A FIRST COURSE

IN

LINEAR ALGEBRA

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MAT2040 Notebook

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Contents

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Foreword

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Preface

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Acronyms

ASTA	Arrivals See Time Averages
BHCA	Busy Hour Call Attempts
BR	Bandwidth Reservation
b.u.	bandwidth unit(s)
CAC	Call / Connection Admission Control
CBP	Call Blocking Probability(-ies)
CCS	Centum Call Seconds
CDTM	Connection Dependent Threshold Model
CS	Complete Sharing
DiffServ	Differentiated Services
EMLM	Erlang Multirate Loss Model
erl	The Erlang unit of traffic-load
FIFO	First in - First out
GB	Global balance
GoS	Grade of Service
ICT	Information and Communication Technology
IntServ	Integrated Services
IP	Internet Protocol
ITU-T	International Telecommunication Unit – Standardization sector
LB	Local balance
LHS	Left hand side

LIFO	Last in - First out
MMPP	Markov Modulated Poisson Process
MPLS	Multiple Protocol Labeling Switching
MRM	Multi-Retry Model
MTM	Multi-Threshold Model
PASTA	Poisson Arrivals See Time Averages
PDF	Probability Distribution Function
pdf	probability density function
PFS	Product Form Solution
QoS	Quality of Service
r.v.	random variable(s)
RED	random early detection
RHS	Right hand side
RLA	Reduced Load Approximation
SIRO	service in random order
SRM	Single-Retry Model
STM	Single-Threshold Model
TCP	Transport Control Protocol
TH	Threshold(s)
UDP	User Datagram Protocol

7.2. Thursday

7.2.1. Review

- **Eigenvalue and eigenvectors**: If for **square** matrix $\boldsymbol{A}$ we have

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

  where $\boldsymbol{x} \neq \mathbf{0}$, then we say λ is the *eigenvalue*, $\boldsymbol{x}$ is the *eigenvector* associated with λ.

- **How to compute eigenvalues and eigenvectors?** To solve the eigenvalue problem for matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, you should follow these steps:
  - *Compute the characteristic polynomial of* $\lambda \boldsymbol{I} - \boldsymbol{A}$. The determinant is a polynomial in λ of degree n.
  - *Find the roots of this polynomial*, by solving $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$. The n roots are the n eigenvalues of $\boldsymbol{A}$. They make $\boldsymbol{A} - \lambda \boldsymbol{I}$ singular.
  - For each eigenvalue λ, *solve* $(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \mathbf{0}$ *to find a corresponding eigenvector* $\boldsymbol{x}$.

7.2.2. Similarity

The similar matrices have the same eigenvalues:

Definition 7.3 [Similar] If there exists a *nonsingular* matrix $\boldsymbol{S}$ such that

$$\boldsymbol{B} = \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S},$$

then we say $\boldsymbol{A}$ is **similar** to $\boldsymbol{B}$.

Proposition 7.3 Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices. If $\boldsymbol{B}$ is *similar* to $\boldsymbol{A}$, then $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same eigenvalues.

Proofidea. Since eigenvalues are the roots of the *characteristic polynomial*, so it suffices to prove these two polynomials are the same.

Proof. The *characteristic polynomial* for $\boldsymbol{B}$ is given by

$$\begin{aligned}P_{\boldsymbol{B}}(\lambda) &= \det(\lambda \boldsymbol{I} - \boldsymbol{B}) \\ &= \det(\lambda \boldsymbol{I} - \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}) = \det(\boldsymbol{S}^{-1}\lambda \boldsymbol{I}\boldsymbol{S} - \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}) \\ &= \det(\boldsymbol{S}^{-1}(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{S}) \\ &= \det(S^{-1})\det(\lambda \boldsymbol{I} - \boldsymbol{A})\det(\boldsymbol{S})\end{aligned}$$

Since $\det(\boldsymbol{S}^{-1})\det(\boldsymbol{S}) = 1$, we obtain:

$$\begin{aligned}P_{\boldsymbol{B}}(\lambda) &= \det(\lambda \boldsymbol{I} - \boldsymbol{A}) \\ &= P_{\boldsymbol{A}}(\lambda).\end{aligned}$$

Since they have the same *characteristic polynomial*, the roots for *characteristic polynomials* of $\boldsymbol{A}$ and $\boldsymbol{B}$ must be same. Therefore they have the same eigenvalues. ■

R What is invarient? In other words, what is not changed during matrix transformation?

