扩展卡尔曼滤波

$$\left.\begin{aligned} X_k &= f(X_{k-1}) + Q_k \\ Y_k &= h(X_k) + R_k \end{aligned}\right\} \xRightarrow{\text{bayes filter}} \begin{cases} f_k^-(x) = \int_{-\infty}^{+\infty} f_Q[x - f(v)] f_{k-1}^+(v) dv \\ f_k^+(x) = \eta \int_{-\infty}^{+\infty} f_R[y_k - h(x)] f_k^-(x) dx \end{cases}$$

$$\eta = \left(\int_{-\infty}^{+\infty} f_R[y_k - h(x)] f_k^-(x) dx\right)^{-1}$$

若 $f(X_{k-1}) = F \cdot X_{k-1}$

$h(X_k) = H \cdot X_k$

$Q_k \sim N(0, Q)$ $R_k \sim N(0, R)$

$\Rightarrow$ bayes filter 的无穷积分有解析解

bayes filter $\Rightarrow$ kalman filter

若 $f(X_{k-1})$ $h(X_k)$ 为非线性函数 particle filter 9-12讲

对 $f(X_{k-1})$ $h(X_k)$ 线性化 EKF

设 X_{k-1} 服从期望为 $\hat{x}_{k-1}^+$，方差为 P_{k-1}^+ 的正态分布

$X_k = f(X_{k-1}) + Q_k$ 对 $f(X_{k-1})$ 泰勒展开

$f(X_{k-1}) \approx f(\hat{x}_{k-1}^+) + f'(\hat{x}_{k-1}^+)(X_{k-1} - \hat{x}_{k-1}^+)$ (大小写)

$\approx f'(\hat{x}_{k-1}^+) X_{k-1} + f(\hat{x}_{k-1}^+) - f'(\hat{x}_{k-1}^+)\hat{x}_{k-1}^+$

设 $A = f'(\hat{x}_{k-1}^+)$ $B = f(\hat{x}_{k-1}^+) - f'(\hat{x}_{k-1}^+)\hat{x}_{k-1}^+$

$f(X_{k-1}) \approx A X_{k-1} + B$

预测方程 $X_k = AX_{k-1} + B + Q_k$ $A = f'(\hat{x}_{k-1}^+)$ $B = f(\hat{x}_{k-1}^+) - f'(\hat{x}_{k-1}^+)\hat{x}_{k-1}^+$

bayes filter 预测步

$$f_k^-(x) = \int_{-\infty}^{+\infty} f_Q[x - f(v)] f_{k-1}^+(v) dv$$

$f(v) = Av + B$

$$\Rightarrow f_k^-(x) = \int_{-\infty}^{+\infty} (2\pi Q)^{-\frac{1}{2}} e^{-\frac{(x - Av - B)^2}{2Q}} (2\pi P_{k-1}^+)^{-\frac{1}{2}} e^{-\frac{(x - \hat{x}_{k-1}^+)}{2P_{k-1}^+}} dx$$

$$= N(A\hat{x}_{k-1}^{+}+B,\ A^2P_{k-1}^{+}+Q) \quad \text{Mathematica}$$

$$A\hat{x}_{k-1}^{+}+B = f'(\hat{x}_{k-1}^{+})\hat{x}_{k-1}^{+} + f(\hat{x}_{k-1}^{+}) - f'(\hat{x}_{k-1}^{+})\hat{x}_{k-1}^{+} = f(\hat{x}_{k-1}^{+})$$

B不用算，算A即可 $\quad X_k^- \sim N(f(\hat{x}_{k-1}^{+}),\ A^2P_{k-1}^{+}+Q)$

① $\hat{x}_k^- = f(\hat{x}_{k-1}^{+})$

② $P_k^- = A^2P_{k-1}^{+}+Q$

预测步完毕

更新步

$y_k = h(X_k)+R_k$ 线性化 $h(X_k) \approx h(\hat{x}_k^-) + h'(\hat{x}_k^-)(X_k-\hat{x}_k^-)$

$$\approx CX_k+D \quad C=h'(\hat{x}_k^-) \quad D=h(\hat{x}_k^-)-h'(\hat{x}_k^-)\hat{x}_k^-$$

bayes filter $\quad f_k^+(x) = \eta\cdot f_R[y_k-h(x)]f_k^-(x)$

$$\approx \eta\cdot f_R[y_k-Cx-D]f_k^-(x)$$

$$= \eta\cdot(2\pi R)^{-\frac{1}{2}}e^{-\frac{(y_k-Cx-D)^2}{2R}}\cdot(2\pi P_k^-)^{-\frac{1}{2}}e^{-\frac{(x-x_k^-)^2}{2P_k^-}}$$

$$\eta = \left(\int_{-\infty}^{+\infty}(2\pi R)^{-\frac{1}{2}}e^{-\frac{(y_k-Cx-D)^2}{2R}}\cdot(2\pi P_k^-)^{-\frac{1}{2}}e^{-\frac{(x-x_k^-)^2}{2P_k^-}}dx\right)^{-1}$$

用 Mathematica $\quad X_k^+ \sim N\left(\underline{\frac{R\hat{x}_k^- + CP_k^-(y_k-D)}{R+C^2P_k^-}},\ \underline{\left(1-\frac{C^2P_k^-}{R+C^2P_k^-}\right)P_k^-}\right)$

$$\Downarrow \qquad\qquad \Downarrow$$

$K = \frac{CP_k^-}{R+C^2P_k^-} \qquad \hat{x}_k^- + K(y_k - C\hat{x}_k^- - D) \qquad (1-KC)P_k^-$

