Circular numbers of certain abelian fields

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Throughout this thesis, we will use the convention that whenever any of the indices i, j, l, h appear on the same line, they are pairwise distinct and moreover $1 \leq i, j, l, h \leq 4$.

1 Basic definitions and assumptions

Let k be a real abelian field with exactly four ramified primes p_1, p_2, p_3, p_4. Let K be the genus field (in the narrow sense) of k and assume $K \neq k$. Let $G := \mathrm{Gal}(K/\mathbb{Q})$, then (by the properties of the genus field) we can make the identification $G = T_1 \times T_2 \times T_3 \times T_4$, where T_i is the inertia group corresponding the ramified prime p_i. Next, we will define:

- $H := \mathrm{Gal}(K/k)$,
- $m := |H|$,
- the canonical projections $\pi_i : G \to T_i$,
- $a_i := [T_i : \pi_i(H)]$,
- $r_i := |H \cap \ker \pi_i|$,
- $s_{ij} := |H \cap \ker(\pi_i \pi_j)|$,
- K_i as the maximal subfield of K ramified only at p_i (so that

$$T_i = \mathrm{Gal}(K/K_j K_l K_h) \cong \mathrm{Gal}(K_i/\mathbb{Q}).)$$

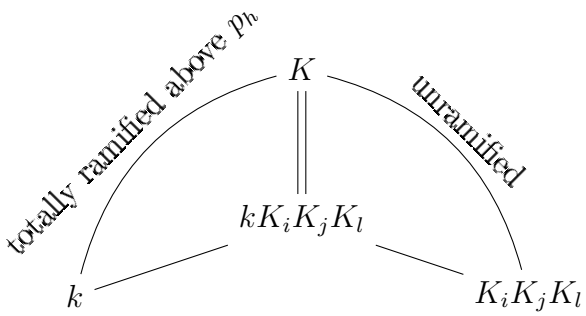
We will assume the following:

- $K \neq k$,
- H is cyclic, generated by τ,
- each T_i is cyclic, generated by σ_i.

2 Auxiliary results

Lemma 1. *We have* $kK_iK_jK_l = K$ *and* $K_1K_2K_3K_4 = K$.

Proof. The extension $K/K_iK_jK_l$ is totally ramified at the prime ideals above p_h, so the same must be true for the extension $K/kK_iK_jK_l$. But since the extension K/k is unramified (by the definition of K), so is $K/kK_iK_jK_l$. Therefore $[K : kK_iK_jK_l] = 1$. The second claim follows from the fact $\mathrm{Gal}(K_i/\mathbb{Q}) = T_i$. □



Proposition 2. *We have* $a_i = [k \cap K_i : \mathbb{Q}]$, $r_i = [K : kK_i]$, $|T_i| = a_i\frac{m}{r_i}$, $s_{ij} = [K : kK_iK_j]$. *Also* $[K_i : k \cap K_i] = \frac{m}{r_i}$, $[K_iK_j : k \cap K_iK_j] = \frac{m}{s_{ij}}$ *and* $[K_iK_jK_l : k \cap K_iK_jK_l] = m$.

Proof. Since

$$\mathrm{Gal}(K/K_i) \cong \mathrm{Gal}(K/\mathbb{Q})/\mathrm{Gal}(K_i/\mathbb{Q}) \cong T_1T_2T_3T_4/T_i \cong T_jT_lT_h$$

and $\mathrm{Gal}(K/k) = H$, it follows that $\mathrm{Gal}(K/k \cap K_i) \cong T_jT_lT_h \cdot H$. Now consider the short exact sequence

$$0 \to H \cap \ker \pi_i \to H \xrightarrow{\pi_i|_H} \pi_i(H) \to 0.$$

It follows that $|\pi_i(H)| = \frac{m}{r_i}$ and

$$\pi_i(H) \cong \frac{H}{H \cap \ker \pi_i} = \frac{H}{H \cap T_jT_lT_h} \cong \frac{T_jT_lT_h \cdot H}{T_jT_lT_h} \cong \frac{\mathrm{Gal}(K/k \cap K_i)}{\mathrm{Gal}(K/K_i)} \cong \mathrm{Gal}(K_i/k \cap K_i).$$

Therefore

$$[k \cap K_i : \mathbb{Q}] = \frac{|\mathrm{Gal}(K_i/Q)|}{|\mathrm{Gal}(K_i/k \cap K_i)|} = \frac{|T_i|}{|\pi_i(H)|} = a_i$$

and

$$[K : kK_i] = \frac{|\mathrm{Gal}(K/k)|}{|\mathrm{Gal}(kK_i/k)|} = \frac{|H|}{|\mathrm{Gal}(K_i/k \cap K_i)|} = \frac{m}{\pi_i(H)} = r_i.$$

Putting everything together, we obtain

$$|T_i| = [K_i : k \cap K_i] \cdot [k \cap K_i : \mathbb{Q}] = a_i|\pi_i(H)| = a_i\frac{m}{r_i}.$$

