A FIRST COURSE

IN

LINEAR ALGEBRA

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LINEAR ALGEBRA

MAT2040 Notebook

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Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

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Library of Congress Cataloging-in-Publication Data:

Survey Methodology / Robert M. Groves . . . [et al.].
p. cm.—(Wiley series in survey methodology)
"Wiley-Interscience."
Includes bibliographical references and index.
ISBN 0-471-48348-6 (pbk.)
1. Surveys—Methodology. 2. Social sciences—Research—Statistical methods. I. Groves, Robert M. II. Series.

HA31.2.S873 2004
001.4'33—dc22 2004044064
Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

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Foreword

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Preface

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Acronyms

ASTA	Arrivals See Time Averages
BHCA	Busy Hour Call Attempts
BR	Bandwidth Reservation
b.u.	bandwidth unit(s)
CAC	Call / Connection Admission Control
CBP	Call Blocking Probability(-ies)
CCS	Centum Call Seconds
CDTM	Connection Dependent Threshold Model
CS	Complete Sharing
DiffServ	Differentiated Services
EMLM	Erlang Multirate Loss Model
erl	The Erlang unit of traffic-load
FIFO	First in - First out
GB	Global balance
GoS	Grade of Service
ICT	Information and Communication Technology
IntServ	Integrated Services
IP	Internet Protocol
ITU-T	International Telecommunication Unit – Standardization sector
LB	Local balance
LHS	Left hand side

LIFO	Last in - First out
MMPP	Markov Modulated Poisson Process
MPLS	Multiple Protocol Labeling Switching
MRM	Multi-Retry Model
MTM	Multi-Threshold Model
PASTA	Poisson Arrivals See Time Averages
PDF	Probability Distribution Function
pdf	probability density function
PFS	Product Form Solution
QoS	Quality of Service
r.v.	random variable(s)
RED	random early detection
RHS	Right hand side
RLA	Reduced Load Approximation
SIRO	service in random order
SRM	Single-Retry Model
STM	Single-Threshold Model
TCP	Transport Control Protocol
TH	Threshold(s)
UDP	User Datagram Protocol

3.2. Thursday

3.2.1. Review

The last lecture you may be confused about how to compute the null space $N(\boldsymbol{A})$, i.e., why we follow the proceed to compute special solutions y_i. Let's review the whole steps for solving rectangular by using block matrix form.

- After converting the matrix $\boldsymbol{A}$ into the rref form $\boldsymbol{R}$, without loss of generality, we could convert the rref into the form $\begin{bmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$ by switching columns.

■ **Example 3.9** Last time our rref is given by:

$$\boldsymbol{R} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We notice that column 3 is pivot column, so we can switch it into the second column. (By switching column 2 and column 3):

$$\boldsymbol{R} \implies \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

■

- Thus our system could be written into the form:

$$\boldsymbol{R}\boldsymbol{x} = \boldsymbol{c} \implies \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{3.1}$$

Since we have changed the columns of $\boldsymbol{R}$, so the row 2 and row 3 of $\boldsymbol{x}$ is also switched respectively. Thus x_1 and x_2 are pivot variables, and x_3 and x_4 are free variables of $\boldsymbol{x}$. From (3.1) we derive:

$$\begin{cases} \boldsymbol{I}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \boldsymbol{B}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ 0 = c_3 \end{cases}$$

- If $c_3 \neq 0$, then there is no solution; so let's preassume $c_3 = 0$. Then *pivot variables* could be expressed as the form of *free variables*:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - \boldsymbol{B}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Suppose $-\boldsymbol{B} = \begin{bmatrix} \hat{\boldsymbol{y}}_{\boldsymbol{1}} & \hat{\boldsymbol{y}}_{\boldsymbol{2}} \end{bmatrix}$, then pivot variables can be expressed as:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + x_3\hat{\boldsymbol{y}}_{\boldsymbol{1}} + x_4\hat{\boldsymbol{y}}_{\boldsymbol{2}}$$

