Modeling Turbulence with Delay Equations

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Project Acknowledgements

This project is a UST CSUMS project funded by the National Science Foundation.

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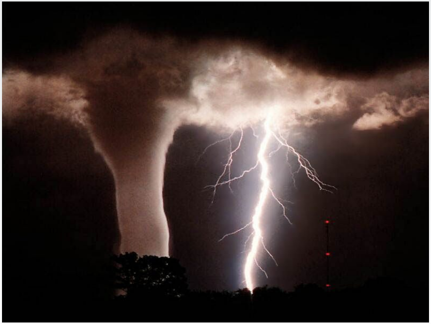
Also, thanks to the Center for Applied Mathematics at UST for continuing the project.

Turbulence

- Turbulent flow is a complicated air pattern that is often observed in nature, including severe weather phenomena.
- In particular, the notion of turbulent flow has been helpful in understanding behaviour of tornadoes which are a main focus of our study.



Tornadoes

- Tornado's occur frequently in a strong super-cell thunderstorm environments accompanied by strong rotation.
- Our project investigates the role of delay in modelling turbulent flow and its contribution to understanding tornado dynamics.

Viscosity

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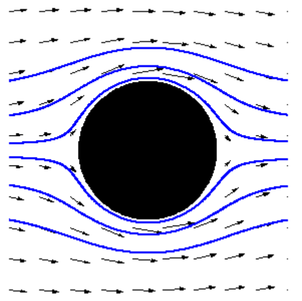
- Viscosity can be thought of as a measure of how friction is affecting a fluid between its own molecules.
- Water vs. honey.
- It is the measure of how much the fluid resists flowing.

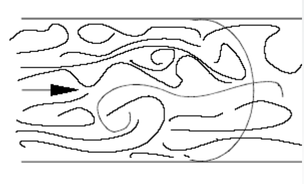
Laminar Flow

- Fluid flow is generally characterized as being either laminar or turbulent.
- Laminar flow or streamline flow occurs when a fluid flows without disruption between layers.
- Laminar flow occurs in fluids with high viscosity.



Turbulent Flow

- Turbulence is a type of fluid flow (air included) that is chaotic and rough.
- Turbulence and chaotic flow is more common in fluids with low viscosity
- This flow is characterized by loops and swirls (eddies) in columns of air.
- Particles are very hard to predict or model



TURBULENT FLOW

Current Knowledge

- The current best description of fluid flow is a set of governing equations.
- These equations say fluid flow depends on: the velocity components in the stream-wise, span-wise and vertical directions (u, v, w);

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Current Knowledge

- The current best description of fluid flow is a set of governing equations.
- These equations say fluid flow depends on: the velocity components in the stream-wise, span-wise and vertical directions (u, v, w); the density ρ; the pressure p; and the temperature θ.

The Governing Equations

The Governing Equations come from

- The Navier-Stokes Equations

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- The Navier-Stokes Equations
- In physics, the Navier-Stokes equations describe the motion of fluid substances.
- These equations arise from applying Newton's second law to fluid motion.

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- Conservation of Mass
- Conservation of Energy

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Boussinesq Approximation - Specialized Equations for the Boundary Layer of the Atmosphere

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + \nu\nabla^2 u \qquad (1)$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu + \nu\nabla^2 v \qquad (2)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g\frac{\theta}{\theta_0} + \nu\nabla^2 w \qquad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (4)$$

$$\frac{D\theta}{Dt} = -w\frac{d\theta_0}{dz} \qquad (5)$$

Navier-Stokes Equations

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + \nu\nabla^2 u \tag{6}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu + \nu\nabla^2 v \tag{7}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g\frac{\theta}{\theta_0} + \nu\nabla^2 w \tag{8}$$

Navier-Stokes Equations

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$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g\frac{\theta}{\theta_0} + \nu\nabla^2 w \tag{8}$$

- The left sides of the equations describes acceleration.