Next, we also have

$$\mathrm{Gal}(K/K_iK_j) \cong \mathrm{Gal}(K/\mathbb{Q})/\mathrm{Gal}(K_iK_j/\mathbb{Q}) \cong T_1T_2T_3T_4/T_iT_j \cong T_lT_h,$$

so that $\mathrm{Gal}(K/k \cap K_iK_j) = T_lT_h \cdot H$. Thus we can consider the short exact sequence

$$0 \to H \cap \ker \pi_i\pi_j \to H \xrightarrow{\pi_i\pi_j|_H} \pi_i\pi_j(H) \to 0$$

to conclude that $|\pi_i\pi_j(H)| = \frac{m}{s_{ij}}$ and

$$\begin{aligned}\pi_i\pi_j(H) &\cong \frac{H}{H \cap \ker \pi_i\pi_j} = \frac{H}{H \cap T_lT_h} \cong \frac{T_lT_h \cdot H}{T_lT_h} \\ &\cong \frac{\mathrm{Gal}(K/k \cap K_iK_j)}{\mathrm{Gal}(K/K_iK_j)} \cong \mathrm{Gal}(K_iK_j/k \cap K_iK_j).\end{aligned}$$

Then it follows that

$$[K : kK_iK_j] = \frac{|\mathrm{Gal}(K/k)|}{|\mathrm{Gal}(kK_iK_j/k)|} = \frac{|H|}{|\mathrm{Gal}(K_iK_j/k \cap K_iK_j)|} = \frac{m}{\pi_i\pi_j(H)} = s_{ij}.$$

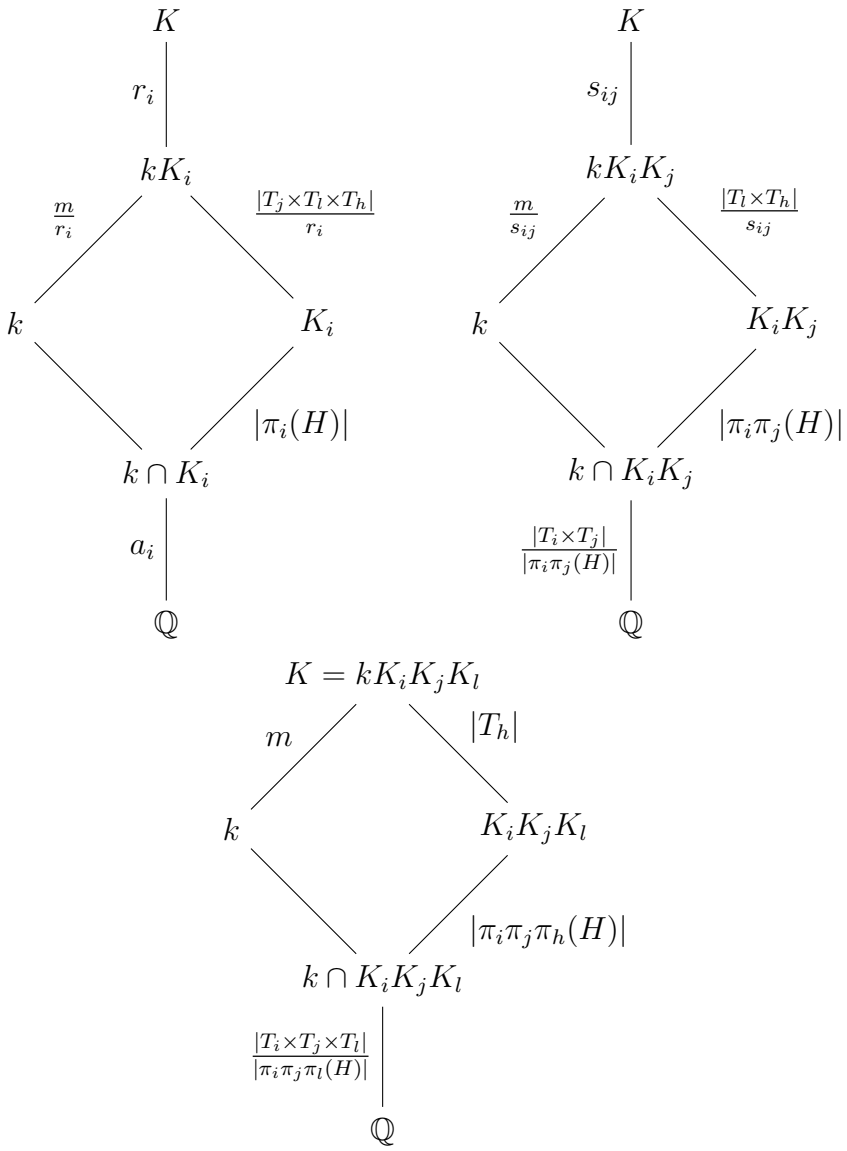
The last part of the statement is a consequence of Lemma 1, since we have

$$\mathrm{Gal}(K_iK_jK_l/k \cap K_iK_jK_l) \cong \mathrm{Gal}(kK_iK_jK_l/k) = \mathrm{Gal}(K/k) = H.$$

Finally note that in the same way as above, we could show that

$$\pi_i\pi_j\pi_l(H) \cong \mathrm{Gal}(K_iK_jK_l/k \cap K_iK_jK_l).$$

□



Corollary 3. *We have* $[k \cap K_iK_j : \mathbb{Q}] = a_ia_j\frac{m}{r_ir_j}s_{ij}$, $[k \cap K_iK_jK_l : \mathbb{Q}] = a_ia_ja_l\frac{m^2}{r_ir_jr_l}$ *and* $[k : \mathbb{Q}] = a_1a_2a_3a_4\frac{m^3}{r_1r_2r_3r_4}$.

