2.1

T_{AB}	T_{BC}	C_{AB}	C_{BC}	T_{AC}	C_{AC}
6	2	1200	300	8	1500
	3		300	9	1500
7	2	1200	300	9	1500
	3		300	10	1500
9	2	550	300	11	850
	3		300	12	850
10	2	550	300	12	850
	3		300	13	850
11	2	550	300	13	850
	3		300	14	850

(a) Sample space of travel time from A to B = {6, 7, 9, 10, 11}
Sample space of travel time from A to C = {8, 9, 10, 11, 12, 13, 14}

(b) Sample space of travel cost from A to C = {850, 1500}

(c) Sample space of T_{AC} and C_{AC}
= {(8, 1500), (9, 1500), (10, 1500), (11, 850), (12, 850), (13, 850), (14, 850)}

2.2

(a) Since the possible values of settlement for Pier 1 overlap partially with those of Pier 2, it is possible that both Piers will have the same settlement. Hence, the minimum differential settlement is zero.

The maximum differential settlement will happen when the settlement of Pier 2 is 10 cm and that of Pier 1 is 2 cm, which yields a differential settlement of 8 cm.

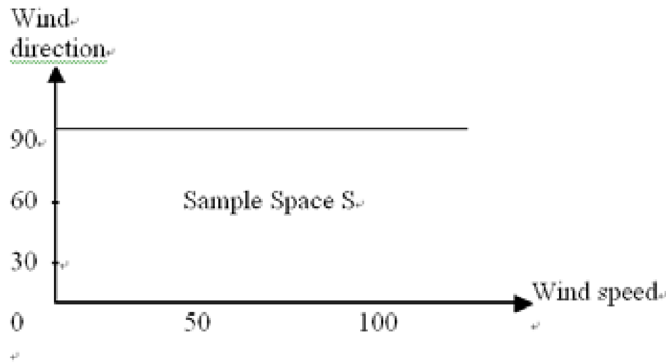
Hence, the sample space of the differential settlement is zero to 8 cm.

(b) If the differential settlement is assumed to be equally likely between 0 and 8 cm, the probability that it will be between 3 and 5 cm is equal to

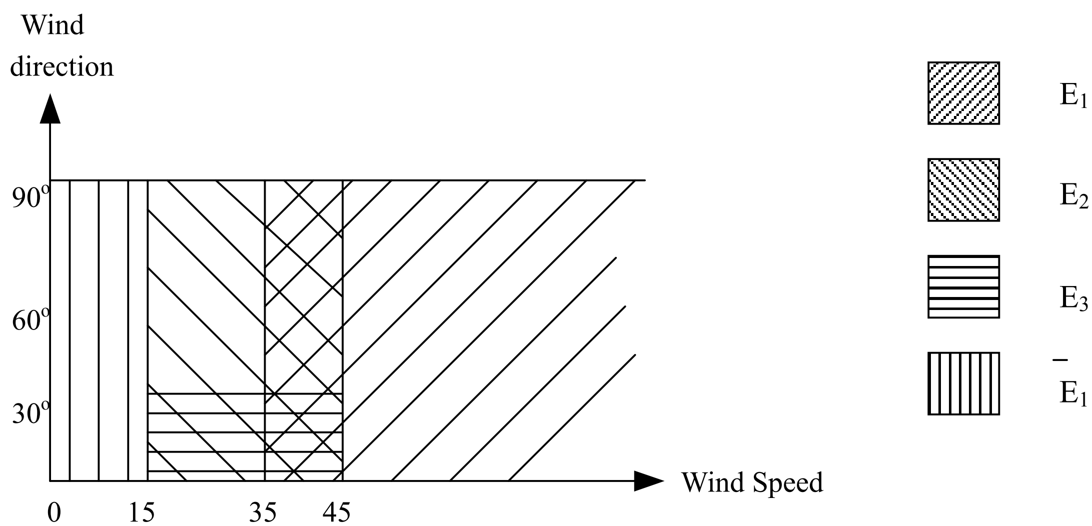
$$P = \frac{5-3}{8-0} = \frac{2}{8} = 0.25$$

2.3

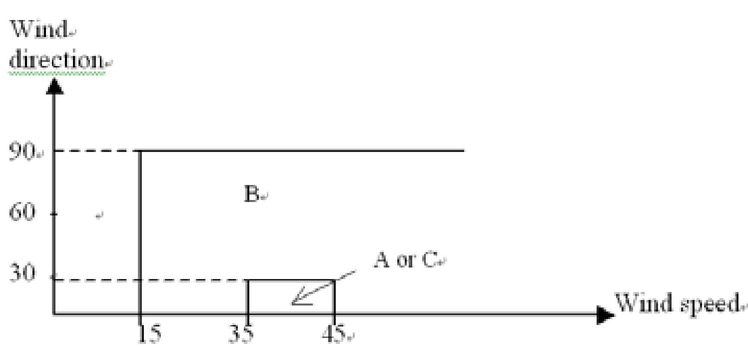
(a)



(b)



(c)



A and B are not mutually exclusive
A and C are not mutually exclusive

2.4

(a)

Inflow	Outflow	Possible water level Inflow – Outflow + 7’
6’	5’	8’
6’	6’	7’
6’	7’	6’
7’	5’	9’
7’	6’	8’
7’	7’	7’
8’	5’	10’
8’	6’	9’
8’	7’	8’

Hence possible combinations of inflow and outflow are
(6’,5’), (6’,6’), (6’,7’), (7’,5’), (7’,6’), (7’,7’), (8’,5’), (8’,6’) and (8’,7’).

(b) The possible water levels in the tank are 6’, 7’, 8’, 9’ and 10’.

(c) Let E = at least 9 ft of water remains in the tank at the end of day. Sample points (7’, 5’), (8’, 5’) and (8’, 6’) are favourable to the event E.

So $P(E) = \frac{3}{9} = \frac{1}{3}$.

2.5

(a)

Locations of W_1	Locations of W_2	Load at B	Load at C	M_A	Probability	E_1	E_2	E_3
–	–	0	0	0	0.15x0.2=0.03			
–	B	500	0	5,000	0.045		X	
–	C	0	500	10,000	0.075	X	X	
B	–	200	0	2,000	0.05		X	X
B	B	700	0	7,000	0.075	X	X	X
B	C	200	500	12,000	0.125	X		
C	–	0	200	4,000	0.12		X	
C	B	500	200	9,000	0.18	X	X	
C	C	0	700	14,000	0.30	X		

(b) E_1 and E_2 are not mutually exclusive because these two events can occur together, for example, when the weight W_2 is applied at C, M_A is 10,000 ft-lb; hence both E_1 and E_2 will occur.

(c) The probability of each possible value of M_A is tabulated in the last column of the table above.

(d) $P(E_1) = P(M_A > 5{,}000) = 0.075 + 0.075 + 0.125 + 0.18 + 0.30 = 0.755$
$P(E_2) = P(1{,}000 \le M_A \le 12{,}000) = 0.045 + 0.075 + 0.05 + 0.075 + 0.12 + 0.18 = 0.545$
$P(E_3) = 0.05 + 0.075 = 0.125$
$P(E_1 \cap E_2) = P(5{,}000 < M_A \le 12{,}000) = 0.075 + 0.075 + 0.18 = 0.33$
$P(E_1 \cup E_2) = 1 - 0.03 = 0.97$

$P(\overline{E_2}) = 0.03 + 0.125 + 0.3 = 0.455$

P($P(\overline{E_2}) = 1 - P(E) = 1 - 0.545 = 0.455$

2.6

(a) Let A_1 = Lane 1 in Route A requires major surfacing
A_2 = Lane 2 in Route A requires major surfacing
B_1 = Lane 1 in Route B requires major surfacing
B_2 = Lane 2 in Route B requires major surfacing

$P(A_1) = P(A_2) = 0.05$ $\qquad$ $P(A_2 | A_1) = 0.15$
$P(B_1) = P(B_2) = 0.15$ $\qquad$ $P(B_2 | B_1) = 0.45$

P(Route A will require major surfacing) = P(A)
$= P(A_1 \cup A_2)$
$= P(A_1) + P(A_2) - P(A_2 | A_1) P(A_1)$
$= 0.05 + 0.05 - (0.15)(0.05)$
$= 0.0925$

P(Route B will require major surfacing) = P(B)
$= P(B_1 \cup B_2)$
$= P(B_1) + P(B_2) - P(B_2 | B_1) P(B_1)$
$= 0.15 + 0.15 - (0.45)(0.15)$
$= 0.2325$

(b) P(route between cities 1 and 3 will require major resurfacing)
$= P(A \cup B)$
$= P(A) + P(B) - P(A)P(B)$
$= 0.0925 + 0.2325 – (0.0925)(0.2325)$
$= 0.302$

2.7

$P(D_i) = 0.1$

Assume condition between welds are statistically independent

(a) $P(\overline{D_1}\ \overline{D_2}\ \overline{D_3}) = P(\overline{D_1})P(\overline{D_2})P(\overline{D_3})$

$= 0.9 \times 0.9 \times 0.9 = 0.729$

(b) P(Exactly two of the three welds are defective)

$= P(\overline{D_1} D_2 D_3 \cup D_1 \overline{D_2} D_3 \cup D_1 D_2 \overline{D_3})$

$= P(\overline{D_1} D_2 D_3) + P(D_1 \overline{D_2} D_3) + P(D_1 D_2 \overline{D_3})$

since the three events are mutually exclusive.

Hence, the probability become

$P = 0.9x0.1x0.1 + 0.1x0.9x0.1 + 0.1x0.1x0.9 = 0.027$

(c) P(all 3 welds defective) $= P(D_1 D_2 D_3) = P^3(D) = (0.1)^3 = 0.001$

2.8

$P(E_1) = 0.8$; $P(E_2) = 0.7$; $P(E_3) = 0.95$

$P(E_3 \mid \overline{E_2}) = 0.6$; assume E_2 and E_3 are statistically independent of E_1

(a) $A = (E_2 \cup E_3)\, E_1$

$B = \overline{(E_2 \cup E_3) E_1} \qquad or\ \overline{E_2}\ \overline{E_3} \cup \overline{E_1}$

(b) $P(B) = P(\overline{E_1} \cup \overline{E_2}\ \overline{E_3})$

$= P(\overline{E_1}) + P(\overline{E_2}\ \overline{E_3}) - P(\overline{E_1}\ \overline{E_2}\ \overline{E_3})$

$= 0.2 + P(\overline{E_3} \mid \overline{E_2})P(\overline{E_2}) - P(\overline{E_1})P(\overline{E_2}\ \overline{E_3})$

$= 0.2 + 0.4x0.3 - 0.2x(0.4x0.3) = 0.296$

(c) P(casting $\mid$ concrete production not feasible at site)

$= P((E_2 \cup E_3)\, E_1 \mid \overline{E_2})$

$$= \frac{P(E_2 \cup E_3)E_1\overline{E_2}}{P(\overline{E_2})} = \frac{P(E_1\overline{E_2}E_3)}{P(\overline{E_2})} = \frac{P(E_1)P(E_3|\overline{E_2})P(\overline{E_2})}{P(\overline{E_2})}$$

$= 0.8 \text{ x } 0.6$

$=0.48$

2.9

E_1, E_2, E_3 denote events tractor no. 1, 2, 3 are in good condition respectively

(a) A = only tractor no. 1 is in good condition

$= E_1 \overline{E_2}\ \overline{E_3}$

B = exactly one tractor is in good condition

$= E_1 \overline{E_2}\ \overline{E_3} \cup \overline{E_1}\ E_2 \overline{E_3} \cup \overline{E_1}\ \overline{E_2}\ E_3$

C = at least one tractor is in good condition

$= E_1 \cup E_2 \cup E_3$

(b) Given $P(E_1) = P(E_2) = P(E_3) = 0.6$

$P(\overline{E_2} \mid \overline{E_1}) = 0.6$ or $P(\overline{E_2} \mid \overline{E_3}) = 0.6$

$P(\overline{E_3} \mid \overline{E_1}\ \overline{E_2}) = 0.8$ or $P(\overline{E_1} \mid \overline{E_2}\ \overline{E_3}) = 0.8$

$P(A) = P(E_1 \overline{E_2}\ \overline{E_3}) = P(E_1 \mid \overline{E_2}\ \overline{E_3})\ P(\overline{E_2}\ \overline{E_3})$

$= [1 – P(\overline{E_1} \mid \overline{E_2}\ \overline{E_3})]\ P(\overline{E_2} \mid \overline{E_3})\ P(\overline{E_3})$

$= (1 - 0.8)(0.6)(0.4) = 0.048$

Since $E_1 \overline{E_2}\ \overline{E_3}$, $\overline{E_1}\ E_2 \overline{E_3}$ and $\overline{E_1}\ \overline{E_2}\ E_3$ are mutually exclusive; also the probability of each of these three events is the same,

$P(B) = 3 \times P(E_1 \overline{E_2}\ \overline{E_3}) = 3 \times 0.048 = 0.144$

$P(C) = P(E_1 \cup E_2 \cup E_3)$

$= 1 – P(\overline{E_1 \cup E_2 \cup E_3})$

$= 1 – P(\overline{E_1}\ \overline{E_2}\ \overline{E_3})$

$= 1 – P(\overline{E_1} \mid \overline{E_2}\ \overline{E_3})\ P(\overline{E_2} \mid \overline{E_3})\ P(\overline{E_3})$

$= 1 – 0.8 \times 0.6 \times 0.4 = 0.808$

2.10

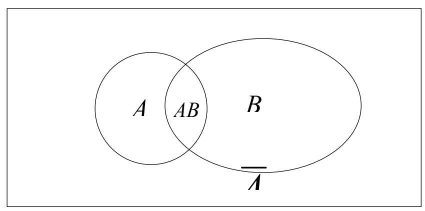
(a) The event "both subcontractors will be available" = AB, hence

$$\text{since } P(A\cup B) = P(A) + P(B) - P(AB)$$
$$\Rightarrow P(AB) = P(A) + P(B) - P(A\cup B)$$
$$= 0.6 + 0.8 - 0.9 = \mathbf{0.5}$$

(b) $P(\text{B is available} \mid \text{A is not available}) = P(B \mid \overline{A})$

$$= \frac{P(B\overline{A})}{P(\overline{A})}$$

while it is clear from the following Venn diagram that $P(B\overline{A}) = P(B) - P(AB)$.



Hence

$$\frac{P(B\overline{A})}{P(\overline{A})} = \frac{P(B) - P(AB)}{1 - P(A)}$$
$$= (0.8 - 0.5)/(1 - 0.6)$$
$$= 0.3/0.4 = \mathbf{0.75}$$

(c)

(i) If A and B are s.i., we must have $P(B \mid A) = P(B) = 0.8$. However, using Bayes' rule,

$$P(B \mid A) = P(AB)/P(A)$$
$$= 0.5/0.6 = 0.8333$$

So A and B are **not s.i.** (A's being available boosts the chances that B will be available)

(ii) From (a), P(AB) is nonzero, hence $AB \neq \varnothing$, i.e. A and B are **not m.e.**

(iii) Given: $P(A\cup B) = 0.9 \Rightarrow A\cup B$ does not generate the whole sample space (otherwise the probability would be 1), i.e. A and B are **not collectively exhaustive.**

2.11

(a) Let L, S_A, S_B denote the respective events "leakage at site", "seam of sand from X to A", "seam of sand from X to B".

Given probabilities:

$$P(L) = 0.01,\ P(S_A) = 0.02,\ P(S_B) = 0.03,\ P(S_B \mid S_A) = 0.2,$$

Also given: independence between leakage and seams of sand, i.e.

$$P(S_B \mid L) = P(S_B),\ P(L \mid S_B) = P(L),\ P(S_A \mid L) = P(S_A),\ P(L \mid S_A) = P(L)$$

The event "water in town A will be contaminated" = $L \cap S_A$, whose probability is

$$\begin{aligned} P(L \cap S_A) &= P(L \mid S_A)\, P(S_A) \\ &= P(L)P(S_A) \\ &= 0.01 \times 0.02 = \mathbf{0.0002}. \end{aligned}$$

(b) The desired event is $(L\, S_A) \cup (L\, S_B)$, whose probability is

$$\begin{aligned} &P(L\, S_A) + P(L\, S_B) - P(L\, S_A\, L\, S_B) \\ &= P(L)P(S_A) + P(L)P(S_B) - P(L\, S_A\, S_B) \\ &= P(L)\, [P(S_A) + P(S_B) - P(S_A\, S_B)] \\ &= P(L)\, [P(S_A) + P(S_B) - P(S_B \mid S_A)\, P(S_A)] \\ &= 0.01\, (0.02 + 0.03 - 0.2 \times 0.02) = \mathbf{0.00046} \end{aligned}$$

2.12

Let A,B,C denote the respective events that the named towns are flooded. Given probabilities: P(A) = 0.2, P(B) = 0.3, P(C) = 0.1, P(B | C) = 0.6, P(A | BC) = 0.8, P($\overline{A}\,\overline{B} \mid \overline{C}$) = 0.9, where an overbar denotes compliment of an event.

(a) P(disaster year) = P(ABC)

$$
\begin{aligned}
&= P(A \mid BC)P(BC) \\
&= P(A \mid BC)P(B \mid C)P(C) \\
&= 0.8\times0.6\times0.1 = \mathbf{0.048}
\end{aligned}
$$

(b) P(C | B) = P(BC) / P(B)

$$
\begin{aligned}
&= P(B \mid C)P(C) / P(B) \\
&= 0.6\times0.1\div0.3 = \mathbf{0.2}
\end{aligned}
$$

(c) The event of interest is $A\cup B\cup C$. Since this is the union of many items, we can work with its compliment instead, allowing us to apply De Morgan's rule and rewrite as

$$
\begin{aligned}
P(A\cup B\cup C) &= 1 - P(\overline{A \cup B \cup C}) \\
&= 1 - P(\overline{A}\,\overline{B}\,\overline{C}) \qquad \text{by De Morgan's rule,} \\
&= 1 - P(\overline{A}\,\overline{B} \mid \overline{C})P(\overline{C}) \\
&= 1 - 0.9\times(1 - 0.1) \\
&= 1 - 0.81 = \mathbf{0.19}
\end{aligned}
$$

2.13

(a) $C = [(L\cup M)\ G]] \cup [(LM)\,\overline{G}\ ]$

(b) Since $(L\cup M)G$ is contained in G, it is mutually exclusive to $(LM)\ \overline{G}$ which is contained in $\overline{G}$. Hence P(C) is simply the sum of two terms,

$P(C) = P[(L\cup M)\ G]] + P[(L\cap M)\ \overline{G}\ ]$

$= P(LG\cup MG) + P(L)P(M\overline{G}\ )$

$= P(LG) + P(MG) - P(LMG) + P(L)P(M|\overline{G}\ )P(\overline{G}\ )$

$= P(L)P(G) + P(M|G)P(G) - P(L)P(M|G)P(G) + P(L)P(M|\overline{G}\ )P(\overline{G}\ )$

$= 0.7\times0.6 + 1\times0.6 - 0.7\times1\times0.6 + 0.7\times0.5\times0.4 =$ **0.74**

(c) $P(\overline{L}|C) = P(\overline{L}C)/P(C) = P(\overline{L}\,\{[(L\cup M)\ G]\} \cup [(LM)\ \ \overline{G}\ ]))\ /\ P(C)$

$= P[\overline{L}\ \ (L\cup M)\ \ G]]\ /\ P(C)$ since $(\overline{L}L)M\overline{G}$ is an impossible event

$= P(\overline{L}MG)\ /\ P(C)$

$= P(\overline{L}\ )P(MG)\ /\ P(C)$

$= P(\overline{L}\ )P(M|G)P(G)\ /\ P(C)$

$= 0.3\times1\times0.6\ /\ 0.74 \cong$ **0.243**

2.14

(a) Note that the question assumes an accident either occurs or does not occur at a given crossing each year, i.e. no more than one accident per year. There were a total of 30 + 20 + 60 + 20 = 130 accidents in 10 years, hence the yearly average is 130 / 10 = 13 accidents among 1000 crossings, hence the yearly probability of accident occurring at a given crossing is

$$\frac{13}{1000} = \mathbf{0.013} \text{ (probability per year)}$$

(b) Examining the data across the "Day" row, we see that the relative likelihood of R compared to S is 30:60, hence

$$P(S \mid D) = 60/90 = \mathbf{2/3}$$

(c) Let F denote "fatal accident". We have 50% of (30 + 20) = 0.5×50 = 25 fatal "run into train" accidents, and 80% of (60 + 20) = 0.8×80 = 64 fatal "struck by train" accidents, hence the total is

$$P(F) = (25 + 64) / 130$$
$$\cong \mathbf{0.685}$$

(d)

(i) D and R are not mutually exclusive; they can occur together (there were 30 run-into-train accidents happened in daytime);

(ii) If D and R are, we must have $P(R \mid D) = P(R)$, but here $P(R \mid D) = 30 / 90 = 1/3$, while $P(R) = (30 + 20) / 130 = 5/13$, so D and R are not statistically independent.

2.15

F = fuel cell technology successfully marketable
S = solar power technology successfully marketable
F and S are statistically independent
Given $P(F) = 0.7$; $P(S) = 0.85$

(i) P(energy supplied) = $P(F \cup S)$

$$= P(F) + P(S) - P(F)P(S)$$
$$= 0.7 + 0.85 - 0.7 \times 0.85$$
$$= 0.955$$

(ii) P(only one source of energy available)

$$= P(F\overline{S} \cup \overline{F}S)$$

$$= P(F\overline{S}) + P(\overline{F}S)$$

$$= (0.7)(1-0.85) + (1-0.7)(0.85)$$
$$= 0.36$$

2.16

a. E_1 = Monday is a rainy day
E_2 = Tuesday is a rainy day
E_3 = Wednesday is a rainy day
Given $P(E_1) = P(E_2) = P(E_3) = 0.3$
$P(E_2 | E_1) = P(E_3 | E_2) = 0.5$
$P(E_3 | E_1E_2) = 0.2$

b. $P(E_1E_2) = P(E_2 | E_1)\, P(E_1) = 0.5\text{x}0.3 = 0.15$

c. $P(E_1E_2\overline{E_3}) = P(\overline{E_3} | E_1E_2)\, P(E_2 | E_1)\, P(E_1)$

$= (1-0.2)(0.5)(0.3)$
$= 0.12$

d. P(at least one rainy day)
$= P(E_1 \cup E_2 \cup E_3)$
$= P(E_1) + P(E_2) + P(E_3) - P(E_1E_2) - P(E_2E_3) - P(E_1E_3) + P(E_1E_2E_3)$
$= 0.3 + 0.3 + 0.3 - 0.3\text{x}0.5 - 0.3\text{x}0.5 – 0.1125 + 0.3\text{x}0.5\text{x}0.2$
$=0.52$

Where $P(E_1E_3)=P(E_3|E_1)P(E_1)=0.375\text{x}0.3=0.1125$

$P(E_3|E_1)= P(E_3|E_2E_1)P(E_2|E_1)+ P(E_3|\overline{E}_2\, E_1)P(\overline{E}_2\, |E_1)$

$=0.15\text{x}0.15+0.3\text{x}0.5$
$=0.375$

2.17

Let A, B, C denote events parking lot A, B and C are available on a week day morning respectively

Given:

P(A) = 0.2;   P(B) = 0.15;   P(C) = 0.8

$P(B \mid \overline{A}) = 0.5$; $P(C \mid \overline{A}\,\overline{B}) = 0.4$

(a) P(no free parking) = $P(\overline{A}\,\overline{B}) = P(\overline{A}) \times P(\overline{B} \mid \overline{A}) = 0.8 \times 0.5 = 0.4$

(b) P(able to park) = 1 – P(not able to park)

$= 1 - P(\overline{A}\,\overline{B}\,\overline{C})$

$= 1 - P(\overline{A}\,\overline{B}) \times P(\overline{C} \mid \overline{A}\,\overline{B})$

= 1 – 0.4x(1-0.4)

= 1 – 0.24

=0.76

(c) P(free parking | able to park)

$P(A \cup B \mid A \cup B \cup C)$

$$= \frac{P[(A \cup B)(A \cup B \cup C)]}{P(A \cup B \cup C)}$$

$$= \frac{P(A \cup B)}{P(A \cup B \cup C)}$$

$$= \frac{1 - P(\overline{A}\overline{B})}{0.76}$$

$$= \frac{1 - 0.4}{0.76} = 0.789$$

2.18

C = Collapse of superstructure
E = Excessive settlement
Given:
$P(E) = 0.1$; $P(C) = 0.05$
$P(C|E) = 0.2$

(a) $P(\text{Failure}) = P(C \cup E)$

$= P(C) + P(E) – P(C|E)P(E)$

$= 0.05 + 0.1 – 0.2\text{x}0.1$

$= 0.13$

(b) $P(EC|E \cup C) = \dfrac{P((EC) \cap (E \cup C))}{P(E \cup C)}$

$= P(EC) / 0.13$

$= 0.2\text{x}0.1 / 0.13$

$= 0.154$

(c) $P(E\overline{C} \cup \overline{E}C) = P(E\overline{C}) + P(\overline{E}C)$

$= P(E \cup C) – P(EC)$

$= 0.13 – 0.02$

$=0.11$

2.19

M = failure of master cylinder
W = failure of wheel cylinders
B = failure of brake pads
Given:
P(M) = 0.02; P(W) = 0.05; P(B) = 0.5
P(MW) = 0.01
B is statistically independent of M or W

(a) $P(W\overline{M}\ \overline{B}) = P(W\overline{M})P(\overline{B})$

$= [P(W) – P(MW)]\ [1 – P(B)]$
$= (0.05 – 0.01)(1 – 0.5)$
$= 0.02$

(b) P(system failure) = 1 – P(no component failure)

$= 1- P(\overline{M}\ \overline{W}\ \overline{B})$

$= 1- P(\overline{M}\ \overline{W})\ P(\overline{B})$

$= 1- \{1 - P(M \cup W)\}P(\overline{B})$

$= 1- \{1-[P(M)+P(W)-P(MW)]\}P(\overline{B})$

$= \{1 - [0.02+0.05-0.01]\}(0.5)$
$= 0.53$

(c) P(only one component failed) = $P(\overline{W}\ \overline{M}\ B) + P(\overline{W}\ M\ \overline{B}) + P(W\overline{M}\ \overline{B})$

$P(W\overline{M}\ \overline{B}) = 0.02$ from part (a)

also $P(\overline{W}\ M\ \overline{B}) = P(\overline{W}\ M) \times P(\overline{B})$

$= [P(M)-P(MW)]P(\overline{B})$

$= (0.02 – 0.01)x0.5$
$= 0.005$

also $P(\overline{W}\ \overline{M}\ B) = P(\overline{W}\ \overline{M}) \times P(\overline{B}) = [1-P(M)-P(W)+P(MW)]P(B)$

$= (1-0.02-0.005+0.01) \times 0.5 = 0.47$

Since the three events $\overline{W}\,\overline{M}\,\mathrm{B}$, $\overline{W}\,\mathrm{M}\,\overline{B}$ and $\mathrm{W}\,\overline{M}\,\overline{B}$ are all within the event of system failure

P(only one component failure | system failure) = (0.47+0.005+0.02) / 0.53 = 0.934

2.20

E_1 = Excessive snowfall in first winter
E_2 = Excessive snowfall in second winter
E_3 = Excessive snowfall in third winter

(a) $P(E_1) = P(E_2) = P(E_3) = 0.1$
$P(E_2 \mid E_1) = 0.4 = P(E_3 \mid E_2)$
$P(E_3 \mid E_1 E_2) = 0.2$

(b) $P(E_1 \cup E_2) = P(E_1)+P(E_2)-P(E_1 E_2)$
$= 0.1 + 0.1 - 0.4x0.1$
$= 0.16$

(c) $P(E_1E_2E_3) = P(E_1)\, P(E_2 \mid E_1)\, P(E_3 \mid E_1 E_2)$
$= 0.1x0.4x0.2$
$=0.008$

(d) $P(\overline{E_2} \mid \overline{E_1}) = ?$

Since $P(\overline{E_1} \cup \overline{E_2}) = P(\overline{E_1})+P(\overline{E_2})-P(\overline{E_1}\ \overline{E_2})$

$= P(\overline{E_1})+P(\overline{E_2})-P(\overline{E_2} \mid \overline{E_1})P(\overline{E_1})$

$= 0.9+0.9-P(\overline{E_2} \mid \overline{E_1})\ 0.9$

and from given relationship,

$P(\overline{E_1} \cup \overline{E_2}) = 1- P(E_1E_2) = 1- P(E_1)\, P(E_2 \mid E_1)$
$= 1-0.1x0.4 = 0.96$

Therefore, $1.8-0.9\, P(\overline{E_2} \mid \overline{E_1}) = 0.96$

and $P(\overline{E_2} \mid \overline{E_1}) = 0.933$

2.21

H_1, H_2, H_3 denote first, second and third summer is hot respectively

Given: $P(H_1) = P(H_2) = P(H_3) = 0.2$

$P(H_2 \mid H_1) = P(H_3 \mid E_2) = 0.4$

(a) $P(H_1H_2H_3) = P(H_1)\, P(H_2 \mid H_1)\, P(H_3 \mid E_2)$

$= 0.2\text{x}0.4\text{x}0.4$

$= 0.032$

(b) $P(\overline{H_2} \mid \overline{H_1}) = ?$

Using the hint given in part (d) of P2.20,

$P(\overline{H_1}) + P(\overline{H_2}) - P(\overline{H_2} \mid \overline{H_1})P(\overline{H_1}) = P(\overline{H_1} \cup \overline{H_2}) = 1 - P(H_1H_2) = 1 - P(H_2 \mid H_1)$

$P(H_1)$

we have,

$0.8 + 0.8 - P(\overline{H_2} \mid \overline{H_1})(0.8) = 1 - 0.4\text{x}0.2$

therefore,

$P(\overline{H_2} \mid \overline{H_1}) = 0.85$

(c) P(at least 1 hot summer)

= 1 – P(no hot summers)

$= 1 - P(\overline{H_1}\ \overline{H_2}\ \overline{H_3})$

$= 1 - P(\overline{H_1})P(\overline{H_2} \mid \overline{H_1})\, P(\overline{H_3} \mid \overline{H_2})$

$= 1 - 0.8\text{x}0.85\text{x}0.85$

$= 0.422$

2.22

A = Shut down of Plant A
B = Shut down of Plant B
C = Shut down of Plant C

Given: $P(A) = 0.05$, $P(B) = 0.05$, $P(C) = 0.1$
$P(B|A) = 0.5 = P(A|B)$
C is statistically independent of A and B

(a) P(complete backout | A)
$= P(BC|A)$
$= P(B|A)P(C)$
$= 0.5 \times 0.1$
$= 0.05$

(b) P(no power)
$= P(ABC)$
$= P(AB)P(C)$
$= P(B|A)P(A)P(C)$
$= 0.5 \times 0.05 \times 0.1$
$= 0.00025$

(c) P(less than or equal to 100 MW capacity)
= P (at most two plants operating)
= 1 – P(all plants operating)

$= 1 - P(\overline{A}\ \overline{B}\ \overline{C})$

$= 1 - P(\overline{C})P(\overline{A}\ \overline{B})$

$= 1 - P(\overline{C})\ [1-\{P(A)+P(B)- P(B|A)P(A)\}]$

$= 1 - 0.9[1-\{0.05+0.05-0.5 \times 0.05\}]$
$= 0.1675$

2.23

P(Damage) = P(D) = 0.02
Assume damages between earth quakes are statistically independent

(a) P(no damage in all three earthquakes)
$= P^3(D)$
$= 0.023$
$= 8 x 10^{-6}$

(b) $P(\overline{D_1}\, D_2) = P(\overline{D_1})P(D_2) = 0.98 x 0.02 = 0.0196$

2.24

(a) Let A, B denote the event of the respective engineers spotting the error. Let E denote the event that the error is spotted, $P(E) = P(A \cup B)$

$$\begin{aligned} &= P(A) + P(B) - P(AB) \\ &= P(A) + P(B) - P(A)P(B) \\ &= 0.8 + 0.9 - 0.8 \times 0.9 = \mathbf{0.98} \end{aligned}$$

(b) "Spotted by A alone" implies that B failed to spot it, hence the required probability is

$$\begin{aligned} P(AB'|E) &= P(AB' \cap E)/P(E) \\ &= P(E|AB')P(AB') / P(E) \\ &= 1 \times P(A)P(B') / P(E) \\ &= 0.8 \times 0.1 / 0.98 \cong \mathbf{0.082} \end{aligned}$$

(c) With these 3 engineers checking it, the probability of not finding the error is

$$P(C_1'C_2'C_3') = P(C_1')P(C_2')P(C_3') = (1 - 0.75)^3 \text{, hence}$$

$$P(\text{error spotted}) = 1 - \quad (1 - 0.75)^3 \cong 0.984,$$

which is higher than the 0.98 in (a), so the team of 3 is better.

(d) The probability that the first error is detected (event D_1) has been calculated in (a) to be 0.98. However, since statistical independence is given, detection of the second error (event D_2) still has the same probability. Hence $P(D_1\, D_2) = P(D_1)P(D_2) = 0.98^2 \cong \mathbf{0.960}$

2.25

L = Failure of lattice structure
A = Failure of anchorage

Given: $P(A) = 0.006$
$P(L|A) = 0.4$
$P(A|L) = 0.3$
Hence, $P(L) = P(L|A)P(A) / P(A|L) = 0.008$

(a) P(antenna disk damage)
$= P(A \cup L)$
$= P(A) + P(L) - P(A)P(L|A)$
$= 0.4 + 0.008 - 0.006 \text{x} 0.4$
$= 0.406$

(b) P(only one of the two potential failure modes)

$= P(A\overline{L}) + P(\overline{A}L)$

$= P(\overline{L}|A)P(A) + P(\overline{A}|L)P(L)(0.008)$

$= (1-0.4)(0.006) + (1-0.3)(0.008)$
$= 0.0036 + 0.0056$
$=0.0092$

(c) $$P(A|D) = \frac{P(A(A \cup L))}{P(A \cup L)} = \frac{P(A)}{P(A \cup L)} = \frac{0.006}{0.406} = 0.0148$$

2.26

$P(F) = 0.01$

$P(A \mid \overline{F}) = 0.1$

$P(A \mid F) = 1$

(a) FA, $F\overline{A}$, $\overline{F}A$, $\overline{F}\,\overline{A}$ are set of mutually exclusive and collectively exhaustive events

(b) $P(FA) = P(A \mid F)P(F) = 1 \text{x} 0.01 = 0.01$

$P(F\overline{A}) = P(\overline{A} \mid F)P(F) = 0 \text{x} 0.01 = 0$

$P(\overline{F}A) = P(A \mid \overline{F})P(\overline{F}) = 0.1 \text{x} 0.99 = 0.099$

$P(\overline{F}\,\overline{A}) = P(\overline{A} \mid \overline{F})P(\overline{F}) = 0.9 \text{x} 0.99 = 0.891$

(c) $P(A) = P(A \mid F)P(F) + P(A \mid \overline{F})P(\overline{F})$

$= 1 \text{x} 0.01 + 0.1 \text{x} 0.99$

$= 0.109$

(d) $P(F \mid A) = \dfrac{P(A|F)P(F)}{P(A)} = \dfrac{0.01}{0.109} = 0.0917$

2.27

Given: P(D) = 0.001

$P(T|D) = 0.85$; $P(T|\overline{D}) = 0.02$

(a) DT, $D\overline{T}$, $\overline{D}T$, $\overline{D}\,\overline{T}$

(b) $P(DT) = P(T|D)P(D) = 0.85x0.001 = 0.00085$

$P(D\overline{T}) = P(\overline{T}|D)P(D) = 0.15x0.001 = 0.00015$

$P(\overline{D}T) = P(T|\overline{D})P(\overline{D}) = 0.02x0.999 = 0.01998$

$P(\overline{D}\,\overline{T}) = P(\overline{T}|\overline{D})P(\overline{D}) = 0.98x0.999 = 0.979$

(c) $$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\overline{D})P(\overline{D})} = \frac{0.00085}{0.00085 + 0.01998} = 0.0408$$

2.28

Given: $P(R_A) = P(G_A) = 0.5$

$P(R_B) = P(G_B) = 0.05$

$P(G_B | G_A) = 0.8$

$P(G_{LT}) = 0.2$

Signal at C is statistically independent of those at A or B.

(a) $E_1 = R_A \cup R_B$

$E_2 = G_{LT}$

$E_3 = R_A G_B \cup G_A R_B$

(b) P(stopped at least once from M to Q)

= 1 – P(no stopping at all from M to Q)

$= 1 - P(G_A G_B G_{LT})$

$= 1 - P(G_{LT})P(G_A)P(G_B | G_A)$

= 1 – 0.2x0.5x0.8

= 0.92

(c) P(stopped at most once from M to N)

= 1 – P(stopped at both A and B)

$= 1 - P(R_A R_B)$

$= 1 - P(R_A)\, P(R_B | R_A)$

From the hint given in P2.20, it can be shown that

$$P(R_B \big| R_A) = \frac{1}{P(R_A)}[1 - P(G_B \big| G_A)P(G_A)] - 1 = \frac{1}{0.5}[1 - 0.8 \times 0.5] - 1 = 0.2$$

Hence P(stopped at most once from M to N) = 1 – 0.5x0.2 = 0.9

2.29

(a) $P(WSC) = P(W)P(SC) = P(W)P(S|C)P(C) = 0.1\times0.3\times0.05 = \mathbf{0.0015}$

(b) $P(W\cup C) = P(W) + P(C) - P(WC) = P(W) + P(C) - P(W)P(C)$

$$= 0.1 + 0.05 - 0.1\times0.05 = \mathbf{0.145}$$

(c)
$$\begin{aligned} P(W\cup C|\overline{S}) &= P[\overline{S}(W\cup C)] / P(\overline{S}) \\ &= P(\overline{S}W\cup\overline{S}C) / P(\overline{S}) \\ &= [\ P(\overline{S}W) + P(\overline{S}C) - P(\overline{S}WC)\ ] / P(\overline{S}) \\ &= [\ P(\overline{S})P(W) + P(\overline{S}|C)P(C) - P(W)P(\overline{S}C)\ ] / P(\overline{S}) \\ &= [0.8\times0.1 + (1-0.3)\times0.05 - 0.1\times(1-0.3)\times0.05] / (1-0.2) \\ &\cong \mathbf{0.139} \end{aligned}$$

(d) P(nice winter day) $= P(\overline{S}\,\overline{W}C) = P(\overline{W})P(\overline{S}C) = P(\overline{W})P(\overline{S}|\ C)P(C) = 0.9\text{x}0.7\text{x}0.05 =$ $\mathbf{0.0315}$

(e) $P(U) = P(U\ |CW)P(CW) + P(U\ |\overline{C}W)P(\overline{C}W) + P(U|C\overline{W})\ P(C|\overline{W}) + P(U||\overline{C}\ \overline{W})\ P(\overline{C}\ \overline{W})$

$= 1\text{x}P(W)P(C) + 0.5\text{x}P(W)P(\overline{C}) + 0.5\text{x}P(C)P(\overline{W}) + 0\text{x}P(\overline{C})P(\overline{W})$

$= 1\text{x}0.1\text{x}0.05 + 0.5\text{x}0.1\text{x}0.95 + 0.5\text{x}0.05\text{x}0.9$

$\mathbf{= 0.075}$

2.30

(a) Let L and B denote the respective events of lead and bacteria contamination, and C denote water contamination.

$$
\begin{aligned}
P(C) &= P(L \cup B) \\
&= P(L) + P(B) - P(LB) \\
&= P(L) + P(B) - P(L)P(B) \because \text{L and B are independent events} \\
&= 0.04 + 0.02 - 0.04 \times 0.02 = \mathbf{0.0592}
\end{aligned}
$$

(b) $P(L\overline{B} \mid C) = P(CL\overline{B}) / P(C)$

$= P(C \mid L\overline{B})P(L\overline{B}) / P(C)$,

but $P(C \mid L\overline{B}) = 1$ ($\because$ lead alone will contaminate for sure)

and $P(L\overline{B}) = P(L)P(\overline{B})$ ($\because$ statistical independence), hence the probability

$= 1 \times P(L)P(\overline{B}) / P(C)$

$= 1 \times 0.04 \times (1 - 0.02) / 0.0592 \cong \mathbf{0.662}$

2.31

C	Water level	W ($\times 10^3$ lb)	F ($\times 10^3$ lb)	Probability
0.1	0	100	10	0.5x0.2=0.1
0.1	10	210	21	0.5x0.4=0.2
0.1	20	320	32	0.5x0.4=0.2
0.2	0	100	20	0.5x0.2=0.1
0.2	10	210	42	0.5x0.4=0.2
0.2	20	320	64	0.5x0.4=0.2

(a) Sample space of F = {10, 20, 21, 32, 42, 64}

(b) Let H be the horizontal force in 10^3 lb

P(sliding) = P(H>F) = P(F<15) = 0.1

(c) P(sliding) = P(F<15) P(H=15) + P(F<20) P(H=20)

= 0.1x0.5 + 0.1x0.1

= 0.06

2.32

Given: $P(W) = 0.9$

$P(H) = 0.3$

$P(E) = 0.2$

$P(W|H) = 0.6$

E is statistically independent of W or H

(a) $I = E(\overline{W} \cup H)$

$II = \overline{E}\,\overline{W}H$

(b) $P[E(\overline{W} \cup H)] = P(E)\,P(\overline{W} \cup H)$

$= P(E) \times [P(\overline{W})+P(H)-P(\overline{W}|H)P(H)]$

$= 0.2x(0.1+0.3-0.4x0.3)$

$= 0.056$

$P(\overline{E}\,\overline{W}H) = P(\overline{E})\,P(\overline{W}|H)\,P(H)$

$= 0.8x0.4x0.3$

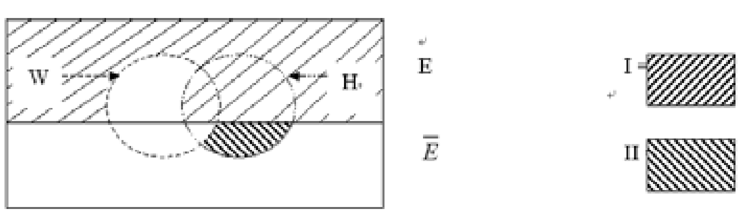
$= 0.096$

(c1) Since $P(W|H) = 0.6 \neq 0$, W and H are <u>not</u> mutually exclusive.

Since $P(W|H) = 0.6 \neq 0.9 = P(W)$, W and H are <u>not</u> statistically independent

(c2) $I\,II = E(\overline{W} \cup H)\,(\overline{E}\,\overline{W}H) = (E\overline{E})(\overline{W} \cup H)(\overline{W}H)$

Since $E\overline{E}$ is an empty set, I and II <u>are</u> mutually exclusive.



From the above Venn Diagram, the union of I and II does not make up the entire sample space. Hence, I and II are <u>not</u> collectively exhaustive.

(d) Since I and II are mutually exclusive,

$$
\begin{aligned}
P(\text{leakage}) &= P(\text{I}) + P(\text{II}) \\
&= 0.056 + 0.096 \\
&= 0.152
\end{aligned}
$$

2.33

Given: $P(E) = 0.15$

$P(G) = 0.1$

$P(O) = 0.2$

$P(E \mid O) = 2x0.15 = 0.3$

G is statistically independent of E or O

(a) P(shortage of all three sources)

$= P(EGO)$

$= P(G)\, P(E \mid O)\, P(O)$

$= 0.1x0.3x0.2$

$= 0.006$

(b) $P(G \cup E) = 1\text{-}P(\overline{G}\ \overline{E})$

$= 1\text{-}\ P(\overline{G})\ P(\overline{E})$

$= 1\text{-}0.9x0.85$

$= 0.235$

(c) $P(GO \mid E) = P(GOE)/P(E) = 0.006/0.15 = 0.04$

(d) P(at least 2 sources will be in short supply)

$= P(EG \cup GO \cup EO)$

$= P(EG)+P(GO)+P(EO)-P(EGO)-P(EGO)-P(EGO)+P(EGO)$

$= P(G)P(E)+P(G)P(O)+P(E \mid O)P(O)-2P(EGO)$

$= 0.1x0.15 + 0.1x0.2 + 0.3x0.2 - 2x0.006$

$= 0.083$

2.34

A, B, C denote failure of component A, B, C respectively
N and H denote normal and ultra-high altitude respectively

Given: $P(A|N) = 0.05$, $P(B|N) = 0.03$, $P(C|N) = 0.02$
$P(A|H) = 0.07$, $P(B|H) = 0.08$, $P(C|H) = 0.03$
$P(N) = 0.6$, $P(H) = 0.4$
$P(B|A) = 2 \text{ x } P(B)$
C is statistically independent of A or B

P(system failure) = P(S)
= $P(S|N)P(N) + P(S|H)P(H)$

$P(S|N) = P(A \cup B \cup C|N)$
$= P(A|N) + P(B|N) + P(C|N) - P(AB|N) - P(BC|N) - P(AC|N) + P(ABC|N)$
$= 0.05 + 0.03 + 0.02 – 2\text{x}0.03\text{x}0.03 – 0.03\text{x}0.02 – 0.05\text{x}0.02 + 0.02\text{x}2\text{x}0.03\text{x}0.03$
$= 0.096636$

Similarly,
$P(S|H)$
$= P(A|H) + P(B|H) + P(C|H) - P(AB|H) - P(BC|H) - P(AC|H) + P(ABC|H)$
$= 0.07 + 0.08 + 0.03 – 2\text{x}0.08\text{x}0.08 – 0.08\text{x}0.03 – 0.07\text{x}0.03 + 0.03\text{x}2\text{x}0.08\text{x}0.08$
$= 0.16308$

Hence P(system failure) = 0.96636x0.6 + 0.16308x0.4 $\cong$ 0.123

$$P(B|S) = \frac{P[B(A \cup B \cup C)]}{P(S)} = \frac{P(B)}{P(S)} = \frac{P(B|N)P(N) + P(B|H)P(H)}{P(S)}$$

$$= \frac{0.03 \times 0.6 + 0.08 \times 0.4}{0.123} = 0.172$$

2.35

Given: $P(C) = 0.5$

$P(W) = 0.3$

$P(W|C) = 0.4$

$U = C \cup W$

(a) Since $P(W|C) = 0.4 \neq 0.3 = P(W)$

W and C are not statistically independent

(b) $P(U) = P(C \cup W)$

$= P(C) + P(W) - P(W|C)\,P(C)$

$= 0.5 + 0.3 - 0.4\text{x}0.5$

$= 0.6$

(c) $P(C\overline{W}) = P(\overline{W}|C)\,P(C)$

$= (1-0.4)\text{x}0.5$

$= 0.3$

(d) $$P(CW|C \cup W) = \frac{P[CW(C \cup W)]}{P(C \cup W)} = \frac{P(CW)}{P(C \cup W)} = \frac{P(W|C)P(C)}{0.6} = \frac{0.4 \times 0.5}{0.6} = 0.333$$

2.36

A, B, C denote route A, B, C are congested respectively

Given: $P(A) = 0.6$, $P(B) = 0.6$, $P(C) = 0.4$

$P(A|B) = P(B|A) = 0.85$

C is statistically independent of A or B

$P(L|ABC) = 0.9$

$P(L|\overline{ABC}) = 0.3$

(a) $P(A\overline{B}\,\overline{C}) + P(\overline{A}B\overline{C}) + P(\overline{A}\,\overline{B}C)$

$= P(\overline{B}|A)P(A)P(\overline{C}) + P(\overline{A}|B)P(B)P(\overline{C}) + P(\overline{A}\,\overline{B})P(C)$

but $P(\overline{A}\,\overline{B}) = 1 - P(A \cup B)$

$= 1 - P(A) - P(B) + P(B|A)P(A)$

$= 1 - 0.6 - 0.6 + 0.85 \text{x} 0.6$

$= 0.31$

Hence, P(exactly one route congested)

$= 0.15\text{x}0.6\text{x}0.6 + 0.15\text{x}0.6\text{x}0.6 + 0.31\text{x}0.4 = 0.232$

(b) $P(\text{Late}) = P(L|ABC)P(ABC) + P(L|\overline{ABC})\,P(\overline{ABC})$

But $P(ABC) = P(AB)P(C)$

$= P(B|A)P(A)P(C)$

$= 0.85\text{x}0.6\text{x}0.4$

$= 0.204$

Hence, $P(L) = 0.9\text{x}0.204 + 0.3\text{x}(1-0.204) = 0.422$

(c) If C is congested, $P(ABC) = 0.85\text{x}0.6\text{x}1 = 0.51$

$P(L) = 0.9\text{x}0.51 + 0.3\text{x}0.49 = 0.606$

2.37

(a) Let subscripts 1 and 2 denote "after first earthquake" and "after second earthquake". Note that (i) occurrence of H_1 makes H_2 a certain event; (ii) H, L and N are mutually exclusive events and their union gives the whole sample space.

$$P(\text{heavy damage after two quakes}) = P(H_2 H_1) + P(H_2 L_1) + P(H_2 N_1)$$
$$= P(H_2 \mid H_1) P(H_1) + P(H_2 \mid L_1)P(L_1) + P(H_2 \mid N_1)P(N_1)$$
$$= 1\times0.05 + 0.5\times0.2 + 0.05\times(1 - 0.2 - 0.05) \cong \mathbf{0.188}$$

(b) The required probability is $[P(H_2 L_1) + P(H_2 N_1)] / P(\text{heavy damage after two quakes})$

$$= (0.5\times0.2 + 0.05\times0.75) / (1\times0.05 + 0.5\times0.2 + 0.05\times0.75)$$
$$\cong \mathbf{0.733}$$

(c) In this case, one always starts from an undamaged state, the probability of <u>not</u> getting heavy damage at any stage is simply (1 – 0.05) = 0.95 (not influenced by previous condition). Hence

$$P(\text{any heavy damage after 3 quakes}) = 1 - P(\text{no heavy damage in each of 3 quakes})$$
$$= 1 - 0.95^3 \quad \cong \mathbf{0.143}.$$

Alternatively, one could explicitly sum the probabilities of the three mutually exclusive events,

$$P(H) = P(H_1) + P(H_2 H_1') + P(H_3 H_2' H_1') = 0.05 + 0.05\times0.95 + 0.05\times0.95\times0.95 \cong 0.143$$

2.38

Given: $P(L) = 0.6$, $P(A) = 0.3$, $P(H) = 0.1$

S denote supply adequate, i.e. no shortage

$P(S|L) = 1$, $P(S|A) = 0.9$, $P(S|H) = 0.5$

(a) $P(S) = P(S|L)P(L)+P(S|A)P(A)+P(S|H)P(H)$

$= 1x0.6 + 0.9x0.3 + 0.5x0.1$

$= 0.92$

(b) $$P(A|\overline{S}) = \frac{P(\overline{S}|A)P(A)}{P(\overline{S})} = \frac{0.1 \times 0.3}{1 - 0.92} = 0.375$$

(c) P(shortage in at least one or of the next two months)

= 1 – P(no shortage in next two months)

= 1 – 0.92x0.92

= 0.154

2.39

(a) P(shipment accepted on a given day)
= P(at most 1 defective panel)
= 0.2 + 0.5
= 0.7

(b) P(exactly one shipment will be rejected in 5 days)
= $5\text{x}P(0.7)^4(0.3)$
= 0.36

(c) P(acceptance of shipment on a given day)
= P(A)
= $P(A|D=0)P(D=0)+ P(A|D=1)P(D=1)+ P(A|D=2)P(D=2)$

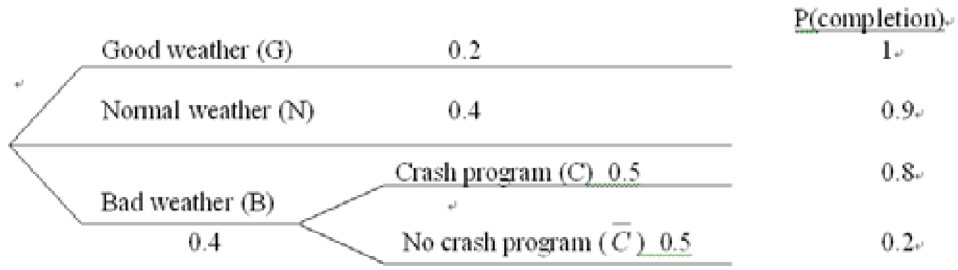
in which D = number of defective panels on a given day

$P(A|D=0) = 1$
$P(A|D=1) = 1$
$P(A|D=2) = 1\text{-}P(\text{both defective panels detected})$
$= 1 – 0.8\text{x}0.8$
$= 0.36$

Hence, P(A) = 0.2 + 0.5 + 0.36x0.3 = 0.808

2.40

The possible scenario of the working condition and respective probabilities are as follows:



(a) P(completion on schedule) = P(E)

$= P(E|G)\,P(G) + P(E|N)\,P(N) + P(E|BC)\,P(BC) + P(E|B\overline{C})\,P(B\overline{C})$

= 1x0.2 + 0.9x0.4 + 0.8x0.4x0.5 + 0.2x0.4x0.5
= 0.76

(b) P(Normal weather | completion)
= P(N | E)
= P(E | N) P(N) / P(E)
= 0.9x0.4 / 0.76
= 0.474

2.41

Let L, N, H denote the event of low, normal and high demand respectively; also O and G denote oil and gas supply is low respectively.

(a) Given normal energy demand, probability of energy crisis

$= P(E|N)$

$= P(O \cup G)$

$= P(O)+P(G)-P(OG) = P(O)+P(G)-P(G|O)P(O)$

$= 0.2 + 0.4 - 0.5 \times 0.2$

$= 0.5$

(b) P(E)

$= P(E|L)\ P(L) + P(E|N)\ P(N) + P(E|H)\ P(H)$

$= P(OG) \times 0.3 + 0.5 \times 0.6 + 1 \times 0.1$

$= P(G|O)P(O) \times 0.3 + 0.3 + 0.1$

$= 0.43$

2.42

Given: $P(H=1) = 0.2$, $P(H=2) = 0.05$, $P(H=0) = 0.75$

Where H = number of hurricanes in a year

$P(J|H=1) = P(D|H=1) = 0.99$

$P(J|H=2) = P(D|H=2) = 0.80$

Where J and D denote survival of jacket and deck substructure respectively

Assume J and D are statistically independent

(a) For the case of one hurricane, i.e. H=1

$$
\begin{aligned}
P(\text{damage}) &= 1-P(JD|H=1) \\
&= 1- P(J|H=1)\, P(D|H=1) \\
&= 1 - 0.99 \text{x} 0.99 \\
&= 0.0199
\end{aligned}
$$

(b) For next year where the number of hurricanes is not known.

$$
\begin{aligned}
P(\text{no damage}) &= P(JD|H=0)P(H=0)+ P(JD|H=1)P(H=1)+ P(JD|H=2)P(H=2) \\
&= 1\text{x}0.75 + 0.99\text{x}0.99\text{x}0.2 + 0.8\text{x}0.8\text{x}0.05 \\
&= 0.75 + 0.196 + 0.032 \\
&= 0.978
\end{aligned}
$$

(c) $$P(H=0|JD) = \frac{P(JD|H=0)P(H=0)}{P(JH)} = \frac{1\times 0.75}{0.978} = 0.767$$

2.43

(a) Sample space = $\{F_A, H_A, E_A, F_B, H_B, E_B\}$

in which F_A, F_B denote fully loaded truck in lane A, B respectively

H_A, H_B denote half loaded truck in lane A, B respectively

E_A, E_B denote empty truck in lane A, B respectively

(b) Assume the events of F, H and E are equally likely

P(full | truck in lane B) = 1/3

(c) Let C denote event of critical stress

$$P(C) = P(C|F_A)P(F_A) + P(C|H_A)P(H_A) + P(C|E_A)P(E_A)$$
$$+P(C|F_B)P(F_B) + P(C|H_B)P(H_B) + P(C|E_B)P(E_B)$$

But $P(C|F_A)=1$, $P(C|H_A)=0.4$, $P(C|E_A)=0$

$P(C|F_B)=0.6$, $P(C|H_B)=0.1$, $P(C|E_B)=0$

$P(F_A)$ = P(fully loaded | truck in lane A) P(truck in lane A) = $\frac{1}{3}\times\frac{5}{6}=\frac{5}{18}$

Similarly $P(F_B)$ = $\frac{1}{3}\times\frac{1}{6}=\frac{1}{18}$, $\quad P(H_B)=\frac{1}{16}$, $\quad P(E_B)=\frac{1}{16}$

$P(H_A)=\frac{5}{18}$, $\quad P(E_A)=\frac{1}{18}$

Hence, P(C) = 1x5/18 + 0.4x5/18 + 0 + 0.6x1/18 + 0.1x1/18 + 0 =0.428

2.44

S denotes shear failure
B denotes bending failure
D denotes diagonal cracks

Given: $P(D|S) = 0.8$
$P(D|B) = 0.1$

5% of failure are in shear mode
95% of failure are in bending mode

(a) $P(D) = P(D|S)P(S) + P(D|B)P(B)$
$= 0.8 \text{x} 0.05 + 0.1 \text{x} 0.95$
$= 0.135$

(b) $$P(S|D) = \frac{P(D|S)P(S)}{P(D)} = \frac{0.8 \times 0.05}{0.135} = 0.296$$

Since the failure in the shear mode was only 29.6%, it does not exceed the threshold value of 75% required, immediate repair is not recommended.

2.45

(a) Let A, D, I denote the respective events that a driver encountering the amber light will accelerate, decelerate, or be indecisive. Let R denote the event that s/he will run the red light. The given probabilities are

$P(A) = 0.10$, $P(D) = 0.85$, $P(I) = 0.05$, also the conditional probabilities
$P(R \mid A) = 0.05$, $P(R \mid D) = 0$, $P(R \mid I) = 0.02$

By the theorem of total probability,

$$\begin{aligned} P(R) &= P(R \mid A)P(A) + P(R \mid D)P(D) + P(R \mid I)P(I) \\ &= 0.05\times0.10 + 0 + 0.02\times0.05 \\ &= 0.005 + 0.001 \\ &= \mathbf{0.006} \end{aligned}$$

(b) The desired probability is $P(A \mid R)$, which can be found by Bayes' Theorem as

$$\begin{aligned} P(A \mid R) &= P(R \mid A)P(A) / P(R) \\ &= 0.05\times0.10 / 0.006 = 0.005 / 0.006 \\ &\cong \mathbf{0.833} \end{aligned}$$

(c) Let V denote "there exists vehicle waiting on the other street", where $P(V) = 0.60$ and $P(V') = 0.40$; and let C denote "Cautious driver in the other vehicle", where $P(C) = 0.80$ and $P(C') = 0.20$. The probability of collision is

$$P(\text{collision}) = P(\text{collision} \mid V)P(V) + P(\text{collision} \mid V')P(V')$$

But the second term is zero (since there is no other vehicle to collide with), while $P(\text{collision} \mid V)$ in the first term is

$$\begin{aligned} &P(\text{collision} \mid C)P(C) + P(\text{collision} \mid C')P(C') \\ &= (1 - 0.95)\times0.80 + (1 - 0.80)\times0.20 \\ &= 0.05\times0.80 + 0.20\times0.20 \\ &= 0.08, \end{aligned}$$

hence

$$\begin{aligned} P(\text{collision}) &= P(\text{collision} \mid V)P(V) \\ &= 0.08\times0.60 \\ &= \mathbf{0.048} \end{aligned}$$

(d) 100000 vehicles × 5% = 5000 vehicles are expected to encounter the yellow light annually. Out of these 5000 vehicles, 0.6% (i.e. 0.006) are expected to run a red light, i.e. 5000×0.006 = 30 vehicles. These 30 dangerous vehicles have 0.048 chance of getting into a collision (i.e. accident), hence 30×0.048 = **1.44** accidents caused by dangerous vehicles can be expected at the intersection per year

2.46

A, B, C denote branch A, B, C will be profitable respectively
E denotes bonus received
$P(A) = 0.7$, $P(B) = 0.7$, $P(C) = 0.6$
$P(A|B) = 0.9 = P(B|A)$
C is statistically independent of A or B

(a) P(Exactly two branches profitable)

$= P(AB\overline{C})+P(A\overline{B}C)+P(\overline{A}BC)$

$= P(A|B)P(B)P(\overline{C})+P(\overline{B}|A)P(A)P(C)+P(\overline{A}|B)P(B)P(C)$

$= 0.9\text{x}0.7\text{x}0.4 + 0.1\text{x}0.7\text{x}0.6 + 0.1\text{x}0.7\text{x}0.6$
$= 0.336$

(b) P(at least two branches profitable)
$= P(T)$
$= P(ABC) + 0.336$
$= P(A|B)P(B)P(C) + 0.336$
$= 0.9\text{x}0.7\text{x}0.6 + 0.336$
$= 0.714$

P(bonus received) = P(E)

$= P(E|T)P(T)+P(E|\overline{T})P(\overline{T})$

$= 0.8\text{x}0.714 + 0.2\text{x}0.286$
$= 0.628$

(c) Given A is not profitable, $P(T|\overline{A}) = P(BC|\overline{A}) = P(B|\overline{A})P(C)$

But $P(B|\overline{A}) = \dfrac{P(\overline{A}|B)P(B)}{P(\overline{A})} = \dfrac{0.1\times 0.7}{0.3} = 0.233$

Hence $P(T|\overline{A}) = 0.233\text{x}0.6 = 0.1398$

P(bonus received) = 0.8x0.1398 + 0.2x0.8602 = 0.284

2.47

$P(D) = P(\text{difficult foundation problem}) = 2/3$
$P(F) = P(\text{project in Ford County}) = 1/3 = 0.333$
$P(I) = P(\text{project in Iroquois County}) = 2/5 = 0.4$

$P(D|I) = 1.0$
$P(D|F) = 0.5$
$P(C) = P(\text{project in Champaign County})$
$= 1 - 0.333 - 0.4 = 0.267$

(a) $P(F\overline{D}) = P(\overline{D}|F)P(F)$

$= 0.5 \text{x} 0.333 = 0.167$

(b) $P(C\overline{D}) = P(\overline{D}|C)P(C)$

But $P(\overline{D}) = P(\overline{D}|C)P(C) + P(\overline{D}|F)P(F) + P(\overline{D}|I)P(I)$

$0.333 = P(\overline{D}|C) \text{x} 0.267 + 0.5 \text{x} 0.333 + 0 \text{x} 0.4$

$\therefore\ P(\overline{D}|C) \text{x} 0.267 = 0.167 \ \rightarrow\ P(\overline{D}|C) = 0.625$

Hence $P(C\overline{D}) = 0.625 \text{x} 0.167 = 0.104$

(c) $$P(I|\overline{D}) = \frac{P(\overline{D}|I)P(I)}{P(\overline{D})} = \frac{0 \times 0.4}{0.104} = 0.0$$

2.48

$P(E) = 0.001,\ P(L) = 0.002,\ P(T) = 0.0015$

$P(L \mid E) = 0.1$

T is statistically independent of E or L

(a) $P(ELT) = P(L \mid E)P(E)P(T)$

$= 0.1 \text{x} 0.001 \text{x} 0.0015$

$= 1.5 \text{x} 10^{-6}$

(b) $P(E \cup L \cup T) = 1\text{-}P(\overline{E}\ \overline{L}\ \overline{T})$

$= 1\text{-}P(\overline{T})P(\overline{E \cup L})$

$= 1 - 0.9985 \text{x}[1\text{-}P(E)\text{-}P(L)+P(L \mid E)P(E)]$

$= 1 - 0.0015[1\text{-}0.001\text{-}0.002+0.1 \text{x} 0.001]$

$= 1 - 0.9985 \text{x} 0.9971$

$= 0.0044$

(c) $P(E \mid E \cup L \cup T) = \dfrac{P(E(E \cup L \cup T))}{P(E \cup L \cup T)} = \dfrac{P(E)}{0.0044} = \dfrac{0.001}{0.0044} = 0.227$

2.49

(a) Let E_1, E_2 denote the respective events of using 100 and 200 units, and S denote shortage of material.

$$\begin{aligned} P(S) &= P(S \mid E_1)P(E_1) + P(S \mid E_2)P(E_2) \\ &= 0.1\times 0.6 + 0.3\times 0.4 \\ &= 0.06 + 0.12 = \mathbf{0.18} \end{aligned}$$

(b) Using Bayes' theorem with the result from part (a),

$$\begin{aligned} P(E_1 \mid S) &= P(S \mid E_1)P(E_1) / P(S) \\ &= 0.06 / 0.18 = \mathbf{1/3} \end{aligned}$$

2.50

Let H and S denote Hard and Soft ground, respectively, and let L denote a successful landing. Given probabilities:

$P(L \mid H) = 0.9$; $P(L \mid S) = 0.5$;

$P(H) = 3P(S)$, but since ground is either hard or soft

$\Rightarrow P(H) = 0.75$, $P(S) = 0.25$

(a) Using theorem of total probability,

$$\begin{aligned} P(L) &= P(L \mid H)P(H) + P(L \mid S)P(S) \\ &= 0.9\times0.75 + 0.5\times0.25 \\ &= \mathbf{0.8} \end{aligned}$$

(b) Let E denote "penetration". Given: $P(E \mid S) = 0.9$; $P(E \mid H) = 0.2$

(i) The "updated" probability of hard ground,

$$\begin{aligned} P'(H) &\equiv P(H \mid E) \\ &= P(E \mid H)P(H) / [P(E \mid H)P(H) + P(E \mid S)P(S)] \quad \text{by Bayes' theorem} \\ &= 0.2\times0.75 / (0.2\times0.75 + 0.9\times0.25) \\ &= \mathbf{0.4} \end{aligned}$$

(ii) Using the updated probabilities $P'(H) = 0.4$, $P'(S) = 1 - 0.4 = 0.6$, the updated probability of a successful landing now becomes

$$\begin{aligned} P'(L) &= P(L \mid H)P'(H) + P(L \mid S)P'(S) \\ &= 0.9\times0.4 + 0.5\times0.6 \\ &= \mathbf{0.66} \end{aligned}$$

2.51

P(C) = 0.1, P(S) = 0.05

Where C, S denote shortage of cement and steel bars respectively

$P(S \mid \overline{C}) = 0.5 \text{x} 0.05 = 0.025$

(a) $P(S \cup C) = P(S) + P(C) - P(C \mid S)P(S)$

But $P(\overline{C} \mid S) = \dfrac{P(S \mid \overline{C})P(\overline{C})}{P(S)} = \dfrac{0.025 \times 0.9}{0.05} = 0.45$

Hence $P(S \cup C) = 0.05 + 0.1 - 0.55 \text{x} 0.05 = 0.1225$

(b) $P(C\overline{S} \cup \overline{C}S) = P(C\overline{S}) + P(\overline{C}S)$

$= P(C \cup S) - P(CS)$ from Venn diagram

$= 0.1225 - P(C \mid S)P(S)$

$= 0.1225 - 0.55 \text{x} 0.05$

$= 0.095$

(c) $P(S \mid S \cup C) = \dfrac{P(S(S \cup C))}{P(S \cup C)} = \dfrac{P(S)}{P(S \cup C)} = \dfrac{0.05}{0.1225} = 0.408$

2.52

Let A and B be water supply from source A and B are below normal respectively

$P(A) = 0.3$, $P(B) = 0.15$

$P(B|A) = 0.3$

$P(S|A\overline{B}) = 0.2$, $P(S|\overline{A}B) = 0.25$, $P(S|\overline{A}\,\overline{B}) = 0$, $P(S|AB) = 0.8$

Where S denotes event of water shortage

(i) $P(A\cup B) = P(A)+P(B)- P(B|A)P(A)$

$= 0.3 + 0.15 - 0.3\text{x}0.3 = 0.36$

(ii) $P(A\overline{B} \cup \overline{A}B) = P(A\cup B) – P(AB)$

$= P(A)+P(B)-2P(B|A)P(A)$

$= 0.3 + 0.15 –2\text{x}0.3\text{x}0.3$

$=0.22$

(iii) $P(S) = P(S|A\overline{B})P(A\overline{B})+P(S|\overline{A}B)P(\overline{A}B)+P(S|\overline{A}\,\overline{B})P(\overline{A}\,\overline{B})+P(S|AB)P(AB)$

But $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)} = \dfrac{0.3\times 0.3}{0.15} = 0.6$

Hence $P(S) = 0.2\text{x}0.7\text{x}0.3 + 0.25\text{x}0.4\text{x}0.15 + 0 + 0.8\text{x}0.3\text{x}0.3 = 0.129$

(iv) $P(AB|S) = \dfrac{P(S|AB)P(AB)}{P(S)} = \dfrac{0.8\times 0.3\times 0.3}{0.129} = 0.558$

(v) $P(\overline{A}|\overline{S}) = P(\overline{A}B|\overline{S})+P(\overline{A}\,\overline{B}|\overline{S})$

$$= \frac{P(\overline{S}|\overline{A}B)P(\overline{A}B)}{P(\overline{S})} + \frac{P(\overline{S}|\overline{A}\overline{B})P(\overline{A}\overline{B})}{P(\overline{S})}$$

But $P(\overline{A}B) = [1-P(A|B)]\,P(B) = 0.4\text{x}0.15 = 0.06$

$P(\overline{A}\,\overline{B}) = 1-P(AB)-P(\overline{A}B)-P(A\overline{B})$

$= 1 – 0.3\text{x}0.3 – 0.06 – 0.7\text{x}0.3$

$= 0.64$

Hence $P(\overline{A}|\overline{S}) = \dfrac{0.75 \times 0.06}{0.871} + \dfrac{1 \times 0.64}{0.871} = 0.786$

2.53

Let W be weather favorable

F be field work completed on schedule

C be computation completed

T_1 be computer 1 available

T_2 be computer 2 available

Given: $P(F \mid W) = 0.9$, $P(F \mid \overline{W}) = 0.5$

$P(\overline{W}) = 0.6$

$P(T_1) = P(T_2) = 0.7$

$P(C \mid T_1 \overline{T_2}) = P(C \mid \overline{T_1} T_2) = 0.6$

$P(C \mid T_1 T_2) = 0.9$, $P(C \mid \overline{T_1}\ \overline{T_2}) = 0.4$

(a) $P(F) = P(F \mid W)P(W) + P(F \mid \overline{W})P(\overline{W})$

$= 0.9 x 0.4 + 0.5 x 0.6$

$= 0.66$

(b) $P(C) = P(C \mid T_1 T_2)\, P(T_1 T_2) + P(C \mid T_1 \overline{T_2})\, P(T_1 \overline{T_2}) +$

$P(C \mid \overline{T_1} T_2)\, P(\overline{T_1} T_2) + P(C \mid \overline{T_1}\ \overline{T_2})\, P(\overline{T_1}\ \overline{T_2})$

$= 0.9 x 0.7 x 0.7 + 0.6 x 0.7 x 0.3 + 0.6 x 0.3 x 0.7 + 0.4 x 0.3 x 0.3 = 0.729$

(c) P(Deadline met) = P(FC)

= P(F)P(C)

= 0.66x0.729

= 0.481

2.54

(a) T = wait time in queue (in min.)
N = number of trucks in queue

$$P(T<5 \mid N=2) = P(\text{loading time for both trucks ahead will take only 2 min. each})$$
$$= 0.5 \text{x} 0.5$$
$$= 0.25$$

(b) $P(T<5) = P(T<5 \mid N=0)P(N=0) + P(T<5 \mid N=1)P(N=1) + P(T<5 \mid N=2)P(N=2)$

Note that if the number of trucks in queue is 3 or more, the waiting time will definitely exceed 5. Hence those items will not contribute any probabilities.

