Solution to Homework Two

Cong Li(PB21081606)

1. Integrate and Fire Model

1.1 when the neuron's potential is strongly refractory

a) *plotting the* $f - I_e$ *curve*:

The Integrate-and-fire model of neuron's firing consists of the following equation for the membrane potential(in dimensionless units):

$$\tau \frac{dV}{dt} = -V + I_e \tag{1}$$

$$\begin{aligned} V(t^-_{spike}) &= 1 \\ V(t^+_{spike}) &= 0 \end{aligned} \tag{2}$$

As the teacher taught in class, the solution to the ODE above is,

$$\begin{aligned} &V(t) = I_e - I_e exp(-t/\tau) \quad t \in [0, T] \\ &V(t) = V(t+T) \end{aligned} \tag{3}$$

The T is the interval between two spikes, which means $V^-(T) = 1$. So we get, $T = -\tau ln(1 - 1/I_e)$

Thus in the normal I-F neuron, the relationship between firing rate f_{norm} and I_e would be,

$$f_{norm} = 1/T = \frac{-1}{\tau ln(1 - 1/I_e)} \tag{4}$$

According to the assumptions of the strongly refractory neuron's potential, we can get,

$$\begin{aligned} &V(t) = I_e - I_e exp(-t/\tau) \quad t \in [0, T_{spike}] \\ &V(t) = 0 \quad t \in (T_{spike}, T_{spike} + \tau_r) \\ &V(t) = V(t+T) \\ &\quad T \stackrel{def}{=} T_{spike} + \tau_r \end{aligned} \tag{5}$$

Thus in the strongly refractory neuron, the relationship between firing rate f_{ref} and I_e would be,

$$f_{ref} = 1/T = \frac{1}{\tau_r - \tau ln(1 - 1/I_e)} \tag{6}$$

According to Eq.4 and Eq.6, I plotted the $f - I_e$ curve shown in Fig.1, with τ set as 20ms and τ_r set as 2ms.

b) *comparsion of their behavior at large I*:

When I_e is large, the $1/I_e$ is close to 0, so $ln(1 - 1/I_e)$ in Eq.4 is close to $-1/I_e$. So we get the

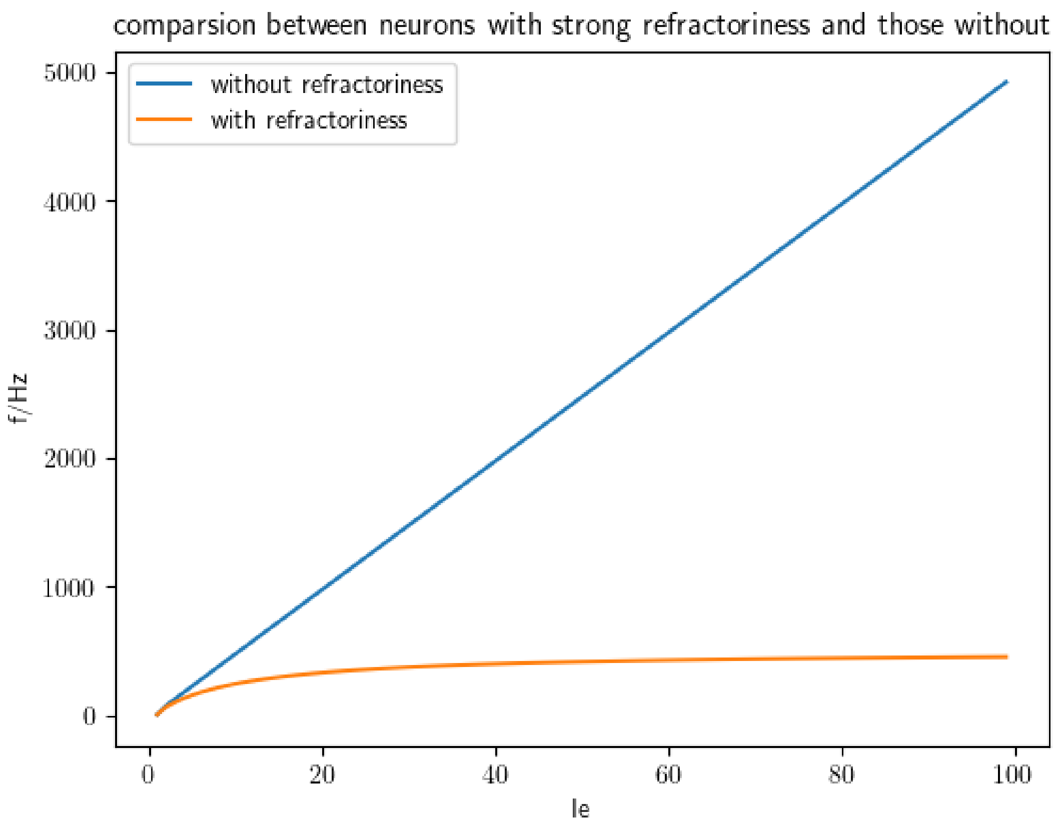

Figura 1. Comparsion between normal I-F neuron and strongly refractory neuron

approximation,

$$f_{norm} \approx \frac{I_e}{\tau} \tag{7}$$

Eq.7 describe the direct proportionality between firing rate and I_e in normal I-F neuron when I_e is large.