Proof. This follows from the computations

$$[k \cap K_iK_j : \mathbb{Q}] = \frac{[K_iK_j : \mathbb{Q}]}{[K_iK_j : k \cap K_iK_j]} = \frac{|T_i| \cdot |T_j|}{m/s_{ij}} = a_ia_j\frac{m}{r_ir_j}s_{ij},$$

$$[k \cap K_iK_jK_l : \mathbb{Q}] = \frac{[K_iK_jK_l : \mathbb{Q}]}{[K_iK_jK_l : k \cap K_iK_jK_l]} = \frac{|T_i| \cdot |T_j| \cdot |T_l|}{m} = a_ia_ja_l\frac{m^2}{r_ir_jr_l}$$

and

$$[k:\mathbb{Q}] = [k\cap K_i:\mathbb{Q}]\cdot[k:k\cap K_i] = a_i\cdot[kK_i:K_i] = a_i\frac{[K:K_i]}{[K:kK_i]}$$
$$= a_i\frac{|T_j|\cdot|T_l|\cdot|T_h|}{r_i} = a_1a_2a_3a_4\frac{m^3}{r_1r_2r_3r_4}.$$

□

Lemma 4. *We have* $s_{ij} = \gcd(r_i, r_j)$, $\gcd(r_i, r_j, r_l) = 1$ *(this is also equivalent to* $\operatorname{lcm}\left(\frac{m}{r_i}, \frac{m}{r_j}, \frac{m}{r_l}\right) = m$ *and to* $\gcd(s_{ij}, r_l) = 1$*) and* $s_{ij}\frac{m}{r_ir_j} = \gcd(\frac{m}{r_i}, \frac{m}{r_j})$.

Proof. It follows from Proposition 2 that $s_{ij} \mid r_i, s_{ij} \mid r_j$ and from its proof that $|\pi_i(H)| = \frac{m}{r_i}$, $|\pi_i\pi_j(H)| = \frac{m}{s_{ij}}$ and $|\pi_i\pi_j\pi_l(H)| = m$. The cyclicity of H then implies

$$\frac{m}{s_{ij}} = |\pi_i\pi_j(H)| = |\langle\pi_i\pi_j(\tau)\rangle| = |\langle\pi_i(\tau)\pi_j(\tau)\rangle| = \operatorname{lcm}\left(\frac{m}{r_i}, \frac{m}{r_j}\right),$$

because $\langle\pi_i(\tau)\rangle = \pi_i(H)$ and the elements $\pi_i(H), \pi_j(H)$ have different non-zero coordinates in G. Now for any common divisor t of r_i, r_j, we have $\frac{m}{s_{ij}} = \operatorname{lcm}\left(\frac{m}{r_i}, \frac{m}{r_j}\right) \mid \frac{m}{t}$, which implies $t \mid s_{ij}$ and we are done.

Similarly, we have

$$m = |\pi_i\pi_j\pi_l(H)| = |\langle\pi_i\pi_j\pi_l(\tau)\rangle| = |\langle\pi_i(\tau)\pi_j(\tau)\pi_l(\tau)\rangle| = \operatorname{lcm}\left(\frac{m}{r_i}, \frac{m}{r_j}, \frac{m}{r_l}\right),$$

so if t is any common divisor of r_i, r_j, r_l, we have $m = \operatorname{lcm}\left(\frac{m}{r_i}, \frac{m}{r_j}, \frac{m}{r_l}\right) \mid \frac{m}{t}$, which implies $t = 1$.

Finally, using the first result, we have $s_{ij}\frac{m}{r_ir_j} = \frac{m}{\operatorname{lcm}(r_i,r_j)}$, which clearly divides both $\frac{m}{r_i}$ and $\frac{m}{r_j}$. Moreover, if t is any common divisor of $\frac{m}{r_i}$ and $\frac{m}{r_j}$, then both r_it and r_jt divide m, hence $t\cdot\operatorname{lcm}(r_i, r_j) = \operatorname{lcm}(r_it, r_jt) \mid m$. Thus $t \mid \frac{m}{\operatorname{lcm}(r_i,r_j)}$ and we are done. □

Proposition 5. *We have*

$$\operatorname{Gal}(k/\mathbb{Q}) \cong \{(\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4})|_k;\ 0 \le x_1 < a_1\frac{m}{r_1}, 0 \le x_2 < a_2\frac{m}{r_2s_{34}},$$
$$0 \le x_3 < a_3\frac{m}{r_3r_4}s_{34}, 0 \le x_4 < a_4\},$$

where each automorphism of k *determines the quadruple* (x_1, x_2, x_3, x_4) *uniquely.*

