ON THERMODYNAMIC BALANCE IN TORNADO THEORY

Pavel Bělík, Augsburg College; Doug Dokken, Kurt Scholz, and Misha Shvartsman, University of St. Thomas

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Introduction

We address the energy balance in a thunderstorm, in particular, how energy is redistributed on a local level inside a tornadic flow.

The Main Goal

To find a correct thermodynamic energy formula in a local form to adequately describe the behavior of an air parcel in a tornado-like flow

Outline

- Motivation and Views from Branches of Thermodynamics
- Rational Vs Mesoscopic Thermodynamics
- Thermodynamic Fluxes and Summary
- Vertical vs Horizontal Scales

Motivation

- Rotunno, [2015]: " ... *there is a strong nexus with **thermodynamics**, because these thunderstorms are driven by the phase change of water vapor. There are lots and lots of things other than pure fluid dynamics in this field. It is a very rich subject.* "
- Doswell et al., [2006]): " ... *forecasters and researchers are seeking a "magic bullet" when they offer up yet another combined variable or index for consideration* ... "

Thermodynamic System

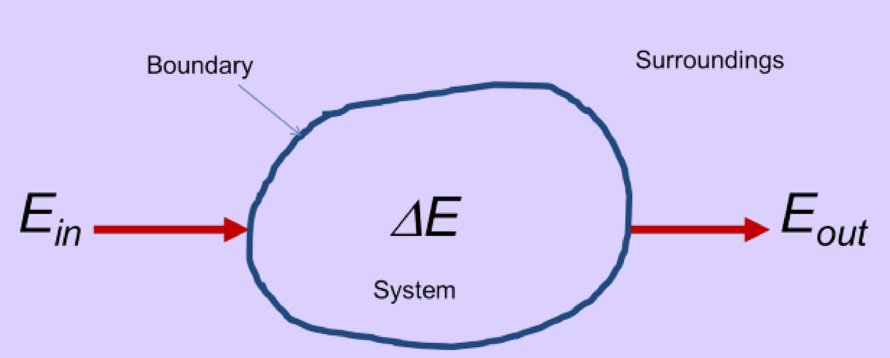

Figure 1: Balance Diagram

Global Reversible Thermodynamics

Equation of state: $S = S(A_1, A_2, \dots A_n)$, S is *entropy*, A_i are the independent extensive variables, that fully determine the state of the system. If the change happens infinitely slowly, then system moves to a different equilibrium state according to

$$dS = \sum F_i \, dA_i$$

F_i are the corresponding conjugate intensive variables (forces), d stands for the (material) change in variables in the system under consideration.

Example: $S = S(U, V)$ where U is the internal energy and V is the specific volume. For the ideal gas, $dS = (1/T)dU + (P/T)dV$ where T is the temperature and P is pressure.

Local Irreversible Thermodynamics

We employ the **Local Equilibrium Hypothesis** where state of air parcel depends on position $\mathbf{x}$ and time t

$$ds(\mathbf{x}, t) = \sum F_i(\mathbf{x}, t) \, da_i(\mathbf{x}, t),$$

s is specific S (per unit mass), and a_i is specific A_i (per unit mass), d is **material** change in variables in the air parcel.

Local Dynamics

Differentiating along the path of the parcel that moves with the velocity field $\mathbf{u}$:

$$\frac{Ds(\mathbf{x}, t)}{Dt} = \sum F_i(\mathbf{x}, t) \frac{Da_i(\mathbf{x}, t)}{Dt},$$
$$D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$$

Extended Irreversible Thermodynamics

The state of the system may include non-equilibrium variables, more precisely, thermodynamic fluxes $b_j(\mathbf{x}, t)$. Then

$$ds(\mathbf{x}, t) = \sum F_i(\mathbf{x}, t) \, da_i(\mathbf{x}, t) + \sum G_j(\mathbf{x}, t) \, db_j(\mathbf{x}, t),$$

where $b_j(\mathbf{x}, t)$ have to satisfy the appropriate transport equations. If the heat flux $\mathbf{q}(\mathbf{x}, t)$ is governed by Cattaneo equation

$$\tau \frac{\partial \mathbf{q}}{\partial t} = -(\mathbf{q} + \lambda \nabla T),$$

where λ is Fourier's Law constant and τ is relaxation time, then

$$ds = \frac{\partial s}{\partial u} du + \frac{\partial s}{\partial \mathbf{q}} \cdot d\mathbf{q} \tag{1}$$

where u is the internal energy density, $\partial s/\partial u = 1/T$ is the reciprocal equilibrium temperature, and $\boldsymbol{\theta} = \partial s/\partial \mathbf{q}$ is the non-equilibrium (vector) "reciprocal" temperature.

