ENGG5781 Matrix Analysis and Computations
Lecture 0: Overview

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Department of Electronic Engineering
The Chinese University of Hong Kong

Course Information

General Information

- Instructor: Wing-Kin Ma
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- Lecture hours and venue:
  - Thursday 11:30am–12:15pm, Lai Chan Pui Ngong Lecture Theatre, Y. C. Liang Hall (LPN-LT)
  - Friday 2:30pm–4:15pm, Room 404, William M. W. Mong Engineering Building (ERB 404).

- Class website: http://www.ee.cuhk.edu.hk/~wkma/engg5781

Course Contents

- This is a foundation course on matrix analysis and computations, which are widely used in many different fields, e.g.,
  - machine learning, computer vision,
  - systems and control, signal and image processing, communications, networks,
  - optimization, and many more...
- **Aim:** covers matrix analysis and computations at an advanced or research level.
- **Scope:**
  - basic matrix concepts, subspace, norms,
  - linear least squares
  - eigendecomposition, singular value decomposition, positive semidefinite matrices,
  - linear system of equations, LU decomposition, Cholesky decomposition
  - pseudo-inverse, QR decomposition
  - (advanced) tensor decomposition, advanced matrix calculus, compressive sensing, non-negative matrix factorization

Learning Resources

- Notes by the instructor will be provided.
- Recommended readings:
  - Gene H. Golub and Charles F. van Loan, *Matrix Computations* (Fourth Edition), John Hopkins University Press, 2013.
  - Roger A. Horn and Charles R. Johnson, *Matrix Analysis* (Second Edition), Cambridge University Press, 2012.
  - Jan R. Magnus and Heinz Neudecker, *Matrix Differential Calculus with Applications in Statistics and Econometrics* (Third Edition), John Wiley and Sons, New York, 2007.
  - Giuseppe Calafiore and Laurent El Ghaoui, *Optimization Models,* Cambridge University Press, 2014.
  - ECE 712 Course Notes by Prof. Jim Reilly, McMaster University, Canada (friendly notes for engineers)

    http://www.ece.mcmaster.ca/faculty/reilly/ece712/course_notes.htm

Assessment and Others

- Assessment:
  - Assignments: 60%
    - may contain MATLAB questions
    - no late submissions would be accepted, except for exceptional cases.
  - Final examination: 40%

- Academic honesty: Students are strongly advised to read the University's guideline on academic honesty (http://www.cuhk.edu.hk/policy/academichonesty/).

- Sitting in is welcome, and please send me your e-mail address to keep you updated with the course.

A Glimpse of Topics

Least Squares (LS)

- **Problem:** given $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{y} \in \mathbb{R}^n$, solve

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2,$$

  where $\| \cdot \|_2$ is the Euclidean norm; i.e., $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$.

- widely used in science, engineering, and mathematics

- assuming a tall and full-rank $\mathbf{A}$, the LS solution is uniquely given by

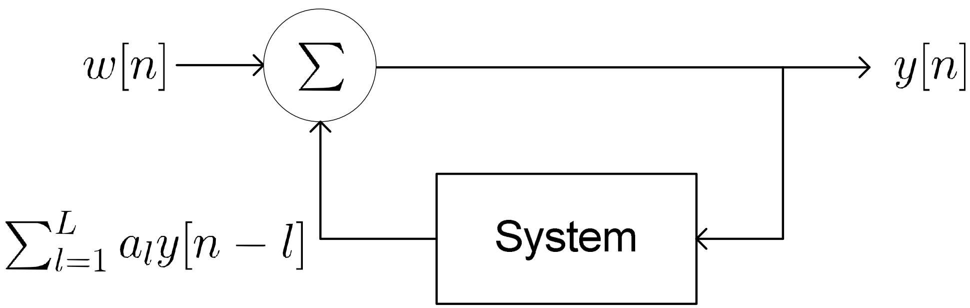
$$\mathbf{x}_{\mathsf{LS}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{y}.$$

Application Example: Linear Prediction (LP)

- let $y[0], y[1], \cdots$ be a time series.
- **Model** (autoregressive (AR) model):

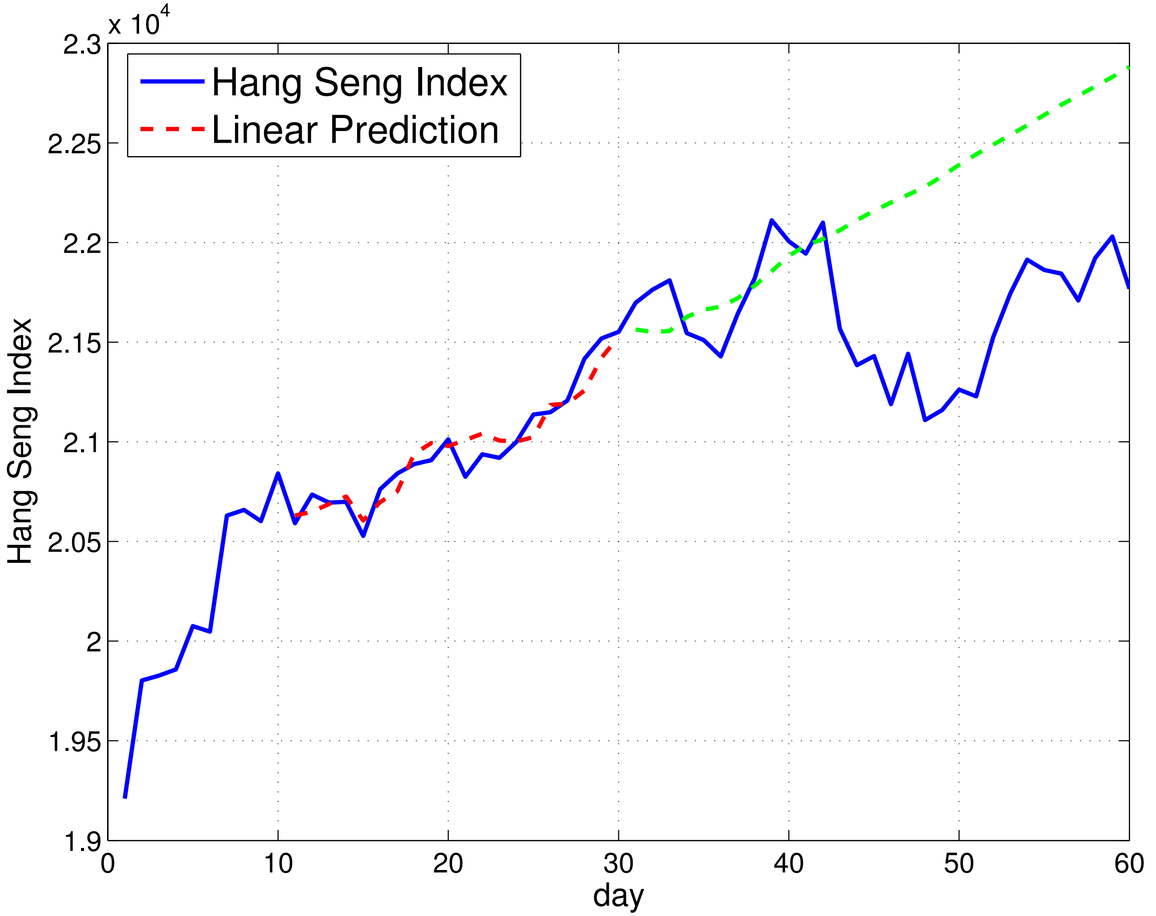
$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \cdots + a_L y[n-L] + w[n], \quad n = 0, 1, 2, \ldots$$

for some coefficients $\{a_i\}_{i=1}^L$, where $w[n]$ is noise or modeling error.



