Applied Graph Theory

Contents

1 Introdcution

A graph is an ordered pair $G = (V, E)$ where V is a nonempty set called the vertex set, whose elements are the vertices of G, and E is a set of unordered pairs of distinct vertices of G, whose elements are the edges of G. The order of G is its number of vertices and the size of G is its number of edges.

Two vertices u and v of G are adjacent, denoted by $u \sim v$, if $(u, v) \in E$ – we may also denote the edge (u, v) as simply uv. Note that $uv = vu$. If there is no edge between u and v, then they are nonadjacent, denoted by $u \nsim v$. The vertex v and the edge uv are said to be incident with each other. The degree of a vertex v is the number of edges incident with it, or equivalently, the number of vertices adjacent with it, and is denoted by $\deg(v)$.

Lemma 1.1 (Handshaking Lemma). *The sum of the degrees of all vertices of a graph is twice its size.*

Proof. Exercise. □

2 Cartesian Products

The Cartesian product of two graphs G and H is the graph denoted by $G \times H$, whose vertex set is $V(G) \times V(H)$ (the Cartesian product of the vertex sets $V(G)$ and $V(H)$, consisting of all ordered pairs (u, v) where u is a vertex of G and v is a vertex of H), in which two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 \sim v_2$ or $u_1 \sim u_2$ and $v_1 = v_2$.

Exercise 2.1. Prove that if G is a graph of order p_1 and size q_1, and H is a graph of order p_2 and size q_2, then $G \times H$ is a graph of order p_1p_2 and size $p_1q_2 + p_2q_1$.

3 Subgraphs

A subgraph of a graph G is a graph H whose vertex and edge sets are, respectively, subsets of the vertex and edge sets of G – i.e., $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph of G is a subgraph containing all the vertices of G.

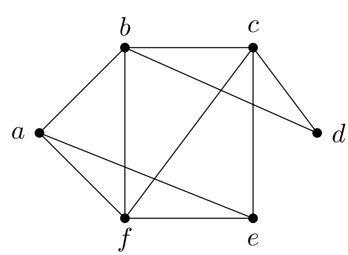
4 Walks, Paths, and Cycles

A walk in a simple graph G is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that $v_i \sim v_{i+1}$ for $i = 1, 2, \ldots, k-1$. This is a walk from v_1 to v_k or between v_1 and v_k, having length $k-1$. We may also say that this is a walk of length $k-1$ starting at v_1 and ending at v_k.

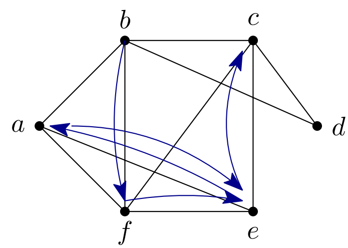
A walk in which no vertex is repeated is a path. Note that the length of a path is the number of edges in it.

A closed walk in G is a walk starting and ending at the same vertex. A closed walk in which no vertex is repeated is a cycle. The length of a cycle is also equal to the number of vertices in it. A cycle of the form $v_1, \ldots, v_{k-1}, v_k$ is usually referred to as "the cycle $v_1, \ldots, v_{k-1}$".

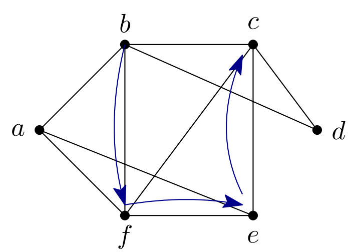
Example 4.1. Consider the graph G shown below.



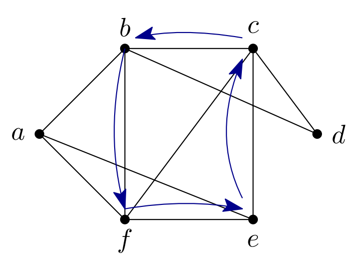
A walk of length 5 from b to c in G is b, f, e, a, e, c.



This walk is not a path, since the vertex e is repeated. An example of a path from b to c in the same graph G is b, f, e, c. The length of this path is 3.



A cycle of length 4 in G is b, f, e, c, b.



