Category Theory for Algebra II

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Contents

1 Examples of Categories

Before formally defining a category, we shall look at some examples of familiar categories. A category consists of a collection of obects and a collection of morphisms satisfying certain conditions that will be discussed later.

1. The category Set has sets as objects and functions as morphisms.
2. The collection of groups and group homomorphisms forms a category denoted by Grp.
3. The collection of vector spaces over a fixed field F, together with F-linear transformations forms the category Vect_F.
4. The collection of sets and relations forms the category Rel.

2 Initial and Final Objects

An initial object in a category C is an object A such that for all objects X of C, there exists a unique morphism from A to X. A final object in C is an object A such that for all objects X of C, there exists a unique morphism from X to A. A zero object is an object that is both initial and final.

Example 2.1.   1. The empty set $\varnothing$ is the initial object and any singleton set is a final object in Set.
2. The trivial group is both the initial and final object in Grp.
3. The zero-dimensional vector space is the zero object in Vect_F.
4. The empty set is the zero object in Rel.

3 Categories

A category C is a collection $\mathrm{Ob}(\mathsf{C})$ of objects and, for every two objects X and Y a collection $\mathrm{Hom}(X,Y)$ of morphisms or arrows, together with operations

$$\circ \colon \mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$$

for all $X, Y, Z \in \mathrm{Ob}(\mathsf{C})$, satisfying the following:

(C1) $\mathrm{Hom}(X_1, Y_1)$ and $\mathrm{Hom}(X_2, Y_2)$ are disjoint unless $X_1 = X_2$ and $Y_1 = Y_2$.

(C2) For each $X \in \mathrm{Ob}(\mathsf{C})$, there exists an identity morphism id_X or 1_X such that if $f \in \mathrm{Hom}(X,Y)$, there $f \circ \mathrm{id}_X = f$ and if $g \in \mathrm{Hom}(Z,X)$, then $\mathrm{id}_X \circ g = g$.

(C3) If $F \in \mathrm{Hom}(X,Y)$, $g \in \mathrm{Hom}(Y,Z)$, and $h \in \mathrm{Hom}(Z,W)$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

This is the law of associativity.

The collection of all morphisms of C is denoted by $\mathrm{Mor}(\mathsf{C})$. If $f \in \mathrm{Hom}(X,Y)$, then we write $f \colon X \to Y$ or $X \xrightarrow{f} Y$ and say that f is a morphism from X to Y, or that $X = \mathrm{dom}\, f$ is the domain and $Y = \mathrm{cod}\, f$ is the codomain of f.

Example 3.1. 1. $\mathsf{C} = \{X, Y\}$, $\mathrm{Mor}(\mathsf{C}) = \{1_X, 1_Y, f\colon X \to Y, g\colon Y \to X\}$ with 1_X and 1_Y as identity morphisms and $f \circ g = 1_Y$, $g \circ f = 1_X$. Graphically,
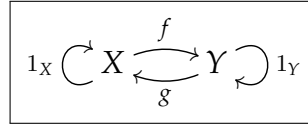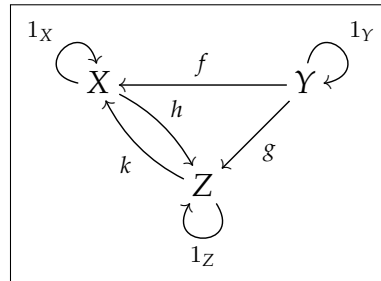

2. $\mathsf{C} = \{\bullet\}$, $\mathrm{Mor}(\mathsf{C}) = \{1\}$.
3. Ring is the category of unital rings and unital ring homomorphisms.
4. Mon is the category of monoids and monoid homomorphisms.
5. Top is the category of topological spaces and continuous maps.
6. AbGrp is the category of Abelian groups and group homomorphisms.

Exercise 3.1. 1. In the category C given below, determine the compositions of all the non-identity morphisms.
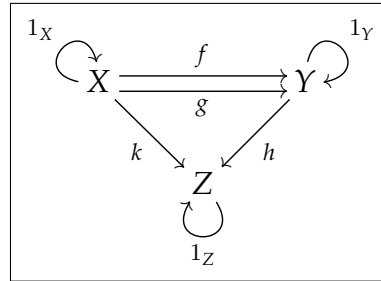
2. In the category C given below, determine the compositions of all the non-identity morphisms.

3. If C is a category with $\mathrm{Ob}(\mathsf{C}) = \{\bullet\}$ and $\mathrm{Mor}(\mathsf{C}) = \{1, f, g\}$, such that $f \circ g = 1$. Then determine $f \circ f$, $g \circ g$, and $g \circ f$.

4. Show that the following is not a valid category.
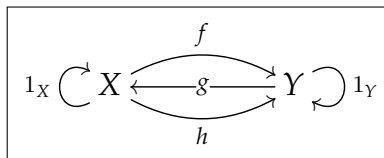
5. Let X be an object of a category C. Denote by C/X the category whose objects are

$$\mathrm{Ob}(\mathsf{C}/X) = \{\, f\colon Y \to X \mid Y \in \mathrm{Ob}(\mathsf{C}) \,\}$$

morphisms are

$$\mathrm{Hom}(f, g) = \{\, h\colon Y \to Z \mid g \circ h = f \,\}$$

for all $Y, Z \in \mathrm{Ob}(\mathsf{C})$, $f\colon Y \to X$, $g\colon Z \to X$, and the composition of $m \in \mathrm{Hom}(f, g)$ and $n \in \mathrm{Hom}(g, h)$ is defined to be the morphism $n \circ m$ of C. Show that C/X is indeed a category.