Hence $P(T<5) = 1\text{x}0.175 + 1\text{x}0.125 + 0.25\text{x}0.3 = 0.375$

2.55

Let O_1 and O_2 denote the events of truck 1 (in one lane) and 2 (in the other lane) being overloaded, respectively, and let D denote the event of bridge damage. The given probabilities are $P(O_1) = P(O_2) = 0.1$; $P(D \mid O_1O_2) = 0.3$; $P(D \mid O_1O_2') = P(D \mid O_1'O_2) = 0.05$; $P(D \mid O_1'O_2') = 0.001$.

(a) P(D) can be determined by the theorem of total probability as

$P(D| O_1O_2)P(O_1O_2) + P(D| O_1O_2')P(O_1O_2') + P(D| O_1'O_2)P(O_1'O_2) + P(D| O_1'O_2')P(O_1'O_2')$

$= 0.3\times0.1\times0.1 + 0.05\times0.1\times(1 - 0.1) + 0.05\times(1 - 0.1)\times0.1 + 0.001\times(1 - 0.1)\times(1 - 0.1)$

$\cong$ **0.0128**

(b) $P(O_1 \cup O_2 \mid D) = 1 - P[\,(O_1 \cup O_2)' \mid D]$

(but De Morgan's rule says $(O_1 \cup O_2)' = O_1'O_2'$)

$= 1 - P(O_1'O_2' \mid D)$

$= 1 - P(D \mid O_1'O_2')P(O_1'O_2') / P(D)$

$= 1 - 0.001\times0.9\times0.9 / 0.01281$

$\cong$ **0.937**

(c) The first alternative reduces all those conditional probabilities in (a) by half, hence the P(D) also reduces by half to 0.0128 / 2 = **0.0064**. If one adopts the second alternative, P(D) becomes

$P(D| O_1O_2)P(O_1O_2) + P(D| O_1O_2')P(O_1O_2') + P(D| O_1'O_2)P(O_1'O_2) + P(D| O_1'O_2')P(O_1'O_2')$

$= 0.3\times0.06\times0.06 + 0.05\times0.06\times(1 - 0.06) + 0.05\times(1 - 0.06)\times0.06 + 0.001\times(1 - 0.06)\times(1 - 0.06)$

$\cong$ **0.0076**

Hence one should take the first alternative (strengthening the bridge) to minimize P(D).

2.56

(a) Let A = "presence of anomaly" and D = "detection of anomaly by geophysical techniques". We are given $P(D \mid A) = 0.5 \Rightarrow P(D' \mid A) = 1 - 0.5 = 0.5$, and also $P(D \mid A') = 0 \Rightarrow P(D' \mid A') = 1$. We need

$$P(A \mid D') = P(D' \mid A)P(A) / P(D') \text{ by Bayes' theorem, in which}$$

$$P(D') = P(D' \mid A)P(A) + P(D' \mid A')P(A') = 0.5\times0.3 + 1\times0.7 = 0.85, \text{ hence}$$

$$P(A \mid D') = 0.5\times0.3 / 0.85 \cong \mathbf{0.176}$$

(b)

(i) With the updated anomaly probability $P^*(A) = 0.15/0.85$ (thus $P^*(A') = 0.7/0.85$), and also a better detection probability $P(D \mid A) = 0.8$ (thus $P(D' \mid A) = 0.2$), the probability of having no anomaly, given no detection, is now

$$P^{**}(A') = P(A' \mid D') = P(D' \mid A')P^*(A') / P(D')$$
$$= 1\times(0.7/0.85) / [1\times(0.7/0.85) + 0.2\times(0.15/0.85)] \cong \mathbf{0.959}$$

(ii) Total probability of a safe foundation $= P(\text{safe} \mid A)P^{**}(A) + P(\text{safe} \mid A')P^{**}(A')$

$$= 0.80\times(1 - 0.959) + 0.9999\times0.959 \cong \mathbf{0.992}$$

(iii) A failure probability of p means that out of a large number (N) of similar systems, p×N of them are expected to fail. The total failure cost would be p×N×\$1000000, which when divided into N gives the "average" or expected cost per system = p×\$1000000. Hence we have the formula

$$\text{Expected loss} = (\text{probability of failure}) \times (\text{failure loss})$$
$$= (1 - 0.992)\times\$1000000 \quad \cong \$8300$$

If the system is known for sure to be anomaly free, however, it has only 1 – 0.9999 = 0.0001 chance of failure, in which case the expected loss would be 0.0001×\$1000000 = \$100. Hence, (\$8300 - \$100) = \$8200 in expected loss is saved when the site can be verified to be anomaly free

2.57

Given: $P(G) = 0.3$, $P(A) = 0.6$, $P(B) = 0.1$

$P(F|G) = 0.001$, $P(F|A) = 0.01$, $P(F|B) = 0.1$

(a) $P(F) = P(F|G)P(G) + P(F|A)P(A) + P(F|B)P(B)$

$= 0.001 \text{x} 0.3 + 0.01 \text{x} 0.6 + 0.1 \text{x} 0.1$

$= 0.0003 + 0.06 + 0.01$

$= 0.0703$

(b) T = passing the test

$P(T|G) = 0.9$, $P(T|A) = 0.7$, $P(T|B) = 0.2$

(i) $$P(G|T) = \frac{P(T|G)P(G)}{P(T|G)P(G) + P(T|A)P(A) + P(T|B)P(B)}$$

$$= \frac{0.9 \times 0.3}{0.9 \times 0.3 + 0.7 \times 0.6 + 0.2 \times 0.1} = \frac{0.27}{0.71} = 0.38$$

(ii) $P'(F) = P(F|G)P'(G) + P(F|A)P'(A) + P(F|B)P'(B)$

But $P'(G) = P(G|T) = 0.38$

$P'(A) = P(A|T) = 0.7 \text{x} 0.6 / 0.71 = 0.59$

$P'(B) = P(B|T) = 0.2 \text{x} 0.1 / 0.71 = 0.03$

Hence $P'(F) = 0.001 \text{x} 0.38 + 0.01 \text{x} 0.59 + 0.1 \text{x} 0.03 = 0.00928$

2.58

Given: $P(L) = 15/(15+4+1) = 15/20 = 0.75$
$P(M) = 4/20 = 0.20$
$P(H) = 1/20 = 0.05$
$P(P) = 0.2$ where P denotes poorly constructed

$P(D|PL) = 0.1$, $P(D|PM) = 0.5$, $P(D|PH) = 0.9$
$P(D|WL) = 0$, $P(D|WM) = 0.05$, $P(D|WH) = 0.2$

(a) $P(D|W) = P(D|WM)P(L)+P(D|WM)P(M)+ P(D|WH)P(H)$
$= 0\text{x}0.75 + 0.05\text{x}0.2 + 0.2\text{x}0.05$
$= 0.02$

(b) $P(D) = P(D|W)P(W)+ P(D|P)P(P)$
But $P(D|P) = P(D|PL)P(L)+P(D|PM)P(M)+ P(D|PH)P(H)$
$= 0.1\text{x}0.75 + 0.5\text{x}0.2 + 0.9\text{x}0.05 = 0.22$

Hence $P(D) = 0.02\text{x}0.8 + 0.22\text{x}0.2 = 0.06$
In other words, 6% of the buildings will be damaged.

(c) $$P(P|D) = \frac{P(D|P)P(P)}{P(D)} = \frac{0.22 \times 0.2}{0.06} = 0.733$$

2.59

(a) Note that any passenger must get off somewhere, so the sum over any row is one. Let O_i denote "Origin at station i" and D_i denote "Departure at station i", where i = 1,2,3,4.

$$P(D_3) = P(D_3 \mid O_1)P(O_1) + P(D_3 \mid O_2)P(O_2) + P(D_3 \mid O_2)P(O_2)$$
$$= 0.3\times0.25 + 0.3\times0.15 + 0.1\times0.25 = \mathbf{0.145}$$

(b) For a passenger boarding at Station 1, his/her trip length would be 4, 9 or 14 minutes for departures at stations 2, 3 or 4, respectively. These lengths have respective probabilities 0.1, 0.3 and 0.6, hence the expected trip length is

$$(4\times0.1 + 9\times0.3 + 14\times0.6) \text{ miles} = \mathbf{11.5\ miles}$$

(c) Let E = "trip exceeds 10 miles",

$$P(E) = P(E \mid O_1)P(O_1) + P(E \mid O_2)P(O_2) + P(E \mid O_3)P(O_3) + P(E \mid O_4)P(O_4)$$
$$= P(D_4 \mid O_1)\ P(O_1) + P(D_1 \mid O_2)\ P(O_2) + P(D_1 \cup D_2 \mid O_3)\ P(O_3) + P(D_2 \cup D_3 \mid O_4)\ P(O_4)$$
$$= 0.6\times0.25 + 0.6\times0.15 + (0.5 + 0.1)\times0.35 + (0.1 + 0.1)\times0.25 = \mathbf{0.5}$$

(d) $P(O_1 \mid D_3) = P(D_3 \mid O_1)\ P(O_1) / P(D_3) = 0.3\times0.25 / 0.145 \cong \mathbf{0.517}$

2.60

$P(A) = 0.6, \quad P(B) = 0.3, \quad P(C) = 0.1$

$P(D|A) = 0.02, \quad P(D|B) = 0.03, \quad P(D|C) = 0.04$

(a) $P(A|D) = \dfrac{P(D|A)P(A)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)}$

$$= \frac{0.02 \times 0.6}{0.02 \times 0.6 + 0.03 \times 0.3 + 0.04 \times 0.1} = 0.48$$

(b) $P(A \cup B|D) = P(A|D) + P(B|D)$

Since $P(B|D) = 0.03 \text{x} 0.3 / 0.025 = 0.36$

$P(A \cup B|D) = 0.48 + 0.36 = 0.84$

2.61

P(winging project A) = P(A) = 0.5
P(B) = 0.3
$P(A|B) = 0.5 \text{x} 0.5 = 0.25$
$P(B|A) = 0.5 \text{x} 0.3 = 0.15$

(a) $P(A \cup B) = P(A)+P(B)-P(A|B)P(B)$
$= 0.5 + 0.3 - 0.25 \text{x} 0.3$
$= 0.725$

(b) $$P(A\overline{B}|A \cup B) = \frac{P(A\overline{B}(A \cup B))}{P(A \cup B)} = \frac{P(A\overline{B})}{P(A \cup B)} = \frac{P(\overline{B}|A)P(A)}{0.725} = \frac{0.85 \times 0.5}{0.725} = 0.586$$

(c) $$P\left(A \middle| A\overline{B} \cup \overline{A}B\right) = \frac{P(A(A\overline{B} \cup \overline{A}B))}{P(A\overline{B} \cup \overline{A}B)} = \frac{P(A\overline{B})}{P(A\overline{B}) + P(\overline{A}B)} = \frac{P(\overline{B}|A)P(A)}{P(A \cup B) - P(AB)}$$
$$= \frac{0.85 \times 0.5}{0.725 - 0.15 \times 0.5} = 0.654$$

(d) $P(C|E) = 0.75$, $P(C|\overline{E}) = 0.5$

Where C denotes completion of project A on time

$P(C) = P(C|E)P(E)+P(C|\overline{E})P(\overline{E})$

$= 0.75 \text{x} 0.5 + 0.5 \text{x} 0.5 = 0.625$

(e) $$P(E|C) = \frac{P(C|E)P(E)}{P(C)} = \frac{0.75 \times 0.5}{0.625} = 0.6$$

2.62

Given: $P(A) = P(\text{poor aggregate}) = 0.2$

$P(W|A) = P(\text{poor workmanship}|\text{poor aggregate}) = 0.3$

$P(A|W) = 015$

(a) $P(W) = P(A)P(W|A) / P(A|W)$

$= 0.2\text{x}0.3 / 0.15 = 0.4$

(b) $P(A \cup W) = P(A)+P(W)- P(W|A)P(A) = 0.2 + 0.4 - 0.3\text{x}0.2 = 0.54$

(c) $P(A\overline{W} \cup \overline{A}W) = P(A\cup W)-P(AW) = 0.54 - 0.3\text{x}0.2 = 0.48$

(d) Given $P(\text{defective concrete } A\overline{W}) = P(D \mid A\overline{W}) = 0.15$

$P(D|\overline{A}W) = 0.2$, $P(D|AW) = 0.8$, $P(D|\overline{A}\,\overline{W}) = 0.05$

$P(D) = P(D \mid A\overline{W})\, P(A\overline{W}) + P(D|\overline{A}W)\, P(\overline{A}W) +$

$P(D|AW)\, P(AW) + P(D|\overline{A}\,\overline{W})\, P(\overline{A}\,\overline{W})$

But $P(A\overline{W}) = 0.7\text{x}0.2 = 0.14$, $P(\overline{A}W) = 0.85\text{x}0.4 = 0.34$

$P(AW) = 0.3\text{x}0.2 = 0.06$, $P(\overline{A}\,\overline{W}) = 1\text{-}0.14\text{-}0.34\text{-}0.06 = 0.46$

Hence $P(D) = 0.15\text{x}0.14 + 0.2\text{x}0.34 + 0.8\text{x}0.06 + 0.05\text{x}0.46 = 0.192$

(e) $$P(AW|D) = \frac{P(D|AW)P(AW)}{P(D)} = \frac{0.048}{0.192} = 0.25$$

2.63

(a) Given probabilities: $P(A \mid S) = 0.2$, $P(A \mid L) = 0.6$, $P(A \mid N) = 0.05$. Since S, L, N are mutually exclusive and collectively exhaustive, $P(N) = 0.7$ and $P(S) = 2P(L)$ lead to $P(S) = 0.2$, $P(L) = 0.1$. Hence

$$P(A) = P(A \mid L)P(L) + P(A \mid S)P(S) + P(A \mid N)P(N)$$
$$= 0.6\times0.1 + 0.2\times0.2 + 0.05\times0.7 = \mathbf{0.135}$$

(b) Let E denote "weak material Encountered by boring"; we are given $P(E \mid L) = 0.8$, $P(E \mid S) = 0.3$, $P(E \mid N) = 0$.

(i) The total probability of NOT encountering any weak material,

$$P(E') = P(E' \mid L)P(L) + P(E' \mid S)P(S) + P(E' \mid N)P(N)$$
$$= (1 - 0.8)\times0.1 + (1 - 0.3)\times0.2 + 1\times0.7 = 0.86,$$

Hence, applying Bayes' theorem,

$$P(L \mid E') = P(E' \mid L)P(L) / P(E') = [1 - P(E \mid L)]P(L) / [1 - P(E)]$$
$$= (1 - 0.8)\times0.1 / 0.86 \cong \mathbf{0.023}$$

This is $P^*(L)$, the updated probability of a large weak zone in the light of new information.

(ii) Similar to (i), $P(S \mid E') = P(E' \mid S)P(S) / P(E') = (1 - 0.3)\times0.2 / 0.86 \cong \mathbf{0.163}$

This is $P^*(S)$, the updated probability of a small weak zone.

(iii) The "updated" probability of A is calculated using $P^*(S)$ and $P^*(L)$,

$$P^*(A) = P(A \mid L)P^*(L) + P(A \mid S)P^*(S) + P(A \mid N)P^*(N)$$

Note that $P(A \mid L)$, $P(A \mid N)$, $P(A \mid S)$ all remains their original values as the settlement-soil condition dependence is not affected by the borings, hence

$$P^*(A) = 0.6\times0.0233 + 0.2\times0.163 + 0.05\times(1 - 0.0233 - 0.163)$$
$$\cong \mathbf{0.087}$$

2.64

Let A, B denote occurrence of earthquake in the respective cities, and D denote damage to dam.

(a) $P(A \cup B) = P(A) + P(B) - P(AB)$

$= P(A) + P(B) - P(A)P(B)$ since A and B are independent events

$= 0.01 + 0.02 - 0.01 \times 0.02 = \mathbf{0.0298}$

(b) Given: $P(D|A\overline{B}) = 0.3$, $P(D|\overline{A}B) = 0.1$, $P(D|AB) = 0.5$, want: $P(D)$

Theorem of total probability gives

$$P(D) = P(D|A\overline{B})P(A\overline{B}) + P(D|\overline{A}B)P(\overline{A}B) + P(D|AB)P(AB)$$

(note: $P(D|\overline{A}\ \overline{B}) = 0$ so it is not included)

$$= P(D|A\overline{B})P(A)P(\overline{B}) + P(D|\overline{A}B)P(\overline{A})P(B) + P(D|AB)P(A)P(B)$$

$$= 0.3 \times 0.01 \times 0.98 + 0.1 \times 0.99 \times 0.02 + 0.5 \times 0.01 \times 0.02 = \mathbf{0.00502}$$

(c) Now that B has dropped out of the picture,

$$P(D) = P(D|A)P(A) + P(D|\overline{A})P(\overline{A})$$

$$= 0.4 \times 0.01 + 0 = 0.004 < 0.00502,$$

the new site is preferred since the yearly probability of incurring damages is about 20% lower.

(d) Let Di denote "damage due to event i", where i = E(arthquake), L(andslide), S(oil being poor).

$$P(D_E \cup D_L \cup D_S) = 1 - P[\overline{D_E \cup D_L \cup D_S}]$$

$= 1 - P(\overline{D}_E \overline{D}_L \overline{D}_S)$ by De Morgan's rule,

$= 1 - P(\overline{D}_E)P(\overline{D}_L)P(\overline{D}_S)$ due to independence,

$= 1 - (1 - 0.004)(1 - 0.002)(1 - 0.001)$

$\cong 0.006986 > 0.00502$,

one should move the site in this case, as the new site has a higher risk.

2.65

A, B, C denote company A, B, C discover oil respectively

$P(A) = 0.4$, $P(B) = 0.6$, $P(C) = 0.2$

$P(A|B) = 1.2 \times 0.4 = 0.48$

C is statistically independent of B or A

(a) $P(A \cup B \cup C) = 1 - P(\overline{A}\ \overline{B}\ \overline{C}) = 1 - P(\overline{A}\ \overline{B})P(\overline{C})$

But $P(\overline{A}\ \overline{B}) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A|B)P(B)$

$= 1 - 0.4 - 0.6 + 0.48 \times 0.6 = 0.288$

Hence $P(A \cup B \cup C) = 1 - 0.288 \times 0.8 = 0.77$

(b) $$P(C|A \cup B \cup C) = \frac{P(C(A \cup B \cup C))}{P(A \cup B \cup C)} = \frac{P(C)}{0.77} = 0.26$$

(c) $P(A\overline{B}\ \overline{C} \cup \overline{A}B\overline{C} \cup \overline{A}\ \overline{B}C) = P(A\overline{B}\ \overline{C}) + P(\overline{A}B\overline{C}) + P(\overline{A}\ \overline{B}C)$

But $$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{0.48 \times 0.6}{0.4} = 0.72$$

Hence, $P(A\overline{B}\ \overline{C} \cup \overline{A}B\overline{C} \cup \overline{A}\ \overline{B}C)$

$= 0.28 \times 0.4 \times 0.8 + 0.52 \times 0.6 \times 0.8 + 0.288 \times 0.2 = 0.397$

2.66

(a) Partition the sample space into AG, AB, PG, PB. We are given

(i) P(D | AG) = 0, P(D | AB) = 0.5; (ii) P(D | PG) = 0.3, P(D | PB) = 0.9; (iii) P(A) = 0.3, P(P) = 0.7;

(iv) P(B | A) = 0.2 ⇒ P(G | A) = 0.8; P(B | P) = 0.1 ⇒ P(G | P) = 0.9;

(v) G and B are mutually exclusive and collectively exhaustive.

Theorem of total probability gives

$P(D) = P(D|AG)P(AG) + P(D \mid AB)P(AB) + P(D \mid PG)P(PG) + P(D \mid PB \mid PB)$

$= P(D|AG)P(G|A)P(A) + P(D|AB)P(B|A)P(A) + P(D| PG)P(G|P)P(P) + P(D|PB)P(B|P)P(P)$

$= 0\times0.8\times0.3 + 0.5\times0.2\times0.3 + 0.3\times0.9\times0.7 + 0.9\times0.1\times0.7$

$=$ **0.282**

(b) Bad weather can happen during AM or PM hours, hence we may rewrite B as the union of two mutually exclusive events, $B = BA\cup BP$, thus

$P(B \mid D) = P(BA \mid D) + P(BP \mid D)$

$= P(BAD) / P(D) + P(BPD) / P(D)$

$= [\ P(D \mid AB)P(B \mid A)P(A) + P(D \mid PB)P(B \mid P)P(P)\ ] / P(D)$

$= (0.5\times0.2\times0.3 + 0.9\times0.1\times0.7) / 0.282 \cong$ **0.330**

(c) Of all morning flights, 20% will encounter bad weather. When they do, there is 50% chance of having a delay. Hence 20%×0.5 = **10%** of morning flights will be delayed (note that good weather guarantees take-off without delay for morning flights). Or, in more formal notation,

$P(D \mid A) = P(DA) / P(A) = [P(D \mid AB)P(AB) + P(D \mid AG)P(AG)] / P(A)$

(where $DA = D(A\cup B) = DAB\cup DAG$ was substituted)

(but $P(D \mid AG) = 0$)

$= P(D \mid AB)P(AB) / P(A)$

$= P(D \mid AB)P(B \mid A)$

$= 0.5\times0.2 = 0.1$ (i.e. 10%)

2.67

$P(M) = 0.05$ where M = malfunction of machinery
$P(W) = 0.08$ where W = carelessness of workers

$P(D \mid M\overline{W}) = 0.1, P(D \mid \overline{M}W) = 0.2, P(D \mid MW) = 0.8$

Assume M and W are statistically independent

(a) $P(D) = P(D \mid M\overline{W})\, P(M\overline{W}) + P(D \mid \overline{M}W)\, P(\overline{M}W)$

$+P(D \mid MW)\, P(MW) + P(D \mid \overline{M}\,\overline{W})\, P(\overline{M}\,\overline{W})$

$= 0.1x0.05x0.92 + 0.2x0.95x0.08 + 0.8x0.05x0.08 + 0 = 0.023$

(b) $P(W \mid D) = P(W\overline{M} \mid D)\, P(WM \mid D)$

$= (0.0152+0.0032) / 0.023 = 0.8$

2.68

Let A, B denote the events that these respective wells observed contaminants, and L denote that there was indeed leakage. Given: P (A | L) = 0.8, P (B | L) = 0.9, P (L) = 0.7, also these wells never false-alarm, i.e. $P(\overline{A} \mid \overline{L}) = 1$, $P(\overline{B} \mid \overline{L}) = 1$.

(a) Given $\overline{A}$, Bayes' theorem yields

$$P(L \mid \overline{A}) = P(\overline{A} \mid L)P(L) / P(\overline{A})$$

where $P(\overline{A}) = P(\overline{A} \mid L)P(L) + P(\overline{A} \mid \overline{L})P(\overline{L}) = 0.2\times0.7 + 1\times0.3 = 0.44$, hence

$$P(L \mid \overline{A}) = 0.2\times0.7 / 0.44 \cong \mathbf{0.318}$$

(b)

(i) It will be convenient to first restrict our attention to within L, the only region in which A or B can happen. In here, $P_L(A) = 0.8$ and $P_L(B) = 0.9$, hence

$$P(A\cup B \mid L) = P_L(A) + P_L(B) - P_L(A)P_L(B)$$
$$= 0.8 + 0.9 - 0.8\times0.9 = 0.98,$$

but this is only a relative probability with respect to L, hence the true probability

$$P(A\cup B) = P(A\cup B \mid L)\times P(L) = 0.98\times0.7 = \mathbf{0.686}$$

(ii) $\overline{A}\,\overline{B}$ is the event "both wells observed no contaminant". Note that, even in L, there is a small piece of A'B', with probability

$$P(\overline{A}\,\overline{B} \mid L)P(L) = (1-0.8)\times(1-0.9)\times0.7 = 0.014,$$

adding this to

$$P(\overline{A}\,\overline{B} \mid \overline{L}) = 0.3 \text{ (since both wells never false alarm)}$$

gives the total probability

$$P(\overline{A}\,\overline{B}) = 0.014 + 0.3 = 0.314, \text{ hence}$$
$$P(\overline{L} \mid \overline{A}\,\overline{B})$$
$$= P(\overline{A}\,\overline{B} \mid \overline{L})P(\overline{L}) / P(\overline{A}\,\overline{B})$$
$$= 1\times0.3 / 0.314 \cong \mathbf{0.955}$$

(a) In terms of the ability to confirm actual existence of leakage, P(B | L) = 0.9 > P(A | L) = 0.8, i.e. B is better. Also, in terms of the ability to infer safety, we have

$P(L \mid \overline{A}) \cong 0.318$ as calculated in part (a), while a similar calculation gives

$$P(L \mid \overline{B}) = P(\overline{B} \mid L)P(L) / P(\overline{B}) = 0.1\times0.7 / (0.1\times0.7 + 1\times0.3) \cong 0.189$$

Hence B gives a better assurance of non-leakage when it observes nothing. In short, **B** is better.

2.69

Let A and B denote having excessive amounts of the named pollutant, and H denote having a health problem due contaminated water. Let us partition the sample space into AB, $\overline{A}$B, A$\overline{B}$, $\overline{A}$ $\overline{B}$. The total probability of having a health problem is

P(H) = P(H | AB)P(AB) + P(H | $\overline{A}$ B)P($\overline{A}$ B) + P(H | A $\overline{B}$)P(A $\overline{B}$) + P(H | $\overline{A}$ $\overline{B}$)P($\overline{A}$ $\overline{B}$)

where P(H | AB) = P(H | A $\overline{B}$) = 1 since existence of pollutant A causes health problem for sure;

P(H | $\overline{A}$ B) = 0.2 since existence of pollutant B affects 20% of all people (even when A is absent)

P(H | $\overline{A}$ $\overline{B}$) = 0 since there is no cause for any health problem; also

P(AB) = P(B | A)P(A) = 0.5×0.1 = 0.05;

P($\overline{A}$ B) = P(B) – P(AB) = 0.2 – 0.05 = 0.15 (this is clear from a Venn diagram);

P(A $\overline{B}$) = P($\overline{B}$ | A)P(A) = (1 – 0.5)×0.1 = 0.05;

P(H) = 1×0.05 + 0.2×0.15 + 1×0.05 = **0.13**

2.70

(a) P(winning at most one job)

= P(winning zero job) + P(winning exactly one job)

= P($\bar{H}_1\bar{H}_2\bar{H}_3\bar{B}_1\bar{B}_2$) + P($H_1\bar{H}_2\bar{H}_3\bar{B}_1\bar{B}_2$) + P($\bar{H}_1 H_2\bar{H}_3\bar{B}_1\bar{B}_2$) + P($\bar{H}_1\bar{H}_2 H_3\bar{B}_1\bar{B}_2$) +

P($\bar{H}_1\bar{H}_2\bar{H}_3 B_1\bar{B}_2$) + P($\bar{H}_1\bar{H}_2\bar{H}_3\bar{B}_1 B_2$)

= $(0.4)^6 + (0.4)^4(0.6)$x5

= 0.0041 + 0.0768

= 0.081

(b) P(win at least 2 jobs)

= 1 – P(win at most one job)

= 1 – 0.081

= 0.919

(c) P(win exactly 1 highway job). P(win zero building jobs)

= P[($\bar{H}_1\bar{H}_2 H_3$)+P($H_1\bar{H}_2\bar{H}_3$)+P($\bar{H}_1 H_2\bar{H}_3$)]. P($\bar{B}_1\bar{B}_2$)

= $(0.4)^2 + (0.6)^4$x3x$(0.4)^2$

= 0.0461

3.1

Total time T = A + B, which ranges from (3 + 4 = 7) to (5 + 6 = 11).
Divide the sample space into A = 3, A = 4, and A = 5 (m.e. & c.e. events)

$$\Rightarrow P(T = 7) = \sum_{n=3,4,5} P(T = 7 \mid A = n)P(A = n)$$

$$= \sum_{n=3,4,5} P(B = 7 - n)P(A = n)$$

$$= P(B = 4)P(A = 3) = 0.2\times0.3$$

$$= \mathbf{0.06}$$

Similarly

$$P(T = 8) = P(B = 5)P(A = 3) + P(B = 4)P(A = 4)$$

$$= 0.6\times0.3 = 0.2\times0.5$$

$$= \mathbf{0.28}$$

$$P(T = 9) = P(B = 6)P(A = 3) + P(B = 5)P(A = 4) + P(B = 4)P(A = 5)$$

$$= 0.2\times0.3 + 0.6\times0.5 + 0.2\times0.2$$

$$= \mathbf{0.4}$$

$$P(T = 10) = P(B = 6)P(A = 4) + P(B = 5)P(A = 5)$$

$$= 0.2\times0.5 + 0.6\times0.2$$

$$= \mathbf{0.22}$$

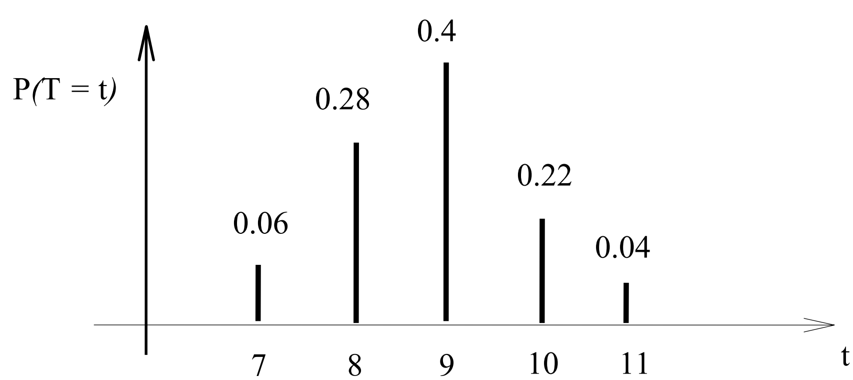
$$P(T = 11) = P(B = 6)P(A = 5)$$

$$= 0.2\times0.2$$

$$= \mathbf{0.04}$$

Check: 0.06 + 0.28 + 0.4 + 0.22 + 0.04 = 1.

The PMF is plotted as follows:



3.2

Let X be the profit (in \$1000) from the construction job.

(a) P(lose money) = $P(X < 0)$

= Area under the PDF where x is negative

= $0.02 \times 10 =$ **0.2**

(b) Given event is $X > 0$ (i.e. money was made), hence the conditional probability,

$P(X > 40 \mid X > 0)$

$= P(X > 40 \cap X > 0) \div P(X > 0)$

$= P(X > 40) \div P(X > 0)$

Let's first calculate $P(X > 40)$: comparing similar triangles formed by the PDF and the x-axis (with vertical edges at $x = 10$ and at $x = 40$, respectively), we see that

$P(X > 40)$ = Area of smaller triangle

$$= \left(\frac{70-40}{70-10}\right)^2 \times \text{Area of larger triangle}$$

$= 0.5^2\ [(70 - 10)\times(0.02) \div 2] = 0.015$

Hence the required probability $P(X > 40 \mid X > 0)$ is

$0.015 \div [10\times0.02 + (70 - 10)\times(0.02) \div 2]$

$= 0.015 \div 0.08 =$ **0.1875**

3.3

(a) Applying the normalization condition $\int_{-\infty}^{\infty} f_X(x)dx = 1$

$$\Rightarrow \int_0^6 c(x - \frac{x^2}{6})dx = 1 \Rightarrow c\left[\frac{x^2}{2} - \frac{x^3}{18}\right]_0^6 = 1$$

$$\Rightarrow c = \frac{18}{9 \times 36 - 6^3} = \mathbf{1/6}$$

(b) To avoid repeating integration, let's work with the CDF of X, which is

$$F_X(x) = \frac{1}{6}\left[\frac{x^2}{2} - \frac{x^3}{18}\right]$$
$$= (9x^2 - x^3)/108 \qquad \text{(for } x \text{ between 0 and 6 only)}$$

Since overflow already occurred, the given event is X > 4 (cms), hence the conditional probability

$$P(X < 5 \mid X > 4) = P(X < 5 \text{ and } X > 4) / P(X > 4)$$
$$= P(4 < X < 5) / [1 - P(X \le 4)]$$
$$= [F_X(5) - F_X(4)] / [1 - F_X(4)]$$
$$= [(9\times5^2 - 5^3) - (9\times4^2 - 4^3)]/ [108 - (9\times4^2 - 4^3)]$$
$$= (100 - 80) / (108 - 80) = 20/28 = 5/7 \cong \mathbf{0.714}$$

(c) Let C denote "completion of pipe replacement by the next storm", where P(C) = 0.6. If C indeed occurs, overflow means X > 5, whereas if C did not occur then overflow would correspond to X > 4. Hence the total probability of overflow is (with ' denoting compliment)

$$P(\text{overflow}) = P(\text{overflow} \mid C)P(C) + P(\text{overflow} \mid C')P(C')$$

$$= P(X > 5)\times0.6 + P(X > 4)\times(1 - 0.6)$$
$$= [1 - F_X(5)]\times0.6 + [1 - F_X(4)]\times0.4$$
$$= (1 - 100/108)\times0.6 + (1 - 80/108)\times0.4 \cong \mathbf{0.148}$$

3.4

(a) The median of X, x_m, is obtained by solving the equation that defines x_m,

$$P(X \le x_m) = 0.5$$
$$\Rightarrow F_X(x_m) = 0.5$$
$$\Rightarrow 1 - (10 \div x_m)^4 = 0.5$$
$$\Rightarrow x_m \approx \mathbf{11.9} \text{ (inches)}$$

(b) First, obtain the PDF of X:

$$f_X(x) = dF_X/dx$$
$$= 4\times10^4 x^{-5} \quad \text{for } x \ge 10\text{, and zero elsewhere.}$$

Hence the expected amount of snowfall in a severe snow storm is

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)dx$$
$$= \int_{10}^{\infty} 4\times10^4 x^{-4} dx$$
$$= -\frac{4}{3}10^4\left[x^{-3}\right]_{10}^{\infty}$$
$$= \frac{4}{3}\times 10\ 4 \approx \mathbf{13.3} \text{ (inches)}$$

(c) P(<u>disastrous</u> severe snowstorm) = $P(X > 15)$

$$= 1 - P(X \le 15)$$
$$= 1 - [1 - (10/15)^4] = (2/3)^4 \approx \mathbf{0.2}$$

∴ About **20%** of snow storms are disastrous.

(d) Let N denote "No disastrous snow storm in a given year". Also, let E_0, E_1, E_2 denote the respective events of experiencing 0,1,2 severe snow storms in a year. In each event, the (conditional) probability of N can be computed:

$P(N \mid E_0) = 1$
(if there's no severe snow storm to begin with, definitely there won't be any disastrous one);
$P(N \mid E_1) = P(X \le 15) = 1 - (10/15)^4 = 1 - (2/3)^4$;
$P(N \mid E_2) = [P(X \le 15)]^2 = [1 - (2/3)^4]^2$ (statistical independence between storms);

Noting that E_0, E_1 and E_2 are m.e. and c.e., the total probability of N can be computed:

$$P(N) = P(N \mid E_0)P(E_0) + P(N \mid E_1)P(E_1) + P(N \mid E_2)P(E_2)$$
$$= 1\times0.5 + [1 - (2/3)^4]\times0.4 + [1 - (2/3)^4]^2\times0.1 \approx \mathbf{0.885}$$

3.5

Let F be the final cost (a random variable), and C be the estimated cost (a constant), hence

$$X = F / C$$

is a random variable.

(a) To satisfy the normalization condition,

$$\int_1^a \frac{3}{x^2}dx = \left[\frac{-3}{x}\right]_1^a = 3 - \frac{3}{a} = 1,$$

hence

$$a = 3/2 = \mathbf{1.5}$$

(b) The given event is "F exceeds C by more than 25%", i.e. "$F > 1.25C$", i.e. "$F / C > 1.25$", whose probability is

$$\begin{aligned} P(X > 1.25) &= \int_{1.25}^{\infty} f_X(x)dx \\ &= \int_{1.25}^{1.5} \frac{3}{x^2}dx = \left[\frac{-3}{x}\right]_{1.25}^{1.5} \\ &= -2 - (-2.4) = \mathbf{0.4} \end{aligned}$$

(c) The mean,

$$E(X) = \int_1^{1.5} x\frac{3}{x^2}dx = [3\ln x]_1^{1.5} \cong \mathbf{1.216},$$

while

$$E(X^2) = \int_1^{1.5} x^2\frac{3}{x^2}dx = 3(1.5 - 1) = 1.5,$$

with these, we can determine the variance

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= 1.5 - 1.216395324^2 = 0.020382415 \end{aligned}$$

$$\therefore \sigma_X = \sqrt{0.020382415} \cong \mathbf{0.143}$$

3.6

(a) $E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^2 x\frac{x}{8}dx + \int_2^8 x\frac{2}{x^2}dx = \left.\frac{x^3}{24}\right|_0^2 + 2\ln x\Big|_2^8$ = 1/3 + 2(ln 8 – ln 2) ≅ **3.106**

(b) $P(X < 3 \mid X > 2) = \dfrac{P(X < 3 \text{ and } X > 2)}{P(X > 2)} = \dfrac{P(2 < X < 3)}{P(X > 2)} = \dfrac{\int_2^3 \frac{2}{x^2}dx}{1 - \int_0^2 \frac{x}{8}dx} = \dfrac{-\left.\frac{2}{x}\right|_2^3}{1 - \left[\frac{x^2}{16}\right]_0^2} =$

$\dfrac{1 - 2/3}{1 - 1/4}$ = **4/9**

3.7

(a) The mean and median of X are **13.3** lb/ft^2 and **11.9** lb/ft^2, respectively (as done in Problem 3-3-3).

(b) The event "roof failure in a given year" means that the annual maximum snow load exceeds the design value, i.e. X > 30, whose probability is

$$\begin{aligned} P(X > 30) &= 1 - P(X \le 30) = 1 - F_X(30) \\ &= 1 - [1 - (10/30)^4] \\ &= (1/3)^4 = 1/81 \cong \mathbf{0.0123} \equiv p \end{aligned}$$

Now for the first failure to occur in the 5th year, there must be four years of non-failure followed by one failure, and the probability of such an event is

$$(1 - p)^4 p = [1 - (3/4)^4]^4 \times (1/3)^4 \cong \mathbf{0.0117}$$

(b) Among the next 10 years, let Y count the number of years in which failure occurs. Y follows a binomial distribution with n = 10 and p = 1/81, hence the desired probability is

$$\begin{aligned} P(Y < 2) &= P(Y = 0) + P(Y = 1) \\ &= (1 - p)^n + n(1 - p)^{n-1} p \\ &= (80/81)^{10} + 10 \times (80/81)^9 \times (1/81) \\ &\cong \mathbf{0.994} \end{aligned}$$

3.8

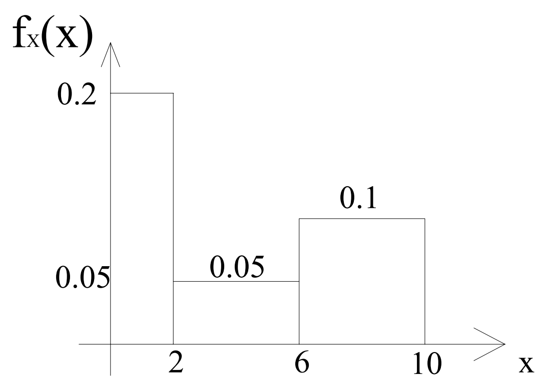
(a) Note that we are given the CDF, $F_X(x) = P(X \leq x)$ and it is a continuous function. Hence

$$P(2 \leq X \leq 8) = F_X(8) - F_X(2) = 0.8 - 0.4 = \mathbf{0.4}$$

(b) The median is defined by the particular x value where $F(x) = 0.5$, which occurs somewhere along $x = 2$ and $x = 6$; precisely, the constant slope there gives

$$(0.5 - 0.4) / (x_m - 2) = (0.6 - 0.4) / (6 - 2)$$
$$\Rightarrow x_m = 2 + 2 = \mathbf{4}$$

(c) The PDF is the derivative (i.e. slope) of the CDF, hence it is the following piecewise constant function:



(d) From the CDF, it is seen that one may view X as a discrete R.V. which takes on the values 1, 4, 8 with respective probabilities 0.4, 0.2, 0.4 (recall probability = area of each strip) for calculating E(X). Hence

$$E(X) = 1\times0.4 + 4\times0.2 + 8\times0.4 = \mathbf{4.4}$$

(integration would give the same result)

3.9

(a) Differentiating the CDF gives the PDF,

$$f_S(s) = \begin{cases} 0 & \text{for } s \le 0 \\ -\dfrac{s^2}{288} + \dfrac{s}{24} & \text{for } 0 < s \le 12 \\ 0 & \text{for } s > 12 \end{cases}$$

The mode $\tilde{s}$ is where f_S has a maximum, hence setting its derivative to

$$f_S'(\tilde{s}) = 0$$

$$\Rightarrow -\ \tilde{s}/144 + 1/24 = 0$$

$$\Rightarrow \text{the mode } \tilde{s} = \mathbf{6}.$$

The mean of S,

$$E(S) = \int_{s=-\infty}^{\infty} sf(s)ds = \int_{0}^{12} \left(\frac{-s^3}{288} + \frac{s^2}{24}\right)ds$$

$$= \frac{-(12^4)}{4 \times 288} + \frac{12^3}{3 \times 24} = = -\ 18 + 24 = \mathbf{6}$$

(b) Dividing the sample space into two regions R = 10 and R = 13, the total probability of failure is

$$P(S > R) = P(S > R \mid R = 10)P(R = 10) + P(S > R)P(R = 13)$$

$$= [1 - F_S(10)]\times 0.7 + [1 - F_S(13)]\times 0.3$$

$$= [1 - (-10^3 / 864 + 10^2 / 48)]\times 0.7 + [1 - 1]\times 0.3$$

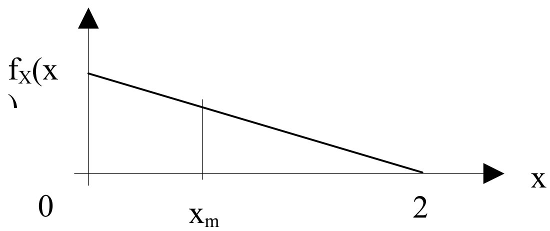
$$= 0.074074074\times 0.7 \cong \mathbf{0.0519}$$

3.10

(a) The only region where $f_X(x)$ is non-zero is between x = 0 and x = 20, where

$$f_X(x) = F'_X(x) = -0.005x + 0.1$$

which is a straight line segment decreasing from y = 0.1 (at x = 0) to y = 0 (at x = 20)



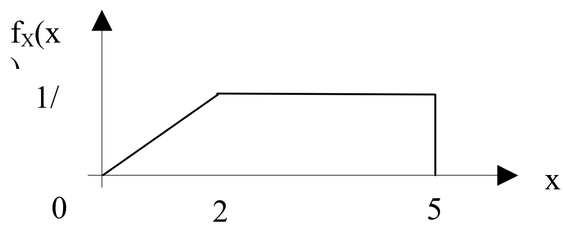
(b) The median divides the triangular area under f_X into two equal parts, hence, comparing the two similar triangles which have area ratio 2:1, one must have

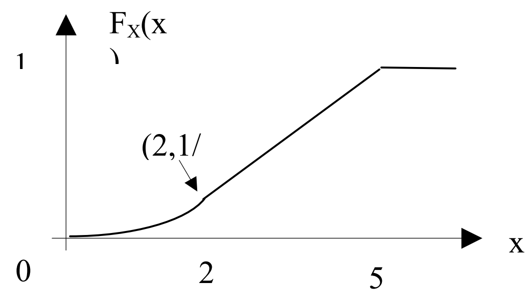
$$(20 - x_m) / 20 = (1 / 2)^{0.5}$$

$$\Rightarrow x_m = 20(1 - 0.5^{0.5}) \cong \mathbf{5.858}$$

3.11

(a) Integration of the PDF gives $F_X(x) = \begin{cases} 0 & x < 0 \\ x^2/16 & 0 < x \le 2 \\ x/4 - 1/4 & 2 < x \le 5 \\ 1 & x > 5 \end{cases}$; sketches follow:





(b) $E(X) = \int_0^2 x(x/8)dx + \int_2^5 x(1/4)dx = \left[\frac{x^3}{24}\right]_0^2 + \left[\frac{x^2}{8}\right]_2^5 = 71/24 \cong$ **2.96** (mm)

(c) $P(X < 4) = 1 - P(X > 4)$
Where $P(X > 4)$ is easily read off from the PDF as the area $(5 - 4)(1/4) = 1/4$, hence

$$P(X < 4) = 1 - 1/4 = 3/4 = \mathbf{0.75}$$

(d) A vertical line drawn at the median x_m would divide the unit area under f_X into two equal halves; the right hand rectangle having area

$$0.5 = (5 - x_m)(1/4)$$
$$\Rightarrow x_m = 5 - 4(0.5) = \mathbf{3} \text{ (mm)}$$

(e) Since each of the four cracks has $p = 0.25$ probability of exceeding 4mm (as calculated in (c)), only one of them exceeding 4mm has the binomial probability (where $n = 4$, $p = 0.25$)

$4\times0.75^3\times0.25 \cong$ **0.422**

3.12

(a) For $3 \le x \le 6$, $F_X(x) = \int_3^x \frac{24}{t^3} dt = 4/3 - 12/x^2$; elsewhere F_X is either 0 (for x < 3) or 1 (for x > 6)

F_X is a parabola going from x = 3 to x = 6, followed by a horizontal line when plotted

(b) $E(X) = \int_3^6 x(\frac{24}{x^3})dx$ = 24(1/3 – 1/6) = **4** (tons)

(c) First calculated the variance, $Var(X) = E(X^2) - [E(X)]^2 = \int_3^6 x^2(\frac{24}{x^3})dx - 16 = 24(\ln 6 - \ln 3) = 24 \ln 2 - 16$

$$\Rightarrow \text{c.o.v.} = \frac{\sigma}{\mu} = \frac{\sqrt{24 \ln 2 - 16}}{4} \cong \mathbf{19.9\%}$$

(d) When the load X exceeds 5.5 tons, the roof will collapse, hence

P(roof collapse) = P(X > 5.5)

= 1 – P(X ≤ 5.5)

$= 1 - F_X(5.5) = 1 - (4/3 - 12/5.5^2) \cong$ **0.063**

3.13

(a) Since the return period, τ, is 200 years, this means the yearly probability of exceedance (i.e. encountering waves exceeding the design value) is

$$p = 1 / \tau = 1 / (200 \text{ years}) = \mathbf{0.005} \text{ (probability per year)}$$

(b) For each year, the probability of non-exceedance is $1 - 0.005 = 0.995$, while the intended lifetime is 30 years. Hence non-exceedance during the whole lifetime has the binomial probability,

$$0.995^{30} \cong \mathbf{0.860}$$

3.14

Design life = 50 years

Mean rate of high intensity earthquake = 1/100 = 0.01/yr

P(no damage within 50 years) = 0.99

Let P = probability of damage under a single earthquake

(a) Using a Bernoulli Sequence Model for occurrence of high intensity earthquakes,

P(quake each year) = 1/100 = 0.01

P(damage each year) = 0.01p

P(no damage in 50 years) = $(1 - 0.01p)^{50} \equiv 0.99$

Hence, $1 - 0.01p = (0.99)^{1/50} = (0.99)^{0.02} = 0.9998$

or $p = 0.02$

(b) Assuming a Poisson process for the quake occurrence,

Mean rate of damaging earthquake = 0.01p = 0.01x0.02 = 0.0002/yr

P(damage in 20 years) = 1 – P(no damage in 20 years)

= 1 – P(no occurrence of damaging quakes in 20 years)

$= 1 - e^{-0.0002x20}$

$= 1 - e^{-0.004}$

= 0.004

3.15

Failure rate = ν = 1/5000 = 0.0002 per hour

(a) P(no failure between inspection) = P(N=0 in 2500 hr)

$= \dfrac{(0.0002 \times 2500)^0 e^{-0.0002\times 2500}}{0!}$ = $e^{-0.5}$ = 0.607

Hence P(failure between inspection) = 1 – 0.607 = 0.393

(b) P(at most two failed aircrafts among 10)

= P(M=0)-P(M=1)-P(M=2)

$= \dfrac{10!}{0!\times 10!}(0.393)^0(0.607)^{10} - \dfrac{10!}{1!\times 9!}(0.393)(0.607)^9 - \dfrac{10!}{2!\times 8!}(0.393)^2(0.607)^8$

= 0.0068 – 0.044 – 0.1281

= 0.179

(c) Let t be the revised inspection/maintenance interval

P(failure between inspection)

= 1-P(no failure between inspection)

= 1-$e^{-0.0002t}$ = 0.05

-0.0002t = ln 0.95

and t = 256.5 $\rightarrow$ 257 hours

3.16

(a) Let C be the number of microscopic cracks along a 20-feet beam. C has a Poisson distribution with mean rate $\nu_C = 1/40$ (number per foot), and length of observation t = 20 feet, hence the parameter $\lambda = (1/40)(20) = 0.5$, thus

$$P(C = 2) = e^{-0.5} (0.5^2 / 2!) \cong \mathbf{0.076}$$

(b) Let S denote the number of slag inclusions along a 20-feet beam. S has a Poisson distribution with mean rate $\nu_S = 1/25$ (number per foot), and length of observation t = 20 feet, hence the parameter $\lambda = (1/25)(20) = 0.8$, thus

$$1 - P(S = 0) = 1 - e^{-0.8} \cong \mathbf{0.551}$$

(c) Let X be the total number of flaws along a 20-feet beam. Along 1000 feet (say) of such a beam, one can expect (1/40)(1000) = 25 cracks and (1/25)(1000) = 40 slag inclusions. Hence the mean rate of flaw would be $\nu = (25 + 40)/1000$ or simply $(1/40) + (1/25) = 0.065$ flaws per foot, which is multiplied to the length of observation, t = 20 feet to get the parameter $\lambda = 1.3$. Hence

$$P(X > 2) = 1 - P(X \le 2) = 1 - e^{-1.3} (1 + 1.3 + 1.3^2 / 2!) = 1 - 0.857112489$$
$$\cong \mathbf{0.143}$$

(d) Thinking of beam rejection as "success", the total number (N) of beams rejected among 4 would follow a binomial distribution with n = 4 and p = answer in part (c). Thus

$$P(N = 1) = \binom{4}{1} 4 \times 0.142887511 \times 0.857^3 \cong \mathbf{0.360}$$

3.17

(a) Let N be the number of poor air quality periods during the next 4.5 months; N follows a Poisson process with mean value (1/month)(4.5 months) = 4.5, hence

$$P(N \le 2) = e^{-4.5}(1 + 4.5 + 4.5^2/2!) \cong \mathbf{0.174}$$

(b) Since only 10% of poor quality periods have hazardous levels, the "hazardous" periods (H) must occur at a mean rate of 1 per month×10% = 0.1 per month, hence, over 3 months, H has the mean

$$\lambda_H = (0.1)(3) = 0.3$$

$$\Rightarrow P(\text{ever hazardous}) = 1 - P(H = 0) = 1 - e^{-0.3} \cong \mathbf{0.259}$$

Alternative approach: use total probability theorem: although there is (1 – 0.1) = 0.9 probability of non-hazardous pollution level during a poor air quality period, during a 3-month period there could be any number (n) such periods and the probability of non-hazardous level reduces to 0.9^n for a given n. Hence the total probability of non-hazardous level during the whole time is

$$\sum_{n=0}^{\infty} 0.9^n P(N = n) = \sum_{n=0}^{\infty} 0.9^n \frac{e^{-(1\times 3)}(1\times 3)^n}{n!} = e^{-3}\sum_{n=0}^{\infty}\frac{(3\times 0.9)^n}{n!} = e^{-3}\, e^{3\times 0.9} = e^{-0.3}$$, hence

P(ever hazardous) = 1 – P(never hazardous) = $1 - e^{-0.3} = 1 - 0.741 \cong \mathbf{0.259}$

3.18

(a) Let E and T denote the number of earthquakes and tornadoes in one year, respectively. They are both Poisson random variables with respective means

$$\lambda_E = \nu_E t = \frac{1}{10 \text{ years}} \times 1 \text{ year} = 0.1; \ \lambda_T = 0.3$$

Also, the (yearly) probability of flooding, P(F) = 1/5 = 0.2, hence, due to statistical independence among E, T, F

$\Rightarrow$ P(good) = P(E = 0)P(T = 0)P(F') = $e^{-0.1}\, e^{-0.3}\,(1 - 0.2) = e^{-0.4} \times 0.8 \cong$ **0.536**

Note: alternatively, we can let D be the combined number of earthquakes or/and tornadoes, with mean rate $\nu_D = \nu_E + \nu_T = 0.1 + 0.3 = 0.4$ (disasters per year), and compute P(D = 0)P(F') = $e^{-0.4} \times 0.8$ instead

(b) In each year, P(good year) $\equiv p \cong 0.536$ (from (a)). Hence P(2 out of 5 years are good)

$$= \binom{5}{2} p^2 (1-p)^3 \cong \mathbf{0.287}$$

(c) Let's work with D as defined in (a).

P(only one incidence of natural hazard) = P(D = 0)P(F) + P(D = 1)P(F')

$= e^{-0.4} \times 0.2 + (e^{-0.4} \times 0.4) \times (1 - 0.2)$

$\cong$ **0.349**

3.19

Mean rate of accident = 1/3 per year

(a) P(N=0 in 5 years) = $e^{-1/3x5}$ = 0.189

(b) Mean rate of fatal accident = 1/3x0.05 = 0.0167 per year
P(failure between inspection)
= 1-P(no fatal accident within 3 years)
= 1-$e^{-0.0167x3}$
= 0.0489

3.20

(a) Let X be the number of accidents along the 20 miles on a given blizzard day. X has a Poisson distribution with $\lambda_X = \dfrac{1}{50\,\text{miles}} \times 20$ miles = 0.4, hence

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-0.4} = 1 - 0\ 670320046 \cong \mathbf{0.33}$$

(b) Let Y be the number of accident-free days among five blizzard days. With n = 5, and p = daily accident-free probability = P(X = 0) ≅ 0 670, we obtain

$$P(Y = 2) = \binom{5}{2} p^2 (1-p)^3 \cong \mathbf{0.16}$$

3.21

(a) Let X be the number of accidents in two months. X has a Poisson distribution with

$$\lambda_X = \frac{3}{12 \text{ months}} \times 2 \text{ months} = 0.5, \text{ hence}$$

$$P(X = 1) = e^{-0.5} \times 0.5 \cong \mathbf{0.303}, \text{ whereas}$$

$$P(2 \text{ accidents in 4 months}) = e^{-(3/12)(4)} [(3/12)(4)]^2 / 2!$$
$$= e^{-1} /2! \cong 0.184$$

No, P(1 accidents in 2 months) and P(2 accidents in 4 months) are not the same.

(b) 20% of all accidents are fatal, so the mean rate of fatal accidents is

$$\nu_F = \nu_x \times 0.2 = 0.05 \text{ per month}$$

Hence the number of fatalities in two months, F has a Poisson distribution with mean

$$\lambda_F = (0.05 \text{ per month})(2 \text{ months}) = 0.1, \text{ hence}$$

$$P(\text{fatalities in two months}) = 1 - P(F = 0) = 1 - e^{-0.1} \cong \mathbf{0.095}$$

3.22

The return period τ (in years) is defined by $\tau = \dfrac{1}{p}$ where p is the probability of flooding *per year*.

Therefore, the design periods of A and B being $\tau_A = 5$ and $\tau_B = 10$ years mean that the respective yearly flood probabilities are

$$P(A) = 1/\,(5 \text{ years}) = 0.2 \text{ (probability per year)}$$
$$P(B) = 1/(10 \text{ years}) = 0.1 \text{ (probability per year)}$$

(a) P(town encounters any flooding in a given year)

$$
\begin{aligned}
&= P(A \cup B) \\
&= P(A) + P(B) - P(AB) \\
&= P(A) + P(B) - P(A)P(B) \qquad (\because \text{A and B s.i.}) \\
&= \frac{1}{\tau_A} + \frac{1}{\tau_B} - \frac{1}{\tau_A \tau_B} \qquad \text{(i)} \\
&= 0.2 + 0.1 - 0.2 \cdot 0.1 \\
&= \mathbf{0.28}
\end{aligned}
$$

(b) Let X be the total number of flooded years among the next half decade. X is a binomial random variable with parameters $n = 5$ and $p = 0.28$.
Hence

$$
\begin{aligned}
P(X \geq 2) &= 1 - P(X < 2) \\
&= 1 - f(0) - f(1) \\
&= 1 - (1 - 0.28)^5 - 5 \times (0.28) \times (1 - 0.28)^4 \\
&\approx \mathbf{0.43}
\end{aligned}
$$

(c) Let τ_A and τ_B be the improved return periods for levees A and B, respectively. Using (i) from part (a), we construct the following table:

New τ_A (and cost)	New τ_B (and cost)	Yearly flooding probability $= \frac{1}{\tau_A} + \frac{1}{\tau_B} - \frac{1}{\tau_A \tau_B}$	Total cost (in million dollars)
10 ($5M)	20 ($10M)	0.145	5 + 10 = **15**
10 ($5M)	30 ($20M)	0.130	5 + 20 = 25
20 ($20M)	20 ($10M)	0.0975	20 + 10 = 30
20 ($20M)	30 ($20M)	0.0817	20 + 20 = 40

Since the goal is to reduce the yearly flooding probability to at most 0.15, all these options will work but the top one is least expensive. Hence the optimal course of action is to **change the return periods of A and B to 10 and 20 years, respectively**.

