Applied Graph Theory

Contents

1 Introduction

A graph is an ordered pair $G = (V, E)$ where V is a nonempty set called the vertex set, whose elements are the vertices of G, and E is a set of unordered pairs of distinct vertices of G, whose elements are the edges of G. The order of G is its number of vertices and the size of G is its number of edges.

Two vertices u and v of G are adjacent, denoted by $u \sim v$, if $uv \in E$ (note that uv here denotes an unordered pair of vertices, and $uv = vu$). If there is no edge between u and v, then they are nonadjacent, denoted by $u \nsim v$. The vertex v and the edge uv are said to be incident with each other. The degree of a vertex v is the number of edges incident with it, or equivalently, the number of vertices adjacent with it, and is denoted by $\deg(v)$.

Lemma 1.1 (Handshaking Lemma). *The sum of the degrees of all vertices of a graph is twice its size.*

Proof. Let G be a graph, and consider the sum of its degrees, $\sum_{v\in V(G)} \deg v$. As each edge is incident with exactly two vertices, each edge is counted once by two terms in this summation. Thus, the sum is equal to twice the number of edges. $\square$

Corollary 1.2. *The number of vertices of odd degree in a graph is always even.*

Proof. Let G be a graph. We know that

$$\sum_{v\in V(G)} \deg v = 2|E(G)|. \tag{1}$$

Considering the vertices of odd and even degrees, we can write (1) as

$$\sum_{v\in V(G)\,:\,\deg v \text{ odd}} \deg v + \sum_{v\in V(G)\,:\,\deg v \text{ even}} \deg v = 2|E(G)|. \tag{2}$$

Both the RHS and the second term on the LHS are even integers, and therefore so is the first term on the LHS. But a sum of odd integers is even only if the number of terms is even. Thus, G has an even number of vertices of odd degree. □

A regular graph is a graph in which all vertices have the same degree. This common value of the degree is the regularity of the graph. A regular graph of regularity k is a k-regular graph. A cubic graph is a 3-regular graph.

Exercise 1.1. Prove that any cubic graph has even order. For which other regular graphs will this statement hold?

Theorem 1.3. *In any graph on two or more vertices, at least two vertices have the same degree.*

Proof. Let G be a graph of order $n \geq 2$. The minimum possible degree of a vertex of G is 0, and the maximum possible degree is $n - 1$ (as there are only n vertices in G). Thus, in order for the n vertices of G to have different degrees, the n different degrees must be exactly $0, 1, \ldots, n - 1$. But if some vertex of G has degree $n - 1$, then it is adjacent to all other vertices, and therefore G cannot have a vertex of degree 0. Thus, at least two vertices of G have the same degree. □

2 Cartesian Products

The Cartesian product of two graphs G and H is the graph denoted by $G \times H$, whose vertex set is $V(G) \times V(H)$ (the Cartesian product of the vertex sets $V(G)$ and $V(H)$, consisting of all ordered pairs (u, v) where u is a vertex of G and v is a vertex of H), in which two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 \sim v_2$ or $u_1 \sim u_2$ and $v_1 = v_2$.

Exercise 2.1. Prove that if G is a graph of order p_1 and size q_1, and H is a graph of order p_2 and size q_2, then $G \times H$ is a graph of order $p_1 p_2$ and size $p_1 q_2 + p_2 q_1$.

Solution. By definition, the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$ of the vertex sets of G and H, and therefore contains $|V(G)| \times |V(H)|$ elements. Thus, the order of $G \times H$ is $p_1 p_2$.

Now, consider any vertex (u, v) of $G \times H$. Corresponding to every vertex u' adjacent to u in G, there is an edge from (u, v) to $(u', v))$ in $G \times H$. Similarly, corresponding to every vertex v' adjacent to v in H, there is an edge from (u, v) from (u, v') in $G \times H$. Thus, the degree of (u, v) in $G \times H$ is $\deg u + \deg v$. Hence, by the Handshaking Lemma, the size of $G \times H$ is half the sum of its vertex degrees, which is

$$\begin{aligned} \frac{1}{2} \sum_{u \in V(G), v \in V(H)} (\deg u + \deg v) &= \frac{1}{2} \sum_{u \in V(G), v \in V(H)} \deg u + \frac{1}{2} \sum_{u \in V(G), v \in V(H)} \deg v \\ &= p_2 \left(\frac{1}{2} \sum_{u \in V(G)} \deg u \right) + p_1 \left(\frac{1}{2} \sum_{v \in V(H)} \deg v \right) \\ &= p_2 q_1 + p_1 q_2. \end{aligned}$$

Exercise 2.2. Show that if G is an r-regular graph and H is an s-regular graph, then $G \times H$ is a regular graph of regularity $r + s$.

3 Subgraphs and Complements

A subgraph of a graph G is a graph H whose vertex and edge sets are, respectively, subsets of the vertex and edge sets of G – i.e. $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph of G is a subgraph containing all the vertices of G. A subgraph H of a graph G is an induced subgraph if for all vertices $u, v \in V(H)$, $uv \in E(H)$ whenever $uv \in E(G)$ – i.e. every edge of G that joins two vertices of H is present in H.

Example 3.1. Fig. 1 shows a graph G and three subgraphs H_1, H_2, and H_3 of G. The subgraph H_1 is neither a spanning subgraph (e.g. it does not contain v_3), nor an induced subgraph (e.g. it contains vertices v_2 and v_5 of G but not the edge v_2v_5). H_2 is a spanning subgraph and H_3 is an induced subgraph of G.

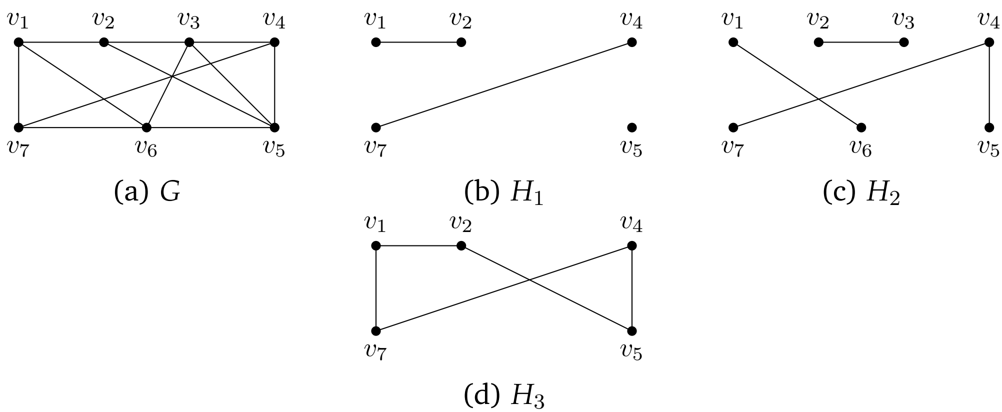


Figure 1: A graph with three of its subgraphs

A graph is complete if all its vertices are adjacent to one another. The complete graph of order n is denoted by K_n. As there is an edge between every pair of distinct vertices of K_n, the size of K_n is $\binom{n}{2}$. The totally disconnected graph of order n, denoted by $\overline{K_n}$ is the graph of order n that has no edges.

