Analízis II

6. Házi feladat

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1. Írja fel az

$$f(x) \coloneqq \sqrt{1+2x} \qquad \left(x \in \left]-\frac{1}{2}, +\infty\right[\right)$$

függvény nulla pont körüli második Taylor-polinomját! Adjon becslést az $|f(x) - T_{2,0}f(x)|$ hibára a $\left[-\frac{5}{18}, \frac{1}{4}\right]$ intervallumon.

Megoldás.

$$f'(x) = \frac{1}{2\sqrt{1+2x}} \cdot 2 = \frac{1}{\sqrt{1+2x}} = (1+2x)^{-\frac{1}{2}}$$
$$f''(x) = -\frac{1}{2} \cdot (1+2x)^{-\frac{3}{2}} \cdot 2 = -(1+2x)^{-\frac{3}{2}}$$

Ekkor,

$$T_{2,0}f(x) = f(0) + \frac{f'(0)}{1!} \cdot x + \frac{f''(0)}{2!} \cdot x^2 = 1 + x - \frac{1}{2}x^2$$
$$f'''(x) = \frac{3}{2} \cdot (1+2x)^{-\frac{5}{2}} \cdot 2 = \frac{3}{\sqrt{(1+2x)^5}}$$
$$|f(x) - T_{2,0}f(x)| = \frac{1}{3!} \cdot \frac{3}{\sqrt{\left(1-\frac{5}{9}\right)^5}} \cdot \left(\frac{1}{4}\right)^3 = \frac{1}{\cancel{6}\ 2} \cdot \frac{\cancel{3^6}\ 3^5}{2^5} \cdot \frac{1}{2^6} = \frac{3^5}{2^{12}}$$

2. Adja meg a következő függvények a pont körüli Taylor-sorát!

a) $f(x) \coloneqq 2^x \quad (x \in \mathbb{R}),\ a = 1$

b) $f(x) \coloneqq \ln(x^2+1) \quad (x \in \mathbb{R}),\ a = 0$

Megoldás.

a)

$$f'(x) = 2^x \cdot \ln 2 \qquad f''(x) = 2^x \cdot \ln^2 2 \qquad f'''(x) = 2^x \cdot \ln^3 2$$

$$T_1 f(x) \coloneqq 2 + 2\ln 2(x-1) + \left(\ln^2 2\right)(x-1)^2 + \frac{\cancel{2}\ \left(\ln^3 2\right)(x-1)^3}{\cancel{6}\ 3} + \cdots$$
$$= \sum_{k=0}^{+\infty} \frac{2 \cdot (\ln 2)^k}{k!} \cdot (x-1)^k$$

b) $f(x) \coloneqq \ln(x^2+1) \quad (x \in \mathbb{R}),\ a = 0$

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\begin{aligned}
f'(x) &= \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1} \\
f''(x) &= \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2} = \frac{2(1-x^2)}{(x^2+1)^2} \\
f'''(x) &= \frac{-4x(x^4+2x^2+1) - (2-2x^2)(4x^3+4x)}{(x^2+1)^4} \\
&= \frac{\cancel{-4x^5} - \cancel{8x^3} - 4x - 8x^3 - 8x + \cancel{8}\ {}^4x^5 + \cancel{8x^3}}{(x^2+1)^4} = \frac{4x^5 - 8x^3 - 12x}{(x^2+1)^4} \\
f^{(4)}(x) &= \frac{(20x^4 - 24x^2 - 12)(x^2+1)^4 - (4x^5 - 8x^3 - 12x)(3(x^2+1)(2x))}{(x^2+1)^8}
\end{aligned}
$$

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\begin{aligned}
T_0 f(x) &= 0 + 0 + x^2 + 0 + \frac{-12}{24}x^4 + 0 + \cdots = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots \\
&= \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} \cdot x^{2k}
\end{aligned}
$$