The Taylor expansion of Eq.6 in $1/I_e$ is,

$$f_{ref} = \frac{1}{\tau_r} - \frac{\tau}{\tau_r^2 I_e} + \ldots \tag{8}$$

Because $1/I_e$ is close to 0, so we can make the approximation,

$$f_{ref} = \frac{1}{\tau_r} - \frac{\tau}{\tau_r^2 I_e} \tag{9}$$

Eq.9 shows that when I_e is large, the firing rate will increase as t increases, but it will not increase indefinitely, instead, it will gradually approach $1/\tau_r$.

c) *explorision of the effect of τ_r on firing rate*

Set the τ as 20ms, and the τ_r as 2ms and 20ms respectively, and we can plot the $f - I_e$ curve with different τ_r, which is shown in the Fig.2

As is shown in the Fig.2, the behavior of two curves with lower I_e is similar, but the curve with higher τ_r will stabilize earlier and with lower maximum.

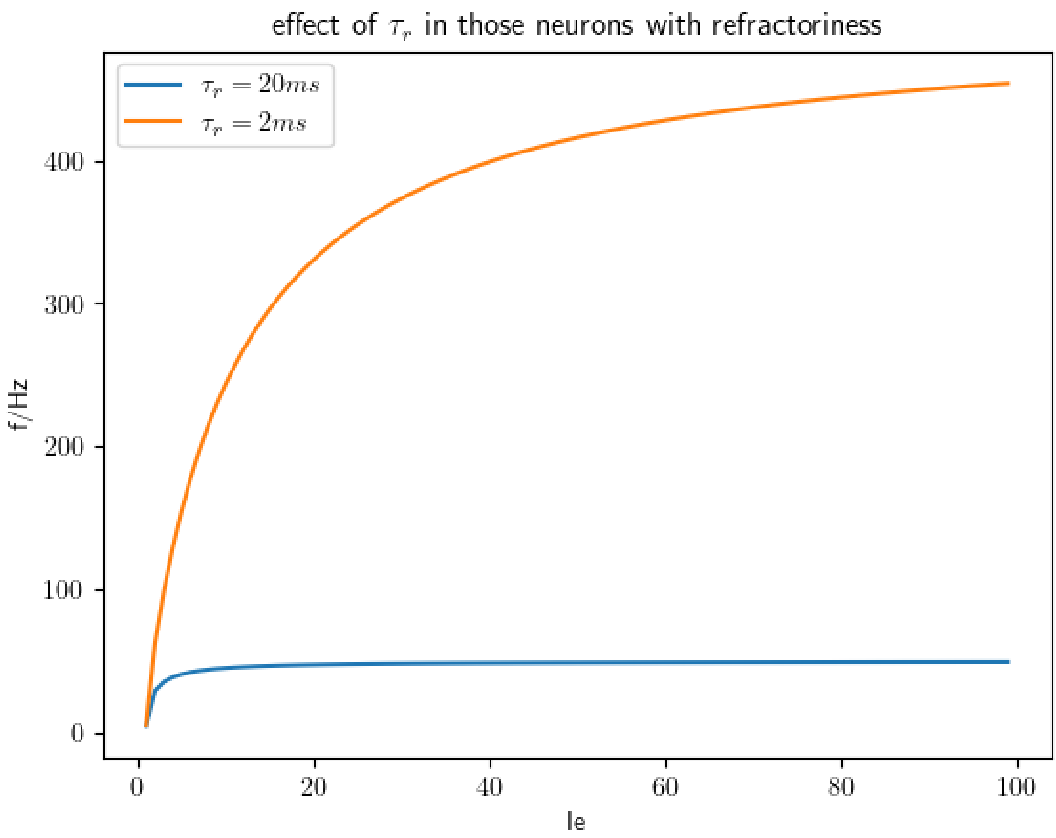

Figura 2. Effect of τ_r on the firing rate of strongly refractory neuron

1.2 when the neuron has negative adaptation current

a) *finding the analytical solution of the stable firing rate*

Now we consider the case that a neuron can introduce a negative adaptation current I_a by itself to decrease the firing rate of subsequent spikes. A simple model of the adaptation current is given by,

$$\tau_a \frac{dI_a}{dt} = -I_a - J_a \tau_a \sum_i \delta(t - t_i^{spike}) \tag{10}$$

where τ_a is the time constant of the adaptation current, J_a is its strength.

And the Eq.1 should be replaced with Eq.11.

$$\tau \frac{dV}{dt} = -V + I_e + I_a \tag{11}$$

We can assume that when in steady state, the neuron fires peridically with an interspike interval T. This means that $V(t) = V(t + T)$, by which we can solve the Eq.10 and get,

$$\begin{aligned} I_a(t) &= -J_a \sum_{n=0}^{\infty} exp(\frac{-nT + t}{\tau_a}) = \frac{-J_a exp(-\frac{t}{\tau_a})}{1 - exp(-\frac{T}{\tau_a})} \quad t \in [0, T] \\ I_a(t) &= I_a(t + T) \end{aligned} \tag{12}$$

So, the ODE in Eq.11 can be solved now.

