PROBABILITY AND STATISTICS – PROBLEM SET 2

1. Very Easy

1.1. If $P(A) = 0.5$, $P(A \cap B) = 0.1$, and $P(A \cup B) = 0.8$, what is $P(B)$?

1.2. If $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{3}$, and $P(AB) = \frac{1}{6}$, compute
   (a) $P(A \cup B)$
   (b) $P(\overline{A} \cup \overline{B})$
   (c) $P(\overline{A} \cup B)$.

1.3. An integer between 1 and n (inclusive) is selected at random. Let A be the event that it is even, and let B be the event that it is **not** divisible by 3. In each of the following cases, check whether A and B are independent. Guess the answer intuitively before checking it computationally.
   (a) $n = 12$.
   (b) $n = 13$.
   (c) $n = 14$.
   (d) $n = 15$.
   (e) $n = 16$.

2. Easy

2.1. A and B are independent events such that $P(A \cup B) = \frac{5}{6}$ and $P(AB) = \frac{1}{4}$. Find the probabilities of A and B.

2.2. Let p and q be distinct prime numbers, and n a positive integer multiple of pq. An integer between 1 and n (inclusive) is selected at random. Let A be the event that it is not divisible by p, and let B be the event that it is not divisible by q. Are A and B independent?

2.3. If $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, show that
   (a) $P(AB) \leq \frac{1}{4}$.
   (b) $\frac{1}{3} \leq P(A \cup B) \leq \frac{7}{12}$

2.4. Of two coins, A and B, coin A is fair while coin B has probability 1/3 for heads. One of the two coins is picked at random and tossed twice, and both the outcomes are seen to be heads. What is the probability that it is coin A?

2.5. Three machines A, B, C produce respectively 50%, 30%, 20% of the total number of items in a factory. The percentages of defective output of these machines are 3%, 4%, and 5% respectively. If an item is selected at random:
 (a) what is the probability that the item is defective?
 (b) given that the item is defective, what is the probability that it was produced by Machine B?

2.6. An evaluation of a small business by an accountant either reveals a problem or does not reveal a problem. The evaluation is either correct or incorrect. The probability that the evaluation is correct is 0.85. The probability that the evaluation is incorrect and it reveals a problem is 0.1. If the probability that the evaluation is correct and it does not reveal a problem is 0.25, what is the probability that the evaluation does not reveal a problem?

2.7. If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present, the radar generates a (false) alarm with probability 0.1. Assume that an aircraft is present with probability 0.05. What is the probability of no aircraft presence and no detection? What is the probability of aircraft presence and detection? What is the probability that an alarm is generated? What is the probability that, if an alarm is generated, an aircraft is present?

3. Normal Difficulty

3.1. If A and B are two events in a sample space, show that A and B are independent if and only if A and $\overline{B}$ are independent.

3.2. Let $p_1,\dots,p_k$ be k distinct primes, and $n = p_1^{r_1} \times \cdots \times p_k^{r_k}$, where $r_1,\dots,r_k$ are positive integers. An integer between 1 and n (inclusive) is selected at random. Let A_i be the event that **it is not divisible by** p_i, $i = 1,\dots,k$. Find $P(A_1A_2\cdots A_k)$, and show that $A_1,\dots,A_k$ are independent.

3.3. In n tosses of a fair coin, let A be the event that there is at least one head and one tail. Let B be the event that there is at most one head. What is the value of n such that A and B are independent?

3.4. A fair 4-sided die is rolled twice. Let X and Y be the outcomes of the first and second rolls, respectively. Determine the conditional probability $P(A \mid B)$, where A is the event that $\max(X,Y) = 3$, and B is the event that $\min(X,Y) = 3$.

3.5. A fair 4-sided die is rolled once. If the result is 1 or 2, it is rolled once more, otherwise not. What is the probability that the total of the outcome(s) is at least 4?

3.6. A patient is suspected to have one of three diseases, A, B, C. The population percentages suffering from these diseases are in the ratio $2:1:1$ (and no person has more than one of the three). There is a single test for these three diseases, which turns out to be positive in 25% of the cases of A, in 50% of the cases of B, and 90% of the cases of C. The test is administered to the patient three times. Given that two of them were positive, compute the probability of the patient having each of these diseases.

3.7. Urn A contains 5 red and 3 blue marbles. Urn B contains 4 red and 6 blue marbles. One marble chosen at random is transferred from A to B. Then, one is chosen at random from B and transferred to A. Finally, one marble is drawn at random from A.
   (a) What is the probability that the marble drawn from A in the third step is red?
   (b) Given that this marble is red, what is the probability that the marble taken from A in the first step is also red?

4. Seemingly Difficult

4.1. An integer between 1 and n (inclusive) is selected at random. Let A be the event that it is even, and let B be the event that it is **not** divisible by 3. For which values of n will A and B be independent?

4.2. A certain family of bacteria is such that every ten minutes, each bacterium either divides into two, with probability $\frac{2}{3}$, or dies. If initially there is a single bacterium of this family, what is the probability that its lineage never dies out?