# Category Theory for Algebra II

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## Contents

# 1 Examples of Categories

Before formally defining a category, we shall look at some examples of familiar categories. A category consists of a collection of obects and a collection of morphisms satisfying certain conditions that will be discussed later.

1. The category Set has sets as objects and functions as morphisms.
2. The collection of groups and group homomorphisms forms a category denoted by Grp.
3. The collection of vector spaces over a fixed field $F$, together with $F$-linear transformations forms the category $\mathsf{Vect}_F$.
4. The collection of sets and relations forms the category Rel.

# 2 Initial and Final Objects

An initial object in a category C is an object $A$ such that for all objects $X$ of C, there exists a unique morphism from $A$ to $X$. A final object in C is an object $A$ such that for all objects $X$ of C, there exists a unique morphism from $X$ to $A$. A zero object is an object that is both initial and final.

**Example 2.1.**  1. The empty set $\varnothing$ is the initial object and any singleton set is a final object in Set.
2. The trivial group is both the initial and final object in Grp.
3. The zero-dimensional vector space is the zero object in $\mathsf{Vect}_F$.
4. The empty set is the zero object in Rel.

# 3 Categories

A category C is a collection $\mathrm{Ob}(\mathsf{C})$ of objects and, for every two objects $X$ and $Y$ a collection $\mathrm{Hom}(X,Y)$ of morphisms or arrows, together with operations

$$\circ\colon \mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$$

for all $X, Y, Z \in \mathrm{Ob}(\mathsf{C})$, satisfying the following:

(C1) $\mathrm{Hom}(X_1, Y_1)$ and $\mathrm{Hom}(X_2, Y_2)$ are disjoint unless $X_1 = X_2$ and $Y_1 = Y_2$.
(C2) For each $X \in \mathrm{Ob}(\mathsf{C})$, there exists an identity morphism $\mathrm{id}_X$ or $1_X$ such that if $f \in \mathrm{Hom}(X,Y)$, there $f \circ \mathrm{id}_X = f$ and if $g \in \mathrm{Hom}(Z,X)$, then $\mathrm{id}_X \circ g = g$.
(C3) If $F \in \mathrm{Hom}(X,Y)$, $g \in \mathrm{Hom}(Y,Z)$, and $h \in \mathrm{Hom}(Z,W)$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

This is the law of associativity.

The collection of all morphisms of C is denoted by $\mathrm{Mor}(\mathsf{C})$. If $f \in \mathrm{Hom}(X,Y)$, then we write $f\colon X \to Y$ or $X \xrightarrow{\ f\ } Y$ and say that $f$ is a morphism from $X$ to $Y$, or that $X = \mathrm{dom}\, f$ is the domain and $Y = \mathrm{cod}\, f$ is the codomain of $f$.

**Example 3.1.** 1. $\mathsf{C} = \{X, Y\}$, $\mathrm{Mor}(\mathsf{C}) = \{1_X, 1_Y, f\colon X \to Y, g\colon Y \to X\}$ with $1_X$ and $1_Y$ as identity morphisms and $f \circ g = 1_Y$, $g \circ f = 1_X$. Graphically,
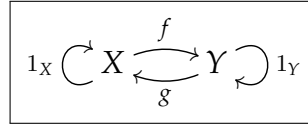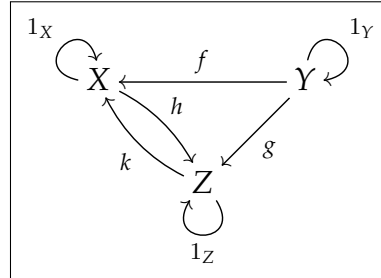


2. $\mathsf{C} = \{\bullet\}$, $\mathrm{Mor}(\mathsf{C}) = \{1\}$.
3. Ring is the category of unital rings and unital ring homomorphisms.
4. Mon is the category of monoids and monoid homomorphisms.
5. Top is the category of topological spaces and continuous maps.
6. AbGrp is the category of Abelian groups and group homomorphisms.

**Exercise 3.1.** 1. In the category $\mathsf{C}$ given below, determine the compositions of all the non-identity morphisms.
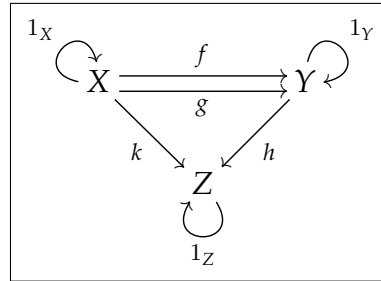


2. In the category $\mathsf{C}$ given below, determine the compositions of all the non-identity morphisms.



3. If $\mathsf{C}$ is a category with $\mathrm{Ob}(\mathsf{C}) = \{\bullet\}$ and $\mathrm{Mor}(\mathsf{C}) = \{1, f, g\}$, such that $f \circ g = 1$. Then determine $f \circ f$, $g \circ g$, and $g \circ f$.

