A FIRST COURSE

IN

LINEAR ALGEBRA

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MAT2040 Notebook

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Foreword

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Preface

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I. R. S.

Acronyms

ASTA	Arrivals See Time Averages
BHCA	Busy Hour Call Attempts
BR	Bandwidth Reservation
b.u.	bandwidth unit(s)
CAC	Call / Connection Admission Control
CBP	Call Blocking Probability(-ies)
CCS	Centum Call Seconds
CDTM	Connection Dependent Threshold Model
CS	Complete Sharing
DiffServ	Differentiated Services
EMLM	Erlang Multirate Loss Model
erl	The Erlang unit of traffic-load
FIFO	First in - First out
GB	Global balance
GoS	Grade of Service
ICT	Information and Communication Technology
IntServ	Integrated Services
IP	Internet Protocol
ITU-T	International Telecommunication Unit – Standardization sector
LB	Local balance
LHS	Left hand side

LIFO	Last in - First out
MMPP	Markov Modulated Poisson Process
MPLS	Multiple Protocol Labeling Switching
MRM	Multi-Retry Model
MTM	Multi-Threshold Model
PASTA	Poisson Arrivals See Time Averages
PDF	Probability Distribution Function
pdf	probability density function
PFS	Product Form Solution
QoS	Quality of Service
r.v.	random variable(s)
RED	random early detection
RHS	Right hand side
RLA	Reduced Load Approximation
SIRO	service in random order
SRM	Single-Retry Model
STM	Single-Threshold Model
TCP	Transport Control Protocol
TH	Threshold(s)
UDP	User Datagram Protocol

3.2. Thursday

3.2.1. Review

Last time you may be confused about how to compute $N(\boldsymbol{A})$ or $\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_{n-r}$ (step2). Now let's review the whole steps for solving rectangular bu using block matrix:

- Firstly, we convert our rref into the form $\begin{bmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$ by switching columns.

Example 3.9 Last time our rref is given by:

$$\boldsymbol{R} = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We notice that column 3 is pivot column, so we can switch it into the second column. (By switching column 2 and column 3):

$$\boldsymbol{R} \implies \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

- Then our system equation is translated (We use 3×4 matrix to show the whole process.):

$$\boldsymbol{R}\boldsymbol{x} = \boldsymbol{c} \implies \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Because we have changed the columns, so here row 2 and row 3 is also switched respectively. And then x_1 and x_2 are pivot variables, x_3 and x_4 are free variables. Then we derive:

$$\implies \begin{cases} \boldsymbol{I}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \boldsymbol{B}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ 0 = c_3 \end{cases}$$

- If $c_3 \neq 0$, then there is no solution; next, let's assume $c_3 = 0$. Then *pivot variables* could be expressed as the form of *free variables*:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - \boldsymbol{B}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

  Hence all solutions to $\boldsymbol{Rx} = \boldsymbol{c}$ is obtained:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \boldsymbol{B}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix}$$

  Suppose $-\boldsymbol{B} = \begin{bmatrix} \hat{\boldsymbol{y}}_1 & \hat{\boldsymbol{y}}_2 \end{bmatrix}$, then pivot variables is equivalent to

$$\implies \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + x_3\hat{\boldsymbol{y}}_1 + x_4\hat{\boldsymbol{y}}_2$$

- So the complete solution to the system is

$$\boldsymbol{x} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_3\hat{\boldsymbol{y}}_1 + x_4\hat{\boldsymbol{y}}_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} \tag{3.1}$$

$$= \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \hat{\boldsymbol{y}}_2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.2}$$

$$= \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix}}_{\boldsymbol{x}_p} + \underbrace{x_3 \begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \hat{\boldsymbol{y}}_2 \\ 0 \\ 1 \end{pmatrix}}_{\boldsymbol{x}_{\text{special}}} \tag{3.3}$$

where x_3 and x_4 could be arbitarary.

- Notice that the block matrix is given by:

$$\begin{pmatrix} \hat{\boldsymbol{y}}_1 & \hat{\boldsymbol{y}}_2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\boldsymbol{B} \\ \boldsymbol{I} \end{pmatrix} \implies \begin{pmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{bmatrix} -\boldsymbol{B} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{B} + \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

If our rectangular matrix is $m \times n(m > n)$, how to solve it?

