A FIRST COURSE

IN

LINEAR ALGEBRA

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MAT2040 Notebook

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Foreword

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Preface

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Acronyms

ASTA	Arrivals See Time Averages
BHCA	Busy Hour Call Attempts
BR	Bandwidth Reservation
b.u.	bandwidth unit(s)
CAC	Call / Connection Admission Control
CBP	Call Blocking Probability(-ies)
CCS	Centum Call Seconds
CDTM	Connection Dependent Threshold Model
CS	Complete Sharing
DiffServ	Differentiated Services
EMLM	Erlang Multirate Loss Model
erl	The Erlang unit of traffic-load
FIFO	First in - First out
GB	Global balance
GoS	Grade of Service
ICT	Information and Communication Technology
IntServ	Integrated Services
IP	Internet Protocol
ITU-T	International Telecommunication Unit – Standardization sector
LB	Local balance
LHS	Left hand side

LIFO	Last in - First out
MMPP	Markov Modulated Poisson Process
MPLS	Multiple Protocol Labeling Switching
MRM	Multi-Retry Model
MTM	Multi-Threshold Model
PASTA	Poisson Arrivals See Time Averages
PDF	Probability Distribution Function
pdf	probability density function
PFS	Product Form Solution
QoS	Quality of Service
r.v.	random variable(s)
RED	random early detection
RHS	Right hand side
RLA	Reduced Load Approximation
SIRO	service in random order
SRM	Single-Retry Model
STM	Single-Threshold Model
TCP	Transport Control Protocol
TH	Threshold(s)
UDP	User Datagram Protocol

2.4. Friday

2.4.1. symmetric matrix

Definition 2.5 [symmetric matrix] A $n \times n$ matrix $\boldsymbol{A}$ is a **symmetric matrix** if we have $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$, which means $a_{ij} = a_{ji}$ for all i, j.

For example, the matrix $\boldsymbol{A}$ shown below is a symmetric matrix:

$$\textbf{symmetric matrix} \quad \boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \boldsymbol{A}^{\mathrm{T}}$$

Definition 2.6 [skew-symmetric matrix] A $n \times n$ matrix $\boldsymbol{A}$ is a **skew-symmetric matrix** or say, **anti-symmetric matrix** if we have $\boldsymbol{A} = -\boldsymbol{A}^{\mathrm{T}}$.

For example, matrix $\boldsymbol{B}$ shown below is a skew-symmetric matrix:

$$\textbf{skew-symmetric matrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\boldsymbol{B}^{\mathrm{T}}$$

Theorem 2.3 Any $n \times n$ matrix can be decomposed as the sum of a *symmetric* and a *skew-symmetric* matrix.

Proofoutline. Given any $n \times n$ matrix $\boldsymbol{A}$, we can write $\boldsymbol{A}$ as:

$$\boldsymbol{A} = \underbrace{\frac{\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}}{2}}_{\text{symmetric}} + \underbrace{\frac{\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}}{2}}_{\text{skew-symmetric}}$$

■

2.4.2. Interaction of inverse and transpose

Proposition 2.5 If A exists, then A^{T} also exists, and $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$.

Proof.

$$(A^{-1}A)^{\mathrm{T}} = A^{\mathrm{T}}(A^{-1})^{\mathrm{T}} = I \implies (A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$$

■

Corollary 2.1 If matrix A is symmetric and invertible, then A^{-1} remains symmetric.

Proof.

$$(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} = A^{-1} \implies A^{-1} \text{ is symmetric.}$$

■

Proposition 2.6 If $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then $M^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{bmatrix}$.

Corollary 2.2 Given matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, matrix M is symmetric if and only if

$$A = A^{\mathrm{T}}, D = D^{\mathrm{T}}, B^{\mathrm{T}} = C.$$

Proposition 2.7 Suppose A is invertible and symmetric. When we do LDU decomposition such that $A = LDU$, U is exactly L^{T}.