Navier-Stokes Equations

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + \nu\nabla^2 u \tag{9}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu + \nu\nabla^2 v \tag{10}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g\frac{\theta}{\theta_0} + \nu\nabla^2 w \tag{11}$$

- The right side of the equation is a summation of body forces

Navier-Stokes Equations

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + \nu\bigtriangledown^2 u \tag{12}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu + \nu\bigtriangledown^2 v \tag{13}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g\frac{\theta}{\theta_0} + \nu\bigtriangledown^2 w \tag{14}$$

- The next section shows where each term comes from.

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Navier-Stokes Equations

$$\frac{\mathbf{Du}}{\mathbf{Dt}} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + \nu\nabla^2 u \quad (15)$$

$$\frac{\mathbf{Dv}}{\mathbf{Dt}} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu + \nu\nabla^2 v \quad (16)$$

$$\frac{\mathbf{Dw}}{\mathbf{Dt}} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g\frac{\theta}{\theta_0} + \nu\nabla^2 w \quad (17)$$

- The bolded terms are the material derivatives for the stream-wise, span-wise, and vertical components of the velocity, and can be thought of as describing acceleration following a particular particle of the fluid.

Navier-Stokes Equations

$$\frac{Du}{Dt} = \left(-\frac{\mathbf{1}}{\rho_{\mathbf{0}}}\frac{\partial \mathbf{p}}{\partial \mathbf{x}}\right) + fv + \nu\nabla^2 u \tag{18}$$

$$\frac{Dv}{Dt} = \left(-\frac{\mathbf{1}}{\rho_{\mathbf{0}}}\frac{\partial \mathbf{p}}{\partial \mathbf{y}}\right) - fu + \nu\nabla^2 v \tag{19}$$

$$\frac{Dw}{Dt} = \left(-\frac{\mathbf{1}}{\rho_{\mathbf{0}}}\frac{\partial \mathbf{p}}{\partial \mathbf{z}}\right) + g\frac{\theta}{\theta_0} + \nu\nabla^2 w \tag{20}$$

- The bolded terms now are the terms describing the effect of the pressure gradient force.

Navier-Stokes Equations

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \mathbf{fv} + \nu\bigtriangledown^2 u \tag{21}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - \mathbf{fu} + \nu\bigtriangledown^2 v \tag{22}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g\frac{\theta}{\theta_0} + \nu\bigtriangledown^2 w \tag{23}$$

- These terms on the right hand of the stream-wise and span-wise equations represent the Coriolis effect.
- This is the effect of the earth's rotation on the airflow.

Navier-Stokes Equations

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + \nu\bigtriangledown^2 u \quad (24)$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu + \nu\bigtriangledown^2 v \quad (25)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \left(\mathbf{g}\frac{\theta}{\theta_\mathbf{0}}\right) + \nu\bigtriangledown^2 w \quad (26)$$

- In the vertical component equation, the bolded term comes from the effect of gravity on the system.
- This term is included because of the difference in density.
- There are no density terms in the other equations because it is assumed density only varies with height.

Navier-Stokes Equations

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + \left(\nu\nabla^2\mathbf{u}\right) \tag{27}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu + \left(\nu\nabla^2\mathbf{v}\right) \tag{28}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g\frac{\theta}{\theta_0} + \left(\nu\nabla^2\mathbf{w}\right) \tag{29}$$

- The last term in each equation represents the effect of viscosity in the stream-wise, span-wise, and vertical direction.

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The Other Equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{30}$$

$$\frac{D\theta}{Dt} = -w\frac{d\theta_0}{dz} \tag{31}$$

- These equations come from the Conservation Laws.
- Equation 30 is Conservation of Mass.
- It assumes that the fluid is incompressible.
- Equation 31 is Conservation of Energy.

The Governing Equations

- So our governing equations produce a system of **five** non-linear partial differential equations for **six** unknowns functions u, v, w, ρ, p, θ that model the atmospheric boundary layer.
- To solve the system, and extra equation must be introduced.
- An example might be an equation that might distinguish viscous from non-viscous fluids.

Direct Numerical Simulation

- Since our equations are non-linear, analytically solving them is close to impossible.
- The only option is to obtain numerical solutions
- To run a direct numerical simulation (DNS), we need to choose initial and boundary conditions, replace the partial differential equations by discretized versions and carry out the computations.
- Having so many independent variables makes this a difficult evaluation.
- Solving this system numerically is computationally expensive.