5 Distances

The distance between two vertices u and v of a graph G is the length of a shortest path between u and v – such a shortest path between u and

v is a geodesic from u to v. Thus, the distance between u and v in G, denoted by $d_G(u,v)$ is the length of a geodesic from u to v in G. When the graph G is clear from the context, we will write $d_G(u,v)$ as $d(u,v)$.

The eccentricity of a vertex v, denoted by $\operatorname{ecc}(v)$, is the maximum of all distances from v to any other vertex. That is,

$$\operatorname{ecc}(v) = \max_{u \in V(G)} d(u,v).$$

The diameter of G is the maximum of the eccentricities of vertices of G, and the radius of G is the minimum of the eccentricities of vertices of G. They are denoted by $\operatorname{diam}(G)$ and $\operatorname{rad}(G)$ respectively. Thus,

$$\begin{aligned}\operatorname{diam}(G) &= \max_{v \in V(G)} \operatorname{ecc}(v) = \max_{u,v \in V(G)} d(u,v)\\ \operatorname{rad}(G) &= \min_{v \in V(G)} \operatorname{ecc}(v) = \min_{v \in V(G)} \max_{u \in V(G)} d(u,v).\end{aligned}$$

The set of all vertices of G having minimum eccentricity, i.e., the set of all $v \in V(G)$ such that $\operatorname{ecc} v = \operatorname{rad} G$, is the centre of G.

6 Connectedness

A graph is connected if there is a path between every two of its vertices. Otherwise, it is disconnected. A (connected) component of a graph G is a maximal connected subgraph of G. Thus, a graph G is connected if and only if it has exactly one component.

Theorem 6.1. *For any graph G, either G or $\overline{G}$ is connected.*

Proof. Consider a graph G, and suppose that G is disconnected. We shall show that $\overline{G}$ is connected, by showing that there is a path between every two vertices of $\overline{G}$.

Let u and v be any two vertices of $\overline{G}$. If they are not adjacent in G, then they are adjacent in $\overline{G}$, and hence there is a path uv from u to v

in $\overline{G}$. If u and v are adjacent in G, then they belong to the same component of G. As G is disconnected, it has at least one more component containing at least one vertex, say w, which is necessarily non-adjacent to both u and v. Hence, in $\overline{G}$, w is adjacent to both u and v. Thus, there is a path uwv from u to v in $\overline{G}$. Therefore, $\overline{G}$ is connected. □

Theorem 6.2. *For any connected graph G, if* $\operatorname{diam} G \geq 3$*, then* $\operatorname{diam} \overline{G} \leq 3$.

Proof. Consider a graph G of diameter at least 3. Then there exist vertices u and v in G such that $d_G(u, v) = 3$. This implies that u and v are non-adjacent in G, and hence adjacent in $\overline{G}$.

Now, consider any two vertices x and y other than u and v. In G, x can be adjacent to at most one of u and v, for otherwise, $d_G(u, v) = 2$. Hence, x is adjacent to at least one of u and v in $\overline{G}$. Similarly, y is adjacent to at least one of u and v in $\overline{G}$. Thus, there is a path of length at most 3 from x to y in $\overline{G}$ (namely xuy, xvy, $xuvy$, or $xvuy$), which implies that $d_{\overline{G}}(x, y) \leq 3$. Therefore, $\operatorname{diam} \overline{G} \leq 3$. □

7 Graph Isomorphism

An isomorphism from a graph G to a graph H is a bijective[1] function $f : V(G) \to V(H)$ such for all vertices u and v of G, $u \sim_G v$ if and only if $f(u) \sim_H f(v)$. In other words, a graph isomorphism is a bijection from the vertex set of the first graph to that of the second, that preserves both edges and non-edges. If there exists an isomorphism from G to H, then G and H are isomorphic. Then we write $G \cong H$.

Isomorphic graphs have exactly the same structure – i.e., all properties of the graph that do not depend on the labelling or drawing will be shared by isomorphic graphs. For example, isomorphic graphs have the same order, size, degree sequence, diameter, and radius. Indeed,

[1]injective and surjective; i.e., 1-1 and onto

if f is an isomorphism from G to H, and v is a vertex of G, then the neighbours of $f(v)$ in H are the images of the neighbours of v in G, and $\deg_G v = \deg_H f(v)$. Similarly, if u and v are two vertices of G, then $d_G(u,v) = d_H(f(u), f(v))$.

