Verification in Isabelle/HOL of Hopcroft's algorithm for minimizing DFAs including runtime analysis

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supervised by Prof. Dr. Tobias Nipkow and Dr. Peter Lammich

Technical University of Munich
Chair for Logic and Verification

September 2023

Outline

Figure: Location of Munich

Figure: Some photos of Munich

Figure: Technical University of Munich (TUM), Garching campus

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DUNKEL
Kloster
Scheyern
SEIT 1119
KLOSTER-GOLD
DUNKEL


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Scheyern
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Dunkel


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Kloster Scheyern
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HERZOGLICH BAYERISCHES
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TEGERNSEE


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DUNKEL


SPEZIAL
BRAUHAUS
TEGERNSEE


WIRTS MADL
Bayrisch Hell


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BRAUHAUS
TEGERNSEE


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HB
HOFBRÄUHAUS

Understanding Refinement

Definition 1

Refinement is a systematic process of refining a high-level abstract specification into a concrete implementation.

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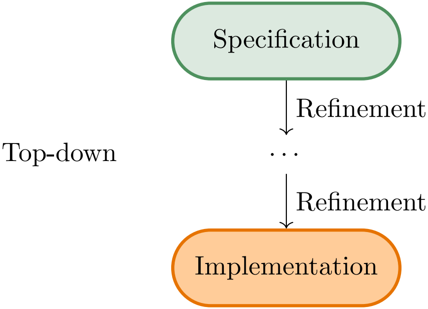

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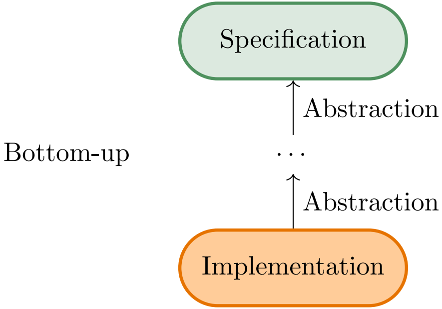
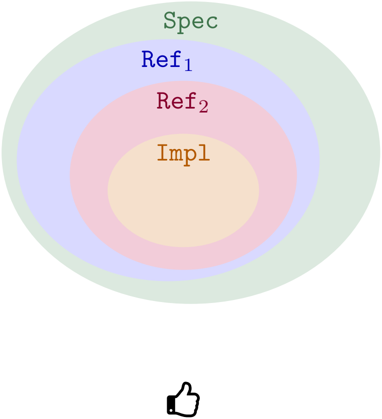
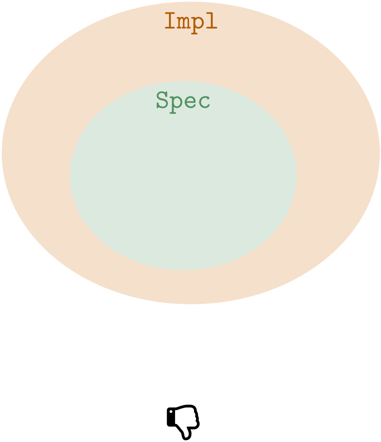
Refinement is a systematic process of refining a high-level abstract specification into a concrete implementation.



Each refinement step preserves the intended behavior.

$$\texttt{Spec} \rightarrow \texttt{Ref}_1 \rightarrow \texttt{Ref}_2 \rightarrow \texttt{Impl}$$



