PROBABILITY AND STATISTICS – PROBLEM SET 1

1. Very Easy

1.1. A box contains 10 paper slips, labelled $1, 2, \ldots, 10$. Find the probability that one slip drawn at random contains:
(a) the number 9.
(b) an even number.
(c) an even number or an odd number.
(d) an even number or a prime number.

1.2. A fair coin is tossed twice. Find the probability that
(a) A head is obtained on the first toss.
(b) A head is obtained on the first toss and a tail on the second.
(c) A head is obtained on at least one of the two tosses.

1.3. A fair, six-sided die is rolled. Find the probability that the outcome is
(a) 2
(b) an odd number.
(c) an odd number or an even number.
(d) an odd number or a composite number.

2. Easy

2.1. A box contains 55 paper slips – one labelled 1, two labelled 2, ..., ten labelled 10 (i.e., k slips labelled k, for each $k = 1, \ldots, 10$). Find the probability that one slip drawn at random contains:
(a) the number 9.
(b) an even number.
(c) an even number or an odd number.
(d) an even number or a prime number.

2.2. A coin with probability $1/3$ for heads and $2/3$ for tails is tossed twice. Find the probability that
(a) A head is obtained on the first toss.
(b) A head is obtained on the first toss and a tail on the second.
(c) A head is obtained on at least one of the two tosses.

2.3. A six-sided die is designed in such a way that the probability of occurrence of each face is proportional to the number on that face. Find the probability that the outcome, when the die is rolled once, is
   (a) 2
   (b) an odd number.
   (c) an odd number or an even number.
   (d) an odd number or a composite number.

2.4. Let m and n denote the two outcomes when two fair dice are rolled. Find the probability that
   (a) $m = 4$ or $n = 4$.
   (b) $\max(m, n) = 4$.
   (c) $\max(m, n) > 4$.

2.5. Three marbles are drawn simultaneously at random from a box containing 2 red, 3 green, and 5 blue marbles. Find the probability that
   (a) all three are green.
   (b) all three are blue.
   (c) all three are red.
   (d) at least one is red.
   (e) each one is green or blue.
   (f) one is red and two are blue.

2.6. A box of 100 lightbulbs manufactured in a factory has 10 defective lightbulbs. An inspector tests 5 lightbulbs selected randomly from the box. What is the probability that a defective one will be found?

2.7. A group of $2n$ boys and $2n$ girls is randomly divided into two equal groups. What is the probability that each group has the same number of boys and girls?

2.8. A box contains n paper slips, labelled $1, 2, \ldots, n$. Find the probability that two slips drawn at random contain consecutive numbers, if they are drawn one after the other
   (a) without replacement.
   (b) with replacement.

3. Normal Difficulty

3.1. A box contains 10 paper slips, labelled $1, \ldots, 10$. Slips are drawn at random without replacement, until 9 is obtained. Find the probability that 9 is obtained
   (a) in the n^{th} draw (for each $n = 1, \ldots, 10$).
   (b) after the n^{th} draw (for each $n = 1, \ldots, 9$). Note: Not necessarily *immediately* after it.
   (c) after 10 is obtained.
   (d) immediately after 10 is obtained.
   (e) immediately before or after 10 is obtained.

3.2. A coin with probability $1/3$ for heads and $2/3$ for tails is tossed until a head is obtained. Find the probability that
  (a) exactly n tosses are required ($n = 1, 2, \ldots$).
  (b) the number of tosses required is even.
  (c) at least n tosses are required.

3.3. A fair, six-sided die is rolled until the same face is obtained twice in succession. Find the probability that
  (a) exactly n rolls are required ($n = 2, 3, \ldots$).
  (b) 2 is obtained on the last two rolls (regardless of number of rolls required).
  (c) 2 is not obtained on any roll.
  (d) 2 is obtained on the last two rolls, but not before.

3.4. Let S be a set of n elements, and $\mathcal{P}(S)$ its power set – the collection of all subsets of S. Let A be a subset of S picked at random from $\mathcal{P}(S)$.
  (a) What is the probability that A has m elements ($0 \leqslant m \leqslant n$)?
  (b) If B is a given subset of S, what is the probability that $A = B$?

3.5. Let $S = \{s_1, s_2, \ldots, s_n\}$ be a set of n elements. Construct a random subset A of S as follows: For each $i = 1, \ldots, n$, toss a fair coin and on heads, include the element s_i in A, and on tails, exclude s_i from A.
  (a) What is the probability that A has m elements ($0 \leqslant m \leqslant n$)?
  (b) If B is a given subset of S, what is the probability that $A = B$?

3.6. Let $S = \{s_1, s_2, \ldots, s_n\}$ be a set of n elements, and consider a coin weighted such that heads occur twice as often as tails. Construct a random subset A of S as follows: For each $i = 1, \ldots, n$, toss the coin and on heads, include the element s_i in A, and on tails, exclude s_i from A.
  (a) What is the probability that A has m elements ($0 \leqslant m \leqslant n$)?
  (b) If B is a given subset of S, what is the probability that $A = B$?

4. Seemingly Difficult

4.1. A box contains n paper slips labelled $1, \ldots, n$ ($n \geqslant 9$). Slips are drawn at random with replacement, until 9 is obtained. What is the minimum value of n such that the probability that more than 10 draws are required is at least 0.5? (Take your time to understand this question correctly.)

4.2. A coin with probability $1/3$ for heads and $2/3$ for tails is tossed n times. What is the minimum value of n such that the probability that at least 2 heads are obtained is at least 0.5?

4.3. A fair, six-sided die is rolled until the total of all outcome 2 is obtained a total of n times (not necessarily in consecutive throws). Find the probability that exactly m throws are required ($m = 1, 2, \ldots$).