A FIRST COURSE

IN

LINEAR ALGEBRA

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MAT2040 Notebook

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Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication Data:

Survey Methodology / Robert M. Groves . . . [et al.].
p. cm.—(Wiley series in survey methodology)
"Wiley-Interscience."
Includes bibliographical references and index.
ISBN 0-471-48348-6 (pbk.)
1. Surveys—Methodology. 2. Social sciences—Research—Statistical methods. I. Groves, Robert M. II. Series.

HA31.2.S873 2004
001.4'33—dc22 2004044064
Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

Contents

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Foreword

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Preface

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Acronyms

ASTA	Arrivals See Time Averages
BHCA	Busy Hour Call Attempts
BR	Bandwidth Reservation
b.u.	bandwidth unit(s)
CAC	Call / Connection Admission Control
CBP	Call Blocking Probability(-ies)
CCS	Centum Call Seconds
CDTM	Connection Dependent Threshold Model
CS	Complete Sharing
DiffServ	Differentiated Services
EMLM	Erlang Multirate Loss Model
erl	The Erlang unit of traffic-load
FIFO	First in - First out
GB	Global balance
GoS	Grade of Service
ICT	Information and Communication Technology
IntServ	Integrated Services
IP	Internet Protocol
ITU-T	International Telecommunication Unit – Standardization sector
LB	Local balance
LHS	Left hand side

LIFO	Last in - First out
MMPP	Markov Modulated Poisson Process
MPLS	Multiple Protocol Labeling Switching
MRM	Multi-Retry Model
MTM	Multi-Threshold Model
PASTA	Poisson Arrivals See Time Averages
PDF	Probability Distribution Function
pdf	probability density function
PFS	Product Form Solution
QoS	Quality of Service
r.v.	random variable(s)
RED	random early detection
RHS	Right hand side
RLA	Reduced Load Approximation
SIRO	service in random order
SRM	Single-Retry Model
STM	Single-Threshold Model
TCP	Transport Control Protocol
TH	Threshold(s)
UDP	User Datagram Protocol

3.3. Friday

3.3.1. Review

Proposition 3.2 Undetermined system $\boldsymbol{Ax} = \boldsymbol{b}$ with $m < n$, i.e., number of equations $<$ number of unknowns, has **no solution** or **infinitely many solutions**.

We want to understand the meaning of rank: number of "real" equations.

Then we introduce definition of *linearly independence* and *linearly dependence.*

The linear dependence has relation with the system:

Proposition 3.3 $\boldsymbol{Ax} = \boldsymbol{0}$ has nonzero solutions if and only if the column vectors of $\boldsymbol{A}$ are dep.

Combining proposition (3.3) with (3.2), we derive the corollary:

Corollary 3.3 Any $(n+1)$ vectors in $\mathbb{R}^n$ are dep.

Proposition 3.4 Undetermined system $\boldsymbol{Ax} = \boldsymbol{b}$ with $m \geq n$, i.e., number of equations $\geq$ number of unknowns may have **no solution** or **unique solution** or **infinitely many solutions**.

From this proposition we derive the corollary immediately:

Corollary 3.4 Any $(n-1)$ vectors in $\mathbb{R}^n$ cannot span the whole space.

Then we introduce the definition of basis:

Definition 3.6 [Basis] A set of ind. vectors that span this space is called the **basis** of this space. ■

Then we introduce a theorem saying that **All basis of a given vector space have the same size.**

Thus we introduce **dimension** to denote the *number of vectors in a basis.*

3.3.2. More on basis and dimension

The basis of a given vector space has to satisfy two conditions:

$$\underbrace{\text{linear independence}}_{\text{not too many}} + \underbrace{\text{span the space}}_{\text{not too few}}$$

The **ind.** constraint let the size of basis not too many. For example, if given 1000 vectors of $\mathbb{R}^3$, they are very likely to be dep.

Spanning the space let the size of basis not too few. For example, given only 3 vectors of $\mathbb{R}^{100}$, they cannot span the whole space obviously.

We claim that:

$$\begin{aligned}\text{A basis } &= \text{maximal ind. set}\\ &= \text{minimal spanning set}\end{aligned}$$

Definition 3.7 [spanning set] $v_1, v_2, \dots, v_n$ is said to be the spanning set of $\boldsymbol{V}$ if

$$\boldsymbol{V} = \text{span}\{v_1, v_2, \dots, v_n\}.$$

■

■ **Example 3.14** $v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is not a basis of $\mathbb{R}^3$.