Proofoutline. Note that

$$A^{\mathrm{T}} = (LDU)^{\mathrm{T}} = U^{\mathrm{T}}D^{\mathrm{T}}L^{\mathrm{T}} = A = LDU.$$

Since D is diagonal matrix, we have $D = D^{\mathrm{T}}$. It follows that

$$U^{\mathrm{T}}DL^{\mathrm{T}} = LDU = A.$$

Since $\boldsymbol{U}^{\mathrm{T}}$ is also a lower triangular matrix, $\boldsymbol{L}^{\mathrm{T}}$ is also an upper triangular matrix, $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{L}^{\mathrm{T}}$ is also the LDU decomposition of $\boldsymbol{A}$.

Due to the uniqueness of LDU decomposition, we obtain $\boldsymbol{U}^{\mathrm{T}} = \boldsymbol{L}, \boldsymbol{L}^{\mathrm{T}} = \boldsymbol{U}$. ■

2.4.3. Vector Space

We move to a new topic: vector spaces.

From Numbers to Vectors. We know matrix calculation(such as $\boldsymbol{A}x = \boldsymbol{b}$) involves many numbers, but they are just linear combinations of n vectors.

Third Level Undetstanding. This topic moves from numbers and vectors to a third level of understanding (the highest level). Instead of individual column vectos, we look at "*spaces*" of vectors. And this topic will end with the "*Fundamental Theorem of Linear Algebra*".

$$\text{Matrix Calculation:} \quad \text{Numbers} \Longrightarrow \text{Vectos} \Longrightarrow \textbf{Spaces}$$

We begin with the typical vector space, which is denoted as $\mathbb{R}^n$.

Definition 2.7 [Real Space] The space $\mathbb{R}^n$ contains all column vectors v such that v has n real number entries.

Notation. We denote vectors as *a column between brackets*, or *along a line using commas and parentheses:*

$$\begin{bmatrix} 4 \\ \pi \end{bmatrix} \text{ is in } \mathbb{R}^2 \quad (1,1,1) \text{ is in } \mathbb{R}^3.$$

Definition 2.8 [vector space] A **vector space** $\boldsymbol{V}$ is a set of vectors such that these vectors satisfy *vector addition* and *scalar multiplication*:

- **vector addition:**If vector v and w is in $\boldsymbol{V}$, then $v + w \in \boldsymbol{V}$.
- **scalar multiplication:**If vector $v \in \boldsymbol{V}$, then $cv \in \boldsymbol{V}$ for any real numbers c.

In other words, the set of vectors is **closed** under *addition* $v + w$ and *multiplication* cv.

In other words,

any linear combination is closed under vector space.

Proposition 2.8 Every vector space must contain the zero vector.

Proof. Given $v \in \boldsymbol{V} \implies -v \in \boldsymbol{V} \implies v + (-v) = \mathbf{0} \in \boldsymbol{V}$. ■

■ **Example 2.3**

$$\boldsymbol{V} = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \end{pmatrix} \middle| \{a_n\} \text{ is infinite length sequences.} \right\}$$

is a vector space.

This is because for any vector $v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \end{pmatrix}, w = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \\ \vdots \end{pmatrix}$, we can define vector addition and scalar multiplication as follows:

$$v + w = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \\ \vdots \end{pmatrix} \quad cv = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \\ \vdots \end{pmatrix} \text{ for any } c \in \mathbb{R}.$$

$$V = \operatorname{span}\left\{ v_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \vdots \\ \frac{1}{2^n} \\ \vdots \end{pmatrix}, v_2 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{9} \\ \vdots \\ \frac{1}{3^n} \\ \vdots \end{pmatrix}, v_3 = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{16} \\ \vdots \\ \frac{1}{4^n} \\ \vdots \end{pmatrix} \right\}$$

$$= \{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$$

is also vector space.

Definition 2.9 [Span] The **span** of a collection of vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in \mathbb{R}^m$ is defined as:

$$\operatorname{span}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\} = \left\{ \boldsymbol{y} \in \mathbb{R}^m \,\middle|\, \boldsymbol{y} = \sum_{i=1}^{n} \alpha_i \boldsymbol{a}_i, \boldsymbol{\alpha} \in \mathbb{R}^n \right\},$$

i.e., it is the set of all linear combinations of $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$.

How to check $\boldsymbol{V}$ is a vector space?

