Circular numbers of certain abelian fields

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Throughout this thesis, we will use the convention that whenever any of the indices i, j, l, h appear on the same line, they are pairwise distinct and moreover $1 \leq i, j, l, h \leq 4$, unless stated otherwise. Also for any $n \in \mathbb{N}$, ζ_n will denote a primitive n-th root of unity (WLOG we can take $\zeta_n = e^{2\pi i/n}$).

0.1 Preliminaries

Definition 0.1.1. An *abelian field* is a finite Galois extension of $\mathbb{Q}$ with an abelian Galois group.

Definition 0.1.2. The *genus field in the narrow sense* of an abelian field is its maximal extension which is abelian over $\mathbb{Q}$ and unramified at all finite primes.

Lemma 1. *If K is the genus field in the narrow sense of an abelian field k and P is the set of ramified primes of k, we have* $\mathrm{Gal}(K/\mathbb{Q}) \cong \prod_{p \in P} T_p$*, where T_p is the inertia subgroup of* $\mathrm{Gal}(K/\mathbb{Q})$ *corresponding to p.*

Proof. □

Theorem 2 (Kronecker-Weber)**.** *Every abelian field is a subfield of some cyclotomic field.*

Proof. See [4], page 319. □

Definition 0.1.3. Let k be an abelian field. The least number $n \in \mathbb{N}$ such that $k \subseteq \mathbb{Q}(\zeta_n)$ is called the conductor of k and denoted by cond k.

Definition 0.1.4. Let G be any group. The (integral) *group ring* $\mathbb{Z}[G]$ is the free $\mathbb{Z}$-module with basis G, which is made into a ring, extending linearly the group law on G.

Definition 0.1.5. An element α of a totally real number field K is called totally positive if for any embedding $\sigma : K \to \mathbb{R}$, we have $\sigma(\alpha) > 0$.

0.2 The group of circular numbers

Let k be a real abelian field, K its the genus field in the narrow sense, P is the set of ramified primes of k, K_p is the maximal subfield of K ramified only at $p \in P$. Since $\mathrm{Gal}(K/\mathbb{Q})$ has a natural action on K (given by evaluating an automorphism on an element), this makes K into a $\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]$-module.

Definition 0.2.1. The group $D(k)$ of circular numbers of k (using Lettl's modification of Sinnott's definition) is given as

$$D := \left\langle \{-1, \eta_I \middle| \emptyset \subsetneq I \subseteq P\} \right\rangle_{\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]},$$

where $\langle \dots \rangle_{\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]}$ means "generated as a $\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]$-submodule of K" and

$$\eta_I = \mathrm{N}_{\mathbb{Q}(\zeta_{\mathrm{cond}(\prod_{i\in I} K_i)})/(\prod_{i\in I} K_i)\cap k}\left(1-\zeta_{\mathrm{cond}\left(\prod_{i\in I} K_i\right)}\right),$$

where N denotes the norm operator and the product of fields denotes their compositum. The subset of totally positive elements of $D(k)$ will be denoted by $D^+(k)$.

Definition 0.2.2. The group $C(k)$ of circular numbers of k is $E(k)\cap D$, where $E(k)$ is the group of units of the ring of algebraic integers of k. The subset of totally positive elements of $C(k)$ will be denoted by $C^+(k)$.

In [2], it is proven that the previous definition of $C(k)$ gives the same group as Sinnott's original definition in [3]. One of the reasons that $C(k)$ is important is the following result, due to Sinnott:

Theorem 3. *The index $[E(k) : C(k)]$ is finite.*

Proof. See [3], Theorem 4.1. □

Lemma 4.

(i) *For $|I| > 1$, we have $\eta_I \in E(k)$.*

(ii) *For $I = \{p\}$, we have $\eta_I \notin E(k)$, but $\eta_I^{1-\sigma} \in E(k)$ for any $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$.*

Proof. This follows from [3], Lemma 4.1. □

Corollary 5. *We have*

$$C(k) = \left\langle \{-1, \eta_I \big| I \subseteq P, |I| \geq 2\} \cup \{\eta_I^{1-\sigma} \big| |I| = 1, \sigma \in \mathrm{Gal}(K/\mathbb{Q}))\} \right\rangle_{\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q}]}.$$

The next result shows that $D^+(k)$ and $C^+(k)$ are free $\mathbb{Z}$-modules.

Lemma 6. *$D^+(k)$ is a subgroup of $D(k)$ given as*

$$D^+(k) = \left\langle \eta_I \big| \emptyset \subsetneq I \subseteq P \right\rangle_{\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]},$$

hence canonically isomorphic to the non-torsion part of $D(k)$. Similarly, $C^+(k)$ is a subgroup of $C(k)$ given as

$$C^+(k) = \left\langle \{\eta_I \big| I \subseteq P, |I| \geq 2\} \cup \{\eta_I^{1-\sigma} \big| |I| = 1, \sigma \in \mathrm{Gal}(K/\mathbb{Q}))\} \right\rangle_{\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q}]}.$$

Proof. □

Proposition 7. *The $\mathbb{Z}$-rank of $D^+(k)$ is $[k : \mathbb{Q}] + |P| - 1$.*

Proof. □

0.3 Notation and assumptions

In the remainder of the thesis, we will fix k to be a real abelian field with exactly four ramified primes p_1, p_2, p_3, p_4 and we will abbreviate $D(k), D^+(k), C(k), C^+(k)$ simply as D, D^+, C, C^+.

Let K be the genus field in the narrow sense of k and let $G := \mathrm{Gal}(K/\mathbb{Q})$. Then by Lemma 1, we can identify G with the direct product $T_1 \times T_2 \times T_3 \times T_4$, where T_i is the inertia group corresponding the ramified prime p_i. Next, we will define:

- $H := \mathrm{Gal}(K/k)$,
- $m := |H|$,
- the canonical projections $\pi_i : G \to T_i$,
- $a_i := [T_i : \pi_i(H)]$,
- $r_i := |H \cap \ker \pi_i|$,
- $s_{ij} := |H \cap \ker(\pi_i \pi_j)|$,
- $n_i := \frac{m}{r_i}$,
- $\eta := \eta_{\{1234\}}$,
- K_i as the maximal subfield of K ramified only at p_i (so that

$$T_i = \mathrm{Gal}(K/K_j K_l K_h) \cong \mathrm{Gal}(K_i/\mathbb{Q}).)$$

We will assume the following:

- $K \neq k$,
- H is cyclic, generated by τ,
- each T_i is cyclic, generated by σ_i.

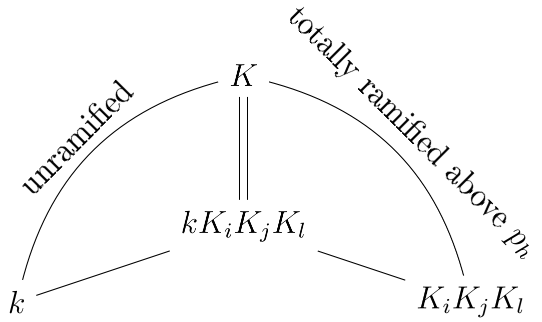
0.4 Auxiliary results

Lemma 8. *Without loss of generality, we can assume $\tau = \sigma_1^{a_1} \sigma_2^{a_2} \sigma_3^{a_3} \sigma_4^{a_4}$.*

Proof. We know that $a_i = [T_i : \pi_i(H)]$, hence $\pi_i(\tau)$ generates a subgroup of T_i of index a_i. The cyclicity of T_i then implies that $\pi_i(\tau)$ must be the a_i-th power of some generator of T_i, WLOG σ_i. The statement now follows, because τ is determined by its four projections. $\square$

Lemma 9. *We have $kK_i K_j K_l = K$ and $K_1 K_2 K_3 K_4 = K$.*

Proof. The extension $K/K_iK_jK_l$ is totally ramified at the prime ideals above p_h, so the same must be true for the extension $K/kK_iK_jK_l$. But since the extension K/k is unramified (by the definition of K), so is $K/kK_iK_jK_l$. Therefore $[K : kK_iK_jK_l] = 1$. The second claim follows from the facts $T_i = \mathrm{Gal}(K/K_jK_lK_h)$ and $G = T_1 \times T_2 \times T_3 \times T_4$. $\square$



Proposition 10. *We have* $a_i = [k \cap K_i : \mathbb{Q}]$, $r_i = [K : kK_i]$, $|T_i| = a_in_i$, $s_{ij} = [K : kK_iK_j]$. *Also* $[K_i : k \cap K_i] = n_i$, $[K_iK_j : k \cap K_iK_j] = \frac{m}{s_{ij}}$ *and* $[K_iK_jK_l : k \cap K_iK_jK_l] = m$.

Proof. Since

$$\begin{aligned}\mathrm{Gal}(K/K_i) &= \mathrm{Gal}(K/K_iK_jK_l \cap K_iK_jK_h \cap K_iK_lK_h)\\ &= \mathrm{Gal}(K/K_iK_jK_l) \cdot \mathrm{Gal}(K/K_iK_jK_h) \cdot \mathrm{Gal}(K/K_iK_lK_h) = T_jT_lT_h\end{aligned}$$

and $\mathrm{Gal}(K/k) = H$, it follows that $\mathrm{Gal}(K/k \cap K_i) = T_jT_lT_h \cdot H$. Now consider the short exact sequence

$$0 \to H \cap \ker \pi_i \to H \xrightarrow{\pi_i|_H} \pi_i(H) \to 0.$$

It follows that $|\pi_i(H)| = \frac{m}{r_i} = n_i$ and

$$\pi_i(H) \cong \frac{H}{H \cap \ker \pi_i} = \frac{H}{H \cap T_jT_lT_h} \cong \frac{T_jT_lT_h \cdot H}{T_jT_lT_h} = \frac{\mathrm{Gal}(K/k \cap K_i)}{\mathrm{Gal}(K/K_i)} \cong \mathrm{Gal}(K_i/k \cap K_i).$$

Therefore

$$[k \cap K_i : \mathbb{Q}] = \frac{|\mathrm{Gal}(K_i/\mathbb{Q})|}{|\mathrm{Gal}(K_i/k \cap K_i)|} = \frac{|T_i|}{|\pi_i(H)|} = a_i$$

and

$$[K : kK_i] = \frac{|\mathrm{Gal}(K/k)|}{|\mathrm{Gal}(kK_i/k)|} = \frac{|H|}{|\mathrm{Gal}(K_i/k \cap K_i)|} = \frac{m}{|\pi_i(H)|} = r_i.$$

Putting everything together, we obtain

$$|T_i| = [K_i : k \cap K_i] \cdot [k \cap K_i : \mathbb{Q}] = a_i|\pi_i(H)| = a_in_i.$$

Next, we also have

$$\begin{aligned}\mathrm{Gal}(K/K_iK_j) &= \mathrm{Gal}(K/K_iK_jK_l \cap K_iK_jK_h)\\ &= \mathrm{Gal}(K/K_iK_jK_l) \cdot \mathrm{Gal}(K/K_iK_jK_h) = T_lT_h\end{aligned}$$

so that $\mathrm{Gal}(K/k \cap K_iK_j) = T_lT_h \cdot H$. Thus we can consider the short exact sequence

$$0 \to H \cap \ker \pi_i\pi_j \to H \xrightarrow{\pi_i\pi_j|_H} \pi_i\pi_j(H) \to 0$$

to conclude that $|\pi_i\pi_j(H)| = \frac{m}{s_{ij}}$ and

$$\begin{aligned}\pi_i\pi_j(H) &\cong \frac{H}{H \cap \ker \pi_i\pi_j} = \frac{H}{H \cap T_lT_h} \cong \frac{T_lT_h \cdot H}{T_lT_h} \\ &\cong \frac{\mathrm{Gal}(K/k \cap K_iK_j)}{\mathrm{Gal}(K/K_iK_j)} \cong \mathrm{Gal}(K_iK_j/k \cap K_iK_j).\end{aligned}$$

Then it follows that

$$[K : kK_iK_j] = \frac{|\mathrm{Gal}(K/k)|}{|\mathrm{Gal}(kK_iK_j/k)|} = \frac{|H|}{|\mathrm{Gal}(K_iK_j/k \cap K_iK_j)|} = \frac{m}{|\pi_i\pi_j(H)|} = s_{ij}.$$

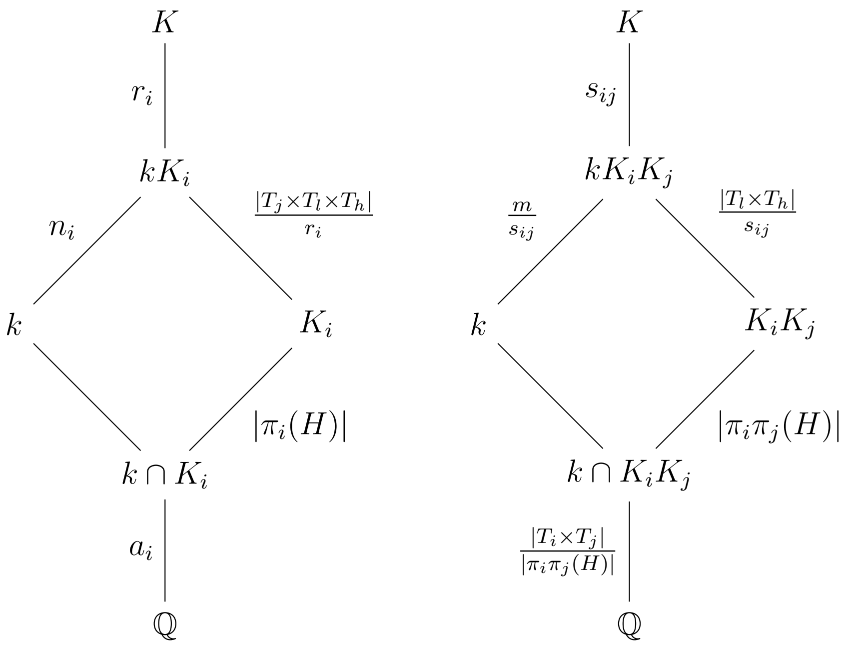
The last part of the statement is a consequence of Lemma 9, since we have

$$\mathrm{Gal}(K_iK_jK_l/k \cap K_iK_jK_l) \cong \mathrm{Gal}(kK_iK_jK_l/k) = \mathrm{Gal}(K/k) = H.$$

Finally note that in the same way as above, we could show that

$$\pi_i\pi_j\pi_l(H) \cong \frac{H}{H \cap T_h} \cong H$$

(since Lemma 9 implies that $|H \cap T_h| = 1$). □



$$\begin{tikzcd}[arrows=dash]
 & K = kK_iK_jK_l \arrow[dl, "m"'] \arrow[dr, "|T_h|"] & \\
k \arrow[dr] & & K_iK_jK_l \arrow[dl, "|\pi_i\pi_j\pi_h(H)|"] \\
 & k \cap K_iK_jK_l \arrow[d, "\frac{|T_i\times T_j\times T_l|}{|\pi_i\pi_j\pi_l(H)|}"'] & \\
 & \mathbb{Q} &
\end{tikzcd}$$

Corollary 11. *We have* $[k \cap K_iK_j : \mathbb{Q}] = a_ia_j\frac{m}{r_ir_j}s_{ij}$, $[k \cap K_iK_jK_l : \mathbb{Q}] = a_ia_ja_l\frac{m^2}{r_ir_jr_l}$ *and* $[k : \mathbb{Q}] = a_1a_2a_3a_4\frac{m^3}{r_1r_2r_3r_4}$.