$$\begin{aligned} V(t) &= V(0)exp(-\frac{t}{\tau}) + \int_0^t exp(-\frac{t-t'}{\tau})(I_e - \frac{-J_a exp(-\frac{t'}{\tau_a})}{1-exp(-\frac{T}{\tau_a})})dt' \\ &= I_e(1-exp(-\frac{t}{\tau})) - \frac{J_a}{\tau(1-exp(-\frac{T}{\tau_a}))}\frac{exp(-\frac{t}{\tau_a})-exp(-\frac{t}{\tau})}{1/\tau - 1/\tau_a} \qquad t \in [0,T] \end{aligned} \tag{13}$$

As T is the interspike interval, $V^-(T)$ should be 1. So we have the following equation T should obey.

$$I_e(1-exp(-\frac{T}{\tau})) - \frac{J_a}{\tau(1-exp(-\frac{T}{\tau_a}))}\frac{exp(-\frac{T}{\tau_a})-exp(-\frac{T}{\tau})}{1/\tau - 1/\tau_a} = 1 \tag{14}$$

Using the root function in the scipy.optimize, I can solve Eq.14 numerically with τ set as 10ms, τ_a set as 200ms. According to the calculation, I plotted the $f - I_e$ curve with the effects of adaptation.

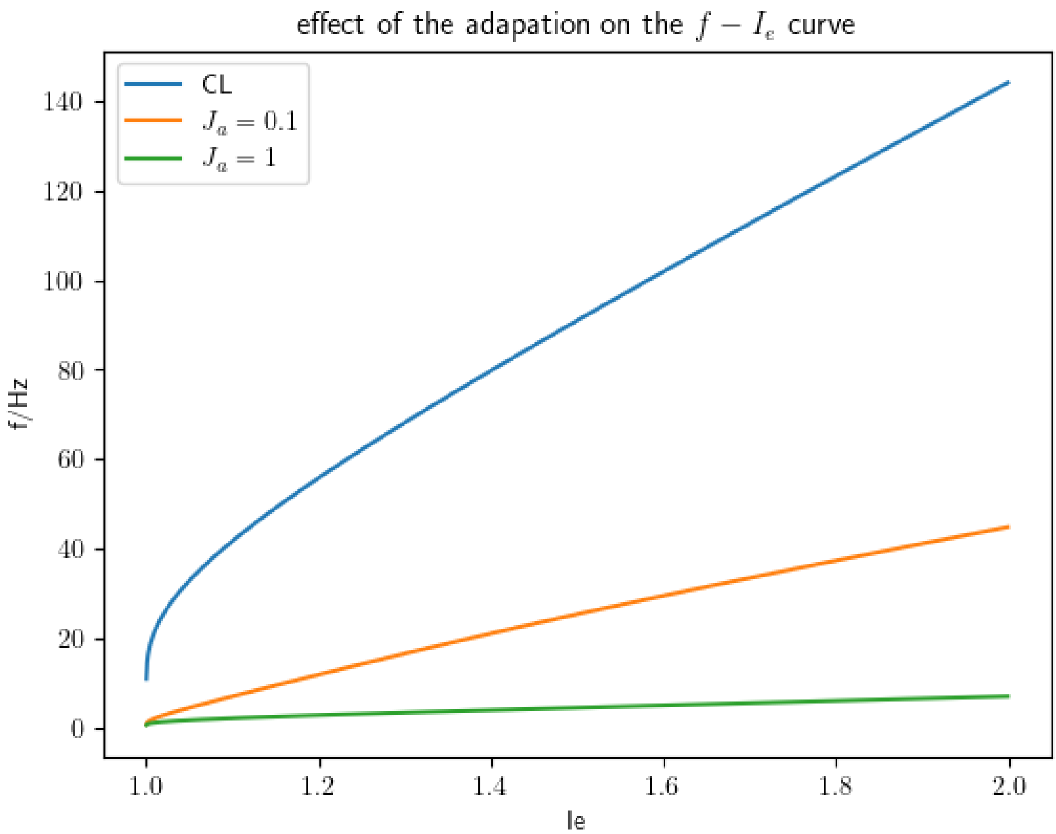

Figura 3. Effect of τ_r on the firing rate of strongly refractory neuron

By comparision, we can find that adapatation does not effect the behavior of firing rate increasing proportionally with I_e when input current is large enough. But it significantly reduce the magnitude of the neuron's response.

b) *modifying the MATLAB code to simulate the I-F model with adapatation*

Owing to my ignorance of MATLAB coding rules, I tranlated the MATLAB code offered by the teacher into a python code with the help of ChatGPT. And you can see it in the jupyter notebook in the annex, and I used a string of $\star$ to distinguish the code I added from the original python code. Also in order to simplify the plotting, I changed the original code into a function.

And you can see the f-I curve with and without the adapatation current in Fig.4, in which J_a was set as 0.1. And the calculation based on simulation behaves similarly to the analytical solution.

