A FIRST COURSE

IN

LINEAR ALGEBRA

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MAT2040 Notebook

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Foreword

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Preface

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Acronyms

ASTA	Arrivals See Time Averages
BHCA	Busy Hour Call Attempts
BR	Bandwidth Reservation
b.u.	bandwidth unit(s)
CAC	Call / Connection Admission Control
CBP	Call Blocking Probability(-ies)
CCS	Centum Call Seconds
CDTM	Connection Dependent Threshold Model
CS	Complete Sharing
DiffServ	Differentiated Services
EMLM	Erlang Multirate Loss Model
erl	The Erlang unit of traffic-load
FIFO	First in - First out
GB	Global balance
GoS	Grade of Service
ICT	Information and Communication Technology
IntServ	Integrated Services
IP	Internet Protocol
ITU-T	International Telecommunication Unit – Standardization sector
LB	Local balance
LHS	Left hand side

LIFO	Last in - First out
MMPP	Markov Modulated Poisson Process
MPLS	Multiple Protocol Labeling Switching
MRM	Multi-Retry Model
MTM	Multi-Threshold Model
PASTA	Poisson Arrivals See Time Averages
PDF	Probability Distribution Function
pdf	probability density function
PFS	Product Form Solution
QoS	Quality of Service
r.v.	random variable(s)
RED	random early detection
RHS	Right hand side
RLA	Reduced Load Approximation
SIRO	service in random order
SRM	Single-Retry Model
STM	Single-Threshold Model
TCP	Transport Control Protocol
TH	Threshold(s)
UDP	User Datagram Protocol

Chapter 3

Week3

3.1. Tuesday

3.1.1. Introduction

3.1.1.1. Motivation of Linear Algebra

So, we raise the question again, why do we learn LA?

- Baisis of AI/ML/SP/etc.

  In information age, *artificial intelligence*, *machine learning*, *structured programming*, and otherwise gains great popularity among researchers. LA is the basis of them, so in order to explore science in modern age, you should learn LA well.

- Solving linear system of equations.

  How to solve linear system of equations efficiently and correctly is the **key** question for mathematicians.

- Internal grace.

  LA is very beautiful, hope you enjoy the beauty of math.

- Interview questions.

  LA is often used for interview questions for phd. The interviewer usually ask difficult questions about LA.

3.1.1.2. Preview of LA

The main branches of Mathematics are given below:

$$\text{mathematics} \begin{cases} \text{Analysis + Calculus} \\ \text{Algebra: foucs on structure} \\ \text{Geometry} \end{cases}$$

All parts of math are based on **axiom systems**. And **LA** is the significant part of *Algebra*, which focus on the linear structure.

3.1.2. Review of 2 weeks

How to solve linear system equations?. The basic method is **Gaussian Elimination**, and the main idea is *induction* to make simpler equations.

- Given one equation $ax = b$, we can easily sovle it:

$$\text{If } a = 0, \text{ there is no solution} \quad \text{otherwise } x = \tfrac{b}{a}.$$

- We could solve 1×1 system. By induction, if we could solve $n \times n$ systems, then we can solve $(n+1) \times (n+1)$ systems.

  In the above process, math notations is needed:

- matrix multiplication
- matrix inverse
- transpose, symmetric matrices

So in first two weeks, we just learn two things:

- *linear system could be solved* ***almost*** *by G.E.*
- *Furthermore, Gaussian Elimination is (almost) LU decomposition.*

But there is a question remained to be solved:

How to solve linear singular system equations?.

- When does the system have no solution, when does the system have infinitely many solutions? (Note that singular system don't has unique solution.)
- If it has infinitely many solutions, how to find and express these solutions?

If we express system into matrix form, the question turns into:

How to solve the rectangular?

3.1.3. Examples of solving equations

- For square case, we often convert the system into $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{c}$, where $\boldsymbol{U}$ is of *row echelon form*.
- However, for rectangular case, *row echelon form*(ref) is not enough, we must convert it into **reduced row echelon form**(rref):

$$\boldsymbol{U}(\text{ref}) = \begin{bmatrix} \boxed{1} & 0 & \times & \times & \times & 0 & \times \\ 0 & \boxed{1} & \times & \times & \times & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \implies \boldsymbol{R}(\text{rref}) = \begin{bmatrix} \boxed{1} & 0 & \times & \times & \times & 0 & \times \\ 0 & \boxed{1} & \times & \times & \times & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

■ **Example 3.1** We discuss how to solve **square** matrix of **rref**:

- If all rows have nonzero entry, we have:

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{c} \implies \boldsymbol{x} = \boldsymbol{c}$$

- But note that *some rows could be all zero*:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \boldsymbol{x} = \boldsymbol{c} \implies \begin{cases} x_1 = c_1 \\ x_2 = c_2 \\ x_3 = c_3 \\ 0 = c_4 \end{cases}$$

  So the solution results have two cases:

  - If $c_4 \neq 0$, we have no solution of this system.
  - If $c_4 = 0$, we have infinitely many solutions, which can be expressed as:

$$x_{\text{complete}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

    where x_4 could be arbitarary number.

