Category Theory for Algebra II

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Contents

1 Examples of Categories

Before formally defining a category, we shall look at some examples of familiar categories. A category consists of a collection of obects and a collection of morphisms satisfying certain conditions that will be discussed later.

1. The category Set has sets as objects and functions as morphisms.
2. The collection of groups and group homomorphisms forms a category denoted by Grp.
3. The collection of vector spaces over a fixed field F, together with F-linear transformations forms the category Vect_F.
4. The collection of sets and relations forms the category Rel.

2 Initial and Final Objects

An initial object in a category C is an object A such that for all objects X of C, there exists a unique morphism from A to X. A final object in C is an object A such that for all objects X of C, there exists a unique morphism from X to A. A zero object is an object that is both initial and final.

Example 2.1.
1. The empty set $\varnothing$ is the initial object and any singleton set is a final object in Set.
2. The trivial group is both the initial and final object in Grp.
3. The zero-dimensional vector space is the zero object in Vect_F.
4. The empty set is the zero object in Rel.

3 Categories

A category C is a collection $\mathrm{Ob}(\mathsf{C})$ of objects and, for every two objects X and Y a collection $\mathrm{Hom}(X, Y)$ of morphisms or arrows, together with operations

$$\circ\colon \operatorname{Hom}(Y,Z)\times\operatorname{Hom}(X,Y)\to\operatorname{Hom}(X,Z)$$

for all $X,Y,Z\in\operatorname{Ob}(\mathsf{C})$, satisfying the following:

(C1) $\operatorname{Hom}(X_1,Y_1)$ and $\operatorname{Hom}(X_2,Y_2)$ are disjoint unless $X_1=X_2$ and $Y_1=Y_2$.
(C2) For each $X\in\operatorname{Ob}(\mathsf{C})$, there exists an identity morphism id_X or 1_X such that if $f\in\operatorname{Hom}(X,Y)$, there $f\circ\operatorname{id}_X=f$ and if $g\in\operatorname{Hom}(Z,X)$, then $\operatorname{id}_X\circ g=g$.
(C3) If $F\in\operatorname{Hom}(X,Y)$, $g\in\operatorname{Hom}(Y,Z)$, and $h\in\operatorname{Hom}(Z,W)$, then

$$h\circ(g\circ f)=(h\circ g)\circ f.$$

This is the law of associativity.

The collection of all morphisms of C is denoted by $\operatorname{Mor}(\mathsf{C})$. If $f\in\operatorname{Hom}(X,Y)$, then we write $f\colon X\to Y$ or $X\xrightarrow{f}Y$ and say that f is a morphism from X to Y, or that $X=\operatorname{dom}f$ is the domain and $Y=\operatorname{cod}f$ is the codomain of f.

Example 3.1. 1. $\mathsf{C}=\{X,Y\}$, $\operatorname{Mor}(\mathsf{C})=\{1_X,1_Y,f\colon X\to Y,g\colon Y\to X\}$ with 1_X and 1_Y as identity morphisms and $f\circ g=1_Y$, $g\circ f=1_X$. Graphically,

$$\begin{tikzcd} X \arrow[loop left, "1_X"] \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \arrow[loop right, "1_Y"] \end{tikzcd}$$

2. $\mathsf{C}=\{\bullet\}$, $\operatorname{Mor}(\mathsf{C})=\{1\}$.
3. Ring is the category of unital rings and unital ring homomorphisms.
4. Mon is the category of monoids and monoid homomorphisms.
5. Top is the category of topological spaces and continuous maps.
6. AbGrp is the category of Abelian groups and group homomorphisms.

Exercise 3.1. 1. In the category C given below, determine the compositions of all the non-identity morphisms.

$$\begin{tikzcd}
X \arrow[loop above, "1_X"] \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr, "k"'] & & Y \arrow[loop above, "1_Y"] \arrow[dl, "h"] \\
& Z \arrow[loop below, "1_Z"] &
\end{tikzcd}$$

2. In the category C given below, determine the compositions of all the non-identity morphisms.

$$\begin{tikzcd}
X \arrow[loop above, "1_X"] \arrow[dr, bend left, "h"] & & Y \arrow[loop above, "1_Y"] \arrow[ll, "f"'] \arrow[dl, "g"] \\
& Z \arrow[loop below, "1_Z"] \arrow[ul, bend left, "k"] &
\end{tikzcd}$$