Proof. This follows from the computations

$$[k \cap K_iK_j : \mathbb{Q}] = \frac{[K_iK_j : \mathbb{Q}]}{[K_iK_j : k \cap K_iK_j]} = \frac{|T_i| \cdot |T_j|}{m/s_{ij}} = a_ia_j\frac{m}{r_ir_j}s_{ij},$$

$$[k \cap K_iK_jK_l : \mathbb{Q}] = \frac{[K_iK_jK_l : \mathbb{Q}]}{[K_iK_jK_l : k \cap K_iK_jK_l]} = \frac{|T_i| \cdot |T_j| \cdot |T_l|}{m} = a_ia_ja_l\frac{m^2}{r_ir_jr_l}$$

and

$$[k : \mathbb{Q}] = \frac{[K : \mathbb{Q}]}{[K : k]} = \frac{|T_1| \cdot |T_2| \cdot |T_3| \cdot |T_4|}{m} = a_1a_2a_3a_4\frac{m^3}{r_1r_2r_3r_4}.$$

□

Lemma 12. *We have*

$$s_{ij} = \gcd(r_i, r_j), \gcd(r_i, r_j, r_l) = 1, \operatorname{lcm}(n_i, n_j, n_l) = m \text{ and } s_{ij}\frac{m}{r_ir_j} = \gcd(n_i, n_j).$$

Proof. It follows from Proposition 10 that $s_{ij} \mid r_i, s_{ij} \mid r_j$ and from its proof that

$$|\pi_i(H)| = n_i, \quad |\pi_i\pi_j(H)| = \frac{m}{s_{ij}} \text{ and } |\pi_i\pi_j\pi_l(H)| = m.$$

The cyclicity of H then implies

$$\frac{m}{s_{ij}} = |\pi_i\pi_j(H)| = |\langle\pi_i\pi_j(\tau)\rangle| = |\langle\pi_i(\tau)\pi_j(\tau)\rangle| = \operatorname{lcm}(n_i, n_j),$$

because $\langle \pi_i(\tau) \rangle = \pi_i(H)$ and any power of the product $\pi_i(\tau)\pi_j(\tau)$ is trivial if and only if the same power of both its factors is (since G is the direct product of the T_i's). Now for any common divisor t of r_i, r_j, we have

$$\frac{m}{s_{ij}} = \mathrm{lcm}\,(n_i, n_j) = \mathrm{lcm}\left(\frac{m}{r_i}, \frac{m}{r_j}\right) \mid \frac{m}{t},$$

which implies $t \mid s_{ij}$. Hence $s_{ij} = \gcd(r_i, r_j)$.

Similarly, we have

$$m = |\pi_i\pi_j\pi_l(H)| = |\langle \pi_i\pi_j\pi_l(\tau)\rangle| = |\langle \pi_i(\tau)\pi_j(\tau)\pi_l(\tau)\rangle| = \mathrm{lcm}(n_i, n_j, n_l),$$

so if t is any common divisor of r_i, r_j, r_l, we have

$$m = \mathrm{lcm}(n_i, n_j, n_l) = \mathrm{lcm}\left(\frac{m}{r_i}, \frac{m}{r_j}, \frac{m}{r_l}\right) \mid \frac{m}{t},$$

which implies $t = 1$. This implies both $m = \mathrm{lcm}(n_i, n_j, n_l)$ and $\gcd(r_i, r_j, r_l) = 1$ (in fact, these are equivalent).

Finally, using the first result, we have

$$s_{ij}\frac{m}{r_ir_j} = \frac{m}{r_ir_j/s_{ij}} = \frac{m}{\mathrm{lcm}(r_i, r_j)},$$

which clearly divides both $\frac{m}{r_i} = n_i$ and $\frac{m}{r_j} = n_j$. Moreover, if t is any common divisor of $n_i = \frac{m}{r_i}$ and $n_j = \frac{m}{r_j}$, then both r_it and r_jt divide m, hence $t \cdot \mathrm{lcm}(r_i, r_j) = \mathrm{lcm}(r_it, r_jt) \mid m$. Thus $t \mid \frac{m}{\mathrm{lcm}(r_i, r_j)}$ and we are done. $\square$

Proposition 13. *We have*

$$\begin{aligned}\mathrm{Gal}(k/\mathbb{Q}) \cong \{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}|_k;\ & 0 \le x_1 < a_1\frac{m}{r_1}, 0 \le x_2 < a_2\frac{m}{r_2s_{34}},\\ & 0 \le x_3 < a_3\frac{m}{r_3r_4}s_{34}, 0 \le x_4 < a_4\},\end{aligned}$$

where each automorphism of k determines the quadruple (x_1, x_2, x_3, x_4) uniquely.

Proof. First note that by Lemma 12, we have

$$a_3\frac{m}{r_3r_4}s_{34} = a_3\gcd(n_3, n_4) \in \mathbb{N}$$

and

$$a_2\frac{m}{r_2s_{34}} = a_2\mathrm{lcm}(r_3, r_4)\frac{m}{r_2r_3r_4} \in \mathbb{N}$$

(this follows from $r_i \mid m$ and $\gcd(r_2, r_3, r_4) = 1$), so the expressions make sense. By Corollary 11, the set on the right hand side has at most $|\mathrm{Gal}(k/\mathbb{Q})|$ elements. Now let ρ be any automorphism of k. If we can show that ρ determines the quadruple (x_1, x_2, x_3, x_4)

belonging to the set on the right hand side uniquely, it will follow that the cardinalities agree and we will be done. Since $\mathrm{Gal}(k \cap K_4/\mathbb{Q})$ is a cyclic group of order a_4 (by Lemma 10) generated by $\sigma_4|_{k \cap K_4}$ (as a quotient of $\mathrm{Gal}(K_4/\mathbb{Q}) = \langle \sigma_4|_{K_4} \rangle$), there must exist a unique $x_4 \in \mathbb{Z}$, $0 \le x_4 < a_4$ such that ρ and $\sigma_4^{x_4}$ have the same restrictions to $k \cap K_4$. Therefore $\rho\sigma_4^{-x_4}\big|_k \in \mathrm{Gal}(k/k \cap K_4)$.

Next, $\mathrm{Gal}(k \cap K_3K_4/k \cap K_4)$ is a cyclic group of order $\frac{[k \cap K_3K_4:\mathbb{Q}]}{[k \cap K_4:\mathbb{Q}]} = a_3 \frac{m}{r_3 r_4} s_{34}$ (by Corollary 11) generated by $\sigma_3|_{k \cap K_3K_4}$ (as it is isomorphic by restriction to $\mathrm{Gal}((k \cap K_3K_4)K_4/K_4)$, which is a quotient of $\mathrm{Gal}(K_3K_4/K_4) = \langle \sigma_3|_{K_3K_4} \rangle$), so there must exist a unique $x_3 \in \mathbb{Z}$, $0 \le x_3 < a_3 \frac{m}{r_3 r_4} s_{34}$ such that $\rho\sigma_4^{-x_4}\big|_k$ and $\sigma_3^{x_3}$ have the same restriction to $k \cap K_3K_4$. Therefore $\rho\sigma_3^{-x_3}\sigma_4^{-x_4}\big|_k \in \mathrm{Gal}(k/k \cap K_3K_4)$.

Following the pattern, $\mathrm{Gal}(k \cap K_2K_3K_4/k \cap K_3K_4)$ is a cyclic group of order

$$\frac{[k \cap K_2K_3K_4 : \mathbb{Q}]}{[k \cap K_3K_4 : \mathbb{Q}]} = a_2 \frac{m}{r_2 s_{34}}$$

(by Corollary 11) generated by $\sigma_2|_{k \cap K_2K_3K_4}$ (as it is isomorphic by restriction to $\mathrm{Gal}((k \cap K_2K_3K_4)K_3K_4/K_3K_4)$, which is a quotient of

$$\mathrm{Gal}(K_2K_3K_4/K_3K_4) = \langle \sigma_2|_{K_2K_3K_4} \rangle),$$

so there must exist a unique $x_2 \in \mathbb{Z}$, $0 \le x_2 < a_2 \frac{m}{r_2 s_{34}}$ such that $\rho\sigma_3^{-x_3}\sigma_4^{-x_4}\big|_k$ and $\sigma_2^{x_2}$ have the same restriction to $k \cap K_2K_3K_4$. Therefore $\rho\sigma_2^{-x_2}\sigma_3^{-x_3}\sigma_4^{-x_4}\big|_k \in \mathrm{Gal}(k/k \cap K_2K_3K_4)$.

Finally, we have

$$\mathrm{Gal}(k/k \cap K_2K_3K_4) \cong \mathrm{Gal}(kK_2K_3K_4/K_2K_3K_4) = \mathrm{Gal}(K_1K_2K_3K_4/K_2K_3K_4) = \langle \sigma_1 \rangle$$

(using Lemma 9), where the isomorphism is given by restriction. Since the order of σ_1 is $a_1 \frac{m}{r_1}$, it follows that there must exist a unique $x_1 \in \mathbb{Z}$, $0 \le x_1 < a_1 \frac{m}{r_1}$ such that $\rho\sigma_2^{-x_2}\sigma_3^{-x_3}\sigma_4^{-x_4}\big|_k$ and $\sigma_1^{x_1}$ have the same restriction to k. Thus $\rho = \sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}|_k$ and the proof is finished.

$$
\begin{tikzcd}[column sep=large]
K_1K_2K_3K_4 = K \arrow[dd, dash, "\langle\sigma_1\rangle"'] \arrow[dr, dash] & \\
 & k \arrow[dd, dash, "a_1\frac{m}{r_1}"] \\
K_2K_3K_4 \arrow[dd, dash, "\langle\sigma_2|_{K_2K_3K_4}\rangle"'] \arrow[dr, dash] & \\
 & k \cap K_2K_3K_4 \arrow[dd, dash, "a_2\frac{m}{r_2 s_{34}}"] \\
K_3K_4 \arrow[dd, dash, "\langle\sigma_3|_{K_3K_4}\rangle"'] \arrow[dr, dash] & \\
 & k \cap K_3K_4 \arrow[dd, dash, "a_3\frac{m}{r_3r_4}s_{34}"] \\
K_4 \arrow[dd, dash, "\langle\sigma_4|_{K_4}\rangle"'] \arrow[dr, dash] & \\
 & k \cap K_4 \arrow[dd, dash, "a_4"] \\
\mathbb{Q} \arrow[dr, dash] & \\
 & \mathbb{Q}
\end{tikzcd}
$$

□

0.5 General strategy

Our goal will be to find a basis of D^+ (it can then be easily modified in order to obtain a basis of C^+). The generators of D^+ are subject to norm relations that correspond to the sum of all elements of the respective inertia groups T_i. Namely, let

$$R_i = \sum_{u=0}^{a_i-1} \sigma_i^u, \; N_i = \sum_{u=0}^{n_i-1} \sigma_i^{ua_i}.$$

Then the norm operators from k to a maximal subfield ramified at three primes can be given as R_iN_i (i.e. the sum of all elements of T_i). If we denote the congruence corresponding to the canonical projection $\mathbb{Z}[G] \to \mathbb{Z}[G/H]$ by $\equiv$, then we have (using Lemma 8)

$$N_4 \equiv \sum_{u=0}^{n_4-1} \sigma_1^{ua_1}\sigma_2^{ua_2}\sigma_3^{ua_3}.$$

Note that any subgroup of k^* is naturally a $\mathbb{Z}[G/H]$-module, since the action of H on k is trivial.

Moreover, we will denote the congruence corresponding to the composition of canonical projections

$$\mathbb{Z}[G] \to \mathbb{Z}[G/H] \to \mathbb{Z}[G/H]/(R_1N_1, R_2N_2, R_3N_3, R_4N_4)$$

by $\sim$, where $(R_1N_1, R_2N_2, R_3N_3, R_4N_4)$ is the ideal generated in $\mathbb{Z}[G/H]$ by the images of the elements R_iN_i. When we apply any element of this ideal to the highest generator η, we will obtain a multiplicative $\mathbb{Z}$-linear combination of circular units belonging to subfields with less ramified primes. We will make use of this extensively.

Lemma 14. *The fields*

$$k \cap K_1K_2K_3, k \cap K_1K_2K_4, k \cap K_1K_3K_4, k \cap K_2K_3K_4$$

satisfy the assumptions of [1].

Proof. It's clear that these fields are all real, abelian (their Galois groups are quotients of G) and ramified at three primes.... $\square$

Using the results in [1], we can thus take the bases of

$$D^+(k \cap K_1K_2K_3), D^+(k \cap K_1K_2K_4), D^+(k \cap K_1K_3K_4), D^+(k \cap K_2K_3K_4)$$

and we will denote their union by B_D. Analogously, we can take bases of

$$C^+(k \cap K_1K_2K_3), C^+(k \cap K_1K_2K_4), C^+(k \cap K_1K_3K_4), C^+(k \cap K_2K_3K_4)$$

and denote their union by B_C.

To construct a basis of D^+ (resp. C^+), we will take the union of B_D (resp. B_C) with a set of suitably chosen conjugates of the highest generator η. In order to have a chance to obtain a basis, this set should contain

$$\begin{aligned} N :&= [k:\mathbb{Q}] + 4 - 1 - |B_D| \\ &= [k:\mathbb{Q}] + 3 - \sum_{i,j,l}([k \cap K_iK_jK_l : \mathbb{Q}] + 2) + \sum_{i,j}([k \cap K_iK_j : \mathbb{Q}] + 1) - \sum_i [k \cap K_i : \mathbb{Q}] \\ &= a_1a_2a_3a_4 \frac{m^3}{r_1r_2r_3r_4} - \sum_{i,j,l} a_ia_ja_l \frac{m^2}{r_ir_jr_l} + \sum_{i,j} a_ia_js_{ij}\frac{m}{r_ir_j} - \sum_i a_i + 1 \end{aligned}$$

by Proposition 7 and using the principle of inclusion and exclusion (due to the fact that these bases were constructed "inductively").

We cannot guarantee at the moment that the union of all these conjugates is not linearly dependent, but if we will show how to obtain all the missing conjugates of η using the relations

$$R_1N_1 \sim 0, R_2N_2 \sim 0, R_3N_3 \sim 0, R_4 \sum_{u=0}^{n_4-1} \sigma_1^{ua_1}\sigma_2^{ua_2}\sigma_3^{ua_3} \sim 0,$$

it will follow that we really have a basis.

We will always refer to the conjugates of η by their coordinates x_1, x_2, x_3, x_4 according to Proposition 13. This allows us to visualise $\mathrm{Gal}(k/\mathbb{Q})$ geometrically as a discrete (at most) four-dimensional cuboid.

