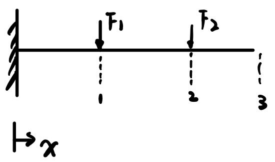
简易有限元自学教程第六讲



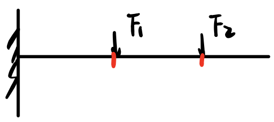
$$u=\begin{cases} a_1x+a_2 & 0<x<1 \\ a_3x+a_4 & 1<x<2 \\ a_5x+a_6 & 2<x<3 \end{cases} \qquad v=\begin{cases} b_1x+b_2 & 0<x<1 \\ b_3x+b_4 & 1<x<2 \\ b_5x+b_6 & 2<x<3 \end{cases}$$

$$E_p=\int \frac{1}{2}\vec{\sigma}\vec{\varepsilon}^T dV - F_1\cdot v(1) - F_2\cdot v(2) = f(a_1, a_2, a_3 \cdots a_6, b_1 \cdots b_6)$$

$$\begin{cases} \dfrac{\partial E_p}{\partial a_1}=0 \\ \dfrac{\partial E_p}{\partial a_2}=0 \\ \vdots \\ \dfrac{\partial E_p}{\partial b_6}=0 \end{cases} \Rightarrow \text{十二个线性方程组，十二个未知数} \Rightarrow \begin{cases} a_1= \\ a_2= \\ \vdots \\ b_6= \end{cases}$$

奇异

没有约束

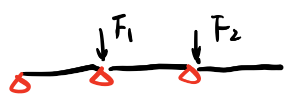


$$u=\begin{cases} a_1x+a_2 \\ a_3x+a_4 \\ a_5x+a_6 \end{cases}$$

单元之间位移要连续

u要满足约束

$\Downarrow$

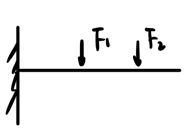


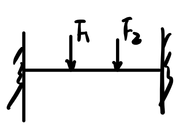
自由运动

确定运动

不允许出现断裂，重叠　变形要协调

u、v的分段一定要慎重



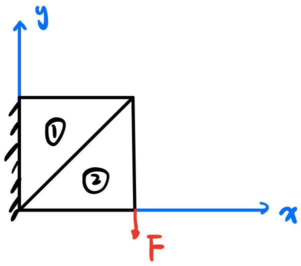


$$E_p=\int \frac{1}{2}\vec{\sigma}\vec{\varepsilon}^T - F_1\cdot v(1) - F_2\cdot v(2)$$

$$u=\begin{cases} a_1x \\ a_2x+a_3 \\ a_4x+a_5 \end{cases} \qquad u=\begin{cases} b_1x \\ b_2x+b_3 \\ b_4(x-3) \end{cases}$$

$a_1 \cdots a_5$ 不完全独立　　　　$b_1 \cdots b_4$ 不完全独立

手算一个单元



$$u(x,y)=\begin{cases} a_2 x & (x,y)\in ① \\ a_4 + a_5 x + a_6 y & (x,y)\in ② \end{cases}$$

$$v(x,y)=\begin{cases} b_2 x & (x,y)\in ① \\ b_4 + b_5 x + b_6 y & (x,y)\in ② \end{cases}$$

约束　$u(0,y)=0$　$v(0,y)=0$　$\Rightarrow$　$a_1=0$,　$a_3=0$,　$b_1=0$,　$b_3=0$

连续　在 $y=x$ 上　$a_2 x = a_4 + a_5 x + a_6 x \Rightarrow a_4=0 \quad a_5+a_6=a_2$

$b_2 x = b_4 + b_5 x + b_6 x \Rightarrow b_4=0 \quad b_5+b_6=b_2$

$$\therefore\ u(x,y)=\begin{cases} a_2 x & (x,y)\in ① \\ a_5 x + (a_2-a_5) y & (x,y)\in ② \end{cases} \qquad v(x,y)=\begin{cases} b_2 x & (x,y)\in ① \\ b_5 x + (b_2-b_5) y & (x,y)\in ② \end{cases}$$

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}=\begin{cases} a_2 & (x,y)\in ① \\ a_5 & (x,y)\in ② \end{cases} \qquad \varepsilon_{yy}=\frac{\partial v}{\partial y}=\begin{cases} 0 & (x,y)\in ① \\ b_2-b_5 & (x,y)\in ② \end{cases}$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\begin{cases} b_2 & (x,y)\in ① \\ a_2-a_5+b_5 & (x,y)\in ② \end{cases}$$

$$\sigma_{xx}=\frac{E}{1-\nu^2}(\varepsilon_{xx}+\nu\varepsilon_{yy})=\begin{cases} \frac{E}{1-\nu^2}a_2 & (x,y)\in ① \\ \frac{E}{1-\nu^2}(a_5-\nu(b_2-b_5)) & (x,y)\in ② \end{cases}$$