The complement of a graph G is the graph $\overline{G}$ with the same vertices as G, in which any two vertices u and v are adjacent if and only if they are non-adjacent in G. Thus, if G is an (n, m)-graph, then $\overline{G}$ is an $\left(n, \binom{n}{2} - m\right)$-graph. Observe that $\overline{K_n}$ is (as the notation suggests) the complement of K_n.

Exercise 3.1. If G is a graph of order n, and $\deg_G v = k$, what is $\deg_{\overline{G}} v$?

Exercise 3.2. Show that if G and $\overline{G}$ are regular graphs of order n with the same regularity, then the regularity is even and $n = 4t + 1$ for some integer t.

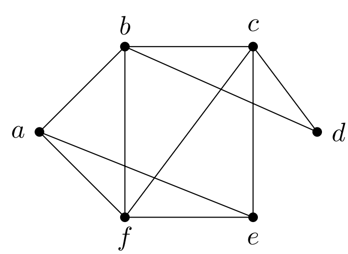
4 Walks, Paths, and Cycles

A walk in a simple graph G is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that $v_i \sim v_{i+1}$ for $i = 1, 2, \ldots, k-1$. This is a walk from v_1 to v_k or between v_1 and v_k, having length $k-1$. We may also say that this is a walk of length $k-1$ starting at v_1 and ending at v_k.

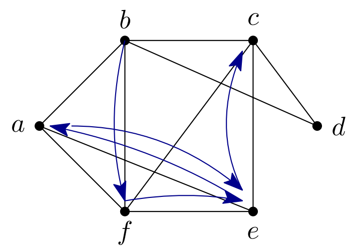
A walk in which no vertex is repeated is a path. Note that the length of a path is the number of edges in it.

A closed walk in G is a walk starting and ending at the same vertex. A closed walk in which no vertex is repeated (except for the start and end vertices being the same) is a cycle. The length of a cycle is also equal to the number of vertices in it. A cycle of the form $v_1, \ldots, v_{k-1}, v_k$ is usually referred to as "the cycle $v_1, \ldots, v_{k-1}$".

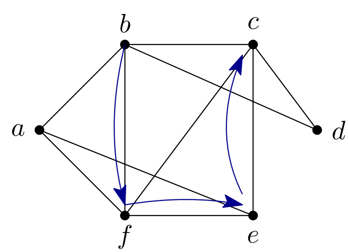
Example 4.1. Consider the graph G shown below.



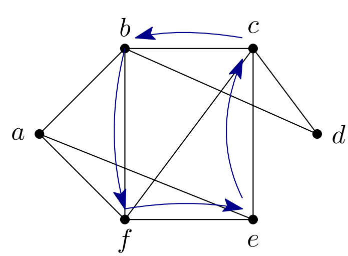
A walk of length 5 from b to c in G is b, f, e, a, e, c.



This walk is not a path, since the vertex e is repeated. An example of a path from b to c in the same graph G is b, f, e, c. The length of this path is 3.



A cycle of length 4 in G is b, f, e, c, b.



5 Distances

The distance between two vertices u and v of a graph G is the length of a shortest path between u and v – such a shortest path between u and

v is a geodesic from u to v. Thus, the distance between u and v in G, denoted by $d_G(u,v)$ is the length of a geodesic from u to v in G. When the graph G is clear from the context, we will write $d_G(u,v)$ as $d(u,v)$.

The eccentricity of a vertex v, denoted by $\operatorname{ecc}(v)$, is the maximum of all distances from v to any other vertex. That is,

$$\operatorname{ecc}(v) = \max_{u \in V(G)} d(u,v).$$

The diameter of G is the maximum of the eccentricities of vertices of G, and the radius of G is the minimum of the eccentricities of vertices of G. They are denoted by $\operatorname{diam}(G)$ and $\operatorname{rad}(G)$ respectively. Thus,

$$\begin{aligned} \operatorname{diam}(G) &= \max_{v \in V(G)} \operatorname{ecc}(v) = \max_{u,v \in V(G)} d(u,v) \\ \operatorname{rad}(G) &= \min_{v \in V(G)} \operatorname{ecc}(v) = \min_{v \in V(G)} \max_{u \in V(G)} d(u,v). \end{aligned}$$

The set of all vertices of G having minimum eccentricity, i.e. the set of all $v \in V(G)$ such that $\operatorname{ecc} v = \operatorname{rad} G$, is the centre of G.

6 Connectedness

A graph is connected if there is a path between every two of its vertices. Otherwise, it is disconnected. A (connected) component of a graph G is a maximal connected subgraph of G. Thus, a graph G is connected if and only if it has exactly one component.

Theorem 6.1. *For any graph G, either G or $\overline{G}$ is connected.*

Proof. Consider a graph G, and suppose that G is disconnected. We shall show that $\overline{G}$ is connected, by showing that there is a path between every two vertices of $\overline{G}$.

Let u and v be any two vertices of $\overline{G}$. If they are not adjacent in G, then they are adjacent in $\overline{G}$, and hence there is a path uv from u to v

in $\overline{G}$. If u and v are adjacent in G, then they belong to the same component of G. As G is disconnected, it has at least one more component containing at least one vertex, say w, which is necessarily non-adjacent to both u and v. Hence, in $\overline{G}$, w is adjacent to both u and v. Thus, there is a path uwv from u to v in $\overline{G}$. Therefore, $\overline{G}$ is connected. $\square$

Theorem 6.2. *For any connected graph G, if $\operatorname{diam} G \geq 3$, then $\operatorname{diam} \overline{G} \leq 3$.*

Proof. Consider a graph G of diameter at least 3. Then there exist vertices u and v in G such that $d_G(u,v) = 3$. This implies that u and v are non-adjacent in G, and hence adjacent in $\overline{G}$.

