HW4 2024.1.3

1. A toy model of neural integrator

(a). 1/ 先来尝试一下用计算机控制基础的知识解答这问题！

$$\tau \frac{dx}{dt} = -ax - 2y + \tau V(t) \quad (1)$$

$$\tau \frac{dy}{dt} = -(3-a)x - y - \tau V(t) \quad (2)$$

x, y 是系统的两个状态变量 记 $\begin{bmatrix} x \\ y \end{bmatrix}$ 是系统的状态向量 $C(t)$

$$C(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

整理 (1) (2)，得

$$\dot{x}(t) = -\frac{a}{\tau}x - \frac{2}{\tau}y + V(t)$$

$$\dot{y}(t) = -\frac{3-a}{\tau}x - \frac{1}{\tau}y - V(t)$$

换言之，记 $V(t)$ 为输入（这很合理），

$$\dot{C}(t) = \begin{bmatrix} -\frac{a}{\tau} & -\frac{2}{\tau} \\ \frac{a-3}{\tau} & -\frac{1}{\tau} \end{bmatrix} C(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} V(t) \quad (3)$$

$x(t)$ 代表了位置，也即速度积分器的输出，则输出 $O(t)$ 有

$$O(t) = [1 \quad 0]\, C(t) \quad (4).$$

(3) (4) 即为系统的状态空间方程描述.

$V(t) = V_0$，意味着这是一个单位阶跃输入 $V_0 1(t)$

用系统的传递函数 $G(s)$ 来分析.

$$Y(s) = G(s) \cdot \frac{V_0}{s}$$

接下来，用 $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ 计算一般的 $G(s)$ 表示：

$$G(s) = [1\ 0]\left(sI - \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$sI - \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} s-\alpha & \beta \\ \gamma & s-\delta \end{bmatrix}$$

求逆：
$$\begin{vmatrix} s-\alpha & \beta \\ \gamma & s-\delta \end{vmatrix}^{-1} \cdot \begin{bmatrix} s-\delta & -\beta \\ -\gamma & s-\alpha \end{bmatrix}$$

$$= \frac{1}{(s-\alpha)(s-\delta) - \beta\gamma} \begin{bmatrix} s-\delta & -\beta \\ -\gamma & s-\alpha \end{bmatrix}$$

$$G(s) = [1\ 0] \frac{1}{s^2 - (\alpha+\delta)s + (\delta\alpha - \beta\gamma)} \begin{bmatrix} s-\delta & -\beta \\ -\gamma & s-\alpha \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{s^2 - (\alpha+\delta)s + (\delta\alpha - \beta\gamma)} [s-\delta\ \ -\beta] \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{s-\delta+\beta}{s^2 - (\alpha+\beta)s + (\delta\alpha - \gamma\beta)} = \frac{s-(\delta-\beta)}{s^2 - (\alpha+\beta)s + (\delta\alpha - \gamma\beta)}$$

① $\delta - \beta = -\frac{1}{\tau} + \frac{2}{\tau} = \frac{2}{\tau}$. ② $\alpha + \beta = -\frac{a+2}{\tau}$.

③ $\delta\alpha = \frac{a}{T^2}$. ④ $\gamma\beta = \frac{6-2a}{T^2}$

$$\delta\alpha - \gamma\beta = \frac{3a-6}{T^2}$$

$$G(s) = \frac{s - \frac{2}{T}}{s^2 + \frac{a+2}{T}s + \frac{3a-6}{T^2}}$$

$$Y(s) = \frac{V_0\left(s - \frac{2}{T}\right)}{\left(s^2 + \frac{a+2}{T}s + \frac{3a-6}{T^2}\right)s}$$

