

Estimation of Parameters

1 Maximum Likelihood Estimation

Let X be a random variable having pdf or pmf $f(x; \theta)$, where θ is an unknown parameter. The *likelihood function* of θ , corresponding to a sample of size n , is

$$L(\theta) = f(x_1; \theta)f(x_2; \theta) \cdots f(x_n; \theta).$$

The value of $\theta = T(x_1, \dots, x_n)$ (say) for which $L(\theta)$ is maximum is the *maximum likelihood estimate* of θ . The corresponding statistic $\hat{\theta} = T(X_1, \dots, X_n)$ is the *maximum likelihood estimator* (mle) of θ .

Where applicable, we maximise $L(\theta)$ by solving the equation $\frac{d}{d\theta}L(\theta) = 0$ for θ . As $L(\theta)$ is the product of $f(x_1; \theta), \dots, f(x_n; \theta)$, it is more often convenient to maximise the *log-likelihood function* $\log L(\theta) = \sum_{i=1}^n \log f(x_i; \theta)$ – which yields the same estimator $\hat{\theta}$, since \log is an increasing function.

Example 1.1. Let X be a random variable with pdf $f(x) = (1 + \theta)x^\theta$, $0 < x < 1$, where $\theta > 0$. The likelihood function, assuming a sample of size n , will be

$$L(\theta) =$$

2 Confidence Intervals

Definition 2.1. If (X_1, \dots, X_n) is a random sample of size n , then the *sample mean* is defined as

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

and the *sample variance* is defined as

$$S^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2.$$

Theorem 2.2. Suppose that $X \sim N(\mu, \sigma^2)$. Then the following hold:

1. $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$.
2. $\frac{\bar{X} - \mu}{S/\sqrt{n-1}} \sim t_{n-1}$.
3. $\frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu)^2 \sim \chi_n^2$.
4. $\frac{nS^2}{\sigma^2} \sim \chi_{n-1}^2$.

Confidence intervals for the mean and variance of a normal population, under different conditions, are as described in the table given below.

Parameter	Condition	Interval
μ	σ^2 known	$\left(\bar{x} - \frac{a\sigma}{\sqrt{n}}, \bar{x} + \frac{a\sigma}{\sqrt{n}} \right)$ where $P[-a < Z < a] = p, Z \sim N(0, 1)$
	σ^2 unknown	$\left(\bar{x} - \frac{bS}{\sqrt{n-1}}, \bar{x} + \frac{bS}{\sqrt{n-1}} \right)$ where $P[-b < T < b] = p, T \sim t_{n-1}$
σ^2	μ known	$\left(\frac{1}{b} \sum_{i=1}^n (x_i - \mu)^2, \frac{1}{a} \sum_{i=1}^n (x_i - \mu)^2 \right)$ where $P[Y < a] = P[Y > b] = \frac{p}{2}, Y \sim \chi_n^2$
	μ unknown	$\left(\frac{ns^2}{b}, \frac{ns^2}{a} \right)$ where $P[Z < a] = P[Z > b] = \frac{p}{2}, Z \sim \chi_{n-1}^2$