0.6 The case $r_1 = r_2 = r_3 = r_4 = 1$

In this case, we have

$$\mathrm{Gal}(k/\mathbb{Q}) \cong \{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}|_k;\ 0 \le x_1 < a_1m, 0 \le x_2 < a_2m, 0 \le x_3 < a_3m, 0 \le x_4 < a_4\},$$
$$s_{12} = s_{13} = s_{14} = s_{23} = s_{24} = s_{34} = 1,$$
$$R_1N_1 \sim 0, R_2N_2 \sim 0, R_3N_3 \sim 0, R_4 \sum_{u=0}^{m-1} \sigma_1^{a_1u}\sigma_2^{a_2u}\sigma_3^{a_3u} \sim 0$$

and

$$\begin{aligned} N &= a_1a_2a_3a_4m^3 - (a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4)m^2 \\ &\quad + (a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)m - a_1 - a_2 - a_3 - a_4 + 1. \end{aligned}$$

We will define B_1 as the set of the following N conjugates $\eta^{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}}$:

Thus we have proven the following theorem:

Theorem 15. *Under the assumptions on page 0.3, the set $B_1 \cup B_D$ forms a basis of D^+ and the set $B_1 \cup B_C$ forms a basis of C^+.*

0.7 The case $r_1 = r_2 = a_3 = r_4 = 1$

In this case, we have

$$\mathrm{Gal}(k/\mathbb{Q}) \cong \{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}|_k;\ 0 \le x_1 < a_1m, 0 \le x_2 < a_2m, 0 \le x_3 < n_3, 0 \le x_4 < a_4\},$$
$$s_{12} = s_{13} = s_{14} = s_{23} = s_{24} = s_{34} = 1,$$
$$R_1N_1 \sim 0, R_2N_2 \sim 0, N_3 \sim 0, R_4 \sum_{u=0}^{m-1} \sigma_1^{a_1u}\sigma_2^{a_2u}\sigma_3^{u} \sim 0$$

and

$$\begin{aligned} N &= a_1a_2a_4\frac{m^3}{r_3} - a_1a_2a_4m^2 - (a_1a_2 + a_1a_4 + a_2a_4)\frac{m^2}{r_3} \\ &\quad + (a_1a_2 + a_1a_4 + a_2a_4)m + (a_1 + a_2 + a_4)\frac{m}{r_3} - a_1 - a_2 - a_4 \\ &= (n_3 - 1)(a_1a_2a_4m^2 - (a_1a_2 + a_1a_4 + a_2a_4)m + a_1 + a_2 + a_4) \\ &= (n_3 - 1)(a_1a_2m^2 - (a_1a_2 + a_1 + a_2)m + a_1 + a_2 + 1) \\ &\quad + (n_3 - 1)(a_4 - 1)(a_1a_2m^2 - a_1m - a_2m + 1) \\ &= (n_3 - 1)(a_4 - 1)(a_1m - 1)(a_2m - 1) + (n_3 - 1)(a_1m - 1)(a_2m - 1 - a_2) + (n_3 - 1)a_1 \end{aligned}$$

We will define B_2 as the set of the following N conjugates $\eta^{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}}$:

- $0 \le x_1 < a_1m - 1, 0 \le x_2 < a_2m - 1, 0 \le x_3 < n_3 - 1, 1 \le x_4 < a_4,$
- $0 \le x_1 < a_1m - 1, a_2 < x_2 < a_2m, 1 \le x_3 < n_3, x_4 = 0,$
- $0 \le x_1 < a_1, x_2 = a_2, 1 \le x_3 < n_3, x_4 = 0.$

First we will recover the cases $0 < x_4 < a_4$, $x_1 = a_1m - 1$ or $x_2 = a_2m - 1$ or $x_3 = n_3 - 1$ using the relations $R_1N_1 \sim 0, R_2N_2 \sim 0, N_3 \sim 0$. From now on, we only need to deal with the cases where $x_4 = 0$.

Next, we will recover the cases

$$x_1 = a_1m - 1, a_2 < x_2 < a_2m, 1 \leq x_3 < n_3$$

using the relation $R_1N_1 \sim 0$ and subsequently the cases

$$0 \leq x_1 < a_1m, a_2 < x_2 < a_2m, x_3 = 0$$

and

$$0 \leq x_1 < a_1, x_2 = a_2, x_3 = 0$$

using the relation $N_3 \sim 0$.

At this moment, we are only missing the conjugates $\eta^{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}}$ with

$$0 \leq x_1 < a_1m, 0 \leq x_2 < a_2, 0 \leq x_3 < n_3$$

and

$$a_1 \leq x_1 < a_1m, x_2 = a_2, 0 \leq x_3 < n_3.$$

Next, we will recover all the cases

$$0 \leq x_1 < a_1m - 1, 1 \leq x_2 < a_2, 0 \leq x_3 < n_3, x_4 = 0$$

and (note that $2a_1 \leq a_1m$, since $m > 1$)

$$a_1 \leq x_1 < 2a_1, x_2 = 0, 0 \leq x_3 < n_3, x_4 = 0$$

using the relation $R_4 \sum_{u=0}^{m-1} \sigma_1^{a_1u}\sigma_2^{a_2u}\sigma_3^{u}$, due to the fact that for any two different conjugates of η used in this relation, the difference of their exponents of σ_2 is divisible by a_2 (and we have already recovered all of them except precisely one).

Finally, we will use induction with respect to $v = 0, 1, \ldots, m-1$ to show we can recover the conjugates $\eta^{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}}$ with

$$va_1 \leq x_1 < (v+1)a_1, x_2 = a_2, 0 \leq x_3 < n_3, x_4 = 0$$

and

$$(v+1)a_1 \leq x_1 < (v+2)a_1, x_2 = 0, 0 \leq x_3 < n_3, x_4 = 0.$$

The basis step $v = 0$ has already been done. Now suppose the statement is true for $0 < v < m - 1$. Then using the relation $R_2N_2 \sim 0$, we can recover the conjugates with

$$(v+1)a_1 \leq x_1 < (v+2)a_1, x_2 = a_2, 0 \leq x_3 < n_3, x_4 = 0$$

and subsequently we can recover the conjugates with

$$(v+2)a_1 \le x_1 < (v+3)a_1, x_2 = 0, 0 \le x_3 < n_3, x_4 = 0$$

using the relation $R_4 \sum_{u=0}^{m-1} \sigma_1^{a_1 u}\sigma_2^{a_2 u}\sigma_3^{u}$, again due to the fact that for any two different conjugates of η used in this relation, the difference of their exponents of σ_2 is divisible by a_2 (and we have already recovered all of them except precisely one). Therefore the induction is complete and we have recovered all conjugates of η. Thus we have proven the following theorem:

Theorem 16. *Under the assumptions on page 0.3, the set $B_2 \cup B_D$ forms a basis of D^+ and the set $B_2 \cup B_C$ forms a basis of C^+.*

0.8 The case $a_1 = a_2 = r_3 = r_4 = 1$

In this case, we have

$$\mathrm{Gal}(k/\mathbb{Q}) \cong \{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}|_k;\ 0 \le x_1 < n_1, 0 \le x_2 < n_2, 0 \le x_3 < a_3 m, 0 \le x_4 < a_4\},$$

$$s_{13} = s_{14} = s_{23} = s_{24} = s_{34} = 1$$

and

$$N_1 \sim 0, N_2 \sim 0, R_3 N_3 \sim 0, R_4 \sum_{u=0}^{m-1} \sigma_1^{u}\sigma_2^{u}\sigma_3^{a_3 u} \sim 0.$$

Moreover, by Lemma 12, we have $s_{12}\frac{m}{r_1 r_2} = \gcd(n_1, n_2)$, hence

$$\begin{aligned}
N &= a_3 a_4 \frac{m^3}{r_1 r_2} - a_3 \frac{m^2}{r_1 r_2} - a_4 \frac{m^2}{r_1 r_2} - a_3 a_4 \left(\frac{m^2}{r_1} + \frac{m^2}{r_2}\right) + s_{12}\frac{m}{r_1 r_2} \\
&+ a_3(n_1+n_2) + a_4(n_1+n_2) + a_3 a_4 m - a_3 - a_4 - 1 \\
&= a_3\left(m n_1 n_2 - n_1 n_2 - m n_1 - m n_2 + n_1 + n_2 + m - 1\right) - n_1 n_2 + n_1 + n_2 - 1 \\
&+ (a_4 - 1)(a_3\left(m n_1 n_2 - m n_1 - m n_2 + m\right) - n_1 n_2 + n_1 + n_2 - 1) + \gcd(n_1, n_2) - 1 \\
&= a_3\left(m-1\right)\left(n_1-1\right)\left(n_2-1\right) - \left(n_1-1\right)\left(n_2-1\right) \\
&+ \left(a_4-1\right)\left(a_3 m\left(n_1-1\right)\left(n_2-1\right) - \left(n_1-1\right)\left(n_2-1\right)\right) + \gcd(n_1,n_2) - 1 \\
&= (n_1-1)(n_2-1)(a_3(m-1)-1) \\
&+ (a_4-1)(n_1-1)(n_2-1)(a_3 m - 1) + \gcd(n_1,n_2) - 1.
\end{aligned}$$

We will define B_3 as the set of the following N conjugates $\eta^{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}}$:

- $0 \le x_1 < n_1 - 1, 0 \le x_2 < n_2 - 1, 0 \le x_3 < a_3 m - 1, 0 < x_4 \le a_4 - 1,$
- $0 \le x_1 < n_1 - 1, 0 \le x_2 < n_2 - 1, a_3 < x_3 < a_3 m, x_4 = 0$
- $1 \le x_1 < \gcd(n_1, n_2), x_2 = 0, x_3 = 0, x_4 = 0.$

First we will recover the cases $0 < x_4 < a_4$, $x_1 = n_1 - 1$ or $x_2 = n_2 - 1$ or $x_3 = a_3 m - 1$ using the relations $N_1 \sim 0, N_2 \sim 0, R_3 N_3 \sim 0$. From now on, we only need to deal with the cases where $x_4 = 0$.

Next, we will recover the cases $x_4 = 0, a_3 < x_3 < a_3 m$, $x_1 = n_1 - 1$ or $x_2 = n_2 - 1$ using the relations $N_1 \sim 0, N_2 \sim 0$. Now note that the exponents of σ_3 in $R_4 \sum_{u=0}^{m-1} \sigma_1^u \sigma_2^u \sigma_3^{a_3 u} \sim 0$ are pairwise congruent modulo a_3. Since for any $1 \leq v < a_3$, we have already recovered all the conjugates with $x_3 \equiv v \pmod{a_3}$ except for one, we can also use the relation $R_4 \sum_{u=0}^{m-1} \sigma_1^u \sigma_2^u \sigma_3^{a_3 u} \sim 0$ several times to recover the cases

$$0 \leq x_1 < n_1, 0 \leq x_2 < n_2, 1 \leq x_3 < a_3, x_4 = 0$$

as well.

At this moment, we are only missing the conjugates $\eta^{\sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{a_3}}$ for all

$$0 \leq x_1 < n_1, 0 \leq x_2 < n_2$$

and among the conjugates $\eta^{\sigma_1^{x_1} \sigma_2^{x_2}}$ we have only those with $0 < x_1 < gcd(n_1, n_2), x_2 = 0$ We will focus on recovering the remaining conjugates $\eta^{\sigma_1^{x_1} \sigma_2^{x_2}}$, because once we have those, we can recover those with $x_3 = a_3, x_4 = 0$ just by using the relation $R_3 N_3 \sim 0$.

Let Q' be the quotient $\mathbb{Z}[G]$-module

$$D^+ / \big\langle \{\eta_I \big| \emptyset \subsetneq I \subsetneq P\} \big\rangle_{\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]}$$

and let Q be the quotient $\mathbb{Z}$-module of Q' by the conjugates we have already recovered, i.e.

$$\begin{aligned} Q := Q' / \big\langle \{\eta^{\sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{x_3} \sigma_4^{x_4}}; \quad & 0 \leq x_1 < n_1, 0 \leq x_2 < n_2, 0 \leq x_3 < a_3 m, 0 < x_4 < a_4, \\ & \text{or } 0 \leq x_1 < n_1, 0 \leq x_2 < n_2, 1 \leq x_3 < a_3 m, x_3 \neq a_3, x_4 = 0, \\ & \text{or } 1 \leq x_1 < \gcd(n_1, n_2), x_2 = x_3 = x_4 = 0\} \big\rangle_{\mathbb{Z}}. \end{aligned}$$

We will write Q additively, denoting the class of η by μ, hence for any $\rho \in \mathrm{Gal}(k/\mathbb{Q})$ or $\rho \in \mathrm{Gal}(K/\mathbb{Q})$, denoting the class of η^ρ in Q by $\rho \cdot \mu$. Showing that we have indeed chosen a basis now amounts to showing that Q is trivial. Since

$$0 = \sigma_1^{x_1} \sigma_2^{x_2} R_3 N_3 \cdot \mu = \sigma_1^{x_1} \sigma_2^{x_2} \cdot \mu + \sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{a_3} \cdot \mu$$

for any $x_1, x_2 \in \mathbb{Z}$, this is equivalent with showing that $\sigma_1^{x_1} \sigma_2^{x_2} \cdot \mu = 0$ for any $0 \leq x_1 < n_1, 0 \leq x_2 < n_2$.

Lemma 17. *In Q, we have*

$$\sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)\cdot\mu = 0$$

for any $x_1, x_2 \in \mathbb{Z}$.

Proof. Using the fact that the order of σ_3 is $a_3 m$, we have

$$\begin{aligned}
0 &\sim \sigma_1^{x_1}\sigma_2^{x_2}\left(R_3R_4\sum_{u=0}^{m-1}\sigma_1^u\sigma_2^u\sigma_3^{a_3u} - \sigma_1\sigma_2R_4R_3N_3\right) = \sigma_1^{x_1}\sigma_2^{x_2}R_3R_4\sum_{u=0}^{m-1}\left(\sigma_1^u\sigma_2^u - \sigma_1\sigma_2\right)\sigma_3^{a_3u} \\
&= \sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)R_3R_4 + \sigma_1^{x_1}\sigma_2^{x_2}R_3R_4\sum_{u=2}^{m-1}\left(\sigma_1^u\sigma_2^u - \sigma_1\sigma_2\right)\sigma_3^{a_3u} \\
&= \sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2) + \sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)R_3\sum_{u=1}^{a_4-1}\sigma_4^u \\
&+ \sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)\sum_{u=1}^{a_3-1}\sigma_3^u + \sigma_1^{x_1}\sigma_2^{x_2}R_3R_4\sum_{u=2}^{m-1}\left(\sigma_1^u\sigma_2^u - \sigma_1\sigma_2\right)\sigma_3^{a_3u}
\end{aligned}$$

Since all the summands in the expression

$$\sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)R_3\sum_{u=1}^{a_4-1}\sigma_4^u + \sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)\sum_{u=1}^{a_3-1}\sigma_3^u + \sigma_1^{x_1}\sigma_2^{x_2}R_3R_4\sum_{u=2}^{m-1}\left(\sigma_1^u\sigma_2^u - \sigma_1\sigma_2\right)\sigma_3^{a_3u}$$

have either $x_4 > 0$ or $1 \leq x_3 < a_3m, x_3 \neq a_3$ (where x_3 and x_4 denote the respective exponents of σ_3 and σ_4 in each term), the result of their action on μ becomes trivial in Q, which yields the result. □

Lemma 18. *For any $0 \leq x_1 < n_1, 0 \leq x_2 < n_2$, we have*

$$\sigma_1^{x_1}\sigma_2^{x_2}\cdot\mu = \begin{cases}\mu & \text{if } x_1 \equiv x_2 \pmod{\gcd(n_1,n_2)} \\ 0 & \text{otherwise.}\end{cases}$$

Proof. First we will prove that for any $1 \leq u < \gcd(n_1, n_2)$ and $0 \leq v < \operatorname{lcm}(n_1, n_2)$, we have

$$\sigma_1^{u+v}\sigma_2^v\cdot\mu = 0 \tag{1}$$

and

$$\sigma_1^v\sigma_2^v\cdot\mu = \mu. \tag{2}$$

We will do so simultaneously by induction on v. For $v = 0$, this follows directly from the definitions of Q and μ. Now suppose that the statements hold for $0 \leq v < \operatorname{lcm}(n_1, n_2) - 1$. Then Lemma 17 implies that

$$\sigma_1^{u+(v+1)}\sigma_2^{v+1}\cdot\mu = \sigma_1^{u+v}\sigma_2^v\cdot\mu = 0$$

and

$$\sigma_1^{v+1}\sigma_2^{v+1}\cdot\mu = \sigma_1^v\sigma_2^v\cdot\mu = \mu$$

by the induction hypothesis, so both statements also hold for $v + 1$ and we are done with the induction.

