. For instance, there are only 8 good abc triples (a, b, c) with $c < 200$. Yet, there are 6 116 abc triples (a, b, c) with $c < 200$. Thus, for $c < 200$, there are only 0.1308% abc triples that are good. The percentage drops even lower as c grows. For instance, only 0.000336% of abc triples (a, b, c) with $c <$ 20 000 are good.

1.1. **History.** The abc conjecture is a famous open problem in number theory which states:

Conjecture 1.4 (The *abc* conjecture, Masser, Oesterlé, 1985)**.** For each $\epsilon > 0$, there are finitely many abc triples (a, b, c) such that

$$\mathrm{rad}(abc)^{1+\epsilon} < c.$$

The abc conjecture has many greater applications if proven, including relevance to Fermat's Last Theorem, Faltings' Theorem, Roth's Theorem, Elliptic curves, and Szpiro's Conjecture [GT02, MM16].

The simplistic abc conjecture, now proven false, was based upon removing epsilon from the statement of the abc conjecture. Notice that by setting $\epsilon = 0$ in the statement of the abc conjecture, we get the statement that there are only finitely many good abc triples.

To produce a counterexample to the simplistic abc conjecture, it suffices to show that there is an infinite sequence of good abc triples – for example, the abc triple $(1, 3^{2^k} - 1, 3^{2^k})$ is good for every integer k. Notice that when $k = 1$, we recover the good abc triple $(1, 8, 9)$. This sequence of good abc triples was among the first of its kind, as depicted in the work of Lang [Lan90]. Building on this sequence, it was shown in [Bar23] that for any odd prime p, the abc triple $(1, p^{(p-1)k} - 1, p^{(p-1)k})$ is good for each positive integer k. Alvarez et al. [ASBHS23] further generalized this sequence by allowing for p to be replaced by any odd integer n.

In fact, Alvarez et al. considered sequences of good abc triples of the form $(1, c - 1, c)$ and deduced two general results that allowed them to recover each of the sequences of good abc triples with $a = 1$ that have previously appeared in the literature. Next, we introduce the *cosocle* of an integer, a fundamental concept prevalent in much of the work of Alvarez et al. In our investigation of analogous statements within this body of work, we identified the *cosocle* as a crucial function as well.

Definition 1.5. Let n be a positive integer. Then, the *cosocle* of n, denoted $\mathrm{cosocle}(n)$, is defined as:

$$\mathrm{cosocle}(n) = \frac{n}{\mathrm{rad}(n)}.$$

For an example of calculating the cosocle, consider

$$\mathrm{cosocle}(214369) = \frac{463^2}{\mathrm{rad}(463^2)} = \frac{463^2}{463} = 463.$$

With this terminology, we include below the two key theorems presented by Alvarez et al.:

Theorem 1.6 ([ASBHS23, Theorem 1])**.** *Let c and m be positive integers with $c > 1$. If m divides $c - 1$ and $\mathrm{cosocle}(m) > \mathrm{rad}(c)$, then*

$$(1, c^k - 1, c^k)$$

is a good abc triple for each positive integer k.

Theorem 1.7 ([ASBHS23, Theorem 2])**.** *Let b and m be positive integers. If m divides $b+1$ and* $\operatorname{cosocle}(m) > \operatorname{rad}(b)$*, then*

$$(1, b^k, b^k + 1)$$

is a good abc triple for each positive odd integer k.

Consequently, they obtain that if $(1, c-1, c)$ is an abc triple, then $(1, c^k - 1, c^k)$ is a good *abc* triple for each positive integer k. Another result from their work that we will later reference is the following lemma:

Lemma 1.8 ([ASBHS23, Lemma 2.3])**.** *Let m and n be positive integers. If m divides n, then* $\operatorname{rad}(n) = \operatorname{rad}\left(\frac{n}{\operatorname{cosocle}(m)}\right)$.

The proofs of these results can be found in the aforementioned article by Alvarez et al.

Our goal for this project was to utilize the previous literature pertaining to a = 1 to create analogous, generalized statements that would allow statements to be made about good *abc* triples for any positive integer $a > 1$. Motivated by [ASBHS23], our investigation was rooted in the following question: if (a, b, c) is a good *abc* triple, is it the case that $(a^k, c^k - a^k, c^k)$ is also a good *abc* triple for each positive integer k?

