Optimal Portfolio Choice with Unknown Benchmark Efficiency

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Overview

Motivation

- The efficiency of benchmark models is unknown out of sample
  - **Whether to include test assets in portfolio optimization optimization**

- In theory, the expected mean and covariance are known $\Rightarrow$ no decision is needed
  - Benchmark is inefficient: **improving the maximum Sharpe ratio**
  - Benchmark is efficient: automatically assigning zero weights to the test assets

- In reality, the expected mean and covariance are unknown $\Rightarrow$ detrimental effect
  - **Adding estimation errors**
  - The effect increases with the number of assets involved (DeMiguel et al., 2009)

- This paper proposes a combining portfolio strategy to **balance the value of including test assets and the effect of estimation errors**.

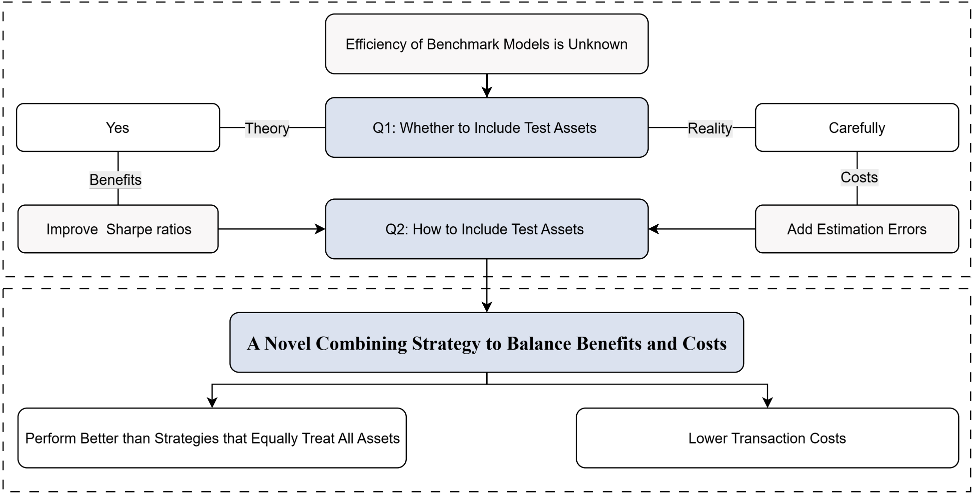
Question

- The efficiency of benchmark models is unknown out of sample
  - Q1: Whether to include the test assets into optimal portfolios
    - When the value in terms of **improved maximum Sharpe ratio** outweighs the cost associated with **additional estimation errors**
  - Q2: How to include the test assets into optimal portfolios
    - Dynamic **weighting** between the all-asset portfolio and the benchmark portfolio

Contribution

- The value of including test assets and the effect of estimation errors
  - Existing: solely focus on one of them
  - **Extension: including them simultaneously in portfolio optimization**
- Treatment of the benchmark portfolios and the test assets
  - Existing: treating them equally
  - **Extension: dynamically weighting the benchmark portfolios and the all asset portfolios**
- Estimation errors
  - Existing: the estimation of the mean and covariance
  - **Extension: the estimation of the weights between the benchmark portfolios and the all asset portfolios**

Framework


Efficiency of Benchmark Models is Unknown
Yes
Theory
Q1: Whether to Include Test Assets
Reality
Carefully
Benefits
Costs
Improve Sharpe ratios
Q2: How to Include Test Assets
Add Estimation Errors
A Novel Combining Strategy to Balance Benefits and Costs
Perform Better than Strategies that Equally Treat All Assets
Lower Transaction Costs

Theoretical optimal portfolio weights

- Let $r_t = [\, r'_{1,t},\ r'_{2,t} \,]'$, where $r_{1,t}$ and $r_{2,t}$ are the excess returns of the **benchmark portfolios** and the **test assets** at time t, respectively:

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \tag{1}$$

- Optimal portfolio weights:

$$w_{s^*} = \frac{1}{\gamma} V_{11}^{-1} \mu_1, \quad w_{p^*} = \frac{1}{\gamma} V^{-1} \mu \tag{2}$$

- w_{s^*}: portfolios containing only the benchmark model
- w_{p^*}: portfolios containing both the benchmark model and the test assets