Proof. First off, the cardinalities of the sets on both sides agree. Now let ρ be any automorphism of k. Since $\operatorname{Gal}(k\cap K_4/\mathbb{Q})$ is a cyclic group of order a_4 (by lemma 2) generated by $\sigma_4|_{k\cap K_4}$ (as a quotient of $\operatorname{Gal}(K_4/\mathbb{Q}) = \langle\sigma_4|_{K_4}\rangle$), there must exist unique

$x_4 \in \mathbb{Z}$, $0 \leq x_4 < a_4$ such that ρ and $\sigma_4^{x_4}$ have the same restrictions to $k \cap K_4$. Therefore $\rho\sigma_4^{-x_4}\big|_k \in \mathrm{Gal}(k/k \cap K_4)$.

Next, $\mathrm{Gal}(k\cap K_3K_4/k\cap K_4)$ is a cyclic group of order $\frac{[k\cap K_3K_4:\mathbb{Q}]}{[k\cap K_4:\mathbb{Q}]} = a_3\frac{m}{r_3r_4}s_{34}$ (by Corollary 3) generated by $\sigma_3|_{k\cap K_3K_4}$ (as a quotient of $\mathrm{Gal}(K_3K_4/K_4) = \langle\sigma_3|_{K_3K_4}\rangle$), so there must exist unique $x_3 \in \mathbb{Z}$, $0 \leq x_3 < a_3\frac{m}{r_3r_4}s_{34}$ such that $\rho\sigma_4^{-x_4}$ and $\sigma_3^{x_3}$ have the same restriction to $k \cap K_3K_4$. Therefore $\rho\sigma_3^{-x_3}\sigma_4^{-x_4}\big|_k \in \mathrm{Gal}(k/k \cap K_3K_4)$.

Following the pattern, $\mathrm{Gal}(k \cap K_2K_3K_4/k \cap K_3K_4)$ is a cyclic group of order

$$\frac{[k \cap K_2K_3K_4 : \mathbb{Q}]}{[k \cap K_3K_4 : \mathbb{Q}]} = a_2\frac{m}{r_2s_{34}}$$

(by Corollary 3) generated by $\sigma_2|_{k\cap K_2K_3K_4}$ (as a quotient of

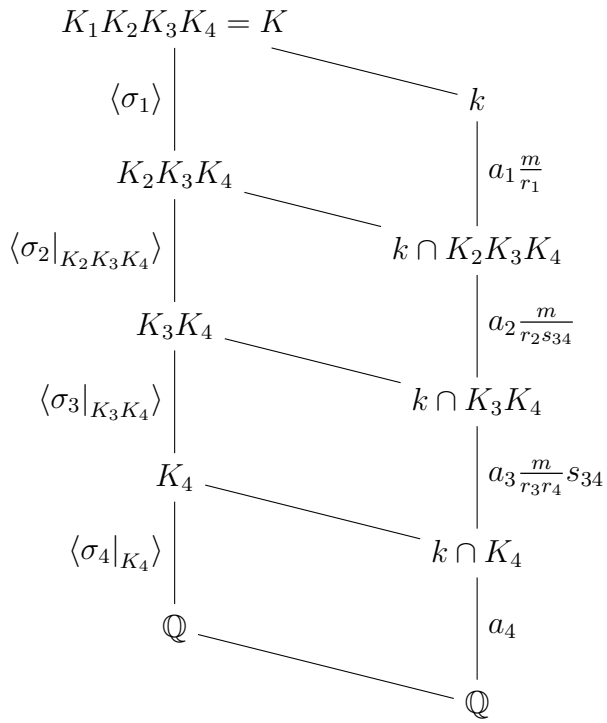
$$\mathrm{Gal}(K_2K_3K_4/K_3K_4) = \langle\sigma_2|_{K_2K_3K_4}\rangle),$$

so there must exist unique $x_2 \in \mathbb{Z}$, $0 \leq x_2 < a_2\frac{m}{r_2s_{34}}$ such that $\rho\sigma_3^{-x_3}\sigma_4^{-x_4}$ and $\sigma_2^{x_2}$ have the same restriction to $k \cap K_2K_3K_4$. Therefore $\rho\sigma_2^{-x_2}\sigma_3^{-x_3}\sigma_4^{-x_4}\big|_k \in \mathrm{Gal}(k/k \cap K_2K_3K_4)$.

Finally, we have

$$\mathrm{Gal}(k/k \cap K_2K_3K_4) \cong \mathrm{Gal}(kK_2K_3K_4/K_2K_3K_4) = \mathrm{Gal}(K_1K_2K_3K_4/K_2K_3K_4) = \langle\sigma_1\rangle$$

(using Lemma 1), where the isomorphism is given by restriction. Since the order of σ_1 is $a_1\frac{m}{r_1}$, it follows that there must exist unique $x_1 \in \mathbb{Z}$, $0 \leq x_1 < a_1\frac{m}{r_1}$ such that $\rho\sigma_2^{-x_2}\sigma_3^{-x_3}\sigma_4^{-x_4}$ and $\sigma_1^{x_1}$ have the same restriction to k. Thus $\rho = \sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}$ and the proof is finished.



