Numerical Solution to the Barotropic Vorticity Equation on an f-plane

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Final Project, Physics 781

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Vorticity

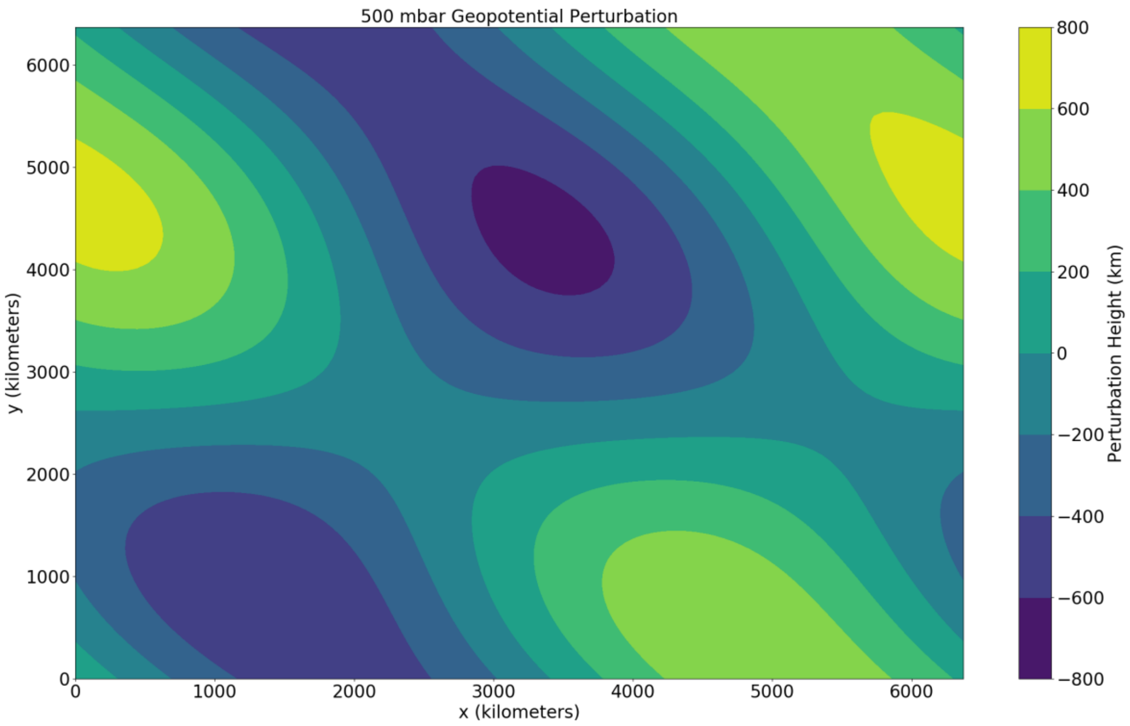
- Describes the local rotation of a fluid near some central point
- Analog to angular momentum for fluids
- Conserved!
- $\eta = \zeta + f$, Absolute vorticity is equal to the relative vorticity plus contribution from Earth's rotation.
- $\frac{\mathrm{d}}{\mathrm{d}t}[\zeta + f] = 0$. Vorticity Conservation
- Helpful to think of a "potential" function, the stream-function: $\nabla^2 \Psi = \zeta$

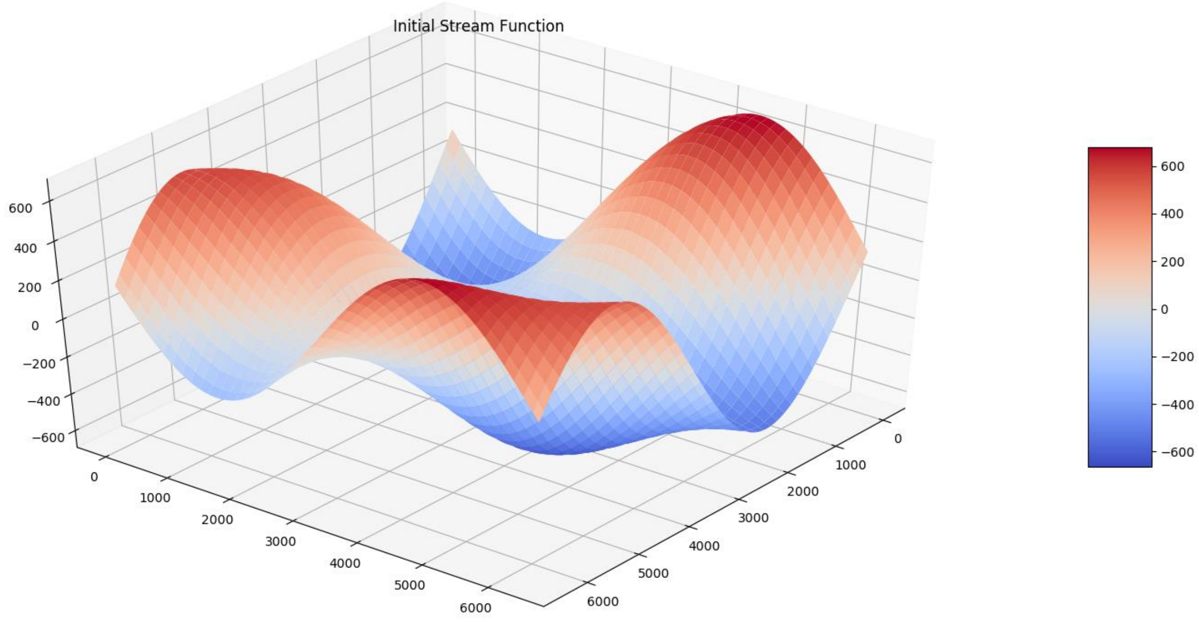
Barotropic Vorticity Equation

$$\frac{d}{dt}(\nabla^2 - F)w + \frac{g}{f}\mathrm{J}(\psi, \nabla^2\psi) + \beta\frac{\partial}{\partial x}w = 0$$

- w is stream function, $w = \frac{g}{f}\Phi$, Φ=Geopotential Height
- g is gravity, f is Coriolis parameter: $2\Omega\sin(\text{latitude})$, Ω is planetary rotation rate
- F is small scale dissipation term

- $\beta = \frac{2\Omega\cos(\text{latitude})}{Radius\ of\ Earth}$
- $\mathrm{J}(w, \nabla^2 w)$ is the Arakawa Jacobian, $\mathrm{J}(w, \nabla^2 w) = \frac{\partial w}{\partial x}\frac{\partial(\nabla^2 w)}{\partial y} - \frac{\partial w}{\partial y}\frac{\partial(\nabla^2 w)}{\partial x}$
- Consider constant latitude at 45 degrees North





Solving the Problem

$$\frac{d}{dt}(\nabla^2 - F)w = -\frac{g}{f}\mathrm{J}(w, \nabla^2 w) - \beta\frac{\partial}{\partial x}w$$

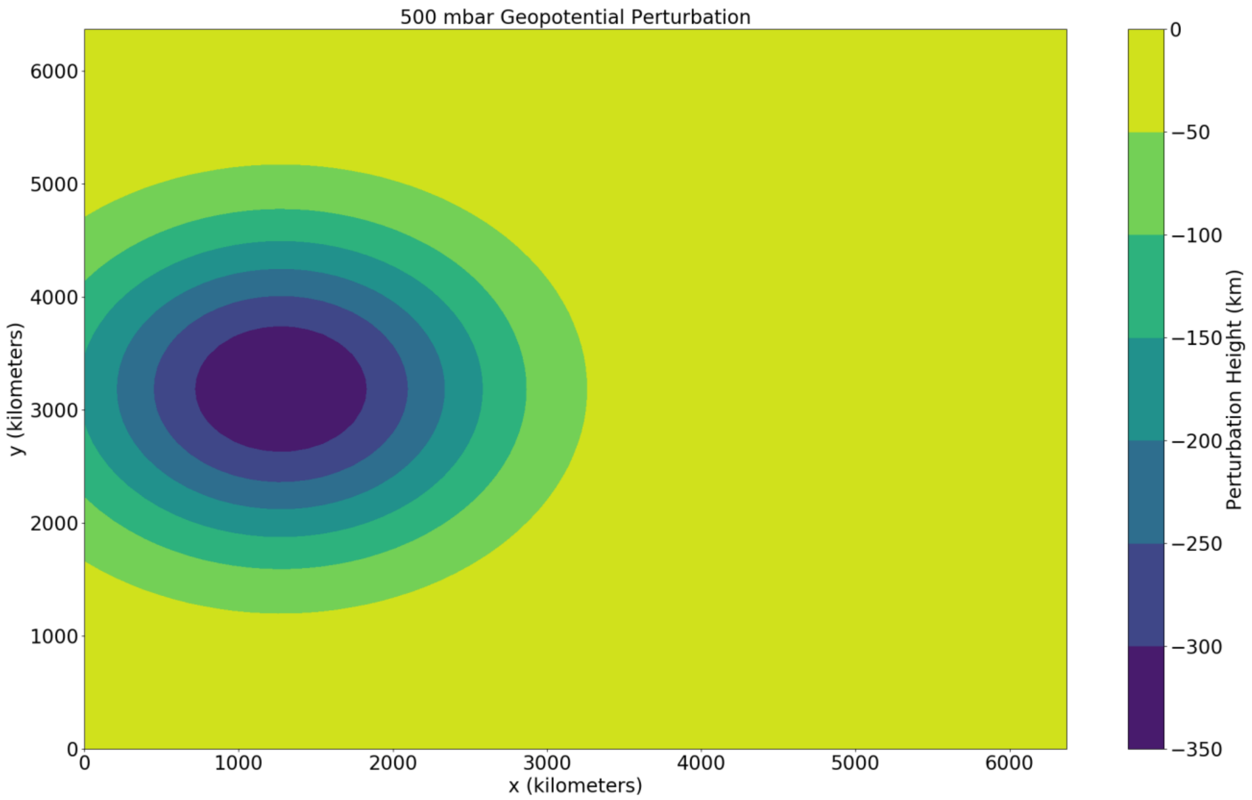
$$\mathrm{J}(w, \nabla^2 w) = \frac{\partial w}{\partial x}\frac{\partial(\nabla^2 w)}{\partial y} - \frac{\partial w}{\partial y}\frac{\partial(\nabla^2 w)}{\partial x}$$

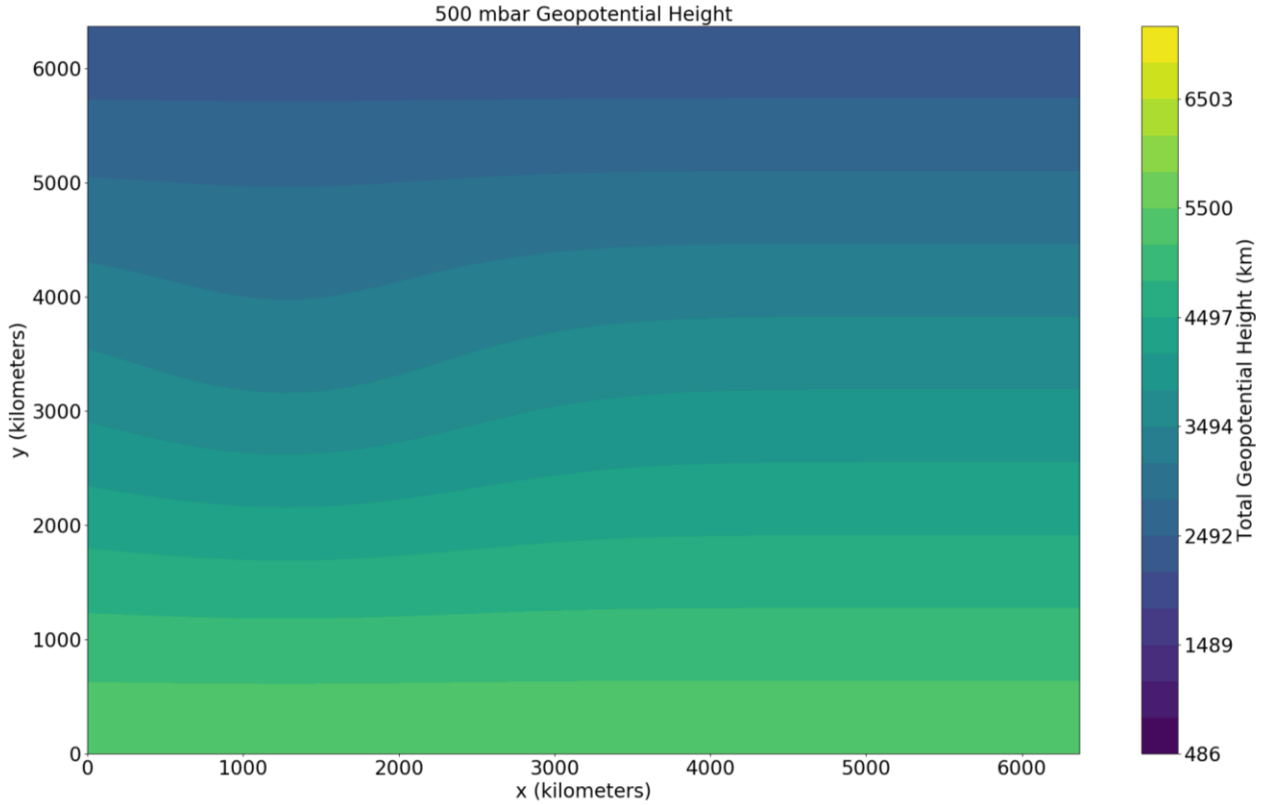
- Define grid
- Generate initial geopotential height perturbation
- Take derivatives using finite-difference method, appropriate to boundary conditions
- Leap-frog forward right side of equation to find left side of equation at new time step
- Use spectral method on left side of equation to find ψ
- Add average horizontal flow
- Repeat

