Final year internship report

Verification in Isabelle/HOL of Hopcroft's algorithm for minimizing DFAs including runtime analysis

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1 Preamble

Since the whole project relies on many deeply theoretical notions such as abstraction, logic or semantics, this section only gives a brief introduction to formal methods in general and their purpose.

Formal methods refer to a field of theoretical computer science whose main purpose is to provide logical links between mathematics and programming languages, so that one can state logical properties about formally described algorithms or mathematical procedures and relate them to actual implementations. The main application of formal methods is to prove correctness of programs with respect to a specification, i.e. a set of rules which must hold throughout the execution of the program. Such a program is then ensured to behave in a way[1] that is intended by its specification.

Such work is usually carried out using a proving assistant such as Isabelle, i.e. a program with an implemented logic allowing to reason with respect to that logic. Isabelle has the advantage to relie on a really light kernel, which is the only part that can only be trusted. Everything else – i.e. librairies, called theories – stems from this kernel. It also has nice interface with LaTeXsyntax adding little overhead to usual mathematical notations and is very flexible in terms of syntax and semantics.

A common way to proceed when we want to prove correctness of an algorithm and generate correct executable code is to follow a top-down strategy via successive refinements. We start from the highest level – i.e. with the highest level of abstraction – and we refine towards the implementation. As an example, if an algorithm manipulates sets at the abstract level, we may use lists or trees in its implementation. When performing a refinement, we have to give links – and prove them – between levels, e.g. that the behavior of the refined one does not contradict the abstract one. In Peter Lammich's Refinement Framework [Lam16], this is done via binary relations and abstraction/concretization functions[2].

If formal methods allow for correctness and termination theorems, it is less trivial to analyze complexity of a program, e.g. in terms of time or memory

[1]In the context of industrial applications, we often differentiate between two sorts of assertions, namely safety properties and liveness properties, so that we can explicitly check that the system works (it is "alive") and that it is safe (it works properly).

[2]A concretization function maps an abstract value to a set of concrete values and conversely for an abstraction function, so that invariants can be carried and preserved through the refinements. For example, we may abstract the set $\mathbb{Z}$ of integers to $\{Neg, Zero, Pos\}$ where Pos (resp. Neg) represents positive (resp. negative) integers and $Zero$ represents the singular set $\{0\}$.

consumption. Since complexity of a program is not a logical property, but a property of the execution of the program, a model of computation must be defined, i.e. a way to execute the program, and then prove that the execution of the program in this model is equivalent to the execution of the program in the real world. Maximilian Haslbeck and Peter Lammich have extended the Refinement Framework to include worst-case time complexity analysis in Isabelle [HL19] and have also implemented a verified framework to analyze algorithms down to LLVM code [HL22]. The principle is to model costs of operations with resources and currencies and embed this new dimension in the program, where operations consume resources.

Since the project also relies on automata theory, it is recommended to have some basic knowledge about automata theory and formal languages.

2 Introduction

This internship report is the result of a 6-month internship at the Chair of Logic and Verification of the Technical University of Munich (TUM) under the supervision of Tobias Nipkow and Peter Lammich. The main goal of this internship was to verify Hopcroft's algorithm for minimizing DFAs in Isabelle/HOL and to analyze its worst-case time complexity using the Refinement Framework. Some considerable work had already been carried out by Peter Lammich and Thomas Türk on the proof of correctness ten years ago in an older version of Isabelle [LT12]. The goal was to port this proof to the latest version of Isabelle and to extend it with a time complexity analysis.

2.1 Hopcroft's algorithm

John E. Hopcroft's algorithm for minimizing deterministic finite automata (DFA) was first presented in his original 1971 paper [Hop71] as a formal algorithm. It has been a major breakthrough in the field of automata theory, since it allowed for minimization in linearithmic[3] time in the worst case[4]. Thanks to this, minimization has become almost costless and thus this algorithm has been widely used in the field of formal verification, e.g. in model checking, since it allows to reduce the size of the automaton to be verified.