- Therefore, the complete solution to the system is given by

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_3\hat{\boldsymbol{y}}_1 + x_4\hat{\boldsymbol{y}}_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} \tag{3.2}$$

$$= \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \hat{\boldsymbol{y}}_2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.3}$$

$$= \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix}}_{\boldsymbol{x}_p} + \underbrace{x_3 \begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \hat{\boldsymbol{y}}_2 \\ 0 \\ 1 \end{pmatrix}}_{\boldsymbol{x}_{\text{special}}} \tag{3.4}$$

where x_3 and x_4 could be arbitarary.

- We can verify our computed special solutions is true by matrix multiplication:

$$\text{Special Solution Matrix: } \begin{pmatrix} \hat{\boldsymbol{y}}_1 & \hat{\boldsymbol{y}}_2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\boldsymbol{B} \\ \boldsymbol{I} \end{pmatrix}$$

$$\text{Verification: } \begin{pmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{bmatrix} -\boldsymbol{B} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{B} + \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

Open Question: If our rectangular matrix is $m \times n (m > n)$, how to solve it?

Answer: Similarly, we do G.E. to get rref. After switching columns, it will be of the form:

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \dots & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

3.2.2. Remarks on solving linear system equations

There are two kinds of linear equations, and the classification criteria depends on m and n:

Theorem 3.1 Let m denotes the number of equations, n denotes the number of variables. For the number of solutions for $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, we obtain:

- $m < n$: either **no solution** or **infinitely many solutions**
- $m \geq n$: **no solution; unique solution** ($N(\boldsymbol{A}) = \boldsymbol{0}$); or **infinitely many solutions**.

We prove for the $m < n$ case first:

Proofoutline for $m < n$ case: Recall that we can convert $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ into $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{c}$. WLOG, we switch columns of $\boldsymbol{R}$ to put pivot columns in the left-most:

$$\begin{bmatrix} 1 & & & \times & \times \\ & \ddots & & \times & \times \\ & & 1 & \times & \times \\ 0 & 0 & 0 & 0 & 0 \\ \dots & & & & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_r \\ c_{r+1} \\ \vdots \\ c_n \end{bmatrix},$$

where $x_1.x_2.\dots,x_r$ are pivot variables. Hence, we have $(n-r)$ free variables, and $N(\boldsymbol{A})$ is spanned by $(n-r)$ special vectors $y_1, y_2, \dots, y_{n-r}$.

It suffices to show that the $m < n$ rectangular system does not have unique solution, i.e., $N(\boldsymbol{A}) > 0$. It suffices to show $n > r$.

Obviously, $r \le m$, and we have $n > m$, so we obtain $n > r$. ■

Equivalently, we obtain the proposition and the corollary below:

Proposition 3.1 For system $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $m < n$, it either has no solution or infinitely many solutions.

Corollary 3.1 For system $\boldsymbol{Ax} = \boldsymbol{0}$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $m < n$, it always has infinitely many solutions.

3.2.2.1. What is r?

We ask the question again, what is r? Let's see some examples before answering this question.

■ **Example 3.10** If we want to solve system of equations of size 1000 as the following:

$$\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + 2x_2 = 6 \\ \cdots \\ 1000x_1 + 1000x_2 = 3000 \end{cases}$$

It seems very difficult when hearding it has 1000 equations, but the remaining 999 equations could be redundant (They actually don't exist):

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ \vdots & \vdots \\ 1000 & 1000 \end{bmatrix} \implies \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

Here we see that only one equation $x_1 + x_2 = 3$ is real, the remaining part is not real. So we claim that r is the number of "real" equations. But what is the definition for "real" equations? Let's discuss the definition for *linear dependence* first.

3.2.3. Linear dependence

Definition 3.2 [linear dependence] The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ in linear space $\boldsymbol{V}$ are **linearly dependent** if there exists $c_1, c_2, \dots, c_n \in \mathbb{R}$ s.t.