3.23

Given: ν = mean flaw rate = 1 (flaw) / 50m^2;

t = area of a panel = 3m x 5m = 15m^2

Let X be the number of flaws found on area t. X is Poisson distributed, i.e.

$$f(x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

with

$$\lambda = \nu t = 15 / 50 = 0.3 \text{ (flaws)}$$

(a)

$$\begin{aligned} P(\text{replacement}) &= 1 - P(0 \text{ or } 1 \text{ flaw}) \\ &= 1 - f(0) - f(1) \\ &= 1 - e^{-0.3} - 0.3\,e^{-0.3} \\ &\approx \mathbf{0.037} \end{aligned}$$

(b) Since the probability of replacement is 0.037 and there are 100 panels, we would expect

$$0.037\times100 = 3.7 \text{ replacements on average,}$$

which gives the expected replacement cost

$$3.7\times\$5000 \cong \mathbf{\$18500}$$

(c) For the higher-grade glass, ν = 1 (flaw) / 80m^2 $\Rightarrow$ λ = 15/80 = 0.1875, with which we can calculate the new probability

$$\begin{aligned} P(\text{replacement}) &= 1 - e^{-0.1875} - 0.1875\,e^{-0.1875} \\ &\approx 0.0155, \end{aligned}$$

which gives rise to an expected replacement cost of

$$100\times0.0155\times\$5100 \approx \$7920$$

(assuming each higher grade panel is also \$100 more expensive to replace)

We can now compare the total costs:

Let C = initial cost of old type panels

Old type: Expected total cost = C + Expected replacement cost

= C + \$18500

New type: Expected total cost = C + Extra initial cost + Expected replacement cost

= C + \$100×100 + \$7920

= C + \$17920,

which is less than that of the old type, $\therefore$ **the higher grade panels are recommended**.

3.24

(a) Let X be the number of trucks arriving in a 5-minute period.

Given: truck arrival has mean rate $\nu = 1$ (truck) / minute; hence with

$t = 5$ minutes $\Rightarrow \lambda = \nu t = 5$ (trucks)

Hence $P(X \geq 2) = 1 - P(X < 2)$

$= 1 - P(X = 0) - P(X = 1)$

$= 1 - e^{-5} - 5e^{-5}$

$\approx$ **0.96**

(b) Given: P(a truck overloads) $\equiv p = 0.1$; number of trucks $\equiv n = 5$. We can use the binomial distribution to compute

P(at most 1 truck overloaded among 5)

$= b(0; n, p) + b(1; n, p)$

$= (1 - 0.1)^5 + 5\cdot(1 - 0.1)^4\cdot(1 - 0.1)$

$\approx$ **0.92**

(c) We will first find probability of the compliment event. During t = 30 minutes, the average number of trucks is $\lambda = \nu t = 1\times30 = 30$ (trucks). Any number (x) of trucks could pass, with the (Poisson) probability

$$f(x) = \lambda^x e^{-\lambda} / x!$$

but if none of them is to be overloaded (event NO), the (binomial) probability is

$$P(NO \mid x \text{ trucks pass}) = 0.9^x$$

Hence the total probability of having no overloaded trucks during 30 minutes is

$$P(NO \mid x = 0)P(x = 0) + P(NO \mid x = 1)P(x = 1) + \ldots\ldots$$

$$= \sum_{x=0}^{\infty} 0.9^x e^{-30} \frac{30^x}{x!}$$

$$= e^{-30} \sum_{x=0}^{\infty} \frac{27^x}{x!}$$

$$= e^{-30} e^{27}$$

$= e^{-3}$, hence the desired probability

P(any overloaded trucks passing) $= 1 - e^{-3} \cong$ **0.95**

3.25

(a) Let X be the number of tornadoes next year. X has a Poisson distribution with

$$\nu = \frac{20 \text{ occurences}}{20 \text{ years}} = 1 \text{ (per year)};$$

$$t = 1 \text{ year}$$

$$\Rightarrow \lambda = \nu t = 1, \text{ hence}$$

$$\begin{aligned} &P(\text{next year will be a tornado year}) \\ &\quad = P(X \geq 1) = 1 - P(X = 0) \\ &\quad = 1 - e^{-\lambda} = 1 - 1/e \\ &\quad \cong \mathbf{0.632} \end{aligned}$$

(b) This is a binomial problem, with $n = 3$ and p = yearly probability of having any tornado(es) = 0.6321. Hence

$$\begin{aligned} &P(\text{two tornado years in three years}) \\ &\quad = \binom{3}{2} 0.632^2 (1 - 0.632)^1 \\ &\quad \cong \mathbf{0.441} \end{aligned}$$

(c)

(i) The tornadoes follow a Poisson model with mean rate of occurrence $\nu = 1$ (per year), hence for an observation period of t = 10 (years), the expected number of tornadoes is $\lambda = \nu t = \mathbf{10}$

(ii) Number of tornado years follows a binomial model with $n = 10$, $p = 0.632120559$, hence the mean (i.e. expected) number of tornado years over a ten-year period is $np \cong$ **6.32**

3.26

(a) Let X be the number of strong earthquakes occurring within the next 20 years. X has a Poisson distribution with λ = (1/50 years)(20 years) = 0.4, hence

$$\begin{aligned} P(X \le 1) &= P(X = 0) + P(X = 1) \\ &= e^{-0.4}(1 + 0.4) \\ &\cong \mathbf{0.938} \end{aligned}$$

(b) Let Y be the total number (maximum 3) of bridges that will collapse during a strong earthquake; Y has a binomial distribution with parameters $n = 3$, $p = 0.3$, hence the required probability is

$$\begin{aligned} P(Y = 1) &= 3 \times 0.3 \times (1 - 0.3)^2 \\ &= \mathbf{0.441} \end{aligned}$$

(c) During a strong earthquake, the probability that all 3 bridges survive is $(1 - 0.3)^3 = 0.7^3 = 0.343$. But since any number of strong earthquakes could happen during the next 20 years, we need to compute the total probability that there is no bridge collapse (with X as defined in part (a)),

$$\begin{aligned} \text{P(no bridge collapses)} &= \sum_{x=0}^{\infty} \text{P(no bridge collapses} \mid x \text{ earthquakes occur in 20 years)} P(X = x) \\ &= \sum_{x=0}^{\infty} (0.343)^x \left[(e^{-0.4})(0.4^x) / x!\right] \\ &= (e^{-0.4}) \sum_{x=0}^{\infty} \left[(0.343)(0.4)\right]^x / x! \\ &= (e^{-0.4})(e^{0.1372}) = e^{-0.2628} \\ &\cong \mathbf{0.769} \end{aligned}$$

3.27

(a) Let X be the total number of excavations along the pipeline over the next year; X has a Poisson distribution with mean λ = (1/50 miles)(100 miles) = 2, hence

$$\begin{aligned} &\text{P(at least two excavations)} \\ &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - e^{-\lambda}(1 + \lambda) = 1 - e^{-2}(3) \\ &\cong \mathbf{0.594} \end{aligned}$$

(b) For each excavation that takes place, the pipeline has 0.4 probability of getting damaged, and hence (1 – 0.4) = 0.6 probability of having no damage. Hence

$$\begin{aligned} &\text{P(any damage to pipeline} \mid X = 2) \\ &= 1 - \text{P(no damage} \mid X = 2) \\ &= 1 - 0.6^2 = 1 - 0.36 \\ &= \mathbf{0.64} \end{aligned}$$

Alternative method: Let D_i denote "damage to pipeline in i-th excavation"; the desired probability is

$$\begin{aligned} P(D_1 \cup D_2) &= P(D_1) + P(D_2) - P(D_1 D_2) \\ &= P(D_1) + P(D_2) - P(D_1 \mid D_2)\, P(D_2) \\ &= P(D_1) + P(D_2) - P(D_1)\, P(D_2) \\ &= 0.4 + 0.4 - 0.4^2 = 0.8 - 0.16 \\ &= \mathbf{0.64} \end{aligned}$$

(c) Any number (x) of excavations could take place, but there must be no damage no matter what x value, hence we have the total probability

$$\begin{aligned} &\sum_{x=0}^{\infty} \text{P(no damage} \mid x \text{ excavations) P(}x\text{ excavations)} \\ &= \sum_{x=0}^{\infty} 0.6^x \frac{e^{-\lambda}\lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(0.6\lambda)^x}{x!} = e^{-\lambda} e^{0.6\lambda} \\ &= e^{-0.4\lambda} = e^{-0.4(2)} = e^{-0.8} \\ &\cong \mathbf{0.449} \end{aligned}$$

Alternative method: recall the meaning of ν in a Poisson process—it is the mean rate, i.e. the true proportion of occurrence over a large period of observation. Experimentally, it would be determined by

$$\nu = \frac{n_E}{N}$$

where n_E is the number of excavations observed over a very large number (N) of miles of pipeline. Since 40% of all excavations are damaging ones, damaging excavations must also occur as a Poisson process, but with the "diluted" mean rate of

$$\nu_D = \frac{0.4 n_E}{N} = 0.4\nu \text{, hence}$$

ν_D = (0.4)(1/50) = (1/125) (<u>damaging</u> excavations per mile)

Hence

$$P(\text{no damaging excavation over a 100 mile pipeline}) = e^{-(1/125 \text{ mi.})(100 \text{ mi.})} = e^{-100/125} = e^{-0.8}$$

$\cong$ **0.449**

3.28

(a) Let X be the number of flaws along the 30-inch weld connection. X has a Poisson distribution with

$$\lambda_X = \frac{0.1}{12 \text{ inches}} \times 30 \text{ inches} = 0.25, \text{ hence}$$

$$P(\text{acceptable connection}) = P(X = 0) = e^{-0.25} \cong \mathbf{0.779}$$

(b) Let Y be the number of acceptable connections among three. Y has a binomial distribution with n = 3 and p = P(an acceptable connection) = 0.778800783, hence the desired probability is

$$\begin{aligned} P(Y \geq 2) &= P(Y = 2) + P(Y = 3) \\ &= 3p^2(1-p) + p^3 \\ &\cong \mathbf{0.875} \end{aligned}$$

(c) The number of flaws (F) among 90 inches of weld has a Poisson distribution with

$$\lambda_F = \frac{0.1}{12 \text{ inches}} \times 90 \text{ inches} = 0.75, \text{ hence}$$

$$\begin{aligned} \text{Hence } P(\text{1 flaw in 3 connections}) &= P(\text{1 flaw in 90 inches of weld}) \\ &= P(F = 1) \\ &= e^{-0.75} \times 0.75 \\ &\cong \mathbf{0.354} \end{aligned}$$

Note: “1 flaw in 3 connections” is not the same as “1 unacceptable connections among 3”, as an unacceptable connection does not necessarily contain only 1 flaw—it could have 2, 3, 4, etc.

3.29

Mean rate of flood occurrence = 6/10 = 0.6 per year

(a) P($1 \le N \le 3$ within 3 years)

$$= [\frac{(0.6\times 3)^1}{1!} + \frac{(0.6\times 3)^2}{2!} + \frac{(0.6\times 3)^3}{3!}]e^{-0.6\times 3}$$

= (1.8+1.62+0.972)x0.165

= 0.725

(b) Mean rate of flood that would cause inundation

= 0.6x0.02

P(treatment plant will not be inundated within 5 years)

= P(no occurrence of inundating flood)

= $e^{-0.02}$

= 0.98

3.30

Given: both I and N are Poisson processes, with $\nu_I = 0.01$/mi and $\nu_N = 0.05$/mi. Hence, along a 50-mile section of the highway, I and N have Poisson distributions with respective means

$$\lambda_I = (0.01/\text{mi})(50\text{ mi}) = 0.5,\ \lambda_N = (0.05/\text{mi})(50\text{ mi}) = 2.5$$

(a) $P(N = 2) = e^{-\lambda_N} (\lambda_N)^2 / 2! = e^{-2.5}\, 2.5^2 / 2 \cong$ **0.257**

(b) A, accident of either type, has the combined mean rate of occurrence $\nu_A = \nu_I + \nu_N = 0.06$, hence A has a Poison distribution with $\lambda_A = (0.06/\text{mi})(50\text{ mi}) = 3$, thus

$$\begin{aligned} P(A \geq 3) &= 1 - P(A = 0) - P(A = 1) - P(A = 2) \\ &= 1 - e^{-3} (1 + 3 + 3^2/2) \\ &\cong \mathbf{0.577} \end{aligned}$$

(c) Let us model this as an n = 2 binomial problem, where each “trial” corresponds to an accident, in which “success” means “injury” and “failure” means “non-injury”. How do we determine p? Consider the physical meaning of the ratio $\nu_I : \nu_N = 0.01 : 0.05$ -- it compares how frequently I occurs relative to N, hence their relative likelihood must be 1 to 5, thus P(N) = 5/6 and P(I) = 1/6 whenever an accident occurs. Therefore, p = 5/6, and

$$P(2\text{ successes among 2 trials}) = p^2 = (1/6)^2 \cong \mathbf{0.028}$$

3.31

(a) Let W and S denote the number of winter and summer thunderstorms, respectively. Their mean occurrence rates (in numbers per month) are estimated based on the observed data as

(i) $\nu_W \cong (173)/(21\times6) \cong$ **1.37**; (ii) $\nu_S \cong (840)/(21\times6) =$ **6.67**

(b) Let us "superimpose" the two months as one, with winter and summer thunderstorms (X) that could happen simultaneously, at a combined mean rate of $\nu_X = (173 + 840)/(21\times6) = 8.03968254$ (storms per month), thus $\lambda_X = (\nu_X)(1 \text{ month}) = 8.03968254$, hence

$$P(X = 4|t = 1 \text{ month}) = e^{-8.040}(8.040^4)/4! \cong \mathbf{0.056}$$

(c) In any particular year, the (Poisson) probability of having no thunderstorm in December is

$$P(W = 0 \mid t = 1 \text{ month}) = e^{-173/126} = 0.253,$$

let's call this p. Having such a "success" in 2 out of the next 5 years has the (binomial) probability

$$\binom{5}{2} p^2(1-p)^3 = 5!/2!/3!\times 0.253^2\times(1 - 0.253)^3 \cong \mathbf{0.267}$$

3.32

(a) Let D denote defects and R denote defects that remain after inspection. Since only 20% of defects remain after inspection, we have

$$\nu_R = \nu_D \times 0.2 = \frac{1}{200 \text{ meters}} \times 0.2 = \mathbf{0.001\ per\ meter}$$

(b) For t = 3000 meters of seams, undetected defects have a mean of $\lambda = \nu_R\, t = (0.001/\text{m})(3000\text{m}) = 3$,

$$\begin{aligned}\Rightarrow P(\text{more than two defects}) &= 1 - P(\text{two or less defects}) \\ &= 1 - [P(D = 0) - P(D = 1) - P(D = 2)] \\ &= 1 - [e^{-3}(1 + 3 + 3^2/2!)] = 1 - 0.423 \\ &\cong \mathbf{0.577}\end{aligned}$$

(c) Suppose the allowable fraction of undetected defects is p ($0 < p < 1$), then the mean rate of undetected defects is $\nu_U = \nu_D \times p$, hence, for 1000 meters of seams, the (Poisson) mean number of defects is

$$\nu_U \times 1000\text{m} = (1/200\text{m})(p)(1000\text{m}) = 5p,$$

thus if we require

$$\begin{aligned}&P(0 \text{ defects}) = 0.95 \\ &\Rightarrow e^{-5p} = 0.95 \\ &\Rightarrow -5p = \ln(0.95) \\ &\Rightarrow p = -\ln(0.95)/5 = 0.0103,\end{aligned}$$

i.e. only about **1%** of defects can go on undetected.

3.33

Let J_1 and J_2 denote the events that John's scheduled connection time is 1 and 2 hours, respectively, where $P(J_1) = 0.3$ and $P(J_2) = 0.7$. Also, let X be the delay time of the flight in hours. Note that

$$P(X > x) = 1 - P(X \le x) = 1 - F(x) = 1 - (1 - e^{-x/0.5})$$
$$= e^{-x/0.5}$$

(a) Let M denote the event that John misses his connection, i.e. the flight delay time exceeded his scheduled time for connection. Using the theorem of total probability,

$$\begin{aligned} P(M) &= P(M \mid J_1)P(J_1) + P(M \mid J_2)P(J_2) \\ &= P(X > 1)\times 0.3 + P(X > 2)\times 0.7 \\ &= e^{-1/0.5} \times 0.3 + e^{-2/0.5} \times 0.7 \\ &= 0.135\times 0.3 + 0.0183\times 0.7 \cong \mathbf{0.053} \end{aligned}$$

(b) Regardless of whether John has a connection time of 1 hour and Mike has 2, or the opposite, for them to both miss their connections the flight must experience a delay of more than two hours, and the probability of such an event is

$$P(X > 2) = e^{-2/0.5} \cong \mathbf{0.018}$$

(c) Since Mary has already waited for 30 minutes, the flight will have a delay time of at least 0.5 hours when it arrives. Hence the desired probability is

$$\begin{aligned} &P(X > 1 \mid X > 0.5) \\ &= P(X > 1 \text{ and } X > 0.5) / P(X > 0.5) \\ &= P(X > 1) / P(X > 0.5) \\ &= e^{-1/0.5} / e^{-0.5/0.5} = e^{-2} / e^{-1} = 1/e \cong \mathbf{0.368} \end{aligned}$$

3.34

1000 rebars delivered with 2% below specification

(a) 20 rebars are tested; hence,

$$N = 1000,\ m = 20,\ n = 20$$

P(all 20 bars will pass test)
= P(zero bars will fail test)

$$= \frac{\binom{20}{0}\binom{980}{20}}{\binom{1000}{20}} = 0.665$$

(b) P(at least two bars will fail)
= 1 – P(at most 1 bar will fail)
= 1 – P(x=0) – P(x=1)

$$= 1 - 0.665 - \frac{\binom{20}{1}\binom{980}{19}}{\binom{1000}{20}}$$

= 1 – 0.665 – 0.277
= 0.058

(c) Let n be the required number of bars tested
P(all n bars will pass test)

$$= \frac{\binom{n}{0}\binom{980}{n}}{\binom{1000}{n}} \equiv 0.10$$

$n \cong 107.712 \Rightarrow 108$ bars required

3.35

(a) "A gap larger than 15 seconds" means that there is sufficient time (15 seconds or more) between the arrival of two successive cars. Let T be the (random) time (in seconds) between successive cars, which is exponentially distributed with a mean of

$$E(T) = (1/10) \text{ minute} = 0.1 \text{ minute} = 6 \text{ seconds},$$

$$\text{hence } P(T > 15) = 1 - P(T \leq 15) = e^{-15/6} = e^{-2.5} \cong \mathbf{0.082}$$

(b) For any one gap, there is $(1 - e^{-2.5})$ chance that it is not long enough for the driver to cross. For such an event to happen 3 times consecutively, the probability is $(1 - e^{-2.5})^3$; followed by a long enough gap that allows crossing, which has probability $e^{-2.5}$. Hence the desired probability is

$$(1 - e^{-2.5})^3 (e^{-2.5}) \cong \mathbf{0.063}$$

More formally, if N is the number of gaps one must wait until the first "success" (i.e. being able to cross), G follows a geometric distribution,

$$P(N = n) = (1 - p)^{n-1} p \quad \text{where } p = e^{-2.5}$$

(c) Since the mean of a geometric distribution is 1/p (see Ang & Tang Table 5.1), he has to wait a mean number of

$$1/p = 1/ e^{-2.5} = e^{2.5}$$

$$\cong \mathbf{12.18} \text{ gaps before being able to cross.}$$

(d) First let us find the probability of the compliment event, i.e. P(none of the four gaps were large enough)

$$= (1 - e^{-2.5})^4$$

Hence

P(cross within the first 4 gaps) $= 1 - (1 - e^{-2.5})^4 = 1 - 0.70992075 \cong \mathbf{0.290}$

3.36

Standard deviation of crack length= $\sqrt{6.25}$ = 2.5

(a) Let X be the length (in micrometers) of any given crack..

$$P(X > 74) = 1 - \Phi\left(\frac{74 - 71}{2.5}\right) = 1 - \Phi(1.2) = 1.2$$

(b) Given that 72 < X , the conditional probability

$$\begin{aligned}P(X > 77 \mid X > 72) &= P(X > 77 \text{ and } X > 72) / P(X > 72)\\ &= P(X > 77) / P(X > 72)\\ &= [1 - \Phi((77 - 71)/2.5)] / [1 - \Phi((72 - 71)/2.5)]\\ &= [\ 1 - \Phi(2.4)] / [1 - \Phi(0.4)] = 0.0082 / 0.8446\\ &\cong \mathbf{0.024}\end{aligned}$$

3.37

(a) P(activity C will start on schedule)

$= P(A<60) \cap P(B<60)$

$= \Phi(\frac{60-50}{10})\Phi(\frac{60-45}{15}) = \Phi(1)\Phi(1)$

$= 0.841^2 = 0.707$

(b) P(Project completed on target) = P(T)

$= P(T|E)P(E) + P(T|\overline{E})P(\overline{E})$

= P(C<90)x0.707 + P(C<90-15)x0.293

$= \Phi(\frac{90-80}{25}) \times 0.707 + \Phi(-0.2) \times 0.293$

= 0.655x0.707 + 0.421x0.293 = 0.586

3.38

Let, T = The present traffic volume = N(200,60)

(a) $P(T>350) = 1 - \Phi((350 - 200)/60) = 1 - \Phi(2.5) = 0.00621$

(b) Let, T_1 = Traffic volume after 10 years.

$\mu_{T_1} = 200 + 0.10\times200\times10 = 400$

$\delta_{T_1} = \delta_T = 60/200 = 0.3$

Hence,

$\sigma_{T_1} = 0.3\times400 = 120$

So,

T_1 is N(400, 120)

$P(T_1>350) = 1 - \Phi((350 - 400)/120) = 1 - \Phi(0.42) = 0.662$

(c) C = Capacity of airport after 10 years.

$P(T_1>C) = 0.00621$

or,

$1 - \Phi((C - 400)/120) = 1 – 0.99379 = 1- \Phi(2.5)$

or,

$(C - 400)/120 = 2.5$

or,

$C = 300 + 400 = 700$

3.39

A = volume of air traffic
C = event of overcrowded

(a) $P(C \mid N\text{-}S) = P(A>120)$

$$= 1-\Phi\left(\frac{120-100}{10}\right) = 1-\Phi(2) = 0.02275$$

(b) $P(C \mid E\text{-}W) = P(A>115)$

$$= 1-\Phi\left(\frac{115-100}{10}\right) = 1-\Phi(1.5) = 0.0668$$

(c) $P(C) = P(C \mid N\text{-}S)P(N\text{-}S)+P(C \mid E\text{-}W)P(E\text{-}W)$

$= 0.02275\text{x}0.8 + 0.0668\text{x}0.2$

$= 0.0315$

3.40

Let X be the settlement of the proposed structure; $X \sim N(\mu_X, \sigma_X)$. Given probability:

$$P(X \leq 2) = 0.95$$

Also, since we're given the c.o.v. $= \frac{\sigma_X}{\mu_X} = 0.2$, $\sigma_X = 0.2\mu_x$

Hence,

$$P(X \leq 2) = \Phi(\frac{2-\mu_x}{0.2\mu_x}) = 0.95$$

or

$$\frac{2-\mu_x}{0.2\mu_x} = \Phi^{-1}(0.95) = 0.95 = 1.645$$

and

$$\mu_x = 1.5$$

$$P(X > 2.5) = 1 - \Phi(\frac{2.5-1.5}{0.2 \times 1.5}) = 1 - \Phi(3.063) = 0.00047$$

3.41

(a) Let X be her cylinder's strength in kips. To be the second place winner, X must be above 70 but below 100, hence

$$
\begin{aligned}
&P(70 < X < 100) \\
&= \Phi(\frac{100-80}{20}) - \Phi(\frac{70-80}{20}) \\
&= \Phi(1) - \Phi(-0.5) \\
&= 0.841 - 0.309 \\
&= 0.532
\end{aligned}
$$

(b) $P(X > 100 \mid X > 90) = P(X > 100 \text{ and } X > 90) / P(X > 90)$

$$
\begin{aligned}
&= P(X > 100) / P(X > 90) \\
&= \{1 - \Phi[(100 - 80)/20]\} / \{1 - \Phi[(90 - 80)/20]\} \\
&= [1 - \Phi(1)] / [1 - \Phi(0.5)] \\
&= 0.159 / 0.309 \cong \mathbf{0.514}
\end{aligned}
$$

(c) Let Y be the boyfriend's cylinder strength in kips, which has a mean of $1.01 \times 80 = 80.8$. Therefore,

$$Y \sim N(80.8, \sigma_Y)$$

Let $D = Y - X$; D is normally distributed with a mean of

$$\mu_D = \mu_Y - \mu_X = 80.8 - 80 = 0.8 > 0$$

This suffices to conclude that the guy's cylinder is more likely to score higher. Mathematically,

$$P(D > 0) = \Phi(\frac{0-0.8}{\sigma_D}) > 0.5$$

hence it is more likely for the guy's cylinder strength to be higher than the girl's.

3.42

H = annual maximum wave height = N(4,3.2)

(a) P(H>6) = $1-\Phi(\frac{6-4}{3.2}) = 1-\Phi(0.625) = 0.266$

(b) Design requirement is

P(no exceedance over 3 years) = 0.8 = $(1-p)^3$

Where p = p(no exceedance in a given year)

Hence, p = $1-(0.8)^{1/3} = 0.0717$

That is, $\Phi(\frac{h-4}{3.2}) = 0.0717$ where h is the designed wave height

$h = 3.2\,\Phi^{-1}(0.0717) + 4$

$= 3.2 \text{x} 0.374 + 4 = 5.2$ m

(c) Probability of damaging wave height in a year

= 0.266x0.4 = 0.1064

Hence mean rate of damaging wave height = 0.1064 per year

P(no damage in 3 years)

= P(no damaging wave in 3 years)

= $e^{-0.1064 \text{x} 3}$

=0.727

3.43

Let X be the daily SO_2 concentration in ppm, where X ~ N(0.03, 0.4×0.03), i.e., X is normal with mean 0.03 and standard deviation 0.012. Thus the weekly mean, $\overline{X}$ ~ N(0.03, 0.012/$\sqrt{7}$). Hence

1. P($\overline{X}$ < 0.04) $=\Phi\left(\dfrac{0.04-0.03}{0.012/\sqrt{7}}\right)=\Phi(2.205)=0.986$, hence

   P_1(violation) = 1 – 0.9863 = **0.0137**, whereas

2. On any one day, the probability of X being under 0.075 is

P(X < 0.075) = $\Phi\left(\dfrac{0.075-0.03}{0.012}\right)$ = Φ(3.75) ≅, hence

P(no violation) = P(0 or 1 day being under 0.075 out of 7 days)

= $1^7 + 7\times1^6\times(1-1)$

= 1

⇒ P_2(violation) = 1 – 1 =0 < P_1(violation),

∴ **criteria 1** is more likely to be violated, i.e. more strict.

3.44

Let F be the daily flow rate; we're given $F \sim N(10,2)$
(a) P(excessive flow rate) = P($F > 14$)

$$= 1 - \Phi\left(\frac{14-10}{2}\right)$$
$$= 1 - \Phi(2)$$
$$= 1 - 0.97725 \approx 0.02275$$

(b) Let X be the total number of days with excessive flow rate during a three-day period. X follows a binomial distribution with $n = 3$ and $p = 0.02275$ (probability of excessive flow on any given day), hence

P(no violation)
= P(zero violations for 3 days)
= P($X = 0$)
= $(1 - p)^3 \cong$ **0.933**

(c) Now, with n changed to 5, while $p = 0.02275$ remains the same,

P(not charged) = P($X = 0$ or $X = 1$)
= P($X = 0$) + P($X = 1$)
= $(1 - p)^5 + 5 \cdot p \cdot (1 - p)^4 \cong 0.995$

which is larger than the answer in (b). Since the non-violation probability is larger, this is a better option.

(d) In this case we work backwards—fix the probability of violation, and determine the required parameter values.

We want

P(violation) = 0.01
$\Rightarrow$ P(non-violation) = 0.99
$\Rightarrow (1 - p)^3 = 0.99$
$\Rightarrow p = 0.00334$,

but recall from part (a) that p is obtained by computing P($F > 14$)

$$\Rightarrow \quad 0.003344 = 1 - \Phi\left(\frac{14-\mu_F}{\sigma_F}\right)$$

hence,

$$\frac{14-\mu_F}{2} = \Phi^{-1}(0.99666) = 2.712 \Rightarrow \mu_F \cong \mathbf{8.58}$$

3.45

(a) The "parameters" λ and ζ of a log-normal R.V. are related to its mean and standard deviation μ and σ as follows:

$$\zeta^2 = \ln(1+\left(\frac{\sigma}{\mu}\right)^2)$$

$$\lambda = \ln\mu - \frac{\zeta^2}{2}$$

Substituting the given values $\delta_T \equiv \frac{\sigma_T}{\mu_T} = 0.4, \mu_T = 80$, we find

$\zeta^2 = 0.14842$ and $\lambda = 4.307817$, hence

$\zeta \approx$ **0.385** and $\lambda \approx$ **4.308**

The importance of these parameters is that they are the standard deviation and mean of the related variable $X \equiv \ln(T)$, while X is normal. Probability calculations concerning T can be done through X, as follows:

(b) $$P(T \le 20) = \Phi\left(\frac{\ln 20 - 4.3078}{0.385}\right) = \Phi(-3.406) = 0.00033$$

(c) "$T > 100$" is the given event, while "a severe quake occurs over the next year" is the event "$100 < T < 101$", hence we seek the conditional probability

P(100 < T < 101 | T > 100)
= P(100 < T < 101 and T > 100) / P(T > 100)
= P(100 < T < 101) / P(T > 100)
= P(ln 100 < ln T < ln 101) / P(ln T > ln 100)

$$= \frac{\Phi\left(\frac{\ln 101 - \lambda}{\zeta}\right) - \Phi\left(\frac{\ln 100 - \lambda}{\zeta}\right)}{1 - \Phi\left(\frac{\ln 100 - \lambda}{\zeta}\right)}$$

$$= \frac{0.787468 - 0.779895}{1 - 0.779895}$$

$= 0.034$

3.46

(a) Let X be the daily average pollutant concentration. The parameters of X are:

$$\zeta \cong \delta = 0.2,$$
$$\lambda \cong \ln 60 - 0.2^2 / 2 \cong 4.074$$

Hence

$$\begin{aligned} P(X > 100) &= 1 - P(X \leq 100) \\ &= 1 - \Phi\left(\frac{\ln 100 - 4.074}{0.2}\right) \\ &= 1 - \Phi(2.654) \\ &= 1 - 0.996024 \\ &\cong \mathbf{0.004} \end{aligned}$$

(b) Let Y be the total number of days on which critical level is reached during a given week. Y follows a binomial distribution with parameters $n = 7$, $p = 0.004$ (and $1 - p = 0.9964$), hence the required probability is

$$\begin{aligned} P(Y = 0) &= 0.996^7 \\ &\cong \mathbf{0.972} \end{aligned}$$

3.47

$$\zeta = \sqrt{\ln(1+\delta^2)} = 0.2936 \text{ H}$$

$$\lambda = \ln \mu - \frac{1}{2}\zeta^2 = 2.953$$

(a) P(25 m long pile will not anchor satisfactorily)

$$= \text{P(H>25-2)} = \text{P(H>23)} = 1 - \Phi\left(\frac{\ln 23 - 2.953}{0.2936}\right) = 1 - \Phi(0.622) = 0.267$$

(b) $$\text{P(H>2T} \mid \text{H>24)} = \frac{P(H > 27 \cap H > 24)}{P(H > 24)} = \frac{P(H > 27)}{P(H > 24)}$$

$$= \frac{1 - \Phi\left(\frac{\ln 27 - 2.953}{0.2936}\right)}{1 - \Phi\left(\frac{\ln 24 - 2.953}{0.2936}\right)} = \frac{1 - \Phi(1.677)}{1 - \Phi(0.767)} = \frac{0.0468}{0.222} = 0.21$$

3.48

(a) Let X be the pile capacity in tons. X is log-normal with parameters

$$\zeta_X \cong \delta_X = 0.2,$$
$$\lambda_X \cong \ln 100 - 0.2^2 / 2$$

$$P(X > 100) = 1 - P(X \le 100) = 1 - \Phi\left(\frac{\ln 100 - 4.585}{0.2}\right) = 1 - \Phi(0.1) = 1 - 0.54 = 0.46$$

(b) Let L be the maximum load applied; L is log-normal with parameters

$$\delta_L = 15/50 = 0.3$$
$$\Rightarrow \zeta_L = [\ln(1 + 0.3^2)]^{1/2} = 0.294$$
$$\Rightarrow \lambda_L = \ln 50 - 0.294^2 / 2 = 3.869$$

It is convenient to formulate P(failure) as

$$P(X < L) = P(X / L < 1)$$
$$= P(\ln(X/L) < \ln 1)$$
$$= P(\ln X - \ln L < 0)$$

but $D = \ln X - \ln L$ is the difference of two normals, so it is again normal, with

$$\mu_D = \lambda_X - \lambda_L = 0.716 \text{ and } \sigma_D = (\zeta_X^{\ 2} + \zeta_L^{\ 2})^{1/2} = 0.355$$
$$\Rightarrow P(D < 0) = \Phi\left(\frac{0 - 0.716}{0.355}\right) = \Phi(-2.017) = 0.0219$$

(c) $P(X > 100 \mid X > 75) = P(X > 100 \text{ and } X > 75) / P(X > 75)$

$$= P(X > 100) / P(X > 75)$$
$$= [\text{answer to (a)}] / [1 - \Phi(\frac{\ln 75 - 4.585}{0.2})]$$
$$= 0.46 / [1 - \Phi(-1.338)]$$
$$= 0.46 / \Phi(1.338)$$
$$= 0.46 / 0.9096 \cong \mathbf{0.506}$$

(d) $P(X > 100 \mid X > 90) = P(X > 100 \text{ and } X > 90) / P(X > 90)$

$$= P(X > 100) / P(X > 90)$$
$$= [\text{answer to (a)}] / [1 - \Phi(\frac{\ln 90 - 4.585}{0.2})]$$
$$= 0.460172104 / [1 - \Phi(-0.427)]$$
$$= 0.46 / \Phi(0.427)$$
$$= 0.46 / 0.665 \cong \mathbf{0.692}$$

3.49

$$\zeta^2 = \ln(1+\left(\frac{\sigma}{\mu}\right)^2)$$

$$\lambda = \ln\mu - \frac{\zeta^2}{2}$$

(a) Let X be the maximum wind velocity (in mph) at the given city. X ~ LN(λ, ζ) where

$$\zeta = \sqrt{\ln(1+\delta^2)} \cong \delta = 0.2\text{, thus}$$

$$\lambda = \ln\mu - \frac{\zeta^2}{2} \cong \ln(90) - 0.2^2 / 2$$

Since ζ and λ are the standard deviation and mean of the normal variate ln(X),

$$P(X > 120) = 1 - \Phi\left(\frac{\ln 120 - 4.48}{0.2}\right) = 1 - \Phi(1.538) = 1 - 0.938 \cong \mathbf{0.062}$$

(b) To design for 100-year wind means the yearly probability of exceeding the design speed (V) is 1/100 = 0.01, i.e.

$$P(X > V) = 0.01\text{, i.e.}$$

$$P(X \le V) = 0.99$$

$$\Phi\left(\frac{\ln V - 4.48}{0.2}\right) = 0.99$$

$$\frac{\ln V - 4.48}{0.2} = \Phi^{-1}(0.99) = 2.33$$

$$V = 141(mph)$$

3.50

T = time until breakdown = gamma with mean of 40 days and standard deviation of 10 days

(a) $k/\nu = 40$, c.o.v. $= 0.25 = \dfrac{1}{\sqrt{k}}$

Hence, $k = 16$, $\nu = 0.4$

Let t be the required maintenance schedule interval

Hence, $P(T<t) \equiv 0.95$ or $I_u(0.4\ t, 16) \equiv 0.95$

By trial and error,

t	Probability
40	0.533
50	0.8435
60	0.9656
58	0.9520
57	0.944

Hence, $t \cong 58$ days

(b) $P(T < 58+7 \mid T > 58)$

$$= 1 - \frac{P(T>65)}{P(T>58)} = 1 - \frac{I_u(0.4\times 65{,}16)}{I_u(0.4\times 58{,}16)} = 1 - \frac{I_u(26{,}16)}{I_u(23.2{,}16)} = 1 - \frac{0.01417}{0.048} = 0.7$$

(c) P(at least one out of three will breakdown)
= 1 – P(none of the three will breakdown)
$= 1 - (0.95)^3$
= 0.143

3.51

The capacity C is gamma distributed with a mean of 2,500 tone and a c.o.v. of 35%

$k/\nu = 2500, \qquad \frac{1}{\sqrt{k}} = 0.35$

Hence, k = 8.16, $\nu = 0.00327$

(a) P(C >3000 | C > 1500)

$$= \frac{P(C > 3000)}{P(C > 1500)} = \frac{1 - I_u(0.00327 \times 3000, \quad 8.16)}{1 - I_u(0.00327 \times 1500, \quad 8.16)} = \frac{1 - I_u(9.81, \quad 8.16)}{I_u(4.905, \quad 8.16)} = \frac{0.745}{0.889} = 0.838$$

(b) P(C<2000) = I_u (0.00327x2000, 8.16) = 0.312

Mean rate of damaging quake = 0.312 x 1/20 = 0.0156

P(no damage over 50 years) = $e^{-0.0156x50}$ = 0.458

(c) P(at least 4 buildings damaged)
= 1 – P(none of the the 4 will be damaged)
= 1 – $(0.688)^4$
= 0.776

3.52

T_i = time for loading/unloading operation of i^{th} ship

T = waiting time of the merchant ship

(a) Given that there are already 2 ships in the queue,

$P(T>5) = P(T>5 \mid T_1 = 2)P(T_1 = 2) + P(T>5 \mid T_1 = 3)P(T_1 = 3)$

$= P(T_2>3)P(T_1 = 2) + P(T_2>2)P(T_1 = 2)$

$= 0\text{x}1/4 + 1/4\text{x}3/4 = 0.1875$

(b) $P(T>5) = P(T>5 \mid N=0)P(N=0) + P(T>5 \mid N=1)P(N=1) + P(T>5 \mid N=2)P(N=2) + P(T>5 \mid N=3)P(N=3)$

Given 0 ship in the queue, $P(T>5) = 0$

Given 1 ship in the queue, $P(T>5) = P(T_1>5) = 0$

Given 2 ships in the queue, $P(T>5) = 0.1875$ from above

Given 3 ships in the queue, the waiting time will be at least 6 days. Hence it is certain that $P(T>5) = 1$

In summary,

$P(T>5) = 0\text{x}0.1 + 0\text{x}0.3 + 0.1875\text{x}0.4 + 1\text{x}0.2 = 0.275$

3.53

Traffic volume= V = Beta between 600 and 1100 vph with mean of 750 vph and c.o.v. of 0.20

A = accidenct

(a) P(Jamming occurs on the bridge) = P(J)

= $P(J \mid A)P(A)+P(J \mid \overline{A})P(\overline{A})$

= 1x0.02 + P(V>1000)x0.98

For the parameters of Beta distribution for V,

$$750 = 600+\frac{q}{q+r}(1100\text{-}600)$$

$$(0.2\text{x}750)^2 = \frac{qr}{(q+r)^2(q+r+1)}(1100\text{-}600)^2$$

Hence, $r \approx 0.95, q \approx 0.41$

P(V>1000) = 1- β_u (0.41, 0.95)

But u = (1000-600)/(1100-600) = 0.8

P(V>1000) = 1- $\beta_{0.8}$ (0.41, 0.95) = 0.097

P(J) = 0.02 + 0.097x0.98 = 0.971

3.54

By using the binomial distribution,
P(2 out of 8 students will fail)

$$= \binom{8}{2}(0.2)^2(0.8)^6$$

$= 28x(0.2)^2(0.8)^6$
$= 0.294$

By using the hyper geometric distribution, we need
number of passing students in a class of 30 = 24
number of failing students in a class of 30 = 6

Hence P(2 out of 8 students will fail)

$$= \frac{\binom{24}{6}\binom{6}{2}}{\binom{30}{8}} = \frac{28 \times 15}{5852925}$$

$=0.00007$

3.55

(a) T = trouble-free operational time follows a gamma distribution with mean of 35 days and c.o.v. of 0.25

$k/\nu = 35, \qquad \frac{1}{\sqrt{k}} = 0.25$

Hence, k = 16, $\nu = 0.457$

$P(T>40) = 1 - I_u(0.457x40, 16) = I_u(18.3, 16) = 0.736$

(b) For the total of 50 road graders, number of graders,

Number of graders with T<40 = 0.1x50 = 5

P(2 among 10 graders selected will have T<40)

$$= \frac{\binom{5}{2}\binom{45}{8}}{\binom{50}{10}} = \frac{10 \times 0.0216 \times 10^{9}}{1.027 \times 10^{10}} = 0.021$$

3.56

Seismic capacity C is lognormal with median of 6.5 and standard deviation of 1.5

λ = ln 6.5

For lognormal distribution, $\mu > x_m$

hence $\sigma / \mu < \sigma / x_m = 1.5 / 6.5 = 0.23$

and $\xi \approx \delta$

Since λ =ln 6.5 = ln $\mu - (1.5/\mu)^2$

By trial and error, $\mu \approx 6.7; and \quad \xi \approx 1.5/6.7 = 0.198$

(a) P(Damage) = P(C<5.5) = $\Phi(\dfrac{\ln 5.5 - \ln 6.5}{0.198}) = \Phi(-0.84) = 0.2$

(b) P(C>5.5 | C>4) = $\dfrac{P(C > 4 \cap C > 5.5)}{P(C > 4)} = \dfrac{P(C > 5.5)}{P(C > 4)}$

$$= \frac{1 - \Phi(\dfrac{\ln 5.5 - \ln 6.5}{0.198})}{1 - \Phi(\dfrac{\ln 4 - \ln 6.5}{0.198})} = \frac{0.907}{0.9929} = 0.915$$

(c) Mean rate of damaging earthquake = 0.2x1/500 = 0.0004

P(building survives 100 years)
= P(no damaging earthquake in 100 years)
= $e^{-0.0004x100} = e^{-0.04}$ = 0.96

(d) P(survival of at least 4 structures during the earthquake)
$= \binom{5}{4}(0.8)^4(0.2)^1 + \binom{5}{5}(0.8)^5(0.2)^0$
= 0.4096+0.3277 = 0.737

mean rate earthquake that causes damage to at most 3 structures
= 0.737x1/500 = 0.00147

P(at least four building will survive 100 years)
= $e^{-0.00147x100} = e^{-0.147}$ = 0.863

3.57

(a) Let A and B denote the pressure at nodes A and B, respectively. Since A is log-normal with mean = 10 and c.o.v. = 0.2 (small), we have

$$\zeta_A \cong 0.2, \text{ and}$$

$$\lambda_A = \ln(\mu_A) - \frac{\zeta_A^{\ 2}}{2} \cong \ln(10) - 0.02 = 2.282585093$$

$\Rightarrow$ P(satisfactory performance at node A)

$$= P(6 < A < 14) = \Phi\left(\frac{\ln 14 - 2.283}{0.2}\right) - \Phi\left(\frac{\ln 6 - 2.283}{0.2}\right) = \Phi(1.788) - \Phi(-2.4560) = 0.955$$

(b) Let N_i denote the event of pressure at node i being within normal range; i = A,B. Given:

$$P(N_B) = 0.9 \Rightarrow P(\overline{N_B}) = 0.1;$$

$$P(\overline{N_B} \mid \overline{N_A}) = 2\times 0.1 = 0.2$$

Hence

P(unsatisfactory water services to the city)

$= P(\overline{N_A} \cdot \overline{N_B})$

$= P(\overline{N_B} \mid \overline{N_A})P(\overline{N_A})$

$= 0.2\times(1 -$ answer to (a))

$= 0.2\times(1 - 0.955319)$

$\cong$ **0.0089**

(c) The options are:

(I) to change the c.o.v. of A to 0.15: repeating similar calculations as done in (a), we get the new values of:

λ_A	ζ_A	lower limit for Z	upper limit for Z	P(normal pressure at A)
2.2913	0.15	–3.33	2.318	0.98934

hence the probability of unsatisfactory water services,

$P(\overline{N_B} \mid \overline{N_A})P(\overline{N_A})$ becomes

$$0.2\times(1 - 0.9893) \cong 0.0021$$

(II) to change $P(\overline{N_B})$ to 0.05, and hence $P(\overline{N_B} \mid \overline{N_A}) = 2\times 0.05 = 0.10$, thus

$$P(\overline{N_B} \mid \overline{N_A})P(\overline{N_A}) = 0.10\times(1 - 0.9556) \cong 0.0044$$

$\Rightarrow$ **Option I** is better since it offers a lower probability of unsatisfactory water services than II.

3.58

(a) $f_X(x)$ is obtained by "integrating out" the independence on y,

$$\therefore f_X(x) = \int_0^1 \frac{6}{5}(x + y^2)dy = \frac{6}{5}\left[xy + \frac{y^3}{3}\right]_0^1$$

$$= \frac{2}{5}(3x + 1) \qquad (0 < x < 1)$$

(b) $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(6/5)(x+y^2)}{(2/5)(3x+1)} = 3\,\frac{x+y^2}{3x+1}$$

Hence $$P(Y > 0.5 \mid X = 0.5) = \int_{0.5}^{1} f_{Y|0.5}(y \mid x = 0.5)dy$$

$$= 3\int_{0.5}^{1} \frac{0.5 + y^2}{1.5+1}dy = (3/2.5)\left[0.5y + \frac{y^3}{3}\right]_{0.5}^{1}$$

$$= \mathbf{0.65}$$

(c) $$E(XY) = \int_0^1\int_0^1 xyf_{X,Y}(x,y)dxdy = \frac{6}{5}\int_0^1\int_0^1 (x^2y + xy^3)dxdy$$

$$= \frac{2}{5}\int_0^1 ydy + \frac{3}{5}\int_0^1 y^3dy = 1/5 + 3/20 = 7/20 = 0.35$$

$\Rightarrow$ $Cov(X,Y) = E(XY) - E(X)E(Y) = 0.35 - (3/5)(3/5) = -0.01$,

while

$$\sigma_X = \{E(X^2) - [E(X)]^2\}^{1/2} = [(13/30) - (3/5)^2]^{1/2} = 0.271$$

$$\sigma_Y = \{E(Y^2) - [E(Y)]^2\}^{1/2} = [(11/25) - (3/5)^2]^{1/2} = 0.283,$$

Hence the correlation coefficient,

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{-0.01}{0.271 \times 0.283} \cong \mathbf{-\ 0.131}$$

3.59

(a) Summing over the last row, 2nd & 3rd columns of the given joint PMF table, we obtain

$$P(X \geq 2 \text{ and } Y > 20) = 0.1 + 0.1 = \mathbf{0.2}$$

(b) Given that X = 2, the new, reduced sample space corresponds to only the second column, where probabilities sum to (0.15 + 0.25 + 0.10) = 0.5, not one, so all those probabilities should now be divided by 0.5. Hence

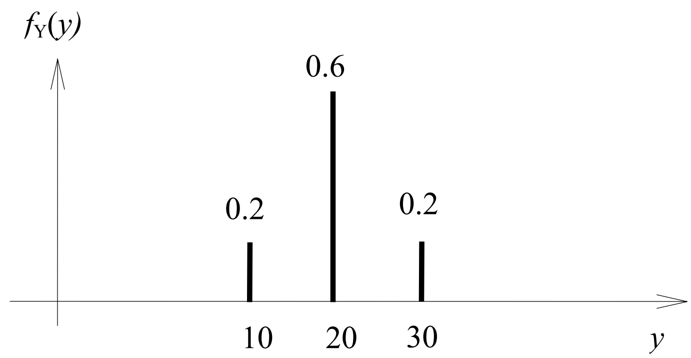
$$P(Y \geq 20 \mid X = 2) = 0.25 / 0.5 + 0.1 / 0.5 = 0.35 / 0.5 = \mathbf{0.7}$$

(c) If X and Y are s.i., then (say) $P(Y \geq 20)$ should be the same as $P(Y \geq 20 \mid X = 2) = 0.7$. However,

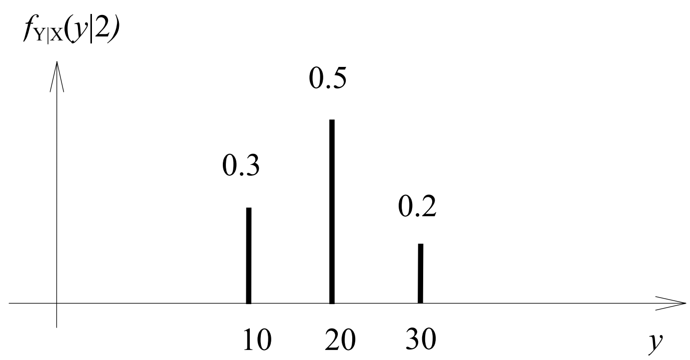
$$P(Y \geq 20) = 0.10 + 0.25 + 0.25 + 0.0 + 0.10 + 0.10 = 0.80 \neq 0.7,$$

hence X and Y are **not s.i.**

(d) Summing over each row, we obtain the (unconditional) probabilities P(Y = 10) = 0.20, P(Y = 20) = 0.60, P(Y = 30) = 0.20, hence the marginal PMF of runoff Y is as follows:



(e) Given that X = 2, we use the probabilities in the X = 2 column, each multiplied by 2 so that their sum is unity. Hence we have P(Y = 10 | X = 2) = 0.15×2 = 0.30, P(Y = 20 | X = 2) = 0.25×2 = 0.50, P(Y = 30 | X = 2) = 0.10×2 = 0.20, and hence the PMF plot:



(f) By summing over each column, we obtain the marginal PMF of X as $P(X = 1) = 0.15$, $P(X = 2) = 0.5$, $P(X = 3) = 0.35$. With these, and results from part (d), we calculate

$E(X) = 0.15\times1 + 0.5\times2 + 0.35\times3 = 2.2$,
$Var(X) = 0.15\times(1 - 2.2)^2 + 0.5\times(2 - 2.2)^2 + 0.35\times(3 - 2.2)^2 = 0.46$, similarly
$E(Y) = 0.2\times10 + 0.6\times20 + 0.2\times30 = 20$,
$Var(Y) = 0.2\times(10 - 20)^2 + 0.6\times(20 - 20)^2 + 0.2\times(30 - 20)^2 = 40$,

Also,
$E(XY) = \sum_{all} x\cdot y\cdot f(x,y)$
$= 1\times10\times0.05 + 2\times10\times0.15 + 1\times20\times0.10 + 2\times20\times0.25 + 3\times20\times0.25 + 2\times30\times0.10 + 3\times30\times0.10$
$= 45.5$

Hence the correlation coefficient is

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$
$$= \frac{45.5 - 2.2\times 20}{\sqrt{0.46\times 40}}$$
$$\cong \mathbf{0.35}$$

4.1

Since it leads to an important result, let us first do it for the general case where $X \sim N(\lambda, \zeta)$, i.e.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta} e^{-\frac{1}{2}\left(\frac{x-\lambda}{\zeta}\right)^2}$$

To obtain PDF $f_Y(y)$ for $Y = e^X$, one may apply Ang & Tang (4.6),

$$f_Y(y) = f_X(g^{-1}) \left|\frac{dg^{-1}}{dy}\right|$$

where

$$y = g(x) = e^x;$$

$\Rightarrow x = g^{-1}(y) = \ln(y)$

$\Rightarrow \left|\frac{dg^{-1}}{dy}\right| = \left|\frac{d}{dy}\ln y\right| = \left|\frac{1}{y}\right|$

$= \frac{1}{y}$ since $y = e^X > 0$

Hence, expressed as a function of y, the PDF $f_Y(y)$ is

$f_X(g^{-1}) \left|\frac{dg^{-1}}{dy}\right| = f_X(\ln(y)) \frac{1}{y}$

$= \frac{1}{\sqrt{2\pi}y\zeta} e^{-\frac{1}{2}\left(\frac{\ln y-\lambda}{\zeta}\right)^2}$ (non-negative y only)

which, by comparison to Ang & Tang (3.29), is exactly what is called a log-normal distribution with parameters λ and ζ, i.e. if $X \sim N(\lambda, \zeta)$, and $Y = e^X$, then $Y \sim LN(\lambda, \zeta)$.

Hence, for the particular case where $X \sim N(2, 0.4)$, Y is LN with parameters λ being 2 and ζ being 0.4.

4.2

To have a better physical feel in terms of probability (rather than probability density), let's work with the CDF (which we can later differentiate to get the PDF) of Y: since Y cannot be negative, we know that $P(Y < 0) = 0$, hence

when $y < 0$:

$$F_Y(y) = 0$$

$$\Rightarrow f_Y(y) = [F_Y(y)]' = 0$$

But when $y \geq 0$,

$$F_Y(y) = P(Y \leq y)$$

$$= P(\frac{1}{2}mX^2 \leq y)$$

$$= P(-\sqrt{\frac{2y}{m}} \leq X \leq \sqrt{\frac{2y}{m}})$$

$$= F_X\left(\sqrt{\frac{2y}{m}}\right) - F_X\left(-\sqrt{\frac{2y}{m}}\right)$$

$$= F_X\left(\sqrt{\frac{2y}{m}}\right) - 0$$

$$= F_X\left(\sqrt{\frac{2y}{m}}\right)$$

Hence the PDF,

$$f_Y(y) = \frac{d}{dy}\ [F_Y(y)]$$

$$= f_X\left(\sqrt{\frac{2y}{m}}\right)\frac{d}{dy}\sqrt{\frac{2y}{m}}$$

$$= \frac{8y}{ma^3\sqrt{\pi}}\exp(-\frac{2y}{ma^2})\ \frac{1}{\sqrt{2my}}$$

$$= \frac{4}{a^3}\sqrt{\frac{2y}{\pi m^3}}\exp(-\frac{2y}{ma^2})$$

Hence the answer is

$$f_Y(y) = \begin{cases} \frac{4}{a^3}\sqrt{\frac{2y}{\pi m^3}}\exp(-\frac{2y}{ma^2}) & y \geq 0 \\ 0 & y < 0 \end{cases}$$

4.3

A = volume of air traffic
C = event of overcrowded

(a) T = total power supply = $N(\mu_T, \sigma_T)$

Where $\mu_T = 100 + 200 + 400$

$\sigma_T = \sqrt{15^2 + 40^2 + 40^2} = 58.5$

(b) P(Normal weather) = P(W) = 2/3

P(Extreme weather) = P(E) = 1/3

P(Power shortage) = P(S) = P(S | W)P(W)+P(S | E)P(E)

= P(T<400)x2/3 + P(T<600)x1/3

$= [\Phi(\frac{400-400}{58.5})]\times\frac{2}{3} + [\Phi(\frac{600-400}{58.5})]\times\frac{1}{3}$

$= [\Phi(0)]\times\frac{2}{3} + [\Phi(3.42)]\times\frac{1}{3}$

= 0.5x0.667 + 0.99968x0.333

= 0.667

(c) $P(W \mid S) = \frac{P(S|W)P(W)}{P(S)} = \frac{0.5\times 0.667}{0.667} = 0.5$

(d) P(all individual power source can meet respective demand)

= P(N>0.15x400)P(F>0.3x400)P(H>0.55x400)

$= [1-\Phi(\frac{60-100}{15})][1-\Phi(\frac{120-200}{40})][1-\Phi(\frac{220-400}{40})]$

= Φ (2.67)x Φ (2)x Φ (4.5)

= 0.09962x0.977x1

= 0.973

P(at least one source not able to supply respective allocation) = 0.027

4.4

(a) Let T_J be John's travel time in (hours); $T_J = T_3 + T_4$ with

$$\mu_{T_J} = 5 + 4 = 9 \text{ (hours), and}$$

$$\sigma_{T_J} = [3^2 + 1^2 + (2)(0.8)(3)(1)]^{1/2} = 3.847 \text{ (hours)}$$

Hence

$$P(T_J > 10 \text{ hours}) = 1 - P\left(\frac{T_J - \mu_{T_J}}{\sigma_{T_J}} \le \frac{10-9}{3.847}\right)$$

$$= 1 - \Phi(0.26) = 1 - 0.603$$

$$\cong \mathbf{0.397}$$

(b) Let T_B be Bob's travel time in (hours); $T_B = T_1 + T_2$ with

$$\mu_{T_B} = 6 + 4 = 10 \text{ (hours), and}$$

$$\sigma_{T_B} = [2^2 + 1^2]^{1/2} = \sqrt{5} \text{ (hours)}$$

Hence

$$P(T_J - T_B > 1) = P(T_B - T_J + 1 < 0),$$

now let $R \equiv T_B - T_J + 1$; R is normal with

$$\mu_R = \mu_{T_B} - \mu_{T_J} + 1 = 10 - 9 + 1 = 2,$$

$$\sigma_R = [\sigma_{T_B}{}^2 + \sigma_{T_J}{}^2]^{1/2} = (5 + 14.8)^{1/2} = \sqrt{19.8},$$

hence

$$P(R < 0) = \Phi\left(\frac{0-2}{\sqrt{19.8}}\right)$$

$$= \Phi(-0.449)$$

$$\cong \mathbf{0.327}$$

(c) Since the lower route (A-C-D) has a smaller expected travel time of $\mu_{T_J} = 9$ hours as compared to the upper (with expected travel time = $\mu_{T_B} = 10$ hours), one should take the **lower** route to minimize expected travel time from A to D.

4.5

(a) To calculate probability, we first need to have the PDF of S. As a linear combination of three normal variables, S itself is normal, with parameters

$$\mu_S = 0.3\times5 + 0.2\times8 + 0.1\times7 = 3.8 \text{ (cm)}$$
$$\sigma_S = \sqrt{0.3^2 1^2 + 0.2^2 2^2 + 0.1^2 1^2} \cong 0.51 \text{ (cm)}$$

Hence

$$\begin{aligned} P(S > 4\text{cm}) &= 1 - \Phi\left(\frac{4-3.8}{0.51}\right) \\ &= 1 - \Phi(0.3922) = 1 - 0.6526 \\ &\cong \mathbf{0.347} \end{aligned}$$

(b) Now that we have a constraint $A + B + C = 20$m, these variables are no longer all independent, for example, we have

$$C = 20 - A - B$$

Hence

$$S = 0.3\,A + 0.2\,B + 0.1(20 - A - B)$$
$$\Rightarrow S = 0.2\,A + 0.1\,B + 2, \text{ with } \rho_{AB} = 0.5.$$

Thus

$$\mu_S = 0.2\times5 + 0.1\times8 + 2 = 3.8 \text{ (cm) as before, and}$$

$$\begin{aligned} \sigma_S &= \sqrt{0.2^2 1^2 + 0.1^2 2^2 + 2\times0.5\times0.1\times1\times2} \\ &= \sqrt{0.12} \cong 0.346 \text{ (cm)} \end{aligned}$$

Hence

$$\begin{aligned} P(S > 4\text{cm}) &= 1 - \Phi\left(\frac{4-3.8}{\sqrt{0.12}}\right) \\ &= 1 - \Phi(0.577) \cong \mathbf{0.282} \end{aligned}$$

4.6

$$
\begin{aligned}
Q &= 4\mathrm{A} + B + 2\mathrm{C} \\
&= 4\mathrm{A} + B + 2(30 - A - B) \\
&= 2\mathrm{A} - B + 60, \text{ hence} \\
\mu_Q &= 2\times 5 - 8 + 60 = 62, \\
\sigma_Q &= [4\times 3^2 + 1\times 2^2 + 2(2)(-1)(-0.5)(3)(2)]^{1/2} \\
&= (36 + 4 + 12)^{1/2} = \sqrt{52}
\end{aligned}
$$

$$
\begin{aligned}
\text{Hence } P(Q < 40) &= P\left(\frac{Q-\mu_Q}{\sigma_Q} < \frac{40-62}{\sqrt{52}}\right) \\
&= \Phi(-3.0508) \\
&\cong \mathbf{0.00114}
\end{aligned}
$$

4.7

(a) Let Q_1 and Q_2 be the annual maximum flood peak in rivers 1 and 2, respectively. We have

$$Q_1 \sim \text{N}(35, 10),\ Q_2 \sim \text{N}(25, 10)$$

The annual max. peak discharge passing through the city, Q, is the sum of them,

$$Q = Q_1 + Q_2, \text{ hence}$$

$$\mu_Q = \mu_{Q_1} + \mu_{Q_2} = 35 + 25 = \mathbf{60} \text{ (m}^3\text{/sec), and}$$

$$\sigma_Q^2 = \sigma_{Q_1}^2 + \sigma_{Q_2}^2 + 2\,\rho_{Q_1Q_2}\sigma_{Q_1}\sigma_{Q_2}$$

$$= 10^2 + 10^2 + 2\times0.5\times10\times10 = 300 \text{ (m}^3\text{/sec)},$$

$$\Rightarrow \sigma_Q = \sqrt{300} \cong \mathbf{17.32} \text{ (m}^3\text{/sec)}$$

(b) The annual risk of flooding, $p = \text{P}(Q > 100)$

$$= 1 - \Phi\left(\frac{100-60}{\sqrt{300}}\right)$$

$$= 1 - \Phi(2.309) = 1 - 0.9895$$

$$= \mathbf{0.0105} \text{ (probability each year)}$$

Hence the return period is $\tau = \dfrac{1}{p}$

$$= \frac{1}{0.0105} = 95.59643882$$

$$\cong \mathbf{95\ years}.$$

(c) Since the yearly risk of flooding is $p = 0.0105$, and we have a course of $n = 10$ years, we adopt a binomial model for X, the total number of flood years over a 10-year period.

P(city experiences (any) flooding) = 1 – P(city experiences no flooding at all)

$$= 1 - \text{P}(X = 0)$$

$$= 1 - (1-p)^{10} = 1 - (1 - 0.0105)^{10}$$

$$= 1 - 0.9895^{10}$$

$$\cong \mathbf{10\%}$$

(d) The requirement on p is, using the flooding probability expression from part (c):

$$1 - (1-p)^{10} = 0.1 \div 2 = 0.05$$

$$\Rightarrow p = 1 - (1 - 0.05)^{1/10} = 0.0051,$$

which translates into a condition on the design channel capacity Q_0, following what's done in (b),

$$1 - \Phi\left(\frac{Q_0 - 60}{\sqrt{300}}\right) = 0.0051$$

$$\Rightarrow \Phi\left(\frac{Q_0 - 60}{\sqrt{300}}\right) = 0.9949$$

$$\Rightarrow Q_0 = 60 + \Phi^{-1}(0.9949)\ \sqrt{300}$$

$$= 60 + 2.57\sqrt{300} = 104.5$$

$\therefore$ Extending the channel capacity to about **104.5 m^3/sec** will cut the risk by half.

