Estimation of Parameters

1 Maximum Likelihood Estimation

Let X be a random variable having pdf or pmf $f(x;\theta)$, where θ is an unknown parameter. The *likelihood function* of θ, corresponding to a sample of size n, is

$$L(\theta) = f(x_1;\theta)f(x_2;\theta)\cdots f(x_n;\theta).$$

The value of $\theta = T(x_1,\dots,x_n)$ (say) for which $L(\theta)$ is maximum is the *maximum likelihood estimate* of θ. The corresponding statistic $\hat{\theta} = T(X_1,\dots,X_n)$ is the *maximum likelihood estimator* (mle) of θ.

Where applicable, we maximise $L(\theta)$ by solving the equation $\dfrac{d}{d\theta}L(\theta) = 0$ for θ. As $L(\theta)$ is the product of $f(x_1;\theta),\dots,f(x_n;\theta)$, it is more often convenient to maximise the *log-likelihood function* $\log L(\theta) = \sum_{i=1}^{n} \log f(x_i;\theta)$ – which yields the same estimator $\hat{\theta}$, since $\log$ is an increasing function.

2 Confidence Intervals

Definition 2.1. If $(X_1,\dots,X_n)$ is a random sample of size n, then the *sample mean* is defined as

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and the *sample variance* is defined as

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2.$$

Theorem 2.2. *Suppose that $X \sim N(\mu,\sigma^2)$. Then the following hold:*

1. $\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$.

2. $\dfrac{nS^2}{\sigma^2} \sim \chi^2_{n-1}$.

3. $\dfrac{\overline{X} - \mu}{S/\sqrt{n-1}} \sim t_{n-1}$.

Confidence intervals for the mean and variance of a normal population, under different conditions, are as described in the table given below.

Parameter	**Condition**	**Interval**
μ	σ^2 known	$\left(\overline{x} - \frac{a\sigma}{\sqrt{n}}, \overline{x} + \frac{a\sigma}{\sqrt{n}}\right)$ where $P[-a < Z < a] = p$, $Z \sim N(0,1)$
	σ^2 unknown	$\left(\overline{x} - \frac{bS}{\sqrt{n-1}}, \overline{x} + \frac{bS}{\sqrt{n-1}}\right)$ where $P[-b < T < b] = p$, $T \sim t_{n-1}$
σ^2	μ unknown	$\left(\frac{ns^2}{b}, \frac{ns^2}{a}\right)$ where $P[Z < a] = P[Z > b] = \frac{p}{2}$, $Z \sim \chi^2_{n-1}$.