Defining Grid

- 6000 x 6000 km grid cartesian grid
- Spatial resolution 50-150 km depending on what one is doing
- Temporal Resolution of 37 seconds to 5 minutes depending on what one is doing
- Coriolis parameter kept constant at 45 degrees for simplicity

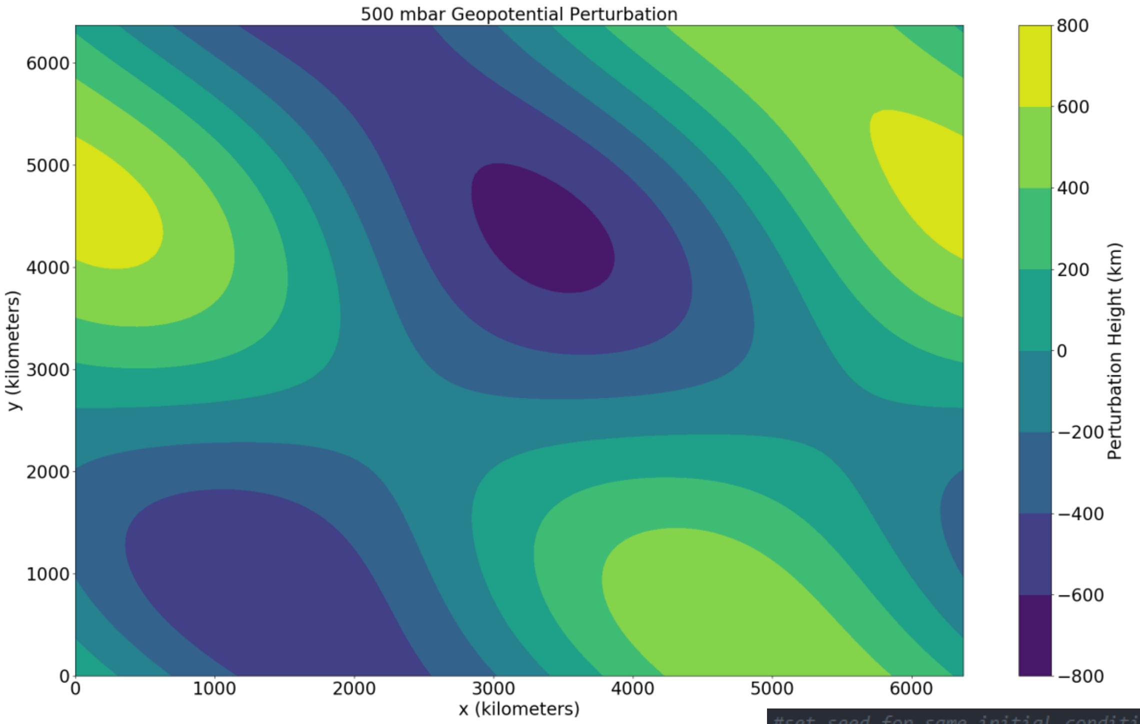
Generating Initial Conditions: Gaussian Perturbation

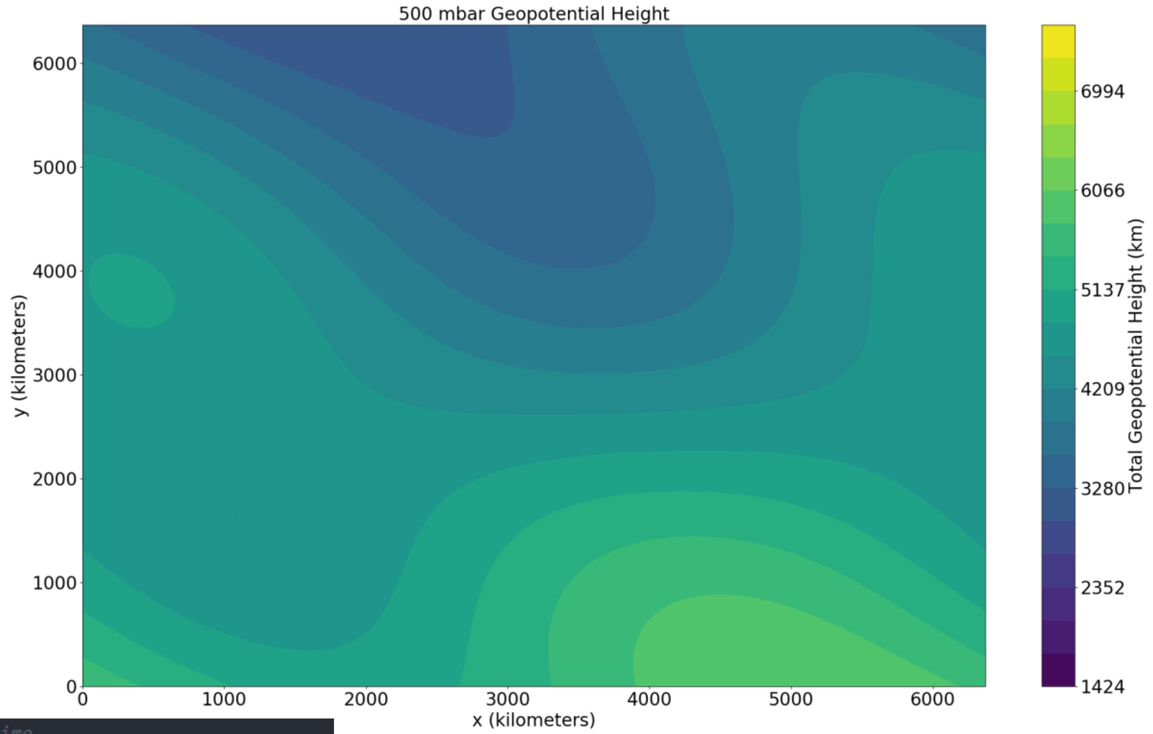





```
Z_0 = zeros((nx+1,ny+1));
term = -350*exp(((-(XX/xlen-.2)**2-(YY/ylen-.5)**2))/.05)
Z_0 = Z_0 + term
Z_0[0:nx+1,ny]=Z_0[0:nx+1,ny-1]
Z_0[0:nx+1,0]=Z_0[0:nx+1,1]
```

Generating Initial Conditions: 2-D waves






```
#set seed for same initial conditions every time
seed(a=2)
#randomize seed for different initial conditions
#seed()
Z_0 = zeros((nx+1,ny+1));

# Specify the number of waves in x and y

Nwavex = 1
Nwavey = 1
Nwaves = (2*Nwavex+1)*(2*Nwavey+1)


for kwave in range(-Nwavex,Nwavex+1):
    for lwave in range(-Nwavey,Nwavey+1):
         Amplitude = (2000.0*2*(random()-0.5)) / Nwaves
         phase= 2*pi*(random()-0.5)
         term = cos(2*pi*(kwave*(XX/xlen)+lwave*(YY/ylen)+phase))
         Z_0 = Z_0 + Amplitude*term
```

Boundary Appropriate Finite Differencing

- We have an N x N grid, when we take a derivative for the bulk of the matrix we use a central difference method (Order h^2)
- At edges we have to be careful and compute derivatives in accordance with our boundary conditions.

	f(x-h)	f(x)	f(x+h)
f(0)	f(h)		f(N*h)

- For a step size h, central difference:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h},\ f''(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

- At periodic boundary $f'(0) = \frac{f(h) - f(N*h)}{2h}, f''(0) = \frac{f(h) + f(N*h) - 2f(0)}{h^2}$

Boundary Appropriate Finite Differencing continued…

- In this method one can use periodic or symmetric boundary conditions in y
- For symmetric boundary conditions y, the derivative must be redefined using backward and forward differencing

- For a step size h, forward difference:

$$f'(y) = \frac{f(y+h) - f(y)}{h},\ f''(y) = \frac{f(y+2h) + f(y) - 2f(y+h)}{h^2}$$

- At symmetric boundary $f'(0) = \frac{f(h) - f(0)}{h}, f''(0) = \frac{f(0) + f(2h) - 2f(h)}{h^2}$

$f(0)$		
$f(0)$		
$f(h)$		
$f(2h)$		$f(N-3h)$
		$f(N-2h)$
		$f(N-h)$
		$f(N-h)$

Leap-frog forward

$\frac{d}{dt}(\nabla^2 - F)w$=$-\frac{g}{f}J(w, \nabla^2 w)$-β$\frac{\partial}{\partial x}w$

R=$-\frac{g}{f}$J$(w, \nabla^2 w)$-β$\frac{\partial}{\partial x}w$

- $(\nabla^2 - F)w$=Q ➔ $\frac{d}{dt}$Q=$\frac{\text{Q(n+1)-Q(n)}}{\Delta t}$ ➔ Q(n+1)=Q(n)+RΔt
- For first time step use Δt/2

Spectral Method to find ψ

Q(n+1)= $(\nabla^2 - F)w$(n+1) ➔ Fourier Transform ➔ Q(n+1)= $(-k^2 - F)w$(n+1) ➔

➔ $\frac{Q(n+1)}{(-k^2-F)} = w(n+1)$ ➔ Inverse Fourier Transform ➔ w(n+1)

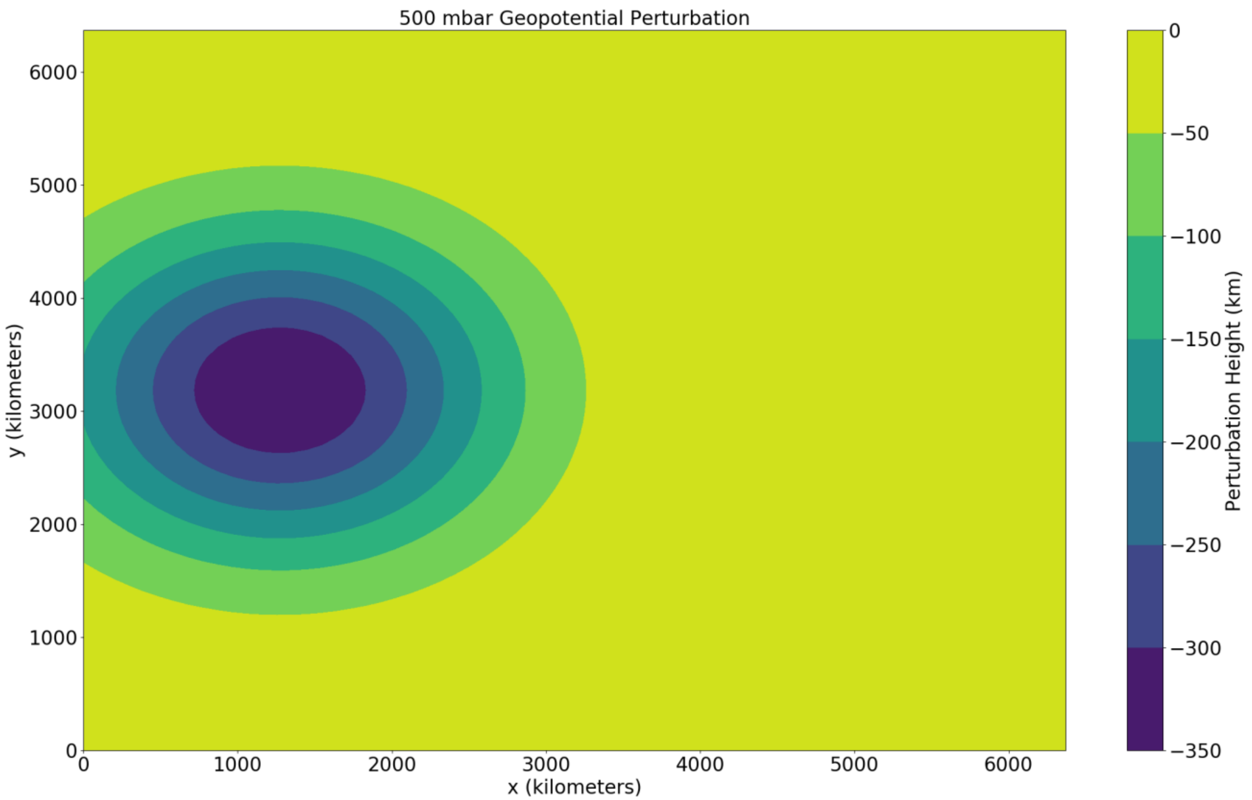
```
#Section 3.3: Solve the Helmholtz Equation (Del^2-F)w = R.Compute the fft of the
#right hand side (strip off additional row and column).
    R[0:nx,0:ny] = Q_np1[0:nx,0:ny]
    R_hat = fft2(R)

#Compute the transform of the solution
    W_hat = R_hat/C_rs
#Compute the inverse transform to get the solution at (n+1)*Delta_t.
    w_new = real(ifft2(W_hat)) # We assume w is real
    w[0:nx,0:ny] = w_new
    w[nx,0:ny] = w[0,0:ny]      # Fill in additional column at east.
    w[0:nx+1,ny]=w[0:nx+1,0]    # Fill in additional row at north.
```