如果不动点具有稳定性，那么 $Y(s)$ 应当可以被分解为 $\frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \frac{k_3}{s}$ 的形式。让我们试一下！

$$s^2 + \frac{a+2}{T}s + \left(\frac{a+2}{2T}\right)^2 + \frac{3a-6}{T^2} - \left(\frac{a+2}{2T}\right)^2 = 0$$

$$\left(s + \frac{a+2}{2T}\right)^2 + \frac{3a-6}{T^2} - \frac{a^2+4a+4}{4T^2} = 0$$

$$\left(s + \frac{a+2}{2T}\right)^2 + \frac{-a^2+8a-28}{4T^2} = 0$$

$$\left(s + \frac{a+2}{2T}\right)^2 = \frac{a^2-8a+28}{4T^2}$$

$$s_1 = \frac{\sqrt{a^2-8a+28}}{2T} - \frac{a+2}{2T}$$

$$s_2 = -\frac{\sqrt{a^2-8a+28}}{2T} - \frac{a+2}{2T}$$

记 $\frac{\sqrt{a^2-8a+28}}{2T} = m$

分母：$\left[s+\left(\frac{a+2}{2\tau}-m\right)\right]\left[s+\left(\frac{a+2}{2\tau}+m\right)\right]s$

$$\frac{k_1}{s+\left(\frac{a+2}{2\tau}-m\right)}+\frac{k_2}{s+\left(\frac{a+2}{2\tau}+m\right)}+\frac{k_3}{s}=Y(s)$$

$$k_1\left[s+\left(\frac{a+2}{2\tau}+m\right)\right]s+k_2\left[s+\left(\frac{a+2}{2\tau}-m\right)\right]s+$$

$$k_3\left[s+\left(\frac{a+2}{2\tau}-m\right)\right]\left[s+\left(\frac{a+2}{2\tau}+m\right)\right]=V_0s-\frac{2}{\tau}V_0$$

$$k_1\left[s^2+\left(\frac{a+2}{2\tau}+m\right)s\right]+k_2\left[s^2+\left(\frac{a+2}{2\tau}-m\right)s\right]$$

$$+k_3\left[s^2+\frac{a+2}{\tau}s+\left(\frac{a+2}{2\tau}\right)^2-m^2\right]=V_0s-\frac{2}{\tau}V_0$$

$$\begin{cases} k_1+k_2+k_3=0 \\ \left(\frac{a+2}{2\tau}+m\right)k_1+\left(\frac{a+2}{2\tau}-m\right)k_2+\frac{a+2}{\tau}k_3=V_0 \\ k_3\left[\left(\frac{a+2}{2\tau}\right)^2-m^2\right]=-\frac{2}{\tau}V_0 \end{cases}$$

$$V_3=-\frac{2}{\tau}V_0\Big/\left[\left(\frac{a+2}{2\tau}\right)^2-m^2\right]$$

...

其实到此为止我们不必算下去了。只要 $\frac{a+2}{2\tau}-m$ 和 $\frac{a+2}{2\tau}+m$ 都小于0，

系统输出最终都是稳定的。

$$\begin{cases} \frac{a+2}{2\tau}-m<0 \\ \frac{a+2}{2\tau}+m<0 \end{cases} \Rightarrow \text{m为正数，} \quad \frac{a+2}{2\tau}<-m$$

$$-\frac{\sqrt{a^2-8a+28}}{2\tau} > \frac{a+2}{2\tau}$$

先考虑临界情况

$$a^2-8a+28=(a+2)^2$$

$$a^2-8a+28=a^2+4a+4$$

$$24=12a$$

$$a=2$$

难以解释为什么会这样，也许我有计算错误；或者，由于不动点附近的稳定性与系统在无穷大处的行为极限并无关系。

让我们重新开始！

$$\begin{cases}\frac{dx}{dt}=0\\ \frac{dy}{dt}=0\end{cases} \Rightarrow \begin{cases}-ax-2y+\tau U_0=0 & ①\\ -(3-a)x-y-\tau U_0=0 & ②\end{cases}$$

① + ②

$$-3x-3y=0 \Rightarrow x=-y. \quad (a-2)y+\tau U_0=0 \Rightarrow y=\frac{\tau U_0}{2-a}$$

$$x=\frac{\tau U_0}{a-2}$$

若在不动点处稳定，由 Jacobi 矩阵，

$$J=\begin{bmatrix}-a & -2\\ a-3 & -1\end{bmatrix}$$

特征方程： $\det(\lambda I - J)$

$$= \begin{vmatrix} \lambda+a & 2 \\ 3-a & \lambda+1 \end{vmatrix}$$

$$= (\lambda+a)(\lambda+1) - 2(3-a)$$

$$= \lambda^2 + (1+a)\lambda + a - 6 + 2a$$

$$= \lambda^2 + (1+a)\lambda + (3a-6)$$

特征值都要位于左半平面.