4. Show that the following is not a valid category.
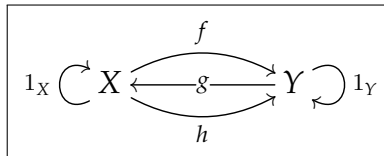


5. Let $X$ be an object of a category $\mathsf{C}$. Denote by $\mathsf{C}/X$ the category whose objects are

$$\mathrm{Ob}(\mathsf{C}/X) = \{\, f\colon Y \to X \mid Y \in \mathrm{Ob}(\mathsf{C}) \,\}$$

morphisms are

$$\mathrm{Hom}(f, g) = \{\, h\colon Y \to Z \mid g \circ h = f \,\}$$

for all $Y, Z \in \mathrm{Ob}(\mathsf{C})$, $f\colon Y \to X$, $g\colon Z \to X$, and the composition of $m \in \mathrm{Hom}(f, g)$ and $n \in \mathrm{Hom}(g, h)$ is defined to be the morphism $n \circ m$ of $\mathsf{C}$. Show that $\mathsf{C}/X$ is indeed a category.

6. Define a category $X/C$ analogous to $C/X$ and show that it is a category.

7. Let C be a category. Define a new category $C^{op}$ whose objects are the same as those of C, with a morphism $f^{op}: Y \to X$ corresponding to every morphism $f: X \to Y$ in C, and composition defined by
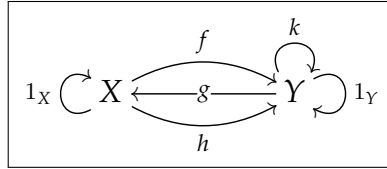
$$f^{op} \circ g^{op} = (g \circ f)^{op}$$

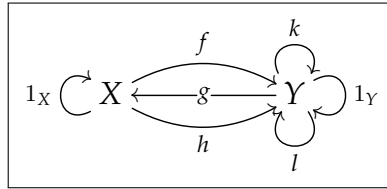   for all $f^{op}: Y \to X$, $g^{op}: Z \to Y$. Show that $C^{op}$ is indeed a category (see Section 4).

8. What is the category $(C/X)^{op}$?

9. Can the following be a category? If so, how?
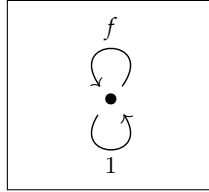


10. Can the following be a category? If so, how?



11. Can the following be a category? If so, how?



A category is small if its collections of objects and morphisms are sets – note that this is equivalent to simply saying that its collection of morphisms, as the objects are in one-to-one correspondence with the identity morphisms. A category that is not small is large. A category is locally small if $\mathrm{Hom}(X, Y)$ is a set for every two objects $X$ and $Y$ of the category. In this case, $\mathrm{Hom}(X, Y)$ is said to be a hom-set.

In Example 3.1, the categories given in 1 and 2 are small (and hence also locally small), whereas the ones given in 3, 4 and 6 are large but locally small.

# 4  Duality

The opposite category of a category C is the category $C^{op}$ with $\mathrm{Ob}(C^{op}) = \mathrm{Ob}(C)$ and for each pair of objects $X$ and $Y$, $\mathrm{Hom}_{C^{op}}(X, Y) = \{ f^{op} \mid f \in \mathrm{Hom}_C(Y, X) \}$, in which composition is defined as

$$g^{op} \circ f^{op} = (f \circ g)^{op}$$

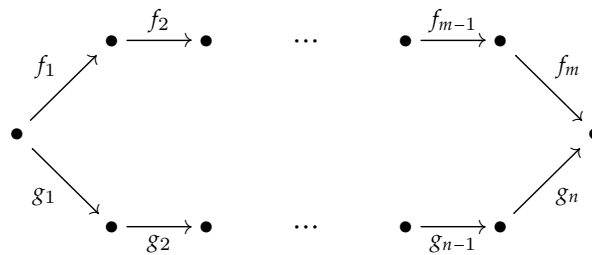for all $f^{op} \in \mathrm{Hom}_{C^{op}}(X, Y)$ and $g^{op} \in \mathrm{Hom}_{C^{op}}(Y, Z)$, for all objects $X$, $Y$, and $Z$.

