PROBABILITY AND STATISTICS – PROBLEM SET 7

1. If X is a discrete random variable with pmf $f(x)$, compute the pmf $g(y)$ of $Y = X^2 - 1$ in each of the following cases:
   (a) $f(x) = \frac{1}{3}$, $x = 1, 2, 3$.
   (b) $f(x) = \frac{1}{3}$, $x = -1, 0, 1$.
   (c) $f(x) = \frac{|x|}{2}$, $x = -1, 0, 1$.
2. If $X \sim U[-1, 1]$, determine the pdfs of $Y = \sin\frac{\pi x}{2}$ and $Z = \cos\frac{\pi x}{2}$.
3. Find the pdf of $Y = -\log X^4$ in each of the following cases:
   (a) $X \sim U[0, 1]$.
   (b) X has pdf $f(x) = 4x^3$, $0 < x < 1$.
4. Show that if X follows the Cauchy distribution with pdf $f(x) = \frac{1}{\pi(1+x^2)}$, then so does $Y = \frac{1}{X}$.
5. Compute the pdf of $Y = \tan X$, if $X \sim U\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
6. If $X \sim U[-1, 1]$, find the pdf of
   (a) $Y = X^3$.
   (b) $Z = X^4$.
   (c) $W = X^n$, where n is any positive integer.
7. Compute the pdf of $Y = X^2$ if $X \sim U[-1, 2]$.
8. If X is a random variable with pdf $f(x) = 2x$, $0 < x < 1$, compute the pdf of $Y = e^{-X}$.
9. Let X be a random variable with pdf $f(x) = \frac{1}{2x^2}$, $|x| > 1$. Then show that $Y = \log X^2$ has an exponential distribution. What is the mean of Y?
10. (X, Y) is a two dimensional random variable having joint pdf $f(x, y) = 3xe^{-(x+3y)}$, $x, y > 0$. If $Z = X$ and $W = 2X + Y$, determine the distribution of (Z, W).
11. Let (X, Y) be uniformly distributed in the unit square $0 \le x, y \le 1$. Find the pdf of $Z = X + Y$.

12. If $X \sim \mathscr{E}(2)$ and $Y \sim \mathscr{E}(1)$ are independent, find the pdf of:
    (a) $Z = X + Y$.
    (b) $Z = X/Y$.
13. If (X,Y) has the joint pdf $f(x,y) = 10xy^2$, $0 < x < y < 1$, determine the pdf of $Z = X/Y$.
14. Find the pdf of $Z = X/Y$, if (X,Y) has joint pdf $f(x,y) = 8xy$, $0 < x < y < 1$.
15. (X,Y) is uniformly distributed over the unit disc $x^2 + y^2 \leq 1$. Find the pdf of $R = \sqrt{X^2 + Y^2}$.
16. Let X be a random variable with pdf $f(x) = \dfrac{5}{x^2}$, $x > 5$. If X_1 and X_2 are two independent random variables following this distribution, find the pdf of $Y = X_1/X_2$.
17. Let $W \sim N(0,1)$ and $V \sim \chi^2_n$. Compute the distribution of $T = \dfrac{W}{\sqrt{V/n}}$.
18. If (X_1,X_2) has the joint pdf $f(x_1,x_2) = 2e^{-(x_1+x_2)}$, $x_1 > x_2 > 0$, find the joint pdf of $Y_1 = X_1 - X_2$, $Y_2 = 2X_2$.
19. If (X,Y) is a two dimensional random variable with joint pdf $f(x,y) = 24xy$, $x > 0$, $y > 0$, $x + y < 1$, find the pdf of $Z = XY$.