6. Define a category X/C analogous to C/X and show that it is a category.
7. Let C be a category. Define a new category C^{op} whose objects are the same as those of C, with a morphism $f^{\mathrm{op}}\colon Y \to X$ corresponding to every morphism $f\colon X \to Y$ in C, and composition defined by

$$f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$$

for all $f^{\mathrm{op}}\colon Y \to X$, $g^{\mathrm{op}}\colon Z \to Y$. Show that C^{op} is indeed a category (see Section 4).
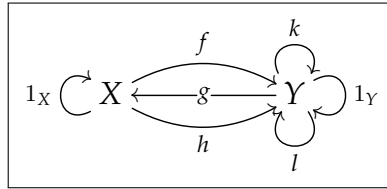8. What is the category $(\mathsf{C}/X)^{\mathrm{op}}$?
9. Can the following be a category? If so, how?
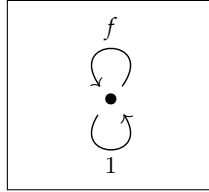
10. Can the following be a category? If so, how?
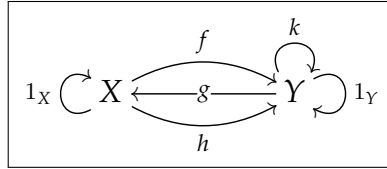
11. Can the following be a category? If so, how?

A category is small if its collections of objects and morphisms are sets – note that this is equivalent to simply saying that its collection of morphisms, as the objects are in one-to-one correspondence with the identity morphisms. A category that is not small is large. A category is locally small if $\mathrm{Hom}(X, Y)$ is a set for every two objects X and Y of the category. In this case, $\mathrm{Hom}(X, Y)$ is said to be a hom-set.

In Example 3.1, the categories given in 1 and 2 are small (and hence also locally small), whereas the ones given in 3, 4 and 6 are large but locally small.

4   Duality

The opposite category of a category C is the category C^{op} with $\mathrm{Ob}(\mathsf{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathsf{C})$ and for each pair of objects X and Y, $\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X, Y) = \{\, f^{\mathrm{op}} \mid f \in \mathrm{Hom}_{\mathsf{C}}(Y, X) \,\}$, in which composition is defined as

$$g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$$

for all $f^{\mathrm{op}} \in \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(X, Y)$ and $g^{\mathrm{op}} \in \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(Y, Z)$, for all objects X, Y, and Z.

The fact that C^{op} is a category gives rise to the principle of duality – if a statement is true in all categories, then its dual statement, obtained by reversing the directions of all morphisms, must also be true in all categories. In fact, every concept defined in category theory has a dual concept. For instance, the concepts of initial and final objects are duals of each other.

We will later prove that any two initial objects in a category must be isomorphic to each other. From this, it immediately follows, by the principle of duality, that any two final objects in a category must be isomorphic to each other.

5 Diagram Chasing

A diagram of objects and morphisms commutes (or is commutative) if, whenever there are two directed paths with the same source and same target, the composition of the morphisms along one path equals the composition of the morphisms along the other. That is, for each pair of paths
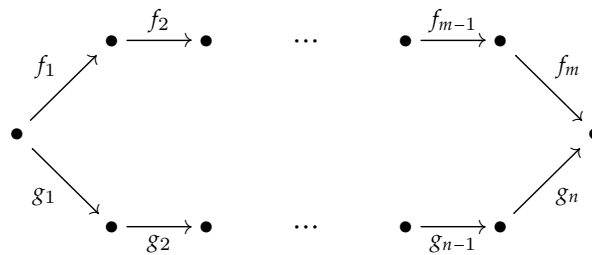

in the diagram, $f_m \circ f_{m-1} \circ \cdots \circ f_2 \circ f_1 = g_n \circ g_{n-1} \circ \cdots \circ g_2 \circ g_1$.

**Example 5.1.** In the diagram below, suppose $h = g \circ f$.
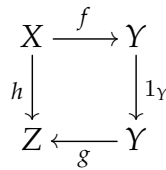
In this diagram, there are two different directed paths from X to Z, one consisting of the morphisms f, 1_Y, and g, and the other consisting of a single morphism h. The composition of the morphisms along the first path is $g \circ 1_Y \circ f = g \circ f = h$, and the composition of the morphisms along the second path is simply h. As both of these are equal, the diagram commutes.

