卡尔曼滤波

$$\begin{cases} X_k = f(X_{k-1}) + Q_k \\ Y_k = h(X_k) + R_k \end{cases}$$

X_0 , $Q_1 \cdots Q_k$, $R_1 \cdots R_k$ 相互独立

Bayes Filter

$$f_k^-(x) = \int_{-\infty}^{+\infty} f_Q[x - f(v)] f_{k-1}^+(v)\, dv$$

$$f_k^+(x) = \eta f_R[y_k - h(x)] f_k^-(x)$$

$$\eta = \left(\int_{-\infty}^{+\infty} f_R[y_k - h(x)] f_k^-(x)\, dx\right)^{-1}$$

Kalman Filter

Bayes Filter :: Kalman Filter ① $f(X_{k-1}) = F \cdot X_{k-1}$, $h(X_k) = h \cdot X_k$

② $f_Q(x) = (2\pi Q)^{-\frac{1}{2}} e^{-\frac{x^2}{2Q}}$, $f_R(x) = (2\pi R)^{-\frac{1}{2}} e^{-\frac{x^2}{2R}}$

$Q \sim N(0, Q)$, $R \sim N(0, R)$

Kalman Filter

设 $X_{k-1} \sim N(\mu_{k-1}^+ , \sigma_{k-1}^+)$

① 预测步 $f_k^-(x) = \int_{-\infty}^{+\infty} f_Q[x - f(v)] f_{k-1}^+(v)\, dv$

$$= \int_{-\infty}^{+\infty} (2\pi Q)^{-\frac{1}{2}} e^{-\frac{(x - Fv)^2}{2Q}} \cdot (2\pi \sigma_{k-1}^+)^{-\frac{1}{2}} e^{-\frac{(v - \mu_{k-1}^+)^2}{2\sigma_{k-1}^+}}\, dv$$

推荐：①数学软件 Mathematica $N(F\mu_{k-1}^+ , F^2\sigma_{k-1}^+ + Q)$

②复变函数，用留数定理算

③用 F.T + 卷积去算

$X_k = F X_{k-1} + Q_k$ X_{k-1} 与 Q_k 独立

$X_{k-1} \sim N(\mu_{k-1}^+ , \sigma_{k-1}^+)$ ，$F X_{k-1} \sim N(F\mu_{k-1}^+ , F^2\sigma_{k-1}^+)$

$Q_k \sim N(0, Q)$

X_k 的 pdf 实际上是 FX_{k-1} 与 Q_k 的卷积，

$h = f * g$ $G(h) = G(f) \cdot G(g)$

$F X_{k-1} \xrightarrow{F.T} g_1(t) = e^{iF\mu_{k-1}^+ t - \frac{F^2\sigma_{k-1}^+}{2} t^2}$

$Q_k \xrightarrow{F.T} g_2(t) = e^{-\frac{Q}{2}t^2}$ $g_1(t) \cdot g_2(t) = e^{iF\mu_{k-1}^+ t - \frac{F^2\sigma_{k-1}^+ + Q}{2} t^2}$

$\xrightarrow{i.F.T} N(\underline{F\mu_{k-1}^+} , \underline{F^2\sigma_{k-1}^+ + Q})$

$N(\mu, \sigma^2) \xrightarrow{F.T} e^{i\mu t - \frac{\sigma^2}{2} t^2}$

设 $f_k^-(x) \sim N(\mu_k^- , \sigma_k^-)$ 有

① $\mu_k^- = F\mu_{k-1}^+$

② $\sigma_k^- = F^2\sigma_{k-1}^+ + Q$ 预测步完毕

二. 更新步

$f_k^-(x) \sim N(\mu_k^- , \sigma_k^-)$

$$f_k^+(x) = \eta\, f_R(y_k - h\cdot x)\cdot f_k^-(x)$$

$$= \eta\,(2\pi R)^{-\frac{1}{2}} e^{-\frac{(y_k - hx)^2}{2R}} \cdot (2\pi\sigma_k^-)^{-\frac{1}{2}} e^{-\frac{(x-\mu_k^-)^2}{2\sigma_k^-}}$$

$$\eta = \left(\int_{-\infty}^{+\infty} (2\pi R)^{-\frac{1}{2}} e^{-\frac{(y_k - hx)^2}{2R}} \cdot (2\pi\sigma_k^-)^{-\frac{1}{2}} e^{-\frac{(x-\mu_k^-)^2}{2\sigma_k^-}}\, dx\right)^{-1}$$

数学软件: $X_k^+ \sim N\left(\frac{h\sigma_k^- y_k + R\mu_k^-}{h^2\sigma_k^- + R}, \frac{R\sigma_k^-}{h^2\sigma_k^- + R}\right)$

$X_k^+ \sim N(\mu_k^+, \sigma_k^+)$ 则

③ $$\mu_k^+ = \frac{h\sigma_k^-}{h^2\sigma_k^- + R} y_k + \frac{\mu_k^-(R + h^2\sigma_k^-) - \mu_k^- h^2\sigma_k^-}{h^2\sigma_k^- + R}$$

$$= \frac{h\sigma_k^-}{h^2\sigma_k^- + R}(y_k - h\mu_k^-) + \mu_k^- \Rightarrow \mu_k^- + K(y_k - h\mu_k^-)$$