□

Lemma 6. *Without loss of generality, we can assume* $\tau = \sigma_1^{a_1}\sigma_2^{a_2}\sigma_3^{a_3}\sigma_4^{a_4}$.

Proof. We know that $a_i = [T_i : \pi_i(H)]$, hence $\pi_i(\tau)$ generates a subgroup of T_i of index a_i. The cyclicity of T_i then implies that $\pi_i(\tau)$ must be the a_i-th power of some generater of T_i, WLOG σ_i. The statement now follows, because τ is determined by its four projections. □

3 The group of circular numbers

Recall that D^+, the non-torsion part of the group D of circular numbers of an abelian field k' (using Lettl's modification of Sinnott's definition), has one generator η_I for each nonempty subset $I \subseteq P$, where P is the set of ramified primes of k'.

More explicitly, if we let K_i' be the largest subfield of K' (the genus field of k') in which p_i is the only ramified prime for any $i \in I$, we have

$$\eta_I = \mathrm{N}_{\mathbb{Q}(\zeta_{\mathrm{cond}\left(\prod_{i\in I} K_i'\right)})/\left(\prod_{i\in I} K_i'\right)\cap k'} \left(1 - \zeta_{\mathrm{cond}\left(\prod_{i\in I} K_i'\right)}\right).$$

It is well known that D^+ is a $\mathbb{Z}[G]$-module of $\mathbb{Z}$-rank $[k : \mathbb{Q}] + |P| - 1$.

(In our case, we have $k' = k, K' = K, K_i' = K_i, |P| = 4$, $\eta := \eta_{\{1,2,3,4\}}$.)

Our goal will be to find a basis of D^+ (it can then be easily modified in order to obtain a basis of the group of circular units). The generators of D^+ are subject to norm relations that correspond to the sum of all elements of the respective inertia groups.

Let

$$R_i = \sum_{u=0}^{a_i-1} \sigma_i^u,\ N_i = \sum_{u=0}^{\frac{m}{r_i}-1} \sigma_i^{au_i}.$$

Then the norm operators from k to the maximal subfield ramified at less primes can be given as R_iN_i (i.e. the sum of all elements of T_i). If we denote the congruence corresponding to the canonical projection $\mathbb{Z}[G] \to \mathbb{Z}[G/H]$ by $\equiv$, then we have $N_4 \equiv \sum_{u=0}^{m-1} \sigma_1^{ua_1}\sigma_2^{ua_2}\sigma_3^{ua_3}$.

Moreover, we will denote the congruence corresponding to the composition of canonical projections $\mathbb{Z}[G] \to \mathbb{Z}[G/H] \to \mathbb{Z}[G/H]/(R_1N_1, R_2N_2, R_3N_3, R_4N_4)$ by $\sim$, where $(R_1N_1, R_2N_2, R_3N_3, R_4N_4)$ is the ideal generated in $\mathbb{Z}[G/H]$ by the images of the elements R_iN_i. When we apply any element of this ideal to the highest generator η, we will obtain a circular unit belonging to a field with less ramified primes. We will make use of this extensively.

To construct the basis of D^+, we can take the union of all bases for the fields

$$k \cap K_1K_2K_3, k \cap K_1K_2K_4, k \cap K_1K_3K_4, k \cap K_2K_3K_4$$

(which have three ramified primes, so we can use the results in [1]) and add in

$$c := [k:\mathbb{Q}] + 3 - \sum_{i,j,l}([k \cap K_iK_jK_l : \mathbb{Q}] + 2) - \sum_{i,j}([k \cap K_iK_j : \mathbb{Q}] + 1) - \sum_i [k \cap K_i : \mathbb{Q}]$$
$$= a_1a_2a_3a_4\frac{m^3}{r_1r_2r_3r_4} - \sum_{i,j,l} a_ia_ja_l\frac{m^2}{r_ir_jr_l} + \sum_{i,j} + a_ia_js_{ij}\frac{m}{r_ir_j} - \sum_i a_i + 1$$

(by the principle of inclusion and exclusion) conjugates of η. Then we will need to show how to obtain the missing conjugates of η using the relations

$$R_1N_1 \sim 0, R_2N_2 \sim 0, R_3N_3 \sim 0, R_4\sum_{u=0}^{m-1}\sigma_1^{ua_1}\sigma_2^{ua_2}\sigma_3^{ua_3} \sim 0.$$

We will always refer to the conjugates of η by their coordinates x_1, x_2, x_3, x_4 according to Proposition 5. This allows for geometric interpretation.

4 The case $r_1 = r_2 = a_3 = r_4 = 1$

(Note that in this case we have $s_{34} = 1$.)