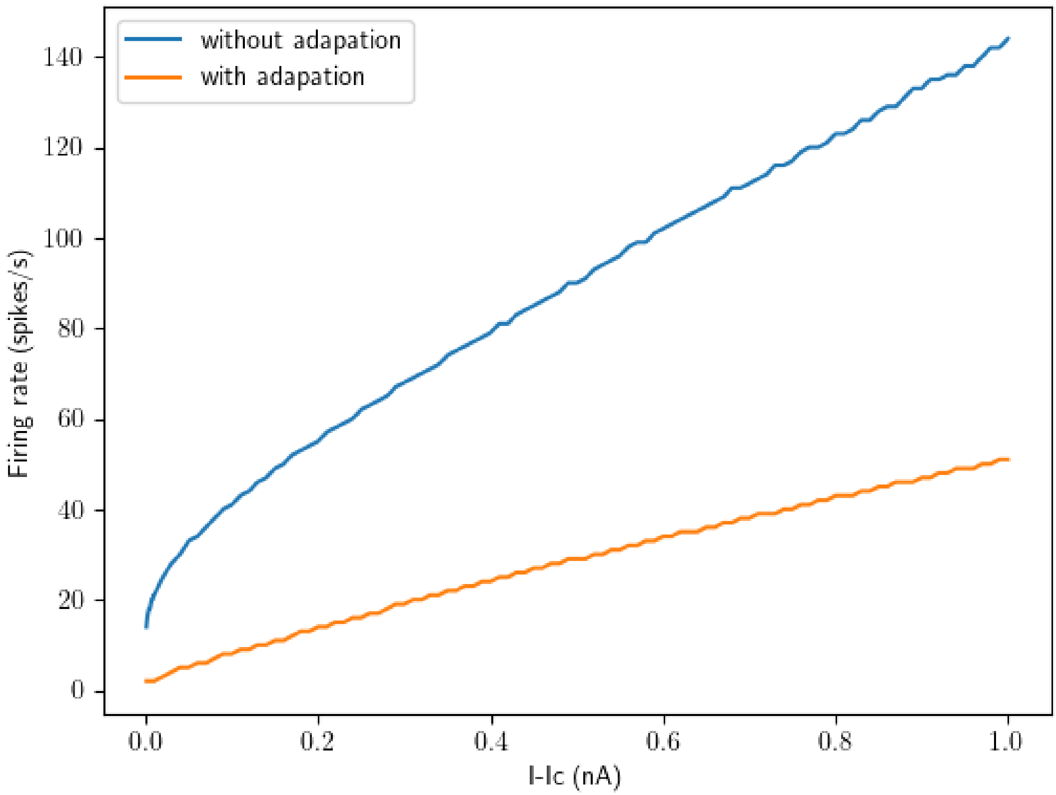

Figura 4. Simulation of I-F model with adapatation

Also I plotted the membrane potential changing with time for 2 different current values in Fig.5. You can find in Fig.5 that before the first spike occurs, the curve with lower J_a behaves similarly to that with higher J_a. But after that, the former shows higher membrane potential changing frequency.

In addition, I drew a figure to compare the simulated result with the analytical solution. As shown in Fig.6, their difference is small especially when input current is low, which proves that the analytical solution is correct. The reason why the simulated result is higher than the analytical solution is that the simulated result is influenced by the behaviors before the model enters stable state and during that time the membrane potential changes faster than in the stable state.

Finally I computed the first interspike interval(ISI) for the different current values, and compared 1/ISI with the steady state firing rate in Fig.7. The 1/ISI is greater than the stable firing rate because the influence of adapatation current has not reached its maximum.

2. Hodgkin and Huxley Model

a) *Simulation of Hodgkin and Huxley Model*

In order to receive a good simulation of the HH model, I set a process of 100ms with time step to be 0.01ms and creat a circulation to let the membrane potential, m, n and h to change a bit each time.

And I initialize this system by the following parameters:

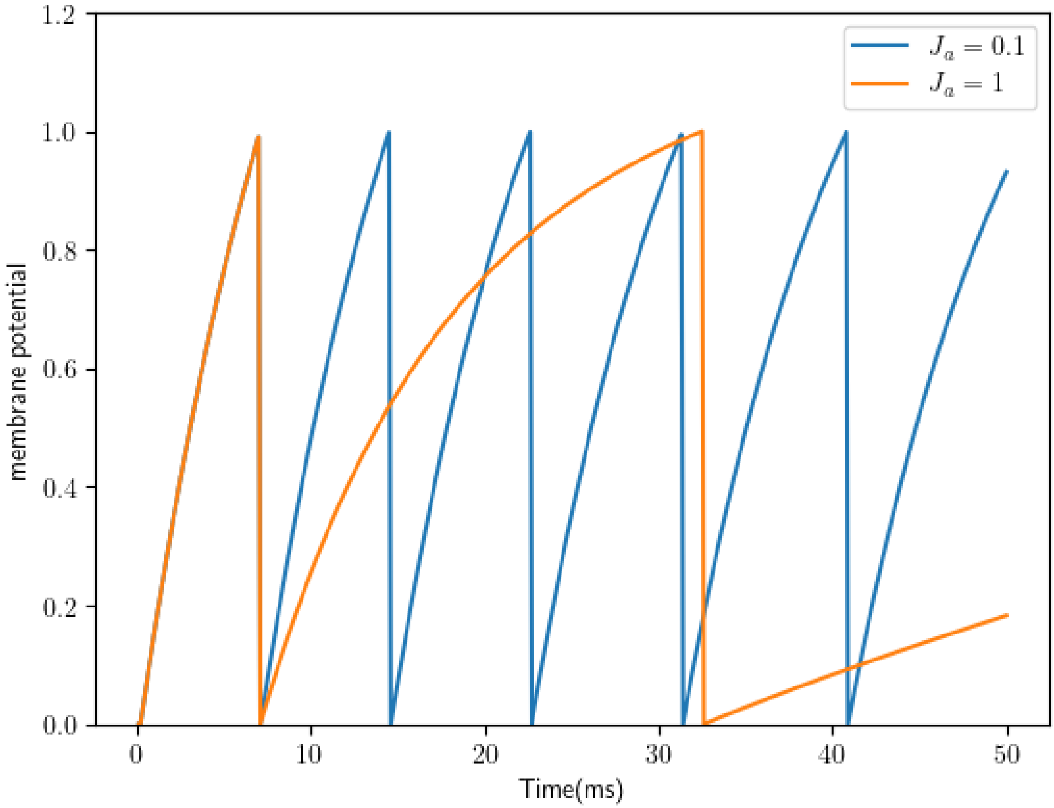

Figura 5. Membrane potential changes with time in different J_a

1. $n = m = 0, h = 1$
2. input current $I = 10.0\mu A/cm^2$
3. initial membrane potential $V(0) = 0$

And I got the membrane potential-Time curve in the Fig.8.

As is shown in the Fig.8, when the input current is set as $10\mu A/cm^2$, the simulation can produce action potentials and its behavior is very similar to the real neuron.

b) *the relationship between frequency of pluses and input current*

Then I set up a series of increasing input currents with a minimum of 0 and a maximum of $60.0\mu A/cm^2$ in steps of $0.5\mu A/cm^2$. For each input current, I carried out the same simulation as above, and detected the time when the peak occurred. Then I calculated the average interval of each peak as the period of action potentials in this simulation, by which I got the frequency of the pulses corresponding to each input current.

I have to mention that in this part, I change the initial membrane potential to be $-65.0\mu A/cm^2$. And n, m, h was changed to 0.3177,0.0529,0.5961 respectively. Because $-65.0\mu A/cm^2$ is close to the resting potential of the simulated neuron and $0.2913, 0.0529, 0.5961$ are the stable values when membrane potential is fixed at $-65.0\mu A/cm^2$(I will proved below), such a setting could help reduce the time for my simulated neuron to reach the stable state, thus decreasing the error in the calculation of spike frequency.