Theorem 7.1. *Graph isomorphism is an equivalence relation on the class of all graphs.*

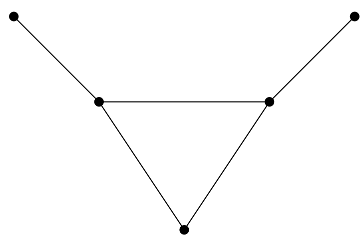
Proof. To show that graph isomorphism is an equivalence relation, we need to show that $\cong$ is a reflexive, symmetric, transitive relation between graphs. First, observe that the identity map on the vertex set of a graph is an isomorphism from the graph to itself – for, if G is a graph and id denotes the identity map on its vertex set then for any two vertices u and v of G, $u \sim v$ if and only if $\mathrm{id}(u) \sim \mathrm{id}(v)$, as $\mathrm{id}(u) = u$ and $\mathrm{id}(v) = v$. Hence, $\cong$ is reflexive.

Next, suppose that $G \cong H$. Then there exists an isomorphism f from G to H. Since $f\colon V(G) \to V(H)$ is bijective, it has an inverse $f^{-1}\colon V(H) \to V(G)$. We claim that f^{-1} is an isomorphism from H to G. We know that f^{-1} is bijective. To see that it preserves edges and non-edges, observe that if x and y are two vertices of H, then $x = f(f^{-1}(x))$ and $y = f(f^{-1}(y))$, which implies that $x \sim_H y$ if and only if $f^{-1}(x) \sim_G f^{-1}(y)$ (since f is an isomorphism from G to H). Therefore, f^{-1} is an isomorphism from H to G. This shows that $H \cong G$. Thus, $\cong$ is symmetric.

Finally, suppose that $G \cong H$ and $H \cong K$. Then there exist isomorphisms f from G to H and g from H to K. We claim that $g \circ f$ is an isomorphism from G to K. Indeed, $g \circ f$ is a function from $V(G)$ to $V(K)$, and being a composition of bijections, is itself a bijection. Now, suppose u and v are two vertices of G. Then $u \sim_G v$ if and only if $f(u) \sim_H f(v)$ (since f is an isomorphism from G to H) if and only if $g(f(u)) \sim_K g(f(v))$ (since g is an isomorphism from H to K). Thus, $g \circ f$ is an isomorphism from G to K. Therefore, $\cong$ is transitive. $\square$

8 Self-Complementary Graphs

A graph is self-complementary if it is isomorphic to its complement – i.e., G is self-complementary if $G \cong \overline{G}$. For example, K_1, P_4 and C_5 are self-complementary graphs of orders 1, 4, and 5 respectively. There is one more self-complementary graph of order 5, namely the bull graph, which can be constructed by adding one new vertex to P_4 and making it adjacent to the two non-pendant vertices of the path. This graph is shown below.



Theorem 8.1. *The order of any self-complementary graph is $4k$ or $4k+1$, for some non-negative integer k.*

Proof. Let G be a self-complementary graph of order n and size m. Then the size of $\overline{G}$ is $\binom{n}{2} - m$. Since $G \cong \overline{G}$, $m = \binom{n}{2} - m$, which implies that $m = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$. As m is an integer, this implies that 4 divides $n(n-1)$, which in turn implies that either one of n and $n-1$ is divisible by 4, or both n and $n-1$ are even. Since the latter is not possible, it follows that $n = 4k$ or $n - 1 = 4k$, i.e., $n = 4k$ or $4k + 1$ for some integer k. □

From Theorem 6.1, we know that a graph and its complement cannot both be disconnected. Thus, if G is a disconnected graph, then $\overline{G}$ must be connected, and therefore it cannot be isomorphic to G. This implies that a self-complementary graph is necessarily connected.

Corollary 8.2. *Every self-complementary graph is connected.* □

Similarly, from Theorem 6.2, we obtain the following corollary about the diameter of self-complementary graphs.

Corollary 8.3. *Every non-trivial self-complementary graph has diameter* 2 *or* 3*.*

Proof. Exercise. □

9 Bipartite Graphs

A graph $G = (V, E)$ is bipartite if its vertex set V can be partitioned into two subsets V_1 and V_2 such that no two vertices in V_i are adjacent, for $i = 1, 2$. That is, there exist two non-empty, disjoint subsets $V_1, V_2 \subseteq V$ such that $V_1 \cup V_2 = V$, and every edge of G (if any) joins a vertex in V_1 with a vertex in V_2.