We can add $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which is ind. of v_1. But v_1, v_2 still don't form a basis.

If we add one more vector $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then v_1, v_2, v_3 form a basis of $\mathbb{R}^3$. ■

Theorem 3.3 Let $\boldsymbol{V}$ be a space of dimension $n > 0$, then

1. Any set of n ind. vectors span $\boldsymbol{V}$.
2. Any n vectors that span $\boldsymbol{V}$ are ind.

Here is the proof outline, but you should complete the proof in detail.

proofoutline.

1. Suppose $v_1, v_2, \ldots, v_n$ are ind. and v is an arbitarary vector in $\boldsymbol{V}$. Firstly, show that $v_1, v_2, \ldots, v_n, v$ is dep., thus derive the equation $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n + c_{n+1} v = \mathbf{0}$. Argue that the scalar $c_{n+1} \neq 0$. Then we can express v in form of $v_1, v_2, \ldots, v_n$, i.e., $v_1, v_2, \ldots, v_n$ span $\boldsymbol{V}$.
2. Suppose $v_1, v_2, \ldots, v_n$ span $\boldsymbol{V}$. Assume $v_1, v_2, \ldots, v_n$ are dep. Then show that v_n could be written as form of other $(n-1)$ vectors, it follows that $v_1, v_2, \ldots, v_{n-1}$ still span $\boldsymbol{V}$. If $v_1, v_2, \ldots, v_{n-1}$ are also dep, we can continue eliminating one vector. We continue this way until we get an ind. spanning set with $k < n$ elements, which contradicts dim($\boldsymbol{V}$)$= n$. Therefore, $v_1, v_2, \ldots, v_n$ must be ind.

■

■ **Example 3.15** $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ are ind. $\implies$ they span $\mathbb{R}^3$. ■

3.3.2.1. Clarification of dimension

Firstly, we need to understand "set":

1. $P \triangleq \{\text{All polynomials}\} = \text{span}\{1, x, x^2, \ldots\} \implies \dim(P) = \infty$.
2. $P_3 \triangleq \{\text{All polynomials with degree} \leq 3\} = \text{span}\{1, x, x^2, x^3\} \implies \dim(P) = 4$.
3. $Q \triangleq \text{span}\{x^2, 1 + x^3 + x^{10}, x^{300}\} \implies \dim(Q) = 3$.

R dim of space $\neq$ dim of the space it lives in.

For example, the line in $\mathbb{R}^{100}$ has dim 1.

3.3.3. What is rank?

Definition 3.8 [Rank] The rank of matrix $\boldsymbol{A}$ is defined as the **number of nonzero pivots of rref of $\boldsymbol{A}$**.

■ **Example 3.16**

$$\boldsymbol{A}=\begin{bmatrix}1 & 3 & 3 & 4\\ 2 & 6 & 9 & 7\\ -1 & -3 & 3 & 4\end{bmatrix}\xrightarrow{\text{row transform}}\boldsymbol{U}=\begin{bmatrix}1 & 3 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$\boldsymbol{U}$ has two pivots, hence $\text{rank}(\boldsymbol{A})=\text{rank}(\boldsymbol{U})=2$.

However, the definition for rank is too complicated, can we define rank of $\boldsymbol{A}$ directly?

Key question: What quantity is not changed under row transformation?

Answer: Dimension of row space.

Definition 3.9 [column space] The **column space** of a matrix is the subspace of $\mathbb{R}^n$ spanned by the columns.

In other words, suppose $\boldsymbol{A}=\begin{bmatrix} a_1 & \Big| & \dots & \Big| & a_n\end{bmatrix}$, the column space of $\boldsymbol{A}$ is given by

$$\mathcal{C}(\boldsymbol{A})=\text{span}\{a_1,a_2,\dots,a_n\}.$$

Definition 3.10 [row space] The **row space** of a matrix is the subspace of $\mathbb{R}^n$ spanned by the rows.

Suppose $A = \begin{bmatrix} a_1 \\ \hline \dots \\ \hline a_n \end{bmatrix}$, the row space of A is given by

$$\mathcal{R}(A) = \text{span}\{a_1, a_2, \dots, a_n\}.$$

The **row space** of A is essentially $\mathcal{R}(A) := \mathcal{C}(A^T)$, i.e., the column space of A^T.

Proposition 3.5 Row transforamtion doesn't change the row space

Proof. After row transformation, **new rows are linear combinations of old rows.**

Hence we have $\mathcal{R}(\text{new rows}) \subset \mathcal{R}(\text{old rows})$.