Our Project

- We want to experiment with working on the system in a way to simplify the DNS problem.
- We looked at properties of fluid mechanics for a way to experiment on the system.
- The property we looked at was the cohesive nature of particles in turbulence.
- This is similar to delay in partial differential systems.

Delay and Delay Equations

- Delay Equations are designed for system with memory.

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- Delay Equations are designed for system with memory.
- Turbulence is characterized by coordinated (coherent) motion of fluid at different scales.
- Delay Equations are differential equations that depend on information from the system's history in order for the system to progress.
- Example:

$$\frac{dy}{dt} = y(t-1) + 4t \tag{32}$$

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Burgers' Equation

- The full Navier-Stokes equations are difficult to work with.
- To start, the project focused on a simpler, one dimensional version of the Navier-Stokes equations (Burgers' Equation).
- Our project worked with the Burgers' Equation, both viscous and inviscid, examining the effects of delay on the system.
- The Burgers' Equation can be thought as a one-dimensional model of fluid flow (with pressure gradient and the Coriolis Effect neglected).

Creating the 1D Burgers' Equation

- To extract the Burgers' Equation from the full Navier-Stokes equation, one of the single equations is simplified.

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + \nu \bigtriangledown^2 u \quad (33)$$

- Burgers' Equation neglects the Coriolis effect and the pressure gradient force, so they are removed from the equation.

$$\frac{Du}{Dt} = \nu \bigtriangledown^2 u \quad (34)$$

Creating the 1D Burgers' Equation

- Expanding out the material derivative on the left side gives:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u = \nu \bigtriangledown^2 u \tag{35}$$

- This is equivalent to the Viscous Burgers' Equation.
- If viscosity is neglected as well, the equation is the Inviscid Burgers' Equation.

Burgers' Equation

- The Inviscid Burgers' Equation is:

$$U_t + UU_x = 0 \tag{36}$$

- The Viscous Burgers' Equation is:

$$U_t + UU_x = \nu U_{xx} \tag{37}$$

- U_t is the partial derivative of the velocity with respect to time
- UU_x is the convective term.

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- U_t is the partial derivative of the velocity with respect to time
- UU_x is the convective term.
- Both of these terms were hidden in the material derivatives of the governing equations.
- νU_{xx} is the diffusive term that describes the effect of viscosity.

Current Mathematica Work

- We are comparing the two different Burgers Equations for a given set of initial conditions.
- For both equations, we are looking at versions with and without delay.
- We are using the Finite Differences numerical method to solve the equation.
- We are also using more than one scheme for the iterative process, varying the discretization method for the derivatives.

Discretization Methods

- Inviscid Burgers Equation

$$U_t\left(x,t\right)+U\left(x,t\right)U_x\left(x,t\right)=0 \tag{38}$$

- Discretized:

$$\frac{U\left[i,j+1\right]-U\left[i,j\right]}{m}+U\left[i,j\right]\frac{U\left[i+1,j\right]-U\left[i,j\right]}{h}=0 \tag{39}$$

$$U[i,j+1]=U[i,j]-mU[i,j]\frac{U[i+1,j]-U[i,j]}{h} \tag{40}$$

Discretization Methods

- Equation for Viscous Burgers' Equation

$$U_t(x,t) + UU_x(x,t) = \nu U_{xx}(x,t) \tag{41}$$

- Discretized:

$$\frac{U[i,j+1]-U[i,j]}{m} + \underline{U^*[i,j]}\,\frac{U[i+1,j]-U[i,j]}{h} = \nu\left(\frac{U[i+1,j]-U[i-1,j]-2U[i,j]}{h^2}\right)$$

- Solved:

$$U[i,j+1] = U[i,j] + \frac{m\nu}{h^2}\left(U[i+1,j]+U[i-1,j]-2U[i,j]\right) - \underline{\mathbf{U^*[i,j]}}\,\frac{m}{h}\left(U[i+1,j]-U[i,j]\right)$$

- The highlighted term is where the different delay discretizations were added to the equation.
- This term is part of the convective term.

Different Discretizations

- There were three different delays that were tried.
- All delays were spatial delays.