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{0}.$$

In other words, it means one of v_i could be expressed as the linear combination of others. Assume $c_n \neq 0$, we can express $\boldsymbol{v}_n$ as:

$$\boldsymbol{v}_n = -\frac{c_1}{c_n}\boldsymbol{v}_1 - \frac{c_2}{c_n}\boldsymbol{v}_2 - \cdots - \frac{c_{n-1}}{c_n}\boldsymbol{v}_{n-1}.$$

Definition 3.3 [linear independence] The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ in linear space $\boldsymbol{V}$ are **linearly independent** if the equation

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{0}$$

only has the trivial solution $c_1 = c_2 = \cdots = c_n = 0$.

In other words, if $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ are not **linearly dependent**, they must be **linearly independent**.

R Note that **only** in this course, if we say vectors are dependent, we mean they are **linearly** dependent. In other courses dependent may have other definitions. In the following lectures, we simplify the noun *dependent* as *dep.*; and the noun *independent* as *ind.*

Here we pick some examples to help you understand dep. and ind.:

■ **Example 3.11** • $\boldsymbol{v}_1 = (1,1)$ and $\boldsymbol{v}_2 = (2,2)$ are **dep.** because

$$(-2) \times \boldsymbol{v}_1 + \boldsymbol{v}_2 = \mathbf{0}.$$

- The only one vector $\boldsymbol{v}_1 = 2$ is **ind.** because

$$c\boldsymbol{v}_1 = \mathbf{0} \Longleftrightarrow c = 0.$$

- The only one vector $\boldsymbol{v}_1 = 0$ is **dep.** because

$$2 \times \boldsymbol{v}_1 = \mathbf{0}$$

- $\boldsymbol{v}_1 = (1,2)$ and $\boldsymbol{v}_2 = (0,0)$ are **dep.** because

$$0 \times \boldsymbol{v}_1 + 1 \times \boldsymbol{v}_2 = \mathbf{0}.$$

- The upper triangular matrix $\boldsymbol{A} = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ has three column vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

  $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are **ind.** because

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0} \Longleftrightarrow c_1 = c_2 = c_3 = 0.(\text{Why? because } \boldsymbol{A} \text{ is invertible})$$

■

3.2.3.1. Remarks

How many solutions meet the linear dependence criteria?. Recall that in last week we have studied that the following statements are equivalent: ()

- Vectors $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ are dep.
- $\exists$ nonzero $\boldsymbol{c}$ s.t. $\sum_{i=1}^{n} c_i a_i = \mathbf{0}$.
- $\exists\, \boldsymbol{c} \neq \mathbf{0}$ s.t.

$$\boldsymbol{A}\boldsymbol{c} := \left[\begin{array}{c|c|c} a_1 & \ldots & a_n \end{array}\right] \boldsymbol{c} = \mathbf{0}$$

For the third statement, if we could choose one $\boldsymbol{c}$, then how many $\boldsymbol{c}$ can we choose? For the $m < n$ case, by corollary (3.1), we obtain:

> **Corollary 3.2** When vectors $a_1, a_2, \ldots, a_n \in \mathbb{R}^m (m < n)$ are dependent, there exists infinitely solutions $c_1, c_2, \ldots, c_n$ such that $\sum_{i=1}^{n} c_i a_i = \mathbf{0}$.

The real equations are essentially those linearly independent equations.

3.2.4. Basis and dimension

> **Definition 3.4** [Basis] The vectors $v_1, \ldots, v_n$ form a **basis** for a vector space $\boldsymbol{V}$ if and only if:
>
> 1. $v_1, \ldots, v_n$ are **linearly independent**.
>
>   and
>
> 2. $v_1, \ldots, v_n$ span $\boldsymbol{V}$.
>
> ▪

> ▪ **Example 3.12** In $\mathbb{R}^3$,
>
> - $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis.