Example 5.2. Consider $f \colon \mathbb{C} \to \mathbb{C}$, $f(z) = -iz$, $g \colon \mathbb{C} \to \mathbb{R}$, $g(z) = \mathrm{Re}(z)$, and $h \colon \mathbb{C} \to \mathbb{R}$, $h(z) = \mathrm{Im}(z)$, for all $z \in \mathbb{C}$. Then the following diagram commutes:
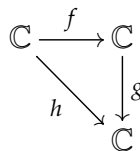
To verify this, we could take an arbitrary element of $\mathbb{C}$ and "chase" it across the diagram to see if it maps to the same element along all paths that lead to the same target. We may represent this visually as given below.
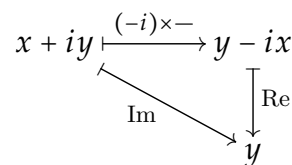
Exercise 5.1. Let $f\colon\mathbb{C}\to\mathbb{C}$, $f(z)=\overline{z}$, $g\colon\mathbb{C}\to\mathbb{C}$, $g(z)=z^2$, and $h\colon\mathbb{C}\to\mathbb{C}$, $h(z)=\mathrm{Re}(z)$. Which of the following is/are commutative?
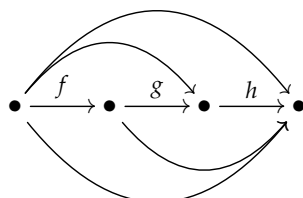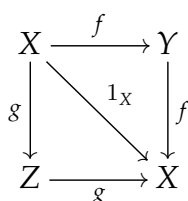
$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\ f\ } & \mathbb{C}\\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g}\\
\mathbb{C} & \xrightarrow[\ f\ ]{} & \mathbb{C}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\ f\ } & \mathbb{C}\\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h}\\
\mathbb{C} & \xrightarrow[\ f\ ]{} & \mathbb{C}
\end{array}
$$

$$\text{(a)}\qquad\qquad\qquad\qquad\text{(b)}$$
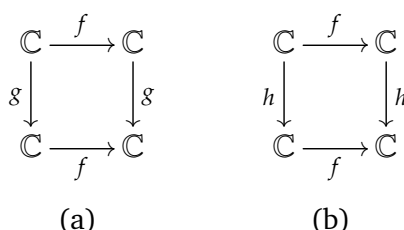
Example 5.3. The law of associativity of composition can be stated in terms of a commutative diagrams as follows. For all morphisms f, g, h with domains and codomains as shown in the diagram below, if each subdiagram with three vertices commutes, then the whole diagram commutes.

The identity morphism of an object X can be defined as the morphism such that all diagrams of the following form commute.

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y\\
{\scriptstyle g}\downarrow & {\scriptstyle 1_X} & \downarrow{\scriptstyle f}\\
Z & \xrightarrow[\ g\ ]{} & X
\end{array}
$$

Very often, the commutativity of a larger diagram can be checked by verifying the commutativity of its smaller components. Commutative diagrams of the forms given below are called commuting triangles and commuting squares, respectively.

Exercise 5.2 ([1, Exercise 1.2])**.** In each of the diagrams given below, show that if the two triangles commute, then the square commutes.

$$\text{(a)}\qquad\qquad\qquad\qquad\qquad\qquad\text{(b)}$$

Solution. In (a), from the commutativity of the upper triangle, the composition of the top and right sides of the square equals the diagonal, which, from the commutativity of the lower

triangle, equals the composition of the left and bottom sides. This shows that the square commutes.

In (b), since the lower triangle commutes, the right side equals the composition of the diagonal and the bottom side. Hence, the composition of the top and right sides equals the composition of the top side, the diagonal, and the bottom side. Similarly, from the commutativity of the upper triangle, the composition of the top side, the diagonal, and the bottom side also equals the composition of the bottom and left sides. Thus, the square commutes.

Note. Arguments such as the above one are much easier to *see* than to understand by reading, but this requires some training. The writing itself could be made considerably simpler by naming the arrows (the morphisms) in the diagram and expressing the argument in the form of equations, but the aim of these exercises is to develop the skill of "diagram chasing". Attempt the remaining exercises in the same manner, avoiding the urge to name arrows, as much as possible.
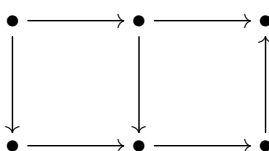
Exercise 5.3 ([1, Exercise 1.2])**.** In the diagram given below, viewed as a tetrahedron, show that if the left, right, and bottom faces commute, then the back face also commutes.
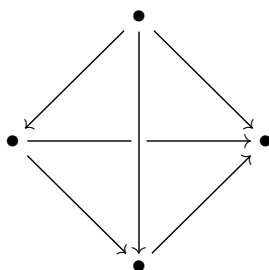
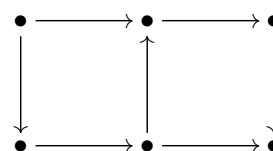**Exercise 5.4.** In each of the diagrams below, show that if both the squares commute, then the outer rectangle commutes.
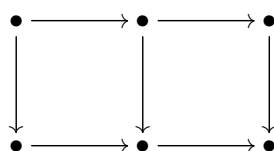
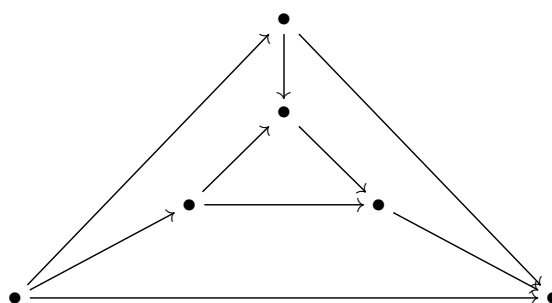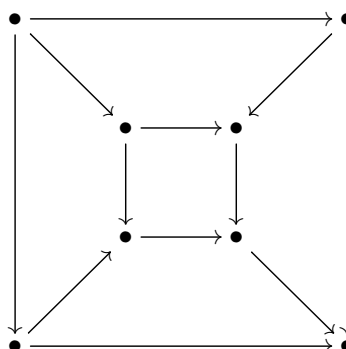(a)   (b)   (c)

