Solving integro-differential equations in MATLAB

Case 1: $\ddot{a} + 2a + 5\int_0^t a(x)\,dx = 1$

Analytical Solution

```
In[•]:= DSolveValue[{eqn = a'[t] + 2 a[t] == 1 - 5 ∫_0^t a[x] dx, a[0] == 0}, a[t], t]
Out[•]= e^-t Cos[t] Sin[t]
```

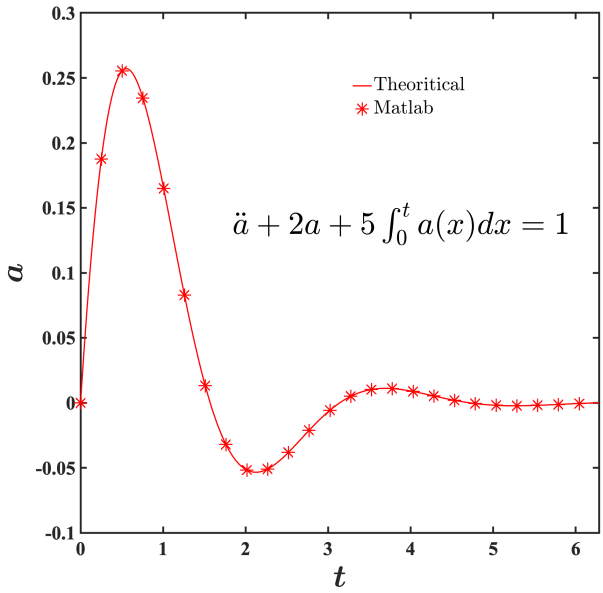
Numerical solution using MATLAB:

We will use the MATLAB ordinary differential equation solver **ode113** to solve this integro-differential equation. **Case1.m (see raw)** contains the relevant codes for this. The idea is:
1.Find an initial guess of the solution. To do this, drop the integral part of the equation and solve.
2. Use the guess value from the first step to evaluate the integral part of the equation. Put this step in a while loop, minimizing the change in solution.

```
%% Testing the IntegroDifferential Equation solver
clc
clear
close all
%% Case 1: a'[t] == 1 - 2*a[t] - 5*\int_0^t a[x]dx With a[0] = 0
tmax = 2*pi;
t = linspace(0,tmax,500);
%% Theoritical Solution (Using MATHEMATICA)
yth = exp(-t).*cos(t).*sin(t);
% ythGuess = 0.5*exp(-2*t).*(-1 + exp(2*t));

figure1 = figure('visible','on','WindowState','fullscreen','Color',[1×1×1]);
axes1 = axes('Parent',figure1);
hold(axes1,'on');
% plot(t, ythGuess,'r-','MarkerSize',20,'LineWidth',2,'DisplayName','Theoritical');
plot(t, yth,'r-','MarkerSize',20,'LineWidth',2,'DisplayName','Theoritical');

%% MATLAB Solution
odeOptions = odeset('AbsTol',1e-8,'RelTol',1e-8);
a0 = 0;
[t,guess] = ode45(@model0,t,a0,odeOptions);
guess = [t guess];
counter = 1; err = 1e4; Tol = 1e-8;
```

```
% [t,a] = ode113(@model, t, a0, odeOptions, guess);
while err > Tol
    [t,a] = ode113(@model, t, a0, odeOptions, guess);
    err = sum((a(:,1)-guess(:,2)).^2);
    fprintf('  %4i: %8.2e\n',[counter err]);
    wt = 0.5;
    guess = [t (1-wt)*guess(:,2) + wt*a(:,1)];
    counter = counter+1;
end

nskip = 20;
% plot(t(1:nskip:end), guess(1:nskip:end),'r*','MarkerSize',20,'LineWidth',2,'DisplayN
plot(t(1:nskip:end), a(1:nskip:end),'r*','MarkerSize',20,'LineWidth',2,'DisplayName','

legend1 = legend(axes1,'show');
set(legend1,...
    'Position',[0.45×0.75×0.263939012799944×0.0847176079734219],...
    'Interpreter','latex',...
    'FontSize',30,...
    'EdgeColor',[1×1×1]);
box(axes1,'on');
set(axes1,'FontName','times new roman','FontSize',30,'FontWeight','bold',...
    'LineWidth',3);
axis square
xlim([0.0 tmax])
% ylim([0.0×1.0])
xlabel('\boldmath{$t$}','LineWidth',2,'FontWeight','bold','FontSize',50,...
            'FontName','times new roman',...
            'Interpreter','latex');
ylabel('\boldmath{$a$}','LineWidth',2,'FontWeight','bold','FontSize',50,...
    'FontName','times new roman',...
    'Interpreter','latex');

annotation(figure1,'textbox',...×
    [0.45×0.53×0.25×0.10],...
    'String','$\ddot{a} + 2a + 5\int_0^ta(x)dx = 1$',...
    'LineStyle','none',...
    'Interpreter','latex',...
    'HorizontalAlignment','center',...
    'FontSize',50,...
    'FitBoxToText','off');

function dadt = model0(~,a)
    % Generating Initial Guess
    dadt = 1.0 - 2.0*a;
end

function dadt = model(t,a,guess)
% Inside the while loop, getting to the actual solution
% Integro-differential equation
```

```
ys = @(s) interp1(guess(:,1), guess(:,2), s);
dadt = 1.0 - 2.0*a - 5*integral(@(s) ys(s), 0, t,'RelTol',1e-8,'AbsTol',1e-13);
end
```