Unlike the $a = 1$ case, it is not necessarily true that the above question holds in general. A counterexample can be found in which $a = 5$. In particular, the triple $(5, 2^5 - 5, 2^5)$ is good, but $(5^k, 2^{5k} - 5^k, 2^{5k})$ is not always a good *abc* triple. In fact, for $k > 1$, the sequence appears to only yield a good *abc* triple when k has a factor of 3. In other words, the sequence $(5^{3k}, 2^{15k} - 5^{3k}, 2^{15k})$ does appear to result in good *abc* triples for $k \geq 1$. Yet, other good triples, such as $(5^{19}, 2^{95} - 5^{19}, 2^{95})$ are not of this desired form. Despite these challenges, generalized statements may still be made with certain given restrictions. In fact, our Corollary 3.1 gives a positive answer to the above question in the case when $a = 2$.

To investigate this question, we utilized the work from the abc@home project [dS]. The abc@home project was a network computing project that was startedin 2006 by the Mathematics Department at Leiden University and the Dutch Kennislink Science Institute with the goal of finding computational evidence for the *abc* conjecture. By 2011, they successfully found all good *abc* triples (a, b, c) with $c < 10^{18}$. This amounted to 14 482 065 good *abc* triples. By the time the project came to an end in 2015, the abc@home project had found 23.8 million good *abc* triples. In our work, we used the data computed by the abc@home project, specifically the 14 482 065 good *abc* triples (a, b, c) with $c < 10^{18}$ to find patterns that good *abc* triples possessed. We utilized SageMath [Dev24] to develop algorithms and databases that allowed us to examine the abc@home data efficiently and effectively. From these resources, we were able to derive the findings and proofs found in this article. Some of the code that we wrote is found in Appendix A.

1.2. **Relevant Notions.** In this section, we introduce the essential definitions that will be referenced in subsequent proofs, forming the basis for the formal arguments and results that follow.

Theorem 1.9 (The Fundamental Theorem of Arithmetic, Euclid, 300 B.C.)**.** *Every integer greater than 1 has a unique prime factorization.*

For example, $2\ 158\ 096 = 2^4 \cdot 19 \cdot 31 \cdot 229$. In particular, the right-hand side is the unique prime factorization of 2 158 096.

Definition 1.10. A *square-free integer* is an integer that is not divisible by any perfect square other than 1.

In other words, a number is square-free if none of its prime factors are repeated. Equivalently, n is square-free if $n = \mathrm{rad}(n)$. For example, 6 is square-free because its prime factorization is $2 \cdot 3$. Whereas 12 is not square-free, as its prime factorization is $2^2 \cdot 3$.

Finally, consider the following lemma which will be used later in our proofs:

Lemma 1.11. *Let a and b be positive relatively prime integers with $a \leq b$. Then $a + b < ab$ if and only if $2 \leq a < b$.*

Proof. Starting with the inequality $a + b < ab$, we observe that

$$\begin{aligned} a + b < ab \qquad &\iff \qquad 0 < ab - a - b, \\ &\iff \qquad 1 < ab - a - b + 1 \qquad \text{by adding 1 to both sides,} \\ &\iff \qquad 1 < (a-1)(b-1) \qquad \text{by factoring the right-hand side.} \end{aligned}$$

The inequality $(a-1)(b-1) > 1$ holds if and only if one of $a - 1 > 1$ or $b - 1 > 1$ with $a, b \neq 1$. Equivalently, $a > 2$ or $b > 2$ with $a, b \neq 1$. By assumption, a and b are relatively prime with $a \leq b$. Since $a, b \neq 1$, it follows that $a < b$ since a and b are relatively prime. Thus, $a + b < ab$ if and only if $2 \leq a < b$. ☐

Lemma 1.12. *Let k be a positive integer. Then, the following factorization holds for any integers x and y:*

$$x^k - y^k = (x - y) \sum_{j=0}^{k-1} x^{k-1-j} y^j.$$

Proof. For a fixed integer k, observe that

$$\begin{aligned} & \sum_{j=0}^{k-1} x^{k-1-j} y^j = x^{k-1} + x^{k-2}y + x^{k-3}y^2 + \cdots + xy^{k-2} + y^{k-1} \\ \implies \qquad & (x-y) \sum_{j=0}^{k-1} x^{k-1-j} y^j = (x-y)\left(x^{k-1} + x^{k-2}y + x^{k-3}y^2 + \cdots + xy^{k-2} + y^{k-1}\right) \\ & \qquad = \left(x^k + x^{k-1}y + \cdots + xy^{k-1}\right) - \left(x^{k-1}y + \cdots + xy^{k-1} + y^k\right) \\ & \qquad = x^k - y^k. \end{aligned}$$

