Zero Divisor Graphs and Domination Numbers
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1 Definitions and Notation

To begin, necessary terms and notations will be introduced in order to understand the rest of the report.

A ring is a set R together with two binary operations, usually called addition and multiplication. The set R is closed under these two operations and satisfy a set of ring axioms [4]. Examples of rings include $\mathbb{Z}$ and $\mathbb{R}$. A type of ring that will be particularly relevant in this report is $\mathbb{Z}_n$, defined as $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ with addition and multiplication modulo n. A *zero-divisor* is an element $r \in R$ for which there exists $t \neq 0 \in R$ such that $rt = 0$. The notation $Z(R)$ is used when referring to the set that consists of all zero-divisors. The notation $Z^*(R)$ is used when referring to the set that consists of only non-zero zero-divisors. For every $r \in R$, the annihilator of r, denoted $\text{ann}(r)$, is the set of all $t \in Z(R)$ such that $rt = 0$.

A *graph* $G = (V(G), E(G))$ consists of a nonempty set of vertices $V(G)$, and a set of edges $E(G)$, where an edge connects two vertices together. There are two types of graphs that are relevant to this paper. First, a *star graph*, shown on the left in Figure 1, is a graph with one central vertex that connects to all other vertices. No other vertex has any connections besides its connection to the central vertex. Second, a *complete bipartite graph*, shown on the right in Figure 1, is a graph with two groups of vertices. A vertex in the graph connects to all vertices in the other group, but it connects to no vertices in its own group.

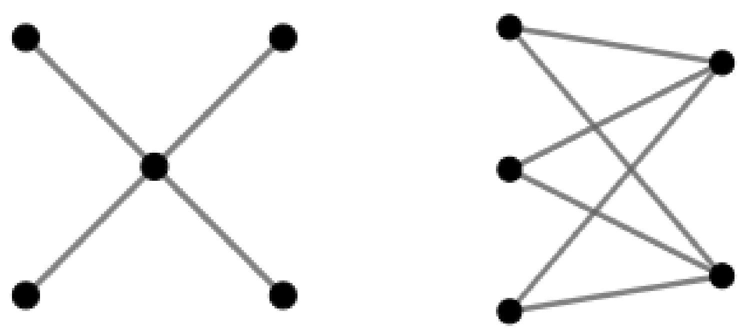

Figure 1: A Star Graph and a Complete Bipartite Graph

A *zero-divisor graph* is a graph in which the vertices are the non-zero zero-divisors of R, and $\{u, v\} \in E(G)$ indicates that $uv = 0$.

A *dominating set* of a graph G is a subset of the vertices, $D \subseteq V(G)$, such that for each $v \notin D$, there is a $d \in D$ such that $\{v, d\} \in E(G)$. That is, v is connected to at least one vertex in D. It is possible for a graph to have more than one dominating set. The *domination number* of a graph G is the number of vertices in a smallest possible dominating set and is denoted $\gamma(G)$. A *minimum dominating set* is a set with the number of elements equal to $\gamma(G)$ and is not necessarily unique.

Similarly a *total dominating set* of a graph G is a subset of the vertices, $D \subseteq V(G)$, such that for each $v \in V(G)$, there is a $d \in D$ such that $\{v, d\} \in E(G)$. That is, v is connected to at least one vertex in D. Also, it is possible for a graph to have more than one total dominating set. The *total* *domination number* of a graph G is the number of vertices in a smallest possible total dominating set and is denoted $\gamma_T(G)$. A *minimum total dominating set* is a set with the number of elements equal to $\gamma_T(G)$ and is not necessarily unique.

2 Problem Statement and Research Goals

This project has the intention of studying the structure of zero-divisor graphs in regards to various graph theory notions of dominance. The goal is to determine if understanding dominating sets and the domination number will reveal information about the original ring.

Creating graphs out of rings and studying the structure of the resulting graphs has been an active area of research for the past 25 years. Only recently has dominance been brought into the fray, meaning there is a lot of work to be done.

3 Results and Methodology

3.1 Results for $\mathbb{Z}_n$

We began this project by specifically examining $\mathbb{Z}_n$ rings. One of our first findings was a known result about zero-divisors of $\mathbb{Z}_n$. Lemma 3.1 is a basic but important result that is used in several of the proofs that follow. Lemma 6 and Proposition 3.2 explore the domination number and total domination number for the zero-divisor graph of rings of the form $\mathbb{Z}_n$, including special cases of the value of n. We then looked at direct products $\mathbb{Z}_n$ rings.

Proposition 3.1 (Zero-Divisors of $\mathbb{Z}_n$ Proposition)**.** *Let $n = p_1 p_2 \dots p_i$ such that $p_1, \dots, p_i$ are primes. The elements of $Z(\mathbb{Z}_n)$ are all the multiples of $p_1, \dots, p_i$.*

Proof. Without loss of generality, take p_1 and $b > 0 \in \mathbb{Z}$ such that $bp_1 \in \mathbb{Z}_n$. Then, $bp_1 \in Z^*(\mathbb{Z}_n)$ since $bp_1(p_2 p_3 \dots p_i) = bn \equiv 0 \mod n$.

Conversely, let $c \in \mathbb{Z}_n$ and $\gcd(c, n) = 1$. By way of contradiction, suppose that $c \in Z^*(\mathbb{Z}_n)$. This implies that there is a w such that $cw \equiv 0 \mod n$. Then, $w = p_1 p_2 \dots p_i \equiv 0 \mod n$ due to the Fundamental Theorem of Arithmetic. This is a contradiction because w must be non-zero. ☃

Since the only prime factor of p is p itself and $p \notin \mathbb{Z}_p$, $\mathbb{Z}_p$ contains no non-zero zero-divisors. Corollary 3.1 follows.

Corollary 3.1. *There are no non-zero zero-divisors of Z_p for p a prime.*

In many of the following proofs, we use Lemma 3.1 when finding the minimum total domination number.

Lemma 3.1. *Given a ring R, let $G = \Gamma(R)$. If D is both a minimum dominating set and a total dominating set of G, then D is also a minimum total dominating set of G.*

Proof. Suppose that D is a minimum dominating set and a total dominating set for G, containing q elements. Then, $D = \{d_1, d_2, \dots, d_q\}$.

