CNeuroFinal

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1 How Granule Cells Sample Inputs

Assume each granule cell can choose K inputs out of all M mossy fibers, the number of possibilities is simply a binomial coefficient C_M^K.

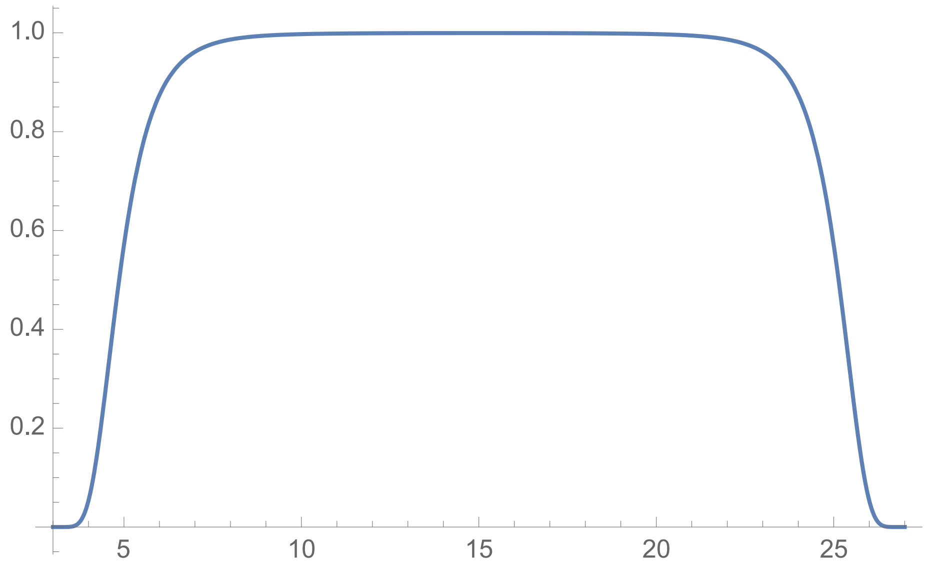
1.1 probability p that granule cells all receive different combinations of inputs

From C_M^K possibilities, one can choose $C_{C_M^K}^N N!$ different combinations of inputs for N granule cells.

Thus, the probability that granule cells all receive different combinations of inputs is

$$p = \frac{C_{C_M^K}^N N!}{(C_M^K)^N} = \left(1 - \frac{1}{C_M^K}\right)\left(1 - \frac{2}{C_M^K}\right) ... \left(1 - \frac{N-1}{C_M^K}\right) \tag{1}$$

1.2 plot p as a function of K and when p reaches its maximum



p is increasing with C_M^K and C_M^K reaches its maximun when $K = \frac{M}{2}$. Thus when $K = \frac{M}{2}$, p reaches its maximum.

1.3 compute K when p approaches 95 percent

$M = 7000, N = 21000$. If $K \geq 3$ then $N/C_M^K \ll 1$. By approximation $ln(1 - x) \approx -x(x \to 0)$, we have

$$p \approx \exp - \sum_{i=1}^{N-1} \frac{i}{C_M^K} = \exp - \frac{N(N-1)}{2C_M^K} \tag{2}$$

So if p approches 95% of its maximum, then K satisfies

$$\frac{1}{C_M^{M/2}} - \frac{1}{C_M^K} = ln(0.95)\frac{2}{N(N-1)} \tag{3}$$

By this formula we can find it is $K = 3$ when p approches 95% of its maximum.

1.4 whether it is beneficial to have small K when M is very large

It is benificial to have small K when M is very large. Because a very small K is enough for p to become very high. Large K is a waste of resource.

2 FitzHugh-Nagumo Model

2.1 (a)

Determine nullclines of the model (when $V' = 0$ or $w' = 0$) and draw the two-dimensional phase portrait (vector field) of the model using MATLAB.

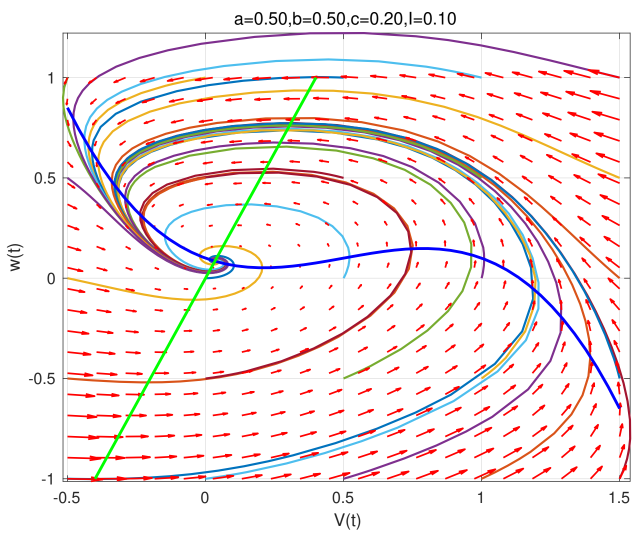
V nullcline:

