Analízis II

7. Házi feladat

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1. Számítsuk ki a következő határozatlan integrálokat!

a)

$$\begin{aligned}\int \frac{(x+1)^2}{\sqrt{x}}\,dx &= \int \frac{x^2+2x+1}{\sqrt{x}}\,dx = \int \frac{x^2}{\sqrt{x}}\,dx + \int \frac{2x}{\sqrt{x}}\,dx + \int \frac{1}{\sqrt{x}}\,dx \\ &= \int x^{2-\frac{1}{2}}\,dx + 2\cdot\int x^{1-\frac{1}{2}}\,dx + \int x^{-\frac{1}{2}}\,dx \\ &= \int x^{\frac{3}{2}}\,dx + 2\cdot\int x^{\frac{1}{2}}\,dx + \int x^{-\frac{1}{2}}\,dx \\ &= \frac{2}{5}\cdot x^{\frac{5}{2}} + 2\cdot\frac{2}{3}\cdot x^{\frac{3}{2}} + 2\cdot x^{\frac{1}{2}} + c = \frac{2}{5}\cdot x^{\frac{5}{2}} + \frac{4}{3}\cdot x^{\frac{3}{2}} + 2\cdot x^{\frac{1}{2}}\end{aligned}$$

b)

$$\begin{aligned}\int \sqrt{1-\cos 2x}\,dx &= \int \sqrt{\sin^2 x + \cos^2 x - (\cos^2 x - \sin^2 x)}\,dx = \int \sqrt{2\sin^2 x}\,dx \\ &= \sqrt{2}\int \sin x\,dx = \sqrt{2}\cdot -\cos x + c = -\sqrt{2}\cos x + c\end{aligned}$$

c)

$$\begin{aligned}\int \frac{1}{1+e^{-x}}\,dx &= \int \frac{-e^{-x}}{1+e^{-x}} + 1\,dx \\ &= \int \underbrace{\frac{-e^{-x}}{1+e^{-x}}}_{\frac{f'}{f}}\,dx + \int 1\,dx \quad \text{(Első helyettesítési szabály)} \\ &= \ln\left(1+e^{-x}\right) + x + c\end{aligned}$$

d)

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\cdot\int \underbrace{\frac{2x}{x^2+4}}_{\frac{f'}{f}}\,dx \quad \text{(Első helyettesítési szabály)}$$

$$= \frac{1}{2}\cdot \ln\left(x^2+4\right)+c$$

e)

$$\int \frac{x}{\sqrt[3]{x^2+4}}\,dx = \frac{1}{2}\cdot\int \frac{2x}{(x^2+4)^{\frac{1}{3}}}\,dx$$

$$= \frac{1}{2}\int 2x\cdot(x^2+4)^{-\frac{1}{3}}\,dx \quad \text{(Első helyettesítési szabály)}$$

$$= \frac{1}{2}\cdot\frac{3}{2}\cdot(x^2+4)^{\frac{2}{3}}+c = \frac{3}{4}\cdot(x^2+4)^{\frac{2}{3}}+c$$

f)

$$\int x^2\cdot\sqrt[3]{6x^3+4}\,dx = \int x^2\cdot(6x^3+4)^{\frac{1}{3}}\,dx$$

$$= \frac{1}{18}\cdot\int 18x^2\cdot(6x^3+4)^{\frac{1}{3}}\,dx \quad \text{(Első helyettesítési szabály)}$$

$$= \frac{1}{18}\cdot\frac{3}{4}\cdot(6x^3+4)^{\frac{4}{3}}+c = \frac{1}{24}\cdot(6x^3+4)^{\frac{4}{3}}+c$$

g)

