Problem Set 4

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Rall's Cable equation

In the last lecture, we introduced to you the cable equation that describes the propagation of the signal in a dendrite. It reads

$$c_m \frac{\partial V}{\partial t} = \frac{a}{2\rho_L} \frac{\partial^2 V}{\partial x^2} - i_a(x,t) - i_e(x,t), \tag{1}$$

where c_m is the specific membrane conductance per unit area, i_a as the membrane conductance per unit area, and i_e is the external current per unit area.

In a linear approximation of the membrane current, we have

$$i_a = g_m(V - E_L),$$

where E_L is the resting potential of the neuron, and $g_m = 1/r_m$ is a constant membrane conductance per unit area, and r_m is specific membrane resistance. Now by introducing a new variable $v = V - E_L$, the cable equation can be simplified as

$$c_m r_m \frac{\partial v}{\partial t} = \frac{a r_m}{2\rho_L} \frac{\partial^2 v}{\partial x^2} - v - i_e(x,t) r_m.$$

Note that r_m has the dimension $\Omega{\cdot}\mathrm{mm}^2$, and ρ_L has the dimension $\Omega{\cdot}\mathrm{mm}$, and thus we could define a critical spatial scale λ, called electrotonic length,

$$\lambda = \sqrt{\frac{a r_m}{2\rho_L}} \tag{2}$$

Moreover, recall the membrane time constant

$$\tau_m = c_m r_m. \tag{3}$$

We could rewrite the cable equation as

$$\tau_m \frac{\partial v}{\partial t} = \lambda^2 \frac{\partial^2 v}{\partial x^2} - v - i_e(x,t) r_m. \tag{4}$$

- Solve a special case for 4 for an infinitely long cable by injecting a point current at point $x = 0$ that does not change over time. Find the steady state solution that is independent of time, that is,

$$\lambda^2 \frac{d^2 v}{dx^2} = v + i_e r_m$$

where

$$i_e = -\frac{I_e}{2\pi a}\delta(x)$$
$$v|_{x=\infty} = 0$$
$$v|_{x=-\infty} = 0$$

- We now consider the membrane potential produced by an instantaneous pulse current injected at $x = 0$ and $t = 0$. We shall derive an analytical solution for the evolution of membrane potential over time and space. More specifically, our pulse current has the following functional form

$$i_e = -\frac{I_e \tau_m}{2\pi a}\delta(x)\delta(t)$$

Again we have the following boundary conditions,

$$v|_{x=\infty} = 0; v|_{x=-\infty} = 0; v|_{t<0} = 0$$

the solution for membrane potential is given by

$$v(x,t) = \frac{I_e r_m}{2\pi a\sqrt{4\pi\lambda^2 t/\tau_m}} \exp\left(-\frac{x^2 \tau_m}{4\lambda^2 t}\right) \exp(-t/\tau_m). \tag{5}$$

Curious student can try to derive this formula, which is not easy. But our homework is easier. At a given position $x \neq 0$, plot the temporal profile of the membrane potential. Describe the physical meaning of Equation 5. In addition, is it possible to define an effective wave propagation speed for the membrane potential? If so, how precisely does the speed depend on τ_m and λ?