- **Problem:** estimate $\{a_i\}_{i=1}^L$ from $\{y[n]\}$; can be formulated as LS
- **Applications:** time-series prediction, speech analysis and coding, spectral estimation. . .

A Toy Demo: Predicting Hang Seng Index

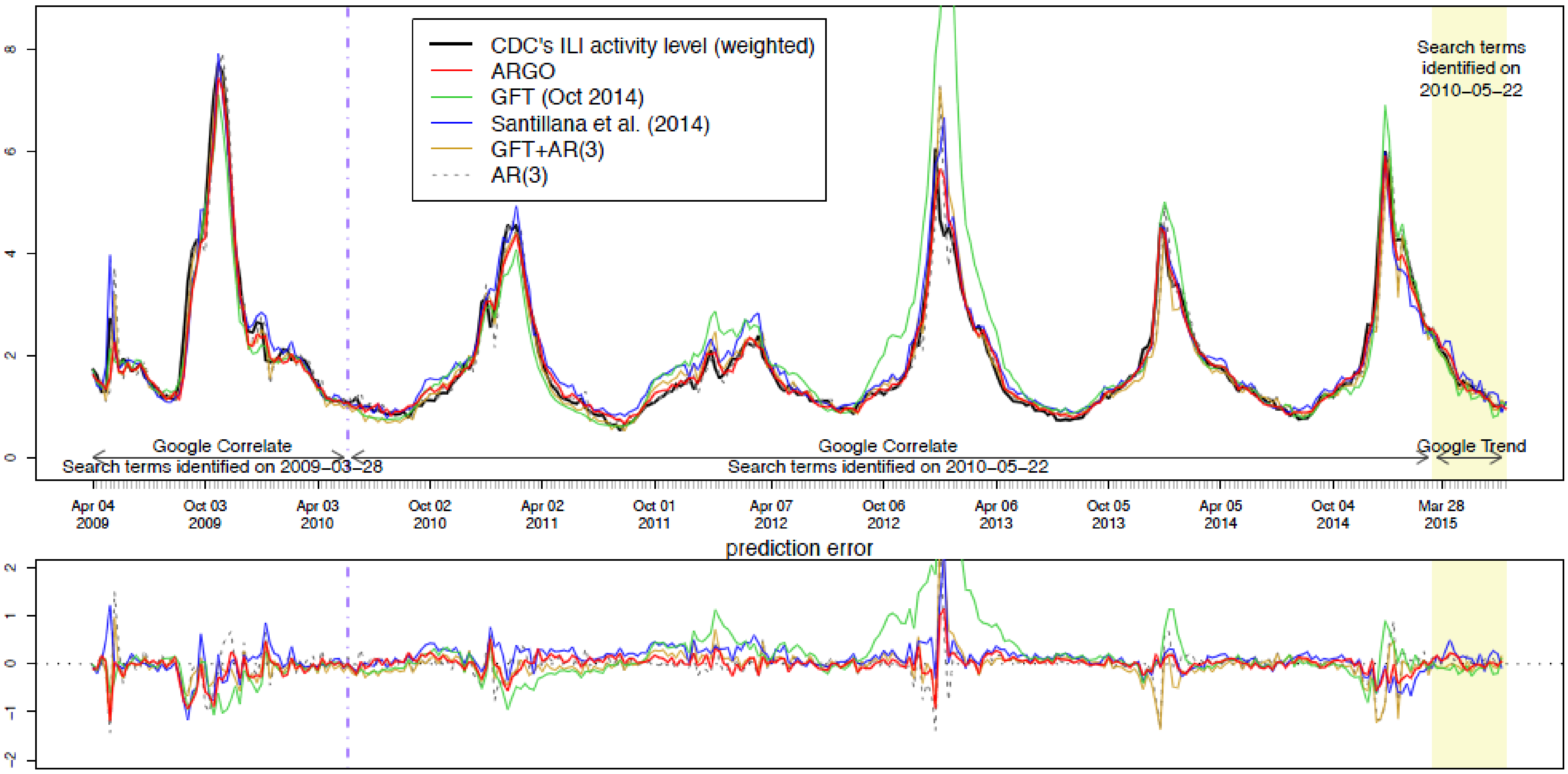


blue— Hang Seng Index during a certain time period.

red— training phase; the line is $\sum_{l=1}^{L} a_l y[n-l]$; $\mathbf{a}$ is obtained by LS; $L = 10$.

green— prediction phase; the line is $\hat{y}[n] = \sum_{l=1}^{L} a_l \hat{y}[n-l]$; the same $\mathbf{a}$ in the training phase.

A Real Example: Real-Time Prediction of Flu Activity



Tracking influenza outbreaks by ARGO — a model combining the AR model and Google search data. Source: **[Yang-Santillana-Kou2015]**.

Eigenvalue Problem

- **Problem:** given $\mathbf{A} \in \mathbb{R}^{n \times n}$, find a $\mathbf{v} \in \mathbb{R}^n$ such that

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}, \qquad \text{for some } \lambda.$$

- **Eigendecomposition:** let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be symmetric; i.e., $a_{ij} = a_{ji}$ for all i, j. It also admits a decomposition

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T,$$

where $\mathbf{Q} \in \mathbb{R}^{n \times n}$ is orthogonal, i.e., $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$; $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$

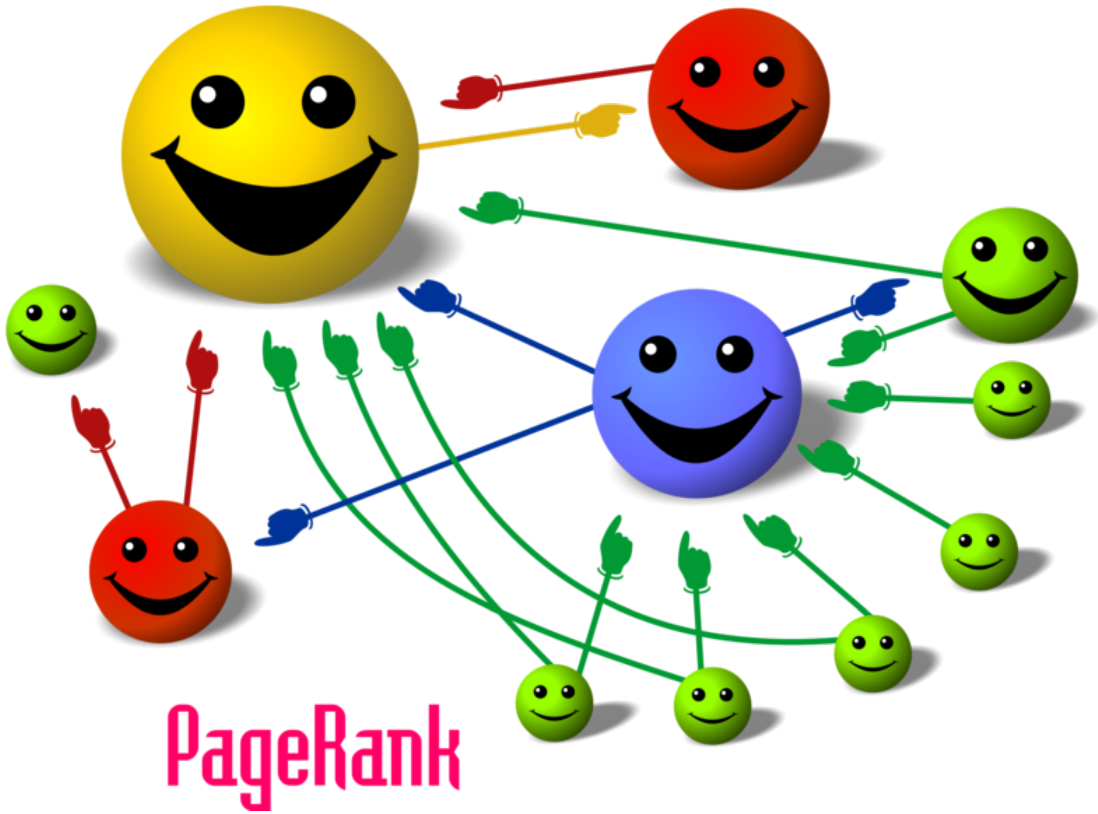
- also widely used, either as an analysis tool or as a computational tool