Given any two vectors u, w in $\boldsymbol{V}$, suppose

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, \quad v = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3,$$

then we obtain:

$$\begin{aligned} \gamma_1 u + \gamma_2 v &= \gamma_1(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3) + \gamma_2(\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3) \\ &= (\gamma_1 \alpha_1 + \gamma_2 \beta_1) v_1 + (\gamma_1 \alpha_2 + \gamma_2 \beta_2) v_2 + (\gamma_1 \alpha_3 + \gamma_2 \beta_3) v_3 \end{aligned}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$. Hence any linear combination of u and w are also in $\boldsymbol{V}$. Hence $\boldsymbol{V}$ is a vector space. ■

■ **Example 2.4** $\boldsymbol{F} = \{f(x) \mid f : [0,1] \mapsto \mathbb{R}\}$ is also a vector space. (verify it by yourself.)

This vector space $\boldsymbol{F}$ contains all real functions defined on $[0,1]$, an it is infinite dimensional.

Given two functions f and g in $\boldsymbol{F}$, the inner product of f and g is defined as:

$$\langle f, g \rangle := \int_0^1 f(x)g(x)\,\mathrm{d}x$$

Also, we can use the span to form a vector space:

$$\boldsymbol{F} = \operatorname{span}\{sinx, x^3, e^x\} = \{\alpha_1 sinx + \alpha_2 x^3 + \alpha_3 e^x \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}.\}$$

This set $\boldsymbol{F}$ is also a vector space.

Example 2.5

$$\boldsymbol{V} = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \middle| a_{ij} \in \mathbb{R} \text{ for } i = 1,2; j = 1,2,3. \right\}$$

is a vector space. Moreover, it is equivalent to the span of six basic vectors:

$$\boldsymbol{V} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

We say that $\boldsymbol{V}$ is 6-dimensional without introducing the definiton of dimension formally.

Example 2.6

$$\boldsymbol{V} = \left\{ \begin{bmatrix} a_{ij} \end{bmatrix}_{3\times 3} \middle| \text{any } 3\times 3 \text{ matrices} \right\}$$

is also a vector space.

Obviously, it is 9-dimensional. We usually denote it as $\dim(\boldsymbol{V}) = 9$.

$$\boldsymbol{V}_1 = \left\{ \begin{bmatrix} a_{ij} \end{bmatrix}_{3\times 3} \middle| \text{any } 3\times 3 \text{ symmetric matrices} \right\}$$

is a special vector space.

Notice that $\boldsymbol{V}_1 \subset \boldsymbol{V}$, so we say $\boldsymbol{V}_1$ is a *subspace* of $\boldsymbol{V}$. In the future we will know $\dim(\boldsymbol{V}_1) = 6 < 9$.

2.4.3.1. The solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$

We can use vector space to discuss the solution to system of equation. Firstly, let's introduce some definitions:

Definition 2.10 [homogeneous equations] A system of linear equations is said to be **homogeneous** if the constants on the righthand side are all zero. In other words, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ is said to be **homogeneous**.

Definition 2.11 [column space] The column space consists of all linear combinations of the columns of matrix $\boldsymbol{A}$. In other words, for the matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ given by $\boldsymbol{A} = \begin{bmatrix} a_1 & \vert & a_2 & \vert & \dots & \vert & a_n \end{bmatrix}$, its column space is denoted as

$$\boldsymbol{C}(\boldsymbol{A}) := \operatorname{span}(a_1, a_2, \dots, a_n) \subset \mathbb{R}^m.$$

Definition 2.12 [null space] The null space of a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ consists of all solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, which can be denoted as

$$\boldsymbol{N}(\boldsymbol{A}) = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\} \subset \mathbb{R}^n.$$

Proposition 2.9 The null space $\boldsymbol{N}(\boldsymbol{A})$ is a vector space.