4.8

$$
\begin{aligned}
P(T<30) &= P(T<30|N=0)P(N=0)+P(T<30|N=1)P(N=1)+P(T<30|N=2)P(N=2) \\
&= \Phi(\frac{30-30}{5})\times 0.2+\Phi(\frac{30-35}{\sqrt{25+3^2}})\times 0.5+\Phi(\frac{30-40}{\sqrt{25+2\times 3^2}})\times 0.3 \\
&= 0.5\times 0.2+\Phi(-0.857)\times 0.5+\Phi(-1.525)\times 0.3 \\
&= 0.217
\end{aligned}
$$

4.9

N = total time for one round trip in normal traffic

$= N(30+20+40, \sqrt{9^2 + 4^2 + 12^2})$

$= N(90, 15.52)$

R = total round trip time in rush hour traffic

$= N(30+30+40, \sqrt{9^2 + 6^2 + 12^2})$

$= N(100, 16.16)$

(a) P(on schedule in normal traffic)

= P(N<20)

$= \Phi(\frac{120-90}{15.52}) = \Phi(1.933) = 0.974$

(b) T_A = Normal time taken for passenger starting from A = $T_2 + T_3$

$= N(20+40, \sqrt{4^2 + 12^2}) = N(60, 12.65)$

$P(T_A < 60) = \Phi(\frac{60-60}{12.65}) = \Phi(0) = 0.5$

(c) Under rush hour traffic

$T_A = N(30+40, \sqrt{6^2 + 12^2}) = N(70, 13.4)$

$T_B = N(40, 12)$

$P(T_A<60) = \Phi(\frac{60-70}{13.4}) = \Phi(-0.746) = 0.227$

$P(T_B<60) = \Phi(\frac{60-40}{12}) = \Phi(1.667) = 0.952$

Percentage of passengers arriving in less than an hour

= 0.227x1/3 + 0.952x2/3

= 0.7

(d) $P(T_B < 60 \mid T_B > 45) = \frac{P(45 < T_B < 60)}{P(T_B > 45)}$

$$= \frac{\Phi(\frac{60-40}{12}) - \Phi(\frac{45-40}{12})}{1 - \Phi(\frac{45-40}{12})} = \frac{\Phi(1.667) - \Phi(0.417)}{1 - \Phi(0.417)} = \frac{0.952 - 0.661}{0.339} = 0.858$$

4.10

(a) $D = S_1 - S_2$

$$\mu_D = \mu_{S_1} - \mu_{S_2} = 2 - 2 = 0$$

$$Var(D) = Var(S_1) + Var(S_2) - 2\rho\sqrt{Var(S_1)Var(S_2)}$$

$$= (0.3 \times 2)^2 + (0.3 \times 2)^2 - 2 \times 0.7(0.3 \times 2)^2$$

$$= .216$$

(b) $P(-.5 < D < .5) = \Phi(\frac{0.5-0}{\sqrt{0.216}}) - \Phi(\frac{-0.5-0}{\sqrt{0.216}}) = \Phi(1.076) - \Phi(-1.076) = 0.718$

4.11

(a) $X_5 = X_1 + X_2 + X_3$

$\mu_5 = \mu_1 + \mu_2 + \mu_3 = 10 + 15 + 20 = 45$

$\sigma_5^2 = \sigma_1^2 + \sigma_2^2 + 2\rho_{1,2}\sigma_1\sigma_2 = 3^2 + 3^2 + 2 \times 0.6 \times 3 \times 3 = 28.8$

$\therefore \sigma_5 = 5.37$

(b) $V_1 = 60X_1$

$V_2 = 60X_2$

$\therefore P(V_2 - V_1 > 400) = P(Z > 400)$

$\therefore \mu_Z = 60\mu_2 - 60\mu_1 = 60 \times 15 - 60 \times 10 = 300$

$\sigma_Z^2 = 60^2\sigma_2^2 + 60^2\sigma_1^2 - 2\rho_{1,2} \times 60^2\sigma_1\sigma_2 = 25920$

$\therefore \sigma_Z = 161$

$$\therefore P(V_2 - V_1 > 400) = 1 - \Phi(\frac{400 - 300}{161}) = 1 - \Phi(.621) = .267$$

(c) $X_5 = X_1 + X_2 + X_3$

$\mu_5 = \mu_1 + \mu_2 + \mu_3 = 10 + 15 + 20 + 3n = 45 + 3n$

$\sigma_5^2 = \sigma_1^2 + \sigma_2^2 + 2\rho_{1,2}\sigma_1\sigma_2 = 3^2 + 3^2 + 2 \times 0.6 \times 3 \times 3 = 28.8$

$\therefore \sigma_5 = 5.37$

$\because P(X_5 > 70) = .05$

$\therefore P(X_5 < 70) = .95$

$$\Phi\left(\frac{70 - 45 - 3n}{5.37}\right) = .95$$

$\Rightarrow 25 - 3n = 5.37\Phi^{-1}(.95) = 5.37 * 1.645 = 8.839$

$\Rightarrow n = 5.4$

4.12

(a) P(X>20∪Y>20)=P(X>20)+P(Y>20)-P(X>20)P(Y>20)

$$= (1-\Phi(\frac{20-20}{4})) + (1-\Phi(\frac{20-15}{3})) - (1-\Phi(\frac{20-20}{4})) \times (1-\Phi(\frac{20-15}{3}))$$

$=0.5+0.0478-0.5\times 0.0478$
$=0.524$

(b) Z in normal distribution with

$\mu_Z = 0.6\times 20 + 0.4\times 15 = 18$

$\sigma_Z = \sqrt{0.6^2 4^2 + 0.4^2 3^2} = 2.683$

$\therefore P(Z > 20) = 1-\Phi(\frac{20-18}{2.683}) = 1-0.7718 = 0.2282$

(c) $\mu_Z = 0.6\times 20 + 0.4\times 15 = 18$

$\sigma_Z = \sqrt{0.6^2 4^2 + 0.4^2 3^2 + 2\times 0.8\times 0.6\times 0.4\times 4\times 3} = 3.436$

$\therefore P(Z > 20) = 1-\Phi(\frac{20-18}{3.436}) = 1-0.72 = 0.28$

4.13

(a) $P(X\geq 2)=1-P(X=0)-P(X=1)$

$$= 1-\binom{5}{0}0.6^0 0.4^5 - \binom{5}{1}0.6^1 0.4^4$$

$$= 1-0.4^5 - 5\times 0.6\times 0.4^4$$

$$= 0.046$$

(b) N_H, N_B are the number of highway and building jobs won

$P(N_H=1\cap N_B=0)=P(N_H=1)P(N_B=0)$

$$=\binom{3}{1}0.6^1 0.4^2 \binom{2}{0}0.6^0 0.4^2$$

$$=0.046$$

(c) $T=H_1+H_2+B$ where H_1, H_2 and B are the profits from the respective jobs.
T in Normal distribution with

$$\mu_T = 100+100+80 = 280$$

$$\sigma_T = \sqrt{40^2+40^2+20^2} = 60$$

$$\therefore P(T>300) = 1-\Phi(\frac{300-280}{60}) = 1-\Phi(0.333) = 0.2695$$

(d)

$$\mu_T = 100+100+80 = 280$$

$$\sigma_T = \sqrt{40^2+40^2+20^2+2\times 0.8\times 40\times 40} = 78.5$$

$$\therefore P(T>300) = 1-\Phi(\frac{300-280}{78.5}) = 1-\Phi(0.255) = 0.4$$

4.14

S: the amount of water available. S follows LogNormal Distribution with

E(S)=1, c.o.v=0.4

$\zeta_S^2 = \ln(1+0.4^2) = 0.14842$

$\lambda_S = \ln\mu_S - \frac{1}{2}\zeta_S^2 = \ln 1 - 0.5\times 0.14842 = -0.07421$

D: the total demand of water.

E(D)=1.5, c.o.v=0.1

$\zeta_D^2 = 0.1^2 = 0.01$

$\lambda_D = \ln\mu_D - \frac{1}{2}\zeta_D^2 = \ln 1.5 - 0.5\times 0.01 = 0.4005$

P(water shortage)=P(D>S)

=P(D/S>1)

=P((Z=ln(D/S))>0)

Z follows Normal Distribution with:

$Var(Z) = \zeta^2{}_D + \zeta_S^2 = 0.1^2 \;\; + 0.148 = 0.158$

$\mu_Z = \lambda_D - \lambda_S = 0.4005 - (-0.07421) = 0.4747$

$\Rightarrow P(Z>0) = 1 - \Phi(\dfrac{0-0.4747}{\sqrt{0.158}}) = 0.883$

4.15

(a) P is lognormally distributed with

$\lambda_P = \lambda_C + \lambda_R + 2\lambda_V - \ln 2$

$= (\ln 1.8 - 0.5\text{x}0.2^2) + (\ln 2.3\text{x}10^{-3} - 0.5\text{x}0.1^2) + 2[\ln 120 - 0.5\text{x}\ln(1+0.45^2)] - \ln 2$

$= 0.568 + (-6.08) + 2(4.6953) - 0.693$

$= 3.186$

ξ_P =

$$\sqrt{\xi_C^2 + \xi_R^2 + 4\xi_V^2} = \sqrt{0.2^2 + 0.1^2 + 4 \times \ln(1+0.45^2)} = \sqrt{0.2^2 + 0.1^2 + 4 \times 0.1844}$$

$= 0.887$

(b) P(P>30) = $1 - \Phi(\frac{\ln 30 - 3.186}{0.887}) = 1 - \Phi(0.243) = 0.596$

(c) C is lognormally distributed with

$\lambda_C = \ln 90 - 0.5\text{x}0.15^2 = 4.4886$

$\xi_C = 0.15$

P(Failure of antenna) = P(C<P) = P(C/P<1)

Define B = C/D

B also follows a lognormal distribution with

$\lambda_B = \lambda_C - \lambda_P = 4.4886 - 3.186 = 1.7$

$\xi_B = \sqrt{\xi_C^2 + \xi_P^2} = \sqrt{0.15^2 + 0.887^2} = 0.9$

Hence, P(Failure of antenna) = P(B<1)

$= \Phi(\frac{\ln 1 - \lambda_B}{\xi_B}) = \Phi(\frac{-1.7}{0.9}) = \Phi(-1.89) = 0.029$

(d) Mean rate of wind storm causing failure

= 1/5 x 0.029 = 0.0058

P(antenna failure in 25 years) = 1-P(no damaging storm in 25 years)

$= 1 - e^{-0.0058\text{x}25} = 0.135$

(e) P(at least two out of 5 antenna failures)

= 1 – P(X=0) – P(X=1)

$= 1 - 0.865^5 - 5(0.135)(0.865)^4$

= 0.138

4.16

(a) P(failure)=P(L>C)=P(C/L<1)=P(Z<1)

Z in LN with $\lambda_Z = \lambda_C - \lambda_L$

$$\zeta_Z = \sqrt{\zeta_C^2 + \zeta_L^2}$$

in which $\zeta_C = 0.2$

$$\lambda_C = \ln 20 - \frac{1}{2}(0.2)^2 = 2.259$$

$$\zeta_L = \sqrt{\ln(1+\delta^2)} = .294$$

$$\lambda_L = \ln 10 - \tfrac{1}{2}(0.294)^2 = 2.259$$

$$\therefore \lambda_Z = 2.976 - 2.259 = 0.717$$

$$\zeta_Z = \sqrt{0.2^2 + 0.294^2} = 0.356$$

$$\therefore P_F = \Phi(\frac{\ln 1 - 0.717}{0.356}) = \Phi(-2.014) = 1 - 0.978 = 0.022$$

(b) $T=C_1+C_2$

$E(T)=E(C_1)+E(C_2)=20+20=40$

$$Var(T) = Var(C_1) + Var(C_2) + 2\rho\sigma_{C_1}\sigma_{C_2} = 57.6$$

$$\delta_T = \frac{\sqrt{57.6}}{40} = 0.19$$

(c) some other distribution

4.17

(a) C=F+B

$$\mu_C = \mu_F + \mu_B = 20 + 30 = 50$$

$$\sigma_C = \sqrt{(0.2 \times 20)^2 + (0.3 \times 30)^2} = 9.85$$

(b) $T = C_1 + C_2 = 2\,\mu_C = 0.197$

$$\sigma_T = \sqrt{9.85^2 + 9.85^2 + 2 \times 0.8 \times 9.85^2} = 18.69$$

$$\therefore \delta_T = {}^{18.69}\!/_{100} = 0.187$$

(c) P(failure)=P(T<L)=P(T-L<0)

$$= \Phi(\frac{0 - \mu_Z}{\sigma_Z})$$

$$= \Phi(\frac{-(100-50)}{\sqrt{18.69^2 + (.3 \times 50)^2}}$$

$$= \Phi(\frac{-50}{23.95})$$

$$= 0.0184$$

4.18

(a) $F = 18 + \sum_{i=1}^{16} A_i$

$\mu_F = 18 + 16 \times 0.1 = 19.6$

$\sigma_F = \sqrt{(0.3 \times 0.1)^2 \times 16} = 0.12$

P(F>20)=1- $\Phi(\frac{20-19.6}{0.12}) = 0.00043$

(b)

i. P(no collapse)=P(C')=P(F<M)=P(F-M<0)=P(Z<0)

$\mu_Z = \mu_F - \mu_M = 19.6 - 20 = -0.4$

$\sigma_Z = \sqrt{(0.12)^2 + (0.01 \times 20)^2} = 0.233$

P(C')= $\Phi(\frac{0-(-0.4)}{0.233}) = 0.957$

ii. F=18+16A

$\mu_F = 18 + 16 \times 0.1 = 19.6$

$\sigma_F = \sqrt{(0.3 \times 0.1)^2 \times 16^2} = 0.48$

$\sigma_Z = \sqrt{(0.48)^2 + (0.01 \times 20)^2} = 0.52$

P(C')= $\Phi(\frac{0-(-0.4)}{0.52}) = 0.779$

4.19

(a) a is lognormal with mean 0.3g and c.o.v. of 25%

W = 200 kips

F = wa/g is also lognormal with

λ_F = ln 200 + λ_a = 5.298 – 1.204 = 4.094

$\xi_F = \xi_a = 0.25$

R = frictional resistance = WC

Where C = coefficient of friction is lognormal with median 0.4 and a c.o.v. of 0.2

Hence R is lognormal with

λ_R = ln 200 + λ_C = ln200 + ln0.4 = 4.382

$\xi_R = \xi_C = 0.2$

P(failure) = P(R<F)

$$= \Phi\left(\frac{-\lambda_R + \lambda_F}{\sqrt{\xi_R^2 + \xi_F^2}}\right) = \Phi\left(\frac{-4.382 + 4.094}{0.32}\right) = \Phi(-0.9) = 0.184$$

(b) P(none out of five tanks will fail)

$= (0.184)^5 = 0.00021$

4.20

For any one car, its average waiting time is μ_T = 5 minutes, with standard deviation σ_T = 5 minutes. Let W be the total waiting time of 50 cars. W is the sum of 50 i.i.d. random variables, hence, according to CLT, W is approximately normal with mean

$$\mu_W = 50\times\mu_T = 50\times5 \text{ min.} = 250 \text{ min.},$$

and standard deviation

$$\sigma_W = \sqrt{50}\times\sigma_T = \sqrt{50}\times5 \text{ min.}$$

Hence the probability

P(W < 3.5 hrs) = P(W < 210 mins)

$= P(\frac{W-\mu_W}{\sigma_W} < \frac{210-250}{5\sqrt{50}})$

= P(Z < -1.13137085) ≅ **0.129**

4.21

(a) First, let's calculate the parameters of F and X:

$$\xi_F = [\ln(1 + \delta_F^2)]^{0.5} = [\ln(1 + 0.2^2)]^{0.5} = [\ln(1.04)]^{0.5}$$
$$\therefore \lambda_F = \ln \mu_F - \xi_F^2/2 = \ln(0.2 / 1.04^{0.5}), \text{ similarly}$$
$$\xi_X = [\ln(1 + \delta_X^2)]^{0.5} = [\ln(1 + 0.3^2)]^{0.5} = [\ln(1.09)]^{0.5}$$
$$\therefore \lambda_x = \ln \mu_x - \xi_x^2/2 = \ln(10 / 1.09^{0.5})$$

Now, let M denote the bending moment at A. Since $M = FX$,

$\ln M = \ln F + \ln X$ (i.e. normal + normal), hence
M is lognormal (because $\ln M$ is normal), with parameters

$$\lambda_M = \lambda_F + \lambda_X = \ln\left(\frac{2}{\sqrt{1.04 \times 1.09}}\right), \text{ and}$$
$$\xi_M = (\xi_F^2 + \xi_X^2)^{0.5} = [\ln(1.04 \times 1.09)]^{0.5}, \text{ hence}$$

$$P(M > 3) = P\left(\frac{\ln M - \lambda_M}{\xi_M} > \frac{\ln 3 - \ln(2/\sqrt{1.04 \times 1.09})}{\sqrt{\ln(1.04 \times 1.09)}}\right)$$
$$= 1 - \Phi(1.322063427) \cong \mathbf{0.093}$$

(b) Let M_i be the moment at A due to the i-th force, where i = 1,2,…,50. Since each M_i is identically distributed as M in part (a), the lognormal parameters of M_i are $\lambda = 0.630447976$ and $\xi = 0.354116378$. From these, we can calculate the mean and standard deviation of M_i as

$$\mu_i = \exp(\lambda + \xi^2/2) = 2,$$
$$\sigma_i = \mu_i\, [\exp(\xi^2) - 1]^{0.5} = 0.731026675$$

The total bending moment at A is $T = M_1 + M_2 + \ldots + M_{50}$, the sum of a large number of identically distributed, independent RVs, hence we may apply the central limit theorem:

$$T \sim N(50 \times 2, \sqrt{50} \times 0.731026675)$$
$$\therefore P(T > 120) = P\left(\frac{T - \mu_T}{\sigma_T} > \frac{120 - 50 \times 2}{\sqrt{50} \times 0.731026675}\right)$$
$$= 1 - \Phi(3.869116163)$$
$$= 1 - 0.999945364$$
$$\cong \mathbf{0.000055}$$

4.22

(a) Let X be the monthly salary (in dollars) of a randomly chosen assistant engineer. X has a uniform distribution between 10000 and 20000, covering an area of $\frac{20000-16000}{20000-10000} \times 1$ = **0.4** from x = 16000 to x = 20000.

(b) Using properties of an uniform random variable,

$$\mu_X = (10000+20000)/2 = 15000,$$

$$\sigma_X = \frac{(20000-10000)}{\sqrt{12}} = \frac{5000}{\sqrt{3}}.$$

Let Y be the mean monthly salary of 50 assistant engineers. By the central limit theorem, Y is approximately normally distributed with

$$\mu_Y = \mu_X = 15000 \text{ and } \sigma_Y = \frac{\sigma_X}{\sqrt{50}} = \frac{5000}{\sqrt{150}},$$

hence the desired probability is

$$P(Y > 16000) = 1 - \Phi\left(\frac{16000-15000}{5000/\sqrt{150}}\right)$$

$$= 1 - \Phi(2.45) \cong \mathbf{0.007}$$

(c) An individual's probability of exceeding \$16000 is much higher than that for the mean of a group of people, as X has a much larger standard deviation than Y, though they have the same mean. Uncertainty is reduced as sample size increases, and "collective behavior" (i.e. average value) becomes very centralized (i.e. almost constantly at \$15000) around the mean, while individual behavior can differ from the mean significantly.

4.23

(a) Let C denote car weight and T denote truck weight. The total vehicle weight (in kips) on the bridge is

$$V = C_1 + C_2 + \dots + C_{100} + T_1 + T_2 + \dots + T_{30}$$

Since V is a linear combination of independent random variables, it mean and variance can be found as

$E(V) = 100\times E(C) + 30\times E(T) = 100\times 5 + 30\times 20 =$ **1100** (kips);

$Var(V) = 100\times Var(C) + 30\times Var(T) = 100\times 2^2 + 30\times 5^2 = 1150$ (kip^2),

Hence the c.o.v.

$\delta_V = 1150^{0.5} / 1100 \cong$ **0.031**

(b) Assuming that the distribution of V approaches normal due to CLT,

$$P(V > 1200) = 1 - \Phi\left(\frac{1200-1100}{\sqrt{1150}}\right)$$

$$= 1 - \Phi(2.948839) \cong \mathbf{0.0016}$$

(c)

(i) Let D denote the total dead load in kips, where $D \sim N(1200, 120)$

The difference $S = V - D$ is again normal,

$S \sim N(1100 - 1200, (1150 + 120^2)^{1/2}) = N(-100, 15550^{0.5})$, hence

$P(V > D) = P(V - D > 0) = P(S > 0)$

$$= P\left(\frac{S-\mu_S}{\sigma_S} > \frac{0-(-100)}{\sqrt{15550}}\right)$$

$$= 1 - \Phi(0.80192694) \cong \mathbf{0.211}$$

(ii) Let T denote the total (dead + vehicle) weight, $T = V + D$. $T \sim N(1100 + 1200, 15550^{0.5})$, hence

$P(V + D > 2500) = P(T > 2500)$

$$= P\left(\frac{T-\mu_T}{\sigma_T} > \frac{2500-2300}{\sqrt{15550}}\right)$$

$$= 1 - \Phi(1.60385388) \cong \mathbf{0.054}$$

4.24

(a) Let B be the total weight of one batch. $\mu_B = 40 \times 2.5$ kg = 100 kg

(b) Since $n > 30$, we may apply the Central Limit Theorem:

B is approximately normal with μ_B = 100 kg and $\sigma_B = \sqrt{40} \times 0.1$ kg, hence

$$P(\text{penalty}) = P(B > 101) = 1 - \Phi\left(\frac{101-100}{\sqrt{40} \times 0.1)}\right)$$
$$= 1 - \Phi(1.581)$$
$$= 1 - 0.9431$$
$$\cong \mathbf{5.69\%}$$

(c) In this case, $P(\text{penalty}) = P(B > 101) = 1 - \Phi\left(\frac{101-100}{\sqrt{40} \times 1)}\right)$
$$= 1 - \Phi(0.158)$$
$$= 1 - 0.562816481$$
$$\cong \mathbf{43.7\%}$$

This penalty probability is much higher, caused by the large standard deviation which makes the PDF occupy more area in the tails. Hence a large standard deviation (i.e. uncertainty) is undesirable.

4.25

Over the 50-year life, the expected number of change of occupancy is 25.

(a) The PDF of the lifetime maximum live load Y_n or Y_{20} in this case is, from Eq. 3.57,

$$f_{Y_n}(y) = f_{Y_{20}}(y) = 20[F_x(y)]^{19} f_x(y)$$

However X is lognormal with parameters λ, ξ whose values are

$$\xi = \sqrt{\ln(1+0.3^2)} = 0.294$$

$$\lambda = \ln 12 - 0.5 \mathrm{x} 0.294^2 = 2.44$$

Hence $$f_X(y) = \frac{1}{\sqrt{2\pi}(0.294)y} \exp[-\frac{1}{2}(\frac{\ln y - 2.44}{0.294})^2]$$

$$F_X(y) = \Phi(\frac{\ln y - 2.44}{0.294})$$

$$f_{Y_{20}}(y) = 20[\Phi\frac{\ln y - 2.44}{0.294}]^{19} \frac{1}{0.737y} \exp[-\frac{1}{2}(\frac{\ln y - 2.44}{0.294})]^2$$

(b) By following the results of Examples 3.34 and 3.37,
$Y_{20}(y)$ will converge asymptotically to the Type I distribution with parameters

$$u_n = 2.94(\sqrt{2\ln 20} - \frac{\ln\ln 20 + \ln 4\pi}{2\sqrt{2\ln 20}}) + 2.44 = 2.187$$

and $$\alpha_n = \frac{\sqrt{2\ln 20}}{0.294} = 8.326$$

$$\nu_n = e^{2.187} = 8.9$$

$$k = 8.326$$

$$f_{Y_{20}}(y) = \frac{8.326}{8.9}(\frac{8.9}{y})^{9.326} \exp[-(\frac{8.9}{y})^{8.326}]$$

4.26

(a) $$f_X(x) = \frac{\nu(\nu x)^{k-1} e^{-\nu x}}{\Gamma(k)}$$

Then,

$$F_X(x) = \int_0^x \frac{\nu(\nu z)^{k-1} e^{-\nu z}}{\Gamma(k)} dz$$

The CDF of the largest value from a sample of size n is (Eq. 4.2):

$$F_{Y_n}(y) = [F_X(y)]^n = [\int_0^x \frac{\nu(\nu z)^{k-1} e^{-\nu z}}{\Gamma(k)} dz]^n$$

whereas, the PDF is (Eq. 4.3):

$$f_{Y_n}(y) = n[F_X(y)]^{n-1} f_x(y) = n[\int_0^x \frac{\nu(\nu z)^{k-1} e^{-\nu z}}{\Gamma(k)} dz]^{n-1} \frac{\nu(\nu x)^{k-1} e^{-\nu x}}{\Gamma(k)}$$

(b) If the distribution of the large value is Type I asymptotic, then according Eq. 4.77,

$$\lim_{x\to\infty} \frac{d}{dx}[\frac{1}{h_n(x)}] = 0$$

The hazard function $h_n(x)$ is (Eq. 4.28):

$$h_n(x) = \frac{f_X(x)}{1 - F_X(x)} = \frac{\nu(\nu x)^{k-1} e^{-\nu x} / \Gamma(k)}{1 - \int_0^x \nu(\nu z)^{k-1} e^{-\nu z} dz / \Gamma(k)}$$

$$= \frac{\nu^k x^{k-1} e^{-\nu x}}{\Gamma(k) - \int_0^x \nu^k z^{k-1} e^{-\nu z} dz} = \frac{x^{k-1} e^{-\nu x}}{\nu^{-k}\Gamma(k) - \int_0^x z^{k-1} e^{-\nu z} dz}$$

$$\lim_{x\to\infty} \frac{d}{dx}[\frac{1}{h_n(x)}] = \lim_{x\to\infty} \frac{d}{dx}[\frac{\nu^{-k}\Gamma(k) - \int_0^x z^{k-1} e^{-\nu z} dz}{x^{k-1} e^{-\nu x}}]$$

Using Leibnitz rule for the derivative of the integral, the above limit becomes,

$$\lim_{x\to\infty}\frac{-x^{k-1}e^{-\nu x}x^{k-1}e^{-\nu x}-[\nu^{-k}\Gamma(k)-\int_0^x z^{k-1}e^{-\nu z}dz]e^{-\nu x}[(k-1)x^{k-2}-\nu x^{k-1}}{(x^{k-1}e^{-\nu x})^2}$$

$$=-1-\lim_{x\to\infty}\frac{[\nu^{-k}\Gamma(k)-\int_0^x z^{k-1}e^{-\nu z}dz](k-1-\nu x)}{x^k e^{-\nu x}}$$

Recall that

$$\int_0^\infty z^{k-1}e^{-\nu z}dz=\frac{\Gamma(k)}{\nu^k}$$

$$\lim_{x\to\infty}\{[\nu^{-k}\Gamma(k)-\int_0^x z^{k-1}e^{-\nu z}dz](k-1-\nu x)\}=0$$

and

$$\lim_{x\to\infty}(x^k e^{-\nu x})=0$$

Therefore, using the L'Hospital's rule:

$$-1-\lim_{x\to\infty}\frac{-x^{k-1}e^{-\nu x}(k-1-\nu x)-\nu[\nu^{-k}\Gamma(k)-\int_0^x z^{k-1}e^{-\nu z}dz]}{e^{-\nu x}x^{k-1}(k-\nu x)}$$

$$=-1-\lim_{x\to\infty}\frac{-(k-1-\nu x)}{k-\nu x}-\lim_{x\to\infty}\frac{-\nu[\nu^{-k}\Gamma(k)-\int_0^x z^{k-1}e^{-\nu z}dz]}{e^{-\nu x}x^{k-1}(k-\nu x)}$$

The second term is of the form $\frac{\infty}{\infty}$. Again, using L'Hospital's rule:

$$\lim_{x\to\infty}\frac{-(k-1-\nu x)}{k-\nu x}=\frac{\nu}{-\nu}=-1$$

The third term is of the form $\frac{0}{0}$, thus,

$$\lim_{x\to\infty}\frac{\nu x^{k-1}e^{-\nu x}}{e^{-\nu x}[-\nu x^{k-1}(k-\nu x)+(k-1)x^{k-2}(k-\nu x)-\nu x^{k-1}]}=0$$

since the order of the polynomial in the denominator is higher than that of the numerator.

Therefore, the distribution of the largest value converges to the Type I distribution.

4.27

(a) Let Z = X-18; Z follows an exponential distribution with mean = 3.2, i.e. λ = 1/3.2
Following the result of Example 3.33, the largest value of an exponentially distributed random variable will approach the Type I distribution.

Also, for large n, the CDF of the largest value

$$F_{Y_n}(y) = \exp(-ne^{-\lambda y})$$

which can be compared with the Type I distribution

$$F_{Y_n}(y) = \exp(-e^{-\alpha_n(y-u_n)})$$

By setting $ne^{-\lambda y} = e^{-\alpha_n(y-u_n)} = e^{-\alpha_n y}e^{\alpha_n u_n}$

We obtain $\alpha_n = \lambda$ and $u_n = \dfrac{\ln n}{\lambda}$

For a 1 year period, the number of axle loads is 1355. Hence mean maximum axle load is

$$u_{Y_{1355}} = 18 + [u_n + \frac{\gamma}{\alpha_n} = \frac{\ln 1355}{1/3.2} + \frac{0.577}{1/3.2}] = 42.9$$

$$\sigma_{Y_{1355}} = \frac{\pi^2}{6\alpha_n^2} = \frac{\pi^2(3.2)^2}{6} = 16.8 \qquad \therefore \delta_{Y_{1355}} = \frac{16.8}{42.9} = 0.39$$

Similarly for period of 5, 10 and 25 years, the number of axle loads are 6775, 13550 and 33875 respectively. The corresponding mean values and c.o.v. are as follows:

$\mu_{Y6775} = 48$	$\mu_{Y13550} = 50.3$	$\mu_{Y33875} = 53.2$
$\sigma_{Y6775} = 16.8$	$\sigma_{Y13550} = 16.8$	$\sigma_{Y33875} = 16.8$ and
$\delta_{Y6775} = 0.35$	$\delta_{Y13550} = 0.34$	$\delta_{Y33875} = 0.316$

(b) For a 20 year period, n = 27100

$$F_{Y_n}(y) = \exp(-e^{-\alpha_n(y-18-u_n)}) = \exp[-e^{-(1/3.2)(y-18-(\ln 27100)3.2)}]$$

Hence the probability that it will subject to an axle load of over 80 tone is

$$1 - F_{Y_{27100}}(80) = 1 - \exp[-e^{-(1/3.2)(80-18-(\ln 27100)3.2)}]$$

$$= 1 - 0.999896 = 0.000104$$

(c) For an exceedance probability of 10%, the “design axle load” L can be obtained from

$$1 - \exp[-e^{-(1/3.2)(L-18-(\ln 27100)3.2)}] \equiv 0.1$$

Hence, $\exp[-e^{-(1/3.2)(L-50.7)}] = 0.9$

L = 57.9 tons

4.28

(a) Daily DO level = N(3, 0.5)

From E 4.18, the largest value for an initial variate of N(μ , σ) follows a Type I Extreme Value distribution.

Because of the symmetry between the left tails of the normal distribution, it can be shown that the monthly minimum DO level also follows a Type I smallest value distribution with

$$u_1 = -\sqrt{2\ln 30} + \frac{\ln\ln 30 + \ln 4\pi}{2\sqrt{2\ln 30}} + 3 = 1.112$$
$$\alpha_1 = \sqrt{2\ln 30} / 0.5 = 5.21$$

Similarly, for the annual minimum DO, it also follows the Type I smallest value distribution with

$$u_1 = -\sqrt{2\ln 365} + \frac{\ln\ln 365 + \ln 4\pi}{2\sqrt{2\ln 365}} + 3 = -3.435 + 1.227 + 3 = 0.792$$
$$\alpha_1 = \sqrt{2\ln 365} / 0.5 = 6.87$$

(b) $P[(Y_1)_{30} < 0.5] = 1 - \exp[-e^{5.21(0.5-1.112)}] = 0.04$

$P[(Y_1)_{365} < 0.5] = 1 - \exp[-e^{6.87(0.5-0.792)}] = 0.125$

(c) For Type I smallest value distribution

$$\mu_1 = u_1 - \frac{0.577}{\alpha_1}; \qquad \sigma_1 = \frac{\pi^2}{6\alpha_1^2}$$

Hence for monthly minimum DO level,

$$\mu_1 = 1.112 - \frac{0.577}{5.21} = 1.0 mg/l; \qquad \sigma_1 = \frac{\pi^2}{6\times 5.21^2} = 0.06 mg/l$$

Similarly for the annual minimum level,

$$\mu_1 = 0.792 - \frac{0.577}{6.87} = 0.708 mg/l; \qquad \sigma_1 = \frac{\pi^2}{6\times 6.87^2} = 0.035 mg/l$$

4.29

Let Y be the maximum wind velocity during a hurricane

Y follows a Type I asymptotic distribution with $\mu_Y = 100$ and $\sigma_Y = 0.45$

From Equations 3.61a and 3.61b, the distribution parameters u_n and α_n can be determined as follows:

$$\frac{\pi^2}{6\alpha_n^2} = (0.45 \times 100)^2 \quad \text{or}$$

$$\alpha_n = \frac{\pi}{\sqrt{6}(0.45 \times 100)} = 0.0285$$

$$u_n = 100 - \frac{0.5772}{0.0285} = 79.75$$

(a) P(Damage during a hurricane)

= P(Y>150)

= $1 - F_Y(150) = 1 - \exp[-e^{-0.0285(150-79.75)}]$

= 0.126

(b) Let D be the revised design

Hence, $1 - F_Y(D) = 1 - \exp[-e^{-0.0285(D-79.75)}] = 0.063$

D = 175.6 kph

(c) Mean rate of damaging hurricane = ν

= (1/200) $\times$ 0.126 = 0.00063

P(Damage to original structure over 100 years)

= 1 – P(no damaging hurricane in 100 years)

= 1 – exp(-0.00063 $\times$ 100)

= 0.061

P(damage to revised structure over 100 years)

= 1 – $e^{-(0.00063/2)(100)}$

= 0.031

(d) P(at least one out of 3 structure with original design will be damaged over 100 years)

= 1 – P(none of the 3 structure damaged)

= 1 – $(1-0.061)^3$

= 0.172

4.30

(a) Daily maximum wind velocity is N(40, 10)

The maximum over 3 months will converge to a Type I largest value distribution whose CDF is (from Example 4.18

$$F_{Y_n}(y) = \exp[-e^{-\alpha_n(y-u_n)}]$$

with $u_n = \sqrt{2\ln n} - \dfrac{\ln\ln n + \ln 4\pi}{2\sqrt{2\ln n}} + \mu$

$$\alpha_n = \sqrt{2\ln n}/\sigma$$

For a 3 month period, use n = 91

Hence $u_n = \sqrt{2\ln 91} - \dfrac{\ln\ln 91 + \ln 4\pi}{2\sqrt{2\ln 91}} + 40 = 42.1$

$$\alpha_n = \sqrt{2\ln 91}/10 = 0.3$$

(b) The most probable wind velocity during the 3-month period is 42.1 mph

P(Damage)

= $P(Y_n > 70)$

= $1 - \exp[-e^{-0.3(70-42.1)}]$

= 0.00023

(c) Mean of $Y_n = u_n + \gamma/\alpha_n = 42.1 + 0.577/0.3 = 44$ mph

σ of $Y_n = \pi/(\sqrt{6}\alpha_n) = \pi/(0.3\sqrt{6}) = 4.277$

and c.o.v. of $Y_n = 4.277/44 = 0.097$

4.31

(a) Daily maximum wind velocity is lognormal with mean 40 and c.o.v. of 25%. The maximum wind velocity over a 3-month period will converge to a Type II, or the Fisher-Tippett distribution, whose CDF is

$$F_{Y_n}(y) = \exp[-(\frac{\nu_n}{y})^k]$$

with the most probable value ν_n and shape parameters k

$$\nu_n = e^{u_n} \qquad and \qquad k = \alpha_n$$

and

$$u_n = \zeta(\sqrt{2\ln n} - \frac{\ln\ln n + \ln 4\pi}{2\sqrt{2\ln n}}) + \lambda$$

$$\alpha_n = \frac{\sqrt{2\ln n}}{\zeta}$$

using the results obtained in Example 4.21

In this case, n = 91 ; $\zeta = 0.25$

$$\lambda = \ln 40 - \frac{1}{2}(0.25)^2 = 3.658$$

Hence the most probable wind velocity during erection period is

$$\nu_n = \exp[0.25(\sqrt{2\ln 91} - \frac{\ln\ln 91 + \ln 4\pi}{2\sqrt{2\ln 91}}) + 3.658$$

$$= \exp[4.18] = 65.4 \text{ mph}$$

Moreover,

$$k = \sqrt{2\ln 91}/0.25 = 12$$

(b) P(Damage)

$= P(Y_n > 70)$

$= 1 - \exp[-(\frac{65.4}{70})^{12}]$

$= 0.36$

4.32

$$h = \frac{LV^2 f}{2Dg}$$

Variable	Mean Value	c.o.v.	std. Deviation
L	100 ft	0.05	5 ft
D	1 ft	0.15	0.15 ft
f	0.02	0.25	0.005
V	9.0 fps	0.2	1.8 fps

(a) Assuming the above random variables are statistically independent, the first order mean and variance of the hydraulic head loss h are, respectively,

$$\mu_h \approx \frac{100(9)^2(0.02)}{2(1)(32.2)} = 2.516\,ft$$

$$\sigma_h^2 \approx \sigma_L^2 \left(\frac{\mu_V^2 \mu_f}{\mu_D (2g)}\right)^2 + \sigma_D^2 \left(\frac{\mu_L \mu_V^2}{\mu_D^2 (2g)}\right)^2 + \sigma_f^2 \left(\frac{\mu_L \mu_V^2}{\mu_D (2g)}\right)^2 + \sigma_V^2 \left(\frac{\mu_L 2\mu_V \mu_f}{\mu_D (2g)}\right)^2$$

=

$$5^2 \left(\frac{9^2 \times 0.02}{1(2\times 32.2)}\right)^2 + 0.15^2 \left(\frac{100\times 9^2}{1^2(2\times 32.2)}\right)^2 + 0.005^2 \left(\frac{100\times 9^2}{1(2\times 32.2)}\right)^2 + 1.8^2 \left(\frac{100\times 2\times 9\times 0.02}{1(2\times 32.2)}\right)^2$$

= 0.0158+355.9433+0.3955+1.01246 = 357.367

Hence, $\sigma_h \approx 18.9$ ft

(b) The second order mean of the hydraulic head loss

$$\mu_h = 2.516 + \frac{1}{2}\sigma_V^2 \left(\frac{2\mu_L \mu_f}{\mu_D (2g)}\right) + \frac{1}{2}\sigma_D^2 \left(\frac{\mu_L \mu_V^2 \mu_f}{\mu_D^3 (2g)}(2)\right)$$

$$= 2.516 + \frac{1}{2}1.8^2 \left(\frac{2\times 100\times 0.02}{1(2\times 32.2)}\right) + \frac{1}{2}0.15^2 \left(\frac{100\times 9^2 \times 0.02}{1^3(2\times 32.2)}(2)\right)$$

= 2.516 + 0.1006 + 0.056

= 2.673 ft

4.33

$$D = \frac{PL^3}{3EI}, \qquad I = \frac{bh^3}{12} \qquad \Rightarrow D = \frac{4PL^3}{Ebh^3}$$

Variable	Mean Value	c.o.v.	std. Deviation
P	500 lb	0.2	100 lb
E	3,000,000 psi	0.25	750,000 psi
b	72 inch	0.05	3.6 inch
h	144 inch	0.05	7.2 inch

L = 15 ft = 180 inch

(a) Assuming the above random variables are statistically independent, the first order mean and variance of the deflection D are, respectively,

$$\mu_D \approx \frac{4 \times 500 \times 180^3}{3000000 \times 72 \times 144^3} = 1.8 \times 10^{-5} inch$$

$$\sigma_D^2 \approx \sigma_P^2 (\frac{4L^3}{\mu_E \mu_b \mu_h^3})^2 + \sigma_E^2 (\frac{4\mu_P L^3}{\mu_E^2 \mu_b \mu_h^3})^2 + \sigma_b^2 (\frac{4\mu_P L^3}{\mu_E \mu_b^2 \mu_h^3})^2 + \sigma_h^2 (\frac{4\mu_P L^3 \times 3}{\mu_E \mu_b \mu_h^4})^2$$

$$+ 2 \times \rho_{bh} \times \sigma_b \times \sigma_h \times \frac{4\mu_P L^3}{\mu_E \mu_b^2 \mu_h^3} \times \frac{4\mu_P L^3 \times 3}{\mu_E \mu_b \mu_h^4}$$

$$= 100^2 (\frac{4 \times 180^3}{3e6 \times 72 \times 144^3})^2 + 750000^2 (\frac{4 \times 500 \times 180^3}{(3e6)^2 \times 72 \times 144^3})^2 + 3.6^2 (\frac{4 \times 500 \times 180^3}{3e6 \times 72^2 \times 144^3})^2$$

$$+ 7.2^2 (\frac{4 \times 500 \times 180^3 \times 3}{3e6 \times 72 \times 144^4})^2 + 2 \times 0.8 \times 3.6 \times 7.2 \times \frac{4 \times 500 \times 180^3 \times 3}{3e6 \times 72^2 \times 144^3} \times \frac{4 \times 500 \times 180^3 \times 3}{3e6 \times 72 \times 144^4}$$

$= 1.308 \times 10^{-11} + 2.044 \times 10^{-11} + 8.176 \times 10^{-13} + 7.359 \times 10^{-12} + 3.925 \times 10^{-12} = 4.562 \times 10^{-11}$

Hence, $\sigma_D \approx 6.75 \times 10^{-6}$ ft

(b) The second order mean of the deflection

$$\mu_D = 1.8 \times 10^{-5} + \frac{1}{2} [\sigma_E^2 (\frac{4\mu_P L^3 (-1)(-2)}{\mu_E^3 \mu_b \mu_h^3}) + \sigma_b^2 (\frac{4\mu_P L^3 (-1)(-2)}{\mu_E \mu_b^3 \mu_h^3})^2 + \sigma_h^2 (\frac{4\mu_P L^3 (-3)(-4)}{\mu_E \mu_b \mu_h^5})]$$

=

$$1.8 \times 10^{-5} + \frac{4 \times 500 \times 180^3}{2} [\frac{750000^2 \times 2}{(3e6)^3 \times 72 \times 144^3} + \frac{3.6^2 \times 2}{3e6 \times 72^3 \times 144^3} + \frac{7.2^2 \times 12}{3e6 \times 72 \times 144^5}]$$

$= 1.94 \times 10^{-5}$ ft

4.34

(a) Using *Ang & Tang* Table 5.1,

$$E(X) = (4 + 2) / 2 = \mathbf{3},\ Var(X) = (4 - 2)^2 / 12 = \mathbf{1/3};$$

$$E(Y) = 0 + (3 - 0) / (1 + 2) = \mathbf{1},\ Var(Y) = 1\times2\times(3 - 0)^2/(1+2)^2(1+2+1) = \mathbf{1/2};$$

Also, for W, since we know its median is 1, i.e.

$$\int_0^1 \lambda e^{-\lambda x}dx = 0.5 \Rightarrow 1 - e^{-\lambda} = 0.5 \Rightarrow \lambda = \ln 2 \text{, hence}$$

$$E(W) = 1/\lambda \approx \mathbf{1.443},\ Var(W) = 1/\lambda^2 \approx \mathbf{2.081}$$

(b) Since $Z = XY^2W^{0.5}$, we compute the approximate values

$E(Z) \approx E(X)\ [E(Y)]^2\ [E(W)]^{0.5} = 3\times1^2\times1.443^{0.5} \approx \mathbf{3.6}$, and

$$Var(Z) \approx \left[\frac{\partial Z}{\partial X}\right]_\mu^2 Var(X) + \left[\frac{\partial Z}{\partial Y}\right]_\mu^2 Var(Y) + \left[\frac{\partial Z}{\partial W}\right]_\mu^2 Var(W)$$

$$= \left[Y^2W^{0.5}\right]_\mu^2 Var(X) + \left[2XYW^{0.5}\right]_\mu^2 Var(Y) + \left[0.5XY^2W^{-0.5}\right]_\mu^2 Var(W)$$

where μ denotes "evaluated at the mean values $X = 3$, $Y = 1$, $W = 1.443$", hence $Var(Z) \approx \mathbf{29.7}$,

Thus the c.o.v. of Z is $\dfrac{\sqrt{29.7}}{3.6} \approx \mathbf{1.51}$

4.35

(a) Let $X' \equiv \ln(X)$. Since X is log-normal, X' is normal with parameters

$$\sigma_{X'} \cong \delta_X = 0.20, \text{ and}$$
$$\begin{aligned}\mu_{X'} &\cong \ln 100 - \delta_X^2 / 2 \\ &= \ln 100 - 0.20^2 / 2 \\ &= 4.585170186\end{aligned}$$

When L is a constant (0.95), we have

$$Y = (1 - 0.95)X = 0.05\,X$$

Taking the log of both sides and letting $Y' \equiv \ln(Y)$

$$\Rightarrow Y' = \ln 0.05 + X',$$

i.e. Y' is a constant plus a normal variate, hence Y' is normal with parameters

$$\mu_{Y'} = \ln 0.05 + \mu_{X'} \cong 1.589437912, \text{ and}$$
$$\sigma_{Y'} = \sigma_{X'} \cong 0.20$$

Hence

$$\begin{aligned}\text{P(acceptable)} &= \text{P}(Y \le 8) \\ &= \text{P}(Y' \le \ln 8) \\ &= \text{P}\left(\frac{Y' - \mu_{Y'}}{\sigma_{Y'}} \le \frac{\ln 8 - 1.589437912}{0.20}\right) \\ &= \Phi(2.450018146) \\ &\cong \mathbf{0.993}\end{aligned}$$

(b)

(i) When L is also a random variable, $Y = (1 - L)X$ is a non-linear function of two random variables, hence we need to estimate its mean and variance by

$$\begin{aligned}\text{E}(Y) &\cong Y(L = \mu_L, X = \mu_X) \\ &= (1 - \mu_L)(\mu_X) = (1 - 0.96)(100) \\ &= \mathbf{4}, \text{ and}\end{aligned}$$

$$\begin{aligned}\text{Var}(Y) &\cong \left(\frac{\partial Y}{\partial L}\right)^2_{\underline{\mu}} Var(L) + \left(\frac{\partial Y}{\partial X}\right)^2_{\underline{\mu}} Var(X) \\ &= \\ &(-X)^2_{X=100, L=0.96}(0.01^2) + (1 - L)^2_{X=100, L=0.96}(100 \times 0.2)^2 \\ &= 10000 \times 0.0001 + 0.0016 \times 400 \\ &= \mathbf{1.64}\end{aligned}$$

(ii) Assuming a log-normal distribution for Y and using its estimated mean and variance from (i), we have the approximate parameter values

$$\delta_Y \cong \frac{\sqrt{1.64}}{4}, \text{ hence}$$

$$\xi_Y = \sqrt{\ln(1+\delta_Y^2)}$$
$$\cong \sqrt{\ln(1+1.64/16)} = 0.312;$$
$$\lambda_Y \cong \ln 4 - 0.312^2 / 2 = 1.338, \text{ thus}$$
$$P(Y \leq 8) = \Phi\left(\frac{\ln 8 - 1.338}{0.312}\right)$$
$$= \Phi(2.375)$$
$$\cong \mathbf{0.991}$$

No, this offer from the contractor does not yield a larger performance acceptance probability.

4.36

(a) $E(Q_c)=0.463E(n)^{-1}E(D)^{2.67}E(S)^{0.5}=0.463*0.015^{-1}*3.0^{2.67}*0.005^{0.5}=41.0\text{ft}^3/\text{sec}$

$$\text{Var}(Q_c)=\left[\frac{\partial Q_c}{\partial n}\right]_\mu^2 Var(n)+\left[\frac{\partial Q_c}{\partial D}\right]_\mu^2 Var(D)+\left[\frac{\partial Q_c}{\partial S}\right]_\mu^2 Var(S)$$

$$=\begin{array}{l}\left(-0.463D^{2.67}S^{0.5}D^{-2}\right)_\mu^2\ Var(n)+\left(0.463n^{-1}\times 2.67D^{1.67}S^{0.5}\right)_\mu^2 Var(D)\\ +\left(0.463n^{-1}D^{2.67}\ \times 0.5S^{-0.5}\right)_\mu^2 Var(S)\end{array}$$

$=22.665(\text{ft}^3/\text{sec})^2$

(b) The percentage of contribution of each random variable to the total uncertainty:

$$\left[\frac{\partial Q_c}{\partial n}\right]_\mu^2 Var(n)/Var(Q_c)=74.2\%$$

$$\left[\frac{\partial Q_c}{\partial D}\right]_\mu^2 Var(D)/Var(Q_c)=21.2\%$$

$$\left[\frac{\partial Q_c}{\partial S}\right]_\mu^2 Var(S)/Var(Q_c)=4.6\%$$

(c) Q_C follows the log-normal distribution with:

$\mu_{Q_C}=41.0$

$\sigma_{Q_C}^2=22.665$

$$\Rightarrow \zeta\approx\frac{\sigma}{\mu}=\frac{\sqrt{22.665}}{41}=0.116$$

$$\lambda=\ln\mu-\tfrac{1}{2}\zeta^2=3.707$$

$$\therefore P(Q_C>30)=1-\Phi\left(\frac{\ln 30-3.707}{0.116}\right)=0.9958$$

4.37

(a) $T=\sum_{i=1}^{50} R_i$

$$\mu_T = 50\times 1 = 50$$

$$\sigma_T = \sqrt{50\times 1^2} = 7.07$$

$$\therefore P(T>60) = 1-\Phi(\frac{60-50}{7.07}) = 1-\Phi(1.414) = 1-0.9213 = 0.078$$

(b)

i. $\overline{V} = \overline{N}\,\overline{R}\ln(\overline{R}+1)$

=1*1*ln(1+1)

=ln2

=0.693

$$\text{Var(V)}=\left(\frac{\partial V}{\partial N}\right)^2 Var(N)+\left(\frac{\partial V}{\partial R}\right)^2 Var(R)$$

$$=\left[\overline{R}\ln(\overline{R}+1)\right]^2(0.5)^2+\left[\overline{N}(\frac{R}{R+1}+\ln(R+1))\right]^2\times 1$$

$=(\ln 2)^2(0.5)^2+(1/2+\ln 2)^2\times 1$

=0.48*0.25+1.424*1

=0.12+1.424

=1.544

$$\therefore \delta_V = \frac{\sqrt{1.544}}{0.693} = 1.793$$

ii. The approximation is not expected to be good because:

a) R and N have large variance;

b) The function is highly nonlinear

Check the second order term for the mean:

$$\frac{1}{2}\frac{\partial^2 V}{\partial R^2}\cdot Var(R) = \frac{1}{2}\left[\overline{N}(\frac{(\overline{R}+1)-\overline{R}}{(\overline{R}+1)^2}+\frac{1}{\overline{R}+1})\right]Var(R) = 0.375$$

which is not that small compared to the first order term.

$\therefore$ the approximations are not good.

4.38

$$w = \sqrt{\frac{K}{M}}$$

(a) M = 100, $\mu_K = 400$, $\sigma_K = 200$

The first order mean and variance of w are

$$\mu_w \approx \frac{1}{10}\sqrt{\mu_k} = \frac{1}{10}\sqrt{400} = 2$$

$$\sigma_w^2 \approx \sigma_K^2 (\frac{1}{10}\frac{1}{2}\mu_K^{-\frac{1}{2}})^2 = 200^2 (\frac{1}{2\sqrt{400}})^2 = 0.25$$

Hence, $\sigma_w = 0.5$

(b) M is also random with $\mu_M = 100$ and $\sigma_M = 20$

The first order mean and variance of w are

$$\mu_w \approx \sqrt{\frac{\mu_K}{\mu_M}} = \sqrt{\frac{400}{100}} = 2$$

$$\sigma_w^2 \approx 0.25 + \sigma_M^2 (\sqrt{\mu_K}\,\mu_M^{-3/2}\,\frac{1}{2})^2 = 0.25 + 20^2 ((20)(100)^{-3/2}(\frac{1}{2}))^2$$

$= 0.25 + 0.04$
$= 0.29$

Hence, $\sigma_w = 0.538$

(c) The second order approximate mean of w is

$$\mu_w \approx 2 + \frac{1}{2}\sigma_K^2 (\frac{\partial^2 w}{\partial K^2})_\mu + \frac{1}{2}\sigma_M^2 (\frac{\partial^2 w}{\partial M^2})_\mu$$

$$= 2 + \frac{1}{2}\sigma_K^2 (-\frac{1}{2})(\frac{1}{2})\mu_K^{-3/2}\mu_M^{-1/2} + \frac{1}{2}\sigma_M^2 (-\frac{1}{2})(-\frac{3}{2})\mu_K^{1/2}\mu_M^{-5/2}$$

$$= 2 - \frac{1}{8}(200)^2(400)^{-3/2}(100)^{-1/2} + \frac{3}{8}(20)^2(400)^{1/2}(100)^{-5/2}$$

$= 2 - 0.0625 + 0.03$
$= 1.968$

5.1

MATHCAD statements:

$D := \mathrm{rnorm}(100000, 4.2, 0.3)$

$\mu L := 6.5 \qquad \sigma L := 0.8 \qquad \delta L := \frac{\sigma L}{\mu L} \qquad \zeta L := \sqrt{\ln\left(1 + \delta L^2\right)}$

$\lambda L := \ln(\mu L) - 0.5 \cdot \zeta L^2 \qquad \lambda L = 1.864$

$L := \mathrm{rlnorm}(100000, \lambda L, \zeta L)$

$\mu w := 3.4 \qquad \sigma w := 0.7$

$\alpha := \frac{\pi}{\sqrt{6} \cdot \sigma w} \qquad \alpha = 1.832$

$\beta := \mu w - \frac{0.57721566}{\alpha} \qquad \beta = 3.085$

$u := \mathrm{runif}(100000, 0, 1)$

$W := \beta - \frac{1}{\alpha} \cdot \ln\left(\ln\left(\frac{1}{u}\right)\right)$

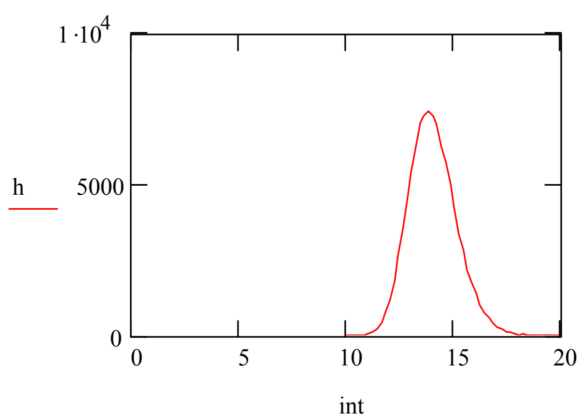
$S := \overrightarrow{(D + L + W)}$

$n := 100$

$j := 0 .. n$

$int_j := 0 + 20 \cdot \frac{j}{n}$

$h := \mathrm{hist}(int, S)$



$\mathrm{mean}(S) = 14.102$

$\mu R := 1.5 \cdot \mathrm{mean}(S) \qquad \mu R = 21.152$

$$\delta R := 0.15$$

$$\zeta R := \sqrt{\ln\left(1 + \delta R^2\right)} \qquad \zeta R = 0.149$$

$$\lambda R := \ln(\mu R) - \frac{1}{2} \cdot \zeta R^2 \qquad \lambda R = 3.041$$

$$R := \text{rlnorm}(100000, \lambda R, \zeta R)$$

$$x := \overrightarrow{[(R - S) < 0]}$$

$$pF := \frac{\sum_{i=1}^{99999} x_i}{100000} \qquad pF = 9.76 \times 10^{-3}$$

The result obtained with Mathcad with a sample size of 100,000 yields the probability of R<S is 0.009

5.2

MATHCAD statements:

$W := \mathrm{rlnorm}(10000, \ln(2000), 0.20)$

Distribution of F is beta

$a := 0$ $\mu := 20$ $b := 2 \cdot \mu$ therefore, $b = 40$

$cov := 0.15$ $\sigma := \mu \cdot cov$ therefore, $\sigma = 3$

$$q := \frac{\frac{(b-a)^2}{2\sigma^2} - 1}{2}$$ therefore, $q = 199.5$

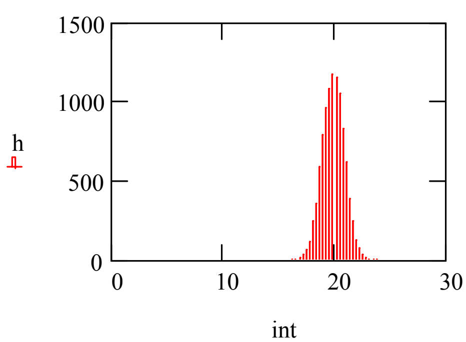
$r := q$

$x1 := \mathrm{rbeta}(10000, q, r)$

$F := (b - a) \cdot x1 + a$

$n := 100$ $j := 0 .. n$ $int_j := 0 + 30 \cdot \frac{j}{n}$

$h := \mathrm{hist}(int, F)$



$E := \mathrm{rnorm}(10000, 1.6, 0.125 \cdot 1.6)$

$$C := \frac{\overrightarrow{W \cdot F}}{\sqrt{E}}$$

$s1 := C > 35000$

$$p1 := \frac{\sum_{i=1}^{9999} s1_i}{10000}$$

therefore, the probability that the annual cost of operating the waste treatment plant will exceed $35,000 is

$$p1 = 0.3224$$

5.3

(a) Define water supply as random variable S, total demand of water is D

MATHCAD statements:

$$\mu S := 1$$

$$\delta S := 0.40 \qquad \sigma S := \mu S \cdot \delta S$$

$$\zeta S := \sqrt{\ln\left(1 + \delta S^2\right)} \qquad \zeta S = 0.385$$

$$\lambda S := \ln(\mu S) - \frac{1}{2}\zeta S^2 \qquad \lambda S = -0.074$$

$$S := \text{rlnorm}(100000, \lambda S, \zeta S)$$

$$\mu D := 1.5 \qquad \delta D := 0.1$$

$$\zeta D := \delta D \qquad \lambda D := \ln(\mu D) - \frac{1}{2}\zeta D^2 \qquad \lambda D = 0.4$$

$$D := \text{rlnorm}(100000, \lambda D, \zeta D)$$

$$x := \overrightarrow{(S < D)}$$

$$pF := \frac{\sum_{i=1}^{99999} x_i}{100000} \qquad pF = 0.884$$

Define random variable Y=S/D, therefore event of water shortage is Y<1

$$\zeta Y := \sqrt{\zeta S^2 + \zeta D^2} \qquad \zeta Y = 0.398$$

$$\lambda Y := \lambda S - \lambda D \qquad \lambda Y = -0.475$$

$$pf := \text{plnorm}(1.0, \lambda Y, \zeta Y) \qquad pf = 0.883$$

Therefore, the result by MCS is very close to the exact solution.

(b) S follows beta distribution

MATHCAD statements:

$$q := 2.00 \qquad r := 4.00$$

$$a := \mu S - q \cdot \sigma S \cdot \sqrt{\frac{q + r + 1}{q \cdot r}} \qquad a = 0.252$$

$$b := a + \sigma S \cdot \sqrt{\frac{q + r + 1}{q \cdot r}} \cdot (q + r) \qquad b = 2.497$$

$$v := \text{rbeta}(100000, q, r)$$

$$S := v \cdot (b - a) + a$$

$$x := \overrightarrow{(S < D)}$$

$$pF := \frac{\sum_{i=1}^{99999} x_i}{100000} \qquad pF = 0.863$$

(C)

MATHCAD statements:

μ_S is uniform distributed.

$$\mu S := \text{runif}(1000, 0.80, 1.1) \qquad \delta S := 0.40$$

$$\zeta S := \sqrt{\ln\left(1 + \delta S^2\right)} \qquad \zeta S = 0.385$$

$$\lambda S := \ln(\mu S) - \frac{1}{2}\zeta S^2$$

$$\mu D := 1.5$$

$$\delta D := 0.1 \qquad \zeta D := \delta D \qquad \lambda D := \ln(\mu D) - \frac{1}{2}\zeta D^2 \qquad \lambda D = 0.4$$

$$pF := \left| \begin{array}{l} \text{for } i \in 0..999 \\ \quad \left| \begin{array}{l} S \leftarrow \text{rlnorm}\left(1000, \lambda S_i, \zeta S\right) \\ D \leftarrow \text{rlnorm}\left(1000, \lambda D, \zeta D\right) \\ x \leftarrow \overrightarrow{(S \le D)} \\ pF_i \leftarrow \dfrac{\sum_{j=1}^{999} x_j}{1000} \\ pF \end{array} \right. \\ pF \end{array} \right.$$

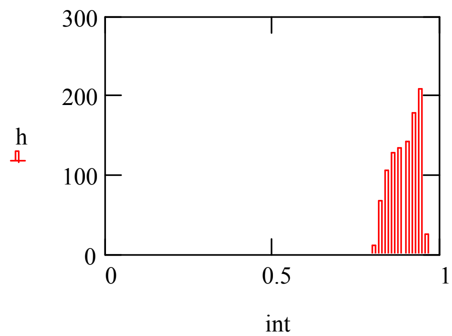
$$\text{mean}(pF) = 0.903 \qquad \text{Stdev}(pF) = 0.04 \qquad \text{skew}(pF) = -0.322$$

$$n := 50$$

$$j := 1..n$$

$$int_j := 0 + 1.0 \cdot \frac{j}{n}$$

$$h := \text{hist}(int, pF)$$



300
200
100
0
h
0
0.5
1
int

5.4

Define the total travel time for one round trip under normal traffic as TR1, the one during rush hours is TR2

(a)

MATHCAD statements:

$T1 := \text{rnorm}(100000, 30, 9)$

$\zeta T2 := 0.2 \qquad \mu T2 := 20$

$\lambda T2 := \ln(\mu T2) - \frac{1}{2}\zeta T2^2 \qquad \lambda T2 = 2.976$

$\zeta T3 := 0.3 \qquad \mu T3 := 40$

$\lambda T3 := \ln(\mu T3) - \frac{1}{2}\zeta T3^2 \qquad \lambda T3 = 3.644$

$T2 := \text{rlnorm}(100000, \lambda T2, \zeta T2)$

$T3 := \text{rlnorm}(100000, \lambda T3, \zeta T3)$

$TR1 := T1 + T2 + T3$

$x := \overrightarrow{(TR1 - 2.0 \cdot 60 > 0.0)}$

$$pf := \frac{\sum_{i=1}^{99999} x_i}{100000} \qquad pf = 0.037$$

(b) MATHCAD statements:

$x := \overrightarrow{[(T2 + T3) < 60]}$

$$pf := \frac{\sum_{i=1}^{99999} x_i}{100000} \qquad pf = 0.55$$

(c) MATHCAD statements:

$\zeta T2 := 0.2 \qquad \mu T2 := 20 \cdot 2$

$\lambda T2 := \ln(\mu T2) - \frac{1}{2}\zeta T2^2 \qquad \lambda T2 = 3.669$

$TRA := T2 + T3$

$$xA := \overrightarrow{[(TRA) < 60]}$$

$$pfA := \frac{\sum_{i=1}^{99999} xA_i}{100000} \qquad pfA = 0.55$$

$$pfB := \mathrm{plnorm}(60, \lambda T3, \zeta T3) \qquad pfB = 0.933$$

$$pfA \cdot \frac{1}{3} + pfB \cdot \frac{2}{3} = 0.806$$

Therefore, 80.7% of passengers will arrive the shopping center during rush hours in less than one hour.

(d) The passenger left Town B at 2:00pm, therefore the traffic is normal

P(T3<60|T3>45)=?