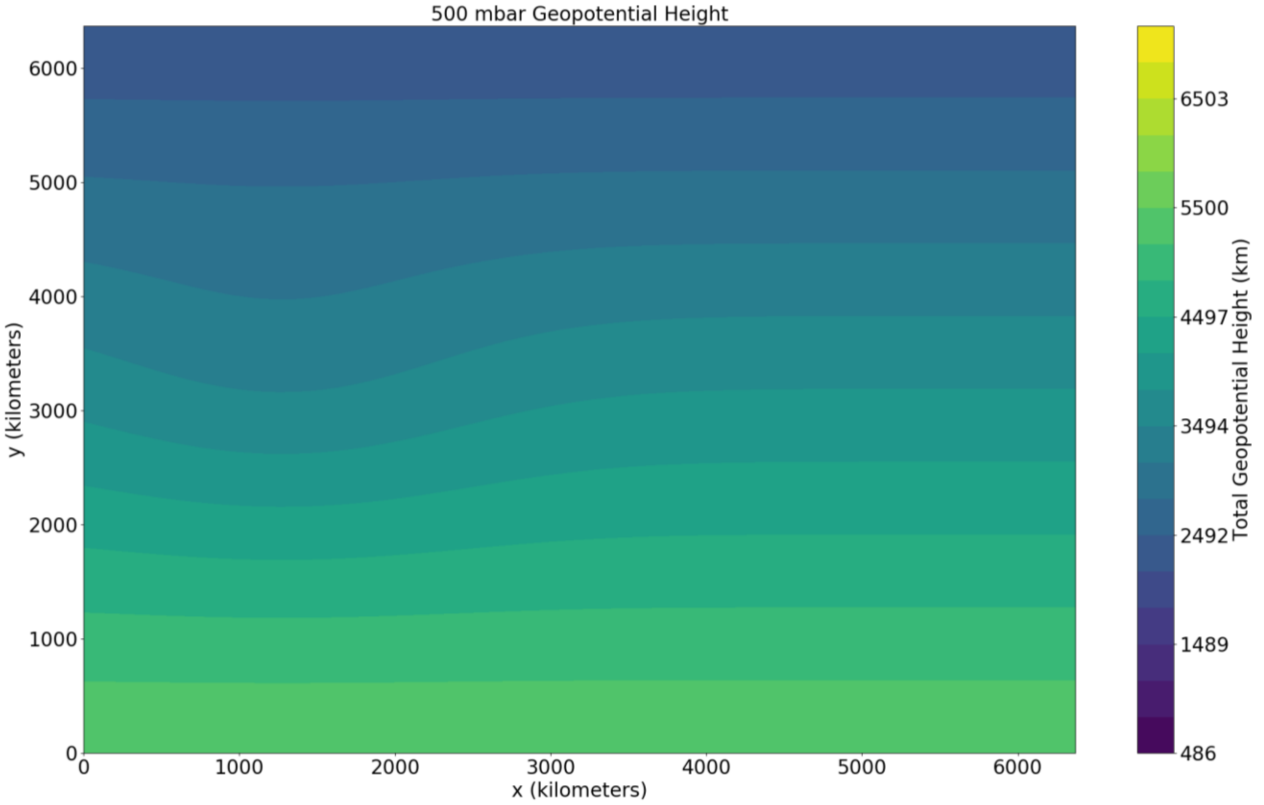
Comparisons and spatial and temporal resolution

$$C = u\frac{\Delta t}{\Delta x}$$

500 mb pressure anomoly



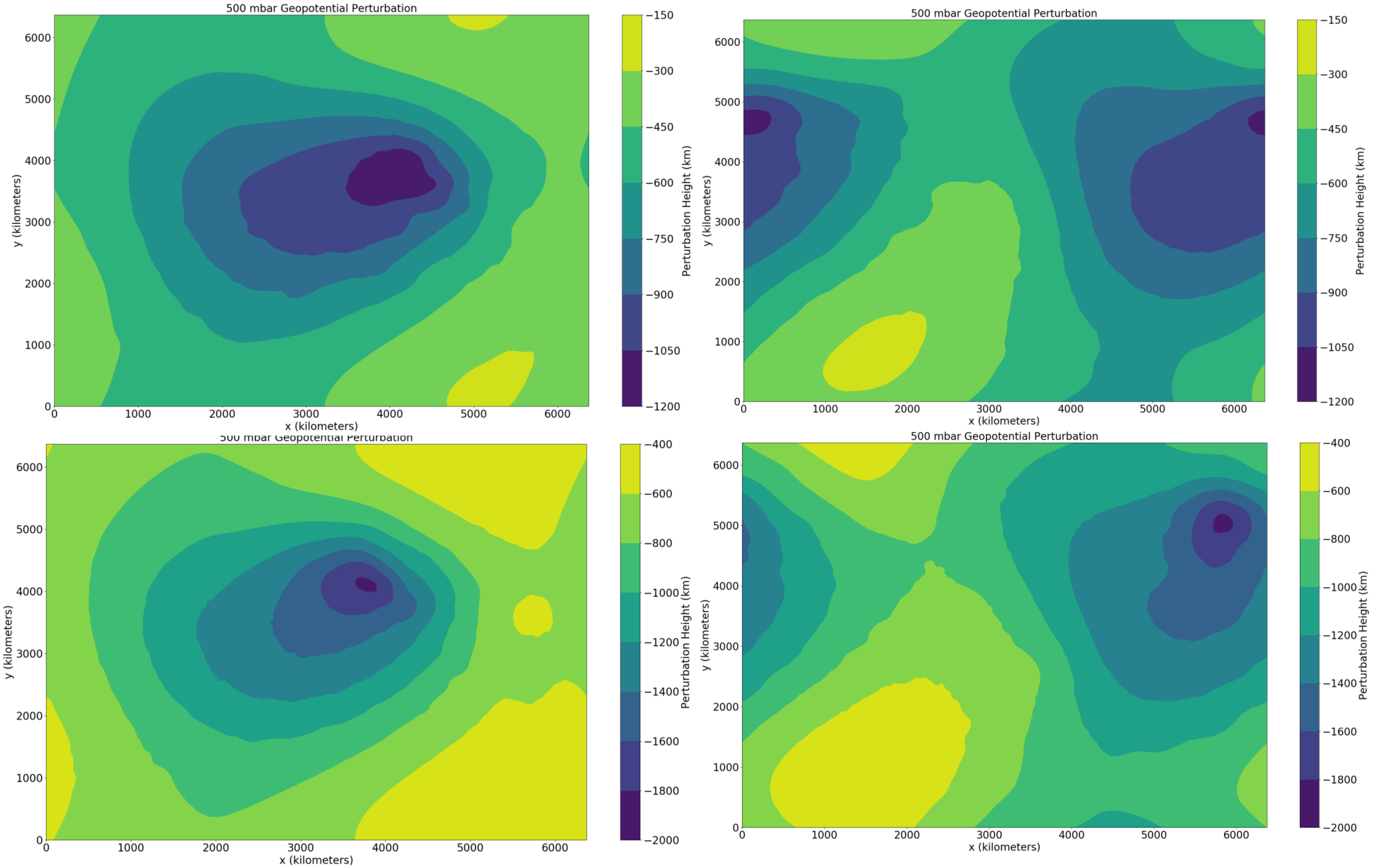
Initial geopotential height field



96.53 km Resolution, 66x66 grid

63.71 km Resolution, 100x100 grid

100x100 grid, 50 m/s Mean zonal wind, Symmetric B.C.

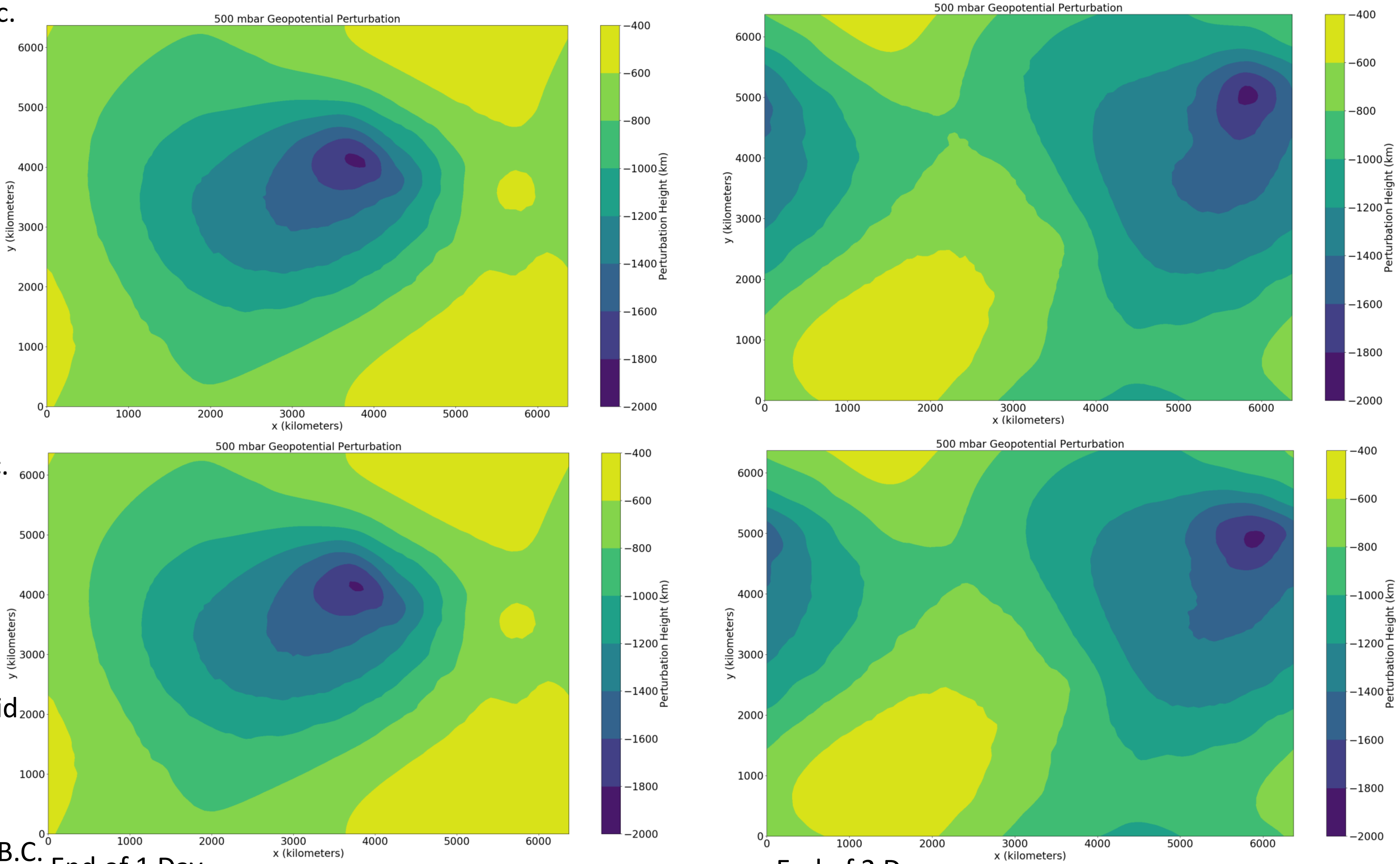


End of 1 Day

End of 2 Days

Δt=100 sec.