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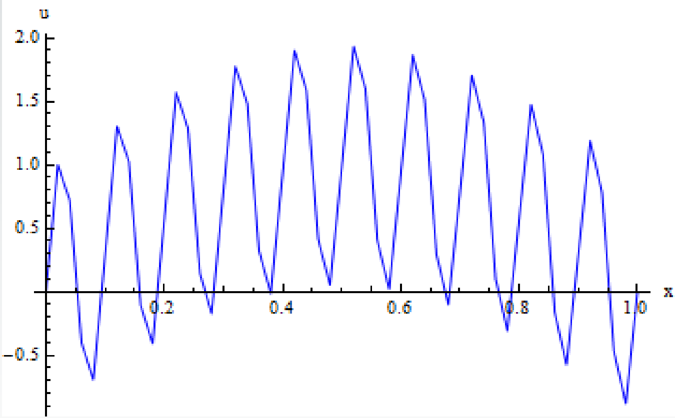
Previous Step Delay $\underline{U^*[i,j]} = U[i-1,j]$

Next Step Delay $\underline{U^*[i,j]} = U[i+1,j]$

Mathematica Run-through

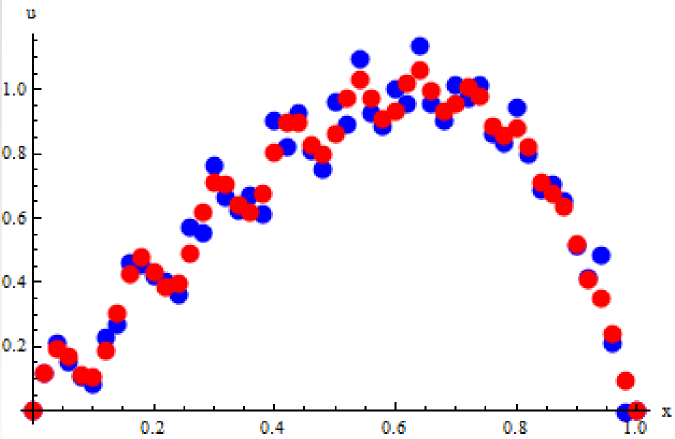
- Initial conditions:
- 50 steps in space, step-size of .02
- 1501 steps in time, step-size of .0001
- Viscosity of .005
- At $t = 0$, our behaviour is $f(x) = sin(\pi x) + sin(20\pi x)$