☐

Definition 1.13. The *Euler totient function*, denoted $\phi(n)$, is defined as the number of positive integers up to n that are relatively prime to n. Formally:

$$\phi(n) = |\{k \mid 1 \leq k \leq n \text{ and } \gcd(k, n) = 1\}|$$

Definition 1.14. Let a, b, and n be integers with n positive. We say that a *is congruent to* b *modulo* n, denoted $a \equiv b \pmod{n}$, if n divides $a - b$. Equivalently, a and b have the same remainder when they are divided by n.

For example, $7 \equiv 15 \pmod 4$ since $15 - 7 = 8$ is divisible by 4. Notice that when 7 and 15 are divided by 4, their remainder is 3.

Theorem 1.15 (Euler's Theorem, 1763)**.** *If a and n are relatively prime integers, then*

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

where $\phi(n)$ is the Euler totient function.

2. Main results

In the following theorems, we explore analogous statements to those found in [ASBHS23], where the case $a = 1$ was considered. When extending these results to arbitrary values of a, we encountered new challenges that required us to impose additional conditions to ensure the validity of the statements. These conditions, which were implicitly satisfied when $a = 1$, are now explicitly stated to maintain the rigor and correctness of the theorems. The necessity of these conditions highlights the delicate balance in generalizing results in number theory, where seemingly minor changes in assumptions can lead to significant shifts in outcomes.

In the following, we will first present a theorem where the analogous case for $a = 1$ extends naturally to arbitrary a without requiring any additional conditions. This result shows that certain properties hold universally, regardless of the value of a. Following this, we will introduce another theorem where the generalization from $a = 1$ to arbitrary a needs the introduction of a specific assumption to preserve the validity of the analogous statement.

Proposition 2.1. *Let a and c be positive integers with $c > a$. Then, $(a, c-a, c)$ is a good abc triple if and only if* $\operatorname{cosocle}(c) > \operatorname{rad}(a) \cdot \operatorname{rad}(c-a)$.

Proof. Let $b = c - a$. Since (a, b, c) is a good *abc* triple, then

$$\begin{aligned}
\operatorname{rad}(a \cdot b \cdot c) < c \qquad &\iff \qquad \operatorname{rad}(a) \cdot \operatorname{rad}(b) \cdot \operatorname{rad}(c) < c \\
&\iff \qquad \operatorname{rad}(a) \cdot \operatorname{rad}(b) \cdot \frac{c}{\operatorname{cosocle}(c)} < c \\
&\iff \qquad \operatorname{rad}(a) \cdot \operatorname{rad}(b) < c \cdot \frac{\operatorname{cosocle}(c)}{c} \\
&\iff \qquad \operatorname{rad}(a) \cdot \operatorname{rad}(b) < \operatorname{cosocle}(c)
\end{aligned}$$

□

Theorem 2.2. *Let a and c be relatively prime positive integers with $c > a > 1$. If $a < \operatorname{rad}(c - a)$, then the following are equivalent:*

(*i*) $\operatorname{cosocle}(c - a) > \operatorname{rad}(a) \cdot \operatorname{rad}(c)$
(*ii*) $\operatorname{cosocle}(c) > \operatorname{rad}(a) \cdot \operatorname{rad}(c - a)$
(*iii*) *$(a, c - a, c)$ is a good abc triple*

Proof. From Proposition 2.1, we know that $\operatorname{cosocle}(c) > \operatorname{rad}(a) \cdot \operatorname{rad}(c-a)$ is equivalent to $(a, c-a, c)$ being a good *abc* triple.

Now suppose that $\operatorname{rad}(a) \cdot \operatorname{rad}(c) < \operatorname{cosocle}(c-a)$. From the definition of cosocle, we deduce that

$$\begin{aligned}
\operatorname{rad}(a) \cdot \operatorname{rad}(c) < \operatorname{cosocle}(c - a) \qquad &\iff \qquad \operatorname{rad}(a) \cdot \frac{c}{\operatorname{cosocle}(c)} < \frac{c - a}{\operatorname{rad}(c - a)} \\
&\iff \qquad \operatorname{rad}(a) \cdot \operatorname{rad}(c - a) < \frac{c - a}{c} \operatorname{cosocle}(c)
\end{aligned}$$

Since $\frac{c-a}{c} < 1$, we have our desired inequality: $\operatorname{rad}(a) \cdot \operatorname{rad}(c - a) < \operatorname{cosocle}(c)$.