The fact that $C^{op}$ is a category gives rise to the principle of duality – if a statement is true in all categories, then its dual statement, obtained by reversing the directions of all morphisms, must also be true in all categories. In fact, every concept defined in category theory has a dual concept. For instance, the concepts of initial and final objects are duals of each other.

We will later prove that any two initial objects in a category must be isomorphic to each other. From this, it immediately follows, by the principle of duality, that any two final objects in a category must be isomorphic to each other.

# 5 Diagram Chasing

A diagram of objects and morphisms commutes (or is commutative) if, whenever there are two directed paths with the same source and same target, the composition of the morphisms along one path equals the composition of the morphisms along the other. That is, for each pair of paths

$$\xymatrix{ & \bullet \ar[r]^{f_2} & \bullet & \cdots & \bullet \ar[r]^{f_{m-1}} & \bullet \ar[dr]^{f_m} & \\ \bullet \ar[ur]^{f_1} \ar[dr]_{g_1} & & & & & & \bullet \\ & \bullet \ar[r]_{g_2} & \bullet & \cdots & \bullet \ar[r]_{g_{n-1}} & \bullet \ar[ur]_{g_n} & }$$

in the diagram, $f_m \circ f_{m-1} \circ \cdots \circ f_2 \circ f_1 = g_n \circ g_{n-1} \circ \cdots \circ g_2 \circ g_1$.

**Example 5.1.** In the diagram below, suppose $h = g \circ f$.

$$\xymatrix{ X \ar[r]^{f} \ar[d]_{h} & Y \ar[d]^{1_Y} \\ Z & Y \ar[l]^{g} }$$
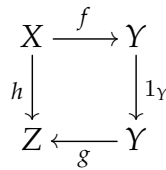
In this diagram, there are two different directed paths from $X$ to $Z$, one consisting of the morphisms $f$, $1_Y$, and $g$, and the other consisting of a single morphism $h$. The composition of the morphisms along the first path is $g \circ 1_Y \circ f = g \circ f = h$, and the composition of the morphisms along the second path is simply $h$. As both of these are equal, the diagram commutes.

**Example 5.2.** Consider $f: \mathbb{C} \to \mathbb{C}$, $f(z) = -iz$, $g: \mathbb{C} \to \mathbb{R}$, $g(z) = \mathrm{Re}(z)$, and $h: \mathbb{C} \to \mathbb{R}$, $h(z) = \mathrm{Im}(z)$, for all $z \in \mathbb{C}$. Then the following diagram commutes:

$$\xymatrix{ \mathbb{C} \ar[r]^{f} \ar[dr]_{h} & \mathbb{C} \ar[d]^{g} \\ & \mathbb{C} }$$
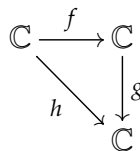
To verify this, we could take an arbitrary element of $\mathbb{C}$ and "chase" it across the diagram to see if it maps to the same element along all paths that lead to the same target. We may represent this visually as given below.

$$\xymatrix{ x + iy \ar@{|->}[r]^{(-i) \times -} \ar[dr]_{\mathrm{Im}} & y - ix \ar[d]^{\mathrm{Re}} \\ & y }$$

5
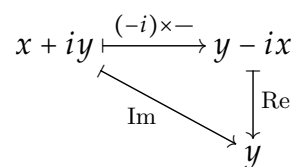
**Exercise 5.1.** Let $f:\mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$, $g:\mathbb{C} \to \mathbb{C}$, $g(z) = z^2$, and $h:\mathbb{C} \to \mathbb{C}$, $h(z) = \mathrm{Re}(z)$. Which of the following is/are commutative?



(a)          (b)
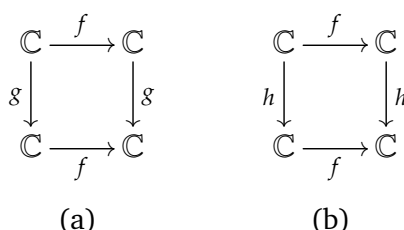
**Example 5.3.** The law of associativity of composition can be stated in terms of a commutative diagrams as follows. For all morphisms $f$, $g$, $h$ with domains and codomains as shown in the diagram below, if each subdiagram with three vertices commutes, then the whole diagram commutes.
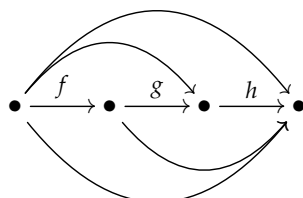


The identity morphism of an object $X$ can be defined as the morphism such that all diagrams of the following form commute.
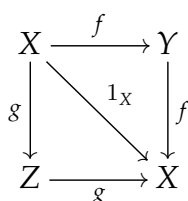


Very often, the commutativity of a larger diagram can be checked by verifying the commutativity of its smaller components. Commutative diagrams of the forms given below are called commuting triangles and commuting squares, respectively.



