Final year intership report

Verification in Isabelle/HOL of Hopcroft's algorithm for minimizing DFAs including runtime analysis

Vincent Trélat

April 13, 2023

Contents

1 Introduction

1.1 Original algorithm

John E. Hopcroft's algorithm for minimizing DFAs was first presented in his original 1971 paper [Hop71] as a formal algorithm. Algorithm 1 is a direct translation of the original algorithm with only slight changes in the notations.

Algorithm 1: Hopcroft's original formal algorithm

Data: Input: a finite DFA $\mathcal{A} = (\mathcal{Q}, \Sigma, \delta, q_0, \mathcal{F})$
Result: Output: the equivalence class of $\mathcal{Q}$ under state equivalence

Construct $\delta^{-1}(q, a) := \{t \in \mathcal{Q} \mid \delta(t, a) = q\}$ for all $q \in \mathcal{Q}$ and $a \in \Sigma$;
Construct $P_1 := \mathcal{F}$, $P_2 := \mathcal{Q} \setminus \mathcal{F}$ and $s_{a,i} := \{q \in P_i \mid \delta^{-1}(q, a) \neq \varnothing\}$ for all $i \in \{1, 2\}$ and $a \in \Sigma$;
Let $k := 3$;
For all $a \in \Sigma$, construct $L_a := \underset{0 \leq i < k}{\arg\min}\ |s_{a,i}|$;
while $\exists a \in \Sigma, L_a \neq \varnothing$ **do**
  Pick $a \in \Sigma$ such that $L_a \neq \varnothing$ and $i \in L_a$;
  $L_a := L_a \setminus \{i\}$;
  forall $j < k, \exists q \in P_j, \delta(q, a) \in s_{a,i}$ **do**
    $P'_j := \{t \in P_j \mid \delta(t, a) \in s_{a,i}\}$ and $P''_j := P_j \setminus P'_j$;
    $P_j := P'_j$ and $P_k := P''_j$; construct $s_{a,j}$ and $s_{a,k}$ for all $a \in \Sigma$ accordingly ;
    For all $a \in \Sigma$, $L_a := \begin{cases} L_a \cup \{j\} & \text{if } j \notin L_a \wedge |s_{a,j}| \leq |s_{a,k}| \\ L_a \cup \{k\} & \text{otherwise} \end{cases}$;
    $k := k + 1$;
  end
end

1.2 Modern formalisation

The algorithm is now usually given in a more mathematical and formalised way[1], as presented below in Algorithm 2. We justify this transformation.

First, instead of storing indices for the blocks, we directly deal with sets and explicitly work with a partition $\mathcal{P}$ of $\mathcal{Q}$. Then, we define splitters.

Definition 1. Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, I, F)$ be a DFA. Let P be a partition of

[1] see for example [EB23]

Q. Let $B \in P$ and $a \in \Sigma$. We say for $C \in P$ that (a, C) splits B if

$$\exists q_1, q_2 \in B \quad \delta(q_1, a) \in C \wedge \delta(q_2, a) \notin C.$$

Remark 1. If (a, C) is a splitter of B, P can be updated to $P \backslash \{B\} \cup \{B', B''\}$, where

$$B' := \{q \in B \mid \delta(q, a) \in C\} \text{ and } B'' := B \setminus B'.$$

Definition 2. For a splitter $s \in \Sigma \times \mathcal{P}$, we define its size $\|s\|$ as follows. If $s = (a, B)$, then $\|s\| := |B|$.

In Algorithm 1, picking an index $i \in L_a$ corresponds to picking a set $s_{a,i}$, i.e. the subset of P_i of states having at least one predecessor. We cannot directly replace L_a and $s_{a,i}$ by a set of splitters (a, P_i) as is, however the next step in the algorithm is to go over all blocks in the partition that have a successor in $s_{a,i}$. If such a block B is found, it is split into two blocks, namely the set of states having a successor in $s_{a,i}$ and the others. Overall, we look for B such that P_i has a predecessor in B. We add the condition that B must also have at least one successor not in P_i to avoid splitting B into $\{B, \varnothing\}$. Since this is equivalent to splitting B with (a, P_i), we can replace L_a by a set of splitters (a, P_i). Since the symbol a is chosen such that $L_a \neq \varnothing$, we can define a workset $\mathcal{W}$ of all splitters, with all symbols of Σ. Therefore, L_a is empty if and only if there is no a-splitter in $\mathcal{W}$. Because testing emptiness is equivalent to testing membership, and because of the existential quantifier in the definition of splitters, the $s_{a,i}$ are no longer needed.