We will add all the conjugates of η to our basis except the following cases:

- $x_1 = a_1m - 1$ or $x_2 = a_2m - 1$ or $x_1 = \frac{m}{r_3} - 1$,
- $a_1 \leq x_1 < a_1m - 1, a_2(m-1) - 1 \leq x_2 < a_2m - 1, 0 \leq x_3 < \frac{m}{r_3} - 1, x_4 = 0$,
- $0 \leq x_1 < a_1, a_2(m-1) \leq x_2 < a_2m - 1, 0 \leq x_3 < \frac{m}{r_3} - 1, x_4 = 0$.

These cases are all disjoint, so it's easy to see that the number of conjugaters of η that we chose is exactly

$$((a_1m-1)a_2(m-2) + (a_1m-1)(a_2-1) + a_1 + (a_4-1)(a_1m-1)(a_2m-1))\left(\frac{m}{r_3} - 1\right) = c.$$

First we will recover the cases $0 < x_4 < a_4$, $x_1 = a_1m - 1$ or $x_2 = a_2m - 1$ or $x_1 = \frac{m}{r_3} - 1$ using the relations $R_1N_1 \sim 0, R_2N_2 \sim 0, R_3N_3 \sim 0$. From now on, we only need to deal with the cases where $x_4 = 0$.

Next, we will recover the cases

$$x_1 = a_1m - 1, 0 \leq x_2 < a_2(m-1) - 1, 0 \leq x_3 < \frac{m}{r_3} - 1$$

using the relation $R_1N_1 \sim 0$ and subsequently the cases

$$0 \leq x_1 < a_1m - 1, 0 \leq x_2 < a_2(m-1) - 1, 0 \leq x_3 < \frac{m}{r_3} - 1$$

and

$$0 \leq x_1 < a_1 - 1, x_2 = a_2(m-1) - 1, 0 \leq x_3 < \frac{m}{r_3} - 1$$

using the relation $R_3 N_3 \sim 0$.

Next, we will sequentially recover all the cases

$$0 \leq x_1 < a_1 m - 1, a_2(m-1) \leq x_2 < a_2 m - 1, 0 \leq x_3 < \frac{m}{r_3} - 1$$

using the relation $R_4 \sum_{u=0}^{m-1} \sigma_1^{a_1 u} \sigma_2^{a_2 u} \sigma_3^u$. We can do this since any two conjugates of η used in this relation differ by at least a_2 in their second coordinate. After this, we can recover the cases

$$0 \leq x_1 < a_1, x_2 = a_2 m - 1, 0 \leq x_3 < \frac{m}{r_3} - 1$$

using the relation $R_2 N_2 \sim 0$.

Finally, we can use the relation $R_4 \sum_{u=0}^{m-1} \sigma_1^{a_1 u} \sigma_2^{a_2 u} \sigma_3^u$ to recover the cases

$$a_1 \leq x_1 < 2a_1, x_2 = a_2 m - 1, 0 \leq x_3 < \frac{m}{r_3} - 1$$

and subsequently $R_4 \sum_{u=0}^{m-1} \sigma_1^{a_1 u} \sigma_2^{a_2 u} \sigma_3^u$ to recover the cases

$$a_1 \leq x_1 < 2a_1, x_2 = a_2(m-1) - 1, 0 \leq x_3 < \frac{m}{r_3} - 1.$$

By repeating these two steps $(m-2)$ more times, increasing the first coordinate by a_1 each time, we will recover all the conjugates.

5 The case $a_1 = a_2 = r_3 = r_4 = 1$

In this case, using Lemma 4, we have

$$\begin{aligned} c = {} & a_3 \left(\frac{m}{r_1} - 1\right)\left(\frac{m}{r_2} - 1\right)(m-1) - \left(\frac{m}{r_1} - 1\right)\left(\frac{m}{r_2} - 1\right) + \gcd\left(\frac{m}{r_1}, \frac{m}{r_2}\right) - 1 \\ & + (a_4 - 1)\left(a_3 \left(\frac{m}{r_1} - 1\right)\left(\frac{m}{r_2} - 1\right) m - \left(\frac{m}{r_1} - 1\right)\left(\frac{m}{r_2} - 1\right)\right). \end{aligned}$$

We will add the following c conjugates of η to our basis:

- $0 \leq x_1 < a_1 \frac{m}{r_1} - 1, 0 \leq x_2 < a_2 \frac{m}{r_2} - 1, 0 \leq x_3 < a_3 m - 1, 0 < x_4 \leq a_4 m,$
- $0 \leq x_1 < a_1 \frac{m}{r_1} - 1, 0 \leq x_2 < a_2 \frac{m}{r_2} - 1, 0 \leq x_3 < a_3(m-1), x_4 = 0,$
- $a_1 \frac{m}{r_1} - (\gcd\left(\frac{m}{r_1}, \frac{m}{r_2}\right) - 1) \leq x_1 \leq a_1 \frac{m}{r_1} - 1, \leq x_2 = a_2 \frac{m}{r_2} - 1, x_3 = a_3 m - 1, x_4 = 0.$

First we will recover the cases $0 < x_4 < a_4$, $x_1 = a_1 m - 1$ or $x_2 = a_2 m - 1$ or $x_1 = \frac{m}{r_3} - 1$ using the relations $N_1 \sim 0, N_2 \sim 0, R_3 N_3 \sim 0$. From now on, we only need to deal with the cases where $x_4 = 0$.

