A FIRST COURSE

IN

LINEAR ALGEBRA

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MAT2040 Notebook

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Foreword

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Preface

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Acronyms

ASTA	Arrivals See Time Averages
BHCA	Busy Hour Call Attempts
BR	Bandwidth Reservation
b.u.	bandwidth unit(s)
CAC	Call / Connection Admission Control
CBP	Call Blocking Probability(-ies)
CCS	Centum Call Seconds
CDTM	Connection Dependent Threshold Model
CS	Complete Sharing
DiffServ	Differentiated Services
EMLM	Erlang Multirate Loss Model
erl	The Erlang unit of traffic-load
FIFO	First in - First out
GB	Global balance
GoS	Grade of Service
ICT	Information and Communication Technology
IntServ	Integrated Services
IP	Internet Protocol
ITU-T	International Telecommunication Unit – Standardization sector
LB	Local balance
LHS	Left hand side

LIFO	Last in - First out
MMPP	Markov Modulated Poisson Process
MPLS	Multiple Protocol Labeling Switching
MRM	Multi-Retry Model
MTM	Multi-Threshold Model
PASTA	Poisson Arrivals See Time Averages
PDF	Probability Distribution Function
pdf	probability density function
PFS	Product Form Solution
QoS	Quality of Service
r.v.	random variable(s)
RED	random early detection
RHS	Right hand side
RLA	Reduced Load Approximation
SIRO	service in random order
SRM	Single-Retry Model
STM	Single-Threshold Model
TCP	Transport Control Protocol
TH	Threshold(s)
UDP	User Datagram Protocol

Chapter 2

Week2

2.1. Tuesday

2.1.1. Review

2.1.1.1. Solving a system of linear Equations

Gaussian Elimination. For the system of equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, it has three cases for its solutions:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \begin{cases} \text{unique solution} \\ \text{no solution} \\ \text{infinitely many solutions} \end{cases}$$

We claim that

> if for this system of equation it has **infinitely** many solutions, then *its columns(or rows) could be linearly combined to zero nontrivially.*

Let's raise an example to explain this statement. Let's use an augmented matrix to represent $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ (Assume $\boldsymbol{A}$ is a 3×3 matrix):

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \Longleftrightarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

When focusing on the columns, we may have the question: in which case does its columns could be linearly combined to zero? That means we need to choose the coefficients c_1, c_2, c_3 such that

$$c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + c_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = 0$$

- It's obvious that when $c_1 = c_2 = c_3 = 0$ we can linearly combine the columns. So $c_1 = c_2 = c_3 = 0$ is the *trival* solution.

- But is there any **nontrival** solution? We claim that if this system of equation has *infinitely* many solutions, we could linearly combine the columns *nontrivally*. We will prove this statement in the end of this lecutre.

  If we focus on the rows, we may have the similar question and conclusion.

Matrix to describe Gaussian Elimination.

1. Firstly let's consider the nonsingular matrix $\boldsymbol{A}$ without row exchange case. We find that postmultiplying elementary matrix has the same effect as doing gaussian elimination. If we finally convert $\boldsymbol{A}$ into *upper triangular matrix* $\boldsymbol{U}$, we can write this process in matrix notation:

$$\boldsymbol{E}_n \dots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{U} \implies \boldsymbol{A} = (\boldsymbol{E}_n \dots \boldsymbol{E}_1)^{-1} \boldsymbol{U} \implies \boldsymbol{A} = \boldsymbol{E}_1^{-1} \dots \boldsymbol{E}_n^{-1} \boldsymbol{U}$$

   (a) If we define $\boldsymbol{L} := \boldsymbol{E}_1^{-1} \dots \boldsymbol{E}_n^{-1}$, which is a lower triangular matrix, then we finally decompose $\boldsymbol{A}$ into the product of two triangular matrix:

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$$

   (b) We can fuirther decompose $\boldsymbol{A}$ into product of three matrices to make the diagonal entries of $\boldsymbol{U}$ and $\boldsymbol{L}$ to be **one**:

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{U}$$

Recall that the LDU decomposition is unique for any matrix.