MATHCAD statements:

$$\zeta T3 := 0.3 \qquad \mu T3 := 40 \qquad \lambda T3 := \ln(\mu T3) - \frac{1}{2}\zeta T3^2$$

$$T3 := \mathrm{rlnorm}(100000, \lambda T3, \zeta T3)$$

$$x := \overrightarrow{(T3 > 45)}$$

$$y := \overrightarrow{(T3 > 45 \wedge T3 < 60)}$$

$$pf := \frac{\sum_{i=1}^{99999} y_i}{\sum_{i=1}^{99999} x_i}$$

$$pf = 0.776$$

Therefore, the probability he will arrive on time is 0.776

Verify the simulation result by directly calculate the probability by cumulative distribution function

$$pf1 := \mathrm{plnorm}(60, \lambda T3, \zeta T3) - \mathrm{plnorm}(45, \lambda T3, \zeta T3)$$

$$pf2 := 1 - \mathrm{plnorm}(45, \lambda T3, \zeta T3)$$

$$pf3 := \frac{pf1}{pf2} \qquad pf3 = 0.773$$

5.5

S1 and S2 follow bivariate normal distribution

(a) MATHCAD statements are:

$u1 := \text{runif}(100000, 0, 1)$

$u2 := \text{runif}(100000, 0, 1)$

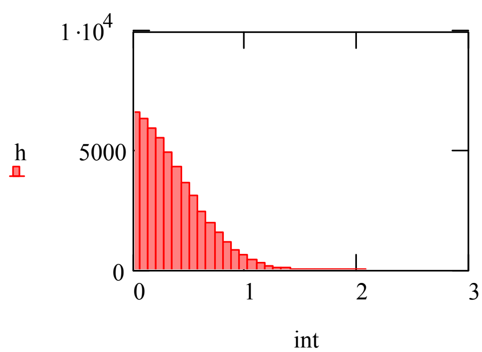
$S1 := \text{qnorm}(u1, 0, 1) \cdot 0.3 \cdot 2 + 2$

$$S2 := \text{qnorm}(u2, 0, 1) \cdot (0.3 \cdot 2) \cdot \sqrt{1 - 0.7^2} + \left[2 + 0.7 \cdot \frac{0.3 \cdot 2}{0.3 \cdot 2} \cdot (S1 - 2)\right]$$

$D := \overrightarrow{S1 - S2}$

$n := 40 \qquad j := 0 .. n \qquad int_j := 0 + 3 \cdot \frac{j}{n}$

$h := \text{hist}(int, D)$



$\text{mean}(D) = 3.77 \times 10^{-4}$

$\text{var}(D) = 0.217$

(b) the probability that D is less than 0.5 inch is

$x := \overrightarrow{(D < 0.5)}$

$$pF := \frac{\sum_{i=1}^{99999} x_i}{100000} \qquad pF = 0.858$$

(c) S1 and S2 are jointly lognormally distributed

$\mu S1 := 2$

$\delta S1 := 0.30$

$\zeta S1 := \sqrt{\ln\left(1 + \delta S1^2\right)} \qquad \zeta S1 = 0.294$

$\lambda S1 := \ln(\mu S1) - \frac{1}{2}\zeta S1^2 \qquad \lambda S1 = 0.65$

$\lambda S2 := \lambda S1 \qquad \zeta S2 := \zeta S1$

$S1 := \mathrm{rlnorm}(100000, \lambda S1, \zeta S1)$

$A := \log(S1)$

$uB := \left[\lambda S2 + 0.7 \cdot \frac{\zeta S2}{\zeta S1} \cdot (A - \lambda S1)\right]$

$sB := \zeta S2 \cdot \sqrt{1 - 0.7^2}$

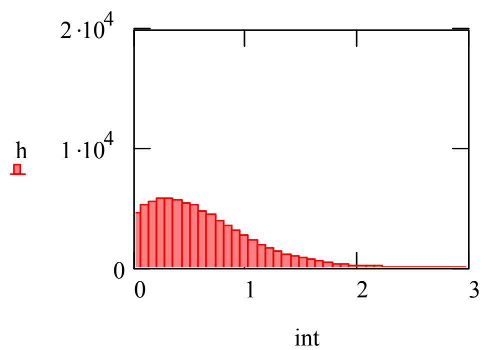
$B := \mathrm{qnorm}(u2, 0, 1) \cdot sB + uB$

$S2 := \exp(B)$

$D := \overrightarrow{S1 - S2}$

$n := 40 \qquad j := 0..n \qquad int_j := 0 + 3 \cdot \frac{j}{n}$

$h := \mathrm{hist}(int, D)$



$\mathrm{mean}(D) = 0.481$

$\mathrm{var}(D) = 0.321$

The probability that D is less than 0.5 inch is:

$x := \overrightarrow{(D < 0.5)}$

$pF := \frac{\sum_{i=1}^{99999} x_i}{100000} \qquad pF = 0.555$

5.6

(a) MATHCAD statements:

$\mu F1 := 35 \qquad \sigma F1 := 10$

$$\zeta F1 := \sqrt{\ln\left(1 + \frac{\sigma F1^2}{\mu F1^2}\right)} \qquad \zeta F1 = 0.28$$

$$\lambda F1 := \ln(\mu F1) - \frac{1}{2}\zeta F1^2 \qquad \lambda F1 = 3.516$$

$\mu F2 := 25 \qquad \sigma F2 := 10$

$$\zeta F2 := \sqrt{\ln\left(1 + \frac{\sigma F2^2}{\mu F2^2}\right)} \qquad \zeta F2 = 0.385$$

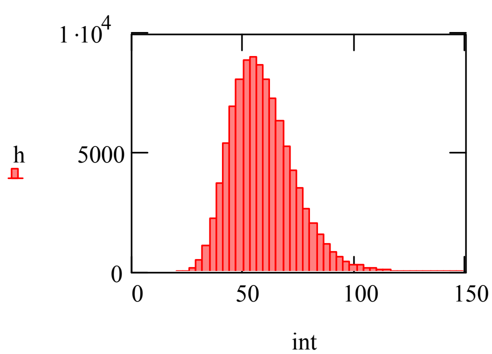
$$\lambda F2 := \ln(\mu F2) - \frac{1}{2}\zeta F2^2 \qquad \lambda F2 = 3.145$$

$F1 := \text{rlnorm}(100000, \lambda F1, \zeta F1)$

$F2 := \text{rlnorm}(100000, \lambda F2, \zeta F2)$

$F := F1 + F2$

$n := 50 \qquad j := 0..n \qquad int_j := 0 + 150\cdot\frac{j}{n} \qquad h := \text{hist}(int, F)$



$\text{mean}(F) = 59.96$

$\text{stdev}(F) = 14.14$

(b) the annual risk that the city will experience flooding is the probability of flooding

MATHCAD statements:

$$C := 100 \qquad x := \overrightarrow{(F > C)}$$

$$PF := \frac{\sum_{i=1}^{99999} x_i}{100000} \qquad PF = 0.011$$

$$RP := \frac{1}{PF} \qquad RP = 92.678$$

The annual risk is 0.011, the corresponding return period is 92.7 years.

(c) If the desired risk is reduced to half of the current one, i.e., 0.0055, then the capacity should be increased.

$$C := \left| \begin{array}{l} \text{for } c \in 100, 101 .. 300 \\ \quad \left| \begin{array}{l} x \leftarrow \overrightarrow{(F > c)} \\ p \leftarrow \dfrac{\sum_{i=1}^{99999} x_i}{100000} \\ c \ \text{ if } \ p > 0.0055 \\ \text{break } \text{ if } \ p \le 0.0055 \\ c \end{array} \right. \\ C \leftarrow c \\ C \end{array} \right. \qquad C = 107$$

Verify the above result:

$$C := 107 \qquad x := \overrightarrow{(F > C)}$$

$$PF := \frac{\sum_{i=1}^{99999} x_i}{100000} \qquad PF = 5.02 \times 10^{-3}$$

Therefore, the channel capacity should be increased to $107 m^3/s$.

(d). Correlated case

(d.a) MATHCAD statements:

$\mu F1 := 35 \qquad \sigma F1 := 10$

$\zeta F1 := \sqrt{\ln\left(1 + \frac{\sigma F1^2}{\mu F1^2}\right)} \qquad \zeta F1 = 0.28$

$\lambda F1 := \ln(\mu F1) - \frac{1}{2}\zeta F1^2 \qquad \lambda F1 = 3.516$

$F1 := \mathrm{rlnorm}(100000, \lambda F1, \zeta F1)$

$A1 := \ln(F1)$

$\mu F2 := 25 \qquad \sigma F2 := 10$

$\zeta F2 := \sqrt{\ln\left(1 + \frac{\sigma F2^2}{\mu F2^2}\right)} \qquad \zeta F2 = 0.385$

$\lambda F2 := \ln(\mu F2) - \frac{1}{2}\zeta F2^2 \qquad \lambda F2 = 3.145$

$\rho := 0.80$

$uB1 := \lambda F2 + \rho \cdot \frac{\zeta F2}{\zeta F1} \cdot (A1 - \lambda F1)$

$sB1 := \zeta F2 \cdot \sqrt{1 - \rho^2}$

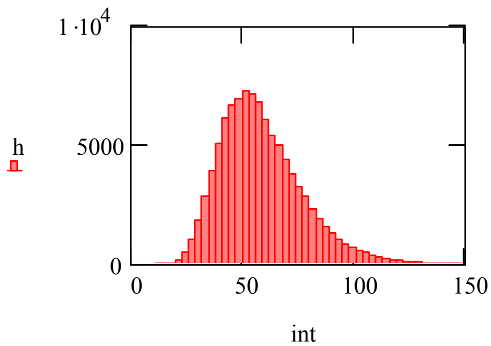
$u1 := \mathrm{runif}(100000, 0.0, 1.0)$

$B1 := \mathrm{qnorm}(u1, 0.0, 1.0) \cdot sB1 + uB1$

$F2 := \exp(B1)$

$F := F1 + F2$

$n := 50 \qquad j := 0 .. n \qquad int_j := 0 + 150 \cdot \frac{j}{n} \qquad h := \mathrm{hist}(int, F)$



$\text{mean}(F) = 60.019 \qquad \text{stdev}(F) = 18.901$

(d.b) the annual risk that the city will experience flooding is the probability of flooding

MATHCAD statements:

$$C := 100 \qquad x := \overrightarrow{(F > C)}$$

$$PF := \frac{\sum_{i=1}^{99999} x_i}{100000} \qquad PF = 0.035$$

$$RP := \frac{1}{PF} \qquad RP = 28.425$$

The annual risk is 0.035, the corresponding return period is 28.8 years.

Compare with the independent case, it shows the annual risk of flooding is increased due to the correlation between the flow rates of the two rivers.

(d.c) If the desired risk is reduced to half of the current one, i.e., 0.0175, then the capacity should be increased.

$$C := \left| \begin{array}{l} \text{for } c \in 100, 101 .. 300 \\ \quad \left| \begin{array}{l} x \leftarrow \overrightarrow{(F > c)} \\ p \leftarrow \dfrac{\sum_{i=1}^{99999} x_i}{100000} \\ c \ \text{ if } p > 0.0175 \\ \text{break } \text{ if } p \le 0.0175 \\ c \end{array} \right. \\ C \leftarrow c \\ C \end{array} \right.$$

$$C = 110$$

Verify the above result:

$$C := 110 \qquad x := \overrightarrow{(F > C)}$$

$$PF := \frac{\sum_{i=1}^{99999} x_i}{100000} \qquad PF = 0.017$$

Therefore, the channel capacity should be increased to 110m^3/s.

5.7

W and g are contant values $W := 200$ $g := 32.2$

Coefficient of friction, k, is beta distributed

$q := 3.0$ $r := 3.0$

As q=r, then the distribution is symmetric. The median value is equal to the mean value.

$\mu k := 0.40$ $\sigma k := 0.08$

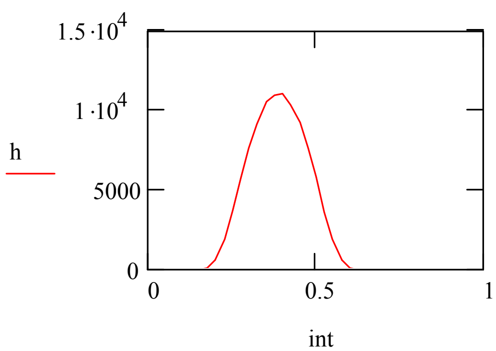
$$ak := \mu k - q \cdot \sigma k \cdot \sqrt{\frac{q + r + 1}{q \cdot r}} \qquad ak = 0.188$$

$$bk := ak + \sigma k \cdot \sqrt{\frac{q + r + 1}{q \cdot r}} \cdot (q + r) \qquad bk = 0.612$$

$x := \mathrm{rbeta}(100000, 3.0, 3.0)$

$k := (bk - ak) \cdot x + ak$

$n := 40$ $j := 0 .. n$ $int_j := 0 + 1.0 \cdot \frac{j}{n}$ $h := \mathrm{hist}(int, k)$



Maximum ground acceleration, a, is a Type I extremal variate

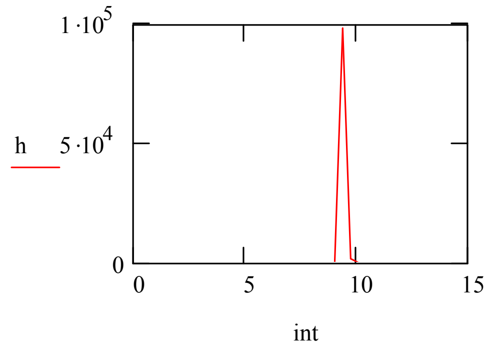
$$\alpha := \frac{\pi}{0.35 \cdot \% \cdot 0.3 \cdot g \cdot \sqrt{6}} \qquad \alpha = 37.934$$

$$\beta := 0.3 \cdot g - \frac{0.577216}{\alpha} \qquad \beta = 9.645$$

$u := \mathrm{runif}(100000, 0, 1)$

$$a := \beta - \frac{1}{\alpha} \ln\left(\ln\left(\frac{1}{u}\right)\right)$$

$n := 40$ $j := 0 .. n$ $int_j := 0.0 + 15 \cdot \frac{j}{n}$ $h := \mathrm{hist}(int, a)$



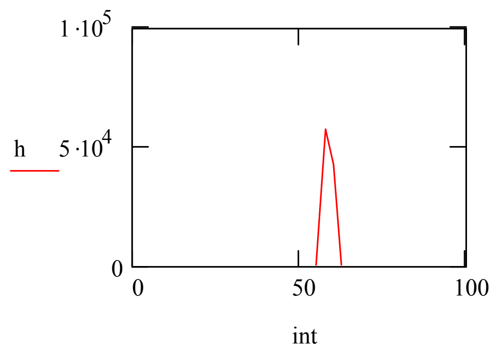
(a)

The resistance by friction R

$$R := k \cdot W$$

$$F := W \cdot \frac{a}{g}$$

$n := 40$ $\quad$ $j := 0..n$ $\quad$ $int_j := 0.0 + 100 \cdot \frac{j}{n}$ $\quad$ $h := hist(int, F)$



$$y := \overrightarrow{(R < F)}$$

$$pf := \frac{\sum_{i=1}^{99999} y_i}{100000} \qquad pf = 1.181 \times 10^{-1}$$

The probability that a tank will slide from its base support is 0.119

(b) The resistance of the anchors for each tank is RA

$$
RA := \left| \begin{array}{l} \text{for } ra \in 0, 0.1 .. 10 \\ \quad \left| \begin{array}{l} R \leftarrow k \cdot W + ra \\ F \leftarrow W \cdot \dfrac{a}{g} \\ x \leftarrow \overrightarrow{(R < F)} \\ pf \leftarrow \dfrac{\sum_{i=1}^{99999} x_i}{100000} \\ ra \text{ if } pf > (0.119 \cdot 0.3) \\ \text{break if } pf \le (0.119 \cdot 0.3) \\ ra \end{array} \right. \\ RA \leftarrow ra \\ RA \end{array} \right.
$$

$$RA = 8.2$$

Verify the above result:

$$R := k \cdot W + RA$$

$$F := W \cdot \frac{a}{g}$$

$$x := \overrightarrow{(R < F)}$$

$$pf := \frac{\sum_{i=1}^{99999} x_i}{100000} \qquad pf = 3.512 \times 10^{-2}$$

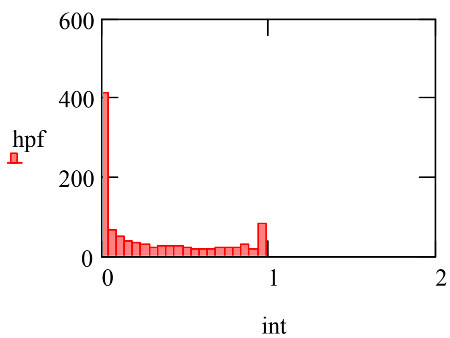
(c) Considering epistemic uncertainties:

assume the error of μk is mk, the error of μa is ma

$$q := 3.0 \qquad r := 3.0 \qquad g := 32.2 \qquad \mu k := 0.40 \qquad \sigma k := 0.08$$

$$
\begin{aligned}
Pf := \; & mk \leftarrow \text{rnorm}(1000, 1.0, 0.25) \\
& ma \leftarrow \text{rnorm}(1000, 1.0, 0.45) \\
& \text{for } \; i \in 0 .. 999 \\
& \quad ak \leftarrow \mu k \cdot mk_i - q \cdot \sigma k \cdot \sqrt{\frac{q + r + 1}{q \cdot r}} \\
& \quad bk \leftarrow ak + \sigma k \cdot \sqrt{\frac{q + r + 1}{q \cdot r}} \cdot (q + r) \\
& \quad x \leftarrow \text{rbeta}(1000, 3.0, 3.0) \\
& \quad k \leftarrow (bk - ak) \cdot x + ak \\
& \quad \alpha \leftarrow \frac{\pi}{(0.35 \cdot \% \cdot 0.3 \cdot g) \cdot ma_i \cdot \sqrt{6}} \\
& \quad \beta \leftarrow 0.3 \cdot g \cdot ma_i - \frac{0.577216}{\alpha} \\
& \quad u \leftarrow \text{runif}(1000, 0.0, 1.0) \\
& \quad a \leftarrow \beta - \frac{1}{\alpha} \ln\left(\ln\left(\frac{1}{u}\right)\right) \\
& \quad R \leftarrow k \cdot W \\
& \quad F \leftarrow W \cdot \frac{a}{g} \\
& \quad y \leftarrow \overrightarrow{(R < F)} \\
& \quad Pf_i \leftarrow \frac{\displaystyle\sum_{j = 1}^{999} y_j}{1000} \\
& \quad \text{continue} \\
& Pf
\end{aligned}
$$

$n := 40$ $j := 0 .. n$ $int_j := 0.0 + 2 \cdot \frac{j}{n}$ $hpf := hist(int, Pf)$



$mean(Pf) = 0.295$ $stdev(Pf) = 0.346$ $skew(Pf) = 0.876$

$v := sort(Pf)$

$Pf90 := v_{900}$ $v_{900} = 0.897$

The 90% value for a conservative probability of sliding is 0.897

5.8

Single pile capacity T1, T2

$\mu T1 := 20$

$\delta T1 := 0.20$

$\zeta T1 := \sqrt{\ln\left(1 + \delta T1^2\right)}$ $\qquad \zeta T1 = 0.198$

$\lambda T1 := \ln(\mu T1) - \frac{1}{2}\zeta T1^2$ $\qquad \lambda T1 = 2.976$

$\lambda T2 := \lambda T1 \qquad \zeta T2 := \zeta T1$

$T1 := \mathrm{rlnorm}(100000, \lambda T1, \zeta T1)$

$x1 := \log(T1)$

$ux2 := \left[\lambda T2 + 0.8 \cdot \frac{\zeta T2}{\zeta T1} \cdot (x1 - \lambda T1)\right]$

$sx2 := \zeta T2 \cdot \sqrt{1 - 0.8^2}$

$u2 := \mathrm{runif}(100000, 0, 1)$

$x2 := \mathrm{qnorm}(u2, 0, 1) \cdot sx2 + ux2$

$T2 := \exp(x2)$

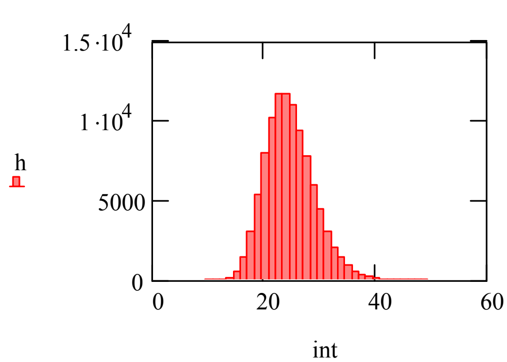
$T := T1 + T2$

$\mathrm{mean}(T) = 25.151$

$\mathrm{stdev}(T) = 4.392$

$covT := \frac{\mathrm{stdev}(T)}{\mathrm{mean}(T)}$ $\qquad covT = 0.175$

$n := 40 \qquad j := 0 .. n \qquad int_j := 0 + 50 \cdot \frac{j}{n} \qquad h := \mathrm{hist}(int, T)$



The external load is L

$\mu L := 10$

$\delta L := 0.30$

$\zeta L := \sqrt{\ln\left(1 + \delta L^2\right)}$ $\qquad \zeta L = 0.294$

$\lambda L := \ln(\mu L) - \frac{1}{2}\zeta L^2$ $\qquad \lambda L = 2.259$

$L := \text{rlnorm}(100000, \lambda L, \zeta L)$

$y := \overrightarrow{(T < L)}$

$$pf := \frac{\sum_{i=1}^{99999} y_i}{100000} \qquad pf = 2.6 \times 10^{-3}$$

Therefore, the probability of failure of this pile group is 0.0026

Problem 5.9

Determine the solution to Problem 3.57 by MCS

(a)

$$f(x,y) := \frac{6}{5}\left(x+y^2\right)$$

$$fX(x) := \int_0^1 f(x,y)\,dy \rightarrow \frac{6}{5}\cdot x + \frac{2}{5}$$

(b)

$$fYx(x,y) := \frac{f(x,y)}{fX(x)} \text{ simplify} \rightarrow 3\cdot\frac{x+y^2}{3\cdot x+1}$$

$$FYx(x,y) := \int fYx(x,y)\,dy \text{ simplify} \rightarrow y\cdot\frac{3\cdot x+y^2}{3\cdot x+1}$$

$$FYx(0.5,y) - u2 \;\Big|\begin{matrix}\text{simplify}\\ \text{solve}, y\end{matrix} \rightarrow \begin{bmatrix} \frac{1}{2}\cdot\left[10\cdot u2+2\cdot\left(2+25\cdot u2^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} - \frac{1}{\left[10\cdot u2+2\cdot\left(2+25\cdot u2^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}} \\ \frac{-1}{4}\cdot\left[10\cdot u2+2\cdot\left(2+25\cdot u2^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \frac{1}{2\cdot\left[10\cdot u2+2\cdot\left(2+25\cdot u2^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}} + \frac{1}{2}\cdot i\cdot 3^{\frac{1}{2}}\cdot\left[\frac{1}{2}\cdot\left[10\cdot u2+2\cdot\left(2+25\cdot u2^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \frac{1}{\left[10\cdot u2+2\cdot\left(2+25\cdot u2^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}}\right] \\ \frac{-1}{4}\cdot\left[10\cdot u2+2\cdot\left(2+25\cdot u2^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \frac{1}{2\cdot\left[10\cdot u2+2\cdot\left(2+25\cdot u2^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}} - \frac{1}{2}\cdot i\cdot 3^{\frac{1}{2}}\cdot\left[\frac{1}{2}\cdot\left[10\cdot u2+2\cdot\left(2+25\cdot u2^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \frac{1}{\left[10\cdot u2+2\cdot\left(2+25\cdot u2^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}}\right] \end{bmatrix}$$

$$u2 := \text{runif}(100000, 0, 1)$$

y should be real value and greater than zero, so

$$IFYx := \frac{1}{2}\cdot\frac{\left[10\cdot u2 + 2\cdot\left(2+25\cdot u2^2\right)^{\frac{1}{2}}\right]^{\frac{2}{3}} - 2}{\left[10\cdot u2 + 2\cdot\left(2+25\cdot u2^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}}$$

$$n := \overrightarrow{(1.0 > IFYx \wedge IFYx > 0.5)}$$

$$p := \frac{\sum_{i=1}^{99999} n_i}{100000} \qquad p = 0.65$$

Verify the simultion result,

$$\int_{0.5}^{1.0} fYx(0.5, y)\, dy \rightarrow .6500000000000000000($$

(c)

$$FX(x) := \int fX(x)\, dx \rightarrow \frac{3}{5}\cdot x^2 + \frac{2}{5}\cdot x$$

$$FX(x) - u1 \left|\begin{matrix}\text{solve}, x\\ \text{simplify}\end{matrix}\right. \rightarrow \begin{bmatrix} \frac{-1}{3} + \frac{1}{3}\cdot(1+15\cdot u1)^{\frac{1}{2}} \\ \frac{-1}{3} - \frac{1}{3}\cdot(1+15\cdot u1)^{\frac{1}{2}} \end{bmatrix}$$

x should be greater than zero, therefore the inverse function of FX is

$u1 := \mathrm{runif}(100000, 0, 1)$

$$IFX := \frac{-1}{3} + \frac{1}{3}\cdot(1 + 15\cdot u1)^{\frac{1}{2}}$$

$$fY(y) := \int_0^1 f(x,y)\,dx \rightarrow \frac{3}{5} + \frac{6}{5}\cdot y^2$$

$$FY(y) := \int fY(y)\,dy \rightarrow \frac{3}{5}\cdot y + \frac{2}{5}\cdot y^3$$

$$FY(y) - u3 \ \left|\begin{array}{l}\text{solve}, y\\ \text{simplify}\end{array}\right. \rightarrow \begin{bmatrix} \dfrac{1}{2}\cdot\dfrac{\left[10\cdot u3 + 2\cdot\left(2 + 25\cdot u3^2\right)^{\frac{1}{2}}\right]^{\frac{2}{3}} - 2}{\left[10\cdot u3 + 2\cdot\left(2 + 25\cdot u3^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}} \\ \dfrac{1}{4}\cdot\dfrac{-\left[10\cdot u3 + 2\cdot\left(2 + 25\cdot u3^2\right)^{\frac{1}{2}}\right]^{\frac{2}{3}} + 2 + i\cdot 3^{\frac{1}{2}}\cdot\left[10\cdot u3 + 2\cdot\left(2 + 25\cdot u3^2\right)^{\frac{1}{2}}\right]^{\frac{2}{3}} + 2\cdot i\cdot 3^{\frac{1}{2}}}{\left[10\cdot u3 + 2\cdot\left(2 + 25\cdot u3^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}} \\ \dfrac{-1}{4}\cdot\dfrac{\left[10\cdot u3 + 2\cdot\left(2 + 25\cdot u3^2\right)^{\frac{1}{2}}\right]^{\frac{2}{3}} - 2 + i\cdot 3^{\frac{1}{2}}\cdot\left[10\cdot u3 + 2\cdot\left(2 + 25\cdot u3^2\right)^{\frac{1}{2}}\right]^{\frac{2}{3}} + 2\cdot i\cdot 3^{\frac{1}{2}}}{\left[10\cdot u3 + 2\cdot\left(2 + 25\cdot u3^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}} \end{bmatrix}$$

Y should be greater than zero, therefore the inverse function of FY is

$u3 := \mathrm{runif}(100000, 0, 1)$

$$IFY := \frac{1}{2}\cdot\frac{\left[10\cdot u3 + 2\cdot\left(2 + 25\cdot u3^2\right)^{\frac{1}{2}}\right]^{\frac{2}{3}} - 2}{\left[10\cdot u3 + 2\cdot\left(2 + 25\cdot u3^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}}$$

$$\mathrm{corr}(IFX, IFYx) = -0.003$$

Yes, there are statistical correlation between X and Y.

Check

$$\mathrm{stdev}(IFYx) = 0.2828$$

$$\mathrm{stdev}(IFY) = 0.2817$$

$$\rho := \sqrt{1 - \left(\frac{\mathrm{stdev}(IFYx)}{\mathrm{stdev}(IFY)}\right)^2} \qquad \rho = 0.09i$$

5.10

MathCAD statements

$$u1 := \mathrm{runif}(100000, 0.0, 1.0)$$

$$X1 := \left| \begin{array}{l} \text{for } i \in 1, 2 .. 99999 \\ \quad \left| \begin{array}{l} X1_i \leftarrow 1 \ \text{ if } \ u1_i \le 0.15 \\ X1_i \leftarrow 3 \ \text{ if } \ u1_i > 0.65 \\ X1_i \leftarrow 2 \ \text{ otherwise} \end{array} \right. \\ X1 \end{array} \right.$$

$$u2 := \mathrm{runif}(100000, 0.0, 1.0)$$

$$Y1 := \left| \begin{array}{l} \text{for } i \in 1, 2 .. 99999 \\ \quad \left| \begin{array}{l} Y1_i \leftarrow 10 \ \text{ if } \ u2_i \le \frac{1}{3} \wedge X1_i = 1 \\ Y1_i \leftarrow 20 \ \text{ if } \ u2_i > \frac{1}{3} \wedge u2_i \le \frac{2}{3} \wedge X1_i = 1 \\ Y1_i \leftarrow 30 \ \text{ if } \ u2_i \ge 1 \wedge X1_i = 1 \\ Y1_i \leftarrow 10 \ \text{ if } \ u2_i \le \frac{0.15}{0.5} \wedge X1_i = 2 \\ Y1_i \leftarrow 20 \ \text{ if } \ u2_i \le \frac{0.4}{0.5} \wedge u2_i > \frac{0.15}{0.5} \wedge X1_i = 2 \\ Y1_i \leftarrow 30 \ \text{ if } \ u2_i \ge \frac{0.4}{0.5} \wedge X1_i = 2 \\ Y1_i \leftarrow 10 \ \text{ if } \ u2_i \le \frac{0}{0.35} \wedge X1_i = 3 \\ Y1_i \leftarrow 20 \ \text{ if } \ u2_i \le \frac{0.25}{0.35} \wedge u2_i > \frac{0}{0.35} \wedge X1_i = 3 \\ Y1_i \leftarrow 30 \ \text{ if } \ u2_i \ge \frac{0.25}{0.35} \wedge X1_i = 3 \end{array} \right. \\ Y1 \end{array} \right.$$

(a)

$$n := \overrightarrow{(X1 \ge 2 \wedge Y1 > 20)}$$

$$PF := \frac{\sum_{i=1}^{99999} n_i}{100000} \qquad PF = 0.2$$

(b)

$$n := \overrightarrow{(X1 = 2)}$$

$$PF1 := \frac{\sum_{i=1}^{99999} n_i}{100000} \qquad PF1 = 0.5$$

$$m := \overrightarrow{(X1 = 2 \wedge Y1 \geq 20)}$$

$$PF2 := \frac{\sum_{i=1}^{99999} m_i}{100000} \qquad PF2 = 0.352$$

$$PF3 := \frac{PF2}{PF1} \qquad PF3 = 0.704$$

The probability of the runoff Y is not less than 20 cfs when the precipitation is 2 inches is 0.7

(c)

$$n := \overrightarrow{(Y1 \geq 20)}$$

$$PF4 := \frac{\sum_{i=1}^{99999} n_i}{100000} \qquad PF4 = 0.752$$

as shown in (b)

P(Y>=20|X=2)=0.7

P(Y>=20)=0.75

Therefore, X and Y are not independent

(d)

$$n := \overrightarrow{(Y1 = 10)}$$

$$Py1 := \frac{\sum_{i=1}^{99999} n_i}{100000} \qquad Py1 = 0.197$$

$$n := \overrightarrow{(Y1 = 20)}$$

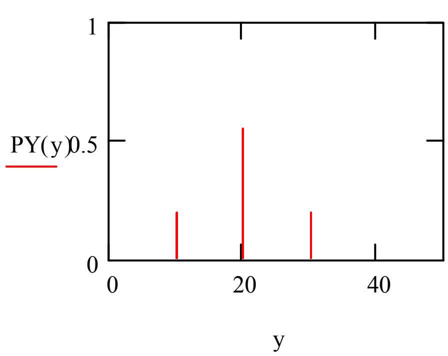
$$Py2 := \frac{\sum_{i=1}^{99999} n_i}{100000} \qquad Py2 = 0.553$$

$$n := \overrightarrow{(Y1 = 30)}$$

$$Py3 := \frac{\sum_{i=1}^{99999} n_i}{100000} \qquad Py3 = 0.2$$

$$PY(y) := \left| \begin{array}{l} PY \leftarrow Py1 \ \text{ if } \ y = 10 \\ PY \leftarrow Py2 \ \text{ if } \ y = 20 \\ PY \leftarrow Py3 \ \text{ if } \ y = 30 \end{array} \right.$$

The marginal PMF of runoff is plotted as follows.



(e)

$$n := \overrightarrow{(Y1 = 10 \land X1 = 2)}$$

$$m := \overrightarrow{(X1 = 2)}$$

$$Py1 := \frac{\sum_{i=1}^{99999} n_i}{\sum_{i=1}^{99999} m_i} \qquad Py1 = 0.296$$

$$n := \overrightarrow{(Y1 = 20 \land X1 = 2)}$$

$$m := \overrightarrow{(X1 = 2)}$$

$$Py2 := \frac{\sum_{i=1}^{99999} n_i}{\sum_{i=1}^{99999} m_i} \qquad Py2 = 0.506$$

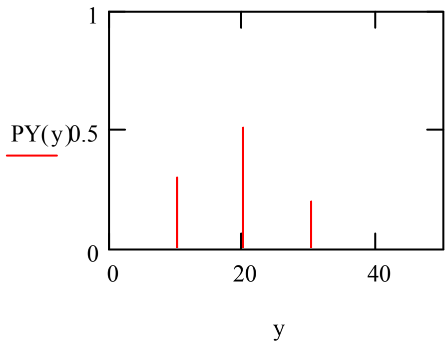
$$n := \overrightarrow{(Y1 = 30 \land X1 = 2)}$$

$$m := \overrightarrow{(X1 = 2)}$$

$$Py3 := \frac{\sum_{i=1}^{99999} n_i}{\sum_{i=1}^{99999} m_i} \qquad Py3 = 0.198$$

$$PY(y) := \left| \begin{array}{l} PY \leftarrow Py1 \ \ \text{if } \ y = 10 \\ PY \leftarrow Py2 \ \ \text{if } \ y = 20 \\ PY \leftarrow Py3 \ \ \text{if } \ y = 30 \end{array} \right.$$

The conditional PMF of runoff with precipitation = 2 inches is shown as follows.



(f)

$\mathrm{Stdev}(X1) = 0.678 \qquad \mathrm{Stdev}(Y1) = 7.671$

$\mathrm{cvar}(X1, Y1) = 2.701 \qquad \rho := \dfrac{\mathrm{cvar}(X1, Y1)}{\mathrm{stdev}(X1) \cdot \mathrm{stdev}(Y1)} \qquad \rho = 0.519$

Or directly by

$\mathrm{corr}(X1, Y1) = 0.519$

Therefore, the correlation coefficient between precipitation and runoff is 0.52.

6.1

(a) $\bar{x} = \dfrac{\sum x_i}{n} = \dfrac{765}{9} = 85 \text{ ton} = \hat{\mu}_x$

$s^2 = \dfrac{\sum (x_i - 85)^2}{n-1} = \dfrac{366}{8} = 45.75$

$\hat{\sigma}_x = \sqrt{45.75} = 6.7643$

(b) one-sided hypothesis test

Null hypothesis H_o: $\mu = 80$ tons

Alternative hypothesis H_A: $\mu < 80$ tons

Without assuming known standard deviation, the estimated value of the test statistic to be

$$t = \frac{\bar{x} - 80}{s / \sqrt{n}} = \frac{85 - 80}{6.764 / \sqrt{9}} = 2.2176$$

with f = 9-1 = 8 d.o.f. we obtain the critical value of t from Table A.2. 2 at the 5% significance level to be $t_\alpha = -1.8595$. Therefore the value of the test statistic is 2.2176 > -1.8595 which is outside the region of rejection; hence the null hypothesis is accepted, and the capacity of the pile remain acceptable.

(c)

$$\langle \mu_x \rangle_{0.98} = (\bar{x} - k_{0.01} \frac{\sigma_x}{\sqrt{n}},\ \bar{x} + k_{0.01} \frac{\sigma_x}{\sqrt{n}})$$

$$= (85 - 2.33 \frac{6.764}{3},\ 85 + 2.33 \frac{6.764}{3})$$

$$= (79.747, 90.253)$$

(d) $$\langle \mu_x \rangle_{0.98} = (\bar{x} - t_{0.01,8} \frac{s_x}{\sqrt{n}},\ \bar{x} + t_{0.01,8} \frac{s_x}{\sqrt{n}})$$

$$= (85 - 2.8965 \frac{6.764}{3},\ 85 + 2.8965 \frac{6.764}{3})$$

$$= (78.47, 91.53)$$

6.2

(a) Since $\bar{x} = 65$, n = 50, σ = 6, a 2-sided 99% interval for the true mean is given by the limits

$$65 \pm k_{0.005}\ \frac{6}{\sqrt{50}}$$

$$\Rightarrow \langle\mu\rangle_{0.99} = (65 - 2.58\times\frac{6}{\sqrt{50}}\ ,\ 65 - 2.58\times\frac{6}{\sqrt{50}})$$

= (**62.81, 67.19**) (in mph)

(b) Requiring $2.58\frac{6}{\sqrt{n}} \le 1 \Rightarrow n \ge (2.58\times6)^2 = 239.6304$

⇒ n = 240

⇒ (240 – 50) = **190** additional vehicles must be observed.

Before we proceed to (c) and (d), let $\bar{X}_J$ and $\bar{X}_M$ denote John and Mary's sample means, respectively. Both are approximately normal with mean μ (unknown true mean speed) and standard deviation $\frac{6}{\sqrt{n}}$, hence their difference

$$D = \bar{X}_J - \bar{X}_M$$

has an (approximate) normal distribution with

$$\text{mean} = \mu - \mu = 0, \text{ and}$$

$$\text{standard deviation} = \sqrt{\left(\frac{6}{\sqrt{n}}\right)^2 + \left(\frac{6}{\sqrt{n}}\right)^2} = 6\sqrt{\frac{2}{n}}\ ,$$

i.e.

$$D \sim N(0, 6\sqrt{\frac{2}{n}})$$

Hence

(c) When n = 10, $P(\bar{X}_J - \bar{X}_M > 2) = P(D > 2)$

$$= P(\frac{D-\mu_D}{\sigma_D} > \frac{2-0}{6/\sqrt{5}})$$

$$= 1 - \Phi(0.745) \cong \mathbf{0.228}$$

(d) When n = 100, $P(D > 2) = 1 - \Phi\left(\frac{2-0}{6/\sqrt{50}}\right) = 1 - \Phi(2.357) \cong \mathbf{0.0092}$

6.3

(a) The formula to use is $<\mu>_{1-\alpha} = [\,\bar{x} - t_{\alpha/2,n-1}\dfrac{s}{\sqrt{n}}, \bar{x} - t_{\alpha/2,n-1}\dfrac{s}{\sqrt{n}}\,]$, where

$$\alpha = 0.1,\ n = 10,\ \bar{x} = 10000 \text{ cfs},\ s = 3000 \text{ cfs, and } t_{\alpha/2,n-1} = t_{0.05,9} = 1.833$$

Hence, the 2-sided 90% confidence interval for the mean annual maximum flow is

$$[(10000 - 1739.044) \text{ cfs}, (10000 + 1739.044) \text{ cfs}]$$
$$\cong [\mathbf{8261\ cfs,\ 11739\ cfs}]$$

(b) Since the confidence level $(1 - \alpha)$ is fixed at 90%, while s is assumed to stay at approximately 3000 cfs, the "half-width" of the confidence interval, $t_{\alpha/2,n-1}\dfrac{s}{\sqrt{n}}$ can be considered as a function of the sample size n only. To make it equal to 1000 cfs, one must have

$$t_{0.05,n-1}\frac{3000}{\sqrt{n}} = 1000$$

$$\Rightarrow \frac{t_{0.05,n-1}}{\sqrt{n}} = 1/3,$$

which can be solved by trial and error. We start with $n = 20$ (say), and increase the sample size until the half width is narrowed to the desired 1000cfs, i.e. until

$$\frac{t_{0.05,n-1}}{\sqrt{n}} \leq 0.33333\ldots.$$

With reference to the following table,

Sample size, n	$t_{0.05,n-1}$	$\dfrac{t_{0.05,n-1}}{\sqrt{n}}$
20	1.7291	0.387
21	1.7247	0.376
22	1.7207	0.367
23	1.7171	0.358
24	1.7138	0.350
25	1.7108	0.342
26	1.7081	0.335
27	1.7056	0.328

We see that a sample size of 27 will do, hence an additional (27 – 10) = **17** years of observation will be required.

6.4

(a)

Pile Test No.	1	2	3	4	5
N = A/P	**1.507**	**0.907**	**1.136**	**1.070**	**1.149**

(b) $\overline{N} = \dfrac{\sum_{i=1}^{5} N_i}{5} \cong$ **1.154**, $S_N^2 = \dfrac{\sum_{i=1}^{5}(N_i - \overline{N})^2}{5-1} \cong 0.219850903^2 \cong$ **0.048**

(c) Substituting n = 5, $\overline{N}$ = 1.15392644, S_N = 0.219850903 into the formula

$$<\mu_N>_{0.95} = \overline{N} \mp t_{0.025,\, n-1} \frac{S_N}{\sqrt{n}},$$

where $t_{0.025,\, n-1} = t_{0.025,4} \cong 2.776$, we have (1.15392644 ∓ 0.27293719) = (**0.881, 1.427**) as a 95% confidence interval for the true mean of N.

(d) With $\sigma = \sqrt{0.045}$ known, and assuming N is normal, to estimate μ_N to ±0.02 with 90% confidence would require

$$k_{0.05} \frac{\sqrt{0.045}}{\sqrt{n}} \leq 0.02$$

$$\Rightarrow n \geq \left(1.645 \frac{\sqrt{0.045}}{0.02}\right)^2 = 304.37$$

⇒ n = 305, hence an additional (305 – 5) = **300** piles should be tested.

(e) Since A = 15N, P(pile failure) = P(A < 12)

$$= P(15N < 12) = P(N < 12/15)$$

$$= \Phi\left(\frac{12/15 - 1.154}{0.22}\right)$$

$$= \Phi(-1.61)$$

$$\cong \mathbf{0.0537}$$

6.5

(a) Let x be the concrete strength. $\bar{x} = \dfrac{\sum_{i=1}^{5} x_i}{5}$ = 3672, $S_x = \sqrt{\dfrac{\sum_{i=1}^{5}(x_i - \bar{x})^2}{5-1}} = \sqrt{\dfrac{589778}{4}}$, hence

$$<\mu_N>_{0.90} = 3672 \mp t_{0.05,\, 5-1}\,\frac{\sqrt{\frac{589778}{4}}}{\sqrt{5}},$$

where $t_{0.05,4} \cong 2.131846486$, we have (3672 ∓ 366.09) = (**3305.91, 4038.09**) as a 90% confidence interval for the true mean of N. Note: our convention is that the subscript p in $t_{p,\,dof}$ always indicates the tail area to the right of $t_{p,\,dof}$

(b) If the "half-width" of the confidence interval, $t_{\alpha/2,\,4}\,\dfrac{\sqrt{\frac{589778}{4}}}{\sqrt{5}}$ is only 300 (psi) wide, it implies

$$t_{\alpha/2,\,4} = 300 \,/\, \left(\frac{\sqrt{\frac{589778}{4}}}{\sqrt{5}}\right) = 1.746996,$$

We need to determine the corresponding α. On the other hand, from t-distribution table we have (at 4 d.o.f.)

$$t_{0.1,\,4} = 1.533,$$
$$t_{0.05,\,4} = 2.132,$$

(and $t_{\alpha/2,\,4}$ increases as α/2 gets smaller in general)

Hence we may use linear interpolation to get an approximate answer: over such a small "x" range (from t = 1.533 to t = 2.132), we treat "y" (i.e. α/2) as decreasing linearly, with slope

$$m = \frac{0.05 - 0.1}{2.132 - 1.533} = -\,0.083472454$$

Hence, as t goes from 1.533 to 1.746996, α/2 should decrease from 0.1 to

$$\alpha/2 = 0.1 + m(1.746996 - 1.533)$$
$$= 0.1 - 0.083472454 \times 0.213996232 = 0.082137209$$
$$\Rightarrow \alpha = 2 \times 0.082137209 = 0.164274419$$

Hence the confidence level is 1 – α = 1 – 0.164274419 ≅ **83.6%**

6.6

(a) N = 30, $\bar{x}$ = 12.5 tons
Assume σ = 3 tons

Assume σ = 3 tons

$$< \mu_x >_{0.99} = \left(\bar{x} - k_{0.005} \frac{\sigma}{\sqrt{n}},\ \bar{x} + k_{0.005} \frac{\sigma}{\sqrt{n}} \right)$$

$$= \left(12.5 - 2.58 \times \frac{3}{\sqrt{30}},\ 85 + 12.5 + 2.58 \times \frac{3}{\sqrt{30}} \right)$$

$$= (12.5 - 2.826,\ 12.5 - 2.826)$$

$$= (9.674,\ 15.326) \text{ tons}$$

(b) Let n' be the total number of observations required for estimating to ± 1 tone with 99% confidence. Hence

$$2.58 \times \frac{3}{\sqrt{n'}} \equiv 1$$

or $n' = (2.58x3)^2 = 59.9$

Therefore, (60 - 30) or 30 additional observations of truck weights would be required.

6.7

$$f_H(h) = \frac{2h}{\alpha^2} e^{\frac{-h^2}{\alpha^2}}$$

From Eq. 6.4, the likelihood function of n observations on a Rayleigh distributinon is

$$L = \prod_{i=1}^{n} \frac{2h_i}{\alpha^2} e^{\frac{-h_i^2}{\alpha^2}}$$

Taking logarithm of both sides

$$\ln L = n \ln 2 - 2n \ln \alpha + \Sigma \ln h_i - \frac{1}{\alpha^2} \Sigma h_i^2 \qquad \text{where } \sum = \sum_{i=1}^{n}$$

$$\frac{\partial \ln L}{\partial \alpha} = -\frac{2n}{\alpha} + \frac{2\Sigma h_i^2}{\alpha^3} = 0 \qquad \rightarrow \qquad \hat{\alpha}^2 = \frac{\Sigma h_i^2}{n} or\, \hat{\alpha} = \pm\sqrt{\frac{\Sigma h_i^2}{n}}$$

From the given data, n = 10,

$\Sigma h_i^2 = (1.5^2 + 2.8^2 + \ldots\ldots 2.3^2) = 80.42$

$\therefore \hat{\alpha} = \pm 2.836$

6.8

(a) $\bar{x}$ = 2.03 mg/l, s = 0.485 mg/l

One-side hypothesis test that the minimum concentration DO is 2.0 mg/l .

Null hypothesis H_o:	μ = 2.0 mg/l
Alternative hypothesis H_A:	μ > 2.0 mg/l

Without assuming known standard deviation, the estimated value of the test statistic to be

$$t = \frac{\bar{x} - 2}{0.485/\sqrt{10}} = \frac{2.03 - 2}{0.485/\sqrt{10}} = 0.0196$$

with f = 10-1 = 9 d.o.f. we obtain the critical value of t from Table A.2. 2 at the 5% significance level to be t_α =1.833. Therefore the value of the test statistic is 0.0196 <1.833, which is outside the region of rejection; hence the stream quality satisfy the EPA standard.

(b)

$$< \mu >_{0.95} = (\overline{DO} - t_{0.025,\,9} \frac{s}{\sqrt{10}},\ \overline{DO} + t_{0.025,\,9} \frac{s}{\sqrt{10}})$$

$$= (\ 2.03 - 2.2622 \frac{0.485}{\sqrt{10}},\ 2.03 + 2.2622 \frac{0.485}{\sqrt{10}})$$

$$= (1.683,\ 2.377) \text{ mg/l}$$

(c)

$$< \mu >_{0.95} = \overline{DO} - t_{0.05,\,9} \frac{s}{\sqrt{10}} = 2.03 - 1.8331 \frac{0.485}{\sqrt{10}} = 1.749 \text{mg/l}$$

6.9

(a) $\overline{L} = \dfrac{124.3 + 124.2 + 124.4}{3} = 124.3\,ft.$

$S_L^{\ 2} = \dfrac{1}{3-1}[(124.3 - 124.3)^2 + (124.2 - 124.3)^2 + (124.4 - 124.3)^2] = 0.01$

Standard error of the mean = $\dfrac{s_L}{\sqrt{n}} = \dfrac{\sqrt{0.01}}{\sqrt{3}} = 0.0577\,ft.$

(b) $\overline{\beta} = \dfrac{40^o 24.6' + ... + 40^o 25.2'}{5} = 40^o 25'$

$S_\beta^{\ 2} = \dfrac{1}{5-1}[(40^o 24.6' - 40^o 25.0')^2 + ... + (40^o 25.2' - 40^o 25.0')^2] = 0.135'$

Standard error of the mean = $\dfrac{s_\beta}{\sqrt{n}} = \dfrac{\sqrt{0.135}}{\sqrt{5}} = 0.164' = 4.77 \times 10^{-5}$ radian

(c) $\overline{h} = 3\,ft; \sigma_{\overline{h}} = 0.01\,ft$

H = h + L tan β

$\overline{H} = \overline{h} + \overline{L} \tan \overline{\beta} = 3 + 124.3 \times \tan 40^o 25' = 108.85\,ft.$

(d) $\sigma_{\overline{H}}^{\ 2}$ = Var ($\overline{h}$) + Var ($\overline{L}$)(tan $\overline{\beta}$)^2 + Var($\overline{\beta}$) x ($\overline{L}$ •sec^2 $\overline{\beta}$)^2

= $(0.01)^2 + (0.0577)^2(\tan 40^o25')^2 + (4.77 \text{ x } 10^{-5})^2 \text{ x } (124.3 \sec^2 40^o25')^2$

= 0.0001+0.0024+0.0001 = 0.0026

$\sigma_{\overline{H}} = 0.051\,ft.$

(e) $P(-k_{\alpha/2} < \dfrac{\overline{H} - H}{\sigma_{\overline{H}}} \le k_{\alpha/2}) = 0.98$

or $P(-2.33 < \dfrac{108.85 - H}{0.051} \le 2.33) = 0.98$

or $P(108.73 \le H < 108.97) = 0.98$

So $\langle H \rangle_{0.98}$ = (108.73, 108.97) ft

6.10

(a) $\overline{r_1} = \dfrac{2.5+2.4+2.6+2.6+2.4}{5} = 2.5cm$

$\overline{r_2} = \dfrac{1.6+1.5+1.6+1.4+1.4}{5} = 1.5cm$

$S_{r_1}{}^2 = \dfrac{1}{5-1}[(2.5-2.5)^2 + (2.4-2.5)^2 \times 2 + (2.6-2.5)^2 \times 2] = 0.01$

$S_{r_1} = 0.1cm$; $\quad S_{\bar{r}_1} = \sqrt{0.01/5} = 0.0447$

Similarly, $S_{\bar{r}2} = \sqrt{0.01/5} = 0.0447$

(b) $\overline{A} = \pi\,(\bar{r}_1{}^2 - \bar{r}_2{}^2) = \pi\,(2.5^2 - 1.5^2) = 12.57\text{cm}^2$

(c) $\sigma_{\bar{A}}{}^2 = S_{\bar{r}_1}{}^2\,(2\pi\,\bar{r}_1)^2 + S_{\bar{r}_2}{}^2\,(-2\pi\,\overline{r_2})^2$

$= (0.0447)^2\,(2\pi \text{ x } 2.5)^2 + (0.0447)^2\,(-2\pi \text{ x } 1.5)^2 = 0.671$

$\sigma_{\bar{A}} = 0.819\text{cm}^2$

(d) $t_{\alpha/2,n-1}\dfrac{s_{r_1}}{\sqrt{n}} = 0.07$

$S_{r_1} = 0.1$

So, n = $(\dfrac{t_{\alpha/2,n-1}}{0.7})^2$ ------ (1)

Assume n=10, $t_{\alpha/2,9} = 3.2498$ so from Equation (1), n = 21.55.

For n = 16, $t_{\alpha/2,15} = 2.9467$ and n = 17.7

For n = 17, $t_{\alpha/2,16} = 2.9208$ and n = 17.4

For n = 18, $t_{\alpha/2,17} = 2.8982$ and n = 17.14

From trial and error, n is found to be 17.

So additional measurements are to be made = 17-5 = 12.

6.11

(a) From the observation data we can get

$\bar{a} = 119.9375$, $\bar{b} = 449.5$, $\bar{\beta} = 59.9583$,

(b) $s_a^2 = 0.131$, $s_b^2 = 0.15$, $s_\beta^2 = 0.1394\%$

(c) The mean area of the triangle is

$$\bar{A} = \frac{1}{2}\bar{a}\bar{b}\sin\bar{\beta} = 0.5 \times 119.9375 \times 449.5 \times \sin 59.9583 = 23334.73\text{m}^2$$

The variance of the area is:

$$\begin{aligned}\sigma_{\bar{A}}^2 &= \frac{1}{2}\bar{b}\sin\bar{\beta}\sigma_{\bar{a}}^2 + \frac{1}{2}\bar{a}\sin\bar{\beta}\sigma_{\bar{b}}^2 + \frac{1}{2}\bar{a}\bar{b}\cos\bar{\beta}\sigma_{\bar{\beta}}^2 \\ &= 0.5 \times 449.5 \times \sin 59.9583 \times 0.131 + 0.5 \times 119.9375 \times \sin 59.9583 \times 0.15 \\ &\quad +0.5 \times 119.9375 \times 449.5 \times \cos 59.9583 \times 0.1394\% \\ &= 52.086\text{m}^4\end{aligned}$$

$$\sigma_{\bar{A}} = 7.217\text{m}^2$$

(d)

$$\begin{aligned}\langle A \rangle_{0.9} &= \left(\bar{A} + k_{0.05} \bullet \sigma_{\bar{A}}, \bar{A} + k_{0.95} \bullet \sigma_{\bar{A}}\right) \\ &= \left(23334.73 - 1.64 \times 7.217, 23334.73 + 1.64 \times 7.217\right) \\ &= \left(23322.89, 23346.57\right)\end{aligned}$$

6.12

(a) From the observation data

$\overline{a} = 80.06$, $s_a^2 = 0.182$; $\overline{b} = 120.52$, $s_b^2 = 0.277$

Thus $\overline{A} = \frac{1}{2}\overline{a}\overline{b} = 0.5 \times 80.06 \times 120.52 = 4824.42\text{m}^2$

(b)

$$\sigma_{\overline{A}}^2 = \frac{1}{2}\overline{b}\sigma_{\overline{a}}^2 + \frac{1}{2}\overline{a}\sigma_{\overline{b}}^2 = 0.5 \times 120.52 \times 0.182 + 0.5 \times 80.06 \times 0.277 = 22.056\text{m}^4$$

$\sigma_{\overline{A}} = 4.696\text{m}^2$

(c)

$$\begin{aligned}\langle A \rangle_{0.95} &= \left(\overline{A} + k_{0.025} \bullet \sigma_{\overline{A}}, \overline{A} + k_{0.975} \bullet \sigma_{\overline{A}}\right) \\ &= \left(4824.42 - 1.96 \times 4.696, 4824.42 + 1.96 \times 4.696\right) \\ &= \left(4815.21, 4833.62\right)\end{aligned}$$

(d) It can be shown that

$$E(C) = E_A\left[E(C \mid A)\right] = 10000 + 15 \times E(A) = 10000 + 15 \times 4824.42 = 82366.3\$$$

and

$$\begin{aligned}Var(C) &= E_A\left[Var(C \mid A)\right] + Var_A\left[E(C \mid A)\right] \\ &= 20000^2 + Var_A\left[10000 + 15A\right] \\ &= 20000^2 + 15^2 \times \sigma_{\overline{A}}^2 \\ &= 20000^2 + 15^2 \times 22.056 \\ &= 400004962.6\$^2\end{aligned}$$

$\sigma_C = 20000.12\$$

$$P(C > 90000) = 1 - P(C < 90000) = 1 - \Phi\left(\frac{90000 - 82366.3}{20000.12}\right) = 0.35$$

6.13

Let x denotes the mileage

(a) $\overline{x} = 37.4$ $s_x = 4.06$

(b) One-sided hypothesis test that the stated mileage is smaller than 35 mpg:

Null hypothesis H_o: $\mu = 35$ mpg

Alternative hypothesis H_A: $\mu < 35$ mpg

Without assuming known standard deviation, the estimated value of the test statistic to be

$$t = \frac{\overline{x} - 35}{4.06/\sqrt{10}} = \frac{37.4 - 35}{4.06/\sqrt{10}} = 1.87$$

with f = 10-1 = 9 d.o.f. we obtain the critical value of t from Table A.2. 2 at the 2% significance level to be t_α =-2.448. Therefore the value of the test statistic is 1.87>-2.448, which is outside the region of rejection; hence the stated mileage is satisfied.

(c)

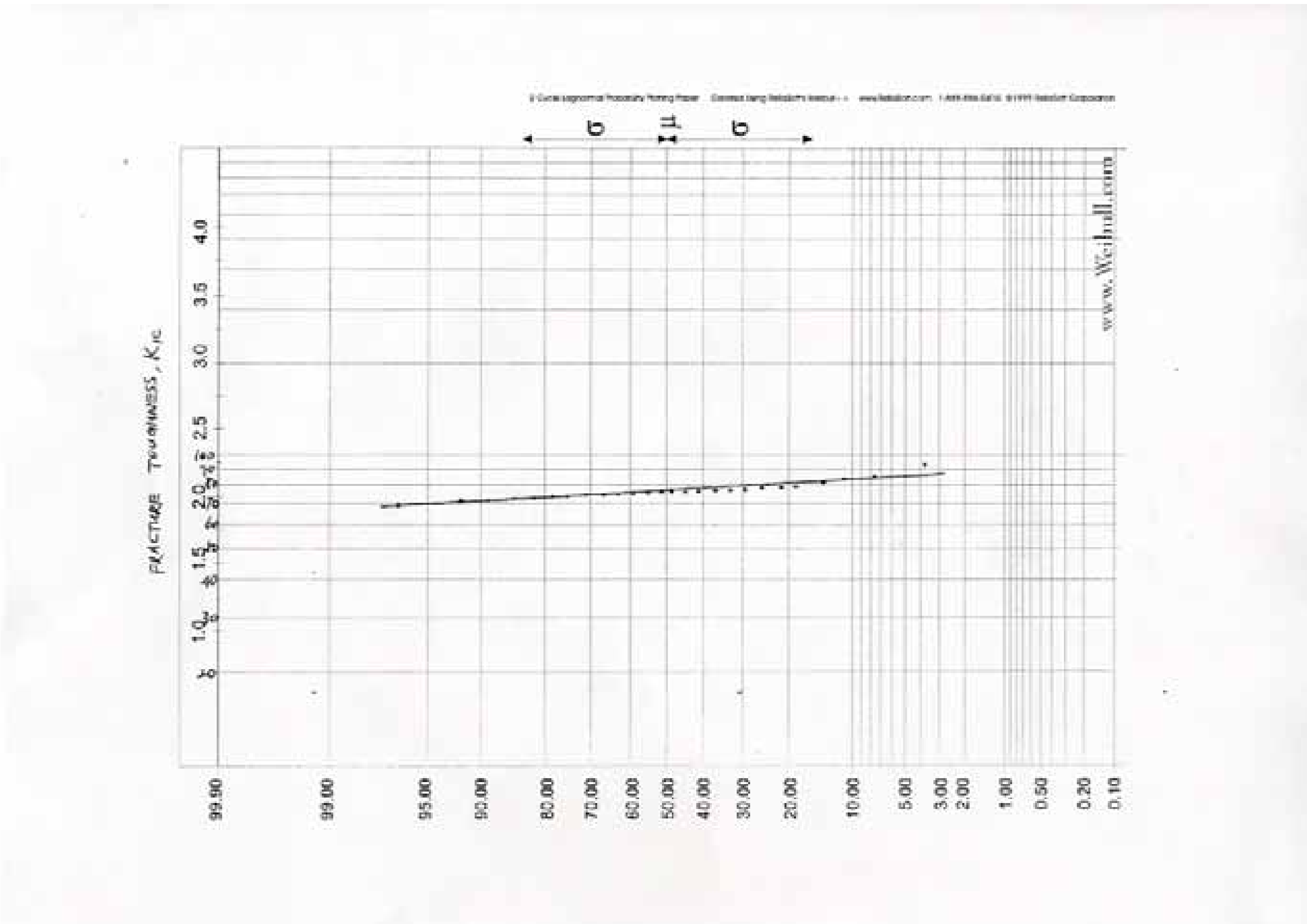
$$\langle x \rangle_{0.95} = \left(\overline{x} - t_{0.025,9} \bullet \frac{s_x}{\sqrt{10}}, \overline{x} + t_{0.025,9} \bullet \frac{s_x}{\sqrt{10}} \right)$$

$$= \left(37.4 - 2.262 \times \frac{4.06}{\sqrt{10}}, 37.4 + 2.262 \times \frac{4.06}{\sqrt{10}} \right)$$

$$= (34.50, 40.30)$$

7.1

Median = 76.9
c.o.v. = ln (81.5 / 76.9) = 0.06



7.2

(a) The CDF of Gumbel distribution is:

$$F_X(x) = \exp\left(-e^{-\alpha(x-\mu)}\right), -\infty < x < \infty$$

The standard variate is $S = -\alpha(X - \mu)$

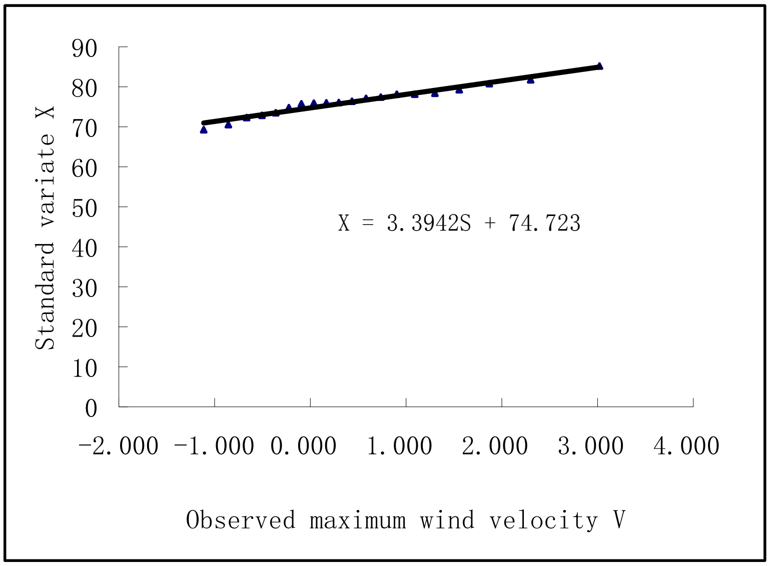
With CDF of: $F_S(s) = \exp\left(-e^{-s}\right)$

If x is observed with frequency of p, then its corresponding standard variate is

$$s = -\ln\left(-\ln(p)\right)$$

Thus, the observed data can be arranged in the right table and plotted in the following figure.

	x	P	s
1	69.3	0.048	-1.113
2	70.6	0.095	-0.855
3	72.3	0.143	-0.666
4	72.9	0.190	-0.506
5	73.5	0.238	-0.361
6	74.8	0.286	-0.225
7	75.8	0.333	-0.094
8	75.9	0.381	0.036
9	76	0.429	0.166
10	76.1	0.476	0.298
11	76.4	0.524	0.436
12	77.1	0.571	0.581
13	77.4	0.619	0.735
14	78.2	0.667	0.903
15	78.2	0.714	1.089
16	78.4	0.762	1.302
17	79.3	0.810	1.554
18	80.8	0.857	1.870
19	81.8	0.905	2.302
20	85.2	0.952	3.020



(b) The linear regression of the data gives the trend line equation of

$$X = 3.3942S + 74.723$$

which gives

$$S = \frac{X - 74.723}{3.3942} = 0.295(X - 74.723)$$

Thus, $\alpha = 0.295$, $\mu = 72.723$

(c) Perform Chi-Square test for Gumbel distribution

Interval	Observed frequency n_i	Theoretical frequency e_i	$(n_i - e_i)^2$	$(n_i - e_i)^2 / e_i$
65-70	1	0.356	0.415	1.164
70-75	5	7.602	6.770	0.891
75-80	11	8.240	7.617	0.924
80-85	3	2.860	0.020	0.007
Σ	20	20		2.986

The degree of freedom for the Gumbel distribution is f=4-1-2=1.

For a significance level $\alpha = 2\%$, $c_{.98,1} = 5.41$. Comparing with the $\Sigma(n_i - e_i)^2 / e_i$ calculated, the Gumbel distribution is a valid model for the maximum wind velocity at the significance level of $\alpha = 2\%$.

7.3

(a)

m	ε_μ	$\frac{m}{N+1}$
1	16.0	0.0625
2	16.1	0.1250
3	16.6	0.1875
4	16.8	0.2500
5	17.0	0.3125
6	17.3	0.3750
7	17.8	0.4375
8	17.9	0.5000
9	18.1	0.5625
10	18.4	0.6250
11	18.6	0.6875
12	18.8	0.7500
13	19.1	0.8125
14	19.4	0.8750
15	20.1	0.9375

From Figs. 7.3a and 7.3b, it can be observed that both models (normal and lognormal) fit the data pretty well.

(b)

$\bar{U} = 17.8 \; s_U^2 = 1.312$

$$\zeta = \sqrt{\ln(1+\frac{\sigma^2}{\mu^2})} = \sqrt{\ln(1+\frac{1.312^2}{17.8^2})} = 0.074$$

$$\lambda = \ln\mu - \frac{1}{2}\zeta^2 = \ln(17.8) - 0.5\times 0.074^2 = 2.876$$

Perform Chi-square test for normal distribution:

	Observed frequency n_i	Theoretical frequency e_i	$(n_i - e_i)^2$	$\frac{(n_i - e_i)^2}{e_i}$
15.5-16.5	3.000	1.816	1.401	0.772
16.5-17.5	3.000	3.730	0.533	0.143
17.5-18.5	4.000	4.404	0.163	0.037
18.5-19.5	4.000	2.989	1.021	0.342
19.5-20.5	1.000	1.166	0.028	0.024
$\sum$	15	15		1.317

Perform Chi-square test for lognormal distribution:

Interval	Observed frequency n_i	Theoretical frequency e_i	$(n_i - e_i)^2$	$\frac{(n_i - e_i)^2}{e_i}$
15.5-16.5	3.000	1.939	1.126	0.581
16.5-17.5	3.000	3.943	0.890	0.226

17.5-18.5	4.000	4.316	0.100	0.023
18.5-19.5	4.000	2.778	1.493	0.537
19.5-20.5	1.000	1.133	0.018	0.016
$\sum$	15	15		1.382

The degree of freedom for the Normal and Lognormal distribution are both f=5-1-2=2.

For a significance level $\alpha = 2\%$, $c_{.98,2} = 7.99$. Comparing with the $\sum \frac{(n_i - e_i)^2}{e_i}$ calculated, the 2 distributions are both valid for the rainfall intensity at the significance level of $\alpha = 2\%$.

As the $\sum \frac{(n_i - e_i)^2}{e_i}$ in Normal distribution is less than that of the Lognormal distribution, the Normal distribution is superior to the Lognormal distribution in this problem.