- **Rank** is invarient under *row transformation*.
- **Eigenvalues** is invarient undet *similar transformation*.
- Unluckily, similar matrices usually don't have the same eigenvectors. It's easy to raise a counterexample.

By using eigenvalues, we have a new proof for $\det(\boldsymbol{S}^{-1}) = \frac{1}{\det(\boldsymbol{S})}$:

Proof. Suppose $\det(\boldsymbol{S}) = \lambda_1\lambda_2\dots\lambda_n$, where λ_i's are eigenvalues of $\boldsymbol{S}$. Then there exists $\boldsymbol{x}_i$ such that

$$\boldsymbol{S}\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i$$

for $i = 1, \dots, n$.

Since $\boldsymbol{S}$ is invertible and all λ_i's are nonzero, we imply that:

$$\boldsymbol{S}\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i \implies \boldsymbol{x}_i = \lambda_i \boldsymbol{S}^{-1}\boldsymbol{x}_i \implies \boldsymbol{S}^{-1}\boldsymbol{x}_i = \frac{1}{\lambda_i}\boldsymbol{x}_i$$

Hence, $\frac{1}{\lambda_i}$'s are eigenvalues of $\boldsymbol{S}^{-1}$. Since $S^{-1} \in \mathbb{R}^{n\times n}$, $\frac{1}{\lambda_i}$'s $(i = 1,\dots,n)$ are the only eigenvalues of $\boldsymbol{S}^{-1}$.

Hence the determinant of $\boldsymbol{S}^{-1}$ is the product of its eigenvalues:

$$\det(\boldsymbol{S}^{-1}) = \frac{1}{\lambda_1}\frac{1}{\lambda_2}\cdots\frac{1}{\lambda_n} = \frac{1}{\det(\boldsymbol{S})}.$$

■

We can also use eigenvalue to proof the statement shown below:

Proposition 7.4 $\boldsymbol{A}$ is singular if and only if $\det(\boldsymbol{A}) = 0$.

Proof. Suppose $\det(\boldsymbol{A}) = \lambda_1\lambda_2\dots\lambda_n$, where λ_i's are eigenvalues of $\boldsymbol{A}$.

Thus

$$\det(\boldsymbol{A}) = 0 \Longleftrightarrow \exists \lambda_i = 0 \Longleftrightarrow \exists \text{ nonzero } \boldsymbol{x} \text{ s.t. } \boldsymbol{A}\boldsymbol{x} = \lambda_i \boldsymbol{x} = 0\boldsymbol{x} = \boldsymbol{0}.$$

Or equivalently, $\boldsymbol{A}$ is singular. ■

7.2.3. Diagonalization

Proposition (7.3) says if $\boldsymbol{A}$ is similar to $\boldsymbol{B}$, then they have the same eigenvalues.

Question 1. What about the reverse direction?

Question 2. We all approve that the simplest form of a matrix to have eigenvalues $\lambda_1,\dots,\lambda_n$ is the diagonal matrix $\operatorname{diag}(\lambda_1,\dots,\lambda_n)$. Suppose $\boldsymbol{A}$ has eigenvalues $\lambda_1,\dots,\lambda_n$, is $\boldsymbol{A}$ similar to the diagonal matrix $\operatorname{diag}(\lambda_1,\dots,\lambda_n)$?

R Why the matrix $\operatorname{diag}(\lambda_1,\dots,\lambda_n)$ has eigenvalues $\lambda_1,\dots,\lambda_n$?

Answer: Let's explain it with $n = 2$:

$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The case for general n is also easy to verify.

The answers to Question 1 and 2 are both **No**! Let's raise a counterexample to explain it:

■ **Example 7.4** We give a counterexample to show that two matrices with the same eigenvalues are not necessarily similar to each other; and $\boldsymbol{A}$ does not necessarily similar to the corresponding diagonal matrix.