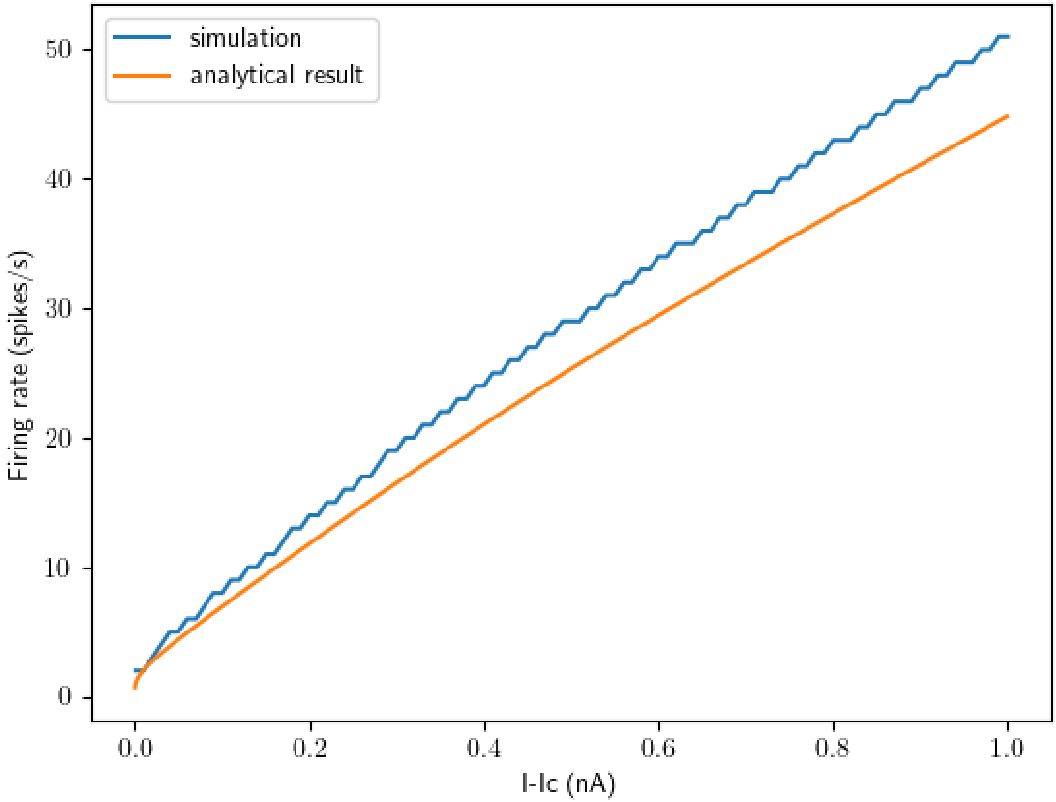

Figura 6. Comparsion between the simulated result and the analytical solution

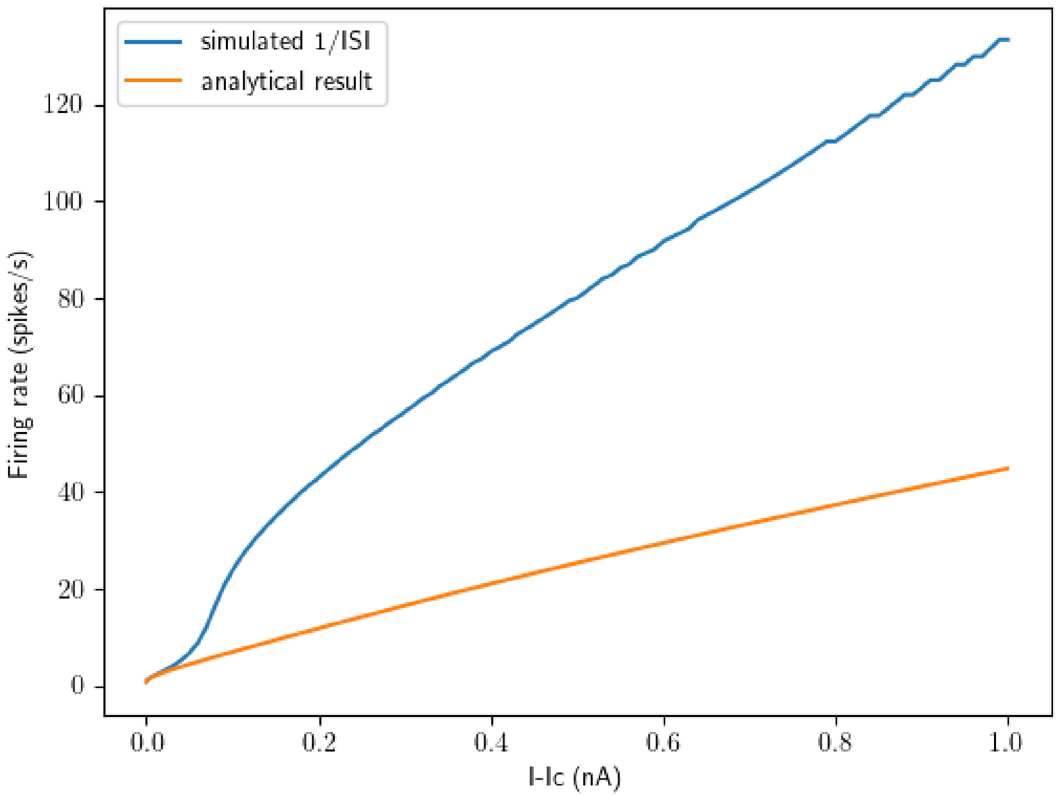

Figura 7. Comparsion between 1/ISI and analytical firing rate

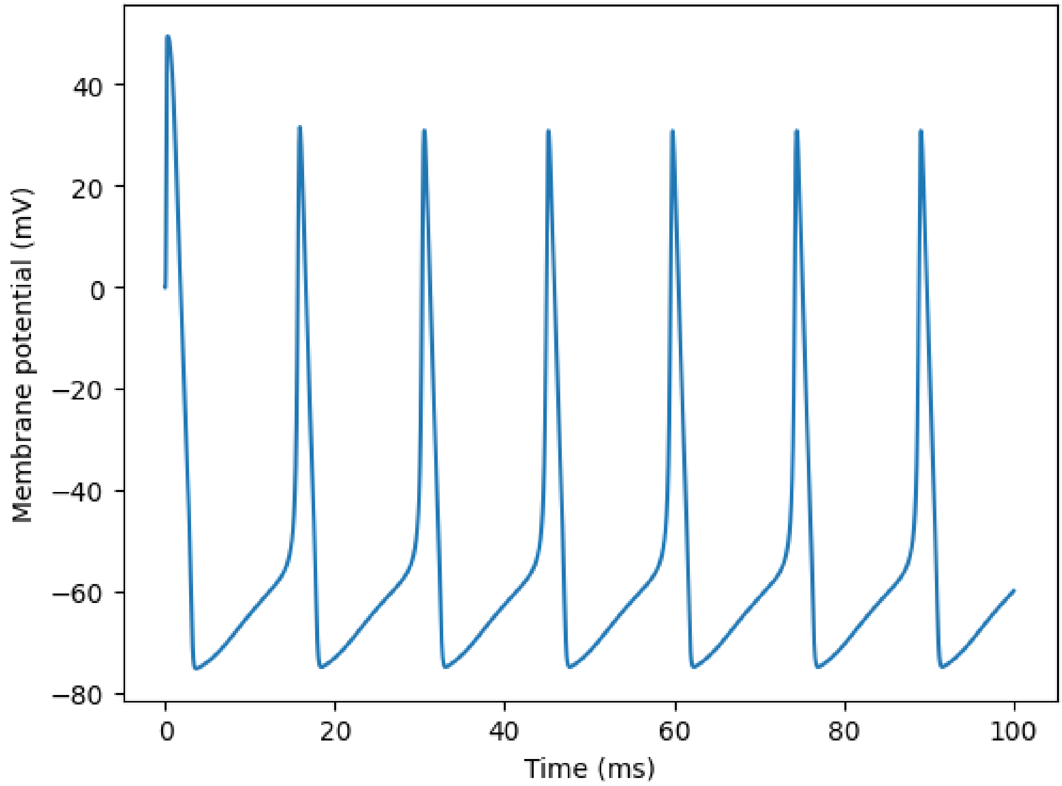

Figura 8. membrane potential changes with time($I = 10.0\mu A/cm^2$)