More specifically, assuming $A \xRightarrow{\text{Row Transfom}} B$, then we have $\mathcal{R}(B) \subset \mathcal{R}(A)$.

Since row transformations are invertible, we also have $B \xRightarrow{\text{Row Transfom}} A$, thus we have $\mathcal{R}(A) \subset \mathcal{R}(B)$.

In conclusion, we obtain $\mathcal{R}(B) = \mathcal{R}(A)$. ∎

Hence $\text{rank}(A) = \text{pivots of } U = \dim(\text{row}(U)) = \dim(\text{row}(A))$.

Hence we have a much simpler definition for rank:

Definition 3.11 [rank] The **dimension of the row space** is the **rank of a matrix**, i.e.,

$$\text{rank}(A) = \dim(\mathcal{R}(A)).$$

In the example (3.15), we find $\dim(\text{row}(A)) = \dim(\text{col}(A)) = 2$, is this a coincidence? *The fundamental theorem of linear algebra* gives this answer:

Theorem 3.4 The row space and column space both have the **same** dimension r.

We call $\dim(\mathcal{C}(A))$ as *column rank*; $\dim(\mathcal{R}(A))$ as *row rank*.

In brevity, **column rank=row rank= rank**, i.e.,

$$\dim(\mathcal{C}(A)) = \dim(\mathcal{R}(A)) = \text{rank}(A), \quad \text{for matrix } A$$

Let's discuss an example to have an idea of proving it.

■ **Example 3.17**

$$A = \begin{bmatrix} 1 & 3 & 3 & 4 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \xRightarrow{\text{row transform}} U = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We notice that **column rank of $A = 2$** and **column rank of $U = 2$**.

Why do they have the same **column space dimension**?

Wrong reason: A and U has the same column space. This is false. For example, the first column of A is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \notin \text{col}(U)$. The column spaces of A and U are **different**, but the dimension of them are **equal**.

Right reason: $Ax = 0$ iff. $Ux = 0$. The same combinations of the columns are zero (or nonzero) for A and U.

In other words, the r pivot columns (for both A and U) are independent; the $(n - r)$ free columns (for both A and U) are dependent.

For example, for U, column 1 and 3 are ind.(pivot columns); column 2 and 4 are dep.(free columns).

For A, column 1 and 3 are also ind.(pivot columns); column 2 and 4 are also dep.(free columns). ■

This example shows that **Row transformation doesn't change independence relations of columns**. We give a formal proof below:

Proposition 3.6 Suppose matrix A is converted into B by row transformation. If a set of columns of A are ind. then so are the corresponding columns of B.

Proof. Assume $A = \left[a_1 \middle| \dots \middle| a_n \right], B = \left[b_1 \middle| \dots \middle| b_n \right]$.

Without loss of generality (We often denote it as "WLOG"), we assume $a_1, a_2, \dots, a_k$ are ind.(We can achieve it by switching columns.)

We define the sub-matrices $\hat{\boldsymbol{A}} = \left[a_1 \middle| \dots \middle| a_k \right]$ and $\hat{\boldsymbol{B}} = \left[b_1 \middle| \dots \middle| b_k \right]$.

1. Notice that $\hat{\boldsymbol{A}}$ could be converted into $\hat{\boldsymbol{B}}$ by row transformation.

   Hence $\hat{\boldsymbol{A}}\boldsymbol{x} = \boldsymbol{0}$ and $\hat{\boldsymbol{B}}\boldsymbol{x} = \boldsymbol{0}$ has the same solutions.

2. On the other hand, $a_1, a_2, \dots, a_k$ are ind. columns.

   Hence $\hat{\boldsymbol{A}}\boldsymbol{x} = \boldsymbol{0}$ has the only zero solution.

Combining (1) and (2), $\hat{\boldsymbol{B}}\boldsymbol{x} = \boldsymbol{0}$ has the only zero solution. Hence $b_1, b_2, \dots, b_k$ are ind. ■

We can answer why the coincidence shown in the example, i.e., $\boldsymbol{A}$ and $\boldsymbol{U}$ has the same column space dimension:

Proposition 3.7 Row transformation doesn't change the column rank.

Proof. Assume $\boldsymbol{A} \xrightarrow{\text{row transform}} \boldsymbol{B}$.

Suppose $\dim(\mathcal{C}(\boldsymbol{A})) = r$, then we pick r ind. columns of $\boldsymbol{A}$. After row transformation, they are still ind. Hence $\dim(\mathcal{C}(\boldsymbol{B})) \geq r = \dim(\text{col}(\boldsymbol{A}))$.