Now consider the map

$$\{0, 1, \ldots, \gcd(n_1, n_2)\} \times \{0, 1, \ldots, \operatorname{lcm}(n_1, n_2)\} \to \{0, 1, \ldots, n_1\} \times \{0, 1, \ldots, n_2\}$$

given by $(u, v) \mapsto (u+v \pmod{n_1}, v \pmod{n_2})$. Suppose that both (u, v) and (u', v') map to the same element. Then, for suitable $q, q' \in \mathbb{Z}$,

$$(u - u') + (v - v') = qn_1$$

and

$$v - v' = q'n_2,$$

hence

$$(u - u') = qn_1 - q'n_2 \equiv 0 \pmod{\gcd(n_1, n_2)}.$$

Since $0 \leq u, u' \leq \gcd(n_1, n_2)$, this implies $u = u'$. Consequently both n_1 and n_2 divide $v - v'$, hence so does $\operatorname{lcm}(n_1, n_2)$ and $v = v'$ (using that $0 \leq v, v' \leq \operatorname{lcm}(n_1, n_2)$). Thus we have shown that the above map is injective, and since both sets have cardinality $n_1 n_2$, it must be a bijection. Therefore for any $0 \leq x_1 < n_1$, $0 \leq x_2 < n_2$, we can write

$$\sigma_1^{x_1} \sigma_2^{x_2} \cdot \mu = \sigma_1^{u+v} \sigma_2^{v} \cdot \mu$$

for unique $0 \leq u < \gcd(n_1, n_2)$ and $0 \leq v < \operatorname{lcm}(n_1, n_2)$, and the equalities (1) and (2) imply that $\sigma_1^{x_1} \sigma_2^{x_2} \cdot \mu = 0$ unless $u = 0$, in which case $\sigma_1^{x_1} \sigma_2^{x_2} \cdot \mu = \mu$. But the congruences

$$x_1 \equiv u + v \pmod{n_1}$$

and

$$x_2 \equiv v \pmod{n_2}$$

imply that

$$x_1 - x_2 \equiv u \pmod{(\gcd(n_1, n_2))},$$

so the condition $u = 0$ is equivalent to

$$x_1 \equiv x_2 \pmod{(\gcd(n_1, n_2))},$$

as needed.

□

Proposition 19. *We have $\mu = 0$.*

Proof. Using the relation $N_1 \sim 0$ and Lemma 18 together with the bijection

$$\{0, 1, \ldots, \gcd(n_1, n_2) - 1\} \times \{0, 1, \ldots, \frac{n_1}{\gcd(n_1, n_2)} - 1\} \to \{0, 1, \ldots, n_1 - 1\}$$

given by $(u, v) \mapsto v \cdot \gcd(n_1, n_2) + u$, we get

$$0 = N_1 \cdot \mu = \sum_{w=0}^{n_1 - 1} \sigma_1^w \cdot \mu = \sum_{u=0}^{\gcd(n_1,n_2)-1} \sum_{v=0}^{\frac{n_1}{\gcd(n_1,n_2)}-1} \sigma_1^{v \cdot \gcd(n_1,n_2)+u} \cdot \mu = \frac{n_1}{\gcd(n_1, n_2)} \cdot \mu,$$

since by Lemma 18, $\sigma_1^{v\cdot\gcd(n_1,n_2)+u}\cdot\mu$ is zero for $u\neq 0$ and equal to μ otherwise.

Analogously, we get

$$0 = N_2\cdot\mu = \sum_{w=0}^{n_2-1}\sigma_2^w\cdot\mu = \sum_{u=0}^{\gcd(n_1,n_2)-1}\sum_{v=0}^{\frac{n_2}{\gcd(n_1,n_2)}-1}\sigma_2^{v\cdot\gcd(n_1,n_2)+u}\cdot\mu = \frac{n_2}{\gcd(n_1,n_2)}\cdot\mu,$$

since by Lemma 18, $\sigma_2^{v\cdot\gcd(n_1,n_2)+u}\cdot\mu$ is zero for $u\neq 0$ and equal to μ otherwise.

Due to the fact that $\frac{n_1}{\gcd(n_1,n_2)}$ and $\frac{n_2}{\gcd(n_1,n_2)}$ are coprime, this implies $\mu = 0$ by Bezout's identity. $\square$

It now follows that Q is trivial, so we have proven the following theorem:

Theorem 20. *Under the assumptions on page 0.3, the set $B_3\cup B_D$ forms a basis of D^+ and the set $B_3\cup B_C$ forms a basis of C^+.*

0.9 The case $a_1 = a_2 = a_3 = r_4 = 1, r_1\neq 1, r_2\neq 1, r_3\neq 1, \gcd(n_1,n_2,n_3) = \gcd(n_1,n_2)$

In this case, we have

$$\mathrm{Gal}(k/\mathbb{Q}) \cong \{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}|_k;\ 0\leq x_1<n_1, 0\leq x_2<n_2, 0\leq x_3<n_3, 0\leq x_4<a_4\},$$

$$s_{23} = s_{24} = s_{34} = 1$$

$$N_1\sim 0, N_2\sim 0, N_3\sim 0, R_4\sum_{u=0}^{m-1}\sigma_1^u\sigma_2^u\sigma_3^u\sim 0.$$

Lemma 21. *For any three integers a, b, c, we have*

$$\mathrm{lcm}(a,b,c) = \frac{abc\cdot\gcd(a,b,c)}{\gcd(a,b)\cdot\gcd(a,c)\cdot\gcd(b,c)}.$$

Proof. Let $d := \gcd(a,b,c)$. Then there exist $a', b', c'\in\mathbb{Z}$ such that $a = da', b = db', c = dc'$ and $\gcd(a',b',c') = 1$. Letting $e := \gcd(a,b), f := \gcd(a,c), g := \gcd(b,c)$, we get that there must exist $a'', b'', c''\in\mathbb{Z}$ such that $a = defa'', b = degb'', c = dfgc''$ and

$$\gcd(a'',b'') = \gcd(a'',c'') = \gcd(b'',c'') = 1.$$

Also the condition $\gcd(a',b',c') = 1$ can be reformulated as

$$\gcd(e,f) = \gcd(e,g) = \gcd(f,g) = 1.$$

Thus we get

$$\mathrm{lcm}(a,b,c) = defga''b''c'' = \frac{abcdefg}{d^3e^2f^2g^2} = \frac{abc\cdot d}{de\cdot df\cdot dg} = \frac{abc\cdot\gcd(a,b,c)}{\gcd(a,b)\cdot\gcd(a,c)\cdot\gcd(b,c)},$$

as needed. $\square$

Lemma 22. *The following are equivalent:*

(i) $\gcd(n_1, n_2, n_3) = \gcd(n_1, n_2)$,

(ii) $\frac{n_1 n_2 n_3}{m} = \gcd(n_1, n_3) \cdot \gcd(n_2, n_3)$,

(iii) $\gcd(n_1, n_2) = \gcd(\gcd(n_1, n_3), \gcd(n_2, n_3))$.

Proof.

"(i) $\Leftrightarrow$ (ii)": Using Lemma 21 together with Lemma 12, we get

$$m = \operatorname{lcm}(n_1, n_2, n_3) = \frac{n_1 n_2 n_3 \cdot \gcd(n_1 n_2 n_3)}{\gcd(n_1, n_2) \cdot \gcd(n_1, n_3) \cdot \gcd(n_2, n_3)},$$

hence

$$\frac{n_1 n_2 n_3}{m} = \frac{\gcd(n_1, n_2) \cdot \gcd(n_1, n_3) \cdot \gcd(n_2, n_3)}{\gcd(n_1 n_2 n_3)}$$

and this equals $\gcd(n_1, n_3) \cdot \gcd(n_2, n_3)$ iff $\gcd(n_1, n_2, n_3) = \gcd(n_1, n_2)$.

"(i) $\Leftrightarrow$ (iii)": It suffices to show that $\gcd(n_1, n_2, n_3) = \gcd(\gcd(n_1, n_3), \gcd(n_2, n_3))$. This is true in general, because any integer is a common divisor of n_1, n_2, n_3 iff it is a common divisor of n_1, n_3 and a common divisor of n_2, n_3 iff it is a common divisor of $\gcd(n_1, n_3)$ and $\gcd(n_2, n_3)$ iff it is a common divisor of $\gcd(\gcd(n_1, n_3), \gcd(n_2, n_3))$.

$\square$

Therefore, using Lemma 22 together with Lemma 12, we get

$$\begin{aligned} N &= a_4 n_1 n_2 n_3 - \frac{n_1 n_2 n_3}{m} - a_4(n_1 n_2 + n_1 n_3 + n_2 n_3) - a_4 - 2 + a_4(n_1 + n_2 + n_3) \\ &+ \gcd(n_1, n_2) + \gcd(n_1, n_3) + \gcd(n_2, n_3) \\ &= (a_4 - 1)(n_1 - 1)(n_2 - 1)(n_3 - 1) + (n_1 - 1)(n_2 - 1)(n_3 - 2) + (n_1 - 1)(n_2 - 1) - 2 \\ &- \gcd(n_1, n_3) \cdot \gcd(n_2, n_3) + \gcd(n_1, n_2) + \gcd(n_1, n_3) + \gcd(n_2, n_3) \\ &= (a_4 - 1)(n_1 - 1)(n_2 - 1)(n_3 - 1) + (n_1 - 1)(n_2 - 1)(n_3 - 2) \\ &+ (n_1 - 1)(n_2 - \gcd(n_2, n_3) + (n_1 - \gcd(n_1, n_3)) \cdot (\gcd(n_2, n_3) - 1) + \gcd(n_1, n_2) - 1 \end{aligned}$$

We will define B_4 as the set of the following N conjugates $\eta^{\sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{x_3} \sigma_4^{x_4}}$:

- $0 \leq x_1 < n_1 - 1, 0 \leq x_2 < n_2 - 1, 0 \leq x_3 < n_3 - 1, 0 < x_4 \leq a_4 - 1$,
- $0 \leq x_1 < n_1 - 1, 0 \leq x_2 < n_2 - 1, 1 < x_3 \leq n_3 - 1, x_4 = 0$,
- $1 \leq x_1 < n_1, \gcd(n_2, n_3) \leq x_2 < n_2, x_3 = 0, x_4 = 0$,
- $\gcd(n_1, n_3) \leq x_1 < n_1, 1 \leq x_2 < \gcd(n_2, n_3), x_3 = 0, x_4 = 0$,
- $0 \leq x_1 < \gcd(n_1, n_2) - 1, x_2 = 0, x_3 = 0, x_4 = 0$.

First we will recover the cases $0 < x_4 < a_4$, $x_1 = n_1 - 1$ or $x_2 = n_2 - 1$ or $x_3 = n_3 - 1$ using the relations $N_1 \sim 0, N_2 \sim 0, N_3 \sim 0$. From now on, we only need to deal with the cases where $x_4 = 0$. Next, we will recover the cases $1 < x_3 \leq n_3 - 1$, $x_1 = n_1 - 1$ or $x_2 = n_2 - 1$ (and always $x_4 = 0$) using the relations $N_1 \sim 0, N_2 \sim 0$ and the cases $x_3 = x_4 = 0$, $\gcd(n_1, n_3) \leq x_1 < n_1, x_2 = 0$ and $x_3 = x_4 = 0$, $x_1 = 0, \gcd(n_2, n_3) \leq x_2 < n_2$ using the relation $N_2 \sim 0$.

At this moment, we are only missing all the cases with $x_3 = 1, x_4 = 0$ and some of those with $x_3 = x_4 = 0$. From now on, we will only focus on recovering those with $x_3 = x_4 = 0$, because once we have those, we can recover those with $x_3 = 1, x_4 = 0$ just by using the relation $N_3 \sim 0$.

Let Q' be the quotient $\mathbb{Z}[G]$-module

$$D^+ / \big\langle \{\eta_I \big| \emptyset \subsetneq I \subsetneq P\} \big\rangle_{\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]}$$

and let Q be the quotient $\mathbb{Z}$-module of Q' by the conjugates we have already recovered, i.e.

$$\begin{aligned} Q := Q' / \Big\langle \{\eta^{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}};\quad & 0 \leq x_1 < n_1, 0 \leq x_2 < n_2, 0 \leq x_3 < n_3, 0 < x_4 < a_4, \\ & \text{or } 0 \leq x_1 < n_1, 0 \leq x_2 < n_2, 1 < x_3 < n_3, x_4 = 0, \\ & \text{or } 0 \leq x_1 < n_1, \gcd(n_2, n_3) \leq x_2 < n_2, x_3 = x_4 = 0, \\ & \text{or } \gcd(n_1, n_3) \leq x_1 < n_1, 0 \leq x_2 < \gcd(n_2, n_3), x_3 = x_4 = 0 \\ & \text{or } 0 \leq x_1 < \gcd(n_1, n_2) - 1, x_2 = x_3 = x_4 = 0\} \Big\rangle_{\mathbb{Z}}. \end{aligned}$$

We will write Q additively, denoting the class of η by μ, hence for any $\rho \in \mathrm{Gal}(k/\mathbb{Q})$ or $\rho \in \mathrm{Gal}(K/Q)$, denoting the class of η^ρ in Q by $\rho \cdot \mu$. Showing that we have indeed chosen a basis now amounts to showing that Q is trivial. Since

$$0 = \sigma_1^{x_1}\sigma_2^{x_2} N_3 \cdot \mu = \sigma_1^{x_1}\sigma_2^{x_2} \cdot \mu + \sigma_1^{x_1}\sigma_2^{x_2}\sigma_3 \cdot \mu$$

for any $x_1, x_2 \in \mathbb{Z}$, this is equivalent with showing that $\sigma_1^{x_1}\sigma_2^{x_2} \cdot \mu = 0$ for each $0 \leq x_1 < n_1, 0 \leq x_2 < n_2$ (and because of the definition of Q, it suffices to show this for each $0 \leq x_1 < \gcd(n_1, n_3), 0 \leq x_2 < \gcd(n_2, n_3)$).

The conjugates with $x_3 = 0$ and $x_4 = 0$ (i.e., those of the form $\eta^{\sigma_1^{x_1}\sigma_2^{x_2}}$) can be visualized as a discrete rectangle with n_1 rows and n_2 columns. Since for each x_4, there are n_3 layers of such rectangles in total, the sum $\eta^{R_4 \sum_{u=0}^{m-1} \sigma_1^u \sigma_2^u \sigma_3^u}$ must contain $\frac{m}{n_3} = r_3$ conjugates in each of these rectangles (and in this case, it can be seen geometrically that these form a regular grid). We will now describe the sum of these.

Let

$$T := \sum_{u=0}^{r_3 - 1} \sigma_1^{un_3}\sigma_2^{un_3}.$$

Lemma 23. *In Q, we have*

$$\sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)T\cdot\mu = 0$$

for any $x_1, x_2 \in \mathbb{Z}$.