Next, we will recover the cases $0 \leq x_3 < a_3(m-1)$, $x_1 = a_1 m - 1$ or $x_2 = a_2 m - 1$ using the relations $N_1 \sim 0, N_2 \sim 0$. Now we can also use the relation $R_4 \sum_{u=0}^{m-1} \sigma_1^u \sigma_2^u \sigma_3^{a_3 u}$ multiple times to recover the cases

$$0 \leq x_1 \leq a_1 \frac{m}{r_1} - 1, 0 \leq x_2 \leq a_2 \frac{m}{r_2} - 1, a_3(m-1) < x_3 < a_3 m - 1.$$

At this moment, we are only missing the cases

$$0 \leq x_1 \leq a_1 \frac{m}{r_1} - 1, 0 \leq x_2 \leq a_2 \frac{m}{r_2} - 1, x_3 = a_3(m-1) = x_3$$

and some of those with $x_3 = a_3 m - 1$. Let's focus on the second kind. The conjugates with $x_3 = a_3 m - 1$ (and $x_4 = 0$) can be visualized as a discrete rectangle with sides $\frac{m}{r_1}$ and $\frac{m}{r_2}$. It is easy to see that such a rectangle can be partitioned into $\gcd(\frac{m}{r_1}, \frac{m}{r_2})$ diagonals, each containing $\operatorname{lcm}(\frac{m}{r_1}, \frac{m}{r_2})$ elements (two conjugates lie in the same diagonal iff one can be obtained from the other as a multiple of $\sigma_1^v \sigma_2^v$ for some $v \in \mathbb{Z}$). Now consider the relations

$$T := -\left(\sigma_3^{a_3-1} R_4 \sum_{u=0}^{m-1} \sigma_1^u \sigma_2^u \sigma_3^{a_3 u}\right) - \sigma_1^{a_1 \frac{m}{r_1} - 2} \sigma_2^{a_2 \frac{m}{r_2} - 2} R_3 N_3$$

and

$$S_v := \sum_{u=0}^{v} \sigma_1^{-u} \sigma_2^{-u} T \text{ for } v \in \mathbb{Z}.$$

Clearly $T \sim 0, S_v \sim 0$ for all $v \in \mathbb{Z}$. Also note that for any v, S_v contains no conjugate with $x_3 = a_3(m-1)$ and contains exactly one conjugate with $x_3 = a_3 m - 1$ that we cannot recover yet minus $\sigma_3^{a_3 m - 1}$, and these two always lie on the same diagonal. Moreover, any conjugate sharing this diagonal can occur as the one with positive sign for suitable $v \in \mathbb{Z}$. Therefore, since we already have the conjugates

$$a_1 \frac{m}{r_1} - (\gcd\left(\frac{m}{r_1}, \frac{m}{r_2}\right) - 1) \leq x_1 \leq a_1 \frac{m}{r_1} - 1, \leq x_2 = a_2 \frac{m}{r_2} - 1, x_3 = a_3 m - 1$$

in our basis, we can recover all the conjugates that share the same diagonal with any of these.

Now we can recover all the conjugates with $x_3 = a_3 m - 1$ except $\operatorname{lcm}(\frac{m}{r_1}, \frac{m}{r_2}$ of them, which lie on the same diagonal. By using the relation $\sigma_1^{\gcd\left(\frac{m}{r_1}, \frac{m}{r_2}\right) - 1} (S_v - S_w) \sim 0$ for suitable $v, w \in \mathbb{Z}$, it is clear that we can generate the difference of any two conjugates lying

on this diagonal. Now note that in each column, there are exactly $\frac{\frac{m}{r_1}}{\gcd\left(\frac{m}{r_1},\frac{m}{r_2}\right)}$ conjugates lying on this diagonal, and in each row, there are exactly $\frac{\frac{m}{r_2}}{\gcd\left(\frac{m}{r_1},\frac{m}{r_2}\right)}$ conjugates lying on this diagonal. Moreover, we have

$$\gcd\left(\frac{\frac{m}{r_1}}{\gcd\left(\frac{m}{r_1},\frac{m}{r_2}\right)},\frac{\frac{m}{r_2}}{\gcd\left(\frac{m}{r_1},\frac{m}{r_2}\right)}\right)=1,$$

so there exists an integer $z>0$ such that

$$\frac{\frac{m}{r_2}}{\gcd\left(\frac{m}{r_1},\frac{m}{r_2}\right)}z\equiv 1 \pmod{\frac{\frac{m}{r_1}}{\gcd\left(\frac{m}{r_1},\frac{m}{r_2}\right)}}.$$