Δt=37.5 sec.



100x100 grid
50 m/s
Mean zonal wind,
Symmetric B.C.

End of 1 Day

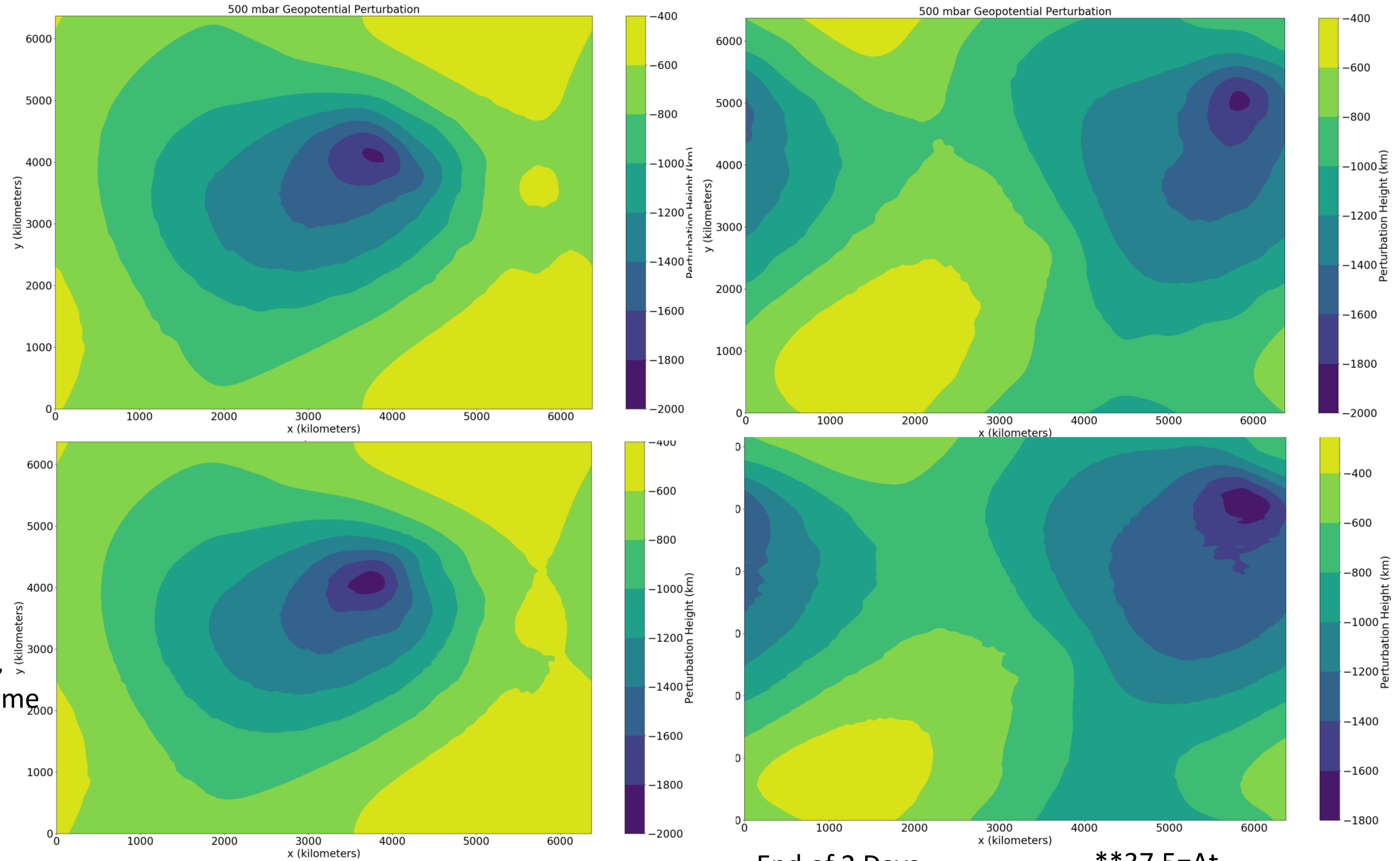
End of 2 Days

Periodic vs Symmetric B.C.’s

Symmetric B.C.

Periodic B.C.

100x100 grid, 100 second time step, 50 m/s Mean zonal wind



End of 1 Day

End of 2 Days

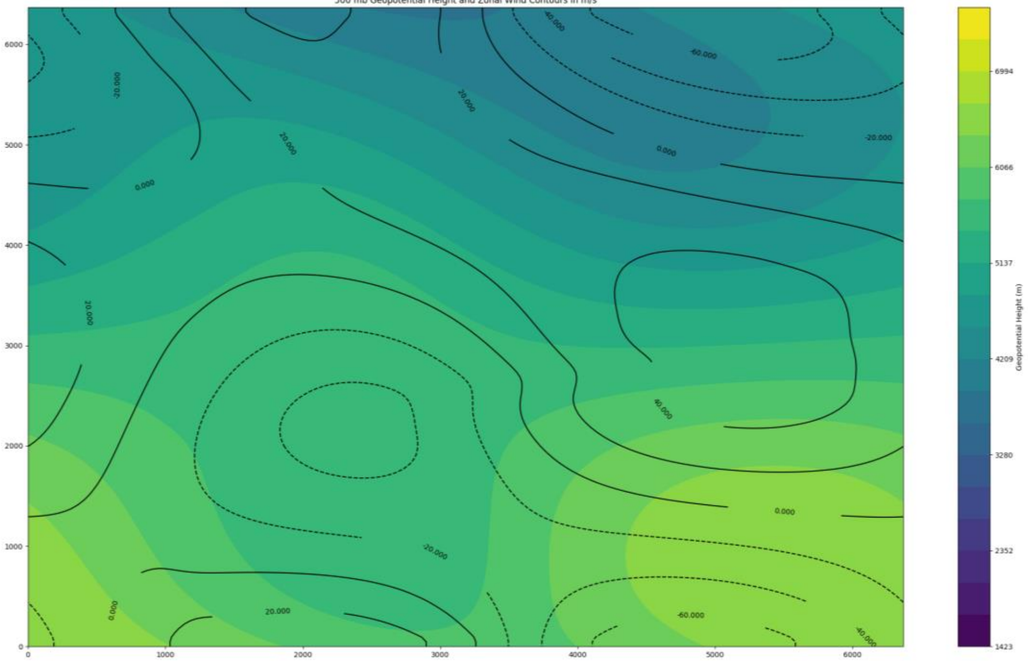
**37.5=Δt

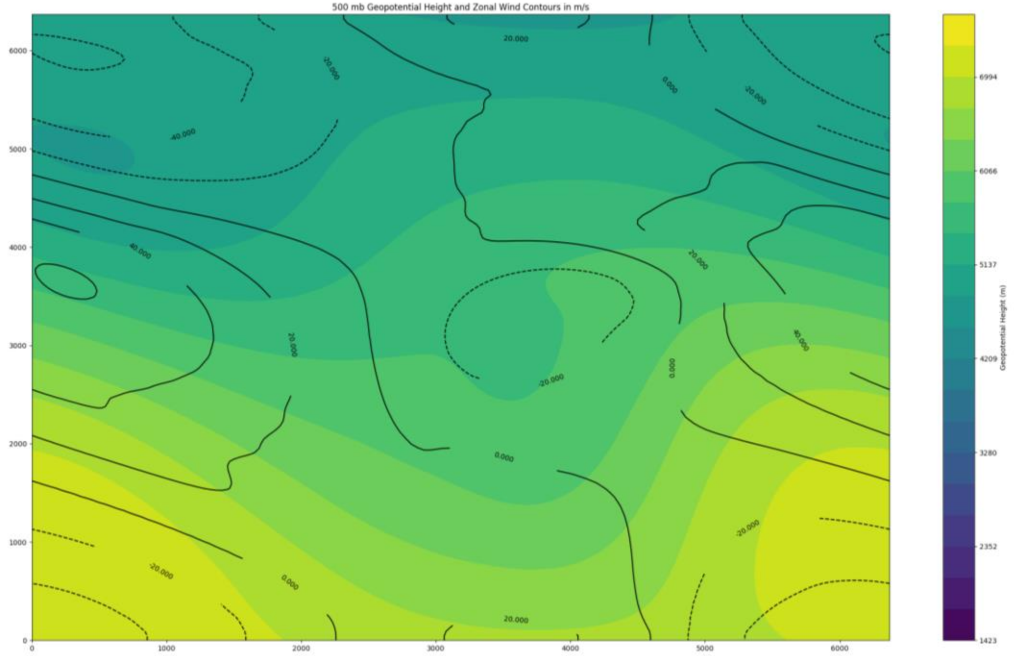
How does the zonal wind average change as the background wind state is modulated?