- no closed form in general; can be numerically computed

Application Example: PageRank

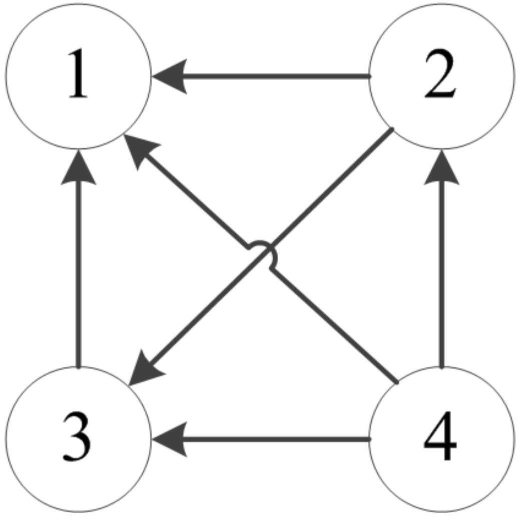
- PageRank is an algorithm used by Google to rank the pages of a search result.

- the idea is to use counts of links of various pages to determine pages' importance.



Source: Wiki.

One-Page Explanation of How PageRank Works



- Model:

$$\sum_{j \in \mathcal{L}_i} \frac{v_j}{c_j} = v_i, \quad i = 1, \ldots, n,$$

where c_j is the number of outgoing links from page j; $\mathcal{L}_i$ is the set of pages with a link to page i; v_i is the importance score of page i.

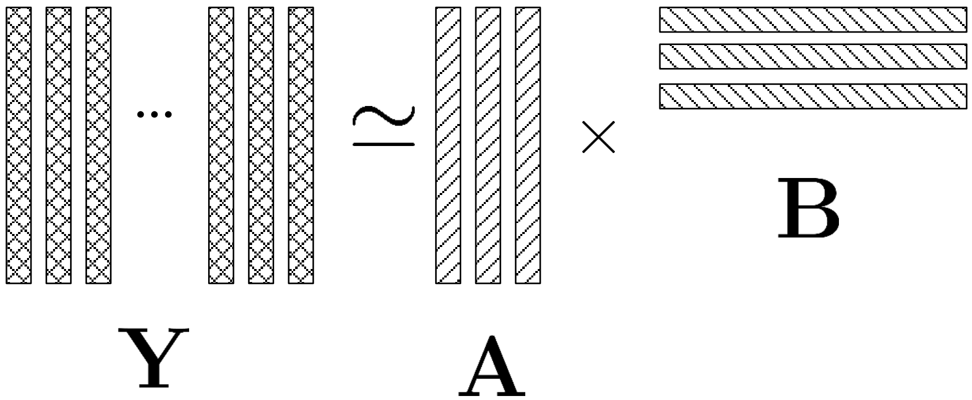
- as an example,

$$\underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}} = \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}}.$$

- finding $\mathbf{v}$ is an eigenvalue problem—with n being of the order of millions!
- further reading: **[Bryan-Tanya2006]**

Low-Rank Matrix Approximation

- **Problem:** given $\mathbf{Y} \in \mathbb{R}^{m \times n}$ and an integer $r < \min\{m, n\}$, find an $(\mathbf{A}, \mathbf{B}) \in \mathbb{R}^{m \times r} \times \mathbb{R}^{r \times n}$ such that either $\mathbf{Y} = \mathbf{AB}$ or $\mathbf{Y} \approx \mathbf{AB}$.



- **Formulation:**

$$\min_{\mathbf{A} \in \mathbb{R}^{m \times r}, \mathbf{B} \in \mathbb{R}^{r \times n}} \|\mathbf{Y} - \mathbf{AB}\|_F^2,$$

where $\|\cdot\|_F$ is the Frobenius, or matrix Euclidean, norm.

- **Applications:** dimensionality reduction, extracting meaningful features from data, low-rank modeling, . . .

Application Example: Image Compression

- let $\mathbf{Y} \in \mathbb{R}^{m \times n}$ be an image.

(a) original image, size= 102×1347

- store the low-rank factor pair $(\mathbf{A}, \mathbf{B})$, instead of $\mathbf{Y}$.

(b) truncated SVD, k= 5

(c) truncated SVD, k= 10

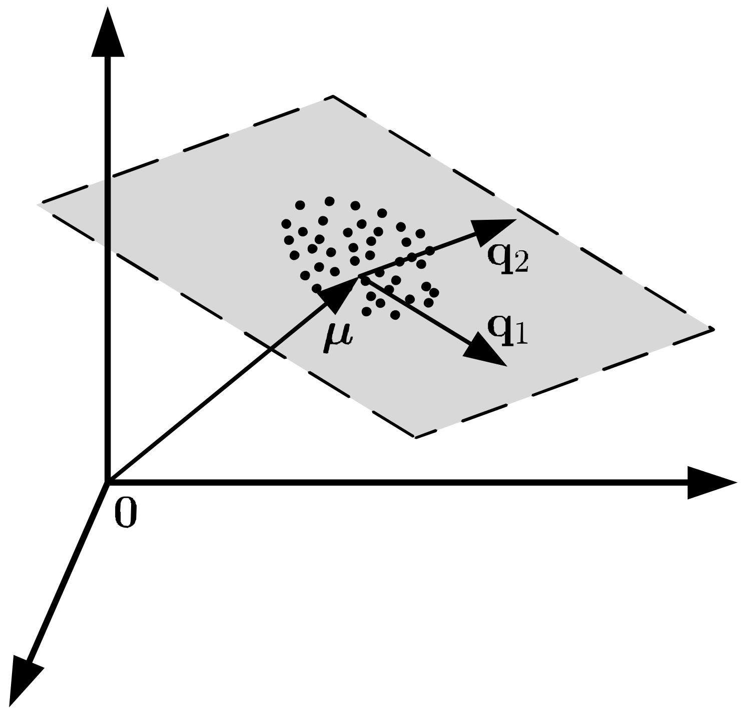
(d) truncated SVD, k= 20

Application Example: Principal Component Analysis (PCA)