2. If we have to do row exchange, the process for converting $\boldsymbol{A}$ into $\boldsymbol{U}$ may be like the form:

$$\boldsymbol{E}\cdots\boldsymbol{EPE}\cdots\boldsymbol{EPE}\cdots\boldsymbol{EA} = \boldsymbol{U},$$

but we can always do row exchange first to combine all elementary matrix together, which means we can convert this process into:

$$\boldsymbol{E}\cdots\boldsymbol{EPA} = \boldsymbol{U} \implies \boldsymbol{PA} = \boldsymbol{LU}$$

Also, we can do LDU decomposition to get $\boldsymbol{PA} = \boldsymbol{LDU}$.

2.1.2. Special matrix multiplication case

Firstly let's introduce a new type of vector named unit vector:

Definition 2.1 [unit vector] An ith unit vector is given by:

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Only in ith row its entry is 1, other entries of e_i are all 0.

Then let's discuss some interesting matrix multiplication cases:

1. (a) Given $m \times n$ matrix $\boldsymbol{A} = \begin{bmatrix} a_{ij} \end{bmatrix}_{m\times n}$, the product $\boldsymbol{A}e_i$ is given by:

$$\boldsymbol{A}e_i = \begin{bmatrix} a_{:i} \end{bmatrix},$$

where $\begin{bmatrix} a_{:i} \end{bmatrix}$ denotes the ith column of $\boldsymbol{A}$. (It is from the MATLAB or Julia language.)

(b) Also, given a row vector $e_j^{\mathrm{T}} := \begin{bmatrix} 0 & 0 & \dots & 1 & \dots & 0 \end{bmatrix}$, the product $e_j^{\mathrm{T}}\boldsymbol{A}$ is given by:

$$e_j^{\mathrm{T}}\boldsymbol{A} = \begin{bmatrix} a_{j:} \end{bmatrix},$$

where $\begin{bmatrix} a_{j:} \end{bmatrix}$ denotes the jth row of $\boldsymbol{A}$.

2. Secondly, we want to compute the product $\mathbf{1}^{\mathrm{T}}\boldsymbol{A}\mathbf{1}$, where $\mathbf{1}$ denotes a column vector that all entries of $\mathbf{1}$ are 1 and $\mathbf{1}^{\mathrm{T}}$ denotes the corresponding row vector.

   Let's first compute $\boldsymbol{A} \times \mathbf{1}$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{1} \in \mathbb{R}^n$:

$$\boldsymbol{A} \times \mathbf{1} = \begin{pmatrix} \sum_{j=1}^n a_{1j} \\ \sum_{j=1}^n a_{2j} \\ \vdots \\ \sum_{j=1}^n a_{mj} \end{pmatrix}$$

   It follows that

$$\mathbf{1}^{\mathrm{T}}\boldsymbol{A}\mathbf{1} = \mathbf{1}^{\mathrm{T}}(\boldsymbol{A}\mathbf{1}) = \mathbf{1}^{\mathrm{T}} \begin{pmatrix} \sum_{j=1}^n a_{1j} \\ \sum_{j=1}^n a_{2j} \\ \vdots \\ \sum_{j=1}^n a_{mj} \end{pmatrix} = \langle \mathbf{1}, \boldsymbol{A}\mathbf{1} \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij},$$

3. For vectors $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, we can compute $x^{\mathrm{T}}\boldsymbol{A}y$:

$$x^{\mathrm{T}}\boldsymbol{A}y = x^{\mathrm{T}} \begin{pmatrix} \sum_{j=1}^n a_{1j}y_j \\ \sum_{j=1}^n a_{2j}y_j \\ \vdots \\ \sum_{j=1}^n a_{mj}y_j \end{pmatrix} = \sum_{i=1}^m x_i (\sum_{i=1}^n a_{ij}y_j) = \sum_{i,j} a_{ij}x_i y_j$$

4. For vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, you should distinguish $x^{\mathrm{T}}y$ and xy^{T}:

$$x^{\mathrm{T}}y = \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

$$xy^{\mathrm{T}} = \begin{bmatrix} x_1y_1 & x_1y_2 & \dots & x_1y_n \\ x_2y_1 & x_2y_2 & \dots & x_2y_n \\ \vdots & & \vdots & \\ x_ny_1 & x_ny_2 & \dots & x_ny_n \end{bmatrix} = \begin{bmatrix} x_iy_j \end{bmatrix}_{n \times n}$$