Proof. Using the fact that every $0 \leq w < m$ can be uniquely written as $un_3 + v$ with $0 \leq u < r_3, 0 \leq v < n_3$ together with the fact that the order of σ_3 is n_3, we get

$$R_4 T\sum_{v=0}^{n_3-1}\sigma_1^v\sigma_2^v\sigma_3^v = R_4\sum_{u=0}^{r_3-1}\sigma_1^{un_3}\sigma_2^{un_3}\sigma_3^{un_3}\cdot\sum_{v=0}^{n_3-1}\sigma_1^v\sigma_2^v\sigma_3^v = R_4\sum_{w=0}^{m-1}\sigma_1^w\sigma_2^w\sigma_3^w \sim 0.$$

Together with $N_3 \sim 0$, this means that

$$\begin{aligned}
0 &\sim \sigma_1^{x_1}\sigma_2^{x_2}\left(R_4T\sum_{v=0}^{n_3-1}\sigma_1^v\sigma_2^v\sigma_3^v - \sigma_1\sigma_2N_3R_4T\right) = \sigma_1^{x_1}\sigma_2^{x_2}R_4T\sum_{v=0}^{n_3-1}\left(\sigma_1^v\sigma_2^v - \sigma_1\sigma_2\right)\sigma_3^v \\
&= \sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)R_4T + \sigma_1^{x_1}\sigma_2^{x_2}R_4T\sum_{v=2}^{n_3-1}\left(\sigma_1^v\sigma_2^v - \sigma_1\sigma_2\right)\sigma_3^v \\
&= \sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)T + \sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)T\sum_{u=1}^{a_4-1}\sigma_4^u + \sigma_1^{x_1}\sigma_2^{x_2}R_4T\sum_{v=2}^{n_3-1}\left(\sigma_1^v\sigma_2^v - \sigma_1\sigma_2\right)\sigma_3^v
\end{aligned}$$

Since all the summands in the expression

$$\sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)T\sum_{u=1}^{a_4-1}\sigma_4^u + \sigma_1^{x_1}\sigma_2^{x_2}R_4T\sum_{v=2}^{n_3-1}\left(\sigma_1^v\sigma_2^v - \sigma_1\sigma_2\right)\sigma_3^v$$

have either $x_4 > 0$ or $x_3 > 1$ (where x_3 and x_4 denote the respective exponents of σ_3 and σ_4 in each term), the result of their action on μ becomes trivial in Q, which yields the result. □

Lemma 24. *For any $0 \leq x_1 < \gcd(n_1, n_3), 0 \leq x_2 < \gcd(n_2, n_3)$, we have*

$$\sigma_1^{x_1}\sigma_2^{x_2}\cdot\mu = \begin{cases}\mu & \text{if } x_1 \equiv x_2 \pmod{\gcd(n_1,n_2)} \\ 0 & \text{otherwise.}\end{cases}$$

Proof. Podobně jako v předchozí sekci, ale některé části bude potřeba trochu upravit. □

Proposition 25. *We have $\mu = 0$.*

Proof. Recall that we have $\gcd(n_1, n_2) = \gcd(\gcd(n_1, n_3), \gcd(n_2, n_3))$ by Lemma 22. Using the relation $N_1 \sim 0$ and Lemma 18 together with the bijection

$$\{0, 1, \ldots, \gcd(n_1, n_2) - 1\} \times \{0, 1, \ldots, \frac{\gcd(n_1, n_3)}{\gcd(n_1, n_2)} - 1\} \to \{0, 1, \ldots, \gcd(n_1, n_3) - 1\}$$

given by $(u, v) \mapsto v \cdot \gcd(n_1, n_2) + u$, we get

$$0 = N_1 \cdot \mu = \sum_{w=0}^{\gcd(n_1,n_3)-1} \sigma_1^w \cdot \mu = \sum_{u=0}^{\gcd(n_1,n_2)-1} \sum_{v=0}^{\frac{\gcd(n_1,n_3)}{\gcd(n_1,n_2)}-1} \sigma_1^{v\cdot\gcd(n_1,n_2)+u} \cdot \mu = \frac{\gcd(n_1,n_3)}{\gcd(n_1,n_2)} \cdot \mu,$$

since by Lemma 18, $\sigma_1^{v\cdot\gcd(n_1,n_2)+u} \cdot \mu$ is zero for $u \neq 0$ and equal to μ otherwise.

Analogously, we get

$$0 = N_2 \cdot \mu = \sum_{w=0}^{\gcd(n_2,n_3)-1} \sigma_2^w \cdot \mu = \sum_{u=0}^{\gcd(n_1,n_2)-1} \sum_{v=0}^{\frac{\gcd(n_2,n_3)}{\gcd(n_1,n_2)}-1} \sigma_2^{v\cdot\gcd(n_1,n_2)+u} \cdot \mu = \frac{\gcd(n_2,n_3)}{\gcd(n_1,n_2)} \cdot \mu,$$

since by Lemma 18, $\sigma_2^{v\cdot\gcd(n_1,n_2)+u} \cdot \mu$ is zero for $u \neq 0$ and equal to μ otherwise.

Due to the fact that $\frac{\gcd(n_1,n_3)}{\gcd(n_1,n_2)}$ and $\frac{\gcd(n_2,n_3)}{\gcd(n_1,n_2)}$ are coprime, this implies $\mu = 0$ by Bezout's identity. $\square$

It now follows that Q is trivial, so we have proven the following theorem:

Theorem 26. *Under the assumptions on page 0.3, the set $B_4 \cup B_D$ forms a basis of D^+ and the set $B_4 \cup B_C$ forms a basis of C^+.*

0.10 The case $a_1 = a_2 = a_3 = r_4 = 1, r_1 \neq 1, r_2 \neq 1, r_3 \neq 1, s_{12} = s_{13} = s_{23} = 1, \gcd(n_1, n_2, n_3) = 1$

In this case, we have

$$\mathrm{Gal}(k/\mathbb{Q}) \cong \{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}|_k; 0 \leq x_1 < n_1, 0 \leq x_2 < n_2, 0 \leq x_3 < n_3, 0 \leq x_4 < a_4\},$$

$$s_{12} = s_{13} = s_{14} = s_{23} = s_{24} = s_{34} = 1$$

and

$$N_1 \sim 0, N_2 \sim 0, N_3 \sim 0, R_4 \sum_{u=0}^{m-1} \sigma_1^u \sigma_2^u \sigma_3^u \sim 0.$$

Note that the condition $r_1 \neq 1, r_2 \neq 1, r_3 \neq 1$ is actually not restrictive, since we have already solved the cases where it is not true. Also r_1, r_2, r_3 must be pairwise distinct, otherwise their coprimality would imply that two of them equal 1.

Lemma 27. *Under the assumptions $s_{12} = s_{13} = s_{23} = 1$, the following are equivalent:*

(i) $\gcd(n_1, n_2, n_3) = 1$,

(ii) $\mathrm{lcm}(r_1, r_2, r_3) = m$,

(iii) $r_1 r_2 r_3 = m$,

(iv) $n_1 = r_2r_3, n_2 = r_1r_3, n_3 = r_1r_2$,

(v) $\frac{n_1n_2n_3}{m} = m$,

(vi) $\gcd(n_1, n_2) = r_3, \gcd(n_1, n_3) = r_2, \gcd(n_2, n_3) = r_1$.

Proof.

"(i) $\Leftrightarrow$ (ii)": For any $t \in \mathbb{Z}$, we have

$$t \mid \gcd(n_1, n_2, n_3) \Leftrightarrow t \mid n_1, t \mid n_2, t \mid n_3 \Leftrightarrow r_1 \mid \frac{m}{t}, r_2 \mid \frac{m}{t}, r_3 \mid \frac{m}{t}$$
$$\Leftrightarrow \operatorname{lcm}(r_1, r_2, r_3) \mid \frac{m}{t} \Leftrightarrow t \mid \frac{m}{\operatorname{lcm}(r_1, r_2, r_3)},$$

from which it follows that $\gcd(n_1, n_2, n_3) = \frac{m}{\operatorname{lcm}(r_1,r_2,r_3)}$.

"(ii) $\Leftrightarrow$ (iii)": Since $s_{12} = s_{13} = s_{23} = 1$, any common multiple of r_1, r_2, r_3 is in fact a multiple of $r_1r_2r_3$, hence $\operatorname{lcm}(r_1, r_2, r_3) = r_1r_2r_3$.

"(iii) $\Leftrightarrow$ (iv)": This follows straight from the definition $n_i = \frac{m}{r_i}$.

"(iii) $\Leftrightarrow$ (v)": We have $\frac{n_1n_2n_3}{m} = \frac{m^2}{r_1r_2r_3}$, which equals m iff $\frac{m}{r_1r_2r_3} = 1$.

"(iv) $\Rightarrow$ (vi)": For $\{i, j, l\} = \{1, 2, 3\}$, we have $\gcd(n_i, n_j) = \gcd(r_jr_l, r_ir_l) = r_ls_{ij} = r_l$.

"(vi) $\Rightarrow$ (i)": Since $\gcd(n_1, n_2, n_3)$ must divide $\gcd(n_1, n_2)$, $\gcd(n_1, n_3)$, $\gcd(n_2, n_3)$ and these are pairwise coprime, it must be equal to 1.

□

Thus $\frac{n_1n_2n_3}{m} = m = r_2n_2 = \gcd(n_1, n_3)n_2$ and we have

$$\begin{aligned}
N &= a_4n_1n_2n_3 - \frac{n_1n_2n_3}{m} - a_4(n_1n_2 + n_1n_3 + n_2n_3) - a_4 - 2 + a_4(n_1 + n_2 + n_3) \\
&+ \gcd(n_1, n_2) + \gcd(n_1, n_3) + \gcd(n_2, n_3) \\
&= (a_4 - 1)(n_1 - 1)(n_2 - 1)(n_3 - 1) + (n_1 - 1)(n_2 - 1)(n_3 - 2) \\
&+ n_1n_2 - (\gcd(n_1, n_3) + 1)n_2 - (n_1 - \gcd(n_1, n_3) - 1) + \gcd(n_2, n_3) + \gcd(n_1, n_2) - 2 \\
&= (a_4 - 1)(n_1 - 1)(n_2 - 1)(n_3 - 1) + (n_1 - 1)(n_2 - 1)(n_3 - 2) \\
&+ (n_2 - 1)(n_1 - r_2 - 1) + r_1 + r_3 - 2.
\end{aligned}$$

We will define B_5 as the set of the following N conjugates $\eta^{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}}$:

- $0 \le x_1 < n_1 - 1, 0 \le x_2 < n_2 - 1, 0 \le x_3 < n_3 - 1, 0 < x_4 \le a_4 - 1$,
- $0 \le x_1 < n_1 - 1, 0 \le x_2 < n_2 - 1, 1 < x_3 \le n_3 - 1, x_4 = 0$,
- $0 \le x_1 < n_1 - r_2 - 1, 0 \le x_2 < n_2 - 1, x_3 = 0, x_4 = 0$,
- $x_1 = n_1 - r_2 - 1, 0 \le x_2 < r_1 + r_3 - 2, x_3 = 0, x_4 = 0$.

(Note that $n_1 - r_2 - 1 = r_2(r_3 - 1) - 1 > 0$ and $r_1 + r_3 - 2 > 0$ since $r_1, r_2, r_3 > 1$.)

First we will recover the cases $0 < x_4 < a_4$, $x_1 = n_1 - 1$ or $x_2 = n_2 - 1$ or $x_3 = n_3 - 1$ using the relations $N_1 \sim 0, N_2 \sim 0, N_3 \sim 0$. From now on, we only need to deal with the cases where $x_4 = 0$. Next, we will recover the cases $1 < x_3 \leq n_3 - 1$, $x_1 = n_1 - 1$ or $x_2 = n_2 - 1$ (and always $x_4 = 0$) using the relations $N_1 \sim 0, N_2 \sim 0$ and the cases $x_3 = x_4 = 0$, $0 \leq x_1 < n_1 - r_2 - 1$, $x_2 = n_2 - 1$ using the relation $N_2 \sim 0$.

At this moment, we are only missing all the cases with $x_3 = 1, x_4 = 0$ and some of those with $x_3 = x_4 = 0$. From now on, we will only focus on recovering those with $x_3 = x_4 = 0$, because once we have those, we can recover those with $x_3 = 1, x_4 = 0$ just by using the relation $N_3 \sim 0$.

From now on, we will write $\overline{z} := z \pmod{r_3}$ for any $z \in \mathbb{Z}$, so that $\overline{z} \in \{0, 1, \ldots, r_3 - 1\}$. We will also define h to be the unique integer satisfying $r_1 \cdot h \equiv r_2 \pmod{r_3}$ and $h \in \{0, 1, \ldots, r_3 - 1\}$ and similarly h' to be the unique integer satisfying $r_2 \cdot h' \equiv r_1 \pmod{r_3}$ and $h' \in \{0, 1, \ldots, r_3 - 1\}$ (both are well defined, since $\gcd(r_1, r_3) = \gcd(r_2, r_3) = 1$). Clearly $h \cdot h' \equiv 1 \pmod{r_3}$.

Let Q' be the quotient $\mathbb{Z}[G]$-module

$$D^+ / \big\langle \{\eta_I \big| \emptyset \subsetneq I \subsetneq P\} \big\rangle_{\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]}$$

and let Q be the quotient $\mathbb{Z}$-module of Q' by the conjugates we have already recovered, i.e.

$$\begin{aligned} Q := Q' / \big\langle \{\eta^{\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}}; \quad & 0 \leq x_1 < n_1, 0 \leq x_2 < n_2, 0 \leq x_3 < n_3, 0 < x_4 < a_4, \\ & \text{or } 0 \leq x_1 < n_1, 0 \leq x_2 < n_2, 1 < x_3 < n_3, x_4 = 0, \\ & \text{or } 0 \leq x_1 < n_1 - r_2 - 1, 0 \leq x_2 < n_2, x_3 = x_4 = 0, \\ & \text{or } x_1 = n_1 - r_2 - 1, 0 \leq x_2 < r_1 + r_3 - 2, x_3 = x_4 = 0\} \big\rangle_{\mathbb{Z}}. \end{aligned}$$

We will write Q additively, denoting the class of η by μ, hence for any $\rho \in \mathrm{Gal}(k/\mathbb{Q})$ or $\rho \in Gal(K/Q)$, denoting the class of η^ρ in Q by $\rho \cdot \mu$. Showing that we have indeed chosen a basis now amounts to showing that Q is trivial. Since

$$0 = \sigma_1^{x_1}\sigma_2^{x_2} N_3 \cdot \mu = \sigma_1^{x_1}\sigma_2^{x_2} \cdot \mu + \sigma_1^{x_1}\sigma_2^{x_2}\sigma_3 \cdot \mu$$

for any $x_1, x_2 \in \mathbb{Z}$, this is equivalent with showing that $\sigma_1^{x_1}\sigma_2^{x_2} \cdot \mu = 0$ for each $0 \leq x_1 < n_1, 0 \leq x_2 < n_2$.

The conjugates with $x_3 = 0$ and $x_4 = 0$ (i.e., those of the form $\eta^{\sigma_1^{x_1}\sigma_2^{x_2}}$) can be visualized as a discrete rectangle with n_1 rows and n_2 columns. Since for each x_4, there are n_3 layers of such rectangles in total, the sum $\eta^{R_4 \sum_{u=0}^{m-1} \sigma_1^u \sigma_2^u \sigma_3^u}$ must contain $\frac{m}{n_3} = r_3$ conjugates in each of these rectangles. We will now describe the sum of these.

Let

$$T := \sum_{u=0}^{r_3 - 1} \sigma_1^{un_3}\sigma_2^{un_3}$$

Lemma 28. *In Q, we have*

$$\sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)T\cdot\mu = 0$$

for any $x_1, x_2 \in \mathbb{Z}$.

