Problem Set 3

Due Wednesday, Dec 16, 2020

A toy model of neural integrator

An animal moves on a one-dimensional track. In order to navigate properly (e.g., to find its way back home) the brain must compute the 'on line' position of the animal given sensory information about its velocity. We will assume that the position is computed by means of a network composed of two neurons, whose input is the animal's velocity. The network obeys the following equations:

$$\begin{aligned} \tau \frac{dx}{dt} &= -ax - 2y + \tau V(t); \\ \tau \frac{dy}{dt} &= -(3-a)x - y - \tau V(t) \end{aligned} \tag{1}$$

where the value of $x(t)$ represents the estimated position of the animal at time t.

(a) Study the dynamics of the network for constant velocity, i.e., $V(t) = V_0$ for all $t > 0$: First, find the fixed point of the network and the region of values of the parameter a for which this fixed point is stable. Next, write down the solution for $x(t)$ given that $(x(0), y(0)) = (0, 0)$ and explain what happens as time increases.

(b) Position Estimation. Assume that at $t = 0$, the animal starts out at the origin, and begins moving at a constant velocity $V(t) = 0.1$ m/s. Use $\tau = 100$ ms. If the system acted as a perfect integrator of the velocity, the expected position at $t = 10$ s would be 1 m. Is there a parameter choice (for parameter a) for which this system acts as a perfect integrator? (If so, take any appropriate limits to prove it.) For what range of parameters does the system act as a leaky integrator, i.e., the readout is within 1 cm error of the estimated position from a perfect integrator after 10 seconds? What happens to $x(t)$ when a is outside this range?

Ring network

Consider the ring network model we discussed in the class

$$\tau \frac{du_i}{dt} = -u_i + F\left(\sum_{j=1..N} w_{ij}u_j + I_i^0\right), \tag{2}$$

where $F(x) = [x]_+$ is a rectified linear function and the weights between the neurons are determined by their angular difference and thus are translational invariant.

$$w_{ij} = \frac{1}{N}J(\theta_i - \theta_j), \tag{3}$$

where $\theta_i = -\pi + \frac{2\pi i}{N}$ and J is a 2π periodic function, and I_i^0 is also a periodic 2π function. In the continuous limit, we have

$$\tau \frac{\partial u(\theta,t)}{\partial t} = -u(\theta,t) + \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} J(\theta - \theta')u(\theta',t)d\theta' + I^0(\theta)\right]_+, \tag{4}$$

(a) If the connectivity matrix J is translational invariant and symmetric, prove that the eigenfunctions of J are Fourier basis, namely

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta' J(\theta - \theta')e_\mu(\theta') = \lambda_\mu e_\mu(\theta) \tag{5}$$

$$e_\mu \sim \cos(\mu\theta) + \sin(\mu\theta), \tag{6}$$

where $\mu = 0, 1, 2, ...$ Find out the correct normalization factor for the eigenfunctions. The corresponding eigenvalues of the the connectivity matrix can be viewed as the Fourier transform $J(\theta)$, namely

$$\lambda_\mu = \frac{1}{2\pi}\int_{-\pi}^{\pi} J(\theta)\cos(\mu\theta)d\theta \tag{7}$$

(b) In the spirit of the linear recurrent network model we discussed in the class, derive the general solution of the *linear* ring network by assuming that $F(x) = x$ instead of a rectified linear network. In the class, we have considered a simple form of the connectivity matrix,

$$J(\theta) = J_0 + J_1\cos\theta \tag{8}$$

$$I_i^0(\theta) = I_0 + I_1\cos(\theta - \theta_0) \tag{9}$$

Discuss the stability of a linear ring network in this simple case.

(c) Consider the special case $I_1 = 0$, so that each neuron receives the same homogeneous inputs and $F(x) = [x]_+$. A naive solution is that the population neural activity is homogeneous : all neurons have the same activity. Show that this naive solution is unstable when $J_1 > 2$.

(d) We show in the class that a marginal stable solution is the emergence of a bump, namely, $u(\theta) = [u_1 \cos(\theta) + u_0]_+$. Please show that the bump can actually appear anywhere, namely $u(\theta) = [u_1 \cos(\theta - \phi) + u_0]_+$, where ϕ is arbitrary.

(e) Draw the phase diagram on the $J_0 - J_1$ plane. Discuss and plot in what region, the system is marginally stable? In what region, the system exhibits homogeneous activity? In what region, the system is unstable?

(f) Now consider some tiny modulatory input with $0 < I_1 \ll I_0$ and $J_1 > 2$. Show that in this case, the location of the peak of the stationary activity is not arbitrary, but aligned to the stimulus angle θ_0.

(g*) Add to the connectivity matrix a term $\frac{J_1}{N}\gamma \sin(\theta_i - \theta_j)$, where $|\gamma| \ll 1$. Show that there is a solution with the form

$$u(\theta, t) = f(\theta - \omega t), \tag{10}$$

where $f(\theta)$ is the steady state activity profile calculated in the class for $\gamma = 0$ and the angular velocity satisfies

$$\omega = \frac{\gamma}{\tau} \tag{11}$$

Last question is a bonus. Here is some specific guide:
A. Assume a traveling profile of the form of 10 (for now just use an arbitrary profile $f(\theta)$) and insert it into the two sides of 2. Express the RHS in terms of the order parameters as was done in the class. Expand the RHS in powers of γ and keep only terms up to linear in γ.
B. Show that if $f(\theta)$ is the profile for $\gamma = 0$ the dynamic equations 2 are satisfied.