3. If C is a category with $\mathrm{Ob}(C) = \{\bullet\}$ and $\mathrm{Mor}(C) = \{1, f, g\}$, such that $f \circ g = 1$. Then determine $f \circ f$, $g \circ g$, and $g \circ f$.
4. Show that the following is not a valid category.

$$\begin{tikzcd}
X \arrow[loop left, "1_X"] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "h"'] & & Y \arrow[loop right, "1_Y"] \arrow[ll, "g" description]
\end{tikzcd}$$

5. Let X be an object of a category C. Denote by C/X the category whose objects are

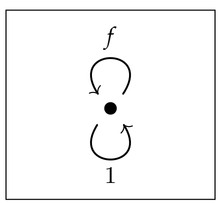
$$\mathrm{Ob}(C/X) = \{\, f : Y \longrightarrow X \mid Y \in \mathrm{Ob}(C) \,\}$$

morphisms are

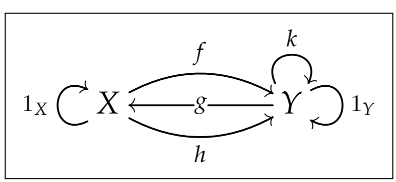
$$\mathrm{Hom}(f, g) = \{\, h\colon Y \to Z \mid g \circ h = f \,\}$$

for all $Y, Z \in \mathrm{Ob}(\mathsf{C})$, $f\colon Y \to X$, $g\colon Z \to X$, and the composition of $m \in \mathrm{Hom}(f, g)$ and $n \in \mathrm{Hom}(g, h)$ is defined to be the morphism $n \circ m$ of C . Show that C/X is indeed a category.

6. Define a category X/C analogous to C/X and show that it is a category.
7. Let C be a category. Define a new category C^{op} whose objects are the same as those of C, with a morphism $f^{\mathsf{op}}\colon Y \to X$ corresponding to every morphism $f\colon X \to Y$ in C, and composition defined by
$$f^{\mathsf{op}} \circ g^{\mathsf{op}} = (g \circ f)^{\mathsf{op}}$$
for all $f^{\mathsf{op}}\colon Y \to X$, $g^{\mathsf{op}}\colon Z \to Y$. Show that C^{op} is indeed a category.
8. What is the category $(\mathsf{C}/X)^{\mathsf{op}}$?
9. Can the following be a category? If so, how?



10. Can the following be a category? If so, how?



11. Can the following be a category? If so, how?

$$
\begin{tikzcd}
X \arrow[loop left, "1_X"] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "h"'] & & Y \arrow[ll, "g" description] \arrow[loop above, "k"] \arrow[loop right, "1_Y"] \arrow[loop below, "l"]
\end{tikzcd}
$$

A category is small if its collections of objects and morphisms are sets – note that this is equivalent to simply saying that its collection of morphisms, as the objects are in one-to-one correspondence with the identity morphisms. A category that is not small is large. A category is locally small if $\mathrm{Hom}(X, Y)$ is a set for every two objects X and Y of the category. In this case, $\mathrm{Hom}(X, Y)$ is said to be a hom-set.

In Example 3.1, the categories given in 1 and 2 are small (and hence also locally small), whereas the ones given in 3, 4 and 6 are large but locally small.

4 Some Classes of Morphisms

A morphism $f\colon X \to Y$ in a category C is an isomorphism if there exists a morphism $g\colon Y \to X$ in C such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Such a morphism g is an inverse of f. If there exists an isomorphism $f\colon X \to Y$, then X is isomorphic to Y, written $X \cong Y$.

Exercise 4.1. 1. Show that each identity morphism is an isomorphism.

2. Show that if f is an isomorphism, then it has a unique inverse, and the inverse is also an isomorphism.
3. Show that the composition of two compatible isomorphisms is also an isomorphism.
4. Show that the isomorphism relation $\cong$ is an equivalence relation on the collection of objects of a category.

Example 4.1. 1. In Set, the isomorphisms are exactly the bijective functions.

2. In Grp, Mon, Ring, and AbGrp, the isomorphisms are the bijective homomorphisms of the respective algebraic structures.
3. In Vect_F, the isomorphisms are the bijective linear transformations.
4. In Top, the isomorphisms are the homeomorphisms. Note that a homeomorphism is not simply a bijective continuous map (i.e. a morphism in Top that is bijective as a function). Why?
5. In the category given in Example 3.1, 1, all the morphisms are isomorphisms.