Since row transformations are invertible, we get $\boldsymbol{B} \xrightarrow{\text{row transform}} \boldsymbol{A}$. Similarly, $\dim(\mathcal{C}(\boldsymbol{A})) \geq \dim(\mathcal{C}(\boldsymbol{B}))$.

Hence $\dim(\mathcal{C}(\boldsymbol{A})) = \dim(\mathcal{C}(\boldsymbol{B}))$. ■

Combining proposition (3.5) and (3.7), we can proof theorem (3.4):

Proof for theorem 3.4. Assume $\boldsymbol{A} \xrightarrow{\text{row transform}} \boldsymbol{U}(\text{rref})$.

- Proposition (3.5) $\implies \dim(\mathcal{R}(\boldsymbol{A})) = \dim(\mathcal{R}(\boldsymbol{U}))$.
- Proposition (3.7) $\implies \dim(\mathcal{C}(\boldsymbol{A}) = \dim(\mathcal{C}(\boldsymbol{U}))$.
- Notice that $\dim(\mathcal{R}(\boldsymbol{U}))$ denotes the number of pivots, $\dim(\mathcal{C}(\boldsymbol{U}))$ denotes the number of pivot columns. Obviously, $\dim(\mathcal{R}(\boldsymbol{U})) = \dim(\mathcal{C}(\boldsymbol{U}))$.

Hence $\dim(\mathcal{R}(\boldsymbol{A})) = \dim(\mathcal{C}(\boldsymbol{A}))$. ■

R $\dim(\mathcal{R}(\boldsymbol{U}))$ essentially denotes the number of "real" equations. $\dim(\mathcal{C}(\boldsymbol{U}))$ denotes the number of "real" variables.

So Theorem 3.4 implies that the number of "real" equations should equal to the number of "real" variables.

3.3.3.1. What is the null space dimension?

Assume the system $\boldsymbol{Ax} = \boldsymbol{b}$ has n variables.

Proposition 3.8 For matrix $\boldsymbol{A}$,

$$\text{rank}(\boldsymbol{A}) + \text{rank}(N(\boldsymbol{A})) = n.$$

Proof. **Number of pivot varibales + Number of free variables = $\boldsymbol{n}$.** ■

Note that $\boldsymbol{b} \in \text{col}(\boldsymbol{A})$ iff. $\boldsymbol{Ax} = \boldsymbol{b}$ for some $\boldsymbol{x}$.

Hence $\mathcal{C}(\boldsymbol{A})$ denotes all possible vectors in the form $\boldsymbol{Ax}$. Hence we call $\mathcal{C}(\boldsymbol{A})$ as "**range space**" of $\boldsymbol{A}$, which is denoted as range($\boldsymbol{A}$).

Equivalently, we have $\dim(\text{range}(\boldsymbol{A})) + \dim(N(\boldsymbol{A})) = n$.

Proposition 3.9 If $\boldsymbol{Ax} = \boldsymbol{b}$ has at least one solution, then $\text{rank}(\boldsymbol{A}) = \text{rank}(\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix})$.

■ **Example 3.18** Suppose $\boldsymbol{A} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$. If $\boldsymbol{Ax} = \boldsymbol{b}$ has at least one solution, then $\text{rank}(\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}) = \text{rank}(\begin{bmatrix} a_1 & a_2 & a_3 & b \end{bmatrix})$. ■

Proofoutline.

$$\boldsymbol{Ax} = \boldsymbol{b} \Longleftrightarrow \boldsymbol{b} \in \mathcal{C}(\boldsymbol{A})$$

Hence $\boldsymbol{b}$ is the linear combination of columns of $\boldsymbol{A}$. Adding one more column $\boldsymbol{b}$ into $\boldsymbol{A}$ doesn't change the dimension of $\mathcal{C}(\boldsymbol{A})$. Hence $\text{rank}(\boldsymbol{A}) = \text{rank}(\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix})$. ■

Proposition 3.10 If $\text{rank}(\boldsymbol{A}) \le n - 1$ for $m \times n$ matrix $\boldsymbol{A}$, then $\boldsymbol{Ax} = \boldsymbol{b}$ has **no solution** or **infinitely many solutions**.