Hence, for square system, does Gaussian Elimination work?

Answer: Almost, except for the "pivot=0"case:

- All pivots $\neq 0 \implies$ the system has unique solution.
- Some pivots $= 0$ (The matrix is singular)
  1. No solution. (When LHS $\neq$ RHS)
  2. Infinitely many solutions.

3.1.3.1. Review of G.E. for Nonsingular case

We use matrix to represent system of equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{23}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{m3}x_n = b_m \end{cases} \implies \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

By postmultiplying $\boldsymbol{E}_{ij}$ or $\boldsymbol{P}_{ij}$, we are essentially doing one step of elimination:

$$\boldsymbol{E}_{ij}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{E}_{ij}\boldsymbol{b} \quad \text{or} \quad \boldsymbol{P}_{ij}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{E}_{ij}\boldsymbol{b}$$

By several steps of elimination, we obtain the final result:

$$\hat{\boldsymbol{L}}\boldsymbol{P}\boldsymbol{A}\boldsymbol{x} = \hat{\boldsymbol{L}}\boldsymbol{P}\boldsymbol{b}$$

where $\hat{\boldsymbol{L}}\boldsymbol{P}\boldsymbol{A}$ represents an upper triangular matrix $\boldsymbol{U}$, $\hat{L}$ is the lower triangular matrix.

Equivalently, we obtain

$$\hat{\boldsymbol{L}}\boldsymbol{P}\boldsymbol{A} = \boldsymbol{U} \implies \boldsymbol{P}\boldsymbol{A} = \hat{\boldsymbol{L}}^{-1}\boldsymbol{U} \triangleq \boldsymbol{L}\boldsymbol{U}$$

Hence, Gaussian Elimination is almost the LU decomposition.

3.1.3.2. Example for solving rectangular system of rref

Recall the definition for rref:

Definition 3.1 [reduced row echelon form] Suppose a matrix has r *nonzero* rows, each row has leading 1 as pivots. If all columns with pivots (call it pivot column) are all zero entries apart from the pivot in this column, then this matrix is said to be **reduced row echelon form**(**rref**).

Next, we want to show how to solve a rectangular system of rref. Note that in last lecture we study the solution to a rectangular system is given by:

$$\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_{\text{special}}.$$

■ **Example 3.2** Solve the system

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} = \boldsymbol{c}.$$

Step 1: Find null space. Firstly we solve for $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{0}$:

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{cases} x_1 + 3x_2 = 0 \\ x_3 + x_4 = 0 \end{cases}$$

Then we express the **pivot variables** in the form of **free variables.**

Note that the pivot columns in $\boldsymbol{R}$ are column 1 and 3, so the pivot variable is x_1 and x_3. The free variable is the remaining variable, say, x_2 and x_4.

The expressions for x_1 and x_3 are given by:

$$\begin{cases} x_1 = -3x_2 \\ x_3 = -x_4 \end{cases}$$

Hence, all solutions to $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{0}$ are

$$\boldsymbol{x}_{\text{special}} = \begin{bmatrix} -3x_2 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

where x_2 and x_4 can be taken arbitararily.

Step 2: Find one particular solution to $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{c}$. The trick for this step is to set $x_2 = x_4 = 0$. (*set free variable to be zero and then derive the pivot variable.*):

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \implies \begin{cases} x_1 = c_1 \\ x_3 = c_2 \\ 0 = c_3 \end{cases}$$

which follows that:

- if $c_3 = 0$, then exists particular solution $\boldsymbol{x}_p = \begin{bmatrix} c_1 \\ 0 \\ c_2 \\ 0 \end{bmatrix}$;
- if $c_3 \neq 0$, then $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{c}$ has no solution.

Final solution. If assume $c_3 = 0$, then all solutions to $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{c}$ are given by:

$$\boldsymbol{x}_{complete} = \boldsymbol{x}_p + \boldsymbol{x}_{\text{special}} = \begin{bmatrix} c_1 \\ 0 \\ c_2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Next we show how to solve a general rectangular:

3.1.4. How to solve a general rectangular

For linear system $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is rectangular, we can solve this system as follows:

Step 1: Gaussian Elimination. With proper row permutaion (postmultiply $\boldsymbol{P}_{ij}$) and row transformation (postmultiply $\boldsymbol{E}_{ij}$), we convert $\boldsymbol{A}$ into $\boldsymbol{R}$(rref), then we only need to solve $\boldsymbol{Rx} = \boldsymbol{c}$.

