PROBABILITY AND STATISTICS – PROBLEM SET 4

1. (X,Y) is a two dimensional random variable with joint pdf $f(x,y)=x+y$, $0<x<1$, $0<y<1$. Compute the following.
   (a) $P[2X+2Y<1]$.
   (b) $P[2X+2Y<3]$.
   (c) $P[X\geq \frac{1}{2}]$.
   (d) $P[X\geq \frac{1}{2}, Y\geq \frac{1}{2}]$.
   (e) $P[Y\geq \frac{1}{2} \mid X\geq \frac{1}{2}]$.
   (f) $P[X+Y>1 \mid X\geq \frac{1}{2}]$.
2. Let $f(x,y)=x^2+\dfrac{xy}{3}$, $0\leq x\leq 1$, $0\leq y\leq 2$. Verify that $f(x,y)$ is a joint pdf, and find $P[X+Y>1]$.
3. Let $f(x,y)=kx(x-y)$, $|y|<x\leq 2$. Find the value k such that $f(x,y)$ is a joint pdf, and compute the marginal pdfs of X and Y.
4. From a box containing r red, g green, and b blue marbles, n marbles are drawn successively **with replacement**. Let X and Y respectively be the number of blue marbles and number of green marbles obtained in these n draws.
   (a) Determine the joint pmf of (X,Y) and compute the marginal pmfs.
   (b) Are X and Y independent?
   (c) For the particular case $r=3$, $g=b=2$, tabulate the joint pmf values and compute the coefficient of correlation between X and Y.
5. The joint pdf of (X,Y) is $f(x,y)=2e^{-x-2y}$, $x,y>0$. Compute
   (a) $P[X>1, Y<1]$.
   (b) $P[X<Y]$.
   (c) $P[X<a]$.
6. Compute the joint pdf of the random variable (X,Y) that is uniformly distributed over the region bounded by the curves $y=x^2$ and $y=x$. Also find the marginal pdfs of X and Y.
7. If $f(x,y)=2(x+y-2xy)$, $0\leq x,y\leq 1$, show that $Y\sim U[0,1]$.
8. Compute the marginal pdfs of X and Y and show that they are independent, given their joint pdf (X,Y).
   (a) $f(x,y)=3-6x-y+2xy$, $0<x<\frac{1}{2}$, $0<y<2$.

(b) $f(x,y) = e^{-2x-\frac{y}{2}}$, $x, y > 0$.

9. For each random variable (X,Y) with joint pdf $f(x,y)$ as given below, compute the coefficient of correlation ρ.
   (a) $f(x,y) = x + y$, $0 < x, y < 1$.
   (b) $f(x,y) = 2x + 2y$, $0 < y < x < 1$.
   (c) $f(x,y) = 8xy$, $0 < x < y < 1$.
   (d) $f(x,y) = 2x - xy$, $0 < x < 1$, $0 < y < 2$.
   (e) $f(x,y) = \frac{1}{\pi}$, $x^2 + y^2 \leq 1$ [Uniform distribution in the unit disc centred at the origin].
   (f) $f(x,y) = \frac{2}{\pi}$, $y \geq 0$, $x^2 + y^2 \leq 1$.
   (g) $f(x,y) = \frac{4}{\pi}$, $x, y \geq 0$, $x^2 + y^2 \leq 1$.
   (h) $f(x,y) = \frac{1}{4}$, $(x,y) \in R$, where R is the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$.
   (i) $f(x,y) = \frac{1}{2}$, $(x,y) \in R$, where R is the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$.

   Are X and Y independent in all the cases where $\rho = 0$?