By definition, $\gamma(G) \leq \gamma_T(G)$. Because D is a minimum dominating set, $\gamma(G) = q \leq \gamma_T(G)$. Additionally, since D is also total dominating that has cardinality q, then $\gamma_T(G) \leq q$. Thus, $\gamma_T(G) = q$, and D is a minimum total dominating set of G. ☃

Lemma 6 is used exclusively to help prove Proposition 3.2. It was not originally among the results in our research, but it had to be added, when we discovered a problem with the original proof for Proposition 3.2.

Lemma 3.2. *(Lower Bound for Domination Number) Let $G = \Gamma(Z_n)$ for $n = p_1^{a_1} p_2^{a_2} \dots p_q^{a_q}$ And n is neither prime nor $2p$ for p a prime, $p \neq 2$. Then $\gamma(G) \geq q$.*

Proof. Proof in Appendix ∎

The result of Proposition 3.2 was found early in our research and is used in the proof of Theorem 3.4. This proof was critical to understanding that result.

Proposition 3.2. *Let $G = \Gamma(\mathbb{Z}_n)$. If $n = 2p$, where prime $p \neq 2$, then $\gamma(G) = 1$ and $\gamma_T(G) = 2$. Otherwise, if $n = p_1^{a_1} p_2^{a_2} \dots p_q^{a_q}$, then $\gamma(G) = \gamma_T(G) = q$.*

Proof. Proof in Appendix ∎

Next, our research investigated the direct product of $\mathbb{Z}_n$ rings. It is a known result that if $m, n > 0 \in \mathbb{Z}$ and $\gcd(m,n) = 1$, then $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$. As we explored this result, we also wondered about the direct product of rings $\mathbb{Z}_m, \mathbb{Z}_n$ where $\gcd(m,n) \neq 1$. Results from that exploration are shown in the following Proposition 3.3 and Theorem 3.1.

Proposition 3.3. *Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_p$, where p is prime, and let $G = \Gamma(R)$. Then, $\gamma(G) = 1$ and $\gamma_T(G) = 2$.*

Proof. By Corollary 3.1, neither $\mathbb{Z}_2$ nor $\mathbb{Z}_p$ has any non-zero zero-divisors. Consequently, the only non-zero zero-divisors in G are $(r_n, 0)$ and $(0, t_n)$. The only element of form $(r_n, 0)$ is $(1, 0)$ because 1 is the only non-zero element in $\mathbb{Z}_2$. Note that $(1, 0)$ connects to every element of the form $(0, t_n)$. As a result, the set $D = \{(1, 0)\}$ is a dominating set for G. Because a dominating set must have at least one element, $\gamma(G) = 1$.

The element $(1, 0)$ does not connect to itself, and no $(0, t_n)$ will connect to any other $(0, t_n)$. Therefore, no element in G connects to every element in G, so $\gamma_T(G)$ must be greater than 1. Now, let $D_T = \{(1, 0), (0, 1)\}$. This is a total dominating set since $(1, 0)(0, 1) = (0, 0)$. By Lemma 3.1 D_T is a minimum total dominating set, so $\gamma_T(G) = 2$. ∎

Theorem 3.1. *Let $R \cong \mathbb{Z}_{{p_1}^{a_1}} \times \mathbb{Z}_{{p_2}^{a_2}} \times \dots \times \mathbb{Z}_{{p_n}^{a_n}}$ with all p_i primes, p_i, p_j not necessarily distinct, and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_p$, for p, a prime. Let $G = \Gamma(R)$. Then, $\gamma(G) = \gamma_T(G) = n$.*

Proof. First, determine elements of R that are non-zero zero-divisors. The non-zero zero-divisors can be represented as one general form, $y_i \in Z^*(R)$:

$$y_i = (l_1, l_2, \dots, l_{i-1}, bp_i, l_{i+1}, \dots, l_n)$$

an arbitrary $l_i \in \mathbb{Z}_{{p_i}^{a_i}}$ and $bp_i \in Z(\mathbb{Z}_{{p_i}^{a_i}})$.

Now, find a dominating set, D, for G. Let $u_i = (0, 0, \dots, 0, p_i^{a_i - 1}, 0, \dots, 0)$, with $u_i \in R$. Every $y_i u_i = (0, \dots, 0)$. Then, the set $D = \{u_1, u_2, \dots u_n\}$ is a dominating set for G since every possible y_i connects to at least one vertex in D. Additionally, D is a total dominating set since $u_i u_j = (0, \dots, 0)$, $i \neq j$. Notice that D contains n elements.

Now, it must be shown that D is a minimum dominating set for G. Consider $v_m, v_n \in Z^*(R)$, in which

$$v_m = (1, 1, \ldots, 1, 0_m, 1, \ldots, 1), \text{ and } v_n = (1, 1, \ldots, 1, 0_n, 1, \ldots, 1)$$

and $m \neq n$. By way of contradiction, assume there exists $w \in Z^*(R)$ such that $wv_m = (0, \ldots, 0)$ and $wv_n = (0, \ldots, 0)$.

Now, if $wv_m = (0, \ldots, 0)$, then w must be a multiple of $(0, 0, \ldots, 0, 1_m, 0, \ldots, 0)$, and if $wv_n = (0, \ldots, 0)$, then w must be a multiple of $(0, 0, \ldots, 0, 1_n, 0, \ldots, 0)$. Therefore, $w = (0, \ldots, 0)$. This is a contradiction because w must be non-zero.

Because no element in G connects to two distinct v_i, there are three possibilities.

Case 1:
There is one element in a minimum dominating set for G for each of the distinct v_i. In this case, there are n elements in a minimum dominating set.

Case 2:
Some but not all of the v_i are in a minimum dominating set. Since no v_i connects to another element of that form, for any v_i not in the minimum dominating set, there must be some element of a different form in the minimum dominating set to connect to it. As previously established, since two distinct v_i do not share any connections, there must be a distinct element in the dominating set for each v_i not in the dominating set. Then, there are at least n elements in a minimum dominating set.

Case 3:
The case in which all v_i are in a minimum dominating set results in the dominating set having at least n elements.

This demonstrates that a dominating set for G must contain at least one distinct element for each $\mathbb{Z}_{{p_i}^{a_i}} \in R$. Therefore, since D has the same number of elements as there are rings in the direct product, D is a minimum dominating set for G. By Lemma 3.1, D is a minimum total dominating set for G. Thus, $\gamma(G) = \gamma_T(G) = n$.