Algorithm 2: Hopcroft's algorithm in a modern style

```
Data: Input: a finite DFA $\mathcal{A} = (\mathcal{Q}, \Sigma, \delta, q_0, \mathcal{F})$
Result: Output: the language partition $P_\ell$
if $\mathcal{F} = \varnothing \vee \mathcal{Q} \setminus \mathcal{F} = \varnothing$ then
    return $\mathcal{Q}$
else
    $\mathcal{P} := \{\mathcal{F}, \mathcal{Q} \setminus \mathcal{F}\}$ ;
    $\mathcal{W} := \{(a, \min\{\mathcal{F}, \mathcal{Q} \setminus \mathcal{F}\}, a \in \Sigma\}$ ;
    while $\mathcal{W} \neq \varnothing$ do
        Pick $(a, B')$ from $\mathcal{W}$ ;
        forall $B \in \mathcal{P}$ do
            Split $B$ with $(a, B')$ into $B_0$ and $B_1$ ;
            $\mathcal{P} := (\mathcal{P} \setminus \{B\}) \cup \{B_0, B_1\}$ ;
            forall $b \in \Sigma$ do
                if $(b, B) \in \mathcal{W}$ then
                    $\mathcal{W} := (\mathcal{W} \setminus \{(b, B)\}) \cup \{(b, B_0), (b, B_1)\}$ ;
                else
                    $\mathcal{W} := \mathcal{W} \cup \{(b, \min\{B_0, B_1\})\}$ ;
                end
            end
        end
    end
end
```

2 Proof of correctness

3 Time complexity analysis

We focus on the original algorithm presented in Algorithm 1 in order to work on the arguments given in [Hop71]. The data structures used at that time were mostly linked lists, but let us rather give some requirements for the data structures instead of actual implementations. The goal is to show that the algorithm can be executed in $O(m \cdot n \log n)$ time, where m is the number of symbols in the alphabet and n is the number of states in the DFA.

The following requirements come directly from [Hop71] and are specific to the algorithm presented in Algorithm 1.

Requirement 1. Sets such as $\delta^{-1}(q, a)$ and L_a must be represented in a way that allows $O(1)$ time for addition and deletion in front position.

Requirement 2. Vectors must be maintained to indicate whether a state is in a given set.

Requirement 3. Sets such as P_i must be represented in a way that allows $O(1)$ time for addition and deletion at any given position.

Requirement 4. For a state q in a set P_i or $s_{a,i}$, its position must be determined in $O(1)$ time.

VT: Maybe not necessary? This should be provable from Req. 2 and Req. 3.

Lemma 1. In Algorithm 1, lines 1 to 4 can be executed in $O(|\Sigma| \cdot |\mathcal{Q}|)$ time.

Proof. The non trivial part is the computation of the inverse transition function $\delta^{-1}(q, a)$, for all $q \in \mathcal{Q}$ and $a \in \Sigma$. This can be done in $O(|\Sigma| \cdot |\mathcal{Q}|)$ time by iterating over Σ and traversing the automaton (e.g. with a DFS) while keeping track of the predecessor at each step. ∎

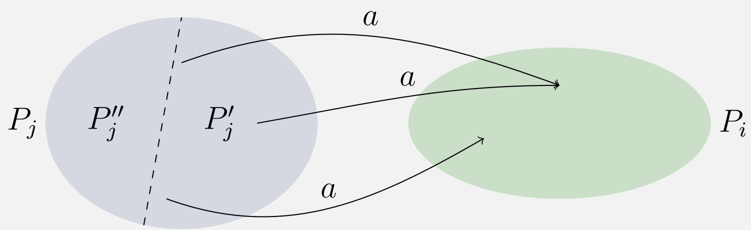
Lemma 2. An iteration of the while loop in both algorithms for a splitter $s = (a, C)$ takes a time proportional to the number of transitions terminating in C and the number of symbols in the alphabet, i.e. $\Theta\left(|\Sigma| \cdot |\delta^{-1}(C, a)|\right)$ time.

Proof. We start by showing it for Algorithm 1. We pick $a \in \Sigma$ such that $L_a \neq \varnothing$ and an $i \in L_a$. We need to examine all $j < k$ such that $\exists q \in P_j, \delta(q, a) \in s_{a,i}$ to construct the sets corresponding to splitting the block P_j w.r.t. a and P_i.

