Thermodynamic Balance in Tornado Theory

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We address the energy balance in a thunderstorm, in particular, how energy is redistributed on a local level inside a tornadic flow.

Outline

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Motivation

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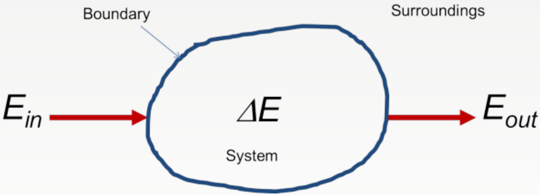
- **Rotunno** [2015], *Weatherwise*, May/June 2015, 56–57: …there is a strong nexus with **thermodynamics**, because these thunderstorms are driven by the phase change of water vapor. There are lots and lots of things other than pure fluid dynamics in this field. It is a very rich subject..



Outline

Global Reversible Thermodynamics

Equation of state: $S = S(A_1, A_2, \ldots A_n)$, S is *entropy*, A_i are the independent extensive variables, that fully determine the state of the system. If the change happens infinitely slowly, then system moves to a different equilibrium state according to

$$dS = \sum F_i \, dA_i$$

F_i are the corresponding conjugate intensive variables (forces), d stands for the (material) change in variables in the system under consideration.

Example: $S = S(U, V)$ where U is the internal energy and V is the specific volume. For the ideal gas, $dS = (1/T)dU + (P/T)dV$ where T is the temperature and P is pressure.

Outline

Local Irreversible Dynamics

We employ the **Local Equilibrium Hypothesis** where state of air parcel depends on position $\mathbf{x}$ and time t (cf. [1])

$$ds(\mathbf{x}, t) = \sum F_i(\mathbf{x}, t)\, da_i(\mathbf{x}, t),$$

s is specific S (per unit mass), and a_i is specific A_i (per unit mass), d is **material** change in variables in the air parcel.

Differentiating along the path of the parcel that moves with local velocity $\mathbf{u}$:

$$\frac{Ds(\mathbf{x}, t)}{Dt} = \sum F_i(\mathbf{x}, t)\, \frac{Da_i(\mathbf{x}, t)}{Dt},$$

$$D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$$

Outline

Extended Irreversible Thermodynamics

The state of the system may include non-equilibrium variables, more precisely, thermodynamic fluxes $b_j(\mathbf{x}, t)$. Then

$$ds(\mathbf{x}, t) = \sum F_i(\mathbf{x}, t)\, da_i(\mathbf{x}, t) + \sum G_j(\mathbf{x}, t)\, db_j(\mathbf{x}, t),$$

where $b_j(\mathbf{x}, t)$ have to satisfy the appropriate transport equations. For example, the heat flux $\mathbf{q}(\mathbf{x}, t)$ may behave according to the Cattaneo equation

$$\tau \frac{\partial \mathbf{q}}{\partial t} = -(\mathbf{q} + \lambda \nabla T)$$

where λ is Fourier's Law constant and τ is relaxation time.

Example:

$$ds = \frac{\partial s}{\partial u} du + \frac{\partial s}{\partial \mathbf{q}} \cdot d\mathbf{q} \tag{1}$$

where $\boldsymbol{\theta} = \frac{\partial s}{\partial \mathbf{q}}$ is non-equilibrium temperature.

Outline

Internal Variables Thermodynamics

Internal Variables $c_k(\mathbf{x}, t)$: compensate for lack of information (no corresponding conjugate computable forces).

$$ds(\mathbf{x}, t) = \sum F_i(\mathbf{x}, t)\, da_i(\mathbf{x}, t) + \sum dc_k(\mathbf{x}, t)$$

No *a priori* constitutive equation. Variables depend on the system non-locally (space and time). Conservation laws plus Clausius–Duhem's apply:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\textbf{stress}) = 0,$$

$$\rho \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = 0,$$

$$\rho \frac{\partial s}{\partial t} + \nabla \cdot \frac{\mathbf{q}}{T} - \frac{\rho r}{T} \geq 0.$$

Outline

Rational Vs Mesoscopic Thermodynamics

Axiom of Local Action: An air parcel is only influenced by its immediate neighborhood in space and time, so higher-order space and time derivatives are excluded from the constitutive relations. Its validity is controversial as it ignores "memory".