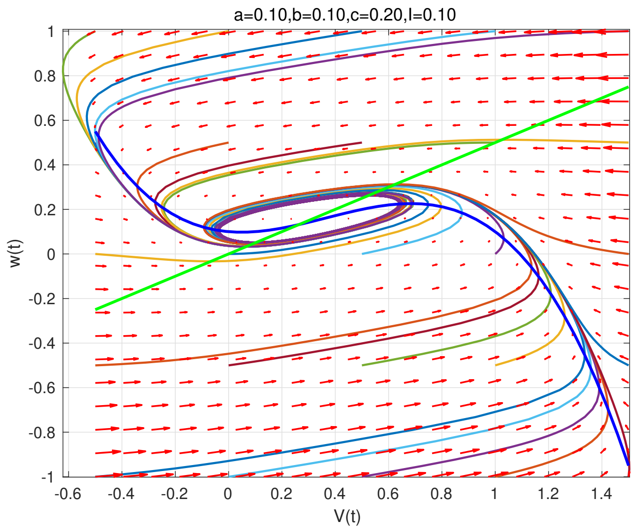
$$V(a - V)(V - 1) - w + I = 0$$

w nullcline:

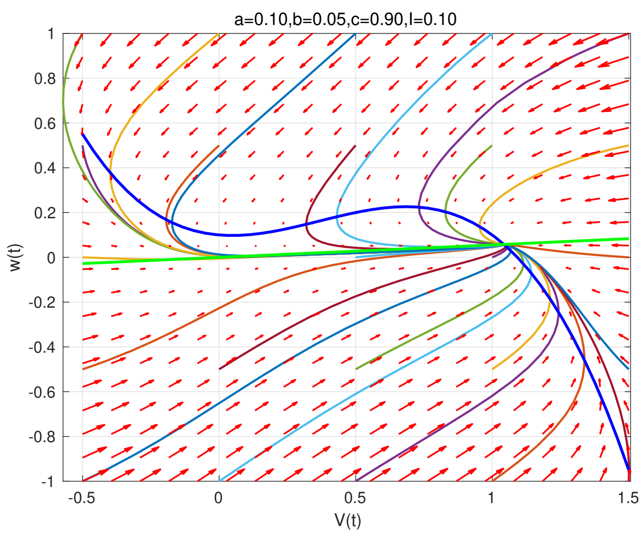
$$bV - cw = 0$$

A Matlab program named 'p2.m' is used to plot the phase portrait, nullclines and vector fields. Here are some sample results with various parameters.

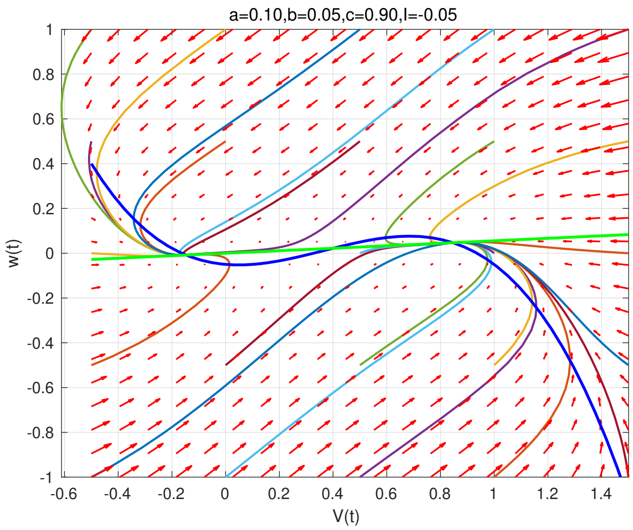




a=0.10,b=0.10,c=0.20,I=0.10
w(t)
V(t)



a=0.10,b=0.05,c=0.90,I=0.10
w(t)
V(t)



2.2 (b)

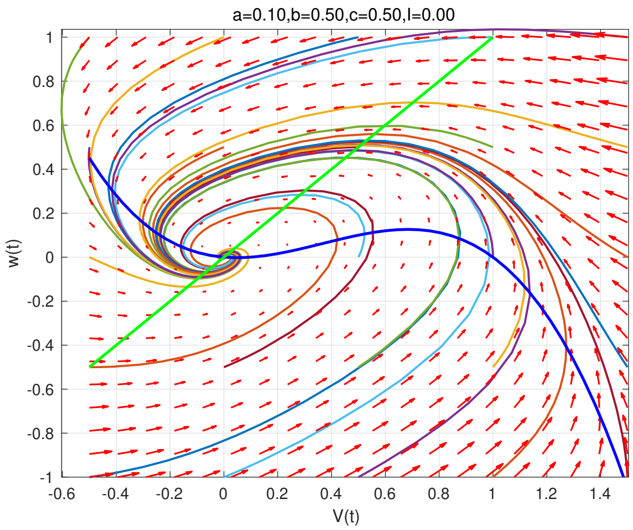
Analyze the stability of the fixed point (0, 0), and how they depends on the above parameters. Plot the phase diagram just like what we did in the class. What happens when the fixed point is not stable?