由劳斯判据，有

$$\begin{array}{lll} \lambda^2 & 1 & 3a-6 \\ \lambda^1 & 1+a & \\ \lambda^0 & 3a-6. & \end{array}$$

$$a > 2$$

令人惊喜的结果！

2/ 当初始条件为松弛条件时，解方程：

$$\begin{cases} T\dfrac{dx}{dt} = -ax - 2y + TV_0 \\ T\dfrac{dy}{dt} = -(3-a)x - y - TV_0 \end{cases}$$

$$\tau \frac{dy}{dt} + y = -(3-a)x - \tau V_0$$

$$\frac{dy}{dt} + \frac{1}{\tau} y = \frac{a-3}{\tau} x - V_0$$

$$\frac{d}{dt}(e^{\frac{1}{\tau}t} y) = e^{\frac{1}{\tau}t}\left(\frac{a-3}{\tau} x - V_0\right)$$

$$e^{\frac{1}{\tau}t} y(t) = \int_0^t e^{\frac{1}{\tau}k}\left(\frac{a-3}{\tau} x - V_0\right) dk$$

$$= \left(\frac{a-3}{\tau} x - V_0\right) \int_0^t e^{\frac{1}{\tau}k} dk$$

$$= \left(\frac{a-3}{\tau} x - V_0\right) \tau e^{\frac{1}{\tau}k}\Big|_0^t$$

$$e^{\frac{t}{\tau}} y(t) = \left(\frac{a-3}{\tau} x - V_0\right)\left(\tau e^{\frac{t}{\tau}} - \tau\right)$$

$$y(t) = \left(\frac{a-3}{\tau} x(t) - V_0\right)\left(\tau e^{\frac{t}{\tau}} - \tau\right) e^{-\frac{t}{\tau}}$$

$$\tau \frac{dx}{dt} = -ax - 2\left(\frac{a-3}{\tau} x(t) - V_0\right)\left(1 - e^{-\frac{t}{\tau}}\right)\tau + \tau V_0$$

$$\frac{dx}{dt} = -\frac{a}{\tau} x - 2\left(\frac{a-3}{\tau} x(t) - V_0\right)\left(1 - e^{-\frac{t}{\tau}}\right) + V_0$$

$$= -\frac{a}{\tau} x - 2\left(1 - e^{-\frac{t}{\tau}}\right)\frac{a-3}{\tau} x(t) + 2V_0\left(1 - e^{-\frac{t}{\tau}}\right) + V_0$$

$$= \left\{-\frac{a}{\tau} - 2\left(1 - e^{-\frac{t}{\tau}}\right)\frac{a-3}{\tau}\right\} x(t) + 3V_0 - 2V_0 e^{-\frac{t}{\tau}}$$

$$= \left\{-\frac{a}{\tau} - \frac{2a-6}{\tau} + \frac{2a-6}{\tau} e^{-\frac{t}{\tau}}\right\} x(t) + 3V_0 - 2V_0 e^{-\frac{t}{\tau}}$$

$$= \left\{\frac{6-3a}{\tau} + \frac{2a-6}{\tau} e^{-\frac{t}{\tau}}\right\} x(t) + 3V_0 - 2V_0 e^{-\frac{t}{\tau}}$$

$$\frac{dx}{dt} + \left\{\frac{3a-6}{\tau} + \frac{6-2a}{\tau} e^{-\frac{t}{\tau}}\right\} x(t) = 3V_0 - 2V_0 e^{-\frac{t}{\tau}}.$$