Mesoscopic approach appreciates limitations of the local equilibrium hypothesis and takes into consideration fluctuations of thermodynamic variables. Statistical entropy $S = k \ln W$, W is the number of microstates corresponding to a macrostate with the specific value of S. The probability of such a macrostate is (**Einstein**, [2])

$$W \approx \exp(S/k).$$

Rational Vs Mesoscopic Thermodynamics

If **fluctuation** is associated with entropy change ΔS we can write

$$\text{Probability of Fluctuation} \approx \exp(\Delta S/k).$$

Einstein's formula underwent a range of generalizations, in particular, for the thermodynamic fluxes to be included it requires for any thermodynamic variable $\rho(\mathbf{x}, t)$ with the associated current $\mathbf{j}(\mathbf{x}, t)$ in the *mobility* matrix χ to be

$$\text{Probability of Fluctuation} \approx \exp\left(-\frac{B}{kT}\right),$$

where

$$B = \int dt \int dx (\mathbf{j} - \mathbf{J}(\rho)) \cdot \chi(\rho)^{-1} (\mathbf{j} - \mathbf{J}(\rho))$$

and where $\mathbf{J}(\rho)$ is a *hydrodynamic* flux of ρ.

Outline

Evolution with Macroscopic Fluctuations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}(t) = 0, \quad \mathbf{j}(t) = \mathbf{J}(t, \rho(t)),$$
$$\mathbf{J}(t, \rho) = -D(\rho)\nabla\rho + \chi(\rho)E(t).$$

So

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \chi(\rho)E(t) = \nabla \cdot \left(D(\rho)\nabla\rho\right).$$

where $D(\rho)$ is a diffusion matrix and $E(t)$ is the external field. Local Equilibrium implies that

$$D(\rho) = \chi(\rho) f''(\rho)$$

where f is the (local) Helmholtz free energy per unit volume.

Outline

Thermodynamic Fluxes and Instability

Helmholtz energy density of an air parcel:

$$\hat{f}(t) = f(\theta(t, \xi_e(t), \boldsymbol{\xi}_i(t)), v(t), t)$$

where θ is non-equilibrium temperature defined in (1), $\boldsymbol{\xi}_i(t)$ is a vector of internal parameters, and $\xi_e(t)$ is an external parameter. Then, neglecting explicit dependence on time t and external parameter $\xi_e(t)$,

$$\frac{df}{dt} = -s\frac{d\theta}{dt} - p\frac{dv}{dt} - S\frac{\partial\theta}{\partial\boldsymbol{\xi}_i}\frac{\partial\boldsymbol{\xi}_i}{\partial t} \tag{2}$$

If $\boldsymbol{\xi}_i$ are linear functions of $\nabla\mathbf{u}$, then (2) is compatible with N-S equations
Solutions of (2) are also particular solutions of Kuramoto–Tsuzuki equation for complex velocity $\Phi = u + iv$

$$\frac{d\Phi}{dt} = \nu_1(1 + ic_1)\Delta\Phi + q\Phi - (\alpha_1(1 + ic_2)|\Phi|^2\Phi$$

that has a plane wave solution

$$\Phi(x, y, t) = R(x, y)\exp(i\omega t + ia(x, y))$$

Outline

Vertical vs Horizontal Scales and Energy Evolution

Non-equilibrium thermodynamics of condensation in the upper portion of the flow would have to take into consideration the pressure drop due to condensation as equations of motion do not explain the downdraft inside the vortex core. In case of infinitely slow simplified model we can use Clausius-Clapeyron relation

$$\Delta P = \frac{L}{T\Delta v}\Delta T$$

where L is the *latent heat* of condensation.

In the cooler upper portion of the flow the vertical speed plays dominating role as radial velocity vanishes at certain height, so the pressure drop is significant, and an additional term has to be added to the internal flux system in (2), more precisely

$$\frac{df}{dt} = -s\frac{d\theta}{dt} - p\frac{dv}{dt} - S\frac{\partial\theta}{\partial\boldsymbol{\xi}_i}\frac{\partial\boldsymbol{\xi}_i}{\partial t} - \frac{d}{dt}\left(v\Delta p\right)$$

where we still can neglect the volume changes due to condensation.

Outline

Summary

We investigated the thermodynamic equation for a tornado-like vortex using the *Gibbs Relations* in various contexts (equilibrium and non-equilibrium case), as well as the theory when the Gibbs Relations are abandoned and *Rational Thermodynamics* is used to form a constitutive behavior of moist air subject to tornado-like behavior of pressure, temperature, and velocity profiles. We also discussed the thermodynamic flux associated with the faster condensation process in the cooler portion of the flow. Our long-term goal is to model the evolution of the Helmholtz free energy density f in the tornado-like vortex consistent with the available soundings.

References

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