Wave Initial Condition - Symmetric B.C., 1/96th time step, 140x140 grid

Gaussian Initial Condition – Symmetric B.C, 1/12th time step, 66x66 grid

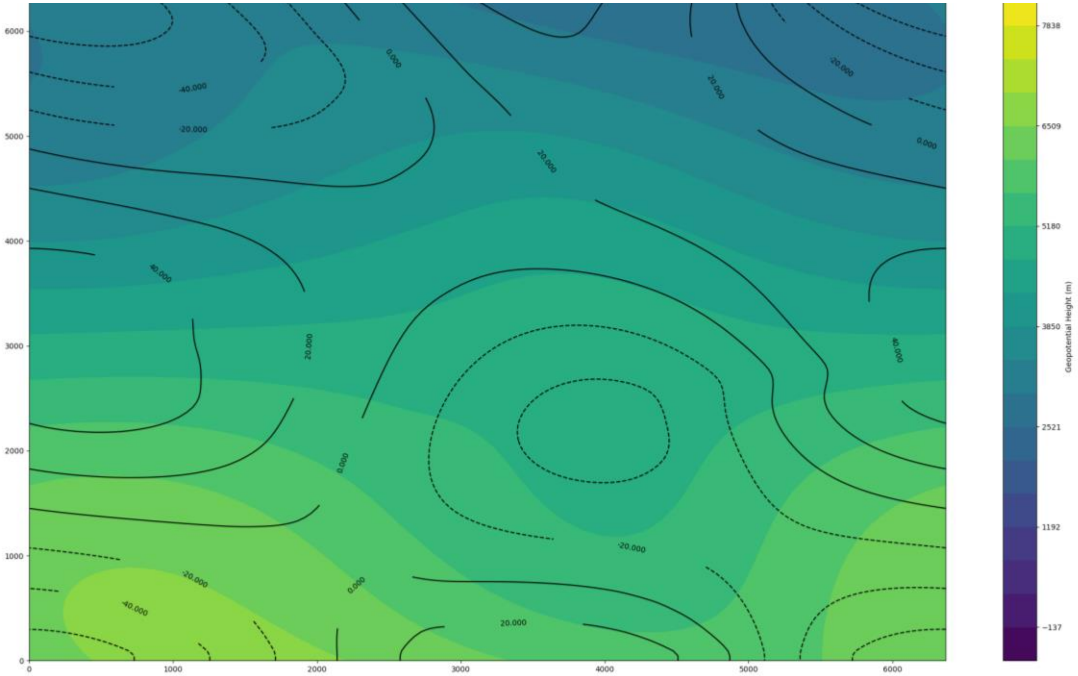
30 m/s

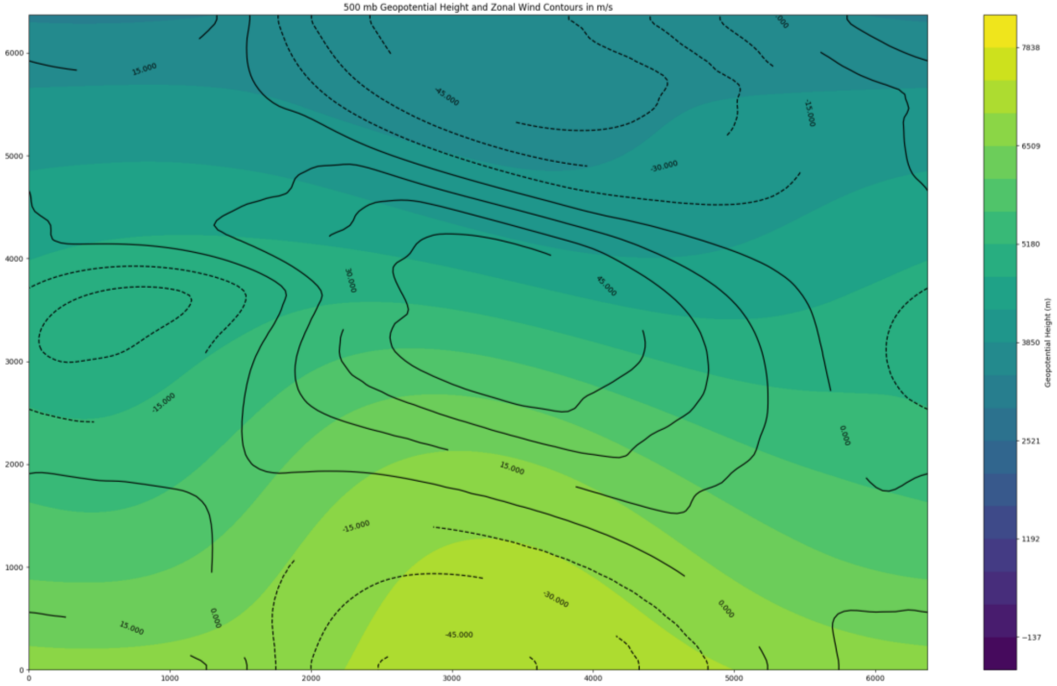




50 m/s

100x100 grid,
50 m/s
Mean zonal
wind

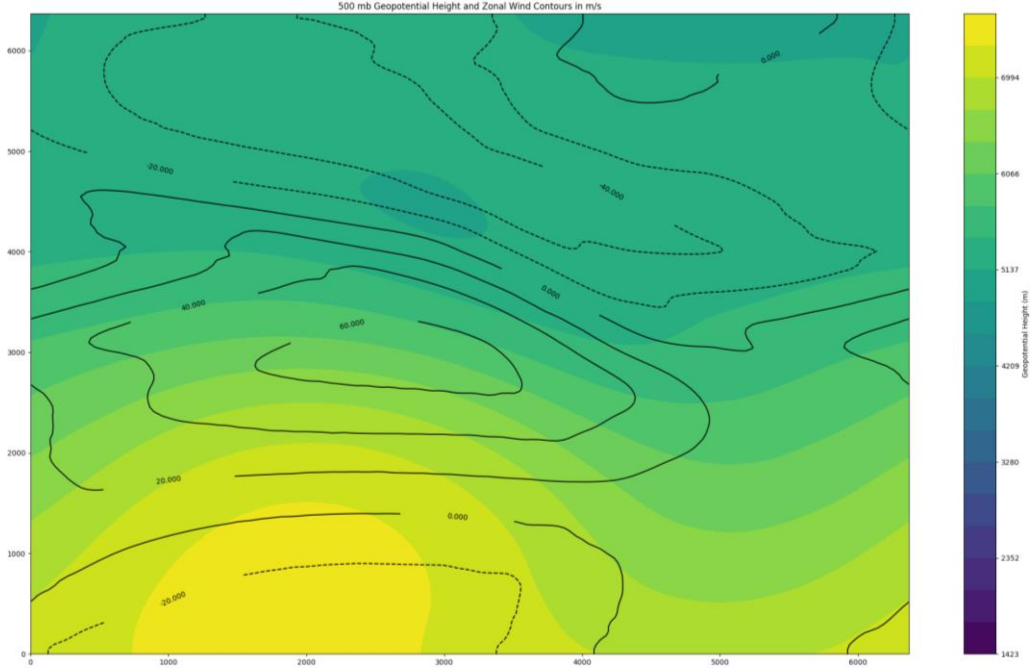


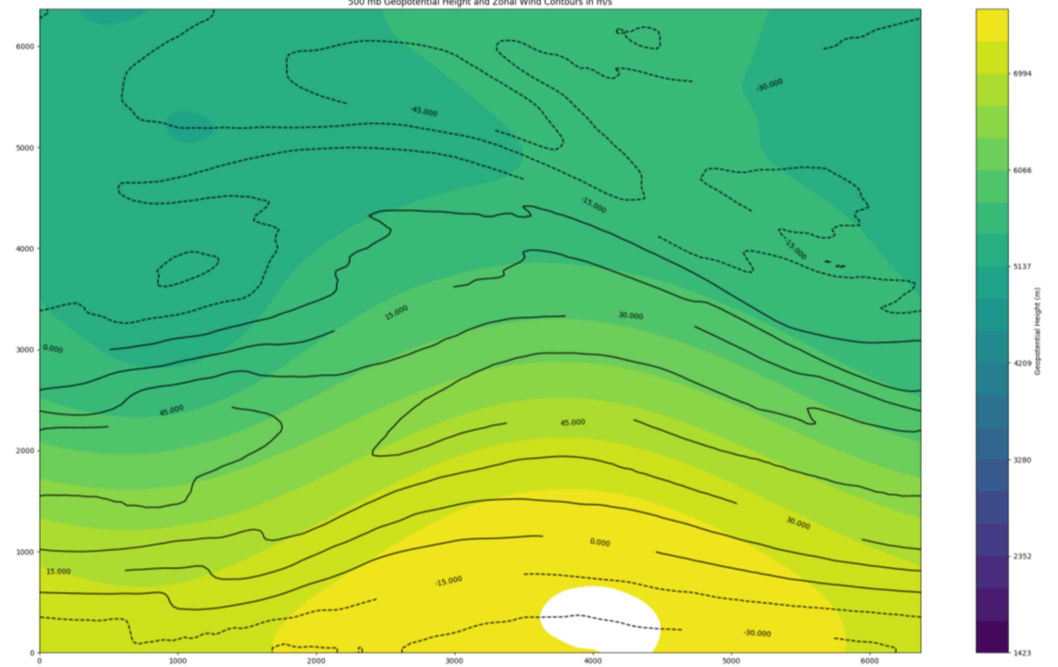


End of 1 Day

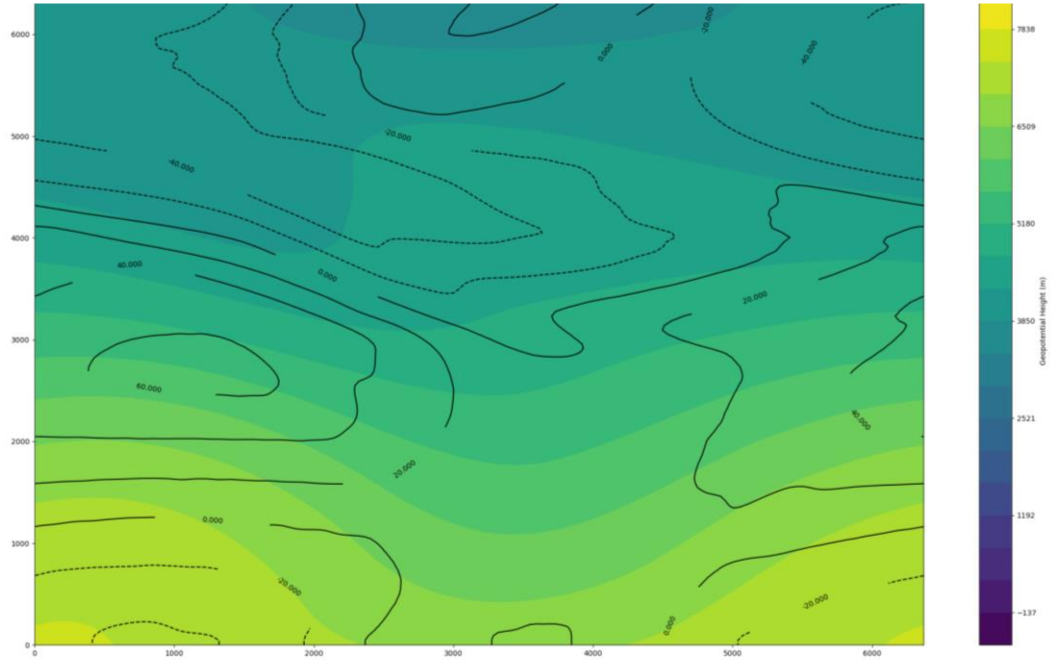