The formal idea behind the algorithm is to partition the set of states of the DFA into equivalence classes, i.e. sets of states that are equivalent[5]

[3] $O(n \log n)$

[4] Other minimization algorithms were presented prior to Hopcroft, e.g. by Moore (quadratic) or Brzozowski (exponential).

[5] Two states are equivalent if they accept the same language.

with respect to the language they recognize. The algorithm starts with two partitions, namely the set of final states and the set of non-final states. Then, it iteratively refines the partitions by splitting them into smaller ones until no more refinement is possible. The algorithm is guaranteed to terminate since the number of partitions is finite and strictly decreases at each iteration. The equivalence class of the set of states of the DFA is then the set of partitions, so that any two different sets of states from that partition recognize different languages.

Algorithm 1: Hopcroft's original formal algorithm

Data: a finite DFA $\mathcal{A} = (\mathcal{Q}, \Sigma, \delta, q_0, \mathcal{F})$
Result: the equivalence class of $\mathcal{Q}$ under state equivalence

Construct $\delta^{-1}(q,a) := \{t \in \mathcal{Q} \mid \delta(t,a) = q\}$ for all $q \in \mathcal{Q}$ and $a \in \Sigma$;
Construct $P_1 := \mathcal{F}$, $P_2 := \mathcal{Q} \setminus \mathcal{F}$ and $s_{a,i} := \{q \in P_i \mid \delta^{-1}(q,a) \neq \varnothing\}$ for all $i \in \{1,2\}$ and $a \in \Sigma$;
Let $k := 3$;
For all $a \in \Sigma$, construct $L_a := \underset{1 \le i < k}{\arg\min} \ |s_{a,i}|$;
while $\exists a \in \Sigma, L_a \neq \varnothing$ **do**
  Pick $a \in \Sigma$ such that $L_a \neq \varnothing$ and $i \in L_a$;
  $L_a := L_a \setminus \{i\}$;
  forall $j < k, \exists q \in P_j, \delta(q,a) \in s_{a,i}$ **do**
    $P'_j := \{t \in P_j \mid \delta(t,a) \in s_{a,i}\}$ and $P''_j := P_j \setminus P'_j$;
    $P_j := P'_j$ and $P_k := P''_j$; construct $s_{a,j}$ and $s_{a,k}$ for all $a \in \Sigma$ accordingly ;
    For all $a \in \Sigma$, $L_a := \begin{cases} L_a \cup \{j\} & \text{if } j \notin L_a \wedge |s_{a,j}| \le |s_{a,k}| \\ L_a \cup \{k\} & \text{otherwise} \end{cases}$;
    $k := k + 1$;
  end
end

Algorithm 1 is a direct translation of the original algorithm from [Hop71] with only slight changes in the notations.

2.2 Modern formalisation

The algorithm is now usually given in a more mathematical and formalised way[6], which loosens up the choice for an implementation. The algorithm is given below in Algorithm 2.

[6] see for example [EB23]

Algorithm 2: Hopcroft's algorithm in a modern style

Data: a finite DFA $\mathcal{A} = (\mathcal{Q}, \Sigma, \delta, q_0, \mathcal{F})$
Result: the language partition P_ℓ

if $\mathcal{F} = \varnothing \vee \mathcal{Q} \setminus \mathcal{F} = \varnothing$ **then**
    return $\mathcal{Q}$
else
    $\mathcal{P} := \{\mathcal{F}, \mathcal{Q} \setminus \mathcal{F}\}$;
    $\mathcal{W} := \{(a, \min\{\mathcal{F}), \mathcal{Q} \setminus \mathcal{F}\}, a \in \Sigma\}$;
    while $\mathcal{W} \neq \varnothing$ **do**
        Pick (a, B') from $\mathcal{W}$ and remove it;
        forall $B \in \mathcal{P}$ **do**
            Split B with (a, B') into B_0 and B_1 ;
            $\mathcal{P} := (\mathcal{P} \setminus \{B\}) \cup \{B_0, B_1\}$;
            forall $b \in \Sigma$ **do**
                if $(b, B) \in \mathcal{W}$ **then**
                    $\mathcal{W} := (\mathcal{W} \setminus \{(b, B)\}) \cup \{(b, B_0), (b, B_1)\}$;
                else
                    $\mathcal{W} := \mathcal{W} \cup \{(b, \min\{B_0, B_1\})\}$;
                end
            end
        end
    end
end

We justify this transformation. First, instead of storing indices for the blocks, we directly deal with sets and explicitly work with a partition $\mathcal{P}$ of $\mathcal{Q}$. Then, we define splitters.