**Exercise 5.2** ([1, Exercise 1.2])**.** In each of the diagrams given below, show that if the two triangles commute, then the square commutes.



(a)                          (b)

*Solution.* In (a), from the commutativity of the upper triangle, the composition of the top and right sides of the square equals the diagonal, which, from the commutativity of the lower

triangle, equals the composition of the left and bottom sides. This shows that the square commutes.

In (b), since the lower triangle commutes, the right side equals the composition of the diagonal and the bottom side. Hence, the composition of the top and right sides equals the composition of the top side, the diagonal, and the bottom side. Similarly, from the commutativity of the upper triangle, the composition of the top side, the diagonal, and the bottom side also equals the composition of the bottom and left sides. Thus, the square commutes.

**Note.** Arguments such as the above one are much easier to *see* than to understand by reading, but this requires some training. The writing itself could be made considerably simpler by naming the arrows (the morphisms) in the diagram and expressing the argument in the form of equations, but the aim of these exercises is to develop the skill of "diagram chasing". Attempt the remaining exercises in the same manner, avoiding the urge to name arrows, as much as possible.
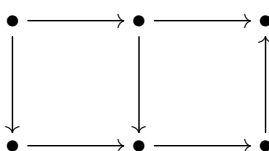
**Exercise 5.3** ([1, Exercise 1.2]). In the diagram given below, viewed as a tetrahedron, show that if the left, right, and bottom faces commute, then the back face also commutes.
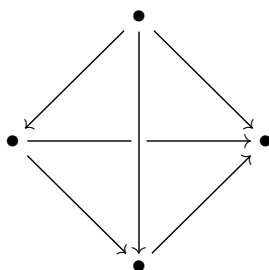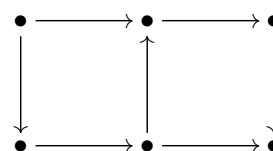


**Exercise 5.4.** In each of the diagrams below, show that if both the squares commute, then the outer rectangle commutes.
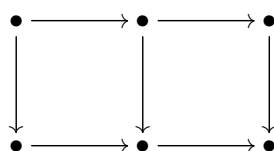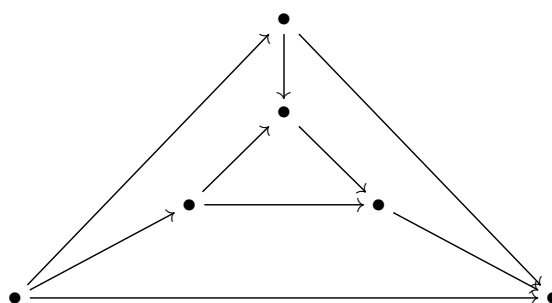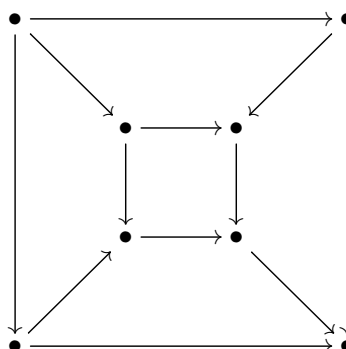


| (a) | (b) | (c) |
|-----|-----|-----|

**Exercise 5.5.** Show that if the inner triangle and the three trapeziums in the diagram below commute, then the outer triangle commutes.
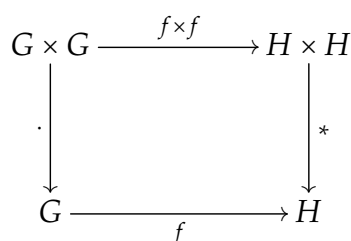


**Exercise 5.6** ([1, Exercise 1.4]). Show that if the inner square and the four trapeziums in

the diagram below commute, then the outer square commutes.