Proof. Using the fact that every $0 \le w < m$ can be uniquely written as $un_3 + v$ with $0 \le u < r_3, 0 \le v < n_3$ together with the fact that the order of σ_3 is n_3, we get

$$R_4T\sum_{v=0}^{n_3-1}\sigma_1^v\sigma_2^v\sigma_3^v = R_4\sum_{u=0}^{r_3-1}\sigma_1^{un_3}\sigma_2^{un_3}\sigma_3^{un_3}\cdot\sum_{v=0}^{n_3-1}\sigma_1^v\sigma_2^v\sigma_3^v = R_4\sum_{w=0}^{m-1}\sigma_1^w\sigma_2^w\sigma_3^w \sim 0.$$

Together with $N_3 \sim 0$, this means that

$$0 \sim \sigma_1^{x_1}\sigma_2^{x_2}\left(R_4T\sum_{v=0}^{n_3-1}\sigma_1^v\sigma_2^v\sigma_3^v - \sigma_1\sigma_2N_3R_4T\right) = \sigma_1^{x_1}\sigma_2^{x_2}R_4T\sum_{v=0}^{n_3-1}\left(\sigma_1^v\sigma_2^v - \sigma_1\sigma_2\right)\sigma_3^v$$

$$= \sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)R_4T + \sigma_1^{x_1}\sigma_2^{x_2}R_4T\sum_{v=2}^{n_3-1}\left(\sigma_1^v\sigma_2^v - \sigma_1\sigma_2\right)\sigma_3^v$$

$$= \sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)T + \sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)T\sum_{u=1}^{a_4-1}\sigma_4^u + \sigma_1^{x_1}\sigma_2^{x_2}R_4T\sum_{v=2}^{n_3-1}\left(\sigma_1^v\sigma_2^v - \sigma_1\sigma_2\right)\sigma_3^v$$

Since all the summands in the expression

$$\sigma_1^{x_1}\sigma_2^{x_2}(1-\sigma_1\sigma_2)T\sum_{u=1}^{a_4-1}\sigma_4^u + \sigma_1^{x_1}\sigma_2^{x_2}R_4T\sum_{v=2}^{n_3-1}\left(\sigma_1^v\sigma_2^v - \sigma_1\sigma_2\right)\sigma_3^v$$

have either $x_4 > 0$ or $x_3 > 1$ (where x_3 and x_4 denote the respective exponents of σ_3 and σ_4 in each term), the result of their action on μ becomes trivial in Q, which yields the result. □

The rest of the proof will be carried out purely algebraically, but perhaps it is helpful (although not strictly required) to see some of its parts geometrically.

We will decompose our rectangle (of conjugates of η having $x_3 = x_4 = 0$) into $r_3 \times r_3$ rectangular blocks of height r_2 and width r_1 in the natural way. In the following, by a big row (resp. big column) we will understand a row of blocks (resp. columns), that is r_3 consecutive blocks next to (resp. above) each other. Since $r_2 \mid n_3, r_1 \mid n_3$ and the conjugates contained in η^T are given by $\eta^{\sigma_1^{qn_3}\sigma_2^{qn_3}}$ for $0 \le q \le r_3 - 1$, the Chinese remainder theorem implies that $\eta^{\sigma_1^{x_1}\sigma_2^{x_2}T}$ contains exactly one conjugate in every big row (resp. big column) for any $0 \le x_1 < n_1, 0 \le x_2 < n_2$, and these have the same relative position in each of the respective blocks (determined only by $\overline{r_1}, \overline{r_2}, x_1, x_2$). We can be even more precise: the horizontal distance between $\eta^{\sigma_1^{qn_3+x_1}\sigma_2^{qn_3+x_2}}$ and $\eta^{\sigma_1^{(q+1)n_3+x_1}\sigma_2^{(q+1)n_3+x_2}}$ for $0 \le q \le r_3 - 1$ and $0 \le x_1 < n_1, 0 \le x_2 < n_2$ is exactly $\overline{r_2}\cdot r_1$, i.e. $\overline{r_2}$ blocks, and the vertical distance between them is exactly $\overline{r_1}\cdot r_2$, i.e. $\overline{r_1}$ blocks (again this follows easily from the Chinese remainder theorem). It follows that the horizontal distance between any two conjugates in η^T with a vertical distance of one block is h blocks.

For all $0 \le u \le n_2$, we will denote $X_u := \sigma_1^{n_1-2}\sigma_2^u \cdot \mu$ and $Y_u := \sigma_1^{r_2(r_3-1)-1}\sigma_2^u \cdot \mu$. It will be convenient to allow any integers in the indices of the X's and Y's and regard them only modulo n_2 (to be more precise, as in the set $\{0, 1, \dots, n_2 - 1\}$). Moreover note that by definition, $Y_u = 0$ for $0 \le u < r_1 + r_3 - 2$.

Lemma 29. *For any $0 \le x_1 < n_1, 0 \le x_2 < n_2$, we have*

$$\sigma_1^{x_1}\sigma_2^{x_2} \cdot \mu = \begin{cases} 0 & \text{if } x_1 < r_2(r_3-1)-1 \\ Y_{x_2} & \text{if } x_1 = r_2(r_3-1)-1 \\ X_{\overline{x_2-x_1-2}} & \text{if } r_2(r_3-1) \le x_1 < n_1 - 1 \\ X_{\overline{x_2-x_1-2}} - Y_{x_2-h\cdot r_1} & \text{if } x_1 = n_1 - 1. \end{cases}$$

Moreover, we have $X_q = X_{q'}$ for any $q \equiv q' \pmod{r_3}$.

Proof. The first case ($x_1 < r_2(r_3-1)-1$) follows directly from the definition of Q and the second case ($x_1 = r_2(r_3-1)-1$) directly from the definition of Y_{x_2}.

Now for every $0 \le u < n_2$, we will prove by induction with respect to $v = 0, 1, \dots, r_2 - 2$ that

$$\sigma_1^{n_1-2-v}\sigma_2^{u-v} \cdot \mu = X_u. \tag{3}$$

The base step $v = 0$ is just the definition of X_u. Now suppose that $0 < v \le r_2 - 2$ and the statement holds for $v - 1$. Then in the equality

$$\left(\sigma_1^{n_1-2-v}\sigma_2^{u-v}(1-\sigma_1\sigma_2)\sum_{w=0}^{r_3-1}\sigma_1^{wn_3}\sigma_2^{wn_3}\right) \cdot \mu = 0, \tag{4}$$

which follows from Lemma 28, we claim that all the terms with $w > 0$ do not contribute anything to the sum. Indeed, all the exponents of σ_1 are pairwise congruent modulo r_2 (since $r_2 \mid n_3$), and since $n_1 - r_2 \le n_1 - 2 - v < n_1 - 2$ and $n_1 - r_2 + 1 \le n_1 - 1 - v < n_1 - 1$, we have

$$\left(\sigma_1^{n_1-2-v}\sigma_2^{u-v}(1-\sigma_1\sigma_2)\sigma_1^{wn_3}\sigma_2^{wn_3}\right) \cdot \mu = 0$$

for any $w > 0$, because r_3 does not divide wn_3 in this case. Hence (4) implies that

$$0 = \left(\sigma_1^{n_1-2-v}\sigma_2^{u-v}(1-\sigma_1\sigma_2)\right) \cdot \mu = \sigma_1^{n_1-2-v}\sigma_2^{u-v} \cdot \mu - \underbrace{\sigma_1^{n_1-2-(v-1)}\sigma_2^{u-(v-1)} \cdot \mu}_{=X_u},$$

therefore $\sigma_1^{n_1-2-v}\sigma_2^{u-v} \cdot \mu = X_u$ by the induction hypothesis. This completes the induction, so (3) holds.

Now for any $0 \le u < n_2$, we will take $v = r_2 - 1$ in (4). Again, since all the exponents of σ_1 are pairwise congruent modulo r_2 (since $r_2 \mid n_3$) in this sum, the only terms which could be nonzero are those arising from $w = 0$ and from w satisfying

$$wn_3 + n_1 - 2 - (r_2 - 1) \equiv n_1 - 1 \pmod{n_1},$$

which is equivalent to $wn_3 \equiv r_2 \pmod{n_1}$, which implies $wn_3 \equiv r_2 \pmod{r_3}$. Together with $wn_3 \equiv 0 \pmod{r_1}$ and the fact that $\gcd(r_1, r_3) = 1$, this means that the only solution to the above congruence is $wn_3 \equiv h \cdot r_1 \pmod{n_2}$.

Thus we have

$$\begin{aligned} 0 &= \left(\sigma_1^{n_1-r_2-1}\sigma_2^{u-r_2+1}(1-\sigma_1\sigma_2) + \sigma_1^{n_1-1}\sigma_2^{u-r_2+1+h\cdot r_1}(1-\sigma_1\sigma_2)\right)\cdot\mu \\ &= \underbrace{\sigma_1^{n_1-r_2-1}\sigma_2^{u-r_2+1}\cdot\mu}_{=Y_{u-r_2+1}} - \underbrace{\sigma_1^{n_1-r_2}\sigma_2^{u-r_2+2}\cdot\mu}_{=X_u \text{ due to (3)}} + \sigma_1^{n_1-1}\sigma_2^{u-r_2+1+h\cdot r_1}\cdot\mu - \underbrace{\sigma_1^{n_1}\sigma_2^{u-r_2+1+h\cdot r_1+1}\cdot\mu}_{=0}. \end{aligned}$$

Therefore

$$\sigma_1^{n_1-1}\sigma_2^{u-r_2+1+h\cdot r_1}\cdot\mu = X_u - Y_{u-r_2+1}. \tag{5}$$

Finally, for any $0 \le u < n_2$, we will take $v = r_2$ in (4). Again, since all the exponents of σ_1 are pairwise congruent modulo r_2 in this sum, we only get nonzero terms for $w = 0$ and for w satisfying

$$wn_3 + n_1 - 2 - r_2 \equiv n_1 - 2 \pmod{n_1},$$

which implies (because we have got the same congruence as above) $wn_3 \equiv h \cdot r_1 \pmod{n_2}$.

Thus we have

$$0 = \underbrace{\sigma_1^{n_1-r_2-2}\sigma_2^{u-r_2}\cdot\mu}_{=0} - \underbrace{\sigma_1^{n_1-r_2-1}\sigma_2^{u-r_2+1}\cdot\mu}_{=Y_{u-r_2+1}} + \underbrace{\sigma_1^{n_1-2}\sigma_2^{u-r_2+h\cdot r_1}\cdot\mu}_{=X_{u-r_2+h\cdot r_1}} - \underbrace{\sigma_1^{n_1-1}\sigma_2^{u-r_2+1+h\cdot r_1}\cdot\mu}_{=X_u - Y_{u-r_2+1} \text{ due to (5)}}.$$

Therefore $X_{u-r_2+h\cdot r_1} = X_u$. Note that

$$h \cdot r_1 - r_2 \equiv 0 \pmod{r_3}$$

and

$$h \cdot r_1 - r_2 \equiv -r_2 \pmod{r_1}.$$

Since $\gcd(-r_2, r_1) = 1$ and $n_2 = r_1 r_3$, this means that for all $q, q' \in \mathbb{Z}$ satisfying $q \equiv q' \pmod{r_3}$, there is some $w \in \mathbb{Z}$ such that

$$q' = w(h \cdot r_1 - r_2) + q \pmod{n_2}.$$

Without loss of generality, we can assume that $w \ge 0$ (otherwise we can just swap q and q'). But then

$$X_q = X_{q+(h\cdot r_1 - r_2)} = X_{q+2(h\cdot r_1-r_2)} = \cdots = X_{q+w(h\cdot r_1-r_2)} = X_{q'}.$$

Now for any x_1, x_2 satisfying $r_2(r_3-1) \le x_1 < n_1 - 1$ and $0 \le x_2 < x_2$, denoting $v = n_1 - 2 - x_1, u = v + x_2$, we get $0 \le v \le r_2 - 2$ and the equality (1) implies

$$\sigma_1^{x_1}\sigma_2^{x_2}\mu = X_{n_1-2-x_1+x_2} = X_{x_2-x_1-2},$$

because $r_3 \mid n_1$.

Similarly, for $x_1 = n_1 - 1$ and any $0 \le x_2 < n_2$, denoting $u = x_2 + r_2 - 1 - h \cdot r_1$, the equality (5) implies that

$$\sigma_1^{x_1}\sigma_2^{x_2} \cdot \mu = X_u - Y_{u-r_2+1} = X_{x_2-x_1-2} - Y_{x_2-h\cdot r_1},$$

since

$$u = x_2 - 1 + r_2 - h \cdot r_1 \equiv x_2 - 1 \equiv x_2 - 2 + 1 - n_1 = x_2 - x_1 - 2 \pmod{r_3}$$

by definition of h and the fact that $r_3 \mid n_1$.

This concludes the proof. $\square$

Thanks to Lemma 29, from now on we will regard the indices of the X's only modulo r_3 (to be more precise, in the set $\{0, 1, 2, \ldots, r_3 - 1\}$) and drop the bar notation for them. The lemma also implies the equality

$$\sigma_1^{n_1-1}\sigma_2^{x_2} \cdot \mu + \sigma_1^{n_1-r_2-1}\sigma_2^{x_2-h\cdot r_1} \cdot \mu = X_{x_2-1} - Y_{x_2-h\cdot r_1} + Y_{x_2-h\cdot r_1} = X_{x_2-1} \tag{6}$$

for any $x_2 \in \mathbb{Z}$, which we will use several times. Another simple observation that will come in handy in the proofs of the following lemmas is that the unary operation of adding a fixed integer induces an automorphism of $\mathbb{Z}/r_3$, which we will not mention explicitly anymore.

To show that Q is trivial, it now suffices to show that $X_u = 0$ for all $0 \le u < r_3$ and $Y_v = 0$ for all $r_1 + r_3 - 2 \le v < n_2$ (knowing already that $Y_v = 0$ for all $0 \le v < r_1 + r_3 - 2$). To achieve this, we will use linear algebra.

Let $\alpha := Y_{r_1+r_3-2} + Y_{r_1+r_3-1} + \cdots + Y_{n_2-1} \in Q$ and $\beta := X_0 + X_1 + \cdots + X_{r_3-1} \in Q$.

Lemma 30. *We have $\alpha = \beta = 0$.*

Proof. Using the relation $N_2 \sim 0$, we have

$$0 = \sigma_1^{r_2(r_3-1)-1} N_2 \cdot \mu = \sum_{x_2=0}^{n_2-1} \sigma_1^{r_2(r_3-1)-1}\sigma_2^{x_2} \cdot \mu = \sum_{x_2=0}^{n_2-1} Y_{x_2} = \alpha$$

and

$$\begin{aligned} 0 = \sigma_1^{r_2(r_3-1)} N_2 \cdot \mu &= \sum_{x_2=0}^{n_2-1} \sigma_1^{r_2(r_3-1)}\sigma_2^{x_2} \cdot \mu = \sum_{x_2=0}^{n_2-1} X_{x_2-r_2(r_3-1)-2} \\ &= \sum_{x_2=0}^{r_1r_3-1} X_{x_2+r_2-2} = \sum_{u=0}^{r_1-1}\sum_{v=0}^{r_3-1} X_{ur_3+v+r_2-2} = r_1 \cdot \sum_{v=0}^{r_3-1} X_{v+r_2-2} = r_1 \cdot \beta, \end{aligned}$$

since each $x_2 \in \{0, 1, \ldots, r_1r_3 - 1\}$ can be uniquely written as $ur_3 + v$, where $0 \le u < r_1$, $0 \le v < r_3$.