Now, consider any two vertices x and y other than u and v. In G, x can be adjacent to at most one of u and v, for otherwise, $d_G(u,v) = 2$. Hence, x is adjacent to at least one of u and v in $\overline{G}$. Similarly, y is adjacent to at least one of u and v in $\overline{G}$. Thus, there is a path of length at most 3 from x to y in $\overline{G}$ (namely xuy, xvy, $xuvy$, or $xvuy$), which implies that $d_{\overline{G}}(x,y) \leq 3$. Therefore, $\operatorname{diam} \overline{G} \leq 3$. $\square$

7 Graph Isomorphism

An isomorphism from a graph G to a graph H is a bijective[1] function $f \colon V(G) \to V(H)$ such for all vertices u and v of G, $u \sim_G v$ if and only if $f(u) \sim_H f(v)$. In other words, a graph isomorphism is a bijection from the vertex set of the first graph to that of the second, that preserves both edges and non-edges. If there exists an isomorphism from G to H, then G and H are isomorphic. Then we write $G \cong H$.

Isomorphic graphs have exactly the same structure – i.e. all properties of the graph that do not depend on the labelling or drawing will be shared by isomorphic graphs. For example, isomorphic graphs have the same order, size, degree sequence, diameter, and radius. Indeed,

[1] injective and surjective; i.e. one-to-one and onto

if f is an isomorphism from G to H, and v is a vertex of G, then the neighbours of $f(v)$ in H are the images of the neighbours of v in G, and $\deg_G v = \deg_H f(v)$. Similarly, if u and v are two vertices of G, then $d_G(u,v) = d_H(f(u), f(v))$.

Theorem 7.1. *Graph isomorphism is an equivalence relation on the class of all graphs.*

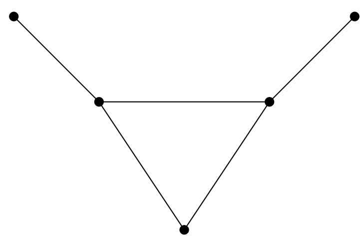
Proof. To show that graph isomorphism is an equivalence relation, we need to show that $\cong$ is a reflexive, symmetric, transitive relation between graphs. First, observe that the identity map on the vertex set of a graph is an isomorphism from the graph to itself – for, if G is a graph and id denotes the identity map on its vertex set then for any two vertices u and v of G, $u \sim v$ if and only if $\mathrm{id}(u) \sim \mathrm{id}(v)$, as $\mathrm{id}(u) = u$ and $\mathrm{id}(v) = v$. Hence, $\cong$ is reflexive.

Next, suppose that $G \cong H$. Then there exists an isomorphism f from G to H. Since $f\colon V(G) \to V(H)$ is bijective, it has an inverse $f^{-1}\colon V(H) \to V(G)$. We claim that f^{-1} is an isomorphism from H to G. We know that f^{-1} is bijective. To see that it preserves edges and non-edges, observe that if x and y are two vertices of H, then $x = f(f^{-1}(x))$ and $y = f(f^{-1}(y))$, which implies that $x \sim_H y$ if and only if $f^{-1}(x) \sim_G f^{-1}(y)$ (since f is an isomorphism from G to H). Therefore, f^{-1} is an isomorphism from H to G. This shows that $H \cong G$. Thus, $\cong$ is symmetric.

Finally, suppose that $G \cong H$ and $H \cong K$. Then there exist isomorphisms f from G to H and g from H to K. We claim that $g \circ f$ is an isomorphism from G to K. Indeed, $g \circ f$ is a function from $V(G)$ to $V(K)$, and being a composition of bijections, is itself a bijection. Now, suppose u and v are two vertices of G. Then $u \sim_G v$ if and only if $f(u) \sim_H f(v)$ (since f is an isomorphism from G to H) if and only if $g(f(u)) \sim_K g(f(v))$ (since g is an isomorphism from H to K). Thus, $g \circ f$ is an isomorphism from G to K. Therefore, $\cong$ is transitive. $\square$

8 Self-Complementary Graphs

A graph is self-complementary if it is isomorphic to its complement – i.e. G is self-complementary if $G \cong \overline{G}$. For example, K_1, P_4 and C_5 are self-complementary graphs of orders 1, 4, and 5 respectively. There is one more self-complementary graph of order 5, namely the bull graph, which can be constructed by adding one new vertex to P_4 and making it adjacent to the two non-pendant vertices of the path. This graph is shown below.



Theorem 8.1. *The order of any self-complementary graph is $4k$ or $4k+1$, for some non-negative integer k.*

Proof. Let G be a self-complementary graph of order n and size m. Then the size of $\overline{G}$ is $\binom{n}{2} - m$. Since $G \cong \overline{G}$, $m = \binom{n}{2} - m$, which implies that $m = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$. As m is an integer, this implies that 4 divides $n(n-1)$, which in turn implies that either one of n and $n-1$ is divisible by 4, or both n and $n-1$ are even. Since the latter is not possible, it follows that $n = 4k$ or $n - 1 = 4k$, i.e. $n = 4k$ or $4k+1$ for some integer k. □

From Theorem 6.1, we know that a graph and its complement cannot both be disconnected. Thus, if G is a disconnected graph, then $\overline{G}$ must be connected, and therefore it cannot be isomorphic to G. This implies that a self-complementary graph is necessarily connected.

Corollary 8.2. *Every self-complementary graph is connected.* □

Similarly, from Theorem 6.2, we obtain the following corollary about the diameter of self-complementary graphs.

Corollary 8.3. *Every non-trivial self-complementary graph has diameter* 2 *or* 3*.*

Proof. Consider a non-trivial, self-complementary graph G, and suppose that it has diameter strictly greater than 3. Then, by Theorem 6.2, $\operatorname{diam} \overline{G} \leq 3$, which implies that $\operatorname{diam} \overline{G} \neq \operatorname{diam} G$, a contradiction, since $G \cong \overline{G}$. Hence, $\operatorname{diam} G \leq 3$. On the other hand, a non-trivial graph cannot have diameter 0, hence $\operatorname{diam} G \geq 1$. But if $\operatorname{diam} G = 1$, then G is a non-trivial complete graph, whose complement is a totally disconnected graph, which is again impossible. Hence, $\operatorname{diam} G = 2$ or 3. □

Exercise 8.1. If G is a regular self-complementary graph of order n, show that $n = 4k + 1$, for some integer k, and $\operatorname{diam} G = 2$.