Definition 1. Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, I, F)$ be a DFA. Let P be a partition of Q. Let $B \in P$. For $a \in \Sigma$ and $C \in P$ we say that (a, C) splits B or is a splitter of B if

$$\exists q_1, q_2 \in B \quad \delta(q_1, a) \in C \wedge \delta(q_2, a) \notin C.$$

Remark 1. If (a, C) is a splitter of B, P can be updated to $P \setminus \{B\} \cup \{B', B''\}$, where

$$B' := \{q \in B \mid \delta(q, a) \in C\} \text{ and } B'' := B \setminus B'.$$

Definition 2. For a splitter $s \in \Sigma \times P$, we define its size $\|s\|$ as the size of its second component, i.e. if $s = (a, C)$, then $\|s\| := |C|$.

In Algorithm 1, picking an index $i \in L_a$ corresponds to picking a set $s_{a,i}$, i.e. – by definition – the subset of P_i of states having at least one predecessor. We cannot directly replace L_a and $s_{a,i}$ by a set of splitters (a, P_i) as is, however the next step in the algorithm is to go over all blocks in the partition that have a successor in $s_{a,i}$. If such a block B is found, it is split into two blocks, namely the set of states having a successor in $s_{a,i}$ and the others. Overall, we look for B such that P_i has a predecessor in B. We add the condition that B must also have at least one successor not in P_i to avoid splitting B into $\{B, \varnothing\}$. Since this is equivalent to splitting B with (a, P_i), we can replace L_a by a set of splitters (a, P_i). Since the symbol a is chosen such that $L_a \neq \varnothing$, we can define a workset $\mathcal{W}$ of all splitters, with all symbols of Σ. Therefore, L_a is empty if and only if there is no a-splitter in $\mathcal{W}$. Because testing emptiness is equivalent to testing membership, and because of the existential quantifier in the definition of splitters, the $s_{a,i}$ are no longer needed.

We can explicitly write the trivial cases where either $\mathcal{F}$ or $\mathcal{Q} \setminus \mathcal{F}$ is empty to improve performance for this specific case. This is not done in Algorithm 1 but this does not produce wrong results. If one of the two sets is empty, one of the $s_{a,1}$ or $s_{a,2}$ will be empty for all $a \in \Sigma$, and the *while* loop will be entered $|\Sigma|$ times but the inner *for* loop will never be entered. This results in a useless $O(|\Sigma|)$ computation. Another difference in that case is that it would return the partition $\min\{\mathcal{F}, \mathcal{Q} \setminus \mathcal{F}\}$ – where min returns the set with the least number of elements – which we avoid in the modern version by just returning $\mathcal{Q}$.

3 Proof of correctness

4 Time complexity analysis

4.1 Paper proof

In a first time, we focus on the original algorithm presented in Algorithm 1 in order to work on the arguments given in [Hop71]. The data structures used at that time were mostly linked lists, but let us rather give some requirements for the data structures instead of actual implementations. The goal is to show that the algorithm can be executed in $O(m \cdot n \log n)$ time, where m is the number of symbols in the alphabet and n is the number of states in the DFA.

The following requirements come directly from [Hop71] and are specific to the algorithm presented in Algorithm 1.

Requirement 1. Sets such as $\delta^{-1}(q, a)$ and L_a must be represented in a way that allows $O(1)$ time for addition and deletion in "front position"[7].

Requirement 2. Vectors must be maintained to indicate whether a state is in a given set.