Exercise 5.5. Show that if the inner triangle and the three trapeziums in the diagram below commute, then the outer triangle commutes.
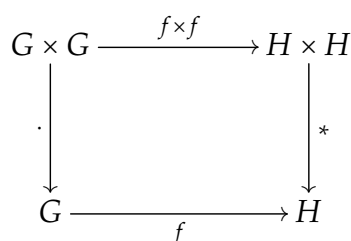

Exercise 5.6 ([1, Exercise 1.4])**.** Show that if the inner square and the four trapeziums in

the diagram below commute, then the outer square commutes.

Exercise 5.7. Let G and H be groups. If $f: G \to H$ is any function, define $f \times f: G \times G \to H \times H$ by $f(x, y) = (f(x), f(y))$, for all $x, y \in G$. Let $\cdot$ and $*$ denote the group operations in G and H, respectively. Then show that f is a homomorphism if and only if the following diagram commutes.

$$
\begin{array}{ccc}
G \times G & \xrightarrow{\ f \times f\ } & H \times H \\
\downarrow{\scriptstyle \cdot} & & \downarrow{\scriptstyle *} \\
G & \xrightarrow{\ f\ } & H
\end{array}
$$

Exercise 5.8. Show that in the diagram below, if all the parallelograms commute, and any one of the two squares commutes, then all directed paths with three arrows are equal under composition.
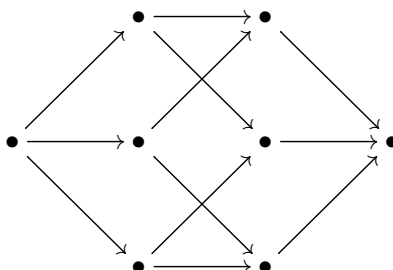
6   Some Classes of Morphisms

A morphism $f: X \to Y$ in a category C is an isomorphism if there exists a morphism $g: Y \to X$ in C such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Such a morphism g is an inverse of f. If there exists an isomorphism $f: X \to Y$, then X is isomorphic to Y, written $X \cong Y$.

Equivalently, we can define an isomorphism as a morphism f such that there exists a morphism g making the following diagram commute.

$$
Y \xrightarrow{\ g\ } X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } X
$$

with arcs 1_Y and 1_X.
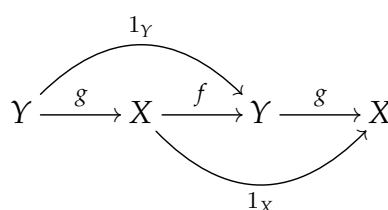
Exercise 6.1.   1. Show that each identity morphism is an isomorphism.

2. Show that if f is an isomorphism, then it has a unique inverse, and the inverse is also an isomorphism.
3. Show that the composition of two compatible isomorphisms is also an isomorphism.
4. Show that the isomorphism relation $\cong$ is an equivalence relation on the collection of objects of a category.

Example 6.1. 1. In Set, the isomorphisms are exactly the bijective functions.
2. In Grp, Mon, Ring, and AbGrp, the isomorphisms are the bijective homomorphisms of the respective algebraic structures.
3. In Vect_F, the isomorphisms are the bijective linear transformations.
4. In Top, the isomorphisms are the homeomorphisms. Note that a homeomorphism is not simply a bijective continuous map (i.e. a morphism of Top that is bijective as a function). Why?
5. In the category given in Example 3.1, 1, all the morphisms are isomorphisms.

We may weaken the definition of isomorphism in two ways as follows. If $r: X \to Y$ and $s: Y \to X$ are morphisms such that $r \circ s = 1_Y$, then r is a retraction and s is a section.

$$Y \xrightarrow{\;s\;} X$$

$$\begin{array}{ccc} Y & \xrightarrow{\;s\;} & X \\ & {}_{1_Y}\searrow & \downarrow{}^{r} \\ & & Y \end{array}$$

We also say that r is a left inverse of s and that s is a right inverse of r. Thus, a retraction is a right-invertible morphism and a section is a left-invertible morphism.

Exercise 6.2. Prove that a morphism is an isomorphism if and only if it is both a retraction and a section.

Note. The statement that a morphism is both a retraction and a section only tells you that it has some right inverse and some left inverse – not that the right and left inverses are the same. That is, if $f: X \to Y$ is both a retraction and a section, you may only assume that there exist morphisms $s: Y \to X$ and $r: Y \to X$ such that $f \circ s = 1_Y$ and $r \circ f = 1_X$.

Hint: Show that $r = s$.