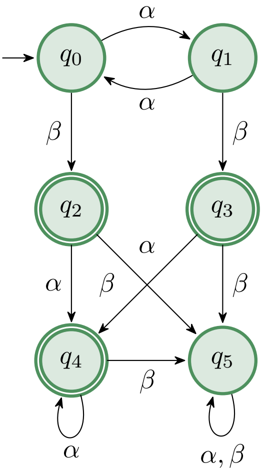


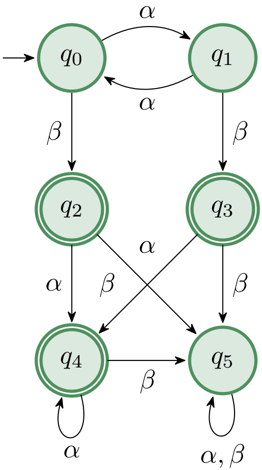
The Isabelle Refinement Framework

Isabelle Refinement Framework

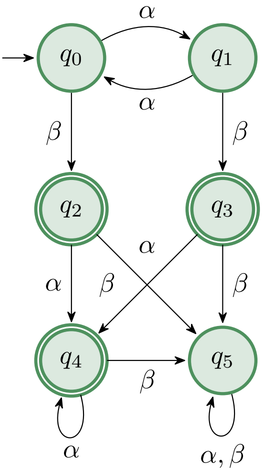
- Stepwise refinement approach to verified program development
- Formal and mathematical
- Ensures correctness at each step

Comes with the Isabelle Collection Framework, which provides an extensive library of reusable verified functional data structures (for data refinement).

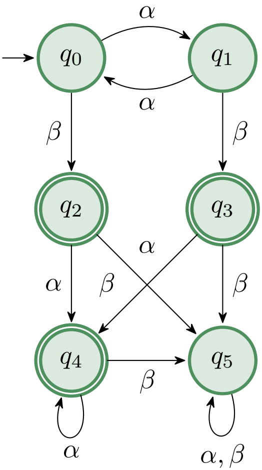

q_0
q_1
q_2
q_3
q_4
q_5
α
β
α, β



- Successively partitions the set of states into equivalence classes



- Successively partitions the set of states into equivalence classes
- Initial partition: accepting and non-accepting states



- Successively partitions the set of states into equivalence classes
- Initial partition: accepting and non-accepting states
- Each iteration: pick a splitter and split all blocks of the current partition

Data: a DFA $\mathcal{A} = (\mathcal{Q}, \Sigma, \delta, q_0, \mathcal{F})$
if $\mathcal{F} = \varnothing \vee \mathcal{Q} \setminus \mathcal{F} = \varnothing$
  return $\mathcal{Q}$
else

$\mathcal{P} := \{\mathcal{F}, \mathcal{Q} \setminus \mathcal{F}\}$
$\mathcal{W} := \{(a, \min\{\mathcal{F}, \mathcal{Q} \setminus \mathcal{F}\}), a \in \Sigma\}$
while $\mathcal{W} \neq \varnothing$ **do**
  Pick (a, C) from $\mathcal{W}$ and remove it
  forall $B \in \mathcal{P}$ **do**
    Split B with (a, C) into B_0 and B_1
    $\mathcal{P} := (\mathcal{P} \setminus \{B\}) \cup \{B_0, B_1\}$
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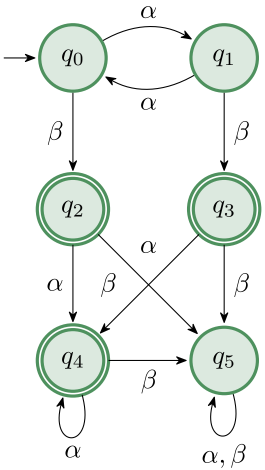

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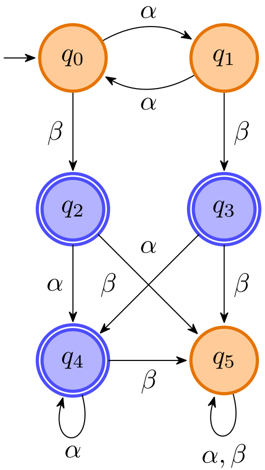

Splitter	Partition	Workset
–	$\{q_0, q_1, q_5\}\{q_2, q_3, q_4\}$	$(\alpha, \{q_0, q_1, q_5\})$ $(\beta, \{q_0, q_1, q_5\})$

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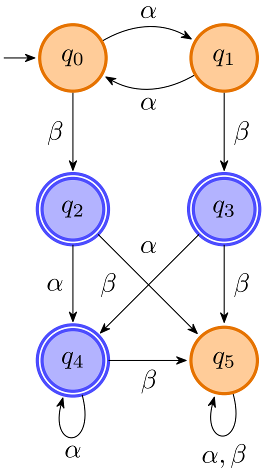

