Final Exam of CNeuro2023

Due Jan 25, 2024

There are 4 problem sets, which cover virtually every topic we have discussed in this course. * indicates that this part has not been covered by our lectures. But solving it would give you extra credit.

How Granule Cells Sample Inputs

We have discussed about the synaptic organization of the cerebellum. Mossy fibers, which are long range projections from various brain regions, make connections with granule cells, the most numerous ($\sim 10^{11}$) neurons in the whole brain. These granule cells have a very small convergence: each of which only receives inputs from a handful mossy fibers. The outputs of granule cells, called parallel fibers, travel along the cerebellar cortex for a few millimeters and synapse onto Purkinje dendrites. Purkinje cells have the highest convergence in the brain, each of which receives inputs from more than 10^5 synapses from parallel fibers. The granule-to-Purkinje projections are what we called in the class "Connecting dense array to sparse array with extreme convergence and divergence".

Why do we need so many granule cells? What can we say about the number of granule cells N, the number of mossy fiber inputs M, and the convergence of a granule cell K? Perhaps each granule cell is sampling a different combination of mossy fiber inputs. The higher the functional diversity, the more powerful computation downstream circuits (e.g., Purkinje dendrites) could perform, such as classification.

Assume each granule cell can choose K inputs out of all M mossy fibers, the number of possibilities is simply a binomial coefficient $\binom{M}{K}$. Now we ask the following questions.

- What is the probability p that granule cells all receive different combinations of inputs?
- For given N and M, plot p as a function of K, and show when p reaches its maximum.
- Using $N = 21000$, $M = 7000$, compute K when p approaches 95 percent of its maximum.

- Discuss whether it is beneficial to have small K when M is very large.

FitzHugh-Nagumo Model

The FitzHugh-Nagumo model

$$\dot{V} = V(a-V)(V-1) - w + I, \tag{1}$$
$$\dot{w} = bV - cw, \tag{2}$$

imitates H-H models or the $I_{Na,p} + I_K$-model by having cubic (N-shaped) nullclines, where parameter $0 < a < 1$ describes the shape of the cubic parabola $V(a-V)(V-1)$, and $b > 0$, $c \geq 0$ describe the kinetics of the recovery variable w. I is the external current.

(a*) Determine nullclines of the model (when $\dot{V} = 0$ or $\dot{w} = 0$) and draw the two-dimensional phase portrait (vector field) of the model using MATLAB.

(b*) Please analyze the stability of the fixed point $(0, 0)$, and how they depends on the above parameters. plot the phase diagram just like what we did in the class. What happens when the fixed point is not stable?

(c) Doing simulation and check when would the FitzHugh-Nagumo model generate oscillation?

(d) Now imagine that we have two such FitzHugh-Nagumo neurons and they are electrically coupled, namely

$$\dot{V_1} = V_1(a-V_1)(V_1-1) - w_1 + I + g(V_2 - V_1), \tag{3}$$
$$\dot{w}_1 = bV_1 - cw_1, \tag{4}$$
$$\dot{V_2} = V_2(a-V_2)(V_2-1) - w_2 + I + g(V_1 - V_2), \tag{5}$$
$$\dot{w}_2 = bV_2 - cw_2, \tag{6}$$

Under what conditions would the two neurons oscillate synchronously? Or do they always oscillate synchronously when $g > 0$?

Oja's rule

We briefly mentioned in the class a modifcation of the basic Hebb rule that dictates how the synaptic weights change with the pre- and post-synaptic neural activity, known as the Oja's rule:

$$\tau_w \frac{d\mathbf{w}}{dt} = r\mathbf{u} - r^2\mathbf{w}, \tag{7}$$

where r is the postsynaptic neural activity and $\mathbf{u}$ is the vector representing presynaptic neural activities such that

$$r = \mathbf{w}^T\mathbf{u} \tag{8}$$

Let us define the covariance matrix of the inputs $\mathbf{u}$ as

$$\mathbf{C} = \langle \mathbf{u}\mathbf{u}^T \rangle$$

Prove that at the fixed point of Eq. 7, (i) the stable weight vector is the *principal component* of the covariance matrix. If you don't know what is PCA, it is now time to learn it. (ii) the weight vector has a unit norm.

Hint:

- For simplicity, assume all eigenvectors are distinct and are positive.
- Linearize the dynamical system around the fixed point.
- Rewrite in the basis of eigenvectors
- Show that all fixed points except the principal eigenvectors is linearly unstable.

Associative Memory with Decaying Memory Traces

Consider a network of binary neurons with the dynamics similar to that of the Hopfield Model.

Network dynamics: The value of each neuron is represented by $S_i = \pm 1$. The input function to each neuron is defined by $h_i(t)$, which stands for the weighted sum of the values of the rest of the neurons in the networks, weighted by the connection matrix W:

$$h_i(t) = \sum_{j=1}^{N} W_{ij} S_j(t) \tag{9}$$

At each time step a single neuron is chosen and its state is updated according to the following rule:

$$S_i(t + \Delta t) = \mathbf{sgn}(h_i(t)) \tag{10}$$

Connectivity Matrix: The memories are stored in a temporal sequence by updating the weight matrix W_{ij} dynamically. We start with $W_{ij} = 0$. As time proceeds, we add a new stored pattern ξ^μ to the existing memory while allowing the other memories to decay exponentially:

$$W_{ij}^t = \lambda W_{ij}^t + \frac{1}{N} \xi_i^\mu \xi_j^\mu \tag{11}$$

($W_{ii} = 0$). We let $\lambda = e^{-\tau}, \tau > 0$. (This kind of memory is dubbed a *palimpsest* — papyrus that has been written on more than once, with the earlier writing incompletely erased and often legible.)

- Show that the connection matrix can be written as:

$$W_{ij}^p = \frac{1}{N} \sum_{\mu=0}^{p} e^{-\mu t} \xi_i^\mu \xi_j^\mu, i \neq j \tag{12}$$

- Consider a steady state situation where the time step p approaches infinity. Estimate how many past memories we can store, assuming all ξ_i^μ are drawn independently with equal probability of ± 1.

Hint: To estimate the memory capacity, use a 'signal-to-noise' analysis similar to that presented in the lecture for the Hopfield Model. A pattern ξ^μ is stable if for all neurons $\xi_i^\mu h_i^\mu > 0$.

1. Derive an expression for the mean and variance of this quantity.

2. Assuming that N and p large, approximate the distribution of $\xi_i^p h_i^p$ by a normal distribution. Find the maximum p for which the probability of error (instability of the stored pattern) is small. Note: For the sake of calculation, it is useful to define new variables $\bar{\tau}$ and $\bar{p}$ with $\tau = \frac{\bar{\tau} \ln N}{N}$, $p = \frac{\bar{p} N}{\ln N}$, (so that in particular p is expected to be large and τ is expected to be small) to capture the expected dependence on N.

3. Plot the maximal value of $\bar{p}$ as a function of $\bar{\tau}$ and discuss the results and their implications on the performance of this model in comparison to the Hopfield model.