A FIRST COURSE

IN

LINEAR ALGEBRA

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MAT2040 Notebook

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Foreword

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Preface

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Acronyms

ASTA	Arrivals See Time Averages
BHCA	Busy Hour Call Attempts
BR	Bandwidth Reservation
b.u.	bandwidth unit(s)
CAC	Call / Connection Admission Control
CBP	Call Blocking Probability(-ies)
CCS	Centum Call Seconds
CDTM	Connection Dependent Threshold Model
CS	Complete Sharing
DiffServ	Differentiated Services
EMLM	Erlang Multirate Loss Model
erl	The Erlang unit of traffic-load
FIFO	First in - First out
GB	Global balance
GoS	Grade of Service
ICT	Information and Communication Technology
IntServ	Integrated Services
IP	Internet Protocol
ITU-T	International Telecommunication Unit – Standardization sector
LB	Local balance
LHS	Left hand side

LIFO	Last in - First out
MMPP	Markov Modulated Poisson Process
MPLS	Multiple Protocol Labeling Switching
MRM	Multi-Retry Model
MTM	Multi-Threshold Model
PASTA	Poisson Arrivals See Time Averages
PDF	Probability Distribution Function
pdf	probability density function
PFS	Product Form Solution
QoS	Quality of Service
r.v.	random variable(s)
RED	random early detection
RHS	Right hand side
RLA	Reduced Load Approximation
SIRO	service in random order
SRM	Single-Retry Model
STM	Single-Threshold Model
TCP	Transport Control Protocol
TH	Threshold(s)
UDP	User Datagram Protocol

2.2. Wednesday

2.2.1. Remarks on Gaussian Elimination

Gaussian Elimination to compute $\boldsymbol{A}^{-1}$ is equivalent to solving n linear systems $\boldsymbol{A}\boldsymbol{x}_i = e_i$, $i = 1, 2, \dots, n$.

Computing Complexity. For each i solving $\boldsymbol{A}\boldsymbol{x_i} = e_i$ takes $O(n^3)$ operations.

- Hence, solving these systems one by one take $O(n^4)$ time.
- However, if we solve $\boldsymbol{A}\boldsymbol{x_i} = e_i$ for $i = 1, 2, \dots, n$ simultaneously (that means we write all b_i at the right side of the Augmented matrix), by Gaussian Elimination, it only takes $O(n^3)$ operations.

Large Scale Inverse Computation. Gaussian Elimination is not a good job for large scale sparse matrix (**sparse matrix** is a matrix in which most of the elements are zero. If given a 1000×1000 sparse matrix, it is expensive to do Gaussian Elimination on this matrix).

Actually, for such matrix we use *iterative method* to solve it.

Gaussian Elimination is just a sequence of matrix multiplications. Given nonsingular matrix $\boldsymbol{A}$, Gaussian Elimination is really a sequence of multiplications by elementary matrices $\boldsymbol{E}$'s and permutation matrix $\boldsymbol{P}$:

$$\boldsymbol{E} \cdots \boldsymbol{E}\boldsymbol{P}\boldsymbol{A} = \boldsymbol{U},$$

where $\boldsymbol{U}$ is an upper triangular matrix.

By postmultiplying $\boldsymbol{U}^{-1}$ we obtain

$$\boldsymbol{U}^{-1}(\boldsymbol{E} \dots \boldsymbol{E}\boldsymbol{P}\boldsymbol{A}) = \boldsymbol{I} \implies (\boldsymbol{U}^{-1}\boldsymbol{E} \dots \boldsymbol{E}\boldsymbol{P})\boldsymbol{A} = \boldsymbol{I}.$$

Furthermore, we could decompose $\boldsymbol{A}$ as the product of a permutation matrix, a lower triangular matrix and an upper triangular matrix:

$$A = P^{-1}(E^{-1} \dots E^{-1})U$$

2.2.2. Properties of matrix

1. If A is a diagonal matrix which is given by

$$A = \begin{bmatrix} d_1 & & 0 \\ & \vdots & \\ 0 & & d_n \end{bmatrix},$$

and $d_1 d_2 d_3 \dots d_n \neq 0$, then A^{-1} exists, and $A^{-1} = \begin{bmatrix} d_1^{-1} & & 0 \\ & \vdots & \\ 0 & & d_n^{-1} \end{bmatrix}$.