Having established that (*ii*) and (*iii*) are equivalent, and that (*i*) implies (*ii*), it suffices to show that (*iii*) implies (*i*) to conclude the proof. To this end, suppose $(a, c - a, c)$ is a good *abc* triple. Then $\operatorname{rad}(a) \cdot \operatorname{rad}(c - a) \cdot \operatorname{rad}(c) < c$. Consequently,

$$c > \operatorname{rad}(a) \cdot \operatorname{rad}(c) \operatorname{rad}(c - a) = \operatorname{rad}(a) \cdot \operatorname{rad}(c) \frac{(c - a)}{\operatorname{cosocle}(c - a)}.$$

This implies that

$$\operatorname{rad}(a) \cdot \operatorname{rad}(c) < \operatorname{cosocle}(c - a) \frac{c}{c - a}.$$

Since $\operatorname{rad}(c)$ is an integer and $\frac{c}{c-a} > 1$, we deduce that $\operatorname{rad}(a)\operatorname{rad}(c) \leq \left\lfloor \operatorname{cosocle}(c-a)\frac{c}{c-a} \right\rfloor$, where $\lfloor x \rfloor$ denotes the floor function. By assumption, $a < \operatorname{rad}(c-a)$. Therefore,

$$\frac{a}{\operatorname{rad}(c-a)} = \operatorname{cosocle}(c-a)\frac{a}{c-a} < 1.$$

We now observe that

$$\left\lfloor \operatorname{cosocle}(c-a)\frac{c}{c-a} \right\rfloor = \left\lfloor \operatorname{cosocle}(c-a) + \operatorname{cosocle}(c-a)\frac{a}{c-a} \right\rfloor = \operatorname{cosocle}(c-a).$$

Lastly, c is relatively prime to $c-a$, and thus $\operatorname{cosocle}(c-a) > \operatorname{rad}(a) \cdot \operatorname{rad}(c)$. □

To conclude this section, we note the following result regarding good abc triples:

Proposition 2.3. *If (a, b, c) is a good abc triple, then c is not square-free, and at most, one of a or b is square-free.*

Proof. Assume for the sake of contradiction that c is square-free. That is, $c = \operatorname{rad}(c)$. Since (a, b, c) is a good abc triple, we have that

$$c > \operatorname{rad}(a)\operatorname{rad}(b)\operatorname{rad}(c) = \operatorname{rad}(a)\operatorname{rad}(b)c.$$

This implies $1 > \operatorname{rad}(a)\operatorname{rad}(b)$, which is a contradiction since a and b are positive integers. Thus, if (a, b, c) is a good abc triple, then c is not square-free.

Next, assume a and b are both square-free for the sake of contradiction. Without loss of generality, we may assume that $a \leq b$. Then $\operatorname{rad}(a) = a$ and $\operatorname{rad}(b) = b$. Since (a, b, c) is a good abc triple, we have that

$$c > \operatorname{rad}(a)\operatorname{rad}(b)\operatorname{rad}(c) = ab\operatorname{rad}(c). \tag{2.1}$$

Now suppose that $a = 1$. Then,

$$\begin{aligned} b\operatorname{rad}(c) < c \quad &\iff \quad b\operatorname{rad}(c) < 1 + b \\ &\iff \quad \operatorname{rad}(c) < \frac{1}{b} + 1 \leq 2, \end{aligned}$$

since $\frac{1}{b} \leq 1$. Thus, $\operatorname{rad}(c) < 2$ which implies that $\operatorname{rad}(c) = 1$. But this is impossible, which is our desired contradiction in the case when $a = 1$. It remains to consider the case when $a > 1$. In particular, $2 \leq a < b$ since a and b are relatively prime. Now observe that by (2.1),

$$c > ab\operatorname{rad}(c) > ab.$$

Since $a + b = c$, we have that $ab < a + b$. But this contradicts Lemma 1.11, which concludes the proof. □

The following is a generalization of [ASBHS23, Theorem 1].