- **Aim:** given a set of data points $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\} \subset \mathbb{R}^n$ and an integer $k < \min\{m, n\}$, perform a low-dimensional representation

$$\mathbf{y}_i = \mathbf{Q}\mathbf{c}_i + \boldsymbol{\mu} + \mathbf{e}_i, \quad i = 1, \ldots, n,$$

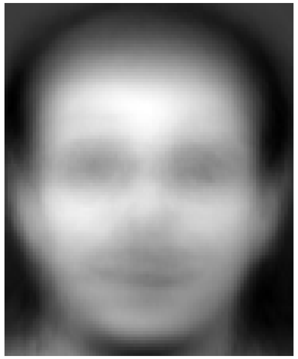
where $\mathbf{Q} \in \mathbb{R}^{m \times k}$ is a basis; $\mathbf{c}_i$'s are coefficients; $\boldsymbol{\mu}$ is a base; $\mathbf{e}_i$'s are errors



Toy Demo: Dimensionality Reduction of a Face Image Dataset

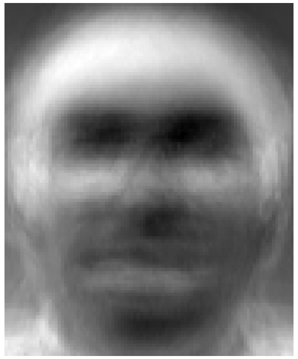
A face image dataset. Image size $= 112 \times 92$, number of face images $= 400$. Each $\mathbf{x}_i$ is the vectorization of one face image, leading to $m = 112 \times 92 = 10304$, $n = 400$.

Toy Demo: Dimensionality Reduction of a Face Image Dataset

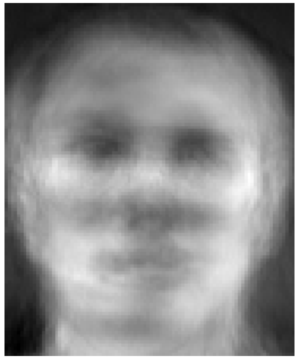


Mean face

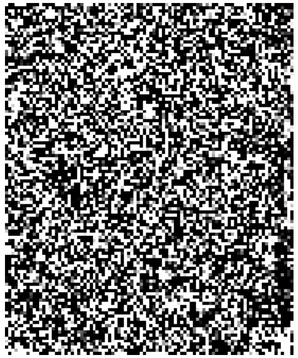
1st principal left singular vector



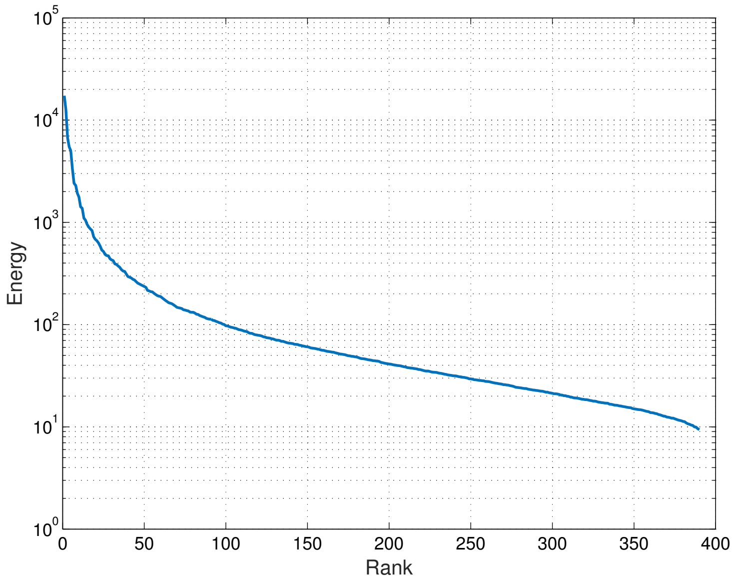
2nd principal left singular vector



3rd principal left singular vector



400th left singular vector



Energy Concentration

Singular Value Decomposition (SVD)

- **SVD:** Any $\mathbf{Y} \in \mathbb{R}^{m \times n}$ can be decomposed into

$$\mathbf{Y} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T,$$

where $\mathbf{U} \in \mathbb{R}^{m \times m}, \mathbf{V} \in \mathbb{R}^{n \times n}$ are orthogonal; $\boldsymbol{\Sigma} \in \mathbb{R}^{m \times n}$ takes a diagonal form.

- also a widely used analysis and computational tool; can be numerically computed

- SVD can be used to solve the low-rank matrix approximation problem

$$\min_{\mathbf{A} \in \mathbb{R}^{m \times r}, \mathbf{B} \in \mathbb{R}^{r \times n}} \|\mathbf{Y} - \mathbf{AB}\|_F^2.$$

Linear System of Equations

- **Problem:** given $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{y} \in \mathbb{R}^n$, solve

$$\mathbf{A}\mathbf{x} = \mathbf{y}.$$

- **Question 1:** How to solve it?
  - don't tell me answers like `x=inv(A)*y` or `x= A\y` on MATLAB!
  - this is about matrix computations
- **Question 2:** How to solve it when n is very large?
  - it's too slow to do the generic trick `x= A\y` when n is very large
  - getting better understanding of matrix computations will enable you to exploit problem structures to build efficient solvers
- **Question 3:** How sensitive is the solution $\mathbf{x}$ when $\mathbf{A}$ and $\mathbf{y}$ contain errors?
  - key to system analysis, or building robust solutions

Why Matrix Analysis and Computations is Important?

