PROBABILITY AND STATISTICS – PROBLEM SET 10

1. If $X_1, \dots, X_n$ is an independent sample of size n from a population defined by the pmf $f(x) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$, where $0 < \theta < 1$, find the maximum likelihood estimator for θ.

2. Find the maximum likelihood estimator for θ in each of the following cases:
   (a) $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, where $\theta > 0$.
   (b) $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$, where $\theta > 0$.
   (c) $f(x;\theta) = e^{-(x-\theta)}$, $x \geqslant \theta$, where θ is any real number.
   (d) $f(x;\theta) = \frac{1}{2}e^{-|x-\theta|}$, where θ is any real number.
   (e) $f(x;\theta) = \dfrac{2x}{\theta^2}$, $0 \leqslant x \leqslant \theta$, where $\theta > 0$.

3. Find the mles of θ_1 and θ_2 in each of the following cases:
   (a) $X \sim N(\theta_1, \theta_2)$
   (b) $f(x;\theta_1,\theta_2) = \dfrac{1}{\theta_2}e^{-\frac{(x-\theta_1)}{\theta_2}}$, $x > \theta_1$
   where θ_1 is any real number and $\theta_2 > 0$.

4. Find the mle of θ in each of the following cases:
   (a) $X \sim B(m, \theta)$.
   (b) $X \sim \mathcal{P}(\theta)$.
   (c) $X \sim N(\mu, \theta)$.

5. Let $X \sim U[0, \theta]$, where $\theta > 0$. Compute the mle for θ.

6. Show that the sample mean is an unbiased and consistent estimator of population mean.

7. Show that sample variance is a biased estimator of the population variance.

8. Show that $Y = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is an unbiased estimator of population variance.

9. If $X \sim N(0, \theta)$, show that $Y = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ is an unbiased estimator of θ and has variance $\frac{2\theta^2}{n}$.

10. Show that $\overline{X}$ is an unbiased estimator of θ if the pdf of X is $f(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$, $x > 0$, where $\theta > 0$. Also show that $\overline{X}$ has variance $\frac{\theta^2}{n}$ and is therefore a consistent estimator.

11. Let Y_n be an unbiased estimator of θ, such that $V(Y_n) \to 0$ as $n \to \infty$. Then show that Y_n is consistent.

12. Let Y_1 and Y_2 be two independent unbiased statistics for θ such that the variance of Y_1 is twice that of Y_2. Find the constants k_1 and k_2 such that $Z = k_1Y_1 + k_2Y_2$ is an unbiased statistic for θ with minimum possible variance for such a linear combination.

13. Find the same size n such that $P[\overline{X} - 1 < \mu < \overline{X} + 1] = 0.9$, given that $X \sim N(\mu, 9)$.

14. If the observed value of the mean of a sample of size 20 from a population having distribution $N(\mu, 80)$ is $\overline{x} = 81.2$, find a 95 percent confidence interval for the population mean.

15. If a random sample of size 17 from a normal distribution $N(\mu, \sigma^2)$ yields $\overline{x} = 4.7$ and $s^2 = 5.76$, determine a 90 percent confidence interval for μ.

16. The observed values of a random sample from a population having distribution $N(8, \sigma^2)$ are 8.6, 7.9, 8.3, 6.4, 8.4, 9.8, 7.2, 7.8, and 7.5. Construct a 90 percent confidence interval for σ^2.

17. A random sample of size from the distribution $N(\mu, \sigma^2)$ yields $s^2 = 7.63$. Determine a 95 percent confidence interval for σ^2.

18. Find an approximate 95 percent confidence interval for the mean of a population having variance 100, if the sample size is 25.

19. A random sample of size 15 from a normal population with unknown mean and variance yields $\overline{x} = 3.2$ and $s^2 = 4.24$. Determine a 95 percent confidence interval for σ^2.

20. If a sample of size 15 from a population with distribution $N(\mu, \sigma^2)$ yields values $\sum_{i=1}^{15} X_i = 8.7$ and $\sum_{i=1}^{15} X_i^2 = 27.3$, obtain a 95 percent confidence interval for σ^2.

21. Suppose that $X \sim N(8, \sigma^2)$, and the observed values of a sample are of size 9 from this population are 8.6, 7.9, 8.3, 6.4, 8.4, 9.8, 7.2, 7.8, and 7.5. Construct a 90 percent confidence interval for σ^2.

22. Suppose that $X \sim N(\mu, 4)$. If $\overline{x} = 78.3$ with $n = 25$, obtain a 99 percent confidence interval for μ.