- $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not a basis, since it doesn't span $\mathbb{R}^3$.
- $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ don't form a basis, since they aren't linearly independent.
- $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ form a basis.

We find that the number of vectors for the basis of $\mathbb{R}^3$ is always 3, is this a coincidence? The theorem below gives the answer.

Theorem 3.2 If $v_1, v_2, \ldots, v_m$ is a basis; and $w_1, w_2, \ldots, w_n$ is another basis for the same vector space $\boldsymbol{V}$, then $n = m$.

In order to proof it, let's try simple case first:

proofoutline. 1. In order to proof it, let's try simple case first:

- Consider $\boldsymbol{V} = \mathbb{R}$ case first: For $\mathbb{R}$, the number 1 forms a basis. Let's show that 2 vectors in $\mathbb{R}$ cannot be a basis:
  - Given any two vectors x and y, they are not a basis for $\mathbb{R}$, since that
    * if $x = 0$ or $y = 0$, they are not ind.
    * otherwise, $y = \frac{y}{x} \times x \implies \frac{y}{x} \times x + (-1) \times y = 0$. So they are not ind.
- Then we consider $\boldsymbol{V} = \mathbb{R}^3$ case:
  For $\mathbb{R}^3$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis. Our goal is to show that if $v_1, v_2, \ldots, v_m$ is a basis, then $m = 3$.
  - Let's show that $m = 4$ is impossible, i.e., 4 vectors in $\mathbb{R}^3$ cannot be a basis.):

* It suffices to show that for $\forall a_1, a_2, a_3, a_4 \in \mathbb{R}^3$ they must be dep.
* Or equivalently, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has nonzero solutions, where $\boldsymbol{A} = \left[\begin{array}{c|c|c|c} a_1 & a_2 & \dots & a_4 \end{array}\right] \in \mathbb{R}^{3\times 4}$, which is true by corollary (3.1).

- Similarly, we could show any basis for $\mathbb{R}^3$ satisfies $m \leq 3$ (i.e., m=4,5,...is impossible).
- Then let's show that $m = 2$ is impossible, i.e., 2 vectors in $\mathbb{R}^2$ cannot be a basis:
  * It suffics to show that for $\forall a_1, a_2 \in \mathbb{R}^3$, they cannot span the whole space.
  * Otherwise, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ must have solution for arbitrary $\boldsymbol{b} \in \mathbb{R}^3$, where $\boldsymbol{A} = \left[\begin{array}{c|c} a_1 & a_2 \end{array}\right] \in \mathbb{R}^{3\times 2}$.
  * However, this kind system may have no solution, which is a contradiction.
- Similarly, we could show any basis for $\mathbb{R}^3$ satisfies $m \geq 3$.

• The same arugment could show any basis for $\mathbb{R}^n$ satisfies $m = n$.

2. Next, let's consider general vector space. We assume that $n < m$ (by contradiction method).
   Given that $v_1, \dots, v_n$ and $w_1, \dots, w_m$ are the basis of $\boldsymbol{V}$, our goal is to construct a contradiction that $w_1, \dots, w_m$ cannot form a basis.
   It suffices to show that $\exists$(construct) $\boldsymbol{c} = \begin{bmatrix} c_1 & c_2 & \dots & c_m \end{bmatrix}^{\mathrm{T}} \neq \boldsymbol{0}$ s.t.

$$c_1 w_1 + c_2 w_2 + \cdots + c_m w_m = 0. \tag{3.5}$$

   Moreover, we can express $w_1, \dots, w_m$ in form of $v_1, \dots, v_n$:

$$\begin{cases} w_1 = a_{11} v_1 + \cdots + a_{1n} v_n \\ \dots \\ w_m = a_{m1} v_1 + \cdots + a_{mn} v_n \end{cases} \tag{3.6}$$