Answer: Also, we do G.E. to get rref, which will be of the form

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ 0 & \dots & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

3.2.2. Remarks on solving linear system equations

The two possibilities for linear equations depend on m and n:

Theorem 3.1 Let m denote number of equations, n denote number of variables. For number of solutions for $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, we obtain:

- $m < n$: either no solution or infinitely many solutions
- $m \geq n$: no solution;unique solution ($N(\boldsymbol{A}) = \boldsymbol{0}$); or infinitely many solutions.

Proofoutline for $m < n$ case: Recall we can convert $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ into $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{c}$:

$$\begin{bmatrix} 1 & & & \times & \times \\ & \ddots & & \times & \times \\ & & 1 & \times & \times \\ 0 & 0 & 0 & 0 & 0 \\ \dots & & & & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_r \\ c_{r+1} \\ \vdots \\ c_n \end{bmatrix}$$

Note that $x_1.x_2\dots,x_r$ is pivot variables (This is because of column switching). Hence we have $(n-r)$ free variables, thus $N(\boldsymbol{A})$ is spanned by $(n-r)$ special vectors $y_1, y_2, \dots, y_{n-r}$.

Hence we only need to show $n > r$ *given the condition* $n > m$:

Obviously, $r \leq m$, and we have $n > m$, so we obtain $n > r$. ■

So we get the proposition immediately:

Proposition 3.1 For system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m < n$, it either has no solution or infinitely many solutions.

Corollary 3.1 For system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m < n$, it always has infinitely many solutions.

3.2.2.1. What is r?

We ask the question again, what is r? Let's see some examples before answering this question.

■ **Example 3.10** If we want to solve system of equations of size 1000 as the following:

$$\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + 2x_2 = 6 \\ \cdots \\ 1000x_1 + 1000x_2 = 3000 \end{cases}$$

It seems very difficult when hearding it has 1000 equations, but the remaining 999 equations could be redundant (They actually don't exist):

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ \vdots & \vdots \\ 1000 & 1000 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

■

Here we see only one equation $x_1 + x_2 = 3$ is true, the remaining part is not true. So we claim that r is the number of "true" equations. But what is the definition for "true" equations? Let's discuss the definition for *linear dependence* first.

3.2.3. Linearly dependence

Definition 3.2 [linearly dependence] The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ in linear space $\boldsymbol{V}$ are **linearly dependent** if there exists $c_1, c_2, \dots, c_n \in \mathbb{R}$ s.t.

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \mathbf{0}.$$

In other words, it means one of v_i could be expressed as linear combination of others. When assuming $c_n \neq 0$, we can express $\boldsymbol{v}_n$ as:

$$\boldsymbol{v}_n = -\frac{c_1}{c_n}\boldsymbol{v}_1 - \frac{c_2}{c_n}\boldsymbol{v}_2 - \cdots - \frac{c_{n-1}}{c_n}\boldsymbol{v}_{n-1}.$$

Definition 3.3 [linearly independence] The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ in linear space $\boldsymbol{V}$ are **linearly independent** if the two statements are equivalent:

- $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{0}$
- All scalars $c_1 = c_2 = \cdots = c_n = 0$.

In other words, if $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ are not **linearly dependent**, they must be **linearly independent**.

R Note that **only** in this course, if we say vectors are dependent, we mean they are **linearly** dependent. And we often express *dependent* as *dep.*; we also sometimes express *linearly dependent* as *lin. dep.*; express *linearly independent* as *lin. ind.*

Here we pick some examples to help you understand lin. dep. and lin. ind.:

Example 3.11

- $\boldsymbol{v}_1 = (1,1)$ and $\boldsymbol{v}_2 = (2,2)$ are **dep.** because

$$(-2) \times \boldsymbol{v}_1 + \boldsymbol{v}_2 = \mathbf{0}.$$

- The only one vector $\boldsymbol{v}_1 = 2$ is **ind.** because

$$c\boldsymbol{v}_1 = \mathbf{0} \Longleftrightarrow c = 0.$$

- The only one vector $\boldsymbol{v}_1 = 0$ is **dep.** because

$$2 \times \boldsymbol{v}_1 = \mathbf{0}$$

- $\boldsymbol{v}_1 = (1,2)$ and $\boldsymbol{v}_2 = (0,0)$ are **dep.** because

$$0 \times \boldsymbol{v}_1 + 1 \times \boldsymbol{v}_2 = \mathbf{0}.$$

- The upper triangular matrix $\boldsymbol{A} = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ has three column vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