(c) Perform the K-S test as follows:

k	x	S	F (Normal)	D (Normal)	F (Lognormal)	D (Lognormal)
1	15.8	0.067	0.064	0.003	0.059	0.008
2	16.0	0.133	0.085	0.048	0.081	0.052
3	16.1	0.200	0.098	$\boxed{0.102}$	0.095	$\boxed{0.105}$
4	16.6	0.267	0.180	0.086	0.184	0.083
5	17.0	0.333	0.271	0.062	0.282	0.052
6	17.3	0.400	0.352	0.048	0.366	0.034
7	17.8	0.467	0.500	0.033	0.517	0.051
8	17.9	0.533	0.530	0.003	0.547	0.014
9	18.1	0.600	0.590	0.010	0.606	0.006
10	18.4	0.667	0.676	0.010	0.688	0.022
11	18.6	0.733	0.729	0.004	0.738	0.005
12	18.8	0.800	0.777	0.023	0.783	0.017
13	19.1	0.867	0.839	0.028	0.840	0.026
14	19.4	0.933	0.889	0.045	0.886	0.047
15	20.1	1.000	0.960	0.040	0.954	0.046
max				0.102		0.105

The K-S test shows that the Normal distribution is more suitable in this case as it gives smaller maximum discrepancy.

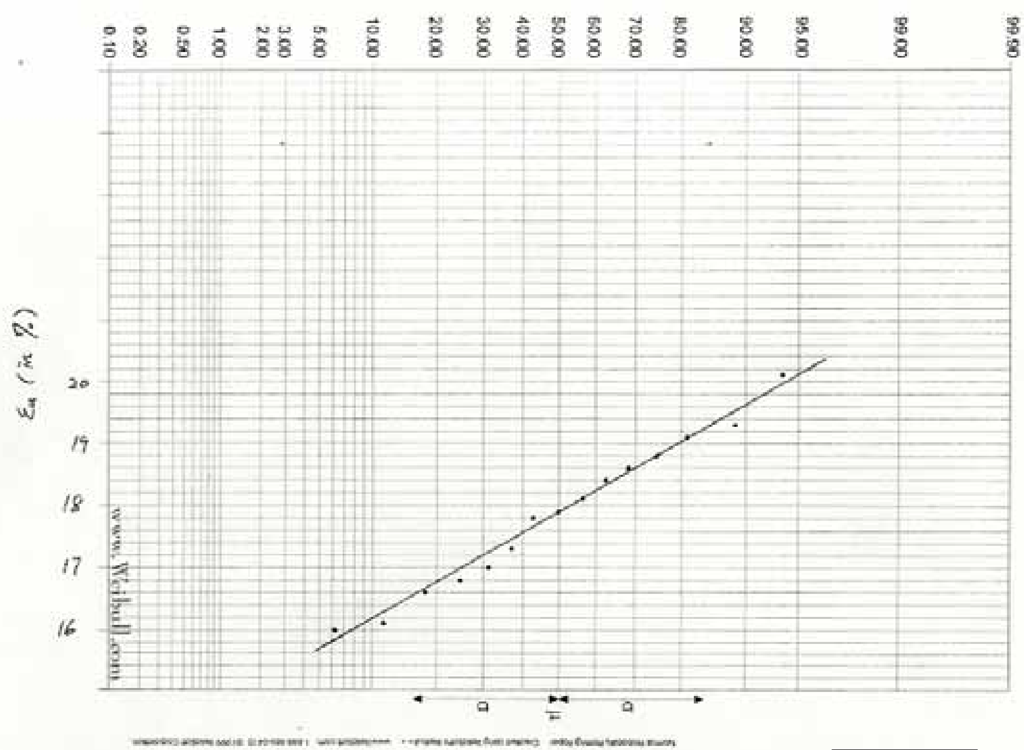
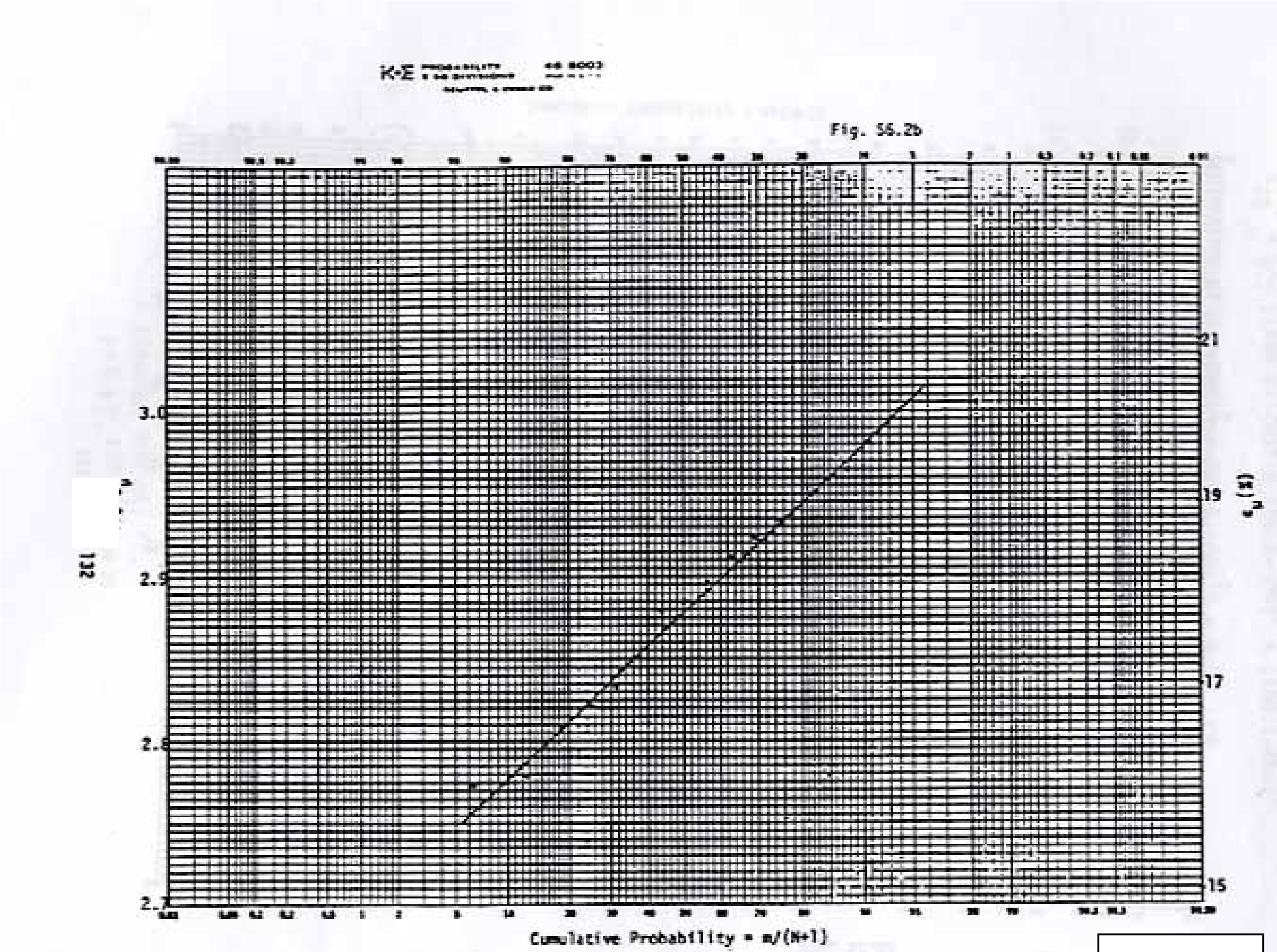

0.10
0.20
0.50
1.00
2.00
3.00
5.00
10.00
20.00
30.00
40.00
50.00
60.00
70.00
80.00
90.00
95.00
99.00
99.90
20
19
18
17
16
www.Weibull.com

Fig. 7.3a



K+E PROBABILITY 46 8003
Fig. 56.2b
3.0
2.9
2.8
2.7
21
19
17
15
Cumulative Probability = m/(N+1)
132

Fig. 7.3b

7.4

(a)

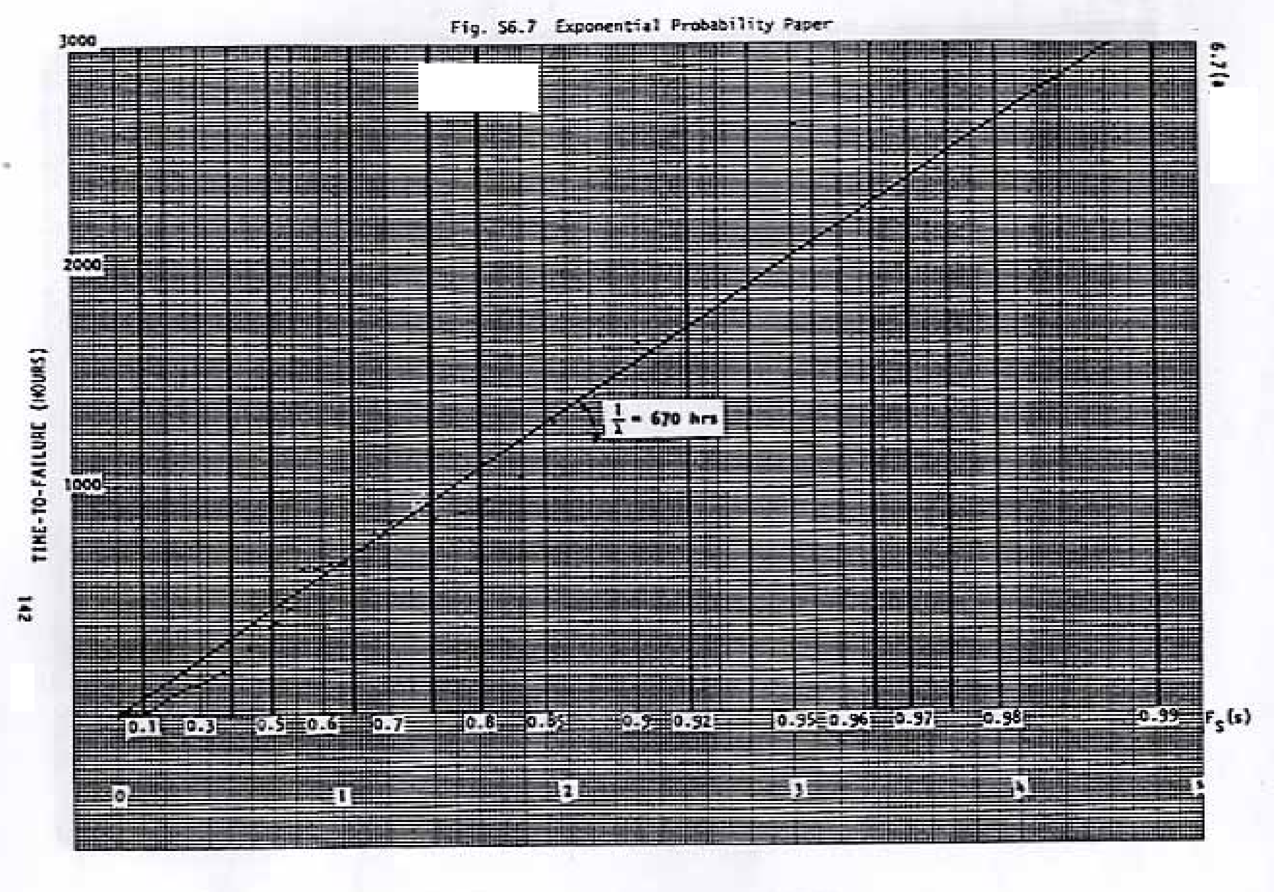


Fig. S6.7 Exponential Probability Paper
3000
2000
1000
TIME-TO-FAILURE (HOURS)
$\frac{1}{\lambda}$ = 670 hrs
0.1
0.3
0.5
0.6
0.7
0.8
0.9
0.92
0.95
0.96
0.97
0.98
0.99
$F_s(s)$
0
1
2
3
4
5
142

(b) The minimum time-to-failure = 0 hr.

Mean time to failure = $\dfrac{1}{\lambda} = 670 \quad hrs.$

(c)

Time-to-failure (hours)	Observed Frequency n_i	Theoritical Frequency e_i	$(n_i-e_i)^2$	$(n_i-e_i)^2 / e_i$
0 – 400	20	17.98	4.08	0.227
400 – 800	7	9.90	8.41	0.849
800 – 1200	7	5.45	2.40	0.441
> 1200	6	6.67	0.45	0.067
Σ	40	40		1.584

x^2 distribution with f = 4 - 2 = 2 degrees of freedom and α = 5% significance level,
$c_{0.95,\,2} = 5.99$

In this case $\sum_{i=1}^{4} \dfrac{(n_i - e_i)^2}{e_i} = 1.584 < 5.99$

Hence, the exponential distribution is a valid distribution model.

(d) Perform K-S test as follows:

k	x	S	F(Normal)	D(Normal)	k	x	S	F(Normal)	D(Normal)
1	0.13	0.025	0.000	0.025	21	412.03	0.525	0.462	0.063
2	0.78	0.050	0.001	0.049	22	470.97	0.550	0.507	0.043
3	2.34	0.075	0.004	0.071	23	633.98	0.575	0.615	0.040
4	3.55	0.100	0.005	0.095	24	646.01	0.600	0.621	0.021
5	8.82	0.125	0.013	0.112	25	656.04	0.625	0.627	0.002
6	14.29	0.150	0.021	0.129	26	658.38	0.650	0.628	0.022
7	29.75	0.175	0.044	0.131	27	672.87	0.675	0.636	0.039
8	39.1	0.200	0.057	0.143	28	678.13	0.700	0.639	0.061
9	54.85	0.225	0.079	0.146	29	735.89	0.725	0.669	0.056
10	62.09	0.250	0.089	0.161	30	813	0.750	0.705	0.045
11	84.09	0.275	0.119	0.156	31	855.95	0.775	0.724	0.051
12	85.28	0.300	0.120	0.180	32	861.93	0.800	0.726	0.074
13	121.58	0.325	0.167	0.158	33	862.93	0.825	0.727	0.098
14	124.09	0.350	0.170	0.180	34	895.8	0.850	0.740	0.110
15	151.44	0.375	0.204	0.171	35	952.65	0.875	0.761	0.114
16	163.95	0.400	0.218	$\boxed{0.182}$	36	1057.57	0.900	0.796	0.104
17	216.4	0.425	0.278	0.147	37	1296.93	0.925	0.858	0.067
18	298.58	0.450	0.362	0.088	38	1407.52	0.950	0.880	0.070
19	380	0.475	0.435	0.040	39	1885.22	0.975	0.941	0.034
20	393.37	0.500	0.446	0.054	40	2959.47	1.000	0.988	0.012
max				0.182					

Where $\overline{t} = 541.19$ $s_t^2 = 359937.2$ The sample size is 40. At the 5% significance level, $D_{40}^{0.05} = 0.21$. As the maximum discrepancy is smaller than the critical value, the exponential distribution is verified.

7.5

(a) From the observation data we can get

$\bar{v} = 1.08$, $s_v^2 = 1.243$

(b)

Nos. of veh/min	Observed Freq. n_i	Total no. of veh. Observed	Theoretical Probability ($v = 1.2$)	Theoretical Frequency e_i	$(n_i-e_i)^2$	$(n_i-e_i)^2 / e_i$
0	6	0	0.301	6.02	0.0004	0.0001
1	8	8	0.361	7.22	0.6084	0.0843
≥ 2	6	2x3 + 3x2 + 4x1 = 16	0.338	6.76	0.5776	0.0854
	Σ20	Σ24	Σ1.0	Σ20.0		Σ0.1698

$$v = \frac{24}{20} = 1.2$$

x^2 distribution with f = 3-2 = 1 degree of freedom and α = 1% significance level, $c_{0.99,1}$ = 6.635

In this case $\sum_{i=1}^{3} \frac{(n_i - e_i)^2}{e_i} = 0.1698 < 6.635$

Hence, the Poisson distribution is a valid distribution model.

7.6

(a) $f_X(x) = \frac{x}{\alpha^2} e^{-\frac{1}{2}(\frac{x}{\alpha})^2} \geq 0$

$$S = \frac{X}{\alpha}$$

$$X = S\alpha, \quad \left|\frac{dX}{dS}\right| = \alpha$$

So $f_S(s) = f_X(s\alpha)\left|\frac{dX}{dS}\right| = s \cdot e^{-\frac{1}{2}s^2}$

$$= s \cdot e^{-\frac{1}{2}s^2}; \qquad s \geq 0$$
$$= 0 \qquad ; \qquad s < 0$$

$$F_S(s) = 1 - e^{-\frac{1}{2}s^2}$$

s	$F_S(s)$	s	$F_S(s)$
0.46	0.10	2.25	0.92
0.67	0.20	2.31	0.93
0.84	0.30	2.37	0.94
1.01	0.40	2.45	0.95
1.18	0.50	2.49	0.955
1.26	0.55	2.54	0.96
1.35	0.60	2.59	0.965
1.45	0.65	2.65	0.97
1.55	0.70	2.72	0.975
1.67	0.75	2.80	0.98
1.79	0.80	2.90	0.985
1.95	0.85	3.03	0.99
2.15	0.90	3.26	0.995

The corresponding probability paper is shown in Fig. 7.6. The slope of a straight line on this paper is,

$$\frac{dX}{ds} = \alpha$$

which is the model value of X.

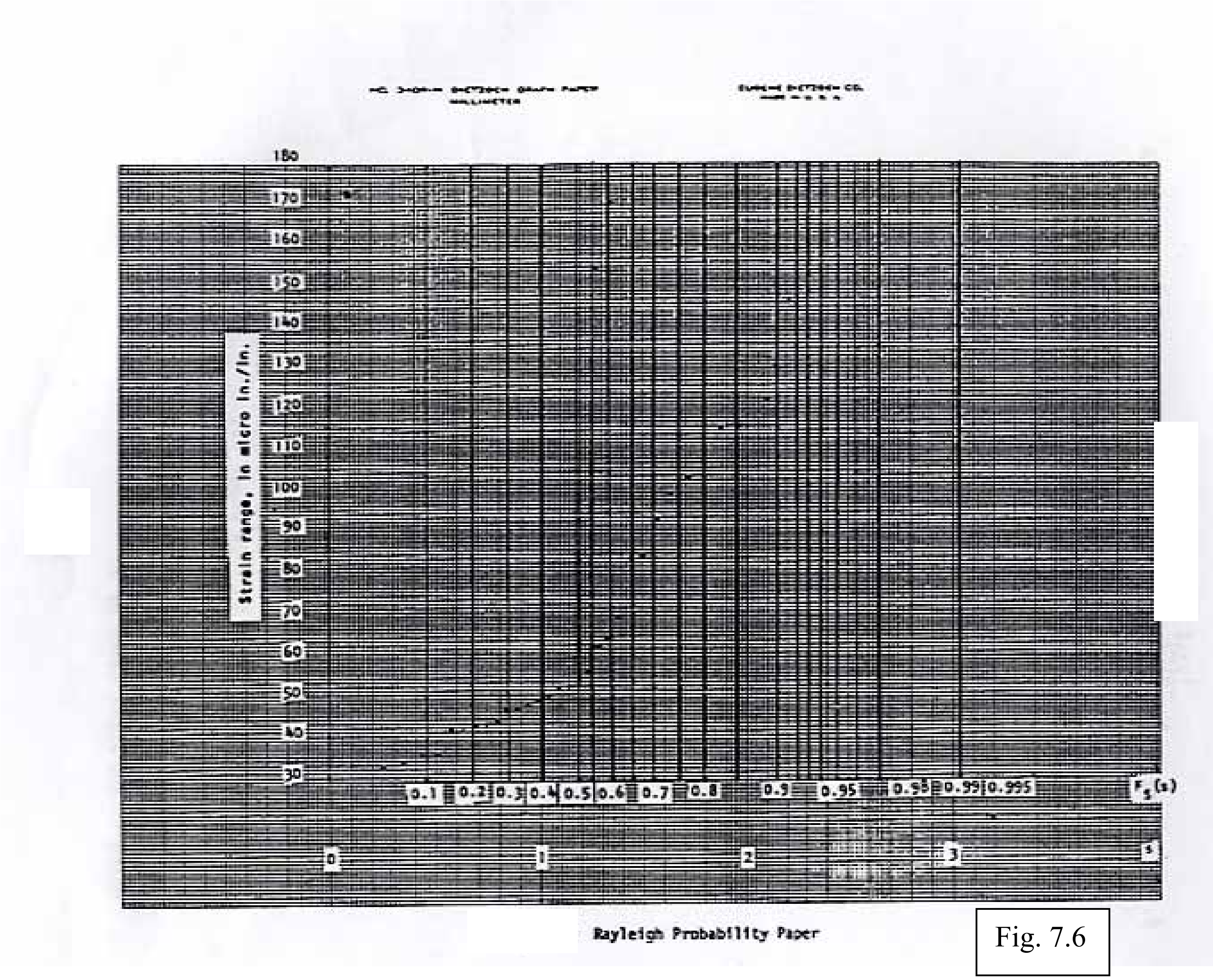


Fig. 7.6

(c) On the basis of the observed data, Rayleigh distribution does not appear to be a suitable model for live-load stress range in highway bridges. However, if the observed data indeed fell on a straight line, the slope of the fitted line will give the most probable stress range.

7.7

.(a) The middle point of each range is used to represent the range.

K_i	No. of observation(n_i)	n_iK_i	$(K_i - \bar{K})^2$	$n_i(K_i - \bar{K})^2$
0.025	1	0.025	0.022	0.022
0.075	11	0.825	0.010	0.107
0.125	20	2.500	0.002	0.048
0.175	23	4.025	0.000	0.000
0.225	15	3.375	0.003	0.039
0.275	11	3.025	0.010	0.113
0.325	2	0.650	0.023	0.046
Σ	83	14.425		0.375
		$\bar{K} = 0.174$		$s_K^2 = 0.0046$

(b)

K (per day)	Observed frequency n_i	Theoretical frequency e_i	$(n_i-e_i)^2$	$(n_i-e_i)^2 / e_i$
< 0.100	12	11.09	0.8281	0.075
0.100 – 0.150	20	19.06	0.8836	0.046
0.150 – 0.200	23	24.56	2.4336	0.099
0.200 – 0.250	15	18.25	10.5625	0.579
> 0.250	13	10.04	8.7616	0.873
Σ	83	83		1.672

x^2 distribution with f = 5-3 = 2 degree of freedom and α = 5% significance level, $c_{0.95,2}$ = 5.991

In this case $\sum_{i=1}^{5} \frac{(n_i - e_i)^2}{e_i} = 1.672 < 5.991$

So the normal distribution is acceptable.

7.8

(a) $S = \frac{X-a}{r}$; $X = Sr + a$; $\left|\frac{dX}{dS}\right| = r$

So $f_s(s) = f_x(sr+a)\left|\frac{dX}{dS}\right| = \frac{2}{r^2}(sr)\cdot r$

$= 2s$; $0 \le s \le 1$
$= 0$; elsewhere

(b) $F_S(s) = S^2$; $0 \le s \le 1$
$= 1$; $S > 1.0$
$= 0$; $S < 0$

s	$F_S(s)$	s	$F_S(s)$
0.224	0.05	0.959	0.92
0.316	0.10	0.964	0.93
0.447	0.20	0.970	0.94
0.548	0.30	0.975	0.95
0.632	0.40	0.977	0.955
0.707	0.50	0.980	0.96
0.742	0.55	0.982	0.965
0.775	0.60	0.985	0.97
0.806	0.65	0.987	0.975
0.837	0.70	0.990	0.98
0.866	0.75	0.992	0.985
0.894	0.80	0.995	0.99
0.922	0.85		
0.949	0.90		

The corresponding probability paper is shown in Fig 7.8.

When $s = 0 = \frac{X-a}{r} \rightarrow \quad x = a$

When $s = 1 = \frac{X-a}{r} \rightarrow \quad x = r + a$

$\therefore$ values of X at $F_S(0)$ and $F_S(1.0)$ correspond to a and r+a which are the minimum and maximum values of X.

(c)

m	X	$\frac{m}{N+1}$
1	18	0.045
2	28	0.091
3	32	0.136
4	34	0.182
5	36	0.227
6	45	0.273
7	48	0.318
8	50	0.364

9	53	0.409
10	53	0.455
11	55	0.500
12	56	0.545
13	58	0.591
14	62	0.636
15	64	0.682
16	66	0.727
17	69	0.773
18	71	0.818
19	71	0.864
20	72	0.909
21	75	0.955

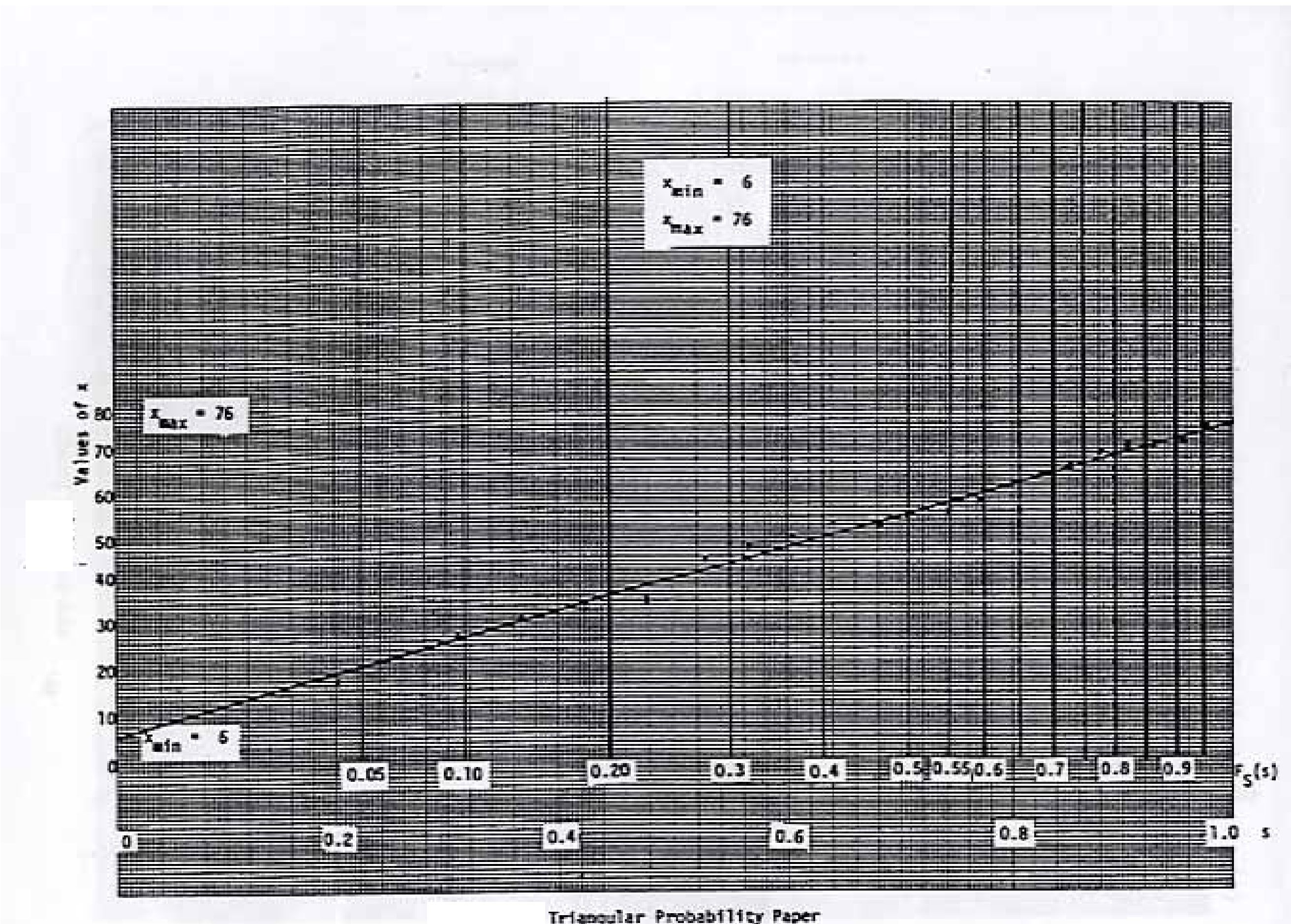
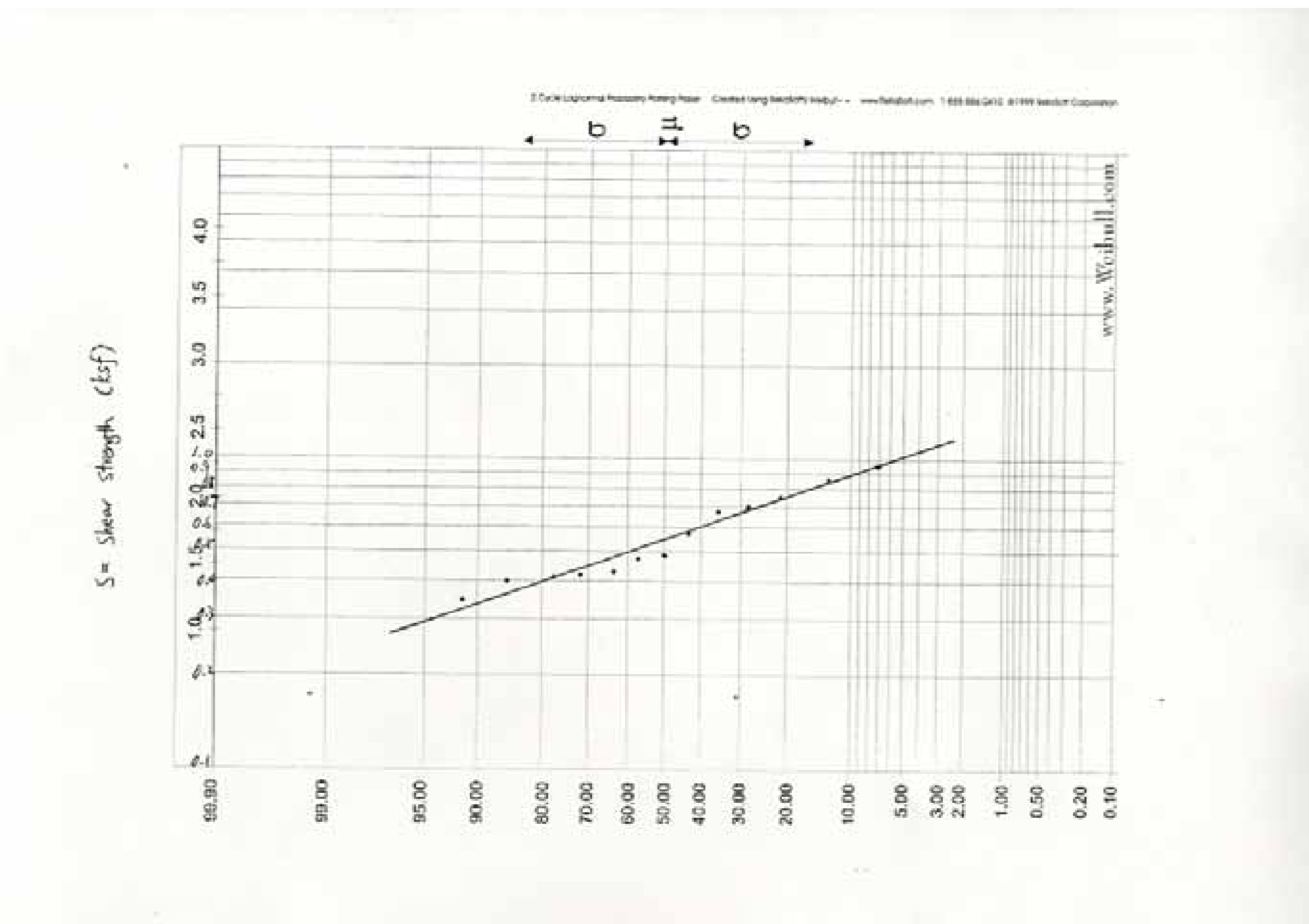


Fig. 7.8

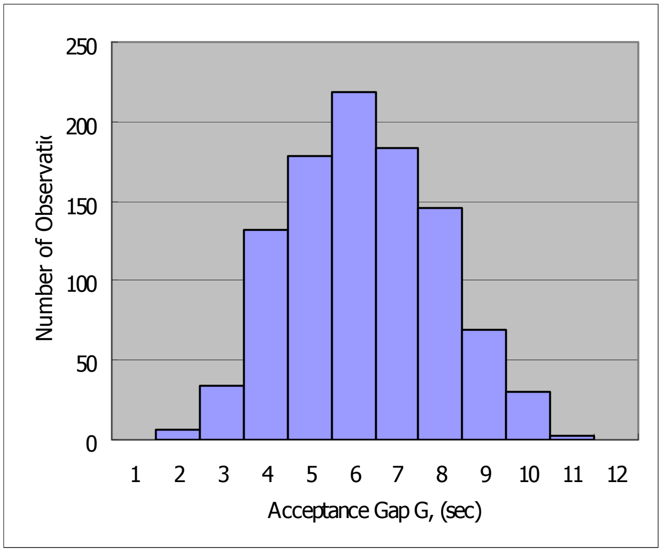
7.9

m	shear strength (S) (Ksf)	$\frac{m}{N+1}$
1	0.35	0.0714
2	0.40	0.1429
3	0.41	0.2143
4	0.42	0.2857
5	0.43	0.3571
6	0.48	0.4286
7	0.49	0.5000
8	0.58	0.5714
9	0.68	0.6429
10	0.70	0.7143
11	0.75	0.7857
12	0.87	0.8571
13	0.96	0.9286



7.10

(a)



(b) The middle point of each range is used to calculated the sample mean and sample variance as follows:

G_i	No. of observation(n_i)	n_iG_i	$(G_i - \overline{G})^2$	$n_i(G_i - \overline{G})^2$
1	0	0	27.542	0.000
2	6	12	18.046	108.273
3	34	102	10.550	358.683
4	132	528	5.054	667.063
5	179	895	1.558	278.793
6	218	1308	0.062	13.408
7	183	1281	0.566	103.487
8	146	1168	3.070	448.148
9	69	621	7.574	522.572
10	30	300	14.078	422.325
11	3	33	22.582	67.745
12	0	0	33.086	0.000
Σ	1000	6248		2990.496
		$\overline{G} = 6.248$		$s_G^2 = 2.993$

Perform the Chi-square test for normal distribution

Interval	Observed frequency n_i	Theoretical frequency e_i	$(n_i - e_i)^2$	$\frac{(n_i - e_i)^2}{e_i}$
0.5-1.5	0	2.586	6.688	2.586
1.5-2.5	6	12.113	37.368	3.085
2.5-3.5	34	40.966	48.520	1.184

3.5-4.5	132	100.063	1019.948	10.193
4.5-5.5	179	176.577	5.870	0.033
5.5-6.5	218	225.150	51.116	0.227
6.5-7.5	183	207.452	597.887	2.882
7.5-8.5	146	138.121	62.074	0.449
8.5-9.5	69	66.443	6.537	0.098
9.5-10.5	30	23.088	47.774	2.069
10.5-11.5	3	5.794	7.804	1.347
11.5-12.5	0	1.050	1.102	1.050
$\sum$	1000			25.204

Perform the Chi-square test for lognormal distribution

Interval	Observed frequency n_i	Theoretical frequency e_i	$(n_i - e_i)^2$	$\frac{(n_i - e_i)^2}{e_i}$
0.5-1.5	0	0.000	0.000	0.000
1.5-2.5	6	0.610	29.047	47.582
2.5-3.5	34	22.354	135.621	6.067
3.5-4.5	132	119.016	168.591	1.417
4.5-5.5	179	227.509	2353.093	10.343
5.5-6.5	218	241.300	542.883	2.250
6.5-7.5	183	179.619	11.434	0.064
7.5-8.5	146	107.248	1501.746	14.003
8.5-9.5	69	55.622	178.971	3.218
9.5-10.5	30	26.330	13.472	0.512
10.5-11.5	3	11.745	76.484	6.512
11.5-12.5	0	5.044	25.441	5.044
$\sum$	1000			97.009

Where the lognormal distribution parameter is calculated as follows:

$$\lambda = \ln \mu - \frac{1}{2}\zeta^2 = \ln(6.248) - 0.5 \times 2.993 = 1.795$$

$$\zeta = \sqrt{\ln(1 + \frac{\sigma^2}{\mu^2})} = \sqrt{\ln(1 + \frac{2.993}{6.248^2})} = 0.272$$

As 25.204<97.009, the normal distribution is more suitable for this problem.

(c)

Acceptance gap size (secs) G_i	No. of Observations n_i	$\frac{G_i \cdot n_i}{\sum n_i}$	$(G_i-\mu_G)^2$	$(G_i-\mu_G)^2 n_i / \Sigma n_i$ $(\times 10^{-3})$
1	0	0	27.56	0
2	6	0.012	18.0625	108.375
3	34	0.104	10.5625	359.125
4	132	0.528	5.0625	668.250
5	179	0.895	1.5625	279.688

6		218	1.308	0.0625	13.625
7		183	1.281	0.5625	102.938
8		146	1.168	3.0625	447.125
9		69	0.621	7.5625	521.813
10		30	0.300	14.0625	421.875
11		3	0.033	22.5625	67.688
12		0	0	33.0625	0
	Σ	1000	6.25		2990.502×10^{-3}

$\mu_G = 6.25$ sec
$Var(G) = 2.99$
$\sigma_G = 1.729$

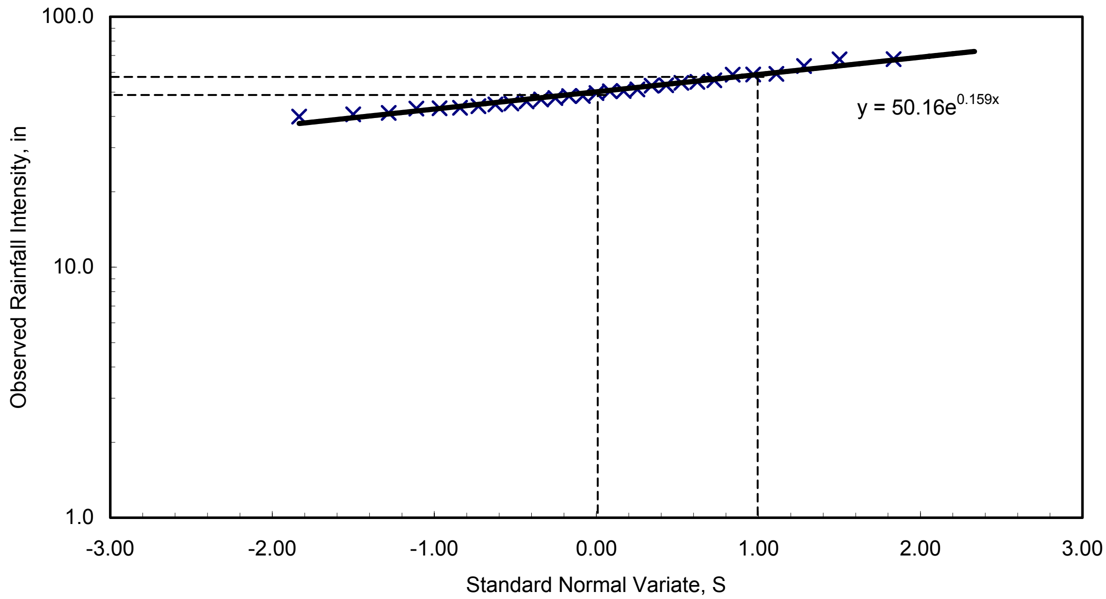
7.11

Rearrange the given data in increasing order, we obtain the following table:

m	Observed Rainfall Intensity X, in	m/N+1	S
1	39.91	0.03	-1.83
2	40.78	0.07	-1.50
3	41.31	0.10	-1.28
4	42.96	0.13	-1.11
5	43.11	0.17	-0.97
6	43.30	0.20	-0.84
7	43.93	0.23	-0.73
8	44.67	0.27	-0.62
9	45.05	0.30	-0.52
10	45.93	0.33	-0.43
11	46.77	0.37	-0.34
12	47.38	0.40	-0.25
13	48.21	0.43	-0.17
14	48.26	0.47	-0.08
15	49.57	0.50	0.00
16	50.37	0.53	0.08
17	50.51	0.57	0.17
18	51.28	0.60	0.25
19	53.02	0.63	0.34
20	53.29	0.67	0.43
21	54.49	0.70	0.52
22	54.91	0.73	0.62
23	55.77	0.77	0.73
24	58.71	0.80	0.84
25	58.83	0.83	0.97
26	59.12	0.87	1.11
27	63.52	0.90	1.28
28	67.59	0.93	1.50
29	67.72	0.97	1.83

(a)

Plot the data points in a lognormal probability paper



(b)

From the data in probability paper

$x_m = 50.16 \Rightarrow \lambda = \ln(x_m) = 3.92$

$x_{0.84} = 58.8 \Rightarrow \zeta = \ln(x_{0.84}/x_m) = 0.159$

Perform Chi-square test for log-normal distribution

Interval (in)	Observed frequency n_i	Theoretical frequency e_i	$(n_i\text{-}e_i)^2$	$\frac{(n_i - e_i)^2}{e_i}$
<40	1	2.118	1.250	0.590
40-45	7	4.784	4.913	1.027
45-50	7	7.018	0.000	0.000
50-55	7	6.629	0.137	0.021
55-60	4	4.495	0.245	0.054
>60	3	3.956	0.915	0.231
Σ	29	29		1.92

The degree of freedom for the lognormal distribution is f=6-1-2=3.

On the basis of the observation data

$\hat{\mu} = 50.354$ $\hat{\sigma}^2 = 61.179$, thus $\hat{\nu} = 0.823$ $\hat{k} = 41.445$

Perform Chi-square test for Gamma distribution

Interval (in)	Observed frequency n_i	Theoretical frequency e_i	$(n_i - e_i)^2$	$\frac{(n_i - e_i)^2}{e_i}$
<40	1	2.454	2.116	0.862
40-45	7	4.947	4.215	0.852

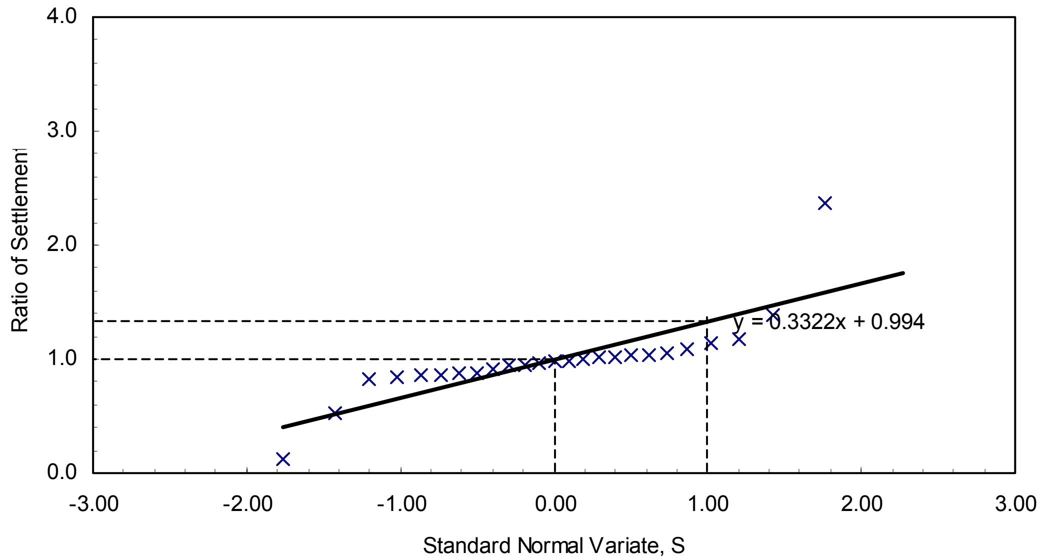
45-50	7	7.173	0.030	0.004
50-55	7	6.741	0.067	0.010
55-60	4	4.422	0.178	0.040
>60	3	3.263	0.069	0.021
$\sum$	29	29		1.790

The degree of freedom for the Gamma distribution is f=6-1-2=3.

For a significance level $\alpha = 5\%$, $c_{.95,3} = 7.81$. Comparing with the $\sum \frac{(n_i - e_i)^2}{e_i}$ calculated, both the Gamma distribution and Lognormal appear to be valid model for the rainfall intensity at the significance level of $\alpha = 5\%$. As the $\sum \frac{(n_i - e_i)^2}{e_i}$ in Gamma distribution is less than that of the lognormal distribution, the Gamma distribution is superior to the lognormal distribution in this problem.

7.12

(a) Plot data on normal probability paper



Linear trend is observed, but there are two data points out of the trend.

(b) Based on the trend line,

$x_m = 0.994 \Rightarrow \mu = 0.994$

$\Rightarrow \sigma = \text{slope} = 0.3322$

(b)

Perform Chi-square test for normal distribution,

Ratio of settlement	Observed frequency n_i	Theoretical frequency e_i	$(n_i\text{-}e_i)^2$	$\frac{(n_i - e_i)^2}{e_i}$
<0.75	2	5.783	14.312	2.475
0.75-1.0	13	6.897	37.246	5.400
1.0-1.25	8	6.808	1.420	0.209
1.25-1.5	1	3.915	8.499	2.171
>1.5	1	1.596425	0.356	0.223
Σ	25	25		10.25> $c_{.95,2}$=5.99

The degree of freedom for the normal distribution is f=5-1-2=2.

For a significance level α=5%, $c_{.95,2}$=5.99. Comparing with the $\sum \frac{(n_i - e_i)^2}{e_i}$ calculated, the normal distribution is not a valid model for the ratio of settlement.

Perform Anderson-Darling test for normal distribution,

i	x_i	F(x_i)	F(x_{n+1-i})	$(2i-1)\{\ln F_X(x_i)+\ln[1-F_X(x_{n+1-i})]\}/n$
1	0.12	0.004	1.000	-0.657
2	0.52	0.077	0.877	-0.560
3	0.82	0.300	0.712	-0.490
4	0.84	0.321	0.670	-0.628
5	0.86	0.343	0.614	-0.727
6	0.86	0.343	0.579	-0.851
7	0.87	0.354	0.555	-0.960
8	0.88	0.366	0.555	-1.089
9	0.92	0.412	0.531	-1.118
10	0.94	0.435	0.519	-1.188
11	0.94	0.435	0.507	-1.293
12	0.97	0.471	0.495	-1.321
13	0.99	0.495	0.495	-1.386
14	0.99	0.495	0.471	-1.447
15	1.00	0.507	0.435	-1.451
16	1.01	0.519	0.435	-1.522
17	1.02	0.531	0.412	-1.536
18	1.04	0.555	0.366	-1.462
19	1.04	0.555	0.354	-1.519
20	1.06	0.579	0.343	-1.509
21	1.09	0.614	0.343	-1.490
22	1.14	0.670	0.321	-1.356
23	1.18	0.712	0.300	-1.253
24	1.38	0.877	0.077	-0.396
25	2.37	1.000	0.004	-0.008
				Σ=-27.22

Calculate the Anderson-Darling (A-D) statistic

$$A^2 = -\sum_{i=1}^{n}\left[(2i-1)\{\ln F_X(x_i)+\ln[1-F_X(x_{n+1-i})]\}/n\right]-n \quad = 27.22\text{-}25 = 2.22$$

For normal distribution, the adjusted A-D statistic for a sample size $n=25$ is,

$$A^* = A^2\left(1.0+\frac{0.75}{n}+\frac{2.25}{n^2}\right) \quad =2.294$$

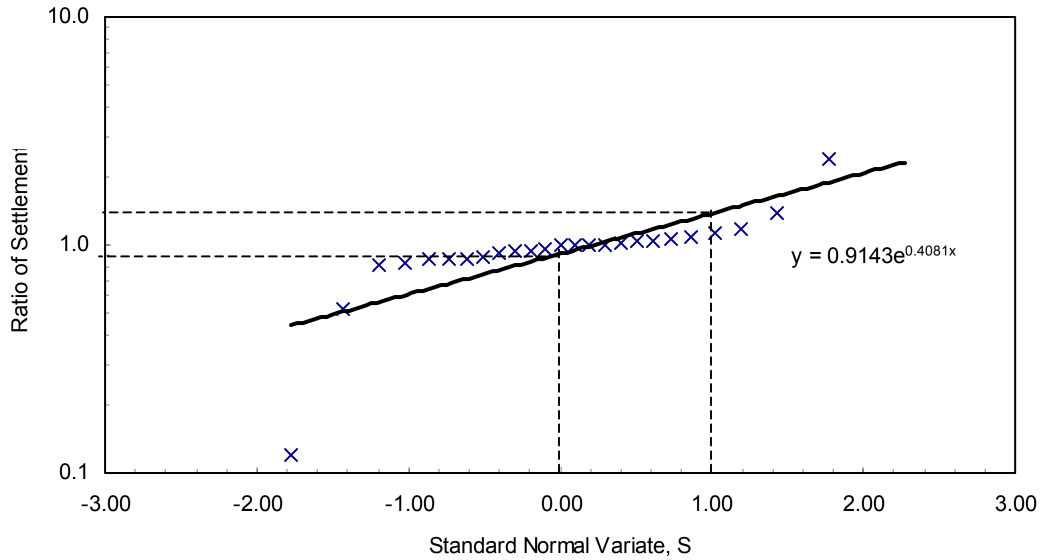
The critical value c_α is given by,

$$c_\alpha = a_\alpha\left(1+\frac{b_0}{n}+\frac{b_1}{n^2}\right) \quad = \ 0.726$$

for a prescribed significance level α = 5%, a_α =0.7514, b_0=-0.795, b_1=-0.89 (Table A.6).

As A^* is greater than c_α, the normal distribution is not acceptable at the significance level α = *5%.*

(c) Plot the data on the lognormal probability paper,



Therefore,
$x_m = 0.9143 \quad \Rightarrow \lambda = \ln(x_m) = -0.0896$
ζ = slope = 0.4081
Perform Chi-square test for lognormal distribution,

Ratio of settlement	Observed frequency n_i	Theoretical frequency e_i	$(n_i - e_i)^2$	$\frac{(n_i - e_i)^2}{e_i}$
<0.75	2	7.843	34.135	4.353
0.75-1.0	13	6.830	38.073	5.575
1.0-1.25	8	4.784	10.341	2.161
1.25-1.5	1	2.730	2.992	1.096
>1.5	1	2.814	3.290	1.169
Σ	25	25		Σ= 13.18> $c_{.95,2}$=5.99

The degree of freedom for the normal distribution is f=5-1-2=2.

For a significance level α=5%, $c_{.95,2}$=5.99. Comparing with the $\sum \frac{(n_i - e_i)^2}{e_i}$ calculated, the lognormal distribution is not a valid model for the ratio of settlement.

Perform Anderson-Darling test for lognormal distribution,

i	x_i	$F(x_i)$	$F(x_{n+1-i})$	$(2i-1)\{\ln F_X(x_i) + \ln[1 - F_X(x_{n+1-i})]\}/n$
1	0.12	0.000	0.990	-0.783
2	0.52	0.083	0.843	-0.521
3	0.82	0.395	0.734	-0.451
4	0.84	0.418	0.706	-0.587

5	0.86	0.440	0.667	-0.691
6	0.86	0.440	0.641	-0.812
7	0.87	0.452	0.624	-0.922
8	0.88	0.463	0.624	-1.049
9	0.92	0.506	0.606	-1.096
10	0.94	0.527	0.596	-1.176
11	0.94	0.527	0.587	-1.281
12	0.97	0.558	0.577	-1.330
13	0.99	0.577	0.577	-1.410
14	0.99	0.577	0.558	-1.474
15	1.00	0.587	0.527	-1.487
16	1.01	0.596	0.527	-1.570
17	1.02	0.606	0.506	-1.593
18	1.04	0.624	0.463	-1.530
19	1.04	0.624	0.452	-1.587
20	1.06	0.641	0.440	-1.598
21	1.09	0.667	0.440	-1.617
22	1.14	0.706	0.418	-1.530
23	1.18	0.734	0.395	-1.461
24	1.38	0.843	0.083	-0.484
25	2.37	0.990	0.000	-0.019
				Σ=-28.06

Calculate the Anderson-Darling (A-D) statistic

$$A^2 = -\sum_{i=1}^{n}\left[(2i-1)\left\{\ln F_X(x_i) + \ln[1-F_X(x_{n+1-i})]\right\}/n\right] - n \quad = 28.06\text{-}25 = 3.06$$

For lognormal distribution (lognormal distribution should be the same as normal), the adjusted A-D statistic for a sample size *n=25* is,

$$A^* = A^2\left(1.0 + \frac{0.75}{n} + \frac{2.25}{n^2}\right) \quad =3.16$$

The critical value c_α is given by,

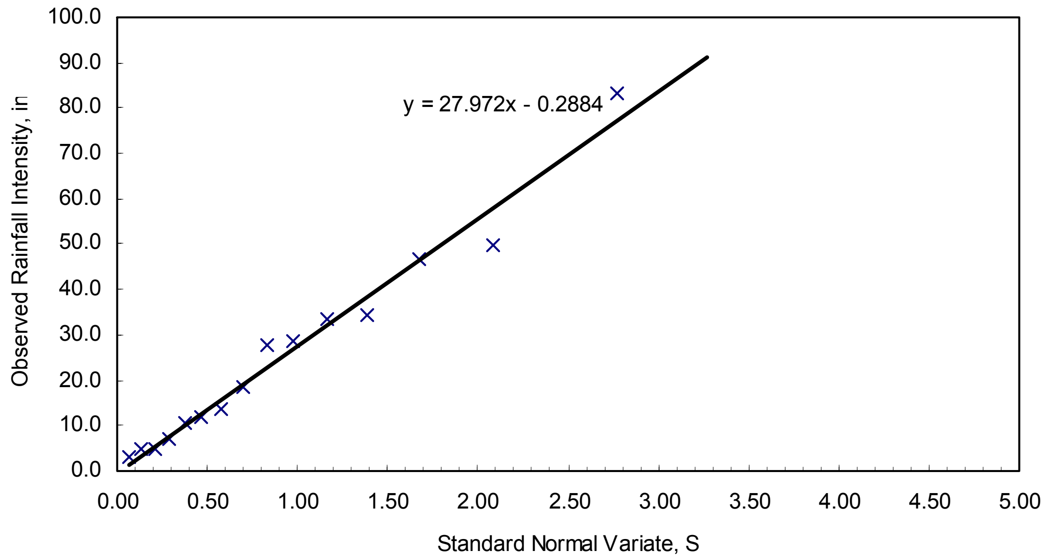
$$c_\alpha = a_\alpha\left(1 + \frac{b_0}{n} + \frac{b_1}{n^2}\right) \quad = 0.726$$

for a prescribed significance level α = 5%, a_α =0.7514, b_0=-0.795, b_1=-0.89 (Table A.6).

As A^* is greater than c_α, the lognormal distribution is not acceptable at the significance level α = *5%.*

7.13

Plot data on (shifted) exponential probability paper,



$\Rightarrow$ a = -0.2884, λ = 1/27.972=0.03575

Perform Chi-square test for shifted exponential distribution,

Mean depth (m)	Observed frequency n_i	Theoretical frequency e_i	$(n_i\text{-}e_i)^2$	$\frac{(n_i - e_i)^2}{e_i}$
<5	3	2.584	0.173	0.067
5-15	4	3.732	0.072	0.019
15-25	1	2.610	2.593	0.993
25-35	4	1.826	4.728	2.590
35-45	0	1.277	1.630	1.277
Σ	15	15		Σ= 4.95 < $c_{.95,2}$=5.99

The degree of freedom for the shifted exponential distribution is f=5-1-2=2.

For a significance level α=5%, $c_{.95,2}$=5.99. Comparing with the $\sum \frac{(n_i - e_i)^2}{e_i}$ calculated, the shifted exponential distribution is an appropriate model for the mean depth of Glacier Lakes.

Perform Anderson-Darling test for exponential distribution,

i	x_i	$F(x_i)$	$F(x_{n+1-i})$	$(2i-1)\{\ln F_X(x_i) + \ln[1 - F_X(x_{n+1-i})]\}/n$
1	2.90	0.108	0.950	-0.348
2	4.70	0.163	0.834	-0.722

3	5.00	0.172	0.815	-1.149
4	7.10	0.232	0.710	-1.259
5	10.40	0.318	0.699	-1.409
6	12.00	0.356	0.644	-1.516
7	13.60	0.391	0.635	-1.686
8	18.60	0.491	0.491	-1.387
9	27.90	0.635	0.391	-1.077
10	28.60	0.644	0.356	-1.114
11	33.30	0.699	0.318	-1.036
12	34.30	0.710	0.232	-0.931
13	46.90	0.815	0.172	-0.656
14	50.00	0.834	0.163	-0.647
15	83.30	0.950	0.108	-0.320
				Σ=-15.26

Calculate the Anderson-Darling (A-D) statistic

$$A^2 = -\sum_{i=1}^{n}\left[(2i-1)\left\{\ln F_X(x_i) + \ln[1-F_X(x_{n+1-i})]\right\}/n\right] - n \quad = 15.26\text{-}15 = 0.26$$

For exponential distribution, the adjusted A-D statistic for a sample size *n=15* is,

$$A^* = A^2(1.0 + \frac{0.6}{\sqrt{n}}) \quad = 0.3$$

The critical value c_α is given in Table A.6b,

c_α = 1.321

for a prescribed significance level $\alpha = 5\%$.

$A^* < c_\alpha$, therefore the exponential distribution is acceptable at the significance level $\alpha = 5\%$.

Therefore, both Chi-square test and Anderson-Darling test have shown the shifted exponential distribution is an appropriate model for the mean depth of Glacier Lakes.

Acceptance gap size (secs) G_i	Observed Frequency n_i	Theoritical Frequency e_i	$(n_i-e_i)^2$	$(n_i-e_i)^2 / e_i$
< 2.5	6	15.00	81.00	5.4
2.5 – 3.5	34	40.91	47.75	1.167
3.5 – 3.5	132	100.33	1002.99	9.997
4.5 – 3.5	179	177.35	2.72	0.015
5.5 – 3.5	218	222.07	16.56	0.075
6.5 – 3.5	183	208.57	653.82	3.135
7.5 – 3.5	146	138.96	49.56	0.357
8.5 – 3.5	69	66.75	5.06	0.076
> 9.5	33	30.06	8.64	0.287
Σ	1000	1000		20.509

x^2 distribution with f = 9-3 = 6 degree of freedom and α = 1% significance level, $c_{0.99,6} = 16.812$

In this case $\sum_{i=1}^{9} \frac{(n_i - e_i)^2}{e_i} = 20.509 > 16.812$

So normal distribution is not a suitable one.