Using the observation above, we can generate $\frac{\frac{m}{r_2}}{\gcd\left(\frac{m}{r_1},\frac{m}{r_2}\right)}z$ differences of conjugates lying on the last diagonal in such a way that we will obtain $\frac{\frac{m}{r_2}}{\gcd\left(\frac{m}{r_1},\frac{m}{r_2}\right)}$ conjugates with a negative sign evenly distributed among the row $x_1=0$, $\frac{\frac{\frac{m}{r_2}}{\gcd\left(\frac{m}{r_1},\frac{m}{r_2}\right)}z-1}{\frac{\frac{m}{r_1}}{\gcd\left(\frac{m}{r_1},\frac{m}{r_2}\right)}}$ conjugates with a positive sign evenly distributed among the column $x_2=0$ and finally one conjugate with a positive sign with

$$x_1=a_1\frac{m}{r_1}-(\gcd\left(\frac{m}{r_1},\frac{m}{r_2}\right)-1)-1, x_2=\frac{m}{r_2}-1.$$

We can keep this last one and get rid of the rest using the relations $N_1\sim 0$, $N_2\sim 0$. Using this last one, we can generate the rest of its diagonal in the same way as above. Hence we have recovered all the conjugates with $x_3=a_3m-1$. Finally, using the relation $R_3N_3\sim 0$, we can now recover all the conjugates with $x_3=a_3(m-1)$ and we are done.

6 The module of relations

7 Construction of suitable abelian fields

Let $m,a_1,a_2,a_3,a_4,r_1,r_2,r_3,r_4$ be positive integers such that

$$m>1, r_i \mid m, \gcd(r_i,r_j,r_l)=1.$$

We will construct an infinite family of fields k that satisfy all of our assumptions such that these integers correspond to the parameters in our problem of the same name.

First, we will fix distinct primes p_1,p_2,p_3,p_4 such that $p_i\equiv 1 \pmod{2a_i\frac{m}{r_i}}$ (by Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many ways of doing

this). Then there exist even Dirichlet characters χ_i of conductors p_i and orders $a_i\frac{m}{r_i}$ (namely, these can be given as $\chi_i := \chi^{\frac{p_i-1}{a_i m_i/r_i}}$, where χ is any generator of the cyclic group $\widehat{(\mathbb{Z}/p_i\mathbb{Z})^\times}$ (note that $p_i > 2$)).

Now let K_i be the field associated to $\langle\chi_i\rangle$. Then K_i is real (because χ_i is even) and $\mathrm{Gal}(K_i/\mathbb{Q})$ is cyclic of order $a_i\frac{m}{r_i}$, say $\mathrm{Gal}(K_i/\mathbb{Q}) = \langle\sigma_i\rangle$. Moreover, since the conductors p_i are coprime, the group $\langle\chi_1,\chi_2,\chi_3,\chi_4\rangle$ corresponds to the compositum field $K = K_1K_2K_3K_4$. By the theory of Dirichlet characters, K is ramified exactly at primes p_i (with inertia subgroups isomorphic to $\mathrm{Gal}(K_i/\mathbb{Q})$) and

$$\mathrm{Gal}(K/\mathbb{Q}) = \mathrm{Gal}(K_1/\mathbb{Q})\mathrm{Gal}(K_2/\mathbb{Q})\mathrm{Gal}(K_3/\mathbb{Q})\mathrm{Gal}(K_4/\mathbb{Q}) = \langle\sigma_1,\sigma_2,\sigma_3,\sigma_4\rangle,$$

so that $[K:\mathbb{Q}] = a_1a_2a_3a_4\frac{m^4}{r_1r_2r_3r_4}$. Now let $\tau := \sigma_1^{a_1}\sigma_2^{a_2}\sigma_3^{a_3}\sigma_4^{a_4}$ and let k be the subfield of K fixed by τ. Since k is a subfield of a compositum of real fields, it must also be real. In order to reach our goal, we now only need to prove the following theorem (it is not hard to see that we could have used the results from Lemma 1 and Proposition 2 as definitions instead).

Theorem 7. *In the above notation, we have* $[K:k] = m$, $[K:kK_i] = r_i$, $[k\cap K_i:\mathbb{Q}] = a_i$ *and* $kK_iK_jK_l = K$ *(i.e. K is the genus field of k).*

Proof. Using Lemma 4 several times, we can compute

$$[K:k] = |\langle\tau\rangle| = \mathrm{lcm}\left(\frac{m}{r_i},\frac{m}{r_j},\frac{m}{r_l}\right) = m,$$

$$[K:kK_i] = |\langle\tau\rangle\cap\langle\sigma_j\sigma_l\sigma_h\rangle| = |\langle\tau^{a_im/r_i}\rangle| = r_i,$$

$$[k\cap K_i:/\mathbb{Q}] = [\langle\sigma_1,\sigma_2,\sigma_3,\sigma_4\rangle:\langle\tau,\sigma_j,\sigma_l,\sigma_h\rangle] = [\langle\sigma_1,\sigma_2,\sigma_3,\sigma_4\rangle:\langle\sigma_i^{a_i},\sigma_j,\sigma_l,\sigma_h\rangle] = a_i$$

and

$$[K:kK_iK_jK_l] = |\langle\tau\rangle\cap\langle\sigma_h\rangle| = |\langle\tau^{\mathrm{lcm}\left(\frac{m}{r_i},\frac{m}{r_j},\frac{m}{r_l}\right)}\rangle| = |\langle\tau^m\rangle| = 1.$$

□