Splitter	Partition	Workset
–	$\{q_0, q_1, q_5\}\{q_2, q_3, q_4\}$	$(\alpha, \{q_0, q_1, q_5\})$ $(\beta, \{q_0, q_1, q_5\})$

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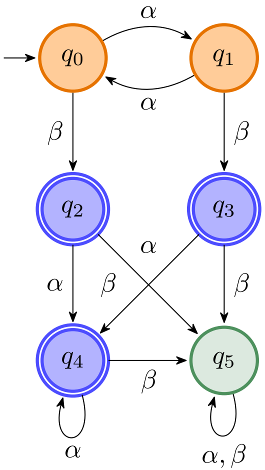

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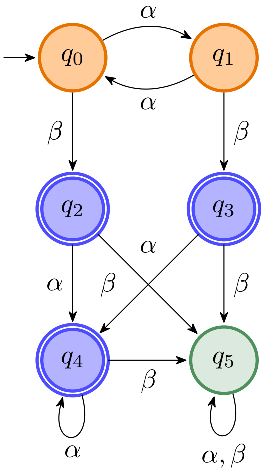

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  return $\mathcal{Q}$
else

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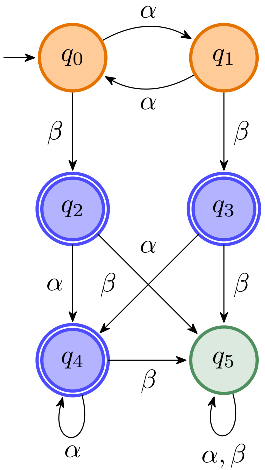

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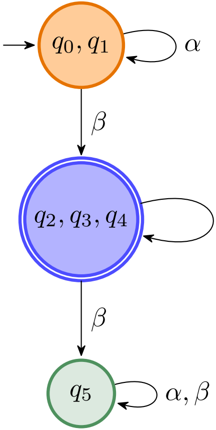
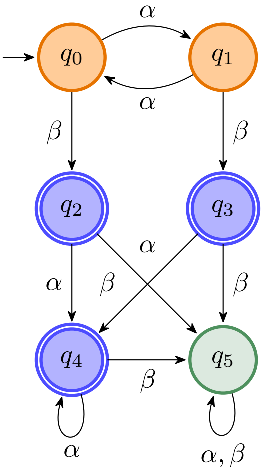
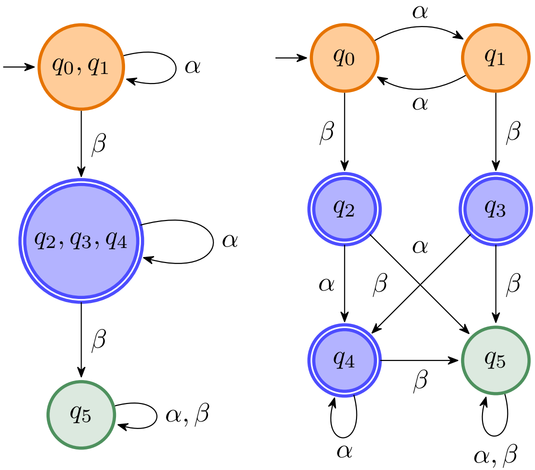


