最小势能原理的简单证明：

平衡方程：

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0$$
$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0$$

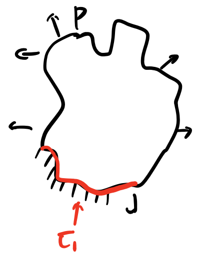
几何方程：

$$\varepsilon_x=\frac{\partial u}{\partial x}\qquad \varepsilon_y=\frac{\partial v}{\partial y}$$
$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

物理方程

$$\varepsilon_x=\frac{1}{E}(\sigma_x-\nu\sigma_y)$$
$$\varepsilon_y=\frac{1}{E}(\sigma_y-\nu\sigma_x)$$
$$\gamma_{xy}=\frac{2(1+\nu)}{E}\tau_{xy}$$

边界条件



$$\sigma_x\cos\alpha+\tau_{xy}\cos\beta=p_x$$
$$\tau_{xy}\cos\alpha+\sigma_y\cos\beta=p_y$$
在 Γ_2 上

$u=0,\ v=0$　　在 Γ_1 上

为了方便，设 $\nu=0$ $\Rightarrow$ $\varepsilon_x=\frac{\sigma_x}{E}$　$\varepsilon_y=\frac{\sigma_y}{E}$　$\gamma_{xy}=\frac{2\tau_{xy}}{E}$

在所有可能的位移中，真实位移使系统的总势能取最小值

设 u, v 为真实位移，$u+u^*$，$v+v^*$，为可能的位移，u^*, v^* 为满足位移边界条件的任意函数

在这里 $u^*, v^*=0$ （在 Γ_1 上）　$\varepsilon_x^*=\frac{\partial u^*}{\partial x}$　$\varepsilon_y^*=\frac{\partial v^*}{\partial y}$　$\gamma_{xy}^*=\frac{\partial u^*}{\partial y}+\frac{\partial v^*}{\partial x}$，$\sigma_x^*=E\varepsilon_x^*$，$\sigma_y^*=E\varepsilon_y^*$　$\tau_{xy}=\frac{E}{2}\gamma_{xy}^*$

真实总势能 $\pi=\int_\Omega \frac{1}{2}(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\tau_{xy}\gamma_{xy})dxdy-\int_{\Gamma_1+\Gamma_2}(p_xu+p_yv)dl$

可能的总势能 $\pi'=\int_\Omega \frac{1}{2}\left[(\sigma_x+\sigma_x^*)(\varepsilon_x+\varepsilon_x^*)+(\sigma_y+\sigma_y^*)(\varepsilon_y+\varepsilon_y^*)+(\gamma_{xy}+\gamma_{xy}^*)(\tau_{xy}+\tau_{xy}^*)\right]dxdy-\int p_x(u+u^*)+p_y(v+v^*)dl$

$\because$ $\sigma_x^*\varepsilon_x=E\varepsilon_x^*\varepsilon_x$，$\sigma_x\varepsilon_x^*=E\varepsilon_x\varepsilon_x^*$　$\therefore$ $\sigma_x^*\varepsilon_x=\sigma_x\varepsilon_x^*$

$$\therefore \pi'=\pi+\int_\Omega\frac{1}{2}(\sigma_x^*\varepsilon_x^*+\sigma_y^*\varepsilon_y^*+\tau_{xy}^*\gamma_{xy}^*)dxdy+\int_\Omega(\sigma_x\varepsilon_x^*+\sigma_y\varepsilon_y^*+\tau_{xy}\gamma_{xy}^*)dxdy-\int(p_xu^*+p_yv^*)dl$$

$$\int_\Omega\frac{1}{2}(\sigma_x^*\varepsilon_x^*+\sigma_y^*\varepsilon_y^*+\tau_{xy}^*\gamma_{xy}^*)dxdy=\int_\Omega\frac{1}{2}\left(E\varepsilon_x^{*2}+E\varepsilon_y^{*2}+\frac{E}{2}\gamma_{xy}^{*2}\right)dxdy\geqslant 0$$ （正定积分）

$$\int_\Omega(\sigma_x\varepsilon_x^*+\sigma_y\varepsilon_y^*+\tau_{xy}\gamma_{xy}^*)dxdy-\int_{\Gamma_1+\Gamma_2}p_xu^*+p_yv^*dl$$
$$=\int_\Omega\left(\sigma_x\frac{\partial u^*}{\partial x}+\sigma_y\frac{\partial v^*}{\partial y}+\tau_{xy}\frac{\partial u^*}{\partial y}+\tau_{xy}\frac{\partial v^*}{\partial x}\right)dxdy-\int(p_xu^*+p_yv^*)dl$$

$$= \int_\Omega \left(\frac{\partial \sigma_x u^*}{\partial x} - \frac{\partial \sigma_x}{\partial x} u^* + \frac{\partial \sigma_y v^*}{\partial y} - \frac{\partial \sigma_y}{\partial y} v^* + \frac{\partial (\tau_{xy} u^*)}{\partial y} - \frac{\partial \tau_{xy}}{\partial y} u^* + \frac{\partial \tau_{xy} v^*}{\partial x} - \frac{\partial \tau_{xy}}{\partial x} v^* \right) dxdy$$

$$- \int (P_x u^* + P_y v^*) dl$$

$$= \int_\Omega \frac{\partial}{\partial x} (\sigma_x u^* + \tau_{xy} v^*) + \frac{\partial}{\partial y} (\sigma_y v^* + \tau_{xy} u^*) dxdy - \int (P_x u^* + P_y v^*) dl$$

$$\underline{- \int_\Omega \left[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) u^* + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right) v^* \right] dxdy}$$

平衡方程，积分为零

高斯公式（曲线积分与曲面积分）

$$\int_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz = \int_C (P\cos\alpha + Q\cos\beta + R\cos\gamma) ds$$

二维 $\int_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy = \int_C (P\cos\alpha + Q\cos\beta) dl$

$$\int_{\Gamma_1 + \Gamma_2} (\sigma_x \cos\alpha \cdot u^* + \tau_{xy} \cos\alpha \cdot v^* + \sigma_y \cos\alpha\, v^* + \tau_{xy} \cos\beta\, u^*) - (P_x u^* + P_y v^*) dl$$

$$= \int_{\Gamma_1 + \Gamma_2} \left[\underline{(\sigma_x \cos\alpha + \sigma_y \cos\beta - P_x)} u^* + \underline{(\tau_{xy} \cos\beta + \sigma_y \cos\alpha - P_y)} v^* \right] dl$$

边界条件

∵ 在 Γ_1 上 $u^* = v^* = 0$

在 Γ_2 上 $\sigma_x \cos\alpha + \sigma_y \cos\beta - P_x = 0$ $\tau_{xy} \cos\beta + \sigma_y \cos\alpha - P_y = 0$

∴ 最终 $\pi' = \pi +$ 正定积分 $+ 0$

最小势能原理得证