Evaluation of out-of-sample portfolio performance

- Certainty equivalent return (CER):

$$CER = \mu - \frac{\gamma}{2}\sigma^2 \tag{3}$$

- μ: out-of sample mean
- σ^2: out-of sample variance
- γ: risk aversion coefficient, set to 3

Data

- Eight benchmark models
  - CAPM, FF3, FF5, FF5-UMD
  - q-factor, DHS, DMNU-7

- Five sets of test assets
  - 10 value-weighted momentum portfolios (MOM-10)
  - 10 value-weighted idiosyncratic volatility portfolios (IVOL-10)
  - 10 value-weighted industry portfolios (IND-10)
  - 30 value-weighted industry portfolios (IND-30)
  - 48 anomalies (DMNU-48) in DeMiguel et al. (2020)

Q1: CER of sample optimal portfolios

- Expected mean and covariance are directly estimated using **historical values**
- Including test assets **undermines** the performance of portfolios
- Additional estimation errors **greatly outweigh** the benefits of including test assets

	CAPM	FF3	Carhart-4	FF5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10								
$\hat{w}_s$	2.83	2.70	5.32	8.73	8.71	14.23	25.05	-34.22
$\hat{w}_p$	-14.88	-27.74	-48.36	-43.02	-81.61	-62.10	-40.85	-247.20
IVOL-10								
$\hat{w}_s$	2.83	2.70	5.32	8.73	8.71	14.23	25.05	-34.22
$\hat{w}_p$	-28.61	-48.20	-55.63	-112.39	-121.00	-103.60	-67.28	-312.96
IND-10								
$\hat{w}_s$	2.83	2.70	5.32	8.73	8.71	14.23	25.05	-34.22
$\hat{w}_p$	-24.06	-27.87	-40.84	-59.51	-72.99	-57.10	-37.59	-224.00
IND-30								
$\hat{w}_s$	2.83	2.70	5.32	8.73	8.71	14.23	25.05	-34.22
$\hat{w}_p$	-184.91	-233.40	-273.61	-396.08	-417.19	-369.99	-298.08	-1141.24
DMNU-48								
$\hat{w}_s$	-0.45	-1.62	-4.41	-3.02	-8.40	-2.44	11.29	-34.22
$\hat{w}_p$	-3424.82	-4384.52	-4679.40	-4934.98	-5387.40	-4588.02	-3800.61	-3424.82

Q1: CER of portfolios with estimation risk reduction strategies

- Estimation risk reduction strategies lack robustness when **the number of assets increases**

	CAPM	FF3	Carhart-4	FF5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10								
$\hat{w}_s$	2.83	2.70	5.32	8.73	8.71	14.23	25.05	-34.22
$\hat{w}_p^{pew}$	9.84	12.03	10.63	19.42	14.14	18.12	24.20	0.64
IVOL-10								
$\hat{w}_s$	2.83	2.70	5.32	8.73	8.71	14.23	25.05	-34.22
$\hat{w}_p^{pew}$	8.05	11.37	20.29	8.98	16.03	5.87	21.51	-27.27
IND-10								
$\hat{w}_s$	2.83	2.70	5.32	8.73	8.71	14.23	25.05	-34.22
$\hat{w}_p^{pew}$	1.72	5.95	11.65	9.65	10.90	13.08	17.30	3.81
IND-30								
$\hat{w}_s$	2.83	2.70	5.32	8.73	8.71	14.23	25.05	-34.22
$\hat{w}_p^{pew}$	-1.78	0.98	7.04	-0.05	2.49	1.72	5.70	-16.53
DMNU-48								
$\hat{w}_s$	-0.45	-1.62	-4.41	-3.02	-8.40	-2.44	11.29	-34.22
$\hat{w}_p^{pew}$	-47.16	-76.25	-77.36	-71.03	-77.12	-72.33	-37.23	-47.16

Q1: CER of portfolios with efficiency improvement Strategies

- Incorporating the efficiency information from the GRS test yields **mediocre** results