Let $j < k$. From the definition of $s_{a,i}$, we obtain the following:

$$\begin{aligned}
\exists q \in P_j, \delta(q, a) \in s_{a,i} &\iff \exists q \in P_j, \delta(q, a) \in P_i \wedge \underbrace{\delta^{-1}(\delta(q, a), a) \neq \varnothing}_{\text{true}} \\
&\iff \exists q \in P_j, \delta(q, a) \in P_i
\end{aligned}$$

Which corresponds to finding states in P_j having an outgoing a-transition to a state in P_i, as represented in the following scheme:



This set of states can be expressed via the inverse transition function:

$$\{q \in P_j \mid \delta(q,a) \in P_i\} = \left(\bigcup_{q \in P_i} \delta^{-1}(q,a)\right) \cap P_j$$

Since δ^{-1} was already computed in the first step of the algorithm, we can determine using req. 3 whether a state of $\bigcup_{q \in P_i} \delta^{-1}(q,a)$ is also in P_j in $\Theta(1)$ time.

Thus, instead of examining P_j for all $j < k$, we rather go through the table of δ^{-1} and for each state q such that $\delta(q,a) \in P_i$, we know from req. 4 that we can determine the index $j < k$ (because there are k blocks) of the block P_j containing q in $\Theta(1)$ time. The sets P'_j and $P''_j = P_k$ resulting from the partition can be constructed on the fly without any additional time cost. The construction of the sets b_j and b_k as well as the update of L_b for all $b \in \Sigma$ can also be done on the fly but require $\Theta(1)$ time for each symbol $b \in \Sigma$ and thus add up to a total of $\Theta(|\Sigma|)$ time.

Overall, one iteration of the loop runs in time

$$\Theta\left(|\Sigma| \cdot \left|\bigcup_{q \in P_i} \delta^{-1}(q,a)\right|\right)$$

We now show the result for Algorithm 2. A splitter $s =: (a, C)$ is picked from $\mathcal{W}$. Then, we iterate over all blocks $B \in \mathcal{P}$ that may be split with s. Let $B \in \mathcal{P}$.

$$\begin{aligned} s \text{ splits } B \iff & \exists q_1, q_2 \in B, \delta(q_1,a) \in C \wedge \delta(q_2,a) \notin C \\ \iff & \left(\bigcup_{q \in C} \delta^{-1}(q,a)\right) \cap B \neq \varnothing \wedge \left(\bigcup_{q \in B} \delta(q,a)\right) \setminus C \neq \varnothing \end{aligned}$$

Likewise, instead of examining all blocks $B \in \mathcal{P}$, we go through δ^{-1} and we update the splitters for all symbols in Σ and one iteration of the loop runs in time

$$\Theta\left(|\Sigma| \cdot \left|\bigcup_{q \in C} \delta^{-1}(q,a)\right|\right)$$

■

For the sake of simplicity, we now denote $\bigcup_{q \in C} \delta^{-1}(q,a)$ by $\overleftrightarrow{C}$.

We will now justify the logarithmic factor in the time complexity. We briefly explain why a logarithm stands out and prove the statement by induction. The idea is that for each symbol $a \in \Sigma$, a state $q \in \mathcal{Q}$ can be in at most one of the splitters in $\mathcal{W}$. When the loop iterates over this splitter, it will split the block and keep the smaller one, whose size will be at most the size of the splitter divided by two. This means that a splitter s can be processed at most $\log \|s\|$ times. We now properly state and prove the property by induction.

Lemma 3. Any splitter in the workset $s \in \mathcal{W}$ is processed – i.e. picked at the beginning of the *while* loop – at most $\lfloor \log \|s\| \rfloor$ times.

Proof. We first show the following statement:

During an iteration, the chosen splitter s will either be removed from the set of splitters or its size will be reduced by at least $\frac{\|s\|}{2}$ after the iteration.

Let $\{P_1, \cdots, P_\ell\}$ be the current partition and let $s =: (a, C)$ be the picked splitter. Thus, s is no longer in the set of splitters $\mathcal{W}$. Since we go over all splittable blocks, C may also be split by s.

- If C is not split by s, then s was already removed from the splitters. Note that it can be added again later if it is split by another splitter.
- If C is split into C' and C'', then $(a, \min\{C', C''\})$ is added to the splitters and its size is at most $\frac{\|s\|}{2}$.

Therefore, since a splitter cannot be empty, s can be processed at most m times where m is such that

$$1 \leq \frac{\|s\|}{2^m} < 2$$

which is equivalent to

$$m \leq \log \|s\| < m + 1 \quad \text{i.e.} \quad \lfloor \log \|s\| \rfloor = m$$

■

Lemma 4. Any block B resulting from a split and not added as a splitter is processed at most $\lfloor \log \frac{|B|}{2} \rfloor$ times.