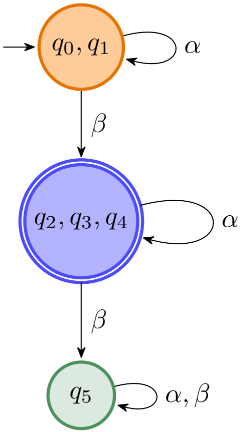


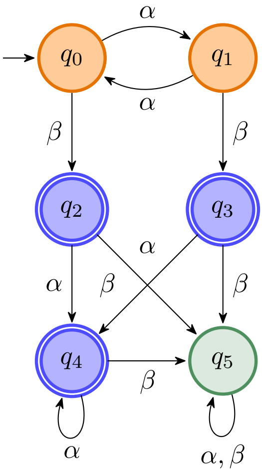
$$\mathcal{L} = \alpha^*\beta\alpha^*$$



$$\mathcal{L} = \alpha^* \beta \alpha^*$$

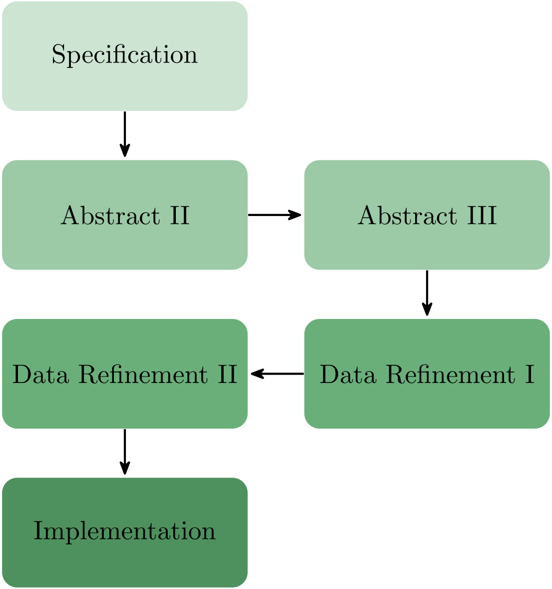
$$O(|\mathcal{Q}||\Sigma| \log(|\mathcal{Q}|))$$

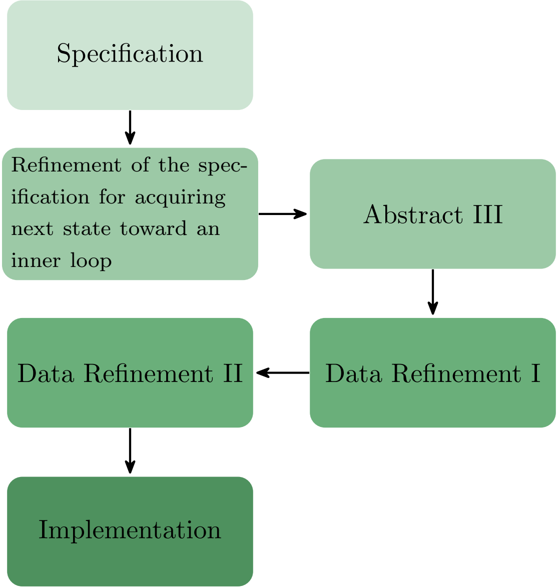


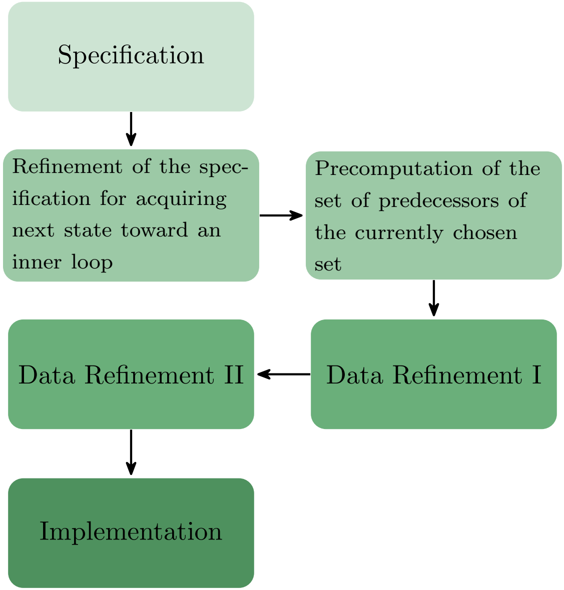


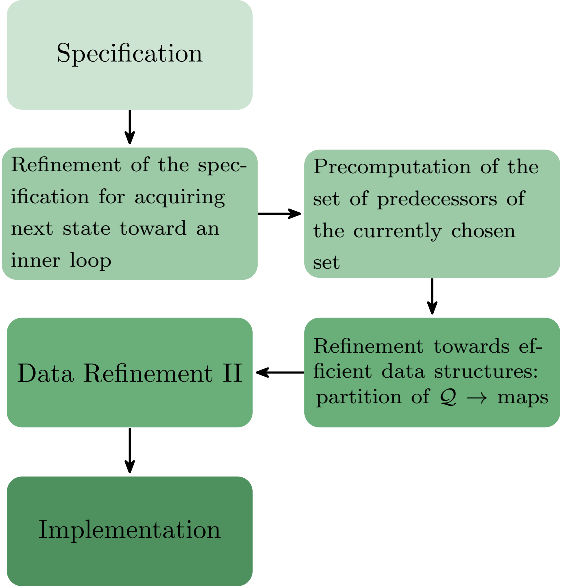
$$\mathcal{L} = \alpha^* \beta \alpha^*$$