Exercise 8.2. Let G be a self-complementary graph, and let H be the graph obtained by taking the disjoint union of H and P_4, and making every vertex of H adjacent to the two non-pendant vertices of this P_4. Then show that H is also self-complementary. What is the graph H obtained in this manner if $G = K_1$?

Exercise 8.3. Construct a self-complementary graph of order 9.

Exercise 8.4. Show that for each positive integer n of the form $4k$ or $4k+1$, where k is an integer, there exists at least one self-complementary graph of order n.

9 Bipartite Graphs

A graph $G = (V, E)$ is bipartite if its vertex set V can be partitioned into two subsets V_1 and V_2 such that no two vertices in V_i are adjacent, for $i = 1, 2$. That is, there exist two non-empty, disjoint subsets $V_1, V_2 \subseteq V$

such that $V_1 \cup V_2 = V$, and every edge of G (if any) joins a vertex in V_1 with a vertex in V_2. Then we say that (V_1, V_2) is a bipartition of G.

The following theorem characterises bipartite graphs in terms of its cycles. A cycle is even if its length is even and odd if its length is odd.

Theorem 9.1. *A graph is bipartite if and only if it contains no odd cycles.*

Proof. First, suppose that G is a bipartite graph with bipartition (V_1, V_2), and let $v_1, v_2, \ldots, v_k$ be a cycle of length k in G. Without loss of generality, suppose that $v_1 \in V_1$. Then, since $v_2 \sim v_1$, $v_2 \in V_2$, which in turn implies that $v_3 \in V_1$, as $v_2 \sim v_3$. Proceeding similarly, we see $v_i \in V_1$ if i is odd and $v_i \in V_2$ if i is even. But $v_k \sim v_1$ and $v_1 \in V_1$ implies that $v_k \in V_2$. Therefore, k (the length of the cycle) is even.

Conversely, suppose G is a graph that has no odd cycles. Without loss of generality, assume that G is connected – otherwise, apply the argument to each component. Let v be any vertex of G. Define subsets V_1 and V_2 of $V(G)$ as follows:

$$V_1 = \{\, u \in V \mid d(u, v) \text{ is even} \,\}$$
$$V_2 = \{\, u \in V \mid d(u, v) \text{ is odd} \,\}.$$

We claim that (V_1, V_2) is a bipartition of G.

Suppose not, and let x and y be adjacent vertices of G, both belonging to V_i for $i = 1$ or 2. Let P and Q be shortest paths from v to x and y respectively. Let w be the last vertex common to both P and Q when traversing P from v to x. Then the portion of P from w to x, the edge (x, y), and the portion of Q from y to w forms a cycle in G, having length, say l, given by

$$\begin{aligned} l &= d(w, x) + d(w, y) + 1 \\ &= [d(v, x) - d(v, w)] + [d(v, y) - d(v, w)] + 1 \\ &= d(v, x) + d(v, y) - 2d(v, w) + 1. \end{aligned}$$

If $x, y \in V_1$, then both $d(v, x)$ and $d(v, y)$ are even, and if $x, y \in V_2$, then both these numbers are odd. In either case, $d(v, x) + d(v, y)$ is even, and hence l is odd, which is a contradiction. Hence, (V_1, V_2) is a bipartition of G as claimed. □

10 Trees

A tree is a connected, acyclic graph. There are several well known characterisations or alternative definitions of trees. We take the given definition as the basic one and prove its equivalence to some others.

Theorem 10.1. *A graph T is a tree if and only if there is a unique path joining every two vertices of T.*

Proof. First, suppose that T is a tree, and let u and v be vertices of T. Since T is connected, there is a path, say P_1, joining u and v. Now we must show that this path is unique. Assume to the contrary that there exists another path P_2 from u to v. When traversing P_1 from u to v, let w be the first vertex that is present on P_1 but not P_2. Let x be the vertex on P_1 preceding w, and note that x is on P_2 as well. Let y be the next vertex common to both P_1 and P_2 when traversing P_1 from x to v. Then the portion of P_1 from x to y together with the portion of P_2 from y to x forms a cycle in the tree T, which is a contradiction. Thus, P_1 is the unique path joining u and v.

Conversely, suppose that T is a graph in which there is a unique path joining any two vertices. Clearly, T is connected. To show that T is acyclic, suppose that $v_1, v_2, \ldots, v_n$ is a cycle in T. Then we get two different paths joining v_1 and v_n, namely the path $v_1, v_2, \ldots, v_n$ and the path v_1, v_n (since $v_1 \sim v_n$ in the cycle). This contradicts our assumption. Thus, T must be acyclic and hence is a tree. □

The next two results show that the size of a tree is always one less than its order, and that conversely, this property together with either connectedness or acyclicity implies that the graph is a tree.

Theorem 10.2. *A (p,q)-graph T is a tree if and only if it is connected and $p = q+1$.*

Proof. Let T be a tree with p vertices and q edges. Then T is connected. We prove that $p = q+1$ by induction. This is clearly true when $p = 1$. Assume it to be true for all trees of order less than p. Now in T, we know that every two vertices are joined by a unique path. Thus, if e is any edge of T, then the graph $T - \{e\}$ obtained by deleting e has exactly two components, say T_1 and T_2. Each one is a tree, since it is connected and acyclic. Let T_i have p_i vertices and q_i edges, $i = 1,2$. Then by the hypothesis, $p_i = q_i + 1$ (since $p_i < p$). But $p = p_1 + p_2$ and $q = q_1 + q_2 + 1$ (since the size of $T - \{e\}$ is one less than that of T). Thus, $p = q_1 + q_2 + 2 = q + 1$.

For the converse, suppose that T is a connected (p,q)-graph with $p = q+1$. We must show that is acyclic. Suppose to the contrary that T has a cycle C with k vertices. Then C has k edges as well. Since T is connected, there is a path from every vertex not on C to some vertex of C. The shortest path from each vertex v not on C to a vertex on C has a unique edge incident with v, which is not part of C. Since there are $p-k$ vertices in T not on C, there are $p-k$ such edges. Thus $q \geq (p-k)+k = p$, which contradicts our assumption that $p = q+1$. Thus, T must be acyclic. □

In the following theorem, the proof of the direct part is identical to that of Theorem 10.2, except for the assertion being about acyclicity rather than connectedness. The proof of the converse part is entirely different.

Theorem 10.3. *A (p,q)-graph T is a tree if and only if it is acyclic and $p = q+1$.*

Proof. Let T be a tree with p vertices and q edges. Then T is acyclic. We prove that $p = q+1$ by induction. This is clearly true when $p = 1$.