$$
\begin{aligned}
\int \frac{5x+3}{2x-3}\,dx &= \int \frac{4x-6}{2x-3} + \frac{x+9}{2x-3}\,dx = \int 2\,dx + \int \frac{x+9}{2x-3}\,dx \\
&= 2x + \int \frac{x}{2x-3}\,dx + \int \frac{9}{2x-3}\,dx \\
&= 2x + \int \frac{x}{2x-3}\,dx + \frac{9}{2}\ln(2x-3)
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{x}{2x-3}\,dx &= \frac{1}{2}\int \frac{2x}{2x-3}\,dx \\
&= \frac{1}{2}\int \frac{2x-3+3}{2x-3}\,dx = \frac{1}{2}\left(\int 1\,dx + \int \frac{3}{2x-3}\,dx\right) \\
&= \frac{1}{2}\left(x + \frac{3}{2}\int \frac{2}{2x-3}\,dx\right) = \frac{1}{2}x + \frac{3}{4}\ln(2x-3)
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{5x+3}{2x-3}\,dx &= 2x + \frac{9}{2}\ln(2x-3) + \frac{1}{2}x + \frac{3}{4}\ln(2x-3) \\
&= 2x + \frac{1}{2}x + \frac{3}{4}\ln(2x-3) + \frac{9}{2}\ln(2x-3) + c \\
&= 2,5x + \frac{21}{4}\ln(2x-3) + c
\end{aligned}
$$

h)

$$\int \frac{x}{1+x^4}\,dx$$

i) A parciális integrálás szabálya alapján:

$$\begin{aligned}
\int x^2 \ln^2 x \, dx &= \int \left(\frac{x^3}{3}\right)' \ln^2 x \, dx \\
&= \frac{x^3}{3} \cdot \ln^2 x - \int \frac{x^3}{3} \cdot \frac{2 \ln x}{x} \, dx = \frac{x^3 \cdot ln^2 x}{3} - \frac{2}{3} \cdot \int x^2 \cdot \ln x \, dx
\end{aligned}$$

$$\begin{aligned}
\int x^2 \cdot \ln x \, dx &= \int \left(\frac{x^3}{3}\right)' \ln x \, dx \\
&= \frac{x^3}{3} \cdot \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx = \frac{x^3 \cdot \ln x}{3} - \frac{1}{3} \cdot \int x^2 \, dx \\
&= \frac{x^3 \cdot \ln x}{3} - \frac{1}{3} \cdot \frac{x^3}{3} = \frac{x^3 \cdot \ln x}{3} - \frac{x^3}{9}
\end{aligned}$$

$$\begin{aligned}
\int x^2 \ln^2 x \, dx &= \frac{x^3 \cdot \ln^2 x}{3} - \frac{2}{3} \cdot \left(\frac{x^3 \cdot \ln x}{3} - \frac{x^3}{9}\right) \\
&= \frac{x^3 \cdot \ln^2 x}{3} - \frac{2x^3 \ln x}{9} + \frac{2x^3}{27} \\
&= \frac{1}{3} x^3 \cdot \ln^2 x - \frac{2}{9} x^3 \cdot \ln x + \frac{2}{27} x^3 + c
\end{aligned}$$

j) A parciális integrálás szabálya alapján:

$$\begin{aligned}\int e^x \cdot \sin(3x+1)\,dx &= \int (e^x)' \cdot \sin(3x+1)\,dx \\ &= e^x \cdot \sin(3x+1) - \int e^x \cdot \cos(3x+1) \cdot 3\,dx \\ &= e^x \cdot \sin(3x+1) - 3\int e^x \cdot \cos(3x+1)\,dx\end{aligned}$$

$$\begin{aligned}\int e^x \cdot \cos(3x+1)\,dx &= \int (e^x)' \cdot \cos(3x+1)\,dx \\ &= e^x \cdot \cos(3x+1) - \int e^x \cdot -\sin(3x+1) \cdot 3\,dx\end{aligned}$$

$$\begin{aligned}\int e^x \cdot \sin(3x+1)\,dx &= e^x \cdot \sin(3x+1) - 3\int e^x \cdot \cos(3x+1)\,dx \\ &= e^x \cdot \sin(3x+1) - 3\left(e^x \cdot \cos(3x+1) - 3\int e^x \cdot -\sin(3x+1)\,dx\right)\end{aligned}$$

Rendezést követően a következő egyenletet kapjuk:

$$\int e^x \cdot \sin(3x+1)\,dx = e^x \cdot \sin(3x+1) - 3e^x \cdot \cos(3x+1) - 9\int e^x \cdot \sin(3x+1)\,dx$$

$$10\int e^x \cdot \sin(3x+1)\,dx = e^x \cdot \sin(3x+1) - 3e^x \cdot \cos(3x+1)$$

Melyből az következik, hogy:

$$\int e^x \cdot \sin(3x+1)\,dx = \frac{1}{10}e^x \cdot \sin(3x+1) - \frac{3}{10}e^x \cdot \cos(3x+1) + c$$