④ $$\sigma_k^- = \frac{\sigma_k^-(R + h^2\sigma_k^-) - \sigma_k^- h^2\sigma_k^-}{h^2\sigma_k^- + R}$$

$$= \left(1 - \frac{h^2\sigma_k^-}{h^2\sigma_k^- + R}\right)\sigma_k^- \Rightarrow (1 - Kh)\sigma_k^-$$

⑤ $$K = \frac{h\sigma_k^-}{h^2\sigma_k^- + R}$$

卡尔曼滤波5个公式

$$\mu_k^- = F\mu_{k-1}^+$$

$$\sigma_k^- = F^2\sigma_{k-1}^+ + Q$$

$$K = \frac{h\sigma_k^-}{h^2\sigma_k^- + R}$$

$$\mu_k^+ = \mu_k^- + K(y_k - h\mu_k^-)$$

$$\sigma_k^+ = (1 - Kh)\sigma_k^-$$

K : 卡尔曼增益 $= \dfrac{h}{h^2 + R/\sigma_k^-}$

当 $R \gg \sigma_k^-$　$K \to 0$　$\mu_k^+ = \mu_k^- + K(y_k - h\mu_k^-) = \mu_k^-$ 相信预测

当 $R \ll \sigma_k^-$　$K \to \frac{1}{h}$　$\mu_k^+ = \mu_k^- + \frac{y_k}{h} - \mu_k^- = \frac{y_k}{h}$　$y_k = h x_k + R$ 相信观测

☆ 矩阵形式的 Kalman Filter

$\mu_k \to \vec{\mu_k}$　$\sigma_k \to \Sigma_k$ 协方差矩阵

F, H 皆为矩阵

类推
$$\vec{\mu_k^-} = F \cdot \vec{\mu_{k-1}^+}$$
$$\vec{\Sigma_k^-} = F \Sigma_{k-1}^+ F^T + Q \quad (F^2 \sigma_{k-1}^- + Q)$$
$$K = \Sigma_k^- H^T (H \Sigma_k^- H^T + R)^{-1}$$
$$\vec{\mu_k^+} = \vec{\mu_k^-} + K(\vec{y_k} - H\mu_k^-)$$
$$\Sigma_k^+ = (I - KH)\Sigma_k^-$$
《概率机器人》

应用:　more is different

纸上得来终觉浅，绝知此事要躬行

① Filter　问题：请用计算机生成一个含正态噪声的信号，并用 KF 滤波

② sensor Fusion

问题：已知 $x = t^2$ 为信号　有2个不同传感器对X进行观测

产生了 y_{a_1}, y_{a_2}, y_{a_3} … y_{a_k}

$y_{b1}, y_{b2}, y_{b3} \cdots y_{bk}$

$Y_{ak} = X_k + R_{ak}$ $\quad Y_{bk} = X_k + R_{bk}$ $\quad R_{ak} \sim N(0.1)$

$R_{bk} \sim N(0.2)$

求 sensor Fusion 下的 $\hat{X}^+$

$\hat{X}^+ = \int_{-\infty}^{+\infty} x f^+(x) dx$ 用代码实现!!!

$y_{ak} = t^2 +$ normrnd(0.1)

$y_{bk} = t^2 +$ normrnd(0.2)

matlab / C / C++ / python

✓

马尔可夫与观测独立

证明若: $\begin{cases} X_k = f(X_{k-1}) + Q_k \\ Y_k = h(X_k) + R_k \end{cases}$, $X_0, Q_1 \cdots Q_k, R_1 \cdots R_k$ 独立

则 $P(X_k < x_k | X_{k-1} = x_{k-1}, X_{k-2} = x_{k-2} \cdots X_0 = x_0) = P(X_k = x_k | X_{k-1} = x_{k-1})$ 马尔可夫

以及 $P(Y_k = y_k | X_k = x_k, X_{k-1} = x_{k-1} \cdots X_0 = x_0) = P(Y_k = y_k | X_k = x_k)$ 观测独立

马尔可夫 $P(X_k < x_k | X_0 = x_0, \cdots X_{k-1} = x_{k-1})$

$= P(X_k - f(X_{k-1}) < x_k - f(x_{k-1}) | X_0 = 0, X_1 - f(X_0) = x_1 - f(x_0), \cdots X_{k-1} - f(X_{k-2}) = x_{k-1} - f(x_{k-2}))$

$= P(Q_k < x_k - f(x_{k-1}) | X_0 = 0, Q_1 = x_1 - f(x_0), \cdots Q_{k-1} = x_{k-1} - f(x_{k-2}))$

$= P(Q_k < x_k - f(x_{k-1}))$

而 $P(X_k < x_k | X_{k-1} = x_{k-1}) = P(X_k - f(X_{k-1}) < x_k - f(x_{k-1}) | X_{k-1} = x_{k-1})$

$$= P(Q_k < x_k - f(x_{k-1}) \mid X_{k-1} = x_{k-1})$$

$$= P(Q_k < x_k - f(x_{k-1}))$$

左边 = 右边　马尔可夫性成立

观测独立性 留作练习

tips：①使用矩阵形式的 KF

② F　H 可以不是方阵，阶数也可以不相同

③ $X_k = X_{k-1} + \dot{X}_{k-1}\,dt + \frac{\ddot{X}_{k-1}}{2!}dt^2 + \cdots\cdots$

学习的关键不在于有没有人来帮你，而在于有没有恒心，勇气与毅力