Similarly, using Lemma 29 together with the relation $N_1 \sim 0$ and the equality (6), we get

$$\begin{aligned}
0 &= \sum_{q=0}^{r_3-1} \sigma_2^{qr_1} N_1 \cdot \mu = \sum_{q=0}^{r_3-1} \left(\sigma_1^{n_1-1} + \sigma_1^{r_2(r_3-1)-1}\right) \sigma_2^{qr_1} \cdot \mu + \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{q=0}^{r_3-1} \sigma_1^{x_1} \sigma_2^{qr_1} \cdot \mu \\
&= \sum_{q=0}^{r_3-1} (\sigma_1^{n_1-1}\sigma_2^{qr_1} + \sigma_1^{r_2(r_3-1)-1}\sigma_2^{(q-h)\cdot r_1}) \cdot \mu + \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{q=0}^{r_3-1} \sigma_1^{x_1}\sigma_2^{qr_1} \cdot \mu \\
&= \sum_{q=0}^{r_3-1} X_{qr_1-1} + \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{q=0}^{r_3-1} X_{qr_1-x_1-2} = \sum_{x_1=r_2(r_3-1)}^{n_1-1} \sum_{q=0}^{r_3-1} X_{qr_1-x_1-2} = r_2 \cdot \beta,
\end{aligned}$$

since for any x_1, all possible remainders modulo r_3 occur exactly once as the indices in the sum $\sum_{q=0}^{r_3-1} X_{qr_1-x_1-2}$ (due to the fact that the order of the class of r_1 is r_3 in $\mathbb{Z}/r_3$, due to their coprimality). Since $\gcd(r_1, r_2) = 1$, this implies $\beta = 0$ by Bezout's identity. $\square$

Next, for $0 \le q \le r_3 - 3$, we will define

$$\Gamma_q := \sum_{u=0}^{r_3-h'-1} \sum_{v=0}^{\overline{r_2}-1} X_{q+v-ur_2-1} \in Q.$$

Lemma 31. *For any $0 \le q \le r_3 - 3$, we have $\Gamma_q = 0$.*

Proof. Using Lemma 29, the relation $N_1 \sim 0$ and the equality (6), we get

$$\begin{aligned}
0 &= \sum_{u=0}^{r_3-h'-1} \sigma_2^{q-uhr_1} N_1 \cdot \mu \\
&= \sum_{u=0}^{r_3-h'-2} \underbrace{\left(\sigma_1^{n_1-1}\sigma_2^{q-uhr_1} + \sigma_1^{r_2(r_3-1)-1}\sigma_2^{q-(u+1)hr_1}\right) \cdot \mu}_{=X_{q-uhr_1-1} \text{ due to (6)}} \\
&\quad + \underbrace{\sigma_1^{r_2(r_3-1)-1}\sigma_2^{q} \cdot \mu}_{=Y_q} + \underbrace{\sigma_1^{n_1-1}\sigma_2^{q-(r_3-h'-1)hr_1} \cdot \mu}_{=X_{q-(r_3-h'-1)hr_1-1}-Y_{q+r_1}} \\
&\quad + \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{u=0}^{r_3-h'-1} \sigma_1^{x_1}\sigma_2^{q-uhr_1} \cdot \mu.
\end{aligned}$$

Now we will use the fact that $q \le r_3 - 3 \le r_1 + r_3 - 3$ (implying $Y_q = 0$) and

$$q - (r_3 - h' - 1)hr_1 - hr_1 = q - r_1r_3h + r_1hh' \equiv q + r_1 \pmod{n_2},$$

since the congruence holds modulo both r_1 and r_3 (and $\gcd(r_1, r_3) = 1$). Also note that $Y_{q+r_1} = 0$, since

$$r_1 \leq q + r_1 \leq r_1 + r_3 - 3,$$

which precisely justifies the bounds on q that we used in the definition of Γ_q and also explains why the upper bound in the first sum was chosen to be $r_3 - h' - 1$.

Continuing with the previous equality and using the congruence $hr_1 \equiv r_2 \pmod{r_3}$ and Lemma 29, we thus have

$$\begin{aligned}
0 &= \left(\sum_{u=0}^{r_3-h'-2} X_{q-uhr_1-1}\right) + X_{q-(r_3-h'-1)hr_1-1} + \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{u=0}^{r_3-h'-1} X_{q-uhr_1-x_1-2} \\
&= \sum_{u=0}^{r_3-h'-1} X_{q-ur_2-1} + \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{u=0}^{r_3-h'-1} X_{q-ur_2-x_1-2} \\
&= \sum_{x_1=r_2(r_3-1)}^{n_1-1} \sum_{u=0}^{r_3-h'-1} X_{q-ur_2-x_1-2}.
\end{aligned}$$

After using the substitution $v = n_1 - 1 - x_1$, this becomes

$$\begin{aligned}
0 &= \sum_{u=0}^{r_3-h'-1} \sum_{v=0}^{r_2-1} X_{q+v-ur_2-1} \\
&= \sum_{u=0}^{r_3-h'-1} \left(\sum_{v=0}^{\overline{r_2}-1} X_{q+v-ur_2-1} + \sum_{v=\overline{r_2}}^{r_2-1} X_{q+v-ur_2-1}\right) \\
&= \sum_{u=0}^{r_3-h'-1} \sum_{v=0}^{\overline{r_2}-1} X_{q+v-ur_2-1} + \sum_{u=0}^{r_3-h'-1} \frac{r_2 - \overline{r_2}}{r_3} \sum_{v=\overline{r_2}}^{\overline{r_2}+r_3-1} X_{q+v-ur_2-1} \\
&= \Gamma_q + \sum_{u=0}^{r_3-h'-1} \frac{r_2 - \overline{r_2}}{r_3} \cdot \beta \\
&= \Gamma_q,
\end{aligned}$$

since $\beta = 0$ by Lemma 30. $\square$

Finally, let

$$\Delta := \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{\overline{r_2}-1} \sum_{w=0}^{\overline{r_1}-1} X_{v+w-ur_2-1} \in Q.$$

Lemma 32. *We have $\Delta = 0$.*

Proof. Using Lemma 29, the relation $N_1 \sim 0$ and the equality (6), we get

$$
\begin{aligned}
0 &= \sum_{u=0}^{r_3-1} u \cdot \sum_{x_2=0}^{r_1-1} \sigma_2^{x_2-uhr_1} N_1 \cdot \mu \\
&= \sum_{u=0}^{r_3-1} u \cdot \sum_{x_2=0}^{r_1-1} \left(\sigma_1^{n_1-1} \sigma_2^{x_2-uhr_1} + \sigma_1^{r_2(r_3-1)-1} \sigma_2^{x_2-uhr_1} \right) \cdot \mu \\
&+ \sum_{u=0}^{r_3-1} u \cdot \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{x_2=0}^{r_1-1} \sigma_1^{x_1} \sigma_2^{x_2-uhr_1} \cdot \mu \\
&= \sum_{u=0}^{r_3-2} \sum_{x_2=0}^{r_1-1} \left(u \cdot \underbrace{\sigma_1^{n_1-1} \sigma_2^{x_2-uhr_1} \cdot \mu}_{=X_{x_2-uhr_1-1} - Y_{x_2-(u+1)hr_1}} + (u+1) \cdot \underbrace{\sigma_1^{r_2(r_3-1)-1} \sigma_2^{x_2-(u+1)hr_1} \cdot \mu}_{=Y_{x_2-(u+1)hr_1}} \right) + \\
&+ \sum_{x_2=0}^{r_1-1} (r_3-1) \cdot \underbrace{\sigma_1^{n_1-1} \sigma_2^{x_2-(r_3-1)hr_1} \cdot \mu}_{=X_{x_2-(r_3-1)hr_1-1} - Y_{x_2-hr_1r_3} =} + \sum_{u=0}^{r_3-1} u \cdot \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{x_2=0}^{r_1-1} \sigma_1^{x_1} \sigma_2^{x_2-uhr_1} \cdot \mu,
\end{aligned}
$$

where we used the fact that

$$x_2 - hr_1r_3 \equiv x_2 \pmod{n_2}$$

and $0 \leq x_2 < r_1$, hence $Y_{x_2-hr_1r_3} = 0$. Also note that for any $r_1 \leq q < n_2$, there exist unique

$$u \in \{0, 1, \ldots, r_3 - 2\}, x_2 \in \{0, 1, \ldots, r_1 - 1\}$$

such that

$$q \equiv x_2 - (u+1)hr_1 \pmod{n_2}$$

by the Chinese remainder theorem, since $\gcd(h, r_3) = 1$ and for $u = r_3 - 1$, we would get $q \equiv r \pmod{n_2}$, where $0 \leq r < r_1$. Thus we get a bijection

$$\{0, 1, \ldots, r_3 - 2\} \times \{0, 1, \ldots, r_1 - 1\} \to \{r_1, r_1 + 1, \ldots, n_2 - 1\},$$

which we will use in a moment to transform a double sum into a simple one.

Continuing with the previous equality and using the congruence $hr_1 \equiv r_2 \pmod{r_3}$, we thus have

$$
\begin{aligned}
0 &= \sum_{u=0}^{r_3-2}\sum_{x_2=0}^{r_1-1} u \cdot X_{x_2-ur_2-1} + \sum_{u=0}^{r_3-2}\sum_{x_2=0}^{r_1-1} Y_{x_2-(u+1)hr_1} + \underbrace{\sum_{q=0}^{r_1-1} Y_q}_{=0} \\
&+ \sum_{x_2=0}^{r_1-1}(r_3-1)\cdot X_{x_2-(r_3-1)r_2-1} + \sum_{u=0}^{r_3-1} u \cdot \sum_{x_1=r_2(r_3-1)}^{n_1-2}\sum_{x_2=0}^{r_1-1} X_{x_2-ur_2-x_1-2} \\
&= \sum_{u=0}^{r_3-1}\sum_{x_2=0}^{r_1-1} u \cdot X_{x_2-ur_2-1} + \underbrace{\sum_{q=r_1}^{n_2-1} Y_q + \sum_{q=0}^{r_1-1} Y_q}_{=\alpha} \\
&+ \sum_{u=0}^{r_3-1} u \cdot \sum_{x_1=r_2(r_3-1)}^{n_1-2}\sum_{x_2=0}^{r_1-1} X_{x_2-ur_2-x_1-2} \\
&= \alpha + \sum_{u=0}^{r_3-1} u \cdot \sum_{x_1=r_2(r_3-1)}^{n_1-1}\sum_{x_2=0}^{r_1-1} X_{x_2-ur_2-x_1-2}
\end{aligned}
$$

After using the equality $\alpha = 0$ by Lemma 30 and the substitutions $v = n_1 - 1 - x_1$,

$w = x_2$. this becomes

$$
\begin{aligned}
0 &= \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{r_2-1} \sum_{w=0}^{r_1-1} X_{v+w-ur_2-1} \\
&= \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{r_2-1} \left(\sum_{w=0}^{\overline{r_1}-1} X_{v+w-ur_2-1} + \sum_{w=\overline{r_1}}^{r_1-1} X_{v+w-ur_2-1} \right) \\
&= \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{r_2-1} \sum_{w=0}^{\overline{r_1}-1} X_{v+w-ur_2-1} + \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{r_2-1} \frac{r_1 - \overline{r_1}}{r_3} \cdot \sum_{w=\overline{r_1}}^{\overline{r_1}+r_3-1} X_{v+w-ur_2-1} \\
&= \sum_{u=0}^{r_3-1} u \cdot \sum_{w=0}^{\overline{r_1}} \sum_{v=0}^{r_2-1} X_{v+w-ur_2-1} + \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{r_2-1} \frac{r_1 - \overline{r_1}}{r_3} \cdot \beta \\
&= \sum_{u=0}^{r_3-1} u \cdot \sum_{w=0}^{\overline{r_1}} \sum_{v=0}^{r_1} X_{v+w-ur_2-1} \\
&= \sum_{u=0}^{r_3-1} u \cdot \sum_{w=0}^{\overline{r_1}} \left(\sum_{v=0}^{\overline{r_2}-1} X_{v+w-ur_2-1} + \sum_{v=\overline{r_2}}^{r_2-1} X_{v+w-ur_2-1} \right) \\
&= \sum_{u=0}^{r_3-1} u \cdot \sum_{w=0}^{\overline{r_1}} \sum_{v=0}^{\overline{r_2}-1} X_{v+w-ur_2-1} + \sum_{u=0}^{r_3-1} u \cdot \sum_{w=0}^{\overline{r_1}-1} \frac{r_2 - \overline{r_2}}{r_3} \cdot \sum_{v=\overline{r_2}}^{\overline{r_2}+r_3-1} X_{v+w-ur_2-1} \\
&= \Delta + \sum_{u=0}^{r_3-1} u \cdot \sum_{w=0}^{\overline{r_1}-1} \frac{r_2 - \overline{r_2}}{r_3} \cdot \beta \\
&= \Delta,
\end{aligned}
$$

since $\beta = 0$ by Lemma 30. □

Since β, Γ_q and Δ are linear combinations of the X_u, they can be written as $\sum_{c=0}^{r_3-1} c_u X_u$, and thus correspond to the r_3-tuples of integer coefficients $(c_1, c_2, \ldots, c_{r_3})$. Using this correspondence, we will now construct a matrix M of size $r_3 \times r_3$ (indexing its dimensions from 0 to $r_3 - 1$) as follows:

- The 0-th row will correspond to β (i.e., it will consist of all 1's).
- The q-th row for $1 \leq q \leq r_3 - 2$ will correspond to Γ_{q-1}.
- The $r_3 - 1$-th row will correspond to Δ.

Since the rows of M are coefficients of valid equalities in Q, we have $M \cdot X' = 0$, where $X = (X_0, X_1, \ldots, X_{r_3-1})$ and $'$ denotes transposition. We will show that M is unimodular, i.e. invertible over $\mathbb{Z}$, from which it will follow that $X = 0$. To do that, we will study the effect of multiplying M by a character matrix (i.e., basically performing the discrete Fourier transform).

Let

$$R(x) := \sum_{q=0}^{r_3-1} x^q \in \mathbb{Z}[x],$$

$$D(x) := \sum_{q=0}^{r_3-1} q \cdot x^q \in \mathbb{Z}[x]$$

and

$$P(x) := -x^{r_2-1} \cdot \sum_{q=0}^{r_1-1} x^q \in \mathbb{Z}[x].$$

Lemma 33. *Let $\zeta \neq 1$ be any r_3-th root of unity. Then we have $R(\zeta) = 0$ and*

$$D(\zeta) \cdot (\zeta - 1) = r_3.$$

Proof. The first assertion is immediate since $R(\zeta) \cdot (\zeta - 1) = \zeta^{r_3} - 1 = 0$, but $\zeta \neq 1$. The second follows from the computation

$$\begin{aligned} D(\zeta) \cdot (\zeta - 1) &= \sum_{q=1}^{r_3-1} q \cdot \zeta^{q+1} - \sum_{q=1}^{r_3-1} q \cdot \zeta^q = \sum_{q=2}^{r_3} (q-1) \cdot \zeta^q - \sum_{q=1}^{r_3-1} q \cdot \zeta^q \\ &= (r_3 - 1)\zeta^{r_3} + \sum_{q=1}^{r_3-1} (q-1) \cdot \zeta^q - \sum_{q=1}^{r_3-1} q \cdot \zeta^q \\ &= r_3 - 1 - \sum_{q=1}^{r_3-1} \zeta^q \\ &= r_3 - R(\zeta) \\ &= r_3. \end{aligned}$$

□

Now let $\mathcal{X}$ be the free $\mathbb{Z}$-module with generators $X_0, X_1, \ldots, X_{r_3-1}$. By abuse of notation, we can consider $\widehat{\beta}, \widehat{\Gamma}_q, \widehat{\Delta} \in \mathcal{X}$ for $0 \leq q \leq r_3 - 3$, which formally look the same as β, Γ_q, Δ. Moreover let ζ be any r_3-th root of unity and consider the $\mathbb{Z}$-module homomorphism from $\mathcal{X}$ to the cyclotomic field $\mathbb{Q}(\zeta)$ given by

$$\sum_{u=0}^{r_3-1} c_u X_u \mapsto \sum_{u=0}^{r_3-1} c_u \zeta^u.$$

We can apply this homomorphism to $\widehat{\beta}, \widehat{\Gamma}_q, \widehat{\Delta}$, and we will denote its respective values by $\beta(\zeta), \Gamma_q(\zeta), \Delta(\zeta) \in \mathbb{Q}(\zeta)$. Note that since $\zeta^{r_3} = 1$, these values depend on the indices of X_u only modulo r_3, so it doesn't matter whether we regard them as in the set $\{0, 1, \ldots, r_3 - 1\}$ or just as integers. This will allow us to use the original definitions of β, Γ_q, Δ for their computation quite easily.

