PROBABILITY AND STATISTICS – PROBLEM SET 4

1. (X,Y) is a two dimensional random variable with joint pdf $f(x,y) = x + y$, $0 < x < 1$, $0 < y < 1$. Compute the following.
   (a) $P[2X + 2Y < 1]$.
   (b) $P[2X + 2Y < 3]$.
   (c) $P[X \geq \frac{1}{2}]$.
   (d) $P[X \geq \frac{1}{2}, Y \geq \frac{1}{2}]$.
   (e) $P[Y \geq \frac{1}{2} \mid X \geq \frac{1}{2}]$.
   (f) $P[X + Y > 1 \mid X \geq \frac{1}{2}]$.

2. Let $f(x,y) = x^2 + \dfrac{xy}{3}$, $0 \leq x \leq 1$, $0 \leq y \leq 2$. Verify that $f(x,y)$ is a joint pdf, and find $P[X + Y > 1]$.

3. Let $f(x,y) = kx(x - y)$, $|y| < x < 2$. Find the value k such that $f(x,y)$ is a joint pdf, and compute the marginal pdfs of X and Y.

4. From a box containing r red, g green, and b blue marbles, n marbles are drawn successively **with replacement**. Let X and Y respectively be the number of blue marbles and number of green marbles obtained in these n draws.
   (a) Determine the joint pmf of (X,Y) and compute the marginal pmfs.
   (b) Are X and Y independent?
   (c) For the particular case $r = 3$, $g = b = 2$, tabulate the joint pmf values and compute the coefficient of correlation between X and Y.

5. The joint pdf of (X,Y) is $f(x,y) = 2e^{-x-2y}$, $x, y > 0$. Compute
   (a) $P[X > 1, Y < 1]$.
   (b) $P[X < Y]$.
   (c) $P[X < a]$.

6. Compute the joint pdf of the random variable (X,Y) that is uniformly distributed over the region bounded by the curves $y = x^2$ and $y = x$. Also find the marginal pdfs of X and Y.

7. If $f(x,y) = 2(x + y - 2xy)$, $0 \leq x, y \leq 1$, show that $Y \sim U[0,1]$.

8. Compute the marginal pdfs of X and Y and show that they are independent, given their joint pdf (X,Y).
   (a) $f(x,y) = 3 - 6x - y + 2xy$, $0 < x < \frac{1}{2}$, $0 < y < 2$.

(b) $f(x,y) = e^{-2x-\frac{y}{2}}$, $x,y > 0$.

9. For each random variable (X,Y) with joint pdf $f(x,y)$ as given below, compute the coefficient of correlation ρ.

(a) $f(x,y) = x + y$, $0 < x, y < 1$.

(b) $f(x,y) = 3x + 3y$, $0 < y < x < 1$.

(c) $f(x,y) = 8xy$, $0 < x < y < 1$.

(d) $f(x,y) = 2x - xy$, $0 < x < 1$, $0 < y < 2$.

(e) $f(x,y) = \dfrac{1}{\pi}$, $x^2 + y^2 \leq 1$ [Uniform distribution in the unit disc centred at the origin].

(f) $f(x,y) = \dfrac{2}{\pi}$, $y \geq 0$, $x^2 + y^2 \leq 1$.

(g) $f(x,y) = \dfrac{4}{\pi}$, $x,y \geq 0$, $x^2 + y^2 \leq 1$.

(h) $f(x,y) = \dfrac{1}{4}$, $(x,y) \in R$, where R is the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$.

(i) $f(x,y) = \dfrac{1}{2}$, $(x,y) \in R$, where R is the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$.

Are X and Y independent in all the cases where $\rho = 0$?