Given $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $P_{\boldsymbol{A}}(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix}$. Hence its eigenvalues are $\lambda_1 = \lambda_2 = 0$.

Hence, $\boldsymbol{A}$ and $\boldsymbol{D} = \text{diag}(0,0)$ have the same eigenvalues. Then we show that $\boldsymbol{A}$ and $\boldsymbol{D}$ are not similar:

Assume they are similar, which means there exists invertible matrix $\boldsymbol{S}$ such that

$$\boldsymbol{A} = \boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{S} = \boldsymbol{S}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\boldsymbol{S} = \boldsymbol{0} \implies \text{contradiction!}$$

■

Suppose $\boldsymbol{A}$ has eigenvalues $\lambda_1, \dots, \lambda_n$, but $\boldsymbol{A}$ and $\text{diag}(\lambda_1, \dots, \lambda_n)$ may not be similar! We are curious about what kind of matrix can be similar to a diagonal matrix:

Definition 7.4 [Diagonalizable] An $n \times n$ matrix $\boldsymbol{A}$ is **diagonalizable** if $\boldsymbol{A}$ is similar to a *diagonal matrix*, that is to say, $\exists$ nonsingular matrix $\boldsymbol{S}$ and diagonal matrix $\boldsymbol{D}$ such that

$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{D} \tag{7.6}$$

We say $\boldsymbol{S}$ *diagonalizes* $\boldsymbol{A}$. ■

R Note that Eq.(7.6) can be equivalently written as $\boldsymbol{AS} = \boldsymbol{SD}$, or in column-by-column form:

$$\boldsymbol{A}\boldsymbol{s}_i = d_i \boldsymbol{s}_i, \quad i = 1, \dots, n, \tag{7.7}$$

where $\boldsymbol{s}_i$ denotes the ith column of $\boldsymbol{S}$, d_i denotes the (i,i)th entry of $\boldsymbol{D}$. The equivalent form Eq.(7.7) also implies that every $(\boldsymbol{s}_i, d_i)$ must be an eigen-pair of $\boldsymbol{A}$. (Proposition (7.5))

Proposition 7.5 Suppose that $\boldsymbol{A}$ is diagonalizable, then the column vectors of the diagonalizing matrix $\boldsymbol{S}$ are eigenvectors of $\boldsymbol{A}$; and the diagonal elements of $\boldsymbol{D}$ are the corresponding eigenvalues of $\boldsymbol{A}$.

Proposition 7.6 The diagonalizing matrix $\boldsymbol{S}$ is not unique.

Proof. Suppose there exists a diagonalizing matrix $\boldsymbol{S}$, verify by yourself that $\alpha\boldsymbol{S}$ is also a a diagonalizing matrix for any $\alpha \neq 0$. ∎

R We know that the reverse of proposition (7.3) is not true. However, if we add one more constraint that all eigenvalues of $\boldsymbol{A}$ are distinct, the reverse is true. We will give a proof of it later.

1. If $\boldsymbol{A}$ is $n \times n$ and A has n distinct eigenvalues, then $\boldsymbol{A}$ is diagonalizable. If the eigenvalues are not distinct, then $\boldsymbol{A}$ may or may not be diagonalizable depending on whether $\boldsymbol{A}$ has n linearly independent eigenvectors.

Why is diagonalizable good?

Theorem 7.3 — Diagonalization. A $n \times n$ matrix $\boldsymbol{A}$ is *diagonalizable* iff $\boldsymbol{A}$ has n independent eigenvectors.

Proof. Necessity. For n eigen-pairs $(\lambda_i, \boldsymbol{x}_i)$ of $\boldsymbol{A}$, suppose that $\boldsymbol{x}_i$'s are independent.