Theorem 2.4. *Let a and c be positive integers with $c > 1$. If m divides $c - a$ and $\operatorname{cosocle}(m) > \operatorname{rad}(a)\operatorname{rad}(c)$, then $\left(a^k, c^k - a^k, c^k\right)$ is a good abc triple for each positive integer k.*

Proof. Since m divides $c-a$, $\operatorname{cosocle}(m)$ divides $c-a$. By assumption, $\operatorname{cosocle}(m) > \operatorname{rad}(a)\operatorname{rad}(c)$. Since the radical ignores the exponent, as it pertains only to the distinct prime factors of an integer, we have that $\operatorname{rad}(a^k) = \operatorname{rad}(a)$ and $\operatorname{rad}(c^k) = \operatorname{rad}(c)$. Now let $b = c^k - a^k$. By the difference of powers formula in Lemma 1.12, we know that

$$b = c^k - a^k = (c-a)\sum_{j=0}^{k-1} c^{k-1-j}a^j.$$

Since m divides $c-a$, we have that for any positive integer k, m divides $c^k - a^k$. By Lemma 1.8, we have that

$$\operatorname{rad}(c^k - a^k) = \operatorname{rad}\left(\frac{c^k - a^k}{\operatorname{cosocle}(m)}\right)$$

Next, observe that our assumption $\operatorname{cosocle}(m) > \operatorname{rad}(a)\operatorname{rad}(c)$ is equivalent to $\frac{\operatorname{rad}(a)\operatorname{rad}(c)}{\operatorname{cosocle}(m)} < 1$. Then,

$$\begin{aligned}
\operatorname{rad}(a^k \cdot (c^k - a^k) \cdot c^k) &= \operatorname{rad}(a)\operatorname{rad}(c)\operatorname{rad}(c^k - a^k) \\
&= \operatorname{rad}(a)\operatorname{rad}(c)\operatorname{rad}\left(\frac{c^k - a^k}{\operatorname{cosocle}(m)}\right) \\
&\le \operatorname{rad}(a)\operatorname{rad}(c)\frac{c^k - a^k}{\operatorname{cosocle}(m)} \\
&< c^k - a^k \qquad \text{since } \frac{\operatorname{rad}(a)\operatorname{rad}(c)}{\operatorname{cosocle}(m)} < 1 \\
&< c^k \qquad\qquad \text{since } a^k > 0.
\end{aligned}$$

Therefore, $\operatorname{rad}(a^k \cdot (c^k - a^k) \cdot c^k) < c^k$, implying that $\left(a^k, c^k - a^k, c^k\right)$ is a good abc triple for all positive integers k. □

3. Sequences of good abc triples

In this section, we consider some consequences from the results in the previous section. We begin with a consequence of Theorem 2.2, Proposition 2.3, and Theorem 2.4.

Corollary 3.1. *If $(2, c-2, c)$ is a good abc triple, then $(2^k, c^k - 2^k, c^k)$ is also a good abc triple for all positive integers k.*

Proof. Assume $(2, c-2, c)$ is a good abc triple. Since $a = 2$ is square-free, then by Proposition 2.3, $c-2$ cannot be square-free. Thus $c - 2 \neq 1$. Since $(2, c-2, c)$ is a good abc, we have that $\gcd(2, c-2) = 1$. This implies that $c-2$ is odd, and thus, $c-2$ has at least one odd prime factor, which further ensures that $2 < \operatorname{rad}(c-2)$. In particular, we satisfy the assumption of Theorem 2.2. This, then implies that $\operatorname{cosocle}(c-2) > \operatorname{rad}(2)\cdot\operatorname{rad}(c) = 2\operatorname{rad}(c)$. Taking $m = c-2$ in Theorem 2.4 allows us to conclude that $(2^k, c^k - 2^k, c^k)$ is a good abc triple for all positive integers k. □

3.1. Explicit sequences of good abc triples.

We now discuss some sequences of good abc triples.

Proposition 3.2. *For each positive integer k, the abc triple $(2^k, 11^{8k} - 2^k, 11^{8k})$ is good.*

Proof. Consider the abc triple $(2^k, 11^{8k} - 2^k, 11^{8k})$ for some integer k. According to Theorem 2.4, it suffices to show that there exists an integer m such that $m \mid 11^8 - 2$ and $\operatorname{cosocle}(m) > \operatorname{rad}(2)\cdot\operatorname{rad}(11^8) = 22$. To demonstrate this, we consider the prime factorization of $11^8 - 2$, which is $7^4 \cdot 73 \cdot 1223$. Now take $m = 7^4$. Then, $m|(11^8 - 2)$ and it is the case that $\operatorname{cosocle}(m) = 7^3 > 22$. Therefore, $(2^k, 11^{8k} - 2^k, 11^{8k})$ is a good abc triple for all positive integers k by Theorem 2.4.