10 Trees

A tree is a connected, acyclic graph. There are several well known characterisations or alternative definitions of trees. We take the given definition as the basic one and prove its equivalence to some others.

Theorem 10.1. *A graph T is a tree if and only if there is a unique path joining every two vertices of T.*

Proof. First, suppose that T is a tree, and let u and v be vertices of T. Since T is connected, there is a path, say P_1, joining u and v. Now we must show that this path is unique. Assume to the contrary that there exists another path P_2 from u to v. When traversing P_1 from u to v, let w be the first vertex that is present on P_1 but not P_2. Let x be the vertex on P_1 preceding w, and note that x is on P_2 as well. Let y be the next vertex common to both P_1 and P_2 when traversing P_1 from x to v. Then the portion of P_1 from x to y together with the portion of P_2 from y to x forms a cycle in the tree T, which is a contradiction. Thus, P_1 is the unique path joining u and v.

Conversely, suppose that T is a graph in which there is a unique path joining any two vertices. Clearly, T is connected. To show that T is acyclic, suppose that $v_1, v_2, \ldots, v_n$ is a cycle in T. Then we get two different paths joining v_1 and v_n, namely the path $v_1, v_2, \ldots, v_n$ and the path v_1, v_n (since $v_1 \sim v_n$ in the cycle). This contradicts our assumption. Thus, T must be acyclic and hence is a tree. □

The next two results show that the size of a tree is always one less than its order, and that conversely, this property together with either connectedness or acyclicity implies that the graph is a tree.

Theorem 10.2. *A (p, q)-graph T is a tree if and only if it is connected and $p = q + 1$.*

Proof. Let T be a tree with p vertices and q edges. Then T is connected. We prove that $p = q + 1$ by induction. This is clearly true when $p = 1$. Assume it to be true for all trees of order less than p. Now in T, we know that every two vertices are joined by a unique path. Thus, if e is any edge of T, then the graph $T - \{e\}$ obtained by deleting e has exactly two components, say T_1 and T_2. Each one is a tree, since it is connected and acyclic. Let T_i have p_i vertices and q_i edges, $i = 1, 2$. Then by the hypothesis, $p_i = q_i + 1$ (since $p_i < p$). But $p = p_1 + p_2$ and $q = q_1 + q_2 + 1$ (since the size of $T - \{e\}$ is one less than that of T). Thus, $p = q_1 + q_2 + 2 = q + 1$.

For the converse, suppose that T is a connected (p, q)-graph with $p = q + 1$. We must show that is acyclic. Suppose to the contrary that T has a cycle C with k vertices. Then C has k edges as well. Since T is connected, there is a path from every vertex not on C to some vertex of C. The shortest path from each vertex v not on C to a vertex on C has a unique edge incident with v, which is not part of C. Since there are $p-k$ vertices in T not on C, there are $p-k$ such edges. Thus $q \geq (p-k)+k = p$, which contradicts our assumption that $p = q+1$. Thus, T must be acyclic. □

In the following theorem, the proof of the direct part is identical to that of Theorem 10.2, except for the assertion being about acyclicity rather than connectedness. The proof of the converse part is entirely different.

Theorem 10.3. *A (p,q)-graph T is a tree if and only if it is acyclic and $p = q+1$.*

Proof. Let T be a tree with p vertices and q edges. Then T is acyclic. We prove that $p = q+1$ by induction. This is clearly true when $p = 1$. Assume it to be true for all trees of order less than p. Now in T, we know that every two vertices are joined by a unique path. Thus, if e is any edge of T, then the graph $T - \{e\}$ obtained by deleting e has exactly two components, say T_1 and T_2. Each one is a tree, since it is connected and acyclic. Let T_i have p_i vertices and q_i edges, $i = 1,2$. Then by the hypothesis, $p_i = q_i + 1$ (since $p_i < p$). But $p = p_1 + p_2$ and $q = q_1 + q_2 + 1$ (since the size of $T - \{e\}$ is one less than that of T). Thus, $p = q_1 + q_2 + 2 = q + 1$.