We after-multiply $\boldsymbol{A}$ with $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots & \boldsymbol{x}_n \end{bmatrix}$. The first column of $\boldsymbol{AS}$ is $\boldsymbol{A}\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$. Hence we obtain the result for the product $\boldsymbol{AS}$:

$$\textbf{A times S} \quad \boldsymbol{AS} = \boldsymbol{A}\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots & \boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\boldsymbol{x}_1 & \lambda_2\boldsymbol{x}_2 & \dots & \lambda_n\boldsymbol{x}_n \end{bmatrix}. \tag{7.8}$$

Note that the right side of Eq.(7.8) is essentially the product $\boldsymbol{SD}$:

$$\textbf{S times D}\quad \begin{bmatrix} \lambda_1\boldsymbol{x}_1 & \lambda_2\boldsymbol{x}_2 & \dots & \lambda_n\boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \boldsymbol{SD}.$$

Hence we obtain $\boldsymbol{AS} = \boldsymbol{SD}$. Since $\boldsymbol{x}_i$'s are independent, there exists the inverse $\boldsymbol{S}^{-1}$.

Therefore, $\boldsymbol{D} = \boldsymbol{S}^{-1}\boldsymbol{AS}$.

Sufficiency. If $\boldsymbol{A}$ is diagonalizable, then there exists $\boldsymbol{S}$ and $\boldsymbol{D}$ such that

$$\boldsymbol{D} = \boldsymbol{S}^{-1}\boldsymbol{AS} \tag{7.9}$$

where $\boldsymbol{S}$ is nonsingular. Suppose $\boldsymbol{D} = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$, and $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots & \boldsymbol{x}_n \end{bmatrix}$, where $\boldsymbol{x}_i$'s are independent.

The Eq.(7.9) can be equivalently written as $\boldsymbol{AS} = \boldsymbol{SD}$, i.e., $\boldsymbol{A}\boldsymbol{x}_i = \lambda_i\boldsymbol{x}_i$ for $i = 1,2,\dots,n$.

Hence $\boldsymbol{x}_i$'s are the independent eigenvectors of $\boldsymbol{A}$ associated with λ_i's. ■

Diagonalizable matrix is very useful. For diagonalizable matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, it follows that its eigenvectors $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\}$ are independent, i.e., form a basis for $\mathbb{R}^n$. Then for any $\boldsymbol{y} \in \mathbb{R}^n$, there exists $(c_1,c_2,\dots,c_n)$ such that

$$\boldsymbol{y} = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_n\boldsymbol{x}_n$$

If we consider matrix $\boldsymbol{A}$ as representation of linear transformation, we obtain

$$\begin{aligned} \boldsymbol{Ay} &= c_1\boldsymbol{A}\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{A}\boldsymbol{x}_n \\ &= c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n \end{aligned}$$

Hence, the linear transformation from $\boldsymbol{y}$ into $\boldsymbol{A}\boldsymbol{y}$ is equivalent to transforming the coordinate coefficients from $(c_1,\dots,c_n)$ into $(c_1\lambda_1,\dots,c_n\lambda_n)$:

$$\boldsymbol{y} \xRightarrow{\boldsymbol{A}} \boldsymbol{A}\boldsymbol{y}$$

$$(c_1,\dots,c_n) \xRightarrow{\boldsymbol{D}=\operatorname{diag}(\lambda_1,\dots,\lambda_n)} (c_1\lambda_1,\dots,c_n\lambda_n) = (c_1,\dots,c_n)\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$$

We are curious about whether there is an useful way to determine whether $\boldsymbol{A}$ is diagonalizable.

Theorem 7.4 If $\lambda_1,\dots,\lambda_k$ are *distinct* eigenvalues of a matrix $\boldsymbol{A}\in\mathbb{R}^{n\times n}(n\ge k)$ with the corresponding eigenvectors $\boldsymbol{x}_1,\dots,\boldsymbol{x}_k$, then $\boldsymbol{x}_1,\dots,\boldsymbol{x}_k$ are linearly independent.

Proof.

