HOMEWORK 2: DUE SEP 5

For this week, please answer the following questions from the text. I've copied the problem itself below and the question numbers for your convenience.

(1) (1.6) Let $a, b, c \in \mathbb{Z}$. Use the definition of divisibility to directly prove the following properties of divisibility.
   (a) If $a \mid b$ and $b \mid c$, then $a \mid c$.
   (b) If $a \mid b$ and $b \mid a$, then $a = \pm b$.
   (c) If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$ and $a \mid (b - c)$.

(2) (1.9.a) Use the Euclidean algorithm to compute $\gcd(291, 252)$ by hand.

(3) (1.10.a) Use the extended Euclidean algorithm to find integers u, v such that

$$291u + 252v = \gcd(291, 252)$$

(4) (1.11) Let a and b be positive integers.
   (a) Suppose that there are integers u and v satisfying $au + bv = 1$. Prove that $\gcd(a, b) = 1$.
   (b) Suppose that there are integers u and v satisfying $au + bv = 6$. Is it necessarily true that $\gcd(a, b) = 6$? If not, give a specific counterexample, and describe in general all the possible values of $\gcd(a, b)$.
   (c) Suppose that (u_1, v_1) and (u_2, v_2) are two integral solutions to the equation $au + bv = 1$. Prove that $a \mid (v_2 - v_1)$ and $b \mid u_2 - u_1$.
   (d) More generally, let $g = \gcd(a, b)$ and (u_0, v_0) be an integral solution to $au + bv = g$. Prove that every other solution has the form $u = u_0 + kg/g$ and $v = v_0 - ka/g$ for some integer k.

(5) (1.15) Let $m \geq 1$ be an integer and suppose that

$$a_1 \equiv a_2 \mod m \text{ and } b_1 \equiv b_2 \mod m.$$

Prove that

$$a_1 \pm b_1 \equiv a_2 \pm b_2 \mod m \text{ and } a_1 \cdot b_1 \equiv a_2 \cdot b_2 \mod m.$$

(6) (1.16) Write out the following tables for $\mathbb{Z}/m\mathbb{Z}$ and $(\mathbb{Z}/m\mathbb{Z})^*$ as done in Fig. 1.4 and 1.5 in the textbook.
   (a) Make addition and multiplication tables for $\mathbb{Z}/3\mathbb{Z}$.
   (b) Make addition and multiplication tables for $\mathbb{Z}/6\mathbb{Z}$.
   (c) Make a multiplication table for the unit group $(\mathbb{Z}/9\mathbb{Z})^*$.
   (d) Make a multiplication table for the unit group $(\mathbb{Z}/16\mathbb{Z})^*$.