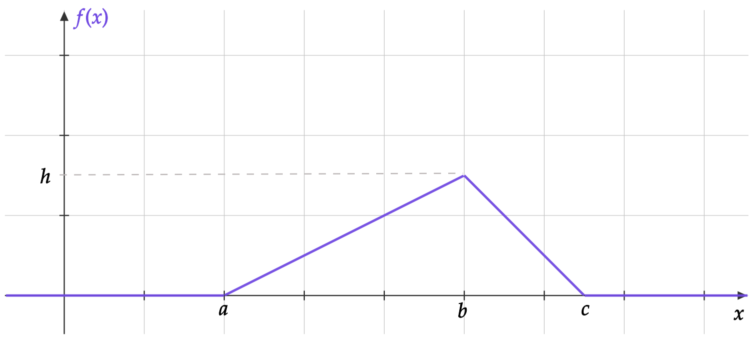
PROBABILITY AND STATISTICS – PROBLEM SET 3

1. $X \sim U[a, b]$, $a < 1 < 2 < b$. If $P[X < 1] = 2\,P[X > 2]$, and $5\,P[X > 1] = 3\,P[X < 2]$, find the values of a and b.
2. X is a discrete random variable taking values $0, 1, 2, \ldots$, whose pmf $f(x)$ is such that $\dfrac{f(x_1)}{f(x_2)} = r^{x_2 - x_1}$ for all x_1, x_2 (where $r \in (0, 1)$ is a constant). Determine $f(x)$ and compute $\mathbb{E}[X]$.
3. Let X be the random variable with pdf $f(x)$ whose graph is shown below (where $a < b < c$ are three real numbers).



   Find h, and compute $\mathbb{E}[X]$.
4. Show that the function

$$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}, x = 0, 1, 2, \ldots$$

   (where $\lambda > 0$ is a constant) is a valid pmf, and compute $\mathbb{E}[X]$ and $\mathbb{V}[X]$.
5. X is a discrete random variable with pmf $f(x) = \dfrac{6}{\pi^2 x^2}$, $x = 1, 2, \ldots$.
   (a) Compute $P[X \geqslant 3]$
   (b) Compute $P[10 \leqslant X \leqslant 15]$
   (c) Show that $\mathbb{E}[X]$ does not exist.
6. X is a continuous random variable with pdf $f(x) = \dfrac{6}{\pi^2}\left(\dfrac{1}{\lfloor x \rfloor}\right)^2$, $x > 1$.
   (a) Compute $P[X \geqslant 3]$

(b) Compute $P[10 \leqslant X \leqslant 15]$
(c) Show that $\mathbb{E}[X]$ does not exist.

7. X has pdf $f(x) = \dfrac{x^2}{3}$, $-1 \leqslant x \leqslant 2$. Compute
   (a) $P\left[X < \frac{3}{2}\right]$
   (b) $P[|X| > 1]$
   (c) $P\left[X < \frac{3}{2} \mid X > \frac{1}{2}\right]$
   (d) $P\left[X < \frac{3}{2} \mid |X| > 1\right]$
   (e) $P\left[|X| < 1 \mid X < \frac{3}{2}\right]$.
8. An urn initially has 1 red and 1 blue marble. A marble is drawn at random from the urn, and if it is blue, it is put back and one red marble is added to the urn. This is continued until a red marble is drawn. Let X denote the total number of draws required to obtain a red marble. Determine the probability distribution of X, and find its mean and variance. **Hint**: Compute $\mathbb{E}[X+1]$ and $\mathbb{E}[X^2-1]$.
9. The *median* of a random variable X is the point c such that $P[X \leqslant c] = P[X \geqslant c] = \frac{1}{2}$. Compute the median of X with pdf $f(x)$ in each of the following cases.
   (a) $f(x) = \dfrac{1}{b-a}$, $a < x < b$.
   (b) $f(x) = ae^{-ax}$, $x > 0$ [where $a > 0$ is a constant].
   (c) $f(x) = \dfrac{1}{\pi(1+x^2)}$ [Note that this distribution has no mean, but has a median].