Requirement 3. Sets such as P_i must be represented in a way that allows $O(1)$ time for addition and deletion at any given position.

Requirement 4. For a state q in the representation of a set P_i or $s_{a,i}$, its "position" – in the implemented data structure – must be determined in $O(1)$ time.

VT: Maybe not necessary? This should be provable from Req. 2 and Req. 3.

Lemma 1. In Algorithm 1, lines 1 to 4 can be executed in $O(|\Sigma| \cdot |\mathcal{Q}|)$ time.

Proof. The non trivial part is the computation of the inverse transition function $\delta^{-1}(q, a)$, for all $q \in \mathcal{Q}$ and $a \in \Sigma$. This can be done in $O(|\Sigma| \cdot |\mathcal{Q}|)$ time by iterating over Σ and traversing the automaton (e.g. with a DFS) while keeping track of the predecessor at each step. ■

Lemma 2. An iteration of the while loop in both algorithms for a splitter $s = (a, C)$ takes a time proportional to the number of transitions terminating in C and the number of symbols in the alphabet, i.e. $\Theta\left(|\Sigma| \cdot |\delta^{-1}(C, a)|\right)$ time.

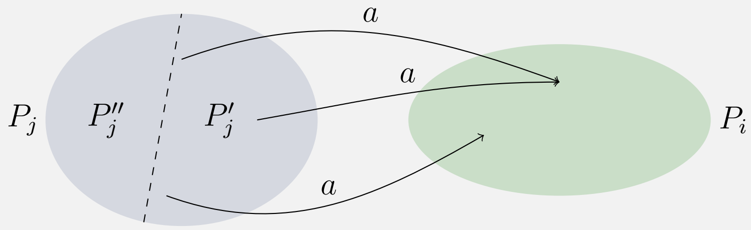
Proof. We start by showing it for Algorithm 1. We pick $a \in \Sigma$ such that $L_a \neq \varnothing$ and an $i \in L_a$. We need to examine all $j < k$ such that $\exists q \in P_j, \delta(q, a) \in s_{a,i}$ to construct the sets corresponding to splitting the block P_j w.r.t. a and P_i.

Let $j < k$. From the definition of $s_{a,i}$, we obtain the following:

$$\begin{aligned} \exists q \in P_j, \delta(q, a) \in s_{a,i} &\iff \exists q \in P_j, \delta(q, a) \in P_i \wedge \underbrace{\delta^{-1}(\delta(q, a), a) \neq \varnothing}_{\text{true}} \\ &\iff \exists q \in P_j, \delta(q, a) \in P_i \end{aligned}$$

Which corresponds to finding states in P_j having an outgoing a-transition to a state in P_i, as represented in the following scheme:

[7] This does not make sense for sets and is specific for a data structure. What is meant here is a bit informal and designates a position directly accessible, e.g. the head for a stack, the value of the root for a tree, etc.



This set of states can be expressed via the inverse transition function:

$$\{q \in P_j \mid \delta(q,a) \in P_i\} = \left(\bigcup_{q' \in P_i} \delta^{-1}(q',a)\right) \cap P_j$$

Since δ^{-1} was already computed in the first step of the algorithm, we can determine using req. 3 whether a state of $\bigcup_{q \in P_i} \delta^{-1}(q,a)$ is also in P_j in $\Theta(1)$ time.

Thus, instead of examining P_j for all $j < k$, we rather go through the table of δ^{-1} and for each state q such that $\delta(q,a) \in P_i$, we know from req. 4 that we can determine the index $j < k$ (because there are k blocks) of the block P_j containing q in $\Theta(1)$ time. The sets P_j' and $P_j'' = P_k$ resulting from the partition can be constructed on the fly without any additional time cost. The construction of the sets $s_{b,j}$ and $s_{b,k}$ as well as the update of L_b for all $b \in \Sigma$ can also be done on the fly but require $\Theta(1)$ time for each symbol $b \in \Sigma$ and thus add up to a total of $\Theta(|\Sigma|)$ time.

VT: the on-the-fly computation part is a little hand-waving...

