PROBABILITY AND STATISTICS – PROBLEM SET 2

1. Very Easy

1.1. If $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{3}$, and $P(AB) = \frac{1}{6}$, compute
   (a) $P(A \cup B)$
   (b) $P(\overline{A} \cup \overline{B})$
   (c) $P(\overline{A} \cup B)$.

1.2. An integer between 1 and n (inclusive) is selected at random. Let A be the event that it is even, and let B be the event that it is **not** divisible by 3. In each of the following cases, check whether A and B are independent. Guess the answer intuitively before checking it computationally.
   (a) $n = 12$.
   (b) $n = 13$.
   (c) $n = 14$.
   (d) $n = 15$.
   (e) $n = 16$.

2. Easy

2.1. A and B are independent events such that $P(A \cup B) = \frac{5}{6}$ and $P(AB) = \frac{1}{4}$. Find $P(A)$ and $P(B)$.

2.2. Let p and q be distinct prime numbers, and n a positive integer multiple of pq. An integer between 1 and n (inclusive) is selected at random. Let A be the event that **it is divisible by** p, and let B be the event that **it is not divisible by** q. Are A and B independent?

2.3. If $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, show that
   (a) $P(AB) \leqslant \frac{1}{4}$.
   (b) $\frac{1}{3} \leqslant P(A \cup B) \leqslant \frac{7}{12}$

3. Normal Difficulty

3.1. If A and B are two events in a sample space, show that A and B are independent if and only if A and $\overline{B}$ are independent.

3.2. Let $p_1, \ldots, p_k$ be k distinct primes, and $n = p_1^{r_1} \times \cdots \times p_k^{r_k}$, where $r_1, \ldots, r_k$ are positive integers. An integer between 1 and n (inclusive) is selected at random.

Let A_i be the event that **it is not divisible by** p_i, $i = 1, \dots, k$. Find $P(A_1 A_2 \cdots A_k)$, and show that $A_1, \dots, A_k$ are independent.

3.3. In n tosses of a fair coin, let A be the event that there is at least one head and one tail. Let B be the event that there is at most one head. What is the value of n such that A and B are independent?

3.4. A patient is suspected to have one of three diseases, A, B, C. The population percentages suffering from these diseases are in the ratio $2 : 1 : 1$ (and no person has more than one of the three). There is a single test for these three diseases, which turns out to be positive in 25% of the cases of A, in 50% of the cases of B, and 90% of the cases of C. The test is administered to the patient three times. Given that two of them were positive, compute the probability of the patient having each of these diseases.

3.5. Urn A contains 5 red and 3 blue marbles. Urn B contains 4 red and 6 blue marbles. One marble chosen at random is transferred from A to B. Then, one is chosen at random from B and transferred to A. Finally, one marble is drawn at random from A.

(a) What is the probability that the marble drawn from A in the third step is red?

(b) Given that this marble is red, what is the probability that the marble taken from A in the first step is also red?

4. Seemingly Difficult

4.1. An integer between 1 and n (inclusive) is selected at random. Let A be the event that it is even, and let B be the event that it is **not** divisible by 3. For which values of n will A and B be independent?

4.2. A certain family of bacteria is such that every ten minutes, each bacterium either divides into two, with probability $\frac{2}{3}$, or dies. If initially there is a single bacterium of this family, what is the probability that its lineage never dies out?