2. If D_1, D_2 are diagonal and their product exists, then we have

$$D_1 D_2 = D_2 D_1$$

3. If A, B are both invertible, then AB is also invertible. The inverse of product AB is

$$(AB)^{-1} = B^{-1} A^{-1}$$

Proofoutline. To see why the order is reversed, firstly multiply AB with $B^{-1}A^{-1}$:

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Similarly, $B^{-1}A^{-1}$ times AB leads to the same result. Hence we draw the conclusion: Inverse come in reverse order. ■

4. The same reverse order applies to three or more matrix:
If A, B, C are nonsingular, then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

5. It's hard to say whether $(\boldsymbol{A}+\boldsymbol{B})$ is invertible, but we have an interesting property:

   When $\boldsymbol{A}$ is "small" (we will explain it later), we have $\quad (\boldsymbol{I}-\boldsymbol{A})^{-1} = \sum_{i=1}^{\infty} \boldsymbol{A}^i$

6. **A triangular matrix is invertible if and only if no diagonal entries are zero.**

   In order to explain it, let's discuss an example:

   ■ **Example 2.2**

   We want to find the inverse of a lower triangular matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

   Thus we do Gaussian Elimination to compute solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{I}$:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

   This result is obtained by three row operations:

   (a) "Add $(-1)\times$ row 3 to row 4";

   (b) "Add $(-1)\times$ row 2 to row 3";

   (c) "Add $(-1)\times$ row 1 to row 2".

   ■

   Proof. Only for a nonzero diagonal lower triangular matrix, we can continue the Gaussian Elimination to convert it into identity matrix. ■

7. Given an invertible lower triangular matrix $\boldsymbol{A}$, the inverse of $\boldsymbol{A}$ remains lower triangular.

8. The LDU decomposition is unique for an invertible matrix. (We assume the existence of the LDU decomposition).

Proof. • Assume the invertible matrix $\boldsymbol{A}$ could be decomposed as:

$$\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1 = \boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2$$

• By aftermultiplying $\boldsymbol{U}_1^{-1}$ and postmultiplying $\boldsymbol{L}_2^{-1}$ for the latter equation, we obtain:

$$\boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1 = \boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2 \implies \boldsymbol{L}_2^{-1}\boldsymbol{L}_1\boldsymbol{D}_1 = \boldsymbol{D}_2\boldsymbol{U}_2\boldsymbol{U}_1^{-1} \tag{2.1}$$

• Note that $\boldsymbol{L}_2^{-1}\boldsymbol{L}_1$ remains lower triangular with unit diagonal, thus $\boldsymbol{L}_2^{-1}\boldsymbol{L}_1\boldsymbol{D}_1$ must be lower triangular matrix. Similarly, $\boldsymbol{D}_2\boldsymbol{U}_2\boldsymbol{U}_1^{-1}$ must be upper triangular matrix. Hence $\boldsymbol{L}_2^{-1}\boldsymbol{L}_1\boldsymbol{D}_1$ and $\boldsymbol{D}_2\boldsymbol{U}_2\boldsymbol{U}_1^{-1}$ must be *diagonal* matrix due to equality (2.1).

• Note that the diagonal of $\boldsymbol{L}_2^{-1}\boldsymbol{L}_1\boldsymbol{D}_1$ is the same as the diagonal of $\boldsymbol{D}_1$ since $\boldsymbol{L}_2^{-1}\boldsymbol{L}_1$ has unit diagonal. Hence

$$\boldsymbol{L}_2^{-1}\boldsymbol{L}_1\boldsymbol{D}_1 = \boldsymbol{D}_1. \tag{2.2}$$

Similarly,

$$\boldsymbol{D}_2\boldsymbol{U}_2\boldsymbol{U}_1^{-1} = \boldsymbol{D}_2. \tag{2.3}$$

Combining (2.1) to (2.3), we derive $\boldsymbol{D}_1 = \boldsymbol{D}_2$.

• Furthermore,

$$\boldsymbol{L}_2^{-1}\boldsymbol{L}_1\boldsymbol{D}_1 = \boldsymbol{D}_1 \implies \boldsymbol{L}_2^{-1}\boldsymbol{L}_1 = \boldsymbol{I} \implies \boldsymbol{L}_1 = \boldsymbol{L}_2$$

Similarly, $\boldsymbol{U}_1 = \boldsymbol{U}_2$.