Proof. Let $P \in \mathcal{P}$ be a block in the partition and $s =: (a, C)$ be a splitter such that s splits P into S and B so that (a, S) is added to the workset and B is not. In order for B to be processed, some (b, B) must be added to the workset.

Since the only way to create a fresh splitter is to split B, this means that there exists some later step in the loop such that some splitter (b, B') splits B into B_1 and B_2 and $(b, \min\{B_1, B_2\})$ is added to the workset.

From lemma 3, we know that this splitter can be processed at most $\lfloor \log |\min\{B_1, B_2\}| \rfloor$ times. Since $|\min\{B_1, B_2\}| \leq \frac{|B|}{2}$ and since there is no splitter containing B in the workset, we obtain that B can be processed at most $\lfloor \log \frac{|B|}{2} \rfloor$ times.

■

Lemma 5. Let us consider some step in the algorithm such that $\mathcal{P}$ is the current partition. The total time spent in the loop until termination is bounded by

$$T := \theta \left(\sum_{(a,C) \in \mathcal{W}} |\overleftarrow{C}| \log |C| + \sum_{(a,B) \in \Sigma \times \mathcal{P} \setminus \mathcal{W}} |\overleftarrow{B}| \log \frac{|B|}{2} \right)$$

where θ is the constant of proportionality that may be obtained from lemma 2.

Proof. We show the result by induction over the steps.

Base case: the current partition is $\{P_1, P_2\}$ and we may assume w.l.o.g. that P_1 is the smaller set, so that $\mathcal{W} = \{(a, P_1), a \in \Sigma\}$. We have to show that the total time spent in the loop is bounded by

$$T = \theta |\Sigma| \left(|\overleftarrow{P_1}| \log |P_1| + |\overleftarrow{P_2}| \log \frac{|P_2|}{2} \right)$$

We know from lemma 2 that an iteration of the loop for (a, P_i) takes $\theta |\overleftarrow{P_i}|$ time.

- From lemma 3, for all $a \in \Sigma$, (a, P_1) can be processed at most $\log \|(a, P_1)\| = \log |P_1|$ times[a], hence the total time for (a, P_1) is bounded by $\theta |\overleftarrow{P_1}| \log |P_1|$ and thus the total time for splitters with block P_1 is bounded by

$$\theta|\Sigma||\overset{\hookleftarrow}{P_1}|\log|P_1|$$

- From lemma 4, for all $a \in \Sigma$, (a, P_2) can be processed at most $\log\frac{\|(a,P_2)\|}{2} = \log\frac{|P_2|}{2}$ times. Thus, the total time for non splitters with block P_2 is bounded by

$$\theta|\Sigma||\overset{\hookleftarrow}{P_2}|\log\frac{|P_2|}{2}$$

Inductive step: Let $\{P_1, \cdots, P_\ell\} := \mathcal{P}$ be the current partition. The induction hypothesis states that the total time spent in the loop until termination for a is bounded by

$$T := \theta\left(\sum_{(a,C)\in\mathcal{W}}|\overset{\hookleftarrow}{C}|\log|C| + \sum_{(a,B)\in\Sigma\times\mathcal{P}\setminus\mathcal{W}}|\overset{\hookleftarrow}{B}|\log\frac{|B|}{2}\right)$$

By going through one more step, some blocks of $\mathcal{P}$ may be split and we define a new time $\hat{T}$ over this new partition and we have to show that $\hat{T} \leq T$.

Suppose some $B \in \mathcal{P}$ is split into B' and B'' by any splitter. We have to consider the cases where (a, B) is a splitter or not for all $a \in \Sigma$. Let $a \in \Sigma$.

- $(a, B) \in \mathcal{W}$: $\mathcal{W}$ is updated as follows:

$$\mathcal{W} := \mathcal{W} \setminus \{(a, B)\} \cup \{(a, B'), (a, B'')\}$$

Thus, instead of taking a time $\theta|\overset{\hookleftarrow}{B}|\log|B|$ time for (a, B), it now takes a time bounded by:

$$\theta|\overset{\hookleftarrow}{B'}|\log|B'| + \theta|\overset{\hookleftarrow}{B''}|\log|B''|$$

Finally, we show the following:

$$|\overset{\hookleftarrow}{B'}|\log|B'| + |\overset{\hookleftarrow}{B''}|\log|B''| \leq |\overset{\hookleftarrow}{B}|\log|B|$$