□

Proposition 3.3. *For each positive integer k, the following are sequences of good abc triples.*

- $(2^k, c^k - a^k, 5^{11k})$
- $(2^k, c^k - a^k, 7^{10k})$
- $(2^k, c^k - a^k, 23^{5k})$
- $(3^k, c^k - a^k, 2^{7k})$
- $(3^k, c^k - a^k, 2^{16k})$
- $(5^k, c^k - a^k, 2^{3k})$
- $(6^k, c^k - a^k, 5^{4k} \cdot 7^{3k})$

- $(10^k, c^k - a^k, 13^{3k})$
- $(10^k, c^k - a^k, 43^{3k} \cdot 127^k)$
- $(10^k, c^k - a^k, 13^k \cdot 43^{4k})$
- $(10^k, c^k - a^k, 7^{3k} \cdot 19^{4k})$
- $(10^k, c^k - a^k, 53^k \cdot 101^{2k})$
- $(11^k, c^k - a^k, 3^{7k})$
- $(25^k, c^k - a^k, 3^{6k})$

Proof. We omit the proof as it is similar to the proof above. □

4. Pythagorean triples

Let a, b, and c be positive integers. If $a^2 + b^2 = c^2$, then we say that we have a Pythagorean triple. If we further require the condition that a and b be relatively prime, then we also have an abc triple. Specifically, a *primitive Pythagorean triple*. Particularly intriguing scenarios arise when a Pythagorean triple also qualifies as a good abc triple. Such cases are rare, as expected, making them of special interest in the study of both Pythagorean triples and good abc triples. In this section, we will explore formulas that generate Pythagorean triples, which are also good abc triples. These formulas were found by using Euclid's classification of primitive Pythagorean triples, which states:

Theorem 4.1 (Euclid, 300 B.C)**.** *Suppose (a^2, b^2, c^2) is an abc triple. Then there exist relatively prime integers p and q of opposite parity with $p > q$ such that*

$$a = 2pq, \qquad b = p^2 - q^2, \qquad c = p^2 + q^2.$$

From the abc@home data, we were able to analyze all good Pythagorean abc triples (a^2, b^2, c^2) with $c^2 < 10^{18}$. We observed that most of these triples satisfy $\frac{p}{q} \approx 1$, and in particular, several of these satisfied the equation $p = q + 1$. This led us to investigate the special case when $q = 2^n$ and $p = q + 1$. In this setting, we concluded the following two results:

Theorem 4.2. *Let n be a positive odd integer that is divisible by 3. If $q = 2^n$ and $p = q + 1$, then $((2pq)^2, (p^2 - q^2)^2, (p^2 + q^2)^2)$ is a good Pythagorean abc triple.*

Proof. Since n is divisible by 3, and never even, we have that $n \equiv 3 \pmod 6$. In particular, $n = 3 + 6k$ for some integer k. Now observe that

$$p = 2^n + 1 = 2^{3+6k} + 1 = 8 \cdot 2^{6k} + 1.$$

By Theorem 1.15, we deduce that $2^{6k} = (2^k)^6 \equiv 1 \pmod 9$ since $\phi(9) = 6$. Thus,

$$2^n + 1 \equiv 8 + 1 \pmod 9 = 0 \pmod 9.$$

Now consider,

$$\begin{aligned}
&a = 2pq = 2^{n+1}(2^n + 1),\\
&b = p^2 - q^2 = 2^{2n} + 2^{n+1} + 1 - 2^{2n} = 2^{n+1} + 1,\\
&c = p^2 + q^2 = 2^{2n} + 2^{n+1} + 1 + 2^{2n} = 2^{2n+1} + 2^{n+1} + 1.
\end{aligned}$$

We want to show that (a^2, b^2, c^2) is a good abc triple. That is $\operatorname{rad}(a^2b^2c^2) = \operatorname{rad}(abc) < c^2$. To this end, observe that

$$\operatorname{rad}(abc) = \operatorname{rad}(a)\operatorname{rad}(b)\operatorname{rad}(c) = \operatorname{rad}(2^{n+1}(2^n + 1))\operatorname{rad}(b)\operatorname{rad}(c) = 2\operatorname{rad}(2^n + 1)\operatorname{rad}(b)\operatorname{rad}(c)$$