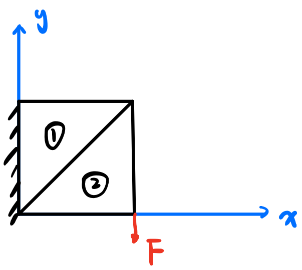
$$\sigma_{yy} = \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx}) = \begin{cases} \frac{E}{1-\nu^2}(\nu a_2) & (x,y)\in ① \\ \frac{E}{1-\nu^2}(b_2 - b_5 + \nu a_5) & (x,y)\in ② \end{cases}$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = \begin{cases} \frac{E}{2(1+\nu)} b_2 & (x,y)\in ① \\ \frac{E}{2(1+\nu)}(a_2 - a_5 + b_5) & (x,y)\in ② \end{cases}$$

$$\therefore\ u_0 = \begin{cases} \frac{1}{2}\left(\frac{E}{1-\nu^2}a_2^2 + \frac{E}{2(1+\nu)}b_2^2\right) & (x,y)\in ① \\ \frac{1}{2}\left(\frac{E}{1-\nu^2}[a_5 - \nu(b_2-b_5)]a_5 + \frac{E}{1-\nu^2}(b_2 - b_5 + \nu a_5)(b_2-b_5) + \frac{E}{2(1+\nu)}(a_2 - a_5 + b_5)^2\right) & (x,y)\in ② \end{cases}$$

$$u_0 = \begin{cases} \frac{1}{2}\frac{E}{1-\nu^2}\left(a_2^2 + \frac{1+\nu}{2}b_2^2\right) & (x,y)\in ① \\ \frac{1}{2}\frac{E}{1-\nu^2}\left[b_2^2 - 2b_2b_5 - (1+\nu)a_5b_5 + \left(\frac{1+\nu}{2}\right)a_2^2 + \left(\frac{3+\nu}{2}\right)a_5^2 - (1+\nu)a_2a_5 + (1+\nu)a_2b_5\right] \end{cases}$$

$$u(x,y) = \begin{cases} a_2 x & (x,y)\in ① \\ a_5 x + (a_2 - a_5)y & (x,y)\in ② \end{cases} \qquad v(x,y) = \begin{cases} b_2 x & (x,y)\in ① \\ b_5 x + (b_2 - b_5)y & (x,y)\in ② \end{cases}$$

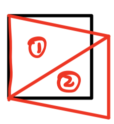


$$E_p = \iint_{①+②} u_0\,dxdy + F b_5$$

$$= \iint_① u_0\,dxdy + \iint_② u_0\,dxdy + \bar{F}\cdot b_5$$

$$= u_0\Big|_{(x,y)\in①}\cdot S_① + u_0\Big|_{(x,y)\in②}\cdot S_② + F\cdot b_5$$

$$S_① = S_② = \frac{1}{2}\ ?$$



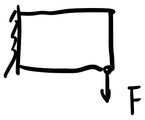
有限元缺陷：不能处理网格变形剧烈的问题 （流体力学）

$$E_p = \frac{1}{2}\cdot\frac{1}{2}\frac{E}{1-\nu^2}(a_2^2 + b_2^2) + \frac{1}{2}\cdot(\qquad\qquad) - F\cdot b_5 = f(a_2, b_2, a_5, b_5)$$

$$\begin{cases} \frac{\partial E_p}{\partial a_2} = 0 \\ \frac{\partial E_p}{\partial a_5} = 0 \end{cases} \Rightarrow \begin{cases} 2a_2 \quad a_5(-1-\nu) + 2a_2\left(\frac{1}{2}+\frac{\nu}{2}\right) + b_5(1+\nu) = 0 \\ a_2(\quad \nu) + b_5(-1-\nu) + 2a_5\left(\frac{3}{2}+\frac{\nu}{2}\right) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial E_p}{\partial b_2} = 0 \\ \dfrac{\partial E_p}{\partial b_5} = 0 \end{cases} \qquad \begin{cases} 2b_2 - 2b_5 + b_2(1+v) = 0 \\ +F + \dfrac{E}{4(1-v^2)}\left(-2b_2 + (-1-v)a_5 + 2b_5\left(\dfrac{3}{2} + \dfrac{v}{2}\right) + a_2(1+v)\right) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_2 = -\dfrac{2F(-1+v)(1+v)(3+v)}{E(7+3v)} & a_5 = +\dfrac{2F(-1+v)(1+v)(3+v)}{E(7+3v)} \\ b_2 = +\dfrac{8F(-1+v)(2+v)}{E(7+3v)} & b_5 = +\dfrac{4F(-1+v)(2+v)(3+v)}{E(7+3v)} \end{cases}$$



$$v(1,0) = b_5 = \frac{4F}{E}\frac{(-1+v)(2+v)(3+v)}{(7+3v)}$$

材料力学 $\Delta = \dfrac{Fl^3}{3EI}$ $I = \dfrac{bh^3}{12}$ $l = b = h = 1$ $\therefore$ $\Delta = \dfrac{4F}{E}$

$v=0$ 时 $b_5 = -\dfrac{4F}{E} \cdot \dfrac{6}{7}$ 同一数量级 网格越密，阶数越高，越精确