■

2.2.3. matrix transpose

We introduce a new matrix, it is the **transpose** of $\boldsymbol{A}$:

Definition 2.3 [Transpose] The **transpose** of matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is denoted as $\boldsymbol{A}^{\mathrm{T}}$. The columns of $\boldsymbol{A}^{\mathrm{T}}$ are the rows of $\boldsymbol{A}$, i.e., $\boldsymbol{A}^{\mathrm{T}}$ means that

$$\boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{n\times n}$$

For example,

- given a column vector $x \in \mathbb{R}^n$, the transpose $x^{\mathrm{T}} = (x_1, x_2, \dots, x_n)$ is row vector.
- When $\boldsymbol{A}$ is $m \times n$ matrix, the transpose is $n \times m$:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \qquad \boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 3 \end{bmatrix} \qquad (\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{A}$$

The entry in row i, column j of $\boldsymbol{A}^{\mathrm{T}}$ comes from row j, column i of the original matrix $\boldsymbol{A}$:

$$\textbf{Exchange rows and columns} \quad (\boldsymbol{A}^{\mathrm{T}})_{ij} = \boldsymbol{A}_{ji}$$

The rules for transposes are very direct:

Proposition 2.4

- **Sum** The transpose of $\boldsymbol{A} + \boldsymbol{B}$ is $\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}$.
- **Product** The transpose of $\boldsymbol{AB}$ is $(\boldsymbol{AB})^{\mathrm{T}} = (\boldsymbol{B})^{\mathrm{T}}(\boldsymbol{A})^{\mathrm{T}}$.

Proofoutline of Product Rule.

- We start with $(\boldsymbol{A}x)^{\mathrm{T}} = x^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$, where x refers to a vector:

  $\boldsymbol{A}x$ combines the columns of $\boldsymbol{A}$; while $x^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$ combines the rows of $\boldsymbol{A}^{\mathrm{T}}$.

  Since they are the same combinations of the same vectors, we obtain $(\boldsymbol{A}x)^{\mathrm{T}} = x^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$.

- Now we can prove the formula $(\boldsymbol{AB})^{\mathrm{T}} = (\boldsymbol{B})^{\mathrm{T}}(\boldsymbol{A})^{\mathrm{T}}$, where $\boldsymbol{B}$ has several columns:

  Assuming $\boldsymbol{B} = \begin{bmatrix} b_1 & | & b_2 & | & \dots & | & b_k \end{bmatrix}$, then Transposing $\boldsymbol{AB} = \begin{bmatrix} \boldsymbol{A}b_1 & | & \boldsymbol{A}b_2 & | & \dots & | & \boldsymbol{A}b_k \end{bmatrix}$ gives

$$(\boldsymbol{AB})^{\mathrm{T}} = \begin{bmatrix} b_1^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} \\ b_2^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} \\ \vdots \\ b_k^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} \end{bmatrix},$$

  which is actually $\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$.

■

2.2.3.1. symmetric matrix

For a *symmetric matrix*, transposing $\boldsymbol{A}$ into $\boldsymbol{A}^{\mathrm{T}}$ makes no change.

Definition 2.4 [symmetric matrix] A matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is **symmetric matrix** if we have $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$. This means that $a_{ij} = a_{ji}$ for all i, j. We usually denote it as $\boldsymbol{A} \in \mathbb{S}^{n\times n}$.

Choose any matrix $\boldsymbol{A}$ (probably rectangular), then postmultiplying $\boldsymbol{A}^{\mathrm{T}}$ for $\boldsymbol{A}$ automatically leads to a square symmetric matrix:

The transpose of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is $\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}$, which is $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

The matrix $\boldsymbol{AA}^{\mathrm{T}}$ is also symmetric. But note that $\boldsymbol{AA}^{\mathrm{T}}$ is a different matrix from $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

R For two vector x and y,

- **The dot product or inner product is denoted as** $x^{\mathrm{T}}y$
- **The rank one product or outer product is denoted as** xy^{T}

$x^{\mathrm{T}}y$ is a number while xy^{T} is a matrix.

We introduce a matrix that seems opposite to symmetric matrix:

Definition 2.5 [Skew-symmetric] For matrix $\boldsymbol{A}$, if we have $\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A}$, then we say $\boldsymbol{A}$ is **skew-symmetric** or **anti-symmetric**. ■

Moreover, any $n \times n$ matrix can be decomposed as the summation of a symmetric and a skew-symmetric matrix. Let's prove it in the next lecture.