Internal Variables Thermodynamics

Internal Variables $c_k(\mathbf{x}, t)$ are introduced to compensate for lack of knowing the behavior of the system. It does not have the corresponding conjugate forces that can be directly calculated. Then

$$ds(\mathbf{x}, t) = \sum F_i(\mathbf{x}, t) \, da_i(\mathbf{x}, t) + \sum dc_k(\mathbf{x}, t),$$

Rational Thermodynamics

In this case we do not assume an *a priori* constitutive equation. Conservation of energy and linear momentum for each control volume remain to be valid. Then the balance equations and Clausius–Duhem's inequality:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$
$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{stress}) = 0,$$
$$\rho c \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{q} = 0,$$
$$\rho \frac{\partial s}{\partial t} + \nabla \cdot \frac{\mathbf{q}}{T} - \frac{\rho r}{T} \geq 0,$$

where ρ is the density, c is the specific heat, and r is the internal heat source.

Axiom of Local Action: An air parcel is only influenced by its immediate neighborhood in space and time, so higher-order space and time derivatives are excluded from the constitutive relations. Its validity is controversial as it ignores "memory".

Mesoscopic Thermodynamics

Mesoscopic approach appreciates limitations of the local equilibrium hypothesis and takes into consideration fluctuations of thermodynamic variables. Statistical entropy $S = k \ln W$, W is the number of microstates corresponding to a macrostate with the specific value of S. The probability of such a macrostate (**Einstein**, [3]) is proportional to

Mesoscopic Thermodynamics

$$W \approx \exp(S/k).$$

If **fluctuation** is associated with entropy change ΔS we can write

$$\text{Probability of Fluctuation} \sim \exp(\Delta S/k).$$

Einstein's formula underwent a range of generalizations, in particular, for the thermodynamic fluxes to be included it requires for any thermodynamic variable $\rho(\mathbf{x}, t)$ - not necessarily density - with the associated current $\mathbf{j}(\mathbf{x}, t)$, and the *mobility* $\chi(\rho)$ to be

$$\text{Probability of Fluctuation} \sim \exp\left(-\frac{B}{kT}\right),$$

where

$$B = \int dt \int dx (\mathbf{j} - \mathbf{J}(\rho)) \cdot \chi(\rho)^{-1} (\mathbf{j} - \mathbf{J}(\rho))$$

and where $\mathbf{J}(\rho)$ is a *hydrodynamic* flux of ρ. The derivation and justification with the references is given in [4]. The evolution of a system subject to macroscopic fluctuations has to satisfy

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}(t) = 0, \quad \mathbf{J}(\rho) = -D(\rho) \nabla \rho,$$
$$\mathbf{j}(t) = \mathbf{J}(\rho) + \chi(\rho) E(t).$$

So

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \chi(\rho) E(t) = \nabla \cdot (D(\rho) \nabla \rho).$$

where $D(\rho)$ is a diffusion matrix and $E(t)$ is the external field. Local Equilibrium implies that

$$D(\rho) = \chi(\rho) f''(\rho)$$

where f is the (local) Helmholtz free energy per unit volume. These equations can be justified using microscopic stochastic dynamics [5].

Thermodynamic Fluxes

According to [6] in a thin horizontal layer of tornado-like flow the Helmholtz free energy density of an air parcel is defined by

$$\hat{f}(t) = f(\theta(t, \boldsymbol{\xi}(t)), v(t), t)$$

where θ is non-equilibrium temperature defined in (1), $\boldsymbol{\xi}(t)$ is a vector of parameters. Then, we have

$$\frac{d\hat{f}}{dt} = -s \frac{\partial \theta}{\partial t} - p \frac{dv}{dt} - s \frac{\partial \theta}{\partial \boldsymbol{\xi}} \frac{\partial \boldsymbol{\xi}}{\partial t} + \frac{\partial f}{\partial t} \tag{2}$$

It can be proven that if we neglect explicit dependence on time t and if $\boldsymbol{\xi}$ are linear functions of $\nabla \mathbf{u}$, then (1) is compatible with Navier-Stokes governing equations [6]. Solutions of (1) also are particular solutions of the Kuramoto–Tsuzuki system of equations of motion in a plane layer [6]:

$$\frac{du_1}{dt} = \nu_1 \Delta u_1 - \nu_2 \Delta u_2 + q u_1 - (\alpha_1 |\mathbf{u}|^2 u_1 - \alpha_2 |\mathbf{u}|^2 u_2) \tag{3}$$
$$\frac{du_2}{dt} = \nu_1 \Delta u_2 + \nu_2 \Delta u_1 + q u_2 - (\alpha_1 |\mathbf{u}|^2 u_2 + \alpha_2 |\mathbf{u}|^2 u_1) \tag{4}$$

where Δ is the Laplacian in 2-D, and ν_1, ν_2, q, α_1, and α_2 are the parameters describing the air parcel at the altitude h. (3) and (4) can be combined in a vector equation for a complex velocity $\Phi = u_1 + iu_2$

$$\frac{d\Phi}{dt} = \nu_1 (1 + ic_1) \Delta \Phi + q \Phi - (\alpha_1 (1 + ic_2) |\Phi|^2 \Phi$$

that has a plane wave solution

$$\Phi(x, y, t) = R(x, y) \exp(i\omega t + ia(x, y))$$

or a *spiral-wave* solution (cf. [6]).