  $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are **ind.** because

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0} \Longleftrightarrow c_1 = c_2 = c_3 = 0.\text{(Why?)}$$

3.2.3.1. Relation between *lin.ind.* and *equations*

The following statements are equivalent:

- Vectors $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ are dep.
- $\exists\ c_i$ not all zero s.t. $\sum_{i=1}^n c_i a_i = \mathbf{0}$.
- $\exists$ some $\boldsymbol{c} \neq \mathbf{0}$ s.t.
$$\boldsymbol{A}\boldsymbol{c} = \left[\begin{array}{c|c|c} a_1 & \ldots & a_n \end{array}\right] \boldsymbol{c} = \mathbf{0}$$

So what if $m < n$, when checking corollary (3.1), we immediately obtain:

> **Corollary 3.2** When vectors $a_1, a_2, \ldots, a_n \in \mathbb{R}^m (m < n)$ are dependent, there exists infinitely solutions $c_1, c_2, \ldots, c_n$ such that $\sum_{i=1}^n c_i a_i = \mathbf{0}$.

So we say the true equations are those linearly independent equations.

3.2.4. Basis and dimension

Definition 3.4 [Basis] The vectors $v_1, \ldots, v_n$ form a **basis** for a vector space $\boldsymbol{V}$ if and only if:

1. $v_1, \ldots, v_n$ are **linearly independent**.
2. $v_1, \ldots, v_n$ span $\boldsymbol{V}$.

■

■ **Example 3.12** In $\mathbb{R}^3$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not a basis, since it doesn't span $\mathbb{R}^3$.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ don't form a basis, since they don'y linearly independent.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ form a basis. ■

We feel that the number of vectors for basis of $\mathbb{R}^3$ is always 3, is this a coincidence? The theorem below gives the answer.

Theorem 3.2 If $v_1, v_2, \ldots, v_m$ is a basis; $w_1, w_2, \ldots, w_n$ is a basis for the same vector space $\boldsymbol{V}$, then $n = m$.

In order to proof it, let's try simple case first:

proofoutline.

- Let's consider $\boldsymbol{V} = \mathbb{R}$ case first:

For $\mathbb{R}$, 1 forms a basis.

Given any two vectors x and y, they are not a basis for $\mathbb{R}$. It is because that

- if $x = 0$ or $y = 0$, they are not ind.
- otherwise, $y = \frac{y}{x} \times x \implies \frac{y}{x} \times x + (-1) \times y = 0$. So they are not ind.

- Then we consider $\boldsymbol{V} = \mathbb{R}^3$ case:

  For $\mathbb{R}^3$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis.

  We want to show if $v_1, v_2, \dots, v_m$ is a basis, then $m = 3$.

  - Let's proof $m = 4$ is impossible (4 vectors in $\mathbb{R}^3$ cannot be a basis.):

    We only need to show for $\forall a_1, a_2, a_3, a_4 \in \mathbb{R}^3$ they must be dep.

    $\Longleftrightarrow \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has nonzero solutions, where $\boldsymbol{A} = \left[\begin{array}{c|c|c|c} a_1 & a_2 & \dots & a_4 \end{array}\right] \in \mathbb{R}^{3\times 4}$.

    By corollary (3.1), it is obviously true.

  - The same argument could show any basis for $\mathbb{R}^3$ satisfies $m \leq 3$.
  - Then let's prove $m = 2$ is impossible (2 vectors in $\mathbb{R}^2$ cannot be a basis):

    We only need to show for $\forall a_1, a_2 \in \mathbb{R}^3$, they cannot span the whole space.

    If this is not true, then $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ must have solution, where $\boldsymbol{A} = \left[\begin{array}{c|c} a_1 & a_2 \end{array}\right] \in \mathbb{R}^{3\times 2}$.

    However, this kind matrix may have no solution, which forms a contradiction.

  - The same arugment could show any basis for $\mathbb{R}^3$ satisfies $m \geq 3$.

- The same arugment could show any basis for $\mathbb{R}^n$ satisfies $m = n$.

- Next, let's consider general vector space:

  We assume $n < m$ (contradiction).

  We have known $v_1, \dots, v_n$ is a basis, our goal is to show $w_1, \dots, w_m$ cannot form a basis.

