PROBABILITY AND STATISTICS – PROBLEM SET 4

1. (X, Y) is a two dimensional random variable with joint pdf $f(x, y) = x+y$, $0 < x < 1$, $0 < y < 1$. Compute the following.
   (a) $P[2X + 2Y < 1]$.
   (b) $P[2X + 2Y < 3]$.
   (c) $P[X \geqslant \frac{1}{2}]$.
   (d) $P[X \geqslant \frac{1}{2}, Y \geqslant \frac{1}{2}]$.
   (e) $P[Y \geqslant \frac{1}{2} \mid X \geqslant \frac{1}{2}]$.
   (f) $P[X + Y > 1 \mid X \geqslant \frac{1}{2}]$.
2. Let $f(x, y) = x^2 + \dfrac{xy}{3}$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$. Verify that $f(x, y)$ is a joint pdf, and find $P[X + Y > 1]$.
3. Let $f(x, y) = kx(x - y)$, $|y| < x \leqslant 2$. Find the value k such that $f(x, y)$ is a joint pdf, and compute the marginal pdfs of X and Y.
4. From a box containing r red, g green, and b blue marbles, n marbles are drawn successively **with replacement**. Let X and Y respectively be the number of blue marbles and number of green marbles obtained in these n draws.
   (a) Determine the joint pmf of (X, Y) and compute the marginal pmfs.
   (b) Are X and Y independent?
   (c) For the particular case $r = 3$, $g = b = 2$, tabulate the joint pmf values and compute the coefficient of correlation between X and Y.
5. The joint pdf of (X, Y) is $f(x, y) = 2e^{-x-2y}$, $x, y > 0$. Compute
   (a) $P[X > 1, Y < 1]$.
   (b) $P[X < Y]$.
   (c) $P[X < a]$.
6. Compute the joint pdf of the random variable (X, Y) that is uniformly distributed over the region bounded by the curves $y = x^2$ and $y = x$. Also find the marginal pdfs of X and Y.
7. If $f(x, y) = 2(x + y - 2xy)$, $0 \leqslant x, y \leqslant 1$, show that $Y \sim U[0, 1]$.
8. Compute the marginal pdfs of X and Y and show that they are independent, given their joint pdf (X, Y).
   (a) $f(x, y) = 3 - 6x - y + 2xy$, $0 < x < \frac{1}{2}$, $0 < y < 2$.
   (b) $f(x, y) = e^{-2x-\frac{y}{2}}$, $x, y > 0$.

9. For each random variable (X, Y) with joint pdf $f(x, y)$ as given below, compute the coefficient of correlation ρ.
   (a) $f(x, y) = x + y$, $0 < x, y < 1$.
   (b) $f(x, y) = 3x + 3y$, $0 < y < x < 1$.
   (c) $f(x, y) = 8xy$, $0 < x < y < 1$.
   (d) $f(x, y) = 2x - xy$, $0 < x < 1$, $0 < y < 2$.
   (e) $f(x, y) = \frac{1}{\pi}$, $x^2 + y^2 \leqslant 1$ [Uniform distribution in the unit disc centred at the origin].
   (f) $f(x, y) = \frac{2}{\pi}$, $y \geqslant 0$, $x^2 + y^2 \leqslant 1$.
   (g) $f(x, y) = \frac{4}{\pi}$, $x, y \geqslant 0$, $x^2 + y^2 \leqslant 1$.
   (h) $f(x, y) = \frac{1}{4}$, $(x, y) \in R$, where R is the square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$.
   (i) $f(x, y) = \frac{1}{2}$, $(x, y) \in R$, where R is the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.

   Are X and Y independent in all the cases where $\rho = 0$?