Overall, one iteration of the loop runs in time

$$\Theta\left(|\Sigma| \cdot \left|\bigcup_{q \in P_i} \delta^{-1}(q,a)\right|\right)$$

We now show the result for Algorithm 2. A splitter $s =: (a, C)$ is picked from $\mathcal{W}$. Then, we iterate over all blocks $B \in \mathcal{P}$ that may be split with s. Let $B \in \mathcal{P}$.

$$\begin{aligned} s \text{ splits } B &\iff \exists q_1, q_2 \in B, \delta(q_1,a) \in C \wedge \delta(q_2,a) \notin C \\ &\iff \left(\bigcup_{q \in C} \delta^{-1}(q,a)\right) \cap B \neq \varnothing \wedge \left(\bigcup_{q \in B} \delta(q,a)\right) \setminus C \neq \varnothing \end{aligned}$$

VT: we may express the right-hand conjunct in terms of δ^{-1} in order to prepare the paragraph that follows (But it wouldn't be the same union because you have to start from states outside C)

VT: The second equation may be a little too quick, especially for the right-hand conjunct.

Likewise, instead of examining all blocks $B \in \mathcal{P}$, we go through δ^{-1} and we update the splitters for all symbols in Σ and one iteration of the loop runs in time

$$\Theta\left(|\Sigma| \cdot \left|\bigcup_{q\in C} \delta^{-1}(q,a)\right|\right)$$

■

For the sake of simplicity, we now denote $\bigcup_{q\in C} \delta^{-1}(q,a)$ by $\overleftrightarrow{C}$.

We will now justify the logarithmic factor in the time complexity. We briefly explain why a logarithm stands out and prove the statement by induction. The idea is that for each symbol $a \in \Sigma$, a state $q \in \mathcal{Q}$ can be in at most one of the splitters in $\mathcal{W}$. When the loop iterates over this splitter, it will split the block and keep the smaller one, whose size will be at most the size of the splitter divided by two. This means that a splitter s can be processed at most $\log \|s\|$ times. We now properly state and prove the property by induction.

We see splitters as updatable program variables. This means that $s_{a,i}$ as a set may differ along the loop, however we know there exists some $q \in \mathcal{Q}$ such that $s_{a,i}$ is the unique set containing q throughout the execution. This gives a way to characterize splitters throughout the whole execution.

Lemma 3. Any splitter in the workset $s \in \mathcal{W}$ is processed – i.e. picked at the beginning of the *while* loop – at most $\lfloor \log \|s\| \rfloor$ times.

Proof. We first show the following statement:

During an iteration, the chosen splitter s will either be removed from the set of splitters or its size will be reduced by at least $\frac{\|s\|}{2}$ after the iteration[a].

Let $\{P_1, \cdots, P_\ell\}$ be the current partition and let $s =: (a, C)$ be the picked splitter. Thus, s is no longer in the set of splitters $\mathcal{W}$. Since we go over all splittable blocks, C may also be split by s.

- If C is not split by s, then s was already removed from the splitters.

Note that it can be added again later if it is split by another splitter.

- If C is split into C' and C'', then $(a, \min\{C', C''\})$ is added to the splitters and its size is at most $\frac{||s||}{2}$.

Therefore, since a splitter cannot be empty, s can be processed at most m times where m is such that

$$1 \leq \frac{||s||}{2^m} < 2$$

which is equivalent to

$$m \leq \log ||s|| < m+1 \quad \text{i.e.} \quad \lfloor \log ||s|| \rfloor = m$$

■

[a] That is a bit informal because, the original splitter is actually removed and another "similar" splitter (cf the paragraph above the statement of the lemma) of size at most $\frac{||s||}{2}$ will be added.

Lemma 4. Any block B resulting from a split and not added as a splitter is processed at most $\lfloor \log \frac{|B|}{2} \rfloor$ times.

Proof. Let $P \in \mathcal{P}$ be a block in the partition and $s =: (a, C)$ be a splitter such that s splits P into S and B so that (a, S) is added to the workset and B is not. In order for B to be processed, some (b, B) must be added to the workset.