3.2 Results for Direct Product of Arbitrary Rings

After the direct product of $\mathbb{Z}_n$ rings, we explored the direct product of any arbitrary ring. We wondered if, given the domination or total domination numbers of two zero-divisor graphs of arbitrary rings, the domination number or total domination number of their direct product could be determined. The following is much of the notation used for these arbitrary rings.

Let R, T be rings with non-zero zero-divisors. The elements of $R \times T$ are of four forms: (r_z, t_z), (r_n, t_z), (r_z, t_n), and (r_n, t_n), where $r_z \in Z(R)$, $r_n \notin Z(R)$, $t_z \in Z(T)$, and $t_n \notin Z(T)$.

Let $G = \Gamma(R)$. There exists a minimum total dominating set for G, $D_G = \{u_1, u_2, \ldots, u_m\}$. Let $H = \Gamma(T)$. There exists a minimum total dominating set for H, $D_H = \{v_1, v_2, \ldots, v_n\}$.

In $Z(R \times T)$, elements are of three forms: $(r_z, t_z), (r_n, t_z)$, and (r_z, t_n). Then, $Z^*(R \times T)$ is all elements of one of these three forms, except $(0, 0)$, which can be further be divided into seven total forms. Observe what is true for each, noting the following notation:

- $r \in R, t \in T$;

- $r_z^*, \hat{r_z} \in Z^*(R)$, $t_z^*, \hat{t_z} \in Z^*(T)$;
- $r_z^* \hat{r_z} = 0_R, t_z^* \hat{t_z} = 0_T$.

Consider elements of the form (r_z, t_z). There are three possibilities:

1. $(0, t_z^*)$ and $\operatorname{ann}(0, t_z^*) = \{(r, \hat{t_z})\} \cup \{(r, 0)\}$
2. $(r_z^*, 0)$ and $\operatorname{ann}(r_z^*, 0) = \{(\hat{r_z}, t)\} \cup \{(0, t)\}$
3. (r_z^*, t_z^*) and $\operatorname{ann}(r_z^*, t_z^*) = \{(\hat{r_z}, \hat{t_z})\} \cup \{(0, \hat{t_z})\} \cup \{(\hat{r_z}, 0)\}$

Note that $\operatorname{ann}(r_z^*, t_z^*) \subseteq \operatorname{ann}(r_z^*, 0)$ because $\{(\hat{r_z}, \hat{t_z})\} \cup \{(0, \hat{t_z})\} \cup \{(\hat{r_z}, 0)\} \subseteq \{(\hat{r_z}, t)\} \cup \{(0, t)\}$.

Consider elements of the form (r_n, t_z). There are two possibilities:

1. $(r_n, 0)$ and $\operatorname{ann}(r_n, 0) = \{(0, t)\}$
2. (r_n, t_z^*) and $\operatorname{ann}(r_n, t_z^*) = \{(0, \hat{t_z})\} \cup \{(0, 0)\}$

Note that $\operatorname{ann}(r_n, t_z^*) \subseteq \operatorname{ann}(r_n, 0)$ because $\{(0, \hat{t_z})\} \cup \{(0, 0)\} \subseteq \{(0, t)\}$.

Consider elements of the form (r_z, t_n). Again there are two possibilites:

1. $(0, t_n)$ and $\operatorname{ann}(0, t_n) = \{(r, 0)\}$.
2. (r_z^*, t_n) and $\operatorname{ann}(r_z^*, t_n) = \{(\hat{r_z}, 0) \cup \{(0, 0)\}$

Note that $\operatorname{ann}(r_z^*, t_n) \subseteq \operatorname{ann}(0, t_n)$ because $\{(\hat{r_z}, 0)\} \cup \{(0, 0)\} \subseteq \{(r, 0)\}$.

Originally, we wanted show there exists a minimum dominating set for the graph of a direct product such that each ordered pair in the dominating set is $(r_z^*, 0)$ or $(0, t_z^*)$. We were unable to prove this result, but we were able to prove an alternative result in which the set is a minimum total dominating set. Lemmas 3.3 and 3.4 are used in proving the main result in Theorem 3.2. Lemma 3.3 establishes the existence of a minimum total dominating set for the zero-divisor graph of a direct product, $\Gamma(R \times T)$, that has the form S_F given below. Lemma 3.4 then establishes what connections must exist in such a minimum total dominating set. The main result, Theorem 3.2, shows that the total domination number of the zero-divisor graph of the direct product of rings with non-zero divisors is the sum of the total domination numbers of the zero-divisor graphs of the component rings.

Lemma 3.3. *Let R, T be rings with non-zero zero-divisors. Let $F = \Gamma(R \times T)$. There exists a minimum total dominating set for F of form*

$$S_F = \{(r_1^*, 0), (r_2^*, 0), \ldots, (r_c^*, 0), (0, t_1^*), (0, t_2^*), \ldots, (0, t_d^*)\}.$$

in which $r_i^* \in \mathbb{Z}^*(R)$ and $t_j^* \in \mathbb{Z}^*(T)$.

Proof. There exists a minimum total dominating set for F, $D_F = \{w_1, w_2, \ldots, w_q\}$.

Without loss of generality, suppose $w_1 = (r_n, 0)$. Replace w_1 with $w_1^* = (r_z^*, 0)$ in the total dominating set. Since $\operatorname{ann}(w_1) \subseteq \operatorname{ann}(w_1^*)$, any elements that connected to w_1 in the original total dominating set will connect to w_1^* in the new total dominating set. Recall that the set D_F is a total dominating set. Because $r_n \notin Z(R)$, $(r_n, 0)(r_n, 0) \neq (0, 0)$. This means that w_1 has a connection $w_i \in D_F \backslash \{w_1\}$. If a $w_i \in D_F$ has been replaced with w_i^*, w_1 will also connect with its replacement. As a result, w_1 still has a connection in the new total dominating set.