Example 6.2. In the category Set, consider the sets $X = \mathbb{R}$ and $Y = \mathbb{Z}$. Define $r: X \to Y$ by $r(x) = \lfloor x \rfloor$ (the greatest integer not exceeding x) for all $x \in X$, and $s: Y \to X$ by $s(y) = y$ for all $y \in Y$. Then $r \circ s = \mathrm{id}_Y$. Note that neither r, nor s is an isomorphism.

Exercise 6.3. 1. Show that f is a retraction in a category C if and only if f^{op} is a section in C^{op}.
2. Show that in Set, every section is an injective function and every retraction is a surjective function.
3. Show that in Set, every injective function is a section.
4. Read about the Axiom of Choice and try to understand its equivalence to the following statement: In Set, every surjective function is a retraction.
5. Show that in Grp, every section is an injective group homomorphism.
6. In Grp, consider the inclusion map $\iota: \mathbb{Z} \to \mathbb{R}$ defined by $\iota(n) = n$ for all $n \in \mathbb{Z}$ (where $\mathbb{Z}$ and $\mathbb{R}$ are the additive groups defined as usual). Show that ι is an injective homomorphism, but is not a section.

The definitions of sections and retractions can be further weakened in the following manner. A monomorphism is a morphism $m: X \to Y$ such that for all Z and $f, g: Z \to X$, if

$m \circ f = m \circ g$, then $f = g$. In other words, m is left-cancellable. We also say that m is monic. Dually, an epimorphism is a morphism $e: X \to Y$ such that for all Z and $f, g: Y \to Z$, if $f \circ e = g \circ e$, then $f = g$. That is, e is right-cancellable. We also say that e is epic.

Exercise 6.4. Show that monomorphisms and epimorphisms are indeed dual concepts.

Observe that the defining property of a monomorphism, namely $m \circ f = m \circ g \implies f = g$, appears similar to that of an injective functions between sets, namely $m(x) = m(y) \implies x = y$ (for elements x and y of the domain). Indeed, monomorphisms are the categorical generalisations of injective functions. This idea is formalised in the statement of the next exercise.

Exercise 6.5. Show that monomorphisms in the category Set are exactly injective functions.

Solution. Consider a function $m: X \to Y$ in Set. First, suppose that m is a monomorphism. That is, if f and g are functions from some set Z to X such that $m \circ f = m \circ g$, then $f = g$. We need to show that m is an injection. That is, if x and y are two elements of X such that $m(x) = m(y)$, then $x = y$. Let Z be a singleton set, say $Z = \{\bullet\}$. Define functions $f, g: Z \to X$ by $f(\bullet) = x$ and $g(\bullet) = y$.
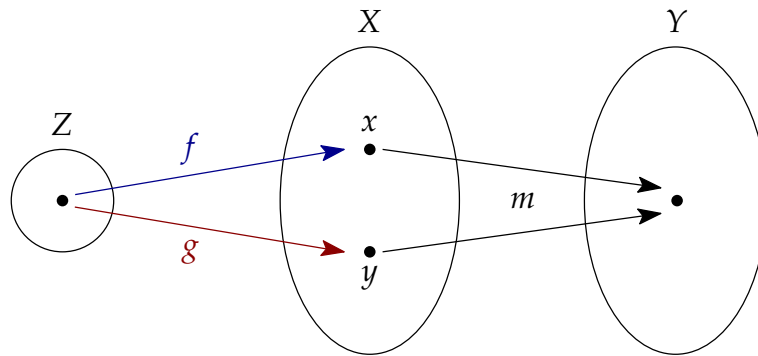
Then, $m(f(\bullet)) = m(x)$, and $m(g(\bullet)) = m(y)$. Now, suppose that $m(x) = m(y)$. Then we have $m(f(\bullet)) = m(g(\bullet))$, which implies that $m \circ f = m \circ g$ (since $\bullet$ is the only element of the domain Z of these two functions). But m is a monomorphism, and therefore we have $f = g$, which implies $f(\bullet) = g(\bullet)$, i.e. $x = y$, as required.

Next, suppose that m is an injection. If $f, g: Z \to X$ are such that $m \circ f = m \circ g$, then for all $z \in Z$, $m(f(z)) = m(g(z))$. As m is injective, this implies that $f(z) = g(z)$, for all $z \in Z$, i.e. $f = g$, showing that m is a monomorphism.

Similarly, epimorphisms are the categorical generalisations of surjective functions, as you are asked to show in the next exercise.

Exercise 6.6. Show that epimorphisms in the category Set are exactly surjective functions.

Exercise 6.7. Let $f: X \to Y$ and $g: Y \to Z$ be morphisms in a category. Prove the following:

1. If f and g are monic, then so is $g \circ f$.
2. If f and g are epic, then so is $g \circ f$.
3. If $g \circ f$ is monic, then so is f.
4. If $g \circ f$ is epic, then so is g.

Exercise 6.8. Prove that every monic retraction is an isomorphism and that every epic section is an isomorphism.