- Let's start with the case $k=2$. Assume that $\lambda_1\ne\lambda_2$ but $\boldsymbol{x}_1,\boldsymbol{x}_2$ are dependent, i.e., $\exists(c_1,c_2)\ne\boldsymbol{0}$ s.t.

$$c_1\boldsymbol{x}_1+c_2\boldsymbol{x}_2=\boldsymbol{0}. \tag{7.10}$$

Postmultiplying $\boldsymbol{A}$ for Eq.(7.10) both sides results in

$$\boldsymbol{A}(c_1\boldsymbol{x}_1+c_2\boldsymbol{x}_2)=\boldsymbol{0} \implies c_1\lambda_1\boldsymbol{x}_1+c_2\lambda_2\boldsymbol{x}_2=\boldsymbol{0}. \tag{7.11}$$

Eq.(7.10)$\times\lambda_2-$Eq.(7.11) results in:

$$(c_1\lambda_2-c_1\lambda_1)\boldsymbol{x}=\boldsymbol{0}. \implies c_1(\lambda_2-\lambda_1)\boldsymbol{x}=\boldsymbol{0}.$$

Since $\lambda_1\ne\lambda_2$ and $\boldsymbol{x}\ne\boldsymbol{0}$, we derive $c_2=0$. Similarly, if we let Eq.(7.10)$\times\lambda_1-$Eq.(7.11) to cancel c_2, then we get $c_1=0$.

Therefore, $(c_1,c_2)=\boldsymbol{0}$ leads to a contradiction!

- How to proof this statement for general k?

Assume there exists $(c_1,\dots,c_k) \neq \mathbf{0}$ s.t.

$$c_1\boldsymbol{x}_1 + \cdots + c_k\boldsymbol{x}_k = \mathbf{0} \tag{7.12}$$

Then we obtain two equations from Eq.(7.12):

$$\boldsymbol{A}(c_1\boldsymbol{x}_1 + \cdots + c_k\boldsymbol{x}_k) = c_1\lambda_1\boldsymbol{x}_1 + c_2\lambda_2\boldsymbol{x}_2 + \cdots + c_k\lambda_k\boldsymbol{x}_k = \mathbf{0}. \tag{7.13}$$

$$\lambda_k(c_1\boldsymbol{x}_1 + \cdots + c_k\boldsymbol{x}_k) = c_1\lambda_k\boldsymbol{x}_1 + c_2\lambda_k\boldsymbol{x}_2 + \cdots + c_k\lambda_k\boldsymbol{x}_k = \mathbf{0}. \tag{7.14}$$

We can let Eq.(7.13)−Eq.(7.14) to cancel $\boldsymbol{x}_k$:

$$c_1(\lambda_1 - \lambda_k)\boldsymbol{x}_1 + \cdots + c_k(\lambda_{k-1} - \lambda_k)\boldsymbol{x}_{k-1} = \mathbf{0}. \tag{7.15}$$

By repeatedly applying the trick from (7.12) to (7.15), we can show that

$$c_1(\lambda_1 - \lambda_k)\dots(\lambda_1 - \lambda_2)\boldsymbol{x}_1 = \mathbf{0} \qquad \text{which forces } c_1 = 0.$$

Similarly every $c_i = 0$ for $i = 1,\dots,n$. Here is the contradiction!

■

Corollary 7.1 If all eigenvalues of $\boldsymbol{A}$ are *distinct*, then $\boldsymbol{A}$ is *diagonalizable*

7.2.4. Powers of $\boldsymbol{A}$

Matrix Powers. If $\boldsymbol{A} = \boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{S}$, then $\boldsymbol{A}^2 = (\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{S})(\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{S}) = \boldsymbol{S}^{-1}\boldsymbol{D}^2\boldsymbol{S}$.

In general, $\boldsymbol{A}^k = (\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{S})\dots(\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{S}) = \boldsymbol{S}^{-1}\boldsymbol{D}^k\boldsymbol{S}$.

Eigenvalues of matrix powers. We may ask if eigenvalues of $\boldsymbol{A}$ are $\lambda_1,\dots,\lambda_n$, then what is the eigenvalues of $\boldsymbol{A}^k$? The answer is intuitive, the eigenvalues of $\boldsymbol{A}^k$ are $\lambda_1^k,\dots,\lambda_n^k$. However, you may use the wrong way to prove this statement:

Proposition 7.7 If eigenvalues of $n \times n$ matrix $\boldsymbol{A}$ are $\lambda_1,\dots,\lambda_n$, then eigenvalues of $\boldsymbol{A}^k$ are $\lambda_1^k,\dots,\lambda_n^k$.