Assume it to be true for all trees of order less than p. Now in T, we know that every two vertices are joined by a unique path. Thus, if e is any edge of T, then the graph $T - \{e\}$ obtained by deleting e has exactly two components, say T_1 and T_2. Each one is a tree, since it is connected and acyclic. Let T_i have p_i vertices and q_i edges, $i = 1, 2$. Then by the hypothesis, $p_i = q_i + 1$ (since $p_i < p$). But $p = p_1 + p_2$ and $q = q_1 + q_2 + 1$ (since the size of $T - \{e\}$ is one less than that of T). Thus, $p = q_1 + q_2 + 2 = q + 1$.

Conversely, suppose that T is an acyclic (p, q)-graph with $p = q+1$. To show that T is connected, we need to prove that it is connected – i.e. it has only one component. Let T have k components $T_1, \ldots, T_k$. Each one is acyclic, and being connected, is a tree. Thus from the first part of the theorem, we know that if p_i and q_i are respectively the order and size of the component T_i, $p_i = q_i + 1$. Now $p = p_1 + \cdots p_k = (q_1 + 1) + \cdots (q_k + 1) = q + k$. But we know that $p = q + 1$. Therefore, $k = 1$. Thus, T is a tree. □

Exercise 10.1. A pendant vertex of a graph is a vertex of degree 1. Prove that every non-trivial tree contains at least two pendant vertices.

Hint: Observe that a non-trivial tree cannot have a vertex of degree zero. Use Handshaking Lemma and assume every degree is at least 2 to get a contradiction.

Exercise 10.2. The centre of a graph G is the set of all vertices of G with minimum eccentricity – i.e. the set of all vertices v of G with $\operatorname{ecc} v = \operatorname{rad} v$. Show that every tree has a centre consisting of either exactly one vertex or exactly two adjacent vertices.

Hint: Observe that deleting all pendant vertices of a tree results in a new tree with the same centre.

Solution. Let T be a tree on n vertices. We prove the result by induction on n.

If $n = 1$ or 2, the result obviously holds. Suppose, for the sake of induction, that the result holds for all trees of order less than n.

Consider T, of order $n \geq 3$. Then T has at least two pendant vertices. Let T' be the tree (of order less than n) obtained by deleting all the pendant vertices of T.

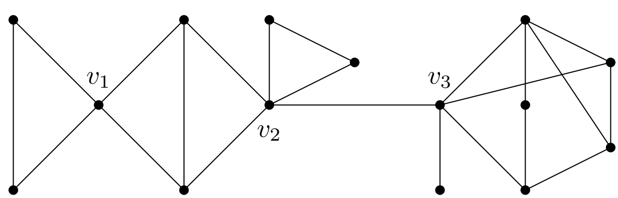
Then the vertices of T' have eccentricities exactly one less than their eccentricities in T, and therefore, T and T' have the same centre. Hence, the result follows by induction.

Exercise 10.3. If G and H are two trees of orders n and m respectively, what is the size of $G \times H$?

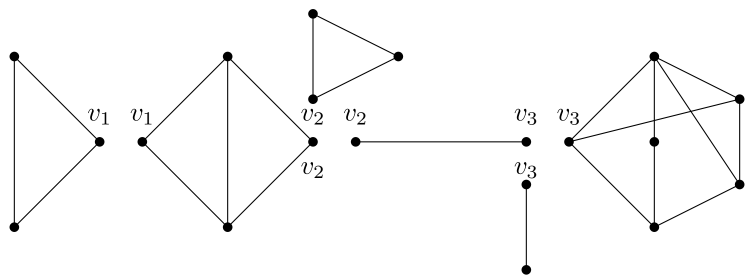
11 Blocks

A cutvertex of a graph is a vertex whose removal increases the number of components, i.e. a vertex v of G such that $G - v$ has more components than G. If G is connected, we can equivalently say that v is a cutvertex if $G - v$ is disconnected. Similarly, a cutedge or bridge of a graph whose removal increases the number of components. A nonseparable graph is a connected, non-trivial graph with no cutvertices. A maximal nonseparable subgraph of a graph is a block of the graph. A nonseparable graph is itself said to be a block as well.

Example 11.1. The graph shown below has 6 blocks and 3 cutvertices (v_1, v_2, v_3).



The 6 blocks of this graph are shown below.



Theorem 11.2. *If G is a connected graph, and v is any vertex of G, then the following are equivalent:*

(i) *v is a cutvertex of G.*
(ii) *There exist vertices u and w of G, distinct from v, such that every u-w path passes through v.*
(iii) *There exists a partition of $V(G) - v$ into two non-empty subsets U and W such that for all $u \in U$ and $w \in W$, every u-w path passes through v.*

Proof. (i) $\implies$ (iii). Since v is a cutvertex, the graph $G - v$ is disconnected, i.e. it has two or more components. Let U be the set of all the vertices in any one of the components, and let W be the set of all the remaining vertices of $G - v$. Clearly, $\{U, W\}$ is a partition of $V(G) - v$. Now, if $u \in U$ and $w \in W$, then u and w are in different components of $G - v$, which implies that any path from u to w must pass through v.
(iii) $\implies$ (ii) is obvious as the latter is a special case of the former.
(ii) $\implies$ (i). Consider the graph $G - v$. As every u-w path passes through v, none of them is present in $G - v$, and therefore $G - v$ is disconnected. Hence, v is a cutvertex of G. □

12 Line Graphs

Let G be a graph with at least one edge. The line graph of G is the graph $L(G)$ whose vertex set is the edge set of G, in which two vertices e and f

are adjacent if the edges e and f of G are adjacent.

Example 12.1. Fig. 2 shows a graph G with eight edges and and its line graph $L(G)$.