Since $2^n + 1 \equiv 0 \pmod 9$, it is the case that $\operatorname{rad}(2^n + 1) = \operatorname{rad}\left(\frac{2^n+1}{3}\right)$. Thus,

$$2\operatorname{rad}(2^n + 1)\operatorname{rad}(b)\operatorname{rad}(c) = 2\operatorname{rad}\left(\frac{2^n + 1}{3}\right)\operatorname{rad}(b)\operatorname{rad}(c).$$

This implies that

$$2\operatorname{rad}\left(\frac{2^n+1}{3}\right)\operatorname{rad}(b)\operatorname{rad}(c) \le \frac{2}{3}(2^n+1)bc.$$

We want to show $\frac{2}{3}(2^n+1)bc < c^2$. This is equivalent to showing $\frac{2}{3}(2^n+1)b < c$. Since n is a positive integer, consider

$$\begin{aligned}
0 < 2^{2n+1}+1 \quad &\iff \quad 0 < 3\cdot 2^{2n+1} - 2\cdot 2^{2n+1} + 1\\
&\iff \quad 2^{2n+2} < 3\cdot 2^{2n+1}+1\\
&\iff \quad 2^{2n+2} + 3\cdot 2^{n+1} + 2 < 3\cdot 2^{2n+1} + 3\cdot 2^{n+1} + 3\\
&\iff \quad \frac{2}{3}(2^{2n+1} + 3\cdot 2^n + 1) < 2^{2n+1} + 2^{n+1} + 1\\
&\iff \quad \frac{2}{3}(2^n+1)(2^{n+1}+1) < 2^{2n+1} + 2^{n+1} + 1\\
&\iff \quad \frac{2}{3}(2^n+1)b < c.
\end{aligned}$$

Since the inequality $\frac{2}{3}(2^n+1)b < c$ holds, we conclude that

$$\frac{2}{3}(2^n+1)bc < c^2,$$

which shows that

$$\operatorname{rad}(abc) < c^2.$$

Thus, the abc triple (a^2, b^2, c^2) is a good Pythagorean abc triple. $\square$

Theorem 4.3. *Let n be a positive integer such that $n \equiv 10 \pmod{20}$. If $q = 2^n$ and $p = q+1$, then $((2pq)^2, (p^2-q^2)^2, (p^2+q^2)^2)$ is a good Pythagorean abc triple.*

Proof. Since $n \equiv 10 \pmod{20}$, we have that $n = 10 + 20k$ for some integer k. So

$$p = 2^n + 1 = 2^{10+20k} + 1 = 2^{10}\cdot 2^{20k} + 1$$

By Theorem 1.15 we deduce that $2^{20k} \equiv 1 \pmod{25}$ since $\phi(25) = 20$. Thus,

$$2^{10}\cdot 2^{20k} + 1 \equiv 1024\cdot 1 + 1 \pmod{25} = 0 \pmod{25}$$

Consider

$$\begin{aligned}
&a = 2pq = 2^{n+1}(2^n+1)\\
&b = p^2 - q^2 = 2^{2n} + 2^{n+1} + 1 - 2^{2n} = 2^{n+1} + 1\\
&c = p^2 + q^2 = 2^{2n} + 2^{n+1} + 1 + 2^{2n} = 2^{2n+1} + 2^{n+1} + 1
\end{aligned}$$

We want to show that (a^2, b^2, c^2) is a good abc triple. That is, $\operatorname{rad}(a^2b^2c^2) = \operatorname{rad}(abc) < c^2$. Observe that

$$\operatorname{rad}(abc) = \operatorname{rad}(a)\operatorname{rad}(b)\operatorname{rad}(c) = \operatorname{rad}(2^{n+1}(2^n+1))\operatorname{rad}(b)\operatorname{rad}(c) = 2\operatorname{rad}(2^n+1)\operatorname{rad}(b)\operatorname{rad}(c)$$

Since $2^n + 1 \equiv 0 \pmod{25}$, we have that $\operatorname{rad}(2^n+1) = \operatorname{rad}(\frac{2^n+1}{5})$. Thus,

$$2\operatorname{rad}(2^n+1)\operatorname{rad}(b)\operatorname{rad}(c) = 2\operatorname{rad}\left(\frac{2^n+1}{5}\right)\operatorname{rad}(b)\operatorname{rad}(c)$$

This implies that

$$2\operatorname{rad}(\frac{2^n+1}{5})\operatorname{rad}(b)\operatorname{rad}(c) \le \frac{2}{5}(2^n+1)bc$$