Similarly, without loss of generality, suppose $w_2 = (0, t_n)$. Replace w_2 with $w_2^* = (0, t_z^*)$ in the total dominating set. Since $\mathrm{ann}(w_2) \subseteq \mathrm{ann}(w_2^*)$, any elements that connected to w_2 in the original total dominating set will connect to w_2^* in the new total dominating set. Recall that the set D_F is a total dominating set. Because $t_n \notin Z(T)$, $(0, t_n)(0, t_n) \neq (0, 0)$. This means that w_2 has a connection $w_i \in D_F \backslash \{w_2\}$. If a $w_i \in D_F$ has been replaced with w_i^*, w_2 will also connect with its replacement. As a result, w_2 still has a connection in the new total dominating set.

Without loss of generality, suppose $w_3 = (r_n, t_z^*)$. Replace w_3 with $w_3^* = (0, t_z^*)$ in the total dominating set. Since $\mathrm{ann}(w_3) \subseteq \mathrm{ann}(w_3^*)$, any elements that connected to w_3 in the original total dominating set will connect to w_3^* in the new total dominating set. Recall that the set D_F is a total dominating set. Because $(r_n, t_z^*)(r_n, t_z^*) \neq (0, 0)$, w_3 has a connection $w_i \in D_F \backslash \{w_3\}$. If a $w_i \in D_F$ has been replaced with w_i^*, w_3 will also connect with its replacement. As a result, w_3 still has a connection in the new total dominating set.

Similarly, without loss of generality, suppose $w_4 = (r_z^*, t_n)$. Replace w_4 with $w_4^* = (r_z^*, 0)$ in the total dominating set. Since $\mathrm{ann}(w_4) \subseteq \mathrm{ann}(w_4^*)$, any elements that connected to w_4 in the original total dominating set will connect to w_4^* in the new total dominating set. Recall that the set D_F is a total dominating set. Because $(r_z^*, t_n)(r_z^*, t_n) \neq (0, 0)$, w_4 has a connection $w_i \in D_F \backslash \{w_4\}$. If a $w_i \in D_F$ has been replaced with w_i^*, w_4 will also connect with its replacement. As a result, w_4 still has a connection in the new total dominating set.

Without loss of generality, suppose $w_5 = (r_z^*, t_z^*)$. Replace w_5 with $w_5^* = (r_z^*, 0)$ in the total dominating set. Since $\mathrm{ann}(w_5) \subseteq \mathrm{ann}(w_5^*)$, any elements that connected to w_5 in the original total dominating set will connect to w_5^* in the new total dominating set. Recall that the set D_F is a total dominating set. This means that w_5 connects to at least one $w_i \in D_F$. There are two cases that result.

Case 1: The element w_5 connects to $w_i \in D_F \backslash \{w_5\}$.
If this is the case, w_5 still has a connection in the new total dominating set, w_i. If that $w_i \in D_F$ has been replaced with w_i^*, w_i will also connect with its replacement. Thus, w_5 will still have a connection in the new total dominating set.

Case 2: The element w_5 only connects to w_5 in D_F.
If w_5 is its only connection in D_F, then $(r_z^*, t_z^*)(r_z^*, t_z^*) = (0, 0)$. Notably, $r_z^* r_z^* = 0$. As a result, w_5 will connect to w_5^* because $(r_z^*, t_z^*)(r_z^*, 0) = (0, 0)$.

As a result, a minimum total dominating set has been created,

$$\hat{D_F} = \{(r_1^*, 0), (r_2^*, 0), \ldots, (r_c^*, 0), (0, t_1^*), (0, t_2^*), \ldots, (0, t_d^*)\}.$$

Note that $c + d = q$.

Lemma 3.4. *Let R, T be rings with non-zero zero-divisors, and let $F = \Gamma(R \times T)$. There exists a minimum total dominating set for F,*

$$S_F = \{(r_1^*, 0), (r_2^*, 0), \ldots, (r_c^*, 0), (0, t_1^*), (0, t_2^*), \ldots, (0, t_d^*)\}$$

in which $r_i^* \in \mathbb{Z}^*(R)$ and $t_j^* \in \mathbb{Z}^*(T)$. For this set, every element in $Z^*(R \times T)$ of the form (r_z, t_n) connects to at least one $(r_z^*, 0)$, every element of the form $(r_n, t_z) \in Z^*(R \times T)$ connects to at least one $(0, t_z^*)$, and every element of the form $(r_z, t_z) \in Z^*(R \times T)$ connects to at least one of both $(r_z^*, 0), (0, t_z^*)$.

Proof. By Lemma 3.3, there exists a minimum total dominating set for F,

$$S_F = \{(r_1^*, 0), (r_2^*, 0), \dots, (r_c^*, 0), (0, t_1^*), (0, t_2^*), \dots, (0, t_d^*)\}.$$

This implies that every element in $Z^*(R \times T)$ connects to at least one element in S_F.

Consider $(r_z, t_n) \in Z^*(R \times T)$. An arbitrary element (r_z, t_n) does not connect to any vertex of the form $(0, t_z^*) \in S_F$, since $t_n t_z^* \neq 0_T$. Therefore, that arbitrary (r_z, t_n) must connect to at least one $(r_z^*, 0) \in S_F$. That is, $(r_z, t_n)(r_z^*, 0) = (0, 0)$. Additionally, because this is true, an arbitrary $(r_z, t_z) \in Z^*(R \times T)$ must also connect to at least one $(r_z^*, 0) \in S_F$, since $(r_z, t_z)(r_z^*, 0) = (0, 0)$.

Similarly, consider $(r_n, t_z) \in Z^*(R \times T)$. An arbitrary element (r_n, t_z) does not connect to any element of the form $(r_z^*, 0) \in S_F$, since $r_n r_z^* \neq 0_R$. Therefore, that arbitrary (r_n, t_z) must connect to at least one $(0, t_z^*) \in S_F$. That is, $(r_n, t_z)(0, t_z^*) = (0, 0)$. Because this is true, an arbitrary $(r_z, t_z) \in Z^*(R \times T)$ must also connect to at least one $(0, t_z^*) \in S_F$, since $(r_z, t_z)(0, t_z^*) = (0, 0)$.

Thus, it is shown that every $(r_z, t_n) \in Z^*(R \times T)$ connects to at least one $(r_z^*, 0) \in S_F$, every $(r_n, t_z) \in Z^*(R \times T)$ to at least one $(0, t_z^*) \in S_F$, and every $(r_z, t_z) \in Z^*(R \times T)$ to at least one of both $(r_z^*, 0), (0, t_z^*) \in S_F$.