Plots



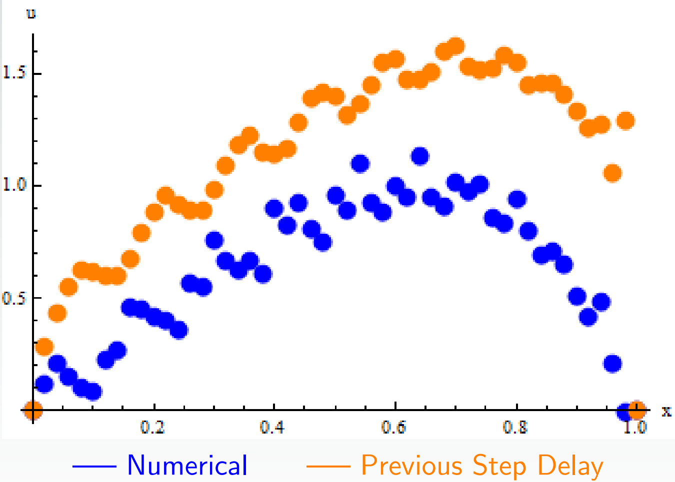
Initial Data

Plots



— Numerical — Average Delay

Plots



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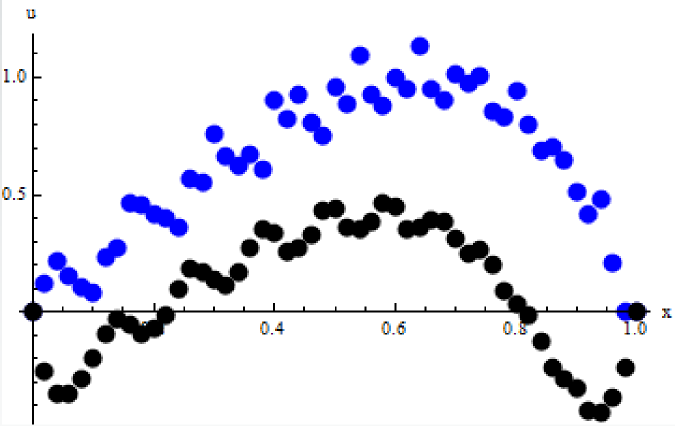
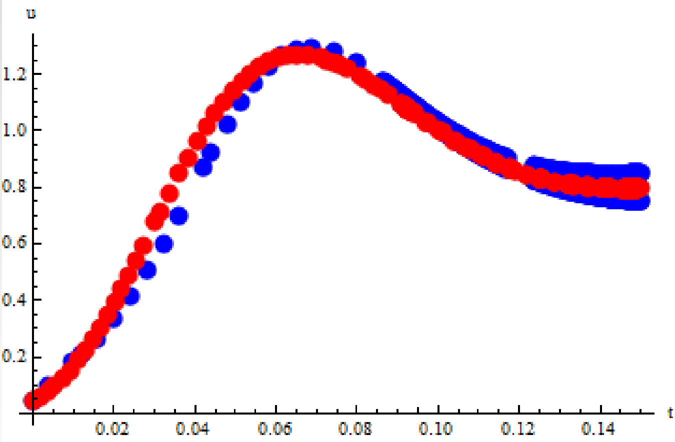
Plots



— Numerical — Next Step Delay

Plots



Numerical — Average Delay

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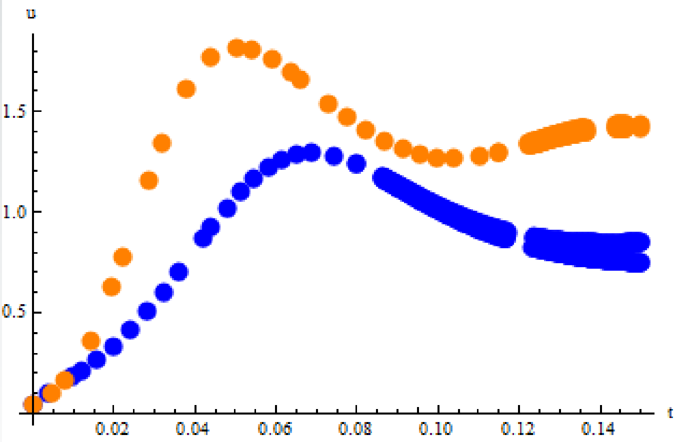
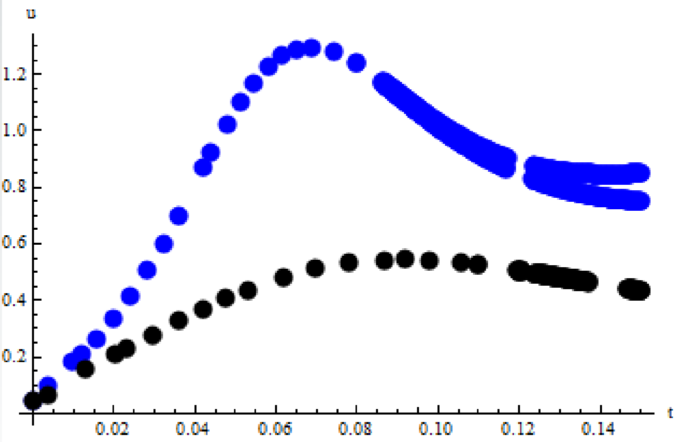