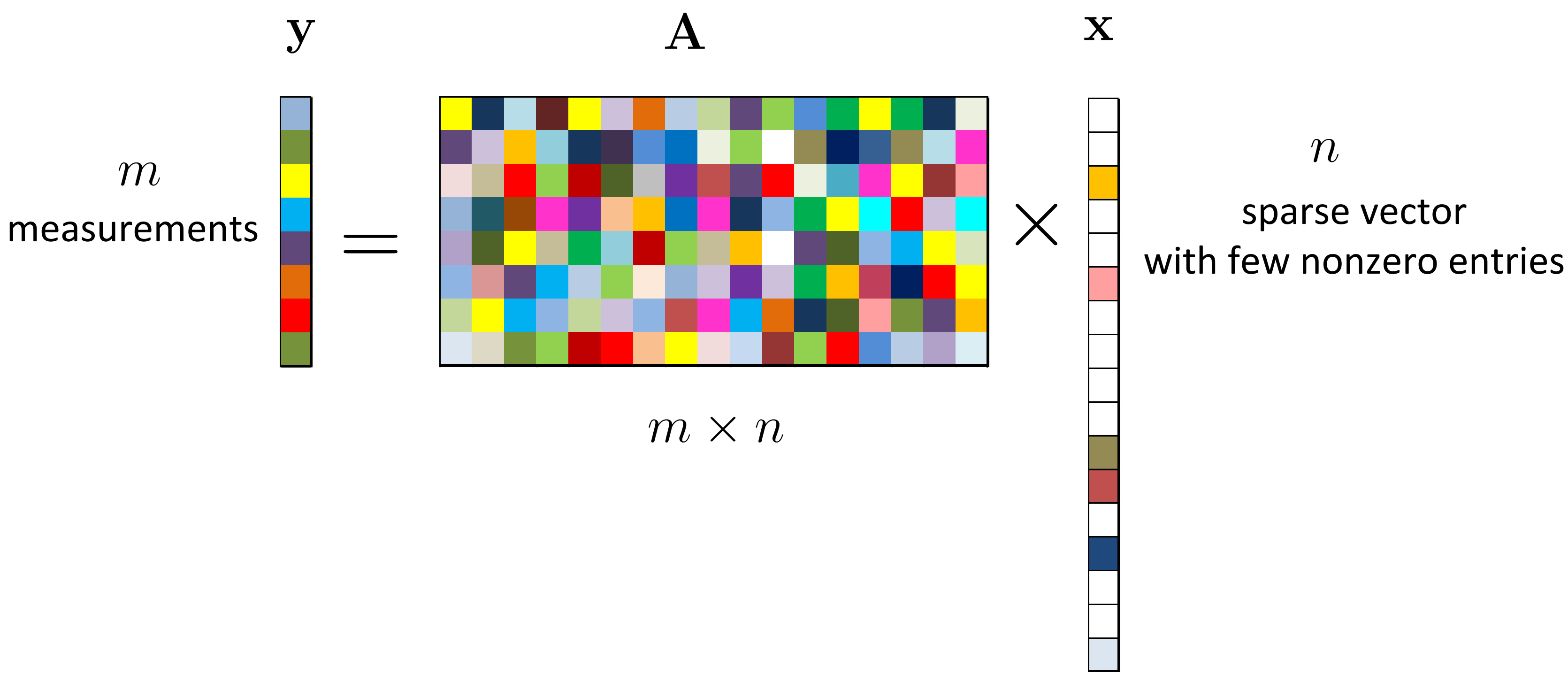
- as said, areas such as signal processing, image processing, machine learning, optimization, computer vision, control, communications, . . ., use matrix operations extensively

- it helps you build the foundations for understanding "hot" topics such as
  - sparse recovery;
  - structured low-rank matrix approximation; matrix completion.

The Sparse Recovery Problem

Problem: given $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m < n$, find a sparsest $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathbf{y} = \mathbf{A}\mathbf{x}.$$

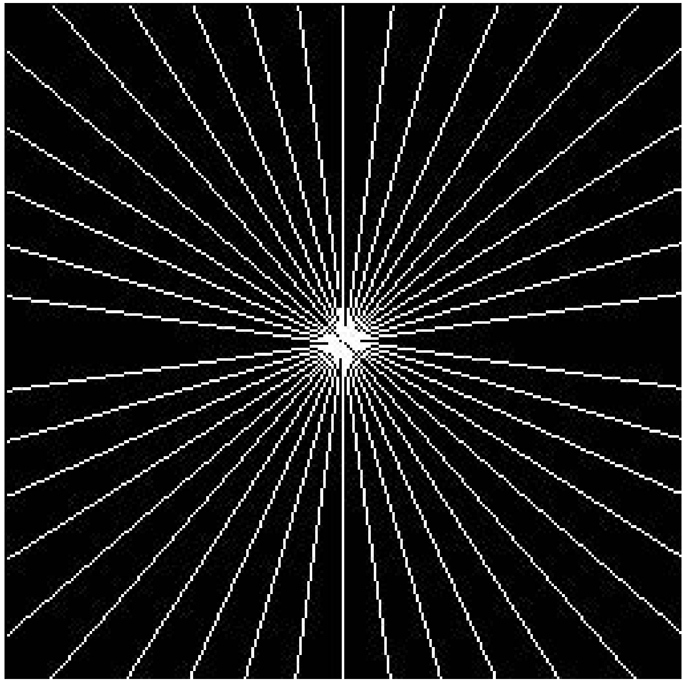


- by sparsest, we mean that $\mathbf{x}$ should have as many zero elements as possible.

Application: Magnetic resonance imaging (MRI)

Problem: MRI image reconstruction.

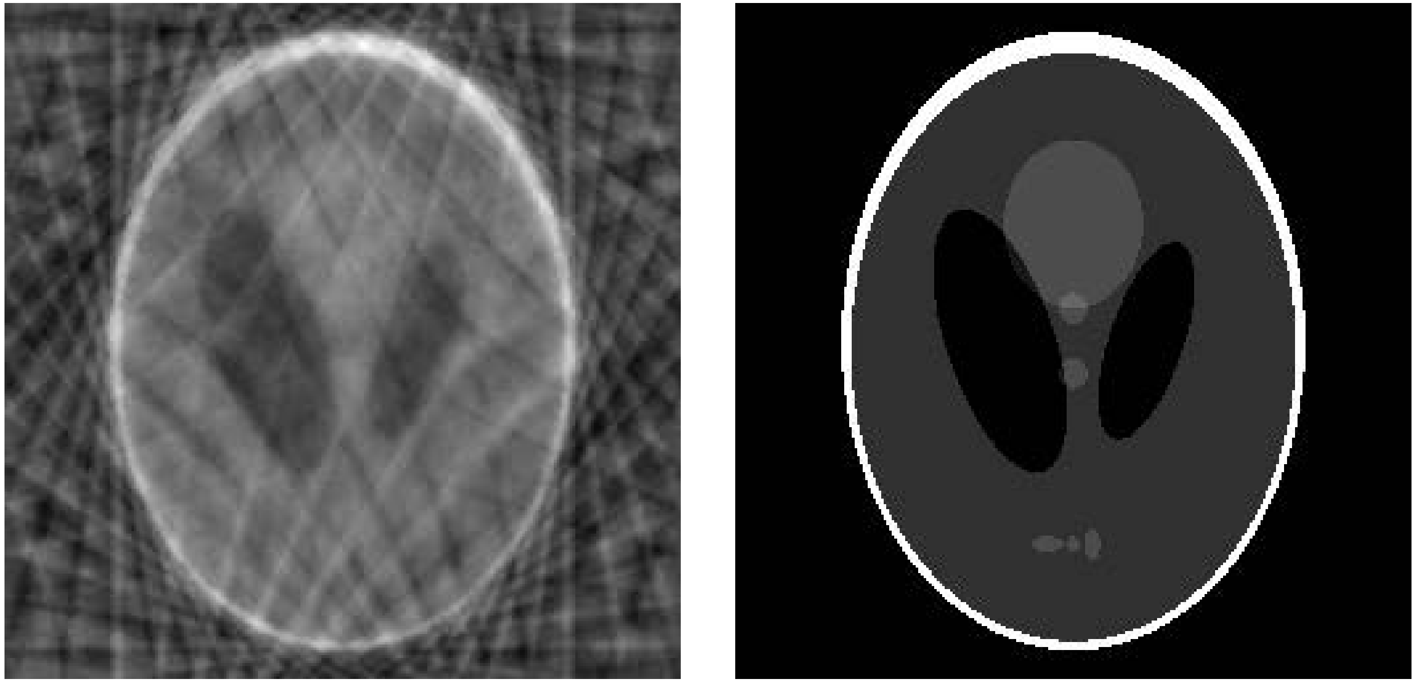
(a)



(b)

Fig. a shows the original test image. Fig. b shows the sampling region in the frequency domain. Fourier coefficients are sampled along 22 approximately radial lines. Source: **[Candès-Romberg-Tao2006]**