Wrong proof 1: Assume $\boldsymbol{A} = \boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{S}$, then $\boldsymbol{A}^k = \boldsymbol{S}^{-1}\boldsymbol{D}^k\boldsymbol{S}$. Suppose $\boldsymbol{D} = \text{diag}(\lambda_1,\dots,\lambda_n)$, then $\boldsymbol{D}^k = \text{diag}(\lambda_1^k,\dots,\lambda_n^k)$. Hence eigenvalues of $\boldsymbol{A}^k$ are $\lambda_1^k,\dots,\lambda_n^k$.

This proof is wrong, because $\boldsymbol{A}$ may not be *diagonalizable*, which means $\boldsymbol{A}$ may not have the form $\boldsymbol{A} = \boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{S}$. ■

Wrong proof 2: If $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, then $\boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A}(\lambda\boldsymbol{x}) = \lambda(\boldsymbol{A}\boldsymbol{x}) = \lambda^2\boldsymbol{x}$.
Hence for general k, $\boldsymbol{A}^k\boldsymbol{x} = \lambda^k\boldsymbol{x}$.

This proof only states that if λ is the eigenvalue of $\boldsymbol{A}$, then λ^k is the eigenvalues of $\boldsymbol{A}^k$. Unfortunately, it still cannot derive this proposition. Because it does not prove that if λ are the eigenvalues with multiplicity m, then λ^k are the eigenvalues of $\boldsymbol{A}^k$ with multiplicity m.
Let's raise a counterexample: Let eigenvalues of $\boldsymbol{A}$ be $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2$; the eigenvalues of $\boldsymbol{A}^2$ could be $1^2, 2^2, 2^2$. Hence $\boldsymbol{A}$ has the eigenvalues 1 with multiplicity 2; while $\boldsymbol{A}^2$ has the eigenvalue 1^2 with multiplicity 1. So this $\boldsymbol{A}$ and $\boldsymbol{A}^2$ is a contradiction for this proof. In other words, this proof fails to determine the multiplicity of eigenvalues. ■

R The proposition(7.7) could be proved using **Jordan form**, i.e., for any matrix $\boldsymbol{A}$ there exists invertible matrix $\boldsymbol{S}$ such that $\boldsymbol{A} = \boldsymbol{S}^{-1}\boldsymbol{U}\boldsymbol{S}$, where $\boldsymbol{U}$ is an upper triangular matrix with diagonal entries $\lambda_1,\dots,\lambda_n$. Then $\boldsymbol{A}^k = \boldsymbol{S}^{-1}\boldsymbol{U}^k\boldsymbol{S}$, where $\boldsymbol{U}^k$ is an upper triangular matrix with diagonal entries $\lambda_1^k,\dots,\lambda_n^k$. Hence the eigenvalues of $\boldsymbol{A}^k$ are $\lambda_1^k,\dots,\lambda_n^k$.

7.2.5. Nondiagonalizable Matrices

Sometimes we face some matrices that have too few eigenvalues. (don't count with multiplicity)

For example, given $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, it's easy to verify that its eigenvalue is $\lambda = 0$ and eigenvectors are of the form $\boldsymbol{x} = \begin{bmatrix} c \\ 0 \end{bmatrix}$.

This 2×2 matrix cannot be diagonalized. Why? Let's introduce a definition first:

Definition 7.5 [Eigenspace] Suppose $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ has k distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Then the eigenspace for $\boldsymbol{A}$ associated with λ_i is the collection of all eigenvectors associated with the eigenvalue λ_i, i.e., the null space $N(\lambda_i \boldsymbol{I} - \boldsymbol{A})$. ■

Why does this 2 by 2 matrix $\boldsymbol{A}$ cannot be diagonalizable? Because the the dimension of its eigenspace is too small, i.e., eigenspace$(\boldsymbol{A}, \lambda = 0) = 1 < 2$. In general, if the dimension of eigenspace associated with the eigenvalue λ_i is less than the multiplicity of this eigenvalue, then this matrix cannot be diagonalizable. We will discuss it in the next lecture.