8.1

(a) B = stress = y / 0.1; A = strain = x / 10

Let, $E(B \mid A = a) = \hat{\alpha} + \hat{\beta} a$

Nos.	a_i	b_i	$a_i b_i$	a_i^2	b_i^2
1	$9\text{x}10^{-4}$	10	$9\text{x}10^{-3}$	$81\text{x}10^{-8}$	100
2	$20\text{x}10^{-4}$	20	$40\text{x}10^{-3}$	$400\text{x}10^{-8}$	400
3	$28\text{x}10^{-4}$	30	$84\text{x}10^{-3}$	$784\text{x}10^{-8}$	900
4	$41\text{x}10^{-4}$	40	$164\text{x}10^{-3}$	$1681\text{x}10^{-8}$	1600
5	$52\text{x}10^{-4}$	50	$260\text{x}10^{-3}$	$2704\text{x}10^{-8}$	2500
6	$63\text{x}10^{-4}$	60	$378\text{x}10^{-3}$	$3969\text{ x}10^{-8}$	3600
Σ	$213\text{x}10^{-4}$	210	$935\text{x}10^{-3}$	$9619\text{x}10^{-8}$	9100

$$\bar{a} = \frac{213\times10^{-4}}{6} = 35.5\times10^{-4}, \qquad \bar{b} = \frac{210}{6} = 35$$

$$\hat{\beta} = \frac{\sum a_i b_i - n\bar{a}\bar{b}}{\sum a_i^2 - n\bar{a}^2} = \frac{935\times10^{-3} - 6\times35\times35.5\times10^{-4}}{9619\times10^{-8} - 6\times35.5^2\times10^{-8}}$$

$= 9.21 \text{ x } 10^3 \text{ k/in}^2$

$$\hat{\alpha} = \bar{b} - \hat{\beta}\bar{a} = 35 - 9.21\times10^3\times35.5\times10^{-4} = 35 - 32.7 = 2.3$$

So Young's modulus = $\hat{\beta} = 9.21 \text{ x } 10^3 \text{ k/in}^2$

(b) Assume $E(B \mid A = a) = \hat{\beta} a$

$$\Delta^2 = \sum_{i=1}^{6} (b_i - \beta a_i)^2$$

Now,

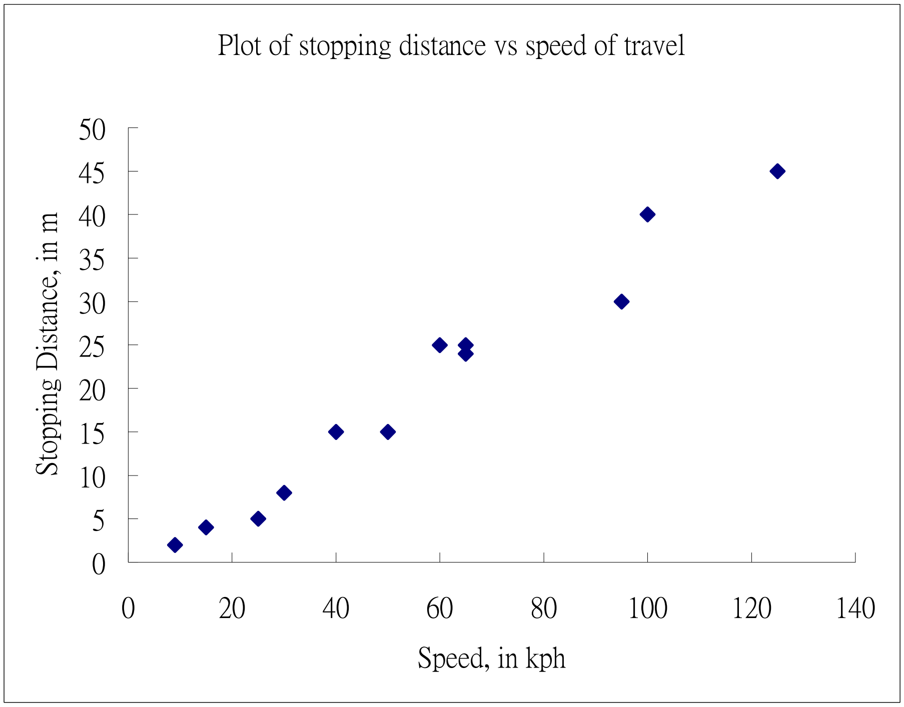
$$\frac{\partial\Delta}{\partial\beta} = \sum_{i=1}^{6} 2(b_i - \beta a_i)(-a_i) = 0$$

or,

$$\hat{\beta} = \frac{\sum a_i b_i}{\sum a_i^2} = \frac{935\times10^{-3}}{9619\times10^{-8}} = 9.72\times10^3\, ksi$$

8.2

(a)



(b)

Vehicle No.	Speed (kph)	Stopping Distance (m)					
	Xi	Yi	Xi^2	Yi^2	$XiYi$	$Yi'=a+bXi$	$(Yi-Yi')^2$
1	40	15	1600	225	600	13.43	2.461
2	9	2	81	4	18	1.46	0.288
3	100	40	10000	1600	4000	36.59	11.598
4	50	15	2500	225	750	17.29	5.252
5	15	4	225	16	60	3.78	0.048
6	65	25	4225	625	1625	23.08	3.676
7	25	5	625	25	125	7.64	6.972
8	60	25	3600	625	1500	21.15	14.804
9	95	30	9025	900	2850	34.66	21.755
10	65	24	4225	576	1560	23.08	0.842
11	30	8	900	64	240	9.57	2.467

	12	125	45	15625	2025	5625	46.25	1.552
Total:		679	238	52631	6910	18953		71.716

On the basis of calculations in the above table we obtain the respective sample means of X and Y as,

$\overline{X}$ = 679/12 = 56.6 kph, $\overline{Y}$ = 238/12 = 19.8 m

and corresponding sample variances,

$$S_x^2 = \frac{1}{11}(52631 - 12 \times 56.6^2) = 1289.84 \quad ,$$

$$S_y^2 = \frac{1}{11}(6910 - 12 \times 19.8^2) = 200.50$$

From Eq. 8.4 & 8.3, we also obtain,

$$\hat{\beta} = \frac{18953 - 12 \times 56.6 \times 19.8}{52631 - 12 \times 56.6^2} = 0.388\,, \qquad \hat{\alpha} = 19.8 - 0.388 \times 56.6 = -2.161$$

From Eq. 8.6a, the conditional variance is,

$$S_{Y|X}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - y_i^{'})^2 = 71.716 / (12-2) = 7.172$$

and the corresponding conditional standard deviation is $S_{Y|x}$ = 2.678 m

From Eq. 8.9, the correlation coefficient is,

$$\hat{\rho} = \frac{1}{n-1}\frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y}}{s_x s_y} = \frac{1}{11}\frac{18953 - 12 \times 56.6 \times 19.8}{\sqrt{1289.84} \cdot \sqrt{200.50}} = 0.98$$

(c) To determine the 90% confidence interval, let us use the following selected values of X_i = 9, 30, 60 and 125, and with $t_{0.95,10}$ = 1.812 from Table A.3, we obtain,

At X_i = 9;

$$< \mu_{Y|X} >_{0.90} = 1.33 \pm 1.812 \times 2.679 \sqrt{\frac{1}{12} + \frac{(9 - 56.6)^2}{(52631 - 12 \times 56.6^2)}} = (-1.063 \to 3.723)m$$

At $X_i = 30$;

$$< \mu_{Y|X} >_{0.90} = 9.48 \pm 1.812 \times 2.679 \sqrt{\frac{1}{12} + \frac{(30 - 56.6)^2}{(52631 - 12 \times 56.6^2)}} = (7.708 \to 11.252)m$$

At $X_i = 60$;

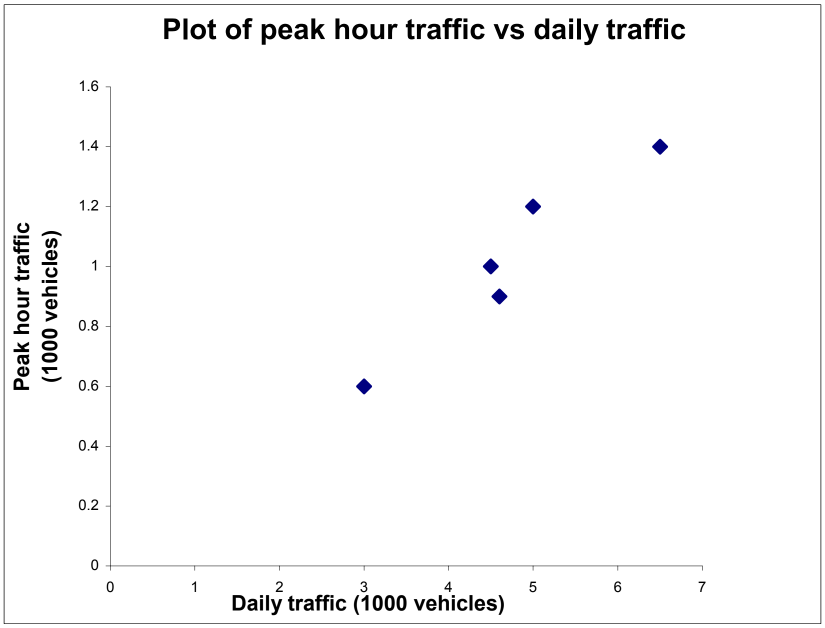
$$< \mu_{Y|X} >_{0.90} = 21.12 \pm 1.812 \times 2.679 \sqrt{\frac{1}{12} + \frac{(60 - 56.6)^2}{(52631 - 12 \times 56.6^2)}} = (19.712 \to 22.528)m$$

At $X_i = 125$;

$$< \mu_{Y|X} >_{0.90} = 46.34 \pm 1.812 \times 2.679 \sqrt{\frac{1}{12} + \frac{(125 - 56.6)^2}{(52631 - 12 \times 56.6^2)}} = (43.220 \to 49.460)m$$

8.3

(a)



(b) Let X be the daily traffic volume and Y the peak hour traffic volume, both in thousand vehicles. By Eq. 8.9, the correlation coefficient between X and Y is estimated by

$$\hat{\rho} = \frac{1}{n-1} \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{s_x s_y}$$

With n = 5, $\sum_{i=1}^{n} x_i y_i$ = 25.54, $\bar{x}$ = 1.02, $\bar{y}$ = 4.72, and the sample standard deviations s_x = 0.303315018 while s_y = 1.251798706,

$\Rightarrow$ $\hat{\rho}$ = (1/4)(1.468 / 0.379689347) ≅ **0.967**

(c)

x_i	y_i	$x_i y_i$	x_i^2	$y_i' = \hat{\alpha} + \hat{\beta} x_i$	$(y_i - y_i')^2$
5	1.2	6	25	1.085577537	0.0130925
4.5	1	4.5	20.25	0.968474793	0.00099384
6.5	1.4	9.1	42.25	1.436885769	0.00136056
4.6	0.9	4.14	21.16	0.991895341	0.00844475

	3	0.6	1.8	9	0.61716656	0.00029469
Sum	23.6	5.1	25.54	117.66		0.02418634
Mean	4.72	1.02				

Since we know the total (daily) traffic count (X), and want a probability estimate about the peak hour traffic (Y), we need the regression of Y on X, i.e. the best estimates of α and β in the model

$$E(Y \mid X = x) = \alpha + \beta x$$

From Eq. 8.4 & 8.3

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$$

$= (25.54 - 5\times4.72\times1.02) / (117.66 - 5\times4.72^2) = 1.468 / 6.268 \cong 0.234$, and

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} = 1.02 - 0.23420549\times4.72 \cong -0.0854$$

Also, by (7.5), using the results in the table above, we estimate the variance of Y (assumed constant)

$$S_{Y|x}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - y_i^{'})^2$$

$= 0.02418634 / (5 - 2) \cong 0.0081$

Hence when the daily traffic is 6000 vehicles, i.e. x = 6, Y has the estimated parameters
$\mu_Y = -0.0854 + 0.234\times6 = 1.319$
$\sigma_Y = 0.008062114^{0.5} = 0.089789278$
Hence the estimated probability of peak hour traffic exceeding 1500 vehicles is

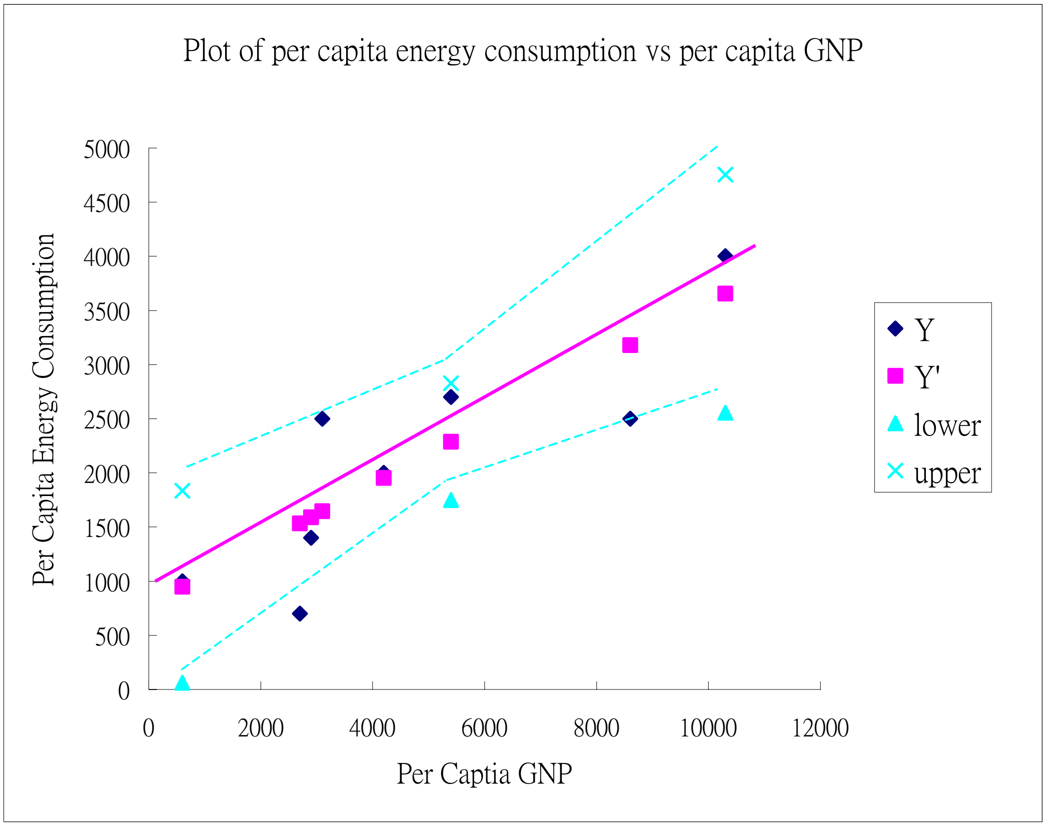
$$P(Y > 1.5) = 1 - P(Y \le 1.5)$$

$$= 1 - \Phi\left(\frac{1.5 - 1.319}{0.09}\right) = 1 - \Phi(2.01)$$

$$\cong \mathbf{0.0222}$$

8.4

(a)



(b)

Country #	Per Capita GNP	Per Capita Energy Consumption					
	Xi	*Yi*	Xi^2	Yi^2	*XiYi*	*Yi'=a+bXi*	$(Yi-Yi')^2$
1	600	1000	360000	1000000	600000	950.37	2463.269
2	2700	700	7290000	490000	1890000	1535.64	698286.734
3	2900	1400	8410000	1960000	4060000	1591.38	36624.478
4	4200	2000	17640000	4000000	8400000	1953.68	2145.238
5	3100	2500	9610000	6250000	7750000	1647.11	727412.944
6	5400	2700	29160000	7290000	14580000	2288.12	169643.906
7	8600	2500	73960000	6250000	21500000	3179.96	462341.123

8	10300	4000	106090000	16000000	41200000	3653.74	119893.103
Total:	37800	16800	252520000	43240000	99980000		2218810.796

On the basis of calculations in the above table we obtain the respective sample means of X and Y as,

$\overline{X} = 37800/8 = 4725$, $\overline{Y} = 16800/8 = 2100$

and corresponding sample variances,

$$S_x^2 = \frac{1}{7}(252520000 - 8 \times 4725^2) = 10559285.71,$$

$$S_y^2 = \frac{1}{7}(43240000 - 8 \times 2100^2) = 1137142.86$$

From Eq. 8.4 & 8.3, we also obtain,

$$\hat{\beta} = \frac{99980000 - 8 \times 4725 \times 2100}{252520000 - 8 \times 4725^2} = 0.279, \hat{\alpha} = 2100 - 0.279 \times 4725 = 781.725$$

(c) From Eq. 8.9, the correlation coefficient is,

$$\hat{\rho} = \frac{1}{n-1} \frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y}}{s_x s_y} = \frac{1}{7} \frac{99980000 - 8 \times 4725 \times 2100}{\sqrt{10559285.71} \cdot \sqrt{1137142.86}} = 0.85$$

(d) From Eq. 8.6a, the conditional variance is,

$$S_{Y|X}^2 = \frac{1}{n-2} \sum_{i=1}^{n} (y_i - y_i^{'})^2 = 2218817.515 / 6 = 369801.799$$

and the corresponding conditional standard deviation is $S_{Y|X} = 608.113$

(e) To determine the 95% confidence interval, let us use the following selected values of X_i = 600, 5400 and 10300, and with $t_{0.975,6} = 2.447$ from Table A.3, we obtain,

At $X_i = 600$;

$$< \mu_{Y|X} >_{0.95} = 950.37 \pm 2.447 \times 608.113 \sqrt{\frac{1}{8} + \frac{(600-4725)^2}{(252520000 - 8 \times 4725^2)}} = (62.263 \rightarrow 1835.997)$$

At $X_i = 5400$;

$$< \mu_{Y|X} >_{0.95} = 2288.12 \pm 2.447 \times 608.113 \sqrt{\frac{1}{8} + \frac{(5400-4725)^2}{(252520000 - 8 \times 4725^2)}} = (1749.407 \rightarrow 2827.253)$$

At $X_i = 10300$;

$$< \mu_{Y|X} >_{0.95} = 3653.74 \pm 2.447 \times 608.113 \sqrt{\frac{1}{8} + \frac{(10300-4725)^2}{(252520000 - 8 \times 4725^2)}} = (2556.391 \rightarrow 4754.469)$$

(f) Similarly, $\widehat{\beta} = 2.588$, $\widehat{\alpha} =$ -709.673 and $S_{X|Y} =$ 1853.080 for predicting the per capita GNP on the basis of the per capita energy consumption.

8.5

(a) Let X be the car weight in kips; X ~ N(3.33, 1.04). Hence

$$P(X > 4.5) = P(\frac{X-\mu}{\sigma} > \frac{4.5-3.33}{1.04})$$

$$= P(Z > 1.125) = 1 - \Phi(1.125) = 1 - 0.8697 \cong \mathbf{0.130}$$

(b) Let Y be the gasoline mileage. For linear regression, we assume E(Y | X = x) = α + βx and seek the best (in a least squares sense) estimates of α and β in the model. Based on the following data,

	x_i (kips)	y_i (mpg)	x_iy_i	x_i^2	$y_i^{'} = \hat{\alpha} + \hat{\beta}x_i$	$(y_i - y_i')^2$
	2.5	25	6.25	62.5	24.33641	0.440358
	4.2	17	17.64	71.4	16.94624	0.002891
	3.6	20	12.96	72	19.55453	0.198442
	3.0	21	9.00	63	22.16283	1.352165
sum	13.3	83.0	45.9	268.9		1.993856
average	3.325	20.75				

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y}}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2}$$

$= (45.9 - 4\times3.325\times20.75) / (268.9 - 4\times3.325^2)$

$= 1.468 / 6.268 \cong - 4.35$, and

$$\hat{\alpha} = \overline{y} - \hat{\beta}\overline{x} = 20.75 - 4.347158218\times3.325 \cong 35.20$$

Also, by Eq. 8.6a, using the results in the preceding table, we estimate the variance of Y (which is assumed constant)

$$S_{Y|x}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - y_i^{'})^2$$

$= 1.993856 / (4 - 2) \cong 0.997$

Hence when the car weighs 2.3 kips, i.e. x = 2.3, Y has the estimated parameters

$\mu_Y = 35.20430108 + (-4.347158218)\times2.3 = 25.206$

$\sigma_Y = 0.996927803^{0.5} = 0.998$

Hence the estimated probability of gas mileage being more than 28 mpg is

$$P(Y > 28) = 1 - P(Y \le 28)$$
$$= 1 - \Phi\left(\frac{28 - 25.206}{0.998}\right)$$
$$= 1 - \Phi(2.8) = 1 - 0.9974$$
$$\cong \mathbf{0.0026}$$

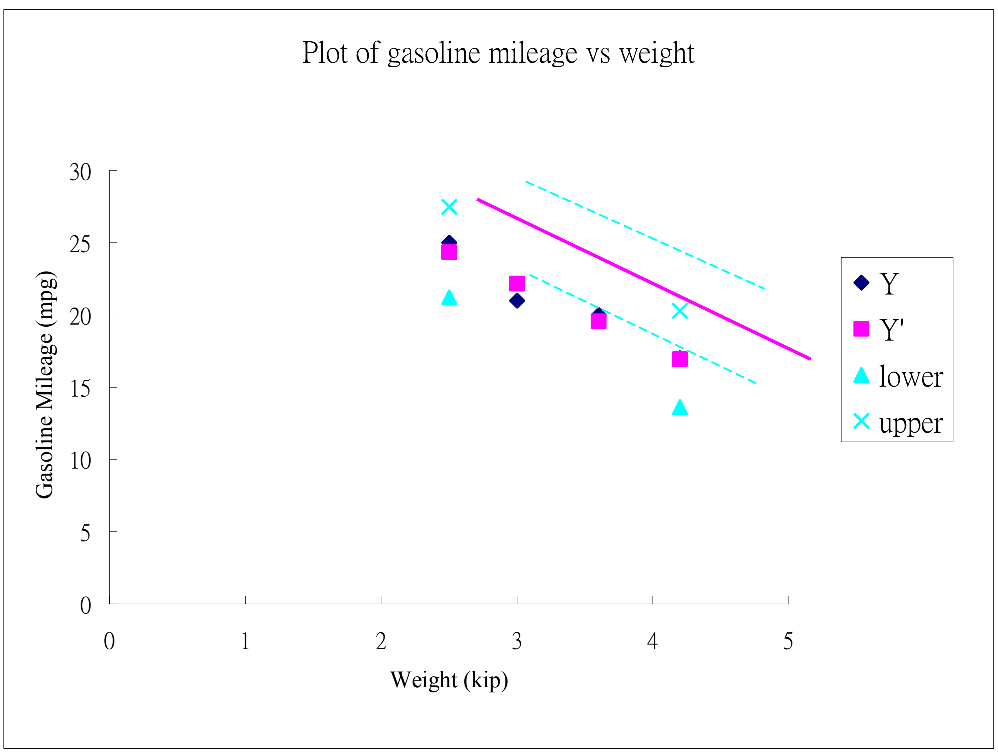
(c) To determine the 95% confidence interval, let us use the following selected values of X_i = 2.5 and 4.2, and with $t_{0.975,2}$ = 4.303 from Table A.3, we obtain,

At X_i = 2.5;

$$< \mu_{Y|X} >_{0.95} = 24.34 \pm 4.303 \times 0.998 \sqrt{\frac{1}{4} + \frac{(2.5-3.3)^2}{(45.85 - 4 \times 3.3^2)}} = (21.215 \to 27.465)mpg$$

At X_i = 30;

$$< \mu_{Y|X} >_{0.95} = 16.95 \pm 4.303 \times 0.998 \sqrt{\frac{1}{4} + \frac{(4.2-3.3)^2}{(45.85 - 4 \times 3.3^2)}} = (13.613 \to 20.287)mpg$$



8.6

(a) Let Y be the number of years of experience, and M be the measurement error in inches. We have the following data:

i	y_i	m_i	$y_i m_i$	y_i^2
1	3	1.5	4.5	9
2	5	0.8	4	25
3	10	1	10	100
4	20	0.8	16	400
5	25	0.5	12.5	625
Σ	63	4.6	47	1159

From Eq. 8.4 & 8.3, we have

$$\hat{\beta} = \frac{\sum y_i m_i - n\bar{y}\bar{m}}{\sum y_i^2 - n\bar{y}^2} = \text{-0.030010953, and}$$

$$\hat{\alpha} = \bar{m} - \hat{\beta}\bar{y} = 1.298138007$$

so our regression model is

$$M_{\text{mean}} = 1.298138007 - 0.030010953Y,$$

Hence the mean measurement error when a surveyor has 15 years of experience is estimated to be

$$1.298138007 + (-0.030010953)(15)$$
$$= 0.847973713,$$

while the estimated variance of M is 0.073026652 (by equation 7.5), hence

$$\text{P}(M < 1 \mid Y = 15) = \Phi\left(\frac{1 - 0.847973713}{\sqrt{0.073026652}}\right)$$

$$= \Phi(0.562571845)$$
$$\cong \mathbf{0.713}$$

(b) No, our data for Y ranges from 3 to 25, so our regression model is only valid for predictions using Y values within this range. 60 is outside the range, where our model may not be correct. In fact, other factors could come into play, such as effects of aging—a 100-year-old surveyor might not be able to see the crosshair anymore!

8.7

(a) Let $E(DO \mid T=t) = \alpha + \beta t$

	T (days)	DO (ppm)					
	Xi	Yi	Xi^2	Yi^2	$XiYi$	$Yi'=a+bXi$	$(Yi-Yi')^2$
1	0.5	0.28	0.25	0.0784	0.14	0.29	0.00012
2	1	0.29	1	0.0841	0.29	0.27	0.00052
3	1.6	0.29	2.56	0.0841	0.464	0.24	0.00260
4	1.8	0.18	3.24	0.0324	0.324	0.23	0.00246
5	2.6	0.17	6.76	0.0289	0.442	0.19	0.00048
6	3.2	0.18	10.24	0.0324	0.576	0.16	0.00027
7	3.8	0.1	14.44	0.01	0.38	0.14	0.00124
8	4.7	0.12	22.09	0.0144	0.564	0.09	0.00074
Total:	19.2	1.61	60.58	0.3647	3.18		0.00842

$\Sigma\, t_i = 19.2$, $\Sigma\, DO_i = 1.61$, $\Sigma\, t_i\, DO_i = 3.180$, $\Sigma\, t_i^2 = 60.58$,

$\Sigma\, DO_i^2 = 0.3647$, $\Delta^2 = \Sigma(DO_i - DO_i')^2 = 84.825 \times 10^{-4}$

$$\bar{t} = \frac{19.2}{8} = 2.4, \qquad \overline{DO} = \frac{1.61}{8} = 0.20$$

$$\hat{\beta} = \frac{\Sigma t_i \cdot DO_i - n\bar{t}\,\overline{DO}}{\Sigma t_i^2 - n\bar{t}^2} = \frac{3.18 - 8 \times 2.4 \times 0.20}{60.58 - 8 \times 2.4^2} = -0.0455$$

$$\hat{\alpha} = \overline{DO} - \hat{\beta}\,\bar{t} = 0.20 + 0.0455 \times 2.4 = 0.3092$$

So the least-squares regression equation is,

$\hat{E}(DO \mid T=t) = 0.3092 - 0.0455\, t$

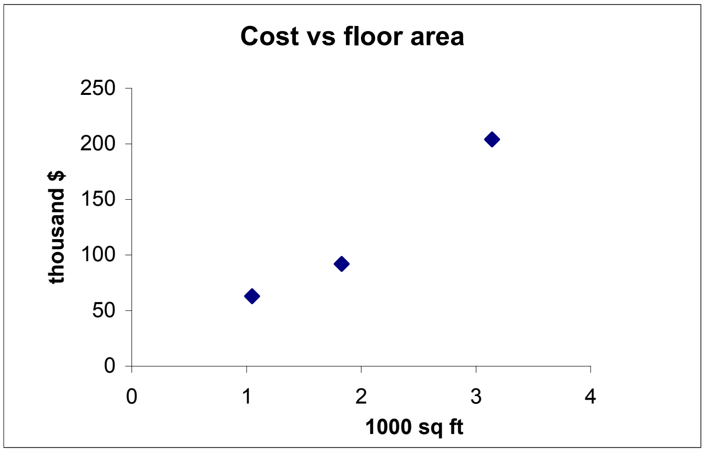
(b) $S_x^2 = \frac{1}{7}(60.58 - 8 \times 2.4^2) = 2.0714$, $\quad S_y^2 = \frac{1}{7}(0.3647 - 8 \times 0.2^2) = 0.0058$

$$\hat{\rho} = \frac{1}{n-1}\frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\,\overline{y}}{s_x s_y} = \frac{1}{7}\frac{3.18 - 8\times 2.4\times 0.2}{\sqrt{2.0714}\cdot\sqrt{0.0058}} = -0.89$$

$$S_{Y|X}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - y_i^{'})^2 = 0.00842 / 6 = 0.0014$$

and the corresponding conditional standard deviation is $S_{Y|X}$ = 0.037 ppm.

8.8



(a) The standard deviation of Y is $\sqrt{0.0025} = 0.05$. When X = 0.35, the mean of Y is

$$E(Y \mid X = 0.35) = 1.12 \times 0.35 + 0.05 = 0.442, \text{ hence}$$
$$P(Y > 0.3 \mid X = 0.35) = 1 - P(Y \le 0.3 \mid X = 0.35)$$
$$= 1 - \Phi\left(\frac{0.3 - 0.442}{0.05}\right) = \mathbf{0.9977}$$

Also, as preparation for part (b), if X = 0.4

$$\Rightarrow E(Y \mid X = 0.4) = 1.12 \times 0.4 + 0.05 = 0.498$$
$$\Rightarrow P(Y > 0.3 \mid X = 0.4) = 1 - P(Y \le 0.3 \mid X = 0.4)$$
$$= 1 - \Phi\left(\frac{0.3 - 0.498}{0.05}\right)$$
$$= 0.999963$$

(b) Theorem of total probability gives

$$P(Y > 0.3) = P(Y > 0.3 \mid X = 0.35)P(X = 0.35) + P(Y > 0.3 \mid X = 0.4)P(X = 0.4)$$
$$= 0.9977 \times (1/5) + 0.999963 \times (4/5) \cong \mathbf{0.9995}$$

(c) Let Y_A and Y_B be the respective actual strengths at A and B. Since these are both normal,

$$Y_A \sim N(0.442, 0.05);\ Y_B \sim N(0.498, 0.05),$$

Hence the difference

$$D = Y_A - Y_B \sim N(0.442 - 0.498, \sqrt{0.05^2 + 0.05^2}), \text{ i.e.}$$

$$D \sim N(-0.056, \sqrt{0.005})$$

Hence $P(Y_A > Y_B) = P(Y_A - Y_B > 0)$

$$= P(D > 0)$$

$$= 1 - \Phi\left[\frac{0 - (-0.056)}{\sqrt{0.005}}\right]$$

$$= 1 - \Phi(0.791959595) = 1 - 0.785807948 \cong \mathbf{0.214}$$

8.9

(a) The formulas to use are

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$$

and

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$$

The various quantities involved are calculated in the following table:

n =	3					
	x_i	y_i	$x_i y_i$	x_i^2	y_i'	$(y_i - y_i')^2$
	1.05	63	66.15	1.1025	53.3363865	93.3854267
	1.83	92	168.36	3.3489	107.417521	237.699949
	3.14	204	640.56	9.8596	198.246093	33.1074492
Sum =	6.02	359	875.07	14.311		364.192825
Mean =	2.00666667	119.666667				
	Beta =	154.676667	/	2.23086667	=	**69.335**
	Alpha =	119.666667	-	139.131807	=	**-19.465**

The regression line

$$y = -19.46514060 + 69.33478768\,x$$

can be plotted on the scattergram, and has the best fit to the data (in a least square sense).

(b) Note the words "for given floor area". In this case, we should <u>not</u> simply calculate the sample standard deviation of the y data, since although the individual y_i's may deviate a lot from their mean, this may be due to changes in x and hence not really a randomness of y. Hence we need to calculate the conditional variance E(Y | x), assuming it is a constant. An unbiased estimate of this is given by Eq. 8.6a as

$$S_{Y|x}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - y_i^{'})^2$$

$$= 364.192825 / (3 - 2) \cong 364.193$$

Thus the standard deviation of construction cost for given floor area is estimated as

$$S_{Y|x} = 364.192825^{0.5} \cong \mathbf{19.08}$$

(c) The individual sample standard deviations are s_x = 1.056140773, s_y = 74.46028024. Plugging these (and beta) into 8.10a,

$$\hat{\rho} = \hat{\beta}\frac{S_x}{S_y}$$

$= 69.3347877 \times 1.056140773 \div 74.46028024$

$\cong$ **0.983**

Since it is close to 1, there is a strong linear relationship between X and Y

(d) When X = 2.5, $\mu_Y = \alpha + \beta(25) = \quad -19.46514060 + 69.33478768(25) \cong 153.8718286$, thus

$Y \sim N(153.8718286, 19.08383675)$

$\Rightarrow P(Y < 180 \mid X = 2.5)$

$= P(Z < \dfrac{180 - 153.8718286}{19.08383675}) = \Phi(1.369125704) \cong$ **0.915**

8.10

(a)

Car #	Rated Mileage (mpg)	Actual Mileage (mpg)					
	Xi	*Yi*	*Xi*^{2}	*Yi*^{2}	*XiYi*	*Yi'=a+bXi*	*(Yi-Yi')*^{2}
1	20	16	400	256	320	15.54	0.215
2	25	19	625	361	475	19.29	0.086
3	30	25	900	625	750	23.05	3.807
4	30	22	900	484	660	23.05	1.100
5	25	18	625	324	450	19.29	1.671
6	15	12	225	144	180	11.78	0.048
Total:	145	112	3675	2194	2835		6.927

On the basis of calculations in the above table we obtain the respective sample means of X and Y as,

$\overline{X}$ =145/6 = 24.2, $\overline{Y}$ = 112/6 = 18.7

and corresponding sample variances,

$$S_x^2 = \frac{1}{5}(3675 - 6\times 24.2^2) = 34.17\,, \quad S_y^2 = \frac{1}{5}(2194 - 6\times 18.7^2) = 20.67$$

From Eq. 8.4 & 8.3, we also obtain,

$$\hat{\beta} = \frac{2835 - 6\times 24.2\times 18.7}{3675 - 6\times 24.2^2} = 0.751\,, \quad \hat{\alpha} = 18.7 - 0.751\times 24.2 = 0.512$$

(b) From Eq. 8.9, the correlation coefficient is,

$$\hat{\rho} = \frac{1}{n-1}\frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y}}{s_x s_y} = \frac{1}{5}\frac{2835 - 6\times 24.2\times 18.7}{\sqrt{34.17}\cdot\sqrt{20.67}} = 0.97$$

From Eq. 8.6a, the conditional variance is,

$$S_{Y|X}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - y_i')^2 = 6.927 / 4 = 1.732$$

and the corresponding conditional standard deviation is $S_{Y|X} = 1.316$ mpg.

(c) Y_Q = actual milage of model Q car
Y_R = actual milage of model R car

$\because$ X_Q = rated milage of model Q car = 22 mpg
$E(Y_Q) = 0.512 + 0.751x22 = 17.03$

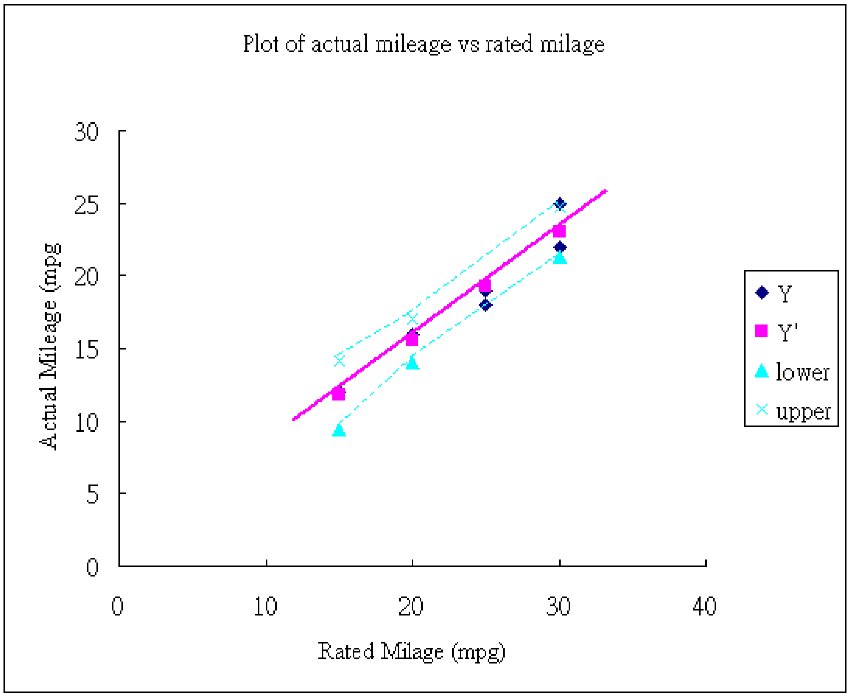
Similarly $E(Y_R) = 0.512 + 0.751x24 = 18.54$

Hence $Y_Q = N(17.03, 1.32)$; $Y_R = (18.54, 1.32)$

Assume Y_Q, Y_R to be statistically independent,
$P(Y_Q > Y_R) = P(Y_R - Y_Q < 0)$

$$= \Phi\frac{-(18.54 - 17.03)}{1.32\sqrt{2}} = \Phi(-0.809) = 0.21$$

(d)



To determine the 90% confidence interval, let us use the following selected values of X_i = 15, 20 and 30, and with $t_{0.95,4} = 2.132$ from Table A.3, we obtain,

At $X_i = 15$;

$$< \mu_{Y|X} >_{0.90} = 11.78 \pm 2.132 \times 1.316 \sqrt{\frac{1}{6} + \frac{(15-24.2)^2}{(3675 - 6 \times 24.2^2)}} = (9.446 \rightarrow 14.114) \text{ mpg}$$

At $X_i = 20$;

$$< \mu_{Y|X} >_{0.90} = 15.54 \pm 2.132 \times 1.316 \sqrt{\frac{1}{6} + \frac{(20-24.2)^2}{(3675 - 6 \times 24.2^2)}} = (14.066 \rightarrow 17.014) \text{ mpg}$$

At $X_i = 30$;

$$< \mu_{Y|X} >_{0.90} = 23.05 \pm 2.132 \times 1.316 \sqrt{\frac{1}{6} + \frac{(30-24.2)^2}{(3675 - 6 \times 24.2^2)}} = (21.331 \rightarrow 24.769) \text{ mpg}$$

8.11

(a)

Case No.	Obs. S'mt.	Cal. S'mt.					
	Yi	*Xi*	Yi^2	Xi^2	*XiYi*	*Xi'=a+bYi*	$(Xi\text{-}Xi')^2$
1	0.7	4.8	0.49	23.04	3.36	-0.93	32.864
2	64	25.1	4096	630.01	1606.4	55.01	894.691
3	5	3.8	25	14.44	19	2.87	0.869
4	25	26.4	625	696.96	660	20.54	34.299
5	29.5	27.8	870.25	772.84	820.1	24.52	10.755
6	3.8	0.5	14.44	0.25	1.9	1.81	1.708
7	5.9	6.5	34.81	42.25	38.35	3.66	8.048
8	38.1	2.6	1451.61	6.76	99.06	32.12	871.498
9	185	174	34225	30276	32190	161.95	145.195
10	28.1	26.7	789.61	712.89	750.27	23.28	11.674
11	0.6	0.6	0.36	0.36	0.36	-1.02	2.628
12	29.2	31.5	852.64	992.25	919.8	24.26	52.484
13	32	31	1024	961	992	26.73	18.233
14	5.4	3.7	29.16	13.69	19.98	3.22	0.229
15	3.6	4	12.96	16	14.4	1.63	5.615
16	35.9	25.9	1288.81	670.81	929.81	30.18	18.291
17	11.6	11.3	134.56	127.69	131.08	8.70	6.757
18	12.7	14.2	161.29	201.64	180.34	9.67	20.495
19	46	42.1	2116	1772.41	1936.6	39.10	8.981
20	3.8	1.58	14.44	2.4964	6.004	1.81	0.052
21	4.9	5	24.01	25	24.5	2.78	4.932
22	11.7	11.8	136.89	139.24	138.06	8.79	9.066
23	1.83	3.39	3.3489	11.4921	6.2037	0.07	11.049
24	9.43	3.24	88.9249	10.4976	30.5532	6.78	12.552
25	6.6	4.3	43.56	18.49	28.38	4.28	0.000
Total:	600.4	491.8	48063.16	38138.51	41546.51		2182.967

On the basis of calculations in the above table we obtain the respective sample means of X and Y as,

$\overline{Y} = 600.4/25 = 24, \qquad \overline{X} = 491.8/25 = 19.7$

and corresponding sample variances,

$$S_Y^2 = \frac{1}{24}(48063.16 - 25 \times 24^2) = 1401.91 \qquad ,$$

$$S_X^2 = \frac{1}{24}(38138.51 - 25 \times 19.7^2) = 1185.98$$

From Eq. 8.4 & 8.3, we also obtain,

$$\hat{\beta} = \frac{41546.51 - 25 \times 24 \times 19.7}{48063.16 - 25 \times 24^2} = 0.884, \qquad \hat{\alpha} = 19.7 - 0.884 \times 24 = -1.551$$

Hence, $E(X \mid y) = \alpha + \beta y = -1.551 + 0.884y$

From Eq. 8.6a, the conditional variance is,

$$S_{X|Y}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(x_i - x_i')^2 = 2182.967 / (25 - 2) = 94.912$$

and the corresponding conditional standard deviation is $S_{X|Y} = 9.742$

(b) From Eq. 8.9, the correlation coefficient is,

$$\hat{\rho} = \frac{1}{n-1}\frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y}}{s_x s_y} = \frac{1}{24}\frac{41546.51 - 25 \times 19.7 \times 24}{\sqrt{1185.98} \cdot \sqrt{1401.91}} = 0.96$$

(c) To determine the 95% confidence interval, let us use the following selected values of Y_i = 64, 25, 5.9, 185, 0.6, 3.6, 11.6 and 46, and with $t_{0.975,23} = 2.069$ from Table A.3, we obtain,

At $Y_i = 64$;

$$<\mu_{X|Y}>_{0.95} = 55.01 \pm 2.069 \times 9.742\sqrt{\frac{1}{25} + \frac{(64-24)^2}{(48063.16 - 25 \times 24^2)}} = (49.048 \rightarrow 60.975)$$

At $Y_i = 25$;

$$<\mu_{X|Y}>_{0.95} = 20.54 \pm 2.069 \times 9.742\sqrt{\frac{1}{25} + \frac{(25-24)^2}{(48063.16 - 25 \times 24^2)}} = (16.511 \rightarrow 24.576)$$

At $Y_i = 5.9$;

$$< \mu_{X|Y} >_{0.95} = 3.66 \pm 2.069 \times 9.742 \sqrt{\frac{1}{25} + \frac{(5.9-24)^2}{(48063.16 - 25 \times 24^2)}} = (-0.833 \to 8.159)$$

At $Y_i = 185$;

$$< \mu_{X|Y} >_{0.95} = 161.95 \pm 2.069 \times 9.742 \sqrt{\frac{1}{25} + \frac{(185-24)^2}{(48063.16 - 25 \times 24^2)}} = (143.806 \to 180.094)$$

At $Y_i = 0.6$;

$$< \mu_{X|Y} >_{0.95} = -1.02 \pm 2.069 \times 9.742 \sqrt{\frac{1}{25} + \frac{(0.6-24)^2}{(48063.16 - 25 \times 24^2)}} = (-5.804 \to 3.761)$$

At $Y_i = 3.6$;

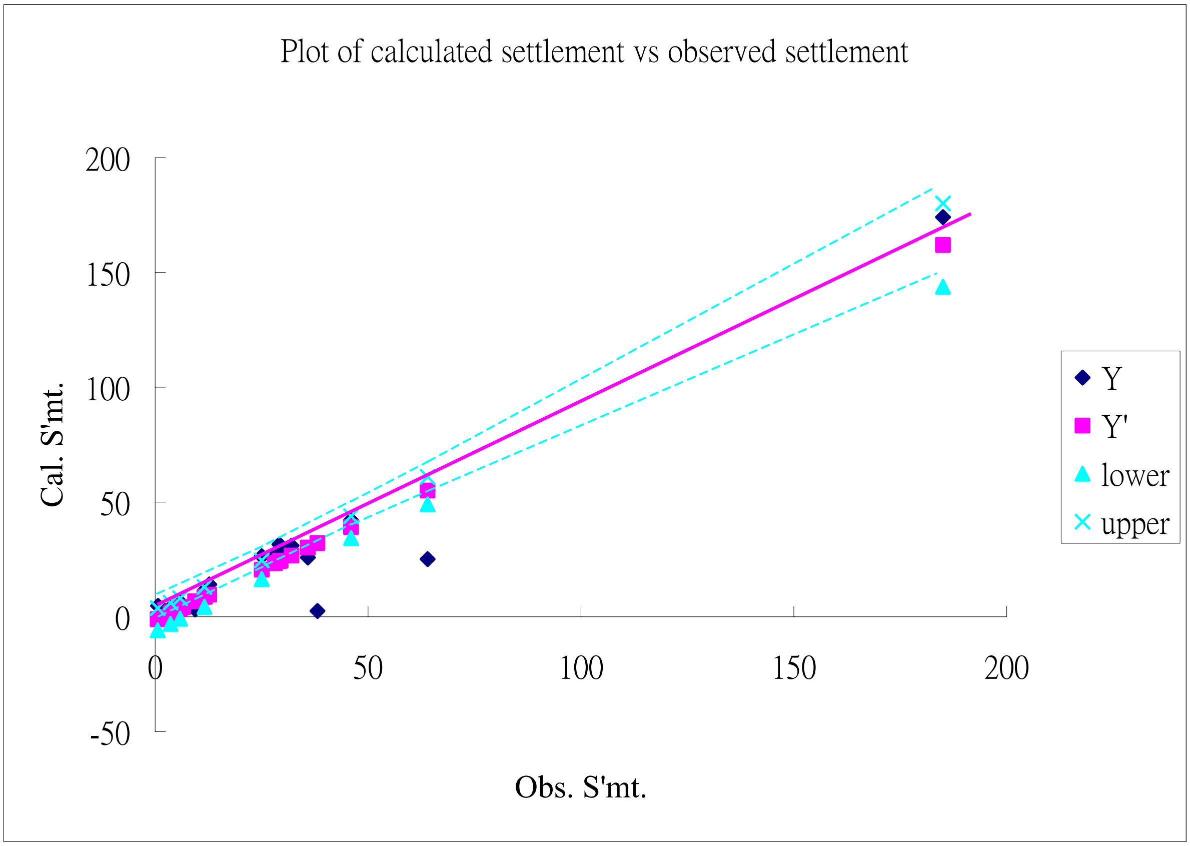
$$< \mu_{X|Y} >_{0.95} = 1.63 \pm 2.069 \times 9.742 \sqrt{\frac{1}{25} + \frac{(3.6-24)^2}{(48063.16 - 25 \times 24^2)}} = (-2.983 \to 6.244)$$

At $Y_i = 11.6$;

$$< \mu_{X|Y} >_{0.95} = 8.70 \pm 2.069 \times 9.742 \sqrt{\frac{1}{25} + \frac{(11.6-24)^2}{(48063.16 - 25 \times 24^2)}} = (4.445 \to 12.957)$$

At $Y_i = 46$;

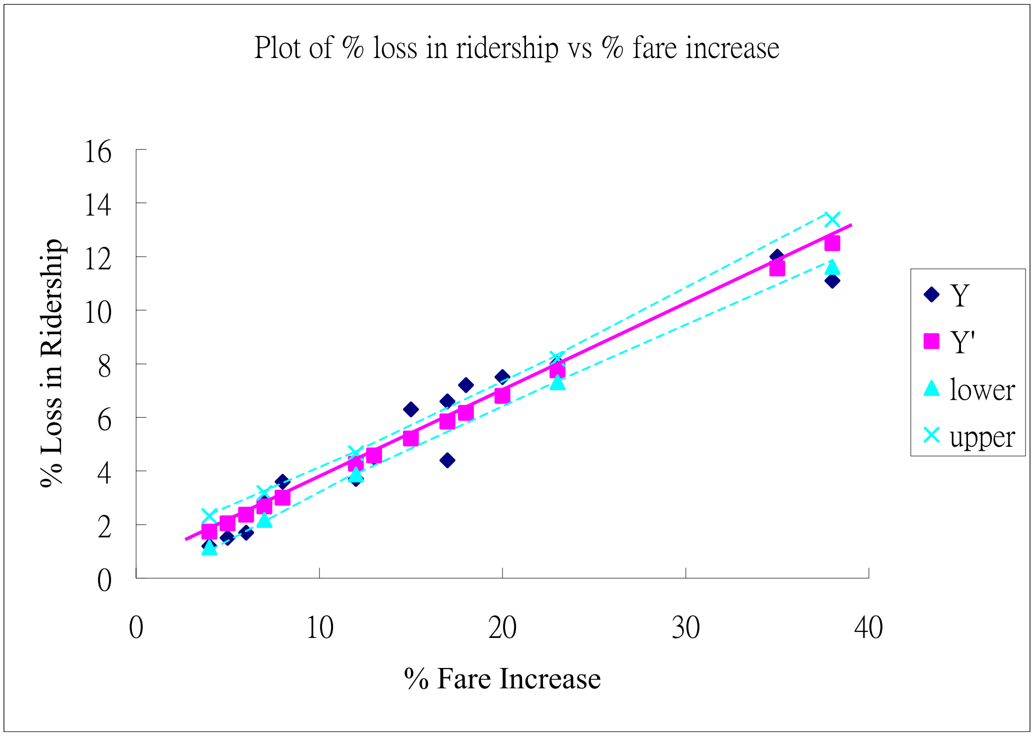
$$< \mu_{X|Y} >_{0.95} = 39.10 \pm 2.069 \times 9.742 \sqrt{\frac{1}{25} + \frac{(46-24)^2}{(48063.16 - 25 \times 24^2)}} = (34.403 \to 43.803)$$



Plot of calculated settlement vs observed settlement
200
150
100
50
0
-50
Cal. S'mt.
0
50
100
150
200
Obs. S'mt.
Y
Y'
lower
upper

8.12

(a)



(b)

	Fare Increase %	Loss in Ridership %					
	Xi	*Yi*	Xi^2	Yi^2	*XiYi*	*Yi'=a+bXi*	$(Yi-Yi')^2$
1	5	1.5	25	2.25	7.5	2.05	0.300
2	35	12	1225	144	420	11.55	0.202
3	20	7.5	400	56.25	150	6.80	0.491
4	15	6.3	225	39.69	94.5	5.22	1.176
5	4	1.2	16	1.44	4.8	1.73	0.282
6	6	1.7	36	2.89	10.2	2.36	0.442
7	18	7.2	324	51.84	129.6	6.17	1.070
8	23	8	529	64	184	7.75	0.063
9	38	11.1	1444	123.21	421.8	12.50	1.962
10	8	3.6	64	12.96	28.8	3.00	0.362
11	12	3.7	144	13.69	44.4	4.27	0.320

12	17	6.6	289	43.56	112.2	5.85	0.564
13	17	4.4	289	19.36	74.8	5.85	2.100
14	13	4.5	169	20.25	58.5	4.58	0.007
15	7	2.8	49	7.84	19.6	2.68	0.014
16	23	8	529	64	184	7.75	0.063
Total:	261.0	90.1	5757.00	667.23	1944.70		9.417

On the basis of calculations in the above table we obtain the respective sample means of X and Y as,

$$\overline{X} = 261/16 = 16.3, \qquad \overline{Y} = 90.1/16 = 5.6$$

and corresponding sample variances,

$$S_x^2 = \frac{1}{15}(5757 - 16\times 16.3^2) = 99.96\,,\; S_y^2 = \frac{1}{15}(667.23 - 16\times 5.6^2) = 10.66$$

From Eq. 8.4 & 8.3, we also obtain,

$$\hat{\beta} = \frac{1944.7 - 16\times 16.3\times 5.6}{5757 - 16\times 16.3^2} = 0.317\,,\; \hat{\alpha} = 5.6 - 0.317\times 16.3 = 0.464$$

(c) From Eq. 8.9, the correlation coefficient is,

$$\hat{\rho} = \frac{1}{n-1}\frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y}}{s_x s_y} = \frac{1}{15}\frac{1944.7 - 16\times 16.3\times 5.6}{\sqrt{99.96}\cdot\sqrt{10.66}} = 0.97$$

From Eq. 8.6a, the conditional variance is,

$$S_{Y|X}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - y_i^{'})^2 = 9.417 / 14 = 0.673$$

and the corresponding conditional standard deviation is $S_{Y|X} = 0.820$

(d) To determine the 90% confidence interval, let us use the following selected values of X_i = 4, 23, 38, 12 and 7, and with $t_{0.95,14} = 1.761$ from Table A.3, we obtain,

At $X_i = 4$;

$$< \mu_{Y|X} >_{0.90} = 1.73 \pm 1.761 \times 0.82 \sqrt{\frac{1}{16} + \frac{(4-16.3)^2}{(5757 - 16 \times 16.3^2)}} = (1.147 \rightarrow 2.315)$$

At X_i =23;

$$< \mu_{Y|X} >_{0.90} = 7.75 \pm 1.761 \times 0.82 \sqrt{\frac{1}{16} + \frac{(23-16.3)^2}{(5757 - 16 \times 16.3^2)}} = (7.311 \rightarrow 8.188)$$

At $X_i = 38$;

$$< \mu_{Y|X} >_{0.90} = 12.5 \pm 1.761 \times 0.82 \sqrt{\frac{1}{16} + \frac{(38-16.3)^2}{(5757 - 16 \times 16.3^2)}} = (11.615 \rightarrow 13.387)$$

At $X_i = 12$;

$$< \mu_{Y|X} >_{0.90} = 4.27 \pm 1.761 \times 0.82 \sqrt{\frac{1}{16} + \frac{(12-16.3)^2}{(5757 - 16 \times 16.3^2)}} = (3.870 \rightarrow 4.661)$$

At $X_i = 7$;

$$< \mu_{Y|X} >_{0.90} = 2.68 \pm 1.761 \times 0.82 \sqrt{\frac{1}{16} + \frac{(7-16.3)^2}{(5757 - 16 \times 16.3^2)}} = (2.181 \rightarrow 3.183)$$

8.13

(a) Given I = 6, E(D) = 10.5 + 15x6 = 100.5

Hence P(D>150) = $1-\Phi\frac{(150-100.5)}{30}=1-\Phi(1.65)=1-0.95=0.05$

(b) Given I = 7, E(D) = 10.5 + 15x7 = 115.5

Given I = 8, E(D) = 10.5 + 15x8 = 130.5

Hence E(D) = E(D$|$6)x0.6 + P(D$|$7)x0.3 + E(D$|$7)x0.1

= 100.5x0.6 + 115.5x0.3 +130.5x0.1 = 108 million $

8.14

(a)

	Load tons *Xi*	Deflection cm *Yi*	Xi^2	Yi^2	*XiYi*	*Yi'=a+bXi*	$(Yi-Yi')^2$
1	8.4	4.8	70.56	23.04	40.32	4.44	0.126
2	6.7	2.9	44.89	8.41	19.43	3.43	0.276
3	4.0	2.0	16	4	8	1.81	0.037
4	10.2	5.5	104.04	30.25	56.1	5.52	0.001
Total:	29.3	15.2	235.49	65.70	123.85		0.440

On the basis of calculations in the above table we obtain the respective sample means of X and Y as,

$$\overline{X} = 29.3/4 = 7.3, \qquad \overline{Y} = 15.2/4 = 3.8$$

and corresponding sample variances,

$$S_x^2 = \frac{1}{3}(235.49 - 4 \times 7.3^2) = 6.96, \quad S_y^2 = \frac{1}{3}(65.70 - 4 \times 3.8^2) = 2.65$$

From Eq. 8.4 & 8.3, we also obtain,

$$\hat{\beta} = \frac{123.85 - 4 \times 7.3 \times 3.8}{235.49 - 4 \times 7.3^2} = 0.599, \quad \hat{\alpha} = 3.8 - 0.599 \times 7.3 = -0.591$$

From Eq. 8.6a, the conditional variance is,

$$S_{Y|X}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - y_i^{'})^2 = 0.44 / 2 = 0.220$$

and the corresponding conditional standard deviation is $S_{Y|X} = 0.469$ cm.

(b) To determine the 90% confidence interval, let us use the following selected values of $X_i = 4$ and 10.2, and with $t_{0.95,2} = 2.92$ from Table A.3, we obtain,

At $X_i = 4$;

$$<\mu_{Y|X}>_{0.90} = 1.81 \pm 2.92 \times 0.469\sqrt{\frac{1}{4} + \frac{(4-7.3)^2}{(235.49 - 4\times 7.3^2)}} = (0.597 \rightarrow 3.017)\,\text{cm}$$

At X_i =10.2;

$$<\mu_{Y|X}>_{0.90} = 5.52 \pm 2.92 \times 0.469\sqrt{\frac{1}{4} + \frac{(10.2-7.3)^2}{(235.49 - 4\times 7.3^2)}} = (4.422 \rightarrow 6.625)\,\text{cm}$$

(c) Under a load of 8 tons, E(Y) = -0.591 + 0.599x8 = 4.20 cm

Let y_75 be the 75-percentile reflection

$$P(Y < y_{75}) = 0.75 = \Phi\frac{y_{75} - 4.20}{0.469}$$

Hence $\dfrac{y_{75} - 4.20}{0.469} = \Phi^{-1}(0.75) = 0.675 \qquad \textit{and} \qquad y_{75} = 4.52\textit{cm}$

8.15

(a)

	Deformation mm	Brinell Hardness kg/mm^2					
	Xi	Yi	Xi^2	Yi^2	$XiYi$	$Yi'=a+bXi$	$(Yi-Yi')^2$
1	6	68	36	4624	408	68.65	0.419
2	11	65	121	4225	715	61.88	9.720
3	13	59	169	3481	767	59.18	0.031
4	22	44	484	1936	968	47.00	9.000
5	28	37	784	1369	1036	38.88	3.543
6	35	32	1225	1024	1120	29.41	6.699
Total:	115	305	2819	16659	5014		29.412

On the basis of calculations in the above table we obtain the respective sample means of X and Y as,

$$\overline{X} = 115/6 = 19.2, \qquad \overline{Y} = 305/6 = 50.8$$

and corresponding sample variances,

$$S_x^2 = \frac{1}{5}(2819 - 6\times 19.2^2) = 122.97\,, \quad S_y^2 = \frac{1}{5}(16659 - 6\times 50.8^2) = 230.97$$

From Eq. 8.9, the correlation coefficient is,

$$\hat{\rho} = \frac{1}{n-1}\frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y}}{s_x s_y} = \frac{1}{5}\frac{5014 - 6\times 19.2\times 50.8}{\sqrt{122.97}\cdot\sqrt{230.97}} = -0.99$$

(b) From Eq. 8.4 & 8.3, we also obtain,

$$\hat{\beta} = \frac{5014 - 6\times 19.2\times 50.8}{2819 - 6\times 19.2^2} = -1.353\,,\ \ \hat{\alpha} = 50.8 + 1.353\times 19.2 = 76.765$$

From Eq. 8.6a, the conditional variance is,

$$S^2_{Y|X} = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - y_i^{'})^2 = 29.412 / 4 = 7.353$$

and the corresponding conditional standard deviation is $S_{Y|X} = 2.712$ kg/mm^2.

(c) E(Y | X=20) = 76.765 – 1.353x20 = 49.7

Hence,
P(40<Y ≦ 50) =
$$\Phi\frac{50-49.7}{2.71} - \Phi\frac{40-49.7}{2.71} = \Phi(0.11) - \Phi(-3.58) = 0.544 - 0.0002 = 0.544$$

8.16

(a)

Car #	Travel Speed mph	Stopping Distance ft					
	Xi	*Yi*	*Xi²*	*Yi²*	*XiYi*	*Yi'=a+bXi*	*(Yi-Yi')²*
1	25	46	625	2116	1150	42.48	12.384
2	5	6	25	36	30	3.29	7.370
3	60	110	3600	12100	6600	111.07	1.152
4	30	46	900	2116	1380	52.28	39.437
5	10	16	100	256	160	13.08	8.502
6	45	75	2025	5625	3375	81.68	44.578
7	15	16	225	256	240	22.88	47.377
8	40	76	1600	5776	3040	71.88	16.993
9	45	90	2025	8100	4050	81.68	69.277
10	20	32	400	1024	640	32.68	0.465
Total:	295	513	11525	37405	20665		247.536

On the basis of calculations in the above table we obtain the respective sample means of X and Y as,

$$\overline{X} = 295/10 = 29.5, \qquad \overline{Y} = 513/10 = 51.3$$

and corresponding sample variances,

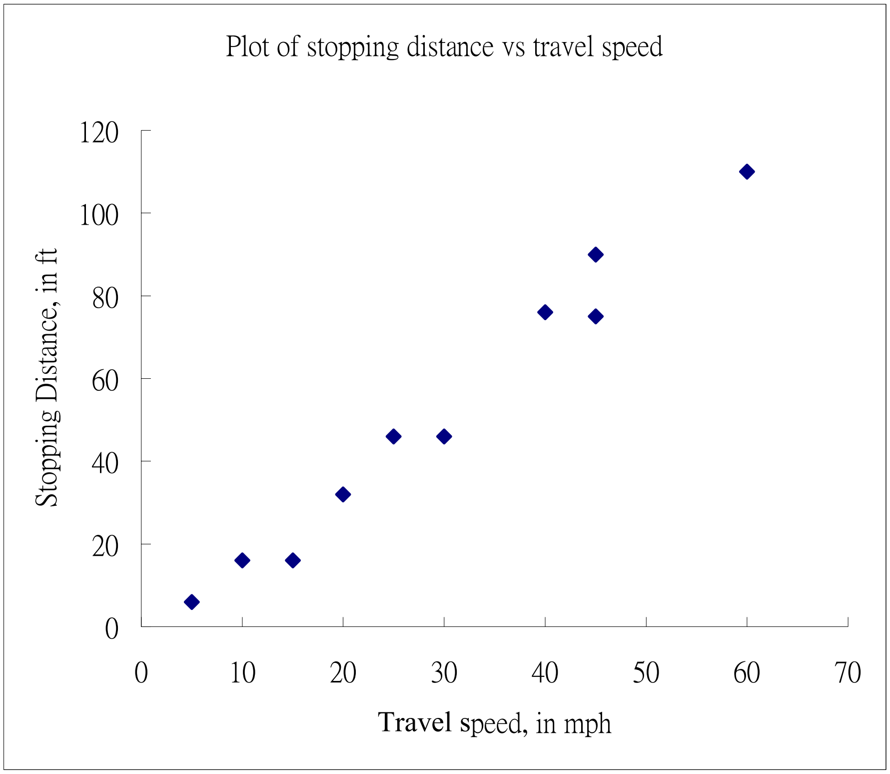
$$S_x^2 = \frac{1}{9}(11525 - 10\times 29.5^2) = 313.61,\; S_y^2 = \frac{1}{9}(37405 - 10\times 51.3^2) = 1232.01$$

From Eq. 8.9, the correlation coefficient is,

$$\hat{\rho} = \frac{1}{n-1}\frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\,\overline{y}}{s_x s_y} = \frac{1}{9}\frac{20665 - 10\times 29.5\times 51.3}{\sqrt{313.61}\cdot\sqrt{1232.01}} = 0.99$$

We can say that there is a reasonable linear relationship between the stopping distance and the speed of travel.