	CAPM	FF3	Carhart-4	FF5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10								
$\alpha = 1\%$	6.12	6.58	7.15	14.95	12.69	11.03	19.27	1.56
$\alpha = 5\%$	8.40	9.51	6.69	15.17	12.38	13.00	17.87	1.34
$\alpha = 10\%$	8.42	10.14	7.17	18.35	11.75	11.43	18.26	-2.48
IVOL-10								
$\alpha = 1\%$	5.18	7.81	13.38	10.12	16.07	3.62	12.78	2.65
$\alpha = 5\%$	5.65	7.83	12.92	8.43	14.33	3.00	18.05	-0.96
$\alpha = 10\%$	6.87	8.89	16.43	6.81	14.23	2.40	18.32	-39.15
IND-10								
$\alpha = 1\%$	0.33	3.23	8.89	8.21	8.28	10.69	28.15	24.98
$\alpha = 5\%$	0.26	1.59	7.03	7.62	9.79	8.64	27.39	10.93
$\alpha = 10\%$	-0.18	1.09	7.33	10.85	10.42	7.21	24.42	4.97
IND-30								
$\alpha = 1\%$	-1.85	-0.34	4.03	5.50	7.60	7.92	29.99	0.34
$\alpha = 5\%$	-2.79	-2.46	2.78	1.22	4.39	-0.86	25.61	-8.27
$\alpha = 10\%$	-2.26	-2.86	3.29	5.11	5.05	-0.24	22.04	-12.62
DMNU-48								
$\alpha = 1\%$	-49.07	-77.67	-80.00	-80.09	-86.40	-77.39	-42.43	-39.27
$\alpha = 5\%$	-49.07	-77.67	-80.00	-73.90	-80.68	-77.39	-42.43	-43.41
$\alpha = 10\%$	-49.07	-77.67	-80.00	-73.90	-80.68	-77.39	-42.43	-48.42

Q2: A new combining strategy: general framework

- Optimal combining weights:

$$\hat{w}_c = \hat{\lambda}_p \hat{w}_p + \hat{\lambda}_s \hat{w}_s \tag{4}$$

- By maximizing the expected out-of-sample utility of the combining portfolio:

$$\hat{\lambda}_p = \frac{\hat{\mu}_p}{\gamma \hat{\sigma}_p^2} \left(\frac{1 - (\hat{\theta}_s / \hat{\theta}_p) \hat{\rho}_{ps}}{1 - \hat{\rho}_{ps}^2} \right), \quad \hat{\lambda}_s = \frac{\hat{\mu}_s}{\gamma \hat{\sigma}_s^2} \left(\frac{1 - (\hat{\theta}_p / \hat{\theta}_s) \hat{\rho}_{ps}}{1 - \hat{\rho}_{ps}^2} \right) \tag{5}$$

- $\hat{\theta}_i$ denotes the estimated Sharpe ratio of portfolio i
- How to balance the value of including test assets and the effect of estimation errors?
  - $\hat{\lambda}_i$: putting **more weight on the more efficient portfolio** $\Rightarrow$ **improves the Sharpe ratio**
  - $\hat{w}_i$: estimated by **existing shrinkage methods** $\Rightarrow$ **alleviates estimation errors**

Q2: A new combining strategy: additional estimation errors

- To alleviate the estimation errors when estimating $\hat{\lambda}_p$ and $\hat{\lambda}_s$, $\hat{w}_c$ should be shrunk:

$$\hat{w}_{c2} = \hat{\eta}\hat{w}_c, \quad \hat{\eta} = \frac{\mathbb{E}[\hat{w}_c'\mu]}{\gamma\,\mathbb{E}[\hat{w}_c' V \hat{w}_c]} \tag{6}$$

- $\hat{\eta}$ is estimated using a threefold cross-validation approach

Q2: A new combining strategy: empirical performance

- The new combining strategy **performs better** than existing methods, especially when **the number of assets is large**