Proofoutline.

$$\dim(\mathcal{C}(\boldsymbol{A})) + \dim(N(\boldsymbol{A})) = n \implies \dim(N(\boldsymbol{A})) \geq 1$$

So we have special solutions for $\boldsymbol{Ax} = \boldsymbol{b}$. For the particular solution, if doesn't exist, then we have no solution, otherwise we have infinitely many solutions. ■

Definition 3.12 [Full Rank] For $m \times n$ matrix $\boldsymbol{A}$, if $\operatorname{rank}(\boldsymbol{A}) = \min(m,n)$, then we say $\boldsymbol{A}$ is **full rank**.

Theorem 3.5 For $n \times n$ matrix $\boldsymbol{A}$, it is invertible iff. $\operatorname{rank}(\boldsymbol{A}) = n$.

Proof. Sufficiency. Assume $\operatorname{rank}(\boldsymbol{A}) = r < n$, then by row transformation, we can convert $\boldsymbol{A}$ into $\boldsymbol{U} := \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$(rref), where $\boldsymbol{B} \in \mathbb{R}^{r\times(n-r)}$. We can represent this process in matrix notation:

$$\boldsymbol{PA} = \boldsymbol{U} := \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},$$

where $\boldsymbol{P}$ is the product of row transformation matrices, which is obviously invertible.

Since $\boldsymbol{A}$ is invertible, we let $\boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \end{bmatrix}_{(r+(n-r))\times n}$. It follows that

$$\boldsymbol{P} = \boldsymbol{PI}_n = \boldsymbol{P}(\boldsymbol{AA}^{-1}) = (\boldsymbol{PA})\boldsymbol{A}^{-1} = \boldsymbol{UA}^{-1} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}_1 + \boldsymbol{BC}_2 \\ \boldsymbol{0} \end{bmatrix}.$$

Since $\boldsymbol{P}$ has $(n-r)$ zero rows as shown above, it is not invertible, which is a contradiction.

Necessity. If $\boldsymbol{A}$ is full rank, then it has n pivots, then by row transformation we can convert it into $\boldsymbol{I}$(rref). We can represent this process in matrix notation:

$$\boldsymbol{PA} = \boldsymbol{I}$$

where $\boldsymbol{P}$ is the product of row transformation matrix. Hence $\boldsymbol{P}$ is the left inverse of $\boldsymbol{A}$, $\boldsymbol{A}$ is invertible. ∎

3.3.3.2. Matrices of rank 1

Example 3.19

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} \xlongequal{\boldsymbol{v}^{\mathrm{T}}=\begin{bmatrix} 2 & 1 & 1 \end{bmatrix}} \begin{bmatrix} \boldsymbol{v}^{\mathrm{T}} \\ 2\boldsymbol{v}^{\mathrm{T}} \\ 4\boldsymbol{v}^{\mathrm{T}} \\ -\boldsymbol{v}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \boldsymbol{v}^{\mathrm{T}} \xlongequal{\boldsymbol{u}=\begin{bmatrix} 1 & 2 & 4 & -1 \end{bmatrix}^{\mathrm{T}}} \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$$

Here $\operatorname{rank}(\boldsymbol{A}) = 1$.

Proposition 3.11 Every rank 1 matrix $\boldsymbol{A}$ has the form $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} =$ column vector $\times$ row vector.

You may prove it directly by SVD decomposition (we will learn it later, but note that most theorems or propositions could be proved by SVD). Alternatively, we have another proof:

Proof. We set

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \vdots \\ \boldsymbol{c}_n \end{bmatrix},$$

where $\boldsymbol{c}_i$ is row vector. WLOG, we set $\boldsymbol{c}_1 \neq \boldsymbol{0}$ and $\boldsymbol{c}_1 = \begin{pmatrix} a_1b_1 & a_1b_2 & \dots & a_1b_n \end{pmatrix}$, where $a_1 \neq 0$, and $b_i (i = 1, \dots, n)$ are not all zero.

Since $\operatorname{rank}(\boldsymbol{A}) = 1$, we have $\dim(\mathcal{R}(\boldsymbol{A})) = 1$. Hence other $\boldsymbol{c}_i$ are dep. with $\boldsymbol{c}_1$. So we set

$$b_i = \frac{a_i}{a_1} \text{ for } i = 1, 2, \dots, n.$$

Thus we construct the form of $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} a_1b_1 & a_1b_2 & \dots & a_1b_n \\ a_2b_1 & a_2b_2 & \dots & a_2b_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & \dots & a_nb_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}$$

■

Question: What about the form of rank 2?

Answer: By SVD, it has the form $\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$.

Enjoy Your Midterm!