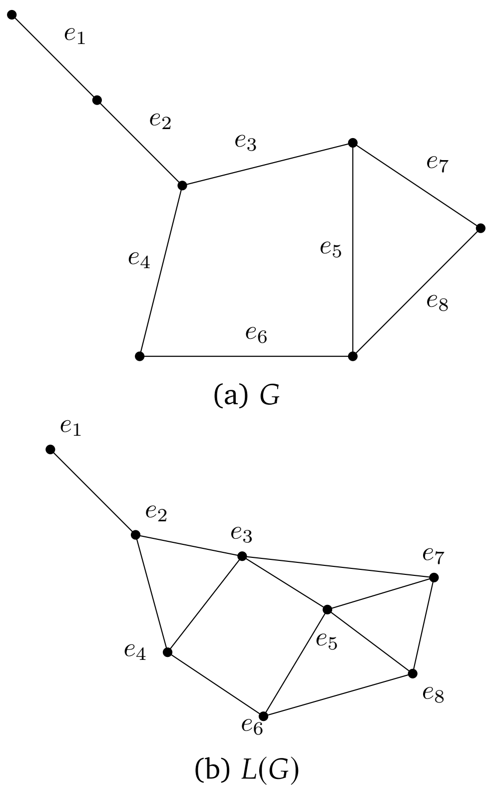


Figure 2: A graph and its line graph

If u and v are edges joined by an edge e in G, then in $L(G)$, the vertex e will be adjacent to all the vertices corresponding to the edges of G other than e that are incident with u and v (see Fig. 3). Thus, $\deg_{L(G)} e = \deg_G u + \deg_G v - 2$.

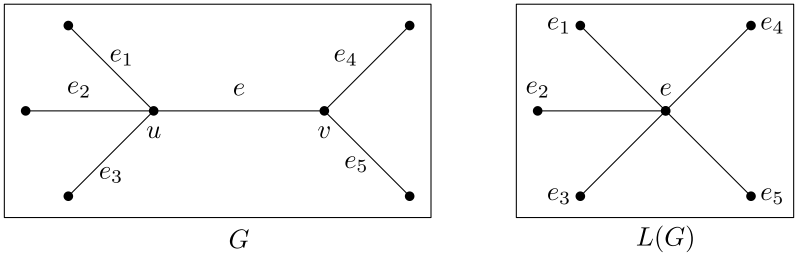


Figure 3: Neighbourhood of a vertex in $L(G)$

Theorem 12.2. *For any graph G of size $m \geq 1$, its line graph $L(G)$ has order m size* $\sum_{v \in V(G)} \binom{\deg_G v}{2}$.

Proof. By definition, the vertices of $L(G)$ are the edges of G, and hence the order of $L(G)$ is the size of G, m.

Now, let $e = uv$ be an edge of G. In $L(G)$, e is a vertex, and it is adjacent to another vertex f if and only if f is an edge of G and is adjacent to e, i.e. f is an edge of G incident with one of the end vertices of e, namely u and v. Thus, the degree of the vertex e of $L(G)$ is equal to the total number of edges that are incident with either u or v in G, except for the edge e itself.

Clearly, u is incident with to $\deg_G u - 1$ edges other than e, and v is incident with $\deg_G v - 1$ edges other than e. Thus, the degree of e in $L(G)$ is $\deg_G u - 1 + \deg_G v - 1$.

By Handshaking Lemma, the size of $L(G)$ is half the sum of its vertex degrees, which is

$$\frac{1}{2} \sum_{uv \in E(G)} (\deg_G u - 1 + \deg_G v - 1).$$

As the summation is over the edges of G, each vertex v of G appears exactly $\deg v$ times in the above summation, which can therefore be written as

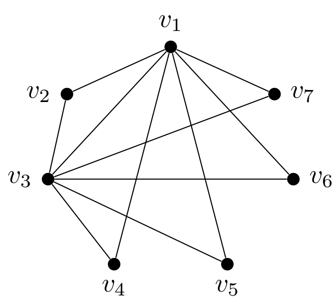
$$\frac{1}{2}\sum_{v\in V(G)}(\deg_G v)(\deg_G v-1)=\sum_{v\in V(G)}\binom{\deg_G v}{2}. \qquad \square$$

13 Adjacency Matrices

The adjacency matrix of a graph G of order n, with vertex set $V = \{v_1, \ldots, v_n\}$, is the $n \times n$ matrix $A = A(G)$ whose (i, j)-entry is

$$a_{ij} = \begin{cases} 1, & v_i \sim v_j \\ 0, & v_i \nsim v_j. \end{cases}$$

Example 13.1. The adjacency matrix of the graph



is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that, as the graphs we discuss are simple graphs and therefore have no self-loops on vertices, no vertex is adjacent to itself – i.e. $a_{ii} = 0$ for all $i = 1, \ldots, n$. Also, since the graphs are undirected, $v_i \sim v_j$ if and only if $v_j \sim v_i$ – i.e. $a_{ij} = a_{ji}$. Thus, we have the following observation.

Observation 13.2. *The adjacency matrix of a (simple, undirected) graph is a symmetric, zero-diagonal,* $(0, 1)$*-matrix.*

In the $i$þrow of the adjacency matrix, for each j, the $j$þentry is 1 if v_j is adjacent to v_i, and 0 otherwise. That is, the number of 1s in the $i$þrow is the number of vertices adjacent to v_i, or in other words, the degree of v_i. Thus, the row sums of A are the vertex degrees. Observe that $A\mathbb{1}$ is the vector of row sums, where $\mathbb{1}$ is the vector (of suitable size) with all entries equal to 1. For instance, with the matrix A given in Example 13.1,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 6 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

From the Handshaking Lemma and the preceding observation, it follows that the sum of all entries of A is twice the number of edges of the graph.

Exercise 13.1. Show that the (i, j)-entry of A^2 is the number of walks of length 2 from v_i to v_j. Hence show that $\operatorname{tr}(A^2) = 2|E(G)|$.

Hint: Recall that if A is any $n \times n$ matrix, then the (i, j)-entry of A^2 is $\sum_{k=1}^{n} a_{ik}a_{kj}$. As A is a 0-1 matrix, each term in this summation is 1 or 0, with the former if and only if $a_{ik} = a_{kj} = 1$. What does this imply about the vertices v_i, v_k, and v_j? Then, as k varies from 1 to n, what does the value of the sum imply about v_i and v_j?

The following result (which generalises the statement in Exercise 13.1) shows that the adjacency matrix can be used to obtain certain information about walks in the graph.

Theorem 13.3. *Let A be the adjacency matrix of a graph G with vertex set $\{v_1, \ldots, v_n\}$. Then the (i,j)-entry of A^m, for any positive integer m, is the number of walks of length m from v_i to v_j.*

Proof. We prove the result by induction on m. For $m = 1$, the (i,j)-entry of $A^1 = A$ is a_{ij}, which is 1 if and only if v_i is adjacent to v_j, i.e. if and only if there is a walk of length 1 (namely, an edge) from v_i to v_j. Thus, the result holds for $m = 1$.