  $\Leftarrow \exists\, \boldsymbol{c} = \begin{bmatrix} c_1 & c_2 & \dots & c_m \end{bmatrix}^{\mathrm{T}} \neq \boldsymbol{0}$ s.t.

$$c_1 w_1 + c_2 w_2 + \cdots + c_m w_m = 0. \tag{3.4}$$

Moreover, we can express $w_1, \dots, w_m$ in form of $v_1, \dots, v_n$:

$$\begin{cases} w_1 = a_{11}v_1 + \cdots + a_{1n}v_n \\ \dots \\ w_m = a_{m1}v_1 + \cdots + a_{mn}v_n \end{cases} \tag{3.5}$$

By (3.5), we can write (3.4) as:

$$\begin{aligned} 0 &= \sum_{j=1}^{m} c_j w_j \\ &= \sum_{j=1}^{m} c_j (\sum_{i=1}^{n} a_{ji} v_i) \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} c_j a_{ji} v_i \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} c_j a_{ji} v_i \\ &= \sum_{i=1}^{n} v_i \times (\sum_{j=1}^{m} c_j a_{ji}) \\ &= v_1 \times (\sum_{j=1}^{m} c_j a_{j1}) + v_2 \times (\sum_{j=1}^{m} c_j a_{j2}) + \cdots + v_n \times (\sum_{j=1}^{m} c_j a_{jn}) \end{aligned}$$

So, in order to let LHS=0, we only need to let each of RHS=0, more specifically, we only need to let $\sum_{j=1}^{m} c_j a_{j1} = \sum_{j=1}^{m} c_j a_{j2} = \cdots = \sum_{j=1}^{m} c_j a_{jn} = 0$.
To write it into matrix form, we only need to let system $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c} = \boldsymbol{0}$ has solution.
where $\boldsymbol{A} = \begin{bmatrix} a_{ij} \end{bmatrix}_{1 \le i \le m; 1 \le j \le n}$ and $\boldsymbol{c} = \begin{bmatrix} c_1 & c_2 & \dots & c_m \end{bmatrix}^{\mathrm{T}}$.
By corollary (3.1), since $\boldsymbol{A}^{\mathrm{T}}$ is $n \times m$ matrix where $n < m$, it has infinitely nonzero solution.

■

During the proof, we face two difficulties:

1. For arbitararily $\boldsymbol{V}$, we write a concrete form to express $w_1, w_2, \dots, w_m$.
2. We write matrix form to express $\sum_{j=1}^{m} c_j a_{j1} = \sum_{j=1}^{m} c_j a_{j2} = \cdots = \sum_{j=1}^{m} c_j a_{jn} = 0$.

Next since all basis contains the same number of vectors, we can define the number of vectors to be dimension:

Definition 3.5 [Dimension] The **dimension** for a vector space is the number of vectors in a basis for it. ■

R Remember that vector space $\{0\}$ has dimension 0.
In order to denote the dimension for a given vector space $\boldsymbol{V}$, we often write it as $\dim(\boldsymbol{V})$.

■ **Example 3.13**
- $\mathbb{R}^n$ has dimension n.
- {All $m \times n$ matrix} has dimension $m \cdot n$.
- {All $n \times n$ symmetric matrix} has dimension $\frac{n(n+1)}{2}$.
- Let $\boldsymbol{P}$ denote the vector space of all polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$.
  $\dim(\boldsymbol{P}) \neq 3$ since $1, x, x^2, x^3$ are ind.
  The same argument can show $\dim(\boldsymbol{P})$ is not equal to any real number, so $\dim(\boldsymbol{P}) = \infty$ ■

Human beings often ask a question: for a line and a plane, which is bigger?

1. Does plane has more point than a line?
   No, Cantor syas they have the same "number" of points by constructing a one-to-one mapping.
   Furthermore, $\mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^n$ has the same number of points.
2. However, the plane has bigger dimension than a line. So from this point of view, a plane is bigger than a line.

You should know some common knowledge for dimension:

1. Programmer lives in **2** dimension world. (They only live with binary.)
2. Engineer lives in **3** dimension world. (They only live with enign.)
3. Physician lives in **4** dimension world. (They discuss time.)
4. String theories states that our world is **11** or **26** dimension, which has been proved by Qingshi Zhu.

5. For 3-body, they can perform dimension attack on you.

3.2.4.1. What is rank?

Finally let's answer the question: What is rank?

rank=dimension of row space of a matrix.

We will discuss it next lecture.