Lemma 34. *For any $b \in \mathbb{N}$ and $y \in \mathbb{C}$, we have the equality*

$$(y-1) \cdot \sum_{u=1}^{b} u \cdot y^u = (b+1)y^{b+1} - \sum_{u=0}^{b} y^{u+1}.$$

Proof. We have

$$\begin{aligned}
(y-1) \cdot \sum_{u=1}^{b} u \cdot y^u &= \sum_{u=1}^{b} u \cdot y^{u+1} - \sum_{u=1}^{b} u \cdot y^u \\
&= \sum_{u=0}^{b} u \cdot y^{u+1} - \sum_{v=0}^{b-1} (v+1) \cdot y^{v+1} \\
&= b \cdot y^{b+1} + \sum_{u=0}^{b-1} (u-(u+1)) \cdot y^{u+1} \\
&= b \cdot y^{b+1} + \underbrace{y^{b+1} - y^{b+1}}_{=0} + \sum_{u=0}^{b-1} -1 \cdot y^{u+1} \\
&= (b+1)y^{b+1} - \sum_{u=0}^{b} y^{u+1}.
\end{aligned}$$

□

Lemma 35. *Let $\zeta \neq 1$ be any r_3-th root of unity. Then for all $0 \leq q < r_3 - 3$, we have*

$$\beta(\zeta) = 0,$$

$$\Gamma_q(\zeta) = \zeta^q \cdot P(\zeta)$$

and

$$\Delta(\zeta) = D(\zeta) \cdot P(\zeta).$$

Proof. Note that $\zeta^{-r_2} \neq 1$, since $\gcd(r_3, -r_2) = 1$ and $\zeta \neq 1$.

From the definitions and Lemma 33, we directly get $\beta(\zeta) = R(\zeta) = 0$. For the second assertion, we have

$$
\begin{aligned}
\Gamma_q(\zeta) &= \sum_{u=0}^{r_3-h'-1} \sum_{v=0}^{\overline{r_2}-1} \zeta^{q+v-ur_2-1} \\
&= \zeta^{q-1} \cdot \sum_{v=0}^{\overline{r_2}-1} \zeta^v \sum_{u=0}^{r_3-h'-1} \zeta^{-ur_2} \\
&= \zeta^{q-1} \cdot (1+\zeta+\cdots+\zeta^{\overline{r_2}-1})(1+\zeta^{-r_2}+\zeta^{-2r_2}+\cdots+\zeta^{-(r_3-h'-1)r_2}) \\
&= \zeta^{q-1} \cdot \frac{\zeta^{\overline{r_2}}-1}{\zeta-1} \cdot \frac{\zeta^{-(r_3-h')r_2}-1}{\zeta^{-r_2}-1} \\
&= \zeta^{q-1} \cdot \frac{\zeta^{r_2}-1}{\zeta^{-r_2}-1} \cdot \frac{\zeta^{r_1}-1}{\zeta-1} \\
&= -\zeta^q \cdot \zeta^{r_2-1} \cdot (1+\zeta+\zeta^2+\cdots+\zeta^{r_1-1}) \\
&= \zeta^q \cdot P(\zeta).
\end{aligned}
$$

Similarly, using Lemma 34 with $y = \zeta^{-r_2}$ and $b = r_3 - 1$, we can see that

$$
\begin{aligned}
\Delta(\zeta) &= \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{\overline{r_2}-1} \sum_{w=0}^{\overline{r_1}-1} \zeta^{v+w-ur_2-1} \\
&= \zeta^{-1} \cdot \sum_{v=0}^{\overline{r_2}-1} \zeta^v \sum_{w=0}^{\overline{r_1}-1} \zeta^w \sum_{u=0}^{r_3-1} u \cdot \zeta^{-ur_2} \\
&= \zeta^{-1}(1+\zeta+\ldots+\zeta^{\overline{r_2}-1})(1+\zeta+\ldots+\zeta^{\overline{r_1}-1})(\zeta^{-r_2}+2\zeta^{-2r_2}+\ldots+(r_3-1)\zeta^{-(r_3-1)r_2}) \\
&= \zeta^{-1} \cdot \frac{\zeta^{\overline{r_2}}-1}{\zeta-1} \cdot \frac{\zeta^{\overline{r_1}}-1}{\zeta-1} \cdot \frac{r_3\zeta^{-r_2r_3} - \sum_{u=0}^{r_3-1} \zeta^{-r_2(u+1)}}{\zeta^{-r_2}-1} \\
&= \zeta^{-1} \cdot \frac{\zeta^{\overline{r_2}}-1}{\zeta-1} \cdot \frac{\zeta^{\overline{r_1}}-1}{\zeta-1} \cdot \frac{r_3(\zeta^{r_3})^{r_2} - \zeta^{-r_2} \cdot R(\zeta^{-r_2})}{\zeta^{-r_2}-1} \\
&= \zeta^{-1} \cdot \frac{\zeta^{r_2}-1}{\zeta-1} \cdot \frac{\zeta^{r_1}-1}{\zeta-1} \cdot \frac{r_3}{\zeta^{-r_2}-1} \\
&= \zeta^{-1} \cdot \frac{r_3}{\zeta-1} \cdot \frac{\zeta^{r_2}-1}{\zeta^{-r_2}-1} \cdot \frac{\zeta^{r_1}-1}{\zeta-1} \\
&= -D(\zeta) \cdot \zeta^{r_2-1} \cdot (1+\zeta+\zeta^2+\cdots+\zeta^{r_1-1}) \\
&= D(\zeta) \cdot P(\zeta).
\end{aligned}
$$

□

Proposition 36. *M is unimodular, hence $X = 0$.*

Proof. Let ζ_{r_3} be a primitive r_3-th root of unity and let C be the corresponding $r_3 \times r_3$ character matrix, i.e. $C = (\zeta_{r_3}^{r \cdot c})_{0 \le r,c < r_3}$. We will use the two previous lemmas together with the fact that multiplying a column of successive powers of ζ_{r_3} by a row of M from the left corresponds to evaluating the polynomial obtained from this row at ζ_{r_3}. Hence we have $M \cdot C = C'$, where $C'_{0,0} = R(1) = r_3$ and the $c-th$ column of C' is

$$\begin{pmatrix} R(\zeta_{r_3}^c) \\ P(\zeta_{r_3}^c) \\ \zeta_{r_3}^c \cdot P(\zeta_{r_3}^c) \\ (\zeta_{r_3}^c)^2 \cdot P(\zeta_{r_3}^c) \\ \vdots \\ (\zeta_{r_3}^c)^{r_3-3} \cdot P(\zeta_{r_3}^c) \\ D(\zeta_{r_3}^c) \cdot P(\zeta_{r_3}^c) \end{pmatrix} = \begin{pmatrix} 0 \\ P(\zeta_{r_3}^c) \\ \zeta_{r_3}^c \cdot P(\zeta_{r_3}^c) \\ \zeta_{r_3}^{2c} \cdot P(\zeta_{r_3}^c) \\ \vdots \\ \zeta_{r_3}^{(r_3-3)c} \cdot P(\zeta_{r_3}^c) \\ D(\zeta_{r_3}^c) \cdot P(\zeta_{r_3}^c) \end{pmatrix}$$

for any $0 < c < r_3$ (we don't need to specify the rest of the 0-th column, since it doesn't influence the determinant of C'). Thus by taking out $P(\zeta_{r_3}^c)$ from each of these columns, we get (using that multiplication by r_1 is an automorphism of $\mathbb{Z}/r_3$, since $\gcd(r_1, r_3) = 1$)

$$|\det C'| = |\det C''| \cdot \left| \prod_{0<c<r_3} P(\zeta_{r_3}^c) \right| = |\det C''| \cdot \left| \prod_{0<c<r_3} -\zeta_{r_3}^{c(r_2-1)} \right| \cdot \left| \prod_{0<c<r_3} \frac{\zeta_{r_3}^{cr_1} - 1}{\zeta_{r_3}^c - 1} \right| = |\det C''|,$$

where

$$C'' = \begin{pmatrix} r_3 & 0 & \dots & 0 & \dots & 0 \\ * & 1 & \dots & 1 & \dots & 1 \\ * & \zeta_{r_3} & \dots & \zeta_{r_3}^c & \dots & \zeta_{r_3}^{r_3-1} \\ * & \zeta_{r_3}^2 & \dots & \zeta_{r_3}^{2c} & \dots & \zeta_{r_3}^{2(r_3-1)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ * & \zeta_{r_3}^{r_3-3} & \dots & \zeta_{r_3}^{(r_3-3)c} & \dots & \zeta_{r_3}^{(r_3-3)(r_3-1)} \\ * & D(\zeta_{r_3}) & \dots & D(\zeta_{r_3}^c) & \dots & D(\zeta_{r_3}^{r_3-1}) \end{pmatrix}.$$

On the other hand, we can take the matrix C, add all of its rows to the $r_3 - 1$-th one (thus creating $\begin{pmatrix} r_3 & 0 & 0 & \dots & 0 \end{pmatrix}$ there) and then, using the equality

$$-\zeta_{r_3}^{(r_3-2)c} + \sum_{u=0}^{r_3-3} (u - r_3 + 1) \cdot \zeta_{r_3}^{uc} = \sum_{u=0}^{r_3-1} u \cdot \zeta_{r_3}^{uc} - (r_3 - 1) \cdot \underbrace{\sum_{u=0}^{r_3-1} \zeta_{r_3}^{uc}}_{=0},$$

multiply the $(r_3 - 2)$-th row by -1 and add the u-th row multiplied by $(u - r_3 + 1)$ for each $0 \le u \le r_3 - 3$, so that the $r_3 - 2$-th row will become

$$\begin{pmatrix} * & D(\zeta_{r_3}) & \dots & D(\zeta_{r_3}^c) & \dots & D(\zeta_{r_3}^{r_3-1}) \end{pmatrix}.$$

Thus we will obtain a matrix with the same determinant as C'' (up to a sign). Since the elementary row operations preserve the determinant up to a sign, it follows that

$$|\det C| = |\det C''| = |\det C'| = |\det M| \cdot |\det C|.$$

Now, C can be seen as a special type of a Vandermonde matrix, so we have

$$\det C = \prod_{0\le r<c<r_3} (\zeta_{r_3}^r - \zeta_{r_3}^c) \neq 0$$

(in fact it is well known that this equals $\pm\sqrt{r_3^{r_3}}$), which implies that $|\det M| = 1$, as needed. □

Corollary 37. *We have $Y_q = 0$ for all $r_1 + r_3 - 2 \le q \le n_2 - 1$.*

Proof. By the Chinese remainder theorem, it suffices to show by induction with respect to $u = 0, 1, \ldots, r_3 - 1$ that for any $0 \le v < r_1$, we have $Y_{v-uhr_1} = 0$. The base case $u = 0$ follows directly from the definition of Y_u. Now suppose the statement is true for $0 \le u < r_3 - 1$. Then using $N_1 \sim 0$ and Lemma 29, we get

$$\begin{aligned} 0 = \sigma_2^{v-uhr_1} N_1 \cdot \mu &= \sum_{x_1=r_2(r_3-1)-1}^{n_1-1} \sigma_1^{x_1}\sigma_2^{v-uhr_1} \cdot \mu \\ &= \underbrace{Y_{v-uhr_1}}_{=0} - Y_{v-uhr_1-hr_1} + \sum_{x_1=r_2(r_3-1)}^{n_1-1} \underbrace{X_{v-uhr_1-x_1-2}}_{=0} = -Y_{v-(u+1)hr_1} \end{aligned}$$

by the induction hypothesis and the fact that $X = 0$. This completes the induction. □

It now follows that Q is trivial, so we have proven the following theorem:

Theorem 38. *Under the assumptions on page 0.3, the set $B_5 \cup B_D$ forms a basis of D^+ and the set $B_5 \cup B_C$ forms a basis of C^+.*

0.11 The module of relations

0.12 Construction of suitable abelian fields

Let $m, a_1, a_2, a_3, a_4, r_1, r_2, r_3, r_4$ be positive integers such that

$$m > 1, r_i \mid m, \gcd(r_i, r_j, r_l) = 1.$$

We will construct an infinite family of fields k that satisfy all of our assumptions such that these integers correspond to the parameters in our problem of the same name (again we will denote $n_i = \frac{m}{r_i}$).

First, we will fix distinct primes p_1, p_2, p_3, p_4 such that $p_i \equiv 1 \pmod{2a_in_i}$ (by Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many ways of doing this). Then there exist even Dirichlet characters χ_i of conductors p_i and orders a_in_i (namely, these can be given as $\chi_i := \chi^{\frac{p_i-1}{a_in_i}}$, where χ is any generator of the cyclic group $\widehat{(\mathbb{Z}/p_i\mathbb{Z})^\times}$ (note that $p_i > 2$)).

Now let K_i be the field associated to $\langle\chi_i\rangle$. Then K_i is real (because χ_i is even) and $\mathrm{Gal}(K_i/\mathbb{Q})$ is cyclic of order $a_i n_i$, say $\mathrm{Gal}(K_i/\mathbb{Q}) = \langle\sigma_i\rangle$. Moreover, since the conductors p_i are coprime, the group $\langle\chi_1,\chi_2,\chi_3,\chi_4\rangle$ corresponds to the compositum field $K = K_1K_2K_3K_4$. By the theory of Dirichlet characters, K is ramified exactly at primes p_i (with inertia subgroups isomorphic to $\mathrm{Gal}(K_i/\mathbb{Q})$) and

$$\mathrm{Gal}(K/\mathbb{Q}) = \mathrm{Gal}(K_1/\mathbb{Q})\mathrm{Gal}(K_2/\mathbb{Q})\mathrm{Gal}(K_3/\mathbb{Q})\mathrm{Gal}(K_4/\mathbb{Q}) = \langle\sigma_1,\sigma_2,\sigma_3,\sigma_4\rangle,$$

so that $[K:\mathbb{Q}] = a_1a_2a_3a_4\frac{m^4}{r_1r_2r_3r_4}$. Now let $\tau := \sigma_1^{a_1}\sigma_2^{a_2}\sigma_3^{a_3}\sigma_4^{a_4}$ and let k be the subfield of K fixed by τ. Since k is a subfield of a compositum of real fields, it must also be real. In order to reach our goal, we now only need to prove the following theorem (it is not hard to see that we could have used the results from Lemma 9 and Proposition 10 as definitions instead).

Theorem 39. *In the above notation, we have $[K:k]=m$, $[K:kK_i]=r_i$, $[k\cap K_i:\mathbb{Q}]=a_i$ and $kK_iK_jK_l = K$ (i.e. K is the genus field in the narrow sense of k).*

Proof. Using Lemma 12 several times, we can compute

$$[K:k] = |\langle\tau\rangle| = \mathrm{lcm}\,(n_i,n_j,n_l) = m,$$

$$[K:kK_i] = |\langle\tau\rangle\cap\langle\sigma_j\sigma_l\sigma_h\rangle| = |\langle\tau^{a_in_i}\rangle| = r_i,$$

$$[k\cap K_i:/\mathbb{Q}] = [\langle\sigma_1,\sigma_2,\sigma_3,\sigma_4\rangle:\langle\tau,\sigma_j,\sigma_l,\sigma_h\rangle] = [\langle\sigma_1,\sigma_2,\sigma_3,\sigma_4\rangle:\langle\sigma_i^{a_i},\sigma_j,\sigma_l,\sigma_h\rangle] = a_i$$

and

$$[K:kK_iK_jK_l] = |\langle\tau\rangle\cap\langle\sigma_h\rangle| = |\langle\tau^{\mathrm{lcm}(n_i,n_j,n_l)}\rangle| = |\langle\tau^m\rangle| = 1.$$

□

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