	CAPM	FF3	Carhart-4	FF5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10								
$\hat{w}_c^{pew}$	9.39	10.82	8.88	20.99	16.60	20.84	26.05	3.25
$\hat{w}_{c2}^{pew}$	10.64	13.31	10.33	23.04	19.72	29.75	33.00	40.56
IVOL-10								
$\hat{w}_c^{pew}$	6.65	8.30	17.61	8.99	16.97	9.13	25.90	-6.25
$\hat{w}_{c2}^{pew}$	11.80	15.01	26.24	26.65	34.22	20.32	38.62	27.44
IND-10								
$\hat{w}_c^{pew}$	-1.51	3.40	8.85	12.70	14.36	12.03	24.60	8.22
$\hat{w}_{c2}^{pew}$	2.03	4.47	10.70	15.89	19.05	21.68	27.17	43.95
IND-30								
$\hat{w}_c^{pew}$	-3.59	-4.00	7.81	6.46	12.17	8.29	24.41	-9.13
$\hat{w}_{c2}^{pew}$	0.31	2.83	9.82	15.35	19.18	20.32	26.17	39.34
DMNU-48								
$\hat{w}_c^{pew}$	-44.72	-76.98	-85.88	-72.07	-86.37	-84.06	-39.57	-38.16
$\hat{w}_{c2}^{pew}$	34.49	40.38	41.73	23.65	22.68	10.63	29.49	48.76

Portfolio turnover

- Modified combining portfolios exhibit **lowest turnover**

	CAPM	FF3	Carhart-4	FF5	FF5-UMD	q-factor	DHS	DMNU-7
MOM-10								
$\hat{w}_p$	25.72	29.15	35.91	31.96	34.50	38.68	41.65	54.66
$\hat{w}_p^{pew}$	11.30	13.29	14.92	16.30	17.61	21.84	24.29	33.62
$\hat{w}_c^{pew}$	12.52	13.77	13.47	13.87	15.01	18.67	20.38	26.54
$\hat{w}_{c2}^{pew}$	8.55	9.08	9.00	9.59	10.90	14.42	15.24	21.36
IVOL-10								
$\hat{w}_p$	27.55	32.49	35.71	45.02	45.89	45.88	47.07	59.55
$\hat{w}_p^{pew}$	13.34	16.85	19.13	25.97	26.03	27.19	29.04	36.74
$\hat{w}_c^{pew}$	14.16	17.28	18.89	24.74	24.88	25.27	26.95	29.09
$\hat{w}_{c2}^{pew}$	9.58	12.21	12.87	19.45	18.30	19.46	19.30	24.00
IND-10								
$\hat{w}_p$	2.29	3.57	6.99	6.83	8.04	11.99	13.99	26.73
$\hat{w}_p^{pew}$	0.91	1.58	3.45	3.47	4.01	6.80	7.99	16.39
$\hat{w}_c^{pew}$	1.10	1.69	3.71	3.79	4.42	7.26	10.37	18.10
$\hat{w}_{c2}^{pew}$	0.74	1.12	2.22	2.90	2.80	5.80	7.62	14.79
IND-30								
$\hat{w}_p$	8.71	11.16	15.87	16.49	17.74	23.00	23.63	45.05
$\hat{w}_p^{pew}$	1.86	2.68	3.95	4.56	4.76	6.86	6.44	15.96
$\hat{w}_c^{pew}$	2.03	2.72	4.43	4.81	5.24	8.08	10.72	19.59
$\hat{w}_{c2}^{pew}$	1.17	1.38	2.27	3.57	3.23	6.24	7.30	15.86
DMNU-48								
$\hat{w}_p$	115.93	127.37	132.54	135.16	141.08	132.96	128.99	115.92
$\hat{w}_p^{pew}$	30.78	31.93	32.07	31.68	31.85	32.04	32.14	30.78
$\hat{w}_c^{pew}$	30.67	31.65	31.39	30.51	30.38	31.15	32.84	28.14
$\hat{w}_{c2}^{pew}$	20.73	19.24	17.66	20.16	18.20	20.74	22.29	20.58

Conclusion

- This paper proposes a **combining strategy** to balance the value of including test assets and the effect of estimation errors
- This method performs **better than** existing estimation risk reduction strategies and efficiency improvement strategies
- This method can **work together** with some existing estimation risk reduction strategies

Extension

- Improving existing estimation risk reduction strategie:
  - Two-fund rule of Kan and Zhou (2007)

- Integrating different portfolio optimization methods

- Constructing factor models that are more effective out of sample:
  - Extend to a higher-dimensional factor setting

- Incorporating machine learning methods when the number of assets is greatly large
  - Employing lasso to identify effective test assets
  - Using nonlinear machine learning to estimate weights and λ