5. For vectors $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, we can compute $x^{\mathrm{T}}\boldsymbol{A}y$ by using block matrix:

   Firstly, We partition $\boldsymbol{A}$ into four parts:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}_{(m_1+m_2)\times(n_1+n_2)}.$$

   Then we partition vector x and y respectively:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{m_1+m_2}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{n_1+n_2},$$

   where x_1 has m_1 rows, x_2 has m_2 rows, y_1 has n_1 rows, y_2 has n_2 rows.

   Then we can compute $x^{\mathrm{T}}\boldsymbol{A}y$:

$$x^{\mathrm{T}}\boldsymbol{A}y = \begin{bmatrix} x_1^{\mathrm{T}} & x_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \sum_{i=1}^{2}\sum_{j=1}^{2} x_i^{\mathrm{T}}\boldsymbol{A}_{ij}y_j.$$

6.

Proposition 2.1 Postmultiplying $\boldsymbol{Q}$ for the vector $v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has the same effect of rotating v in the plane *anticlockwise* by the angle θ, where

$$\boldsymbol{Q} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}.$$

Proof. We convert vector v into the form $v = \begin{bmatrix} \rho cos\varphi \\ \rho sin\varphi \end{bmatrix}$, where $\rho = \sqrt{x_1^2 + x_2^2}$, and $\varphi = arctan(\frac{x_2}{x_1})$. Hence we obtain the product of $\boldsymbol{Q}$ and v:

$$\boldsymbol{Q}v = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} \rho cos\varphi \\ \rho sin\varphi \end{bmatrix} = \begin{bmatrix} \rho cos\theta cos\varphi - \rho sin\theta sin\varphi \\ \rho cos\theta sin\varphi + \rho sin\theta cos\varphi \end{bmatrix} = \begin{bmatrix} \rho cos(\theta + \varphi) \\ \rho sin(\theta + \varphi) \end{bmatrix}$$

This is the form that this vector has been rotated anticlockwise by the angle θ. ■

7. Given $m \times n$ matrix $\boldsymbol{A} = \begin{bmatrix} a_{ij} \end{bmatrix}$, how to flip this matrix vertically? We just need to postmultiply a special matrix:

$$\begin{bmatrix} \boldsymbol{0} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{m1} & a_{m2} & \dots & a_{mn} \\ a_{(m-1)1} & a_{(m-1)2} & \dots & a_{(m-1)n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

If we aftermultiply this matrix for the matrix $\boldsymbol{A}$, we can flip $\boldsymbol{A}$ horizontally:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} a_{1n} & a_{1(n-1)} & \dots & a_{11} \\ a_{2n} & a_{2(n-1)} & \dots & a_{21} \\ \vdots & \vdots & \ddots & \vdots \\ a_{mn} & a_{m(n-1)} & \cdots & a_{m1} \end{bmatrix}$$

2.1.3. Inverse

Let's introduce the definition for inverse matrix:

Definition 2.2 [Inverse matrix] For $n \times n$ matrix $\boldsymbol{A}$, the matrix $\boldsymbol{B}$ is said to be the **inverse** of $\boldsymbol{A}$ if we have $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$. If such $\boldsymbol{B}$ exists, we say matrix $\boldsymbol{A}$ is **invertible** or **nonsingular**.

And inverse matrix has some interesting properties:

Proposition 2.2 **Matrix inverse is Unique**. In other words, if we have $\boldsymbol{AB}_1 = \boldsymbol{B}_1\boldsymbol{A} = \boldsymbol{I}$

and $AB_2 = B_2 A = I$, then we obtain $B_1 = B_2$.

Proof.

$$AB_1 = I \implies B_2 AB_1 = B_2 I \implies B_2 AB_1 = B_2$$

$$\implies (B_2 A)B_1 = IB_1 = B_1 = B_2.$$

■

Proposition 2.3 If we have both $AB = I$ and $CA = I$, then we have $C = B$.