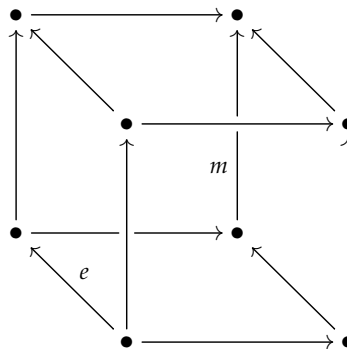**Exercise 6.9** ([1, Exercise 1.4]). Show that in the diagram given below, if e is epic, m, is monic, and the outer square and the four trapeziums commute, then the inner square commutes.
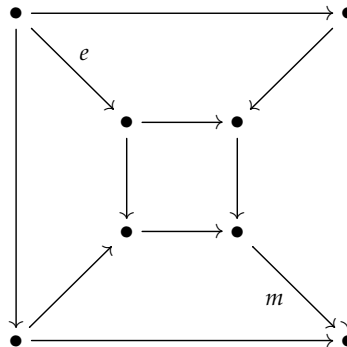
Exercise 6.10 ([1, Exercise 1.3]). Consider the cube given below.


1. Show that if e is epic and if the left, top, front, right, and bottom faces commute, then the back face also commutes.
2. Show that if m is monic and if the back, right, front, top, and left faces commute, then the bottom face also commutes.

7 Products and Coproducts

Familiar mathematical structures such as groups, rings, vector spaces, and topological spaces have some notion of a product structure. In all these cases, where the mathematical structures themselves are sets with some additional structure, the product is defined as Cartesian product of the underlying sets, together with some additional structure. Recall that the Cartesian product of two sets A and B is the set $A \times B = \{(a, b) \mid a \in A, b \in B\}$. Such a definition cannot be given in an arbitrary category, since an object of the category may not have any "underlying set". Instead, we define products (and their duals, coproducts) in a category by using universal properties.

The product of two objects X and Y of a category C is an object P, together with morphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$, such that for every object Z and morphisms $f : Z \to X$ And $g : Z \to Y$, there exists a unique morphism $h : Z \to P$ that makes the following diagram
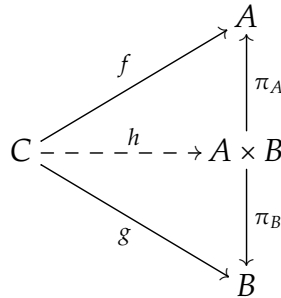
commute:
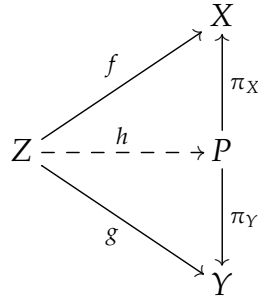
If such a triple (P, π_A, π_B) does not exist, then A and B have no product in C.

Note. The product is not merely the object P, but the triple (P, π_X, π_Y). In particular categories, these maps may be obvious, and therefore we may usually refer to only P as the product. But in an arbitrary category, the actual properties of the product are carried by these two morphisms rather than by the product object itself.

Example 7.1. First, let us see how this definition specialises to the Cartesian product in the category Set. For any two sets A and B, there are projection maps $\pi_A: A \times B \to A$ and $\pi_B: A \times B \to B$ from the Cartesian product $A \times B$ to the sets A and B respectively, defined by $\pi_A(a, b) = a$ and $\pi_B(a, b) = b$ for all $(a, b) \in A \times B$. We shall therefore verify that the triple $(A \times B, \pi_A, \pi_B)$ satisfies the universal property of the product.

Let C be any set and $f: C \to A$ and $g: C \to B$ two functions from C to A and B, respectively. We must show the existence of a unique function $h: C \to A \times B$ such that

commutes – i.e. such that $\pi_A \circ h = f$ and $\pi_B \circ h = g$.

Define h by $h(c) = (f(c), g(c))$, for all $c \in C$. Note that indeed, $h(c) \in A \times B$, since $f(c) \in A$ and $g(c) \in B$.

Next, observe that $\pi_A(h(c)) = f(c)$, and $\pi_B(h(c)) = g(c)$, which shows that $\pi_A \circ h = f$ and $\pi_B \circ h = g$, as required.

Finally, to see that h is the unique map satisfying these conditions, suppose that $h(c) = (a, b) \in A \times B$. Then from $\pi_A \circ h = f$, we have $\pi_A(a, b) = f(c)$, i.e. $a = f(c)$, and similarly $b = g(c)$. Therefore, $h(c) = (f(c), g(c))$ necessarily.

Exercise 7.1. Verify that the triple $(B \times A, \pi_B, \pi_A)$ (with notation as given in Example 7.1) also satisfies the definition of the product of A and B. Conclude that the product of two objects need not be unique. Are $A \times B$ and $B \times A$ isomorphic to each other? (Recall that an isomorphism in Set is merely a bijection).

Example 7.2. Let G and H be two groups. The direct product of G and H is the group defined by the Cartesian product $G \times H$ together with componentwise operations – i.e. the group operation is defined as $(x_1, y_1)(x_2, y_2) = (x_1 x_2, y_1 y_2)$ for all $(x_1, y_1), (x_2, y_2) \in G \times H$.

As the underlying set of the direct product is the Cartesian product $G \times H$, the projection maps π_G and π_H from $G \times H$ to G and H are naturally defined. But in order for $(G \times H, \pi_G, \pi_H)$

to be the product of G and H in the category Grp, the maps π_G and π_H must be morphisms in Grp, i.e. they must be group homomorphisms. It is easily verfied that they are.