End of 2 Days

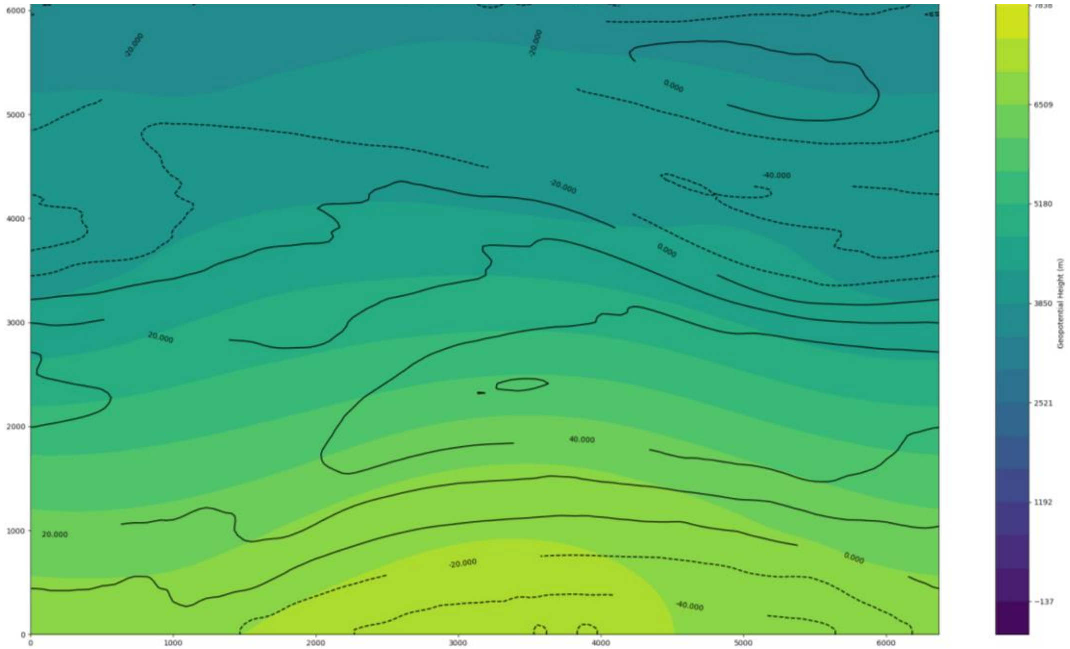
30 m/s





50 m/s





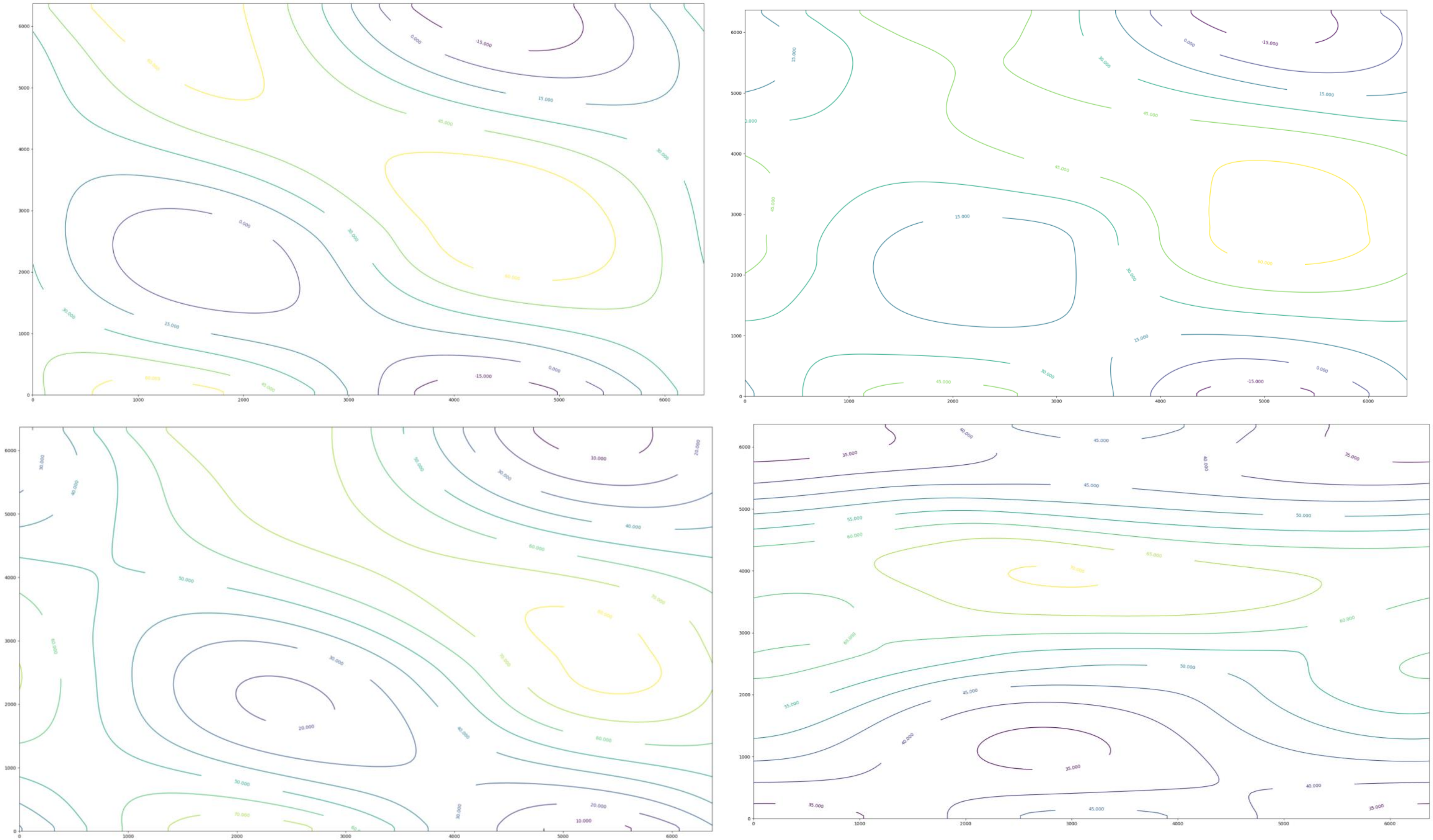
100x100 grid,
50 m/s
Mean zonal
wind

End of 3 Days

End of 4 Days

30 m/s

50 m/s



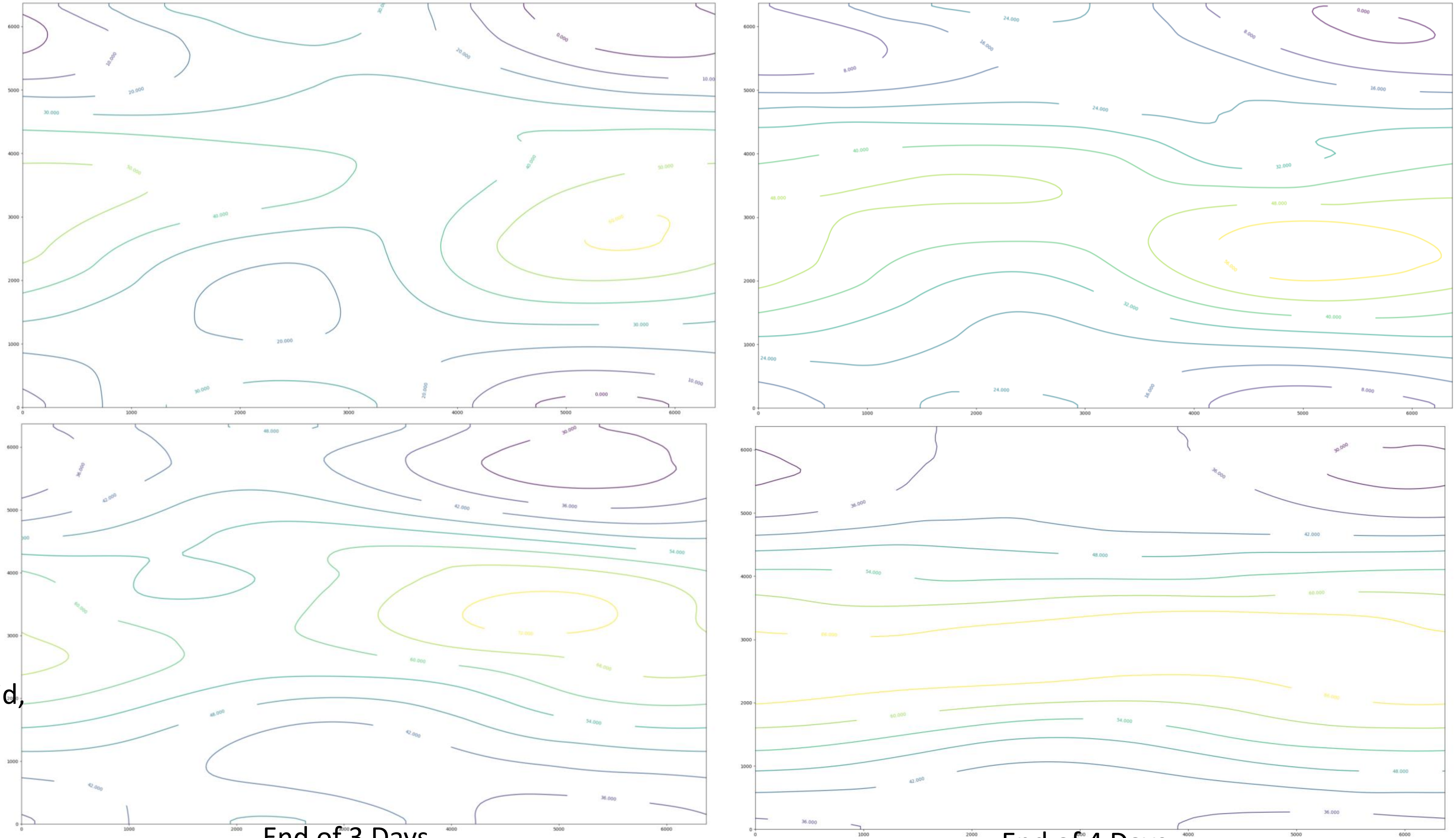
100x100 grid,
50 m/s
Mean zonal
wind

End of 1 Day

End of 2 Days

30 m/s

50 m/s

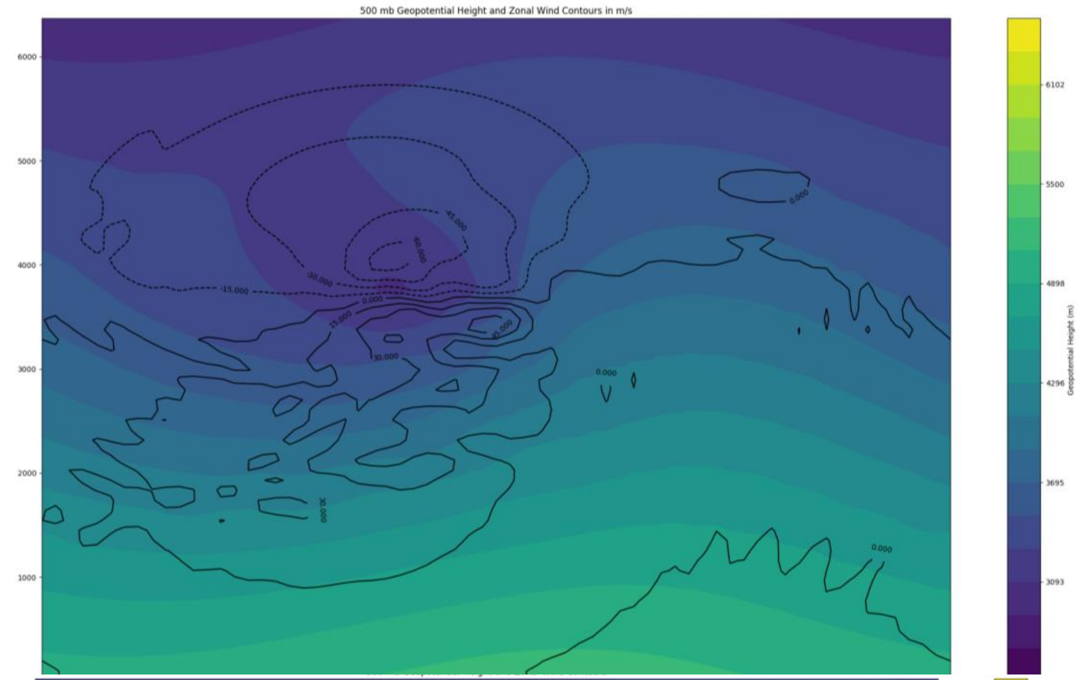


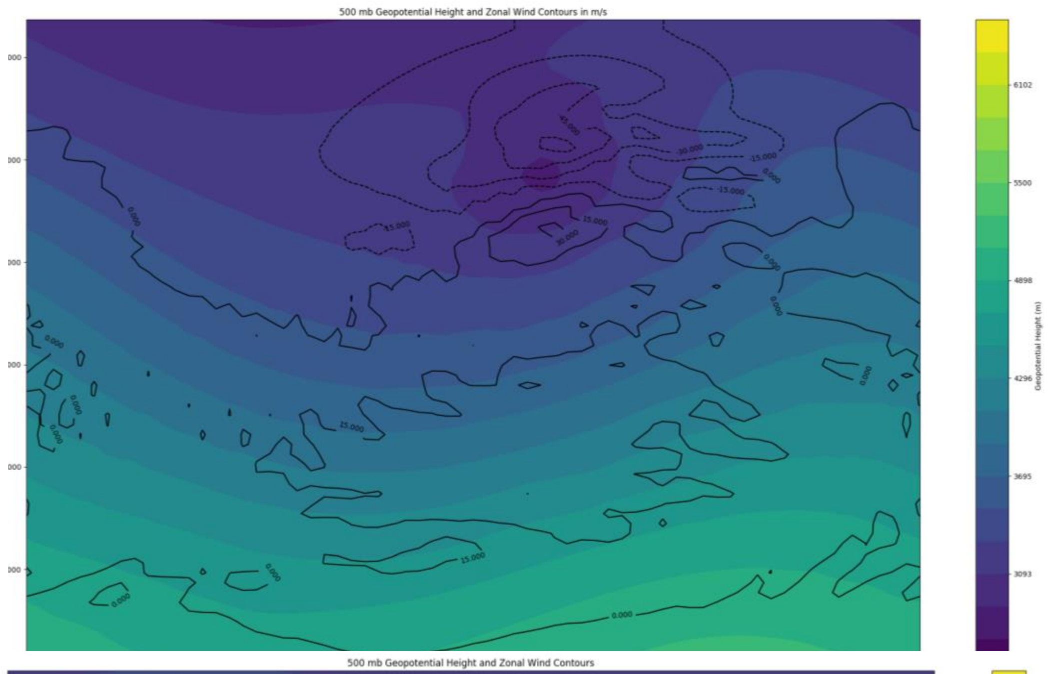