By (3.6), we can write (3.5) as:

$$\begin{aligned}
0 &= \sum_{j=1}^{m} c_j w_j \\
&= \sum_{j=1}^{m} c_j (\sum_{i=1}^{n} a_{ji} v_i) \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} c_j a_{ji} v_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} c_j a_{ji} v_i \\
&= \sum_{i=1}^{n} v_i \times (\sum_{j=1}^{m} c_j a_{ji}) \\
&= v_1 \times (\sum_{j=1}^{m} c_j a_{j1}) + v_2 \times (\sum_{j=1}^{m} c_j a_{j2}) + \cdots + v_n \times (\sum_{j=1}^{m} c_j a_{jn})
\end{aligned}$$

So, in order to let LHS=0, we only need to let each of RHS=0, i.e.,

$$\sum_{j=1}^{m} c_j a_{j1} = \sum_{j=1}^{m} c_j a_{j2} = \cdots = \sum_{j=1}^{m} c_j a_{jn} = 0. \tag{3.7}$$

In order to construct c_j, we write (3.7) into matrix form:

$$\boldsymbol{A}^{\mathrm{T}} \boldsymbol{c} = \mathbf{0}, \text{ where } \quad \boldsymbol{A} = \begin{bmatrix} a_{ij} \end{bmatrix}_{1 \le i \le m; 1 \le j \le n}, \quad \boldsymbol{c} = \begin{bmatrix} c_1 & c_2 & \dots & c_m \end{bmatrix}^{\mathrm{T}}.$$

The system $\boldsymbol{A}^{\mathrm{T}} \boldsymbol{c} = \mathbf{0}$ has infinitie nonzero solutions by corollary (3.1). Hence we could construct infinitely such c_j.

■

During the proof, we face two difficulties:

1. For arbitararily $\boldsymbol{V}$, we write a concrete form to express $w_1, w_2, \dots, w_m$.
2. We write into matrix form to express $\sum_{j=1}^{m} c_j a_{j1} = \sum_{j=1}^{m} c_j a_{j2} = \cdots = \sum_{j=1}^{m} c_j a_{jn} = 0$.

Since any basis for $\boldsymbol{V}$ contains the same number of vectors, we can define the number of vectors to be dimension:

Definition 3.5 [Dimension] The **dimension** for a vector space is the number of vectors in a basis for it. ■

(R) Remember that vector space $\{0\}$ has dimension 0.

In order to denote the dimension for a given vector space $\boldsymbol{V}$, we often write it as $\dim(\boldsymbol{V})$.

■ **Example 3.13**

- $\mathbb{R}^n$ has dimension n.
- $\{\text{All } m \times n \text{ matrix}\}$ has dimension $m \cdot n$.
- $\{\text{All } n \times n \text{ symmetric matrix}\}$ has dimension $\frac{n(n+1)}{2}$.
- Let $\boldsymbol{P}$ denote the vector space of all polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$.

  $\dim(\boldsymbol{P}) \neq 3$ since $1, x, x^2, x^3$ are ind.

  The same argument can show $\dim(\boldsymbol{P})$ doesn't equal to any real number, so $\dim(\boldsymbol{P}) = \infty$

■

Human beings often ask a question: for a line and a plane, which is bigger?

Does plane has more point than a line?. No, Cantor syas they have the same "number" of points by constructing one-to-one mapping.

Furthermore, $\mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^n$ has the same number of points.

Plane and line have different dimensions. However, a plane has more dimensions than a line. So from this point of view, a plane is bigger than a line.

You should know some common knowledge for dimension:

1. Programmer lives in **2** dimension world. (They only live with binary.)
2. Engineer lives in **3** dimension world. (They only live with enign.)
3. Physician lives in **4** dimension world. (They discuss time.)
4. String theories states that our world is **11** or **26** dimension, which has been proved by Qingshi Zhu.

What is rank?. Finally let's answer the question: What is rank?

rank = dimension of row space of a matrix.

We will discuss it in the next lecture.