■ **Example 3.3** The first example is a 3×4 matrix with two pivots:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}$$

Clearly $a_{11} = 1$ is the first pivot, then we clear row 2 and row 3 of this matrix:

$$\boldsymbol{A} \xrightarrow[\boldsymbol{E}_{31}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}]{\boldsymbol{E}_{21}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \xrightarrow[\boldsymbol{E}_{32}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}]{\boldsymbol{E}_{12}=\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If we want to solve $\boldsymbol{Ax} = \boldsymbol{b}$, firstly we should convert $\boldsymbol{A}$ into $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (rref). ■

Then we should identify **pivot variables** and **free variables**. we can follow the proceed below:

$$\text{pivots} \implies \text{pivot columns} \implies \text{pivot variables}$$

■ **Example 3.4** we want to identify **pivot variables** and **free variables** of $\boldsymbol{R}$:

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & \times & \times & \times & 0 & \times \\ 0 & 1 & \times & \times & \times & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & 1 & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot are r_{11}, r_{22}, r_{36}. So the pivot columns are column $1, 2, 6$. So the *pivot variables* are x_1, x_2, x_6; the *free variables* are x_3, x_4, x_5, x_7. ■

Step2: Compute null space $N(\boldsymbol{A})$. In order to find $N(\boldsymbol{A})$, it suffices to compute $N(\boldsymbol{R})$. The space $N(\boldsymbol{R})$ has $(n-r)$ dimensions, so it suffices to get $(n-r)$ special solutions first:

- For each of the $(n-r)$ free variables,
  - set the value of **it** to be 1;
  - set the value of other **free variables** to be 0;
  - Then solve $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{0}$ (to get the value of pivot variables) to get the special solution.

■ **Example 3.5** Continue with 3×4 matrix example:

$$\boldsymbol{R} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We want to find special solutions to $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{0}$:

1. Set $x_2 = 1$ and $x_4 = 0$. Solve $\boldsymbol{Rx} = \mathbf{0}$, then $x_1 = -1$ and $x_3 = 0$.

   Hence one special solution is $y_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

2. Set $x_2 = 0$ and $x_4 = 1$. Solve $\boldsymbol{Rx} = \mathbf{0}$, then $x_1 = -1$ and $x_3 = -1$.

   Then another special solution is $y_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

■

- Then $N(\boldsymbol{A})$ is the collection of linear combinations of these special solutions:

$$N(\boldsymbol{A}) = \text{span}(y_1, y_2, \dots, y_{n-r}).$$

■ **Example 3.6** We continue the example above, when we get all special solutions

$$y_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad y_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

the null space contains all linear combinations of the special solutions:

$$\boldsymbol{x}_{\text{special}} = \text{span}(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}) = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

where x_2, x_4 here could be arbitarary. ■

Step3: Compute a particular solution $\boldsymbol{x}_p$. The easiest way is to "read" from $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{c}$:

- **Guarantee the existence of the solution**. Suppose $\boldsymbol{R} \in \mathbb{R}^{m\times n}$ has $r(\leq m)$ pivot variables, then it has $(m-r)$ zero rows and $(n-r)$ free variables. For the existence of solutions, the value of entries of $\boldsymbol{c}$ which correspond to zero rows in $\boldsymbol{R}$ must also be zero.

■ **Example 3.7** If $\boldsymbol{R}\boldsymbol{x} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{x} = \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, then in order to have a solution, we must let $c_3 \neq 0$. ■

- If the condition above is not satisfied, then the system has no solution. Let's preassume the satisfaction of such a condition. To compute a particular solution $\boldsymbol{x}_p$, we set *the value for all* ***free variables*** *of $\boldsymbol{x}_p$ to be zero, and the value for the pivot variables are from $\boldsymbol{c}$.*

  More specifically, the first entry in $\boldsymbol{c}$ is exactly the value for the first pivot variable;the second entry in $\boldsymbol{c}$ is exactly the value for the second pivot variable......, and the remaining entries of $\boldsymbol{x}_p$ are set to be zero.

■ **Example 3.8** If $\boldsymbol{R}\boldsymbol{x} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{x} = \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix}$, we want to compute particular solution

$$\boldsymbol{x}_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

As we know x_2, x_4 are free variable, $x_2 = x_4 = 0$; and x_1, x_3 are pivot variable, so we have $\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

Hence the solution for $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{c}$ is

$$\boldsymbol{x}_p = \begin{bmatrix} c_1 \\ 0 \\ c_2 \\ 0 \end{bmatrix}.$$

Final step: Obtain complete solutions. All solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ are

$$\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_{\text{special}},$$

where $x_{\text{special}} \in N(\boldsymbol{A})$. Note that $\boldsymbol{x}_p$ is defined in step3, $\boldsymbol{x}_{\text{special}}$ is defined in step2.

However, where does the number r come? r denotes the **rank** of a matrix, which will be discussed in the next lecture.