

PROBABILITY AND STATISTICS – PROBLEM SET 2

1. VERY EASY

- 1.1. If $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{3}$, and $P(AB) = \frac{1}{6}$, compute
- (a) $P(A \cup B)$
 - (b) $P(\overline{A} \cup \overline{B})$
 - (c) $P(\overline{A} \cup B)$.
- 1.2. An integer between 1 and n (inclusive) is selected at random. Let A be the event that it is even, and let B be the event that it is **not** divisible by 3. In each of the following cases, check whether A and B are independent. Guess the answer intuitively before checking it computationally.
- (a) $n = 12$.
 - (b) $n = 13$.
 - (c) $n = 14$.
 - (d) $n = 15$.
 - (e) $n = 16$.

2. EASY

- 2.1. A and B are independent events such that $P(A \cup B) = \frac{5}{6}$ and $P(AB) = \frac{1}{4}$. Find $P(A)$ and $P(B)$.
- 2.2. Let p and q be distinct prime numbers, and n a positive integer multiple of pq . An integer between 1 and n (inclusive) is selected at random. Let A be the event that **it is divisible by** p , and let B be the event that **it is not divisible by** q . Are A and B independent?
- 2.3. If $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, show that
- (a) $P(AB) \leq \frac{1}{4}$.
 - (b) $\frac{1}{3} \leq P(A \cup B) \leq \frac{7}{12}$

3. NORMAL DIFFICULTY

- 3.1. If A and B are two events in a sample space, show that A and B are independent if and only if A and \overline{B} are independent.
- 3.2. Let p_1, \dots, p_k be k distinct primes, and $n = p_1^{r_1} \times \dots \times p_k^{r_k}$, where r_1, \dots, r_k are positive integers. An integer between 1 and n (inclusive) is selected at random.

- Let A_i be the event that **it is not divisible by** p_i , $i = 1, \dots, k$. Find $P(A_1 A_2 \cdots A_k)$, and show that A_1, \dots, A_k are independent.
- 3.3. In a family of n children, let A be the event that there is at least one girl and one boy. Let B be the event that there is at most one girl. Assuming that each child has equal probability of being a boy or a girl, what is the value of n such that A and B are independent?
- 3.4. A patient is suspected to have one of three diseases, A, B, C . The population percentages suffering from these diseases are in the ratio $2 : 1 : 1$ (and no person has more than one of the three). There is a single test for these three diseases, which turns out to be positive in 25% of the cases of A , in 50% of the cases of B , and 90% of the cases of C . The test is administered to the patient three times. Given that two of them were positive, compute the probability of the patient having each of these diseases.
- 3.5. Urn A contains 5 red and 3 blue marbles. Urn B contains 4 red and 6 blue marbles. One marble chosen at random is transferred from A to B . Then, one is chosen at random from B and transferred to A . Finally, one marble is drawn at random from A .
- What is the probability that the marble drawn from A in the third step is red?
 - Given that this marble is red, what is the probability that the marble taken from A in the first step is also red?

4. SEEMINGLY DIFFICULT

- 4.1. An integer between 1 and n (inclusive) is selected at random. Let A be the event that it is even, and let B be the event that it is **not** divisible by 3. For which values of n will A and B be independent?
- 4.2. A certain family of bacteria is such that every ten minutes, each bacterium either divides into two, with probability $\frac{2}{3}$, or dies. If initially there is a single bacterium of this family, what is the probability that its lineage never dies out?