Now suppose, for the sake of induction, that the result holds for some $m \geq 1$, and consider A^{m+1}. The (i,j)-entry of A^{m+1} is

$$(A^{m+1})_{ij} = \sum_{k=1}^{n} (A^m)_{ik} a_{kj}.$$

First, note that $a_{kj} = 1$ if and only if $v_k \sim v_j$. Therefore, the above sum is equal to the sum of all $(A^m)_{ik}$ where $v_k \sim v_j$. Now, by the induction hypothesis, $(A^m)_{ik}$ is the number of walks of length m from v_i to v_k. If v_j is adjacent to v_j, then each walk of length m from v_i to v_k, together with the edge from v_j to v_j, forms a walk of length $m+1$ from v_i to v_j. Thus, for each k such that $v_k \sim v_j$, $(A^m)_{ik} a_{kj} = (A^m)_{ik}$ is the number of walks of length $m+1$ from v_i to v_j that pass through k. Summing over all k, this gives the total number of walks of length $m+1$ from v_i to v_j. Hence the result follows by induction. □

Theorem 13.4. *Let A be the adjacency matrix of a graph G with vertex set $\{v_1, \ldots, v_n\}$. Then*

(i) $\operatorname{tr}(A^2) = 2|E(G)|$
(ii) $\operatorname{tr}(A^3) = 6c_3(G)$

where $c_3(G)$ denotes the number of triangles in G.

Proof. We know that $(A^m)_{ij}$ is the number of walks of length m from v_i to v_j.

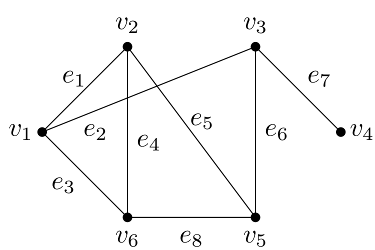
(i) Hence, the i^{th} diagonal entry of A^2, viz. $(A^2)_{ii}$, is the number of walks of length 2 from v_i to itself. Any walk of length 2 from v_i to itself is of the form $v_i v_j v_i$, where v_j is a vertex adjacent to v_i. Thus, the number of such walks is equal to the number of vertices adjacent to v_i, i.e. $\deg v_i$. Hence, $\operatorname{tr}\left(A^2\right) = \sum_{i=1}^{n} \deg v_i = 2|E(G)|$, by Handshaking Lemma.

(ii) Similarly, $(A^3)_{ii}$ is the number of walks of length 3 from v_i to itself. Any such walk is of the form $v_i v_j v_k v_i$, which implies that v_i, v_j, and v_k form a triangle in G. Moreover, each such triangle corresponds to two distinct walks from v_i to itself, when traversed in the two opposite directions. Thus, $(A^3)_{ii}$ is twice the number of triangles having v_i as one of its vertices. Therefore, $\operatorname{tr}\left(A^3\right)$ is $6c_3(G)$, since each triangle contains three vertices, each of which counts the triangle twice in the sum $\operatorname{tr}\left(A^3\right)$. □

14 Incidence Matrices

Consider an (n, m)-graph G having vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{e_1, \ldots, e_m\}$. The incidence matrix of G is the $n \times m$ matrix $B = B(G)$ whose (i, j)-entry is 1 if the vertex v_i is incident with the edge e_i, and 0 otherwise.

Example 14.1. The incidence matrix of the graph



is

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Each column of the incidence matrix corresponds to an edge of the graph, and the only non-zero entries in the column correspond to the end-vertices of the edge – thus, each column contains exactly two 1s. In a simple graph, there is at most one edge between a given pair of vertices, and hence no two columns of the incidence matrix can be equal. We therefore have the following observation.

Observation 14.2. *The incidence matrix of a (simple, undirected) graph is a* 0-1 *matrix in which every column has exactly two* 1*s, and no two columns are equal.*

Theorem 14.3. *If B is the incidence matrix of a graph G, then the adjacency matrix of its line graph $L(G)$ is*

$$A(L(G)) = B^T B - 2I$$

where I is the identity matrix of order $|E(G)|$.

Proof. Let G be an (n,m)-graph with edge set $E = \{e_1, \ldots, e_m\}$. The incidence matrix $B = B(G)$ is of order $n \times m$, and therefore $B^T B - 2I$ (say) is of order $m \times m$.

Observe that the (i,j) entry of $B^T B$ is the dot product of the i^{th} and j^{th} columns of B. As each column of B has exactly two non-zero entries, both equal to 1, the (i,i) entry of $B^T B$ is 2, and therefore the diagonal entries of $B^T B - 2I$ are all zero.

For $i \neq j$, the dot product of the i^{th} and j^{th} columns of B will be 1 if the edges e_i and e_j are adjacent (i.e. have one vertex in common), and

0 otherwise – note that the dot product cannot be 2, as no two distinct columns of B can be equal. Thus, for $i \neq j$, the (i,j)-entry of $B^T B$, which is equal to the (i,j) entry of $B^T B - 2I$, is 1 if and only if the edges e_i and e_j are adjacent in G, or equivalently, the vertices e_i and e_j of $L(G)$ are adjacent. Thus, $B^T B - 2I$ is the adjacency matrix of G. □

15 Dijkstra's Algorithm

Let G be a directed graph on n vertices $v_1, \ldots, v_n$, in which each (directed) edge has a positive weight attached to it (say, representing the cost of traversing that edge). Let W be the weighted adjacency matrix of G, defined as follows. W is an $n \times n$ matrix with rows and columns indexed by the vertices of G, and having (i,j)-entry

$$w_{ij} = \begin{cases} 0, & i = j \\ \text{Weight of the edge } (v_i, v_j), & v_i \sim v_j \\ \infty, & v_i \nsim v_j. \end{cases}$$

Dijkstra's algorithm is a procedure to determine the shortest paths (and hence distances) from a given source vertex s of G to all the other vertices.