100x100 grid, 50 m/s Mean zonal wind

End of 3 Days

End of 4 Days

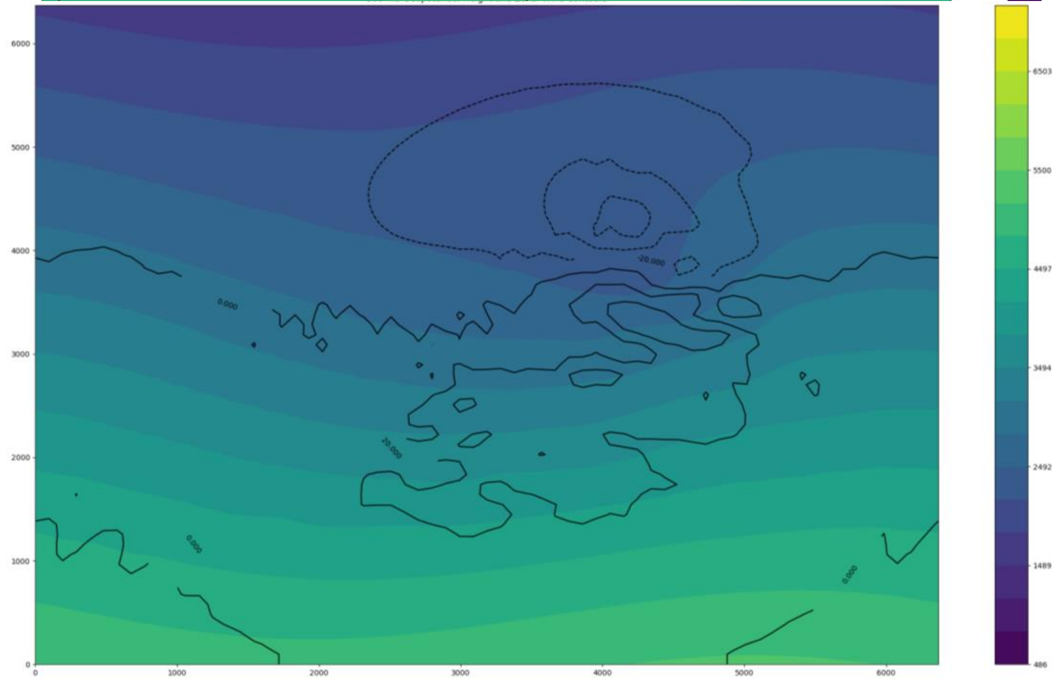
30 m/s

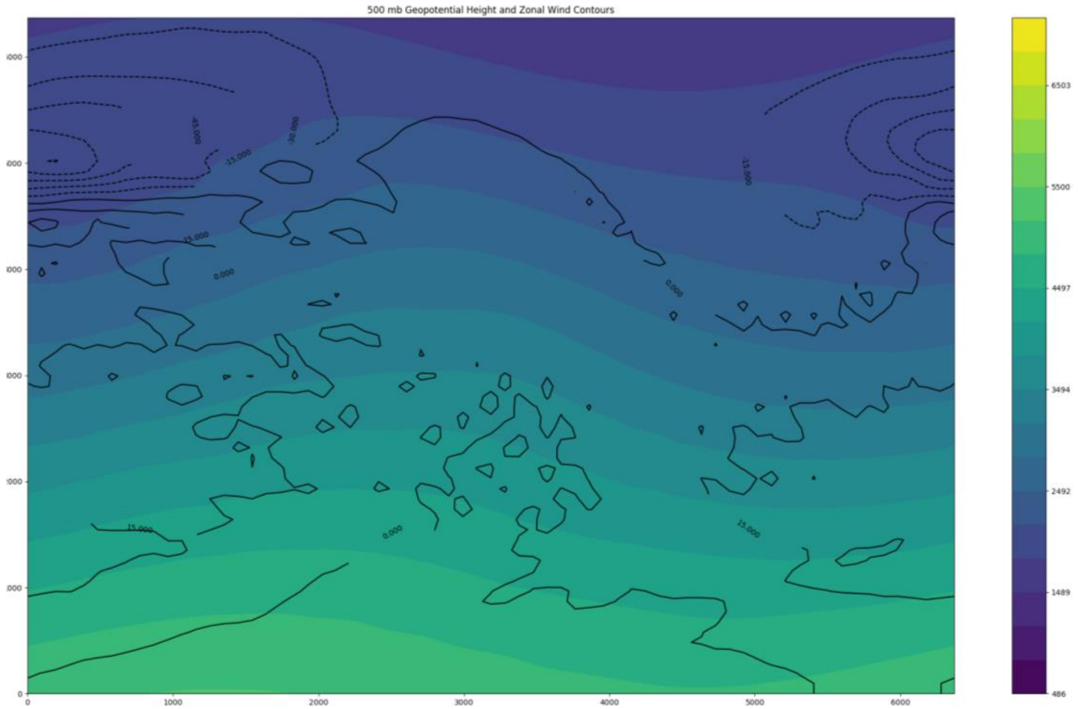




50 m/s

100x100 grid,
50 m/s
Mean zonal
wind

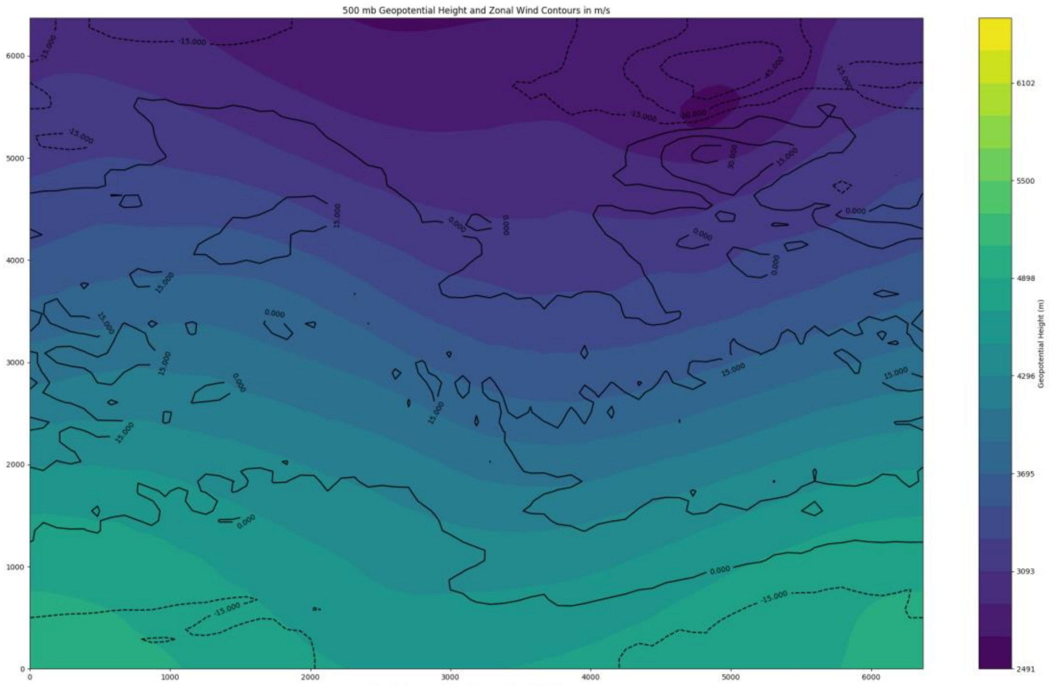


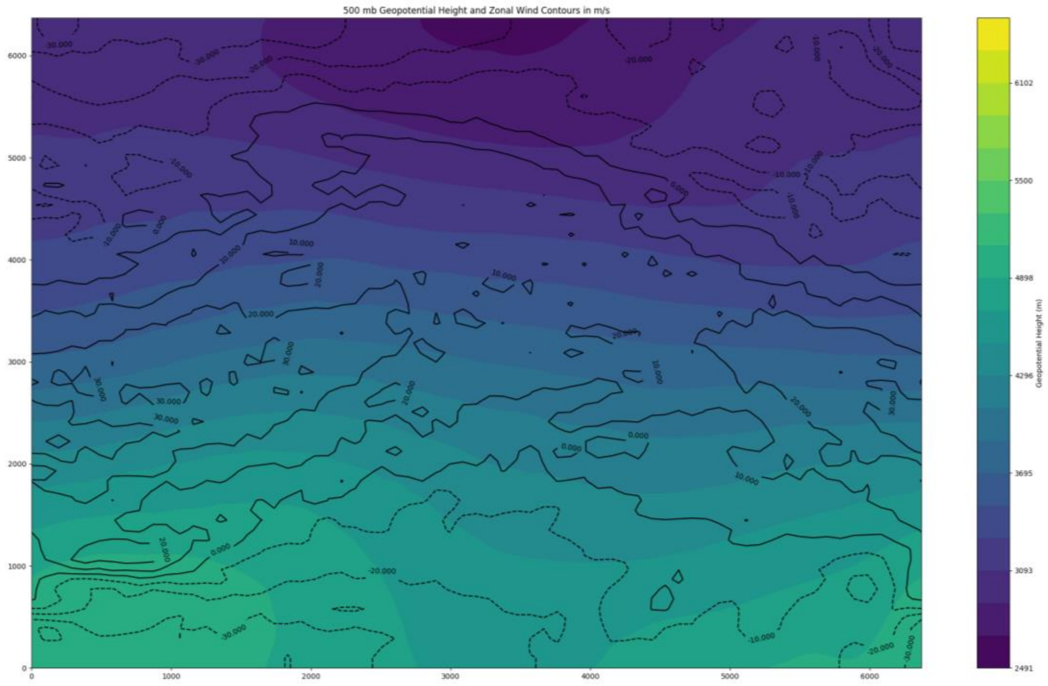


End of 1 Day

End of 2 Days

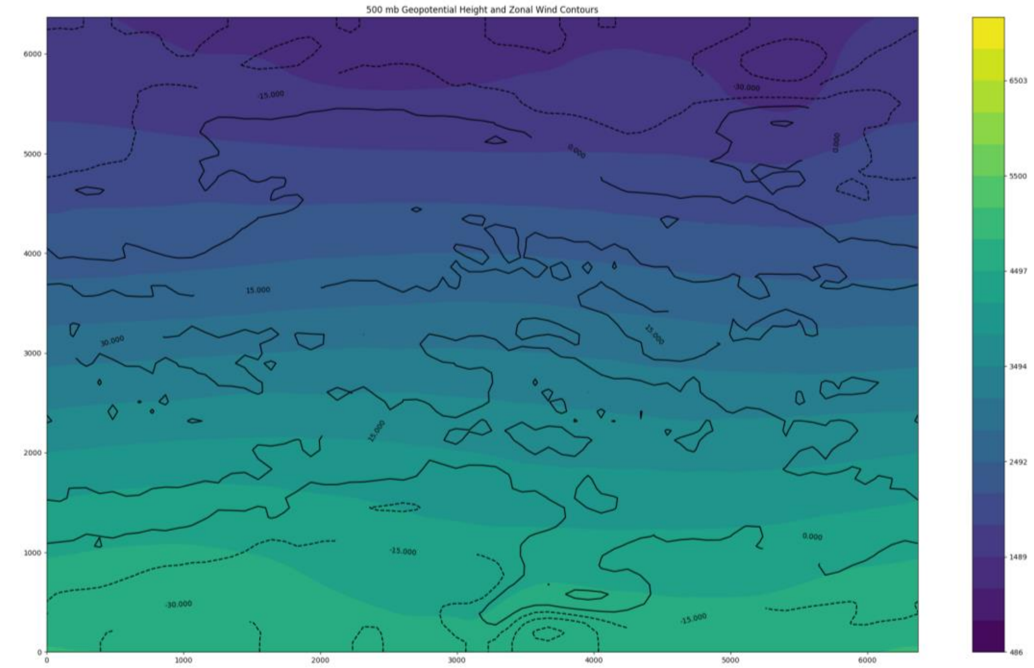
30 m/s

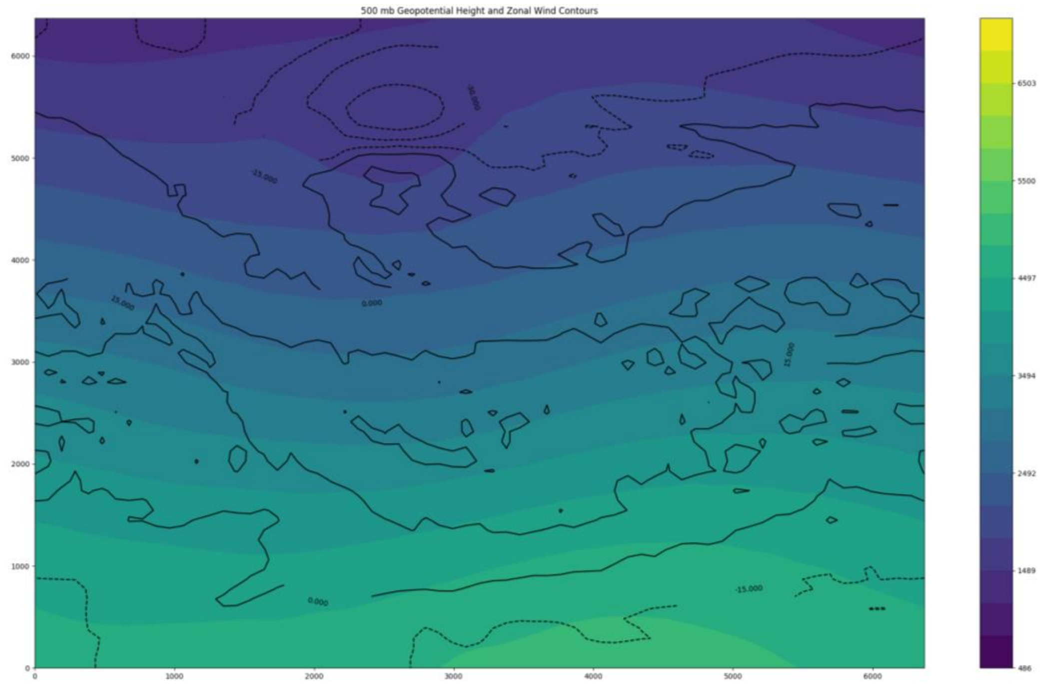




50 m/s

100x100 grid,
50 m/s