This is our most significant result relating to the direct product of rings. We had hoped to find the result of the domination number of the zero-divisor graph of the direct product, but we were able to prove the result of the total domination number of the zero-divisor graph of the direct product.

Theorem 3.2. *Let R, T be rings with non-zero zero-divisors, and let $G = \Gamma(R)$ and $H = \Gamma(T)$. Let $F = \Gamma(R \times T)$. The total domination number of F is equal to the sum of the total domination numbers of G and H.*

Proof. By Lemma 3.3, there exists a minimum total dominating set for F,

$$S_F = \{(r_1^*, 0), (r_2^*, 0), \dots, (r_c^*, 0), (0, t_1^*), (0, t_2^*), \dots, (0, t_d^*)\}$$

Consider all elements in $Z^*(R \times T)$ that are not a part of S_F. These vertices will be of three forms: $(r_z, t_n), (r_n, t_z)$, and (r_z, t_z). It is shown in Lemma 3.4 that any (r_z, t_n) connects to at least one $(r_z^*, 0) \in S_F$, any (r_n, t_z) connects to at least one $(0, t_z^*) \in S_F$, and any (r_z, t_z) connects to at least one of both $(r_z^*, 0)$, $(0, t_z^*) \in S_F$.

Now, let $D_r^* = \{r_1^*, r_2^*, \dots, r_c^*\}$. By Lemma 3.4, any arbitrary $(r_z, t_n) \in Z^*(R \times T)$ connects to at least one $(r_z^*, 0) \in S_F$. Notably, for every $r_z^* \in Z^*(R)$, there exists an $(r_z^*, t_n) \in Z^*(R \times T)$. Because every (r_z^*, t_n) has a connection to a $(r_z^*, 0) \in S_F$, then every r_z^* will have a connection in D_r^*. Thus, D_r^* is a total dominating set for G. Now, it can be shown that D_r^* is a minimum total dominating set for G.

For sake of contradiction, suppose D_r^* is not a minimum total dominating set for G. Then, let $\overline{D}_r = \{\overline{r}_1, \overline{r}_2, \dots, \overline{r}_k\}$ be a minimum total dominating set for G, such that each $\overline{r}_j \in \overline{D}_r$ is a non-zero zero-divisor, and $k < c$.

Now, let $\overline{S}_F = \{(\overline{r}_1, 0), (\overline{r}_2, 0), \dots, (\overline{r}_k, 0), (0, t_1^*), (0, t_2^*), \dots, (0, t_d^*)\}$. Elements in $Z^*(R \times T)$ of the forms (r_z, t_z) and (r_n, t_z) still connect to some $(0, t_z^*) \in \overline{S}_F$. Since $\overline{D}_r$ is a total dominating set, every $(r_z, t_n) \in Z^*(R \times T)$ connects to at least one $(\overline{r}_z, 0) \in \overline{S}_F$. Additionally, any element of the form $(\overline{r}_z, 0)$ will connect to any element of the form $(0, t_z^*)$, making $\overline{S}_F$ a total dominating set for F.

However, $\overline{S}_F$ contains $k+d$ elements, while S_F contains $c+d$ elements. Observe that $k+d < c+d$ since $k < c$. Consequently, $\overline{S}_F$ is a total dominating set with fewer elements than S_F, which is a contradiction. As a result, D_r^* is a minimum total dominating set. Thus, the amount of elements of the form $(r_z^*, 0) \in S_F$ is $c = \gamma_T(G)$.

The same argument can be made for elements of the form $(0, t_z^*)$ to created a set D_t^*. As a result, the number of elements of the form $(0, t_z^*) \in S_F$ is $d = \gamma_T(H)$.

It has been shown that $\gamma_T(G) = c$, and $\gamma_T(H) = d$, so S_F contains $\gamma_T(G) + \gamma_T(H)$ elements. Because S_F is a minimum total dominating set for F, $\gamma_T(F) = \gamma_T(G) + \gamma_T(H)$.

3.3 Polynomial Rings, $\mathbb{Z}_n[x]$

The ring $\mathbb{Z}_n[x]$ is defined as the infinite set of all polynomials only containing coefficients in $\mathbb{Z}_n$, utilizing addition and multiplication modulo n. After researching $\mathbb{Z}_n$ and the direct product of rings, we thought it would be interesting to investigate $\mathbb{Z}_n[x]$.

The following known result is the first discovery we made.

Proposition 3.4 (Zero-Divisors in $\mathbb{Z}_p[x]$). *There are no non-zero zero-divisors in $\mathbb{Z}_p[x]$, in which p is prime.*

Proof. Take two arbitrary non-zero elements $f(x), g(x) \in \mathbb{Z}_p[x]$. These elements can be written as

$$f(x) = a_0x^0 + a_1x^1 + a_2x^2 + \cdots + a_rx^r, a_r \neq 0$$

$$g(x) = b_0x^0 + b_1x^1 + b_2x^2 + \cdots + b_tx^t, b_t \neq 0$$

Let a_i and b_j, where $0 \leq i \leq r$ and $0 \leq j \leq t$, be the first non-zero coefficients of their respective polynomials. These two elements can be rewritten as

$$f(x) = a_ix^i + a_{i+1}x^{i+1} + \cdots + a_rx^r$$

$$g(x) = b_jx^j + b_{j+1}x^{j+1} + \cdots + b_tx^t$$

When multiplied, the product $f(x)g(x)$ is

$$a_ib_jx^{i+j} + (a_ib_{j+1} + a_{i+1}b_j)x^{i+j+1} + \cdots + a_rb_tx^{r+t}$$

By Corollary 3.1, $a_ib_j \not\equiv 0 \mod p$ since no element in $\mathbb{Z}_p$ is a non-zero zero-divisor. Thus, since at least one term of the product of $f(x)g(x)$ is not zero, $f(x)g(x) \not\equiv 0 \mod p$. Therefore, there are no non-zero zero-divisors in $\mathbb{Z}_p[x]$.

This is a well-known result which we attempted to prove but were unsuccessful in doing so and is referenced in Proposition 3.5. The proof in the appendix is taken from [3].

Theorem 3.3 (McCoy's Theorem). *A non-zero element $f(x) \in \mathbb{Z}_n[x]$ is a non-zero zero-divisor if and only if there exists $c \in \mathbb{Z}_n \setminus \{0\}$ such that $cf(x) = 0$.*

Proof. Proof in Appendix

This is a proposition we spent a lot of time trying to prove but were unsuccessful in proving until we had Theorem 3.3.