If $(0, 0)$ is a fixed point, then I must be 0. The linearization of the system at $(0, 0)$ becomes:

$$dV/dt = -aV - w$$
$$dw/dt = bV - cw$$

The eigenvalues of the coefficient matrix are negetive. So the fixed point $(0, 0)$ must be stable.

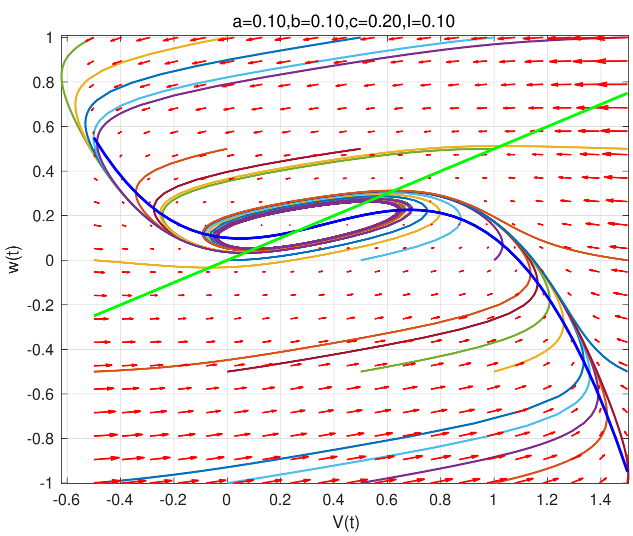
Here is the phase portrait:



2.3 (c)

Doing simulation and check when would the FitzHugh-Nagumo model generate oscillation?

We can find that here is a situation that there exists oscillation.



But I have no idea what are the all possibilities.

3 Oja's Rule

The Oja's Rule can be written into

$$\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}\mathbf{u}^\mathbf{T}\mathbf{w} - \mathbf{w}^\mathbf{T}\mathbf{u}\mathbf{u}^\mathbf{T}\mathbf{w}\mathbf{w} \tag{4}$$

Considering the average, it is

$$\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{C}\mathbf{w} - \mathbf{w}^\mathbf{T}\mathbf{C}\mathbf{w}\mathbf{w} \tag{5}$$

where $\mathbf{C}$ is the covariance matrix of $\mathbf{u}$.

Suppose that $\mathbf{w_0}$ is a fixed point, then

$$\mathbf{C}\mathbf{w_0} = (\mathbf{w_0^T}\mathbf{C}\mathbf{w_0})\mathbf{w_0} \tag{6}$$

Thus the fixed point w_0 must be an eigenvalue of $\mathbf{C}$, i.e. a principal componet.

Multiply $\mathbf{w_0^T}$ to each side of the equation, we can find that the norm of $\mathbf{w_0}$ is 1.

$$\mathbf{w_0^T}\mathbf{C}\mathbf{w_0} = (\mathbf{w_0^T}\mathbf{C}\mathbf{w_0})\mathbf{w_0^T}\mathbf{w_0} \tag{7}$$

Now let's prove that only the first principal component is a stable fixed point while others are not stable.

Suppose that $\mathbf{C}$ has n different eigenvalues $\lambda_1 > \lambda_2 > ... > \lambda_n > 0$, with eigenvectors $\mathbf{e_1}, \mathbf{e_2}, ..., \mathbf{e_n}$.

Denote $v_i = \mathbf{w} \cdot \mathbf{e_i}$. Then equation (5) becomes a differential equations system:

$$\tau_w \frac{dv_i}{dt} = \lambda_i v_i - (\lambda_1 v_1^2 + ... + \lambda_n v_n^2) v_i \tag{8}$$

To linearize the dynamical system around the first principal component, i.e. $(v_1, v_2, ..., v_n) = (1, 0, ..., 0)$, we compute the Jacobian matrix:

$$\begin{pmatrix} -2\lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 - \lambda_1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n - \lambda_1 \end{pmatrix} \tag{9}$$

$-2\lambda_1, \lambda_2 - \lambda_1, ..., \lambda_n - \lambda_1 < 0$, So the first principal component is stable.

