

## PROBABILITY AND STATISTICS – PROBLEM SET 10

1. If  $X_1, \dots, X_n$  is an independent sample of size  $n$  from a population defined by the pmf  $f(x) = \theta^x(1 - \theta)^{1-x}$ ,  $x = 0, 1$ , where  $0 < \theta < 1$ , find the maximum likelihood estimator for  $\theta$ .
2. Find the maximum likelihood estimator for  $\theta$  in each of the following cases:
  - (a)  $f(x; \theta) = \theta x^{\theta-1}$ ,  $0 < x < 1$ , where  $\theta > 0$ .
  - (b)  $f(x; \theta) = \theta e^{-\theta x}$ ,  $x > 0$ , where  $\theta > 0$ .
  - (c)  $f(x; \theta) = e^{-(x-\theta)}$ ,  $x \geq \theta$ , where  $\theta$  is any real number.
  - (d)  $f(x; \theta) = \frac{1}{2}e^{-|x-\theta|}$ , where  $\theta$  is any real number.
  - (e)  $f(x; \theta) = \frac{2x}{\theta^2}$ ,  $0 \leq x \leq \theta$ , where  $\theta > 0$ .
3. Find the mles of  $\theta_1$  and  $\theta_2$  in each of the following cases:
  - (a)  $X \sim N(\theta_1, \theta_2)$
  - (b)  $f(x; \theta_1, \theta_2) = \frac{1}{\theta_2} e^{-\frac{(x-\theta_1)}{\theta_2}}$ ,  $x > \theta_1$   
where  $\theta_1$  is any real number and  $\theta_2 > 0$ .
4. Find the mle of  $\theta$  in each of the following cases:
  - (a)  $X \sim B(m, \theta)$ .
  - (b)  $X \sim \mathcal{P}(\theta)$ .
  - (c)  $X \sim N(\mu, \theta)$ .
5. Let  $X \sim U[0, \theta]$ , where  $\theta > 0$ . Compute the mle for  $\theta$ .
6. Show that the sample mean is an unbiased and consistent estimator of population mean.
7. Show that sample variance is a biased estimator of the population variance.
8. Show that  $Y = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$  is an unbiased estimator of population variance.
9. If  $X \sim N(0, \theta)$ , show that  $Y = \frac{1}{n} \sum_{i=1}^n X_i^2$  is an unbiased estimator of  $\theta$  and has variance  $\frac{2\theta^2}{n}$ .
10. Show that  $\bar{X}$  is an unbiased estimator of  $\theta$  if the pdf of  $X$  is  $f(x; \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$ ,  $x > 0$ , where  $\theta > 0$ . Also show that  $\bar{X}$  has variance  $\frac{\theta^2}{n}$  and is therefore a consistent estimator.

11. Let  $Y_n$  be an unbiased estimator of  $\theta$ , such that  $V(Y_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Then show that  $Y_n$  is consistent.
12. Let  $Y_1$  and  $Y_2$  be two independent unbiased statistics for  $\theta$  such that the variance of  $Y_1$  is twice that of  $Y_2$ . Find the constants  $k_1$  and  $k_2$  such that  $Z = k_1 Y_1 + k_2 Y_2$  is an unbiased statistic for  $\theta$  with minimum possible variance for such a linear combination.
13. Find the same size  $n$  such that  $P[\bar{X} - 1 < \mu < \bar{X} + 1] = 0.9$ , given that  $X \sim N(\mu, 9)$ .
14. If the observed value of the mean of a sample of size 20 from a population having distribution  $N(\mu, 80)$  is  $\bar{x} = 81.2$ , find a 95 percent confidence interval for the population mean.
15. If a random sample of size 17 from a normal distribution  $N(\mu, \sigma^2)$  yields  $\bar{x} = 4.7$  and  $s^2 = 5.76$ , determine a 90 percent confidence interval for  $\mu$ .
16. The observed values of a random sample from a population having distribution  $N(8, \sigma^2)$  are 8.6, 7.9, 8.3, 6.4, 8.4, 9.8, 7.2, 7.8, and 7.5. Construct a 90 percent confidence interval for  $\sigma^2$ .
17. A random sample of size 9 from the distribution  $N(\mu, \sigma^2)$  yields  $s^2 = 7.63$ . Determine a 95 percent confidence interval for  $\sigma^2$ .
18. Find an approximate 95 percent confidence interval for the mean of a population having variance 100, if the sample size is 25.
19. A random sample of size 15 from a normal population with unknown mean and variance yields  $\bar{x} = 3.2$  and  $s^2 = 4.24$ . Determine a 95 percent confidence interval for  $\sigma^2$ .
20. If a sample of size 15 from a population with distribution  $N(\mu, \sigma^2)$  yields values  $\sum_{i=1}^{15} X_i = 8.7$  and  $\sum_{i=1}^{15} X_i^2 = 27.3$ , obtain a 95 percent confidence interval for  $\sigma^2$ .
21. Suppose that  $X \sim N(8, \sigma^2)$ , and the observed values of a sample are of size 9 from this population are 8.6, 7.9, 8.3, 6.4, 8.4, 9.8, 7.2, 7.8, and 7.5. Construct a 90 percent confidence interval for  $\sigma^2$ .
22. Suppose that  $X \sim N(\mu, 4)$ . If  $\bar{x} = 78.3$  with  $n = 25$ , obtain a 99 percent confidence interval for  $\mu$ .