Application: Magnetic resonance imaging (MRI)



(c) (d)

Fig. c is the recovery by filling the unobserved Fourier coefficients to zero. Fig. d is the recovery by a sparse recovery solution. Source: **[Candès-Romberg-Tao2006]**

Low-Rank Matrix Completion

- **Application:** recommendation systems
  - in 2009, Netflix awarded \$1 million to a team that performed best in recommending new movies to users based on their previous preference[1] .
- let $\mathbf{Z}$ be a preference matrix, where z_{ij} records how user i likes movie j.

$$\begin{array}{cc} & \text{movies} \\ \mathbf{Z} = & \begin{bmatrix} 2 & 3 & 1 & ? & ? & 5 & 5 \\ 1 & ? & 4 & 2 & ? & ? & ? \\ ? & 3 & 1 & ? & 2 & 2 & 2 \\ ? & ? & ? & 3 & ? & 1 & 5 \end{bmatrix} \text{ users} \end{array}$$

  - some entries z_{ij} are missing, since no one watches all movies.
  - $\mathbf{Z}$ is assumed to be of low rank; research shows that only a few factors affect users' preferences.
- **Aim:** guess the unkown z_{ij}'s from the known ones.

[1] www.netflixprize.com

Low-Rank Matrix Completion

- The 2009 Netflix Grand Prize winners used low-rank matrix approximations **[Koren-Bell-Volinsky2009]**.

- **Formulation** (oversimplified):

$$\min_{\mathbf{A}\in\mathbb{R}^{m\times r},\mathbf{B}\in\mathbb{R}^{r\times n}} \sum_{(i,j)\in\Omega} |z_{ij} - [\mathbf{A}\mathbf{B}]_{i,j}|^2$$

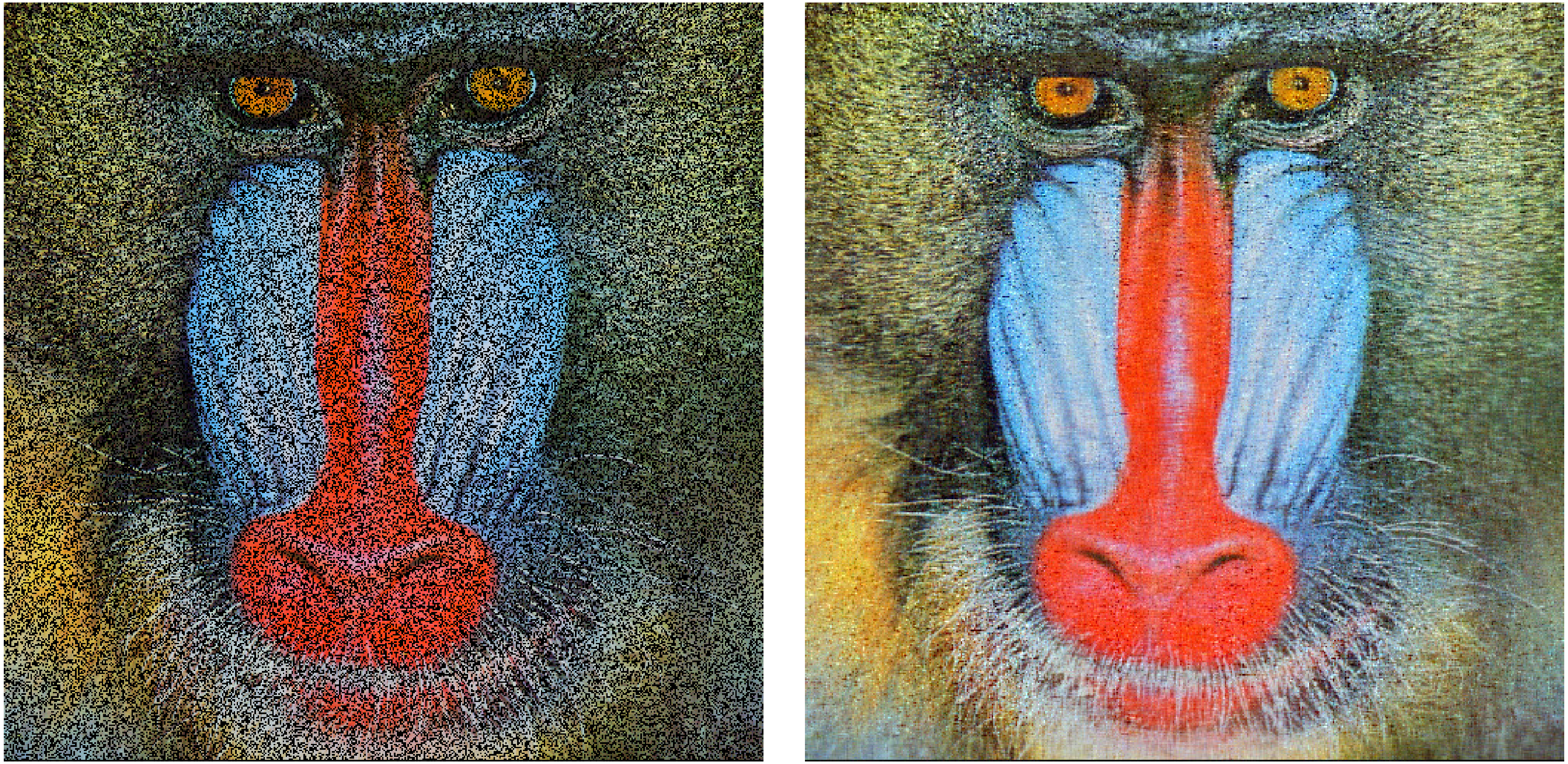
  where Ω is an index set that indicates the known entries of $\mathbf{Z}$.

- cannot be solved by SVD

- in the recommendation system application, it's a large-scale problem

- alternating LS may be used

Toy Demonstration of Low-Rank Matrix Completion



Left: An incomplete image with 40% missing pixels. Right: the low-rank matrix completion result. $r = 120$.

Nonnegative Matrix Factorization (NMF)

- **Aim:** we want the factors to be non-negative

- **Formulation:**

$$\min_{\mathbf{A}\in\mathbb{R}^{m\times r},\mathbf{B}\in\mathbb{R}^{r\times n}} \|\mathbf{Y}-\mathbf{A}\mathbf{B}\|_F^2 \quad \text{s.t. } \mathbf{A}\geq\mathbf{0}, \mathbf{B}\geq\mathbf{0},$$

  where $\mathbf{X}\geq\mathbf{0}$ means that $x_{ij}\geq 0$ for all i,j.