Since the only way to create a fresh splitter is to split B, this means that there exists some later step in the loop such that some splitter (b, B') splits B into B_1 and B_2 and $(b, \min\{B_1, B_2\})$ is added to the workset.

From lemma 3, we know that this splitter can be processed at most $\lfloor \log |\min\{B_1, B_2\}| \rfloor$ times. Since $|\min\{B_1, B_2\}| \leq \frac{|B|}{2}$ and since there is no splitter containing B in the workset, we obtain that B can be processed at most $\lfloor \log \frac{|B|}{2} \rfloor$ times.

■

Lemma 5. Let us consider some step in the algorithm such that $\mathcal{P}$ is the current partition. The total time spent in the loop until termination is bounded by

$$T := \theta \left(\sum_{(a,C)\in\mathcal{W}} |\overleftarrow{C}| \log |C| + \sum_{(a,B)\in(\Sigma\times\mathcal{P})\setminus\mathcal{W}} |\overleftarrow{B}| \log \frac{|B|}{2} \right)$$

where θ is the constant of proportionality that may be obtained from lemma 2.

Proof. We show the result by induction over the steps.

Base case: the current partition is $\{P_1, P_2\}$ and we may assume w.l.o.g. that P_1 is the smaller set, so that $\mathcal{W} = \{(a, P_1), a \in \Sigma\}$. We have to show that the total time spent in the loop is bounded by

$$T = \theta|\Sigma| \left(|\overleftarrow{P_1}| \log |P_1| + |\overleftarrow{P_2}| \log \frac{|P_2|}{2} \right)$$

We know from lemma 2 that an iteration of the loop for (a, P_i) takes $\theta|\overleftarrow{P_i}|$ time.

- From lemma 3, for all $a \in \Sigma$, (a, P_1) can be processed at most $\log \|(a, P_1)\| = \log |P_1|$ times[a], hence the total time for (a, P_1) is bounded by $\theta|\overleftarrow{P_1}| \log |P_1|$ and thus the total time for splitters with block P_1 is bounded by

$$\theta|\Sigma||\overleftarrow{P_1}| \log |P_1|$$

- From lemma 4, for all $a \in \Sigma$, (a, P_2) can be processed at most $\log \frac{\|(a,P_2)\|}{2} = \log \frac{|P_2|}{2}$ times. Thus, the total time for non splitters with block P_2 is bounded by

$$\theta|\Sigma||\overleftarrow{P_2}| \log \frac{|P_2|}{2}$$

Inductive step: Let $\{P_1, \cdots, P_\ell\} := \mathcal{P}$ be the current partition. The induction hypothesis states that the total time spent in the loop until termination for a is bounded by

$$T := \theta \left(\sum_{(a,C)\in\mathcal{W}} |\overset{\hookleftarrow}{C}| \log |C| + \sum_{(a,B)\in\Sigma\times\mathcal{P}\setminus\mathcal{W}} |\overset{\hookleftarrow}{B}| \log \frac{|B|}{2} \right)$$

By going through one more step, some blocks of $\mathcal{P}$ may be split and we define a new time $\hat{T}$ over this new partition and we have to show that $\hat{T} \leq T$.

Suppose some $B \in \mathcal{P}$ is split into B' and B'' by any splitter. We have to consider the cases where (a, B) is a splitter or not for all $a \in \Sigma$. Let $a \in \Sigma$.

- $(a, B) \in \mathcal{W}$: $\mathcal{W}$ is updated as follows:

  $$\mathcal{W} := \mathcal{W} \setminus \{(a, B)\} \cup \{(a, B'), (a, B'')\}$$

  Thus, instead of taking a time $\theta|\overset{\hookleftarrow}{B}| \log |B|$ time for (a, B), it now takes a time bounded by:

  $$\theta|\overset{\hookleftarrow}{B'}| \log |B'| + \theta|\overset{\hookleftarrow}{B''}| \log |B''|$$