Mean zonal
wind



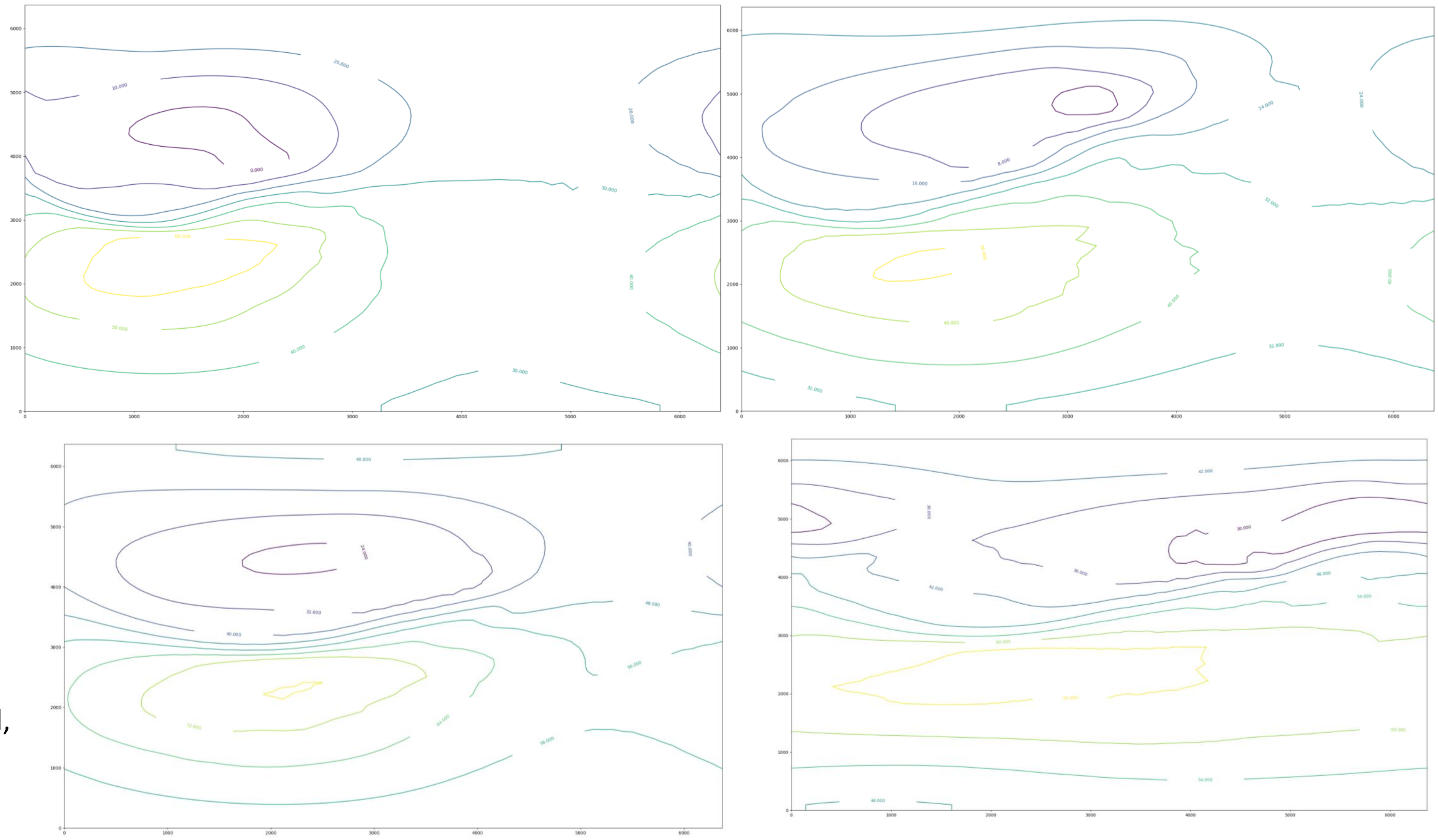


End of 3 Days

End of 4 Days

30 m/s

50 m/s

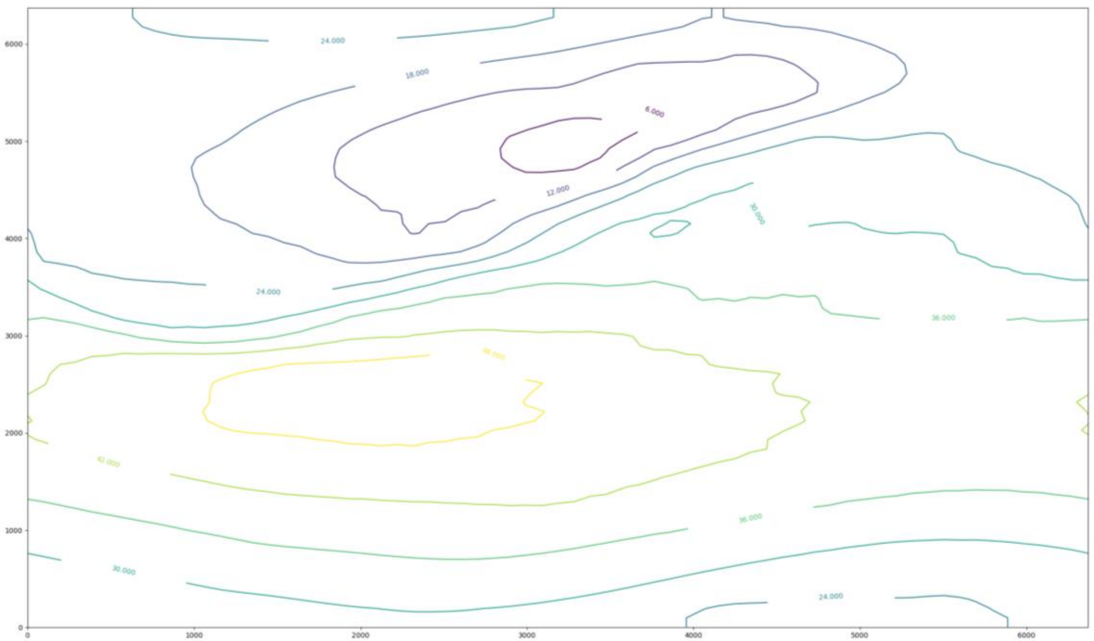


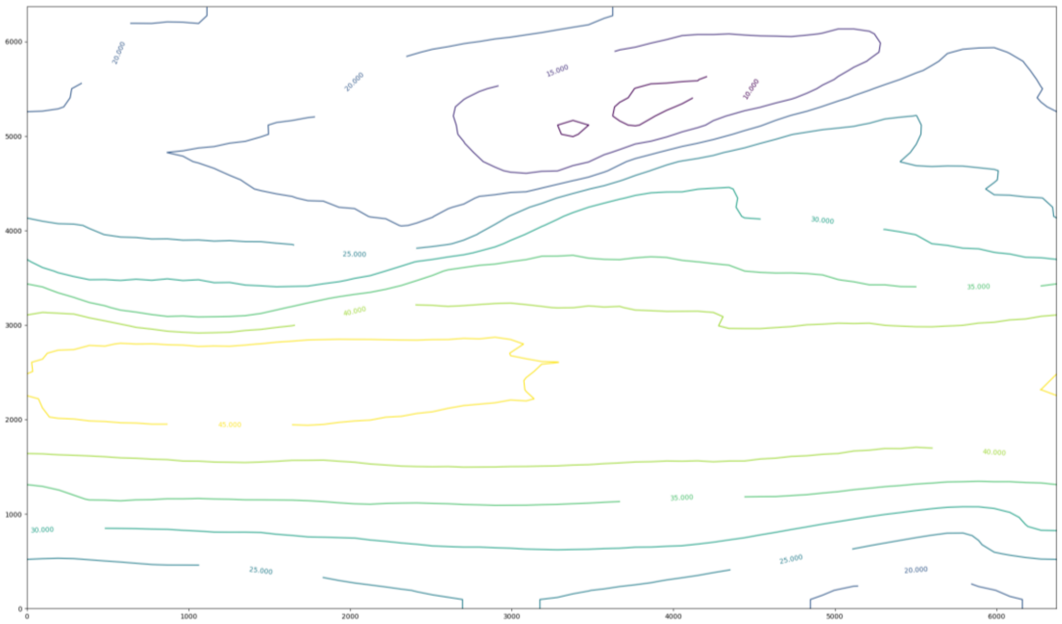
100x100 grid,
50 m/s
Mean zonal
wind

End of 1 Day

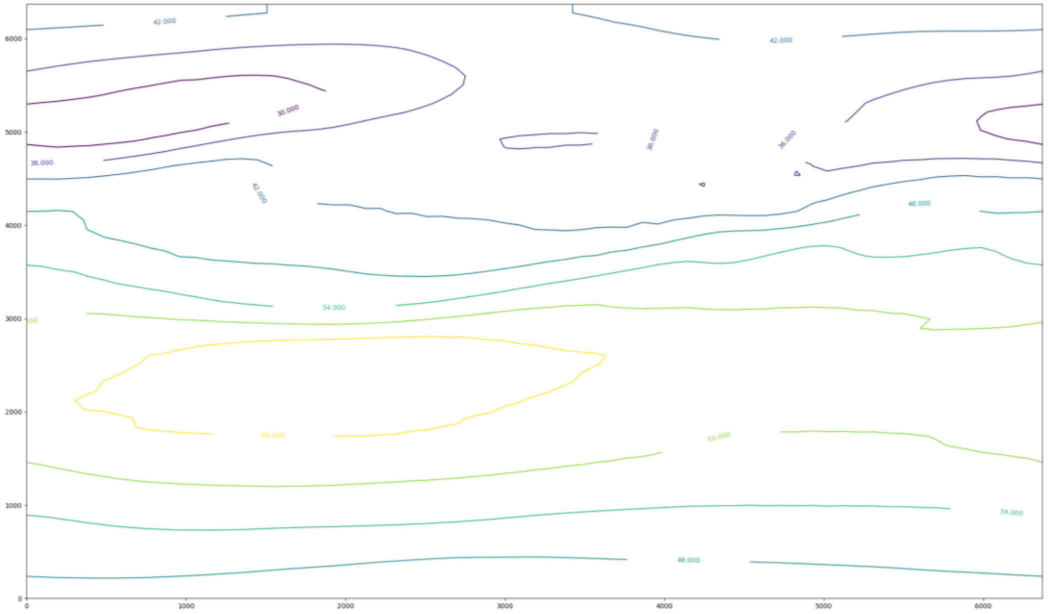
End of 2 Days

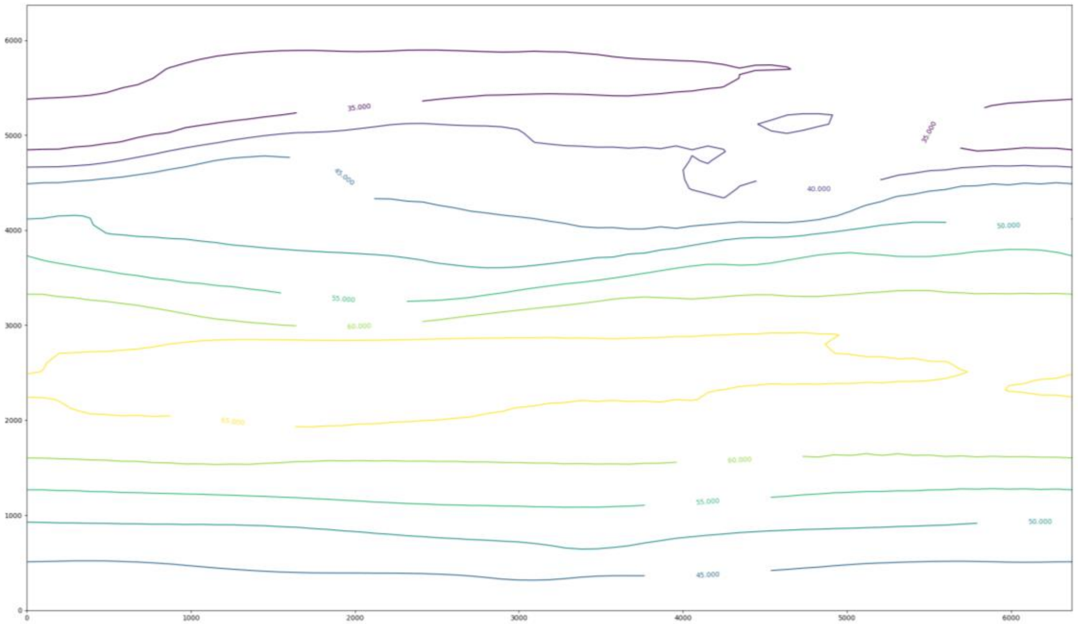
30 m/s





50 m/s





100x100 grid
50 m/s
Mean zonal
wind

End of 3 Day

End of 4 Days