Conversely, suppose that T is an acyclic (p,q)-graph with $p = q+1$. To show that T is connected, we need to prove that it is connected – i.e., it has only one component. Let T have k components $T_1, \ldots, T_k$. Each one is acyclic, and being connected, is a tree. Thus from the first part of the theorem, we know that if p_i and q_i are respectively the order and size of the component T_i, $p_i = q_i + 1$. Now $p = p_1 + \cdots p_k = (q_1+1) + \cdots (q_k+1) = q + k$. But we know that $p = q+1$. Therefore, $k = 1$. Thus, T is a tree. □

Exercise 10.1. A pendant vertex of a graph is a vertex of degree 1. Prove that every non-trivial tree contains at least two pendant vertices.

Hint: Observe that a non-trivial tree cannot have a vertex of degree zero. Use Handshaking Lemma and assume every degree is at least 2 to get a contradiction.

Exercise 10.2. The centre of a graph G is the set of all vertices of G with minimum eccentricity – i.e., the set of all vertices v of G with $\operatorname{ecc} v = \operatorname{rad} v$. Show that every tree has a centre consisting of either exactly one vertex or exactly two adjacent vertices.

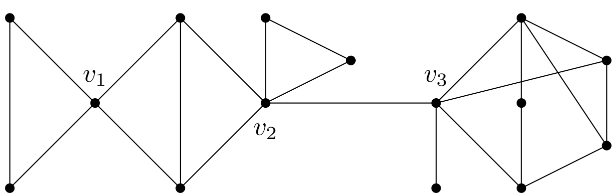
Hint: Observe that deleting all pendant vertices of a tree results in a new tree with the same centre.

Exercise 10.3. If G and H are two trees of orders n and m respectively, what is the size of $G \times H$?

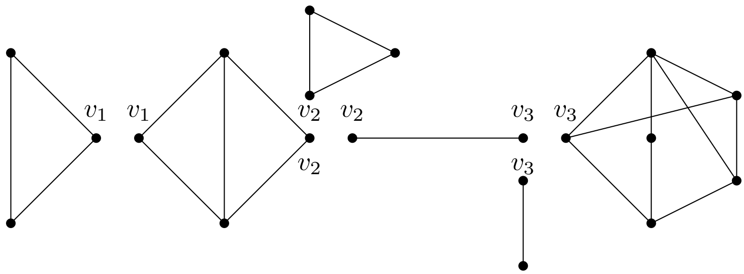
11 Blocks

A cutvertex of a graph is a vertex whose removal increases the number of components, i.e., a vertex v of G such that $G-v$ has more components than G. If G is connected, we can equivalently say that v is a cutvertex if $G-v$ is disconnected. Similarly, a cutedge or bridge of a graph whose removal increases the number of components. A nonseparable graph is a connected, non-trivial graph with no cutvertices. A maximal nonseparable subgraph of a graph is a block of the graph. A nonseparable graph is itself said to be a block as well.

Example 11.1. The graph shown below has 6 blocks and 3 cutvertices (v_1, v_2, v_3).



The 6 blocks of this graph are shown below.



Theorem 11.2. *If G is a connected graph, and v is any vertex of G, then the following are equivalent:*

(i) v is a cutvertex of G.
(ii) There exist vertices u and w of G, distinct from v, such that every u-w path passes through v.
(iii) There exists a partition of $V(G) - v$ into two non-empty subsets U and W such that for all $u \in U$ and $w \in W$, every u-w path passes through v.

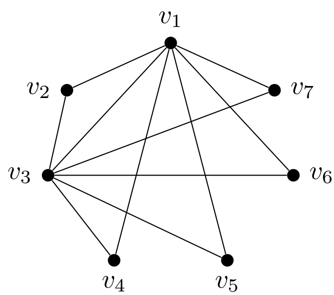
Proof. (i) $\implies$ (iii). Since v is a cutvertex, the graph $G - v$ is disconnected, i.e., it has two or more components. Let U be the set of all the vertices in any one of the components, and let W be the set of all the remaining vertices of $G - v$. Clearly, $\{U, W\}$ is a partition of $V(G) - v$. Now, if $u \in U$ and $w \in W$, then u and w are in different components of $G - v$, which implies that any path from u to w must pass through v.
(iii) $\implies$ (ii) is obvious as the latter is a special case of the former.
(ii) $\implies$ (i). Consider the graph $G-v$. As every u-w path passes through v, none of them is present in $G - v$, and therefore $G - v$ is disconnected. Hence, v is a cutvertex of G. □

