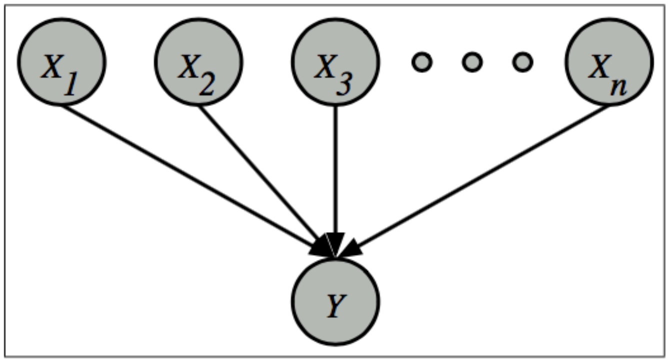
6.2 EM algorithm for noisy-OR

Consider the belief network on the right, with binary random variables $X \in \{0,1\}^n$ and $Y \in \{0,1\}$ and a noisy-OR conditional probability table (CPT). The noisy-OR CPT is given by:

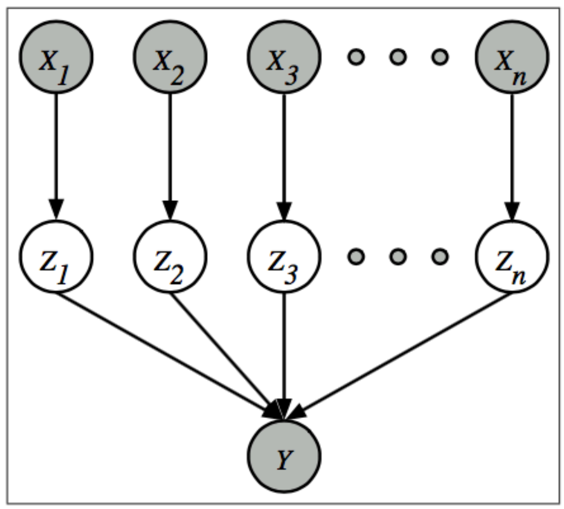


$$P(Y=1|X) \;=\; 1 - \prod_{i=1}^{n} (1-p_i)^{X_i},$$

which is expressed in terms of the noisy-OR parameters $p_i \in [0,1]$.

In this problem, you will derive and implement an EM algorithm for estimating the noisy-OR parameters p_i. It may seem that the EM algorithm is not suited to this problem, in which all the nodes are observed, and the CPT has a parameterized form. In fact, the EM algorithm can be applied, but first we must express the model in a different but equivalent form.

Consider the belief network shown to the right. In this network, a binary random variable $Z_i \in \{0,1\}$ intercedes between each pair of nodes X_i and Y. Suppose that:



$$\begin{aligned} P(Z_i=1|X_i=0) &= 0, \\ P(Z_i=1|X_i=1) &= p_i. \end{aligned}$$

Also, let the node Y be *determined* by the logical-OR of Z_i. In other words:

$$P(Y=1|Z) \;=\; \begin{cases} 1 \text{ if } Z_i=1 \text{ for any } i, \\ 0 \text{ if } Z_i=0 \text{ for all } i. \end{cases}$$

(a) Show that this "extended" belief network defines the same conditional distribution $P(Y|X)$ as the original one. In particular, starting from

$$P(Y=1|X) \;=\; \sum_{Z\in\{0,1\}^n} P(Y=1, Z|X),$$

show that the right hand side of this equation reduces to the noisy-OR CPT with parameters p_i. To perform this marginalization, you will need to exploit various conditional independence relations.

(b) Consider estimating the noisy-OR parameters p_i to maximize the (conditional) likelihood of the observed data. The (normalized) log-likelihood in this case is given by:

$$\mathcal{L} \;=\; \frac{1}{T}\sum_{t=1}^{T} \log P(Y=y^{(t)}|X=\vec{x}^{(t)}),$$

where $(\vec{x}^{(t)}, y^{(t)})$ is the tth joint observation of X and Y, and where for convenience we have divided the overall log-likelihood by the number of examples T. From your result in part (a), it follows that we can estimate the parameters p_i in either the original network or the extended one (since in both networks they would be maximizing the same equation for the log-likelihood).

Notice that in the extended network, we can view X and Y as observed nodes and Z as hidden nodes. Thus in this network, we can use the EM algorithm to estimate each parameter p_i, which simply defines one row of the "look-up" CPT for the node Z_i.

Compute the posterior probability that appears in the E-step of this EM algorithm. In particular, for joint observations $x \in \{0,1\}^n$ and $y \in \{0,1\}$, use Bayes rule to show that:

$$P(Z_i\!=\!1, X_i\!=\!1|X\!=\!x, Y\!=\!y) \;=\; \frac{y x_i p_i}{1-\prod_j (1-p_j)^{x_j}}$$

(c) For the data set $\{\vec{x}^{(t)}, y^{(t)}\}_{t=1}^{T}$, show that the EM update for the parameters p_i is given by:

$$p_i \;\leftarrow\; \frac{1}{T_i}\sum_t P\left(Z_i\!=\!1, X_i\!=\!1|X\!=\!x^{(t)}, Y\!=\!y^{(t)}\right),$$

where T_i is the number of examples in which $X_i\!=\!1$. (You should derive this update as a special case of the general form presented in lecture.)

(d) Download the data files on the course web site, and use the EM algorithm to estimate the parameters p_i. The data set[1] has $T\!=\!267$ examples over $n\!=\!23$ inputs. To check your solution, initialize all $p_i\!=\!\frac{1}{n}$ and perform 256 iterations of the EM algorithm. At each iteration, compute the log-likelihood shown in part (b). (If you have implemented the EM algorithm correctly, this log-likelihood will always increase from one iteration to the next.) Also compute the number of mistakes $M \leq T$ made by the model at each iteration; a mistake occurs either when $y_t=0$ and $P(y_t=1|\vec{x}_t) \geq 0.5$ (indicating a false positive) or when $y_t=1$ and $P(y_t=1|\vec{x}_t) \leq 0.5$ (indicating a false negative). The number of mistakes should generally decrease as the model is trained, though it is not guaranteed to do so at each iteration. Complete the following table:

iteration	number of mistakes M	log-likelihood $\mathcal{L}$
0	195	-1.04456
1	60	
2		-0.41076
4		
8		
16		
32		
64	37	
128		
256		-0.31016

You may use the already completed entries of this table to check your work.