HOMEWORK

For this week, please answer the following questions from the text. I've copied the problem itself below and the question numbers for your convenience.

(1) (3.17) The function $\pi(X)$ counts the number of primes between 2 and X.
   (a) Compue the values of $\pi(20), \pi(30)$, and $\pi(100)$.
   (b) Write a program to compute $\pi(X)$ and use it to compute $\pi(X)$ and the ration $\pi(X)/(X/\ln X)$ for $X = 100, 1000, 10000$, and 100000. Does your list of ratios make the prime number theorem plausible?

(2) (3.19) We noted in Sect. 3.4 that it really makes no sense to say that the number n has probability $1/\ln n$ of being prime. Any particular number that you choose either will be prime or will not be prime; there are no numbers that are 35 % prime and 65 % composite! In this exercise you will prove a result that gives a more sensible meaning to the statement that a number has a certain probability of being prime. You may use the prime number theorem (Theorem 3.21) for this problem.
   (a) Fix a (large) number N and suppose that Bob chooses a random number n in the interval $N/2 \leq n \leq 3N/2$. If he repeats this process many times, prove that approximately $1/\ln N$ of his numbers will be prime. More precisely, define

$$\begin{aligned} P(N) &:= \frac{\text{number of primes between } N/2 \text{ and } 3N/2}{\text{number of integers between } N/2\ 3N/2} \\ &= \begin{bmatrix} \text{Probability that a random integer } n \text{ in the interval} \\ N/2 \leq n \leq 3N/2 \text{ is a prime number} \end{bmatrix} \end{aligned}$$

   and prove that

$$\lim_{N\to\infty} \frac{P(N)}{N/\ln N} = 1$$

   (b) More generally, fix two numbers c_1 and c_2 satisfying $c_1 > c_2 > 0$. Bob chooses random numbers n in the interval $c_1N \leq n \leq c_2N$. Keeping c_1 and c_2 fixed, let

$$P(c_1, c_2; N) := \begin{bmatrix} \text{Probablity that an integer } n \text{ in the interval} \\ c_1N \leq n \leq c_2N \text{ is a prime number} \end{bmatrix}$$

   In the following formula, fill in the gox with a simple function of N so that the statement is true.

$$\lim_{N\to\infty} \frac{P(c_1, c_2; N)}{\boxed{}} = 1.$$

(3) (3.23) A prime of the form $2^n - 1$ is called a *Mersenne prime.*
   (a) Factor each of the numbers $2^n - 1$ for $n = 2, 3, \ldots, 10$. Which ones are Mersenne primes?
   (b) Find the first seven Mersenne primes. (You may need a computer.)
   (c) If n is even and $n > 2$, prove that $2^n - 1$ is not prime.
   (d) If $3 \mid n$ and $n > 3$, prove that $2^n - 1$ is not prime.
   (e) More generally, prove that if n is a composite number, then $2^n - 1$ is not prime. Thus all Mersenne primes have the form $2^p - 1$ with p a prime number.

(f) What is the largest known Mersenne prime? Are there any larger primes known? (You can find out at the "Great Internet Mersenne Prime Search" web site `www.mersenne.org/prime.htm`.)