From $B = B' \cup B''$, we get $|B| = |B'| + |B''|$ and $|\overset{\hookleftarrow}{B}| = |\overset{\hookleftarrow}{B'}| + |\overset{\hookleftarrow}{B''}|$, hence:

$$
\begin{aligned}
&|\overset{\hookleftarrow}{B'}| \log |B'| + |\overset{\hookleftarrow}{B''}| \log |B''| - |\overset{\hookleftarrow}{B}| \log |B| \\
&= |\overset{\hookleftarrow}{B'}| \log |B'| + (|\overset{\hookleftarrow}{B}| - |\overset{\hookleftarrow}{B'}|) \log (|B| - |B'|) - |\overset{\hookleftarrow}{B}| \log |B| \\
&= |\overset{\hookleftarrow}{B'}| \log \frac{|B'|}{|B| - |B'|} + |\overset{\hookleftarrow}{B}| \log \frac{|B| - |B'|}{|B|} \\
&\leq |\overset{\hookleftarrow}{B'}| \log \frac{|B|}{|B| - |B'|} + |\overset{\hookleftarrow}{B}| \log \frac{|B| - |B'|}{|B|} \\
&= (|\overset{\hookleftarrow}{B}| - |\overset{\hookleftarrow}{B'}|) \log \underbrace{\frac{|B| - |B'|}{|B|}}_{\leq 1} \leq 0
\end{aligned}
$$

- $(a, B) \notin \mathcal{W}$: $(a, \min\{B', B''\})$ is added to the set of splitters. We may assume w.l.o.g. that B' is the smaller set. Thus, the corresponding term in the sum $|\overset{\hookleftarrow}{B}| \log \frac{|B|}{2}$ is updated as follows:

$$
\underbrace{\theta |\overset{\hookleftarrow}{B'}| \log |B'|}_{\text{lemma 3}} + \underbrace{\theta |\overset{\hookleftarrow}{B''}| \log \left(\frac{|B''|}{2} \right)}_{\text{lemma 4}}
$$

Since $B = B' \cup B''$, the sum above can be written as:

$$
\theta |\overset{\hookleftarrow}{B'}| \log |B'| + \theta (|\overset{\hookleftarrow}{B}| - |\overset{\hookleftarrow}{B'}|) \log \left(\frac{|B| - |B'|}{2} \right)
$$

By using the fact that $|B'| \leq \dfrac{|B|}{2}$, we have:

$$
\begin{aligned}
&|\overset{\hookleftarrow}{B'}| \log |B'| + (|\overset{\hookleftarrow}{B}| - |\overset{\hookleftarrow}{B'}|) \log \left(\frac{|B| - |B'|}{2} \right) - |\overset{\hookleftarrow}{B}| \log \frac{|B|}{2} \\
&\leq |\overset{\hookleftarrow}{B'}| \log \frac{|B|}{2} + (|\overset{\hookleftarrow}{B}| - |\overset{\hookleftarrow}{B'}|) \log \left(\frac{|B| - |B'|}{2} \right) - |\overset{\hookleftarrow}{B}| \log \frac{|B|}{2} \\
&\leq (|\overset{\hookleftarrow}{B}| - |\overset{\hookleftarrow}{B'}|) \log \left(\frac{|B| - |B'|}{|B|} \right) \leq 0
\end{aligned}
$$

Overall, splitting any block of $\mathcal{P}$ does not increase the total time spent over the loop. By iterating over all blocks, we obtain that the total time $\hat{T}$ spent in the loop from this new iteration until termination is such that $\hat{T} \leq T$.

■

[a]We drop the floor operator for simplicity. Note that since we are giving upper bounds, this is completely valid.

Theorem 1. Algorithm 1 runs in $O\left(|\Sigma| \cdot |\mathcal{Q}| \log |\mathcal{Q}|\right)$ time.

Proof. From lemma 5, we obtain in particular that for the initial partition the time spent in the loop until termination is bounded by:

$$T = \theta|\Sigma| \left(|\overleftarrow{P_1}| \log |P_1| + |\overleftarrow{P_2}| \log \frac{|P_2|}{2} \right)$$

Thus, the following holds:

$$T \leq \theta|\Sigma| (\underbrace{|\overleftarrow{P_1}| + |\overleftarrow{P_2}|}_{\leq 2|\overleftarrow{\mathcal{Q}}|}) \log(\underbrace{|P_1| + |P_2|}_{=|\mathcal{Q}|}) \leq 2\theta|\Sigma||\mathcal{Q}| \log |\mathcal{Q}|$$

■

References

[EB23] Javier Esparza and Michael Blondin. *Automata Theory: An Algorithmic Approach.* 2023.

[Hop71] John E. Hopcroft. *An n Log n Algorithm for Minimizing States in a Finite Automaton.* Stanford University, Stanford, CA, USA, 1971.