Laplace:

$$sX(s) + \left[\frac{3a-6}{\tau s} + \frac{6-2a}{\tau(s+\frac{1}{\tau})}\right] X(s) = 3V_0\frac{1}{s} - 2V_0\frac{1}{s+\frac{1}{\tau}}.$$

$$X(s)\frac{s\cdot\tau s\cdot(\tau s+1) + (3a-6)s(\tau s+1) + (6-2a)\tau s^2}{\tau s(\tau s+1)} = \frac{3V_0}{s} - \frac{2V_0\tau}{\tau s+1}$$

$$= \frac{3V_0\tau s + 3V_0 - 2V_0\tau s}{s(\tau s+1)}$$

$$X(s)\frac{\tau^2 s^3 + \tau s^2 + (3a-6)(\tau s^2+s) + (6-2a)\tau s^2}{\tau s(\tau s+1)} = \frac{V_0\tau s + 3V_0}{s(\tau s+1)}$$

$$X(s)\frac{\tau^2 s^3 + (a+1)\tau s^2 + (3a-6)s}{\tau s(\tau s+1)} = X(s)\frac{\tau^2 s^2 + (a+1)\tau s + (3a-6)}{\tau(\tau s+1)}$$

$$X(s)\frac{\tau^2 s^2 + (a+1)\tau s + (3a-6)}{\tau} = \frac{V_0\tau s + 3V_0}{s}$$

$$X(s) = \frac{\tau V_0(\tau s+3)}{s[\tau^2 s^2 + (a+1)\tau s + (3a-6)]}$$

(不打算了).

(b). $V(t) = 0.1$

$\Rightarrow V_0 = 0.1$

$I = 0.1$

那么 $$Y(s) = \frac{0.1(s-20)}{[s^2+10(a+2)s+100(3a-6)]s}$$

若 10s 后稳态误差为0，将条件放松为 t趋于无穷时的状况.*

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0^+} sY(s) = \frac{-2}{100(3a-6)} = 1$$

$100(3a-6) = -2$

$3a-6 = -0.02$

$3a = 5.98$

$a = \frac{598}{300}$. 但它比2小，说明不存在这样的a.

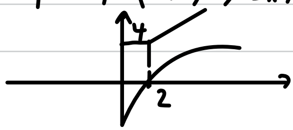
同理，若 $\frac{-2}{100(3a-6)} \in [0.99, 1.01]$

$a \in [2, 2.00673]$

若a超出这个范围，比2小会不稳定，比 2.00673 大会使误差变大.

*. 只要系统的阻尼系数 $\xi \geq 1$，我们的这个放松就是有效的，

即 $10(a+2) \geq 2\times 10\sqrt{3a-6} \Rightarrow a+2 \geq 2\sqrt{3a-6}$.



这个条件总成立.

2. Hopfield model

(1). $\langle \cos\psi \rangle = \frac{1}{N} \left\langle \sum_i V_i^{(1)} V_i^{(2)} \right\rangle$

$= \langle V_i^{(1)} V_i^{(2)} \rangle$

$= 0$

$Var(\cos\psi) = \frac{1}{N} \cdot N \cdot Var(V_i^{(1)} V_i^{(2)})$

$= 1$

(2). $S_i(t+\Delta t) = sgn\left(\sum_j W_{ij} S_j\right)$ 由于 $p \ll N$，所以可以视为随机选择的.

$W_{ij} = \frac{1}{N} \sum_{\mu=}^{p} \xi_i^\mu \xi_j^\mu$

$E(w_{ij}) = \frac{p}{N} \quad E(\xi_i \xi_j) = 0$

$S_i(t+\Delta t) = sgn\left(\sum_j E(w_{ij}) S_j\right) = 1$

ξ 是一个不动点.

(3). 要证 ξ^{mix} 是不动点，只要证 $S_i(t+\Delta t) = S_i(t)$ 即可.

$S_i(t+\Delta t) = sgn\left(\sum_j w_{ij} S_j\right)$

$W_{ij} = \frac{1}{N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu$

$$S_i(t+\Delta t) = \mathrm{sgn}\left(\frac{1}{N}\sum_{\mu=1}^{p}(\xi_{1i}^{\mu}+\xi_{2i}^{\mu}+\xi_{3i}^{\mu})(\xi_{1j}^{\mu}+\xi_{2j}^{\mu}+\xi_{3j}^{\mu})\right)$$

由于奇数个二值变量相加 其绝对值至少为1，

故而 S_i 的符号完全由 ξ_i^{mix} 决定.（奇数个 ξ 之间的影响互相抵消）

故而 $S_i(t+\Delta t)$ 是不动点.

(4). 没时间答了.