Prove: the stable value of n/m/h when membrane potential is fixed

Use the case of n as an example. It is obvious when the $\frac{dn}{dt} = 0$, the n becomes stable.
So we get $n_{stable} = \frac{\alpha_n}{\alpha_n+\beta_n}$.
When $V = -65.0\mu A/cm^2$, $\alpha_n = 0.05820$, $\beta_n = 0.1416$. So $n_{stable} = 0.2913$

The result is shown in Fig.9 which clearly shows that there is a threshold current(around $5.85\mu A/cm^2$) and only when the input current is above this value can the simulted neuron produce periodic action potentials. And surrounding this value, the frequency has a huge step jump. After that, the frequency of impulse will increase with the input current increasing but the increasing rate is slower and slower.

I was surprised to find when the input current is greater than certein value, **the simulated neuron stops to produce periodic action potentials**!

After researching some documents, I found this phenomenon had been found in not only the Hodgkin-Huxley Model, but also the real neuron, and was called as depolarization block. To investigate what happened in earth, I plotted the membrane potential-time curve when input current $I = 58.0\mu A/cm^2$, which is shown in the Fig.10. And it turns out that the frequency is so high that the membrane doesn't have enough time to depolarize and cannot be detected as a peak in my program. In the real world, if the neuron doesn't have enough time to depolarize, it will not reach the threshold membrane potential to let the potential-gated sodium ion channel to open and then cannot induce an action potential.

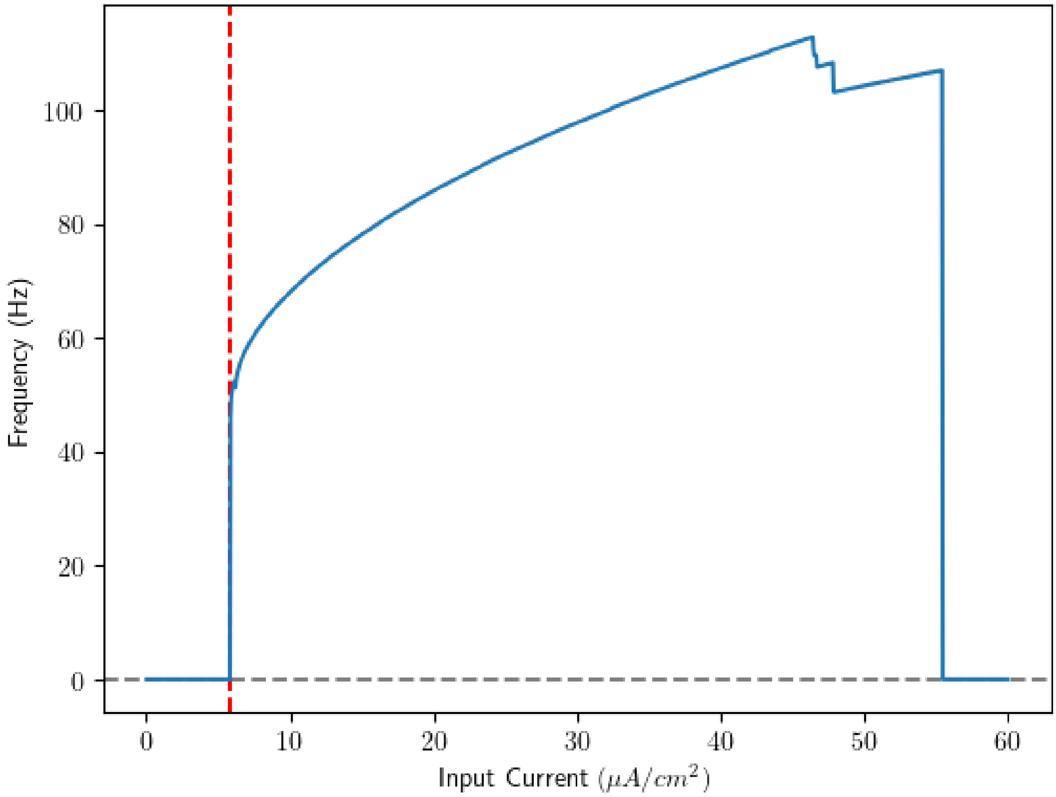

Figura 9. frequency changes with input current

So it is reasonable to deduce that this phenomenon would disappear if I decrease the height requirement to be detected as a peak, just as what I did in Fig.11. I changed the 'height' parameter in function 'scipy.signal.find_peaks' from 10 to 5.

3. Acknowledgement

The successful completion of the work would be impossible without the help of ChatGPT.