$$O(n \log(n)) \quad \text{where} \quad n = |\mathcal{Q}|$$


Specification
Abstract II
Abstract III
Data Refinement II
Data Refinement I
Implementation


Specification
Refinement of the specification for acquiring next state toward an inner loop
Abstract III
Data Refinement I
Data Refinement II
Implementation


Specification
Refinement of the specification for acquiring next state toward an inner loop
Precomputation of the set of predecessors of the currently chosen set
Data Refinement I
Data Refinement II
Implementation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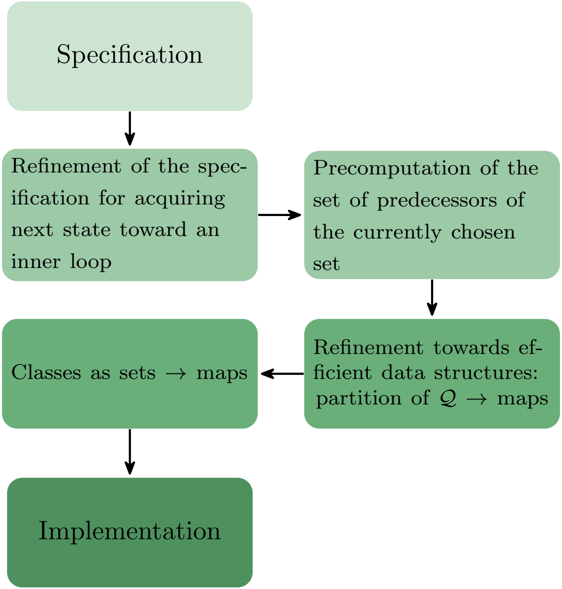
Precomputation of the set of predecessors of the currently chosen set
Refinement towards efficient data structures: partition of $\mathcal{Q}$ → maps
Data Refinement II
Implementation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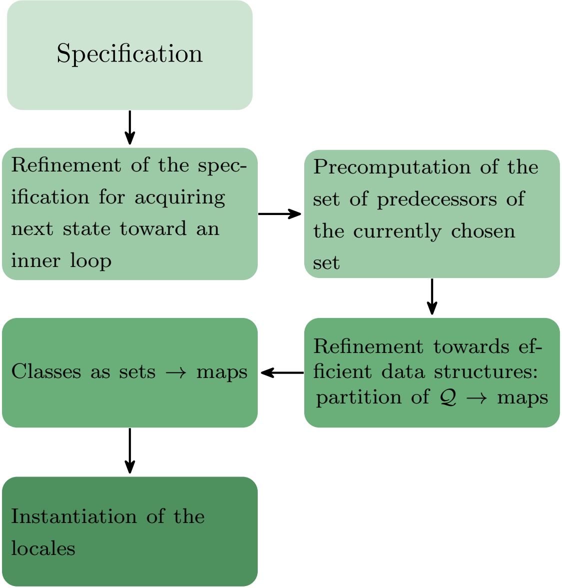
Refinement towards efficient data structures: partition of $\mathcal{Q}$ → maps
Classes as sets → maps
Implementation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Refinement towards efficient data structures: partition of $\mathcal{Q} \to$ maps
Classes as sets $\to$ maps
Instantiation of the locales

Specification

Refinement of the specification for acquiring next state toward an inner loop

Precomputation of the set of predecessors of the currently chosen set

What I had time to do

Refinement towards efficient data structures: partition of $\mathcal{Q} \rightarrow$ maps

Classes as sets $\rightarrow$ maps

Instantiation of the locales