Case 2: $\ddot{a} = \int_0^t a(x)\,dx$

Analytical Solution

In[•]:=
```
DSolveValue[{eqn = a''[t] == ∫_0^t a[x] dx, a[0] == 0, a'[0] == 1}, a[t], t]
```

Out[•]=
$$\frac{1}{3} e^{-t/2}\left(e^{3t/2} - \text{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] + \sqrt{3}\,\text{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]\right)$$

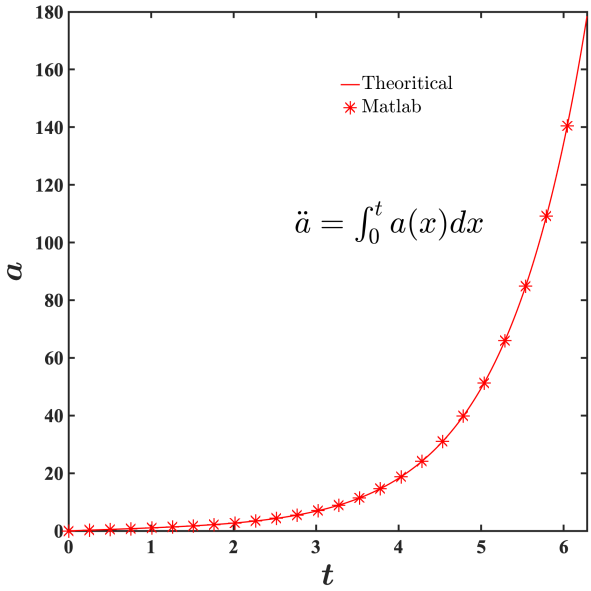
Numerical Solution using MATLAB

Case2.m (see raw) contains the relevant codes for this.

```
%% Testing the IntegroDifferential Equation solver
clc
clear
close all
%% Case 2: a''[t] == a[t] - \int_0^t a[x]dx With a[0] = 0
tmax = 2*pi;
t = linspace(0,tmax,500);
%% Theoritical Solution (Using MATHEMATICA)
% ythGuess = t;
yth = (1/3) * exp(-t/2).*(exp(3*t/2) - cos(sqrt(3)*t/2) + sqrt(3)*sin(sqrt(3)*t/2));
```

```
figure1 = figure('visible','on','WindowState','fullscreen','Color',[1×1×1]);
axes1 = axes('Parent',figure1);
hold(axes1,'on');
% plot(t, ythGuess,'r-','MarkerSize',20,'LineWidth',2,'DisplayName','Theoritical');
plot(t, yth,'r-','MarkerSize',20,'LineWidth',2,'DisplayName','Theoritical');

%% MATLAB Solution
odeOptions = odeset('AbsTol',1e-8,'RelTol',1e-8);
IC = [0; 1];  % IC = [a(0); a'(0)]
[t,guess] = ode45(@model0,t,IC,odeOptions);
guess = [t guess];
counter = 1; err = 1e4; Tol = 1e-8;
% [t,a] = ode113(@model, t, IC, odeOptions, guess);
while err > Tol
    [t,a] = ode113(@model, t, IC, odeOptions, guess);
    err = sum((a(:,1)-guess(:,2)).^2);
    fprintf('  %4i: %8.2e\n',[counter err]);
    wt = 0.5;
    guess = [t (1-wt)*guess(:,2) + wt*a(:,1) guess(:,3)];
    counter = counter+1;
end

nskip = 20;
% plot(t(1:nskip:end), guess(1:nskip:end,1),'r*','MarkerSize',20,'LineWidth',2,'Displa
plot(t(1:nskip:end), a(1:nskip:end,1),'r*','MarkerSize',20,'LineWidth',2,'DisplayName'

legend1 = legend(axes1,'show');
set(legend1,...
    'Position',[0.45×0.75×0.263939012799944×0.0847176079734219],...
    'Interpreter','latex',...
    'FontSize',30,...
    'EdgeColor',[1×1×1]);
box(axes1,'on');
set(axes1,'FontName','times new roman','FontSize',30,'FontWeight','bold',...
    'LineWidth',3);
axis square
xlim([0.0 tmax])
% ylim([0.0×1.0])
xlabel('\boldmath{$t$}','LineWidth',2,'FontWeight','bold','FontSize',50,...
            'FontName','times new roman',...
            'Interpreter','latex');
ylabel('\boldmath{$a$}','LineWidth',2,'FontWeight','bold','FontSize',50,...
    'FontName','times new roman',...
    'Interpreter','latex');

annotation(figure1,'textbox',...×
    [0.45×0.53×0.25×0.10],...
    'String','$\ddot{a} = \int_0^ta(x)dx$',...
    'LineStyle','none',...
    'Interpreter','latex',...
```

```
    'HorizontalAlignment','center',...
    'FontSize',50,...
    'FitBoxToText','off');

function dadt = model0(~,a)
    % Generating Initial Guess
    dadt = [a(2); 0];
end

function dadt = model(t,a,guess)
% Inside the while loop, getting to the actual solution
% Integro-differential equation
ys = @(s) interp1(guess(:,1), guess(:,2), s);
dadt = [a(2); integral(@(s) ys(s), 0, t,'RelTol',1e-8,'AbsTol',1e-13)];
end
```