To linearize the dynamical system around the other principal component, for example $(v_1, v_2, ..., v_n) = (0, 1, ..., 0)$, we compute the Jacobian matrix:

$$\begin{pmatrix} \lambda_1 - \lambda_2 & 0 & ... & 0 \\ 0 & -2\lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n - \lambda_2 \end{pmatrix} \tag{10}$$

$\lambda_1 - \lambda_2 > 0$, so the second principal component is not a stable fixed point.

Similarly, except the first principal componet, other fixed points are not stable.

4 Associative Memory with Decaying Memory Traces

Starting with $W_{ij=}0$, and use $(\lambda = e^{-\tau})$

$$W_{ij}^{t} = \lambda W_{ij}^{t-1} + \frac{1}{N}\xi^{\mu}\xi_j^{\mu} \tag{11}$$

We have

$$\begin{aligned}
W_{ij}^{0} &= \frac{1}{N}\xi_i^0\xi_j^0 \\
W_{ij}^{1} &= e^{-\tau}\frac{1}{N}\xi_i^1\xi_j^1 + \frac{1}{N}\xi_i^0\xi_j^0 \\
W_{ij}^{2} &= e^{-2\tau}\frac{1}{N}\xi_i^2\xi_j^2 + e^{-\tau}\frac{1}{N}\xi_i^1\xi_j^1 + \frac{1}{N}\xi_i^0\xi_j^0 \\
&\quad\ldots\ldots \\
W_{ij}^{p} &= \frac{1}{N}\sum_{\mu=0}^{p} e^{-\mu\tau}\xi_i^{\mu}\xi_j^{\mu}
\end{aligned}$$

We have renumbered each time a new memory was added, and the latest memory number is 0.

The fixed point condition for storing memory pattern ξ_{ν} requires that

$$\begin{aligned}
\xi_i^{\nu} &= \mathbf{sgn}\left(e^{-\nu\tau}\xi_i^{\nu} + \frac{1}{N}\sum_{j=1}^{N}\sum_{\mu=0,\mu\neq\nu}^{P} e^{-\mu\tau}\xi_i^{\mu}\xi_j^{\mu}\xi_j^{\nu}\right) \\
&= \mathbf{sgn}\left(\xi_i^{\nu} + \frac{1}{N}\sum_{j=1}^{N}\sum_{\mu=0,\mu\neq\nu}^{P} e^{\nu\tau}e^{-\mu\tau}\xi_i^{\mu}\xi_j^{\mu}\xi_j^{\nu}\right)
\end{aligned} \tag{12}$$

Multiply the crosstalk term by $-\xi_i^{\nu}$, to define:

$$C_i^{\nu} = -\frac{1}{N}\sum_{j=1}^{N}\sum_{\mu=0,\mu\neq\nu}^{P} e^{\nu\tau}e^{-\mu\tau}\xi_i^{\mu}\xi_j^{\mu}\xi_i^{\nu}\xi_j^{\nu} \tag{13}$$

If $C_i^{\nu} > 1$ then the node i would become unstable.

C_i^{ν} is a random variable with 0 mean value. And its variance is

$$\sigma^2 = \frac{e^{2\nu\tau}}{N}\frac{1-e^{-2p\tau}}{1-e^{-2\tau}}$$

Approximate its distribution by a normal distribution.

So the error probability

$$P_{err} = P(C_i^{\nu} > 1) = \frac{1}{2}(1 - erf(\frac{1}{\sqrt{2\sigma^2}})) \approx \frac{\sqrt{2\sigma^2}}{\sqrt{\pi}}e^{-\frac{1}{2\sigma^2}} \tag{14}$$

To make the error probability small enough, i.e. $P_{err} < 0.01/N$, we need (we have set $\nu = p$, means the oldest memory pattern)

$$\log(P_{err}) < \log(0.01) - \log N \tag{15}$$

i.e. (considering $N \gg p$, omit small terms and take leading term)

$$-\frac{1}{2\sigma^2} < -\log N \tag{16}$$

i.e. ($\tau \ll 1$)

$$e^{2\tilde{p}\tilde{\tau}} - 1 < \frac{N(1 - e^{-2\tau})}{2\log N} \approx \tilde{\tau} \tag{17}$$

where we set $\tilde{\tau} = \frac{\tau N}{\log N}$ and $\tilde{p} = \frac{\log N p}{N}$.

So the maximal value of $\tilde{p}$ is

$$\frac{\log(1 + \tilde{\tau})}{2\tilde{\tau}}$$

It goes back to $p < \frac{N}{2\log N}$ when $\tau \to 0$, the situation of the Hopfield network.

As shown in the figur below, $\frac{\log(1+\tilde{\tau})}{2\tilde{\tau}} < \frac{1}{2}$ when $\tau > 0$, which means the number of maximal memory pattern p is less than the Hopfield network.