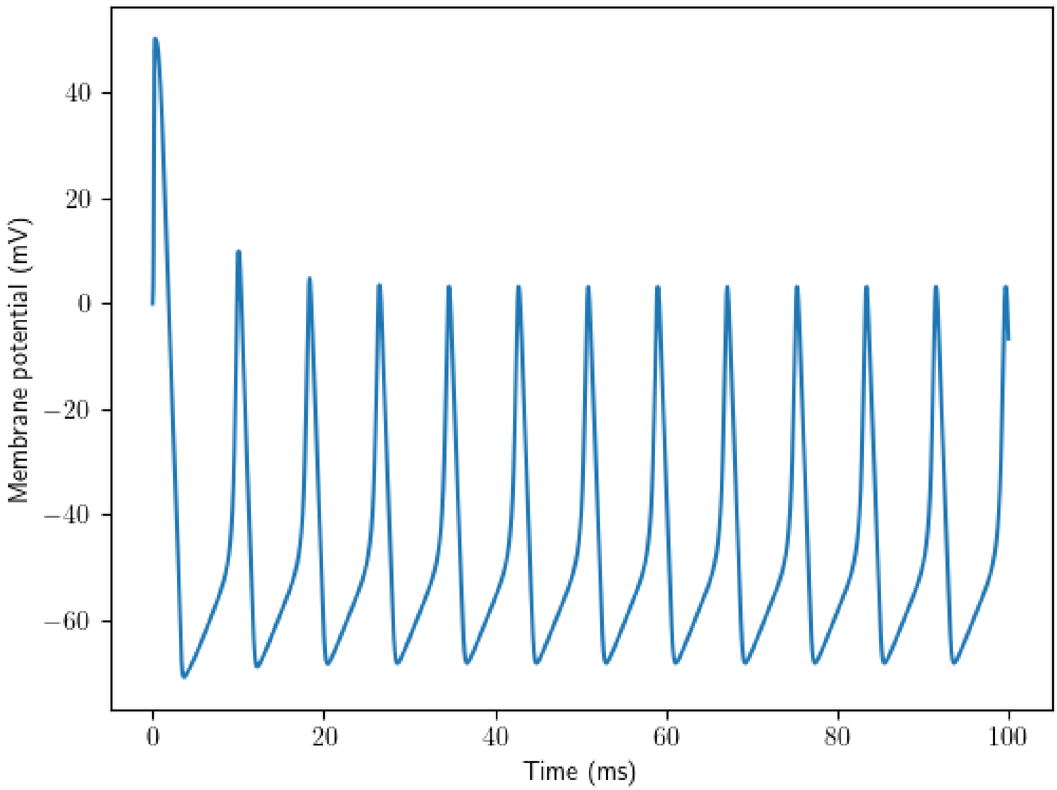

Figura 10. membrane potential changes with time($I = 58.0\mu A/cm^2$)

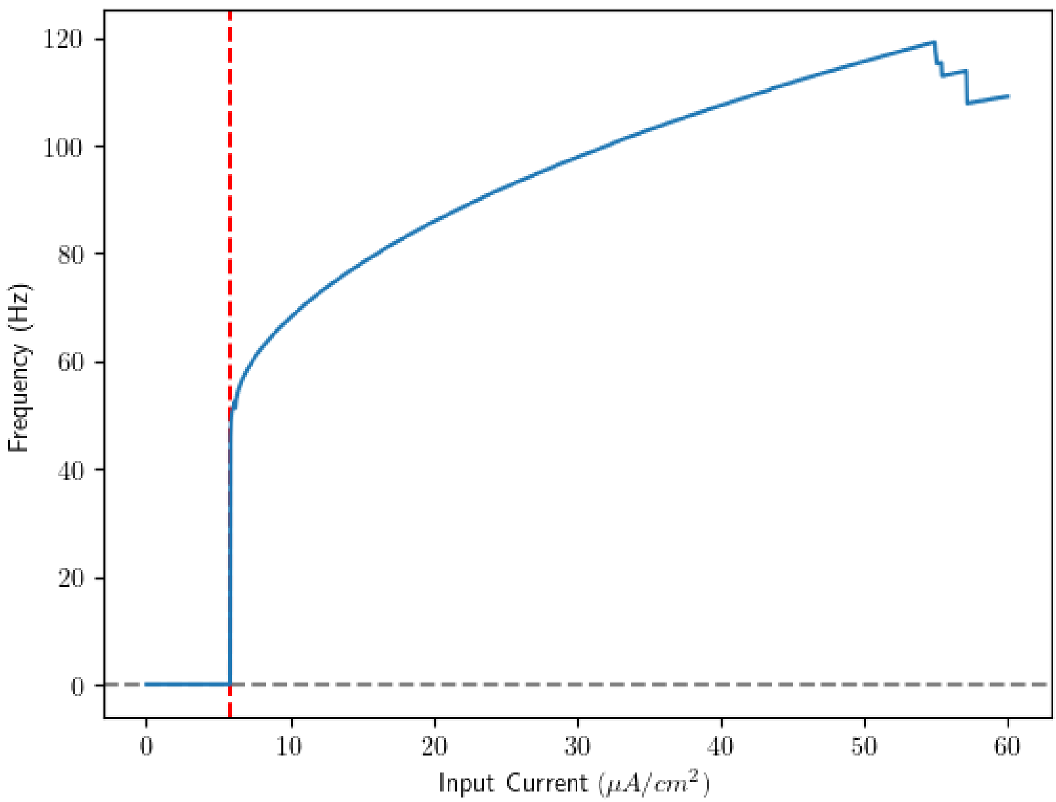

Figura 11. frequency changes with input current(peak detection changed)