Proposition 3.5. *Given composite $n = p_1^{a_1} p_2^{a_2} \ldots p_q^{a_q}$, then the polynomial $f(x) \in Z^*(\mathbb{Z}_n[x])$ if and only if there exist $w \in Z^*(\mathbb{Z}_n)$ and some $g(x) \in \mathbb{Z}_n[x]$, such that $f(x) = wg(x)$.*

Proof. ($\Rightarrow$): Let $f(x) \in Z^*(\mathbb{Z}_n[x])$, with $f(x) = b_0x^0 + b_1x^1 + \cdots + b_mx^m$. By Theorem 3.3, there exists some $v \in Z^*(\mathbb{Z}_n)$, such that $vf(x) = 0$. This implies that $vb_0 \equiv vb_1 \equiv \cdots \equiv vb_m \equiv 0 \mod n$, and thus, all of $b_k \in Z(\mathbb{Z}_n)$ for $k = 0, 1, \ldots m$.

Since $v \in Z^*(\mathbb{Z}_n)$, v is a multiple of at least one of the prime factors of n. Then, let $v = cp_1^{\hat{a}_1} p_2^{\hat{a}_2} \ldots p_i^{\hat{a}_i}$, with $1 \leq i \leq q$, $1 \leq \hat{a}_l \leq a_l$ for $l = 1, 2, \ldots i$, and $c \in \mathbb{Z}^+$. Note that it is impossible for both $i = q$ and all of $\hat{a}_l = a_l$ to be true, since then, $v = cn \equiv 0 \mod n$.

Now, any element that connects to v in G must be some multiple of

$$\frac{n}{p_1^{\hat{a}_1} p_2^{\hat{a}_2} \ldots p_i^{\hat{a}_i}} = p_1^{a_1 - \hat{a}_1} p_2^{a_2 - \hat{a}_2} \ldots p_i^{a_i - \hat{a}_i} p_{i+1}^{a_{i+1}} \ldots p_q^{a_q} = w.$$

But then, each b_k is some multiple of w. Now, we can factor out w from $f(x)$. Thus, for all $f(x) \in Z^*(\mathbb{Z}_n[x])$, there exist $w \in Z^*(\mathbb{Z}_n)$ and some $g(x) \in \mathbb{Z}_n[x]$, such that $f(x) = wg(x)$.

($\Leftarrow$): Let $f(x) \in \mathbb{Z}_n[x]$, and $f(x) = wg(x)$, with $w \in Z^*(\mathbb{Z}_n)$ and $g(x) \in Z_n[x]$. Then, there exists $\tau \in Z^*(\mathbb{Z}_n)$, such that $\tau w = 0$. But then, $\tau f(x) = \tau wg(x) = 0$. Thus, $f(x) \in Z^*(\mathbb{Z}_n[x])$. ☙

This is one of our most significant results. It utilizes several of the previous lemmas and propositions.

Theorem 3.4. *Given composite $n = p_1^{a_1} p_2^{a_2} \ldots p_q^{a_q}$, let $G = \Gamma(\mathbb{Z}_n[x])$ and $N = \Gamma(\mathbb{Z}_n)$. Then, $\gamma(G) = \gamma_T(G) = \gamma_T(N) = q$.*

Proof. By Proposition 3.5, for any $w \in Z^*(\mathbb{Z}_n)$ and $h(x) \in \mathbb{Z}_n[x]$ with $w\,h(x) \neq 0$, the polynomial $w\,h(x) \in Z^*(\mathbb{Z}_n)$. Notice that $\text{ann}(w) \subseteq \text{ann}(w\,h(x))$, since anything that annihilates w also annihilates $w\ h(x)$. Also by Proposition 3.5, there does not exists a zero-divisor in $\mathbb{Z}_n[x]$ that is not a multiple of some $w \in Z^*(\mathbb{Z}_n)$.

Case 1:
Now, consider the case $n = 2p$ with prime $p \neq 2$. By Proposition 3.2, a minimum dominating set for N is $\{p\}$. However, it can be shown that this is not a dominating set for G. Consider the polynomial $j(x) = p + px$. By Proposition 3.5, the polynomial $j(x) \in Z^*(\mathbb{Z}_n[x])$, since $j(x) = p(1 + x)$. But $pj(x) = p^2(1+x) \neq 0$, so $\{p\}$ does not dominate G. The minimum dominating set for G must have at least as many elements as the minimum dominating set for N, since all elements in $\mathbb{Z}_n \in \mathbb{Z}_n[x]$.

Consider a minimum total dominating set for N, $\{2, p\}$. It can be shown that this set dominates G. By Proposition 3.5, every $f(x) \in Z^*(\mathbb{Z}_n[x])$ is a multiple of at least one element in $Z^*(\mathbb{Z}_n)$. By Proposition 3.1, $Z(\mathbb{Z}_n)$ is the set of multiples of at least one of the prime factors of n. Then, every element in $Z^*(\mathbb{Z}_n[x])$ is either a multiple of 2 or p. But 2 annihilates all multiples of p, and p annihilates all multiples of 2. Thus, $\{2, p\}$ is a minimum dominating set for G. Also, since $2p \equiv 0 \mod n$, the set is total dominating. Thus, by Lemma 3.1, the set is also minimum total dominating. Therefore, when $n = 2p$, $\gamma(G) = \gamma_T(G) = \gamma_T(N) = 2$.

Case 2:
Now consider all cases where $n \neq 2p$. First, find a minimum dominating set for N. By Proposition 3.2, let $D = \{u_1, u_2, \ldots, u_q\}$, with $u_i = \frac{n}{p_i}$ for $i = 1, 2, \ldots, q$. Then D is a minimum dominating set for N. That is, every $w \in Z^*(\mathbb{Z}_n)$ is annihilated by at least one u_i. By Proposition 3.5, every $f(x) \in Z^*(\mathbb{Z}_n[x])$ is a multiple of at least one element in $Z^*(\mathbb{Z}_n)$. Because of this, every

$f(x) \in Z^*(\mathbb{Z}_n[x])$ is annihilated by at least one u_i. As previously stated, the minimum dominating set for G must have at least as many elements as the minimum dominating set for N, since $\mathbb{Z}_n \subseteq \mathbb{Z}_n[x]$. Therefore, since D dominates G, it is a minimum dominating set.