Proofoutline. For any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{N}(\boldsymbol{A})$, we have $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{A}\boldsymbol{y} = \boldsymbol{0}$.

$$\implies \boldsymbol{A}(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = \alpha(\boldsymbol{A}\boldsymbol{x}) + \beta(\boldsymbol{A}\boldsymbol{y}) = \alpha\boldsymbol{0} + \beta\boldsymbol{0} = \boldsymbol{0} \quad \alpha, \beta \in \mathbb{R}.$$

Since the linear combination of $\boldsymbol{x}$ and $\boldsymbol{y}$ is also in $\boldsymbol{N}(\boldsymbol{A})$, $\boldsymbol{N}(\boldsymbol{A})$ is a vector space. ■

■ **Example 2.7** Describe the null space of $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 5 & 0 \\ 2 & 3 \end{bmatrix}$.

Obviously, converting matrix into linear system of equation we obtain:

$$\begin{cases} x_1 + 0x_2 = 0 \\ 5x_1 + 4x_2 = 0 \\ 2x_1 + 3x_2 = 0 \end{cases}$$

We can easily obtain the solution $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$. Hence the null space is $\boldsymbol{N}(\boldsymbol{A}) = \boldsymbol{0}$. ■

■ **Example 2.8** Describe the null space of $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}$.

In the next lecture we will know its null space is a line.

We find that $\boldsymbol{A} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \boldsymbol{0}$, so $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is a special solution.

Note that *the null space contains all linear combinations of special solutions.* Hence the null space is $\boldsymbol{N}(\boldsymbol{A}) = \left\{ c \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \middle| c \in \mathbb{R} \right\}$. ■

2.4.3.2. The complete solution to $\boldsymbol{Ax} = \boldsymbol{b}$

In order to find all solutions of $\boldsymbol{Ax} = \boldsymbol{b}$, ($\boldsymbol{A}$ may not be square matrix), let's introduce two kinds of solutions:

Definition 2.13 [Particular & Special Solution] For the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$, there are two kinds of solutions:

$\boldsymbol{x}_{\text{particular}}$ **The particular solution that solves $\boldsymbol{Ax} = \boldsymbol{b}$**

$\boldsymbol{x}_{\text{nullspace}}$ **The special solutions that solves $\boldsymbol{Ax} = \boldsymbol{0}$**

There is a theorem that helps us to obtain the complete solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Theorem 2.4 Any solution to $\boldsymbol{Ax} = \boldsymbol{b}$ can be represented as $\boldsymbol{x}_{complete} = \boldsymbol{x_p} + \boldsymbol{x_n}$.

Proof. *Sufficiency.* Given $\boldsymbol{x}_{complete} = \boldsymbol{x_p} + \boldsymbol{x_n}$, it suffices to show $\boldsymbol{x}_{complete}$ is the solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Note that

$$\boldsymbol{Ax}_{complete} = \boldsymbol{A}(\boldsymbol{x_p} + \boldsymbol{x_n}) = \boldsymbol{Ax_p} + \boldsymbol{Ax_n} = \boldsymbol{b} + \boldsymbol{0} = \boldsymbol{b}.$$

Hence $\boldsymbol{x}_{complete}$ is the solution to $\boldsymbol{Ax} = \boldsymbol{b}$. ■

Necessity. Suppose $\boldsymbol{x}^*$ is the solution to $\boldsymbol{Ax} = \boldsymbol{b}$, it suffices to show $\boldsymbol{x}^*$ could be represented as $\boldsymbol{x_p} + \boldsymbol{x_n}$.

It suffices to show $\boldsymbol{x}^* - \boldsymbol{x_p} \in \boldsymbol{N}(\boldsymbol{A})$.

Notice that $\boldsymbol{A}(\boldsymbol{x}^* - \boldsymbol{x_p}) = \boldsymbol{Ax}^* - \boldsymbol{Ax_p} = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0} \implies \boldsymbol{x}^* - \boldsymbol{x_p} \in \boldsymbol{N}(\boldsymbol{A})$. ■

■

■ **Example 2.9** Let's study a system that has $n = 2$ unknowns but only $m = 1$ equation:

$$x_1 + x_2 = 2.$$

It's easy to check that the particular solution is $\boldsymbol{x_p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the special solutions are $\boldsymbol{x_n} = c\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, c can be taken arbitararily.