CAPE, 06/17/2010, Minneapolis

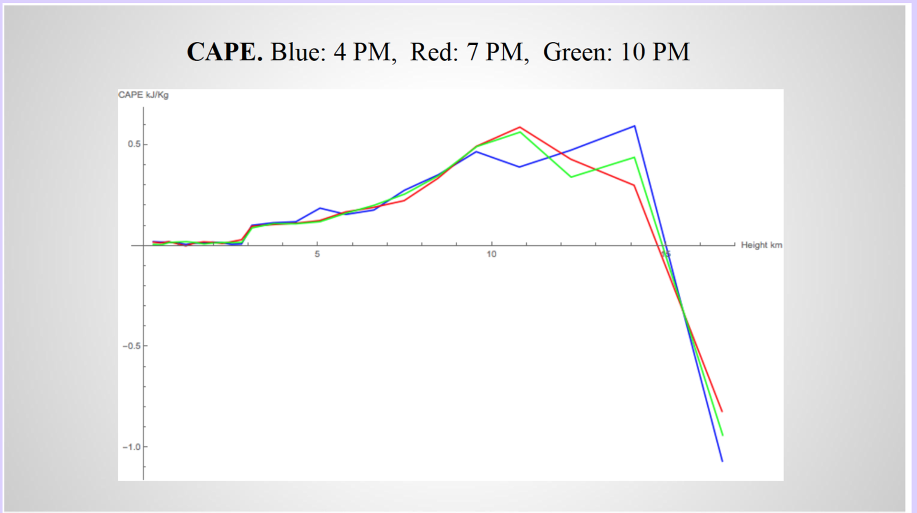

Figure 2: CAPE evolution

SRH, 06/17/2010, Minneapolis

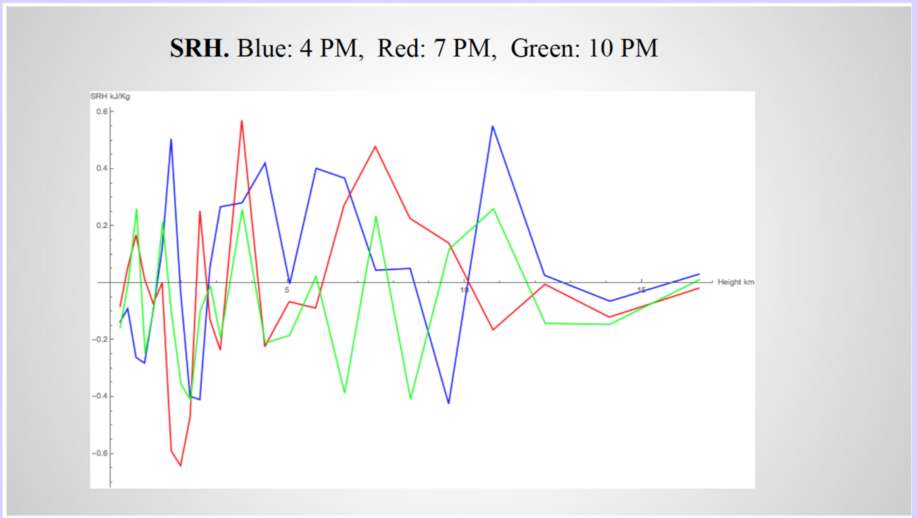

Figure 3: SRH evolution

Vertical vs Horizontal Scales

Non-equilibrium thermodynamics of condensation in the upper portion of the flow would have to take into consideration the pressure drop due to condensation as equations of motion do not explain the downdraft inside the vortex core ([7]). In case of infinitely slow simplified model we can use Clausius-Clapeyron relation $\Delta p = L\Delta\theta/(\theta \Delta v)$ where L is the *latent heat* of condensation. So with the additional term (1) becomes

$$\frac{d\hat{f}}{dt} = -s \frac{\partial \theta}{\partial t} - p \frac{dv}{dt} - s \frac{\partial \theta}{\partial \boldsymbol{\xi}} \frac{\partial \boldsymbol{\xi}}{\partial t} + \frac{\partial f}{\partial t} - \frac{d}{dt}(v \Delta p)$$

Summary

We described the integrated parameters CAPE and SRH as thermodynamic fluxes associated with a non-equilibrium air parcel in a thunderstorm and their contribution to the free energy density formula. It is not yet clear if this formula incorporates all macroscopic fluctuations associated with a non-equilibrium state.

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