Also, D is a total dominating set since, without loss of generality, $u_1u_2 = \frac{n}{p_1}\frac{n}{p_2} = \frac{n}{p_1p_2}n \equiv 0 \mod n$. Thus, D is a minimum total dominating set for G, and $\gamma(G) = \gamma_T(G) = \gamma_T(N) = q$. ☃

4 Future Directions

If we had more time, we would like to do more research into polynomial rings. We did not get any results for polynomial rings until towards the end of our research, and if given more time, it would be interesting to see what other discoveries we would be able to find. Additionally, we would do more research into the direct product of rings. We were able to find some results, but we would like to be able to find more, specifically relating to domination number of the graph of the direct product of rings. Below is Conjecture 4.1, which is a conjecture we believe is provable and would like to look into.

Conjecture 4.1. *Given rings R, T with non-zero zero-divisors, let $G = \Gamma(R)$ and $H = \Gamma(T)$. Additionally, let $F = \Gamma(R \times T)$. Then, $\gamma(F) = \gamma_T(G) + \gamma_T(H)$.*

5 Bibliography

References

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6 Appendix

Proof of **Lemma 6**.

(Lower Bound for Domination Number) Let $G = \Gamma(Z_n)$ for $n = p_1^{a_1} p_2^{a_2} \dots p_q^{a_q}$ And n is neither prime nor $2p$ for p a prime, $p \neq 2$. Then $\gamma(G) \geq q$.

Proof. Recall that n is neither prime nor $2p$ for a prime $p \neq 2$.

Consider that $n = p^a$. Since $Z^*(\mathbb{Z}_n) \neq \emptyset$, $\gamma(G) \geq 1$.

Consider that $n = p_1^{a_1} p_2^{a_2}$. We claim that no element in the graph connects to both p_1, p_2. Assume for sake of contradiction that there exists $w \in Z^*(\mathbb{Z}_n)$ that connects to both p_1, p_2. To connect to p_1, w is some multiple of $\frac{n}{p_1} = p_1^{a_1 - 1} p_2^{a_2}$. To connect to p_2, w is some multiple of $\frac{n}{p_2} = p_1^{a_1} p_2^{a_2 - 1}$. But then w is some multiple of $n \equiv 0 \mod n$. This is a contradiction since w must be non-zero. Therefore, no element in G connects to both p_1, p_2.

There are three possibilities.

Case 1:
Neither p_1 nor p_2 is in a minimum dominating set. Thus, there is one element in the minimum dominating set that connects to p_1 and one element that connects to p_2. In this case, there are at least two elements in the minimum dominating set.

Case 2:
Without loss of generality, p_1 is and p_2 is not in a minimum dominating set.
Subcase A:
If without loss of generality, $a_1 > 1$, then $p_1 p_2 \not\equiv 0 \mod n$. Then, having only one of p_1, p_2 in the set will not make it dominating. So there will be at least two elements in the minimum dominating set.
Subcase B:
Otherwise, if $a_1 = a_2 = 1$, then p_1 connects to all multiples of p_2. But it does not connect to any multiples of p_1. There will be at least two multiples of p_1 in G, since $(2p_1) \in Z^*(\mathbb{Z}_n)$. So, there must be at least two elements in the minimum dominating set.

Case 3:
If p_1, p_2 are both in the minimum dominating set, then the minimum dominating set has at least two elements. Thus, $\gamma(G) \geq 2$.

Consider that $n = p_1^{a_1} p_2^{a_2} \dots p_q^{a_q}$ with $q \geq 3$. We claim that no element in the graph connects to both of, without loss of generality, p_1, p_2. Assume for sake of contradiction that there exists $w \in Z^*(\mathbb{Z}_n)$ that connects to both p_1, p_2. To connect to p_1, w is some multiple of $\frac{n}{p_1} = p_1^{a_1 - 1} p_2^{a_2} \dots p_q^{a_q}$. To connect to p_2, w is some multiple of $\frac{n}{p_2} = p_1^{a_1} p_2^{a_2 - 1} \dots p_q^{a_q}$. But then w is some multiple of $n \equiv 0 \mod n$. This is a contradiction since w must be non-zero. Therefore, no element in G connects to both p_1, p_2.

Since this is true, there are three possibilities.

Case 1:
No p_i is in a minimum dominating set. Then, there is one element in the minimum dominating set for each of the q distinct primes of n. In this case, there are at least q elements in the minimum dominating set.

Case 2:
At least one but not all p_i are in a minimum dominating set. Then, there is at least one p_j not in a minimum dominating set. Since $p_i p_j \not\equiv 0 \mod n$, for every p_j there must be some element in the minimum dominating set that is not a prime factor of n. Since two distinct prime factors of n do not connect to any of the same vertices in G, there must be a distinct element in a minimum dominating set for each p_j not in a minimum dominating set. Then, there are at least q elements in a minimum dominating set.

Case 3:
All p_i are in a minimum dominating set. Then, that minimum dominating set has at least q elements. ☙

Proof of **Proposition 3.2**.

Let $G = \Gamma(\mathbb{Z}_n)$. If $n = 2p$, where prime $p \neq 2$, then $\gamma(G) = 1$ and $\gamma_T(G) = 2$. Otherwise, if $n = p_1^{a_1} p_2^{a_2} \dots p_q^{a_q}$, then $\gamma(G) = \gamma_T(G) = q$.

Proof. Consider different cases below.

Case 1: Let $n = 2p$ with $p \neq 2$.

In this case, G is a star graph (See Figure 1). That is, G is a graph with one central vertex that is the sole connection to all other vertices. The central vertex of G is p, and the rest of the vertices are the even numbers between 2 and $n-2$, inclusive. Since p connects to all vertices besides itself, $D = \{p\}$ is a dominating set for G. Since a dominating set must have at least one element, D is a minimum dominating set for G, so $\gamma(G) = 1$.