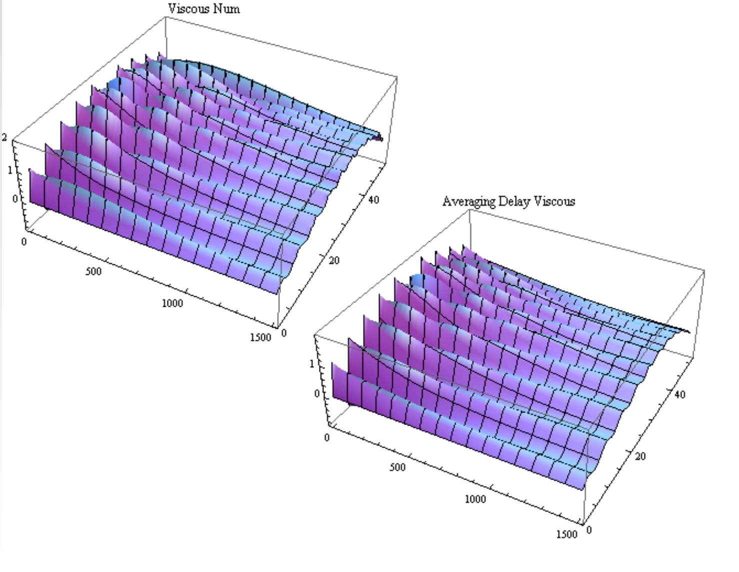
—— Numerical —— Previous Step Delay

Plots



Numerical   Next Step Delay

Plots


Viscous Num
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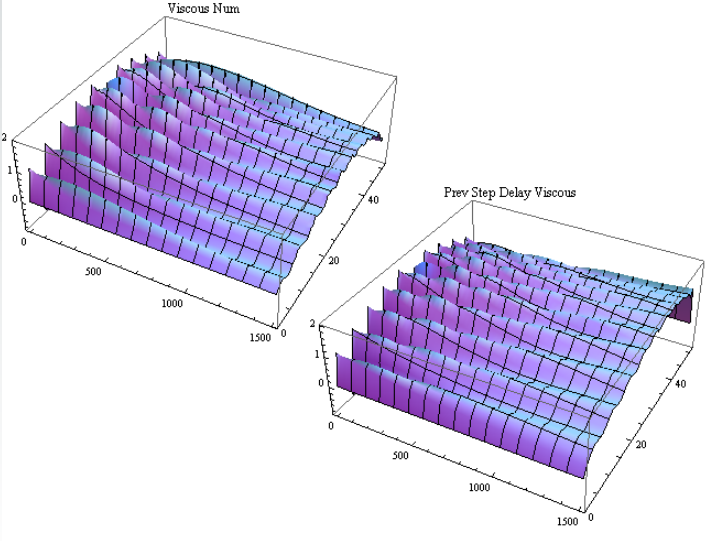
Plots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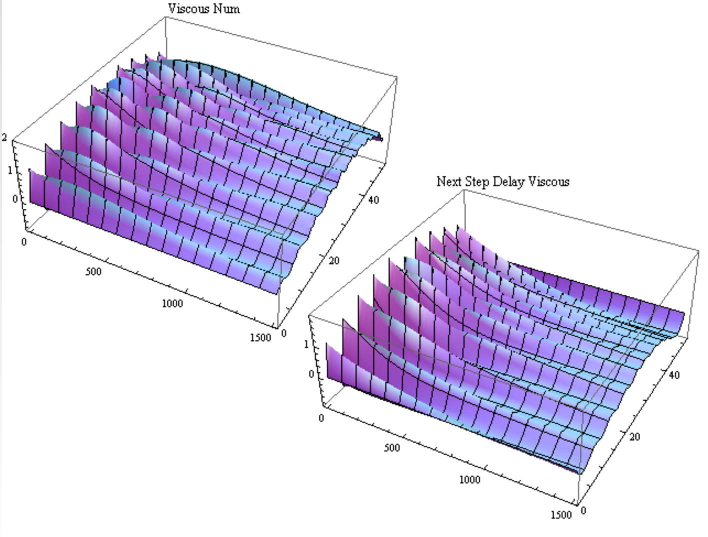
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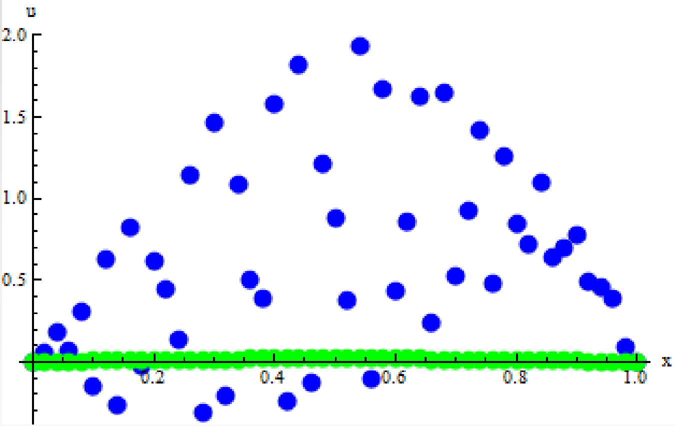
Comparison to Established Method

- The above methods all are forward difference methods.
- They are naturally unstable.

