Problem Set 1

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Hassentein-Reichardt correlator

For a grating stimulus with defined spatial frequency ($k = 2\pi/\lambda$) and temporal frequency ω_0, the light intensity signal received by two neighboring channels (i.e., two photoreceptors) have the following form:

$$\begin{aligned} s_1(t) &= \Delta I \sin(\omega_0 t) = \mathrm{Im}\left[\Delta I e^{i\omega_0 t}\right]; \\ s_2(t) &= \Delta I \sin(\omega_0 t - k\Delta x) = \mathrm{Im}\left[\Delta I e^{i(\omega_0 t - k\Delta x)}\right]. \end{aligned} \tag{1}$$

In the simplest model, we can think that the response of a neuron is a low-pass filter of the sensory input with some Kernel $D_1(t)$ and $D_2(t)$. As a result, the response function might be written as

$$\begin{aligned} r_1(t) &= \int_{-\infty}^{\infty} s_1(t-\tau) D_1(\tau) d\tau; \\ r_2(t) &= \int_{-\infty}^{\infty} s_2(t-\tau) D_2(\tau) d\tau. \end{aligned} \tag{2}$$

Similar responses could be written for $r_3(t)$ and $r_4(t)$. The motion detection output signal is defined as

$$R(t) = r_1(t) r_2(t) - r_3(t) r_4(t) \tag{3}$$

And the steady state solution is given by averaging over the time period $2\pi/\omega_0$.

- Show that the general response has the following functional form*:

  $$\langle R \rangle_t = \sin[\phi_1(\omega_0) - \phi_2(\omega_0)] \sin(k\Delta x) \|\tilde{D_1}(\omega_0)\| \|\tilde{D_2}(\omega_0)\| \tag{4}$$

  where the fourier transform of the kernels are defined as

  $$\tilde{D_1}(\omega_0) = \|\tilde{D_1}(\omega_0)\| e^{i\phi_1(\omega_0)},$$

  $$\tilde{D_2}(\omega_0) = \|\tilde{D_2}(\omega_0)\| e^{i\phi_2(\omega_0)}$$

- Consider a simple kernel $D_1(t) = \frac{1}{\tau}\exp(-t/\tau)$, and $D_2(t) = \delta(t)$, we find $\tilde{D_1}(\omega_0) = \frac{1}{1+i\omega_0\tau}$, and show that

$$\langle R \rangle = \frac{\omega_0 \tau}{\omega_0^2\tau^2 + 1} \tag{5}$$

  Furthermore, show that this function has a maximum when $\omega_0 = 1/\tau$.

- If the filters on both arms are first-order low-pass, so that $D_1(t) = \frac{1}{\tau}\exp(-t/\tau_1)$, $D_2(t) = \frac{1}{\tau_2}\exp(-t/\tau_2)$, then the steady state response is given by

$$\langle R \rangle = \frac{\omega(\tau_2 - \tau_1)}{(1+\omega^2\tau_1^2)(1+\omega^2\tau_2^2)} \tag{6}$$

Hint: The analytical form of $r_1(t)$ and $r_2(t)$ can be computed by taking the fourier transform of the convolution, and then performing an inverse fourier transform. As a first step:

$$\begin{aligned} \tilde{r_1}(\omega) &= \sqrt{2\pi}\delta(\omega - \omega_0)\tilde{D_1}(\omega); \\ \tilde{r_2}(\omega) &= e^{-ik\Delta x}\sqrt{2\pi}\delta(\omega - \omega_0)\tilde{D_2}(\omega) \end{aligned} \tag{7}$$

Note that problem with * is optional.