Notice that no element in G connects to all elements in the graph, including itself. Although p connects to all multiples of 2, it does not connect to itself, since, for a product to be equivalent to $0 \mod n$, that product must be a multiple of 2 and p. Since p is a prime not equal to 2, $p^2 \not\equiv 0 \mod n$. Therefore, a total dominating set must contain at least two elements. Now, select any even number from $Z^*(\mathbb{Z}_n)$ and append to D to make D_T, a total dominating set for G. The set $D_T = \{p, 2\}$ is a total dominating set, and a minimum total dominating set because it is of length two. Then, $\gamma_T(G) = 2$.

Case 2: Let $n = p^a$ with $a > 1$.
By Corollary 1, $n = p$ has no non-zero zero-divisors. Assume $a > 1$. An arbitrary element $x \in Z^*(\mathbb{Z}_n)$ is of the form $x = bp$, $b \in \mathbb{Z}_n$. Now, let $u = \frac{n}{p} = p^{a-1}$. In G, u connects to every $x \in Z^*(\mathbb{Z}_n)$ since $(bp)u = (bp)p^{a-1} = bp^a = bn \equiv 0 \mod n$. Then, $D = \{u\}$ is a dominating set for G. The vertex u also connects to itself, since $u^2 = \frac{n}{p}\frac{n}{p} = \frac{n}{p^2}n \equiv 0 \mod n$, so D is also a total dominating set for G.

Since a dominating set must have at least one vertex, there cannot be a smaller dominating set than $D = \{u\}$. Therefore, D is a minimum dominating set. By Lemma 3.1, D is a minimum total dominating set. As a result, $\gamma(G) = \gamma_T(G) = 1$.

Case 3: Let $n = p_1^{a_1} p_2^{a_2}$, $n \neq 2p$.

Subcase A: $n = p_1 p_2$, $p_1, p_2 \neq 2$
In this case, G will be a complete bipartite graph (See Figure 1). The vertices on one side of the graph will be of the form bp_1, and the vertices on the other side will be of the form cp_2, where b,

$c \in \mathbb{Z}_n$. Also, without loss of generality, notice that any $(bp_1)(cp_2) = bcn \equiv 0 \mod n$. However, $(b_1p_1)(b_2p_1) \not\equiv 0 \mod n$, for any $b_1, b_2 \in \mathbb{Z}_n$. This demonstrates that there cannot be a dominating set of size 1 for G. So, if there exist two vertices that form a dominating set for G, that set will also be a minimum dominating set.

Let $D = \{p_1, p_2\}$. Notice that $p_1(cp_2) \equiv 0 \mod n$, and $(bp_1)p_2 \equiv 0 \mod n$. Thus, D is a dominating set for G. Since it is of length 2, it is a minimum dominating set for G. Further, since $p_1p_2 \equiv 0 \mod n$, D is also a total dominating set, and thus, by Lemma 3.1, a minimum total dominating set for G. Therefore, $\gamma(G) = \gamma_T(G) = 2$.

Subcase B: $n = p_1^{a_1}p_2^{a_2}$, without loss of generality $a_1 > 1$
First, find a dominating set, D, for G. Let $u_1 = \frac{n}{p_1} = p_1^{a_1-1}p_2^{a_2}$, and $u_2 = \frac{n}{p_2} = p_1^{a_1}p_2^{a_2-1}$. Then, $D = \{u_1, u_2\}$ is a dominating set for G because $u_1(b_1p_1) \equiv 0 \mod n$ and $u_2(b_2p_2) \equiv 0 \mod n$. Also, D is a total dominating set for G since $u_1u_2 \equiv 0 \mod n$.

By Lemma 6, since we have D, a dominating set with the same number of elements as there are distinct prime factors of n, D is a minimum dominating set. Since it is also total dominating, by Lemma 3.1, D is a minimum total dominating set. Thus, $\gamma(G) = \gamma_T(G) = 2$.

Case 4: $n = p_1^{a_1}p_2^{a_2}\dots p_q^{a_q}$, $q \geq 3$
Let $u_i = \frac{n}{p_i}$. Now, $D = \{u_1, u_2, \dots, u_q\}$ is a dominating set for G since without loss of generality, u_1 connects to all multiples of p_1. Also, D is a total dominating set for G because without loss of generality, u_1 connects to every other u_i.

By Lemma 6, since we have D, a dominating set with the same number of elements as there are distinct prime factors of n, D is a minimum dominating set. Since it is also total dominating, by Lemma 3.1, D is a minimum total dominating set. Thus, $\gamma(G) = \gamma_T(G) = q$.

Proof for **Theorem 3.3**.

Proof. ($\Leftarrow$): Trivial.

($\Rightarrow$): By contradiction, assume that $f(x) \in Z^*(\mathbb{Z}_n[x])$, but for every $c \in \mathbb{Z}_n\backslash\{0\}$, $cf(x) \neq 0$. Pick a polynomial, $g(x)$ of smallest degree, m, such that $f(x)g(x) = 0$. Notice that because a constant multiplied by $f(x)$ is not 0, $g(x)$ must be of at least degree 1.

Let $f(x) = a_0 + a_1x + \dots + a_qx^q$, and let $g(x) = b_0 + b_1x + \dots + b_mx^m$. Then,

$$(a_0 + a_1x + \dots + a_qx^q)(b_0 + b_1x + \dots + b_mx^m) = 0$$

Observe that, since $b_mf(x) \neq 0$, then $b_ma_i \neq 0$ for some $0 \leq i \leq q$. This implies that $a_ig(x) \neq 0$. Select j maximal with $a_jg(x) \neq 0$. Then, $a_{j+1}g(x) = a_{j+2}g(x) = \dots = a_qg(x) = 0$.

Consider the $j + m$ term of $f(x)g(x)$. That is, $(a_jb_m + a_{j+1}b_{m-1} + \dots)x^{j+m} = 0$. Because every $a_kg(x) = 0$, with $k \geq j + 1$, the $j + m$ term of $f(x)g(x) = a_jb_mx^{j+m} = 0$. This implies that $a_jb_m = 0$.

Now, consider the polynomial $a_jg(x)$. This polynomial is of lesser degree than $g(x)$. Furthermore, $f(x)(a_jg(x)) = 0$. But this is a contradiction because $g(x)$ was a polynomial of smallest degree such that $f(x)g(x) = 0$.

Thus, a polynomial $f(x) \in Z^*(\mathbb{Z}_n[x])$ if and only if there exists $c \in \mathbb{Z}_n\backslash\{0\}$ such that $cf(x) = 0$.