**Exercise 5.7.** Let $G$ and $H$ be groups. If $f: G \to H$ is any function, defined $f \times f: G \times G \to H \times H$ by $f(x, y) = (f(x), f(y))$, for all $x, y \in G$. Let $\cdot$ and $*$ denote the group operations in $G$ and $H$, respectively. Then show that $f$ is a homomorphism if and only if the following diagram commutes.

$$
\begin{array}{ccc}
G \times G & \xrightarrow{\ f \times f\ } & H \times H \\
\downarrow{\scriptstyle \cdot} & & \downarrow{\scriptstyle *} \\
G & \xrightarrow{\ f\ } & H
\end{array}
$$

**Exercise 5.8.** Show that in the diagram below, if all the parallelograms commute, and any one of the two squares commutes, then all directed paths with three arrows are equal under composition.
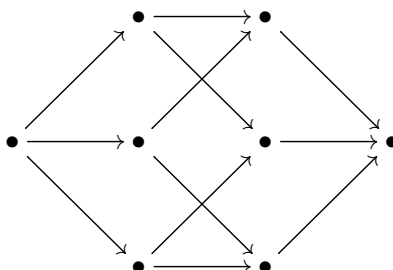


# 6   Some Classes of Morphisms

A morphism $f: X \to Y$ in a category $\mathsf{C}$ is an isomorphism if there exists a morphism $g: Y \to X$ in $\mathsf{C}$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Such a morphism $g$ is an inverse of $f$. If there exists an isomorphism $f: X \to Y$, then $X$ is isomorphic to $Y$, written $X \cong Y$.

Equivalently, we can define an isomorphism as a morphism $f$ such that there exists a morphism $g$ making the following diagram commute.

$$
Y \xrightarrow{\ g\ } X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } X
$$

with $1_Y$ over the left arc and $1_X$ under.
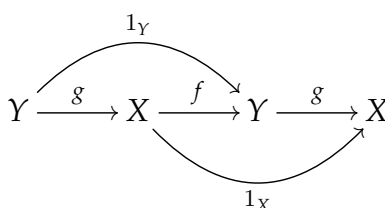
**Exercise 6.1.**   1. Show that each identity morphism is an isomorphism.

2. Show that if $f$ is an isomorphism, then it has a unique inverse, and the inverse is also an isomorphism.
3. Show that the composition of two compatible isomorphisms is also an isomorphism.
4. Show that the isomorphism relation $\cong$ is an equivalence relation on the collection of objects of a category.

**Example 6.1.**  1. In Set, the isomorphisms are exactly the bijective functions.
2. In Grp, Mon, Ring, and AbGrp, the isomorphisms are the bijective homomorphisms of the respective algebraic structures.
3. In $\mathsf{Vect}_F$, the isomorphisms are the bijective linear transformations.
4. In Top, the isomorphisms are the homeomorphisms. Note that a homeomorphism is not simply a bijective continuous map (i.e. a morphism of Top that is bijective as a function). Why?
5. In the category given in Example 3.1, 1, all the morphisms are isomorphisms.

We may weaken the definition of isomorphism in two ways as follows. If $r\colon X \to Y$ and $s\colon Y \to X$ are morphisms such that $r \circ s = 1_Y$, then $r$ is a retraction and $s$ is a section.

$$Y \xrightarrow{\ s\ } X$$
$$\begin{array}{cc} & \downarrow r \\ 1_Y \searrow & \\ & Y \end{array}$$

We also say that $r$ is a left inverse of $s$ and that $s$ is a right inverse of $r$. Thus, a retraction is a right-invertible morphism and a section is a left-invertible morphism.

**Exercise 6.2.** Prove that a morphism is an isomorphism if and only if it is both a retraction and a section.

Note. The statement that a morphism is both a retraction and a section only tells you that it has some right inverse and some left inverse – not that the right and left inverses are the same. That is, if $f\colon X \to Y$ is both a retraction and a section, you may only assume that there exist morphisms $s\colon Y \to X$ and $r\colon Y \to X$ such that $f \circ s = 1_Y$ and $r \circ f = 1_X$.

Hint: Show that $r = s$.

**Example 6.2.** In the category Set, consider the sets $X = \mathbb{R}$ and $Y = \mathbb{Z}$. Define $r\colon X \to Y$ by $r(x) = \lfloor x \rfloor$ (the greatest integer not exceeding $x$) for all $x \in X$, and $s\colon Y \to X$ by $s(y) = y$ for all $y \in Y$. Then $r \circ s = \mathrm{id}_Y$. Note that neither $r$, nor $s$ is an isomorphism.

