LATTICE THEORY PROBLEMS

1. If x and y are two elements of a lattice, show that $x \wedge y = y$ if and only if $x \vee y = x$.
2. If x, y, and z are elements of a lattice, show that

$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$$

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z).$$

3. Show that in an algebraic system $(L, \oplus, \otimes)$, where $\oplus$ and $\otimes$ are binary operations satisfying the absorption law, $\oplus$ and $\otimes$ are idempotent.
4. Let a, b, c be elements in a lattice $(L, \leq)$. Show that $a \leq b$ if and only if

$$a \vee (b \wedge c) \leq b \wedge (a \vee c).$$

5. Show that a lattice L is distributive if and only if for all elements $x, y, z \in L$, $(x \vee y) \wedge z \leq x \vee (y \wedge z)$.
6. Show that every chain is a distributive lattice. Which chains are Boolean lattices?
7. Let L be a distributive lattice. Show that for $a, b \in L$, if there exists an element $x \in L$ such that $a \vee x = b \vee x$ and $a \wedge x = b \wedge x$, then $a = b$.
8. Give an example of a complemented lattice that is not distributive.
9. Does the lattice $(\mathbb{N}, |)$ (where $\mathbb{N} = \{1, 2, 3, \ldots\}$) contain
   (a) a universal lower bound?
   (b) a universal upper bound?
10. Does the lattice $(\mathbb{N}_0, |)$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) contain a universal upper bound?
11. Show that every finite lattice contains a universal upper bound and a universal lower bound.
12. Show that if a lattice contains both 0 and 1, then they are the unique complements of each other.
13. Compute the CNF and DNF of the expression $E(a, b, c) = \overline{\left(a \wedge \left(\overline{b} \vee (\overline{c} \wedge a)\right)\right)}$ over the 2-valued Boolean algebra.