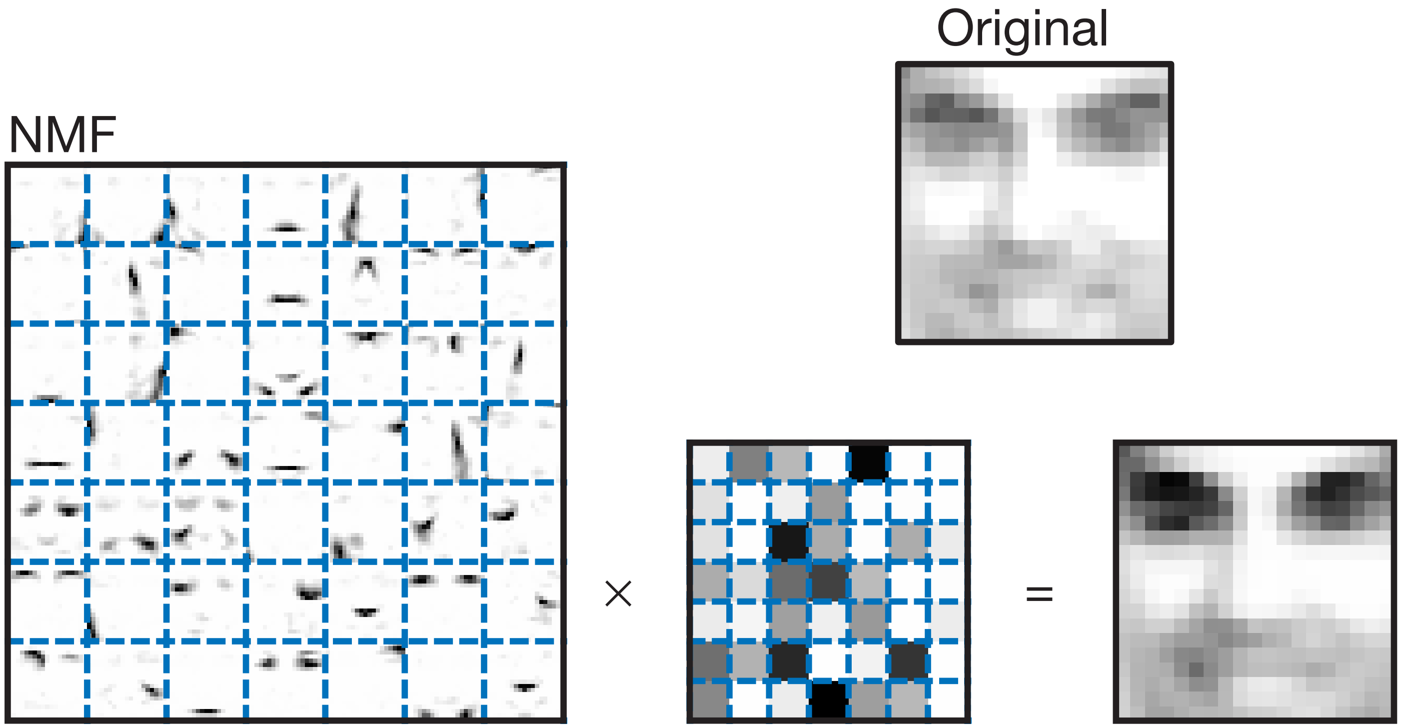
- arguably a topic in optimization, but worth noticing

- found to be able to extract meaningful features (by empirical studies)

- numerous applications, e.g., in machine learning, signal processing, remote sensing

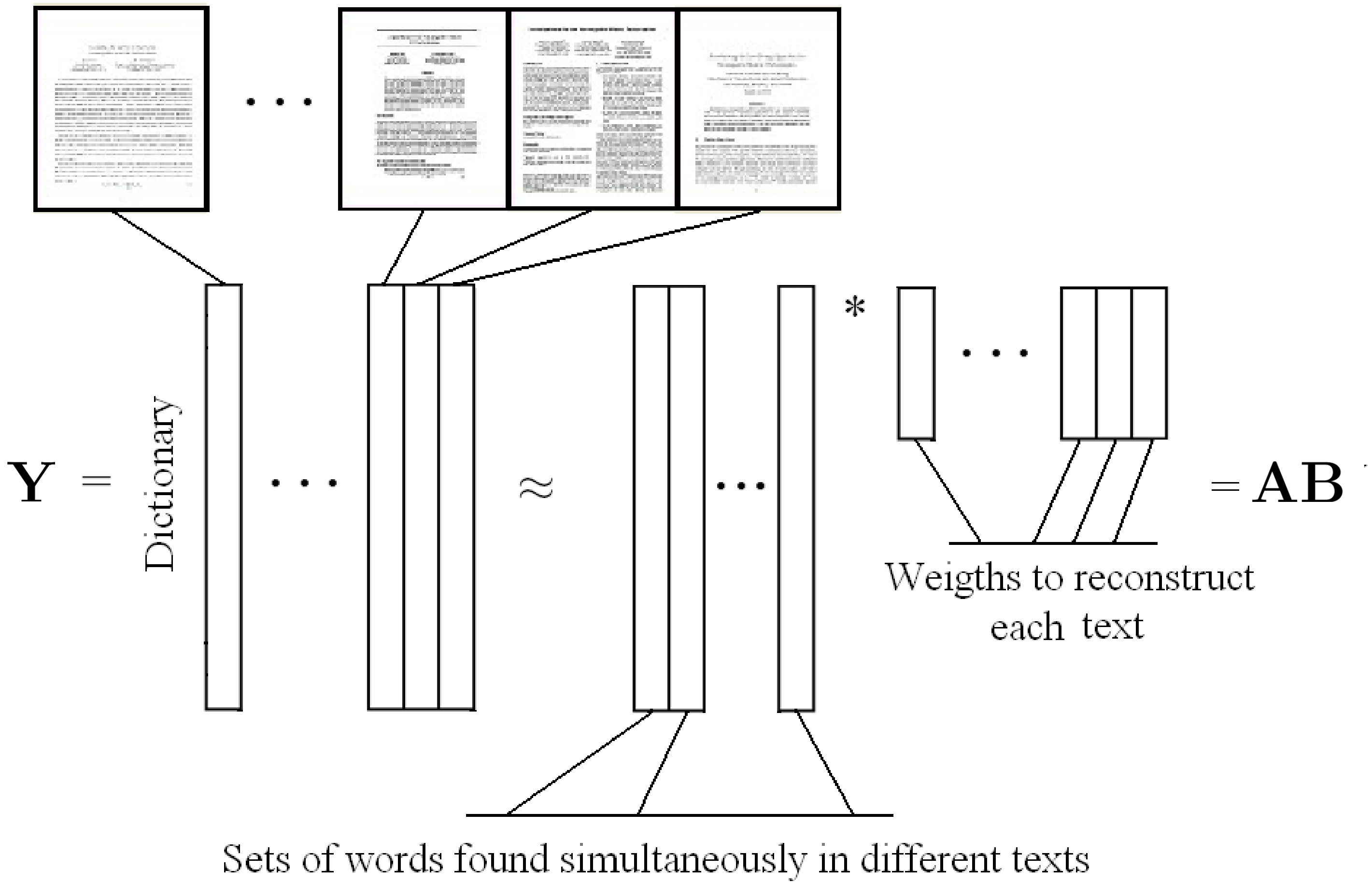
NMF Examples

- **Image Processing:**



The basis elements extract facial features such as eyes, nose and lips. Source: **[Lee-Seung1999]**.