  Finally, we show the following:

  $$|\overset{\hookleftarrow}{B'}| \log |B'| + |\overset{\hookleftarrow}{B''}| \log |B''| \leq |\overset{\hookleftarrow}{B}| \log |B|$$

  From $B = B' \cup B''$, we get $|B| = |B'| + |B''|$ and $|\overset{\hookleftarrow}{B}| = |\overset{\hookleftarrow}{B'}| + |\overset{\hookleftarrow}{B''}|$, hence:

  $$\begin{aligned}
  &|\overset{\hookleftarrow}{B'}| \log |B'| + |\overset{\hookleftarrow}{B''}| \log |B''| - |\overset{\hookleftarrow}{B}| \log |B| \\
  &= |\overset{\hookleftarrow}{B'}| \log |B'| + (|\overset{\hookleftarrow}{B}| - |\overset{\hookleftarrow}{B'}|) \log (|B| - |B'|) - |\overset{\hookleftarrow}{B}| \log |B| \\
  &= |\overset{\hookleftarrow}{B'}| \log \frac{|B'|}{|B| - |B'|} + |\overset{\hookleftarrow}{B}| \log \frac{|B| - |B'|}{|B|} \\
  &\leq |\overset{\hookleftarrow}{B'}| \log \frac{|B|}{|B| - |B'|} + |\overset{\hookleftarrow}{B}| \log \frac{|B| - |B'|}{|B|} \\
  &= (|\overset{\hookleftarrow}{B}| - |\overset{\hookleftarrow}{B'}|) \log \underbrace{\frac{|B| - |B'|}{|B|}}_{\leq 1} \leq 0
  \end{aligned}$$

- $(a, B) \notin \mathcal{W}$: $(a, \min\{B', B''\})$ is added to the set of splitters. We may assume w.l.o.g. that B' is the smaller set. Thus, the corresponding term in the sum $|\overleftarrow{B}| \log \frac{|B|}{2}$ is updated as follows:

$$\theta \underbrace{|\overleftarrow{B'}| \log |B'|}_{\text{lemma 3}} + \theta \underbrace{|\overleftarrow{B''}| \log \left(\frac{|B''|}{2}\right)}_{\text{lemma 4}}$$

Since $B = B' \cup B''$, the sum above can be written as:

$$\theta |\overleftarrow{B'}| \log |B'| + \theta(|\overleftarrow{B}| - |\overleftarrow{B'}|) \log \left(\frac{|B| - |B'|}{2}\right)$$

By using the fact that $|B'| \leq \dfrac{|B|}{2}$, we have:

$$\begin{aligned}
&|\overleftarrow{B'}| \log |B'| + (|\overleftarrow{B}| - |\overleftarrow{B'}|) \log \left(\frac{|B| - |B'|}{2}\right) - |\overleftarrow{B}| \log \frac{|B|}{2} \\
&\leq |\overleftarrow{B'}| \log \frac{|B|}{2} + (|\overleftarrow{B}| - |\overleftarrow{B'}|) \log \left(\frac{|B| - |B'|}{2}\right) - |\overleftarrow{B}| \log \frac{|B|}{2} \\
&\leq (|\overleftarrow{B}| - |\overleftarrow{B'}|) \log \left(\frac{|B| - |B'|}{|B|}\right) \leq 0
\end{aligned}$$

Overall, splitting any block of $\mathcal{P}$ does not increase the total time spent over the loop. By iterating over all blocks, we obtain that the total time $\hat{T}$ spent in the loop from this new iteration until termination is such that $\hat{T} \leq T$.

■

[a] We drop the floor operator for simplicity. Note that since we are giving upper bounds, this is completely valid.

Theorem 1. Algorithm 1 and Algorithm 2 run in $O(|\Sigma| \cdot |\mathcal{Q}| \log |\mathcal{Q}|)$ time.