Comparison to Established Method

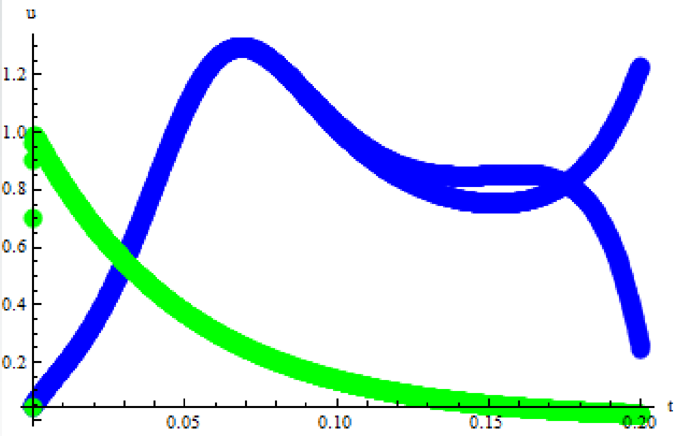
- The above methods all are forward difference methods.
- They are naturally unstable.
- Comparison to Lax-Friedrich's in further time steps.

Plots



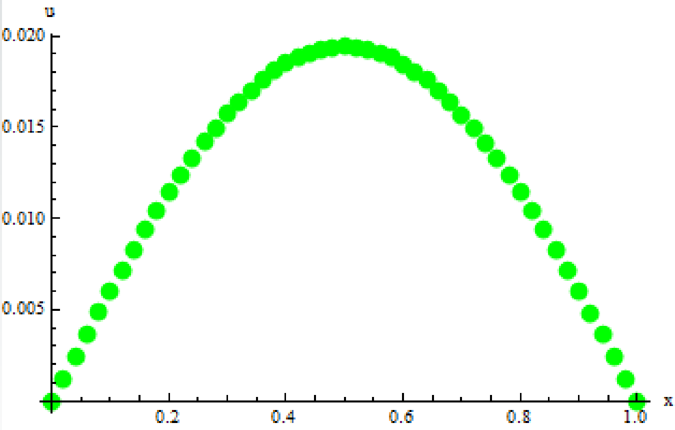
—— Numerical —— Lax-Friedrich's

Plots

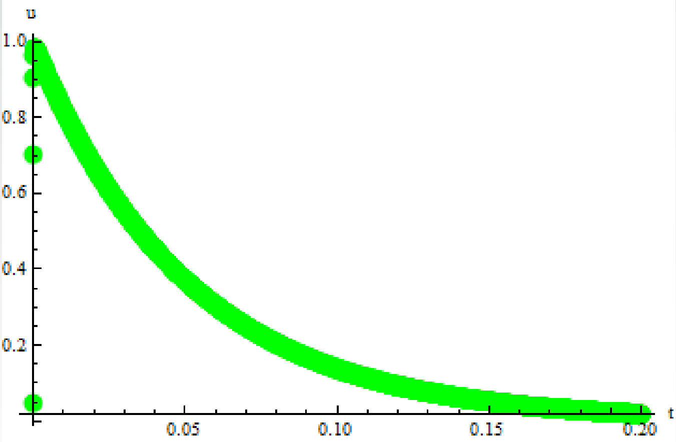


Numerical   Lax-Friedrich's

Plots



Plots



Comparison to Established Method

- Lax-Friedrichs method for Inviscid case.
- Uses averages.
- More stable than forward differences.

Future Plans

- Work on a stopping criteria to stop and reset data values for the forwards difference method.
- We want to work with similar code in Matlab.
- Expand beyond the one dimensional model into higher dimensions.