12 Adjacency Matrices

The adjacency matrix of a graph G of order n, with vertex set $V = \{v_1, \ldots, v_n\}$, is the $n \times n$ matrix $A = A(G)$ whose (i, j)-entry is

$$a_{ij} = \begin{cases} 1, & v_i \sim v_j \\ 0, & v_i \nsim v_j. \end{cases}$$

Example 12.1. The adjacency matrix of the graph



is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that, as the graphs we discuss are simple graphs and therefore have no self-loops on vertices, no vertex is adjacent to itself – i.e., $a_{ii} = 0$ for all $i = 1, \ldots, n$. Also, since the graphs are undirected, $v_i \sim v_j$ if and only if $v_j \sim v_i$ – i.e., $a_{ij} = a_{ji}$. Thus, we have the following observation.

Observation 12.2. *The adjacency matrix of a (simple, undirected) graph is a symmetric, zero-diagonal,* 0-1 *matrix.*

In the i^{th} row of the adjacency matrix, for each j, the j^{th} entry is 1 if v_j is adjacent to v_i, and 0 otherwise. That is, the number of 1s in the i^{th} row is the number of vertices adjacent to v_i, or in other words, the degree of v_i. Thus, the row sums of A are the vertex degrees. Observe that $A\mathbb{1}$ is the vector of row sums, where $\mathbb{1}$ is the vector (of suitable size) with all entries equal to 1. For instance, with the matrix A given in Example 12.1,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 6 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

From the Handshaking Lemma and the preceding observation, it follows that the sum of all entries of A is twice the number of edges of the graph.

Exercise 12.1. Show that the (i, j)-entry of A^2 is the number of walks of length 2 from v_i to v_j. Hence show that $\operatorname{tr}(A^2) = 2|E(G)|$.

Hint: Recall that if A is any $n \times n$ matrix, then the (i, j)-entry of A^2 is $\sum_{k=1}^{n} a_{ik}a_{kj}$. As A is a 0-1 matrix, each term in this summation is 1 or 0, with the former if and only if $a_{ik} = a_{kj} = 1$. What does this imply about the vertices v_i, v_k, and v_j? Then, as k varies from 1 to n, what does the value of the sum imply about v_i and v_j?

The following result (which generalises the statement in Exercise 12.1) shows that the adjacency matrix can be used to obtain certain information about walks in the graph.

Theorem 12.3. *Let A be the adjacency matrix of a graph G with vertex set $\{v_1, \ldots, v_n\}$. Then the (i, j)-entry of A^m, for any positive integer m, is the number of walks of length k from v_i to v_j.*

Proof. We prove the result by induction on m. For $m = 1$, the (i, j)-entry of $A^1 = A$ is a_{ij}, which is 1 if and only if v_i is adjacent to v_j, i.e., if and only if there is a walk of length 1 (namely, an edge) from v_i to v_j. Thus, the result holds for $m = 1$.

Now suppose, for the sake of induction, that the result holds for some $m \geq 1$, and consider A^{m+1}. The (i, j)-entry of A^{m+1} is

$$(A^{m+1})_{ij} = \sum_{k=1}^{n} (A^m)_{ik} a_{kj}.$$

First, note that $a_{kj} = 1$ if and only if $v_k \sim v_j$. Therefore, the above sum is equal to the sum of all $(A^m)_{ik}$ where $v_k \sim v_j$. Now, by the induction hypothesis, $(A^m)_{ik}$ is the number of walks of length m from v_i to v_k. If v_j is adjacent to v_j, then each walk of length m from v_i to v_k, together with the edge from v_j to v_j, forms a walk of length $m + 1$ from v_i to v_j. Thus, for each k such that $v_k \sim v_j$, $(A^m)_{ik} a_{kj} = (A^m)_{ik}$ is the number of walks of length $m + 1$ from v_i to v_j that pass through k. Summing over all k, this gives the total number of walks of length $m + 1$ from v_i to v_j. Hence the result follows by induction. $\square$