- **Text Mining**:

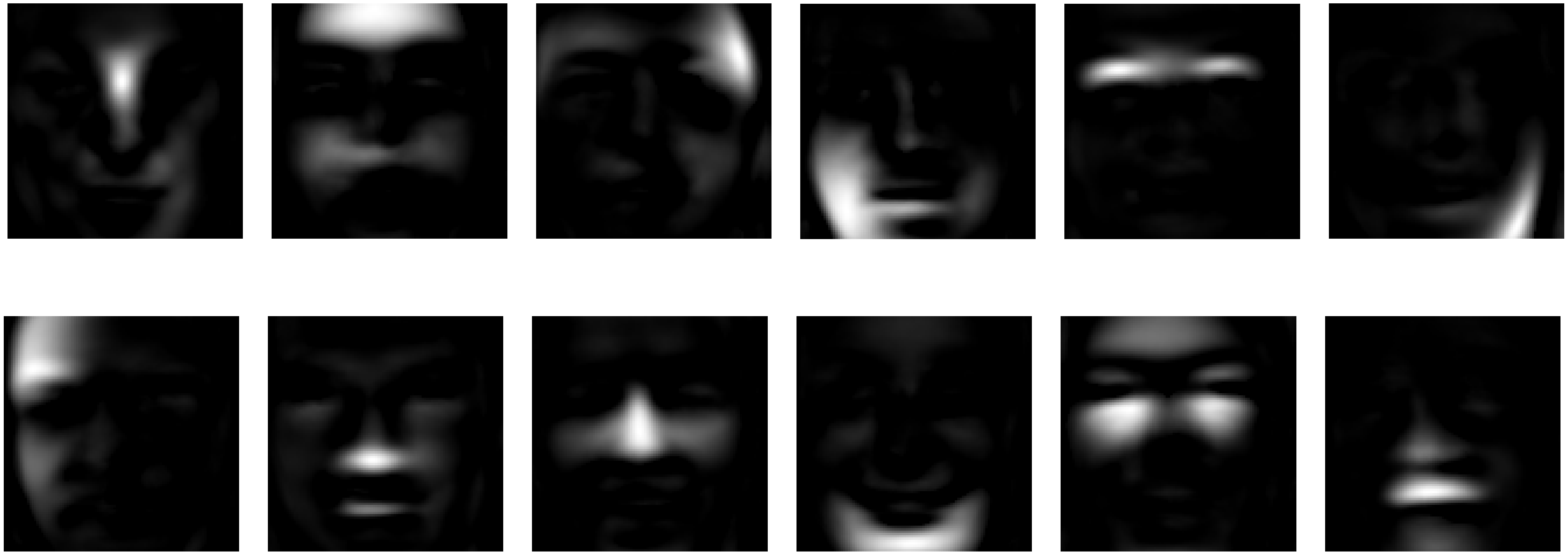


  - basis elements allow us to recover different topics;
  - weights allow us to assign each text to its corresponding topics.

Toy Demonstration of NMF

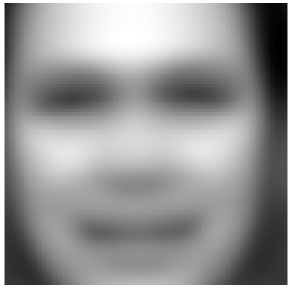
A face image dataset. Image size $= 101 \times 101$, number of face images $= 13232$. Each $\mathbf{x}_i$ is the vectorization of one face image, leading to $m = 101 \times 101 = 10201$, $n = 13232$.

Toy Demonstration of NMF: NMF-Extracted Features

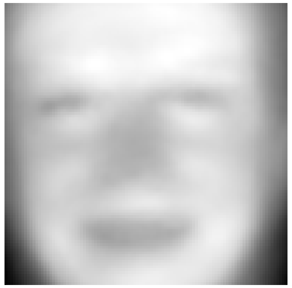


NMF settings: $r = 49$, Lee-Seung multiplicative update with 5000 iterations.

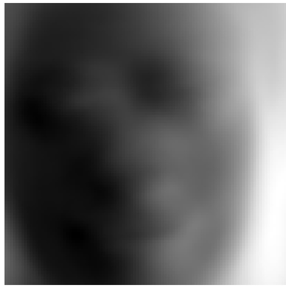
Toy Demonstration of NMF: Comparison with PCA



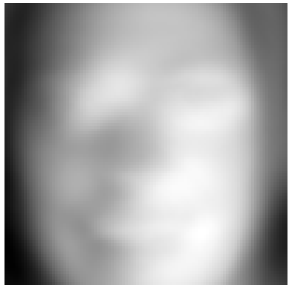
Mean face



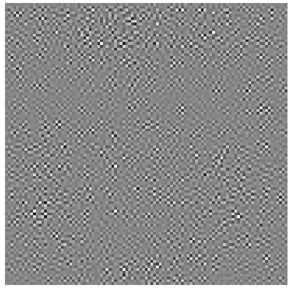
1st principal left singular vector



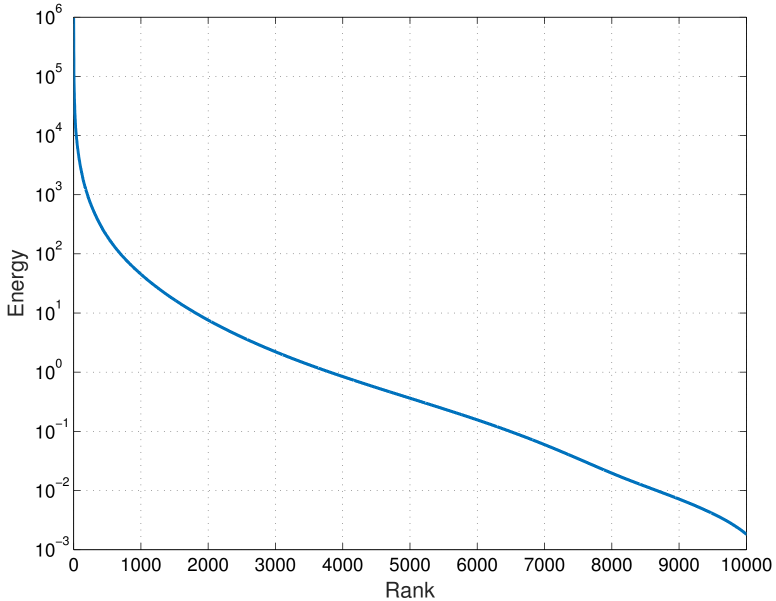
2nd principal left singular vector



3th principal left singular vector



last principal left singular vector



Energy Concentration

A Few More Words to Say

- things I hope you will learn
  - how to read how people manipulate matrix operations, and how you can manipulate them (learn to use a tool);
  - what applications we can do, or to find new applications of our own (learn to apply a tool);
  - deep analysis skills (why is this tool valid? Can I invent new tools? Key to some topics, should go through at least once in your life time)
- feedbacks are welcome; closed-loop systems often work better than open-loop

References

[Yang-Santillana-Kou2015] S. Yang, M. Santillana, and S. C. Kou, "Accurate estimation of influenza epidemics using Google search data via ARGO," *Proceedings of the National Academy of Sciences*, vol. 112, no. 47, pp. 14473–14478, 2015.

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[Candès-Romberg-Tao2006] E. J. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Trans. Information Theory*, vol. 52, no. 2, pp. 489–509, 2006.

[Koren-Bell-Volinsky2009] B. Koren, R. Bell, and C. Volinsky, "Matrix factorization techniques for recommender systems," *IEEE Computer*, vol. 42 no. 8, pp. 30–37, 2009.

[Lee-Seung1999] D. D. Lee and H. S. Seung, "Learning the parts of objects by non-negative matrix factorization," *Nature*, vol. 401, no. 6755, pp. 788–791, 1999.