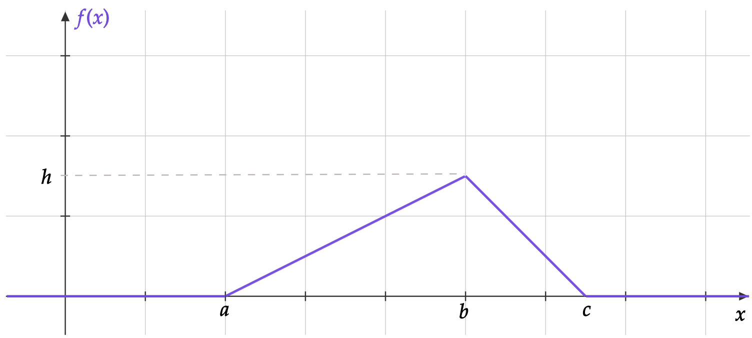
PROBABILITY AND STATISTICS – PROBLEM SET 3

1. X is a random variable with pdf $f(x) = k$, $a < x < b$, where $a < 1 < 2 < b$.
   (a) Find the value of k (in terms of a and b).
   (b) If $\mathrm{P}[X < 1] = 2\,\mathrm{P}[X > 2]$, and $5\,\mathrm{P}[X > 1] = 3\,\mathrm{P}[X < 2]$, find the values of a and b.
   (c) Compute $\mathbb{E}[X]$, $\mathbb{V}[X]$, and $M_X(t)$.
2. Let X be the outcome when a fair six-sided die is rolled. Compute $M_X(t)$.
3. Let X be the number of heads obtained when a coin with probability $\frac{1}{4}$ for heads is tossed three times. Compute $f(x)$, $M_X(t)$, $\mathbb{E}[X]$, and $\mathbb{V}[X]$.
4. Let X be a random variable with pdf $f(x) = ke^{-ax}$, $x \geq 0$, where $a > 0$.
   (a) Find the value of k.
   (b) Compute $F(x)$.
   (c) Compute $\mathrm{P}[1 < X < 2]$.
   (d) Compute $M_X(t)$.
   (e) Compute $\mathbb{V}[X]$.
5. A fair coin is tossed three times. A player wins \$1 if the first toss results in a head, but loses \$1 if the first toss results in a tail. Similarly, the player wins \$2 if the second toss results in a head, but loses \$2 if it results in a tail, and wins or loses \$3 according to the result of the third toss. Let X be the total winnings after the three tosses (possibly a negative value if losses are incurred).
   (a) Compute the probability mass function.
   (b) Compute the cumulative distribution function.
   (c) What is the most likely value of X?
6. The diameter of an electric cable X is assumed to be a continuous random variable with pdf $f(x) = kx(1-x)$, $0 \leq x \leq 1$.
   (a) Find the value of k.
   (b) Find the cdf of X.
   (c) Determine b such that $P(X < b) = 2P(X \geq b)$.
7. A student takes a multiple choice test consisting of three problems. The first question has 3 possible answers, the second has 5 possible answers, and the third question has 4 possible answers. The student chooses at random one answer as the right one from each of the three problems. Let X be the number of right answers. Find $\mathbb{E}[X]$ and $\mathbb{V}[X]$.

8. X is a discrete random variable taking values $0,1,2,\ldots$, whose pmf $f(x)$ is such that $\dfrac{f(x_1)}{f(x_2)} = r^{x_1-x_2}$ for all $x_1,x_2 = 0,1,2,\ldots$ (where $r \in (0,1)$ is a constant). Determine $f(x)$ and compute $\mathbb{E}[X]$.
9. A contestant on a quiz show is presented with two questions, Questions 1 and 2, which she is to attempt to answer in some order she chooses. If she decides to try Question i first, then she will be allowed to go on to question j, $j \neq i$, only if her answer to question i is correct. If her initial answer is incorrect, she is not allowed to answer the other question.

   If she is 60 percent certain of answering Question 1, worth 200 dollars, correctly and she is 80 percent certain of answering Question 2, worth 100 dollars, correctly, then which question should she attempt to answer first so as to maximise her expected winnings? Assume that the events E_i, $i = 1,2$, that she knows the answer to question i are independent events.
10. Let X be the random variable with pdf $f(x)$ whose graph is shown below (where $a < b < c$ are three real numbers).



    Find h, and compute $\mathbb{E}[X]$.
11. Show that the function
$$f(x) = \frac{\lambda^x}{x!}e^{-\lambda}, x = 0,1,2,\ldots$$
    (where $\lambda > 0$ is a constant) is a valid pmf, and compute $\mathbb{E}[X]$, $\mathbb{V}[X]$, and $M_X(t)$.
12. X is a discrete random variable with pmf $f(x) = \dfrac{6}{\pi^2 x^2}$, $x = 1,2,\ldots$.
    (a) Compute $\mathrm{P}[X \geq 3]$
    (b) Compute $\mathrm{P}[10 \leq X \leq 15]$
    (c) Show that $\mathbb{E}[X]$ does not exist.

13. X is a continuous random variable with pdf $f(x) = \frac{6}{\pi^2}\left(\frac{1}{\lfloor x \rfloor}\right)^2$, $x > 1$.
    (a) Compute $P[X \geq 3]$
    (b) Compute $P[10 \leq X \leq 15]$
    (c) Show that $\mathbb{E}[X]$ does not exist.
14. X has pdf $f(x) = \frac{x^2}{3}$, $-1 \leq x \leq 2$. Compute
    (a) $P\left[X < \frac{3}{2}\right]$
    (b) $P[|X| > 1]$
    (c) $P\left[X < \frac{3}{2} \mid X > \frac{1}{2}\right]$
    (d) $P\left[X < \frac{3}{2} \mid |X| > 1\right]$
    (e) $P\left[|X| < 1 \mid X < \frac{3}{2}\right]$.
15. The amount of time in hours that a computer functions before breaking down is a continuous random variable with probability density function given by $f(x) = ke^{\frac{-x}{100}}$, $x \geq 0$.
    What is the probability that
    (a) the computer will function between 50 and 150 hours before breaking down?
    (b) it will function for fewer than 100 hours?
16. An urn initially has 1 red and 1 blue marble. A marble is drawn at random from the urn, and if it is blue, it is put back and one red marble is added to the urn. This is continued until a red marble is drawn. Let X denote the total number of draws required to obtain a red marble. Determine the probability distribution of X, and find its mean and variance. **Hint**: Compute $\mathbb{E}[X+1]$ and $\mathbb{E}[X^2-1]$.
17. The *median* of a random variable X is the point c such that $P[X \leq c] = P[X \geq c] = \frac{1}{2}$. Compute the median of X with pdf $f(x)$ in each of the following cases.
    (a) $f(x) = \frac{1}{b-a}$, $a < x < b$.
    (b) $f(x) = ae^{-ax}$, $x > 0$ [where $a > 0$ is a constant].
    (c) $f(x) = \frac{1}{\pi(1+x^2)}$ [Note that this distribution has no mean, but has a median].
18. Suppose that if you are s minutes early for an appointment, then you incur the cost cs, and if you are s minutes late, then you incur the cost ks. Suppose also that the travel time from where you presently are to the location of your appointment is a continuous random variable having probability density function f. Determine the time at which you should depart if you want to minimise your expected cost.