**Exercise 6.3.**  1. Show that $f$ is a retraction in a category C if and only if $f^{\mathrm{op}}$ is a section in $\mathsf{C}^{\mathrm{op}}$.
2. Show that in Set, every section is an injective function and every retraction is a surjective function.
3. Show that in Set, every injective function is a section.
4. Read about the Axiom of Choice and try to understand its equivalence to the following statement: In Set, every surjective function is a retraction.
5. Show that in Grp, every section is an injective group homomorphism.
6. In Grp, consider the inclusion map $\iota\colon \mathbb{Z} \to \mathbb{R}$ defined by $\iota(n) = n$ for all $n \in \mathbb{Z}$ (where $\mathbb{Z}$ and $\mathbb{R}$ are the additive groups defined as usual). Show that $\iota$ is an injective homomorphism, but is not a section.

The definitions of sections and retractions can be further weakened in the following manner. A monomorphism is a morphism $m\colon X \to Y$ such that for all $Z$ and $f, g\colon Z \to X$, if

$m \circ f = m \circ g$, then $f = g$. In other words, $m$ is left-cancellable. We also say that $m$ is monic. Dually, an epimorphism is a morphism $e: X \to Y$ such that for all $Z$ and $f, g: Y \to Z$, if $f \circ e = g \circ e$, then $f = g$. That is, $e$ is right-cancellable. We also say that $e$ is epic.

**Exercise 6.4.** Show that monomorphisms and epimorphisms are indeed dual concepts.

Observe that the defining property of a monomorphism, namely $m \circ f = m \circ g \implies f = g$, appears similar to that of an injective functions between sets, namely $m(x) = m(y) \implies x = y$ (for elements $x$ and $y$ of the domain). Indeed, monomorphisms are the categorical generalisations of injective functions. This idea is formalised in the statement of the next exercise.

**Exercise 6.5.** Show that monomorphisms in the category Set are exactly injective functions.

*Solution.* Consider a function $m: X \to Y$ in Set. First, suppose that $m$ is a monomorphism. That is, if $f$ and $g$ are functions from some set $Z$ to $X$ such that $m \circ f = m \circ g$, then $f = g$. We need to show that $m$ is an injection. That is, if $x$ and $y$ are two elements of $X$ such that $m(x) = m(y)$, then $x = y$. Let $Z$ be a singleton set, say $Z = \{\bullet\}$. Define functions $f, g: Z \to X$ by $f(\bullet) = x$ and $g(\bullet) = y$.
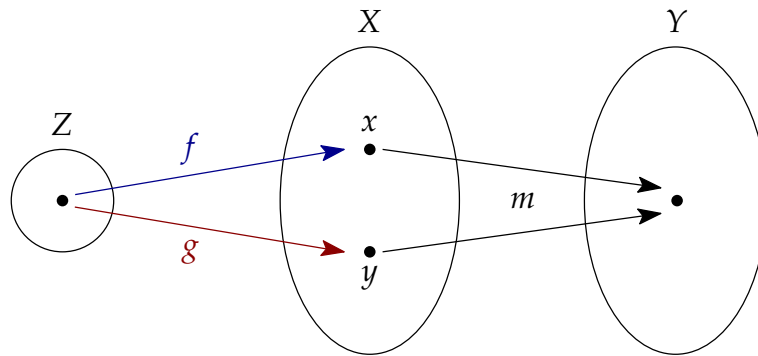


Then, $m(f(\bullet)) = m(x)$, and $m(g(\bullet)) = m(y)$. Now, suppose that $m(x) = m(y)$. Then we have $m(f(\bullet)) = m(g(\bullet))$, which implies that $m \circ f = m \circ g$ (since $\bullet$ is the only element of the domain $Z$ of these two functions). But $m$ is a monomorphism, and therefore we have $f = g$, which implies $f(\bullet) = g(\bullet)$, i.e. $x = y$, as required.

Next, suppose that $m$ is an injection. If $f, g: Z \to X$ are such that $m \circ f = m \circ g$, then for all $z \in Z$, $m(f(z)) = m(g(z))$. As $m$ is injective, this implies that $f(z) = g(z)$, for all $z \in Z$, i.e. $f = g$, showing that $m$ is a monomorphism.

Similarly, epimorphisms are the categorical generalisations of surjective functions, as you are asked to show in the next exercise.

**Exercise 6.6.** Show that epimorphisms in the category Set are exactly surjective functions.

# References

[1] A. Schalk and H. Simmons. An introduction to category theory in four easy movements, 2005. Notes for a course offered as part of the MSc in Mathematical Logic, Manchester University.