Hence the complete solution for the equations could be written as

$$\boldsymbol{x}_{complete} = \boldsymbol{x_p} + \boldsymbol{x_n} = \begin{pmatrix} c+1 \\ -c+1 \end{pmatrix}.$$

So we summarize that if there are n unknowns and m equations such that $m < n$, then $\boldsymbol{Ax} = \boldsymbol{b}$ is **underdetermined** (It may have infinitely many solutions since the special solutions could be infinite).

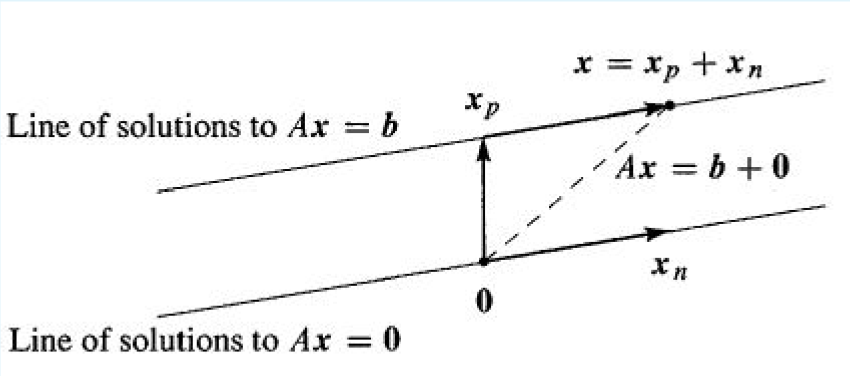

Figure 2.1: Complete solution = one particular solution + all nullspace solutions

2.4.3.3. Row-Echelon Matrices

Given $m \times n$ rectangular matrix $\boldsymbol{A}$, we can still do Gaussian Elimination to convert $\boldsymbol{A}$ into $\boldsymbol{U}$, where $\boldsymbol{U}$ is of **Row Echelon form**. The whole process could be expressed as:

$$\boldsymbol{PA} = \boldsymbol{LDU}.$$

where $\boldsymbol{L}$ is $m \times m$ **lower triangular** matrix, $\boldsymbol{U}$ is $m \times n$ matrix that is of **row echelon form**.

Example 2.10 Here is a 4×7 row echelon matrix with the three pivots $\mathbf{1}$ highlighted in blue:

$$U = \begin{bmatrix} \boxed{1} & \times & \times & \times & \times & \times & \times \\ 0 & \boxed{1} & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Columns 3,4,5,7 have no pivots, and we say the free variables are x_3, x_4, x_5, x_7.
- Columns 1,2,6 have pivots, and we say the pivot variables are x_1, x_2, x_6.

Moreover, we can continue Gaussian Elimination to convert $\boldsymbol{U}$ into $\boldsymbol{R}$ that is of **reduced row echelon form**:

$$R = \begin{bmatrix} \boxed{1} & 0 & \times & \times & \times & 0 & \times \\ 0 & \boxed{1} & \times & \times & \times & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced row echelon matrix $\boldsymbol{R}$ has zeros above the pivots as well as below. Zeros above the pivots come from upward elimination. ■

R Remember the two steps (forward and back elimination) in solving $\boldsymbol{Ax} = \boldsymbol{b}$:

1. **Forward Elimination** takes $\boldsymbol{A}$ to $\boldsymbol{U}$. (or its reduced form $\boldsymbol{R}$)
2. **Back Elimination** in $\boldsymbol{Ux} = \boldsymbol{c}$ or $\boldsymbol{Rx} = \boldsymbol{d}$ produces $\boldsymbol{x}$.

2.4.3.4. Problem Size Analysis

When faced with $m \times n$ matrix $\boldsymbol{A}$, notice that $\boldsymbol{m}$ refers to the **number of equations**, $\boldsymbol{n}$ refers to the **number of variables**. Assume $\boldsymbol{r}$ denotes **number of pivots**, then we know $\boldsymbol{r}$ is also the **number of pivot variables**, $\boldsymbol{n} - \boldsymbol{r}$ is the **number of free variables**. Finally we have $\boldsymbol{m} - \boldsymbol{r}$ **redundant equations** and $\boldsymbol{r}$ **irredundant equations**. In next lecture, we will introduce the definition for $\boldsymbol{r}$ formally (rank).