Again, it is clear that if K is any group and $p: K \to G$ and $q: K \to H$ are group homomorphisms, then there is a unique group homomorphism $m: K \to G \times H$ defined by $m(k) = (p(k), q(k))$ for all $k \in K$, such that $\pi_G \circ m = p$ and $\pi_H \circ m = q$.

The coproduct of two objects X and Y of a category C is an object S, together with morphisms $\iota_X: X \to S$ and $\iota_Y: Y \to S$, such that for every object Z and morphisms $f: X \to Z$ And $g: Y \to Z$, there exists a unique morphism $h: S \to Z$ that makes the following diagram commute:
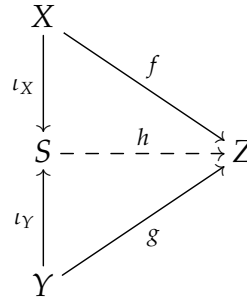
If no such triple (S, ι_X, ι_Y) exists, then X and Y have no coproduct in C.

Example 7.3. Consider two sets A and B. Their coproduct in the category Set is the disjoint union

$$A \sqcup B = A \times \{1\} \cup B \times \{2\} = \{(a,1) \mid a \in A\} \cup \{(b,2) \mid b \in B\}.$$

The injections $\iota_A: A \to A \cup B$ and $\iota_B: B \to A \cup B$ are given by $\iota_A(a) = (a,1)$ for all $a \in A$, and $\iota_B(b) = (b,2)$ for all $b \in B$.

Note. Intuitively, the disjoint union of two sets is simply their union considering the two sets to be disjoint. That is, we relabel an element a from A as $(a,1)$ to distinguish it, as an element of $A \sqcup B$, from an element coming from B, which would be labelled as $(b,2)$. For example, if $A = \{x,y,z\}$ and $B = \{y,w\}$, then $A \sqcup B = \{(x,1),(y,1),(z,1),(y,2),(w,2)\}$. Even though y is common to both A and B, in the disjoint union we have two "copies" of y, labelled as $(y,1)$ and $(y,2)$ to identify where each copy comes from.

Exercise 7.2. Verify that the disjoint union of two sets as defined in Example 7.3 is indeed their coproduct.

Exercise 7.3. Let A and B be two Abelian groups, considered as objects in the category AbGrp of Abelian groups and group homomorphisms.

1. Verify that the product of A and B is their direct product $A \times B$, as defined in the category Grp itself.
2. Define $\iota_A: A \to A \times B$ by $\iota_A(a) = (a, e_B)$ for all $a \in A$ (where e_B) denotes the identity element of B). Verify that ι_A is a group homomorphism. Similarly define $\iota_B: B \to A \times B$ and observe that it is a group homomorphism.
3. Suppose that C is an Abelian group and $f: A \to C$ and $g: B \to C$ are group homomorphisms. Show that there exists a group homomorphism $h: A \times B \to C$ such that $h \circ \iota_A = f$ and $h \circ \iota_B = g$.
   Hint: Define h as $h(a,b) = f(a)g(b)$.
4. Verify that h is the unique such homomorphism.
5. Conclude that $A \times B$ is the coproduct of A and B in AbGrp.

8 Subcategories

As with most mathematical structures, categories also have substructures. A subcategory of a category C is a category D such that $\mathrm{Ob}(\mathsf{D}) \subseteq \mathrm{Ob}(\mathsf{C})$ and $\mathrm{Mor}(\mathsf{D}) \subseteq \mathrm{Mor}(\mathsf{C})$. The latter is equivalent to the statement that if X and Y are objects of D, then $\mathrm{Hom}_{\mathsf{D}}(X,Y) \subseteq \mathrm{Hom}_{\mathsf{C}}(X,Y)$. Clearly, every category is a subcategory of itself, and the empty category, which has no objects and no morphisms, is a subcategory of every category.

A subcategory D of a category C is wide if all objects of C are also present in D. Similarly, D is a full subcategory of C if it is a subcategory of C such that for all objects $X, Y \in \mathrm{Ob}(\mathsf{D})$, $\mathrm{Hom}_{\mathsf{D}}(X,Y) = \mathrm{Hom}_{\mathsf{C}}(X,Y)$.

Example 8.1.   1. Set is a subcategory of Rel – both share the same objects, and every morphism in Set (a function) is a morphism in Rel (a relation). Hence, it Set is a wide subcategory of Rel, but not a full subcategory, since not every relation is a function.
  2. AbGrp is a subcategory of Grp, which is itself a subcategory of Mon. In each case, the subcategory is full, but not wide.
  3. Mon is a subcategory of Semigrp, the category of semigroups and semigroup homomorphisms. Note that it is neither a wide subcategory (obvious), nor a full subcategory (why?).
  4. For any category C, the collection $\mathsf{C}^{\mathsf{epi}}$ consisting of all objects of C and all (and only) the epimorphisms of C forms a (wide) subcategory of C. The subcategories $\mathsf{C}^{\mathsf{mono}}$ and $\mathsf{C}^{\mathsf{iso}}$ can be defined similarly.
  5. For any category C, the collection of all objects of C together with only the identity morphisms is a wide subcategory of C.
  6. For any object X of a category C, the collection consisting of just X and all morphisms from X to itself is a full subcategory of C.