(b)



(c) E(Y| X) = a + bx + cx^2 ; let x' = x^2

So E(Y| X) = a + bx + cx'

$$\Sigma x_i = 295, \quad \Sigma x'_i = 11525, \quad \Sigma y_i = 492$$

$$\Sigma(x_i - \overline{x_i})^2 = 2822.5, \quad \Sigma(x'_i - \overline{x'})^2 = 1186.06 \times 10^4$$

$$\Sigma(x_i - \overline{x_i})(x'_i - \overline{x'}) = 177387.5,$$

$$\Sigma(x_i - \overline{x_i})(y_i - \overline{y}) = 5641$$

$$\Sigma(x'_i - \overline{x'})(y_i - \overline{y}) = 35829.5$$

$\overline{x} = 29.5, \quad \overline{x'} = 1152.5, \quad \hat{\alpha} = \overline{y} = 49.2$

6x2822.5 + cx177387.5 = 5641 and bx177387.5 + cx1186.06x10^4 = 358295

$$b = \frac{\det\begin{vmatrix} 5641 & 177387.5 \\ 358295 & 1186.06\times10^4 \end{vmatrix}}{\det\begin{vmatrix} 2822.5 & 177387.5 \\ 177387.5 & 1186.06\times10^4 \end{vmatrix}} = \frac{3.34859\times10^9}{2.01022\times10^9} = 1.666$$

$$c = \frac{\det\begin{vmatrix} 2822.5 & 5641 \\ 177387.5 & 358295 \end{vmatrix}}{2.01022\times10^9} = \frac{1.064475\times10^7}{2.01022\times10^9} = 0.0053$$

$\hat{\alpha}$ = 49.2 – 1.666x29.5 – 0.0053x1152.5 = 49.2 – 49.147 – 6.10825 = -6.055

So

E(Y| X) = -6.055 + 1.666x + 0.0053 x^2

$\Delta^2 = \Sigma(y_i - y_i')^2 = 376.551$

$$s_{Y|X}^2 = \frac{376.551}{10-2-1} = 53.793 \quad ; \quad s_{Y|X} = 7.334\,ft$$

(d) At a speed of 50 mph, the expected stopping distance is

E(Y) = -6.055 + 1.66x50 + 0.053x50^2 = 90.2

Let y_{90} be the distance allowed

$$P(Y < y_{90}) = \Phi\left(\frac{y_{90} - 90.2}{7.334}\right) \equiv 0.9$$

Hence,

$$\frac{y_{90} - 90.2}{7.334} = \Phi^{-1}(0.9) = 1.28 \qquad and \qquad y_{90} = 7.334\times1.28 + 90.2 = 99.6\,ft$$

8.17

(a)

	Population	Total Consumption 10^6 gal/day					
	Xi	*Yi*	*Xi*^{2}	*Yi*^{2}	*XiYi*	*Yi'=a+bXi*	*(Yi-Yi')*^{2}
1	12,000	1.2	144000000	1.44	14400	-0.10	1.682
2	40,000	5.2	1600000000	27.04	208000	4.97	0.055
3	60,000	7.8	3600000000	60.84	468000	8.58	0.610
4	90,000	12.8	8100000000	163.84	1152000	14.00	1.452
5	120,000	18.5	14400000000	342.25	2220000	19.43	0.862
6	135,000	22.3	18225000000	497.29	3010500	22.14	0.025
7	180,000	31.5	32400000000	992.25	5670000	30.28	1.498
Total:	637000.0	99.3	78469000000.00	2084.95	12742900.00		6.185

On the basis of calculations in the above table we obtain the respective sample means of X and Y as,

$$\overline{X} = 637000/7 = 91000, \qquad \overline{Y} = 99.3/7 = 14.2$$

and corresponding sample variances,

$$S_x^2 = \frac{1}{6}(78469000000 - 7 \times 91000^2) = 3417000000,$$

$$S_y^2 = \frac{1}{6}(2084.95 - 7 \times 14.2^2) = 112.72$$

From Eq. 8.4 & 8.3, we also obtain,

$$\hat{\beta} = \frac{12742900 - 7 \times 91000 \times 14.2}{78469000000 - 7 \times 91000^2} = 0.000181, \; \hat{\alpha} = 14.2 - 0.000181 \times 91000 =$$

$$- 2.266$$

(b) From Eq. 8.6a, the conditional variance is,

$$S_{Y|X}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - y_i^{'})^2 = 6.185 / 5 = 1.237$$

and the corresponding conditional standard deviation is $S_{Y|X} = 1.112$

(c) To determine the 98% confidence interval, let us use the following selected values of X_i = 12000, 90000 and 180000, and with $t_{0.99,5}$ = 3.365 from Table A.3, we obtain (in the unit "x10^6 gal/day"),

At X_i = 12000;

$$<\mu_{Y|X}>_{0.98} = -0.10 \pm 3.365 \times 1.112\sqrt{\frac{1}{7} + \frac{(12000-91000)^2}{(78469000000 - 7\times 91000^2)}} = (-2.600 \to 2.406)$$

At X_i =90000;

$$<\mu_{Y|X}>_{0.98} = 14 \pm 3.365 \times 1.112\sqrt{\frac{1}{7} + \frac{(90000-91000)^2}{(78469000000 - 7\times 91000^2)}} = (12.590 \to 15.420)$$

At X_i = 180000;

$$<\mu_{Y|X}>_{0.98} = 30.28 \pm 3.365 \times 1.112\sqrt{\frac{1}{7} + \frac{(180000-91000)^2}{(78469000000 - 7\times 91000^2)}} = (27.554 \to 32.999)$$

(d) $E(Y | X=100{,}000) = -2.266 + 0.00181\text{x}100000 = 15.8$

$P(Y > 17 | X=100{,}000) = 1 - \Phi(\frac{17-15.8}{1.112}) = 1 - \Phi(1.08) = 0.14$

8.18

$$E(X \mid V,H) = \alpha V^{\beta_1} H^{\beta_2}$$

Taking the logarithm on both sides of the above equations,

$$\ln E(X \mid V,H) = \ln\alpha + \beta_1 \ln V + \beta_2 \ln H$$

Let $\ln E(X \mid V,H) = Y$

$$\ln\alpha = \beta_0, \quad \ln V = X_1, \quad \ln H = X_2$$

Then

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

To find β_0, β_1, and β_2

$$\Sigma x_{1i} = 12.9, \quad \Sigma x_{2i} = 19.67, \quad \Sigma y_i = -0.021$$

$$\Sigma(x_{1i} - \bar{x}_i)^2 = 4889 \times 10^{-4}, \quad \Sigma(x_{2i} - \bar{x}_2)^2 = 25388.9 \times 10^{-4}$$

$$\Sigma(x_{1i} - \bar{x}_i)(x_{2i} - \bar{x}_2) = -558.5 \times 10^{-4},$$

$$\Sigma(x_{1i} - \bar{x}_i)(y_i - \bar{y}) = 5076.35 \times 10^{-4}$$

$$\Sigma(x_{2i} - \bar{x}_2)(y_i - \bar{y}) = -38680.89 \times 10^{-4}$$

$$\bar{x}_1 = 1.075, \quad \bar{x}_2 = 1.639, \quad \bar{y} = -0.00175$$

$$\hat{\beta}_1 \times 4889.0 \times 10^{-4} - \hat{\beta}_2\, 558.5 \times 10^{-4} = 5076.35 \times 10^{-4}$$

$$-\hat{\beta}_1 \times 558.5 \times 10^{-4} + \hat{\beta}_2\, 25388.92 \times 10^{-4} = -38680.89 \times 10^{-4}$$

$$\hat{\beta}_1 = \frac{\begin{vmatrix} 5076.35 & -558.5 \\ -38680.89 & 25388.92 \end{vmatrix}}{\begin{vmatrix} 4889 & -558.5 \\ -558.5 & 25388.92 \end{vmatrix}} = \frac{1.0728}{1.2381 \times 10^{8}} = 0.866$$

$$\hat{\beta}_2 = \frac{\begin{vmatrix} 4889 & 5076.35 \\ -558.5 & -38680.89 \end{vmatrix}}{1.2381\times10^8} = \frac{-1.8628\times10^8}{1.2381\times10^8} = -1.50$$

$\hat{\beta}_0$ = -0.00175 – 0.866x1.075 + 1.5x1.639 = 1.53

Now $\ln \alpha = B_0$

So $\hat{\alpha} = e^{1.53} = 4.618$

So $\hat{\alpha} = 4.618$, $\hat{\beta}_1 = 0.866$ and $\hat{\beta}_2 = -1.50$

	Mean Velocity (fps)	Mean Depth (ft)	Mean Oxygenation Rate (ppm/day)		
	Vi	*Hi*	*Xi*	*Xi'*	*(Xi-Xi')*^{2}
1	3.07	3.27	2.272	2.06	0.044
2	3.69	5.09	1.44	1.25	0.038
3	2.1	4.42	0.981	0.94	0.001
4	2.68	6.14	0.496	0.71	0.047
5	2.78	5.66	0.743	0.83	0.008
6	2.64	7.17	1.129	0.56	0.327
7	2.92	11.41	0.281	0.30	0.000
8	2.47	2.12	3.361	3.27	0.008
9	3.44	2.93	2.794	2.68	0.012
10	4.65	4.54	1.568	1.81	0.057
11	2.94	9.5	0.455	0.40	0.003
12	2.51	6.29	0.389	0.65	0.068
Total:	35.89	68.54	15.909		0.612

From Eq. 8.27, $S_{X|V,H} = \sqrt{\frac{\Delta^2}{n-k-1}} = \sqrt{\frac{0.612}{12-2-1}} = 0.26\, ppm/day$

8.19

(a) $Var(Y|X) = \sigma^2 x^4$; $\quad$ *Thus* $\quad w_i = \dfrac{1}{x_i^4}$

$$\hat{\beta} = \frac{\Sigma w_i(\Sigma w_i x_i y_i) - (\Sigma w_i y_i)(\Sigma w_i x_i)}{\Sigma w_i(\Sigma w_i x_i^2) - (\Sigma w_i x_i)^2}$$

$$= \frac{40.18\times10^{-14}\times157.76\times10^{-7} - 76.03\times10^{-10}\times75.60\times10^{-11}}{40.18\times10^{-14}\times166.67\times10^{-8} - (75.60)^2\times10^{-22}}$$

= 6.02

$$\hat{\alpha} = \frac{\Sigma w_i y_i - \hat{\beta}\Sigma w_i x_i}{\Sigma w_i} = \frac{76.03\times10^{-10} - 6.02\times75.60\times10^{-11}}{40.18\times10^{-14}}$$

= 7595.5

To determine $\hat{\alpha}$ and $\hat{\beta}$;

x_i ($\times10^3$)	y_i ($\times10^4$)	w_i ($\times10^{-14}$)	$w_i x_i$ ($\times10^{-11}$)	$w_i y_i$ ($\times10^{10}$)	$w_i x_i y_i$ ($\times10^{-7}$)	$w_i x_i^2$ ($\times10^{-8}$)	$w_i(y_i - \hat{\alpha} - \hat{\beta} x_i)^2$ ($\times10^{-6}$)
1.4	1.6	26.03	36.44	41.65	58.31	51.02	0
2.2	2.3	4.27	9.39	9.82	21.60	20.66	1.2599
2.4	2.0	3.01	7.22	6.02	14.45	17.36	0.1204
2.7	2.2	1.88	5.08	4.14	11.18	13.72	0.0609
2.9	2.6	1.41	4.09	3.67	10.64	11.89	0.0114
3.1	2.6	1.08	3.35	2.81	8.71	10.41	0.0010
3.6	2.1	0.60	2.16	1.26	4.54	7.72	0.4133
4.1	3.0	0.35	1.44	1.05	4.31	5.95	0.0185
3.4	3.0	0.75	2.55	2.25	7.65	8.65	0.0271
4.3	3.8	0.29	1.25	1.10	4.73	5.41	0.0587
5.1	5.1	0.15	0.77	0.77	3.93	3.84	0.2419
5.9	4.2	0.08	0.47	0.34	2.01	2.87	0.0010
6.4	3.8	0.06	0.38	0.23	1.47	2.44	0.0394
4.6	4.2	0.22	1.01	0.92	4.23	4.73	0.0988
Σ		40.18×10^{-14}	75.60×10^{-11}	76.03×10^{-10}	157.76×10^{-7}	166.67×10^{-8}	2.3523×10^{-6}

So, $\quad E(Y|X) = 7595.5 + 6.02X$

(b) $s^2{}_{Y|X} = \dfrac{2.3523\times10^{-6}}{14-2}x^4 = (0.1960\times10^{-6})x^4$

$s_{Y|X} = (0.4427\times10^{-3})x^2$

(c) To determine the 98% confidence interval, let us use the following selected values of x_i = 1.5, 2.5, 3.5 and 4.5, and with $t_{0.99,12}$ = 2.681 from Table A.3, we obtain,

At x_i = 1.5;

$$< \mu_{Y|X} >_{0.98} = (7595.5 + 6.02\times1.5\times10^3)$$
$$\pm 2.681\times(0.4427\times10^{-3}\times(1.5\times10^3)^2)\sqrt{\frac{1}{14}+\frac{(1.5\times10^3-3721.4)^2}{(220630000-14\times3721.4^2)}}$$
$$= (15274.53 \rightarrow 17976.47)$$

At x_i = 2.5;

$$< \mu_{Y|X} >_{0.98} = (7595.5 + 6.02\times2.5\times10^3)$$
$$\pm 2.681\times(0.4427\times10^{-3}\times(2.5\times10^3)^2)\sqrt{\frac{1}{14}+\frac{(2.5\times10^3-3721.4)^2}{(220630000-14\times3721.4^2)}}$$
$$= (19999.82 \rightarrow 25291.18)$$

At x_i = 3.5;

$$< \mu_{Y|X} >_{0.98} = (7595.5 + 6.02\times3.5\times10^3)$$
$$\pm 2.681\times(0.4427\times10^{-3}\times(3.5\times10^3)^2)\sqrt{\frac{1}{14}+\frac{(3.5\times10^3-3721.4)^2}{(220630000-14\times3721.4^2)}}$$
$$= (24730.18 \rightarrow 32600.82)$$

At x_i = 4.5;

$$< \mu_{Y|X} >_{0.98} = (7595.5 + 6.02\times4.5\times10^3)$$
$$\pm 2.681\times(0.4427\times10^{-3}\times(4.5\times10^3)^2)\sqrt{\frac{1}{14}+\frac{(4.5\times10^3-3721.4)^2}{(220630000-14\times3721.4^2)}}$$
$$= (27313.05 \rightarrow 42057.95)$$

(d) If x = 3500

$E(Y \mid X) = 7595.5 + 6.02 \text{x} 3500 = 2.8665 \text{x} 10^{-4}$

$$s_{Y|X} = 0.4427 \times 10^{-3} \cdot (3500)^2 = 5423$$

Assuming Y is $N(2.8665 \text{ x } 10^4, 5423)$

$P(Y > 30{,}000) = 1 - \phi[(30000\text{-}28665) / 5423]$

$= 1 - \phi(0.25)$

$= 0.40$

9.1

(a) Let p = Probability that the structure survive the proof test.

$P(p=0.9) = 0.70$, $P(p=0.5) = 0.25$, $P(p=0.10) = 0.05$

P(survival of the structure) = P(S)

$= P(S \mid p=0.9)\, P(p=0.9) + P(S \mid p=0.5)\, P(p=0.5) + P(S \mid p=0.1)\, P(p=0.1)$

$= 0.9\times0.7 + 0.5\times0.25 + 0.1\times0.05$

$= 0.76$

(b) Let E = One structure survive the proof test.

$P''(p=0.9) = P(p=0.9 \mid E) = P(E \mid p=0.9)\times P(p=0.9) / P(E)$

$= 0.9\times0.7 / 0.76 = 0.83$

Similarly,

$P''(p=0.5) = 0.5\times0.25 / 0.76 = 0.16$

$P''(p=0.10) = 0.10\times0.05 / 0.76 = 0.01$

(c) $P(p=0.9 \mid E) = \sum_i p_i\, P''(p=p_i) = 0.9\times0.83 + 0.5\times0.16 + 0.1\times0.01$

$= 0.747 + 0.080 + 0.001 = 0.828$

(d) Let A = event of two survival and one failure in three tests.

So,

$P(p_i \mid A) = P(A \mid p_i)\, P(p_i) / P(A)$

$P(A) = P(A \mid p = 0.9)\, P(p=0.9) + P(A \mid p=0.5)\, P(p=0.5) + P(A \mid p=0.1)\, P(p=0.1)$

$$= \binom{3}{2} (0.9)^2\times0.1\times0.7 + \binom{3}{2}(0.5)^2\times0.5\times0.25 + \binom{3}{2}(0.1)^2\times0.9\times0.05$$

$= 0.1701 + 0.09375 + 0.00135 = 0.2652$

So, $P(P=0.9 \mid A) = 0.1701 / 0.2652 = 0.64$

$P(P=0.5 \mid A) = 0.09375 / 0.2652 = 0.35$

$P(P=0.1 \mid A) = 0.00135 / 0.2652 = 0.01$

$E(P \mid A) = 0.9\times0.64 + 0.5\times0.35 + 0.1\times0.01$

$= 0.576 + 0.175 + 0.001 = 0.752$

9.2

(a) Let E_1 = The output will be acceptable in the first day.

Applying Bayes' Theorem,

$P(p_i \mid \overline{E_1}) = P(E_1 \mid p_i) \times P(p_i) / P(\overline{E_1})$

$P(E_1) = P(E_1 \mid p=0.4)\ P(p=0.4) + P(E_1 \mid p=0.6)\ P(p=0.6) + P(E_1 \mid p=0.8)\ P(p=0.8) + P(E_1 \mid p=1.0)\ P(p=1.0)$

$= 0.6\times0.25 + 0.4\times0.25 + 0.2\times0.25 + 0\times1.0 = 0.30$

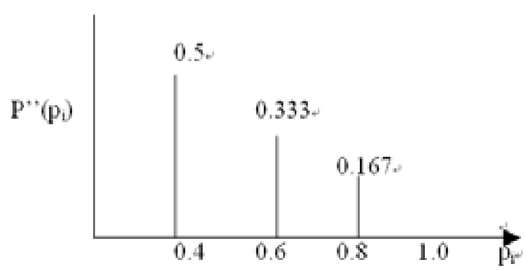
So $P''(p=0.4) = P(p=0.4 \mid E_1) = 0.6\times0.25 / 0.30 = 0.50$

$P''(p=0.6) = 0.4\times0.25 / 0.30 = 0.333$

$P''(p=0.8) = 0.2\times0.25 / 0.30 = 0.167$

$P''(p=1.0) = 0$

$E(p \mid E_1) = 0.4\times0.5 + 0.6\times0.333 + 0.8\times0.167 = 0.533$



(b) Let, E_2 = one unacceptable out of three trials.

$P(E_2) = P(E_2 \mid p=0.4)\ P(p=0.4) + P(E_2 \mid p=0.6)\ P(p=0.6) + P(E_2 \mid p=0.8)\ P(p=0.8) + P(E_2 \mid p=1.0)\ P(p=1.0)$

$= \binom{3}{2}(0.4)^2\times0.6\times0.25 + \binom{3}{2}(0.6)^2\times0.4\times0.25 + \binom{3}{2}(0.8)^2\times0.2\times0.25 + \binom{3}{2}(1.0)^2\times0\times0.25$

$= 3\times0.25\ (0.096 + 0.144 + 0.128) = 0.276$

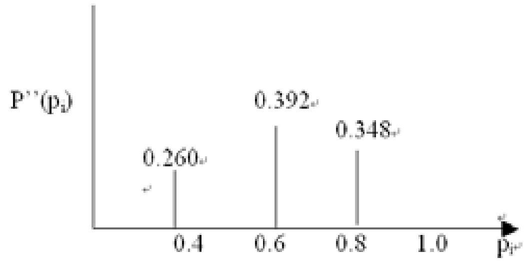
So $P''(p=0.4) = P(p=0.4 \mid E_2) = 3\times0.096\times0.25 / 0.276 = 0.260$

$P''(p=0.6) = 3\times0.144\times0.25 / 0.276 = 0.392$

$P''(p=0.8) = 3\times0.128\times0.25 / 0.276 = 0.348$

$P''(p=1.0) = 0$

$E(p \mid E_2) = 0.26\times0.4 + 0.392\times0.6 + 0.348\times0.8 + 0 = 0.617$



(c) Let E_3 = Out of three trial, first two are satisfactory and the last one is unsatisfactory.

$P(E_3) = P(E_3 | p=0.4)\ P(p=0.4) + P(E_3 | p=0.6)\ P(p=0.6) + P(E_3 | p=0.8)\ P(p=0.8) + P(E_3 | p=1.0)\ P(p=1.0)$

$= (0.4)^2 \times 0.6 \times 0.25 + (0.6)^2 \times 0.4 \times 0.25 + (0.8)^2 \times 0.2 \times 0.25 + 0$

$= 0.25\ (0.096 + 0.144 + 0.128) = 0.092$

So $P''(p=0.4) = P(p=0.4 | E_3) = 0.096 \times 0.25 / 0.092 = 0.260$

$P''(p=0.6) = 0.144 \times 0.25 / 0.092 = 0.392$

$P''(p=0.8) = 0.128 \times 0.25 / 0.092 = 0.348$

$P''(p=1.0) = 0$

$E(p | E_3) = 0.26 \times 0.4 + 0.392 \times 0.6 + 0.348 \times 0.8 + 0$

$= 0.617$

The posterior distribution will be same as shown in part (b) since the likelihood function of the observed events E_2 and E_3 are the same.

9.3

$P(H_AH_F) = P(H_AL_F) = 0.3$, $P(L_AH_F) = P(L_AL_F) = 0.2$

(a) $P(\overline{H}_A\ \overline{H}_F) = P(\overline{H}_A\ \overline{H}_F \mid H_AH_F)\, P(H_AH_F) + P(\overline{H}_A\ \overline{H}_F \mid H_AL_F)\, P(H_AL_F) + P(\overline{H}_A\ \overline{H}_F \mid L_AH_F)$

$P(L_AH_F) + P(\overline{H}_A\ \overline{H}_F \mid L_AL_F)\, P(L_AL_F)$

$= 0.3\times0.3 + 0.4\times0.3 + 0.2\times0.2 + 0.25\times0.2$
$= 0.3$

(b) $P(H_AH_F \mid \overline{H}_A\ \overline{H}_F) = P(\overline{H}_A\ \overline{H}_F \mid H_AH_F)\, P(H_AH_F) / P(\overline{H}_A\ \overline{H}_F) = 0.3\times0.3 / 0.3 = 0.3$

(c) $P(H_AH_F \mid \overline{L}_A\ \overline{L}_F) = P(\overline{L}_A\ \overline{L}_F \mid H_AH_F)\, P(H_AL_F) / P(\overline{L}_A\ \overline{L}_F)$

Now using similar method as in (a),

$P(\overline{L}_A\ \overline{L}_F) = 0.2\times0.3 + 0.1\times0.3 + 0.1\times0.2 + 0.25\times0.2 = 0.16$

So, $P(H_AH_F \mid \overline{L}_A\ \overline{L}_F) = 0.2\times0.3 / 0.16 = 0.375$

Similarly,

$P(H_AL_F \mid \overline{L}_A\ \overline{L}_{F\ F}) = 0.10\times0.3 / 0.16 = 0.1875$

$P(L_AH_F \mid \overline{L}_A\ \overline{L}_F) = 0.10\times0.2 / 0.16 = 0.125$

$P(L_AL_F \mid \overline{L}_A\ \overline{L}_F) = 0.25\times0.2 / 0.16 = 0.3125$

9.4

(a) $P(X=2) = P(X=2 \mid m=0.4)\, P(m=0.4) + P(X=2 \mid m=0.8)\, P(m=0.8)$

$= 0.4\times0.5 + 0.8\times0.5 = 0.6$

$P(m=0.4 \mid X=2) = 0.4\times0.5 / 0.6 = 0.333 = P''(m=0.4)$

$P(m=0.8 \mid X=2) = 0.8\times0.5 / 0.6 = 0.667 = P''(m=0.8)$

(b) Let, N = Number of accurate measurements.

$P(N \geq 2) = P(N \geq 2 \mid m=0.4)\, P(m=0.4) + P(N \geq 2 \mid m=0.8)\, P(m=0.8)$

$$P(N \geq 2 \mid m=0.4) = P(N \geq 2 \mid 3, 0.4) = \binom{3}{2}\times (0.4)^2\times0.6 + \binom{3}{3}\times (0.4)^3\times(0.6)^0 = 0.352$$

Similarly,

$$P(N \geq 2 \mid m=0.8) = P(N \geq 2 \mid 3, 0.8) = \binom{3}{2}\times (0.8)^2\times0.2 + \binom{3}{3}\times (0.8)^3\times(0.2)^0 = 0.896$$

Considering posterior distribution of m

$P(N \geq 2) = 0.352\times0.333 + 0.896\times0.667 = 0.714$

9.5

P(λ = 4) = 0.4, P(λ = 5) = 0.6

(a) Let E = Observed test results of bending capacity for the two tests.
P(E) = P(E | λ = 4) P(λ = 4) + P(E | λ = 5) P(λ = 5)

P(E | λ = 4) = $\{\frac{4.5}{4^2} e^{-\frac{1}{2}(\frac{4.5}{4})^2} \cdot dm\} \{\frac{5.2}{4^2} e^{-\frac{1}{2}(\frac{5.2}{4})^2} \cdot dm\}$

= 0.02085 dm^2

where dm = small increment of m around m.

and P(E | λ = 5) = $\{\frac{4.5}{5^2} e^{-\frac{1}{2}(\frac{4.5}{5})^2} \cdot dm\} \{\frac{5.2}{5^2} e^{-\frac{1}{2}(\frac{5.2}{5})^2} \cdot dm\}$

So, P(E) = (0.02085×0.4 + 0.01454×0.6) dm^2 = 0.01706 dm^2

So, P''(λ = 4 | E) = P''(λ = 4) = 0.02085×0.4 / 0.01706 = 0.489
P''(λ = 5) = 0.01454×0.6 / 0.01706 = 0.511

(b) $f''_M(m) = f_M(m \mid \lambda = 4)\, P''(\lambda = 4) + f_M(m \mid \lambda = 5)\, P''(\lambda = 5)$

$$= \frac{m}{4^2} e^{-\frac{1}{2}(\frac{m}{4})^2} \times 0.489 + \frac{m}{5^2} e^{-\frac{1}{2}(\frac{m}{5})^2} \times 0.511$$

$$= 0.03056m \cdot e^{-\frac{1}{2}(\frac{m}{4})^2} + 0.02044m \cdot e^{-\frac{1}{2}(\frac{m}{5})^2}$$

(c) $P(M<2) = \int_0^2 0.03056m \cdot e^{-\frac{1}{2}(\frac{m}{4})^2}\, dm + \int_0^2 0.02044m \cdot e^{-\frac{1}{2}(\frac{m}{5})^2}\, dm$

$$= [-16 \times 0.03056 \cdot e^{-\frac{1}{2}(\frac{m}{4})^2}]_0^2 + [-25 \times 0.02044 \cdot e^{-\frac{1}{2}(\frac{m}{5})^2}]_0^2$$

= 0.09675

9.6

(a) $P(\alpha=2) = P(\alpha=3) = 0.5$

Let F = errors found from two measurements.

$$P(F) = P(F \mid \alpha=2)\, P(\alpha=2) + P(F \mid \alpha=3)\, P(\alpha=3)$$

$$P(F \mid \alpha=2) = \{\frac{2}{2}(1-\frac{1}{2})de\}\ \{\frac{2}{2}(1-\frac{2}{2})de\} = 0$$

$$P(F \mid \alpha=3) = \{\frac{2}{3}(1-\frac{1}{3})de\}\ \{\frac{2}{3}(1-\frac{2}{3})de\} = \frac{8}{81}de^2$$

Where de = small increment of e

So, $P(F) = 0 + \frac{8}{81}\times 0.5\, de^2 = \frac{4}{81}de^2$

$$P''(\alpha=2 \mid F) = P''(\alpha=2) = 0$$

and

$$P''(\alpha=3) = 1$$

$$E(\alpha \mid F) = 2\times 0 + 3\times 1 = 3 \text{ cm}$$

(b) $f''(\alpha) = k \bullet L(\alpha) \bullet f'(\alpha)$

where

$f''(\alpha) = 1.0 \quad ; \quad 2 \le \alpha \le 3$

$L(\alpha) = \{\frac{2}{\alpha}(1-\frac{1}{\alpha})\}\ \{\frac{2}{\alpha}(1-\frac{2}{\alpha})\}$

So,

$$f''(\alpha) = k \cdot \frac{4}{\alpha^2}(1-\frac{1}{\alpha})(1-\frac{2}{\alpha})\cdot 1 \quad ; \quad 2 \le \alpha \le 3$$

$= 0$, else where

To find the constant k,

$$\int_2^3 k\frac{4}{\alpha^2}(1-\frac{1}{\alpha})(1-\frac{2}{\alpha})\cdot d\alpha = 1.0$$

or, $4k[-\frac{1}{\alpha} + \frac{3}{2\alpha^2} - \frac{2}{3\alpha^3}]_2^3 = 1.0$

or, $k = 14.71$

So,

$$f''(\alpha) = \frac{58.84}{\alpha^2}(1-\frac{1}{\alpha})(1-\frac{2}{\alpha}) \quad ; \quad 2 \le \alpha \le 3$$

$= 0$, elsewhere

$$E(\alpha \mid F) = \int_2^3 \alpha f''(\alpha) d\alpha = \int_2^3 \frac{58.84}{\alpha}(1-\frac{1}{\alpha})(1-\frac{2}{\alpha}) d\alpha$$

$$= 58.85 \quad [\ln(\alpha) + \frac{3}{\alpha} - \frac{1}{\alpha^2}]_2^3 \; = 2.61 \text{ cm.}$$

9.7

Since, f”(ν) = kL(ν) f’(ν)
where

$$L(\nu) = \frac{e^{-\frac{\nu}{12}}(\frac{\nu}{12})^1}{1!}, f'(\nu) = \frac{0.271}{\nu}; \ \ 0.5 \le \nu \le 20$$

= 0, elsewhere

So,

$$f''(\nu) = k \cdot e^{-\frac{\nu}{12}} \times \frac{\nu}{12} \times \frac{0.271}{\nu}; \quad 0.5 \le \nu \le 20$$

= 0, elsewhere

To determine the constant k,

$$\int_{0.5}^{20} k \cdot \frac{0.271}{12} e^{-\frac{\nu}{12}} d\nu = 1.0 \longrightarrow k = 4.79$$

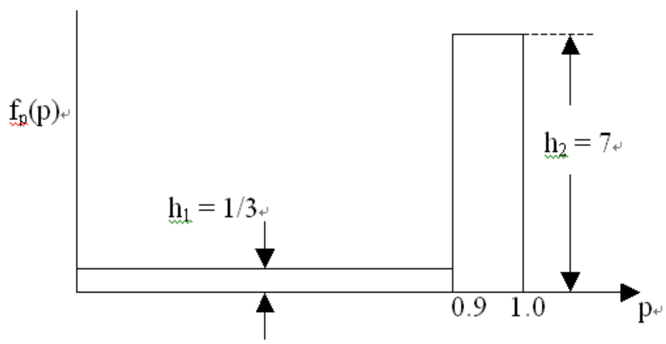
So the posterior distribution of ν is,

$$f''(\nu) = 4.79 \times \frac{0.271}{12} e^{-\frac{\nu}{12}} = 0.1082 e^{-\frac{\nu}{12}}; \qquad 0.5 \le \nu \le 20$$

= 0, elsewhere

9.8

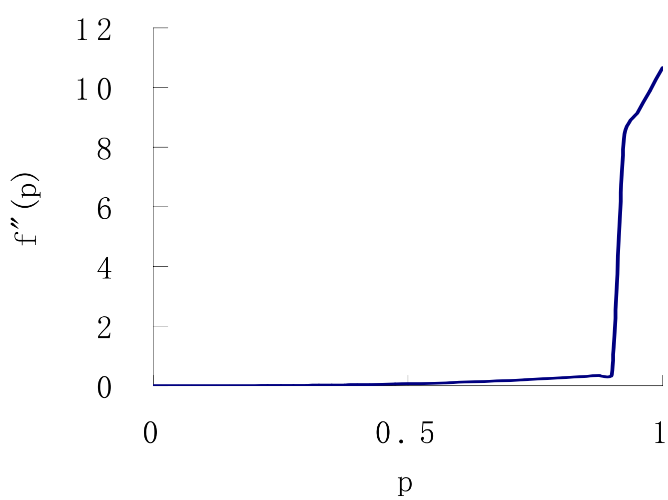
(a) P(p > 0.9) = 0.7 and P(p > 0.9) = 0.3



(b) f''(p) = k × L(p) × f'(p)

$L(p) = \binom{3}{3} p^3 \times (1-p)^0 = p^3$

So, $f''(p) = k \times p^3 \times 1/3; \quad 0 \le p \le 0.9$
$= k \times p^3 \times 7; \quad 0.9 \le p \le 1.0$
=0, elsewhere



To find k,

$$\int_0^{0.9} \frac{1}{3} k \times p^3 dp + \int_{0.9}^{1.0} k \times 7 \cdot p^3 dp = 1.0$$

or,

$$\frac{1}{3} k [\frac{p^4}{4}]_0^{0.9} + 7k[\frac{p^4}{4}]_{0.9}^{1.0} = 1.0 \longrightarrow k = 1.523$$

So,

$f''(p) = 0.508\ p^3 \quad ; \quad 0 \le p \le 0.9$
$= 10.663\ p^3 \quad ; \quad 0.9 \le p \le 1.0$
=0, elsewhere

(c) $$E''(p) = \int_0^{0.9} 0.508 p^4 dp + \int_{0.9}^{1.0} 10.663 p^4 dp$$

$$= 0.508\left[\frac{p^5}{5}\right]_0^{0.9} + 10.663\left[\frac{p^5}{5}\right]_{0.9}^{1.0}$$

$$= 0.933$$

9.9

P(ν=20) = 2/3, P(ν=15) = 1/3

(a) Let X = Number of occurrences of fire in the next year.

P(X=20) = P(X=20 | ν=15) P(ν=15) + P(X=20 | ν=20) P(ν=20)

$$= \frac{e^{-15}(15)^{20}}{20!} \times \frac{1}{3} + \frac{e^{-20}(20)^{20}}{20!} \times \frac{2}{3}$$

= 0.0139 + 0.0592 = 0.0731

(b) $P''(\nu=15) = \dfrac{P(X=20|\nu=15)\cdot P(\nu=15)}{P(X=20)} = \dfrac{0.0139}{0.0731} = 0.19$

Similarly,

$P''(\nu=20) = \dfrac{0.0592}{0.0731} = 0.81$

(c) P(X=20) = P(X=20 | ν=15) P''(ν=15) + P(X=20 | ν=20) P''(ν=20)

$$= \frac{e^{-15}(15)^{20}}{20!} \times 0.19 + \frac{e^{-20}(20)^{20}}{20!} \times 0.81$$

= 0.0079 + 0.0720 = 0.0799

9.10

λ is the parameter to be updated with the prior distribution of:

$$P'(\lambda = 1/5) = 1/3 \qquad P'(\lambda = 1/10) = 2/3$$

(a) Let ε =accidents were reported on days 2 and 5. The probability to observe ε is

$$\begin{aligned} P(\varepsilon) &= P(\varepsilon \mid \lambda = 1/5)P'(\lambda = 1/5) + P(\varepsilon \mid \lambda = 1/10)P'(\lambda = 1/10) \\ &= \frac{1}{5}\times\exp(-\frac{2}{5})\times\frac{1}{5}\times\exp(-\frac{3}{5})\times\frac{1}{3} + \frac{1}{10}\times\exp(-\frac{2}{10})\times\frac{1}{10}\times\exp(-\frac{3}{10})\times\frac{2}{3} \\ &= 8.949\times10^{-3} \end{aligned}$$

Combining the observation with the prior distribution of λ with Bayesian's theorem, the posterior distribution of λ can be calculated as follows:

$$P''(\lambda = 1/5) = \frac{P(\varepsilon \mid \lambda = 1/5)P'(\lambda = 1/5)}{P(\varepsilon)} = \frac{\frac{1}{5}\times\exp(-\frac{2}{5})\times\frac{1}{5}\times\exp(-\frac{3}{5})\times\frac{1}{3}}{8.949\times10^{-3}} = 0.548$$

$$P''(\lambda = 1/10) = 1 - 0.548 = 0.452$$

(b) Let ε =no accidents will occur for at least 10 days after second accident.

$$\begin{aligned} P(\varepsilon) &= P(T > 10) \\ &= 1 - P(T \le 10) \\ &= 1 - P(T \le 10 \mid \lambda = 1/5)P''(\lambda = 1/5) - P(T \le 10 \mid \lambda = 1/10)P''(\lambda = 1/10) \\ &= 1 - (1 - e^{-\frac{10}{5}})\times 0.548 - (1 - e^{-\frac{10}{10}})\times 0.452 \\ &= 0.240 \end{aligned}$$

(c) The updated distribution of λ as prior distribution in this problem:

$$P'(\lambda = 1/5) = 0.548 \qquad P'(\lambda = 1/10) = 0.452$$

Let ε = after the second accident in 10th day no accident was observed until the 15th day. The probability to observe Z is:

$$
\begin{aligned}
P(\varepsilon) &= P(T > 5) \\
&= [1 - P(T \le 5 \,|\, \lambda = 1/5)]P'(\lambda = 1/5) + [P(T \le 5 \,|\, \lambda = 1/10)]P'(\lambda = 1/10) \\
&= [1 - (1 - e^{-\frac{5}{5}})] \times 0.548 + [1 - (1 - e^{-\frac{5}{10}})] \times 0.452 \\
&= 0.202 + 0.274 \\
&= 0.476
\end{aligned}
$$

Combining the observation with the prior distribution of λ with Bayesian's theorem, the posterior distribution of λ can be calculated as follows:

$$P''(\lambda = 1/5) = \frac{[1 - P(T \le 5 \,|\, \lambda = 1/5)]P'(\lambda = 1/5)}{P(Z)} = \frac{0.202}{0.476} = 0.424$$

$$P''(\lambda = 1/10) = 1 - 0.424 = 0.575$$

(d)

$$
\begin{aligned}
P(n = n_c) &= P(n = n_c \,|\, \lambda = \frac{1}{5})P(\lambda = \frac{1}{5}) + P(n = n_c \,|\, \lambda = \frac{1}{10})P(\lambda = \frac{1}{10}) \\
&= \frac{2^{n_c}}{n_c!} \times e^{-2} \times 0.548 + \frac{1^{n_c}}{n_c!} \times e^{-1} \times 0.425 \\
&= 0.074 \times \frac{2^{n_c}}{n_c!} + 0.156 \times \frac{1^{n_c}}{n_c!}
\end{aligned}
$$

$$P''(\lambda = \frac{1}{5}) = \frac{0.449 \times \dfrac{0.2^{n_c}}{n_c!}}{P(n = n_c)}$$

$$P''(\lambda = \frac{1}{10}) = \frac{0.385 \times \dfrac{0.1^{n_c}}{n_c!}}{P(n = n_c)}$$

Set $P''(\lambda = \frac{1}{5}) = P''(\lambda = \frac{1}{10})$

We get $n_c = 1.076$.

Since n_c is an integer, $n_c = 2$.

(e) Non-informative prior for λ is used here.

Likelihood function is: $L(\lambda) = \lambda e^{-2\lambda} \lambda e^{-3\lambda} = \lambda^2 e^{-5\lambda}$

Posterior distribution:

$$f''(\lambda) \propto L(\lambda) = \lambda^2 e^{-5\lambda}$$

As $\int_0^{+\infty} \lambda^2 e^{-5\lambda} \mathrm{d}\lambda = \frac{1}{25}\int_0^{+\infty} (5\lambda)^2 e^{-(5\lambda)} \mathrm{d}(5\lambda) = \frac{\Gamma(3)}{25} = \frac{2}{25}$

Thus the posterior distribution is:

$$f''(\lambda) = 12.5\lambda^2 e^{-5\lambda}$$

9.11

M is the parameter to be updated. It's convenient to prescribe a conjugate prior to the Poisson process. From the information given in the problem, the mean and variance of the gamma distribution of M is:

$$E'(\mu) = \frac{k'}{v'} = 10 \quad , \quad \delta'(\mu) = \frac{\sqrt{k'/v'^2}}{k'/v'} = 0.4$$

Thus, $k' = 6.25\,, v' = 0.625$

(a) It follows that the posterior distribution of M is also gamma distributed. From the relationship given in Table 9.1 between the prior and posterior statistics, and the sample data, the posterior distribution parameter of M can be estimated as:

$$k'' = k' + 1 = 7.25 \qquad v'' = v' + 0.6 = 1.225$$

$$E''(M) = \frac{k''}{v''} = \frac{7.25}{1.225} = 5.918$$

$$\delta'(M) = \frac{\sqrt{k''/v''^2}}{k''/v''} = 0.371$$

(b)

$$P(X=0) = \int_0^{+\infty} P(X=0\,|\,m) f'(m) dm$$

$$= \int_0^{+\infty} \frac{(0.4m)^0}{0!} \times e^{-0.4m} \times \frac{v''(v''m)^{k''-1}}{\Gamma(k'')} \times e^{-mv''} dm$$

$$= \int_0^{+\infty} e^{-0.4m} \times \frac{1.225(1.225m)^{7.25-1}}{\Gamma(7.25)} \times e^{-1.225m} dm$$

$$= 0.209$$

9.12

Let X = compression index

X is N(μ , 0.16)

Sample mean $\bar{x} = \frac{1}{4}(0.75 + 0.89 + 0.91 + 0.81) = 0.84$

(a) From Equation 8.13,

f''(μ) = $N(\bar{x}, \frac{\sigma}{\sqrt{n}})$

= N(0.84, 0.16/2) = N(0.84, 0.08)

(b) f'(μ) = N(μ'_μ , σ'_μ) = N(0.8, 0.2)

L(μ) = N(0.84, 0.08) = $N(\bar{x}, \frac{\sigma}{\sqrt{n}})$

From Equation 9.14, 9.15,

$$\mu''_\mu = \frac{\bar{x}(\sigma'_\mu)^2 + \mu'_\mu \sigma^2 / n}{(\sigma'_\mu)^2 + \sigma^2 / n} = \frac{0.84 \times (0.2)^2 + 0.8 \times (0.08)^2}{0.2^2 + 0.08^2}$$

= 0.834

and,

$$\sigma''_\mu = \sqrt{\frac{(\sigma'_\mu)^2 (\sigma^2 / n)}{(\sigma'_\mu)^2 + (\sigma^2 / n)}} = \sqrt{\frac{0.2^2 \times 0.08^2}{0.2^2 + 0.08^2}} = 0.0743$$

So,

f''(μ) = N(0.834, 0.0743)

(c) P(μ < 0.95) = $\phi(\frac{0.95 - 0.834}{0.0743}) = \phi(1.56) = 0.94$

9.13

(a) T is N(μ , 10)

Sample mean = $\bar{t}$ = 65 min.

n=5

So the posterior distribution of μ ,

$$f''(\mu) = N(\bar{t}, \frac{\sigma}{\sqrt{n}}) = N(65, \frac{10}{\sqrt{5}}) = N(65,4.47) \text{ min.}$$

(b) We know,

f'(μ) = N(65,4.47) min.

L(μ) = $N(60, \frac{10}{\sqrt{10}})$ = N(60, 3.16) min.

From Equation 8.14 and 8.15,

$$\mu''_{\mu_T} = \frac{65 \times 3.16^2 + 60 \times 4.47^2}{4.47^2 + 3.16^2} = 61.7 \text{ min}.$$

$$\sigma''_{\mu_T} = \sqrt{\frac{4.47^2 \times 3.16^2}{4.47^2 + 3.16^2}} = 2.58 \text{ min}.$$

f''(μ_T) = N(61.7, 2.58) min.

(c) From Equation 9.17,

$f_T(t)$ = N(61.7, $\sqrt{10^2 + 2.58^2}$)

= N(61.7, 10.33) min.

P(T > 80) = $1 - \phi(\frac{80 - 61.7}{10.33}) = 1 - \phi(1.77) = 0.0384$

9.14

(a) The sample mean $\overline{\theta} = \frac{1}{6}(32^o 04' + 31^o 59' + 32^o 01' + 32^o 05' + 31^o 57' + 32^o 00') = 32^o 01'$

$$s_\theta{}^2 = \frac{1}{6-1}\{(3')^2 + (2')^2 + 0 + (4')^2 + (4')^2 + (1')^2\} = 9.2$$

$$s_\theta = 3.03' = 0.05^o$$

The actual value of the angle is $N(32^o 01', \frac{0.05^o}{\sqrt{6}})$

or,

$N(32.0167^o, 0.0204^o)$

The value of the angle would be 32.0167 ± 0.0204^o

or $32^o 01' \pm 1.224'$

(b) The prior distribution of θ can be modeled as $N(32^o, 2')$ or $N(32^o, 0.0333^o)$. Applying Eqs. 9.14 and 9.15, the Bayesian estimate of the elevation is,

$$\theta'' = \frac{32 \times 0.0204^2 + 32.0167 \times 0.0333^2}{0.0204^2 + 0.0333^2}$$

$= 32.0121^o = 32^o 0.726'$

and the corresponding standard error is,

$$\sigma_\theta'' = \sqrt{\frac{0.0204^2 \cdot 0.0333^2}{0.0204^2 + 0.0333^2}} = 0.01740^o \; or \;\; 1.044'$$

Hence,1.044'

θ is $32^o 0.726' \pm 1.044'$

9.15

Assume, First measurement → L=N(2.15, σ)

Second measurement → L=N(2.20, 2σ)

and Third measurement → L=N(2.18, 3σ)

Applying Eqs. 9.14 and 9.15 and considering First and Second measurements,

$$L'' = \frac{2.20 \times \sigma^2 + 2.15 \times (2\sigma)^2}{\sigma^2 + (2\sigma)^2} = 2.16km$$

and,

$$\sigma_L^{''} = \sqrt{\frac{\sigma^2 \times (2\sigma)^2}{\sigma^2 + (2\sigma)^2}} = 0.8944\sigma$$

Now consider L'' with the third measurement and apply Eq. 9.14 again,

$$L''' = \frac{2.16 \times (3\sigma)^2 + 2.18 \times (0.8944\sigma)^2}{(3\sigma)^2 + (0.8944\sigma)^2} = 2.1616km$$

9.16

Assume the prior distribution of p to be a Beta distribution, then,

$$E'(p) = \frac{q'}{q'+r'} = 0.1 \quad \text{----------------------------- (1)}$$

and

$$Var'(p) = \frac{a'r'}{(q'+r')^2(q'+r'+1)} = 0.06^2 \text{ ---------(2)}$$

From Equations 1 and 2, we get,

q' = 2.4 and r' = 9×2.4 = 21.6

From Table 9.1, for a binomial basic random variable,

q'' = q' + x = 2.4 + 1 = 3.4

v'' = v' + n-x = 21.6 + 12 – 1 = 32.6

So E''(p) = 3.4 / (3.4 + 32.6) = 0.0945

$$Var''(p) = \frac{3.4 \times 32.6}{(3.4+32.6)^2(3.4+32.6+1)} = 0.00231$$

9.17

μ is the parameter to be updated with the prior distribution parameter

$$\mu'_{\mu} = 3 \qquad \sigma'_{\mu} = 0.2 \times 3 = 0.6$$

(a) From the observation we get

$$\bar{t} = N(\frac{2+3}{2}, \frac{0.5}{\sqrt{2}}) = N(2.5, 0.354)$$

From the relationship given in Table 9.1,

$$\mu''_{\mu} = \frac{0.354^2 \times 3 + 0.6^2 \times 2.5}{0.6^2 + 0.354^2} = 2.629 \qquad \sigma''_{\mu} = \sqrt{\frac{0.354^2 \times 0.6^2}{0.6^2 + 0.354^2}} = 0.305$$

(b) From equation 9.17 the distribution of T is

$$T = N(\mu''_{\mu}, \sqrt{\sigma^2 + \sigma''^{2}_{\mu}}) = N(2.629, \sqrt{0.6^2 + 0.305^2}) = N(2.629, 0.673)$$

$$P(T < 1) = P(\frac{T - 2.629}{0.673} < \frac{1 - 2.629}{0.673}) = \Phi(-2.421) = 0.774\%$$

9.18

(a) The parameter needs to be updated is p. It's suitable to set a conjugate prior for the problem. Thus, suppose p is Beta distributed with parameter p' and q'. With the information presented in the problem, we can get

$$E'(p) = \frac{q'}{q'+r'} = 0.5 \quad , \quad Var'(p) = \frac{q'r'}{(q'+r')^2(q'+r'+1)} = 0.05$$

Thus, $q' = r' = 2$

(b) Upon the observation the posterior distribution of p can be updated in terms of p and q as:

$$q" = q'+2 = 2+2 = 4 \quad r" = r'+n-x = 2+1-2 = 1$$

$$f"(p) = \frac{\Gamma(5)}{\Gamma(4)\Gamma(1)} p^3 = 4p^3$$

(c) Let ε =win the next two bids

$$p(\varepsilon) = \int_0^1 P(\varepsilon \mid p) f"(p) dp = \int_0^1 p^2 \bullet 4p^3 dp = 2/3$$

9.19

Using conjugate distributions, we assume Gamma distribution as prior distribution of λ.
From Table 9.1, for an exponential basic random variable,

$$E'(\lambda) = \frac{k'}{v'} = 0.5; \qquad Var'(\lambda) = \frac{k'}{v'^2} = \frac{0.5}{v'} = (0.5 \times 0.2)^2$$

So, v' = 50
and, k' = 25

Now, v'' = v' + $\sum x_i = 50 + (1 + 1.5) = 52.5$

k'' = k' + n = 25 + 2 =27
E''(λ) = k''/v'' = 27/52.5 = 0.514
Var''(λ) = k''/v''^{2} = 27/52.5^2 = 9.796 x 10^{-3}
σ''(λ) = 0.099

$$\delta_\lambda'' = \frac{0.099}{0.514} = 0.193$$

9.20

λ is the parameter to be updated. Let X denote the crack length.
The probability of crack length larger than 4 is:

$$P(X > 4) = 1 - P(X < 4) = e^{-4\lambda}$$

The probability of crack length smaller than 6 is:

$$P(X < 6) = 1 - e^{-6\lambda}$$

Likelihood function is

$$L(\lambda) = \lambda e^{-3\lambda} \bullet \lambda e^{-5\lambda} \bullet (1 - e^{-6\lambda}) \bullet \lambda e^{-4\lambda} \bullet e^{-4\lambda} \bullet \lambda e^{-8\lambda} = \lambda^4 e^{-24\lambda}(1 - e^{-6\lambda})$$

non-informative prior is used, thus

$$f''(\lambda) \propto L(\lambda) = \lambda^4 e^{-24\lambda}(1 - e^{-6\lambda})$$

$$\int_0^{+\infty} \lambda^4 e^{-24\lambda}(1 - e^{-6\lambda}) \mathrm{d}\lambda = 1/23415$$

thus $f''(\lambda) = 23415\lambda^4 e^{-24\lambda}(1 - e^{-6\lambda})$

9.21

(a) The occurrence rate of the tornado is estimated from the historical record as 0.1/year.
The probability of x occurrences during time t (in years) is

$$P(X = x) = \frac{(0.1t)^x}{x!} e^{-0.1t}$$

The probability that the tornado hits the town during the next 5 years is

$$P(X > 0) = 1 - P(X = 0) = 1 - e^{-0.5} = 0.393$$

The prior distribution parameter of mean maximum wind speed is:

$$\mu_\mu^{'} = \frac{145 + 175}{2} = 160$$

$\dfrac{\mu_\mu^{'} - 145}{\sigma_\mu^{'}} = 1.96$, thus $\sigma_\mu^{'} = \dfrac{\mu_\mu^{'} - 145}{1.96} = \dfrac{15}{1.96} = 7.653$

The updated the distribution parameter of mean maximum wind speed is

$$\mu_\mu^{''} = \frac{160 \times \frac{15^2}{2} + 175 \times 7.653^2}{\frac{15^2}{2} + 7.653^2} = 164.854$$

$$\sigma_\mu^{''} = \sqrt{\frac{\frac{15^2}{2} \times 7.653^2}{\frac{15^2}{2} + 7.653^2}} = 6.143$$

From equation 9.17, the updated the distribution of maximum wind speed is

$$v = N(\mu_\mu^{''}, \sqrt{\sigma_\mu^{''2} + \sigma^2}) = N(164.854, \sqrt{15^2 + 6.143^2}) = N(164.854, 16.209)$$

$$P(v > 220) = 1 - P(v \le 220) = 1 - P\left(\frac{T - 164.854}{16.209} < \frac{220 - 164.854}{16.209}\right) = 1 - \Phi(3.402) = 0.033\%$$

(3)

The probability that the house will be damaged in the next 5 years is

$$P(v > 120, x > 0) = P(v > 220)P(x > 0) = 0.876 \times 0.393 = 0.344$$

where

$$P(v > 120) = 1 - P(v \le 120) = 1 - P\left(\frac{T - 161.376}{35.756} < \frac{120 - 161.376}{35.756}\right) = 1 - \Phi(-1.157) = 0.876$$

$$P(x > 0) = 1 - P(x = 0) = 1 - e^{-0.1 \times 5} = 0.393$$

The probability that the house will not be damaged is $1 - 0.344 = 0.656$

9.22

h is the parameter to be updated.

(a)

$$E(x)=\int_0^{10} E(x\mid h)f(h)dh=\int_0^{10} 0.5h \bullet 0.003h^2=3.75$$

where

$$E(x\mid h)=\int_0^h xf(x\mid h)dx=\int_0^h \frac{x}{h}dh=0.5h$$

(b)

(i) the range of h is 4<h<10 upon the observation

(ii) likelihood:

$$L(h\mid x=4)=p(x=4\mid h)=\frac{1}{h}$$

Prior:

$$f'(h)=0.003h^2$$

Thus the posterior is

$$f''(h)\propto 0.003h^2 \bullet \frac{1}{h}=0.003h$$

Since $\int_4^{10} 0.003h^2 \bullet \frac{1}{h}dh=0.132$, the posterior is:

$$f''(h)=\frac{1}{0.132}\bullet 0.003h=0.0238h$$

(iii) The probability that the hazardous zone will not exceed 4km in the next accident is:

$$P(X<4)=\int_0^4 f(x)dx=0.1428\times 4=0.5712$$

Where

$$f(x)=\int_4^{10} f(x\mid h)f''(h)dh=\int_4^{10}\frac{1}{h}\bullet 0.0238hdh=0.1428$$

9.23

(a) Let x denote the fraction of grouted length. Assumptions: the grouted length is normally distributed; the mean grouted length is normally distributed; the variance of fraction of length grouted to the constant and can be approximated by the observation data.

From the information in the problem, we have

$$\mu_\mu^{'} = \frac{0.75+0.95}{2} = 0.85 \quad , \quad \frac{\mu_\mu^{'} - 0.8}{\sigma_\mu^{'}} = 1.96$$

Thus, $\sigma_\mu^{'} = \dfrac{\mu_\mu^{'} - 0.75}{1.96} = \dfrac{0.85-0.75}{1.96} = 0.05$

$$\sigma \approx s_L = 0.06$$

$$\mu_\mu^{''} = \frac{0.85 \times \frac{0.06^2}{4} + 0.89 \times 0.05^2}{\frac{0.06^2}{4} + 0.05^2} \approx 0.88$$

$$\sigma_\mu^{''} = \sqrt{\frac{\frac{0.06^2}{4} \times 0.05^2}{\frac{0.06^2}{4} + 0.05^2}} = 0.026$$

The posterior covariance is 0.026/0.88=3%

$$X = N(\mu_\mu^{''}, \sqrt{\sigma^2 + \sigma''^2_\mu}) = N(0.88, 0.0654)$$

(b) The probability the sample average grouted length is lessen than 0.85 is

$$P(\bar{X} < 0.85) = \Phi(\frac{0.85-0.88}{0.0654/\sqrt{2}}) = 0.258$$

The probability that the minimum grouted length is smaller than 0.83 is

$$1 - P(X_1 > 0.83)P(X_2 > 0.83) = 1 - 0.778^2 = 0.395$$

So the second requirement is more stringent.

9.24

(1) Prior

Non-informative prior is used as (P.M. Lee. Bayesian Statistics: An Introduction. Edward Arnold, 1989)

$$f'(\mu_T, \sigma_T^2) \propto \frac{1}{\sigma_T^2}$$

(2) Likelihood

$$p(t_1, t_2, \cdots, t_n \mid \mu_T, \sigma_T)$$
$$= \frac{1}{\sqrt{2\pi}\sigma_T} \exp\left[-\frac{(t_1-\mu_T)^2}{2\sigma_T^2}\right] \bullet \frac{1}{\sqrt{2\pi}\sigma_T} \exp\left[-\frac{(t_2-\mu_T)^2}{2\sigma_T^2}\right] \bullet \cdots \bullet \frac{1}{\sqrt{2\pi}\sigma_T} \exp\left[-\frac{(t_n-\mu_T)^2}{2\sigma_T^2}\right]$$
$$= \frac{1}{\left(\sqrt{2\pi}\sigma_T\right)^n} \exp\left[-\frac{\sum (t_i-\mu_T)^2}{2\sigma_T^2}\right]$$
$$= \frac{1}{\left(\sqrt{2\pi}\sigma_T\right)^n} \exp\left[-\frac{(n-1)s^2 + n(\overline{t}-\mu)^2}{2\sigma_T^2}\right]$$

where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(t_i - \overline{t})^2$

(3) Joint posterior is:

$$f''(\mu_T, \sigma_T^2) \propto \sigma_T^{-n-2} \exp\left[-\frac{(n-1)s^2 + n(\overline{t}-\mu)^2}{2\sigma_T^2}\right]$$

(4) Marginal posterior of σ_T^2

$$f''(\sigma_T^2)$$
$$= \int f''(\mu_T, \sigma_T^2) d\mu_T$$
$$\propto \int \sigma_T^{-n-2} \exp\left[-\frac{(n-1)s^2 + n(\overline{y}-\mu)^2}{2\sigma_T^2}\right] d\mu_T$$
$$\propto \left(\sigma_T^2\right)^{-n-1/2} \exp\left[-\frac{(n-1)s^2}{2\sigma_T^2}\right]$$

Which is a scaled inverse-χ^2 density

So $\left(\sigma_T^2\right)'' = Inv - \chi^2(n-1, s^2)$

(5) Marginal posterior of μ_T

Similarly, the posterior of μ_T can be written as

$$
\begin{aligned}
& f''(\mu_T) \\
& = \int f''(\mu_T, \sigma_T^2) d\sigma_T^2 \\
& \propto \int \sigma_T^{-n-2} \exp\left[-\frac{(n-1)s^2 + n(\overline{t} - \mu)^2}{2\sigma_T^2} \right] d\sigma_T^2 \\
& \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz \\
& \propto \left[(n-1)s^2 + n(\mu_T - \overline{t})^2 \right]^{-n/2} \\
& \propto \left[1 + \frac{n(\mu_T - \overline{t})^2}{(n-1)s^2} \right]^{-n/2}
\end{aligned}
$$

which is the $t_{n-1}(\overline{t}, s^2/n)$ density.

Or it can be written as

$$\frac{\mu_T'' - \overline{t}}{s/\sqrt{n}} = t_{n-1}$$

Where $z = \dfrac{A}{2\sigma_T^2}$, $A = (n-1)s^2 + n(\mu_T - \overline{t})^2$

(6) We need both sample mean and sample standard deviation in this problem. From the information available in the problem, only the first set of data with 5 flight time has both sample mean and sample standard deviation. So only this set of data will be used.

$s = 10, n = 5, \overline{t} = 65, S = (n-1)s^2 = (5-1) \times 10^2 = 400$

$\mu_T'' = t_{n-1}(\overline{t}, s^2/n) = t_4(65, 4.472)$

$E(\mu_T'') = 65$, $Var(\mu_T'') = \dfrac{4}{4-2} \times 4.472^2 = 39.998$

$\left(\sigma_T^2\right)'' = Inv - \chi^2(n-1, s^2) = Inv - \chi^2(4, 100)$

$E\left[\left(\sigma_T^2\right)''\right] = \dfrac{4}{4-2} \times 100 = 200$

$$Var\left[\left(\sigma_T^2\right)''\right] = \frac{2 \times 4^2}{(4-2)^2(4-4)} \times 100 = \infty$$

9.25

Let x denotes the rated value, and y denotes the actual value.

Based on 9.24 to 9.27, $\alpha = 0.512$, $\beta = 0.751$, $\sigma^2 = 1.732$, $s_x^2 = 34.167$

From equation 9.26, $E(Y \mid x) = 0.512 + 0.751x$

The variance can be calculated by equation 9.34 as

$$Var(Y \mid x) = \frac{6-1}{6-3} \times \left\{ 1 + \frac{1}{6}\left[1 + \frac{6}{6-1}\frac{(x-24.167)^2}{34.167} \right] \right\} \times 1.732 = 3.368 + 0.0169(x-24.167)^2$$

$$E(Y \mid x = 24) = 0.512 + 0.751 \times 24 = 18.536$$

$$Var(Y \mid x = 4) = 3.368 + 0.0169(24 - 24.167)^2 = 3.368$$

$$P(Y < 18 \mid x = 24) = \Phi(\frac{18 - 18.536}{\sqrt{3.368}}) = 0.385$$

if the value of the basic scatter σ^2 is equal to 1.73, the variance of Y at $x = 24$ becomes 1.73.

Then

$$P(Y < 18 \mid x = 24) = \Phi(\frac{18 - 18.536}{\sqrt{1.73}}) = 0.34$$

10.1

(a) Apply Equation 10.2

$g(0.10) \leq 0.05$

For n = 20, and r = 1

$$g(0.10) = \binom{20}{0}(0.10)^0(0.90)^{20} + \binom{20}{1}(0.10)^1(0.90)^{19}$$

$= 0.1216 + 0.2702$

$= 0.05$

The welds should not be accepted.

(b) $$g(0.10) = \sum_{x=0}^{r}\binom{n}{x}(0.03)^x(0.97)^{n-x} = 0.90$$

and from part (a)

$$g(0.10) = \sum_{x=0}^{r}\binom{n}{x}(0.10)^x(0.90)^{n-x} = 0.05$$

From trial and error, and using Table of Cumulative Binomial Probabilities, n = 105, r = 5.

(c) AOQ = pg(p)

Here, n = 25 and r = 1

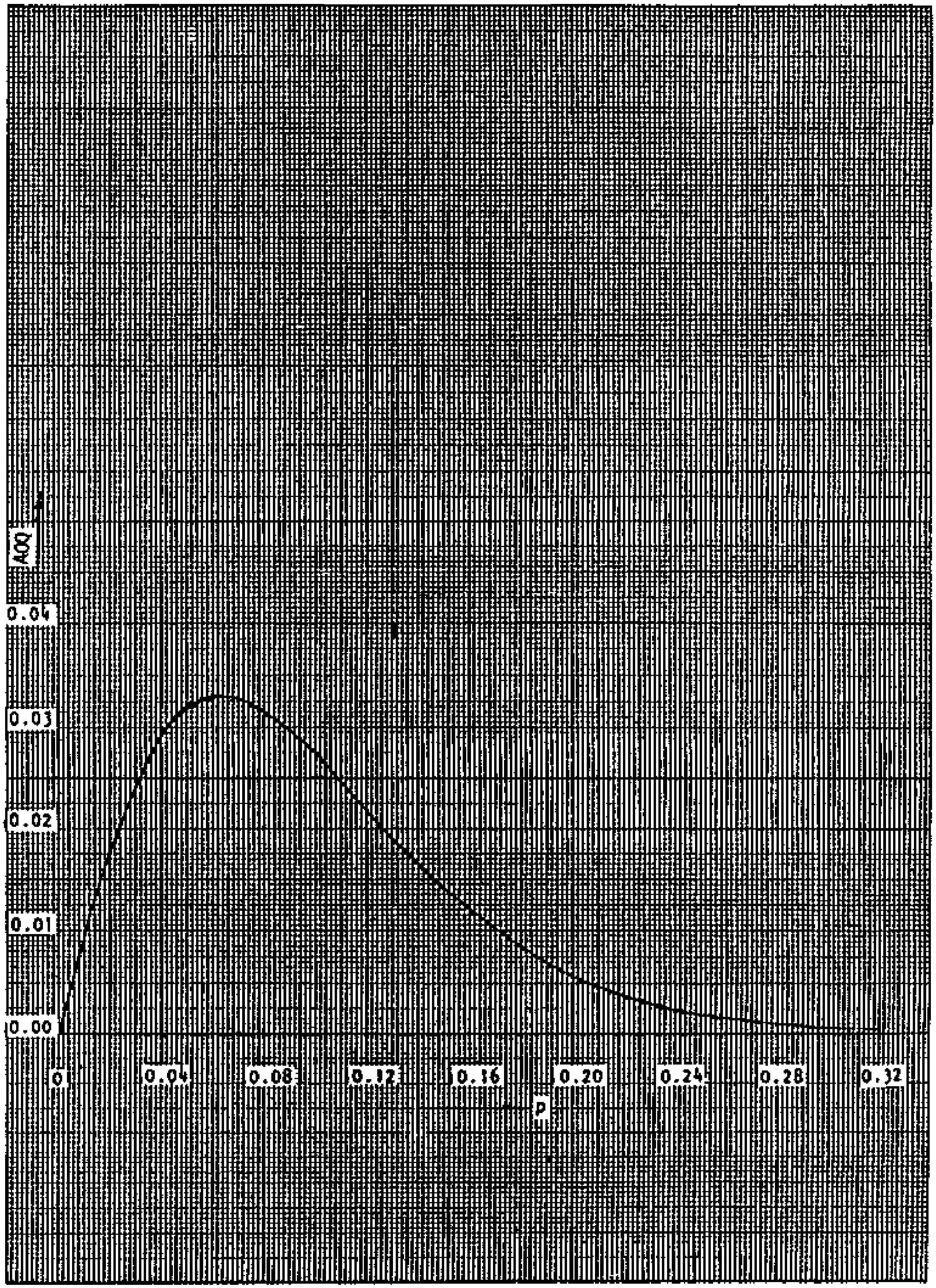
So,

$$\text{AOQ} = p\sum_{x=0}^{1}\binom{25}{x}p^x(1-p)^{25-x} = p\cdot(1-p)^{25} + p^2\times 25\times(1-p)^{24}$$

This is plotted in Fig 10.1. From the graph,

AOQL = 0.0332

Fig. 10.1 AOQ Curve



10.2

$P(SO_2 < 0.1 \text{ unit}) \geq 0.90$ with 95% confidence
i.e. to achieve a reliability of 0.9 with 95% confidence

$$n = \frac{\ln(0.05)}{\ln(0.90)} = 28.4 \approx 29 days$$

10.3

(a) $R^n = 1 - C$

Here,

$n = 10$ and $R = 1 - 0.15 = 0.85$

So,

$C = 1 - R^n = 1 - (0.85)^{10} = 0.8031$

(b) C = 0.99, R = 0.85, $n = \dfrac{\ln(1-C)}{\ln R} = \dfrac{\ln(0.01)}{\ln(0.85)} = 28.3 \approx 29$

So 19 additional borings should be made.

10.4

$$n = \frac{\ln(1-C)}{\ln R}$$

Here,

C = 0.95 and R = 0.99

So,

$$n = \frac{\ln(1-0.95)}{\ln(0.99)} = 298$$

298 consecutive successful starts would be required to meet the standard.

10.5

$$g(0.05) = \sum_{x=0}^{r} \binom{n}{x} (0.05)^x (0.95)^{n-x} = 0.94$$

and,

$$g(0.20) = \sum_{x=0}^{r} \binom{n}{x} (0.20)^x (0.80)^{n-x} = 0.10$$

The values of n and r are to be determined by trial and error. Alternatively, from fig. 10.1e, n = 30 and r = 3 approximately satisfy the above two equations.

10.6

(a) p = fraction defective = 0.30

n = 5, r = 0

From Equation 10.2,

$$g(0.3) = \binom{5}{0}(0.3)^0(0.7)^5 = (0.7)^5 = 0.168$$

= consumer risk

(b) $$g(0.1) = \binom{5}{0}(0.1)^0(0.9)^5 = 0.59$$

$\therefore$ P(rejection) = 1 – g(0.1) = 0.41

10.7

(a) α = Producer's risks = $P(\overline{X} < L|\mu_a) = \Phi(\dfrac{L-\mu_a}{\frac{\sigma}{\sqrt{n}}}) = \Phi(\dfrac{118-120}{\frac{4}{\sqrt{10}}}) = \Phi(-1.58) = 0.057$

β = Consumer's risks = $P(\overline{X} > L|\mu_t) = 1-\Phi(\dfrac{L-\mu_t}{\frac{\sigma}{\sqrt{n}}}) = 1-\Phi(\dfrac{118-114}{\frac{4}{\sqrt{10}}}) = 1-\Phi(3.16) = 0.000792$

(b) α = 0.10 = $\Phi(\dfrac{L-120}{\frac{4}{\sqrt{n}}})$ *or* $(\dfrac{L-120}{4})\sqrt{n} = -1.28$

β = 0.05 = $1-\Phi(\dfrac{L-114}{\frac{4}{\sqrt{n}}})$ *or* $\sqrt{n}(\dfrac{L-114}{4}) = 1.64$

$\therefore \dfrac{L-120}{L-114} = \dfrac{-1.28}{1.64} \rightarrow L = 117.37$

From this,
n= 3.8, so take n = 4.0 and L = 117.34 lb/ft^3

(c) Using equations 10.10 and 10.11,

$\dfrac{\sqrt{n}(L-120)}{4} = -t_{0.10,n-1}$ *and* $\dfrac{\sqrt{n}(L-114)}{4} = t_{0.05,n-1}$

After a trial-and-error procedure, n is found to be 6 and the corresponding value of L becomes 117.59 lb/ft^3 or 117.29 lb/ft^3. So take the average L in order to be fair to both consumer and producer and L = 117.44 lb/ft^3.

10.8

$\alpha = 0.10$, $\beta = 0.10$
$P_a = 0.02$, $P_t = 0.10$
The required sample size n is,

$$n = \left[\frac{\Phi^{-1}(1-\beta) - \Phi^{-1}(\alpha)}{\Phi^{-1}(P_t) - \Phi^{-1}(P_a)}\right]^2 = \left[\frac{\Phi^{-1}(0.90) - \Phi^{-1}(0.10)}{\Phi^{-1}(0.10) - \Phi^{-1}(0.02)}\right]^2$$

$$= \left[\frac{1.28 - (-1.28)}{-1.28 - (-2.06)}\right]^2 = \left[\frac{2.56}{0.78}\right]^2 = 10.77 \approx 11$$

The corresponding tolerable fraction defective M is

$$M = \Phi\left[\Phi^{-1}(P_a) - \frac{\Phi^{-1}(\alpha)}{\sqrt{n}}\right] = \Phi\left[\Phi^{-1}(0.02) - \frac{\Phi^{-1}(0.10)}{\sqrt{11}}\right]$$

$$= \Phi\left[-2.06 + \frac{1.28}{\sqrt{11}}\right] = \Phi(-1.674) = 0.047$$

10.9

$\mu_a = 0.90$, $\mu_t = 0.80$, α = 0.05, β = 0.05, σ = 0.04

$$\Phi\left(\frac{L-0.90}{\frac{0.04}{\sqrt{n}}}\right) = 0.05 \quad or \quad \sqrt{n}\left(\frac{L-0.90}{0.04}\right) = -1.64$$

and $$1-\Phi\left(\frac{L-0.80}{\frac{0.04}{\sqrt{n}}}\right) = 0.05 \quad or \quad \sqrt{n}\left(\frac{L-0.80}{0.04}\right) = 1.64$$

Solving the above two equations, n = 2 and L = 0.85.