Proof. From lemma 5, we obtain in particular that for the initial partition the time spent in the loop until termination is bounded by:

$$T = \theta |\Sigma| \left(|\overleftarrow{P_1}| \log |P_1| + |\overleftarrow{P_2}| \log \frac{|P_2|}{2} \right)$$

Thus, the following holds:

$$T \leq \theta|\Sigma|(\underbrace{|\overleftarrow{P_1}| + |\overleftarrow{P_2}|}_{\leq 2|\overleftarrow{\mathcal{Q}}|}) \log(\underbrace{|P_1| + |P_2|}_{=|\mathcal{Q}|}) \leq 2\theta|\Sigma||\mathcal{Q}| \log |\mathcal{Q}|$$

■

4.2 Towards a formal proof

The objective of the project is to obtain a formal proof for the time complexity of the algorithm using the Refinement Framework with its extension to runtime analysis. We follow a top-down strategy, from the abstract level to the implementation.

4.2.1 Abstract level

The abstract algorithm is defined below:

```
definition Hopcroft_abstract 𝒜 =
(if (𝒬 𝒜 = {}) then RETURN {} else (
if (ℱ 𝒜 = {}) then RETURN {𝒬 𝒜} else (
  do {
    (P, _) ← WHILEIT (abstract_invar 𝒜) abstract_b (abstract_f 𝒜
       ) (abstract_init 𝒜);
    RETURN P
  })))
```

It makes use of some auxiliary definitions, like `abstract_invar` which states that the current partition and the set of splitters are well formed and satisfy some properties on language equivalence. The boolean `abstract_b` is the termination condition of the loop, namely it checks emptiness of the set of splitters, and the initial partition is defined by `abstract_init`. The body of the loop is defined by `abstract_f` and is given below:

```
definition abstract_f 𝒜 =
  (λ(P, L). do {
    ASSERT (abstract_invar 𝒜 (P, L));
    ASSERT (L ≠ {});
    (a,p) ← SPEC (λ(a,p). (a,p) ∈ L);
    (P', L') ← SPEC (λ(P', L'). update_splitters_pred 𝒜 p a P L
       L' ∧ (P' = split 𝒜 p a {} P));
    RETURN (P', L')
  })
```

Again, this definition makes use of auxiliary definitions that we do not detail here. Recall from [HL22] that spending time is modeled by using resources via resource functions, namely $\$_s\ n$ is a resource function using n coins of the currency s.

We first analyze the body of the loop, i.e. we define parameters or generic costs for each operation. We define a cost function for `abstract_f` called $\texttt{abstract\_f}_{\texttt{cost}}$.

For the first two assertions, we can define a resource function $\$_{\texttt{check\_invar}}$ and a resource function $\$_{\texttt{check\_emptiness}}$ which will be useful for other assertions. For the first `SPEC`, we define $\$_{\texttt{pick\_splitter}}$ and for the second `SPEC`, we define $\$_{\texttt{split\_and\_update}}$. We must not split the latter into two separate costs for the moment since both operations will be performed on the fly, see the proof of lemma 2 for more details. To write it informally, we have:

```
definition abstract_f_cost = $_check_invar + $_check_emptiness + $_pick_splitter +
    $_split_and_update
```

Informally again, the complexity lies in $\$_{\texttt{split\_and\_update}}$ and the other costs will be constant eventually. The goal will be to eventually show – respectively to lemma 2 – that this $\$_{\texttt{split\_and\_update}}$ taken for a splitter (a, C) is bounded by `card (Σ 𝒜) * card ((λ q. δ⁻¹(q, a)) ` C) * $_c` where $\$_c$ is some constant, e.g. representing the cost of a comparison or any basic operation.

NOTE (DRAFT): We want something like this for $\$_{\texttt{split\_and\_update}}$:

```
definition split_and_update_cost where
  split_and_update_cost = 𝒜 PL s = let (a, C) = s in ...
```

The "..." should refer to operations performed in functions `split` and `update_splitters_pred`.

We will enventually write something like:

```
abstract_spec ≡ SPEC (λ P. P = Myhill_Nerode_partition 𝒜)

E_abs ≡ 0(abstract := abstract_cost)

Hopcroft_abstract 𝒜 ≤ ⇓_C E_abs (abstract_spec 𝒜 ($_abstract))
```

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