$C\hat{x}_k^- + D = h(\hat{x}_k^-)$ 不用算 D

③ $K = \frac{CP_k^-}{R+C^2P_k^-}$

④ $\hat{x}_k^+ = \hat{x}_k^- + K(y_k - h(\hat{x}_k^-))$

⑤ $P_k^+ = (1-KC)P_k^-$

⑥ $A = f'(\hat{x}_{k-1}^{+})$

⑦ $C = h'(\hat{x}_k^-)$

EKF 算法

设 $x_{k-1} \sim N(\hat{x}_{k-1}^{+}, P_{k-1}^{+})$

预测步:

① $A = f'(\hat{x}_{k-1}^{+})$

② $\hat{x}_k^{-} = f(\hat{x}_{k-1}^{+})$

③ $P_k^{-} = A^2 P_{k-1}^{+} + Q$

更新步

④ $C = h'(\hat{x}_k^{-})$

⑤ $K = \dfrac{C P_k^{-}}{C^2 P_k^{-} + R}$

⑥ $\hat{x}_k^{+} = \hat{x}_k^{-} + K[y_k - h(\hat{x}_k^{-})]$

⑦ $P_k^{+} = (1 - KC) P_k^{-}$

矩阵形式的 EKF

矩阵求导, 《矩阵论》

$\vec{X_k} = f(\vec{X_{k-1}}) + \vec{Q_k}$

$\vec{Y_k} = h(\vec{X_k}) + \vec{R_k}$

设 $\vec{X_{k-1}} \sim N(\hat{X}_{k-1}^{+}, \Sigma_{k-1}^{+})$

① $$A = \left.\begin{pmatrix} \dfrac{\partial f_1}{\partial x_{k-1}^1} & \dfrac{\partial f_1}{\partial x_{k-1}^2}, & \cdots & \dfrac{\partial f_1}{\partial x_{k-1}^n} \\ \dfrac{\partial f_2}{\partial x_{k-1}^1} & \dfrac{\partial f_2}{\partial x_{k-2}^2} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_{k-1}^1} & \dfrac{\partial f_n}{\partial x_{k-1}^2} & \cdots & \dfrac{\partial f_n}{\partial x_{k-1}^n} \end{pmatrix}\right|_{\vec{X_{k-1}} = \hat{X}_{k-1}^{+}}$$

② $\vec{X_k^{-}} = f(\hat{\vec{X}}_{k-1}^{+})$

③ $\Sigma_k^{-} = A \Sigma_{k-1}^{+} A^T + Q$ 　　Q 为 $\vec{Q_k}$ 的协方差矩阵

④

$$C = \left.\begin{pmatrix} \dfrac{\partial h_1}{\partial x_k^1} & \dfrac{\partial h_1}{\partial x_k^2} & \dfrac{\partial h_1}{\partial x_k^3} & \cdots & \dfrac{\partial h_1}{\partial x_k^n} \\ \vdots & & & & \\ \dfrac{\partial h_m}{} & \dfrac{\partial h_m}{} & & \cdots & \dfrac{\partial h_m}{} \end{pmatrix}\right|$$

$$\left. \cdots \frac{}{\partial x_k^1} \quad \frac{}{\partial x_k^2} \quad \cdots \quad \frac{}{\partial x_k^n} \right) \Big|_{x_k = x_k^-}$$

⑤ $K = \Sigma_k^- C^T (C \Sigma_k^- C^T + R)^{-1}$

⑥ 观测到一个数据 $\vec{Y}$

⑦ $\vec{x}_k^+ = \vec{x}_k^- + K[\vec{Y} - h(\vec{x}_k^-)]$

⑧ $\Sigma_k^+ = (I - KC)\Sigma_k^-$

$x_k = f(x_{k-1})$

$x_k^1 = (x_{k-1}^1)^2 \quad \Rightarrow \quad f_1 = (x_{k-1}^1)^2$

$x_k^2 = (x_{k-1}^2)^2 \quad \Rightarrow \quad f_2 = (x_{k-1}^2)^2$

$x_k^3 = (x_{k-1}^3)^2 \quad \Rightarrow \quad f_3 = (x_{k-1}^3)^2$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_{k-1}^1} & \frac{\partial f_1}{\partial x_{k-1}^2} & \frac{\partial f_1}{\partial x_{k-1}^3} \\ \frac{\partial f_2}{\partial x_{k-1}^1} & \frac{\partial f_2}{\partial x_{k-2}^2} & \frac{\partial f_2}{\partial x_{k-1}^3} \\ \frac{\partial f_3}{\partial x_{k-1}^1} & \frac{\partial f_3}{\partial x_{k-1}^2} & \frac{\partial f_3}{\partial x_{k-1}^3} \end{pmatrix} = \begin{pmatrix} 2x_{k-1}^1 & 0 & 0 \\ 0 & 2x_{k-1}^2 & 0 \\ 0 & 0 & 2x_{k-1}^3 \end{pmatrix}$$

$$\hat{\vec{x}}_{k-1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad A = \left(\quad \right) \Big|_{\vec{x}_{k-1} = \hat{x}_{k-1}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$\vec{Y} = h(\vec{x}_k)$

$Y_1 = (x_k^1)^2 \quad \Rightarrow \quad h_1$

$Y_2 = (x_k^2)^2 + (x_k^3)^2 \quad \Rightarrow \quad h_2$

$$C = \begin{pmatrix} \frac{\partial h_1}{\partial x_k^1} & \frac{\partial h_1}{\partial x_k^2} & \frac{\partial h_1}{\partial x_k^3} \\ \frac{\partial h_2}{\partial x_k^1} & \frac{\partial h_2}{\partial x_k^2} & \frac{\partial h_2}{\partial x_k^3} \end{pmatrix} = \begin{pmatrix} 2x_k^1 & 0 & 0 \\ 0 & 2x_k^2 & 2x_k^3 \end{pmatrix}$$