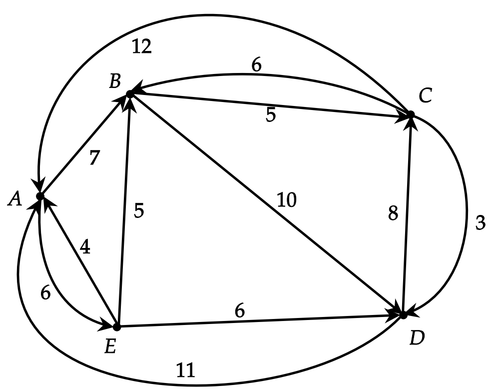
The input to the algorithm is the set of vertices V and the weighted adjacency matrix W. Initially, the algorithm takes the weights of the edges from s to the other vertices as the tentative best distances to those vertices. It also maintains a set K of vertices to which the shortest paths from s have been found (so that no further improvement is possible), and a set U of vertices to which shorter paths may yet be found, passing through some vertex in K. In each step, the vertex $t \in U$ with minimum tentative best distance is selected – it is guaranteed that no shorter path exists from s to this vertex (since any such path would have to pass through some other vertex of U, but the distances to such a vertex is

Algorithm 1 Dijkstra's algorithm

$K \leftarrow s$, $U \leftarrow V(G) - \{s\}$
$\mathrm{bestDTo}(u) \leftarrow w_{su}, \forall u \in U$
$\mathrm{tree}(u) \leftarrow s, \forall u \in U$
while $|U| > 1$ **do**
  $t \leftarrow u \in U$ such that $\mathrm{bestDTo}(u)$ is minimum
  $U \leftarrow U - \{t\}$ ▷ Remove t from U and
  $K \leftarrow K \cap \{t\}$ ▷ add it to K
  for $u \in U$ **do**
    $du_t \leftarrow \mathrm{bestDTo}(t) + w_{tu}$ ▷ Distance to u through t
    if $du_t < \mathrm{bestDTo}(u)$ **then**
      $\mathrm{bestDTo}(u) \leftarrow dsu_t$
      $\mathrm{tree}(u) \leftarrow t$
    end if
  end for
end while

larger than the distance to t). This vertex t is removed from U and added to K, and then for each vertex u remaining in U, the distance from s to u *through* t is compared with the current best distance to u. If the former is found to be smaller, the best distance to u is updated to that value and t is marked as the vertex through which u is to be reached in the shortest path – this is indicated by $\mathrm{tree}(u)$ in the algorithm. This procedure is repeated until there is only vertex remaining in U (i.e. until $|U| = 1$).

Example 15.1. In the graph shown below, find the shortest paths from B to all the other vertices.

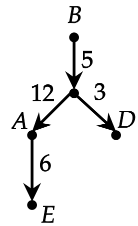


The weighted adjacency matrix of the graph is

$$W = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 7 & \infty & \infty & 6 \\ \infty & 0 & 5 & 10 & \infty \\ 12 & 6 & 0 & 3 & \infty \\ 11 & \infty & 8 & 0 & \infty \\ 4 & 5 & \infty & 6 & 0 \end{bmatrix} \end{array}.$$

The results of applying the algorithm are tabulated below and the tree of shortest paths is also shown.

	A	C	D	E
$B(0)$	∞_B	$\boxed{5_B}$	10_B	∞_B
$C(5)$	17_C	–	$\boxed{8_C}$	∞_B
$D(8)$	$\boxed{17_C}$	–	–	∞_B
$A(17)$	–	–	–	23_A
B	17_C	5_B	8_C	23_A



16 Prim's Algorithm

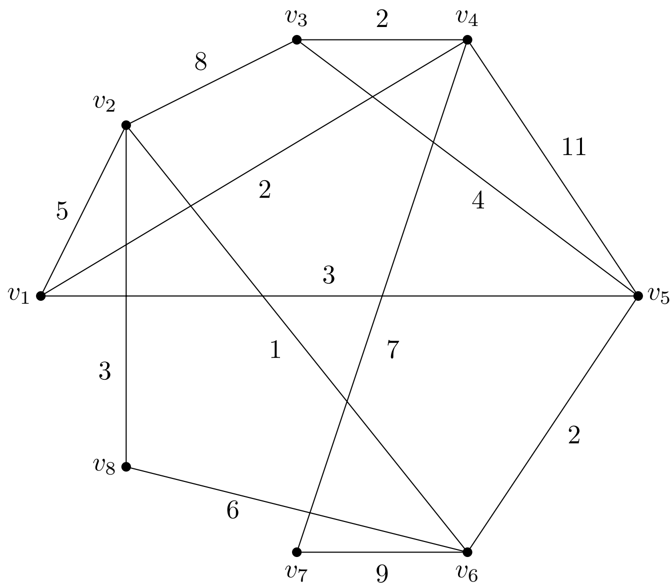
Let G be a weighted, undirected, connected graph with positive edge weights. A minimal spanning tree of G is a spanning tree T of G such that the sum of the weights of its edges is minimum among all the spanning trees of G.

Algorithm 2 Prim's algorithm

Input: Weighted, undirected, connected graph G, source vertex $v \in V(G)$

$V(T) \leftarrow \{v\}$, $E(T) \leftarrow \emptyset$
while $V(T) \neq V(G)$ **do**
  $e \leftarrow$ minimum-weight edge with one end-vertex in $V(T)$ and one end-vertex not in $V(T)$
  $E(T) \leftarrow E(T) \cup \{e\}$
  $V(T) \leftarrow V(T) \cup \{x, y\}$ where $e = xy$
end while

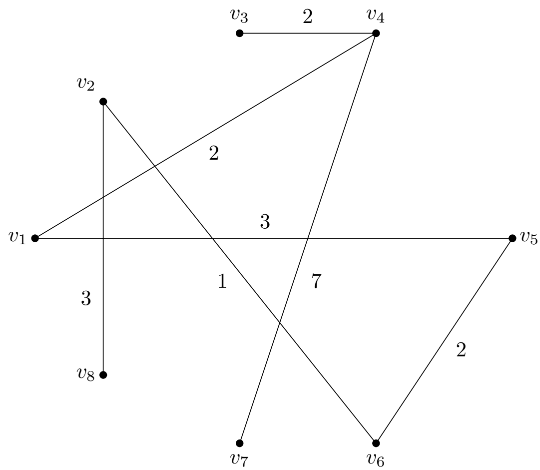
Example 16.1. Apply Prim's algorithm to the graph given below to obtain a minimal spanning tree. Take v_1 as the initial vertex.



The step-by-step result of the algorithm is tabulated below.

Step	Edge added	Vertex added	Total weight
1	–	v_1	0
2	(v_1, v_4)	v_4	2
3	(v_4, v_3)	v_3	4
4	(v_1, v_5)	v_5	7
5	(v_5, v_6)	v_6	9
6	(v_6, v_2)	v_2	10
7	(v_2, v_8)	v_8	13
8	(v_4, v_7)	v_7	20

Therefore, the minimal spanning tree is:


v_3
2
v_4
v_2
2
3
v_1
v_5
1
7
3
2
v_8
v_7
v_6