9 Functors

Categories themselves are mathematical structures, and so we are naturally led to consider categories whose objects are themselves categories[1]. What should be the morphisms in such a category? Just as the structure of an algebraic object (such as a group or a vector space) is considered to be determined by its operation(s), the structure of a category is determined by how its morphisms are composed. A structure-preserving map between categories should therefore preserve compositions and identity morphisms.

A (covariant) functor from a category C to a category D is a function $F \colon \mathsf{C} \to \mathsf{D}$ satisfying the following properties:

  (i) F maps each object X of C to an object FX of D.
  (ii) F maps each morphism $f \colon X \to Y$ in C to a morphism $F(f) \colon FX \to FY$ of D.
  (iii) If $f \colon X \to Y$ and $g \colon Y \to Z$ are morphisms in C, then $F(g \circ f) = F(g) \circ F(f)$.
  (iv) For each object X of C, $F(1_X) = 1_{FX}$.

These properties imply that F preserves commutativity of diagrams – i.e. given a commutative diagram in C, the diagram obtained by replacing each object X in the diagram by FX and each morphism f in the diagram by $F(f)$ is a commutative diagram in D.

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[1]There are set-theoretic problems in attempting to define a *category of all categories*, but this is easily overcome, for instance, by only considering the category of all *small* categories.

A contravariant functor F from C to D is a covariant functor from C^{op} to D. To make this evident, we shall write $F: C^{op} \to D$. Thus, a contravariant functor $F: C^{op} \to D$ is a function satisfying the following properties:

(i) F maps each object X of C to an object FX of D.
(ii) F maps each morphism $f: X \to Y$ in C to a morphism $F(f): FY \to FX$ of D.
(iii) If $f: X \to Y$ and $g: Y \to Z$ are morphisms in C, then $F(g \circ f) = F(f) \circ F(g)$.
(iv) For each object X of C, $F(1_X) = 1_{FX}$.

Obviously, a contravariant functor also preserves commutativity of diagrams. We shall write *functor* to mean a covariant functor, and a contravariant functor will always be explicitly stated to be so.

Example 9.1.   1. For any category C, there is an identity functor $I: C \to C$ that maps each object to itself and each morphism to itself.
2. Let C be Grp (or Ring or $Vect_F$ or Top, etc.). Then there is a functor $U: C \to Set$ that maps each object X to its underlying set UX, and any morphism $f: X \to Y$ to the function $f: UX \to UY$. This functor is said to be a forgetful functor, as it "forgets" some of the structure but otherwise leaves the object unchanged.
3. There are also forgetful functors from AbGrp to Grp, from Grp to Mon, and from Mon to Semigrp, defined in the obvious way.
4. There is a contravariant functor $(-)^*$ from $Vect_F$ to $Vect_F$ that maps each vector space V to its dual space V^*, and each linear transformation $T: V \to W$ to its dual transformation $T^*: W^* \to V^*$. Recall that T^* is defined by $T^*(w^*) = (v \mapsto w^*(T(v)))$.
5. The covariant power set functor $\mathcal{P}: Set \to Set$ maps each set X to its power set $\mathcal{P}(X)$, and each function $f: X \to Y$ to the function $\mathcal{P}(f): \mathcal{P}(X) \to \mathcal{P}(Y)$ defined by

$$\mathcal{P}(f)(A) = \{ f(a) \mid a \in A \}$$

for all $A \subseteq X$. In other words, $\mathcal{P}(f)(A)$ is the image of A under f.
6. The contravariant power set functor $\overline{\mathcal{P}}: Set^{op} \to Set$ maps each set X to its power set $\overline{\mathcal{P}}(X) = \mathcal{P}(X)$ and each function $f: X \to Y$ to the function $\overline{\mathcal{P}}(Y) \to \overline{\mathcal{P}}(X)$ defined by

$$\overline{\mathcal{P}}(f)(B) = \{ x \in X \mid f(x) \in B \}$$

for all $B \subseteq Y$. In other words, $\overline{\mathcal{P}}(B)$ is the preimage of B under f.
7. There is a functor $F: Set \to Vect_F$ that maps each set X to the freely generated vector space with basis X. Such a functor is said to be a free functor.
8. The functor $M_n: CRing \to Mon$ maps each commutative ring R To the monoid $M_n(R)$ of $n{\times}n$ matrices over R, and each function $f: R \to S$ to the induced function $M_n(f): M_n(R) \to M_n(S)$ defined by $M_n(f)([a_{ij}]) = [f(a_{ij})]$ for all matrices $[a_{ij}] \in M_n(R)$.

Exercise 9.1.   1. Let $F: C \to D$ be a functor and f a morphism in C. Prove that if f is a section (respectively, a retraction) in C, then $F(f)$ is a section (respectively, a retraction) in D. Hence show that functors preserve isomorphisms.
2. Show, by means of a counterexample, that a functor need not reflect isomorphisms – i.e. if $F: C \to D$ is a functor and f is a morphism in C such that $F(f)$ is an isomorphism in D, then f need not be an isomorphism in C.

References

[1] A. Schalk and H. Simmons. An introduction to category theory in four easy movements, 2005. Notes for a course offered as part of the MSc in Mathematical Logic, Manchester University.