We want to show $\frac{2}{5}(2^n+1)bc < c^2$. That is $\frac{2}{5}(2^n+1)b < c$. Since n is a positive integer, consider

$$\begin{aligned}
0 < 3\cdot 2^{2n+1} + 2^{n+2} + 3 \quad &\iff \quad 0 < 5\cdot 2^{2n+1} - 2\cdot 2^{2n+1} + 2^{n+2} + 3\\
&\iff \quad 2^{2n+2} < 5\cdot 2^{2n+1} + 2^{n+2} + 3\\
&\iff \quad 2^{2n+2} + 2 < 5\cdot 2^{2n+1} + 2^{n+2} + 5\\
&\iff \quad 2^{2n+2} + 3\cdot 2^{n+1} + 2 < 5\cdot 2^{2n+1} + 5\cdot 2^{n+1} + 5\\
&\iff \quad \frac{2}{5}(2^{2n+1} + 3\cdot 2^n + 1) < 2^{2n+1} + 2^{n+1} + 1\\
&\iff \quad \frac{2}{5}(2^n+1)(2^{n+1}+1) < 2^{2n+1} + 2^{n+1} + 1\\
&\iff \quad \frac{2}{5}(2^n+1)b < c
\end{aligned}$$

Since the inequality $\frac{2}{5}(2^n+1)b < c$ holds, we conclude that

$$\frac{2}{5}(2^n+1)bc < c^2.$$

This implies that

$$\mathrm{rad}(abc) < c^2.$$

Thus, (a^2, b^2, c^2) is a good Pythagorean abc triple. □

5. Conclusion

Future research on abc triples offers several promising directions. Prospective avenues to expand upon our research could include investigating generalizing statements such as [ASBHS23, Theorem 2], which pertains to good abc triples of the form $(1, b^k, b^k+1)$. Further, we believe that further exploration of Pythagorean triples may yield additional sequences and results based on patterns we have observed. We note that the proofs of our findings related to Pythagorean triples are similar, which could imply that a more generalized statement could be made.

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Appendix A.

Below, we include sample SageMath code [Dev24] that we used throughout the summer. We note that while we do not include all the code that utilized in our research project, the omitted code can be obtained from modifications to the code below.

Listing 1. SageMath Code for Checking Powers of Two and Three

```
import csv
filename = "abctriples.csv"

def is_power_of_three(b):
    n = 0
    while 3**n < b:
        n += 1
    return 3**n == b, n

def is_power_of_two(a):
    if a <= 0:
        return False, None
    k = 0
    while a % 2 == 0:
        a = a // 2
        k += 1
    return a == 1, k if a == 1 else None
```

Listing 2. CSV Processing Script

```
from math import log
from sage.all import*
import csv

def quality(a,b):
    c = a + b
    return round(log(c) / log(radical(a*b*c)), 8)

# Function to process a chunk of rows
def process_chunk(chunk, writer):
    for row in chunk:
        A, B, C, q = row
        a, b, c = ZZ(A), ZZ(B), ZZ(C)
        if a == 5 and c < 10**8:
            for k in range(1, 30):
                qk = quality(a**k, c**k - a**k)
                writer.writerow([a, k, factor(a**k),
                factor(c**k - a**k), factor(c**k), qk])

# File names
```

```
input_csv_file = 'abctriples.csv'
output_csv_file = 'exponent-a-is-30.csv'

# Define the chunk size
chunk_size = 100000

# Open the output CSV file in write mode and write the header
with open(output_csv_file, 'w', newline='') as out_csvfile:
    writer = csv.writer(out_csvfile)
    writer.writerow(['a', 'k', 'a^k', 'b^k', 'c^k', 'quality'])

    # Open the input CSV file in read mode
    with open(input_csv_file, 'r') as in_csvfile:
        reader = csv.reader(in_csvfile)

        # Skip the header row
        next(reader)

        # Initialize an empty chunk
        chunk = []

        for row in reader:
            chunk.append(row)
            if len(chunk) >= chunk_size:
                # Process and write the chunk
                process_chunk(chunk, writer)
                # Clear the chunk to free up memory
                chunk = []

        # Process and write any remaining rows
        if chunk:
            process_chunk(chunk, writer)

print("Processing complete. The new CSV file has been created.")
```

Listing 3. Modulo Calculation Script

```
Z = []
for k in range(49):
    Z.append(mod(Integer(11)**(8*k) - Integer(2)**k, Integer(49)))
print(set(Z))
```

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