Proof. On the one hand, we have

$$CAB = C(AB) = CI = C$$

On the other hand, we obtain:

$$CAB = (CA)B = IB = B$$

Hence we have $C = B$. ■

2.1.3.1. How to compute inverse? When does it exist?

Assuming the inverse of $n \times n$ matrix A exists, and we define it to be

$$A^{-1} := X = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = \begin{bmatrix} x_{ij} \end{bmatrix}$$

By definition, we have $AX = I$. We write it into block columns:

$$AX = A \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = I = \begin{bmatrix} e_1 & e_2 & \dots & e_n \end{bmatrix},$$

where $e_1, e_2, \dots, e_n$ are all unit vectors.

Hence we obtain

$$A \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = \begin{bmatrix} Ax_1 & Ax_2 & \dots & Ax_n \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \dots & e_n \end{bmatrix}.$$

Thus we only need to compute n system of equations $\boldsymbol{A}x_i = e_i, i = 1, \dots, n$ to get the columns of the inverse matrix $\boldsymbol{X}$. Or equivalently, we need to do Gaussian Elimination to convert the augmented matrix $\begin{bmatrix} \boldsymbol{A} & \big| & \boldsymbol{I} \end{bmatrix}$ into the form $\begin{bmatrix} \boldsymbol{I} & \big| & \boldsymbol{X} \end{bmatrix}$. Once we have done that, we get the inverse of $\boldsymbol{A}$ immediately. Let's discuss an example to show how to achieve it:

■ **Example 2.1** Assuming we have only 3 systems of equations to solve. And we put them altogehter into one Augmented matrix. And the right side of augmented matrix is an identity matrix

$$\begin{bmatrix} \boldsymbol{A} & \big| & e_1 & \big| & e_2 & \big| & e_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right] \xRightarrow[\boldsymbol{E}_{21}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}]{\boldsymbol{E}_{31}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\xRightarrow{\boldsymbol{E}_{32}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xRightarrow[\boldsymbol{E}_{13}=\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}]{\boldsymbol{E}_{23}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xRightarrow{\boldsymbol{E}_{12}=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

The final augmented matrix is equivalent to the system $\boldsymbol{I}\boldsymbol{X} = \begin{bmatrix} \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{bmatrix}$.

Hence we obtain the inverse: $\boldsymbol{A}^{-1} = \boldsymbol{X} = \begin{bmatrix} \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{bmatrix}$. ■

Then let's study in which case does the inverse exist:

Theorem 2.1 The inverse of $n \times n$ matrix $\boldsymbol{A}$ exists if and only if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.

Proofoutline. The inverse of $n \times n$ matrix $\boldsymbol{A}$ exists

$\Leftrightarrow$ none pivot values of $\boldsymbol{A}$ is zero. $\Leftrightarrow$ $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$. ■

At the end, let's prove the claim at the beginning of the lecture:

Theorem 2.2 Let $\boldsymbol{A}$ be $n \times n$ matrix, the following statements are equivalent:

1. Columns of $\boldsymbol{A}$ can be linearly combined to zero nontribally.
2. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has infinitely many solutions.
3. Row vectors of $\boldsymbol{A}$ can be linearly combined to zero nontrivally.

Proofoutline. The following statements are equivalent:

- Columns of $\boldsymbol{A}$ can be linearly combined to zero nontribally.
- Given $\boldsymbol{A} = \left[\begin{array}{c|c|c|c} a_1 & a_2 & \dots & a_n \end{array}\right]$, then there exists x_i's that are not all zero such that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0.$$

- $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has a nonzero solution $\overline{\boldsymbol{x}}$.

- $2\overline{\boldsymbol{x}}, 3\overline{\boldsymbol{x}}, \dots$ are also solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$.
- $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has infinitely many solutions.
- $\boldsymbol{A}^{-1}$ does not exist. (otherwise we will only have unique solution $\boldsymbol{A}^{-1} \times \boldsymbol{0} = \boldsymbol{0}$.)
- Gaussian Elimination breaks down, i.e., there exists zero row in the row echelon form.
- Row vectors of $\boldsymbol{A}$ can be linearly combined to zero nontrivally.

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