Case 3: $\ddot{a} = \int_0^t \dot{a}(x)\, d\, x$

In[ ]:= `DSolveValue[{eqn = b'[t] == ∫_0^t b[x] dx, b[0] == 1}, b[t], t]`

Out[ ]= `Cosh[t]`

In[ ]:= `DSolveValue[{eqn = a'[t] == Cosh[t], a[0] == 0}, a[t], t]`

Out[ ]= `Sinh[t]`

Numerical Solution using MATLAB

Case3.m (see raw) contains the relevant codes for this.

```
%% Testing the IntegroDifferential Equation solver
clc
```

```
clear
close all
%% Case 3: a''[t] == \int_0^t a'[x]dx With a[0] = 0
tmax = 2*pi;
t = linspace(0,tmax,500);
%% Theoritical Solution (Using MATHEMATICA)
% ythGuess = t;
yth = sinh(t);

figure1 = figure('visible','on','WindowState','fullscreen','Color',[1×1×1]);
axes1 = axes('Parent',figure1);
hold(axes1,'on');
% plot(t, ythGuess,'r-','MarkerSize',20,'LineWidth',2,'DisplayName','Theoritical');
plot(t, yth,'r-','MarkerSize',20,'LineWidth',2,'DisplayName','Theoritical');

%% MATLAB Solution
odeOptions = odeset('AbsTol',1e-8,'RelTol',1e-8);
IC = [0; 1];  % IC = [a(0); a'(0)]
[t,guess] = ode45(@model0,t,IC,odeOptions);
guess = [t guess];
counter = 1; err = 1e4; Tol = 1e-8;
% [t,a] = ode113(@model, t, IC, odeOptions, guess);
while err > Tol
    [t,a] = ode113(@model, t, IC, odeOptions, guess);
    err1 = sum((a(:,1)-guess(:,2)).^2);
    err2 = sum((a(:,2)-guess(:,3)).^2);
    err = max(err1,err2);
    fprintf('  %4i: %8.2e\n',[counter err]);
    wt = 0.5;
    guess = [t (1-wt)*guess(:,2) + wt*a(:,1) a(:,2)];
    counter = counter+1;
end

nskip = 20;
% plot(t(1:nskip:end), guess(1:nskip:end,1),'r*','MarkerSize',20,'LineWidth',2,'Displa
plot(t(1:nskip:end), a(1:nskip:end,1),'r*','MarkerSize',20,'LineWidth',2,'DisplayName'

legend1 = legend(axes1,'show');
set(legend1,...
    'Position',[0.45×0.75×0.263939012799944×0.0847176079734219],...
    'Interpreter','latex',...
    'FontSize',30,...
    'EdgeColor',[1×1×1]);
box(axes1,'on');
set(axes1,'FontName','times new roman','FontSize',30,'FontWeight','bold',...
    'LineWidth',3);
axis square
xlim([0.0 tmax])
% ylim([0.0×1.0])
xlabel('\boldmath{$t$}','LineWidth',2,'FontWeight','bold','FontSize',50,...
```

```
            'FontName','times new roman',...
            'Interpreter','latex');
ylabel('\boldmath{$a$}','LineWidth',2,'FontWeight','bold','FontSize',50,...
    'FontName','times new roman',...
    'Interpreter','latex');

annotation(figure1,'textbox',...
    [0.45 0.53 0.25 0.10],...
    'String','$\ddot{a} = \int_0^t\dot{a}(x)dx$',...
    'LineStyle','none',...
    'Interpreter','latex',...
    'HorizontalAlignment','center',...
    'FontSize',50,...
    'FitBoxToText','off');

function dadt = model0(~,a)
    % Generating Initial Guess
    dadt = [a(2); 0];
end

function dadt = model(t,a,guess)
% Inside the while loop, getting to the actual solution
% Integro-differential equation
ys = @(s) interp1(guess(:,1), guess(:,3), s);
dadt = [a(2); integral(@(s) ys(s), 0, t,'RelTol',1e-8,'AbsTol',1e-13)];
end
```